Holonomy fields and random matrices : invariance by braids and permutations

Franck Gabriel

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Pierre et Marie Curie

Laboratoire de Probabilités
et Modèles Aléatoires

École Doctorale de Science Mathématiques de Paris Centre

THÈSE DE DOCTORAT

Discipline : Mathématiques

présentée par

Franck Gabriel

Champs d'holonomies et matrices aléatoires : symétries de tressage et de permutation.

---

dirigée par Thierry LÉVY

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Résumé

Cette thèse porte sur plusieurs questions liées aux mesures de Yang-Mills planaires et aux champs markoviens d'holonomies planaires. Les problèmes sont de deux sortes : étude des champs markoviens d'holonomies planaires pour un groupe de structure donné et l'étude asymptotique des mesures de Yang-Mills lorsque la dimension du groupe tend vers l'infini.

Dans le chapitre nommé "Champs markoviens d'holonomies", on définit la notion de champs markoviens d'holonomies planaires qui axiomatise la notion de mesures de Yang-Mills planaires. En utilisant une nouvelle symétrie en théorie des probabilités, l'invariance par tresse, on construit, caractérise et classifie les champs markoviens d'holonomies planaires. En particulier, nous montrons que tout champ markovien d'holonomies planaire est associé à un processus de Lévy qui satisfait une condition de symétrie et vice-versa. Ceci nous permet de caractériser, pour les surfaces sphériques, les champs markoviens d'holonomies tels que définis précédemment par Thierry Lévy.

Lorsque le processus de Lévy est à valeurs dans le groupe symétrique $\mathfrak{S}(N)$ on peut construire le champ markovien d'holonomies planaire associé grâce à un modèle de revêtements aléatoires. On s'intéresse alors à la convergence des monodromies de ce revêtement aléatoire dans le chapitre nommé "Revêtements ramifiés" en s'appuyant sur l'étude générale de l'asymptotique des matrices aléatoires invariantes par conjugaison par le groupe symétrique développée dans les chapitres nommés "Partitions et géométrie" et "Matrices aléatoires invariantes par le groupe symétrique". Ceci permet d'étendre les techniques développées par Thierry Lévy, pour l'étude de la mesure de Yang-Mills sur le groupe unitaire en grande dimension, afin d'étudier la mesure de Yang-Mills sur le groupe des permutations en grande dimension.

Les chapitres de thèse "Partitions et géométrie" et "Matrices aléatoires invariantes par le groupe symétrique", dans lesquels sont étudiées les matrices aléatoires invariantes par conjugaison par des sous-groupes du groupe unitaire, peuvent se lire indépendemment du reste de la thèse.

Mots-clefs : Mesures de Yang-Mills, Champs d'holonomies, Processus de Lévy, Symétries, Groupe unitaire, Groupe de permutations, Matrices aléatoires, Dualité de Schur-Weyl-Jones, Grande dimension, Partitions, Combinatoire, Liberté asymptotique, Cumulants, Revêtements ramifiés.

Abstract

This thesis focuses on planar Yang-Mills measures and planar Markovian holonomy fields. We consider two different questions : the study of planar Markovian holonomy fields with fixed structure group and the asymptotic study of the planar Yang-Mills measures when the dimension of the structure group grows.

In the chapter called "Champs markoviens d'holonomies", we define the notion of planar Markovian holonomy fields which generalizes the concept of planar Yang-Mills measures. We construct, characterize and classify the planar Markovian holonomy fields by introducing a new symmetry : the invariance under the action of braids. In particular, this shows that there is a bijection between planar Markovian holonomy fields and some equivalent classes of Lévy processes. Finally, we use these results in order to characterize Markovian holonomy fields on spherical surfaces.

The Markovian holonomy fields with $\mathfrak{S}(N)$ structure group can be constructed using random ramified coverings. We prove in the chapter "Revêtements ramifiés" that the monodromies of these models of random ramified coverings converge as the number of sheets of the covering goes to infinity. To prove this, we develop general tools in order to study the limits of families of random matrices invariant by the symmetric group. These tools can be found in the chapter "Partitions et géométrie" and in the chapter "Matrices aléatoires invariantes par le groupe symétrique". This allows us to generalize ideas, developped by Thierry Lévy in order to study the planar Yang-Mills measure with $U(N)$ structure group, to the setting where the structure group is $\mathfrak{S}(N)$.

The chapters "Partitions et géométrie" and "Matrices aléatoires invariantes par le groupe symétrique", in which we study random matrices invariant by conjugation by some subgroups of the unitary group, can be read independently from the other chapters.

Keywords : Yang-Mills measures, Holonomy fields, Lévy processes, Symmetries, Unitary group, Permutation group, Random matrices, Schur-Weyl-Jones duality, High dimension, Partitions, Combinatorics, Asymptotic freeness, Cumulants, Ramified coverings.

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“Rien ne sert de tirer sur un brin d’herbe pour le faire pousser.”

Proverbe zen,
via L. Capriglione

INTRODUCTION

Cette thèse admet trois fils directeurs et donc trois lectures différentes : l'étude des mesures de Yang-Mills bidimensionelles, l'étude des processus de Lévy matriciels et l'étude de certaines symétries en probabilité. Ces trois fils directeurs possèdent chacun deux aspects : l'étude fini-dimensionelle s'opposant à l'étude en grande dimension pour les deux premiers, une symétrie géométrique et une symétrie de jauge pour le dernier.

Les chapitres sont écrits en anglais puisqu'ils ont vocation à être publiés. En appendice, nous avons inclus, en guise de digestif, une présentation d'une preuve du théorème de Cayley-Hamilton basée sur le même type d'objets que ceux étudiés dans cette thèse. Ayant conscience d'une part de la longueur de cette thèse, qui pourrait rebuter certains lecteurs, d'autre part du caractère peu courant de certains concepts, l'accent est mis dans cette introduction sur l'acquisition d'une intuition qui facilitera la lecture des articles. Bien que cette introduction constitue un essai de vulgarisation des travaux présentés dans cette thèse en adoptant un point de vue personnel sur les sujets abordés, nous donnerons tout de même une idée précise des résultats principaux[(1)] obtenus.

[(1)] En particuliers le lecteur peut se reporter à la dernière section de cette introduction pour une liste plus exhaustive des résultats.

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1. Processus de Lévy

Considérons G un groupe de Lie compact, et soit e l'élément neutre de G. On munit G d'une métrique Riemannienne bi-invariante d_G si la dimension de G est strictement positive. Si G est un groupe fini, on le munit de la distance $d_G(x, y) = \delta_{x,y}$. Par exemple, on peut considérer $G = U(1)$ le groupe des nombres complexes de module égal à 1, ou $G = SO(3)$ l'ensemble des rotations de l'espace tridimensionnel. Il est naturel de s'intéresser à des processus aléatoires dans G indexés par le temps, c'est-à-dire des familles de variables aléatoires indexées par les réels positifs, à valeurs dans G. Par exemple un processus à valeurs dans $U(1)$, indexé par les réels positifs représente-t-il le déplacement aléatoire d'une particule sur un cercle. Dans cette thèse, on s'intéresse particulièrement aux processus de Lévy.

1.1. Définition et exemples. — Les processus de Lévy sont des processus dont les incréments vérifient certaines propriétés d'indépendance et de stationnarité. En reprenant l'image de la particule sur le cercle, celle-ci se déplace alors indépendamment de son passé et toujours avec le même comportement aléatoire.

Le groupe G n'étant pas nécessairement commutatif, il existe deux définitions d'incréments. Si $(X_t)_{t\geq 0}$ est un processus à valeurs dans G, si $s < t$ sont deux réels positifs, l'incrément à droite de $(X_t)_{t\geq 0}$ sur l'intervalle $[s, t]$ est la donnée de $X_s^{-1}X_t$, tandis que l'incrément à gauche de $(X_t)_{t\geq 0}$ sur l'intervalle $[s, t]$ est la donnée de $X_tX_s^{-1}$.

Définition 1.1. — *Un processus de Lévy (à droite ou à gauche) $(X_t)_{t\geq 0}$ à valeurs dans G est un processus càdlàg, tel que $X_0 = e$ et dont les accroissements (respectivement à gauche ou à droite) sont indépendants et stationnaires :*

- indépendance *: pour tous $0 < t_1 < ... < t_n$, les incréments (respectivement à gauche ou à droite) sur les intervalles $[0, t_1], ..., [t_{n-1}, t_n]$ sont indépendants,*
- stationnarité *: pour tous $s < t$, les incréments (respectivement à gauche ou à droite) sur les intervalles $[0, t-s]$ et $[s, t]$ ont même loi.*

L'ensemble des processus de Lévy (à droite ou à gauche) est en bijection avec l'ensemble des semi-groupes continus de convolution $(\mu_t)_{t\geq 0}$ tels que $\lim_{t\to 0} \mu_t = \mu_0 = \delta_0$.

En réalité, nous n'avons besoin que de la continuité stochastique du processus $(X_t)_{t\geq 0}$ puisqu'alors il en existera une version càdlàg. De plus, la distinction entre processus de Lévy à gauche ou à droite n'est pas d'une grande importance pour la suite : nous oublierons donc de spécifier si le processus est à gauche ou à droite.

Un exemple de processus de Lévy est donné par le processus brownien sur $U(1)$ qui modélise le déplacement sans saut d'une particule sur le cercle.

Exemple 1.1. — *Il existe un processus gaussien réel $(B_t)_{t\geq 0}$, presque sûrement continu, appelé mouvement brownien réel, tel que pour tous réels $0 \leq s \leq t$, $\mathsf{Cov}(B_s, B_t) = s$ et pour tout réel $t \geq 0$, $\mathbb{E}[B_t] = 0$. Le processus $\left(U_t = e^{iB_t}\right)_{t\geq 0}$ est un processus brownien sur $U(1)$. Remarquons que $(U_t)_{t\geq 0}$ peut être défini par l'équation différentielle stochastique au sens d'Itô donnée par :*

$$\begin{cases} dU_t &= i(dB_t)U_t - \frac{1}{2}U_t dt, \\ U_0 &= 1, \end{cases}$$

la partie $-\frac{1}{2}U_t dt$ forcant le processus $(U_t)_{t\geq 0}$ à rester sur le cercle puisqu'alors :

$$\begin{aligned} d\left(U_t\overline{U_t}\right) &= dU_t\overline{U_t} + U_t d\overline{U_t} + dU_t d\overline{U_t} \\ &= \left(i(dB_t)U_t - \frac{1}{2}U_t dt\right)\overline{U_t} + U_t\left(-i(dB_t)\overline{U_t} - \frac{1}{2}\overline{U_t}dt\right) \\ &\qquad + (i(dB)_t U_t)(-i(dB_t)\overline{U_t}) \\ &= 0. \end{aligned}$$

Tous les processus de Lévy ne sont pas presque sûrement continus, puisqu'un autre exemple de processus de Lévy, qui est un processus à sauts, est la marche par transpositions sur le groupe symétrique.

Exemple 1.2. — *Étant donné un entier positif N, on considère $\mathfrak{S}(N)$ le groupe des permutations sur $\{1, ..., N\}$. L'ensemble des transpositions est dénoté par $\mathcal{T}_N$. La marche aléatoire par transpositions sur le groupe symétrique est le processus de*

Markov $(S_t)_{t\geq 0}$ dont le générateur H est donné par le fait que pour toute fonction $f \in \mathbb{R}^{\mathfrak{S}(N)}$, pour toute permutation $\sigma_0 \in \mathfrak{S}(N)$:

$$Hf(\sigma_0) = \frac{d}{dt}_{|t=0}\mathbb{E}[f(S_t\sigma_0)] = \frac{1}{\#\mathcal{T}_N}\sum_{\sigma\in\mathcal{T}_N}\big[f(\sigma\sigma_0) - f(\sigma_0)\big].$$

Une façon de construire une marche par transpositions sur le groupe symétrique est de considérer une suite $(X_i)_{i\in\mathbb{N}}$ de variables aléatoires indépendantes et uniformes dans $\mathcal{T}_N$, puis de considérer une suite $(T_i)_{i\in\mathbb{N}^*}$ d'exponentielles indépendantes, de paramètre 1 et indépendantes aussi de $(X_i)_{i\in\mathbb{N}}$. Alors le processus $(S_t)_{t\geq 0}$, tel que $S_0 = \mathrm{id}_N$ et pour tout $t \geq 0$:

$$S_t = \sum_{i=0}^{\infty} \mathbb{1}_{t\in]T_i,T_{i+1}]} X_i...X_1, \tag{1}$$

où par convention $T_0 = 0$ et $X_0...X_1$ est égal à l'identité id_N, est une marche aléatoire par transpositions sur le groupe symétrique.

En utilisant la formulation donnée par l'équation (1), étant donné que pour tout $\sigma \in \mathfrak{S}(N)$ et toute transposition τ, $\sigma\tau\sigma^{-1}$ est une transposition, et vu que $\tau \mapsto \sigma\tau\sigma^{-1}$ est une permutation de $\mathcal{T}_N$, on déduit que la marche aléatoire par transpositions sur le groupe symétrique est invariante par conjugaison par le groupe symétrique.

Définition 1.2. — *Un processus de Lévy $(X_t)_{t\geq 0}$ défini sur G est invariant par conjugaison par un sous-ensemble H de G si pour tout $h \in H$, $(hX_th^{-1})_{t\geq 0}$ a la même loi que $(X_t)_{t\geq 0}$.*

1.2. Mouvements browniens anti-hermitien et unitaire. — Un processus de Lévy qui nous intéressera par la suite est le mouvement brownien unitaire qui est une généralisation du mouvement brownien sur $U(1)$. L'espace des matrices unitaires $U(N)$ est un groupe de Lie donné par :

$$U(N) = \{M \in \mathcal{M}_N(\mathbb{C}), M^*M = \mathrm{Id}\},$$

où $M^* = \overline{M^t}$. L'algèbre de Lie associée $\mathfrak{u}(N)$ est donnée par l'ensemble des matrices anti-hermitiennes :

$$\mathfrak{u}(N) = \{M \in \mathcal{M}_N(\mathbb{C}),\ M^* = -M\}.$$

Dans l'exemple 1.1, nous avons construit le processus brownien sur $U(1)$ par l'exponentiation, via une équation différentielle stochastique, d'un processus brownien sur la droite réelle. Cette construction se généralise à l'espace des matrices unitaires $U(N)$. On considère un mouvement brownien $(K_t)_{t\geq 0}$ sur l'ensemble des matrices anti-hermitiennes. Vu que $\mathfrak{u}(N)$ est un espace vectoriel, $(K_t)_{t\geq 0}$ se construit en considérant un processus dont les coordonnées sont des

mouvements browniens réels indépendants. Cependant il faut choisir une base de $\mathfrak{u}(N)$.

Définition 1.3. — *Considérons $(e_i)_{i=1}^d$ une base orthonormée de $\mathfrak{u}(N)$ pour le produit scalaire $\langle M_1, M_2\rangle = N\mathsf{Tr}(M_1^* M_2)$. Le processus $K_t = \sum_{i=1}^d B_t^i e_i$ est un processus brownien anti-hermitien, sa loi ne dépend pas de la base orthonormée choisie.*

Remarquons tout de même que nous avons fait un choix non canonique de produit scalaire sur l'espace des matrices anti-hermitiennes : nous aurions pu choisir tout simplement le produit scalaire $\mathsf{Tr}(M_1^* M_2)$ cependant cela n'induirait qu'un changement linéaire du temps.

Le processus à valeurs dans $\mathcal{M}_N(\mathbb{C})$, solution de l'équation différentielle stochastique :

$$\left\{ \begin{array}{ll} dU_t & = dK_t U_t - \frac{1}{2} U_t dt, \\ U_0 & = I_N \end{array} \right. \tag{2}$$

est appelé processus brownien sur $U(N)$. C'est un processus presque sûrement continu, qui modélise, via les valeurs propres associées, N particules browniennes sur le cercle unité qui évoluent en se repoussant les unes les autres.

1.3. Autre présentation des processus de Lévy. — On peut donner une autre présentation, nouvelle dans cette thèse, des processus de Lévy qui permet d'avoir une première intuition sur les champs markoviens d'holonomies planaires. Pour ce faire, nous avons besoin d'introduire quelques notations.

Considérons $\mathcal{D}(\mathbb{R})$ l'ensemble des densités lisses strictement positives sur $\mathbb{R}$ et intégrables. Tout homéomorphisme croissant ψ de $\mathbb{R}$ induit une application mesurable de $G^{\mathbb{R}}$ sur lui-même que l'on notera aussi ψ :

$$\forall (g_t)_{t\in\mathbb{R}} \in G^{\mathbb{R}},\ \psi((g_t)_{t\in\mathbb{R}}) = (g_{\psi(t)})_{t\in\mathbb{R}}.$$

Le processus canonique de projections défini sur $G^{\mathbb{R}}$ sera noté $(g_t)_{t\in\mathbb{R}}$.

Définition 1.4. — *Un champ markovien d'holonomies unidimensionnel est une famille de mesures de probabilités $(\mathbb{E}_{\mathsf{vol}})_{\mathsf{vol}\in\mathcal{D}(\mathbb{R})}$ sur $G^{\mathbb{R}}$ qui satisfait trois propriétés :*

-Invariance par homéomorphismes croissants qui préservent l'aire : *Pour tout couple* vol *et* vol′ *d'éléments de $\mathcal{D}(\mathbb{R})$ et pour l' unique homéomorphisme croissant ψ de $\mathbb{R}$ qui envoie* vol *sur* vol′,

$$\mathbb{E}_{\mathsf{vol}} = \mathbb{E}_{\mathsf{vol}'} \circ \psi^{-1}.$$

-Indépendance : *Soit* vol $\in \mathcal{D}(\mathbb{R})$ *et considérons deux intervalles disjoints $[s_0, t_0]$ et $[s_1, t_1]$. Sous la mesure $\mathbb{E}_{\mathsf{vol}}$, $\sigma\big((g_t g_s^{-1}), s_0 \leq s < t \leq t_0\big)$ est indépendante de $\sigma\big((g_t g_s^{-1}), s_1 \leq s < t \leq t_1\big)$.*

-Localité : *Pour tout couple* vol *et* vol' *d'éléments de* $\mathcal{D}(\mathbb{R})$*, pour tout réel* t_0 *tel que* $\mathsf{vol}_{|]-\infty,t_0]} = \mathsf{vol}'_{|]-\infty,\mathsf{t_0}]}$*, la loi de* $(g_t)_{t\leq t_0}$ *est la même sous* $\mathbb{E}_{\mathsf{vol}}$ *que sous* $\mathbb{E}_{\mathsf{vol}'}$*.*

Étant donné un processus de Lévy $\mathsf{X} = (X_t)_{t\geq 0}$ dans G, il est facile de construire un tel champ markovien d'holonomies unidimensionnel. En effet, considérons une densité $\mathsf{vol} \in \mathcal{D}(\mathbb{R})$. On définit alors la mesure $\mathbb{E}^{\mathsf{X}}_{\mathsf{vol}}$ sur $G^{\mathbb{R}}$ telle que, sous $\mathbb{E}^{\mathsf{X}}_{\mathsf{vol}}$, le processus canonique de projections $(g_t)_{t\in\mathbb{R}}$ a la même loi que $\left(X_{\mathsf{vol}(]-\infty,\mathsf{t}])}\right)_{t\in\mathbb{R}}$. Il n'est alors pas difficile de constater la validité la proposition suivante.

Proposition 1.1. — *La famille de mesures* $\left(\mathbb{E}^{\mathsf{X}}_{\mathsf{vol}}\right)_{\mathsf{vol}\in\mathcal{D}(\mathbb{R})}$ *est un champ markovien d'holonomies unidimensionnel.*

Se pose alors la question suivante : l'application qui à un processus de Lévy X associe $\left(\mathbb{E}^{\mathsf{X}}_{\mathsf{vol}}\right)_{\mathsf{vol}\in\mathcal{D}(\mathbb{R})}$ est-elle une bijection? Étant donné un champ markovien d'holonomies unidimensionnel $(\mathbb{E}_{\mathsf{vol}})_{\mathsf{vol}\in\mathcal{D}(\mathbb{R})}$, peut-on retrouver le processus de Lévy X tel que $\left(\mathbb{E}^{\mathsf{X}}_{\mathsf{vol}}\right)_{\mathsf{vol}\in\mathcal{D}(\mathbb{R})} = (\mathbb{E}_{\mathsf{vol}})_{\mathsf{vol}\in\mathcal{D}(\mathbb{R})}$?

Ceci est possible quand le champ markovien d'holonomies unidimensionnel $(\mathbb{E}_{\mathsf{vol}})_{\mathsf{vol}\in\mathcal{D}(\mathbb{R})}$ est *stochastiquement continu*. On supposera donc que pour tout $\mathsf{vol} \in \mathcal{D}(\mathbb{R})$, pour toute suite de réels $(t_n)_{n\in\mathbb{N}}$ convergeant vers t, $\mathbb{E}_{\mathsf{vol}}\left[d(g_{t_n}, g_t)\right]$ converge vers 0 quand n tend vers l'infini et pour toute suite de réels $(t_n)_{n\in\mathbb{N}}$ convergeant vers $-\infty$, $\mathbb{E}_{\mathsf{vol}}\left[d(g_{t_n}, e)\right]$ converge vers 0 quand n tend vers l'infini, d étant une distance sur G compatible avec sa structure de groupe de Lie compact.

Considérons un champ markovien d'holonomies unidimensionnel $(\mathbb{E}_{\mathsf{vol}})_{\mathsf{vol}\in\mathcal{D}(\mathbb{R})}$ stochastiquement continu. Prenons une bijection lisse $\phi :]0, \infty[\to \mathbb{R}$, et considérons une densité lisse vol_0 strictement positive telle que :

$$\mathsf{vol}_0(]\infty, \phi(t)]) = t.$$

Cette densité lisse n'est pas intégrable, mais grâce à la localité, on peut étendre la définition de $(\mathbb{E}_{\mathsf{vol}})_{\mathsf{vol}\in\mathcal{D}(\mathbb{R})}$ à $\mathbb{E}_{\mathsf{vol}_0}$. Considérons alors le processus $\mathsf{X} = (X_t)_{t>0}$ de même loi que $(g_{\phi(t)})_{t>0}$ sous $\mathbb{E}_{\mathsf{vol}_0}$, et posons $X_0 = e$. En utilisant l'invariance par homéomorphismes croissants qui préservent l'aire, on peut montrer que X a des incréments stationnaires. À cause de la continuité stochastique de $(\mathbb{E}_{\mathsf{vol}})_{\mathsf{vol}\in\mathcal{D}(\mathbb{R})}$, X est stochastiquement continu. De plus, bien que la propriété d'indépendance ne permette au premier abord que de montrer l'indépendance d'incréments disjoints, la continuité stochastique permet d'étendre cette propriété à tous les incréments. Nous obtenons alors un processus de Lévy, et il n'est pas dur de voir que $\left(\mathbb{E}^{\mathsf{X}}_{\mathsf{vol}}\right)_{\mathsf{vol}\in\mathcal{D}(\mathbb{R})} = (\mathbb{E}_{\mathsf{vol}})_{\mathsf{vol}\in\mathcal{D}(\mathbb{R})}$. Ainsi nous obtenons le résultat suivant.

Proposition 1.2. — *Il y a une correspondance bijective entre les processus de Lévy et les champs markoviens d'holonomies unidimensionnels stochastiquement continus.*

On verra par la suite que l'on peut définir une notion de champs markoviens d'holonomies planaires, qui est une généralisation des champs markoviens d'holonomies unidimensionnels, et pour laquelle un résultat analogue à la proposition 1.2 existe : il existe une surjection entre une classe de processus de Lévy et les champs markoviens d'holonomies planaires stochastiquement continus. Ce résultat sur les champs markoviens d'holonomies planaires est l'un des principaux théorèmes obtenus dans le chapitre de thèse "Champs markoviens d'holonomies planaires" [Champs].

2. Introduction aux champs markoviens planaires

2.1. Une généralisation du temps. — Jusqu'à présent, nous avons considéré des processus à valeurs dans un groupe de Lie compact G et indexés par les heures d'un temps unidimensionel $\mathbb{R}^+$. Ainsi à chaque point de $\mathbb{R}^+$ on a associé une variable aléatoire à valeurs dans G. Il est intéressant de remarquer que la notion de temps n'est pas naturellement liée à une idée de structure algébrique vu que la demi-droite réelle n'est considérée que comme un simple ensemble ordonné, au mieux comme un espace topologique si des considérations de continuité sont prises en compte.

Que se passe-t-il si l'on essaie de généraliser la notion de temps : d'espace unidimensionnel, faisons le passer à un espace bidimensionnel. Le temps devient alors l'espace topologique $\mathbb{R}^2$, le plan, et deux perspectives s'ouvrent à nous. La première, assez habituelle, est de considérer que la notion d'heure dans cette nouvelle notion de temps est toujours de dimension 0 : c'est un point. Dans ce cadre, on considère des processus indexés par les points du plan : $(X_{(t,s)})_{(t,s)\in\mathbb{R}^2}$. Il est une autre possibilité, moins connue, qui consiste à considérer qu'ayant grossi d'une dimension l'espace dans lequel vit le temps, on devrait alors grossir la dimension d'une heure donnée : dans ce cadre, on considère donc des processus indexés par des chemins dans le plan : $(X_p)_{p\in\mathsf{P}(\mathbb{R}^2)}$, où $\mathsf{P}(\mathbb{R}^2)$ est l'ensemble des chemins réguliers tracés dans le plan, c'est-à-dire dans cette thèse l'ensemble des chemins de longueur finie. Nous ne détaillerons pas la notion de chemins de longueur finie dans cette introduction[(2)].

Dans cette thèse, nous nous focaliserons sur cette dernière possibilité qui est apparue avec l'émergence d'une théorie physique appelée théorie de Yang-Mills. Remarquons que $\mathsf{P}(\mathbb{R}^2)$ est maintenant naturellement muni d'une structure algébrique. En effet, si on considère deux chemins p_1 et p_2, et si p_1 finit là où commence p_2, on peut considérer le chemin p_1p_2 qui est la concaténation de p_1 avec p_2. De plus, en renversant l'orientation d'un chemin p, en allant du point d'arrivée au point de départ, on définit un nouveau chemin p^{-1}. La structure d'ordre total permettant l'écoulement du temps dans le cas unidimensionnel est

[(2)] Définitions 2.1 et 2.2 dans le chapitre de thèse [Champs].

ainsi perdue dans le cas du temps bidimensionnel, les "heures" balayant alors le plan en se composant mais en ne s'écoulant pas. Il devient alors naturel de restreindre les processus considérés à la classe de processus qui sont compatibles avec la structure temporelle. Une trajectoire non-aléatoire est de ce point de vue une fonction multiplicative $h \in \mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$, c'est à dire une fonction h allant des chemins dans G telle que pour tous chemins p_1 et p_2 pouvant se concaténer, et pour tout chemin p :

$$h_{p_1 p_2} = h_{p_2} h_{p_1}$$
$$h_{p^{-1}} = (h_p)^{-1}.$$

On peut alors munir cet espace de trajectoires $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$ de la trace de la tribu Borélienne produit sur $G^{\mathsf{P}(\mathbb{R}^2)}$, ce qui en fait un espace mesurable. Une mesure de probabilité sur $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$ est appelée un **champ d'holonomies aléatoires planaire**, ou plus simplement dans cette introduction, **champ d'holonomies aléatoires**. Dans le cadre d'un temps unidimensionnel, l'espace des temps $\mathbb{R}^+$ était muni d'une structure topologique donnnée par la distance habituelle sur $\mathbb{R}^+$. De même, on peut considérer $\mathsf{P}(\mathbb{R}^2)$ comme un espace topologique en le munissant de la distance :

$$d_l(p_1, p_2) = \inf \sup_{t \in [0,1]} d_{\mathbb{R}^2}(p_1(t), p_2(t)) + |\ell(p_1) - \ell(p_2)|.$$

où l'infimum porte sur les paramétrisations de p_1 et de p_2, et où $\ell(p)$ est la longueur de p. La notion de continuité pour un champ d'holonomies aléatoires découle de cette définition de distance, *à ceci près que l'on se restreint à des suites de chemins qui ont les mêmes points d'arrivée et de départ.* Ainsi, un champ d'holonomies aléatoires $\mathbb{E}$ est dit *stochastiquement continu* si pour toute suite $(p_n)_{n\in\mathbb{N}}$ de chemins réguliers qui convergent vers p, si pour tout n, p_n a les mêmes points de départ et d'arrivée que p, alors $\mathbb{E}\left[d(h_{p_n}, h_p)\right]$ converge vers 0 quand n tend vers l'infini[(3)].

2.2. Des connexions aléatoires. — Nous avons introduit les champs d'holonomies aléatoires comme des généralisations des processus habituels. On peut aussi interpréter les champs d'holonomies comme un certain type d'expérience où une particule interagit avec un champ aléatoire présent dans une pièce.

En effet on peut considérer qu'un chemin dans le plan est la trajectoire d'une particule-test qui aurait une caractéristique à valeurs dans G. Lors de son déplacement, un champ aléatoire présent dans la pièce agit sur la caractéristique de façon aléatoire et de façon multiplicative. Cette expérience permet alors de

(3)La restriction liée aux points d'arrivée et de départ est cruciale puisque sans elle la majorité des champs d'holonomies aléatoires étudiés jusqu'à présent ne seraient pas stochastiquement continus.

mesurer le champ par la façon dont il agit sur les caractéristiques des particules lors de leur déplacement.

Cette image est proche de l'idée sous-jacente à la théorie de Yang-Mills, théorie qui décrit une généralisation des champ markovien d'holonomies planaires. En effet cette théorie, introduite dans les travaux de Yang et Mills en 1954 ([**36**]), est une théorie décrivant des connexions aléatoires très singulières sur un fibré principal. À un niveau formel, la mesure de Yang-Mills est une mesure sur l'espace des connexions de la forme :

$$\frac{1}{Z}e^{-\frac{1}{2}S_{YM}(A)}DA, \tag{3}$$

où $S_{YM}(A)$ est l'action de Yang-Mills d'une connexion A qui est le carré de la norme L^2 de la courbure de A, et DA est une mesure invariante par translation sur l'espace des connexions. Cependant cette formulation est problématique puisque l'espace des connexions aléatoires ne possède pas de mesure invariante par translation à cause de son caractère infini-dimensionnel.

Une idée qui permet de contourner cette difficulté en deux dimensions est alors d'étudier les connexions de façon duale : au lieu de s'intéresser à la connexion, il est préférable d'étudier son holonomie qui est un élement de $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$.

Regardons de plus près ce que sont une connexion et un fibré principal. Nous nous plaçons dans le cadre d'un fibré principal sur le plan, cadre dans lequel la notion se trouve être simplifiée et qui nous intéressera tout le long de la thèse. Un fibré principal P sur le plan ayant pour groupe de structure G est tout simplement une variété différentielle isomorphe à $\mathbb{R}^2 \times G$ munie d'une application projection $\pi : P \to \mathbb{R}^2$ et d'une action à droite de G sur P, qui est libre et transitive sur les fibres. Au-dessus de chaque point du plan se trouve donc une fibre isomorphe à G sur laquelle le groupe G agit de façon lisse. Pour tout $x \in P$, pour tout $g \in G$, on note $x.g$ le résultat de l'action à droite de g sur x : l'élément $x.g$ appartient à $\pi^{-1}(\pi(x))$. Dans le cas du fibré principal $\mathbb{R}^2 \times G$ l'identification de la fibre à G est canonique et permet de définir une notion de "niveau", cependant il faut garder à l'esprit que de façon générale, il n'existe pas d'identification canonique entre $\pi^{-1}(\pi(x))$ et G. Etant donné un point $(a, g) \in \mathbb{R}^2 \times G$, étant donné un chemin $p = (p_t)_{t\in[0,1]}$ dans $\mathbb{R}^2$ partant de a, il est facile de comprendre ce que signifie le fait de marcher horizontalement au-dessus de p : il suffit de suivre le chemin $((p_t, g))_{t\geq 0}$. Cependant dans le cas général, étant donné qu'il n'existe pas d'identification canonique de chaque fibre avec G, il n'existe pas de notion canonique de niveau et donc nous ne pouvons pas marcher horizontalement dans P. C'est ce qui rend nécessaire la notion de connexion. Une connexion est la donnée d'une notion d'horizontalité en chaque point du fibré, c'est la donnée d'une distribution horizontale qui est compatible avec l'action de G sur P. Étant donné une connexion ω sur P, étant donné un point x de P, on peut relever tout chemin p du plan qui commence en $\pi(x)$ en un chemin p' de P qui commence en x en suivant la direction horizontale donnée par la connexion ω. La compatibilité de ω

avec l'action de G sur P implique alors que si on relève p à partir de $x.g$, on finit alors en $p'_1.g$. De plus si le chemin p est une boucle alors le chemin p' finit dans la fibre au-dessus de $\pi(x)$: il existe un unique élément $g \in G$ tel que $p'_1 = p'_0.g$.

On voit alors qu'à tout chemin régulier p du plan, on peut associer une application $\Phi_p : \pi^{-1}(\underline{p}) \to \pi^{-1}(\overline{p})$, qui va de la fibre au-dessus du point de départ $\underline{p}$ de p dans la fibre au-dessus du point d'arrivée $\overline{p}$ de p. Elle vérifie pour tout $x \in \pi^{-1}(\underline{p})$ et pour tout $g \in G$:

$$\Phi_p(x.g) = \Phi_p(x).g, \tag{4}$$

De plus pour tous chemins p_1 et p_2 pouvant se concaténer, on voit facilement que : $\Phi_{p_1 p_2} = \Phi_{p_2}\Phi_{p_1}$ et $\Phi_{p_1^{-1}} = \Phi_{p_1}^{-1}$.

Supposons que l'on ne sache mesurer que des éléments de G : il faut alors introduire la notion de jauge. Une jauge j est une section de π : c'est la donnée d'un isomorphisme entre chaque fibre de P et G qui respecte l'action de G. Pour tout point du plan v on se donne un isomorphisme $j_v : G \to \pi^{-1}(v)$ tel que pour tous g_1, g_2 dans G :

$$j_v(g_1).g_2 = j_v(g_1 g_2). \tag{5}$$

Se donner une jauge permet alors de mesurer l'action d'une connexion sur chaque chemin : à chaque chemin p, on associe le résultat de la mesure égal à $h_p^{j,\omega} = j_{\overline{p}}^{-1}\left(\Phi_p(j_{\underline{p}}(e))\right)$ qui est bien un élément de G. Il est facile de vérifier que $h^{j,\omega}$ est un élément de $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$. En effet, étant donné deux chemins p_1 et p_2 qui sont concaténables, en utilisant les équations (4) et (5) ainsi que la multiplicativité de Φ on a :

$$\begin{aligned} h_{p_2}^{j,\omega} h_{p_1}^{j,\omega} = j_{\overline{p_2}}^{-1}\left(\Phi_{p_2}(j_{\underline{p_2}}(e))\right) h_{p_1}^{j,\omega} = j_{\overline{p_2}}^{-1}\left(\Phi_{p_2}(j_{\underline{p_2}}(h_{p_1}^{j,\omega}))\right) &= j_{\overline{p_2}}^{-1}\left(\Phi_{p_2}\Phi_{p_1} j_{\underline{p_1}}(e)\right) \\ &= j_{\overline{p_2}}^{-1}\left(\Phi_{p_1 p_2} j_{\underline{p_1}}(e)\right) = h_{p_1 p_2}^{j,\omega}. \end{aligned}$$

De même, on prouve que $\left(h_p^{j,\omega}\right)^{-1} = h_{p^{-1}}^{j,\omega}$.

On voit donc qu'un champ d'holonomies aléatoires peut donc provenir, au moins de façon informelle, d'une connexion aléatoire : le champ dont on parlait au début de la section 2.2 est en quelque sorte une connexion aléatoire.

Remarquons qu'afin d'obtenir un champ d'holonomies à partir d'une connexion, nous avons dû faire un choix de mesure : un choix de jauge. Les propriétés physiques auxquelles nous nous intéressons doivent être invariantes par ce choix de jauge qui n'est, dans un sens, que le choix de l'expérimentateur. Que se passe-t-il si nous changeons de jauge ? Il est facile de voir qu'un changement de jauge induit un changement sur h de la forme :

$$h_p^{j_2,\omega} = \left[\left(j_{1\overline{p}}\right)^{-1} j_{2\overline{p}}\right]^{-1} h_p^{j_1,\omega} \left[\left(j_{1\underline{p}}\right)^{-1} j_{2\underline{p}}\right].$$

On voit donc apparaître le pendant de la notion de jauge dans le contexte des champs d'holonomies : dans ce contexte et pour la suite de la présentation, une transformation de jauge sera une fonction $j : \mathbb{R}^2 \to G$, et son action sur une fonction $h \in \mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$ est donnée par le fait que pour tout chemin p :

$$(j \bullet h)_p = j_{\overline{p}}^{-1} h_p j_{\underline{p}}.$$

Toute propriété physique se doit d'être invariante par l'action des transformations de jauges sur $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$, c'est pourquoi on ne s'intéressera qu'aux champs d'holonomies aléatoires invariants par les transformations de jauge.

On cherchera donc à construire la mesure de Yang-Mills comme une mesure sur $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$ invariante par l'action des transformations de jauge, c'est à dire par l'action de $G^{\mathbb{R}^2}$ sur $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$. Une façon de comprendre la mesure de Yang-Mills serait d'étudier de façon informelle les propriétés de la mesure (3). Nous ne prendrons pas ce chemin-là, étant donné qu'il restreindrait la classe de mesures que nous allons étudier.

2.3. Définition. — Dans le chapitre de thèse nommé "Champs markoviens d'holonomies planaires", on s'intéresse à des champs d'holonomies aléatoires dits markoviens qui possèdent des propriétés supplémentaires qui s'inspirent des propriété des champs markoviens d'holonomies unidimensionnels présentées dans la définition 1.4 et qui généralisent la notion de mesure de Yang-Mills. Reprenons un peu l'idée qu'un champ d'holonomies représente l'expérience des mesures d'un champ aléatoire dans un laboratoire. Quelles conditions naturelles pouvons-nous supposer sur ce type d'expérience ? Comme nous l'avons vu précédemment, afin de le mesurer, il a fallu fixer un choix de jauge arbitraire, et donc il est naturel de demander que ce champ aléatoire soit invariant par changement de jauge. De plus on fera la supposition que la particule ne ressent que le champ qu'il y a à l'intérieur de sa trajectoire, cela rend alors naturelle la supposition que, si l'on fait l'expérience dans deux pièces disjointes, les résultats doivent être indépendants. Enfin une invariance par changement de coordonnées préservant l'aire sera demandée : en particulier une particule entourant une seule fois un ouvert simplement connexe d'aire donnée doit être affectée par le champ d'une façon indépendante de la forme de sa trajectoire[(4)]. C'est ainsi que dans le chapitre de thèse [Champs] apparaît la notion de champ markovien d'holonomies planaire[(5)]. On donne par la suite une définition légèrement différente afin de simplifier l'exposé d'introduction.

Considérons $\mathcal{D}(\mathbb{R}^2)$ l'ensemble des densités lisses et strictement positives sur $\mathbb{R}^2$. Le processus canonique des projections défini sur $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$ sera noté $(h_p)_{p \in \mathsf{P}(\mathbb{R}^2)}$ ou $(h(p))_{p \in \mathsf{P}(\mathbb{R}^2)}$.

(4) Ceci pourrait être justifié à partir de l'expression (3) de la mesure de Yang-Mills.
(5) Définition 4.1 dans [Champs].

Définition 2.1. — *Un champ markovien d'holonomies planaire à valeurs dans G est la donnée, pour tout $\mathsf{vol} \in \mathcal{D}(\mathbb{R}^2)$, d'un champ d'holonomies planaire $\mathbb{E}_{\mathsf{vol}}$, invariant par transformation de jauge, tel que :*

$\mathbf{P_1}$: *Soient vol et vol' deux densités dans $\mathcal{D}(\mathbb{R}^2)$. Soit $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ un homéomorphisme qui envoie vol sur vol' (i.e. $\mathsf{vol}' = \mathsf{vol} \circ \psi^{-1}$). Pour tout entier $i \in \{1, ..., n\}$, soit p_i un chemin régulier et soit p'_i son image par ψ. Soit $f : G^n \to \mathbb{R}$ une fonction continue. Supposons que l'une des conditions suivantes est vérifiée :*

1. *ψ est bi-lipschitzienne,*
2. *pour tout $i \in \{1, ..., n\}$, p'_i est régulier et les familles $(p_i)_{i=1}^n$ et $(p'_i)_{i=1}^n$ peuvent être tracées sur deux graphes planaires,*

alors :

$$\mathbb{E}_{\mathsf{vol}}\Big[f\big(h_{p_1}, \ldots, h_{p_n}\big)\Big] = \mathbb{E}_{\mathsf{vol}'}\Big[f\big(h_{p'_1}, \ldots, h_{p'_n}\big)\Big].$$

$\mathbf{P_2}$: *Pour toute densité $\mathsf{vol} \in \mathcal{D}(\mathbb{R}^2)$, pour tous disques fermés D_1 et D_2 disjoints, sous $\mathbb{E}_{\mathsf{vol}}$, les familles $\{h_p, p$ chemin régulier inclus dans $D_1\}$ et $\{h_p, p$ chemin régulier inclus dans $D_2\}$ sont indépendantes.*

$\mathbf{P_3}$: *Pour toutes densités vol et vol' dans $\mathcal{D}(\mathbb{R}^2)$, si D est un disque fermé sur lequel vol et vol' sont égales alors pour toute famille de chemins réguliers $p_1, ..., p_n$ inclus dans D, $(h_{p_1}, ..., h_{p_n})$ a la même loi sous $\mathbb{E}_{\mathsf{vol}}$ et sous $\mathbb{E}_{\mathsf{vol}'}$.*

Il existe une définition de champs markoviens d'holonomies définis sur toute surface bidimensionnelle compacte donnée par T. Lévy (Définition 3.1.2 of [**23**]) qui se compose de sept axiomes. Nous ne la présenterons pas dans l'introduction mais on l'a reprise (avec de légères modifications) dans la Définition 12.5 du chapitre de thèse [Champs].

Remarquons que dans l'axiome P_2, qui diffère légèrement de l'axiome original donné dans [Champs], les deux disques D_1 et D_2 sont des *disques fermés disjoints*. C'est ce qui fait toute la difficulté de la caractérisation des champs markoviens d'holonomies planaires prouvée dans le chapitre de thèse [Champs]. En effet si l_1 et l_2 sont deux boucles basées en 0, étant donné qu'elles partagent le même point d'origine, on ne peut jamais trouver deux disques fermés disjoints D_1 et D_2 tels que l_1 et l_2 soient respectivement incluses dans D_1 et D_2. Ainsi même s'il existe deux disques D_1 et D_2 d'intérieurs disjoints tels que l_1 et l_2 soient respectivement incluses dans D_1 et D_2, on ne peut pas conclure quant à l'indépendance de h_{l_1} et h_{l_2}.[(6)] Le lecteur pourrait penser qu'en rajoutant la condition que le champ markovien d'holonomies planaire doit être stochastiquement continu, on pourrait utiliser un argument de continuité tout comme dans le cadre des champs markoviens d'holonomies unidimensionnels. Cependant la notion de continuité

[(6)] En réalité, on montre dans [Champs] que ce n'est pas le cas généralement.

stochastique pour les champs d'holonomies aléatoires, rappelée à la fin de la section 2.1, ne permet pas de modifier les points d'arrivée ou de départ des chemins : on ne peut donc pas "détacher" la boucle l_2 de 0.

2.4. Des exemples simples : l'indice et les revêtements ramifiés. — Dans les exemples ci-dessous, nous ne définissons pas les champs d'holonomies sur $\mathsf{P}(\mathbb{R}^2)$, mais sur $\mathsf{L}_0(\mathbb{R}^2)$ qui est l'ensemble des boucles régulières basées en 0.

2.4.1. Indice. — Il existe une application naturelle définie sur les boucles basées en 0, à valeurs dans $U(1)$ et qui est multiplicative. Étant donné une boucle lisse l dans le plan et étant donné un point du plan qui n'appartient pas à l'image de l, l'indice de l en x est donné par :

$$\mathfrak{n}_l(x) = \frac{1}{2i\pi} \oint_l \frac{dz}{z-x}.$$

Pour toute boucle continue, on peut définir $\mathfrak{n}_l(x)$ pour tout $x \in \mathbb{R}^2 \setminus l([0,1])$ en approximant l par des boucles lisses. Si l est de longueur finie, d'après l'inégalité de Banchoff-Pohl[(7)], la fonction $\mathfrak{n}_l(x)$ est de carré intégrable. En effet pour une telle boucle :

$$4\pi \int_{\mathbb{R}^2} \mathfrak{n}_l(x)^2 dx \leq \ell(l)^2,$$

où $\ell(l)$ est la longueur de l. La fonction $\mathfrak{n}_l(x)$ est donc intégrable puisqu'à valeurs dans $\mathbb{N}$. En utilisant la propriété d'additivité des intégrales curvilignes, on sait que :

$$\mathfrak{n}_{l_1 l_2} = \mathfrak{n}_{l_1} + \mathfrak{n}_{l_2}.$$

L'application définie sur l'ensemble des boucles basées en 0 :

$$\Phi : l \mapsto \Phi_l = e^{i \int_{\mathbb{R}^2} \mathfrak{n}_l(x) dx}$$

est alors une fonction multiplicative dans le sens ou pour toutes boucles l_1, l_2 basées en 0, $\Phi_{l_1 l_2} = \Phi_{l_2}\Phi_{l_1}$ et $\Phi_{l_1^{-1}} = (\Phi_{l_1})^{-1}$. On peut naturellement étendre la définition de cette application afin de définir un champ markovien d'holonomies planaire $(\mathbb{E}_{\mathsf{vol}})_{\mathsf{vol} \in \mathcal{D}(\mathbb{R}^2)}$ tel que sous $\mathbb{E}_{dx}$, pour toute famille $l_1, ..., l_n$ dans $\mathsf{L}_0(\mathbb{R}^2)$, les variables $h_{l_1}, ..., h_{l_n}$ sont non aléatoires et égales à $\Phi(l_1), ..., \Phi(l_n)$[(8)].

2.4.2. Processus de Poisson et transformation particules/champ. — Dans le cas où G est un groupe fini, d'autres exemples de champs markoviens d'holonomies planaires faciles à se représenter sont donnés par l'utilisation de processus de Poisson. Considérons un groupe fini G, muni d'une mesure μ invariante par conjugaison sous l'action du groupe G.

(7) Une preuve élémentaire est donnée par A. Vogt dans [**32**].
(8) Définition 4.9 dans la section 4.1.1 de [Champs].

Considérons un sous-ensemble R de $\mathbb{R}^2 \times G$ fini ou localement fini, dont on notera R_1 la projection sur $\mathbb{R}^2$. Supposons que 0 n'appartient pas à R_1. L'ensemble R_1 représente la localisation de particules possédant des caractéristiques dans G : ces particules ne sont pas de même nature que les particules-tests introduites dans la section 2.2 puisqu'elles vont générer le champ à étudier. En effet, on peut définir une correspondance particules/champ qui permet d'associer à une telle configuration une fonction multiplicative définie sur les éléments de $\mathsf{L}_0(\mathbb{R}^2 \setminus \mathsf{R}_1)$, à valeurs dans G. Pour ce faire, afin de simplifier quelque peu l'explication, nous allons supposer que :

- deux particules ne peuvent pas être au même endroit, c'est-à-dire que la projection de R sur $\mathbb{R}^2$ est injective. Pour $r \in \mathsf{R}_1$ on désignera par g_r le seul élément de G tel que $(r, g_r) \in \mathsf{R}$,
- l'application $\mathsf{R}_1 \to [0, 2\pi[$ qui à une particule associe l'argument de sa position est injective, en particulier $0 \notin R_1$.

Considérons l une boucle dans $L_0(\mathbb{R}^2 \setminus \mathsf{R}_1)$. Pour chaque $r \in \mathsf{R}_1$, on considère une boucle l_r comme celle représentée dans l'image 1, dont l'indice est égal à 1 dans un cercle centré en r et égal à 0 à l'extérieur, et qui n'entoure aucun autre point de R_1.

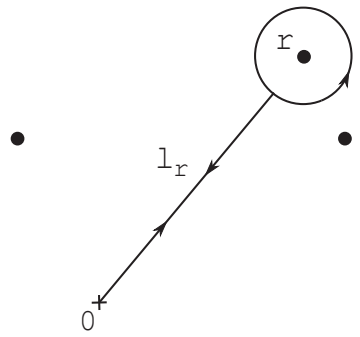

FIGURE 1. Boucle l_r

Cette construction permet de définir une famille $\{\mathsf{l}_r, r \in \mathsf{R}_1\}$ de boucles basées en 0 et qui sont disjointes. Pour toute boucle b dans $\mathsf{L}_0(\mathbb{R}^2 \setminus \mathsf{R}_1)$ basée en 0, on note $[b]$ la classe de b dans le groupe fondamental de $\mathbb{R}^2 \setminus \mathsf{R}_1$ basé en 0.

Théorème 2.1. — *La famille* $\{[\mathsf{l}_r], r \in \mathsf{R}_1\}$ *est un système libre et générateur du groupe fondamental de* $\mathbb{R}^2 \setminus \mathsf{R}_1$ *basé en* 0. *Ce groupe est donc isomorphe à un groupe libre.*

Ainsi il existe un unique mot w_l en $\{[\mathsf{l}_r], [\mathsf{l}_r^{-1}] \mid r \in \mathsf{R}_1\}$ tel que :

$$[l] = w_l \left(\{[\mathsf{l}_r], [\mathsf{l}_r^{-1}] \mid r \in \mathsf{R}_1\}\right).$$

On considère alors le mot w_l^{op} qui est le mot qui s'obtient en lisant de droite à gauche le mot w_l. Par exemple $aaba^{op}$ est le mot $abaa$. On peut alors définir h_l par :

$$h_l = w_l^{op}\left(\{g_r, g_r^{-1} \mid r \in \mathsf{R}_1\}\right).$$

Par exemple supposons que $[l] = [l_{r_1}][l_{r_2}]^2[l_{r_3}]$ alors $h_l = g_{r_3} g_{r_2}^2 g_{r_1}$. Ceci permet donc de définir une fonction $h : \mathsf{L}_0(\mathbb{R}^2 \setminus \mathsf{R}_1) \to G$. On peut alors vérifier que pour toute boucle $l \in \mathsf{L}_0(\mathbb{R}^2 \setminus \mathsf{R}_1)$, $h_{l^{-1}} = (h_l)^{-1}$ et pour tout couple de boucles l_1 et l_2 dans $\mathsf{L}_0(\mathbb{R}^2 \setminus \mathsf{R}_1)$, $h_{l_1 l_2} = h_{l_2} h_{l_1}$.

Expliquons brièvement cette dernière égalité : il existe deux mots w_{l_1} et w_{l_2} tels que $[l_1] = w_{l_1}(\{[\mathsf{l}_r], [\mathsf{l}_r^{-1}] \mid r \in \mathsf{R}_1\})$ et $[l_2] = w_{l_2}(\{[\mathsf{l}_r], [\mathsf{l}_r^{-1}] \mid r \in \mathsf{R}_1\})$. Alors :

$$\begin{aligned} h_{l_1 l_2} &= (w_{l_1} w_{l_2})^{op}\left(\{g_r, g_r^{-1} \mid r \in \mathsf{R}_1\}\right) \\ &= w_{l_2}^{op}\left(\{g_r, g_r^{-1} \mid r \in \mathsf{R}_1\}\right) w_{l_1}^{op}\left(\{g_r, g_r^{-1} \mid r \in \mathsf{R}_{1\omega}\}\right) = h_{l_2} h_{l_1}. \end{aligned}$$

Nous avons donc transformé un ensemble de particules en un champ d'holonomies non aléatoires, c'est-à-dire en une fonction multiplicative de $\mathsf{L}_0(\mathbb{R}^2 \setminus \mathsf{R}_1)$ dans G. Nous appellerons par la suite cette procédure la transformation particules/champ : comme nous le verrons par la suite ce champ est en réalité le champ de monodromie d'un G-fibré ramifié ; ainsi, dans cette transformation les particules s'interprètent comme des singularités. Remarquons de plus que si l et l' ont même image dans le groupe fondamental de $\mathbb{R}^2 \setminus \mathsf{R}_1$ alors par construction $h(l) = h(l')$: le champ obtenu est en réalité un élément de $\mathsf{Hom}\left(\pi_1\left(\mathbb{R}^2 \setminus \mathsf{R}_1\right), G^{op}\right)$ où G^{op} est le groupe dont l'ensemble sous-jacent est G et dont la multiplication est donnée par $x._{op} y = yx$.

Dans la discussion précédente, les boucles ne pouvaient intersecter un certain ensemble localement fini R_1, ce qui n'est pas le cas pour un champ markovien d'holonomies planaire. Afin de supprimer cet interdit et obtenir un champ markovien d'holonomies planaire, il suffit de rendre aléatoire l'ensemble R. Considérons une mesure de densité $\mathsf{vol} \in \mathcal{D}(\mathbb{R}^2)$. Considérons un processus de Poisson R dans $\mathbb{R}^2 \times G$ de mesure d'intensité $\mathsf{vol} \otimes \mu$. C'est un ensemble aléatoire presque sûrement localement fini, qui vérifie les deux hypothèses considérées lors de l'exposé de la transformation particules/champ. Considérons une boucle $l \in \mathsf{L}_0(\mathbb{R}^2)$: presque sûrement $l \in \mathbb{R}^2 \setminus \mathsf{R}_1$. Ainsi on peut appliquer la transformation particules/champ : on obtient alors un processus $(h_l)_{l \in \mathsf{L}_0(\mathbb{R}^2)}$ indexé par l'ensemble $\mathsf{L}_0(\mathbb{R}^2)$ qui est multiplicatif dans le sens où pour toutes boucles l_1 et l_2 dans $\mathsf{L}_0(\mathbb{R}^2)$, presque sûrement on a :

$$\begin{aligned} &h_{l_1 l_2} = h_{l_2} h_{l_1}, \\ &h_{l_1^{-1}} h_{l_1}^{-1}. \end{aligned}$$

Remarquons que le "presque sûrement" est après le "pour toutes boucles", or pour un champ d'holonomies c'est l'inverse. En utilisant la compacité de G[9], on peut en réalité inverser les deux groupes de mots : on montre alors qu'il existe une unique mesure $\mathbb{E}_{\mathsf{vol}}$ sur $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$ qui soit invariante par transformation de jauge et telle que pour toute famille de boucles $l_1, ..., l_n$ basées en 0, sous $\mathbb{E}_{\mathsf{vol}}$, $(h_{l_1}, ..., h_{l_n})$ a la même loi que les variables $(h_{l_1}, ..., h_{l_n})$ précédemment construites. On peut montrer alors que la famille de mesures $(\mathbb{E}_{\mathsf{vol}})_{\mathsf{vol}\in\mathcal{D}(\mathbb{R}^2)}$ est alors un champ markovien d'holonomies planaire.

2.4.3. Revêtements ramifiés aléatoires. — Dans la section 2.2, nous avons expliqué qu'informellement, les champs d'holonomies aléatoires représentaient les holonomies de connexions aléatoires sur des fibrés principaux : ce sont les holonomies d'objets géométriques. Bien que cette interprétation ne soit pas rigoureuse, il s'avère que dans le cas où G est un groupe fini, la construction précédente a une interprétation géométrique. En effet, comme expliqué par T. Lévy dans [**23**], les G-fibrés ramifiés aléatoires permettent de construire des exemples de champs markoviens d'holonomies planaires. Nous n'expliquerons pas le cadre général des G-fibrés ramifiés aléatoires, nous n'expliquerons que le cas où $G = \mathfrak{S}(N)$: dans ce cadre, nous pouvons remplacer l'étude de G-fibrés ramifiés aléatoires par l'étude de revêtements ramifiés aléatoires.

Dans cette section, nous nous limiterons à des processus indexés par $\mathsf{L}_0(\mathbb{D})$, l'ensemble des boucles basées en 0 et définies dans $\mathbb{D}$ le cercle ouvert de rayon 1. Considérons vol une densité lisse sur $\mathbb{D}$ ainsi que μ une mesure sur $\mathfrak{S}(N)$ invariante par conjugaison par le groupe symétrique.

Tout comme pour la section précédente, considérons un processus de Poisson R dans $\mathbb{R}^2 \times \mathfrak{S}(N)$ de mesure d'intensité $\mathsf{vol} \otimes \mu$. Pour tout aléa ω, nous avons donc un ensemble fini de points $\mathsf{R}_1(\omega)$ dans le disque unité qui ne contient presque sûrement pas 0 et à chacun d'entre eux est associée une permutation. Afin de construire un objet géométrique, il faudrait pouvoir mettre en bijection l'ensemble $\mathfrak{S}(N)^{\mathsf{R}_1(\omega)}$ avec un ensemble d'objets géométriques : ce sont les revêtements du disque à N feuillets ramifiés au-dessus de $\mathsf{R}_1(\omega)$ étiquetés en 0.

Expliquons de façon informelle ce qu'est un revêtement ramifié du disque à N feuillets ramifiés au-dessus d'un ensemble fini de points Y dans $\mathbb{D}$ et étiqueté en 0[10]. Pour cela, nous avons besoin de la notion d'hélice "bouclée" : prenons un disque, coupons-le le long d'un rayon, puis vrillons-le comme dans la figure 2, après k tours recollons les deux bords, nous obtenons une hélice "bouclée" d'ordre $k+1$.

Un revêtement ramifié du disque à N feuillets ramifié au-dessus d'un ensemble fini de points Y dans $\mathbb{D}$ est la donnée d'une surface R munie d'une application continue de "projection" de R dans le disque $\mathbb{D}$, qui permet de voir R localement

[9] Et en utilisant la Proposition 2.1 de [Champs].

[10] Pour une définition plus formelle, consulter les définitions 4.1 et 4.2 du chapitre de thèse "Revêtements ramifiés" [Revêtements].

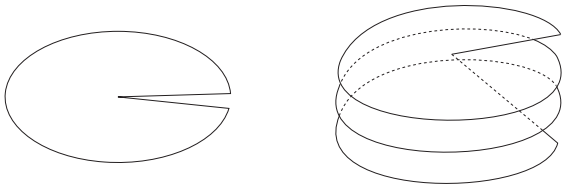

FIGURE 2. Hélice "bouclée" d'ordre trois

comme un ensemble disjoints de N feuilles homéomorphes à un disque au-dessus de tout point, sauf au-dessus des points de ramifications Y. Au-dessus de ces points de ramification, R ressemble à une union d'hélices "bouclées" dont la somme des ordres vaut N.

Supposons que 0 n'appartient pas à Y. Un étiquetage au-dessus du point 0 est la donnée d'une numérotation des feuillets de R au-dessus du point 0. Prenons une boucle l basée en 0 : en suivant la boucle l, on peut propager la numérotation au-dessus de tout point de l, ainsi on se retrouve avec une nouvelle numérotation des feuillets en 0 après avoir parcouru toute la boucle l. La monodromie de ce revêtement étiqueté en 0 le long de l est la donnée de l'unique permutation σ qui envoie la numérotation initiale sur la numérotation finale. Comme annoncé, les revêtements ramifiés étiquetés en 0 permettent de visualiser géométriquement toute fonction de Y dans $\mathfrak{S}(N)$.

Théorème 2.2. — *L'ensemble des revêtements ramifiés du disque à N feuillets ramifiés au-dessus de Y et étiquetés en* 0 *est en bijection avec* $(\mathfrak{S}(N))^Y$.

On peut donc voir la construction d'un champ markovien d'holonomies planaire, esquissée à la fin de la section 2.4.2 via la transformation particules/champ, comme la création d'un revêtement ramifié aléatoire que l'on étudie via sa monodromie aléatoire.

Pour résumer, on peut donc, dans un certain sens, construire des champs markoviens d'holonomies planaires en considérant les monodromies de revêtements ramifiés aléatoires, mais aussi inversement, on peut étudier les monodromies de revêtements ramifiés aléatoires via l'étude des champs markoviens d'holonomies planaires. Ce point de vue est développé dans le chapitre de thèse nommé "Revêtements ramifiés" où nous montrons que des observables de ces monodromies aléatoires convergent quand le nombre de feuillets N tend vers l'infini, à condition de choisir pour chaque N la bonne mesure μ sur $\mathfrak{S}(N)$.

2.5. Les constructions antérieures. — Dans le chapitre de thèse [Champs], nous expliquons comment construire tous les champs markoviens d'holonomies planaires. Pour ce faire on utilise de façon importante les deux symétries : l'invariance par transformation de jauge, ainsi que l'invariance par homéomorphisme

préservant l'aire. Ainsi l'objet obtenu vérifie, par construction, ces deux propriétés. Les constructions antérieures ne permettaient que de construire certains exemples de champs markoviens d'holonomies planaires.

Afin d'illustrer les trois constructions de la mesure de Yang-Mills, faisons un parallèle avec le mouvement brownien. Il existe au moins trois constructions du mouvement brownien :

- méthode stochastique : elle consiste à considérer un bruit blanc F sur $L^2(\mathbb{R}^+)$, c'est-à-dire un isomorphisme de $L^2(\mathbb{R}^+)$ dans un espace gaussien $\mathcal{G}$. On définit le mouvement brownien par l'équation : $B_t = \mathsf{F}([0,t])$,
- méthode physique statistique : elle consiste à discrétiser $\mathbb{R}^+$ et à considérer un système uniforme pondéré par une certaine énergie, on prend une discrétisation de plus en plus fine et on montre que le système converge, lorsque les paramètres sont ajustés, vers une limite continue : c'est la méthode de la construction du mouvement brownien comme limite de marches aléatoires discrètes,
- méthode projective : elle consiste à définir, pour toute famille $\{t_1, ..., t_n\}$ de réels positifs, une mesure $\mu_{\{t_1,...,t_n\}}$ sur $\mathbb{R}^{\{t_1,...,t_n\}}$ telle que les projections canoniques $(X_t)_{t\in\{t_1,...,t_n\}}$ vérifient les deux propriétés suivantes. Les variables $X_{t_1}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ sont indépendantes et pour tout $i \in \{1, ..., n\}$, $X_{t_i} - X_{t_{i-1}}$ est une variable gaussienne centrée de variance $t_i - t_{i-1}$. Ces mesures sont compatibles entre elles dans le sens où pour tous ensembles finis $T_1 \subset T_2 \subset \mathbb{R}^+$, la restriction de la mesure μ_{T_2} à $\mathbb{R}^{T_1}$ est égale à μ_{T_1}. On peut alors, par un théorème général, définir la limite projective de ces mesures.

Il existe une autre méthode qui est l'intersection de la méthode projective et la méthode de la physique statistique : on considère pour tous réels positifs t_1, ..., t_n les mesures à densité par rapport à la mesure uniforme :

$$e^{-\frac{1}{2}S(x_{t_1},...,x_{t_n})} \prod_{i=1}^{n} dx_{t_i},$$

où $S(x_{t_1}, ..., x_{t_n}) = \sum_{i=1}^{n}(x_{t_i} - x_{t_{i-1}})^2$, avec la convention que $x_{t_0} = 0$. Ces mesures sont alors compatibles entre elles et on peut, tout comme dans la méthode projective, prendre la limite projective. On appellera cette méthode la méthode physique statistique exacte.

Pour la construction de la mesure de Yang-Mills et de certains exemples de champs markoviens d'holonomie, la méthode physique statistique permet, à partir de systèmes discrets connus, d'en déduire l'éventuelle limite continue. Ainsi, le système physique statistique discret approximant la mesure de Yang-Mills est défini sur des sous-réseaux finis de $\epsilon\mathbb{Z}^2$: à chaque arrête est associé un élément du groupe G, la mesure uniforme étant donné alors par le produit des mesures de Haar, et l'énergie étant de la forme $e^{-\frac{1}{2}S_W}$ où S_W, l'action de Wilson, est donnée par la

somme sur toutes les faces du réseau :

$$S_W = -2\mathsf{dim}(\rho)\sum_F \epsilon^{-2}\Re\left[1 - \frac{1}{\mathsf{dim}(\rho)}\mathsf{Tr}\left(\rho(h(e_1)...h(e_n)\right)\right],$$

où $h(e_1)...h(e_n)$ est le produit des éléments associés aux arêtes bordant la face F.

Cependant, il a été établi par la suite[11] que la limite de ces systèmes en deux dimensions avaient la même limite que ceux utilisant l'action de Villain, S_V. Cette action est en réalité le logarithme de la densité du mouvement brownien sur G. Cette action a la bonne propriété d'être exacte au sens où la restriction à un réseau, du système continu limite a la même loi que le système discret défini sur ce même réseau grâce à l'action de Villain. L'action de Villain joue donc le même rôle que $-\frac{x^2}{2}$ dans le cas du mouvement brownien.

À partir de ce moment là, la limite liée au mouvement brownien sur G étant identifiée, les mathématiciens ont pu construire directement l'objet continu. Nous présentons par la suite la construction de la mesure de Yang-Mills gaussienne, via la méthode stochastique[12], proposée par L. Gross, B.K. Driver, A.N. Sengupta, etc, ainsi que la méthode physique statistique exacte[13], achevée par T. Lévy, dans la continuité de A. A. Migdal, S. Albeverio-R. Høegh-Krohn-H. Holden.

2.5.1. La méthode stochastique. — Par la suite, nous allons voir la méthode stochastique qui va permettre d'introduire la notion de lassos et de se familiariser avec les transformations de jauge.

On peut associer une notion de courbure à une connexion régulière. Reprenons l'exemple de $P = \mathbb{R}^2 \times G$ comme G-fibré principal sur $\mathbb{R}^2$, et donnons-nous une connexion A sur P. Pour tout point a du plan, nous avons vu comment nous pouvions relever une boucle l basée en a en un chemin $l_{a,g}$ dans P de façon horizontale à partir de n'importe quel point $(a, g) \in P$. En particulier, en prenant la transformation de jauge triviale $\mathbb{R}^2 \to \mathbb{R}^2 \times G$ qui à a associe (a, e), où e est l'élément neutre de G, ceci nous a permis de définir la notion d'holonomie comme étant l'unique élément h_l tel que $l_{a,e}(1) = l_{a,e}(0).h_l$. La courbure est alors la dérivée de l'holonomie dans le sens où elle nous informe localement de la déformation liée à la connexion. Prenons a un point du plan, et prenons deux vecteurs v et w. Considérons le chemin $p_a^{v,w}(\epsilon)$ basé en a et bordant le parallélogramme dont le premier côté est égal à ϵv puis ϵw comme dessiné dans la figure 3. La courbure au point a appliquée en v et w, notée $\Omega_a(v, w)$, est un élément de l'algèbre de Lie $\mathfrak{g}$ de G défini par le fait que l'holonomie obtenue en parcourant $p_a^{v,w}(\epsilon)$ est de la forme :

$$h\left(p_a^{v,w}(\epsilon)\right) \simeq \exp(\epsilon^2\Omega_a(v, w) + o(\epsilon^2)). \tag{6}$$

[11] Par exemple, voir le théorème 8.5 de [**11**].
[12] Méthode stochastique : [**16**], [**17**], [**11**], [**12**], [**29**], [**30**].
[13] Méthode physique statistique exacte : [**33**], [**34**], [**2**], [**4**], [**3**], [**21**], [**23**].

Ceci définit une forme bilinéaire définie en tout point du plan et à valeurs dans $\mathfrak{g}$, l'algèbre de Lie de G.

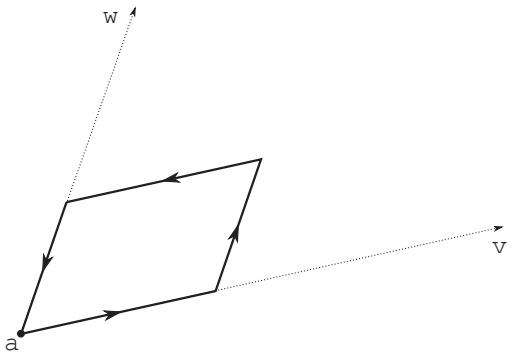

FIGURE 3. Chemin $p_a^{v,w}(\epsilon)$

Remarquons que $\Omega_a(w,v) = -\Omega_a(v,w)$: en effet le chemin $p_a^{w,v}(\epsilon)$ est tout simplement le chemin inverse de $p_a^{v,w}(\epsilon)$, ainsi l'holonomie $h_{p_a^{w,v}(\epsilon)}$ est égale à $\left(h_{p_a^{v,w}(\epsilon)}\right)^{-1} = \exp(-\epsilon^2\Omega_a(v,w) + o(\epsilon^2))$, d'où le fait que Ω soit antisymétrique.

Il existe alors une fonction F_A définie sur le plan à valeurs dans $\mathfrak{g}$ telle qu'en tout point (x,y) du plan :

$$\Omega_{(x,y)} = F_A(x,y)dx \wedge dy.$$

L'idée derrière la méthode stochastique est de remarquer que seule la courbure apparaît dans l'action de Yang-Mills : en effet, l'action de Yang-Mills est donnée par :

$$\int_{\mathbb{R}^2} \|F_A(x,y)\|^2 dxdy,$$

où $\|.\|$ est la norme sur $\mathfrak{g}$ associée à un produit scalaire invariant sur $\mathfrak{g}$. En choisissant un bon choix de jauge, on peut montrer que le jacobien de la transformation qui à une connection associe sa courbure est constante dans le cas de la dimension 2. Ceci permet d'interpréter la mesure de Yang-Mills comme une connexion aléatoire dont la courbure serait un bruit blanc à valeurs dans l'algèbre de Lie $\mathfrak{g}$. Cependant la connaissance de la courbure ne nous donne qu'une information microscopique localisée autour de chaque point x. Il faut pouvoir recoller ou intégrer ces informations afin d'obtenir une information macroscopique comme l'holonomie de la connexion sous-jacente le long d'un chemin ou d'une boucle.

Définition 2.2. — *Soit $a \in \mathbb{R}^2$. Un lasso régulier d'échelle ϵ est la donnée d'une boucle de la forme $pp_a^{v,w}(\epsilon)p^{-1}$ où p est un chemin reliant 0 à a.*

Ces lassos[14] sont importants puisqu'ils permettent d'approximer l'holonomie de toute boucle $\mathsf{L}_0(\mathbb{R}^2)$. En effet, considérons une fonction $h \in \mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$ régulière, dans le sens où pour toute suite de boucles l_n basée en 0 qui converge[15] vers $l \in \mathsf{L}_0(\mathbb{R}^2)$ alors $h(l_n)$ converge vers $h(l)$ quand n tend vers l'infini. Alors pour toute boucle $l \in \mathsf{L}_0(\mathbb{R}^2)$, il existe pour tout $\epsilon > 0$ un produit de lassos réguliers d'échelle ϵ, dénoté l_ϵ tel que $h(l_\epsilon)$ converge vers $h(l)$ quand ϵ tend vers 0. On peut donc dans un sens approximer toute boucle régulière par un produit de lassos réguliers.

Nous donnons un exemple de décomposition de boucle en lassos réguliers dans la figure 5. Pour comprendre cette figure, on doit se rappeler que si un chemin est de la forme $p = p_1 e e^{-1} p_2$ alors $h(p) = h(p_2)h_e^{-1}h_e h(p_1) = h(p_2)h(p_1) = h(p_1 p_2)$: tant que l'on travaille au niveau des holonomies, on peut supprimer les allers-retours consécutifs le long d'un même chemin.

Ainsi on peut donc retrouver l'holonomie d'une boucle connaissant l'holonomie le long de lassos réguliers. Cependant l'information microscopique fournie par la courbure, comme indiqué par l'équation (6), est le comportement, quand ϵ est petit, de l'holonomie $h\left(p_x^{v,w}(\epsilon)\right)$ et non le comportement asymptotique le long de lassos réguliers d'échelle ϵ. Ainsi, pour un lasso régulier de la forme $pp_x^{v,w}(\epsilon)p^{-1}$, nous ne connaissons pas le comportement de $h(pp_x^{v,w}(\epsilon)p^{-1})$, il nous manque pour cela la valeur de $h(p)$.

Nous avons tout de même le choix de jauge pour mesurer l'holonomie : nous pouvons toujours supposer, par invariance par transformation de jauge de la mesure à construire, que les connexions considérées ont une holonomie égale à e pour tout chemin horizontal, c'est à dire parallèle à l'axe (Ox), ainsi que pour tout chemin inclus dans l'axe des ordonnées. En effet prenons une fonction $h \in \mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$, et considérons la transformation de jauge j telle que pour tout point (x, y) du plan :

$$j(x, y) = h\left((0, 0) \to (0, y) \to (x, y)\right),$$

où $(a, b) \to (c, d)$ représente le chemin rectiligne entre (a, b) et (c, d). Alors pour tout chemin p, si p est horizontal ou inclus dans l'axe des ordonnées :

$$(j \bullet h)(p) = e.$$

En effet dans le cas, par exemple, où p est horizontal, de la forme $(x_1, y) \to (x_2, y)$ avec $x_1 < x_2$, et donc $(j \bullet h)(p)$ est égal à $j((x_2, y))^{-1}h((x_1, y) \to (x_2, y))j((x_1, y))$ donc à

$$h\left((0, 0) \to (0, y) \to (x_2, y)\right)^{-1} h((x_1, y) \to (x_2, y))h\left((0, 0) \to (0, y) \to (x_1, y)\right).$$

En utilisant le fait que $(0, y) \to (x_2, y)$ est égal à la concaténation de $(0, y) \to (x_1, y)$ et $(x_1, y) \to (x_2, y)$, et en utilisant la multiplicativité de h, on obtient que

[14] Pour une des première utilisation des lassos, voir [**12**].
[15] Pour la distance rappelée à la fin de la section 2.1.

$(j \bullet h)(p)$ est égal à :

$$\begin{aligned} &h\left((0,0) \to (0,y) \to (x_2,y)\right)^{-1} h\left((0,0) \to (0,y) \to (x_1,y) \to (x_2,y)\right) \\ &\quad = h\left((0,0) \to (0,y) \to (x_2,y)\right)^{-1} h\left((0,0) \to (0,y) \to (x_2,y)\right) \\ &\quad = e. \end{aligned}$$

Ainsi on peut se ramener à des holonomies neutres le long des chemins horizontaux et le long de l'axe des ordonnées.

Proposition 2.1. — *Un L-lasso régulier est un lasso régulier de la forme* $pp_x^{v,w}(\epsilon)p^{-1}$ *où* p *est un chemin composé d'un segment vertical partant de* 0 *et d'un segment horizontal arrivant à* x*. L'ensemble des concaténations de L-lassos réguliers est dense dans* $\mathsf{L}_0(\mathbb{R}^2)$ *dans le sens des holonomies régulières, c'est-à-dire que pour toute boucle* $l \in \mathsf{L}_0(\mathbb{R}^2)$*, pour tout* $\epsilon > 0$*, il existe un produit* l_ϵ *de L-lassos réguliers d'échelle* ϵ *tels que pour tout* $h \in \mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$ *continue pour la convergence définie à la fin de la section 2.1,* $h(l_\epsilon)$ *converge vers* $h(l)$ *quand* ϵ *tend vers* 0*.*

Or pour un L-lasso régulier $pp_x^{v,w}(\epsilon)p^{-1}$ donné, pour toute fonction régulière $h \in \mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$ qui satisfait le choix de jauge qui rend neutre les holonomies le long des chemins horizontaux et le long de l'axe des ordonnées, $h\left(pp_x^{v,w}(\epsilon)p^{-1}\right) = h(p_x^{v,w}(\epsilon))$. On peut donc espérer recoller l'information microscopique donnée par la courbure afin d'obtenir une information macroscopique qu'est l'holonomie.

Considérons un bruit blanc sur $(\mathbb{R}^2, dxdy)$ à valeurs dans $\mathfrak{g}$, c'est-à-dire une collection de variables aléatoires gaussiennes centrées $(\mathsf{F}(f))_{f \in L^2_{\mathsf{dx}}(\mathbb{R}^2)}$ à valeurs dans $\mathfrak{g}$ telle que pour toutes fonctions f et g dans $L^2_{dxdy}(\mathbb{R}^2)$, tout réel α, presque sûrement :

$$\mathsf{F}(\alpha f + g) = \alpha \mathsf{F}(f) + \mathsf{F}(g),$$

et pour tous X, Y dans l'algèbre de Lie $\mathfrak{g}$ de G :

$$\mathbb{E}\left[\langle \mathsf{F}(f), X\rangle_{\mathfrak{g}} \langle \mathsf{F}(g), Y\rangle_{\mathfrak{g}}\right] = \left(\int_{\mathbb{R}^2} f(x)g(y)dxdy\right)\langle X, Y\rangle_{\mathfrak{g}}.$$

Reprenons notre discussion sur la courbure : nous aimerions construire une connexion dont l'holonomie vérifie pour tout L-lasso régulier : $h_{pp^{e_1,e_2}_{(x,y)}(\epsilon)p^{-1}} \simeq \exp(\epsilon^2 F_A(x,y))$ où (e_1, e_2) est la base canonique de $\mathbb{R}^2$ et $F_A(x,y)$ serait un bruit blanc formel, c'est-à-dire une collection de variables aléatoires indépendantes gaussiennes centrées réduites dans $\mathfrak{g}$. Considérons $C^{e_1,e_2}_{(x,y)}(\epsilon)$ le parallélogramme bordé par $p^{e_1,e_2}_{(x,y)}(\epsilon)$. En utilisant le bruit blanc F, qui lui est bien défini, on a $\mathsf{F}\left(\mathbb{1}_{C^{e_1,e_2}_{(x,y)}(\epsilon)}\right) \simeq \epsilon^2 F_A(x,y)$. Ainsi nous voulons construire une connexion dont l'holonomie vérifie :

$$h_{pp^{e_1,e_2}_{(x,y)}(\epsilon)p^{-1}} \simeq \exp(\mathsf{F}(C^{e_1,e_2}_{(x,y)}(\epsilon)))$$

pour tout L-lasso régulier $pp_{(x,y)}^{e_1,e_2}(\epsilon)p^{-1}$.

Prenons par exemple un chemin vertical $p = (x, y_1) \to (x, y_2)$, paramétré par l'intervale $[0, 1]$. Nous aimerions définir l'holonomie de la connexion aléatoire le long de p. Or par le choix de jauge, nous savons que l'holonomie le long de p est égale à l'holonomie le long de l, où l est dessinée dans la figure 4. En effet nous n'avons rajouté que des portions de chemins horizontaux ou inclus dans l'axe des ordonnées.

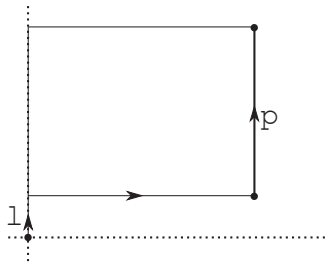

FIGURE 4. Chemin et boucle associée

On peut décomposer le lacet en produit de chemins comme dans la figure 5. La décomposition donnée suit un certain ordre assez évident : on remplit ligne par ligne en allant de gauche à droite puis de bas en haut. L'holonomie aléatoire $h(l)$ est donc approximée par un produit de plus en plus grand d'exponentielles d'un mouvement brownien sur $\mathfrak{g}$ à des temps de plus en plus en plus petits : en faisant cela, on obtient la loi d'un mouvement brownien sur G à un temps fixé, en l'occurrence ici le volume entouré par la boucle l.

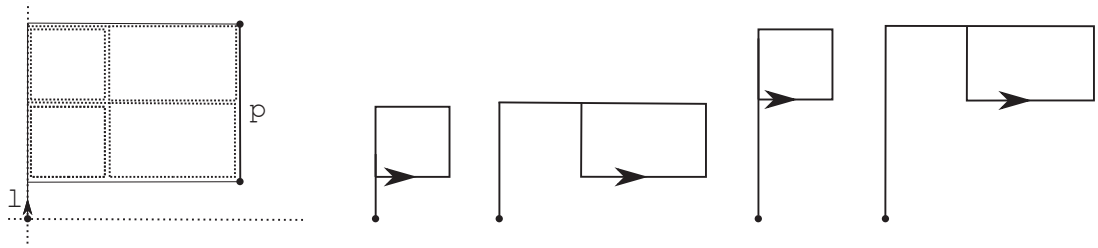

FIGURE 5. Décomposition en L-lassos réguliers

Considérons une bande B_ϵ de hauteur ϵ apparaissant dans la décomposition du lacet : elle se décompose elle-même en L-lassos réguliers, et on peut approximer le produit sur ces lassos L-réguliers $\prod \exp(\mathsf{F}(C_{(x,y)}^{e_1,e_2}(\epsilon)))$ par l'exponentielle de la

somme[16] :

$$\exp\left(\sum_{(x,y)} \mathsf{F}\left(C_{(x,y)}^{e_1,e_2}(\epsilon)\right)\right) \simeq \exp\left(\mathsf{F}(B_\epsilon)\right).$$

On doit maintenant multiplier par rapport aux bandes : on mutiplie de plus en plus d'exponentielles indépendantes d'un mouvement brownien sur $\mathfrak{g}$ à des temps de plus en plus petits[17]. Ceci revient à résoudre l'équation de Stratonovich :

$$dh_t = d\mathsf{F}(A_t)h_t,$$

où A_t est le rectangle compris entre l'axe des ordonnées et la courbe $(p_s)_{t\in[0,t]}$: l'holonomie cherchée $h(l)$ est donc égale à h_1.

On peut alors généraliser ceci pour tout chemin $(p_t)_{t\in[0,1]}$ vertical dans le sens où sa dérivée dans la direction y ne s'annule pas. On associe, pour tout $t \in [0,1]$, le domaine A_t compris entre l'axe des ordonnées et $(p_s)_{s\in[0,t]}$. L'holonomie associée à p, $h(p)$ est alors donnée par h_1 où (h_t) est la solution de l'équation stochastique au sens de Stratonovich :

$$dh_t = d\mathsf{F}(A_t)h_t.$$

On peut alors considérer l'holonomie aléatoire le long de concaténations de chemins verticaux et de lignes horizontales par la propriété de multiplicativité de l'holonomie et en se rappelant du choix de jauge.

En choisissant une autre mesure de densité vol au lieu de la mesure de Lebesgue $dxdy$, on peut construire une famille de champs d'holonomies planaires. B. K. Driver et A. N. Sengupta ont montré que cette famille satisfaisait les trois axiomes présentés dans la définition 2.1 : c'est donc un champ markovien d'holonomies planaire.

Dans les travaux de L. Gross, B. K. Driver et A. N. Sengupta, l'holonomie n'était pas étendue à une classe plus générale et plus naturelle de chemins, les chemins de longueur finie. De plus, l'inconvénient de cette méthode est la difficulté d'étendre cette construction dans le cas où l'on ne considère plus des processus indexés par des chemins sur le plan, mais sur une surface quelconque. Cette généralisation, donnée par Sengupta[18], utilise un habile conditionnement.

2.5.2. La méthode physique statistique exacte. — Nous allons maintenant nous intéresser à la méthode physique statistique exacte. On retrouvera, dans la construction des champs markoviens d'holonomies présentée dans [Champs], l'utilisation de certains arguments clefs de la méthode physique statistique exacte. Dans la méthode stochastique, nous nous sommes restreints à des variables

[16] Le nombre de lassos est d'ordre ϵ^{-1} mais chacun est de taille ϵ^2.
[17] Mais l'approximation précédente ne marche plus car le nombre de bandes est d'ordre ϵ^{-1} et chaque bande est de taille ϵ.
[18] Dans la monographie [**30**].

gaussiennes ; dans la méthode physique statistique exacte, nous allons présenter un cas plus général.

Afin de mener à bien la méthode développée par A. A. Migdal, S. Albeverio-R. Høegh-Krohn-H. Holden, et T. Lévy, on doit se donner des discrétisations de l'espace d'indexation $\mathsf{P}(\mathbb{R}^2)$. Soit $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ un graphe fini plongé dans $\mathbb{R}^2$ dont l'ensemble des sommets est $\mathbb{V}$, l'ensemble des arêtes est $\mathbb{E}$ et l'ensemble des faces est $\mathbb{F}$. Supposons que les arêtes de $\mathbb{G}$ sont régulières. Une discrétisation de $\mathsf{P}(\mathbb{R}^2)$ est donnée par l'ensemble $\mathsf{P}(\mathbb{G})$ des chemins obtenus par concaténation d'arêtes de $\mathbb{G}$.

Cependant, afin de construire une mesure sur $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$, il faut pouvoir prendre la limite projective des mesures qui seront définies sur les ensembles de la forme $\mathcal{M}ult(\mathsf{P}(\mathbb{G}), G)$. Cela restreint la classe de graphes finis que l'on peut considérer. En effet, si les arêtes sont seulement de longueur finie alors on peut trouver deux graphes $\mathbb{G}_1$ et $\mathbb{G}_2$ tels qu'il n'existe pas de troisième graphe fini plus fin que $\mathbb{G}_1$ et que $\mathbb{G}_2$: ceci est dû au fait que des arêtes de longueurs finies peuvent s'intersecter une infinité de fois. Afin de pallier ce problème, on peut se restreindre à des graphes finis dont les arêtes sont des chemins réel-analytiques par morceaux[19] ou plus simplement, comme le propose T. Lévy[20], des graphes finis dont les arêtes sont des chemins affines par morceaux. Une seconde difficulté apparaît alors : lorsque l'on prend la limite projective des mesures sur $\mathcal{M}ult(\mathsf{P}(\mathbb{G}), G)$, où $\mathbb{G}$ est un graphe fini dont les arêtes sont affines par morceaux, on obtient une mesure définie sur l'ensemble des fonctions multiplicatives des chemins affines par morceaux $\mathsf{Aff}(\mathbb{R}^2)$ dans G, c'est-à-dire sur $\mathcal{M}ult(\mathsf{Aff}(\mathbb{R}^2), G)$. T. Lévy a démontré un théorème du type théorème de Kolmogorov [21] qui permet d'étendre certaines mesures de $\mathcal{M}ult(\mathsf{Aff}(\mathbb{R}^2), G)$ à $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$. Ce théorème sera expliqué plus en détail dans la section 9.2.2.

Donnons-nous un processus de Lévy $\mathsf{X} = (X_t)_{t\geq 0}$ sur G, invariant par conjugaison dont la loi possède en tout temps une densité, dénotée Q_t^{X}, par rapport à la mesure de Haar sur G. Comme vu précédemment, la construction proposée par T. Lévy procède en trois étapes :

1. **Définition d'une mesure sur $\mathcal{M}ult(\mathsf{P}(\mathbb{G}), G)$ pour tout graphe fini $\mathbb{G}$ plongé dans le plan dont les arêtes sont affines par morceaux.** Notons $\mathbb{F}^b$ l'ensemble des faces bornées de $\mathbb{G}$. Donnons-nous une orientation $\mathbb{E}^+$ de $\mathbb{E}$, c'est-à-dire un sens pour chaque arête de $\mathbb{G}$. Il est facile de comprendre que $\mathcal{M}ult(\mathsf{P}(\mathbb{G}), G) \simeq G^{\mathbb{E}^+}$: il est équivalent de se donner une fonction multiplicative sur $\mathsf{P}(\mathbb{G})$ ou de se donner une fonction sur $\mathbb{E}^+$ à valeurs dans G. En effet, étant donné $(h_e)_{e\in\mathbb{E}^+} \in G^{\mathbb{E}^+}$, on définit $h_{e^{-1}} = h_e^{-1}$ pour toute arête dans $\mathbb{E}^+$, et par multiplicativité, on étend h à tout chemin dans $\mathsf{P}(\mathbb{G})$: si $p =$

[19] Comme J.C. Baez dans [**7**].
[20] Dans la monographie [**23**].
[21] Théorème 3.3.1 dans [**23**].

$e_1...e_n$, $h_p = h_{e_n}...h_{e_1}$. Ainsi on peut définir une mesure sur $\mathcal{M}ult(\mathsf{P}(\mathbb{G}), G)$ en définissant une mesure sur $G^{\mathbb{E}^+}$. Or il existe une mesure uniforme sur ce dernier espace : le produit des mesures de Haar $\prod_{e\in\mathbb{E}^+} dh_e$.

Pour toute face bornée F, on considère une boucle, notée ∂F, qui borde F et qui parcourt la frontière de F dans le sens horaire. Étant donné une densité lisse $\mathsf{vol} \in \mathcal{D}(\mathbb{R}^2)$, on peut alors définir :

$$\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathbb{G}} = \prod_{F\in\mathbb{F}^b} Q^{\mathsf{X}}_{\mathsf{vol}(F)}\left(h_{\partial F}\right) \prod_{e\in\mathbb{E}^+} h_e,$$

qui ne dépend ni du choix des boucles ∂F, ni de l'orientation $\mathbb{E}^+$ choisie. Par exemple, supposons que pour une face bornée F nous choisissions un autre choix de boucle l que celui considéré ∂F : on peut décomposer $\partial F = p_1p_2$ de sorte que $l = p_2p_1$. Or $(X_t)_{t\geq 0}$ étant invariant par conjugaison, sa densité est aussi invariante par conjugaison :

$$Q^{\mathsf{X}}_{\mathsf{vol}(F)}(h_{\partial F}) = Q^{\mathsf{X}}_{\mathsf{vol}(F)}(h_{p_2}^{-1}h_{\partial F}h_{p_2}) = Q^{\mathsf{X}}_{\mathsf{vol}(F)}(h_{p_1}h_{p_2}) = Q^{\mathsf{X}}_{\mathsf{vol}(F)}(h_l).$$

2. **Limite projective des mesures $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathbb{G}}$.** Les mesures $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathbb{G}}$ sont compatibles entre elles. Prenons $\mathbb{G}$ et $\mathbb{G}'$ deux graphes finis plongés dans $\mathbb{R}^2$ dont les arêtes sont affines par morceaux. Supposons que $\mathbb{G}'$ est plus fin que $\mathbb{G}$: il existe une application de restriction $\rho_{\mathsf{P}(\mathbb{G}),\mathsf{P}(\mathbb{G}')}$ qui restreint les applications multiplicatives sur $\mathsf{P}(\mathbb{G}')$ en applications multiplicatives sur $\mathsf{P}(\mathbb{G})$. L'image de la mesure $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathbb{G}'}$ par $\rho_{\mathsf{P}(\mathbb{G}),\mathsf{P}(\mathbb{G}')}$ est égale à $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathbb{G}}$: $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathbb{G}'} \circ \rho^{-1}_{\mathsf{P}(\mathbb{G}),\mathsf{P}(\mathbb{G}')} = \mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathbb{G}}$. Il existe, par un théorème général[(22)], une unique mesure $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathsf{Aff}}$ définie sur $\mathcal{M}ult(\mathsf{Aff}(\mathbb{R}^2), G)$ telle que pour tout graphe $\mathbb{G}$ fini plongé dans le plan dont les arêtes sont affines par morceaux,$\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathbb{G}}$ est l'image de $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathsf{Aff}}$ par l'application de restriction $\rho_{\mathsf{P}(\mathbb{G}),\mathsf{Aff}(\mathbb{R}^2)}$.
3. **Extension des chemins affines aux chemins de longueur finie.** La mesure obtenue $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathsf{Aff}}$ satisfait une estimée de la forme :

$$\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathsf{Aff}}\left[d(e, h_l)\right] \leq K\sqrt{\mathsf{vol}(D)}$$

pour toute boucle simple l bordant un domaine d'aire D. Cette estimée est du même type que celle apparaissant dans le théorème de continuité de Kolmogorov. Or dans le contexte des champs d'holonomies, T. Lévy a démontré que cette condition permettait justement d'étendre par continuité la mesure $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathsf{Aff}}$ en une mesure $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol}}$ définie sur $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$.

La famille $\left(\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol}}\right)_{\mathsf{vol}\in\mathcal{D}(\mathbb{R}^2)}$ est un champ markovien d'holonomies planaire. Ce résultat se déduit par exemple du fait que cette construction est un cas particulier de la nouvelle construction des champs markoviens d'holonomies planaires que l'on donne dans le chapitre de thèse [Champs] et que l'on explique dans la section

[(22)] Proposition 2.1 de [Champs] et proposition 2.2.2 de [**23**]

5.2. Si X est un mouvement brownien sur G, il s'avère que le champ markovien d'holonomies planaire $\left(\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol}}\right)_{\mathsf{vol}\in\mathcal{D}(\mathbb{R}^2)}$ coïncide avec celui obtenu avec la méthode stochastique.

L'avantage de la construction via la méthode physique statistique exacte est qu'elle permet de considérer n'importe quel groupe de Lie G compact. De plus, il est facile d'étendre la construction afin de définir des champs d'holonomies sur toute surface compacte bidimensionelle M. En effet, pour ce faire, il suffit de considérer des graphes plongés dans M plutôt que dans le plan. Cependant, cela ne permet que de définir des champs markoviens d'holonomies planaires associés à des processus de Lévy $(X_t)_{t\geq 0}$ qui vérifient deux conditions :

1. ils doivent être invariants par conjugaison par le groupe G,
2. pour tout temps $t \geq 0$, la loi de X_t doit être à densité par rapport à la mesure de Haar sur G.

Comme nous le verrons dans la section 5.2, aucune de ces deux conditions n'est nécessaire afin de définir un champ markovien d'holonomies planaire associé à $(X_t)_{t\geq 0}$. La preuve passera seulement par un changement de la première étape : elle se rapprochera de la méthode projective illustrée auparavant dans le cas du mouvement brownien.

3. Invariance par homéomorphisme et tresses

Dans la définition des champs markoviens d'holonomies apparaissent deux notions de symétries. Ce sont en réalité ces deux symétries qui sont les deux piliers de la thèse. La première est l'invariance par une classe d'homéomorphismes, tandis que la seconde est la symétrie par transformation de jauge. Dans cette section, nous traitons de l'invariance par certains homéomorphismes, et de la symétrie qu'elle induit en probabilité : l'invariance par tressage.

3.1. Les tresses. — Le groupe de tresses possède de nombreuses facettes qui sont d'égale importance dans ce travail. Ainsi on peut l'approcher du point de vue combinatoire, du point de vue géométrique et du point de vue algébrique.

3.1.1. Facette géométrique. — Les tresses généralisent la notion de permutations. Prenons un certain nombre d'objets, numérotés par $1, 2, ..., n$, et posés sur une table. Que signifie permuter ces objets ? Habituellement, une permutation en mathématiques n'est que le résultat de l'action de permuter les objets $1, ..., n$. Ainsi une permutation de $1, 2, 3, 4$ est donnée par exemple par $4, 2, 3, 1$ et nous oublions alors comment nous avons permuté les éléments. Les objets 2 et 3 sont-ils restés tout le temps à leur place ? La notion de tresse permet de corriger cet oubli : une tresse géométrique à n brins est la donnée de n fonctions $(\gamma_j)_{j=1}^n$ continues de $[0, 1]$ dans $\mathbb{R}^2$ qui ne s'intersectent pas et telles que pour tout j dans $\{1, ..., n\}$, $\gamma_j(0) = (j, 0)$ et l'ensemble des valeurs prises par $\gamma_j(1)$ est égal à $\{(1, 0), ..., (n, 0)\}$. En regardant l'image 6, nous voyons donc qu'une tresse géométrique à n brins est l'action de permuter n objets posés en ligne. Deux tresses géométriques sont

équivalentes par isotopie si on peut transformer l'une en l'autre de façon continue sans faire passer un brin au travers d'un autre : une tresse est alors la donnée d'une classe d'isotopie d'une tresse géométrique.

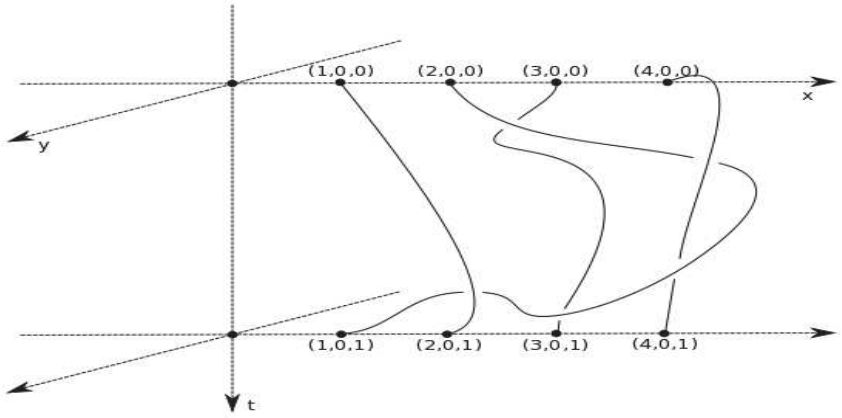

FIGURE 6. Tresse géométrique

En voyant une tresse comme la permutation de n objets en ligne, on voit que l'on peut composer deux tresses : il suffit de permuter les objets en suivant la seconde tresse puis de continuer à permuter en utilisant la première tresse. Un exemple est donné dans la figure 7. On obtient l'inverse d'une tresse en permutant dans l'autre sens les n objets. Ceci permet de munir l'ensemble des tresses à n brins d'une structure de groupe : c'est le groupe des tresses $\mathcal{B}_n$.

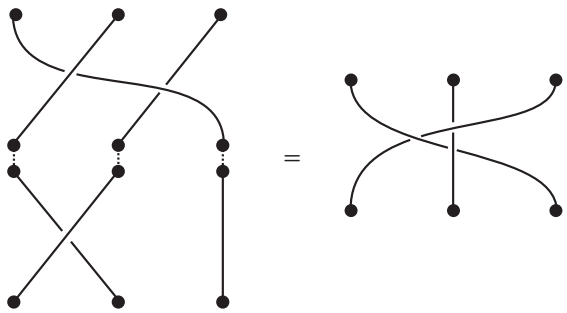

FIGURE 7. Multiplication de deux tresses

Nous avons vu que la notion de permutation en mathématiques ne formalise pas la notion de permutation physique d'objets. C'est en partie dû au fait que l'on perd, en considérant une permutation mathématique, la dynamique ayant échangée les particules. Les tresses permettent de conserver une trace de l'évolution des particules, le temps s'écoulant le long de la tresse de haut en bas : la tresse est dans un sens un objet dynamique.

Nous allons expliquer comment nous pouvons coder, de façon statique, une permutation physique de particules. Pour cela, considérons $\mathcal{P}_1$ et $\mathcal{P}_2$ deux ensembles de particules sur la droite d'équation $y = 1$, et supposons que l'on ait obtenu $\mathcal{P}_2$ en permutant l'ordre des particules dans l'ensemble $\mathcal{P}_1$. Remarquons que, quitte à bouger les particules sans changer leur ordre sur la droite, on peut supposer que $\mathcal{P}_1 = \mathcal{P}_2 = \{(i, 1), i \in \{1, ..., n\}\}$. Considérons alors le disque $\mathbb{D}$ de centre 0 et de rayon $n + 1$. Une permutation physique est codée par un difféomorphisme ϕ de $\mathbb{D} \setminus \mathcal{P}_1$ dans $\mathbb{D} \setminus \mathcal{P}_2$ qui ne modifie que l'intérieur de $\mathbb{D}$.

En effet, étant donné un tel difféomorphisme, un lemme appelé "truc d'Alexander" permet de montrer qu'il existe une dynamique continue ϕ_t de difféomorphismes, de $\mathbb{D}$ dans lui-même, telle que ϕ_0 soit l'identité, et ϕ_1 soit égale à ϕ. On obtient donc une permutation physique des points de $\mathcal{P}_1$. En considérant alors l'évolution des points dans $\mathcal{P}_1$ par $(\phi_t)_{t\geq 0}$, on voit apparaitre la figure 6, c'est à dire une tresse géométrique. Il s'avère que le choix de la dynamique $(\phi_t)_{t\geq 0}$ n'influence pas la classe d'isotopie de la tresse géométrique obtenue, c'est à dire que l'on obtient la même tresse. A tout difféomorphisme de $\mathbb{D} \setminus \mathcal{P}_1$ dans $\mathbb{D} \setminus \mathcal{P}_2$ qui ne modifie que l'intérieur de $\mathbb{D}$, on peut alors associer une tresse.

3.1.2. Facette algébrique. — Il est aussi intuitif de voir que $\mathcal{B}_n$ est engendré par les classes d'équivalence des tresses qui ne font que permuter deux objets adjacents, comme dessiné dans la figure 8. La tresse permutant i et $i + 1$ en faisant passer i derrière $i + 1$ sera notée β_i. De plus, en oubliant comment nous avons permuté, et en ne conservant que le résultat final, nous obtenons pour toute tresse une permutation : cela définit un morphisme $\mathsf{p} : \mathcal{B}_n \to \mathfrak{S}_n$ qui envoie β_i sur la transposition $(i, i + 1)$.

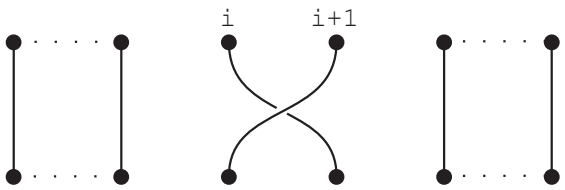

FIGURE 8. La tresse β_i

Il sera important par la suite de considérer une action du groupe de tresses $\mathcal{B}_n$ sur les n-uplets d'éléments de G. En effet, nous considèrerons des systèmes de particules possédant des caractéristiques associées à chacune d'entre elles, et pour lesquelles passer derrière une autre particule affecte la caractéristique de cette dernière. Il s'agit de déterminer comment sont affectées les caractéristiques lors d'un réarrangement des particules.

Définition 3.1. — *L'action naturelle de $\mathcal{B}_n$ sur G^n est l'unique action telle que pour tout $i \in \{1, ..., n-1\}$, et tout n-tuple $(x_i)_{i=1}^n$ dans G^n,*

$$\beta_i \bullet (x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n) = \left(x_1, ..., x_{i-1}, x_i x_{i+1} x_i^{-1}, x_i, ..., x_n\right).$$

Considérons μ une mesure sur G^n ainsi que $(X_1, ..., X_n)$ de loi μ. Considérons β une tresse dans $\mathcal{B}_n$. La loi de $\beta \bullet (X_1, ..., X_n)$ est dénotée par $\beta \bullet \mu$.

De façon informelle, lorsque la particule i passe derrière la particule $i+1$, la particule $i+1$ voit sa caractéristique conjuguée par celle portée par la particule i, cette dernière ne subissant aucune modification. Remarquons que, quand la particule $i+1$ passe derrière la particule i, c'est-à-dire quand on applique la tresse β_i^{-1}, la caractéristique de la particule $i+1$ reste inchangée et celle portée par la particule i est conjugée, elle devient $x_{i+1}^{-1} x_i x_{i+1}$. On peut remarquer que le produit dans l'ordre croissant des caractéristiques des particules reste inchangé par l'action d'une tresse, par exemple lorsque l'on applique β_i :

$$x_1...x_{i-1}(x_i x_{i+1} x_i^{-1}) x_i ... x_n = x_1 ... x_n.$$

La transformation induite par les tresses possède donc une quantité conservée : le produit des caractéristiques. En réalité cette remarque est importante puisqu'elle permet d'introduire la facette algébrique du groupe des tresses. En effet, par un théorème d'Artin, on peut aussi voir le groupe des tresses $\mathcal{B}_n$ comme l'ensemble des automorphismes a du groupe libre $\mathbb{F}^n$, engendré par $\{e_1, ..., e_n\}$, tels que pour tout $i \in \{1, ..., n\}$:

$\mathsf{a}(e_i)$ est conjugué à un élément de la famille $\{e_1, ..., e_n\}$,

le produit dans l'ordre décroissant est conservé : $\mathsf{a}\,(e_n...e_1) = \mathsf{a}(e_n)...\mathsf{a}(e_1)$.

En particulier, il existe une action du groupe des tresses $\mathcal{B}_n$ sur le groupe libre $\mathbb{F}^n$: celle-ci est donnée par le fait que pour tout $i \in \{1, ..., n-1\}$:

$$\begin{aligned} \beta_i(e_j) &= e_j \text{ si } j \notin \{i, i+1\}, \\ \beta_i(e_i) &= e_{i+1}, \\ \beta_i(e_{i+1}) &= e_{i+1} e_i e_{i+1}^{-1}. \end{aligned}$$

3.2. Système bosonique/ Système anyonique. — Afin de donner des intuitions sur les champs markoviens d'holonomies planaires, il est apparu que les notions de systèmes bosoniques et anyoniques étaient d'une grande aide. Ces deux notions, développées pour les besoins d'un exposé, ne doivent pas être comparées aux notions provenant de la littérature, mais doivent être utilisées comme des "modèles-jouets".

3.2.1. Système bosonique : théorème de de Finetti. — Dans cette section, on considère un système naturel qui permet d'introduire la notion d'invariance par le groupe symétrique. Considérons un nombre dénombrable de particules indistinguables qui possèdent une caractéristique aléatoire modélisée par un élément d'un groupe compact topologique G, par exemple des boussoles dont les flèches indiquent un élément de $U(1)$. Peut-on comprendre la loi des caractéristiques aléatoires associées aux particules observées ? Le théorème de de Finetti répond exactement à cette question : les caractéristiques sont conditionnellement indépendantes. La modélisation d'un tel système de particules est donnée dans la définition suivante.

Définition 3.2. — *Un système bosonique est la donnée pour tout sous-ensemble fini $\mathcal{P}$ de points dans le plan d'une mesure $\mathbb{E}_{\mathcal{P}}$ sur $G^{\mathcal{P}}$ telle que :*

1. **Compatibilité :** *Si $\mathcal{P}_1 \subset \mathcal{P}_2$ sont deux sous-ensembles finis de points dans le plan, considérons l'application de restriction : $r_{\mathcal{P}_1,\mathcal{P}_2} : G^{\mathcal{P}_2} \to G^{\mathcal{P}_1}$. On a :*
$$\mathbb{E}_{\mathcal{P}_1} = \mathbb{E}_{\mathcal{P}_2} \circ r^{-1}_{\mathcal{P}_1,\mathcal{P}_2}.$$
2. **Indistinguabilité :** *Pour tous sous-ensembles finis $\mathcal{P}_1$ et $\mathcal{P}_2$ de points dans le plan et tout homéomorphisme $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ tel que $\phi(\mathcal{P}_1) = \mathcal{P}_2$,*
$$\mathbb{E}_{\mathcal{P}_1} = \mathbb{E}_{\mathcal{P}_2} \circ \Phi^{-1},$$
où Φ est ici l'application de $G^{\mathcal{P}_2}$ dans $G^{\mathcal{P}_1}$ induite par l'homéomorphisme ϕ.

Le théorème de de Finetti dans sa version mathématique s'exprime de la façon suivante.

Théorème 3.1 **(Théorème de de Finetti)**. — *Soit $(X_n)_{n\in\mathbb{N}^*}$ une suite de variables aléatoires à valeurs dans G. Supposons que $(X_n)_{n\in\mathbb{N}^*}$ est échangeable, c'est-à-dire pour tout entier $n \in \mathbb{N}^*$, pour toute permutation $\sigma \in \mathfrak{S}_n$, $(X_k)_{k\in\{1,\dots,n\}}$ a la même loi que $(X_{\sigma(k)})_{k\in\{1,\dots,n\}}$. Alors conditionnellement à la tribu queue de $(X_n)_{n\in\mathbb{N}^*}$, $\mathcal{T}$, définie par $\mathcal{T} = \bigcap_{n\in\mathbb{N}} \sigma(X_k, k \geq n)$, la suite $(X_n)_{n\in\mathbb{N}}$ est indépendante. En particulier, il existe une mesure aléatoire μ qui est $\mathcal{T}$ mesurable telle que la loi de $(X_n)_{n\in\mathbb{N}}$ est égale à $\mathbb{E}[\mu^{\otimes\infty}]$. La réciproque est aussi vérifiée.*

Nous pouvons reformuler le théorème de de Finetti dans le cadre des systèmes bosoniques.

Théorème 3.2. — *Pour tout système bosonique $(\mathbb{E}_{\mathcal{P}})_{\mathcal{P}\subset\mathbb{R}^2}$, il existe une mesure aléatoire μ sur G telle que pour tout sous-ensemble fini $\mathcal{P}$ du plan, $\mathbb{E}_{\mathcal{P}} = \mathbb{E}\left[\mu^{\otimes\mathcal{P}}\right]$. Inversement à toute mesure aléatoire μ sur G on peut associer de la sorte un système bosonique.*

On voit qu'il existe un type spécial de systèmes bosoniques, les systèmes bosoniques purs, pour lesquels la mesure aléatoire μ associée est en réalité non-aléatoire : les caractéristiques des particules sont indépendantes de loi μ.

3.2.2. Système anyonique. — Quelle est la motivation derrière l'axiome d'indistinguabilité dans la définition d'un système bosonique ? Deux hypothèses sont nécessaires :

1. d'une part lorsqu'on permute deux particules, les particules n'interagissent pas et échangent simplement leur place,
2. d'autre part, une fois que l'on a permuté les particules, étant donné qu'elles sont indistinguables, les systèmes avant et après permutation doivent avoir la même loi.

On va considérer maintenant un nouveau système naturel qui permet de faire apparaître l'invariance par le groupe de tresses. Pour ceci, on remet en cause la première hypothèse. En effet, nous avons vu que l'action de permuter dans le sens commun est en réalité une tresse, et nous avons défini une action de tresse en supposant que les particules interagissent entre elles via leurs caractéristiques. En remplaçant l'axiome de non-interaction par l'interaction via l'action des tresses, on aboutit à un système que l'on appellera un système anyonique de particules[23].

Au lieu d'étudier un tel système anyonique de particules, nous allons axiomatiser le système de champ associé via la transformation particules/champs. Ces systèmes seront aussi appelés systèmes anyoniques et sont définis de la façon suivante.

Définition 3.3. — *Un système anyonique est la donnée pour tout sous-ensemble fini $\mathcal{P}$ de points dans le plan ne contenant pas* 0 *d'une mesure $\mathbb{E}_{\mathcal{P}}$ sur $\mathsf{Hom}\left(\pi_1(\mathbb{R}^2 \setminus \mathcal{P}), G^{op}\right)$, où $\pi_1(\mathbb{R}^2 \setminus \mathcal{P})$ est le groupe fondamental basé en* 0*, telle que :*

1. **Compatibilité :** *Soient $\mathcal{P}_1 \subset \mathcal{P}_2$ deux sous-ensembles finis de points dans le plan ne contenant pas* 0*. Considérons l'application de restriction :*

$$r_{\mathcal{P}_1,\mathcal{P}_2} : \mathsf{Hom}\left(\pi_1(\mathbb{R}^2 \setminus \mathcal{P}_2), G^{op}\right) \to \mathsf{Hom}\left(\pi_1(\mathbb{R}^2 \setminus \mathcal{P}_1), G^{op}\right).$$

On a :

$$\mathbb{E}_{\mathcal{P}_1} = \mathbb{E}_{\mathcal{P}_2} \circ r^{-1}_{\mathcal{P}_1,\mathcal{P}_2}.$$

[23] Un système anyonique de particules est la donnée pour tout sous-ensemble fini $\mathcal{P}$ de points sur la droite d'équation $y = 1$ d'une mesure $\mathbb{E}_{\mathcal{P}}$ sur $G^{\mathcal{P}}$ telle que :

1. Compatibilité : la condition de compatibilité énoncée dans la définition 3.2 est vérifiée,
2. Indistinguabilité : soient $\mathcal{P}_1$ et $\mathcal{P}_2$ deux sous-ensembles finis de points sur la droite $y = 1$. Soit ϕ une permutation physique de $\mathcal{P}_1$ dans $\mathcal{P}_2$, vue comme difféomorphisme, et considérons β_ϕ sa tresse associée comme expliqué dans la fin de la section 3.1.1. Alors :

$$\mathbb{E}_{\mathcal{P}_2} = \beta_\phi \bullet \mathbb{E}_{\mathcal{P}_1}$$

où nous avons utilisé l'isomorphisme canonique $G^{\mathcal{P}} \sim G^{\#\mathcal{P}}$ défini en utilisant l'ordre naturel de gauche à droite.

2. **Indistinguabilité :** *Pour tous sous-ensembles dénombrables* $\mathcal{P}_1$ *et* $\mathcal{P}_2$ *de points dans le plan ne contenant pas* 0 *et tout homéomorphisme* $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ *tel que* $\phi(\mathcal{P}_1) = \mathcal{P}_2$,

$$\mathbb{E}_{\mathcal{P}_1} = \mathbb{E}_{\mathcal{P}_2} \circ \Phi^{-1},$$

où Φ *est l'application naturelle induite par* ϕ *allant de* $\mathsf{Hom}\,(\pi_1(\mathbb{R}^2 \setminus \mathcal{P}_2), G^{op})$ *dans* $\mathsf{Hom}\,(\pi_1(\mathbb{R}^2 \setminus \mathcal{P}_1), G^{op})$.

Montrons, sur un exemple, que l'inverse de la transformation particules/champs permet d'associer un système anyonique de particules à un système anyonique.

Considérons un système anyonique $(\mathbb{E}_{\mathcal{P}})_{\mathcal{P}}$ et posons $\mathcal{P} = \{(-1,1),(1,1)\}$. L'ensemble $\mathcal{P}$ représente la position de deux particules numérotées $\mathsf{1}$ et $\mathsf{2}$, respectivement en position $(-1,1)$ et $(1,1)$. Nous voulons définir des caractéristiques aléatoires associées à ces particules. Soit h un homéomorphisme aléatoire dans $\mathsf{Hom}(\pi_1(\mathbb{R}^2\setminus\mathcal{P}), G^{op})$ ayant pour loi $\mathbb{E}_{\mathcal{P}}$. Considérons les deux lacets l_1 et l_2 dessinés dans la figure 9. On définit alors les variables aléatoires $(Y_{\mathsf{1}}, Y_{\mathsf{2}}) = (h(l_1), h(l_2))$. Par l'inverse de la transformation particules/champs, la caractéristique aléatoire de $\mathsf{1}$, respectivement de $\mathsf{2}$, est Y_{1}, respectivement Y_{2}.

Considérons la permutation physique ϕ des particules $\mathsf{1}$ et $\mathsf{2}$ telle que représentée dans la figure 9. La tresse associée fait passer la particule $\mathsf{1}$ devant la particule $\mathsf{2}$: c'est donc la tresse β_1^{-1} telle que définie dans le début de la section 3.1.2. Il nous faut donc montrer que $(Y_{\mathsf{1}}, Y_{\mathsf{2}})$ a la même loi que $\beta_1^{-1} \bullet (Y_{\mathsf{1}}, Y_{\mathsf{2}}) = (Y_{\mathsf{2}}, Y_{\mathsf{2}}^{-1} Y_{\mathsf{1}} Y_{\mathsf{2}})$.

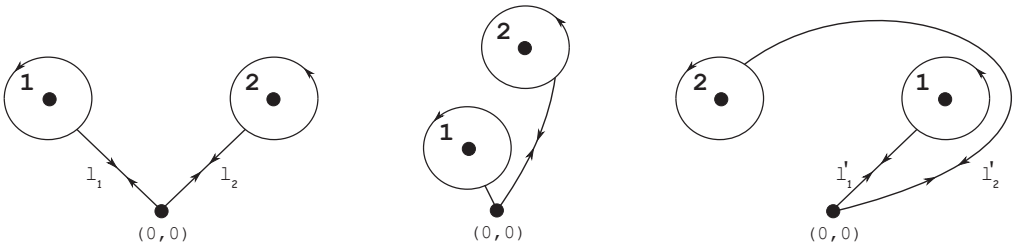

FIGURE 9. Permutation d'un système de particules anyoniques

En utilisant la propriété d'indistinguabilité du système anyonique, nous savons que :

$$\mathbb{E}_{\mathcal{P}} = \mathbb{E}_{\mathcal{P}} \circ \Phi^{-1},$$

où Φ^{-1} est l'application naturelle induite par la permutation physique ϕ vue comme un homéomorphisme. Ainsi, sous $\mathbb{E}_{\mathcal{P}}$, la loi de $(h(l_1), h(l_2))$ est la même que la loi de $(h(l_1'), h(l_2'))$, où l_1' et l_2' sont dessinées dans la figure 9.

Il est bon de remarquer que dans $\pi_1(\mathbb{R}^2 \setminus \{\mathsf{1},\mathsf{2}\})$:

$$l_1' = l_2 \text{ et } l_2' = l_2 l_1 l_2^{-1},$$

ainsi :

$$h(l_1') = h(l_2) = Y_2, \text{ et } h(l_2') = h(l_2)^{-1}h(l_1)h(l_2) = Y_2^{-1}Y_1Y_2,$$

d'où le fait que (Y_1, Y_2) a la même loi que $(Y_2, Y_2^{-1}Y_1Y_2)$. En généralisant ce résultat, on peut donc voir que l'inverse de la transformation particules/champs permet de définir un système anyonique de particules à partir d'un système anyonique.

Le théorème de de Finetti pour l'invariance par le groupe des tresses s'exprime de la façon suivante.

Théorème 3.3 **(Théorème de de Finetti-version tresses[25])**

Soit $(X_n)_{n\in\mathbb{N}^*}$ une suite de variables aléatoires à valeurs dans G. Supposons que $(X_n)_{n\in\mathbb{N}^*}$ est tressable, c'est-à-dire pour tout entier $n \in \mathbb{N}^*$, pour toute tresse $\beta \in \mathcal{B}_n$, $\beta\bullet(X_k)_{k\in\{1,\dots,n\}}$ a la même loi que $(X_k)_{k\in\{1,\dots,n\}}$. Alors conditionnellement à la tribu queue de $(X_n)_{n\in\mathbb{N}^*}$, $\mathcal{T}$, définie par $\mathcal{T} = \bigcap_{n\in\mathbb{N}} \sigma(X_k, k \geq n)$, la suite $(X_n)_{n\in\mathbb{N}}$ est indépendante. De plus conditionnellement à $\mathcal{T}$, presque sûrement la loi de X_1 est invariante par conjugaison par tout élément de son support.

En particulier, il existe une mesure aléatoire μ, presque sûrement invariante par conjugaison par tout élément de son support, qui est $\mathcal{T}$ mesurable telle que la loi de $(X_n)_{n\in\mathbb{N}}$ est égale à $\mathbb{E}[\mu^{\otimes\infty}]$. La réciproque est aussi vérifiée.

Nous pouvons de nouveau formuler le théorème de de Finetti dans le cadre des systèmes anyoniques.

Théorème 3.4. — *Pour tout système anyonique $(\mathbb{E}_\mathcal{P})_{\mathcal{P}\subset\mathbb{R}^2}$, il existe une mesure aléatoire μ sur G, presque sûrement invariante par conjugaison par tout élément de son support, telle que pour tout ensemble fini $\mathcal{P} \subset \mathbb{R}^2$, $\mathbb{E}_\mathcal{P}$ est la loi du champ associé à $\mathbb{E}\left[\mu^{\otimes\mathcal{P}}\right]$ via la transformation particules/champs. Inversement à toute mesure aléatoire μ sur G, invariante par conjugaison par tout élément de son support, on peut associer de la sorte un système anyonique.*

En utilisant la discussion que l'on a eue dans la fin de la section 2.4.2, on voit que lorsque l'on considère un système anyonique et que l'on rend aléatoire l'ensemble $\mathcal{P}$, de sorte qu'il ait la loi d'un processus de Poisson d'intensité vol, on obtient un champ markovien d'holonomies planaire. On peut donc s'attendre à un résultat similaire au théorème 3.4 pour les champs markoviens d'holonomies planaires : c'est ce que l'on expliquera dans la section 5.3.

4. Invariance par transformation de jauge et boucles

Les champs markoviens d'holonomies planaires satisfont deux symétries : l'invariance par une classe d'homéomorphismes dont nous avons exploré les

[25] Proposition 7.2 et théorème 9.1 dans [Champs].

conséquences, ainsi que l'invariance par transformation de jauge. Afin de construire et caractériser les champs markoviens d'holonomies planaires, nous avons besoin d'étudier les conséquences de cette dernière symétrie[26]. Afin d'ancrer un peu plus ces notions, définissons rigoureusement l'invariance par transformation de jauge. Rappelons que le groupe des transformations de jauge est simplement l'ensemble des fonctions de $\mathbb{R}^2$ dans G :

$$\mathcal{J} = G^{\mathbb{R}^2}.$$

Il agit sur l'ensemble des fonctions multiplicatives sur les chemins réguliers. Si $j \in \mathcal{J}$ et si h est une fonction multiplicative sur $\mathsf{P}(\mathbb{R}^2)$ alors :

$$(j \bullet h)(p) = j_{\overline{p}}^{-1} h(p) j_{\underline{p}}.$$

Ainsi une fonction mesurable F sur $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$ est invariante par transformation de jauge si pour toute transformation de jauge j et tout $h \in \mathcal{M}ult(\mathbb{P}(\mathbb{R}^2), G)$, $F(j \bullet h) = F(h)$. Un champ d'holonomies planaire μ est invariant par transformation de jauge si pour toute transformation de jauge j :

$$\int_{\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2),G)} F(j \bullet h)\mu(dh) = \int_{\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2),G)} F(h)\mu(dh).$$

Concentrons-nous sur l'exemple type de fonctions invariantes par conjugaison par le groupe des transformations de jauge. Étant donné que nous avons muni $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$ de la trace de la tribu borélienne cylindrique sur $G^{\mathsf{P}(\mathbb{R}^2)}$, les fonctions continues cylindriques sont les fonctions types mesurables sur $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$. Considérons une telle fonction :

$$\begin{aligned} \mathsf{F}_{(p_i)_{i=1}^k, f} : \mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G) &\to \mathbb{R}^2 \\ h &\mapsto f\left(h(p_1), ..., h(p_k)\right), \end{aligned}$$

où p_1, ..., p_k sont k chemins réguliers, et f est une fonction continue. Une remarque très importante est la suivante[27].

Lemme 4.1. — *Si la fonction $\mathsf{F}_{(p_i)_{i=1}^k, f}$ est invariante par l'action du groupe des transformations de jauge, alors il existe une façon plus simple de l'écrire : il existe un fonction continue f' définie sur G^k invariante par l'action par conjugaison diagonale de G sur G^k, c'est-à-dire que $f'(g^{-1}g_1g, ..., g^{-1}g_kg) = f'(g_1, ..., g_k)$ pour tous $g, g_1, ..., g_k$ dans G, et il existe k boucles l_1,...,l_k dans $\mathsf{L}_0(\mathbb{R}^2)$ telles que :*

$$\mathsf{F}_{(p_i)_{i=1}^k, f} = \mathsf{F}_{(l_i)_{i=1}^k, f'}$$

[26] Pour un traitement plus formel, on peut se référer aux sections 2.2.3 et 2.2.4 de [Champs].
[27] Lemme 2.1.5 dans la monographie [**23**].

Preuve. — Il suffit de prendre pour tout point extrémal[28] de $p_1, p_2, ...$ ou p_k un chemin reliant 0 à x, que l'on notera c_x. Considérons $h \in \mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G)$ et considérons alors la transformation de jauge qui vaut $j(x) = h(c_x)$ si x est un point extrémal de $p_1, p_2, ...$ ou p_k, et $j_x = e$ sinon. Alors en utilisant l'invariance par transformation de jauge ainsi que la mutiplicativité de h, on obtient :

$$\begin{aligned}\mathsf{F}_{(p_i)_{i=1}^k,f}(h) = \mathsf{F}_{(p_i)_{i=1}^k,f}(j \bullet h) &= f(j \bullet h(p_1), ..., j \bullet h(p_k)) \\ &= f\left(j_{\overline{p_1}}^{-1} h(p_1) j_{\underline{p_1}}, ..., j_{\overline{p_k}}^{-1} h(p_k) j_{\underline{p_1}}\right) \\ &= f\left(h(c_{\underline{p_1}} p_1 c_{\overline{p_1}}^{-1}), ..., h(c_{\underline{p_1}} p_1 c_{\overline{p_1}}^{-1})\right) \\ &= f\left(h(l_1), ..., h(l_k)\right)\end{aligned}$$

où l'on a dénoté par l_i la boucle basée en 0 égale à $c_{\underline{p_1}} p_1 c_{\overline{p_1}}^{-1}$. De plus, si on prend la transformation de jauge constante égale à g, on obtient $\mathsf{F}_{(p_i)_{i=1}^k,f}(h) = f\left(g^{-1}h(l_1)g, ..., g^{-1}h(l_k)g\right)$. En intégrant par rapport à la mesure de Haar de G et en notant f' la fonction sur G^k égale à :

$$f'(h_1, ..., h_k) = \int_G f\left(g^{-1}h_1 g, ..., g^{-1}h_k g\right) dg,$$

on obtient alors que $\mathsf{F}_{(p_i)_{i=1}^k,f}(h) = \mathsf{F}_{(l_i)_{i=1}^k,f'}(h)$ avec f' invariante par l'action par conjugaison diagonale de G sur G^k et l_1,...,l_k étant k boucles basées en 0. □

Rappelons que l'ensemble des boucles régulières basées en 0 est noté $\mathsf{L}_0(\mathbb{R}^2)$. On peut alors munir $\mathcal{M}ult(\mathsf{L}_0(\mathbb{R}^2), G)$ d'une tribu $\mathcal{I}$ dite invariante qui est la tribu engendrée par l'ensemble des applications de la forme $\mathsf{F}_{(l_i)_{i=1}^k,f'}$ avec f' invariante par conjugaison diagonale de G sur G^k et l_1,...,l_k étant k boucles dans $\mathsf{L}_0(\mathbb{R}^2)$.

Tout champ d'holonomies peut se restreintre à des processus indexés par $\mathsf{L}_0(\mathbb{R}^2)$ et permet de définir une mesure sur $(\mathcal{M}ult(\mathsf{L}_0(\mathbb{R}^2), G), \mathcal{I})$. Le lemme 4.1 permet alors de montrer qu'il est suffisant de connaître la restriction d'un champ d'holonomies sur $(\mathcal{M}ult(\mathsf{L}_0(\mathbb{R}^2), G), \mathcal{I})$ afin de le caractériser : deux champs d'holonomies ayant même restriction sont égaux.

Une remarque simple, mais qui n'était alors pas utilisée jusqu'à maintenant dans la littérature et démontrée dans [Champs], est que toute mesure sur $(\mathcal{M}ult(\mathsf{L}_0(\mathbb{R}^2), G), \mathcal{I})$ provient d'un unique champ d'holonomies planaire invariant par transformation de jauge. Ainsi, afin de construire un champ d'holonomies aléatoires planaire invariant par transformation de jauge, il suffit de construire une mesure sur $(\mathcal{M}ult(\mathsf{L}_0(\mathbb{R}^2), G), \mathcal{I})$[29]. Ceci est l'un des principes fondamentaux qui permet de généraliser l'étape 1 dans la construction de T. Lévy.

[28] Un point extrémal d'un chemin p étant soit le point de départ soit le point d'arrivée de p.
[29] Proposition 2.5 de [Champs]. En réalité, il faut que la mesure sur $(\mathcal{M}ult(\mathsf{L}_0(\mathbb{R}^2), G), \mathcal{I})$ satisfasse une condition de pré-multiplicativité décrite dans la Définition 2.11 de [Champs].

5. Champs markoviens d'holonomies planaires

Nous allons passer à l'étude plus spécifique des champs markoviens d'holonomies planaires. Dans [Champs], nous définissons ces objets, les construisons, les caractérisons puis les classifions avant d'appliquer les résultats à l'étude des champs markoviens d'holonomies non-planaires de T. Lévy.

Pour ce faire, nous avons besoin d'introduire certains résultats sur le groupe fondamental d'un graphe fini planaire.

5.1. Groupe fondamental d'un graphe fini planaire. — Dans l'étude des systèmes anyoniques, nous avions pu utiliser les tresses puisque nous devions étudier l'action d'homéomorphismes sur le groupe fondamental du plan privé d'un ensemble fini de points $\mathcal{P}$.

Cependant, l'étude des champs markoviens d'holonomies fait intervenir l'ensemble de tous les lacets sur le plan, et non des classes d'homotopies de chemins sur le plan privé d'un ensemble fini de points. On peut cependant se ramener à un cadre similaire à celui des systèmes anyoniques en introduisant un graphe fini planaire et en étudiant les chemins tracés sur ce graphe[(30)].

Donnons-nous $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ un graphe fini plongé dans $\mathbb{R}^2$ dont les arêtes sont régulières. Nous noterons $\mathbb{F}^b$ l'ensemble des faces bornées dans $\mathbb{F}$. De plus, on supposera que $\mathbb{G}$ est connexe. Considérons v un sommet de $\mathbb{G}$, le groupe fondamental de $\mathbb{G}$ basé en v est noté $\mathsf{RL}_v(\mathbb{G})$: c'est le groupe des lacets réduits basés en v. La donnée d'un morphisme de $\mathsf{RL}_v(\mathbb{G})$ dans G^{op} permet de définir canoniquement une application multiplicative de $\mathsf{L}_v(\mathbb{G})$ dans G, $\mathsf{L}_v(\mathbb{G})$ étant l'ensemble des boucles dans $\mathsf{P}(\mathbb{G})$ basées en v. En effet, si l et l' sont dans $\mathsf{L}_v(\mathbb{G})$ et sont homotopes, alors on peut passer de l à l' en effaçant ou rajoutant des allers-retours le long d'arêtes. Or si h est une fonction multiplicative de $\mathsf{P}(\mathbb{G})$ dans G, et si $l = ab$ et $l' = aee^{-1}b$, alors

$$h(l) = h(ab) = h(b)h(a) = h(b)h(e)^{-1}h(e)h(a) = h(l').$$

Ainsi toute fonction multiplicative sur $\mathsf{P}(\mathbb{G})$ dans G est constante sur les classes d'homotopies des boucles dans $\mathsf{L}_v(\mathbb{G})$.

Prenons, dans l'intérieur de chaque face bornée F de $\mathbb{G}$, un point x_F et notons $\mathcal{P}_{\mathbb{G}}$ l'ensemble $\{x_F, F \in \mathbb{F}^b\}$. Le groupe fondamental $\pi_v(\mathbb{G})$ est isomorphe au groupe fondamental $\pi_v(\mathbb{R} \setminus \mathcal{P}_{\mathbb{G}})$.

Dans la transformation particules/champs d'holonomie, étant donné un sous-ensemble fini $\mathcal{P}$ du plan, nous avons utilisé un choix de base de $\pi_1(\mathbb{R}^2 \setminus \mathcal{P})$ assez canonique afin de relever une application de $G^{\mathcal{P}}$ en un homéomorphisme de $\pi_1(\mathbb{R}^2 \setminus \mathcal{P})$ dans G^{op}. On peut se douter de la démarche à suivre : nous allons définir une mesure sur $G^{\mathcal{P}_{\mathbb{G}}} \simeq G^{\mathbb{F}^b}$ que nous allons relever en une mesure sur les

(30) Introduction à la section 6 de [Champs].

homéomorphismes de $\mathsf{RL}_v(\mathbb{G})$ dans G. Pour ce faire, il nous faut une base de $\pi_1\left(\mathbb{R}^2 \setminus \mathcal{P}_{\mathbb{G}}\right)$ et pour cela nous avons besoin de trois objets[31] :

1. un sommet v_0 de $\mathbb{V}$,
2. un arbre couvrant T,
3. pour toute face bornée F de $\mathbb{G}$, une boucle c_F dans $\mathsf{P}(\mathbb{G})$ qui borde $\mathbb{F}$ et qui parcourt la frontière de $\mathbb{F}$ dans le sens anti-horaire.

Pour tout sommet v de $\mathbb{G}$, notons $[v_0, v]_T$ l'unique chemin injectif dans T qui va de v_0 dans v. Pour toute face bornée F de $\mathbb{G}$, on note l_F la boucle $[v_0, \overline{c_F}]_T c_F [v_0, \overline{c_F}]_T^{-1}$ (Figure 10).

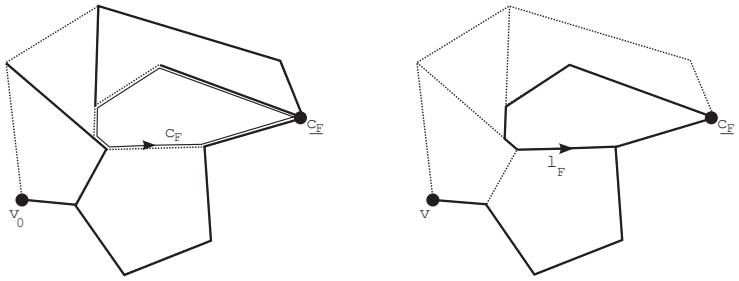

FIGURE 10. Un élément d'une base de lassos

On nommera toute famille $(\mathsf{l}_F)_{F \in \mathbb{F}^b}$ obtenue de cette façon une base de lassos pour $\mathbb{G}$: toute base de lassos pour $\mathbb{G}$ est une famille libre et génératrice du groupe $\mathsf{RL}_v(\mathbb{G})$ qui est un groupe libre de dimension $\#F^b$ [32]. Ainsi l'application :

$$\begin{aligned} \Phi_{T,(c_F)_{F\in\mathbb{F}}} : \mathcal{M}ult(L_{v_0}(\mathbb{G}), G) &\to G^{\mathbb{F}^b} \\ h &\mapsto \left(h\left(\mathsf{l}_F\right)\right)_{F\in\mathbb{F}^b}, \end{aligned}$$

est une bijection et pour tout $l \in L_{v_0}(\mathbb{G})$, il existe un mot $w\left(\left(\mathsf{l}_F\right)_{F\in\mathbb{F}^b}, \left(\mathsf{l}_F^{-1}\right)_{F\in\mathbb{F}^b}\right)$ en les lettres $\left(\mathsf{l}_F\right)_{F\in\mathbb{F}^b}$ et $\left(\mathsf{l}_F^{-1}\right)_{F\in\mathbb{F}^b}$ tel que $h_l = w\left(\left(h_{\mathsf{l}_F}\right)_{F\in\mathbb{F}^b}, \left(h_{\mathsf{l}_F^{-1}}\right)_{F\in\mathbb{F}^b}\right)$.

Ceci permet donc d'appliquer une transformation similaire à la transformation particules/champs et permet de relever une mesure de $G^{\mathbb{F}^b}$ en une mesure sur $\mathcal{M}ult(L_{v_0}(\mathbb{G}), G)$. En restreignant cette mesure à la tribu invariante et en utilisant les remarques expliquées à la fin de la section 4, ceci suffit à construire une mesure sur $\mathcal{M}ult(\mathsf{P}(\mathbb{G}), G)$ invariante par transformation de jauge. Cette mesure dépend en général de la base de lassos considérée sauf si, par exemple, la mesure μ de départ sur $G^{\mathbb{F}^b}$ est tressable, c'est à dire si le processus des projections canoniques sur $(G^{\mathbb{F}^b}, \mu)$ est invariant en loi par l'action des tresses. Cette propriété sera vérifiée dans le cadre de la construction des champs de Yang-Mills planaires esquissée dans la section 5.2.

[31] Voir la section 6.3 de [Champs].
[32] Proposition 6.1 de [Champs].

5.2. Construction. — Afin d'expliquer comment modifier l'étape 1 dans la construction de T. Lévy exposée dans la section 2.5.2 afin de la généraliser[(33)], il nous faut définir une symétrie pour les processus de Lévy.

Définition 5.1. — *Pour tout processus de Lévy* $\mathsf{X} = (X_t)_{t\geq 0}$ *à valeurs dans* G, *le support de* X *est le plus petit sous-groupe fermé* H_{X} *de* G *tel que pour tout réel positif,* X_t *est presque sûrement dans* H. *Un processus de Lévy est dit auto-invariant par conjugaison s'il est invariant par conjugaison par tout élément de* H_{X}.

Donnons-nous $\mathsf{X} = (X_t)_{t\geq 0}$ un processus de Lévy auto-invariant par conjugaison, nous pouvons alors construire un champ markovien d'holonomies planaire associé. Pour cela, on change l'étape 1 dans la construction de T. Lévy de la façon suivante. Rappelons que $\mathsf{vol} \in \mathcal{D}(\mathbb{R}^2)$. La mesure $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathbb{G}}$ est la mesure sur $\mathcal{M}ult(\mathsf{P}(\mathbb{G}), G)$ découlant du choix de mesure sur $G^{\mathbb{F}^b}$ donné par :

$$\int_{G^{\mathbb{F}^b}} f\left((h_{\mathsf{l}_F})_{F\in\mathbb{F}^b}\right)\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathbb{G}}(dh) = \int_{G\times G^{\mathbb{F}^b}} f\left((g^{-1}h_{\mathsf{l}_F}g)_{F\in\mathbb{F}^b}\right)dg\otimes\bigotimes_{F\in\mathbb{F}^b}\mathcal{L}_{X_{\mathsf{vol}(F)}}(dh_{\mathsf{l}_F}), \tag{7}$$

où pour tout $t \geq 0$, $\mathcal{L}_{X_t}$ est la loi de X_t, et pour tout $F \in \mathbb{F}^b$, h_{l_F} est la projection canonique associée à F définie sur $G^{\mathbb{F}^b}$.

Supposons que X est invariant par conjugaison par G, alors sous $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathbb{G}}$, les variables aléatoires $(h_{\mathsf{l}_F})_{F\in\mathbb{F}}$ sont indépendantes, et pour toute face bornée F, h_{l_F} a la même loi que $X_{\mathsf{vol}(F)}$. Dès que X n'est plus invariant par conjugaison par G, les variables aléatoires $(h_{\mathsf{l}_F})_{F\in\mathbb{F}}$ *ne sont plus indépendantes.*

Il s'avère que pour toute base de lassos, il existe une façon de l'ordonner de telle sorte que le produit des éléments de la base représente la frontière de la face non-bornée de $\mathbb{G}$. On voit donc qu'une application envoyant une base de lassos en une autre base de lassos, à permutation près des deux bases, conserve le produit des deux bases. Cette transformation provient alors d'une tresse d'après le théorème d'Artin exposé à la fin de la section 3.1.

Théorème 5.1 **(Proposition** 7.1 **de [Champs])**. — *Étant donné deux bases de lassos pour* $\mathbb{G}$ *basées au même point* v_0, *on peut trouver une tresse qui transforme la première en la seconde.*

Ceci permet de montrer que la mesure obtenue $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol},\mathbb{G}}$ ne dépend pas du choix de l'arbre couvrant et des boucles faciales considérées. Grâce à l'invariance par transformation de jauge, on montre qu'elle ne dépend pas non plus du choix du sommet v_0.

On continue alors avec l'étape 2 (la preuve du fait que $\mathsf{YM}^{\mathsf{X}}_{\mathbb{G}'}\circ\rho^{-1}_{\mathsf{P}(\mathbb{G}),\mathsf{P}(\mathbb{G}')} = \mathsf{YM}^{\mathsf{X}}_{\mathbb{G}}$ est différente mais utilise aussi de façon fondamentale le fait que X est un processus de Lévy) et avec l'étape 3 de la construction de T. Lévy. On montre alors, par des

(33) Ce que nous faisons dans la section 8 de [Champs].

preuves distinctes de celles de T. Lévy, puisqu'utilisant l'invariance par tresses, que le champ d'holonomies est un champ markovien d'holonomies planaire. On aboutit donc au théorème suivant.

***Théorème 5.2* (Théorème** 8.1 **de [Champs]).** — *Pour tout processus de Lévy auto-invariant par conjugaison* $\mathsf{X} = (X_t)_{t\geq 0}$*, il existe un unique champ markovien d'holonomies planaire stochastiquement continu* $\left(\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol}}\right)_{\mathsf{vol}\in\mathcal{D}(\mathbb{R}^2)}$ *tel que pour tout graphe* $\mathbb{G}$*, pour tout choix de base de lassos* $(\mathsf{l}_F)_{F\in\mathbb{F}^b}$ *pour* $\mathbb{G}$*, pour toute fonction continue sur* $G^{\mathbb{F}}$*, l'égalité (7) est vérifiée.*

Cette construction généralise la construction présentée par T. Lévy dans [**23**]. En appliquant ce théorème au cas où X est un processus de Lévy invariant par conjugaison qui possède une densité, on retrouve sur le champ markovien d'holonomies planaire construit par T. Lévy.

Remarque 5.1. — *Une remarque importante est que si l'on considère une mesure de densité* $\mathsf{vol} \in \mathcal{D}(\mathbb{R}^2)$*, deux disques fermés* D_1 *et* D_2 *disjoints sauf en* 0*, et si on considère deux boucles* $l_1 \subset D_1$ *et* $l_2 \subset D_2$ *basées en* 0*, alors en général, sous* $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol}}$*, les variables* $h(l_1)$ *et* $h(l_2)$ *ne sont pas indépendantes. Ceci explique pourquoi dans l'axiome* $\boldsymbol{P}_2$ *de la définition 2.1, les disques* D_1 *et* D_2 *sont complètement disjoints. Ainsi, en utilisant l'image de l'expérience permettant de mesurer un champ grâce à des particules-test, deux particules-test partant d'un même émetteur inséré dans un mur et parcourant chacune un trajet dans deux pièces distinctes voient leurs caractéristiques être modifiées et ce, de façon non indépendante.*

Définition 5.2. — *Un champ markovien d'holonomies planaire de la forme* YM^{X} *est appelé un champ de Yang-Mills planaire.*

5.3. Caractérisation des champs markoviens d'holonomies planaires réguliers. — Maintenant que l'on a construit des exemples de champs markoviens d'holonomies planaires stochastiquement continus, nous aimerions montrer[(34)] que nous les avons tous construits. Dans cette section, on supposera toujours que les champs markoviens d'holonomies planaires considérés sont stochastiquement continus. On a alors le théorème de caractérisation suivant[(35)].

Théorème 5.3. — *Tout champ markovien d'holonomies planaire est un champ de Yang-Mills planaire.*

Donnons une idée de la preuve : considérons $(\mathbb{E}_{\mathsf{vol}})_{\mathsf{vol}\in\mathcal{D}(\mathbb{R}^2)}$ un champ markovien d'holonomies planaire. La preuve repose sur les intuitions et résultats développés pour les systèmes anyoniques, ainsi que sur l'idée de se ramener à l'étude de

(34) Ce que l'on fait dans la section 10 de [Champs].
(35) Qui correspond au théorème 10.1 de [Champs].

mesures sur $G^{\mathbb{F}^b}$ grâce à l'introduction d'un graphe fini $\mathbb{G}$ et d'une base de lassos pour $\mathbb{G}$. Ce faisant, on arrive à montrer qu'étant donné $\mathsf{vol} \in \mathcal{D}(\mathbb{R}^2)$ et une base de lassos $(\mathsf{l}_F)_{F\in\mathbb{F}^b}$ d'un graphe $\mathbb{G}$ dont les faces sont de volume 1, les projections $(h_{l_F})_{F\in\mathbb{F}}$ sont conditionnellement indépendantes, ainsi il existe une mesure aléatoire μ telle que la loi de $(h_{l_F})_{F\in\mathbb{F}^b}$ sous $\mathbb{E}_{\mathsf{vol}}$ est égale à $\mathbb{E}\left[\mu^{\otimes\mathbb{F}}\right]$. En utilisant les axiomes $\mathbf{P_1}$ et $\mathbf{P_2}$, on montre[36] qu'il existe une mesure de probabilité μ_0 invariante par conjugaison par tout élément de son support telle que pour toute fonction f continue sur G, on a :

$$\mathbb{E}_{\mathsf{vol}}\left[f((h_{\mathsf{l}_F})_{F\in\mathbb{F}^b})\right] = \int_{G\times G^{\otimes\mathbb{F}^b}} f\left((g^{-1}h_{\mathsf{l}_F}g)_{F\in\mathbb{F}^b}\right) dg \otimes \mu_0^{\otimes\mathbb{F}^b}(dh_1, ..., dh_{\#\mathbb{F}^b}).$$

On retrouve alors l'équation (7) dans le cas de faces de même volume. On montre alors en utilisant la propriété de multiplicativité des champs markoviens d'holonomies que μ est la loi au temps 1 d'un processus de Lévy auto-invariant par conjugaison.

Bien sûr, beaucoup de détails techniques ne sont pas présentés dans cette introduction mais sont donnés dans le chapitre de thèse [Champs]. Ceux-ci sont inhérents au fait que les champs markoviens d'holonomies planaires sont invariants par une classe d'homéomorphismes préservant l'aire et non pas, à l'instar des systèmes anyoniques, par tous les homéomorphismes du plan.

5.4. Classification. — Le problème initial qui m'a conduit à l'écriture du chapitre de thèse "Champs markoviens d'holonomies planaires" était de caractériser les champs markoviens d'holonomies (non-planaires) définis par T. Lévy. Ce sont des familles de mesures indexées par les surfaces bidimensionnelles compactes M munies d'une densité lisse positive, chaque mesure étant définie sur l'ensemble des fonctions multiplicatives des chemins réguliers dessinés sur M, dans G. La partie sphérique d'un champ markovien d'holonomie est alors la restriction de cette famille de mesures à l'ensemble des surfaces compactes de genre zéro. La donnée d'un champ markovien d'holonomies (non-planaire) permet alors, en ne conservant que sa partie sphérique, de construire un champ markovien d'holonomies planaire associé[37].

T. Lévy a construit dans [**23**] une certaine classe de champs markoviens d'holonomies, et conjecturé que l'on obtient de la sorte tous les champs markoviens d'holonomies réguliers. En supposant cette conjecture vraie, alors, pour tous disques fermés D_1 et D_2 disjoints sauf en 0, et toutes boucles $l_1 \subset D_1$ et $l_2 \subset D_2$ basées en 0, sous le champ markovien d'holonomies planaire associé à n'importe quel champ markovien d'holonomies, les variables $h(l_1)$ et $h(l_2)$ sont indépendantes.

Comme nous l'avons vu précédemment, ce n'est généralement pas le cas pour tout champ markovien d'holonomies planaire sauf si, par exemple, le processus

[36] Grâce au théorème 9.2 de [Champs].
[37] Définition 12.12 et théorème 12.4 de [Champs].

de Lévy sous-jacent est invariant par conjugaison par G. Afin de répondre partiellement à la conjecture de T. Lévy, il est alors naturel de classifier les champs de Yang-Mills planaires selon le degré de symétrie du processus de Lévy sous-jacent[38].

Définition 5.3. — *Soit* X *un processus de Lévy auto-invariant par conjugaison à valeurs dans* G. *Le processus* X *et le champ de Yang-Mills planaire* $\mathsf{YM}^{\mathsf{X}} = \left(\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol}}\right)_{\mathsf{vol}\in\mathcal{D}(\mathbb{R}^2)}$ *sont dit purs si* X *est invariant par conjugaison par* G *et mixte sinon. Ils sont non-dégénérés si le support de* X *est égal à* G *et dégénérés sinon.*

Par définition, un processus de Lévy auto-invariant par conjugaison et le champ de Yang-Mills planaire associé sont purs s'ils sont non-dégénérés.

Considérons un champ markovien d'holonomies planaire régulier. Par le théorème 5.3 de caractérisation, c'est un champ de Yang-Mills planaire noté YM^{X}. En utilisant la théorie des représentations des groupes topologiques compacts, on montre l'existence des trois critères suivants[39] :

1. dans le cas où le champ est à valeur dans un groupe fini G, si pour toute densité $\mathsf{vol} \in \mathcal{D}(\mathbb{R}^2)$, et toute boucle simple l, le support de $h(l)$ sous $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol}}$ est le groupe G, alors YM^{X} est non-dégénérée et pure. Ainsi, dans le cas ou le champ est à valeurs dans un groupe fini, si pour toute trajectoire simple, la caractéristique de la particule-test peut être modifiée par n'importe quel élément de G, on a affaire à un champ de Yang-Mills non-dégénéré et pur.
2. si pour toute densité $\mathsf{vol} \in \mathcal{D}(\mathbb{R}^2)$, la loi de h_l sous $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol}}$ tend vers la mesure de Haar sur G quand l est une boucle basée en 0, simple et bordant un domaine de plus en plus grand, alors YM^{X} est non-dégénérée et pure. Ainsi, si l'action du champ sur une particule test décrivant un cercle est de plus en plus aléatoire quand le cercle est de plus en plus grand, alors on a affaire à un champ de Yang-Mills planaire non-dégénéré et pur.
3. s'il existe un processus de Lévy Z tel que pour toute densité $\mathsf{vol} \in \mathcal{D}(\mathbb{R}^2)$, pour toute boucle l simple bordant un domaine D, $h(l)$ sous $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol}}$ a la même loi que $Z_{\mathsf{vol}(D)}$, alors YM^{X} est pure et $\mathsf{X} = \mathsf{Z}$.

Ces résultats proviennent en réalité de résultats similaires sur les processus de Lévy. Avant de présenter ces résultats, revenons à notre discussion sur les champs markoviens d'holonomies (non-planaires). Nous avons vu qu'ils induisent un champ markovien d'holonomies planaire, qui se trouve être un champ de Yang-Mills planaire. D'après un théorème de T. Lévy, on montre alors que le critère n°3 est vérifié : le champ markovien d'holonomies planaire associé est donc un champ

[38] Cette classification donnée dans la section 11 de [Champs] s'appuie sur des résultats de classification donnée dans les sections 9.3 et 9.4 du même chapitre.
[39] Théorèmes 11.1, 11.2 et 11.3 de [Champs].

de Yang-Mills planaire pur. Ceci permet alors de caractériser totalement la partie sphérique des champs markoviens d'holonomies[40].

5.5. Processus de Lévy auto-invariants par conjugaison. — Le problème de classification des champs de Yang-Mills planaires est équivalent à un problème de classification des processus de Lévy auto-invariants par conjugaison. Soit $\mathsf{X} = (X_t)_{t\geq 0}$ un tel processus de Lévy et soit U une variable aléatoire de Haar indépendante de X, on définit le processus $\mathsf{Y} = (UX_tU^{-1})_{t\geq 0}$.

En prenant une suite de boucles $(l_t)_{t\geq 0}$ basées en 0 qui grossissent, et dont l'intérieur au temps t est de volume t, en utilisant l'équation (7), on voit que la loi de $(h_{l_t})_{t\geq 0}$ sous $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol}}$ est égale à la loi du processus Y défini ci-dessus. Ainsi lorsque l'on considère le champ de Yang-Mills planaire $\mathsf{YM}^{\mathsf{X}}_{\mathsf{vol}}$, nous n'avons pas accès directement au processus de Lévy X mais à Y. Il s'avère qu'en utilisant les travaux de T. Lévy, on peut connaitre la loi des marginales du processus Y lorsque le champ de Yang-Mills planaire provient d'un champs markovien d'holonomie, mais pas la loi de Y en temps que processus.

Le problème de classification des champs de Yang-Mills planaires est alors équivalent au problème suivant. Étant donné que l'*on a connaissance seulement des loi des marginales du processus* Y :

1. Comment pouvons-nous savoir que le support de X, c'est à dire le plus petit sous-groupe fermé qui contient presque sûrement toute la trajectoire $(X_t)_{t\geq 0}$, est égal ou non à G ?
2. Comment pouvons-nous savoir que le processus X est invariant en loi par conjugaison par le groupe G ?

5.5.1. Processus auto-invariants non-dégénérés. — Lorsque le groupe G est fini, afin de répondre à la première question il suffit de regarder le comportement de Y_t lorsque t tend vers l'infini. On supposera que l'élément neutre e est dans le support de Y_t pour tout $t \geq 0$. Il est alors facile de voir qu'il existe un temps t_0 à partir duquel le support de X_{t_0} est égal au support H du procesuss X. Ainsi, à partir de ce temps t_0, le support de Y_t est égal à $\cup_{g\in G} gHg^{-1}$. Or un théorème de Jordan affirme que pour tout sous-groupe H de G,

$$\cup_{g\in G} gHg^{-1} = G$$

si et seulement si $H = G$. Pour savoir si le support de X est égal à G il suffit donc de regarder si le support de Y_t est égal à G pour t assez grand.

Lorsque le groupe G est infini, la réponse est légèrement différente. On suppose toujours que $e \in \mathsf{Supp}(Y_t)$ pour tout réel positif t.

[40] Ce résultat est explicité dans le théorème 12.3 de [Champs].

***Proposition 5.1* (Proposition 9.8 de [Champs])**. — *Le support du processus de Lévy* X *est égal à* G *si et seulement si la loi de* Y_t *converge vers la mesure de Haar de* G *quand* t *tend vers l'infini.*

La preuve de ce théorème repose sur deux généralisations des arguments exposés dans le cas où G est fini.

Premièrement, un théorème d'Itô-Kawada permet de comprendre la limite de la loi de X_t quand t tend vers l'infini : c'est la mesure de Haar λ_H sur le support H de X. Ainsi la limite de la loi de Y_t quand t tend vers l'infini est donnée par :

$$\int_{g\in G} \lambda_{g^{-1}Hg}\lambda_G(dg),$$

où $\lambda_{g^{-1}Hg}$ est la mesure de Haar sur $g^{-1}Hg$ et λ_G la mesure de Haar sur G.

Deuxièmement, le théorème de Jordan admet une généralisation qui permet de conclure la preuve de la proposition 5.1.

***Proposition 5.2* (Proposition** 9.3 **de [Champs])**. — *Soit* G *un groupe de Lie compact, soit* H *un sous-groupe fermé. Si :*

$$\int_{g\in G} \lambda_{g^{-1}Hg}\lambda_G(dg) = \lambda_G,$$

alors $H = G$.

5.5.2. Processus auto-invariants purs. — Nous aimerions aussi avoir un critère afin de pouvoir répondre à la seconde question : comment pouvons-nous savoir que le processus X est invariant en loi par conjugaison par le groupe G ? La réponse est très simple.

***Théorème 5.4* (Théorème** 9.8 **de [Champs])**. — *Le processus* X *est invariant en loi par conjugaison par le groupe* G *si et seulement si il existe un procesus de Lévy* $\mathsf{Z} = (Z_t)_{t\geq 0}$ *invariant par conjugaison par le groupe* G *tel que pour tout réel positif* t, Z_t *a la même loi que* Y_t. *Dans ce cas* X *a la même loi que* Z.

Ce théorème peut légèrement déstabiliser : mal compris il peut donner l'impression de n'être qu'une tautologie. La condition d'auto-invariance par conjugaison supposée par le processus de Lévy X est cruciale : sans cette supposition, la conclusion du théorème précédent est fausse. En effet, on peut construire[(41)] un processus de Lévy X qui n'est pas auto-invariant par conjugaison, et pour lequel il existe un processus de Lévy Z invariant par conjugaison par le groupe G tel que UX_tU^{-1} a la même loi que Z_t pour tout réel positif t. On ne peut donc pas distinguer X de Z en considérant les observables centrales des marginales uni-dimensionnelles.

[(41)] La construction est donnée dans la proposition 9.6 de [Champs].

6. Invariance par transformation de jauge et transformée de Wilson

Nous avons défini, construit et caractérisé les champs markoviens d'holonomies planaires. Nous allons dans cette section nous pencher sur l'étude plus détaillée des observables. Comme nous l'avons vu, l'invariance par transformation de jauge est indipensable afin de caractériser les champs markoviens d'holomies planaires. Elle sert aussi à réduire le nombre d'observables dans la théorie des champs markoviens d'holomies planaires à valeurs dans les groupes orthogonaux $O(N)$ ou unitaires $U(N)$. Dans cette section, en nous basant sur l'article [**31**] de T. Lévy, nous allons expliquer comment l'invariance par transformation de jauge permet de voir la transformée de Wilson comme une notion intéressante dans le cadre des champs d'holomies planaires à valeurs dans $U(N)$.

Étant donné une mesure de probabilité μ sur l'ensemble des nombres réels, on peut définir sa transformée de Fourier comme étant la fonction :

$$\begin{aligned}\hat{\mu} : \mathbb{R} &\to \mathbb{C} \\ t &\mapsto \int_{x\in\mathbb{R}} e^{itx}\mu(dx).\end{aligned}$$

Deux mesures de probabilités sur $\mathbb{R}$ ayant la même transformée de Fourier sont égales, et une suite de mesures de probabilité $(\mu_N)_N$ sur $\mathbb{R}$ converge vers une mesure de probabilité μ si la transformée de Fourier de μ_N converge simplement vers la transformée de Fourier de μ.

Cependant, lorsqu'on étudie des mesures à support compact, il est quelque fois plus simple d'étudier les moments :

$$\begin{aligned}\mathsf{M}_\mu : \mathbb{N} &\to \mathbb{R} \\ n &\mapsto \int_{\mathbb{R}} x^n\mu(dx).\end{aligned}$$

Ainsi, si on étudie les mesures de probabilité à support inclus dans $[a, b]$, le théorème de Stone-Weierstrass permet de démontrer le théorème suivant.

Théorème 6.1. — *Soient μ_1 et μ_2 deux mesures de probabilités à support dans $[a, b]$, si $\mathsf{M}_{\mu_1} = \mathsf{M}_{\mu_2}$ alors μ_1 et μ_2 sont égales.*

Soient $(\mu_N)_{N\in\mathbb{N}}$ des mesures de probabilité à support inclus dans $[a, b]$, la convergence de M_{μ_N} implique la convergence de μ_N vers une mesure de probabilité μ.

On peut affaiblir la condition sur le support, en rajoutant une condition de décroissance rapide. Le même résultat est vérifié si les mesures considérées sont définies sur le cercle unité $\mathbb{U}$.

Comme nous le verrons, la transformée de Wilson est l'équivalent de la transformée M dans le cadre des champs markoviens d'holonomies planaires.

Supposons que l'on considère un champ d'holonomies aléatoires défini non plus sur l'ensemble des chemins réguliers $\mathsf{P}(\mathbb{R}^2)$ mais défini sur une seule boucle l : ce

champ n'est que la donnée d'une variable aléatoire h_l à valeurs dans G. Dans ce cas, l'invariance par transformation de jauge se traduit en une invariance par conjugaison par le groupe G. Ainsi pour tout élément g de G, $g^{-1}h_l g$ a la même loi que h_l. Dans ce cadre particulier, l'étude des champ d'holonomies se réduit à l'étude des variables aléatoires dans G invariantes par conjugaison par G.

6.1. Transformées pour des mesures définies sur G. —

6.1.1. Peter-Weyl. — Donnons-nous un groupe topologique compact G. Nous pouvons l'étudier en regardant comment celui-ci agit sur des espaces vectoriels de dimension finie : c'est la théorie des représentations des groupes compacts. Cette théorie repose en grande partie sur le théorème de Peter-Weyl, que nous allons présenter dans cette section.

Rappelons que tout groupe topologique compact admet une unique mesure de probabilité borélienne invariante par multiplication à gauche et à droite : c'est sa mesure de Haar. Ainsi G est muni d'une mesure de probabilité, que l'on notera dg, telle que pour toute fonction continue $f : G \to \mathbb{C}$, pour tout $h \in G$:

$$\int_G f(hg)dg = \int_G f(gh)dg = \int_G f(g)dg.$$

Une représentation fini-dimensionnelle du groupe G est la donnée d'un espace vectoriel V de dimension finie et d'un morphisme continu ρ de G dans les endomorphismes bijectifs de V. Tout élément $g \in G$, vu par le prisme de cette représentation, devient une application linéaire bijective $\rho(g)$ définie sur V et pour deux éléments g_1 et g_2 de G, $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$. Dorénavant, on supposera toujours que $\mathsf{dim}(V) < \infty$.

Une représentation est dite irréductible si les seuls sous-espaces vectoriels de V stables par $\rho(G)$ sont $\{0\}$ et V. En réalité on ne s'intéresse qu'aux représentations à isomorphisme près : deux représentations (ρ_1, V_1) et (ρ_2, V_2) sont isomorphes s'il existe un isomorphisme de V_1 dans V_2 noté f tel que pour tout $g \in G$, $f \circ \rho_1(g) = \rho_2(g) \circ f$. Pour toute classe d'isomorphisme de représentations irréductibles, on choisit un représentant, et on note $\hat{G}$ l'ensemble de ces représentants. Dans la théorie des groupes compacts, les représentations irréductibles sont les briques élémentaires puisque le théorème de Peter-Weyl implique que toute représentation fini-dimensionnelle de G s'écrit comme somme directe de représentations irréductibles.

Théorème 6.2. — *Soit G un groupe topologique compact. Pour toute représentation (ρ, V) de G, il existe des représentations irréductibles $((\rho_i, V_i))_{i=1}^n$ telles que $V = \bigoplus_{i=1}^n V_i$ et pour tout $g \in G$,*

$$\rho(g) = \bigoplus_{i=1}^{n} \rho_i(g).$$

Il s'avère que les représentations permettent de construire des fonctions de G dans $\mathbb{C}$: ce sont les éléments de matrices. Considérons (ρ, V) une représentation de G. L'espace V étant de dimension finie, on peut considérer la trace habituelle définie sur les endomorphismes de V. Si A est un endomorphisme de V, on peut définir la fonction :

$$\begin{aligned} f_{\rho,A} : G &\to \mathbb{C} \\ g &\mapsto \mathsf{Tr}(\rho(g)A). \end{aligned}$$

La fonction $f_{\rho,A}$ est un élément de matrice irréductible si ρ est une représentation irréductible. D'après le théorème 6.2, tout élément de matrice s'exprime comme combinaison linéaire d'éléments de matrices irréductibles. Remarquons que l'ensemble des éléments de matrice forme une algèbre, puisque $f_{\rho_1,A_1} f_{\rho_2,A_2} = f_{\rho_1\otimes\rho_2, A_1\otimes A_2}$, ce qui fait de l'ensemble des éléments de matrice l'algèbre engendrée par les éléments de matrice irréductibles.

Soit μ une mesure de probabilité sur G, définissons la transformée de Peter-Weyl de μ par :

$$\begin{aligned} \Psi_\mu : \bigoplus_{(\rho,V)\in\hat{G}} \mathsf{End}(V) &\to \mathbb{C} \\ A &\mapsto \int_G f_{\rho,A}(g)\mu(dg). \end{aligned}$$

Le théorème de Peter-Weyl affirme aussi que la transformée de Peter-Weyl caractérise la mesure μ.

Théorème 6.3. — *L'algèbre engendrée par les éléments de matrice irréductibles est dense dans l'espace des fonctions continues sur G à valeurs dans $\mathbb{C}$ pour la norme uniforme. En particulier, la transformée de Peter-Weyl caractérise les mesures de probabilités et leur convergence : si μ_1 et μ_2 sont deux mesures de probabilités sur G telles que $\Psi_{\mu_1} = \Psi_{\mu_2}$ alors $\mu_1 = \mu_2$; si (μ_N) est une suite de mesures de probabilité sur G, si μ est une mesure de probabilité sur G, si Φ_{μ_N} converge simplement vers Φ_μ alors μ_N converge vers μ.*

Si le groupe G est un groupe de matrices, alors ce théorème, tout comme le théorème des moments 6.1, est une simple application du théorème de Stone-Weiesrtrass.

6.1.2. Mesures invariantes par conjugaison par le groupe G. — Supposons maintenant que les mesures considérées sont invariantes par conjugaison par le groupe G, c'est-à-dire que pour toute fonction continue f sur G, $\int_G f(g'^{-1}gg')\mu(dg) = \int_G f(g)\mu(dg)$ pour tout $g' \in G$. Dans ce cas, la transformée de Peter-Weyl se simplifie. En effet, par invariance, pour tout $g' \in G$, toute représentation irreductible

(ρ, V) et tout endomorphisme $A \in \mathsf{End}(V)$:

$$\int_G \mathsf{Tr}(\rho(g)A)\mu(dg) = \int_G\int_G \mathsf{Tr}(\rho(g'gg'^{-1})A)dg'\mu(dg)$$
$$= \int_G \mathsf{Tr}\left(\rho(g)\int_G \rho(g'^{-1})A\rho(g')dg'\right)dg'\mu(dg) = \int_G \mathsf{Tr}(\rho(g)\hat{A})\mu(dg),$$

où $\hat{A} = \int_G \rho(g'^{-1})A\rho(g')dg'$ est un élément de $\mathsf{End}^G(V)$ c'est-à-dire un élement de $\mathsf{End}(V)$ qui commute à l'action de G sur V. Ainsi la transformée de Peter-Weyl peut être restreinte à $\bigoplus_{(\rho,V)\in\hat{G}} \mathsf{End}^G(V)$. Le lemme de Schur, fondamental dans la théorie des représentations permet de décrire cet espace.

***Théorème 6.4* (Lemme de Schur).** — *Si (ρ, V) est une représentation irréductible de G, alors* $\mathsf{End}^G(V) = \mathbb{C}\mathrm{Id}_V$.

Ainsi les mesures invariantes par conjugaison sur G sont caractérisées par une transformée, que l'on appellera la transformée des caractères donnée par :

$$\mathsf{C}_\mu : \quad \hat{G} \to \mathbb{C}$$
$$(\rho, V) \mapsto \int_G \chi_\rho(g)\mu(dg),$$

où l'on a posé pour tout $g \in G$, $\chi_\rho(g) = \mathsf{Tr}(\rho(g))$.

6.2. Schur-Weyl : Cas unitaire. — Passons maitenant aux champs d'holonomies invariants par transformation de jauge. Afin de montrer que la transformée de Wilson caractérise ceux-ci, on va utiliser la dualité de Schur-Weyl que l'on va maintenant présenter.

6.2.1. Enoncé de la dualité. — Le groupe des matrices unitaires $U(N)$ agit naturellement sur $\mathbb{C}^N$ et on peut étendre cette action sur $\left(\mathbb{C}^N\right)^{\otimes k}$ pour tout entier positif k. Pour tout $U \in U(N)$, l'action de U sur $\left(\mathbb{C}^N\right)^{\otimes k}$ est donnée par :

$$\rho_{U(N)}(U) : \quad \left(\mathbb{C}^N\right)^{\otimes k} \to \left(\mathbb{C}^N\right)^{\otimes k}$$
$$x_1 \otimes ... \otimes x_k \mapsto Ux_1 \otimes ... \otimes Ux_k.$$

Le groupe symétrique $\mathfrak{S}_k$ agit aussi naturellement sur $\left(\mathbb{C}^N\right)^{\otimes k}$ par l'action :

$$\rho^{\mathfrak{S}}(\sigma) : \quad \left(\mathbb{C}^N\right)^{\otimes k} \to \left(\mathbb{C}^N\right)^{\otimes k}$$
$$x_1 \otimes ... \otimes x_k \mapsto x_{\sigma^{-1}(1)} \otimes ... \otimes x_{\sigma^{-1}(k)}.$$

Il est alors facile de voir que les deux actions commutent : si $U \in U(N)$ et $\sigma \in \mathfrak{S}_k$:

$$\rho_{U(N)}(U) \circ \rho^{\mathfrak{S}}(\sigma)(x_1 \otimes ... \otimes x_k) = Ux_{\sigma^{-1}(1)} \otimes ... \otimes Ux_{\sigma^{-1}(k)}$$
$$= \rho^{\mathfrak{S}}(\sigma) \circ \rho_{U(N)}(U)(x_1 \otimes ... \otimes x_k).$$

Notons $\rho^{\mathfrak{S}}(\mathbb{C}[\mathfrak{S}_k])$ (respectivement $\rho_{U(N)}(\mathbb{C}[U(N)])$) la sous-algèbre de $\mathsf{End}\left(V^{\otimes k}\right)$ engendrée par $\{\rho^{\mathfrak{S}}(\sigma) \mid \sigma \in \mathfrak{S}_k\}$ (respectivement $\{\rho_{U(N)}(U) \mid U \in U(N)\}$). Notre remarque sur la commutation des actions permet d'affirmer que $\rho^{\mathfrak{S}}(\mathbb{C}[\mathfrak{S}_k])$ est inclus dans $\left(\rho_{U(N)}(\mathbb{C}[U(N)])\right)'$, l'ensemble de $\mathsf{End}\left(V^{\otimes k}\right)$ qui commute à $\rho_{U(N)}(\mathbb{C}[U(N)])$. Le théorème de dualité de Schur-Weyl renforce cet énoncé.

***Théorème 6.5* (Dualité de Schur-Weyl).** — *Pour tous entiers strictement positifs k et N,*

$$\rho^{\mathfrak{S}}(\mathbb{C}[\mathfrak{S}_k]) = \left(\rho_{U(N)}(\mathbb{C}[U(N)])\right)'.$$

De plus si $N \geq k$, $\rho^{\mathfrak{S}}$ est injective.

Ainsi dès lors que $N \geq k$ et que l'on travaille avec un endomorphisme E de $\left(\mathbb{C}^N\right)^{\otimes k}$ tel que pour tout $U \in U(N)$ on ait

$$U^{\otimes k} E \left(U^{-1}\right)^{\otimes k} = E, \tag{8}$$

il existe une famille unique de nombres complexes $(E_\sigma)_{\sigma\in\mathfrak{S}_k}$ tels que :

$$E = \sum_{\sigma\in\mathfrak{S}_k} E_\sigma \rho^{\mathfrak{S}}(\sigma). \tag{9}$$

Remarque 6.1. — *Pour toute permutation $\sigma \in \mathfrak{S}_k$, notons $\ell(\sigma)$ le nombre de cycles de σ. En utilisant les travaux de B. Collins et P. Śniady*[(42)]*, on peut expliciter la famille $(E_\sigma)_{\sigma\in\mathfrak{S}_k}$:*

$$E = \sum_{\substack{\sigma\in\mathfrak{S}_k \\ \tilde{\sigma}\in\mathfrak{S}_k}} \mathsf{Tr}(E\rho_{\mathfrak{S}}(\sigma^{-1}))\mathsf{Wg}(\tilde{\sigma}\sigma^{-1})\rho^{\mathfrak{S}}(\tilde{\sigma}),$$

où Wg *est la fonction de Weingarten, définie par le fait que $\sum_{\sigma\in\mathfrak{S}_k} \mathsf{Wg}(\sigma)\sigma$ est l'inverse de $\sum_{\sigma\in\mathfrak{S}_k} N^{\ell(\sigma)}\sigma$ dans $\mathbb{C}[\mathfrak{S}_k]$. Cependant, dans cette thèse, nous n'aurons jamais besoin de connaître explicitement la famille $(E_\sigma)_{\sigma\in\mathfrak{S}_k}$, son existence est suffisante.*

Remarque 6.2. — *Une façon de démontrer la dualité de Schur-Weyl, comme expliqué par A. Dahlqvist dans* [**1**]*, est d'utiliser le calcul stochastique. Considérons un endomorphisme $E \in End\left((\mathbb{C}^N)^{\otimes k}\right)$ qui commute à l'action du groupe unitaire sur $(\mathbb{C}^N)^{\otimes k}$. Cela implique en particulier que pour tout réel positif t :*

$$\mathbb{E}\left[U_t^{\otimes k} \circ E \circ (U_t^*)^{\otimes k}\right] = E,$$

où $(U_t)_{t\geq 0}$ est un mouvement brownien sur $U(N)$. En faisant un calcul explicite basé sur la formule d'Itô, il est facile de voir que l'élément $\mathbb{E}\left[U_t^{\otimes k} \otimes (U_t^*)^{\otimes k}\right]$ est dans $\rho^{\mathfrak{S}}(\mathbb{C}[\mathfrak{S}_k])$. En prenant la limite quand t tend vers l'infini dans les équations différentielles obtenues et en utilisant le fait que la loi d'un mouvement brownien

(42) Par exemple dans l'article [**6**].

sur $U(N)$ converge vers la mesure de Haar sur $U(N)$ en temps long, on peut alors montrer que seul un certain type de permutations apparaît dans la décomposition de $\mathbb{E}\left[U^{\otimes k}\otimes(U^*)^{\otimes k}\right]$, où U a comme loi la mesure de Haar sur $U(N)$, ceci sans calculer explicitement la décomposition. En notant Tr_2 l'application linéaire :

$$\begin{gathered}\mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right)\otimes\mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right)\to\mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right)\\ A\otimes B\mapsto\mathsf{Tr}(B)A.\end{gathered}$$

et par $(1,2)$ l'élément de :

$$\mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right)\otimes\mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right)\simeq\mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\otimes\left(\mathbb{C}^N\right)^{\otimes k}\right),$$

qui permute les deux copies de $\left(\mathbb{C}^N\right)^{\otimes k}$,

$$E=\mathbb{E}\left[U^{\otimes k}\circ E\circ(U^*)^{\otimes k}\right]=Tr^2\left(\mathbb{E}\left[U^{\otimes k}\otimes(U^*)^{\otimes k}\right]\circ E\otimes\mathrm{Id}^{\otimes k}\circ(1,2)\right).$$

En utilisant le fait que seules certaines permutations apparaissent dans la décomposition de $\mathbb{E}\left[U^{\otimes k}\otimes(U^*)^{\otimes k}\right]$, on déduit de l'égalité précédente que E se décompose en somme de permutations (à travers la représentation du groupe symétrique sur $(\mathbb{C}^N)^{\otimes k}$).

6.3. Partitions et observables. — Avant de finir avec la dualité de Schur-Weyl et les concepts qui l'entourent, faisons un petit calcul très instructif qui s'avère être fondamental par la suite. Donnons-nous $(M_1,...,M_k)$ un k-uplet de matrices de mêmes tailles et prenons une permutation $\sigma\in\mathfrak{S}_k$. Calculons $Tr\left(M_1\otimes...\otimes M_k\circ\rho^{\mathfrak{S}}(\sigma)\right)$; c'est égal à :

$$\begin{aligned}&\sum_{i_1,...,i_k=1}^{N}\left\langle e_{i_1}\otimes...\otimes e_{i_k},\left(M_1\otimes...\otimes M_k\right)\rho^{\mathfrak{S}}(\sigma)(e_{i_1}\otimes...\otimes e_{i_k})\right\rangle\\ &\quad=\sum_{i_1,...,i_k=1}^{N}\left\langle e_{i_1}\otimes...\otimes e_{i_k},\left(M_1\otimes...\otimes M_k\right)\left(e_{i_{\sigma^{-1}(1)}}\otimes...\otimes e_{i_{\sigma^{-1}(k)}}\right)\right\rangle\\ &\quad=\sum_{i_1,...,i_k=1}^{N}(M_1)_{i_1,i_{\sigma^{-1}(1)}}...(M_k)_{i_k,i_{\sigma^{-1}(k)}}\\ &\quad=\prod_{\mathsf{c}=(j_1,...,j_l)\text{ cycle de }\sigma}\ \sum_{i_{j_1},...,i_{j_l}=1}^{N}(M_{j_l})_{i_{j_l},i_{j_{l-1}}}...(M_{j_1})_{i_{j_1},i_{j_l}}\\ &\quad=\prod_{\mathsf{c}=(j_1,...,j_l)\text{ cycle de }\sigma}\mathsf{Tr}(M_{j_l}...M_{j_1}).\end{aligned}$$

Lemme 6.1. — *Pour tout k-uplet de matrices de mêmes tailles* $(M_1, ..., M_k)$, *pour toute permutation* $\sigma \in \mathfrak{S}_k$:

$$Tr\left(M_1 \otimes ... \otimes M_k \circ \rho^{\mathfrak{S}}(\sigma)\right) = \prod_{\mathsf{c}=(j_1,...,j_l) \text{ cycle de } \sigma} \mathsf{Tr}(M_{j_l}...M_{j_1}). \tag{10}$$

6.4. La tranformée de Wilson. — Le but de cette section est de définir une transformée pour les champs d'holonomies invariants par les transformations de jauge qui les caractérise dans le cas où le groupe de structure est $U(N)$.

Dans la section 2.2, où nous avons expliqué l'invariance par transformation de jauge, nous avons fait la supposition que nous ne pouvions mesurer que des éléments dans le groupe de structure, ici $U(N)$. Supposons que nous ne voulions maintenant que mesurer des nombres complexes : quelles observables simples, invariantes par transformation de jauge, pouvons-nous utiliser ? Une réponse est donnée par les boucles de Wilson : étant donné une boucle $l_0 \in \mathsf{P}(\mathbb{R}^2)$, la boucle de Wilson associée à l_0 est l'observable :

$$\begin{aligned} \mathsf{W}(l_0) : \mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), U(N)) &\to \mathbb{C} \\ (h_p)_{p\in\mathsf{P}(\mathbb{R}^2)} &\mapsto \mathsf{Tr}(h_{l_0}). \end{aligned}$$

La boucle de Wilson $\mathsf{W}(l_0)$ est bien invariante par transformation de jauge puisque pour toute transformation de jauge $j \in \mathcal{J}(\mathbb{R}^2)$, pour toute fonction $h \in \mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), U(N))$:

$$\mathsf{W}(l_0)(j \bullet h) = \mathsf{Tr}\left((j \bullet h)_{l_0}\right) = \mathsf{Tr}\left(j_{\overline{l_0}}^{-1} h(l_0) j_{\underline{l_0}}\right) = \mathsf{Tr}\left(h(l_0)\right),$$

par cyclicité de la trace.

Donnons-nous un champ d'holonomies à valeurs dans $U(N)$, c'est-à-dire une mesure μ sur $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), U(N))$. La transformée (renormalisée) de Wilson de μ est donnée par :

$$\begin{aligned} W_\mu : \quad & \bigcup_{k\in\mathbb{N}} \mathsf{L_0}^k\left(\mathbb{R}^2\right) \to \mathbb{C} \\ & (l_1, ..., l_k) \mapsto \frac{1}{N^k} \int_{\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2),U(N))} \prod_{i=1}^{k} \mathsf{W}(l_i)\mu(dh). \end{aligned}$$

Théorème 6.6. — *Soient* μ_1 *et* μ_2 *deux champs d'holonomies planaires invariants par transformation de jauge, alors* $W_{\mu_1} = W_{\mu_2}$ *implique que* $\mu_1 = \mu_2$.

Donnons un bref aperçu de la preuve[(43)]. Supposons que $W_{\mu_1} = W_{\mu_2}$ et montrons que $\mu_1 = \mu_2$.

[(43)] Donnée par T. Lévy dans l'article [**31**].

1. Puisque μ_1 et μ_2 sont deux champs d'holonomies planaires invariants par transformation de jauge, il suffit que leur valeur coincide sur l'algèbre des fonctions continues de la forme :
$$\begin{aligned}\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), U(N)) &\to \mathbb{C}\\ h &\mapsto f\left(h(l_1), ..., h(l_k)\right),\end{aligned}$$
où f est une fonction invariante par conjugaison diagonale par $U(N)$ et l_1,...,l_k sont dans $\mathsf{L}_0(\mathbb{R}^2)$.
2. Ainsi, on se ramène à des mesures définies sur $U(N)^k$ invariantes par conjugaison diagonale par $U(N)$: il suffit de montrer que deux mesures μ_1 et μ_2, définies sur $U(N)^k$ invariantes par conjugaison diagonale par $U(N)$ qui coïncident sur l'algèbre $\mathcal{A}$ engendrée par les applications :
$$\begin{aligned}f_{\mathsf{w}} : \qquad U(N)^k &\to \mathbb{C}\\ (U_1, ..., U_n) &\mapsto \mathsf{Tr}\left(\mathsf{w}(U_1, U_1^{-1}, ..., U_n, U_n^{-1})\right)\end{aligned}$$
où w est un mot $2n$ variables, sont égales.
3. Il suffit donc de montrer que l'algèbre $\mathcal{A}$ est dense dans l'espace des fonctions continues invariantes par conjugaison diagonale par $U(N)$.
4. Par le théorème de Peter-Weyl, appliqué à $U(N)^k$, l'algèbre engendrée par les éléments de matrice irréductibles est dense dans l'espace des fonctions continues sur $U(N)^k$ à valeurs dans $\mathbb{C}$ pour la norme uniforme.
5. Toute représentation irréductible de $U(N)^k$ est de la forme $\alpha_1 \otimes ... \otimes \alpha_k$, où $(\alpha_i)_{i=1}^k$ est une famille irréductible de représentations irréductibles de $U(N)$. Les éléments de matrice irréductibles sont donc des fonctions de la forme :
$$f_{\alpha,J} : (U_1, ..., U_k) \mapsto \mathsf{Tr}\left(\alpha_1(U_1) \otimes ... \otimes \alpha_k(U_k) \circ J\right),$$
où $J \in \mathsf{End}\left(\bigotimes_{i=1}^k V_i\right)$.
6. En moyennant J par conjugaison par l'action diagonale de G, l'algèbre des fonctions de la forme
$$f_{\alpha,J} : (U_1, ..., U_k) \mapsto \mathsf{Tr}\left(\alpha_1(U_1) \otimes ... \otimes \alpha_k(U_k) \circ J\right)$$
où $J \in \mathsf{End}\left(\bigotimes_{i=1}^k V_i\right)$ vérifie pour tout $U \in U(N)$ l'égalité :
$$\left(\alpha_1(U) \otimes ... \otimes \alpha_k(U)\right) J \left(\alpha_1(U)^{-1} \otimes ... \otimes \alpha_k(U)^{-1}\right) = J,$$
est dense dans l'espace des fonctions continues sur $U(N)^k$ à valeurs dans $\mathbb{C}$ qui sont invariantes par l'action diagonale de $U(N)$.
7. Nous avons toujours affaire à des représentations quelconques du groupe $U(N)$: en réalité, il est possible de se ramener à des représentations du type :
$$\mathsf{Reg}^{\otimes n} \otimes (\mathsf{Reg}^{\vee})^{\otimes m},$$

où Reg est la représentation régulière définie sur $\mathbb{C}^N$ donnée pour tout U dans $U(N)$ par :

$$\mathsf{Reg}(U) : x \mapsto Ux,$$

et $\mathsf{Reg}^\vee$ est la représentation contragrédiente de Reg définie sur le dual $(\mathbb{C}^N)^*$ et donnée pour tout $U \in U(N)$ par :

$$\mathsf{Reg}(U) : \phi \mapsto \phi \circ U^{-1}.$$

Ceci est dû au fait[44] que toute représentation irréductible de $U(N)$ est une sous-représentation de $\mathsf{Reg}^{\otimes n} \otimes (\mathsf{Reg}^\vee)^{\otimes m}$

8. Nous avons donc à étudier l'algèbre engendrée par les fonctions de la forme :

$$f_{(n_1,m_1,...,n_k,m_k)} : (U_1, ..., U_k) \mapsto \mathsf{Tr}\left(\bigotimes_{i=1}^{k} \mathsf{Reg}^{\otimes n_i}(U_i) \otimes (\mathsf{Reg}^\vee)^{\otimes m_i}(U_i) \circ J\right),$$

où J commute à l'action de G sur $\bigotimes_{i=1}^k (\mathbb{C}^N)^{\otimes n_i} \otimes (\mathbb{C}^{N*})^{\otimes m_i}$ donnée par $\bigotimes_{i=1}^k \mathsf{Reg}^{\otimes n_i} \otimes (\mathsf{Reg}^\vee)^{\otimes m_i}$.

9. En considérant l'isomorphisme de $U(N)$-modules

$$\mathsf{End}\left((\mathbb{C}^N)^{\otimes n_i} \otimes (\mathbb{C}^{N*})^{\otimes m_i}\right) \simeq \mathsf{End}\left((\mathbb{C}^N)^{\otimes n_i + m_i}\right),$$

l'endomorphisme $\mathsf{Reg}(U_i)^{n_i} \otimes \mathsf{Reg}^\vee(U_i)^{m_i}$ correspond à $\mathsf{Reg}(U_i)^{n_i} \otimes \mathsf{Reg}(U_i^{-1})^{m_i}$ et on se ramène à l'étude de l'algèbre engendrée par les fonctions de la forme :

$$f_{\mathsf{n},\mathsf{m},J} : (U_1, ..., U_k) \mapsto \mathsf{Tr}\left(\bigotimes_{i=1}^{k} U_i^{\otimes n_i} \otimes \left(U_i^{\otimes m_i}\right)^{-1} \circ J\right),$$

où $J \in \mathsf{End}((\mathbb{C}^N)^{\otimes \sum_{i=1}^k (n_i+m_i)})$ vérifie pour tout $U \in U(N)$:

$$U^{\otimes \sum_{i=1}^k (n_i+m_i)} J (U^{-1})^{\otimes \sum_{i=1}^k (n_i+m_i)} = J.$$

10. Un tel endomorphisme J est, d'après la dualité de Schur-Weyl, une combinaison linéaire d'endomorphismes de la forme $\rho^{\mathfrak{S}}(\sigma)$. Or pour toute permutation σ, comme nous l'avons vu dans l'équation (10),

$$f_{(\mathsf{n},\mathsf{m},\rho^{\mathfrak{S}}(\sigma))} : (U_1, ..., U_k) \mapsto \mathsf{Tr}\left(\bigotimes_{i=1}^{k} U_i^{\otimes n_i} \otimes \left(U_i^{\otimes m_i}\right)^{-1} \circ \rho^{\mathfrak{S}}(\sigma)\right),$$

est un produit d'élements de la forme f_{w} tels que définis dans le point numéro 3. Par exemple, si $(1, 2)$ est la permutation dans $\mathfrak{S}_3$ qui permute 1 et 2 en laissant 3 fixe,

$$\mathsf{Tr}(U_1 \otimes U_2 \otimes U_3^{-1} \circ \rho^{\mathfrak{S}}((1, 2))) = \mathsf{Tr}(U_2 U_1)\mathsf{Tr}(U_3).$$

[44] Expliqué dans la proposition 5.2 de [**31**] et prouvé par une courte preuve analytique.

D'où le fait que l'algèbre engendrée par les fonctions f_{w} est dense dans l'espace des fonctions continues invariantes par conjugaison diagonale par $U(N)$: le point 3 est donc vérifié et le théorème est démontré.

Comme expliqué par T. Lévy dans [**31**], ce théorème se généralise dans le cadre des groupes orthogonaux et sympleptiques, sans avoir à changer la définition des boucles de Wilson. Dans le cas où les champs d'holonomies planaires sont à valeurs dans le groupe symétrique $\mathfrak{S}(N)$, il faut généraliser la définition même de transformée de Wilson. Nous nous arrêterons cependant là, mais le lecteur peut facilement voir quelle serait la définition de la transformée de Wilson en reconsidérant la preuve exposée ainsi que la dualité de Jones que nous allons expliquer dans la section 8.6.2.

Grâce au théorème 6.6, un champ d'holonomies planaire invariant par transformation de jauge à valeurs dans $U(N)$ peut être vu, non comme une mesure de probabilité sur $\mathcal{M}ult\left(\mathsf{P}(\mathbb{R}^2), U(N)\right)$ mais comme une fonction sur $\cup_{k\in\mathbb{N}}\mathsf{L}_0(\mathbb{R}^2)^k$.

Dans cette formulation, toute occurrence de $U(N)$ a disparu. Étant donné une suite de champs d'holonomies planaires invariants par transformation de jauge $(\mu_N)_{N\in\mathbb{N}}$, tels que pour tout entier N, μ_N est une mesure sur les fonctions multiplicatives à valeurs dans $U(N)$, on peut alors se demander naturellement si la suite $(\mu_N)_{N\in\mathbb{N}}$ converge ou non, c'est-à-dire si les transformées de Wilson W_{μ_N} convergent simplement quand N tend vers l'infini. Nous allons par la suite montrer que c'est le cas, et nous étendrons ce résultat quand $U(N)$ est remplacé par le groupe symétrique $\mathfrak{S}(N)$. Nous devons donc étudier des systèmes dont la dimension tend vers l'infini. En effet, à chaque boucle, le champ d'holonomies planaire μ_N permet de définir une matrice aléatoire à valeurs dans $U(N)$ et nous voulons comprendre la limite de la trace normalisée de cette matrice aléatoire quand la dimension N tend vers l'infini.

7. Limites en grandes dimensions

7.1. Loi des grands nombres et matrices aléatoires. — Considérons un phénomène aléatoire décrit par un certain nombre de phénomènes élémentaires, ou un certain nombre de particules. Dans de nombreuses situations, quand le nombre de particules augmente, la théorie devient plus simple, l'aléatoire disparaît et la théorie obtenue à la limite ne dépend plus que de quelques paramètres. La loi des grands nombres en théorie des probabilités illustre ce comportement.

Théorème 7.1. — *Soit $(X_i)_{i=1}^{\infty}$ une suite de variables aléatoires indépendantes, de même loi et intégrables. La moyenne temporelle des variables $(X_i)_{i=1}^{\infty}$ converge presque sûrement vers l'espérance de X_1. Ainsi presque sûrement,*

$$\frac{1}{N}\sum_{i=1}^{N} X_i \underset{N\to\infty}{\longrightarrow} \mathbb{E}[X_1].$$

Comme nous allons voir, la propriété d'indépendance n'est pas une condition nécessaire afin d'observer un comportement limite. Dans cette thèse nous nous limitons à l'étude des valeurs propres de matrices aléatoires. Étant donné une matrice complexe M de taille N, on peut considérer ses valeurs propres complexes $\{\lambda_1, ..., \lambda_N\}$.

Définition 7.1. — *La mesure empirique des valeurs propres de M est la mesure :*

$$\mu_M = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}.$$

Cette mesure vérifie que pour tout entier positif k :

$$\int_{\mathbb{C}} z^k \mu_M(dz) = \frac{1}{N}\sum_{i=1}^{N} \lambda_i^k = \frac{1}{N}\mathsf{Tr}\left(M^k\right).$$

Ainsi les moments de la mesure empirique des valeurs propres de M sont simplement les traces de M. Or si on étudie des mesures à support compact dans $\mathbb{R}$, ou satisfaisant une condition de sous-gaussianité uniforme, ou encore définies sur le cercle unité $\mathbb{U}$, la convergence des moments suffit à montrer la convergence des mesures. Cela justifie la méthode moments qui consiste à montrer la convergence des moments d'une matrice aléatoire afin de montrer la convergence de la mesure empirique. Cette méthode ne s'applique pas à des matrices dont les valeurs propres ne seraient pas sur la droite réelle ou sur le cercle unité, cependant toutes les matrices étudiées dans cette thèse sont soit hermitiennes, soit unitaires.

Si M est aléatoire, on peut s'intéresser à la mesure empirique moyenne des valeurs propres de M donnée simplement par $\mathbb{E}\mu_M$, qui vérifie alors que pour tout entier positif k, $\int_{\mathbb{C}} z^k \mathbb{E}\mu_M(dz) = \mathbb{E}\left[\frac{1}{N}\mathsf{Tr}(M^k)\right]$. Ainsi la discussion que l'on vient d'avoir s'applique aussi à la mesure empirique moyenne des valeurs propres de M.

Nous pouvons ainsi reformuler la loi des grands nombres en termes de matrices aléatoires. Soit $(X_i)_{i=1}^{\infty}$ une suite de variables aléatoires indépendantes, de même loi et bornées, nous pouvons considérer pour tout entier N une matrice M_N dont les coefficients sont nuls sauf sur la diagonale, le $i^{\text{ème}}$ coefficient sur la diagonale étant égal à X_i. Par exemple, M_3 est donnée par :

$$M_3 = \begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{pmatrix}.$$

Vu que la trace de $(M_N)^k$ est égale à $\sum_{i=1}^{N} X_i^k$, par la loi des grands nombres, on obtient que presque sûrement :

$$\int_{[-L,L]} x^k \mu_{M_N}(dx) = \frac{1}{N}\mathsf{Tr}\left(M_N^k\right) \underset{N\to\infty}{\longrightarrow} \mathbb{E}\left[X_1^k\right] = \int_{[-L,L]} z^k d\mathbb{P}_{X_1}(dx),$$

où $\mathbb{P}_{X_1}$ est la loi de X_1.

Théorème 7.2. — *La mesure empirique des valeurs propres de M_N converge presque sûrement vers la loi de X_1.*

Une façon de généraliser la loi des grands nombres est alors de considérer d'autres modèles de matrices aléatoires, et d'étudier la convergence des mesures empiriques des valeurs propres.

7.2. Matrices de Wigner gaussiennes. — Une généralisation due à E. Wigner consiste à étudier le spectre d'une grande matrice aléatoire hermitienne à coefficients gaussiens. Étant donné un entier N strictement positif, on considère M_N une matrice de taille N telle que $(M_N)_{i,j} = \overline{(M_N)_{j,i}}$, pour tous $1 \leq i \leq j \leq N$, et telle que les coefficient sur-diagonaux sont indépendants de loi gaussienne complexe $(M_N)_{i,j} \sim \mathcal{N}_{\mathbb{C}}\left(0, \frac{1}{N}\right)$ pour $1 \leq i < j \leq N$ et gaussienne réelle $(M_N)_{i,i} \sim \mathcal{N}_{\mathbb{R}}\left(0, \frac{1}{N}\right)$ pour tout $1 \leq i \leq N$. Une telle matrice aléatoire sera appelée une matrice de Wigner gaussienne de taille N.

Théorème 7.3 **(Théorème de Wigner)**. — *Pour tout entier N strictement positif, soit μ_{M_N} la mesure empirique des valeurs propres de M_N. La suite $(\mu_{M_N})_{N\in\mathbb{N}^*}$ converge en probabilité vers μ_c, la loi du demi-cercle portée par la droite réelle et donnée par :*

$$\mu_c(dx) = \mathbb{1}_{|x|\leq 2}\frac{\sqrt{4-x^2}}{2\pi}dx.$$

Considérons l'espace des matrices hermitiennes et munissons-le du produit scalaire $\langle H_1, H_2\rangle = N\mathsf{Tr}(H_1^* H_2)$. Une base orthogonale pour ce produit scalaire est alors donnée par la famille $\mathcal{E}$ égale à :

$$\left\{\sqrt{\frac{1}{N}}E_{i,i}\right\}_{1\leq i\leq N} \cup \left\{\sqrt{\frac{1}{2N}}\left(E_{i,j}+E_{j,i}\right)\right\}_{1\leq i<j\leq N} \cup \left\{\sqrt{\frac{1}{2N}}\left(iE_{i,j}-iE_{j,i}\right)\right\}_{1\leq i<j\leq N}.$$

où $E_{i,j}$ est la matrice remplie de zéros sauf en position (i,j). Si $((B_t^e)_{t\geq 0})_{e\in\mathcal{E}}$ est une famille de mouvements browniens réels indépendants, alors $B_1^e \sim \mathcal{N}_{\mathbb{R}}(0,1)$ et donc $\sum_{e\in\mathcal{E}} B_1^e e$ est une matrice de Wigner gaussienne de taille N. On aboutit donc au fait qu'une matrice de Wigner gaussienne a la même loi qu'un mouvement brownien au temps 1 sur l'espace des matrices hermitiennes. On montrera par la suite que non seulement les marginales convergent mais qu'en réalité tout le processus converge quand la dimension tend vers l'infini.

En multipliant les matrices de Wigner gaussiennes par le nombre complexe i, on obtient ainsi des matrices ayant la même loi qu'un mouvement brownien au temps 1 sur l'espace des matrices anti-hermitiennes. Or, comme nous l'avons vu dans la section 1.2, le mouvement brownien unitaire est l'exponentielle stochastique du mouvement brownien anti-hermitien. Il est alors naturel de se demander si le théorème 7.3 reste vrai en remplaçant la matrice de Wigner gaussienne par le

mouvement brownien unitaire pris à un certain temps : c'est bien le cas et nous allons voir une des preuves de ce résultat.[45].

7.3. Équations de Schwinger-Dyson, méthode stochastique et mouvement brownien unitaire en grande dimension. —

7.3.1. Les équations de Schwinger-Dyson (S.D.) et la méthode stochastique. — Pour tout entier strictement positif N, on considère un mouvement brownien $\left(U_t^N\right)_{t\geq 0}$ sur le groupe unitaire $U(N)$. Afin de montrer que les mesures empiriques des valeurs propres de U_t^N convergent, il suffirait de pouvoir calculer les observables que sont $\mathbb{E}\left[\left(U_t^N\right)^k\right]$ lorsque k est un entier positif ou du moins de les caractériser par une propriété qui converge quand N tend vers l'infini.

Il existe une méthode développée par les physiciens[46] qui permet de calculer ou du moins de caractériser certaines observables : ce sont les équations de Schwinger-Dyson. Ces équations apparaissent dès lors qu'un modèle est à densité par rapport à un autre modèle satisfaisant une symétrie continue. Cette méthode est liée à une autre méthode plus probabiliste, la méthode stochastique, qui rejoint dans certains cas la méthode de Schwinger-Dyson et qui se base sur la définition de générateur infinitésimal d'un processus de Markov.

Afin d'illuster ces deux méthodes, considérons le calcul des moments d'une gaussienne centrée réduite, c'est-à-dire d'une variable aléatoire réelle de densité $e^{-\frac{x^2}{2}}dx$.

Théorème 7.4. — *Soit X une variable aléatoire gaussienne centrée réduite. Pour tout entier positif n :*

$$\mathbb{E}[X^{n+1}] = n\mathbb{E}[X^{n-1}]. \tag{11}$$

Donnons trois preuves de ce théorème.

1-Schwinger-Dyson (version 1) : En utilisant une intégration par parties :

$$\begin{aligned}\mathbb{E}[X^{n+1}] = \int_{\mathbb{R}} x^{n+1}\exp^{-\frac{x^2}{2}}dx &= \int_{\mathbb{R}} x^n x\exp^{-\frac{x^2}{2}}dx \\ &= n\int_{\mathbb{R}} x^{n-1}\exp^{-\frac{x^2}{2}}dx = n\mathbb{E}[X^{n-1}].\end{aligned}$$

[45]Ce résultat concernant le mouvement brownien unitaire a été démontré par P. Biane dans [**8**] et F. Xu dans [**35**] et étendu par T. Lévy dans [**20**] et [**22**] par une méthode que l'on présentera par la suite.

[46]Voir par exemple l'utilisation dans [**13**] par G. Borot, B. Eynard et N. Orantin des équations de S.D. comme cas particuliers des équations à boucles abstraites, ou l'utilisation mathématique par A. Guionnet et E. Maurel-Segala dans [**18**].

2-Schwinger-Dyson(version 2) : Une autre façon, équivalente mais appréciée des physiciens, est de dire que la mesure de Lebesgue est invariante par translation :

$$\int_{\mathbb{R}}(x+\epsilon)^n \exp^{-\frac{(x+\epsilon)^2}{2}} dx = \int_{\mathbb{R}}(x^n + n\epsilon x^{n-1} + O(\epsilon^2)) \exp^{-\frac{x^2}{2}}(1 - x\epsilon + O(\epsilon^2))dx$$
$$= \int_{\mathbb{R}} x^n \exp^{-\frac{x^2}{2}} dx.$$

Ainsi nous obtenons $\int_{\mathbb{R}}(-x^{n+1}+nx^{n-1}) \exp^{-\frac{x^2}{2}} dx = 0$ ce qui n'est rien d'autre que le résultat attendu.

3-Méthode stochastique : Considérons un mouvement brownien $(X_t)_{t\geq 0}$. Pour toute fonction raisonnable, $\frac{d}{dt}_{|t=t_0}\mathbb{E}[f(X_t)] = \frac{1}{2}\mathbb{E}[f''(X_{t_0})]$. Ceci peut se retrouver par exemple en utilisant l'équation d'Itô, et en prenant l'espérance. On obtient donc l'équation différentielle suivante :

$$\frac{d}{dt}_{|t=t_0}\mathbb{E}[X_t^{n+1}] = \frac{n(n+1)}{2}\mathbb{E}\left[X_{t_0}^{n-1}\right]. \tag{12}$$

Cependant il s'avère qu'étant donné une variable aléatoire gaussienne centrée réduite X, pour tout réel t, X_t a la loi de $t^{\frac{1}{2}}X$. Ainsi l'équation différentielle précédente implique l'égalité $\mathbb{E}[X^{n+1}] = n\mathbb{E}\left[X^{n-1}\right]$.

Les deux méthodes sont bien distinctes de par la nature même des équations obtenues : équations linéaires pour la méthode de S.D., équation différentielle linéaire pour la méthode stochastique. Si nous pouvons montrer l'équivalence directe entre le système donné par (12) et celui donné par (11), c'est bien à cause du phénomène suivant : la loi d'un mouvement brownien en tout temps est un multiple d'une loi gaussienne centrée réduite. Ainsi dès que cette condition d'invariance d'échelle disparaît, par exemple lorsque l'on considère un mouvement brownien sur le cercle unité, les deux méthodes fournissent deux types d'équations dont il est plus difficile de montrer l'équivalence directe.

7.3.2. Convergence des marginales du mouvement brownien unitaire. — En utilisant la méthode stochastique, T. Lévy a donné une preuve du résultat suivant dans les articles [**22**] et [**20**] qui se généralise au cas des mouvements browniens orthogonal et symplectique.

Théorème 7.5. — *Pour tout entier positif N, considérons $\left(U_t^N\right)_{t\geq 0}$ un mouvement brownien unitaire. Pour tout entier r, pour tous entiers positifs $l_1, ..., l_r$, pour tout réel $t \geq 0$:*

$$\mathbb{E}\left[\frac{1}{N}\mathsf{Tr}\left((U_t^N)^{l_1}\right)...\frac{1}{N}\mathsf{Tr}\left((U_t^N)^{l_r}\right)\right] \underset{N\to\infty}{\longrightarrow} \prod_{i=1}^{r} m_{l_i}(t),$$

où l'on a posé :

$$m_k(t) = e^{-\frac{kt}{2}} \sum_{l=0}^{k-1} \frac{(-t)^l}{l!} k^{l-1} \binom{k}{l+1}. \tag{13}$$

Ainsi, pour tout réel positif t, l'espérance de la mesure empirique des valeurs propres de U_t^N converge vers une mesure μ_t sur le cercle $\mathbb{U}$, déterminée par la valeur de ses moments : pour tout entier positif k, $\int_{\mathbb{U}} z^k \mu_t(dz)$ est égal à $m_k(t)$ tel que défini dans (13).

En réalité, on peut montrer que la mesure empirique des valeurs propres converge en probabilité vers la mesure μ_t.

L'idée de la preuve de T. Lévy est de travailler au niveau des observables, donc des traces, ainsi que d'utiliser une indexation adéquate de ces observables. Dans l'exemple 8.3, on verra une autre preuve, découlant des résultats de cette thèse, ayant beaucoup de points communs avec la preuve de T. Lévy, mais dans laquelle nous travaillons dans un système de coordonnées au lieu de travailler au niveau dual des observables. Pour l'instant, expliquons la preuve proposée par T. Lévy, cela nous permettra d'introduire l'équation (14) qui nous sera utile par la suite.

Considérons une permutation $\sigma \in \mathfrak{S}_k$ qui possède r cycles de taille $l_1, ..., l_r$ et donnons-nous un réel positif t. Appliquons l'égalité (10) :

$$\mathsf{Tr}\left((U_t^N)^{\otimes k} \circ \rho_N^{\mathfrak{S}_k}(\sigma)\right) = \prod_{i=1}^{k} \mathsf{Tr}\left((U_t^N)^{l_i}\right).$$

D'observables polynomiales sur l'espace $\mathcal{M}_N(\mathbb{C})$, nous sommes passés grâce à la tensorisation, à des observables linéaires sur l'espace $\left(\mathcal{M}_N^{\otimes k}(\mathbb{C})\right)^{\otimes k}$:

$$M_\sigma\left((U_t^N)^{\otimes k}\right) = \mathsf{Tr}((U_t^N)^{\otimes k} \circ \rho_N^{\mathfrak{S}_k}(\sigma)).$$

Rappelons que r est le nombre de cycles de σ. Il est alors intéressant de définir :

$$m_\sigma\left((U_t^N)^{\otimes k}\right) = \frac{1}{N^r} M_\sigma\left((U_t^N)^{\otimes k}\right).$$

Le théorème 7.5 se réduit donc à montrer que :

$$m_\sigma\left((U_t^N)^{\otimes k}\right) \underset{N\to\infty}{\longrightarrow} \prod_{i=1}^{r} m_{l_i}(t),$$

où $(l_1, ..., l_{\#\sigma})$ est le vecteur des longueurs des cycles de σ. On notera $m_\sigma(t)$ le produit $\prod_{i=1}^{r} m_{l_i}(t)$.

Cependant, vu que $\left(U_t^N\right)_{t\geq 0}$ est un mouvement brownien unitaire, en utilisant le système (2) ainsi que l'équation d'Itô, on peut prouver que :

$$\frac{d}{dt}_{t=t_0} \mathbb{E}\left[\left(U_t^N\right)^{\otimes k}\right] = \left[-\frac{k}{2}\mathrm{Id}^{\otimes k} + \frac{1}{N} \sum_{1\leq i<j\leq k} \rho_N^{\mathcal{P}_k}\left((i,j)\right)\right] \mathbb{E}\left[\left(U_{t_0}^N\right)^{\otimes k}\right]. \tag{14}$$

Ainsi, en composant avec $\rho_N^{\mathfrak{S}_k}(\sigma)$ puis en prenant la trace, nous obtenons :

$$\frac{d}{dt}_{|t=t_0} M_\sigma\left(\left(U_t^N\right)^{\otimes k}\right) = -\frac{k}{2} M_\sigma\left(\left(U_{t_0}^N\right)^{\otimes k}\right) + \frac{1}{N}\sum_{1\leq i<j\leq k} M_{\sigma\circ(i,j)}\left(\left(U_{t_0}^N\right)^{\otimes k}\right). \tag{15}$$

Partitionnons l'ensemble des transpositions $\mathcal{T}_k$, en deux sous-ensembles :

$$\mathcal{T}_\sigma^- = \{(i,j)\in\mathcal{T}_k, \#(\sigma\circ(i,j)) = \#\sigma - 1\}$$
$$\mathcal{T}_\sigma^+ = \{(i,j)\in\mathcal{T}_k, \#(\sigma\circ(i,j)) = \#\sigma + 1\},$$

où pour toute permutation $\sigma\in\mathfrak{S}(N)$, $\#\sigma$ est le nombre de cycles de σ. En divisant par $N^{\#\sigma}$ l'équation (15), on obtient :

$$\frac{d}{dt}_{|t=t_0} m_\sigma\left((U_t^N)^{\otimes k}\right) = -\frac{k}{2} m_\sigma\left((U_{t_0}^N)^{\otimes k}\right) + \sum_{(i,j)\in\mathcal{T}_\sigma^+} m_{\sigma\circ(i,j)}\left((U_{t_0}^N)^{\otimes k}\right) + \frac{1}{N^2}\sum_{(i,j)\in\mathcal{T}_\sigma^-} m_{\sigma\circ(i,j)}\left((U_{t_0}^N)^{\otimes k}\right).$$

Ainsi, lorsque N tend vers l'infini, le système d'équations portant sur les moments converge vers le système :

$$\frac{d}{dt}_{|t=t_0} f_\sigma(t) = -\frac{k}{2} f_\sigma(t_0) + \sum_{(i,j)\in\mathcal{T}_\sigma^+} f_{\sigma\circ(i,j)}(t_0),$$

avec comme conditions initiales le fait que $f_\sigma(0) = 1$ pour toute permutation σ dans $\mathfrak{S}_k$. La solution de ce système est unique et est donnée par la famille $(m_\sigma(t))_{t\geq 0}$, d'où le théorème.

Remarque 7.1. — *On peut donner une preuve dans le formalisme à la Schwinger-Dyson de la convergence du mouvement brownien unitaire. En effet, l'égalité bien connue :*

$$\frac{d}{dt}\mathbb{E}\left[f(U_t^N)\right] = \frac{1}{2}\mathbb{E}\left[\Delta f(U_t^N)\right],$$

peut s'obtenir par intégration par parties, en utilisant le fait que le mouvement brownien unitaire U_t^N possède une densité Q_t qui vérifie l'équation $\partial_t Q_t = \frac{1}{2}\Delta Q_t$.

Cela permet tout de même de mettre en lumière deux faits. Premièrement, il peut être utile d'appliquer plusieurs fois Schwinger-Dyson, vu que dans le cadre du mouvement brownien il faut faire deux intégrations par parties afin d'obtenir le laplacien. De plus, il peut être intéressant de grossir l'espace des observables afin d'obtenir des équations fermées lorsque l'on applique la méthode de Schwinger-Dyson : dans notre cas, on regarde non seulement les espérances de traces, mais aussi les espérances de produits de traces et aussi les dérivées de ces observables.

Remarquons enfin que la méthode de Schwinger-Dyson ne s'adapte plus lorsque l'on considère des processus de Lévy unitaires généraux ou des marches aléatoires

sur le groupe symétrique, tandis que la méthode stochastique peut se généraliser à ces cas.

8. Relations entre systèmes : indépendance, liberté et $\mathcal{P}$-liberté

Nous avons vu, au théorème 7.5, que dans un certain sens, les marginales unidimensionnelles du mouvement brownien unitaire convergent quand la dimension N des matrices augmente. Nous aimerions maintenant "recoller" les convergences des incréments afin d'obtenir une convergence de tout le mouvement brownien unitaire.

On appellera système de matrices aléatoires indexé par un ensemble I, une suite de familles de matrices de la forme $(\mathcal{F}_N)_{N\in\mathbb{N}}$, où $\mathcal{F}_N$ est une famille de matrices aléatoires de tailles N indexée par I. On notera souvent $\mathcal{F}_N = \left(M_i^N\right)_{i\in I}$, l'indice N n'étant pas une puissance mais un rappel de la dimension de la matrice.

Un système de matrices aléatoires converge en distribution non-commutative si pour tout polynôme non-commutatif P en les variables $(X_i)_{i\in I}$,

$$\mathbb{E}\left[\frac{1}{N}\mathsf{Tr}\left(P\left((M_i^N)_{i\in I}\right)\right)\right]$$

converge quand N tend vers l'infini, vers un nombre complexe que l'on notera $\tau_{\mathcal{F}}(P)$. Il semble plus intéressant de considérer la convergence en $\mathfrak{S}$-distribution[47] où l'on considère comme observables des produits de traces. Cette convergence, plus forte, est très souvent vérifiée par les systèmes de matrices aléatoires étudiés. On dira donc qu'un système de matrices aléatoires converge en $\mathfrak{S}$-distribution si pour tous polynômes non-commutatifs $P_1, ..., P_k$ en les variables $(X_i)_{i\in I}$,

$$\mathbb{E}\left[\prod_{l=1}^{k}\frac{1}{N}\mathsf{Tr}\left(P_l\left((M_i^N)_{i\in I}\right)\right)\right]$$

converge quand N tend vers l'infini, vers un nombre complexe que l'on notera $\tau_{\mathcal{F}}(P_1, ..., P_k)$. Si un système de matrices aléatoires converge en $\mathfrak{S}$-distribution, il converge en distribution non-commutative mais il n'est pas généralement vrai que pour tous polynômes non-commutatifs $P_1, ..., P_k$, en les variables $(X_i)_{i\in I}$, $\tau_{\mathcal{F}}(P_1, ..., P_k)$ est égal à $\prod_{l=1}^{k}\tau_{\mathcal{F}}(P_l)$. Si cette condition est vérifiée alors le système vérifie la condition dite de $\mathfrak{S}$-factorisation asymptotique[48]. Nous en reparlerons plus tard, mais donnons un exemple simple où la condition de $\mathfrak{S}$-factorisation asymptotique n'est pas vérifiée.

[47] Comme défini dans la Définition 2.2 du chapitre de thèse "Matrices aléatoires invariantes par le groupe symétrique" [Matrices].

[48] Voir la définition 2.3 de [Matrices] ; pour plus d'informations sur les conséquences de la $\mathfrak{S}$-factorisation asymptotique voir le théorème 2.2 du même chapitre.

Exemple 8.1. — *Le système le plus simple de système de matrices indexé par I est donné par le système suivant. Considérons $(Z_i)_{i\in I}$ une famille de variables aléatoires quelconques, et considérons pour tout entier positif N et tout $i \in I$, $M_i^N = Z_i \mathrm{Id}_N$. Ce système de matrices converge en $\mathfrak{S}$-distribution quand N tend vers l'infini. Par exemple :*

$$\mathbb{E}\left[\frac{1}{N}\mathsf{Tr}\left(M_{i_1}^N M_{i_2}^N\right)\frac{1}{N}\mathsf{Tr}(M_{i_1}^N)\right] = \mathbb{E}[Z_{i_1}Z_{i_2}Z_{i_1}],$$

qui converge trivialement quand N tend vers l'infini. Cependant, il ne vérifie pas la condition de $\mathfrak{S}$-factorisation asymptotique puisque :

$$\tau_{\mathcal{F}}(X_{i_1}, X_{i_2}) = \mathbb{E}[Z_{i_1}Z_{i_2}] \neq \mathbb{E}[Z_{i_1}]\mathbb{E}[Z_{i_2}] = \tau_{\mathcal{F}}(X_{i_1})\tau_{\mathcal{F}}(X_{i_2}).$$

Lorsque $\mathcal{F}$ est de la forme $(M_N)_{N\in\mathbb{N}}$, c'est-à-dire quand l'ensemble I n'est qu'un singleton, la convergence en distribution non-commutative de $\mathcal{F}$ est équivalente à la convergence des observables $\left(\mathbb{E}\left[\frac{1}{N}\mathsf{Tr}(M^k)\right]\right)_{k\in\mathbb{N}}$. D'après la discussion que l'on a eue dans la section 7.1, sous certaines hypothèses cela implique la convergence de la mesure empirique moyenne des valeurs propres. On peut donc considérer la convergence en distribution non-commutative ou en $\mathfrak{S}$-distribution comme une généralisation de la convergence en espérance de la mesure empirique des valeurs propres.

Revenons au mouvement brownien unitaire et au problème de départ : étant donné $\mathcal{F}$ et $\mathcal{F}'$ deux systèmes de matrices aléatoires qui convergent en distribution non-commutative, nous voulons savoir sous quelles conditions un couplage de ces deux systèmes converge toujours en distribution non-commutative ou en $\mathfrak{S}$-distribution et comment calculer alors $\tau_{\mathcal{F}\cup\mathcal{F}'}$ en fonction de $\tau_{\mathcal{F}}$ et de $\tau_{\mathcal{F}'}$. Ceci nous permettrait de"recoller" les convergences des incréments afin d'obtenir une convergence de tout le mouvement brownien unitaire.

Nous allons débuter avec un rappel de l'indépendance, qui permet de répondre à cette question lorsque l'on considère des systèmes de variables aléatoires, c'est-à-dire des familles de variables aléatoires, au lieu de systèmes de matrices aléatoires.

8.1. Indépendance. —

8.1.1. Définition. — Un notion fondamentale en théorie des probabilités est la notion d'indépendance. Prenons deux expériences X et Y à valeurs bornées, la propriété d'indépendance nous dit simplement que la probabilité que la mesure X soit égale à une certaine valeur ne dépend pas de la valeur prise par la mesure Y. Mathématiquement, cela se traduit par l'égalité :

$$\mathbb{P}[X \in A \mid Y \in B] = \mathbb{P}[X \in A],$$

pour tous ensembles mesurables A et B tels que $\mathbb{P}(Y \in B) > 0$.

Cette formulation est très dépendante de la notion de mesures de probabilités et, formulée de la sorte, il semble difficile de la généraliser dans des cadres plus

abstraits. Cependant toute la théorie des probabilités peut se formuler seulement en termes d'espérance, qui se trouve n'être qu'une forme linéaire positive vérifiant le théorème de Beppo Levi. Ainsi étant donné une espérance $\mathbb{E}$, c'est-à-dire une mesure qui permet de calculer la moyenne de chaque observable, on peut reformuler la notion d'indépendance de la façon suivante : pour toutes fonctions mesurables f et g bornées,

$$\mathbb{E}\left[f(X)g(Y)\right] = \mathbb{E}\left[f(X)\right]\mathbb{E}\left[g(Y)\right],$$

et plus généralement :

$$\mathbb{E}\left[F(X,Y)\right] = \mathbb{E}\left[F_X(Y)\right],$$

où $F_X(y) = \mathbb{E}[F(X,y)]$. L'indépendance, lorsqu'elle est supposée, permet ainsi de connaître la loi du couple (X,Y) connaissant la loi de X et la loi de Y. Cette notion se généralise à des familles de variables aléatoires, ce qui permet pour toutes familles $\mathcal{F}_1$ et $\mathcal{F}_2$ de variables aléatoires d'avoir une notion d'indépendance entre $\mathcal{F}_1$ et $\mathcal{F}_2$.

8.1.2. Cumulants classiques. — Si X et Y sont supposées indépendantes, la covariance de X et Y, définie par $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$ est nulle. La nullité de la covariance n'implique pas en général l'indépendance des variables X, Y. La notion de cumulants permet de généraliser la notion de covariance : la nullité des cumulants implique alors l'indépendance des variables aléatoires. Pour simplifier l'exposé, on supposera que X et Y sont des variables aléatoires complexes bornées.

Définition 8.1. — *Soient $(U_i)_{i=1}^n \in \{X,Y\}^n$. Pour toute partition $\mathcal{V}$ de $\{1,...,n\}$, on définit l'observable :*

$$\mathbb{E}_{\mathcal{V}}[U_1,...,U_n] = \prod_{V\in\mathcal{V}} \mathbb{E}\left[\prod_{i\in V} U_i\right]. \tag{16}$$

On notera $\mathcal{P}(\{1,...,n\})$ l'ensemble des partitions de $\{1,...,n\}$. Les cumulants classiques sont alors définis par une relation de récurence.

Définition 8.2. — *Soient $(U_i)_{i=1}^n \in \{X,Y\}^n$. La famille des cumulants classiques de degré n de U_1, ..., U_n est l'unique famille de nombres :*

$$(\kappa_{\mathcal{V}}[U_1,...,U_n])_{\mathcal{V}\in\mathcal{P}(\{1,...,n\})}$$

telle que pour toute partition $\mathcal{V} \in \mathcal{P}(\{1,...,n\})$, on ait :

$$\mathbb{E}_{\mathcal{V}}[U_1,...,U_n] = \sum_{\mathcal{V}' \text{ plus fine que } \mathcal{V}} \kappa_{\mathcal{V}'}[U_1,...,U_n]. \tag{17}$$

Calculons par exemple $\kappa_{\{1,2\}}[X,Y]$. Par définition, on a :

$$\mathbb{E}_{\{1,2\}}[X,Y] = \kappa_{\{1,2\}}[X,Y] + \kappa_{\{\{1\},\{2\}\}}[X,Y],$$
$$\mathbb{E}_{\{\{1\},\{2\}\}}[X,Y] = \mathbb{E}[X]\mathbb{E}[Y] = \kappa_{\{\{1\},\{2\}\}}[X,Y],$$

d'où le fait que $\kappa_{\{1,2\}}[X,Y]$ est la covariance de X et Y, égale à $\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]$.

Étant donné que $\mathbb{E}_{\mathcal{V}}$ a une forme multiplicative (16), on peut vérifier que $\kappa_{\mathcal{V}}$ a aussi forme multiplicative :

$$\kappa_{\mathcal{V}} = \prod_{V\in\mathcal{V}} \kappa_V[(U_i)_{i\in V}]. \tag{18}$$

Théorème 8.1. — *Les variables X et Y sont indépendantes si et seulement si pour tout entier n, pour tout $(U_1,...,U_n)\in\{X,Y\}^n$ tels qu'il existe i et j tels que $U_i = X$ et $U_j = Y$:*

$$\kappa_{\{1,...,n\}}(U_1,...,U_n) = 0.$$

On dit alors que les cumulants mixtes de X et Y sont nuls.

Par exemple, supposons que les cumulants mixtes des deux variables bornées X et Y sont nuls. Montrer que pour toutes fonctions continues f et g, $\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\,\mathbb{E}[g(Y)]$ revient à montrer que pour tous entiers positifs n et m, $\mathbb{E}[X^nY^m] = \mathbb{E}[X^n]\mathbb{E}[Y^m]$, ou encore :

$$\mathbb{E}_{\{\{1,...,n+m\}\}}[U_1,...,U_{n+m}] = \mathbb{E}_{\{\{1,.,n\}\}}[X,...,X]\,\mathbb{E}_{\{\{1..,m\}\}}[Y,...,Y],$$

où $(U_1,...,U_{n+m})$ est le vecteur constitué de n occurrences successives de X suivi de m occurrences successives de Y. Or, en utilisant la définition des cumulants :

$$\mathbb{E}_{\{\{1,...,n+m\}\}}[U_1,...,U_{n+m}] = \sum_{\mathcal{V}\text{ plus fine que }\{\{1,...,n+m\}\}} \kappa_{\mathcal{V}}[U_1,...,U_{n+m}].$$

En réalité toute partition est plus fine que $\{\{1,...,n+m\}\}$. En utilisant le fait que les cumulants croisés sont nuls, les seules partitions qui apparaîtront dans la somme sont celles qui sont plus fines que $\{\{1,.,n\},\{n+1..,n+m\}\}$. On obtient donc :

$$\mathbb{E}_{\{\{1,...,n+m\}\}}[U_1,...,U_{n+m}] = \sum_{\mathcal{V}\text{ plus fine que }\{\{1,.,n\},\{n+1..,n+m\}\}} \kappa_{\mathcal{V}}[U_1,...,U_{n+m}].$$

En utilisant alors la multiplicativité des cumulants donnée par (18) et en utilisant de nouveau la définition des cumulants, on en déduit que :

$$\begin{aligned}\mathbb{E}_{\{\{1,...,n+m\}\}}[U_1,...,U_{n+m}] &= \sum_{\substack{\mathcal{V}_1\text{ plus fine que }\{\{1,.,n\}\}\\ \mathcal{V}_2\text{ plus fine que }\{\{1..,m\}\}}} \kappa_{\mathcal{V}_1}[X,...,X]\,\kappa_{\mathcal{V}_2}[Y,...,Y]\\ &= \mathbb{E}_{\{\{1,.,n\}\}}[X,...,X]\,\mathbb{E}_{\{\{1..,m\}\}}[Y,...,Y].\end{aligned}$$

Ceci montre que la nullité des cumulants mixtes pour X et Y implique l'indépendance des deux variables. Il est à noter que les cumulants sont aussi liés au développement en série de la fonction $\log\mathbb{E}\left[e^{t_1X+t_2Y}\right]$ ce qui permettrait de montrer ce résultat d'une autre façon. Récapitulons ce que nous avons vu :

1. nous avons défini une *notion d'indépendance* afin de calculer la loi jointe de deux variables aléatoires connaissant la loi de chacune des deux variables aléatoires,
2. les observables dans ce modèle ont pu être indexées par un *ensemble ordonné* : les partitions munies de l'ordre dans lequel $\mathcal{V}' \leq \mathcal{V}$ si et seulement si $\mathcal{V}'$ est plus fin que $\mathcal{V}'$,
3. on a défini les *cumulants* comme une *transformation triangulaire des* observables :

$$\mathbb{E}_{\mathcal{V}} = \sum_{\mathcal{V}' \leq \mathcal{V}} \kappa_{\mathcal{V}'},$$

4. la notion d'indépendance est alors équivalente à la *nullité des cumulants mixtes*. On peut donc la définir non pas en utilisant l'étape 1, mais en la définissant directement comme étant la nullité des cumulants mixtes.

C'est en suivant ces étapes, en changeant la notion d'observables, l'ensemble ordonné ainsi que l'ordre lui même, que l'on définira d'autres notions d'indépendance.

8.2. Liberté. —

8.2.1. Définition. — L'indépendance est une condition qui permet de connaître la distribution globale d'un système de variables aléatoires connaissant la distribution des familles indépendantes qui le composent. Cependant, cette notion n'est pas adaptée pour les systèmes de matrices aléatoires : pour ces systèmes nous allons introduire la notion de liberté au sens de Voiculescu.

Considérons A et B deux systèmes de matrices de la forme $(A_N)_{N\in\mathbb{N}}$ et $(B_N)_{N\in\mathbb{N}}$. Un couplage naturel entre les deux systèmes de matrices est de supposer que pour tout entier positif N, A_N est indépendant de B_N, dans le sens où la famille des coefficients de A_N est indépendante de celle des coefficients de B_N. Ce couplage assure-t-il la convergence en distribution non-commutative de $(A_N)_{N\in\mathbb{N}}$ et $(B_N)_{N\in\mathbb{N}}$? Si oui, comment déterminer la distribution non-commutative $\tau_{A,B}$ connaissant τ_A et τ_B ? On supposera par la suite que les deux systèmes de matrices sont couplés de la sorte.

Le fait que l'indépendance ne soit pas la bonne notion pour ces systèmes de matrices provient du fait que les observables ne sont plus de la forme $\mathbb{E}[\bullet]$ mais de la forme $\mathbb{E}\left[\mathsf{Tr}(\bullet)\right]$, ainsi que le fait que les matrices ne commutent pas, ce qui leur confère un autre comportement. Ainsi pour tout entier positif N,

$$\mathbb{E}[\mathsf{Tr}(A_N B_N)] \neq \mathbb{E}[\mathsf{Tr}(A_N)]\mathbb{E}[\mathsf{Tr}(B_N)].$$

En réalité, à la limite, l'égalité précédente est vérifiée. Cependant, on peut montrer que $\mathbb{E}\left[\frac{1}{N}\mathsf{Tr}(A_N B_N A_N B_N)\right]$ converge quand N tend vers l'infini et que de plus :

$$\tau_{\{A,B\}}(X_1X_2X_1X_2) \neq \tau_A\left(X^2\right)\tau_B\left(X^2\right), \tag{19}$$

égalité dans laquelle nous avons supposé que le système A est indexé par 1 et le système B par 2 dans le système $\{A, B\} := (\{A_N, B_N\})_{N\in\mathbb{N}}$. On voit alors que la propriété d'indépendance n'est pas adaptée à la description de la distribution non-commutative du système $\{A, B\}$: c'est en réalité la notion de liberté qui est importante dans ce cadre, et qui s'applique *si l'un des systèmes est invariant par conjugaison en loi par le groupe unitaire.*

Théorème 8.2 **(Liberté asymptotique de systèmes indépendants et invariants par le groupe unitaire**(50)**)**

Si un des deux systèmes est invariant par conjugaison par le groupe unitaire, c'est-à-dire si par exemple pour tout entier positif N, pour tout $U \in U(N)$, les matrices A_N et UA_NU^{-1} ont même loi, et si les deux systèmes convergent en $\mathfrak{S}$-distribution et vérifient la propriété de $\mathfrak{S}$-factorisation asymptotique alors le système $\{A, B\}$ converge en distribution non-commutative et le système $\{A, B\}$ vérifie la propriété de $\mathfrak{S}$-factorisation asymptotique. De plus A et B sont asymptotiquements libres au sens de Voiculescu, c'est-à-dire que pour tous polynômes P_1^A, ..., P_k^A, P_{k+1}^A $P_1^B, ..., P_k^B$, si pour tout $i \in \{1, ..., k+1\}$, $\tau_M[P_i^A] = 0$ et si pour tout $i \in \{1, ..., k\}$, $\tau_M[P_i^B] = 0$, alors :

$$\tau_{\{A,B\}}\left[P_1^A(X_1)P_1^B(X_2)...P_k^A(X_1)P_k^B(X_2)\right] = 0,$$
$$\tau_{\{A,B\}}\left[P_1^A(X_1)P_1^B(X_2)...P_k^A(X_1)P_k^B(X_2)P_{k+1}^A(X_1)\right] = 0.$$

Ainsi tout produit alterné de polynômes en A et B a une moyenne nulle si tant est que les polynômes eux-mêmes ont une moyenne nulle. La relation de liberté asymptotique (ou liberté tout court) permet donc de retrouver la distribution non-commutative $\tau_{\{A,B\}}$ connaissant τ_A et τ_B puisque pour tous polynômes P_1^A, ..., P_k^A, $P_1^B, ..., P_k^B$,

$$\tau_{\{A,B\}}\left[\prod_{i=1}^{k}\left(P_i^A(X_1) - \tau_A[P_i^A]\right)\left(P_i^B(X_2) - \tau_A[P_i^B]\right)\right] = 0.$$

En développant l'expression précédente, on obtient une relation de récurrence qui permet alors de calculer, par récurrence sur le nombre de polynômes, la valeur de $\tau_{\{A,B\}}\left[P_1^A(X_1)P_1^B(X_2)...P_k^A(X_1)P_k^A(X_2)\right]$. Ainsi, pour tous polynômes P_1^A et P_2^B,

$$\tau_{\{A,B\}}\left[\left(P_1^A(X_1) - \tau_A\left[P_1^A\right]\right)\left(P_2^B(X_2) - \tau_A\left[P_2^A\right]\right)\right] = 0$$

permet d'établir l'égalité :

$$\tau_{\{A,B\}}\left[P_1^A(X_1)P_1^B(X_2)\right] = \tau_A\left[P_1^A\right]\tau_B\left[P_1^B\right].$$

(50)Pour des références, voir [**6**], [**5**], [**9**], [**10**]. L'appendice de T. Lévy dans [**20**] permet d'avoir un aperçu clair et rapide des preuves et des concepts sous-jacents. Le théorème 7.10 de [Matrices] généralise ce théorème et sa courte preuve utilise les cumulants fini-dimensionels que l'on introduira par la suite.

En utilisant cette dernière égalité, on peut calculer alors $\tau_{\{A,B\}}(X_1X_2X_1X_2)$. Vu que :

$$\tau_{\{A,B\}}((X_1-\tau_A(X))(X_2-\tau_B(X))(X_1-\tau_A(X))(X_2-\tau_B(X))=0,$$

en développant, on obtient :

$$\tau_{\{A,B\}}(X_1X_2X_1X_2)=\tau_A(X^2)\tau_B(X)^2+\tau_A(X)^2\tau_B(X^2)-\tau_A(X)^2\tau_B(X)^2,$$

qui est bien différent de $\tau_A(X^2)\,\tau_B(X^2)$ en général comme vu précédemment dans l'équation (19).

Remarque 8.1. — *Remarquons que le théorème 8.2 nécessite la propriété de* $\mathfrak{S}$*-factorisation asymptotique. Prenons le cas des systèmes* $(A_N=\mathrm{Id}_N)_{N\in\mathbb{N}}$ *et* $(B_N=\mathrm{Id}_N)_{N\in\mathbb{N}}$ *invariants par conjugaison par le groupe symétrique. Nous avons vu dans l'exemple 8.1 que les systèmes* A *et* B *convergent en* $\mathfrak{S}$*-distribution mais aucun ne satisfait la propriété de* $\mathfrak{S}$*-factorisation asymptotique : le théorème précédent ne peut pas s'appliquer. Cependant il est évident que le système* $\{A,B\}$ *converge en* $\mathfrak{S}$*-distribution.*

C'est pourquoi, pour pallier ce fait étrange, dans le chapitre de thèse dénommé "Matrices aléatoires invariantes par le groupe symétrique", on définit une notion de $\mathfrak{S}$*-liberté*[(51)] *qui se réduit à la liberté de Voiculescu dans le cas où les systèmes considérés satisfont la propriété de* $\mathfrak{S}$*-factorisation asymptotique. On montre alors*[(52)] *que le théorème précédent reste valide si on enlève l'hypothèse de* $\mathfrak{S}$*-factorisation asymptotique tout en remplaçant la liberté au sens de Voiculescu par la* $\mathfrak{S}$*-liberté. Ainsi, si deux variables aléatoires* X *et* Y *sont indépendantes, les deux systèmes de matrices aléatoires* $(X\mathrm{Id}_N)_{N\in\mathbb{N}}$ *et* $(Y\mathrm{Id}_N)_{N\in\mathbb{N}}$ *sont* $\mathfrak{S}$*-libres.*

La notion de liberté permet donc sous certaines hypothèses sur le couplage, de calculer la distribution jointe de deux systèmes de matrices aléatoires connaissant la distribution de chacune. Pour continuer le programme esquissé dans la section traitant de l'indépendance, il nous faut indexer les observables par un ensemble ordonné.

8.2.2. Partitions non-croisées et cumulants libres. — La vision combinatoire de la liberté, élaborée par R. Speicher, utilise comme ensemble d'indexation les partitions non-croisées[(53)]. Nous verrons par la suite que l'on peut aussi indexer les observables par l'ensemble des permutations, mais dans cette section nous nous concentrerons sur les partitions non-croisées.

Considérons k un entier strictement positif. Nous avions noté $\mathcal{P}(\{1,...,k\})$ l'ensemble des partitions de $\{1,...,k\}$. Notons alors $\mathsf{NC}(k)$ l'ensemble des partitions non-croisées de $\mathcal{P}(\{1,...,k\})$, c'est-à-dire l'ensemble des partitions $\mathcal{V}$ dans

(51) Voir la définition 7.1 de [Matrices].
(52) En considérant, dans le théorème 7.10 de [Matrices], le cas où $A=\mathfrak{S}$ et $G=U$.
(53) Voir le livre [**28**] très complet écrit par R. Speicher et A. Nica.

$\mathcal{P}(\{1,...,k\})$ telles que pour tout $1 \leq i_1 < i_2 < i_3 < i_4 \leq k$, i_1, i_3 ne peut pas être inclus dans le même bloc si i_2 et i_4 le sont, et vice-versa. Dessinons une partition $p \in \mathcal{P}(\{1,...,k\})$ en dessinant k points alignés et en reliant les points de chaque bloc dans l'ordre croissant. Alors, une partition non croisée est une partition qui, lorsqu'on la dessine, ne possède aucun croisement, d'où le nom de partitions non-croisées. Un exemple est dessiné dans la figure 11.

FIGURE 11. Une partition à gauche et une partition non croisée à droite

Étant donné que l'ensemble des partitions non-croisées est inclus dans l'ensemble des partitions, on peut restreindre à $\mathsf{NC}(k)$ la relation de plus grande finesse.

Il nous faut maintenant associer à toute partition non croisée une observable : de nouveau, vu que $\mathsf{NC}(k) \subset \mathcal{P}_k$, il existe un candidat tout naturel donné par la définition suivante qui s'inspire de la définition 8.1. Pour simplifier les notations, nous notons $\tau[P(A,B)]$ l'observable $\tau_{\{A,B\}}[P(X_1,X_2)]$.

Définition 8.3. — *Soient $(U_i)_{i=1}^n \in \{A,B\}^n$. Pour toute partition non croisée $\mathcal{V} \in \mathsf{NC}(k)$, on définit l'observable :*

$$\tau_{\mathcal{V}}[U_1,...,U_n] = \prod_{V \in \mathcal{V}} \tau\left[\prod_{i \in V} U_i\right], \tag{20}$$

où le produit des U_i est pris dans l'ordre croissant.

Par exemple :

$$\tau_{\{\{1,2\},\{3\}\}}[A,B,A] = \tau[AB]\tau[A].$$

Les cumulants alors sont définis de la même façon que dans le cas classique.

Définition 8.4. — *Soient $(U_i)_{i=1}^n \in \{X,Y\}^n$. La famille des cumulants libres de degré n de $U_1,...,U_n$ est donnée par l'unique famille de nombres*

$$(\kappa_{\mathcal{V}}[U_1,...,U_n])_{\mathcal{V} \in \mathsf{NC}_n}$$

telle que pour toute partition $\mathcal{V} \in \mathsf{NC}_n$ on ait

$$\tau_{\mathcal{V}}[U_1,...,U_n] = \sum_{\substack{\mathcal{V}' \in \mathsf{NC}(n) \\ \mathcal{V}' \text{ plus fine que } \mathcal{V}}} \kappa^{\mathsf{nc}}_{\mathcal{V}'}[U_1,...,U_n].$$

Étant donné que $\tau_{\mathcal{V}}$ a une forme multiplicative (20), on peut vérifier que $\kappa^{\mathsf{nc}}_{\mathcal{V}}$ a ausis une forme multiplicative :

$$\kappa_{\mathcal{V}} = \prod_{V \in \mathcal{V}} \kappa^{\mathsf{nc}}_V[(U_i)_{i \in V}]. \tag{21}$$

R. Speicher a montré que la liberté au sens de Voiculescu est équivalente à la nullité des cumulants libres mixtes.

Théorème 8.3. — *Considérons un système de matrices aléatoires de la forme $(A_N, B_N)_{N\in\mathbb{N}}$ qui converge en distribution non-commutative et qui vérifie la propriété de $\mathfrak{S}$-factorisation asymptotique.*

Les systèmes de matrices $(A_N)_{N\in\mathbb{N}}$ et $(B_N)_{N\in\mathbb{N}}$ sont libres au sens de Voiculescu si et seulement si pour tout entier n, pour tout $(U_1, ..., U_n) \in \{A, B\}^n$ tels qu'il existe i et j tels que $U_i = X$ et $U_j = Y$:

$$\kappa^{\mathsf{nc}}_{\{1,...,n\}}(U_1, ..., U_n) = 0.$$

On dit alors que les cumulants mixtes non commutatifs sont nuls.

On peut ainsi définir la liberté au sens de Voiculescu seulement en termes de cumulants libres.

8.2.3. Convergence des mouvements browniens. — Nous avons vu que les marginales unidimensionnelles du mouvement brownien unitaire convergent en distribution non-commutative quand la dimension N des matrices augmente. Ainsi par stationnarité des incréments, pour tous $t_1 < t_2$ l'incrément sur l'intervalle $[t_1, t_2]$ converge en distribution. Les incréments du mouvement brownien unitaire sont invariants par conjugaison par le groupe unitaire puisque le processus lui-même est invariant par conjugaison par le groupe unitaire. De plus, en considérant la forme multiplicative de la limite dans le théorème 7.5, la propriété de $\mathfrak{S}$-factorisation asymptotique est vérifiée par les incréments. En utilisant le théorème 8.2, et sa généralisation à une famille de systèmes de matrices aléatoires, on en déduit la convergence en distribution non-commutative du mouvement brownien unitaire. Les mêmes arguments s'appliquent au mouvement brownien hermitien.

Théorème 8.4. — *Pour tout entier positif N, considérons $\left(H_t^N\right)_{t\geq 0}$ et $\left(U_t^N\right)_{t\geq 0}$ deux mouvements browniens respectivement hermitien et unitaire de taille N. Les systèmes de matrices $\left(\left(H_t^N\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ et $\left(\left(U_t^N\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ convergent en distribution et satisfont la propriété de $\mathfrak{S}$-factorisation asymptotique.*

En réalité, on peut montrer que la convergence a lieu dans en probabilité : pour cela il suffit de montrer que les systèmes de matrices $\left(\left(H_t^N\right)_{t\geq 0} \cup \left((H_t^N)^*\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ et $\left(\left(U_t^N\right)_{t\geq 0} \cup \left((U_t^N)^*\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ convergent en distribution et satisfont la propriété de $\mathfrak{S}$-factorisation asymptotique.

8.3. Les ensembles ordonnés. — À la base des théories considérées on retrouve un ensemble ordonné. Nous allons dans la section suivante rappeler des bases simples mais importantes sur ces objects. Un ensemble ordonné est la

donnnée d'un ensemble E muni d'une relation d'ordre $\leq$ qui permet de comparer deux éléments de E. Cette relation d'ordre vérifie trois axiomes, la réflexivité, l'antisymétrie et la transitivité.

Ainsi, l'ensemble des partitions $\mathcal{P}(\{1, ..., n\})$ ou l'ensemble des partitions non-croisées, lorsque munis de la relation de plus grande finesse, sont des ensembles ordonnés. Il existe au moins deux façons de représenter un ensemble ordonné que nous allons exposer par la suite : le diagramme de Hasse et la matrice d'ordre.

8.3.1. Le diagramme de Hasse. — C'est un graphe dont les sommets représentent chacun un élément différent de E. Il existe alors une arête orientée du sommet représentant x à celui représentant $y \neq x$ si et seulement si x est directement plus petit que y, c'est-à-dire si $x \leq y$ et il n'existe pas d'autre élément z tel que $x \leq z \leq y$. Par exemple, un diagramme de Hasse de $\mathcal{P}(\{1, 2, 3\})$ est donné par la figure 12.

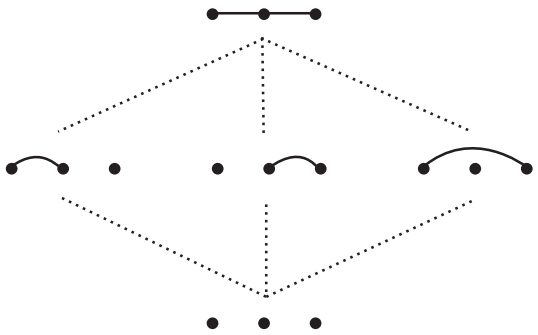

FIGURE 12. Diagramme de Hasse de $\mathcal{P}(\{1, 2, 3\})$

8.3.2. La matrice d'ordre. — Il existe une autre représentation tout aussi intéressante d'un ensemble ordonné $(E, \leq)$: la matrice d'ordre. Il est souvent coutume d'appeler matrice (finie) réelle un ensemble de nombres réels $M_{i,j}$ indexés par un indice i appartenant à $\{1, ..., n\}$ et un indice j appartenant à $\{1, ..., m\}$. Malheureusement cette vision est trop étroite. Une matrice réelle est la donnée de deux ensembles finis d'indexations I et J et d'une application $I \times J$ dans $\mathbb{R}$, I est l'ensemble de sortie, et J est l'ensemble d'entrée. On peut alors multiplier deux matrices si l'ensemble de sortie de la deuxième est égale à l'ensemble d'entrée de la première et alors $(I, J, M_1)(J, K, M_2) = (I, K, M_1 M_2)$ où :

$$M_1 M_2(i, k) = \sum_{j \in J} M_1(i, j) M_2(j, k).$$

Une matrice qui joue un rôle important est la matrice identité donnée par (I, I, Id_I) définie par le fait quel $Id_I(i, j) = \delta_{i=j}$.

Supposons que I est muni d'un ordre $\preceq$. Une matrice (I, I, M) est strictement triangulaire inférieure si pour tous $i \preceq i'$, $M(i, i') = 0$. Toute matrice (I, I, M)

strictement triangulaire inférieure est nilpotente : il existe un entier positif n tel que $M^n = 0$. En effet, si on considère $n = \#I$, alors pour tous i et j dans I:

$$M^n(i,j) = \sum_{(i_1,...,i_{n-1})\in I^{n-1}} M_{i,i_1}...M_{i_{n-1},j}.$$

Cependant, la matrice M étant strictement triangulaire inférieure, j doit être strictement plus grand que i_{n-1}, lui même strictement plus grand que i_{n-2} et ainsi de suite. Étant donné que l'on ne peut avoir une relation du type $i > k > i$, les éléments $i, i_1, ..., i_{n-1}$ et j doivent être distincts. Ce qui n'est pas possible vu que l'on demande à ce que $\#I + 1$ éléments de I soient distincts : $M^{\#I} = 0$.

Étant donné un ensemble ordonné $(E, \leq)$, on peut alors définir la matrice d'ordre associée à E. C'est la donnée de la matrice (E, E, M) telle que pour tous x et y de E,

$$M_{(E,\leq)}(x,y) = \delta_{y\leq x}.$$

8.3.3. Transformation triangulaire. — La matrice d'ordre intervient dans les transformations triangulaires que l'on a rencontrées lorsque l'on définissait les cumulants classiques et libres. Par exemple, considérons $M_{\mathcal{P}(\{1,...,n\})}$ la matrice d'ordre associé aux partitions $\mathcal{P}(\{1, ..., n\})$ munies de la relation de plus grande finesse, c'est à dire $p \leq p'$ si p est plus fine que p'.

L'équation (17) qui définit les cumulants classiques peut alors s'écrire de façon plus compacte :

$$\mathcal{O} = M_{\mathcal{P}(\{1,...,n\})}\mathcal{K},$$

où $\mathcal{O} = (\mathbb{E}_{\mathcal{V}}[U_1, ..., U_N])_{\mathcal{V}\in\mathcal{P}(\{1,...,n\})}$ et $\mathcal{K} = (\kappa_{\mathcal{V}}[U_1, ..., U_n])_{\mathcal{V}\in\mathcal{P}(\{1,...,n\})}$. On peut faire de même pour les cumulants non-commutatifs. On voit qu'il est alors important, afin de calculer les cumulants, de connaître l'inverse de la matrice. Par exemple, dans le cadre des des cumulants classiques, $\mathcal{K} = \left(M_{\mathcal{P}(\{1,...,n\})}\right)^{-1}\mathcal{O}$.

Revenons au cas général où l'on considère un ensemble ordonné $(E, \leq)$ ainsi que sa matrice d'ordre M. La matrice M peut se décomposer en $Id_E + N_E$ avec N_E une matrice strictement triangulaire inférieure donc nilpotente. Cette décomposition permet alors de calculer la matrice inverse de M :

$$M^{-1} = (\mathrm{Id}_E + N_E)^{-1} = \sum_{l=0}^{\infty}(-1)^l N_E^l,$$

d'où la formule de Rota :

$$M^{-1}(x,y) = \sum_{l=0}^{\infty}(-1)^l \#C_l(x,y),$$

où $C_l(x,y)$ est l'ensemble des suites de longueur l strictement décroissantes entre x et y, c'est-à-dire :

$$C_l(x,y) = \{(i_0, ..., i_l), x = i_0 \neq ... \neq i_l = y, i_0 \geq ... \geq i_l\}.$$

Ceci permet alors de calculer la fonction de Möbius, définie simplement par :

$$\mu_{(E,\leq)}(x,y) = M^{-1}(y,x).$$

8.4. Ensemble de partitions : le cadre général de $\mathcal{P}_X$. — Nous avons vu que l'indépendance était une notion intéressante pour des variables aléatoires réelles, de même la notion de liberté était la notion centrale lorsque l'on étudie des systèmes de matrices aléatoires invariantes par conjugaison par le groupe unitaire. Quel ensemble ordonné doit être utilisé afin de définir de nouvelles observables, de nouveaux cumulants et une nouvelle liberté pour des systèmes de matrices aléatoires invariantes par conjugaison par le groupe symétrique ? Nous allons voir par la suite que l'ensemble ordonné au centre de l'étude de ces systèmes de matrices aléatoires est aussi un ensemble de partitions muni d'un nouvel ordre.

8.4.1. L'ensemble $\mathcal{P}_X$; distance et ordre géodésique. — Les ensembles ordonnés considérés jusqu'à maintenant afin d'indexer des observables étaient des sous-ensembles de partitions : $\mathcal{P}_{\{1,\dots,n\}}$ et $\mathsf{NC}(n)$. Nous allons définir un ensemble ordonné qui contient une copie de $\mathcal{P}_{\{1,\dots,n\}}$ et de $\mathsf{NC}(n)$ d'intersection égale à un singleton. Cet ensemble nous servira à indexer de nouvelles observables sur des systèmes de matrices aléatoires invariantes par conjugaison par le groupe symétrique. Donnons-nous un ensemble fini X, et considérons l'ensemble des partitions de X que l'on notera $\mathcal{P}_X$[(54)].

Tout élément dans $\mathcal{P}_X$ peut être représenté graphiquement en considérant des points indexés par les éléments de X : un graphe sur cet ensemble de points représente une partition $p \in \mathcal{P}_X$ si les composantes connexes du graphe sont les blocs de p.

L'ensemble $\mathcal{P}_X$ est aussi naturellement muni d'une notion d'ordre : p est plus fine que la partition p' si tout bloc de p est inclus dans un bloc de p'. Pour toutes partitions p et p', on peut définir le supremum $p \vee p'$ de p et p' comme étant la plus petite partition qui est plus grossière que p et que p'. Graphiquement, on obtient un graphe qui représente $p \vee p'$ en considérant deux graphes qui représentent p et p' dessinés sur le même ensemble de points représentant X et en les mettant l'un sur l'autre. Remarquons qu'il existe une fonction naturelle, dite de hauteur, $\mathsf{nc} : \mathcal{P}_X \to \mathbb{N}$ qui est croissante pour l'ordre défini précédemment : c'est la fonction qui, à une partition p, associe le nombre de blocs de p. On peut définir aussi l'infimum $p \wedge p'$ de p et p' comme étant la plus grosse partition qui est plus fine que p et que p'.

Dans le chapitre de thèse [Partitions], est définie une nouvelle structure géométrique sur l'ensemble $\mathcal{P}_X$, de laquelle découle une nouvelle notion d'ordre.

(54) Nous présentons par la suite une généralisation de la section 3.1 du chapitre "Partitions et géométrie" [Partitions].

Définition 8.5. — *Pour tout couple de partitions (p, p'), on pose :*

$$d(p, p') = \frac{\mathsf{nc}(p) + \mathsf{nc}(p')}{2} - \mathsf{nc}(p \vee p') \tag{22}$$

Le fait que d soit une distance découle du fait qu'on puisse la voir comme une distance géodésique d'un graphe dont les sommets sont les partitions $p \in \mathcal{P}_X$. Ce résultat ne doit pas être minimisé étant donné qu'une part non négligeable de résultats concernant les matrices aléatoires invariantes par conjugaison par le groupe symétrique, comme la $\mathcal{P}$-liberté asymptotique de systèmes invariants et indépendants, reposent sur ce fait.

Théorème 8.5 **(Proposition** 3.2 **et Corollaire** 3.1 **de [Partitions])**
Pour toutes partitions p et p', l'entier $2d(p, p')$ est le nombre minimum de recollement et découpage de blocs que l'on doit faire pour passer de p à p'[55]*. L'application $d : \mathcal{P}_X \times \mathcal{P}_X \to \mathbb{R}^+$ définit une distance sur $\mathcal{P}_X$.*

Une succession de partitions p_1,..., p_n obtenues à chaque étape en recollant ou découpant un bloc est appelé un chemin dans $\mathcal{P}_X$. On peut alors définir une notion d'intervalle géodésique dans $\mathcal{P}_X$. Une géodésique entre p et p' est un chemin qui relie de façon la plus rapide p à p' et l'intervalle géodésique entre p et p', dénoté $[p, p']_{\mathcal{P}_X}$, est l'union des géodésiques reliant p à p'. Heureusement il existe un critère afin de savoir si une partition $\tilde{p}$ est dans l'intervalle géodésique $[p, p']_{\mathcal{P}_X}$. En effet, $\tilde{p}$ est dans l'intervalle géodésique $[p, p']$ si et seulement si $d(p, \tilde{p}) + d(\tilde{p}, p') = d(p, \tilde{p})$. Ceci permet de redéfinir la notion d'intervalle géodésique.

Définition 8.6. — *Soient p et p' deux partitions dans $\mathcal{P}_X$. L'intervalle géodésique $[p, p']_{\mathcal{P}_X}$ est défini par :*

$$[p, p']_{\mathcal{P}_X} = \left\{\tilde{p} \in \mathcal{P}_X, d(p, \tilde{p}) + d(\tilde{p}, p') = d(p, p')\right\}.$$

Cette définition d'intervalle géodésique n'aurait pas autant d'importance s'il ne permettait pas de définir un nouvel ordre sur $\mathcal{P}_X$.

Définition 8.7 **(Définition** 3.4 **de [Partitions])**. — *Choisissons une partition $b \in \mathcal{P}_X$ appellée partition base. On définit alors l'ordre $\leq_b$ sur $\mathcal{P}_X$ par le fait que $p_1 \leq_b p_2$ si et seulement si :*

$$p_1 \in [b, p_2]_{\mathcal{P}_X}.$$

On peut comprendre pourquoi nous avons appelé b la partition de base : l'ordre se base sur la donnée de b et b est le plus petit élément pour l'ordre $\leq_b$.

[55] Proposition 3.2 de [Partitions].

Remarque 8.2. — *Les ordres d'être plus fine ou d'être plus grossière sont en réalité des cas particuliers d'ordre géodésique. En effet, si on prend comme partition base* $0_X = \{\{x\}, x \in X\}$*, alors* $\leq_{0_X}$ *est la relation de plus grande finesse sur* $\mathcal{P}_X$*, c'est-à-dire* $p \leq_{0_X} p'$ *si* p *est plus fine que* p'*. En effet,* $p \leq_{0_X} p'$ *est équivalent à* $d(b,p) + d(p,p') = d(b,p')$ *ou encore à* $\mathsf{nc}(p') = \mathsf{nc}(p \vee p')$ *et encore au fait que* p *est plus fin que* p'*.*

De même, si on considère $1_X = \{\{x, x \in X\}\}$*, alors* $\leq_{1_X}$ *est l'ordre d'être plus grossière sur sur* $\mathcal{P}_X$*, c'est-à-dire* $p \leq_{0_X} p'$ *si* p *est plus grossière que* p'*.*

Dans le chapitre de thèse [Partitions], on caractérise le diagramme de Hasse, la matrice d'ordre ainsi que la fonction de Möbius de ces nouveaux ordres. C'est ce que nous allons voir par la suite.

8.4.2. L'ordre géodésique : diagramme de Hasse. —

Théorème 8.6. — *Le diagramme de Hasse de* $(\mathcal{P}_X, \leq_b)$ *est caractérisé de la façon suivante*[56]*. Il existe une arête orientée reliant* p *à* p' *si l'une de deux conditions est vérifiée :*

- $\mathsf{nc}(p \vee b) \neq \mathsf{nc}(p' \vee b)$ *et on peut passer de* p *à* p' *en recollant deux blocs de* p*, dans ce cas on écrit* $p \sqsupset_b p'$*. Une telle flèche dans le diagramme de Hasse est de type* $\sqsupset$*.*
- $\mathsf{nc}(p \vee b) = \mathsf{nc}(p' \vee b)$ *et on peut passer de* p *à* p' *en découpant un bloc de* p*, dans ce cas on écrit* $p \dashv_b p'$*. Une telle flèche dans le diagramme de Hasse est de type* $\dashv$*.*

De par la définition même d'un diagramme de Hasse, on voit alors que pour toutes partitions p et p', $p \leq_b p'$ si et seulement si on peut passer, dans le diagramme de Hasse, de p à p'. Un diagramme de Hasse est donnée par la figure 13 : c'est le diagramme de Hasse de l'ordre géodésique sur les partitions de quatres points, la partition de base choisie étant la partition dessinée tout en bas de la figure. Dans ce dessin, les flèches bleues représentent les arêtes de type $\sqsupset$ et les arêtes rouges représentent les arêtes de type $\dashv$.

On voit que la définition du diagramme de Hasse fait intervenir deux nouveaux ordres sur $\mathcal{P}_X$:

1. l'ordre engendré par la relation $\dashv_b$: $p \dashv_b p'$ s'il existe un chemin $p = p_1 \dashv_b ... \dashv_b p_n = p'$, on dira que p est un recollement admissible de p'.
2. l'ordre engendré par la relation $\sqsupset_b$: $p \sqsupset_b p'$ s'il existe un chemin $p = p_1 \sqsupset_b ... \sqsupset_b p_n = p'$, on dira que p est un découpage admissible de p'.

Ces relations, qui ne sont qu'une modification de la relation de plus grande finesse ou de moins grande finesse, permettrons de décomposer l'ordre géodésique comme un produit de ces deux relations plus élémentaires.

[56] Nous présentons ici aussi une généralisation de la section 3.2 de [Partitions] et en particulier une généralisation du théorème 3.2 du même chapitre.

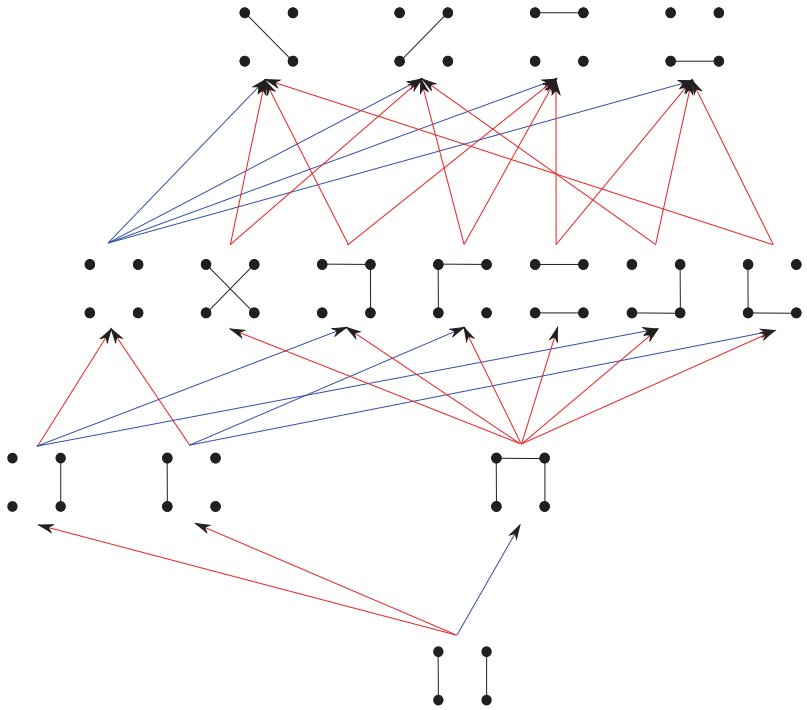

FIGURE 13. Diagramme de Hasse de l'ordre géodésique

8.4.3. L'ordre géodésique : matrice d'ordre. — Passons maintenant à la représentation matricielle de $(\mathcal{P}_X, \leq_b)$[57]. C'est la donnée de la matrice $(\mathcal{P}_X, \mathcal{P}_X, G_{\leq_b})$ telle que pour toutes partitions p et p' :

$$(G_{\leq_b})_{p,p'} = \delta_{p' \leq p}.$$

Considérons alors les matrices d'ordre C_b et S_b respectivement pour $(\mathcal{P}_k, \dashv_b)$ et $(\mathcal{P}_k, \sqsupset_b)$. Elles permettent de décomposer la matrice d'ordre $G_{\leq_b}$: c'est un des résultats principaux de [Partitions].

Theorem 8.1 **(Théorème 3.3 de [Partitions])**. — *Les trois matrices d'ordre $G_{\leq_b}$, C_b, S_b sont reliées par la relation :*

$$G_{\leq_b} = C_b S_b.$$

Soient p et p' deux partitions dans $\mathcal{P}_X$, le théorème précédent nous dit que $p' \leq_b p$ si et seulement si il existe une partition $\tilde{p}$, unique, telle que $\tilde{p}$ est un recollement admissible de p et p' est un découpage admissible de $\tilde{p}$. On peut montrer[58] que dans ce cas, $\tilde{p} = p \vee p'$.

[57] Nous généralisons, ici encore, la section 3.3 de [Partitions].
[58] Théorème 3.1 de [Partitions].

Le théorème précédent permet alors de calculer l'inverse de $G_{\leq_b}$ en utilisant la formule de Rota et des résultats connus pour la relation de plus grande finesse. Pour toutes partitions p_1 et p_2 telles que p_1 est plus fine que p_2, posons :

$$\mu_f(p_1, p_2) = (-1)^{\mathsf{nc}(p_1)-\mathsf{nc}(p_2)} \prod_{i=3}^{\mathsf{nc}(p_1)} ((i-1)!)^{r_i},$$

où r_i est le nombre de blocs de p_2 qui contient exactement i blocs de p_1.

Theorem 8.2 **(Théorème** 3.4 **de [Partitions])**. — *L'inverse de la matrice de l'ordre géodésique $\leq_b$ est donnée par :*

$$(G^{-1}_{\leq_b})_{p,p'} = \delta_{p\wedge p' \sqsupset_b p}\delta_{p' \dashv_b p\wedge p'}\mu_f(p\wedge p', p)\mu_f(p\wedge p', p'), \tag{23}$$

Remarque 8.3. — *Dans le cas où b est la partition $\mathsf{0}_X$, les ordres $\sqsupset_b$ et $\dashv_b$ deviennent assez simples, puisque $p \sqsupset_{\mathsf{0}_X} p'$ est équivalent au fait que p est plus fine que p', et $p \dashv_{\mathsf{0}_X} p'$ est équivalent au fait que $p = p'$. Ainsi $(G^{-1}_{\mathsf{0}_X})_{p,p'} = \delta_{p' \text{ plus fine que } p}\mu_f(p', p)$, ce à quoi on pouvait s'attendre vu que $\leq_{\mathsf{0}_X}$ est la relation de plus grande finesse et que μ_f est la fonction de Möbius pour cet ordre. La même remarque s'applique dans le cas où l'on choisit $\mathsf{1}_X$ comme partition base.*

8.4.4. La $\leq_b$-transformée. — Étant donné que nous avons défini plusieurs ordres sur $\mathcal{P}_X$, on peut définir autant de transformations triangulaires. On ne s'intéressera dans cette intoduction qu'à la transformation liée à l'ordre géodésique $\leq_b$, cependant, celle liée à l'ordre $\dashv_b$ est tout aussi importante[(59)].

Définition 8.8. — *La $\leq_b$-transformée d'une famille de nombres complexes $(m_p)_{p\in\mathcal{P}_X}$ est l'unique famille $(\kappa^p_{\leq_b})_{p\in\mathcal{P}_X}$ de nombres complexes tels que pour tout $p \in \mathcal{P}_X$:*

$$m_p = \sum_{p'\leq p} \kappa^{p'}_{\leq_b}.$$

De nouveau, en posant $\kappa = (\kappa^p_{\leq_b})_{p\in\mathcal{P}_X}$ et $m = (m_p)_{p\in\mathcal{P}_X}$, la relation précédente s'inverse : $\kappa = G^{-1}_{\leq_b} m$. En utilisant la formule (23), on peut écrire explicitement la valeur de la $\leq_b$-transformée en fonction de $(m_p)_{p\in\mathcal{P}_X}$.

[(59)] On définit des observables dans la section 4.3.2 de [Partitions], appelés les moments exclusifs, qui peuvent être vus, comme montré dans la proposition 4.2. du même chapitre, comme résultant d'une transformation triangulaire liée à $\dashv_b$. Ces observables ont un rôle très important dans [Matrices] et [Revêtements].

8.5. L'ensemble des partitions $\mathcal{P}_k$. — Nous allons voir un cas particulier d'ensemble de partitions qui sera important pour l'étude de matrices aléatoires invariantes par conjugaison par le groupe symétrique. Nous avons présenté le cas général dans la section précédente 8.4.1, car dans l'étude de tenseurs aléatoires invariants par action par le groupe symétrique, il est nécessaire d'avoir étudié le cas général $(\mathcal{P}_X, \leq_b)$. Par exemple, si on étudie des tenseurs aléatoires à trois indices, l'ensemble X est égal à $\{1, 1', 1'', 2, 2', 2'', ..., k, k', k''\}$ et la partition de base est $b = \{\{1, 1', 1''\}, \{2, 2', 2''\}, ..., \{k, k', k''\}\}$.

Dans le cas qui nous intéresse ici, à savoir les matrices aléatoires, l'ensemble X sera une union de couples de points. Choisissons un entier positif k. On note $\mathcal{P}_k$ l'ensemble $\mathcal{P}_{\{1,...,k\}\cup\{1',...,k'\}}$ des partitions de $\{1, ..., k\} \cup \{1', ..., k'\}$. Polariser l'ensemble sous-jacent en deux ensembles permet d'une part d'injecter $\mathcal{P}_{\{1,...,k\}}$ et $\mathsf{NC}(k)$ dans cet ensemble de partitions, d'autre part de munir cet ensemble de partitions d'une structure algébrique.

Afin de représenter graphiquement les partitions de $\mathcal{P}_k$, on considère, comme ensemble indexé par $\{1, ..., k\} \cup \{1', ..., k'\}$, deux rangées horizontales de points. Les points de la rangée du haut sont indexés de 1 à k de gauche à droite, tandis que les points de la rangée du bas sont indexés de $1'$ à k' de gauche à droite. Un exemple est donné dans la figure 14.

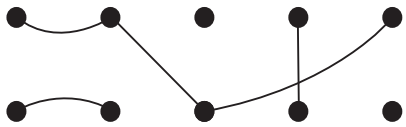

FIGURE 14. Partition $p = \{\{1', 2'\}\{1, 2, 3', 5\}\{3\}\{4', 4\}\{5'\}\}$

Cette polarisation permet de définir naturellement une partition assez spéciale, que l'on reverra par la suite : l'identité. C'est la partition donnée par :

$$\mathrm{id}_k = \{\{1, 1'\}, ..., \{k, k'\}\},$$

dont un exemple est dessiné dans la figure 15.

FIGURE 15. L'élément identité id_5

On ne considérera à partir de maintenant que l'ordre géodésique de partition base égale à id_k. On notera $\leq$ l'ordre $\leq_{\mathrm{id}_k}$.

8.5.1. Sous-ensembles remarquables. — Il existe des sous-ensembles remarquables de $\mathcal{P}_k$ tels que :

- $\mathcal{D}_k$, l'ensemble des éléments blocs. Ce sont les éléments $p \in \mathcal{P}_k$ plus grossiers que id_k,
- $\mathfrak{S}_k$, l'ensemble des permutations. Ce sont les éléments $p \in \mathcal{P}_k$ tels que pour tout bloc s de p, $\#(s \cap \{1, ..., k\}) = \#(s \cap \{1', ..., k'\}) = 1$. À toute permutation σ vu comme fonction de $\{1, ..., k\}$ dans lui-même, on peut associer la partition $\sigma \in \mathfrak{S}_k$ égale à:
$$\sigma = \big\{\{i, \sigma(i)'\} \mid i \in \{1, ..., k\}\big\}.$$
- $\mathcal{B}_k$, l'ensemble des éléments de Brauer. Ce sont les éléments $p \in \mathcal{P}_k$ tels que pour tout bloc s de p, $\#s = 2$.

La liste n'est pas exhaustive, et dans le chapitre de thèse "Matrices aléatoires invariantes par conjugaison par le groupe symétrique", nous considérons par exemple l'ensemble des partitions dont les blocs sont de tailles plus petite que deux. Les deux premiers ensembles sont en réalité bien connus puisque qu'ils permettent l'injection attendue de $\mathcal{P}_{\{1,...,k\}}$ et de $\mathsf{NC}(k)$ dans $\mathcal{P}_k$. Remarquons que toute permutation plus grossière que id_k est égale à id_k : $\mathcal{D}_k \cap \mathfrak{S}_k = \{\mathrm{id}_k\}$. La seconde partie du théorème suivant est bien connu.

Théorème 8.7. — *L'ensemble ordonné $(\mathcal{D}_k, \leq)$ est en bijection avec l'ensemble ordonné $\mathcal{P}_{\{1,...,k\}}$ muni de la relation de plus grande finesse.*

L'ensemble ordonné $([id_k, (1, ..., k)]_{\mathcal{P}_k} \cap \mathfrak{S}_k, \leq)$ est en bijection avec l'ensemble ordonné $\mathsf{NC}(k)$ muni de la relation de plus grande finesse.

On peut ainsi considérer les cumulants classiques et libres comme résultant d'une transformation triangulaire sur les familles indexées respectivement par $\mathcal{D}_k$ et par $\mathfrak{S}_k$.

8.5.2. Structure algébrique. — L'ensemble $\mathcal{P}_k$ peut être enrichi d'une structure algébrique[(60)]. Pour cela on procède de la même manière que pour les tresses, ou pour les permutations. En effet, considérons deux partitions p_1 et p_2, et mettons le diagramme représentant p_2 au-dessus de celui représentant p_1. On identifie les points du bas de p_2 avec les points en haut de p_1. En considérant les composantes connexes du nouveau diagramme cela nous permet de définir une nouvelle partition que l'on appellera $p_1 \circ p_2$: ce faisant, on oublie les composantes connexes ne reliant aucun des points.

En restreignant l'opération $\circ$ à $\mathfrak{S}_k \times \mathfrak{S}_k$, on retombe sur le produit de permutations. Un exemple est donné dans la figure 16.

Cependant, comme nous l'avons remarqué, si les partitions ne sont pas dans $\mathfrak{S}_k$, il se peut que nous ayons oublié des composantes connexes lors de la construction de $p_1 \circ p_2$ comme illustré dans la figure 17. Certes celles-ci ne relient aucun sommet,

(60) Pour plus d'informations sur la structure algébrique associée à $\mathcal{P}_k$, on peut se référer à [**19**].

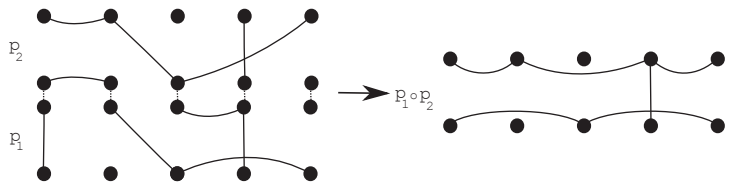

FIGURE 16. Concaténation de deux partitions : $p_1 \circ p_2$

mais elles auront beaucoup d'importance par la suite. Notons $\kappa(p_1, p_2)$ le nombre de telles composantes connexes. Afin de garder une trace de ces composantes connexes, il faut nous donner un paramètre, par exemple un entier positif N. On définit alors le produit de paramètre N de p_1 par p_2 par :

$$p_1 p_2 = N^{\kappa(p_1,p_2)} p_1 \circ p_2.$$

En particulier, le produit de deux partitions n'est plus une partition mais est un élément de $\mathbb{C}[\mathcal{P}_k]$, l'espace vectoriel engendré par les partitions. La définition du produit de paramètre N définit alors une structure d'algèbre sur $\mathbb{C}[\mathcal{P}_k]$: nous noterons $\mathbb{C}[\mathcal{P}_k(N)]$ l'algèbre ainsi obtenue. Remarquons que l'élément neutre pour cette multiplication est le même quel que soit le paramètre N choisi ; c'est la partition identité id_k.

Remark 8.1. — *Le produit de paramètre N n'est pas l'unique produit intéressant que l'on peut définir sur les partitions. Dans le chapitre [Partitions], nous modifions la définition de ce produit afin de définir une algèbre renormalisée que l'on dénote $\mathbb{C}[\mathcal{P}_k(N, N)]$. Nous verrons la définition de cette algèbre dans la section 8.5.4.*

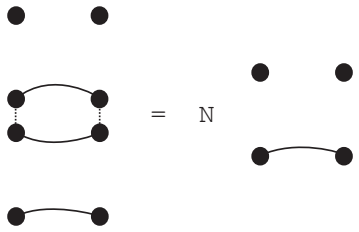

FIGURE 17. Produit faisant intervenir $\kappa(p_1, p_2)$

8.5.3. Représentation. — Dans la section 6.2.1, nous avons vu que le groupe symétrique $\mathfrak{S}_k$ possède une représentation sur $(\mathbb{C}^N)^{\otimes k}$, qui permet d'obtenir une dualité de Schur-Weyl pour le groupe unitaire. Il en est de même pour $\mathbb{C}[\mathcal{P}_k(N)]$.

À la différence de $\mathfrak{S}_k$, nous devons fixer une base de $\mathbb{C}^N$ afin de définir une représentation de $\mathbb{C}[\mathcal{P}_k(N)]$. Considérons la base canonique $(e_i)_{i=1}^N$ de $\mathbb{C}^N$. On définit par E_i^j l'endomorphisme de $\mathbb{C}^N$ qui envoie e_j sur e_i et tout autre élément

de la base sur 0. Pour toute fonction $F : X \to Y$, on note $\mathsf{Ker}(F)$ la partition de Y donnée par $\{F^{-1}(y), y \in Y\}$. Pour toute partition $p \in \mathcal{P}_k$, on définit l'action de p sur $(\mathbb{C}^N)^{\otimes k}$ par :

$$\rho_N^{\mathcal{P}_k}(p) = \sum_{F:\{1,\dots,k,1',\dots,k'\}\to\{1,\dots,N\} | \mathsf{Ker}(F) \text{ plus grossière que } p} E^{F(1)}_{F(1')} \otimes \dots \otimes E^{F(k)}_{F(k')}.$$

Théorème 8.8. — *L'application $\rho_N^{\mathcal{P}_k}$ est une représentation de $\mathbb{C}[\mathcal{P}(N)]$ sur $(\mathbb{C}^N)^{\otimes k}$.*

Il est important d'avoir le même paramètre N dans l'algèbre de partitions considérée et dans l'espace sur lequel elle agit. Afin d'illustrer ce théorème faisons un simple calcul.

$$\left(\sum_{a,b=1}^{N} E_a^b \otimes E_a^b\right)\left(\sum_{a,b=1}^{N} E_a^b \otimes E_a^b\right) = \sum_{a,a',b'=1}^{N} E_a^{a'} E_{a'}^{b'} \otimes E_a^{a'} E_{a'}^{b'} = \sum_{a,a',b'=1}^{N} E_a^{b'} \otimes E_a^{b'}$$
$$= N \sum_{a,b=1}^{N} E_a^b \otimes E_a^b.$$

Ainsi en considérant $[1,2] = \{\{1,2\},\{1',2'\}\}$ comme un élément de $\mathbb{C}[\mathcal{P}_k(N)]$, on obtient :

$$\rho_N^{\mathcal{P}_k}([1,2])\rho_N^{\mathcal{P}_k}([1,2]) = \rho_N^{\mathcal{P}_k}([1,2][1,2]).$$

Remarquons qu'en restreignant cette action à des permutations, on retrouve l'action du groupe symétrique que l'on avait définie auparavant dans la section 6.2.1. Par exemple, en considérant, dans $\mathfrak{S}_2$, la transpositition $(1,2)$:

$$\rho_N^{\mathcal{P}_2}\big((1,2)\big) = \sum_{F:\{1,2,1',2'\}\to\{1,\dots,N\} | \mathsf{Ker}(F) \text{ plus grossière que } (1,2)} E^{F(1)}_{F(1')} \otimes E^{F(2)}_{F(2')}$$
$$= \sum_{a,b\in\{1,\dots,N\}} E_a^b \otimes E_b^a,$$

or pour tous éléments e_i et e_j de la base canonique, $\left(\sum_{a,b\in\{1,\dots,N\}} E_a^b \otimes E_b^a\right)(e_i \otimes e_j)$ est égal à :

$$\sum_{a,b\in\{1,\dots,N\}} \delta_{b=i}\delta_{a=j} e_a \otimes e_b = e_j \otimes e_i = \rho^{\mathfrak{S}}\left((1,2)\right)(e_i \otimes e_j).$$

Nous verrons que la représentation $\rho_N^{\mathcal{P}_k}$ est très importante car elle permet de définir de nombreuses observables sur les systèmes de matrices aléatoires, mais aussi via la dualité de Schur-Weyl-Jones, elle permet de transformer des objets

algébriques en objets combinatoires. Il existe une autre utilisation[61], plus simple, de cette représentation qui permet l'inverse : transformer des quantités combinatoires en quantités algébriques.

Lemme 8.1. — *Pour toutes partitions p et p' dans $\mathcal{P}_k$,*

$$\mathsf{nc}(p \vee p') = \ln_N \left[\mathsf{Tr} \left(\rho_k^{\mathcal{P}_k}(p) \left(\rho_k^{\mathcal{P}_k}(p') \right)^t \right) \right],$$

où $\ln_N$ est le logarithme en base N.

Ceci permet donc de transformer un problème consistant à calculer un nombre de blocs en un calcul de trace. Aussi simple que cela semble, cette égalité simplifie grandement l'étude combinatoire de $\mathcal{P}_k$ menée dans le chapitre de thèse [Partitions].

8.5.4. Distance et produit. — Dans le cas où on se restreint au sous-ensemble des permutations $\mathfrak{S}_k$, la distance devient plus simple : on obtient la distance géodésique sur le graphe de Cayley du groupe symétrique. Si σ et σ' sont deux permutations, alors $d(\sigma, \sigma') = k - \mathsf{nc}(\sigma \vee \sigma')$.

Ainsi, si on considère une troisième permutation σ'',

$$\begin{aligned} d(\sigma''\sigma, \sigma''\sigma') = k - \mathsf{nc}(\sigma''\sigma \vee \sigma''\sigma') &= k - \ln_N \left[\mathsf{Tr} \left(\rho_k^{\mathcal{P}_k}(\sigma''\sigma) \left(\rho_k^{\mathcal{P}_k}(\sigma''\sigma') \right)^t \right) \right] \\ &= k - \ln_N \left[\mathsf{Tr} \left(\sigma(\sigma')^t (\sigma'')^t \sigma'' \right) \right], \end{aligned}$$

où pour simplifier nous avons omis d'utiliser la représentation $\rho_k^{\mathcal{P}_k}$, ou de façon équivalente ici $\rho^{\mathfrak{S}}$, dans le dernier terme. Or $\rho^{\mathfrak{S}}(\sigma'')^t = \left(\rho^{\mathfrak{S}}(\sigma'')\right)^{-1}$, d'où finalement $d(\sigma''\sigma, \sigma''\sigma') = k - \ln_N \left[\mathsf{Tr} \left(\sigma(\sigma')^t \right) \right] = d(\sigma, \sigma')$.

Lemme 8.2. — *La restriction de d à $\mathfrak{S}_k$ est invariante par multiplication à gauche et à droite par multiplication par n'importe quelle permutation. Elle est aussi invariante par inversion : pour toutes permutations σ et σ' dans $\mathfrak{S}_k$, $d(\sigma^{-1}, \sigma'^{-1}) = d(\sigma, \sigma')$.*

En particulier pour toutes permutations σ et σ' dans $\mathfrak{S}_k$,

$$d(\sigma, \sigma\sigma') = d(\mathrm{id}_k, \sigma').$$

Que se passe-t-il quand on considère toutes les partitions $\mathcal{P}_k$ et non pas les permutations ?[62] La dernière égalité n'est plus vérifiée en général, mais on montre alors qu'on a un contrôle sur $d(p, p \circ p')$.

[61] Prouvée dans la preuve du Lemme 3.1 de [Partitions].

[62] C'est ce que l'on étudie dans la section 6 de [Partitions].

Théorème 8.9 **(Proposition** 6.2. **de [Partitions])**. — *Soient p et p' deux partitions dans $\mathcal{P}_k$,*

$$d(p, p \circ p') \leq d(id, p') - \frac{k + \mathsf{nc}(p \circ p') - \mathsf{nc}(p) - \mathsf{nc}(p')}{2} - \kappa(p, p').$$

Cette inégalité permet de définir une notion de complément de Kreweras pour toute partition $p \in \mathcal{P}_k$, mais aussi elle permet de montrer une amélioration de l'inégalité triangulaire.

Théorème 8.10 **(Proposition** 6.1 **de [Partitions])**. — *Soient p et p' deux partitions dans $\mathcal{P}_k$,*

$$d(p', p) \leq d(p', id_k) + d(id_k, p) - k - \mathsf{nc}(p \circ^t p') + \mathsf{nc}(p) + \mathsf{nc}(p') - 2\kappa(p, {}^t p').$$

En réalité ces considérations géométriques proviennent de considérations algébriques. En effet, dans le chapitre [Partitions], on définit une algèbre renormalisée de partitions.

Définition 8.9. — *Soit N un entier positif, l'algèbre déformée des partitions est la structure d'algèbre sur $\mathbb{C}[\mathcal{P}_k]$ découlant du produit noté $._N$ et défini par :*

$$p_1 \cdot_N p_2 = N^{d(id, p_1 \circ p_2) - d(id, p_1) - d(id, p_2) + \frac{k + \mathsf{nc}(p_1 \circ p_2) - \mathsf{nc}(p_1) - \mathsf{nc}(p_2)}{2}} N^{\kappa(p_1, p_2)} (p_1 \circ p_2)$$

On la dénote $\mathbb{C}[\mathcal{P}_k(N, N)]$.

En utilisant le théorème 8.9, on prouve alors un des résultats principaux de [Partitions].

Théorème 8.11. — *L'algèbre déformée $\mathbb{C}[\mathcal{P}_k(N, N)]$ converge vers une algèbre, dénotée $\mathbb{C}[\mathcal{P}_k(\infty, \infty)]$ quand N tend vers l'infini.*

Dans cette introduction, nous ne nous focaliserons pas sur cette algèbre déformée $\mathbb{C}[\mathcal{P}_k(\infty, \infty)]$, mais elle joue un rôle très important dans l'étude, développée dans cette thèse, des matrices aléatoires invariantes par conjugaison par le groupe symétrique.

8.5.5. Les $\mathcal{P}$-cumulants. — Dans le cas où l'on considère $\mathcal{P}_k$, la $\leq$-transformée sera appellée la famille des $\mathcal{P}$-cumulants.

Definition 8.1. — *La famille des $\mathcal{P}$-cumulants $(\kappa_{\mathcal{P}}^p)_{p \in \mathcal{P}_k}$ associée à $(m_p)_{p \in \mathcal{P}_k}$ est l'unique famille de nombres complexes tels que pour tout $p \in \mathcal{P}_k$,*

$$m_p = \sum_{p' \leq p} \kappa_{\mathcal{P}}^{p'}.$$

Le théorème 8.7, qui stipule l'identification de l'ensemble ordonné $\mathcal{D}_k$ avec l'ensemble $\mathcal{P}(\{1, ..., k\})$ et de $[\mathrm{id}_k, (1, ..., k)]_{\mathcal{P}_k} \cap \mathfrak{S}_k$ avec $\mathsf{NC}(k)$, permet alors d'interpréter les cumulants classiques et libres comme des cas particuliers de $\mathcal{P}$-cumulants.

8.6. Cumulants et matrices aléatoires. — Maintenant que nous avons défini un ensemble ordonné ainsi qu'une notion de cumulants, nous devons définir pour toute partition $p \in \mathcal{P}_k$ une observable sur les systèmes de matrices aléatoires.

8.6.1. Signature d'une matrice aléatoire et observables de partitions. — Reprenons à la base notre discussion sur les matrices aléatoires : si M et M' sont deux matrices aléatoires indépendantes à valeurs complexes et de taille N alors si $N = 1$:

$$\mathbb{E}[\mathsf{Tr}(MM')] = \mathbb{E}[\mathsf{Tr}(M)]\mathbb{E}[\mathsf{Tr}(M')],$$

relation qui devient fausse lorsque $N > 1$. Le caractère multiplicatif des observables qui était donné par l'indépendance en dimension 1 n'est plus vérifié : c'est en partie de là que proviennent la richesse et la difficulté en matière de matrices aléatoires. Cependant, on peut remédier ce problème, en considérant la signature de M :

$$\mathsf{S}(M) = \bigoplus_{k=0}^{\infty} \mathbb{E}\left[M^{\otimes k}\right]. \tag{24}$$

Lemme 8.3. — *Soient M et M' deux matrices aléatoires indépendantes de mêmes tailles, on a :*

$$\mathsf{S}(MM') = \mathsf{S}(M)\mathsf{S}(M').$$

La signature de M permet de coder de façon intéressante toutes les observables polynomiales en M. En effet, en utilisant l'identité (10), pour tout l-uplet d'entiers positifs $(n_1, ..., n_l)$ de somme égale à k et toute permutation ayant l cycles de tailles $(n_1, ..., n_l)$:

$$\mathbb{E}\left[\frac{1}{N^l}\prod_{i=1}^{l}\mathsf{Tr}(M^{n_i})\right] = \frac{1}{N^l}\mathbb{E}\left[\mathsf{Tr}\left(M^{\otimes k}\circ\rho_N^{\mathcal{P}_k}(\sigma)^t\right)\right] = \frac{1}{N^{\mathsf{nc}(\sigma\vee \mathrm{id}_k)}} Tr\left(\mathsf{S}(M)\circ\rho_N^{\mathcal{P}_k}(\sigma)^t\right),$$

où dans la dernière identité, on considère $\rho_N^{\mathcal{P}_k}(\sigma)$ comme agissant naturellement sur le même espace que $\mathbb{E}[M^{\otimes k}]$ et agissant comme l'endomorphisme nul sur les autres espaces.

Définition 8.10. — *Pour toute partition $p \in \mathcal{P}_k$, on associe l'observable :*

$$\mathbb{E}m_p(M) = \frac{1}{N^{\mathsf{nc}(p\vee id_k)}}\mathsf{Tr}\left(\mathsf{S}(M)\circ\rho_N^{\mathcal{P}_k}(p)^t\right).$$

Si p est un élément de Brauer, c'est à dire un élément de $\mathcal{B}_k$, l'observable $\mathbb{E}m_p(M)$ peut encore se comprendre comme un produit de traces de produits faisant intervenir M et sa transposée M^t. Cependant si p n'est pas un élément de Brauer, on ne peut plus voir cette observable simplement de cette façon[(63)].

[(63)] Il existe une façon graphique alternative de visualiser ces observables, comme montré dans l'article en préparation [**15**] : ce sont des graphes étiquetés comme introduits dans [**24**] par C.

Définition 8.11. — *Un système de matrices aléatoires de la forme $(M_N)_{N\in\mathbb{N}}$ converge en $\mathcal{P}$-distribution si pour toute partition p, $\mathbb{E}m_p(M_N)$ converge quand N tend vers l'infini. On note $\mathbb{E}m_p(M)$ la limite de $\mathbb{E}m_p(M_N)$.*

Considérons maintenant deux matrices aléatoires M et L de même taille. On généralise la notion de signature tout simplement en posant :

$$\mathsf{S}(M,L) = \bigoplus_{k\in\mathbb{N},(X_1,...,X_k)\in\{M,L\}^k} \mathbb{E}\left[\bigotimes_{i=1}^{k} X_i\right].$$

On peut alors de nouveau définir une notion d'observables associées aux partitions : pour tout $p \in \mathcal{P}_k$, pour tout $(X_1, ..., X_k) \in \{M, L\}^k$, on pose :

$$\mathbb{E}m_p(X_1, ..., X_k) = \frac{1}{N^{\mathsf{nc}(\sigma\vee id)}}\mathsf{Tr}\left(\mathsf{S}(M,L)\circ\rho_N^{\mathcal{P}_k}(p)^t\right),$$

où l'on a de nouveau considéré $\rho_N^{\mathcal{P}_k}(p)$ comme agissant naturellement sur le même espace que $\mathbb{E}\left[\bigotimes_{i=1}^{k} X_i\right]$ et agissant comme l'endomorphisme nul sur les autres espaces.

On peut alors définir une notion de convergence en espérance en $\mathcal{P}$-moments pour tout couple de famille de matrices $(M_N)_{N\in\mathbb{N}}$ et $(L_N)_{N\in\mathbb{N}}$. On peut aussi généraliser ceci à un nombre quelconque de familles de matrices[(64)]. Ceci permet de définir des observables indexées par des partitions sur les systèmes de matrices aléatoires qui convergent en $\mathcal{P}$-distribution comme limite des observables sur les matrices de tailles N. On peut donc définir une notion de $\mathcal{P}$-cumulants sur les systèmes de matrices aléatoires qui convergent en $\mathcal{P}$-distribution. Ce sont simplement les $\mathcal{P}$-cumulants associés à la famille des observables[(65)]. Ceci permet alors de définir une notion de $\mathcal{P}$-liberté asymptotique comme une condition de nullité des $\mathcal{P}$-cumulants mixtes[(66)].

Finalement, que reste-t-il à faire ? Il reste à montrer que l'on peut coupler deux systèmes de matrices aléatoires qui convergent en $\mathcal{P}$-distribution, de telle sorte à ce que le système obtenu converge encore en $\mathcal{P}$-distribution. Pour cela, on aura besoin seulement de supposer une symétrie, plus faible que celle considérée jusqu'à présent : l'invariance par conjugaison par le groupe symétrique. Cette symétrie permettra d'utiliser la dualité de Schur-Weyl-Jones.

Male dans sa théorie des traffics développée précédemment pour étudier les matrices aléatoires invariantes par conjugaison par le groupe symétrique. En particuliers, sous condition de $\mathcal{P}$-factorisation, on montre que la $\mathcal{P}$-liberté est équivalente à la liberté de traffics: le travail de cette thèse fournit des cumulants pour la théorie des traffics.

(64) Définition 2.2 dans le chapitre [Matrices].

(65) Définition 4.1 dans le chapitre [Matrices].

(66) Définition 7.1 dans le chapitre [Matrices].

8.6.2. Symétries et dualité de Schur-Weyl-Jones. — Il est courant que les matrices aléatoires étudiées vérifient des symétries. Ainsi la matrice de Wigner gaussienne est invariante par conjugaison par le groupe unitaire : si M est une matrice de Wigner gaussienne de taille N, pour tout $U \in U(N)$, $U^{-1}MU$ a la même loi que M. De même le mouvement brownien unitaire est invariant par conjugaison par le groupe unitaire. C'est cette symétrie qui a été utilisée dans le couplage des systèmes de matrices aléatoires afin d'obtenir la liberté asymptotique des systèmes couplés.

Cependant pour certains modèles *cette symétrie est trop forte*. Ainsi le groupe des matrices de permutations n'est même pas normal dans $U(N)$; demander l'invariance par conjugaison par $U(N)$ pour une matrice de permutation n'a pas de sens. Dans ce cas, on peut postuler une invariance par conjugaison par le groupe des permutations. Rappelons qu'une matrice aléatoire M est invariante par conjugaison par le groupe symétrique ou par le groupe des permutations si pour tout $S \in \mathfrak{S}(N)$, SMS^{-1} a la même loi que M. Par la suite toute matrice aléatoire considérée est invariante par conjugaison par le groupe symétrique. Dans ce cas, pour tout $S \in \mathfrak{S}(N)$,

$$S^{\otimes k}\mathbb{E}\left[M^{\otimes k}\right]S^{-1^{\otimes k}} = \mathbb{E}\left[(SMS^{-1})^{\otimes k}\right] = \mathbb{E}\left[M^{\otimes k}\right].$$

On voit que $\mathbb{E}\left[M^{\otimes k}\right]$ commute à $S^{\otimes k}$ pour tout $S \in \mathfrak{S}(N)$. L'utilisation des partitions permet d'étendre la dualité de Schur-Weyl en la dualité de Jones, qui stipule que l'on peut remplacer dans la dualité de Schur-Weyl le groupe $U(N)$ par le groupe symétrique $\mathfrak{S}(N)$, vu comme groupe de matrices, si l'on remplace alors l'algèbre des permutations $\mathbb{C}[\mathfrak{S}_k]$ par l'algèbre des partitions $\mathbb{C}[\mathcal{P}_k]$. Ainsi, si on définit par $\rho_{\mathfrak{S}(N)}$ l'action donnée pour tout $S \in \mathfrak{S}(N)$ par :

$$\begin{aligned}\rho_{\mathfrak{S}(N)}(S) : \qquad \left(\mathbb{C}^N\right)^{\otimes k} &\to \left(\mathbb{C}^N\right)^{\otimes k}\\ x_1 \otimes ... \otimes x_k &\mapsto Sx_1 \otimes ... \otimes Sx_k,\end{aligned}$$

la dualité de Schur-Weyl-Jones prend la forme suivante.

Théorème 8.12. — *Pour tous entiers strictement positifs k et N :*

$$\rho_N^{\mathcal{P}_k}\left(\mathbb{C}[\mathcal{P}_k(N)]\right) = \left(\rho_{\mathfrak{S}(N)}(\mathbb{C}[\mathfrak{S}(N)])\right)'.$$

De plus, si $N \geq 2k$, l'application $\rho_N^{\mathcal{P}_k}$ est injective.

Tout endomorphisme $E \in \mathsf{End}\left((\mathbb{C}^N)^{\otimes k}\right)$ tel que pour tout $S \in \mathfrak{S}_N$ on ait $(S^{-1})^{\otimes k}ES^{\otimes k} = E$ peut donc s'écrire de la forme :

$$E = \sum_{p\in\mathcal{P}_k} a_p \rho_N^{\mathcal{P}_k}(p),$$

avec unicité de la famille $(a_p)_{p\in\mathcal{P}_k}$ si $N \geq 2k$. Dorénavant, dans cette introduction, nous oublierons cette condition et travaillerons comme si $\rho_N^{\mathcal{P}_k}$ était injective pour tous entiers positifs N et k.

La preuve de ce théorème est simple, elle est en réalité beaucoup plus simple que la preuve du théorème de Schur-Weyl dans le cas unitaire. En effet, considérons un élément E commutant à l'action de $\mathfrak{S}(N)$. En prenant les mêmes notations que celles prises pour définir l'action $\rho_N^{\mathcal{P}_k}$, on peut décomposer E :

$$E = \sum_{F:\{1,1',...,k,k'\}\to\{1,...,N\}} E_F \left(E_{F_{1'}}^{F_1} \otimes ... \otimes E_{F_{k'}}^{F_k} \right).$$

Cependant, si E commute à l'action de $\mathfrak{S}(N)$, cela nous dit que $E_F = E_{F'}$ si $\mathsf{Ker}(F) = \mathsf{Ker}(F')$. Ainsi, en notant E_{p^c} la valeur de E_F pour n'importe quelle fonction F telle que $\mathsf{Ker}(F) = p$, on obtient :

$$E = \sum_{p\in\mathcal{P}_k} E_{p^c} \sum_{F|\mathsf{Ker}(F)=p} E_{F_{1'}}^{F_1} \otimes ... \otimes E_{F_{k'}}^{F_k}$$

En notant $\rho_N^{\mathcal{P}_k}(p^c) = \sum_{F|\mathsf{Ker}(F)=p} E_{F_{1'}}^{F_1} \otimes ... \otimes E_{F_{k'}}^{F_k}$, on a donc :

$$E = \sum_{p\in\mathcal{P}_k} E_{p^c} \rho_N^{\mathcal{P}_k}(p^c).$$

Or pour toute partition p, $\rho_N^{\mathcal{P}_k}(p)$ est simplement égal à $\sum_{p' \text{ plus grossière que } p} \rho_N^{\mathcal{P}_k}(p'^c)$. Ainsi en inversant la relation triangulaire on peut exprimer $\rho_N^{\mathcal{P}_k}(p'^c)$ en fonction de $\left(\rho_N^{\mathcal{P}_k}(p)\right)_{p \text{ plus grossière que } p'}$, et donc il existe une famille de complexes $(a_p)_{p\in\mathcal{P}_k}$ telle que :

$$E = \sum_{p\in\mathcal{P}_k} a_p \rho_N^{\mathcal{P}_k}(p).$$

8.6.3. Cumulants fini-dimensionnels. — À la fin de la section 8.6.1, nous avons esquissé la définition des $\mathcal{P}$-cumulants pour des systèmes de matrices aléatoires qui convergent en $\mathcal{P}$-distribution. En utilisant la dualité de Schur-Weyl-Jones, nous allons voir que l'on peut aussi définir des cumulants pour des familles de matrices de tailles N invariantes par conjugaison par le groupe symétrique. Considérons une matrice aléatoire M de taille N et invariante par conjugaison par le groupe symétrique. Pour tout entier positif k, nous avons que

$$S^{\otimes k}\mathbb{E}\left[M^{\otimes k}\right](S^{-1})^{\otimes k} = \mathbb{E}\left[M^{\otimes k}\right].$$

On peut donc appliquer la dualité de Schur-Weyl-Jones : la signature $\mathsf{S}(M)$ peut être vue comme un élément de $\bigoplus_{k=0}^{\infty} \mathbb{C}\left[\mathcal{P}_k(N)\right]$.

Or, si $(M_N)_{N\in\mathbb{N}}$ converge en $\mathcal{P}$-distribution, la signature S vue comme un élément de $\bigoplus_{k=0}^{\infty} \mathbb{C}\left[\mathcal{P}_k(N)\right]$ tend vers 0. C'est donc que la signature n'est pas la bonne notion pour comprendre le comportement asymptotique de la famille $(M_N)_{N\in\mathbb{N}}$. Afin de faire émerger un objet qui permet l'étude asymptotique de la famille $(M_N)_{N\in\mathbb{N}}$, on introduit, dans le chapitre de thèse "Matrices

aléatoires invariantes par conjugaison par le groupe symétrique", les cumulants fini-dimensionnels par une renormalisation bien choisie de la signature[67].

Définition 8.12. — *Si M est une matrice aléatoire de taille N, pour toute partition p, le p-cumulant de M est :*

$$\mathbb{E}\kappa^p(M) = N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)} \left(\mathsf{S}(M)\right)_p .$$

où $\mathsf{S}(M)$ est la signature de M définie par l'équation (24).

On montre alors le théorème-clef suivant qui découle du fait que la fonction d définie par l'égalité (22) est une distance sur $\mathcal{P}_k$.

Théorème 8.13 **(Théorème** 4.1 **de [Matrices])**. — *Une famille $(M_N)_{N\in\mathbb{N}}$ converge en $\mathcal{P}$-distribution si et seulement si pour toute partition p le p-cumulant de M_N converge quand N tend vers l'infini.*

On note $\mathbb{E}\kappa^p(M)$ la limite de $\mathbb{E}\kappa^p(M_N)$ quand N tend vers l'infini. Rappelons que $\mathbb{E}m_p(M)$ est la notation pour la limite de $\mathbb{E}m_p(M_N)$.

Théorème 8.14 **(Suite du théorème** 4.1 **de [Matrices])**
La famille $\left(\mathbb{E}\kappa^p(M)\right)_p$ est la famille des $\mathcal{P}$-cumulants associés à $\left(\mathbb{E}m_p(M)\right)_p$, c'est-à-dire que pour toute partition $p \in \mathcal{P}_k$, $\mathbb{E}m_p(M) = \sum_{p'\leq p} \mathbb{E}\kappa^{p'}(M)$. Ce sont les $\mathcal{P}$-cumulants du système de matrices aléatoires $(M_N)_{N\in\mathbb{N}}$.

Ce résultat découle du fait que la distance définie sur $\mathcal{P}_k$ vérifie l'inégalité triangulaire. De ce théorème, et en utilisant la dualité de Schur-Weyl dans le cas unitaire, on en déduit plusieurs résultats, dont une liste non-exhaustive est donnée ci-dessous :

- si pour tout N, la matrice M_N est invariante par conjugaison par le groupe unitaire, et si l'observable $\mathbb{E}m_\sigma(M_N)$ converge pour toute permutation σ, alors la famille $(M_N)_{N\in\mathbb{N}}$ converge en $\mathcal{P}$-moments[68].
- si pour tout N, la matrice M_N est invariante par conjugaison par le groupe unitaire, et si l'observable $\mathbb{E}m_\sigma(M_N)$ converge pour toute permutation σ, on sait relier l'asymptotique de l'espérance de produits de coefficients de la matrice M aux $\mathcal{P}$-cumulants[69]. Par exemple :

$$\mathbb{E}\left[(M_N)^{i_2}_{i_1}(M_N)^{i_1}_{i_2}(M_N)^{i_1}_{i_3}(M_N)^{i_3}_{i_1}\right] = N^{-2}\left(\mathbb{E}\kappa^{(1,2)(3,4)}(M) + o(1)\right).$$

[67] Voir la définition 4.1 de [Partitions] dans un cadre abstrait et définition 4.3 de [Matrices] pour le cadre des matrices aléatoires.
[68] Corollaire 4.2 de [Matrices].
[69] Théorème 5.4 de [Matrices].

8.6.4. La $\mathcal{P}$-liberté. — Comme précédemment expliqué, tout comme on pouvait définir l'indépendance et la liberté grâce aux cumulants classiques et libres, on peut définir la notion de $\mathcal{P}$-liberté en demandant à ce que les $\mathcal{P}$-cumulants mixtes soient nuls.

Ainsi, deux familles de matrices aléatoires $(M_N)_{N\in\mathbb{N}}$ et $(L_N)_{N\in\mathbb{N}}$ sont asymptotiquement $\mathcal{P}$-libres si la famille $(M_N, L_N)_{N\in\mathbb{N}}$ converge en $\mathcal{P}$-distribution, et si les cumulants mixtes s'annulent, ou de façon équivalente si les cumulants fini-dimensionnels mixtes tendent vers 0 quand N tend vers l'infini. Le théorème 8.2 se généralise alors à de la façon suivante.

***Théorème 8.15* (Théorème 7.10 de [Matrices]).** — *Considérons deux systèmes de matrices $(M_N)_{N\in\mathbb{N}}$ et $(L_N)_{N\in\mathbb{N}}$ qui convergent en $\mathcal{P}$-distribution. Si un des deux systèmes est invariant par conjugaison par le groupe symétrique, et si pour tout N, M_N est indépendant de L_N alors le système $\{M, N\}$ converge en $\mathcal{P}$-distribution. De plus M et L sont (asymptotiquement) $\mathcal{P}$-libres.*

Ce théorème découle facilement de l'utilisation des cumulants fini-dimensionnels. Donnons un exemple de calcul qui permet de justifier ce résultat : prenons la transposition $(1, 2)$: le cumulant fini-dimensionnel $\mathbb{E}\kappa^{(1,2)}(M_N, L_N)$ est donné par une renormalisation de la coordonnée de $\mathbb{E}(M_N \otimes L_N)$ sur $(1, 2)$ via la dualité de Schur-Weyl-Jones. Or par indépendance,

$$\mathbb{E}(M_N \otimes L_N) = \mathbb{E}(M_N) \otimes \mathbb{E}(L_N).$$

Le terme de droite se décompose alors, via la dualité de Schur-Weyl-Jones, en partitions ne pouvant relier aucun élément de $\{1, 1'\}$ avec un élément de $\{2, 2'\}$. Or la permutation $(1, 2)$ vue comme partition est la partition $\{\{1, 2'\}, \{1', 2\}\}$. Ceci montre que le cumulant fini-dimensionnel $\mathbb{E}\kappa^{(1,2)}(M_N, L_N)$ est nul, donc sa limite est aussi nulle.

L'utilisation des cumulants fini-dimensionnels permet d'obtenir facilement d'autres résultats dont on donne une liste non-exhaustive :

- on peut affaiblir les hypothèses dans le théorème précédent via l'introduction de la notion d'invariance forte asymptotique[(70)],
- si (M_N) est une famille de matrices aléatoires invariantes par le groupe unitaire qui converge en distribution non-commutative, alors $(M_N)_{N\in\mathbb{N}}$ et sa famille transposée $(M_N^t)_{N\in\mathbb{N}}$ sont asymptotiquement libres au sens de Voiculescu[(71)],

[(70)] Ce que l'on fait dans la section 7.1.9 et le théorème 7.11 de [Matrices].

[(71)] Voir le théorème 7.4 dans [Matrices]. Ce théorème avait été obtenu auparavant par J.A. Mingo et M. Popa dans [**25**] par des méthodes plus sophistiquées.

– on peut étendre la notion de $\mathcal{P}$-liberté à des sous ensembles des partitions, tels que $\mathcal{B}$, $\mathcal{D}$, ...[72] On étudie alors les liens entre ces libertés et les libertés connues[73]. En particulier, on montre que sous certaines hypothèses, deux systèmes de matrices aléatoires invariants par conjugaison par le groupe bistochastique[74] et indépendants seront asymptotiquement libres[75],
– on introduit une notion de $\mathcal{R}$-transformée[76], qui permet de prouver un théorème central limite $\mathcal{P}$-non-commutatif[77]. Grâce aux inclusions de $\mathcal{D}_k$ et de $[\mathrm{id}_k, (1, ..., k)]_{\mathcal{P}_k} \cap \mathfrak{S}_k$ dans $\mathcal{P}_k$, on retrouve le théorème central limite usuel, ainsi que le théorème central limite libre.
– étant donné deux systèmes de matrices aléatoires $(M_N)_{N\in\mathbb{N}}$ et $(L_N)_{N\in\mathbb{N}}$ qui sont asymptotiquement $\mathcal{P}$-libres, on calcule les limites des observables des familles $(M_N + L_N)_{N\in\mathbb{N}}$ et $(M_N L_N)_{N\in\mathbb{N}}$. En particuliers, on montre que la $\mathcal{R}$-transformée de la somme est la somme (au sens d'une opération $\boxplus$ définie dans [Partitions]) des $\mathcal{R}$-transformées de $(M_N)_{N\in\mathbb{N}}$ et $(L_N)_{N\in\mathbb{N}}$. De même la $\mathcal{R}$-transformée de la somme est le produit (au sens d'une opération $\boxtimes$ définie dans [Partitions]) des $\mathcal{R}$-transformées de $(M_N)_{N\in\mathbb{N}}$ et $(L_N)_{N\in\mathbb{N}}$.[78],

Il semble un peu restrictif de se limiter à l'étude de la limite d'espérances de la forme $\frac{1}{N^{\mathsf{nc}(p\vee \mathrm{id}_k)}}\mathbb{E}\left[\mathsf{Tr}\left(M^{\otimes k}\circ p\right)\right]$ au lieu d'étudier la limite des observables aléatoires de la forme $\frac{1}{N^{\mathsf{nc}(p\vee \mathrm{id}_k)}}\mathsf{Tr}\left(M^{\otimes k}\circ p\right)$. Cependant, on peut définir de nouveau une notion de $\mathcal{P}$-factorisation asymptotique[79] pour les systèmes de matrices, qui, si elle est vérifiée, implique la convergence en probabilité des observables de la forme $\frac{1}{N^{\mathsf{nc}(p\vee \mathrm{id}_k)}}\mathbb{E}\left[\mathsf{Tr}\left(M^{\otimes k}\circ p\right)\right]$ vers une limite non aléatoire.

8.7. Les processus de Lévy invariants par conjugaison par le groupe symétrique. — Les objets qui ont motivé la généralisation de la symétrie vérifiée par les systèmes de matrices aléatoires sont les marches aléatoires sur le groupe symétrique qui interviennent dans l'étude de la limite des mesures de Yang-Mills sur $\mathfrak{S}(N)$. Plus généralement, considérons un processus de Lévy $\mathsf{X} = (X_t)_{t\geq 0}$ à valeurs dans les matrices de tailles N et invariant par conjugaison par le groupe symétrique[80].

[72] La définition 7.1 dans [Matrices] est assez générale pour s'appliquer à d'autres ensembles que $\mathcal{P}_k$.
[73] Section 7.13 dans [Matrices].
[74] Le groupe bistochastique est le groupe de matrices orthogonales dont les sommes des éléments de chaque ligne et chaque colonne sont égales à 1.
[75] Voir le théorème 7.5 ainsi que la section 7.1.4 dans [Matrices].
[76] Définition 6.1 et 6.4 dans [Matrices].
[77] Théorème 8.1 dans [Matrices] qui généralise un théorème central limite de C. Male.
[78] Théorème 7.8 de [Matrices].
[79] Voir la définition 2.3 et le théorème 2.2 de [Matrices].
[80] Nous expliquons brièvement la section 10 de [Matrices].

Supposons que X est un processus de Lévy multiplicatif : pour tous $s < t$, $X_t X_s^{-1}$ est indépendant de X_s et a la même loi que X_{t-s}. Ainsi pour tous $s < t$:

$$\mathsf{S}(X_t) = \mathsf{S}((X_t X_s^{-1})X_s) = \mathsf{S}(X_{t-s})\mathsf{S}(X_s).$$

La famille des signatures $(\mathsf{S}(X_t))_{t\geq 0}$ est donc un semigroupe. Étant donné une suite de processus de Lévy multiplicatifs X^N, invariants par conjugaison par le groupe symétrique, on peut montrer qu'il suffit d'étudier l'asymptotique du générateur de $\left(\mathsf{S}(X_t^N)\right)_{t\geq 0}$ pour étudier le comportement asymptotique de X^N. De façon équivalente, ce qui nous intéresse est la famille $(G_k^N)_{k\in\mathbb{N}}$ donnée par :

$$G_k^N = \frac{d}{dt}_{|t=0} \mathbb{E}\left[X_t^{\otimes k}\right].$$

Pour tout entier positif k, pour tout entier positif N, G_k^N est un endomorphisme sur $(\mathbb{C}^N)^{\otimes k}$ qui commute à l'action de $\mathfrak{S}(N)$: d'après la dualité de Schur-Weyl-Jones, on peut le voir comme un élément de $\mathbb{C}\left[\mathcal{P}_k(N)\right]$. En utilisant la même renormalisation que celle utilisée pour définir les cumulants fini-dimensionnels, on peut définir une notion de convergence pour la famille $(G_k^N)_k$ quand N tend vers l'infini. On montre alors les deux théorèmes importants suivants.

Theorem 8.3 **(Théorèmes** 10.1 **et** 10.2 **de [Matrices]**[(82])

La convergence du générateur, c'est-à-dire de $(G_k^N)_{k\in\mathbb{N}}$, quand N tend vers l'infini, implique alors la convergence en $\mathcal{P}$-distribution du processus de Lévy $\mathsf{X_N}$.

On sait calculer les limites des $\mathcal{P}$-moments et les $\mathcal{P}$-cumulants : ceux-ci satisfont un système d'équations différentielles linéaires.

La convergence en $\mathcal{P}-$distribution permet d'affimer que les espérences de certaines observables convergent quand N tend vers l'infini. Afin de savoir si les observables convergent en réalité en probabilité, nous avons prouvé le théorème suivant.

Theorem 8.4 **(Théorèmes** 10.3**,** 10.5 **et Lemme** 10.2 **de [Matrices])**

Il existe un critère (nécessaire et suffisant) sur la limite de $(G_k^N)_{k\in\mathbb{N}}$ qui implique la convergence en probabilité des observables de $\mathsf{X_N}$.

Les mêmes résultats sont aussi valables pour une suite de processus de Lévy additifs. Afin de motiver ces résultats par des exemples simples, nous allons considérer le cas des mouvements browniens hermitien[(83]) et unitaire[(84]).

[(82])On montre, dans le théorème 10.5 du même chapitre, un résultat équivalent pour le système $(\mathsf{X}_N, \mathsf{X}_N^*)_{N\in\mathbb{N}}$.

[(83])On présente la preuve donnée dans [Matrices] du théorème 10.10.

[(84])On présente la preuve donnée dans [Matrices] du théorème 10.14.

***Exemple 8.2* (Mouvement brownien hermitien).** — *On considère un mouvement brownien hermitien de taille N, c'est-à-dire en reprenant les notations de la section 7.2, un processus de la forme :*

$$H_t = \sum_{e\in\mathcal{E}} B_t^e e,$$

avec $(B_t^e)_{e\in\mathcal{E}}$ une famille de browniens réels standards indépendants. D'après le calcul d'Itô :

$$dB_t^e dB_t^f = \delta_{e=f} dt.$$

Ainsi :

$$dH_t \otimes dH_t = \sum_{e,f\in\mathcal{E}} dB_t^e dB_t^f \, (e\otimes f) = \left(\sum_{e\in\mathcal{E}} e\otimes e\right) dt,$$

et $dH_t^{\otimes k} = 0$ si $k \geq 3$. Le calcul de $\sum_{e\in\mathcal{E}} e\otimes e$ est très simple :

$$\begin{aligned}\sum_{e\in\mathcal{E}} e\otimes e &= \frac{1}{N}\sum_{i=1}^N E_{i,i}\otimes E_{i,i} \; + \frac{1}{2N}\sum_{1\leq i<j\leq N} (E_{i,j}+E_{j,i})\otimes(E_{i,j}+E_{j,i}) \\ &\qquad + \frac{1}{2N}\sum_{1\leq i<j\leq N} (iE_{i,j}-iE_{j,i})\otimes(iE_{i,j}-iE_{j,i}) \\ &= \frac{1}{N}\sum_{i=1}^N E_{i,i}\otimes E_{i,i} + \frac{1}{N}\sum_{1\leq i<j\leq N} E_{i,j}\otimes E_{j,i} \\ &= \frac{1}{N}\sum_{i,j=1}^N E_i^j \otimes E_j^i = \frac{1}{N}\rho_N^{\mathcal{P}_2}((1,2)),\end{aligned}$$

où $(1,2)$ est la transposition dans $\mathfrak{S}_2$. Vu que pour tout entier positif k :

$$\begin{aligned}d\mathbb{E}\left[H_t^{\otimes k}\right] &= \mathbb{E}\left[(H_t+dH_t)^{\otimes k}\right] - \mathbb{E}\left[H_t^{\otimes k}\right] \\ &= \sum_{1\leq i<j\leq k} \mathbb{E}\left[H_t^{\otimes(i-1)}\otimes dH_t\otimes H_t\otimes ...\otimes H_t\otimes dH_t\otimes H_t^{\otimes n-j}\right],\end{aligned}$$

et que $H_0 = 0$, on voit que pour tout entier positif k différent de 2, $\frac{d}{dt}_{|t=0}\mathbb{E}\left[H_t^{\otimes k}\right] = 0$, et pour $k=2$,

$$\frac{d}{dt}_{|t=0}\mathbb{E}\left[H_t^{\otimes 2}\right] = \frac{1}{N}\rho_N^{\mathcal{P}_2}((1,2)).$$

Ainsi seul $k=2$ est intéressant afin de comprendre la convergence du mouvement brownien hermitien. Pour cela, nous devons renormaliser les coordonnées

de $\frac{d}{dt}_{|t=0}\mathbb{E}\left[H_t^{\otimes 2}\right]$ *tout comme nous l'avions fait pour la signature dans la définition 8.12. Notons* $\tilde{G}_2^N$ *l'élément obtenu de la sorte :*

$$\tilde{G}_2^N = N^{\mathsf{nc}((1,2))-\mathsf{nc}((1,2)\vee id)}\frac{1}{N}\rho_N^{\mathcal{P}_2}((1,2)).$$

La partition $(1,2)$ *a deux blocs et un cycle, donc pour tout entier positif* N *:* $\tilde{G}_2^N = \rho_N^{\mathcal{P}_2}((1,2))$*. Ainsi* $\tilde{G}_2^N$ *converge, le générateur du mouvement brownien hermitien converge et donc le mouvement brownien hermitien converge* $\mathcal{P}$*-distribution quand* N *tend vers l'infini. On peut montrer que la convergence est vérifiée en probabilité.*

Étant donné que la matrice de Wigner gaussienne a la même loi qu'un mouvement brownien hermitien au temps 1*, on en déduit la convergence des matrices de Wigner gaussiennes.*

Exemple 8.3 **(Mouvement brownien unitaire)**. — *Expliquons comment la preuve de T. Lévy expliquée dans la section 7.3.2 se transforme dans ce nouveau formalisme. Considérons* $(U_t)_{t\geq 0}$ *un mouvement brownien unitaire de taille* N*. Nous devons renormaliser les coordonnées de* $\frac{d}{dt}_{|t=0}\mathbb{E}\left[U_t^{\otimes k}\right]$ *tout comme nous l'avions fait pour la signature et tout comme nous l'avions fait pour le mouvement brownien hermitien. Notons* $\tilde{G}_k^N$ *l'élément obtenu de la sorte, en utilisant l'équation (14), on obtient :*

$$\tilde{G}_k^N = \rho_N^{\mathcal{P}_k}\left(\frac{k}{2}id_k + \sum_{1\leq i<j\leq k}(i,j)\right).$$

Ainsi $\tilde{G}_k^N$ *converge quand* N *tend vers l'infini : le mouvement brownien unitaire converge en* $\mathcal{P}$*-distribution quand* N *tend vers l'infini. De même on peut montrer de plus que la convergence est vérifiée en probabilité.*

Dans les exemples précédents, nous avons calculé explicitement le générateur afin de montrer sa convergence. Cependant, dans des cas plus généraux, il peut être fastidieux de donner une expression explicite du générateur à N fini. On peut alors se contenter de calculer certaines observables liées au générateur, et montrer que ces observables convergent. C'est la méthode que l'on utilise pour montrer la convergence de certains modèles de processus de Lévy, et la convergence des marches aléatoires sur le groupe symétrique.

9. $U(N)$-Yang-Mills quand N tend vers l'infini

Rappelons-nous le chemin parcouru jusqu'à maintenant. Nous avons défini les champs markoviens d'holonomies planaires comme processus indexés par des chemins dans le plan qui satisfont plusieurs axiomes dont le fait d'être invariants par transformation de jauge. Cette symétrie nous avait amenés à définir la transformée de Wilson qui, dans le cas où le groupe de structure est le groupe $U(N)$,

caractérise la loi des champs d'holonomies planaires invariants par transformation de jauge. Ainsi à la fin de la section 6.4 à nous poser la question de la convergence de la transformée de Wilson des champs markoviens d'holonomies planaires à valeurs dans $U(N)$ quand N devient de plus en plus grand. Il nous avait donc fallu comprendre le comportement de systèmes de matrices aléatoires et nous avions alors pu montrer la convergence en distribution du mouvement brownien unitaire $\mathsf{U}^N = (U_t^N)_{t\geq 0}$ sur $U(N)$ (exemple 8.3 et théorème 8.4). Or par le théorème 5.2, on peut construire un champ markovien holonomies planaire $\left(\mathsf{YM}_{\mathsf{vol}}^{\mathsf{U}^N}\right)_{\mathsf{vol}\in\mathcal{D}(\mathbb{R}^2)}$ associé à U^N. Dorénavant, nous n'étudierons que le cas où la mesure de densité vol est la mesure de Lebesgue dx, ainsi nous noterons $\mathsf{YM}^{\mathsf{U}^N}$ la mesure $\mathsf{YM}_{dx}^{\mathsf{U}^N}$. Nous pouvons donc espérer que les mesures $\mathsf{YM}^{\mathsf{U}^N}$ convergent quand N tend vers l'infini au sens de la transformée de Wilson : en effet c'est un théorème dû à T. Lévy et démontré dans [**20**].

Afin de simplifier l'exposé, on va coupler de façon quelconque les mesures $\mathsf{YM}^{\mathsf{U}^N}$: on considère un espace de probabilités $(\Omega, \mathcal{A}, \mathbb{P})$ sur lequel sont définies des variables $(h^N(l))_{l\in\mathsf{L}_0(\mathbb{R}^2)}$ telles que pour tout entier positif N, $(h^N(l))_{l\in\mathsf{L}_0(\mathbb{R}^2)}$ a la même loi que le processus de projection canonique $(h(l))_{l\in\mathsf{L}_0(\mathbb{R}^2)}$ défini sur $\left(\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), G), \mathsf{YM}^{\mathsf{U}^N}\right)$. Posons pour tout entier strictement positif N et pour toute boucle $l \in \mathsf{L}_0(\mathbb{R}^2)$,

$$\mathsf{W}^N(l) = \frac{1}{N}\mathsf{Tr}\left(h^N(l)\right).$$

Théorème 9.1. — *Il existe une fonction ϕ, appellée champ maître, de L_0 dans $\mathbb{R}$, continue pour la convergence à points extrémaux fixés définie à la fin de la section 2.1, telle que pour toute boucle $l \in \mathsf{L}_0(\mathbb{R}^2)$:*

$$\mathsf{W}^N(l) \underset{N\to\infty}{\longrightarrow} \phi(l)$$

dans $L^1(\Omega, \mathcal{A}, \mathbb{P})$.

Ce théorème implique que les transformées de Wilson des mesures $\mathsf{YM}^{\mathsf{U}^N}$ convergent. En effet, la convergence L^1 implique la convergence en probabilité des variables. Ainsi, pour toute famille de boucles $(l_i)_{i=1}^k$ dans $\mathsf{L}_0(\mathbb{R}^2)$, $\prod_{i=1}^k \mathsf{W}^N(l_i)$ converge en probabilité vers $\prod_{i=1}^k \phi(l_i)$. Puisque pour toute boucle $W^N(l)$ est la trace d'une matrice unitaire, elle est bornée par 1 : par théorème de convergence dominée, $\mathbb{E}\left[\prod_{i=1}^k \mathsf{W}^N(l_i)\right]$ converge vers $\prod_{i=1}^k \phi(l_i)$. Or, par définition même,

$$\mathbb{E}\left[\prod_{i=1}^k \mathsf{W}^N(l_i)\right] = W_{\mathsf{YM}^{\mathsf{U}^N}}(l_1, ..., l_k),$$

où la transformée de Wilson $W_{\mathsf{YM}^{\mathsf{U}^N}}$ avait été définie dans la section 6.4. Ainsi les transformées de Wilson convergent et la transformée limite que l'on notera $W_{\mathsf{YM}^{\mathsf{U}}}$

vérifie de plus la propriété de factorisation suivante : pour toute famille $(l_1, ..., l_k)$ de boucles dans $\mathsf{L}_0(\mathbb{R}^2)$,

$$W_{\mathsf{YM}^\cup}(l_1, ..., l_k) = \prod_{i=1}^{k} W_{\mathsf{YM}^\cup}(l_1).$$

Par la suite, nous allons expliquer comment démontrer ce résultat. Nous présenterons des arguments proches de ceux proposés par T. Lévy, puis nous développerons de nouveaux arguments obtenus dans l'article en préparation [**14**] qui permettent plus facilement la généralisation du théorème 9.1 à l'étude asymptotique d'autres champs de Yang-Mills planaires.

9.1. Convergence sur $\mathsf{Aff}_0(\mathbb{R}^2)$. — La première étape consiste à montrer que pour toute boucle $l \in \mathsf{Aff}_0(\mathbb{R}^2)$ (c'est-à-dire affine par morceaux), la variable aléatoire $\mathsf{W}^N(l)$ converge bien vers une limite non-aléatoire que l'on notera $\phi(l)$. C'est la partie la plus simple du théorème : toute boucle $l \in \mathsf{Aff}_0(\mathbb{R}^2)$ peut se voir comme une boucle tracée sur un graphe fini $\mathbb{G}$. Or on a vu que dans ce cas, la variable $h^N(l)$ peut s'écrire comme un produit des variables $\left(h^N_{\mathsf{l}_F}\right)_{F\in\mathbb{F}^b}$ et $\left(h^N_{\mathsf{l}_F^{-1}}\right)_{F\in\mathbb{F}^b}$ où $(\mathsf{l}_F)_{F\in\mathbb{F}^b}$ est une base de lassos basée en 0. Ainsi $\mathsf{W}^N(l)$ est un la trace normalisée d'un produit de matrices. Rappelons-nous l'équation (10) : $\mathbb{E}\left[\mathsf{W}^N(l)\right]$ est typiquement une des observables que l'on considérait auparavant et mesure le système fini-dimensionel de matrices aléatoires $\left(h^N_{\mathsf{l}_F}\right)_{F\in\mathbb{F}^b} \cup \left(h^N_{\mathsf{l}_F^{-1}}\right)_{F\in\mathbb{F}^b}$.

Au lieu de montrer que $\mathsf{W}^N(l)$ converge, nous allons expliquer comment prouver que $\mathbb{E}\left[\mathsf{W}^N(l)\right]$ converge. Pour cela, il suffit de montrer que le système de matrices aléatoires $\left(\left(h^N_{\mathsf{l}_F}\right)_{F\in\mathbb{F}^b} \cup \left(h^N_{\mathsf{l}_F^{-1}}\right)_{F\in\mathbb{F}^b}\right)_{N\in\mathbb{N}}$ converge en distribution non-commutative.

Le mouvement brownien étant invariant par conjugaison par le groupe unitaire $U(N)$, pour toute face bornée $h^N_{\mathsf{l}_F}$ a la même loi qu'un mouvement brownien arrété au temps $dx(F)$ et les variables $\left(h^N_{\mathsf{l}_F}\right)_{F\in\mathbb{F}^b}$ sont indépendantes. Ainsi $\left(h^N_{\mathsf{l}_F}\right)_{F\in\mathbb{F}^b} \cup \left(h^N_{\mathsf{l}_F^{-1}}\right)_{F\in\mathbb{F}^b}$ est le couplage indépendant de systèmes de matrices invariants par conjugaison par le groupe unitaire, chacun étant de la forme $\left(h^N_{\mathsf{l}_F}, h^N_{\mathsf{l}_F^{-1}}\right)_{N\in\mathbb{N}}$. D'après le théorème 8.2, il suffit donc de montrer que le système de matrices aléatoires $\left(h^N_{\mathsf{l}_F}, h^N_{\mathsf{l}_F^{-1}}\right)_{N\in\mathbb{N}}$ converge en distribution non-commutative. Vu qu'il a la même loi que $\left(\mathsf{U}^N_{dx(F)}, \left(\mathsf{U}^N_{dx(F)}\right)^*\right)_{N\in\mathbb{N}}$ qui converge en distribution non-commutative (voir notre discussion après le théorème 8.4), on peut conclure quant à la convergence de ce système et donc de $\mathbb{E}\left[\mathsf{W}^N(l)\right]$.

En utilisant le fait que $\left(\mathsf{U}^N_{dx(F)}, \left(\mathsf{U}^N_{dx(F)}\right)^*\right)_{N\in\mathbb{N}}$ vérifie la propriété de $\mathfrak{S}$-factorisation, on peut montrer que $\left(h^N(l), (h^N(l))^*\right)_{N\in\mathbb{N}}$ vérifie aussi la propriété de $\mathfrak{S}$-factorisation, ce qui implique alors que $\mathsf{W}^N(l)$ converge en probabilité vers une variable non-aléatoire. De nouveau, étant donné que l'on considère des variables aléatoires bornées, cela prouve que $\mathsf{W}^N(l)$ converge dans $L^1(\Omega, \mathcal{A}, \mathbb{P})$ vers un nombre réel $\phi(l)$.

Il reste maintenant à étendre cette convergence pour toute boucle $l \in \mathsf{L}_0(\mathbb{R}^2)$.

9.2. Convergence sur $\mathsf{L}_0(\mathbb{R}^2)$. — Considérons une boucle $l \in \mathsf{L}_0(\mathbb{R}^2)$ et considérons une approximation de l donnée par une suite de boucles $(l_n)_{n\in\mathbb{N}}$, affines par morceaux et basées en 0. À la vue des résultats expliqués jusqu'à présent, nous pouvons dessiner un tableau de doubles limites (dans l'espace $L^1(\Omega, \mathcal{A}, \mathbb{P})$) :

$$\begin{array}{ccc} \mathsf{W}^N(l_n) & \xrightarrow{n\to\infty} & \mathsf{W}^N(l) \\ \Big\downarrow{\scriptstyle N\to\infty} & & \vdots\,{\scriptstyle n\to\infty} \\ \phi(l_n) & \xrightarrow{n\to\infty} & \phi(l) \end{array}$$

dans lequel, pour l'instant, aucune des deux flèches en pointillés n'a été démontrée. Pour ce faire, il nous suffit de montrer que la convergence le long d'une des deux flèches pleines est une convergence uniforme, c'est-à-dire, montrer soit :

$$\sup_n \mathbb{E}\left[\,|\,\mathsf{W}^N(l_n) - \phi(l_n)\,|\,\right] \underset{N\to\infty}{\longrightarrow} 0 \tag{25}$$

soit :

$$\sup_N \mathbb{E}\left[\,|\,\mathsf{W}^N(l_n) - \mathsf{W}^N(l)\,|\,\right] \underset{n\to\infty}{\longrightarrow} 0. \tag{26}$$

9.2.1. Uniformité en n. — Dans [**20**], T. Lévy démontre une estimée qui lui permet de montrer (25). En effet, il montre que pour toute boucle b, affine par morceaux :

$$\mathbb{E}\left[\,|\,\mathsf{W}^N(b) - \phi(b)\,|\,\right] \leq \frac{1}{N}\left[\ell(b)e^{\frac{1}{2}\ell(b)^2} + \ell(b)^2 e^{\ell(b)^2}\right], \tag{27}$$

où on rappelle que $\ell(b)$ est la longueur de b. Or si $(l_n)_{n\in\mathbb{N}}$ converge vers l, la longueur de l_n converge vers celle de l, en particulier $\sup_n \ell(l_n) < \infty$, ce qui permet de conclure quant à la véracité de (25).

9.2.2. Uniformité en N. — Dans l'article en préparation [**14**], et dans le chapitre de thèse "Revêtements ramifiés" dans lequel sont réutilisés les mêmes arguments, nous nous sommes trouvés confrontés au problème que, dans le cas où le processus de Lévy sous-jacent n'était pas un processus de brownien, l'inégalité (27) semblait difficile à obtenir, voire était peut-être fausse. Il reste donc à montrer l'uniformité

en N des convergences des boucles de Wilson lorsque n tend vers l'infini, c'est-à-dire la convergence (26). Pour ce faire, on utilise la version quantitative d'un résultat de T. Lévy, le théorème de T. Lévy-Kolmogorov bidimensionnel que l'on a évoqué dans la section 2.5.2 et que nous allons expliquer dans la suite.

Théorème 9.2. — *Notons dx la mesure de Lebesgue sur le plan. Considérons Γ un groupe métrique complet tel que les translations et l'inversion ($g \mapsto g^{-1}$) sont des isométries. Donnons-nous H, une fonction multiplicative sur les boucles affines basées en 0 à valeurs dans Γ : $\mathsf{H} \in \mathcal{M}ult(\mathsf{Aff}_0(\mathbb{R}^2), \Gamma)$. Supposons qu'il existe une constante $K \geq 0$ telle que pour toute boucle simple $l \in \mathsf{Aff_0}(\mathbb{R}^2)$ de longueur $\ell(l) \leq K^{-1}$ et qui entoure un domaine D, l'inégalité :*

$$d_\Gamma(1, \mathsf{H}(l)) \leq K\sqrt{dx(D)} \tag{28}$$

est vérifiée. Alors la fonction H_N admet une unique extension sur $\mathsf{P}(\mathbb{R}^2)$ qui est multiplicative, c'est-à-dire dans $\mathcal{M}ult(\mathsf{P}(\mathbb{R}^2), \Gamma)$ et qui est régulière.

De plus, pour toute boucle $l \in \mathsf{L}_0(\mathbb{R}^2)$, il existe une approximation de l par des boucles affines par morceaux $(l_n)_{n\in\mathbb{N}}$ telle que $\mid \ell(l) - \ell(l_n) \mid \underset{n\to\infty}{\longrightarrow} 0$ et :

$$d_\Gamma\big[\mathsf{H}(l), \mathsf{H}(l_n)\big] \leq K\ell(l)^{\frac{3}{4}} \mid \ell(l) - \ell(l_n) \mid^{\frac{1}{4}} . \tag{29}$$

Nous donnons la version quantitative ici, c'est-à-dire l'inégalité (29), qui apparaît dans la preuve du théorème de T. Lévy-Kolmogorov bidimensionnel mais pas dans son énoncé[(85)]. Grâce à ce théorème, on peut étendre un champ d'holonomies planaire défini sur les chemins affines par morceaux en un champ d'holonomies planaire régulier. Pour cela, étant donné une mesure de probabilité μ sur $\mathcal{M}ult(\mathsf{Aff}_0(\mathbb{R}^2), G)$ on applique le théorème de T. Lévy-Kolmogorov bidimensionnel avec :

- $\Gamma = L^1(\mathcal{M}ult(\mathsf{Aff}_0(\mathbb{R}^2), G), \mu; G)$, c'est-à-dire les variables intégrables définies sur $(\mathcal{M}ult(\mathsf{Aff}_0(\mathbb{R}^2), G)$ à valeurs dans G.
- $d_\Gamma(X, Y) = \mu\left(d(X, Y)\right)$, où X et Y sont dans Γ et d_G est une distance sur G invariante par translations et inversion.
- H est l'application qui à une boucle affine par morceaux l basée en 0 associe la variable aléatoire :

$$\begin{aligned} \mathsf{H}(l) : \mathcal{M}ult(\mathsf{Aff}_0(\mathbb{R}^2), G) &\to G \\ h &\mapsto h(l). \end{aligned}$$

Si le champ d'holonomies μ vérifie :

$$d_\Gamma\left(1, \mathsf{H}(l)\right) = \int_{\mathcal{M}ult(\mathsf{Aff}_0(\mathbb{R}^2), G)} d_G(e, h(l))\mu(dh) \leq K\sqrt{dx(D)},$$

(85) Théorème 3.3.1 dans [**23**].

pour toute boucle simple affine par morceaux basée en 0 et entourant un domaine D, alors on peut étendre H continûment. Pour toute boucle $l \in L_0(\mathbb{R}^2)$ on peut définir une variable aléatoire $\mathsf{H}(l)$, définie sur un espace de probabilité (en l'occurrence $\mathcal{M}ult(\mathsf{Aff}_0(\mathbb{R}^2), G)$ muni de μ) telle que pour toutes boucles l_1 et l_2, presque sûrement : $\mathsf{H}(l_1 l_2) = \mathsf{H}(l_2)\mathsf{H}(l_1)$ et $\mathsf{H}(l_1^{-1}) = (\mathsf{H}(l_1))^{-1}$. Ceci permet alors par un argument de limite projective à la Kolmogorov-Carathéodory, de définir une mesure de probabilité, notée aussi μ, sur $\mathcal{M}ult(\mathsf{L}_0(\mathbb{R}^2), G)$ telle que pour toute famille de boucles, la famille des projections canoniques $(h(l_1), ..., h(l_n))$ a la même loi que $(\mathsf{H}(l_1), ..., \mathsf{H}(l_n))$. La continuité de H implique que pour toute famille de boucles $(l_n)_{n\in\mathbb{N}}$ basées en 0 et convergeant vers $l \in \mathsf{L}_0(\mathbb{R}^2)$,

$$\int_{\mathcal{M}ult(\mathsf{L}_0(\mathbb{R}^2),G)} d_G\left(h(l_n), h(l)\right) \mu(dh) \underset{n\to\infty}{\longrightarrow} 0.$$

La version quantitative permet quant à elle de montrer l'uniformité (26). En effet, en utilisant les mêmes notations :

$$\begin{aligned}
\sup_N \mathbb{E}\left[\,|\, \mathsf{W}^N(l_n) - \mathsf{W}^N(l)\,|\,\right] &\leq \sup_N \mathbb{E}\left[\left|\frac{1}{N}\mathsf{Tr}(\mathsf{H}(l_n)) - \frac{1}{N}\mathsf{Tr}(\mathsf{H}(l))\right|\right] \\
&\leq \sup_N \mathbb{E}\left[\left|d_{U(N)}\left(\mathsf{H}(l_n), \mathsf{H}(l)\right)\right|\right] \qquad \text{(Cauchy-Schwarz)} \\
&\leq \sup_N d_\Gamma\left(\mathsf{H}(l_n), \mathsf{H}(l)\right),
\end{aligned}$$

où $d_{U(N)}$ est la distance sur $U(N)$ donnée par

$$d_{U(N)}\left(U_1, U_2\right) = \frac{1}{N}\mathsf{Tr}\left((U-V)(U-V)^*\right).$$

On voit alors qu'un contrôle uniforme des constantes K_N nécessaires à l'application de l'inéquation (29)[(86)], permet alors d'obtenir l'uniformité voulue des convergences. Ceci permet alors de boucler le tableau de doubles limites et permet aussi de montrer que la fonction ϕ est continue.

Remarque 9.1. — *Comme nous le montrons dans l'article en préparation* [**14**], *on peut construire directement le champ maître, c'est-à-dire* $(\phi(l))_{l\in\mathsf{L}_0(\mathbb{R}^2)}$ *en suivant la construction des champs markoviens d'holonomies expliquée dans le chapitre de thèse [Champs] mais en utilisant un mouvement brownien libre (voire un processus de Lévy libre multiplicatif) au lieu d'un processus de Lévy, et en remplaçant la notion d'indépendance par la notion de liberté.*

[(86)] Que l'on obtient dans [**14**] ainsi que dans le chapitre de thèse "Revêtements ramifiés" dans le cadre où $U(N)$ est remplacé par $\mathfrak{S}(N)$.

10. Marches aléatoires sur $\mathfrak{S}(N)$, $\mathfrak{S}(N)$-Yang-Mills pour N grand et revêtements ramifiés aléatoires

Dans le chapitre de thèse dénommé "Revêtements ramifiés", nous étudions la limite en grande dimension de champs de Yang-Mills planaires lorsque le groupe de structure est le groupe symétrique $\mathfrak{S}(N)$ ainsi que ses conséquences sur les revêtements aléatoires. Grâce à la méthode introduite dans la section 9, la convergence des champs de Yang-Mills planaires en grande dimension se déduit de la convergence des processus de Lévy sous-jacents.

Lorsque le groupe de structure est $\mathfrak{S}(N)$, nous avons donc à étudier des processus de Lévy invariants par conjugaison par le groupe symétrique. Les outils développés dans les sections 8 et 8.7 sont alors adaptés à ce contexte et permettent d'obtenir la convergence de marches aléatoires sur les groupes symétriques.

10.1. Marche sur le groupe symétrique en grande dimension. —

10.1.1. Définition. — Pour tout entier positif N, considérons une classe de conjugaison λ_N dans $\mathfrak{S}(N)$, c'est-à-dire que si σ_N est une permutation dans λ_N :

$$\lambda_N = \{\sigma^{-1}\sigma_N\sigma, \sigma \in \mathfrak{S}(N)\}.$$

Étant donné $\sigma \in \lambda_N$, on notera $\lambda_N(1^c)$ la taille du support de σ. Cette taille ne dépend pas du choix de la permutation σ. Nous pouvons généraliser l'exemple 1.2 afin de définir la marche aléatoire dans le groupe symétrique associée à λ_N.

Celle-ci est définie de la façon suivante : partons de l'élément neutre id_N, attendons en id_N un certain temps aléatoire de loi exponentielle de paramètre $\frac{N}{\lambda_N(1^c)}$, lorsque le temps est écoulé on passe à σ_1 où σ_1 est choisi uniformément dans λ_N. Puis on attend de nouveau un certain temps aléatoire de loi exponentielle de paramètre $\frac{N}{\lambda_N(1^c)}$ indépendant du reste, puis lorsque le temps est écoulé on saute à $\sigma_2\sigma_1$ où σ_2 est choisi uniformément dans λ_N indépendamment du reste. Et ainsi de suite.

On obtient alors un processus de Lévy, noté $\left(S_t^{\lambda_N}\right)_{t\geq 0}$, qui est un processus de Markov dont le générateur H_{λ_N} est donné par le fait que pour toute fonction $f \in \mathbb{R}^{\mathfrak{S}(N)}$, pour toute permutation $\sigma_0 \in \mathfrak{S}(N)$:

$$H_{\lambda_N} f(\sigma_0) = \frac{d}{dt}_{|t=0} \mathbb{E}[f(S_t\sigma_0)] = \frac{N}{\lambda_N(1^c)} \frac{1}{\#\lambda_N} \sum_{\sigma\in\lambda_N} \big[f(\sigma\sigma_0) - f(\sigma_0)\big].$$

Le fait d'avoir considéré des sauts répartis uniformément dans une classe de conjugaison implique que le processus associé est invariant par conjugaison.

Lemme 10.1. — *Pour toute permutation $\sigma \in \mathfrak{S}(N)$, le processus $\left(\sigma^{-1}S_t^{\lambda_N}\sigma\right)_{t\geq 0}$ a la même loi que $\left(S_t^{\lambda_N}\right)_{t\geq 0}$.*

En effet, étant donné σ une permutation dans $\mathfrak{S}(N)$ on obtient le processus $\left(\sigma^{-1}S_t^{\lambda_N}\sigma\right)_{t\geq 0}$ en restant sur place pendant des temps exponentiels indépendants, et en sautant par multiplication par des permutations indépendantes de loi $\sigma S\sigma^{-1}$ avec S une permutation uniformément choisie dans λ_N. Or $\sigma S\sigma^{-1}$ est aussi une permutation uniformément choisie dans λ_N.

10.1.2. Convergence. — Nous avons construit un processus de Lévy à valeurs dans les permutations. Cependant comme nous l'avons toujours fait de façon implicite, nous pouvons voir le groupe des permutations $\mathfrak{S}(N)$ comme un sous-groupe du groupe des matrices de tailles N. En effet, considérons $\mathbb{C}^N$ muni de sa base canonique $(e_1, ..., e_N)$. Alors on a une représentation du groupe symétrique donnée par le fait que pour toute permutation $\sigma \in \mathfrak{S}(N)$ et tout $i \in \{1, ..., n\}$:

$$\sigma(e_i) = e_{\sigma(i)}.$$

De fait, on peut donc considérer une marche aléatoire sur le groupe symétrique comme un processus de Lévy matriciel invariant par conjugaison par $\mathfrak{S}(N)$. Les résultats exposés dans la section 8.7 s'appliquent donc à l'étude en grande dimension de ces marches aléatoires.

Afin d'obtenir des résultats de convergence, il faut cependant que les classes de conjugaison choisies convergent dans un certain sens. Remarquons qu'une classe de conjugaison de $\mathfrak{S}(N)$ est simplement la donnée d'un découpage de N en N entiers positifs[87], c'est-à-dire d'un vecteur d'entiers positifs $(\lambda_N(i))_{i=1}^N$ tel que $\sum_{i=1}^N \lambda_N(i) = N$. En effet si σ et σ' sont deux permutations dans λ_N, alors pour tout entier $i \in \{1, ..., N\}$,

$$\#\{j \in \{1, ..., N\} \mid i = \min\{i' \geq 1 \mid \sigma^i(j) = j\}\}$$

est égal à :

$$\#\{j \in \{1, ..., N\} \mid i = \min\{i' \geq 1 \mid \sigma'^i(j) = j\}\}.$$

On notera $\lambda_N(i)$ cet entier commun : c'est le nombre d'éléments dans $\{1, ..., N\}$ de période i pour $\sigma \in \lambda_N$. Ce codage des classes de conjugaison de $\mathfrak{S}(N)$ a permis de définir dans [Revêtements] la notion suivante de convergence pour une suite de classes de conjugaison de $\mathfrak{S}(N)$.

Définition 10.1. — *Une suite de classes de conjugaison $(\lambda_N)_{N\in\mathbb{N}}$ converge quand N tend vers l'infini si :*

1. *pour tout entier positif N, $\lambda_N \subset \mathfrak{S}(N)$,*

[87] Nous attirons l'attention du lecteur sur le fait que ceci n'est pas la représentation la plus courante des permutations, puisqu'il est assez courant de représenter une classe de conjugaison de $\mathfrak{S}(N)$ par la suite décroissante des longueurs des cycles d'un représentant. Cela permet de coder une classe de conjugaison de $\mathfrak{S}(N)$ par une partition de N.

2. *pour tout entier* $i \geq 1$, $\frac{\lambda_N(i)}{N-\lambda_N(1)}$ *converge vers un réel* $\lambda(i)$ *quand* N *tend vers l'infini,*
3. *la fraction* $\frac{\lambda_N(1)}{N}$ *converge vers un réel* $1-\alpha$ *quand* N *tend vers l'infini.*

En étudiant la limite du générateur H_{λ_N} et en utilisant les résultats exposés dans la section 8.7, on montre que la convergence des classes de conjugaison $(\lambda_N)_{N\in\mathbb{N}}$ implique la convergence des marches aléatoires associées.

Théorème 10.1 **(Théorèmes** 2.1 **et** 2.2 **de [Revêtements])**
Supposons que les classes de conjugaison convergent quand N *tend vers l'infini, alors :*

1. *pour tout réel positif* t*, les mesures empiriques des valeurs propres de* $S_t^{\lambda_N}$ *convergent en loi vers une mesure aléatoire* μ_t^λ*. Il existe un système d'équations différentielles qui permet de calculer la loi de* μ_t^λ *pour tout réel* t*,*
2. *le processus* $\left(S_t^{\lambda_N}\right)_{t\geq 0}$ *converge en* $\mathcal{P}$*-distribution et définit à la limite un processus de Lévy* $\mathcal{P}$*-libre : les incréments multiplicatifs sont asymptotiquement* $\mathcal{P}$*-libres et stationnaires.*

On peut alors distinguer deux cas de figure : si α est nul, c'est-à-dire si $\frac{\lambda_N(1)}{N}$ tend vers 1, alors le support de toute permutation dans λ_N est de taille négligeable devant N et l'on dit que $(\lambda_N)_{N\in\mathbb{N}}$ est évanescente. Si ce n'est pas le cas, on dit que $(\lambda_N)_{N\in\mathbb{N}}$ est macroscopique.

Théorème 10.2 **(Théorèmes** 2.3 **et** 2.4 **de [Revêtements])**
Supposons que les classes de conjugaison convergent quand N *tend vers l'infini. Deux comportements différents sont possibles :*

1. *si* $(\lambda_N)_{N\in\mathbb{N}}$ *est macroscopique alors, pour tout réel positif* t*,* μ_t^λ *est vraiment aléatoire,*
2. *si* $(\lambda_N)_{N\in\mathbb{N}}$ *est évanescente alors, pour tout réel positif* t*,* μ_t^λ *est non aléatoire, la convergence a lieu en probabilité. Asymptotiquement* $\left(S_t^{\lambda_N}\right)_{t\geq 0}$ *n'est pas un processus de Lévy libre multiplicatif au sens de Voiculescu. Dans ce cas, la mesure* μ_t^λ *est donnée par :*

$$\mu_t^\lambda = \sum_{n\in\mathbb{N}^*}\sum_{k=0}^{n-1} \frac{m_{n^c}(t)}{n}\delta_{e^{\frac{2ik\pi}{n}}} + m_{\infty^c}(t)\lambda_{\mathbb{U}},$$

où $\lambda_{\mathbb{U}}$ *est la mesure de Haar sur* $\mathbb{U}$*, où pour tout entier positif* n*,*

$$m_{n^c}(t) = e^{-nt}\sum_{k=0}^{n-1} t^k \frac{n^{k-1}}{k!} \sum_{(i_1,\ldots,i_k)\in\mathbb{N}^*,\sum_{j=1}^k i_j = n-1} \prod_{j=1}^{k}\lambda(i_j+1)$$

et où $m_{\infty^c}(t) = 1 - \sum_{k=0}^{\infty} m_{n^c}(t)$*.*

10.1.3. Transition de phase. — Lorsque la suite de classes de conjugaison $(\lambda_N)_{N\in\mathbb{N}}$ est évanescente, on voit que la mesure limite possède une partie totalement atomique et une partie proportionnelle à la mesure de Haar. Cependant, la partie absolument continue n'apparaît qu'à partir d'un certain temps. Nous donnons une preuve de ce résultat basée sur les équations fonctionnelles qui sont vérifiées par les fonctions génératrices des nombres $e^{nt}m_{n^c}(t)$.

***Théorème 10.3* (Théorème 2.5 de [Revêtements]).** — *On suppose que $(\lambda_N)_{N\in\mathbb{N}}$ converge et est évanescente. En utilisant les notations de la définition 10.1, on pose :*

$$t_c^{\lambda} = \delta_{\sum_{j=2}^{\infty}\lambda(j)=1}\frac{1}{\sum_{j=2}^{\infty}(j-1)\lambda(j)},$$

alors pour tout $0 \leq t \leq t^{\lambda}$, $m_{\infty^c}(t) = 0$ et pour tout $t > t^{\lambda}$, $m_{\infty^c}(t) > 0$.

Cette transition de phase est liée à la transition de phase observée auparavant par Beresticky et Durrett[(88)]. Ceux-ci ont montré que la distance de la marche aléatoire à l'identité subit une transition de phase, lorsque N est grand, au temps t_c : la marche décélère subitement.

10.1.4. Log-cumulants. — Dans l'étude des processus de Lévy libres multiplicatifs, la notion de log-cumulants permet de caractériser facilement ces processus particuliers. On peut calculer alors[(89)], pour toute suite de classes de conjugaison $(\lambda_N)_{N\in\mathbb{N}}$ qui converge et qui est évanescente, la fonctionnelle des log-cumulants du processus de Lévy $\mathcal{P}$-libre obtenu en considérant la limite de $(S_t^{\lambda_N})_{t\geq 0}$ quand N tend vers l'infini.

10.2. $\mathfrak{S}(\infty)-$Yang-Mills et revêtements aléatoires. — Dans la section 9, nous avions pu montrer la convergence des transformées de Wilson du champ de Yang-Mills planaire associé au mouvement brownien unitaire grâce à trois étapes :

1. convergence en distribution non-commutative du processus sous-jacent arrêté à un temps donné, qui permettait d'obtenir la convergence de la transformée de Wilson pour chaque élément d'une base de lasso,
2. convergence des couplages indépendants de matrices aléatoires invariantes par conjugaison par le groupe de structure, qui permettait d'obtenir la convergence de la transformée de Wilson pour toute boucle affine par morceaux,
3. uniformité dans la vitesse du processus sous-jacent (contrôle du K dans l'inégalité (28)) qui permettait d'étendre la convergence à toute boucle dans $\mathsf{L}_0(\mathbb{R}^2)$.

(88) Dans [**27**] par Beresticky et Durrett pour la marche aléatoire par transpositions, puis par Beresticky dans [**26**] dans le cas de marches aléatoire par une classe de conjugaison fixée.
(89) Ce résultat est présenté dans le théorème 2.7 de [Revêtements].

Nous avons montré dans la section précédente que certains systèmes de marches aléatoires sur le groupe symétrique convergeaient en distribution non-commutative. De plus l'étape 2, dans le cas où le groupe de structure est égal au groupe symétrique, est fournie par l'application directe du théorème 8.15 de $\mathcal{P}$-liberté. En prouvant que l'on peut mener à bien l'étape 3[(90)], on aboutit alors à la convergence des transformées de Wilson pour les mesures de Yang-Mills de groupe de structure $\mathfrak{S}(N)$[(91)].

Théorème 10.4. — *Considérons $(\lambda_N)_{N\in\mathbb{N}}$ une suite de classes de conjugaison qui converge et qui est macroscopique. Considérons la marche aléatoire S^{λ_N} associée à λ_N, ainsi que le champ de Yang-Mills planaire $\mathsf{YM}^{\mathsf{S}^{\lambda_N}}$ associée à S^{λ_N} et à la mesure de Lebesgue dx. La transformée de Wilson de $\mathsf{YM}^{\mathsf{S}^{\lambda_N}}$ converge quand N tend vers l'infini, et sa limite satisfait la même propriété de factorisation que la transformée de Wilson limite associée aux mouvements browniens unitaires.*

Dans la section 2.4.3, nous avons vu le lien entre revêtements aléatoires et champs markoviens d'holonomies. Ainsi nous avons vu que les monodromies de modèles de revêtements aléatoires permettaient de définir des champs markoviens d'holonomies. Cela permet inversement d'obtenir des informations sur les monodromies de modèles de revêtements aléatoires en étudiant les champs markoviens d'holonomies associés.

On dit qu'un revêtement ramifié est simple si, en tout point de ramification, il n'y a qu'une seule "hélice bouclée" au-dessus de lui qui est de plus d'ordre 2. Considérons, par exemple, le modèle de revêtement aléatoire suivant : on tire aléatoirement un ensemble fini de points dans le disque R (selon un processus de Poisson d'intensité $\frac{N}{2}dx$), puis parmi tous les revêtements ramifiés simples à N feuillets, d'ensemble de ramification R, étiquetés en 0, prenons-en un uniformément. Ceci nous donne un revêtement aléatoire simple et étiqueté à N feuillets que l'on notera $\mathcal{R}_N$. Pour toute boucle $l \in \mathsf{L}_0(\mathbb{R}^2)$, on peut considérer la monodromie aléatoire de $\mathcal{R}_N$ le long de l que l'on notera $\mathsf{m}_N(l)$. Le champ des monodromies $(\mathsf{m}_N(l))_{l\in\mathsf{L}_0(\mathbb{R}^2)}$ est alors un champ d'holonomies planaire, régulier, provenant d'un champ de Yang-Mills planaire dont le processus de Lévy associé est la marche par transpositions sur $\mathfrak{S}(N)$ définie dans l'exemple 1.2. En transposant le théorème 10.4 à ce modèle de revêment ramifié aléatoire, on en déduit le résultat suivant.

Théorème 10.5. — *Il existe une application ϕ allant de $\mathsf{L}_0(\mathbb{R}^2)$ dans $\mathbb{R}$ qui est continue et telle que pour toute boucle $l \in \mathsf{L}_0(\mathbb{R}^2)$:*

$$\frac{1}{N}\mathsf{Tr}(\mathsf{m}_N(l)) \underset{N\to\infty}{\longrightarrow} \phi(l)$$

dans $L^1(\Omega, \mathcal{A}, \mathbb{P})$.

(90) Par une généralisation simple du lemme 3.1 de [Revêtements].

(91) Le théorème 10.4 est une généralisation du théorème 3.2 présenté dans [Revêtements].

11. Results obtained in this thesis and their localization

In this section, we give the locations and a short summary of the results proved in this thesis. Since we used a wide range of theories in this thesis, we include not only the main results, but also intermediate results proved in this thesis. We do so in order to help the reader to have a quick overview on the results and to help him to read the long articles.

11.1. Chapter [Champs]. —

11.1.1. Planar graphs and reduced loops group. —

Definition 3.13 For any finite planar graphs with piecewise affine edges $\mathbb{G}$ and $\mathbb{G}'$, we define the notion of $\mathbb{G} - \mathbb{G}'$ piecewise diffeomorphisms.

Proposition 3.2 We prove that for any two homeomorphic simple connected finite graphs $\mathbb{G}$ and $\mathbb{G}'$ with piecewise affine edges, the set of $\mathbb{G}-\mathbb{G}'$ piecewise diffeomorphisms is not empty.

Corollary 3.1 It is shown that given a finite connected planar graph $\mathbb{G}$ with piecewise affine edges, if $\mathbb{G}$ is generic (that means that every vertex of $\mathbb{G}$ belongs at most to four edges) then there exists a subgraph $\mathbb{G}'$ of the graph $\mathbb{N}^2$ such that the set of $\mathbb{G}$-$\mathbb{G}'$ piecewise diffeomorphisms is not empty.

Definition 6.5 We generalize a definition of T. Lévy, and define for any finite planar graph $\mathbb{G}$, for any spanning tree of $\mathbb{G}$, any vertex v of $\mathbb{G}$ any family of facial cycles, a family of facial lassos for $\mathbb{G}$.

Proposition 6.1 We prove that the family of facial lassos defined in Definition 6.5 freely generates the group of reduced loops in $\mathbb{G}$ based at a vertex v. This group is the fondamental group of $\mathbb{G}$ based at v.

Proposition 7.1 We show that two families of facial lassos obtained by applying Definition 6.5 are linked via an action of the braid group.

11.1.2. Braids and probability. —

Definition 7.4 Definition of the notion of invariance by braids for any n-tuple $(X_1, ..., X_n)$ of random variables in a group G.

Theorem 9.1 We prove a theorem for infinite sequences of random variables in G which are invariant by braids, similar to the de Finetti-Ryll-Nardzewski's theorem: any such sequence is a mixture of sequences of independent random variables which are invariant by conjugation in law by any element of their support.

11.1.3. Mesures which are invariant by conjugation by any element of its support. —

Proposition 9.3 A generalization of Jordan's theorem is proved for any compact Lie group G : given a close subgroup H of G, the Haar measure of G can not be written as $\int_G \lambda_{g^{-1}Hg} dg$ where $\lambda_{g^{-1}Hg}$ is the Haar measure on gHg^{-1} and dg is the Haar measure on G.

Let μ be a measure on a compact Lie group G which is invariant by conjugation by any element of its support. Let $\xi = (\xi_i)_{i\in\mathbb{N}}$ be a sequence of identically and independent random variables of law μ. Let U be a Haar random variable on G which is independent from ξ.

Proposition 9.2 The support of μ is G if and only if $U\xi_1...\xi_n U^{-1}$ converges to a random Haar variable on G.

Theorem 9.5 The measure μ is invariant by conjugation by G if and only if there exists a measure ν invariant by conjugation by G such that for any integer n, the law of $U\xi_1...\xi_n U^{-1}$ is ν^{*n}. [(92)]

11.1.4. Holonomy fields. —

Definition 2.13 Definition of holonomy fields on a set of paths P as a measure on the multiplicative functions from P to a compact group G.

Proposition 2.1 A simpler proof of a theorem proved by T. Lévy in [**23**].

Definition 2.17 Definition of the notion of $\mathcal{I}$-independence.

Proposition 2.2 Any holonomy field allows to define, in a canonical way, a unique gauge-invariant holonomy field.

Proposition 2.5 Given a set P of paths which satisfy some conditions and given the set L of loops in P based at a given endpoint of P, we show that a kind of gauge-invariant random holonomy field on L can be extended in a unique way as a gauge-invariant random holonomy field on P. Thus the construction of gauge-invariant holonomy fields on paths is equivalent to the construction of gauge-invariant holonomy fields on loops.

Proposition 2.7 At the beginnig of the section 2.2.5 we explain how, given a closed subgroup $H \subset G$, one can use a gauge-invariant holonomy field which is H-valued in order to define a gauge-invariant holonomy field which is G-valued. In the proposition 2.7 we prove that any G-valued gauge-invariant holonomy field such that for any loop l the field on this loop belongs to H, one can find a *non-unique* H-valued gauge-invariant holonomy field which is in some sense the restriction of the first holonomy field.

[(92)] To explain why this theorem is not as trivial at it seems, let us remark that one can construct a measure μ on G and a measure ν invariant by conjugation by G such that the law of $U\xi_1...\xi_n U^{-1}$ is ν^{*n} but μ is not invariant by conjugation by G. Of course, in the example given in [*Proposition* 9.6] , μ is not invariant by conjugation by all the elements of its support.

11.1.5. Planar markovian holonomy fields. —

Section 4.1 We define different notions of planar Markovian holonomy fields: strong, weak, strong discrete and weak discrete.

Definition 4.8 Since it is not obvious that the restriction to graphs of planar Markovian holonomy fields allows us to define discrete planar Markovian holonomy fields, we define the notion of *constructible* planar Markovian holonomy field for which this property holds.

Proposition 4.2 We state a non trivial result[93], a generalization of Proposition 2.7, which allows us to restrict the structure group from G to a subgroup H when one considers a stochastically continous in law discrete planar Markovian holonomy field for which the random variable associated to any loop l is in H.

Theorem 5.1 We prove that any weak planar Markovian holonomy field is constructible: thus in order to characterize regular weak planar Markovian holonomy fields, it is enough to characterize regular weak discrete planar Markovian holonomy fields.

Proposition 8.4 For any Lévy process Y which is invariant by conjugation, we construct a constructible stochastically continuous strong planar Markovian holonomy field: this is called the planar Yang-Mills field associated with Y. This result is then generalized [*Theorem* 8.1] to the case where the Lévy process is only invariant by conjugation by its own support.

Proposition 8.5 We give an expression of the law of the planar Yang-Mills field obtained when the Lévy process has a density: this agrees with known expressions given by T. Lévy.

Definition 8.5 The classification of Yang-Mills fields is given: they can be either pure or mixed and either degenerate or non-degenerate.

Theorem 10.1 It states that any stochastically continuous in law weak discrete planar Markovian holonomy field is the restriction of a discrete planar Yang-Mills field.

Theorem 10.3 This is a consequence of Theorem 10.1. For a strong discrete planar Markovian holonomy field, stochastically continuity in law is equivalent to locally stochastically $\frac{1}{2}$-Hölder continuity and area-dependent continuity. For a weak discrete planar Markovian holonomy field, stochastically continuity in law is equivalent to locally stochastically $\frac{1}{2}$-Hölder continuity.

Theorem 10.4 Any stochastically continuous strong planar Markovian holonomy field is constructible.

[93] This is a consequence of a stronger result, namely the characterization of stochastically continuous in law weak discrete planar Markovian holonomy fields.

Section 11 We give three conditions in order to know if a stochastically continuous strong planar Markovian holonomy field is pure, mixed, degenerate or non-degenerate.

11.1.6. Markovian holonomy fields. —

Definition 12.12 For any Markovian holonomy field, we associate a free boundary condition expectation on $\mathbb{R}^2$.

Theorem 12.4 The free boundary condition expectation associated to a regular Markovian Holonomy field is a stochastically continuous strong planar Markovian holonomy field.

Theorem 12.3 We characterize in this theorem the spherical part of any regular Markovian holonomy field.

11.2. Chapter [Partitions]. — We give a summary of the results and new definitions of the chapter [Partitions]. For the sake of clarity, we state them only for the set of partitions $\mathcal{P}_k$, yet, one interesting thing with the approch is that one can often replace the set of partitions $\mathcal{P}_k$ by many others subset of partitions, for example the set of permutations $\mathfrak{S}_k$, the set of Brauer element $\mathcal{B}_k$, ...

11.2.1. Geometry on partitions. —

Definition 3.2 We define two graphs on the set of partitions on $2k$ elements, which generalize the Cayley graph on permutations.

Proposition 3.2 The two geodesic distances on these two graphs are equal, and we compute them. This defines a new distance on partitions which simplest form is given in [*Lemma* 3.1].

Proposition 6.2 For any partitions p and p', one can obtain a bound on $d(p, p \circ p')$ using $d(id, p')$. This bound generalizes in some sense the usual invariance by multiplication for the Cayley distance on the symmetric group.

Definition 6.2 The last proposition allows us to define a Kreweras complement for partitions.

Proposition 6.1 An improved triangular inequality is proved for any triangle of the form (p, id, p').

Proposition 8.2 We prove a relation between the default in $\mathcal{P}_{2k}$ for the triangular inequality, the default in $\mathcal{P}_k$ for the triangular inequality and the default in the improved triangular inequality. This relation is important to compute the $\mathcal{R}_{\mathcal{P}}$ transform of a product of asymptotically $\mathcal{P}$-free elements.

11.2.2. Order and partitions. —

Definition 3.4 Using the distance newly defined on partitions, we define a new order on partitions: the geodesic order.

Theorem 3.2 We characterizate the Hasse diagram of the geodesic order.

Theorem 3.3 By introducing two new orders on partitions, which are slight modifications of the coarser and finer relations, we characterize the matric order of the geodesic order as a product of two simpler matrices.

Theorem 3.4 Computation of the Möbius function for the geodesic order: this allows to write cumulants as an explicit function of the observables.

11.2.3. Coordinate numbers, observables. — We consider elements of $\mathbb{C}[\mathcal{P}_k(N)]$ and want to define several observables.

Definition 4.1 We define the coordinate numbers of any element in the algebra generated by partitions, namely $\mathbb{C}[\mathcal{P}_k(N)]$. We define observables (called $\mathcal{P}$-moments) on the algebra $\mathbb{C}[\mathcal{P}_k(N)]$ [*Definition* 4.3].

Definition 4.5 Definition of the exclusive coordinate numbers. We also define other observables (called $\mathcal{P}$-exclusive moments) on the algebra $\mathbb{C}[\mathcal{P}_k(N)]$ [*Definition* 4.6].

11.2.4. Convergence. — We consider elements of the algebra $\prod_{N\in\mathbb{N}}\mathbb{C}[\mathcal{P}_k(N)]$. An element of $\prod_{N\in\mathbb{N}}\mathbb{C}[\mathcal{P}_k(N)]$ is a sequence of elements $(E_N)_{N\in\mathbb{N}}$ such that $E_N \in \mathbb{C}[\mathcal{P}_k(N)]$. We want to define and understand some notions of convergence.

Theorem 4.1 We show that a sequence of elements $(E_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}}\mathbb{C}[\mathcal{P}_k(N)]$, converges in $\mathcal{P}$-moments if and only if the family of coordinate numbers converge. We give the relation between the limiting moments and coordinate numbers.

Theorem 4.4 We prove that the convergence of exclusive coordinate numbers is equivalent to the convergence of coordinate numbers, and we give the relation between the limiting objects.

Proposition 4.2 We prove that the convergence of exclusive moments is equivalent to the convergence of coordinate numbers, and we give the relation between the limiting objects.

Theorem 4.5 The limit of $\mathcal{P}$-exclusive moments and the limit of exclusive coordinate numbers of a sequence $(E_N)_{N\in\mathbb{N}}$ which converges in $\mathcal{P}$-moments are equal.

Theorem 4.2 If $(E_N)_{N\in\mathbb{N}}$ belongs to the subalgebra $\prod_{N\in\mathbb{N}}\mathbb{C}\left[\mathcal{B}_k(N)\right]$ (respectively to $\prod_{N\in\mathbb{N}}\mathbb{C}\left[\mathfrak{S}_k(N)\right]$), if the $\mathcal{B}$-moments (resp. the $\mathfrak{S}$-moments) of $(E_N)_{N\in\mathbb{N}}$ converge, then the $\mathcal{P}$-moments of $(E_N)_{N\in\mathbb{N}}$ converge. We show how to compute the limit of the $\mathcal{P}$-moments using the limit of the $\mathcal{B}$-moments (resp. $\mathfrak{S}$-moments).

Corollary 4.1 For any sequence $(E_N)_{N\in\mathbb{N}}$ which converges in $\mathcal{P}$-moments, for any partition p which is in $\mathcal{B}_k$, the limit of p-exclusive moments of $(E_N)_{N\in\mathbb{N}}$ is equal to the limit of p-coordinate numbers of $(E_N)_{N\in\mathbb{N}}$.

Corollary 4.2 Let us suppose that $(E_N)_{N\in\mathbb{N}}$ is in the subalgebra $\prod_{N\in\mathbb{N}}\mathbb{C}\left[\mathcal{B}_k(N)\right]$ and converges in $\mathcal{P}$-moments. We give a simple formula which allows us to compute the limit of the $\mathcal{P}$-exclusive moments of $(E_N)_{N\in\mathbb{N}}$ using the limit of its coordinate numbers.

11.2.5. Convergence and operations. —

Proposition 4.3 Any sequence $(E_N)_{N\in\mathbb{N}}$ in $\prod_{N\in\mathbb{N}}\mathbb{C}[\mathcal{P}_k(N)]$ can be seen as an element of $\prod_{N\in\mathbb{N}}\mathsf{End}\left[(\mathbb{C}^N)^{\otimes k}\right]$. For any $N \in \mathbb{N}$, we consider U_N a random Haar variable on $U(N)$. If the sequence $(E_N)_{N\in\mathbb{N}}$ converges, the sequence $\left(U_N^{\otimes k}E_N(U_N^{-1})^{\otimes k}\right)_{N\in\mathbb{N}}$ can be seen as a sequence of $\prod_{N\in\mathbb{N}}\mathbb{C}[\mathfrak{S}_k(N)]$ and is shown to converge in $\mathcal{P}$-moments. This allows to define natural and useful projections on $\mathbb{C}[\mathcal{P}_k]$.

Theorem 7.1 If two sequences $(E_N)_{N\in\mathbb{N}}$ and $(F_N)_{N\in\mathbb{N}}$ in $\prod_{N\in\mathbb{N}}\mathbb{C}[\mathcal{P}_k(N)]$ converge, their product converges. We compute the limit of the coordinate numbers and the limit of the $\mathcal{P}$-moments of $(E_NF_N)_{N\in\mathbb{N}}$.

Theorem 7.2 If $(H_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}}\mathbb{C}[\mathcal{P}_k(N)]$ converges, the semigroup generated by $(H_N)_{N\in\mathbb{N}}$ converges. We compute the limit of the coordinate numbers and the limit of the $\mathcal{P}$-moments of $(E_NF_N)_{N\in\mathbb{N}}$ of this semi-group.

11.2.6. Deformed partition algebra. —

Proposition 5.1 This proposition allows us to define a deformed algebra $\mathbb{C}[\mathcal{P}_k(N,N)]$ in which the product is a deformation of the usual product on $\mathbb{C}[\mathcal{P}_k(N)]$.

Theorem 6.1 We prove that the deformed algebra $\mathbb{C}[\mathcal{P}_k(N,N)]$ converges as N goes to infinity to an algebra denoted by $\mathbb{C}[\mathcal{P}_k(\infty,\infty)]$.

11.2.7. Algebraic fluctuations. —

Section 9 We generalize the study of converging elements in $\prod_{N\in\mathbb{N}}\mathbb{C}[\mathcal{P}_k(N)]$: definition of algebraic fluctuations of moments, of cumulants...

11.2.8. The study of the $\prod_{k=0}^{\infty}\mathbb{C}[\mathcal{P}_k(\infty,\infty)]$. — For the sake of clarity, we set, as in Section 10, $\mathfrak{E}_g[\mathcal{P}] = \prod_{k=0}^{\infty}\mathbb{C}[\mathcal{P}_k(\infty,\infty)]$.

Definition 10.2 Definition of two operations on $\mathfrak{E}_g[\mathcal{P}]$: an additive and a multiplicative convolution $\boxplus$ and $\boxtimes$.

Definition 10.4 We define the notion of multiplicative elements in $\mathfrak{E}_g[\mathcal{P}]$.

Theorem 10.1 The set of $\mathfrak{ME}[\mathcal{P}]$ of multiplicative elements in $\mathfrak{E}_g[\mathcal{P}]$ [*Definition* 10.4] is stable by the two operations $\boxplus$ and $\boxtimes$.

Theorem 10.2 We compute the "Lie algebras" of $(\mathfrak{E}_g[\mathcal{P}],\boxplus)$ and $(\mathfrak{E}_g[\mathcal{P}],\boxtimes)$: two characterizations of these Lie algebras are given in [*Theorem* 10.3] and [*Theorem* 10.4].

11.2.9. The $\mathcal{R}_{\mathcal{P}}$ *transform.* —

Definition 10.11 Definition of a combinatorial general $\mathcal{R}$-transform on $\mathfrak{E}_g[\mathcal{P}]$.

Proposition 10.3 This $\mathcal{R}$-transform is a bijection between multiplicative elements of $\mathfrak{E}_g[\mathcal{P}]$.

Theorem 10.5 The general $\mathcal{R}$-transform is a generalization of the usual $\mathcal{R}$-transform.

11.3. Chapter [Matrices]. —

11.3.1. Observables and cumulants. — Some of the definitions depend on a choice of type of partitions, that we denote A. For example if A is $\mathfrak{S}$, this means that we are considering the set of permutations.

Definition 2.1 We define the mean $\mathcal{P}$-normalized moments for a family of matrices of size N.

Definition 5.1 Definition of the mean exclusive normalized moments and definition of the mean exclusive normalized A-moments in [*Definition* 5.2].

Definition 4.3 Definition of N-dimensional A-cumulants for any family of matrices of size N.

Definition 4.1 Definition of A-cumulants for any family of matrices which converges in mean A-normalized moments.

11.3.2. Convergence, general facts. —

Definition 2.2 Definition of the convergence in A-moments when the size N of the matrices goes to infinity: when A is the set of permutations, this convergence generalizes the convergence in non-commutative distribution.

Theorem 4.1 We prove that the convergence of the A-moments is equivalent to the convergence of the N-dimensional A-cumulants.

Theorem 4.2 For any family of matrices which converges in mean A-normalized moments, the limit of the N-dimensional A-cumulants is equal to the A-cumulants.

Proposition 5.1 We prove that the convergence of the mean exclusive normalized A-moments is equivalent to the convergence in mean A-normalized moments. We give the formula in order to compute the limit of one family of observables knowing the other one.

Theorem 5.1 This gives the relation between the limit of mean exclusive A-normalized moments and A-cumulants.

11.3.3. Convergence and invariance. — We consider systems of matrices which are invariant by conjugation by some family of group $G = (G(N))_{N\in\mathbb{N}}$ of matrices such that $\mathfrak{S}(N) \subset G(N) \subset U(N)$. The list of groups are given in [Matrices].For each choice of family of groups G there exists a type A of partitions associated to it.

Theorem 4.5 If a system of matrices is invariant by G, and if it converges in mean A-normalized moments, then it converges in mean $\mathcal{P}$-normalized moments. We know how to compute the limits of the mean $\mathcal{P}$-normalized moments.

Theorem 5.4 If a system of matrices is invariant by U, and if it converges in mean $\mathfrak{S}$-normalized moments, we can compute explicitely the first order of the asymptotic behavior of any moment of the entries of the matrices of the system.

11.3.4. Equalities between observables. —

Theorem 5.2 This allows us to say that for any partition p in $\mathcal{B}_k$, the limit of the p-mean A-exclusive normalized moments is equal to the limit of the A-cumulants associated to p.

Theorem 5.3 If A is equal to $\mathfrak{S}$ or $\mathcal{B}$, we prove that there exists a simple equation between the limit of mean A-exclusive normalized moments and the limit of A-cumulants.

11.3.5. Factorization. —

Definition 2.3 We define the notion of asymptotic A-factorization property.

Theorem 2.2 The asymptotic A-factorization property for a family of random matrices whose size goes to infinity allows to prove that the observables converge in probability and not only in expectation.

Proposition 4.1 We give an equivalent condition in order to characterize the asymptotic A-factorization property.

11.3.6. $\mathcal{R}$-transform. —

Section 6 Definition of the $\mathcal{R}_A$-transform and definition of the A-law for a system of random matrices which converges in mean A-normalized moments.

Theorem 7.9 We prove that, given $(M_N)_{N\in\mathbb{N}}$ and $(L_N)_{N\in\mathbb{N}}$ two systems of matrices whose size goes to infinity, which converge in mean A-normalized moments and which are asymptotically A-free, then the $\mathcal{R}_A$-transform of $(M_N + L_N)_{N\in\mathbb{N}}$ is the sum, for the convolution, $\boxplus$, defined in [Partitions], of the $\mathcal{R}_A$-transforms of $(M_N)_{N\in\mathbb{N}}$ and $(L_N)_{N\in\mathbb{N}}$. We prove a similar result for the product $(L_N M_N)_{N\in\mathbb{N}}$.

11.3.7. A-freeness, general results. — We recall that G is some family of groups of matrices such that $\mathfrak{S}(N) \subset G(N) \subset U(N)$ (the list of the groups we consider are given in [Matrices]), and that for each choice of G there exists a type A of partitions associated to it.

Definition 7.1 We define the notion of asymptotic A-freeness.

Theorem 7.7 This theorem allows us to characterize asymptotic A-freeness in term of mean exlusive A-normalized moments.

Theorem 7.2 If we consider two systems of matrices which are $\mathcal{P}$-asymptotically free, and if the union of the two systems is invariant by conjugation by G then the two systems are A-asymptotically free.

Definition 7.2 We define the notion of asymptotic G invariance and we caracterize it by considering the A-laws. ([Lemme 7.2])

Theorem 7.3 This theorem generalizes the property of "freeness rigidity" of C. Male. If two systems of matrices are asymptotically $\mathcal{P}$-free and one is asymptotically G invariant then the two systems of matrices are asymptotically A-free.

Theorem 7.4 We give a new proof of a result of Mingo and Popa.

Section 7.1.3 We explain the links between the notions of freeness, $\mathfrak{S}$-freeness, $\mathfrak{B}$-freeness and $\mathcal{P}$-freeness.

Theorem 7.5 Let $B(N)$ be the bistochastic group of size N, the type of partitions associated to this group is the type $\mathcal{B}s$ of partitions which have blocks of size less or equal to two. We prove that $\mathcal{B}s$-freeness implies $\mathfrak{S}$-freeness. This implies that two independent systems of matrices which are invariant by conjugation by the bistochastic group, which converge in mean $\mathcal{B}s$-normalized moments and which satisfy the $\mathfrak{S}$-factorization property, are asymptotically free in the sense of Voiculescu.

11.3.8. A-freeness and computations. —

Theorem 7.8 Given two systems of matrices which are asymptotically $\mathcal{P}$-free, we compute the $\mathcal{P}$ law of sums and products of elements of these two systems. The computation of the $\mathcal{P}$-law for the product uses the new definition of Kreweras complements for partitions in $\mathcal{P}_k$.

11.3.9. A-freeness and invariance by conjugation. — We recall again that G is some family of groups of matrices such that $\mathfrak{S}(N) \subset G(N) \subset U(N)$ (the list of the groups we consider are given in [Matrices]), and that for each choice of G there exists a type A of partitions associated to it.

Theorem 7.6 Two independent systems of matrices which are invariant by conjugation by the bistochastic group, which converge in mean $\mathcal{B}s$-moments and which satisfy the $\mathfrak{S}$-factorization property, are asymptotically free in the sense of Voiculescu.

Theorem 7.10 If one considers two independent systems of matrices which converge in mean A-normalized moments, if one is invariant by conjugation by G, then the two systems of matrices are asymptotically $\mathcal{P}$-free .

Theorem 7.11 We prove that one can replace in the last theorem, the condition of G-invariance by the condition of asymptotically strong G-invariance defined in Definition 7.3.

11.3.10. Lévy processes, general theorem. —

Definition 7.4 We define the notion of A-free matricial Lévy processes.

We give general theorems in order to prove the convergence in mean $\mathcal{P}$-normalized moments of a family of Lévy processes $\left((X_t^N)_{t\geq 0}\right)_{N\in\mathbb{N}}$ which are invariant by conjugation by the symmetric groups.

Theorem 10.1 If the family of generators of the Lévy processes converges in $\mathcal{P}$-normalized moments or in coordinate numbers, then the family of Lévy processes converges in mean $\mathcal{P}$-normalized moments.

Theorem 10.2 We give differential equation which is satisfied by the $\mathcal{R}_{\mathcal{P}}$ transform of the family of Lévy processes.

Theorem 10.3 This theorem asserts that the family of Lévy processes satisfies the asymptotic factorization property if and only if the family of generators satisfies a property called asymptotic (weak or strong) condensation. convergence in mean $\mathcal{P}$-normalized

Section 10.2.2 We generalize the three last results for the family $\left((X_t^N)_{t\geq 0}, ((X_t^N)^*)_{t\geq 0}\right)_{N\in\mathbb{N}}$. This allows to have a criterion in order to prove that the convergence obtained holds in probability.

11.3.11. Lévy processes, first studies. — We give new proofs of some known results about convergence in non-commutative distribution of some matricial Brownian motions.

Theorem 10.11 New proof of Wigner's theorem.

Theorem 10.13 We prove a matricial Wick's theorem.

Theorem 10.14 We give a new proof of the convergence in non-commutative distribution of unitary and orthogonal Brownian motions.

We also give new proofs of some known results about convergence in non-commutative distribution of some matricial unitary Lévy processes, and generalize the results for matricial orthogonal Lévy processes.

Theorem 10.24 We give a short proof of a theorem about convergence in non-commutative distribution of Hermitian Lévy processes.

Theorem 10.25 We generalize Theorem 10.24 to symmetric Lévy processes.

Theorem 10.26 We give a new proof of a theorem of G.Cebron about convergence in non-commutative distribution of unitary Lévy processes. The Theorem 10.27 shows that the convergence holds in probability.

Theorem 10.28 We generalize Theorem 10.26 to orthogonal Lévy processes.

11.3.12. Algebraic fluctuation. —

Definition 2.4 We define the algebraic fluctuations of observables, and also, in [*Definition* 4.2], we define the higher order A-cumulants.

Definition 4.5 We define the algebraic fluctuations of the finite dimensional A-cumulants.

Theorem 4.3 We prove that the existence of algebraic fluctuations for the observables is equivalent to the existence of algebraic fluctuations for the finite dimensional A-cumulants. In this case, the algebraic fluctuations for the finite dimensional A-cumulants are equal to the higher A-order cumulants.

Proposition 4.6 If a system of matrices is invariant by G, and if there exist algebraic fluctuations for its mean A-normalized moments, then there exist algebraic fluctuations for the mean $\mathcal{P}$-normalized moments. Again, we know how to compute the algebraic fluctuations for the mean $\mathcal{P}$-normalized moments.

Definition 7.6 We define a notion of high order A-freeness: the mixed higher order A-cumulants must vanish.

Theorem 7.13 We generalize Theorem 7.8: we compute the algebraic fluctuations of product and sum of random matrices which are asymptotically A-free up to a given order of fluctuations.

Section 10.2.3 We generalize Section 10.2.1, and we give theorems about the existence of algebraic fluctuations for family of Lévy processes.

11.3.13. A-non-commutative central limit theorem. —

Theorem 8.1 We prove a $\mathcal{P}$-non-commutative central limit theorem.

11.3.14. Probabilities. —

Theorem 9.2 We prove that classical cumulants can be seen as $\mathcal{P}$-cumulants.

11.4. Chapter [Revêtements]. —

11.4.1. Random walks and eigenvalues distributions. —

Definition 2.1 Definition of a good normalization for the generator of random walks on the symmetric group driven by any conjugacy class of the symmetric group.

Definition 2.2 Definition of the notions of convergence, evanescence and macroscopic behavior for a sequence of conjugacy classes.

Theorem 2.1 Conververgence of the mean eigenvalues distribution of random walks on the symmetric group at any positive time t and characterization of the limit via a system of differential equations.

Theorem 2.3 Convergence in law of the eigenvalues distribution of random walks on the symmetric group if the sequence of conjugacy class converges. If this sequence is evanescent, the limiting measure is non-random and the observables converge in probability; if it is macroscopic, the limiting measure is random.

Theorem 2.4 Computation of the limit of the eigenvalues distribution of random walks on the symmetric group when the converging sequence of conjugacy class is evanescent.

Theorem 2.5 When the sequence of conjugacy class is evanescent we prove the existence of a phase transition which appears at a time t_c that we compute explicitely: before this time the limiting measure of the eigenvalues distribution is purely atomic, after this time it is a sum of a purely atomic measure and a multiple of the Lebesgue measure.

Corollary 2.1 We get a new calculation, in a slightly more general setting, of the limit of the renormalized distance of the random walk to the identity.

11.4.2. Random walks seen as processes. —

Theorem 2.2 Convergence of the random walks on the symmetric group to a $\mathcal{P}$-free multiplicative Lévy process when the sequence of conjugacy classes converges. This Lévy process is shown not to be a free Lévy process in the sense of Voiculescu when the sequence of conjugacy classes is evanescent. Convergence at any order of algebraic fluctuations.

Theorem 2.6 Generalization of the previous convergence for a family of independent random walks on the the symmetric group.

Definition 2.4 Definition of the log-cumulants for $\mathcal{P}$-free multiplicative Lévy processes.

Theorem 2.7 Computation of the log-cumulants of the $\mathcal{P}$-free multiplicative Lévy process which is the limit of random walks on the symmetric group.

11.4.3. Large N $\mathfrak{S}(N)$-Yang-Mills measure and ramified coverings of the disk. —

Theorem 3.2 Convergence in probability of the Wilson loop observables of $\mathfrak{S}(N)$-Yang-Mills on the plane to a continuous function and asymptotic factorization of the Wilson loop observables.

Theorem 4.2 Convergence in probability of the monodromy process of a model of random ramified coverings of the disk with N sheets as N goes to infinity.

Bibliographie

[1] A. Dahlqvist. Integration formula for Brownian motion on classical compact Lie groups. *http://arxiv.org/abs/1212.5107*.

[2] S. Albeverio, R. Høegh-Krohn, and H. Holden. Random fields with values in Lie groups and Higgs fields. *Lecture notes in Physics, Editors: S. Albeverio, G. Casati, D. Merlini*, 262:1–13, 1986.

[3] S. Albeverio, R. Høegh-Krohn, and H. Holden. Stochastic Lie group-valued measures and their relations to stochastic curve integrals, gauge fields and Markov cosurfaces. *Stochastic processes—mathematics and physics, Editors: S. Albeverio, Ph. Blanchard, L. Streit*, 1986.

[4] S. Albeverio, R. Høegh-Krohn, and H. Holden. Stochastic multiplicative measures, generalized Markov semigroups, and group-valued stochastic processes and fields. *J. Funct. Anal.*, 78(1):154–184, 1988.

[5] B. Collins. Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability. *Int. Math. Res. Not.*, 17:953–982, 2003.

[6] B. Collins and P. Śniady. Integration with respect to the Haar measure on unitary, orthogonal and symplectic group. *Communications in Mathematical Physics*, 264(3):773–795, 2004.

[7] J.C. Baez. Generalized measures in gauge theory. *Letters in Mathematical Physics*, 31(3):213–223, 1994.

[8] P. Biane. Free Brownian motion, free stochastic calculus and random matrices. *Free probability theory (Waterloo, ON, 1995)*, 12 of Field Inst. Commun.:1–19, 1997.

[9] D. V. Voiculescu and K. J. Dykema and A Nica. Free Random Variables. *CRM Monograph Series X, American Mathematical Society*, 1:114, 1992.

[10] D. Voiculescu. Limit laws for random matrices and free products. *Invent. Math.*, 104.1:201–220, 1991.

[11] B.K. Driver. YM_2 : continuum expectations, lattice convergence, and lassos. *Comm. Math. Phys.*, 123(4):575–616, 1989.

[12] B.K. Driver. Two-dimensional Euclidean quantized Yang-Mills fields. In *Probability models in mathematical physics*, pages 21–36. World Sci. Publ., 1991.

[13] N. Orantin G. Borot, B. Eynard. Abstract loop equations, topological recursion, and applications. *http://arxiv.org/abs/1303.5808*, 2013.

[14] G. Cébron and A. Dahlqvist and F. Gabriel. The general Lévy Master Field. *http://arxiv.org/abs/1601.00214*.

[15] G. Cébron and A. Dahlqvist and F. Gabriel and C. Male. Traffic-partition correspondence. *Work in progress*.

[16] L. Gross. A Poincaré lemma for connection forms. *J. Funct. Anal.*, 63(1):1–46, 1985.

[17] L. Gross. The Maxwell equations for Yang-Mills theory. In *CMS Conf. Proc., Editors: J.S. Feldman and L.M. Rosen*, volume 9, pages 193–203. Amer. Math. Soc., 1988.

[18] Alice Guionnet and Edouard Maurel-Segala. Combinatorial aspects of matrix models. *ALEA LAT. AM. J. PROBAB. MATH. STAT*, page 2249657, 2005.

[19] T. Halverson and A. Ram. Partition algebras. *European Journal of Combinatorics*, 26:869–921, 2005.

[20] T. Lévy. The master field on the plane. *arxiv.org/abs/1112.2452*.

[21] T. Lévy. *Yang-Mills Measure on compact surfaces*. Number 790. American Mathematical Society, 2003.

[22] T. Lévy. Schur-Weyl duality and the heat kernel measure on the unitary group. *Advances in Mathematics*, 218, no. 2:537–575, 2008.

[23] T. Lévy. *Two-dimensional Markovian holonomy fields*. Number 329. Astérisque, 2010.

[24] C. Male. The distributions of traffics and their free product. *arxiv.org/abs/1111.4662*, 2011.

[25] J.A. Mingo and M. Popa. Freeness and the transposes of unitarily invariant random matrices. *J. of Func. Anal.*, 271, no. 4:883–921, 2016.

[26] N. Berestycki. Emergence of giant cycles and slowdown transition in random transpositions and k-cycles. *Electr. J. Probab.*, 16:152–173, 2011.

[27] N. Berestycki and R. Durrett. Phase transition in the random transposition random walk. *Probab. Theor. Rel. Fields*, 136:203–233, 2006.

[28] A. Nica and R. Speicher. *Lectures on the Combinatorics of Free Probability*. Number 335. Lecture Note Series, London Mathematical Society, Cambridge University Press, 2006.

[29] A.N. Sengupta. The Yang-Mills measure for S^2. *J. Funct. Anal.*, 108(2):231–273, 1992.

[30] A.N. Sengupta. Gauge theory on compact surfaces. *Mem. Amer. Math. Soc.*, 126(600):viii+85, 1997.

[31] T. Lévy. Wilson loops in the light of spin networks. *J. Geom. Phys.*, 52 no. 4:382–397, 2004.

[32] A. Vogt. The isoperimetric inequality for curves with self-intersections. *Canad. Math. Bull.*, 24:161–167, 1981.

[33] E. Witten. On quantum gauge theories in two dimensions. *Comm. Math. Phys.*, 1(1991):153–209, 141.

[34] E. Witten. Two-dimensional gauge theories revisited. *J. Geom. Phys.*, 9(4):303–368, 1992.

[35] F. Xu. A random matrix model from two-dimensional Yang-Mills theory. *Comm. Math. Phys.*, 190, no. 2:287–307, 1997.

[36] C.N. Yang and R.L. Mills. Conservation of isotopic spin and isotopic gauge invariance. *Physical Rev.*, 96(2):191–195, 1954.

“Ne t’inquiète pas, tu vas rentrer chez toi, tu vas t’asseoir et tu auras la solution.”

T. Lévy

CHAMPS MARKOVIENS D'HOLONOMIES PLANAIRES

Abstract. — We study planar random holonomy fields which are processes indexed by paths on the plane which behave well under the concatenation and orientation-reversing operations on paths. We define the planar Markovian holonomy fields as planar random holonomy fields which satisfy some independence and invariance by area-preserving homeomorphisms properties.

We use the theory of braids in the framework of classical probabilities: for finite and infinite random sequences the notion of invariance by braids is defined and we prove a new version of the de-Finetti's theorem.

This allows us to construct a family of planar Markovian holonomy fields, the Yang-Mills fields, and we prove that any regular planar Markovian holonomy field is a planar Yang-Mills field. This family of planar Yang-Mills fields can be partitioned into three categories according to the degree of symmetry: we study some equivalent conditions in order to classify them.

Finally, we recall the notion of Markovian holonomy fields and construct a bridge between the planar and non-planar theories. Using the results previously proved in the article, we compute the spherical part of any regular Markovian holonomy field.

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1. Introduction

Yang-Mills theory is a theory of random connections on a principal bundle, the law of which satisfies some local symmetry: the gauge symmetry. It was introduced in the work of Yang and Mills, in 1954, in [**30**]. Since then, mathematicians have tried to formulate a proper quantum Yang-Mills theory. The construction on a four dimensional manifold for any compact Lie group is still a challenge: we will focus in this article on the 2-dimensional quantum Yang-Mills theory. On a formal level, a Yang-Mills measure is a measure on the space of connections which looks like:

$$e^{-\frac{1}{2}S_{\mathsf{YM}}(A)}DA,$$

where $S_{\mathsf{YM}}(A)$ is the Yang-Mills action of the connection A, which is the L^2 norm of the curvature, and DA is a translation invariant measure on the space of connections. Yet, many problems arise with this formulation, the main of which is that the space of connections can not be endowed with a translation invariant measure. It took some time to understand which space could be endowed by a well-defined measure.

One possibility to handle this difficulty in a probabilistic way is to consider holonomies of the random connections along some finite set of paths: thus, after the works of Gross [**14**], [**15**], Driver [**12**], [**13**] and Sengupta [**23**], [**24**] who constructed the Yang-Mills field for a small class of paths but for any surface, it was well understood that the Yang-Mills measure was a process indexed by some nice paths. Their construction uses the fact that the holonomy process under the Yang-Mills measure should satisfy a stochastic differential equation driven by a Brownian white-noise curvature. The Yang-Mills measure has to be constructed on the multiplicative functions from the set of paths to a Lie group, that is the set of functions which have a good behavior under

concatenation and orientation-inversion of paths. This idea was already present in the precursory work of Albeverio, Høegh-Krohn and Holden ([**1**], [**3**], [**4**], [**2**]).

In [**19**], [**20**] and [**21**], Lévy gave a new construction. This construction allowed him to consider any compact Lie groups, any surfaces and any rectifiable paths. Besides, it allowed him to generalize the definition of Yang-Mills measure to the setting where, in some sense, the curvature of the random connection is a conditioned Lévy noise. The idea was to establish the rigorous discrete construction, as proposed by E. Witten in [**28**] and [**29**] and to show that one could take a continuous limit.

The discrete construction was defined by considering a perturbation of a uniform measure, the Ashtekar-Lewandowski measure, by a density. The continuous limit was established using the general Theorem 3.3.1 in [**21**]. This theorem must be understood as a two-dimensional Kolmogorov's continuity theorem and one should consider it as one of the most important theorem in the theory of two-dimensional holonomy fields. In the article [**10**] in preparation, G. Cébron, A. Dahlqvist and the author show how to use this theorem in order to construct generalizations of the master field constructed in [**5**] and [**18**].

In the seminal book [**21**], Lévy defined also Markovian holonomy fields. This is the axiomatic point of view on Yang-Mills measures, seen as families of measures, indexed by surfaces which have a good behavior under chirurgical operations on surfaces and are invariant under area-preserving homeomorphisms. The importance of this notion is that Yang-Mills measures are Markovian holonomy fields. It is still unknown if any regular Markovian holonomy field is a Yang-Mills measure. This paper is a first step in order to prove so.

The axiomatic formulation of the Markovian holonomy fields allows us to understand Lévy processes as one-dimensional planar Markovian holonomy fields.

1.1. Lévy processes and planar Markovian holonomy fields. — Let G be a compact Lie group. If $\mathsf{dim}(G) \geq 1$, we endow the group G with a bi-invariant Riemannian metric d_G. If G is a finite group, we endow it with the distance $d_G(x,y) = \delta_{x,y}$. There exist two notions of Lévy processes depending on the definitions of the increments: left increments $Y_t Y_s^{-1}$ or right increments $Y_s^{-1} Y_t$. We will fix the following convention: in this article, a Lévy process on G is a càdlàg process with independent and stationary right increments which begins at the neutral element. In fact one can use a weaker definition and forgot about the càdlàg property and define a Lévy process as a continuous in probability family of random variables $(Y_t)_{t\in\mathbb{R}^+}$ such that for any $t > s \geq 0$:

- $Y_s^{-1} Y_t$ has same law as Y_{t-s},
- $Y_s^{-1} Y_t$ is independent of $\sigma(Y_u, u < s)$,
- $Y_0 = e$ a.s.

Let Y be a Lévy process on G. Let us denote by $\mathcal{D}(\mathbb{R})$ the set of integrable smooth densities on $\mathbb{R}$. For any $\mathsf{vol} \in \mathcal{D}(\mathbb{R})$, one can define a measure $\mathbb{E}_{vol}$ on $G^{\mathbb{R}}$ such that, under $\mathbb{E}_{vol}$, the canonical projection process $(X_t)_{t\in\mathbb{R}}$ has the law of $\left(Y_{vol(]-\infty,t])}\right)_{t\in\mathbb{R}}$. The family $\left(\mathbb{E}_{vol}\right)_{vol}$ satisfies three properties:

-Area-preserving increasing homeomorphism invariance : Let ψ be an increasing homeomorphism of $\mathbb{R}$. Let vol and vol' be two smooth densities in $\mathcal{D}(\mathbb{R})$. Let us suppose that ψ sends vol on vol'. The mapping ψ induces a measurable

mapping from $G^{\mathbb{R}}$ to itself which we will denote also by ψ and which is defined by:

$$\psi((x_t)_{t\in\mathbb{R}}) = (x_{\psi(t)})_{t\in\mathbb{R}}.$$

It is then easy to see that $\mathbb{E}_{vol} = \mathbb{E}_{vol'} \circ \psi^{-1}$. For example, for any real $t \in \mathbb{R}$ and any bounded function f on G:

$$\mathbb{E}_{vol'}\big[f(X_{\psi(t)})\big] = \mathbb{E}\big[f(Y_{vol'(]-\infty,\psi(t)])})\big] = \mathbb{E}\big[f(Y_{vol(]-\infty,t])})\big] = \mathbb{E}_{vol}\big[f(X_t)\big].$$

-Independence : Let vol be a smooth density in $\mathcal{D}(\mathbb{R})$. Let $[s_0, t_0]$ and $[s_1, t_1]$ be two disjoint intervals. Under $\mathbb{E}_{vol}$, $\sigma\big((X_s^{-1}X_t), s_0 \leq s < t \leq t_0\big)$ is independent of $\sigma\big((X_s^{-1}X_t), s_1 \leq s < t \leq t_1\big)$.

-Locality property : Let vol and vol' be two smooth densities in $\mathcal{D}(\mathbb{R})$. Let t_0 be a real such that $vol_{]]-\infty,t_0]} = vol'_{]]-\infty,t_0]}$. The law of $(X_t)_{t\leq t_0}$ is the same under $\mathbb{E}_{vol}$ as under $\mathbb{E}_{vol'}$.

Let us consider a family of measures $(\mathbb{E}_{vol})_{vol\in\mathcal{D}(\mathbb{R})}$ on $G^{\mathbb{R}}$; we say that it is stochastically continuous if, for any $vol \in \mathcal{D}(\mathbb{R})$, for any sequence $(t_n)_{n\in\mathbb{N}}$, if t_n converges to $t \in \mathbb{R} \cup \{-\infty\}$, $\mathbb{E}_{vol}\left(d_G(X_{t_n}, X_t)\right) \underset{n\to\infty}{\longrightarrow} 0$, where we recall that $(X_t)_{t\in\mathbb{R}}$ is the canonical projection process and where we set X_∞ the constant function which is equal to the neutral element e of G. If $(\mathbb{E}_{vol})_{vol\in\mathcal{D}(\mathbb{R})}$ is stochastically continuous and satisfies the three axioms stated above then there exists a Lévy process $(Y_t)_{t\in\mathbb{R}^+}$ such that, for any smooth density vol in $\mathcal{D}(\mathbb{R})$, the canonical projection process $(X_t)_{t\in\mathbb{R}}$ has the law of $\big(Y_{vol(]-\infty,t])}\big)_{t\in\mathbb{R}}$.

With these axioms in mind, looking at the definitions in Section 4.1 of planar Markovian holonomy fields, the reader can now understand why we can consider Lévy processes as one dimensional planar Markovian holonomy fields. The surprising fact that we will prove in this paper is that the family of regular two-dimensional planar Markovian holonomy fields is not bigger than the set of gauge invariant one-dimensional planar Markovian holonomy fields.

1.2. Braids. — The most innovative idea of this paper is to introduce for the very first time the braid group in the study of Yang-Mills theory. This is also one of the main ingredient in the upcoming article [**10**].

The braid group is an object which possesses different facets: a combinatorial, a geometric and an algebraic one. One can introduce the braid group using geometric braids: this construction allows us to have a graphical and combinatorial framework to work with. Since it is the most intuitive construction, we quickly present it so that the reader will be familiar with these objects.

Proposition 1.1. — *For any $n \geq 2$, let the configuration space $C_n(\mathbb{R}^2)$ of n indistinguishable points in the plane be $\big((\mathbb{R}^2)^n \setminus \Delta\big)/\mathfrak{S}_n$ where Δ is the union of the hyperplanes $\{\mathbf{x} \in (\mathbb{R}^2)^n,\ \mathbf{x}_i = \mathbf{x}_j\}$. The fundamental group of the configuration space $C_n(\mathbb{R}^2)$ is the braid group with n strands $\mathcal{B}_n$:*

$$\mathcal{B}_n = \pi_1\big(C_n(\mathbb{R}^2)\big).$$

Every continuous loop γ in $C_n(\mathbb{R}^2)$ parametrized by $[0,1]$ and based at $\big((1,0),...,(n,0)\big)$ can be seen as n continuous functions $\gamma_j \in \mathcal{C}\big([0,1],\mathbb{R}^2\big)$ such that, if we set $\sigma : j \mapsto \gamma_j(1)$

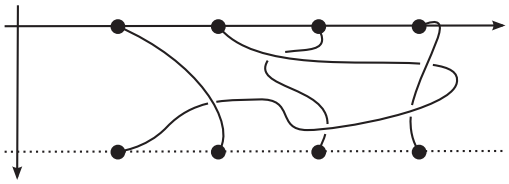


FIGURE 1. A physical braid β.

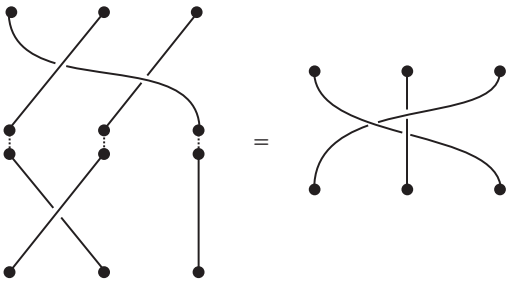


FIGURE 2. The multiplication of two braids.

for any $j \in \{1, \dots, n\}$, the following conditions hold:

$$\begin{aligned}
&1\text{-}\ \forall j \in \{1, \dots, n\}, \gamma_j(0) = (j, 0),\\
&2\text{-}\ \sigma \in \mathfrak{S}_n,\\
&3\text{-}\ \forall\ t \in [0, 1], \forall j \neq j', \gamma_j(t) \neq \gamma_{j'}(t).
\end{aligned}$$

The function γ_j is given by the image of γ by the natural projection $\pi_j : \left(\mathbb{R}^2\right)^n \to \mathbb{R}^2$. We call γ a geometric braid since if we draw the $(\gamma_j)_{j=1}^n$ in $\mathbb{R}^3$, we obtain a physical braid. One can look at Figure 1 to have an illustration of this fact.

With this point of view, the composition of two braids is just obtained by gluing two geometric braids, taking then the equivalence class by isotopy of the new braid as shown in Figure 2. In this paper, we will take the convention that, in order to compute $\beta_1\beta_2$, one has to put the braid β_2 above the braid β_1.

As we see in Figure 3, one can represent a braid by a two dimensional diagram (or, to be correct, classes of equivalence of two-dimensional diagrams) that we call n-diagrams. This representation can remind the reader the representation of any permutation by a diagram, yet, in this representation of braids, one remembers which string is above an other at each crossing. It is a well-known result that any n-diagram represents a unique braid with n-strands. Thus, in order to construct a braid, we only have to construct a n-diagram. Besides, every computation can be done with the n-diagrams.

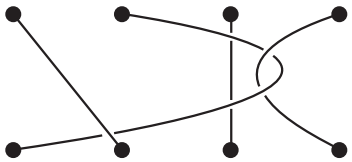


FIGURE 3. A diagram associated with β.

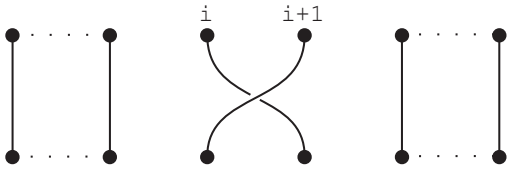


FIGURE 4. The elementary braid β_i.

For any $i \in \{1, \dots, n-1\}$, let β_i be the equivalence class of $(\gamma_j^i)_{j=1}^n$ defined by:

$$\begin{aligned}
&\forall k \in \{1, \dots, n\} \setminus \{i, i+1\},\ \forall t \in [0,1],\ \gamma_k^i(t) = (k, 0),\\
&\forall t \in [0,1],\ \gamma_i^i(t) \quad = \left(i + \frac{1}{2}\right) - \frac{1}{2}e^{i\pi t},\\
&\forall t \in [0,1],\ \gamma_{i+1}^i(t) = \left(i + \frac{1}{2}\right) + \frac{1}{2}e^{i\pi t},
\end{aligned}$$

with the usual convention $\mathbb{R}^2 \simeq \mathbb{C}$. As any braid can be obtained by braiding two adjacent strands, we get that $(\beta_i)_{i=1}^{n-1}$ generates $\mathcal{B}_n$.

1.3. Layout of the article. — In the present paper, as the theory of Markovian holonomy fields is a newborn theory which mixes geometry, representation, probabilities, we tried to recall all the tools we need and make it accessible to any people from any domain of mathematics. This paper must be regarded in the same time as an introduction to [**21**] and as a sequel to the same book. The reader shouldn't be surprised that we copy some of the definitions of [**21**] as any reformulation wouldn't have been as good as Lévy's formulation.

In Section 2, we recall the classical notions: paths, multiplicative functions, ... Besides, we supply a lack in [**21**]: we decided to develop the notion of random holonomy fields, as it might be possible, in the future, that some general random holonomy fields of interest would not be Markovian holonomy fields. Thus, any proposition in [**21**] that could be applied to random holonomy fields is stated in this setting. We also show how to project a random holonomy field on the set of gauge-invariant random holonomy fields and we answer to the questions of restriction and extension of the structure group in the gauge-invariant setting. At last, we develop the loop paradigm which, in particular, implies the new Proposition 2.6.

The Section 3 is devoted to the theory of planar graphs and the notion of $\mathbb{G} - \mathbb{G}'$ piecewise diffeomorphisms. One of the main result is Corollary 3.1 which states that one can send, by a $\mathbb{G} - \mathbb{G}'$ diffeomorphism, any generic planar graph in the $\mathbb{N}^2$-graph.

Using the previous sections, we can define in Section 4, four different notions of planar Markovian holonomy fields. Under some regularity condition, it will be proved in the paper that the four notions are essentially equivalent. These objects are processes, indexed by paths drawn on the plane, which are gauge-invariant, invariant under area-preserving homeomorphisms, which satisfy a weak independence property and a locality property. In the remaining of the section, we show how to add a drift to a planar Markovian holonomy field by using the index field. We also consider the questions of restriction and extension of the structure group for planar Markovian holonomy fields.

The equivalence between the notions of weak discrete and weak continuous planar Markovian holonomy field is then proved in Section 5, using a theorem of Moser and Dacorogna proved in [**11**].

The loop paradigm explained in Section 2 would not be powerful if the group of reduced loops was not introduced in Section 6. In this section, we define the group of reduced group as Lévy did in [**21**]. Then, we obtain a generalization of Lévy's work in the planar case: this allows us to exhibit general families of loops which generate the group of reduced loops of any planar graph. The proof differs greatly from the proof in the original Lévy's work as we use a splitting/recurrence argument.

Two sections are devoted to the link between braids and probabilities: Section 7 and Section 9. If one is reading the article only for the interactions between the braid group and probabilities, one should go directly to these sections. In the first one, we explain an algebraic definition of the braid group which allows us to recall the reader some natural actions of $\mathcal{B}_n$ on the free group of rank n and on G^n. We show how the Artin's theorem can be applied on the group of reduced loops and we define the notion of invariance by braid for finite sequences of random variables. Section 9, is devoted to the geometric point of view on braids and to a de-Finetti-Ryll-Nardzewski's theorem for random infinite sequences which are braid-invariant. Then we show that, under an assumption of independence of the diagonal-conjugacy classes, one can characterize the braidable sequences which are sequences of i.i.d. random variables. The end of the section consists in an application of these results to processes.

In Sections 8, 10 and 11 the main results in the theory of planar Markovian holonomy fields are proved. The results of Section 7 on finite braid-invariant sequences of random variables allow us in Section 8 to construct, for any Lévy process which is self-invariant by conjugation, a planar Yang-Mills field associated with it. This construction differs from all the previous constructions since it uses neither the notion of uniform or Ashtekar-Lewandowski measure nor the notion of stochastic differential equations. This allows us to consider any self-invariant by conjugation Lévy processes, where before, one had to consider Lévy processes with density with respect to the Haar measure and which were invariant by conjugation by the structure group G. In Section 10 and 11, using the results of Section 9, we prove that any regular planar Markovian holonomy field is a planar Yang-Mills field. Besides, we show that one can characterize them according to the law of simple loops.

Since we proved that any regular planar Markovian holonomy field is a planar Yang-Mills field, is it possible to show that any Markovian holonomy field is a Yang-Mills field? In Section 12, we answer partly to this question. First we recall the notion of Markovian holonomy fields. We construct the free boundary condition expectation which is a bridge

between Markovian holonomy fields and planar Markovian holonomy fields. Using the results shown previously, we are able to prove our last main result, namely Theorem 12.3: the spherical part of a Markovian holonomy field is equal to the spherical part of a Yang-Mills field.

In order to get a more accurate idea of the results shown in this article and the different notions defined in it, one can refer to the diagram page 227.

1.4. Notations. — Throughout this paper, M is either a smooth compact surface, possibly with boundary, or the plane $\mathbb{R}^2$. We denote by G a compact Lie group, with the usual convention that a compact Lie group of dimension 0 is a finite group. The Lie algebra of G is denoted by $\mathfrak{g}$. The neutral element will be denoted either by 1 or e. We endow G with a bi-invariant distance d_G. We denote by $\mathcal{M}(G)$ the space of finite Borel positive measures on G.

For each $n \geq 1$, the group G acts by diagonal conjugation on G^n:

$$g.(g_1, \dots, g_n) = (g^{-1}g_1g, \dots, g^{-1}g_ng),$$

for any $g \in G$ and any n-tuple $(g_1, \dots, g_n)$ of elements of G. We denote by $\big[(g_1, \dots, g_n)\big]$ the equivalence class of $(g_1, \dots, g_n)$ in G^n under the diagonal conjugation action.

Each time we will have to use a constant function which is equal to 1, we will denote it by $\mathbb{1}$. Besides, if μ is a finite measure on a measurable space $(\Omega, \mathcal{A})$ and if $\mathcal{B} \subset \mathcal{A}$ is a sub-σ-field, by $\mu_{|\mathcal{B}}$, we denote the image of μ by the identity map: $(\Omega, \mathcal{A}) \to (\Omega, \mathcal{B})$.

2. Backgrounds: paths, random multiplicative functions on paths

2.1. Paths. — Let M be either a smooth compact surface (possibly with boundary) or the plane $\mathbb{R}^2$. A *measure of area* on M is a smooth non-vanishing density on M, that is, a Borel measure which has a smooth positive density with respect to the Lebesgue measure in any coordinate chart. It will often be denoted by *vol*. We call (M, vol) a *measured surface*. We endow M with a Riemannian metric γ and we will denote by γ_0 the standard Riemannian metric on $\mathbb{R}^2$.

Definition 2.1. — *A parametrized path on M is a continuous curve $c : [0, 1] \to M$ which is either constant or Lipschitz continuous with speed bounded below by a positive constant.*

Two parametrized paths can give the same drawing on M but with different speed and we will only consider equivalence classes of paths.

Definition 2.2. — *Two parametrized paths on M are equivalent if they differ by an increasing bi-Lipschitz homeomorphism of $[0, 1]$. An equivalence class of parametrized paths is called a path and the set of paths on M is denoted by $P(M)$.*

Actually, the notion of path does not depend on γ since the distances defined by two different Riemannian metric are equivalent. Two parametrized paths pp_1 and pp_2 which represent the same path p share the same endpoints. It is thus possible to define the endpoints of p as the endpoints of any representative of p. If p is a path, by $\underline{p}$ (resp. $\overline{p}$)

we denote the starting point (resp. the arrival point) of p. From now on, we will not make any difference between a path p and any parametrized path $pp \in p$.

Definition 2.3. — *A path is simple either if it is injective on* $[0,1]$ *or if it is injective on* $[0,1[$ *and* $\underline{p} = \overline{p}$.

Later, we will need the following subset of paths.

Definition 2.4. — *We define* $\mathsf{Aff}_\gamma(M)$ *to be the set of paths on* M *which are piecewise geodesic paths with respect to* γ.

The set of paths $\mathsf{Aff}_{\gamma_0}(\mathbb{R}^2)$ will be simply denoted by $\mathsf{Aff}(\mathbb{R}^2)$. An other set of paths will be very important for our study: the set of loops.

Definition 2.5. — *A loop* l *is a path such that* $\underline{l} = \overline{l}$. *A smooth loop is a loop whose image is an oriented smooth* 1*-dimensional submanifold of* M. *The set of loops is denoted by* $L(M)$. *Let* m *be a point of* M. *A loop* l *is based at* m *if* $\underline{l} = m$. *The set of loops based at* m *is denoted by* $L_m(M)$.

Let p be a path, let pp_1 be a representative of p. The inverse of p, denoted by p^{-1}, is the equivalence class of the parametrized path $t \mapsto pp_1(1-t)$. Concatenation is also defined for paths. Let p_1 and p_2 be two paths such that $\overline{p_1} = \underline{p_2}$, let pp_1 (resp. pp_2) be a representative of p_1 (resp. p_2). The concatenation of p_1 and p_2 denoted by p_1p_2 is the equivalence class of the parametrized path:

$$pp_1.pp_2 : t \mapsto \begin{cases} pp_1(2t) & \text{if } t \leq 1/2, \\ pp_2(2t-1) & \text{if } t > 1/2. \end{cases}$$

The notion of concatenations allows us to set the following definitions.

Definition 2.6. — *A set of paths* P *is connected if any couple of endpoints of elements of* P *can be joined by a concatenation of elements of* P.

Definition 2.7. — *Let* P *be a set of paths. Two loops* l *and* l' *are elementarily equivalent in* P *if there exist three paths,* $a, b, c \in P$ *such that* $\{l, l'\} = \{ab, acc^{-1}b\}$. *We say that* l *and* l' *are equivalent in* P *if there exists a finite sequence* $l = l_0, ..., l_n = l'$ *such that* l_i *is elementarily equivalent to* l_{i+1} *for any* $i \in \{0, ..., n-1\}$. *We will write it* $l \simeq_P l'$.

Definition 2.8. — *A lasso is a loop* l *such that one can find a simple loop* m*, the meander, and a path* s*, the spoke, such that* $l = sms^{-1}$.

A loop has a well-defined origin and orientation. A cycle is a loop in which one forgets about the endpoint. In a non-oriented cycle, the endpoint and the orientation are forgotten.

Definition 2.9. — *We say that two loops* l_1 *and* l_2 *are related if and only if they can be decomposed as:* $l_1 = cd$, $l_2 = dc$, *with* c *and* d *two paths.*

The set of equivalence classes for the relation defined on $L(M)$ *is the set of cycles. The operation of inversion is compatible with this equivalence. A non-oriented cycle is a pair* $\{l, l^{-1}\}$ *where* l *is a cycle.*

A cycle is simple if any loop which represents it is simple and it is said smooth if any loop which represents it is smooth.

We need a notion of convergence of paths in order to define the continuity of random holonomy fields. The definition makes use of the Riemannian metric γ, yet the notion of convergence with fixed endpoints will not depend on the choice of the Riemannian metric. We denote by d_γ the distance on M which is associated with γ.

Definition 2.10. — *Let p_1 and p_2 be two paths of M. Let $\ell(p_1)$ (resp. $\ell(p_2)$) be the length of the path p_1 (resp. p_2). We define the distance between p_1 and p_2 as:*

$$d_l(p_1,p_2) = \inf_{pp_1\in p_1, pp_2\in p_2} \sup_{t\in[0,1]} \Big[d_\gamma(pp_1(t), pp_2(t)) \Big] + |\ell(p_1) - \ell(p_2)|.$$

The topology induced by d_l does not depend on the choice of γ.

Let $(p_n)_{n\geq 0}$ be a sequence of paths on M. Let p be a path on M. The sequence $(p_n)_{n\geq 0}$ converges to p with fixed endpoints if and only if:

- *$d_l(p_n, p) \to 0$ as $n \to +\infty$,*
- *$\forall n \geq 0$, $\overline{p_n} = \overline{p}$ and $\underline{p_n} = \underline{p}$.*

We will see that the convergence with fixed endpoints behaves well when one considers images of paths by bi-Lipschitz homeomorphisms. Let us consider ψ a locally bi-Lipschitz homeomorphism from $\mathbb{R}^2$ to itself.

Lemma 2.1. — *Let p be a path on the plane, the image of p, $\psi(p)$ is also a path.*

Proof. — We only have to prove that $\psi(p)$ has a finite length. This is a consequence of the fact that if $c : [0,1] \to \mathbb{R}^2$ is a continuous function, the length of c is given by $\sup_{\{t_1,\dots,t_n | t_1<\dots<t_n\}\subset[0,1]} \sum_{i=1}^{n-1} |c(t_i) - c(t_{i+1})|$. □

Lemma 2.2. — *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of paths which converges to a path p with fixed endpoints. The sequence $(\psi(p_n))_{n\in\mathbb{N}}$ converges to $\psi(p)$ with fixed endpoints.*

Proof. — Let us consider $(p_n)_{n\in\mathbb{N}}$ and p which satisfy the conditions of the lemma. Lévy proved in Lemma 1.2.17 of [**21**], that p_n converges to p uniformly when the paths are parametrized at constant speed. Let us denote by $(p_n(t))_{t\in[0,1]}$ and $(p(t))_{t\in[0,1]}$ these parametrized paths. Since ψ is locally Lipschitz, $(\psi(p_n(t)))_{t\in[0,1]}$ converges uniformly to $(\psi(p(t)))_{t\in[0,1]}$ when n goes to infinity and there exists a real R such that the speed of $\psi(p_n)$ for any integer n and the speed of $\psi(p)$ are bounded by R. An application of Lemma 1.2.18 of [**21**] and the triangular inequality allows us to assert that the length of $\psi(p_n)$ converges to the length of $\psi(p)$. This proves that $d_l(\psi(p_n), \psi(p))$ converges to zero when n goes to infinity. □

With this notion of convergence comes the notion of density and the following interesting lemma (Proposition 1.2.12 of [**21**]).

Lemma 2.3. — *The set of paths $\mathsf{Aff}_\gamma(M)$ is dense in $P(M)$ for the convergence with fixed endpoints.*

One has to be careful when working with the convergence with fixed endpoints. For example, the set of paths whose images are concatenation of horizontal and vertical segments is *not* dense in $P(\mathbb{R}^2)$. Indeed, one condition in order to have the convergence with fixed endpoints is that the length of the paths converges to the length of the limit

path. But, for any path p which can be written as a concatenation of horizontal and vertical segments, the following inequality holds:

$$\ell(p) \geq ||\overline{p} - \underline{p}||_1,$$

where $||.||_1$ is the usual L^1 norm on $\mathbb{R}^2$. This inequality does not hold for a general path p.

2.2. Measures on the set of multiplicative functions. —

2.2.1. Definitions. — Let P be a subset of $P(M)$ and let L be a set of loops in P.

Definition 2.11. — *A function h from P to G is multiplicative if and only if:*
- *$h(c^{-1}) = h(c)^{-1}$ for any path c in P such that $c^{-1} \in P$,*
- *$h(c_1c_2) = h(c_2)h(c_1)$ for any paths c_1 and c_2 in P which can be concatenated and such that $c_1c_2 \in P$.*

We denote by $\mathcal{M}ult(P, G)$ the set of multiplicative functions from P to G.
A function from L to G is pre-multiplicative over P if and only if:
- *it is multiplicative,*
- *for any l and l' in L which are equivalent in P, we have: $h(l) = h(l')$.*

We denote by $\mathcal{M}ult_P(L, G)$ the set of pre-multiplicative functions over P.

We will often make the following slight abuse of notation.

Notation 2.1. — *Let c be a path in P. If a multiplicative function $h \in \mathcal{M}ult(P, G)$ is not specified in a formula, $h(c)$ will stand for the function on $\mathcal{M}ult(P, G)$ which is the evaluation on c:*

$$\begin{aligned} h(c) : \mathcal{M}ult(P, G) &\to G \\ h &\mapsto h(c). \end{aligned}$$

The notion of equivalence of loop, as stated in Definition 2.7, is important due to the following remark.

Remark 2.1. — *Let h be in $\mathcal{M}ult(P, G)$ and let l, l' be loops in P. A simple induction and the multiplicative property of h imply that if $l \simeq_P l'$ then $h(l) = h(l')$.*

Let P be a set of paths and let Q be a freely generating subset of P in the sense that:
- any path in P is a finite concatenation of elements of Q,
- no element of Q can be written as a non-trivial finite concatenation of paths in $Q \cup Q^{-1}$,
- $Q \cap Q^{-1} = \emptyset$.

Then we have the identification:

$$\mathcal{M}ult(P, G) \simeq G^Q. \tag{1}$$

This is the *edge paradigm* for multiplicative functions. The novelty of the approach we have in this paper is to put the emphasis on the *loop paradigm* for gauge-invariant random holonomy fields. The first paradigm is interesting for general random holonomy fields

on surface, yet the second seems to be more appropriate for gauge-invariant random holonomy fields on the plane.

Remark 2.2. — *All the following definitions and propositions are about multiplicative functions on a set of paths P. All of them extend to G^T, with $T \subset \mathbb{R}$. Indeed, if $P = \cup_{r\in T}\{c_r, c_r^{-1}\}$, with c_r being the path on the plane based at 0 and going clockwise once around the circle of center $(0, r)$ and radius r, we have:*

$$\mathcal{M}ult(P, G) \simeq G^T.$$

We will now endow the space of multiplicative functions with a σ-field in order to be able to speak about measures on $\mathcal{M}ult(P, G)$.

Definition 2.12. — *The Borel σ-field $\mathcal{B}$ on $\mathcal{M}ult(P, G)$ is the smallest σ-field such that for any paths $c_1, ..., c_n$ and any continuous function $f : G^n \to \mathbb{R}$, the mapping $h \mapsto f\big(h(c_1), ..., h(c_n)\big)$ is measurable.*

Definition 2.13. — *A random holonomy field μ on the set P is a measure on $\big(\mathcal{M}ult(P, G), \mathcal{B}\big)$.*

If $P = P(M)$, we call it a random holonomy field on M. Let μ be a random holonomy field on P. We define the weight of μ as $\mu(\mathbb{1})$. One can define a regularity notion for random holonomy fields.

Definition 2.14. — *A random holonomy field on P is stochastically continuous if for any sequence $(p_n)_{n\geq 0}$ of elements of P which converges with fixed endpoints to $p \in P$,*

$$\int_{\mathcal{M}ult(P,G)} d_G\big(h(p_n), h(p)\big)\mu(dh) \underset{n\to\infty}{\longrightarrow} 0. \tag{2}$$

The measure μ is locally stochastically $\frac{1}{2}$-Hölder continuous if for any compact set $S \subset M$, for any measure of area vol on M, there exists $K > 0$ such that for any simple loop $l \in P$ bounding a disk D such that $l \subset S$:

$$\int_{\mathcal{M}ult(P,G)} d_G\big(e, h(l)\big)\mu(dh) \leq K\sqrt{vol(D)}, \tag{3}$$

where e is the neutral element of G.

A family of random holonomy fields, $(\mu)_{\mu\in\mathcal{F}}$, with each μ defined on some set P_μ, is uniformly locally stochastically $\frac{1}{2}$-Hölder continuous if the constant K in equation (3) is independent of the random holonomy field in $\mathcal{F}$.

2.2.2. Construction of random holonomy fields I. —

Notation 2.2. — *Let J and K be two subsets of $P(M)$ such that $J \subset K$. The restriction function from $\mathcal{M}ult(K, G)$ to $\mathcal{M}ult(J, G)$ will be denoted by $\rho_{J,K}$. If $M \subset M'$ are two surfaces, we denote by $\rho_{M,M'}$ the restriction function $\rho_{P(M),P(M')}$.*
The notation is set such that for any $J \subset K \subset L \subset P(M)$, $\rho_{J,K} \circ \rho_{K,L} = \rho_{J,L}$.

The fact that G is a compact group allows us to construct measures on the set of multiplicative functions by taking projective limits of random holonomy fields on finite subsets of paths. This behavior is very different from what can be observed for gaussian measures on Banach spaces, as for the Gaussian free field.

Indeed, in [**21**], Proposition 2.2.3, Lévy proved, when $\mathcal{F}$ is a collection of finite subsets of P, the next proposition using an application of Carathéodory's extension theorem. We give a proof based on the Riesz-Markov's theorem, proof which shows clearly why we only consider compact groups.

Proposition 2.1. — *Let $\mathcal{F}$ be a collection of subsets of paths on M. We denote by P their union. Suppose that, when ordered by the inclusion, $\mathcal{F}$ is directed: for any J_1 and J_2 in $\mathcal{F}$, there exists $J_3 \in \mathcal{F}$ such that $J_1 \cup J_2 \subset J_3$. For any $J \in \mathcal{F}$, let m_J be a probability measure on $\big(\mathcal{M}ult(J,G), \mathcal{B}\big)$. Assume that the probability spaces $\big(\mathcal{M}ult(J,G), \mathcal{B}, m_J\big)$ endowed with the restriction mappings $\rho_{J,K}$ for $J \subset K$ form a projective system. This means that for any J_1 and J_2 in $\mathcal{F}$ such that $J_1 \subset J_2$, one has*

$$m_{J_1} = m_{J_2} \circ \rho_{J_1,J_2}^{-1}.$$

Then there exists a unique probability measure m on $\big(\mathcal{M}ult(P,G), \mathcal{B}\big)$ such that for any $J \in \mathcal{F}$,

$$m_J = m \circ \rho_{J,P}^{-1}.$$

Proof. — We endow G^P with the product topology. As an application of Tychonoff's theorem it is a compact space. A consequence of this is that $\mathcal{M}ult(P,G)$, endowed with the restricted topology, is also a compact space as it is closed in G^P. Besides, the σ-field $\mathcal{B}$ is the Borel σ-field on $\mathcal{M}ult(P,G)$. Let us consider A the set of cylinder continuous functions, that is the set of functions $\mathbf{f} : \mathcal{M}ult(P,G) \to \mathbb{R}^+$ of the form:

$$\mathbf{f} : h \mapsto f\big(h(p_1), \dots, h(p_n)\big),$$

for some $n \in \mathbb{N}$, some $p_1, \dots, p_n \in P$ and some continuous function $f : G^n \to \mathbb{R}$.

The set A is a subalgebra of the algebra $C\big(\mathcal{M}ult(P,G), \mathbb{R}\big)$ of real-valued continuous functions on $\mathcal{M}ult(P,G)$. This subalgebra separates the points of $\mathcal{M}ult(P,G)$ and contains a non-zero constant function. Due to the Stone-Weierstrass's theorem, A is dense in $C\big(\mathcal{M}ult(P,G), \mathbb{R}\big)$. Any function $\mathbf{f}$ in A depends only on a finite number of paths, so that there exists some $J \in \mathcal{F}$ such that $\mathbf{f}$ can be seen as a continuous function on $\mathcal{M}ult(J,P)$. We define:

$$m(\mathbf{f}) = m_J(\mathbf{f}),$$

which does not depend on the chosen $J \in \mathcal{F}$ thanks to the projectivity and multiplicative properties.

We have defined a positive linear functional m on A, the norm of which is bounded by the total weight of any of the measures $(m_J)_{J\in\mathcal{F}}$. Thus m can be extended on $C\big(\mathcal{M}ult(P,G), \mathbb{R}\big)$ and an application of the Riesz-Markov's theorem allows us to consider m as a measure on $\big(\mathcal{M}ult(P,G), \mathcal{B}\big)$. This is the projective limit of $(m_J)_{J\in\mathcal{F}}$. □

The notion of locally stochastically $\frac{1}{2}$-Hölder continuity allows us to have an extension theorem from some subsets of paths to their closure, as shown in the proof of Corollary 3.3.2 of [**21**].

Theorem 2.1. — *Let $\mu_{\mathsf{Aff}_\gamma(M)}$ be a random holonomy field on $\mathsf{Aff}_\gamma(M)$. If it is locally stochastically $\frac{1}{2}$-Hölder continuous then there exists a unique stochastically continuous*

random holonomy field μ on M such that:

$$\mu_{\mathsf{Aff}_\gamma(M)} = \mu \circ \rho^{-1}_{\mathsf{Aff}_\gamma(M),P(M)}.$$

2.2.3. Gauge-invariance. — For any subset P of $P(M)$, a natural group acts on $\mathcal{M}ult(P,G)$: the *gauge group* that we are going to describe. Let us fix a subset P of $P(M)$ which will stay fixed until Section 2.2.4.

Definition 2.15. — *Let $V = \{x \in M, \exists\, p \in P, x = \underline{p} \text{ or } x = \overline{p}\}$ be the set of endpoints of P. We define the partial gauge group associated with P by setting $J_P = G^V$. If $P = P(M)$, this group is called the gauge group of M. The group J_P acts by gauge transformations on the space $\mathcal{M}ult\big(P,G\big)$: if $j \in J_P$, the action of j on $h \in \mathcal{M}ult(P,G)$ is given by:*

$$\forall c \in P, (j \bullet h)(c) = j^{-1}_{\overline{c}} h(c) j_{\underline{c}}.$$

Let $P = \{c_1, ..., c_n\}$ be a finite set of paths on M. Looking only at the evaluation on c_i for $i \in \{1, ..., n\}$, we have the inclusion: $\mathcal{M}ult(P,G) \subset G^n$. The gauge action of J_P on $\mathcal{M}ult(P,G)$ extends naturally to an action on G^n by $(j \bullet \mathbf{g})_i = j^{-1}_{\overline{c_i}} \mathbf{g}_i j_{\underline{c}_i}$ for any $i \in \{1, \ldots, n\}$.

Remark 2.3. — *If $l_1, ..., l_n$ are loops based at a point m, the partial gauge group is nothing but G and the corresponding action on G^n is the diagonal conjugation:*

$$j \bullet (g_1, ..., g_n) = (j^{-1} g_1 j, ..., j^{-1} g_n j).$$

We now define a sub-σ-field of $\mathcal{B}$: the *invariant σ-field.*

Definition 2.16. — *On $\mathcal{M}ult(P,G)$, the invariant σ-field, denoted by $\mathcal{I}$, is the smallest σ-field such that for any paths $c_1, ..., c_n$ in P and any continuous function $f : G^n \to \mathbb{R}$ invariant under the action of $J_{\{c_1,...,c_n\}}$ on G^n defined after Definition 2.15, the mapping $h \mapsto f\big(h(c_1), ..., h(c_n)\big)$ is measurable.*

Let us remark that if M is the disjoint union of two smooth compact surfaces, $M = M_1 \sqcup M_2$ then $\mathcal{M}ult\big(P(M),G\big) \simeq \mathcal{M}ult(P(M_1),G) \times \mathcal{M}ult(P(M_2),G)$. Besides, let $\mathcal{I}$ (respectively $\mathcal{I}_1$, $\mathcal{I}_2$) be the invariant σ-field on $\mathcal{M}ult\big(P(M),G\big)$ (respectively $\mathcal{M}ult\big(P(M_1),G\big)$, $\mathcal{M}ult\big(P(M_2),G\big)$). We have $\mathcal{I} \simeq \mathcal{I}_1 \otimes \mathcal{I}_2$.

Diffeomorphisms between surfaces give rise to some examples of functions which are measurable with respect to the Borel and the invariant σ-fields. Given M and M' two smooth compact surfaces, suppose that we are given a diffeomorphism ψ from M to M', we can construct, for any h in $\mathcal{M}ult\big(P(M'),G\big)$, a natural multiplicative function $\psi^* h$ on M:

$$\big(\psi^* h\big)(p) = h\big(\psi(p)\big), \forall p \in P(M).$$

This defines a function $\psi^* : \mathcal{M}ult\big(P(M'),G\big) \to \mathcal{M}ult\big(P(M),G\big)$. The function ψ^* is measurable for the Borel and the invariant σ-fields. From now on, we denote also by ψ the application ψ^*.

Definition 2.17. — *Let μ be a random holonomy field on P. Let P_1 and P_2 be two families of paths in P. We will say that $\big(h(p)\big)_{p\in P_1}$ and $\big(h(p)\big)_{p\in P_2}$ are $\mathcal{I}$-independent if and only if, for any finite family $(p_{1,i})_{i=1}^n$ in P_1, any finite family $(p_{2,i})_{i=1}^m$ in P_2 and any continuous function $f_1 : G^n \to \mathbb{R}$ (resp. $f_2 : G^m \to \mathbb{R}$) invariant under the action of $J_{\{p_{1,1},\dots,p_{1,n}\}}$ (resp. $J_{\{p_{2,1},\dots,p_{2,m}\}}$), the following equality holds:*

$$\mu\Big[f_1\Big((h(p_{1,i}))_{i=1}^n\Big)f_2\Big((h(p_{2,j}))_{j=1}^m\Big)\Big] = \mu\Big[f_1\Big((h(p_{1,i}))_{i=1}^n\Big)\Big]\mu\Big[f_2\Big((h(p_{2,j}))_{j=1}^m\Big)\Big]. \tag{4}$$

This is equivalent to say that under μ, the two σ-fields $\sigma\big(h(p) : p \in P_1\big) \cap \mathcal{I}$ and $\sigma\big(h(p) : p \in P_2\big) \cap \mathcal{I}$ are independent.

Remark 2.4. — *One has to be careful when dealing with the invariant σ-fields. For instance, let us consider the diagonal action of G by conjugation on G^2. The invariant σ-field on G^2 which we denote by $\mathcal{I}_{(2)}$ is different from the product $\mathcal{I}\otimes\mathcal{I}$ where $\mathcal{I}$ is the invariant σ-field of G. To see why, we only have to consider the symmetric group $\mathfrak{S}_3$. If it was true that $\mathcal{I}_{(2)} = \mathcal{I}\otimes\mathcal{I}$, we would have the equality:*

$$\#\Big\{\big[(\sigma,\sigma')\big], (\sigma,\sigma') \in \mathfrak{S}_3^2\Big\} = \Big(\#\Big\{\big[\sigma\big], \sigma \in \mathfrak{S}_3\Big\}\Big)^2.$$

It is fairly easy to see that the l.h.s. is equal to eleven and the r.h.s. to nine.

Let us consider the measurable space $\big(G^2, \mathcal{I}_{(2)}\big)$ and let π_1 and π_2 be the two canonical projections from $\big(G^2, \mathcal{I}_{(2)}\big)$ to $(G,\mathcal{I})$. Let (X,Y) be a random vector in $\big(G^2, \mathcal{I}_{(2)}\big)$ such that X is independent of Y in the sense that for any real valued functions f and g on $(G,\mathcal{I})$ such that $f\circ\pi_1$ and $g\circ\pi_2$ are measurable on $\big(G^2, \mathcal{I}_{(2)}\big)$, $\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[f(Y)]$. In the light of the discussion we just had, we see that the knowledge of the laws of X and Y as $(G,\mathcal{I})$ random variables does not allow us *to reconstruct the law of the couple (X,Y).*

On the invariant σ-field on $\mathcal{M}ult(P,G)$, any measure is of course invariant by the gauge transformations. Explicitly, for any measure μ on $\big(\mathcal{M}ult(P,G),\mathcal{I}\big)$, for any measurable continuous function f from $\big(\mathcal{M}ult(P,G),\mathcal{I}\big)$ to $\mathbb{R}$ and for any $j \in J_P$:

$$\int_{\mathcal{M}ult(P,G)} f(j\bullet h)d\mu(h) = \int_{\mathcal{M}ult(P,G)} f(h)d\mu(h). \tag{5}$$

The following definition is less trivial as the following class of gauge-invariant measures is not equal to the collection of all measures.

Definition 2.18. — *Let μ be a random holonomy field on P. We say that μ is invariant under gauge transformations if and only if the Equality (5) holds for any continuous function f from $\big(\mathcal{M}ult(P,G),\mathcal{B}\big)$ to $\mathbb{R}$ and for any $j \in J_P$.*

Remark 2.5. — *Let μ be a gauge-invariant random holonomy field on P. Let p a path in P which is not a loop: $\underline{p} \neq \overline{p}$. Then under $\frac{\mu}{\mu(\mathbb{1})}$, $h(p)$ has the law of a Haar random variable. Indeed, applying the gauge transformation which is equal to* 1 *everywhere except at $\underline{p}$ or $\overline{p}$, where its value is set to be an arbitrary element of G, we see that the law of $h(p)$ is invariant by left- and right-multiplication.*

There exists a one-to-one correspondence between measures on $\big(\mathcal{M}ult(P,G),\mathcal{I}\big)$ and gauge-invariant measures on $\big(\mathcal{M}ult(P,G),\mathcal{B}\big)$. This proposition is an analog of the results of [**9**]. For any positive integer n, for any continuous function f on G^n and any set of paths $\{c_1,\dots,c_n\}$ in P, we define the function $\hat{f}_{J_{c_1,\dots,c_n}}$ such that, for any $g_1,\dots,g_n$ in G:

$$\hat{f}_{J_{c_1,\dots,c_n}}(g_1,\dots,g_n)=\int_{J_{c_1,\dots,c_n}} f\Big(j\bullet(g_1,\dots,g_n)\Big)dj, \tag{6}$$

where dj is the Haar measure on $J_{c_1,\dots,c_n}$.

Proposition 2.2. — *For any measure μ on $\big(\mathcal{M}ult(P,G),\mathcal{I}\big)$, there exists a unique gauge-invariant random holonomy field on P which will be denoted either by $\hat{\mu}$ or $\widehat{\mu}$, such that:*

$$\hat{\mu}_{|\mathcal{I}}=\mu.$$

Proof. — The uniqueness of $\hat{\mu}$ follows from the upcoming Proposition 2.4. Let us prove its existence. We will define $\hat{\mu}$ by the fact that for any measurable function $f:G^n\to\mathbb{R}^+$ and any n-tuple $c_1,\dots,c_n$ of elements of P:

$$\hat{\mu}\Big(f\big(h(c_1),...,h(c_n)\big)\Big)=\mu\left(\hat{f}_{J_{c_1,\dots,c_n}}\big(h(c_1),...,h(c_n)\big)\right).$$

Let us consider a finite set of paths in P, $P_1=\{c_1,...,c_n\}$. Let us consider the natural inclusion $\iota:\mathcal{M}ult\,(P_1,G)\subset G^n$ given by the evaluations on $c_1,\dots,c_n$. The equalities $\hat{\mu}_{P_1}(f)=\mu\left(\hat{f}_{J_{c_1,\dots,c_n}}\left(h(c_1),...,h(c_n)\right)\right)$ for any continuous function on G^n define a linear positive functional on $C(G^n)$. By compactness of G^n, applying the theorem of Riesz-Markov, it gives a measure $\hat{\mu}_{P_1}$ on G^n, the support of which is easily seen to be a subset of $\iota\big(\mathcal{M}ult\,(P_1,G)\,\big)$. We can thus look at the induced measure on $\mathcal{M}ult\,(P_1,G)$ named $\hat{\mu}_{|\mathcal{M}ult(P_1,G)}$. The family of measures $\big(\hat{\mu}_{|\mathcal{M}ult(P_1,G)}\big)_{P_1\subset P,\#P_1<\infty}$ forms a projective family of measures for the inclusion of sets. Thus, by Proposition 2.1, it defines a measure on $\big(\mathcal{M}ult\,(P,G)\,,\mathcal{B}\big)$. □

In the following remark, we will see that in some special cases, the $\mathcal{I}$-independence is equivalent to the independence.

Remark 2.6. — *Let μ be a gauge-invariant random holonomy field on P and let P_1 and P_2 be two sets of paths such that their sets of endpoints V_{P_1} and V_{P_2} are disjoint. The two families $\big(h(p)\big)_{p\in P_1}$ and $\big(h(p)\big)_{p\in P_2}$, defined on $(\mathcal{M}ult(P,G),\mathcal{B},\mu)$, are $\mathcal{I}$-independent if and only if they are independent. Indeed, let us suppose that they are $\mathcal{I}$-independent. If f and g are real-valued continuous functions on G^n and G^m respectively, we denote by $f\otimes g$ the function from $G^n\times G^m$ to $\mathbb{R}$ defined by:*

$$f\otimes g(x_1,...,x_n,x_{n+1},...,x_{n+m})=f(x_1,...,x_n)g(x_{n+1},\dots,x_{n+m}).$$

With this notation and the notation of (6), since the two families P_1 and P_2 have disjoint sets of endpoints,

$$\widehat{\big(f\otimes g\big)}_{J_{P_1\cup P_2}}=\widehat{f}_{J_{P_1}}\otimes\widehat{g}_{J_{P_2}},$$

where the partial gauge group was defined in Definition 2.15. Thus, using the gauge-invariance of μ,

$$\begin{aligned}\mu\Big[f\big((h(p))_{p\in P_1}\big)g\big((h(p))_{p\in P_2}\big)\Big] &= \mu\Big[\big(f\otimes g\big)\big((h(p))_{p\in P_1},(h(p))_{p\in P_2}\big)\Big]\\ &= \mu\left[\widehat{(f\otimes g)}_{J_{P_1\cup P_2}}\big((h(p))_{p\in P_1},(h(p))_{p\in P_2}\big)\right]\\ &= \mu\left[\widehat{f}_{J_{P_1}}\otimes\widehat{g}_{J_{P_2}}\big((h(p))_{p\in P_1},(h(p))_{p\in P_2}\big)\right]\\ &= \mu\left[\widehat{f}_{J_{P_1}}\big((h(p))_{p\in P_1}\big)\right]\mu\left[\widehat{g}_{J_{P_2}}\big((h(p))_{p\in P_2}\big)\right]\\ &= \mu\left[f\big((h(p))_{p\in P_1}\big)\right]\mu\left[g\big((h(p))_{p\in P_2}\big)\right].\end{aligned}$$

This proves that the two families $\big(h(p)\big)_{p\in P_1}$ and $\big(h(p)\big)_{p\in P_2}$ are independent.

Let us introduce the main ingredient in order to construct gauge-invariant random holonomy fields: the *loop paradigm* for multiplicative functions.

Lemma 2.4. — *Let us suppose that P is connected, stable by concatenation and inversion. Let m be an endpoint of P. Let L_m be the set of loops in P based at m. The* loop paradigm *for the multiplicative functions is the fact that:*

$$\mathcal{M}ult(P,G)/J_P \simeq \mathcal{M}ult_P(L_m,G)/J_{L_m}. \tag{7}$$

Proof. — There exists a natural restriction function:

$$r : \mathcal{M}ult(P,G)/J_P \to \mathcal{M}ult_P(L_m,G)/J_{L_m}.$$

Let us show that there exists an application

$$\iota : \mathcal{M}ult_P(L_m,G)/J_{L_m} \to \mathcal{M}ult(P,G)/J_P,$$

such that $r\circ\iota = id$ and $\iota\circ r = id$.

For any endpoint v of P, let q_v be a path in P joining m to v. This is possible since we supposed that P was connected. We set q_m to be the trivial path. Then, for any path p in P we define $l(p) = q_{\underline{p}}pq_{\overline{p}}^{-1}$. One can look at the Figure 5 to have a better understanding of $l(p)$. For any h in $\mathcal{M}ult_P(L_m,G)$, we define for any path p,

$$\iota(h)(p) = h(l(p)).$$

This is a multiplicative function. Let us show, for example, that it is compatible with the concatenation operation. For any $h\in\mathcal{M}ult_P(L_m,G)$ and any paths p and p' in P such that $\overline{p} = \underline{p'}$, the following sequence of equalities holds:

$$\begin{aligned}\iota(h)(pp') &= h\big(l(pp')\big) = h(q_{\underline{p}}pp'q_{\overline{p'}}^{-1}) = h(q_{\underline{p}}pq_{\overline{p}}^{-1}q_{\underline{p'}}p'q_{\overline{p'}}^{-1})\\ &= h(q_{\underline{p'}}p'q_{\overline{p'}}^{-1})h(q_{\underline{p}}pq_{\overline{p}}^{-1}) = h\big(l(p')\big)h\big(l(p)\big) = \iota(h)(p')\iota(h)(p),\end{aligned}$$

where in the third equality we used the fact that h is an element of $\mathcal{M}ult_P(L_m,G)$ and not only in $\mathcal{M}ult(L_m,G)$.

Thus ι is an application from $\mathcal{M}ult_P(L_m,G)$ to $\mathcal{M}ult(P,G)$. This application ι defines a function, that we will also call ι from $\mathcal{M}ult_P(L_m,G)/J_{L_m}$ to $\mathcal{M}ult(P,G)/J_P$. Indeed,

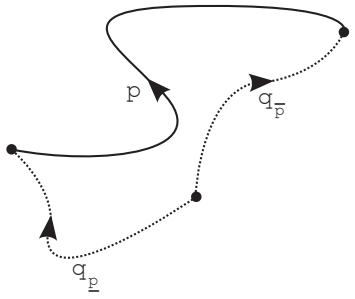


FIGURE 5. Construction of the loop $l(p)$.

if $j \in J_{L_m} \simeq G$, $h \in \mathcal{M}ult_P(L_m, G)$ and $p \in P$:

$$\iota(j \bullet h)(p) = j \bullet h(l(p)) = j(m)^{-1} h(l(p)) j(m) = \tilde{j} \bullet \iota(h)(p),$$

where $\tilde{j}$ is the constant function which is equal to j. Let us show that $\iota \circ r = id$: for any $h \in \mathcal{M}ult(P, G)$,

$$\begin{aligned}(\iota(r(h))(p))_{p\in P} = r(h)(l(p)) &= \left(h(q_{\underline{p}} p q_{\overline{p}}^{-1})\right)_{p\in P} \\ &= \left(h(q_{\overline{p}})^{-1} h(p) h(q_{\underline{p}})\right)_{p\in P},\end{aligned}$$

thus, in $\mathcal{M}ult(P, G)/J_P$, we have the equality $(\iota(r(h))(p))_{p\in P} = (h(p))_{p\in P}$. The equality $r \circ \iota = id$ is even easier. □

From the proof of Lemma 2.4, one also gets the following lemma.

Lemma 2.5. — *Suppose that P is connected, stable by concatenation and inversion. Let m be an endpoint of P and let L_m be the subset of loops in P based at m. There exists an application:*

$$\iota : \mathcal{M}ult_P(L_m, G) \to \mathcal{M}ult(P, G)$$

which is measurable for the Borel σ-field and such that, for any loop $l \in L_m$, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{M}ult_P(L_m, G) & \xrightarrow{\quad\iota\quad} & \mathcal{M}ult(P, G) \\ & {\scriptstyle h(l)} \searrow \quad \swarrow {\scriptstyle h(l)} & \\ & G & \end{array}$$

One consequence of Lemma 2.4 is Lemma 2.1.5 in [**21**].

Lemma 2.6. — *Suppose that P is connected, stable by concatenation and inversion. Let m be an endpoint of P. Then for any paths $c_1, \ldots, c_n$ in P and any measurable function $f : G^n \to \mathbb{R}$ invariant under the action of $J_{c_1,\ldots,c_n}$ on G^n, there exist n loops $l_1, \ldots, l_n$ in P based at m and a measurable function $\tilde{f} : G^n \to \mathbb{R}$ invariant under the diagonal action of G such that:*

$$f\big(h(c_1), \ldots, h(c_n)\big) = \tilde{f}\big(h(l_1), \ldots, h(l_n)\big).$$

Lemma 2.6 allows us to reduce the family of variables that $\mathcal{I}$ has to make measurable: we only have to look at finite collections of loops based at the same point, which leads us to the Definition 2.1.6 of [**21**].

Proposition 2.3. — *Let us suppose that P is connected, stable by concatenation and inversion. Let m be an endpoint of P. The invariant σ-field $\mathcal{I}$ on $\mathcal{M}ult(P,G)$ is the smallest σ-field such that for any positive integer n, any loops $l_1, ..., l_n$ based at m and any continuous function $f : G^n \to \mathbb{R}$ invariant under the diagonal action of G, the mapping $h \mapsto f(h(l_1), ..., h(l_n))$ is measurable.*

Another consequence of Lemma 2.6 is the following proposition.

Proposition 2.4. — *Let us suppose that P is connected, stable by concatenation and inversion. Let m be an endpoint of P and L_m the set of loops in P based at m. If μ and ν are two stochastically continuous gauge-invariant random holonomy fields on P, the two following assertions are equivalent:*

1. *μ and ν are equal,*
2. *there exist an endpoint m of P and A_m a dense subset of L_m for the convergence with fixed endpoints, such that for any integer n, any n-tuple of loops $l_1, ..., l_n$ in A_m and any continuous function $f : G^n \to \mathbb{R}$ invariant under the diagonal action of G,*

$$\int_{\mathcal{M}ult(P,G)} f\big(h(l_1), ..., h(l_n)\big) d\mu(h) = \int_{\mathcal{M}ult(P,G)} f\big(h(l_1), ..., h(l_n)\big) d\nu(h).$$

If the random holonomy fields are not stochastically continuous, the proposition still holds if one replaces A_m by L_m.

Remark 2.7. — *The first consequence of this proposition is the* change of base point invariance property *of gauge-invariant random holonomy fields. For the sake of simplicity, let us consider μ a gauge-invariant random holonomy field on M. Let us consider a bijection $\psi : M \to M$ and let us consider for any point x of M, p_x a path from $\psi(x)$ to x. Then the random holonomy field which has the law of:*

$$(h(p_x))_{x\in M} \bullet [h(p)]_{p\in P}$$

under μ, is still gauge-invariant. The last proposition shows that μ and the new random holonomy field are equal: for any paths $p_1, \dots p_n$, we have the equality in law:

$$\Big(\Big[(h(p_x))_{x\in M} \bullet [h(p)]_{p\in P}\Big]\,(p_i)\Big)_{i=1}^{n} = (h(p_i))_{i=1}^{n}$$

under μ. For example, if $l_1, \dots, l_n$ are n loops based at m and if s is a path from m' to m, under a gauge-invariant measure μ, $\big(h(sl_1s^{-1}), \dots, h(sl_ns^{-1})\big)$ has the same law as $\big(h(l_1), \dots, h(l_n)\big)$.

2.2.4. Construction of random holonomy fields II: the gauge-invariant case. — In this section, for the sake of simplicity, we will suppose that M is connected. However, all results could be easily extended to the non-connected case. Thanks to Lemma 2.4 and Proposition 2.2, constructing a gauge-invariant random holonomy field μ becomes easier. Let P be a connected set of paths, stable by concatenation and inversion.

Proposition 2.5. — *Let m be an endpoint of P. Suppose that for any finite subset L of loops in P based at m, we are given a gauge-invariant measure μ_L on $\mathcal{M}ult_P(L,G)$ such that, when endowed with the natural restriction functions, $\big((\mathcal{M}ult_P(L,G),\mathcal{B}),\mu_L\big)$ is a projective family. Then there exists a unique gauge-invariant random holonomy field μ on P such that for any finite subset L of loops in P based at m, one has:*

$$\mu_L = \mu \circ \rho_{L,P}^{-1}.$$

Proof. — The uniqueness of such a measure comes from a direct application of Proposition 2.4.

Let us prove the existence of the measure μ. Let L_m be the set of loops in P based at m. Using a slight modification of Proposition 2.1, we can consider the projective limit μ_{L_m} of $(\mu_{L_f})_{L_f \subset L_m, \# L_f < \infty}$, defined on $\big(\mathcal{M}ult_P(L_m,G),\mathcal{B}\big)$ and which is gauge-invariant. The set P satisfies the assumptions of Lemma 2.5: let us consider a measurable application ι from $\mathcal{M}ult_P(L_m,G)$ to $\mathcal{M}ult(P,G)$ given by this lemma. We define the measure:

$$\mu = \big((\mu_{L_m} \circ \iota^{-1})_{|\mathcal{I}}\big)\widehat{\ },$$

where we remind the reader that $(\ )\widehat{\ }$ is the notation for the extension of measures from the invariant σ-field to the Borel σ-field given by Proposition 2.2. By definition, it is defined on the Borel σ-field on $\mathcal{M}ult(P,G)$ and it is gauge-invariant.

If L is a finite subset of loops in P based at m, thanks to the definitions of ι and μ_{L_m}, $(\mu_{L_m} \circ \iota^{-1}) \circ \rho_{L,P}^{-1} = \mu_L$. The gauge-invariance of μ_L implies that $\big((\mu_L)_{|\mathcal{I}}\big)\widehat{\ } = \mu_L$. This leads us to the conclusion: $\mu_L = \mu \circ \rho_{L,P}^{-1}$. □

In fact, what will be used is the combination of this last proposition and Theorem 2.1.

Proposition 2.6. — *Let γ be a Riemannian metric on M, let m be a point of M. Suppose that for any finite subset L of loops in $\mathsf{Aff}_\gamma(M)$ based at m, we are given a gauge-invariant measure μ_L on $\mathcal{M}ult_P(L,G)$ such that $\Big(\big(\mathcal{M}ult_P(L,G),\mathcal{B}\big),\mu_L\Big)$ is a projective family of uniformly locally stochastically $\frac{1}{2}$-Hölder continuous random holonomy fields. Then there exists a unique stochastically continuous gauge-invariant random holonomy field μ on M such that for any finite subset L of loops in $P(M)$ based at m, one has:*

$$\mu_L = \mu \circ \rho_{L,P(M)}^{-1}.$$

2.2.5. Restriction and extension of the structure group. — Let P be a connected set of paths, stable by concatenation and inversion. Let m be an endpoint of P and let L_m be the set of loops in P based at m. Let H be a closed subgroup of G. There exists a natural injection:

$$i_P : \big(\mathcal{M}ult(P,H),\mathcal{B}\big) \to \big(\mathcal{M}ult(P,G),\mathcal{B}\big).$$

Thus, we can always push forward any H-valued random holonomy field by i_P in order to create a G-valued random holonomy field. Of course, if a G-valued random holonomy field on P, say μ, is such that there exists a closed group $H \subset G$ such that for any path $p \in P$, one has $h(p) \in H$ μ a.s., then we can restrict the group to H: for any finite $P_f \subset P$ it defines a measure on $\mathcal{M}ult(P_f,H)$ and we can take the projective limit thanks to Proposition 2.1.

In the gauge-invariant setting, what can be done ? First of all, if μ is a H-valued gauge-invariant random holonomy field, $\mu \circ i_P^{-1}$ is not in general a G-valued gauge-invariant random holonomy field. The simplest counterexample is to consider P to be reduced to a single loop: a G-valued random variable can be H-invariant but not G-invariant by conjugation. Thus, in order to extend the structure group from H to G of a H-gauge-invariant random holonomy field μ, one has to consider:

$$\left((\mu \circ i_P^{-1})_{|\mathcal{I}}\right)\widehat{\ } \tag{8}$$

the gauge-invariant extension (see Proposition 2.2) to $\mathcal{B}$ of the restriction on the invariant σ-field $\mathcal{I}$ of $\mu \circ i_P^{-1}$. Thus, the natural injection is replaced by the following map:

$$\mu \mapsto \left((\mu \circ i_P^{-1})_{|\mathcal{I}}\right)\widehat{\ }.$$

Notation 2.3. — *In the following, we will denote by* $\mu \circ \hat{i}_P^{-1}$ *the measure* $\left((\mu \circ i_P^{-1})_{|\mathcal{I}}\right)\widehat{\ }$.

Now, let us consider the problem of restricting a gauge-invariant random holonomy field μ. Thanks to Lemma 2.6, we know that the only important objects are loops based at m. Hence the question: what can be done with a G-valued random holonomy field such that for any loop or for any simple loop $l \in L_m$, μ a.s. $h(l) \in H$? An important remark is that it does not imply that for any path $p \in P$, μ a.s. $h(p) \in H$. Indeed, as we have seen in Remark 2.5, for any p such that $\overline{p} \neq \underline{p}$, under $\mu/\mu(\mathbb{1})$, $h(p)$ has the law of a Haar random variable on G. Nevertheless, the following result is true.

Proposition 2.7. — *Let μ be a G-valued gauge-invariant random holonomy field such that for any loop $l \in L_m$, $h(l) \in H$, μ a.s. Then there exists an H-valued gauge-invariant random holonomy field μ_H such that:*

$$\mu = \mu_H \circ \hat{i}_P^{-1}.$$

Let M be a smooth compact surface and let us suppose that P is $P(M)$ and that μ is stochastically continuous. The result is still true if for any lasso l based at m, $h(l) \in H$, μ a.s.

Remark 2.8. — *An important remark is that μ_H is* not *unique. Besides, using the group of reduced loops (Section 6.3 and Section 2.4 of* [**21**]*), one can show in the last case that it is enough that $h(l) \in H$, μ a.s., for any simple loop l based at m. This is due to the fact that for any graph, there exists a family of generators of the group of reduced loops which can be approximated, for the convergence with fixed endpoints, by simple loops.*

We give below the loop-erasure lemma used in the proof of Proposition 2.7 and which is taken from Proposition 1.4.9 in [**21**].

Lemma 2.7. — *Let (M, γ) be a Riemannian compact surface and let c be a loop in $\mathsf{Aff}_\gamma(M)$. There exists in $\mathsf{Aff}_\gamma(M)$ a finite sequence of lassos $l_1, \ldots, l_p$ and a simple loop d with the same endpoints as c such that:*

$$c \simeq l_1 \ldots l_p d.$$

Proof of Proposition 2.7. — Let us prove the second assertion, when $P = P(M)$ and μ is stochastically continuous. The first assertion can be proved using similar arguments.

Let us suppose that for any simple lasso $l \in L_m(M)$, $h(l) \in H$, μ a.s. Let us consider γ a Riemannian mertric on M. As a consequence of Lemma 2.7, for any loop $l \in \mathsf{Aff}_\gamma(M)$ based at m, $h(l) \in H$, μ a.s. Thus, by Lemma 2.3, using the stochastic continuity of μ and the fact that H is closed, for any $l \in L_m(M)$, $h(l) \in H$, μ a.s.

By restricting the measure μ, one can define, for any finite subset L_f of $L_m(M)$, a gauge-invariant measure μ_{L_f} on $\mathcal{M}ult_{P(M)}(L_f, H)$. As a consequence of Proposition 2.5, there exists a unique H-valued gauge-invariant random holonomy field μ_H on M such that for any finite subset L_f of $L_m(M)$, $\mu_{L_f} = \mu_H \circ \rho^{-1}_{L_f,P(M)}$. This H-valued gauge-invariant random holonomy field μ_H satisfies the equality: $\mu = \mu_H \circ \hat{i}^{-1}_{P(M)}$. □

3. Graphs

3.1. Definitions and simple facts. — Later we will need the notion of graphs for the construction of special families of random fields: the planar Markovian holonomy fields. The graphs we look at are not only combinatorial ones: we insist that the faces are homeomorphic to an open disk of $\mathbb{R}^2$. Let M be a smooth compact surface with boundary.

Definition 3.1. — *A pre-graph on M is a triple $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ such that:*

- *$\mathbb{E}$, the set of edges, is a non-empty finite set of simple paths on M, stable by inversion, such that two edges which are not each other's inverse meet, if at all, only at some of their endpoints,*
- *$\mathbb{V}$, the set of vertices, is the finite subset of M given by $\bigcup_{e\in\mathbb{E}} \{\overline{e}, \underline{e}\}$,*
- *$\mathbb{F}$, the set of faces, is the set of the connected components of $M \setminus \bigcup_{e\in\mathbb{E}} e\big([0,1]\big)$.*

Any pre-graph $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ whose bounded faces $F \in \mathbb{F}$ are homeomorphic to an open disk of $\mathbb{R}^2$ is called a graph on M.

Remark 3.1. — *By Proposition 1.3.10 in* [**21**]*, if $\mathbb{G}$ is a graph on M then ∂M can be represented by a concatenation of edges in $\mathbb{E}$.*

Due to the last definition, any pre-graph $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ is characterized by its set of edges $\mathbb{E}$. Thus, in order to construct a pre-graph, we will only define its set of edges. We will often use the following graph.

Exemple 3.1. — *Let l be a simple loop on $\mathbb{R}^2$. We denote by $\mathbb{G}(l)$ the graph on $\mathbb{R}^2$ composed of l and l^{-1} as unique edges.*

When M is homeomorphic to a sphere, we will consider that $(\{m\}, \emptyset, M \setminus \{m\})$ is a graph for any $m \in M$.

Definition 3.2. — *A graph is connected if and only if any two points of $\mathbb{V}$ are the endpoints of the same path in $P(\mathbb{G})$. A connected finite graph on $\mathbb{R}^2$ will be also called a* finite planar graph*; its set of faces is composed of one unbounded face denoted by F_∞ and a set $\mathbb{F}^b$ of bounded faces.*

Definition 3.3. — *Let $\mathbb{G}$ be a graph on M, $P(\mathbb{G})$ is the set of paths obtained by concatenating edges of $\mathbb{G}$. The set of loops in $P(\mathbb{G})$ is denoted by $L(\mathbb{G})$ and if $v \in \mathbb{V}$, $L_v(\mathbb{G})$ is the set of loops in $L(\mathbb{G})$ based at v.*

For any smooth connected compact surface with boundary M embedded in $\mathbb{R}^2$, a graph on M can be considered as a finite planar graph. This kind of graphs, of interest later, will be called embedded graphs on $\mathbb{R}^2$.

Definition 3.4. — *An embedded graph on $\mathbb{R}^2$ is a graph on a smooth connected compact surface with boundary M embedded in $\mathbb{R}^2$.*

The two definitions of graphs on $\mathbb{R}^2$ seen here are in fact almost equivalent. An embedded graph is obviously a graph on $\mathbb{R}^2$ and a direct consequence of Propositions 1.3.24 and 1.3.26 of [**21**] is the following result.

Proposition 3.1. — *Every finite planar graph on $\mathbb{R}^2$ is a subgraph of an embedded graph.*

The intersection of a graph $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ with a subset A of $\mathbb{R}^2$ is the pre-graph $(\mathbb{V}', \mathbb{E}', \mathbb{F}')$, denoted by $\mathbb{G} \cap A$, such that $\mathbb{E}' = \{e \in \mathbb{E}, e \cap A \nsubseteq \{\underline{e}, \overline{e}\}\}$. This allows us to define the notion of a planar graph. For any positive real r, let $\mathbb{D}(0, r)$ be the closed ball of center $(0, 0)$ and radius r in $\mathbb{R}^2$.

Definition 3.5. — *A planar graph $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ is a triple of sets which represent the vertices, the edges and the faces which are linked by the same relations as in Definition 3.1 and for which there exists an increasing unbounded sequence of positive reals $(r_n)_{n\in\mathbb{N}}$ such that for each integer n, $\mathbb{G} \cap \mathbb{D}(0, r_n)$ is a finite planar graph.*

Exemple 3.2. — *We consider $\mathbb{N}^2$ as a planar graph, the edges being the vertical and horizontal segments between nearest neighbors.*

Notation 3.1. — *Sometimes, one wants to consider connected graphs whose edges are in a given subset A of $P(M)$. We denote by $\mathcal{G}(A)$ the set of connected graphs $\mathbb{G} = \{\mathbb{V}, \mathbb{E}, \mathbb{F}\}$ such that $\mathbb{E} \subset A$.*

In the notions of graph exposed above, the edges are non-oriented, which means that there is no preference between e and e^{-1} for any edge e. We will need later to use an orientation on the graph.

Definition 3.6. — *An orientation on a graph $\mathbb{G}$ is the data of a subset $\mathbb{E}^+$ of $\mathbb{E}$ such that $\mathbb{E}^+ \cap (\mathbb{E}^+)^{-1} = \emptyset$ and $\mathbb{E}^+ \cup (\mathbb{E}^+)^{-1} = \mathbb{E}$. Given an orientation $\mathbb{E}^+$ on $\mathbb{G}$, for each subset J of $\mathbb{E}$, we denote by J^+ the set $J \cap \mathbb{E}^+$.*

3.2. Graphs and homeomorphisms. — In the following we will need to understand the action of orientation-preserving homeomorphisms on the set of graphs.

Definition 3.7. — *Let $\mathbb{G}$ and $\mathbb{G}'$ be two finite planar graphs. They are homeomorphic if there exists an orientation-preserving homeomorphism ψ which sends $\mathbb{G}$ on $\mathbb{G}'$. We will denote it by $\psi(\mathbb{G}) = \mathbb{G}'$ and by definition, this means that ψ induces a bijection $\mathcal{S}_{\mathbb{G}}^{\psi}$*

from the set $\mathbb{V}$ of vertices of $\mathbb{G}$ to the set $\mathbb{V}'$ of vertices of $\mathbb{G}'$ and a bijection $\mathcal{E}^{\psi}_{\mathbb{G}}$ from the set $\mathbb{E}$ of edges of $\mathbb{G}$ to the set $\mathbb{E}'$ of edges of $\mathbb{G}'$. These bijections are defined by:

$$\mathcal{S}^{\psi}_{\mathbb{G}}(v) = \psi(v), \text{ for any } v \in \mathbb{V},$$
$$\mathcal{E}^{\psi}_{\mathbb{G}}(e) = \psi(e), \text{ for any } e \in \mathbb{E}.$$

Definition 3.8. — *Let ψ and ψ' be two orientation-preserving homeomorphisms of $\mathbb{R}^2$ which send $\mathbb{G}$ on $\mathbb{G}'$. The homeomorphisms ψ and ψ' are equivalent on $\mathbb{G}$ if and only if $\mathcal{S}^{\psi}_{\mathbb{G}} = \mathcal{S}^{\psi'}_{\mathbb{G}}$ and $\mathcal{E}^{\psi}_{\mathbb{G}} = \mathcal{E}^{\psi'}_{\mathbb{G}}$.*

We would like to have an easy way to know if two finite planar graphs are homeomorphic. For that, an important notion introduced in [**21**] by Lévy is the cyclic order of the outgoing edges at a vertex.

Definition 3.9. — *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ be a finite planar graph. Let v be a vertex and let $\mathbb{E}_v$ be the set of edges $e \in \mathbb{E}$ such that $\underline{e} = v$. For any $e \in \mathbb{E}_v$, let e_p be a parametrized path which represents e. We define:*

$$r_0 = \min\left\{ \left|\left| v - e_p\left(\frac{1}{2}\right) \right|\right|, e \in \mathbb{E}_v \right\}.$$

Let $r \in]0, r_0[$. For each $e \in \mathbb{E}_v$, we define $s_e(r) \in \left[0, \frac{1}{2}\right]$ as the first time e_p hits the boundary of $\mathbb{D}(0, r)$:

$$s_e(r) = \inf\left\{ t \in \left[0, \frac{1}{2}\right], ||v - e_p(t)|| = r \right\}.$$

The cyclic permutation of $\mathbb{E}_v$, corresponding to the cyclic order of the points $\{e_p(s_e(r)), e \in \mathbb{E}_v\}$ on the circle $\partial\mathbb{D}(0, r)$ oriented anti-clockwise, does not depend on the chosen $r \in]0, r_0[$: it is the cyclic order of the edges at the vertex v denoted by σ_v.

A consequence of Jordan-Schönfliess theorem is the Heffter-Edmonds-Ringel rotation principle, stated in Theorem 3.2.4 of [**22**]. Using the notions early defined, we can state it as follows.

Theorem 3.1. — *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ and $\mathbb{G}' = (\mathbb{V}', \mathbb{E}', \mathbb{F}')$ be two finite planar graphs such that the following assertions hold:*

1. *there exists a bijection $\mathcal{S} : \mathbb{V} \to \mathbb{V}'$,*
2. *there exists a bijection $\mathcal{E} : \mathbb{E} \to \mathbb{E}'$ such that for any $e \in \mathbb{E}$, $\mathcal{E}\left(e^{-1}\right) = \mathcal{E}(e)^{-1}$,*
3. *for any edge $e \in \mathbb{E}$, $\mathcal{S}\left(\underline{e}\right) = \underline{\mathcal{E}(e)}$,*
4. *for any vertex $v \in \mathbb{V}$, $\sigma_{\mathcal{S}(v)} = \mathcal{E} \circ \sigma_v \circ \mathcal{E}^{-1}$.*

Then there exists an orientation-preserving homeomorphism $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\psi(\mathbb{G}) = \mathbb{G}'$ and ψ induces the two bijections $\mathcal{S}$ and $\mathcal{E}$.

If one considers only piecewise affine edges, the theorem can be applied to pre-graph with affine edges.

Later we will need the notion of diffeomorphisms at infinity. The motivation will appear in Lemma 12.3 where we show that the free boundary condition expectation on the plane associated with a Markovian holonomy field is a planar Markovian holonomy

field. In the following definition, $\mathbb{D}(0,R)^c$ is the complement set of the closed disk centered at 0 and of radius R.

Definition 3.10. — *A homeomorphism ψ of $\mathbb{R}^2$ is a diffeomorphism at infinity if there exists a real R such that $\psi_{|\mathbb{D}(0,R)^c}$ is a diffeomorphism.*

In fact, each time we consider a homeomorphism from a close domain delimited by a Jordan curve to an other domain delimited by an other Jordan curve, we can extend it as a diffeomorphism at infinity. Using the Carathéodory's theorem for Jordan curves, we can suppose that both domains are the unit disk. In this case, the result follows from the following lemma.

Lemma 3.1. — *Let $\mathbb{D}$ be the closed disk of center 0 and radius 1. Let $\psi : \partial\mathbb{D} \to \partial\mathbb{D}$ be a homeomorphism. There exists a diffeomorphism $\Psi : \mathbb{D}^c \to \mathbb{D}^c$ such that for any $x \in \partial\mathbb{D}$,*

$$\lim_{y\to x} \Psi(y) = \psi(x).$$

Besides, if ψ preserves the orientation, Ψ will also preserve the orientation.

Proof. — Let η be a smooth even positive function supported on $[-1,1]$. Let us consider for any real $r > 1$ the function $\eta_r(.) = (r-1)^{-1}\eta((r-1)^{-1}.)$. The family $(\eta_r)_{r>1}$ is a smooth even approximation to the identity when r goes to 1.

There is a natural bijection Φ between the set of homeomorphisms of $\partial\mathbb{D}$ and the set $Hom^{\mathbb{R}}_{\partial\mathbb{D}}$ of strictly increasing or decreasing continuous functions f from $\mathbb{R}$ to $\mathbb{R}$ such that $f - \mathsf{Id}$ is 1-periodic. Let $\psi : \partial\mathbb{D} \to \partial\mathbb{D}$ be a homeomorphism of the circle. We define the smooth function Ψ by:

$$\begin{aligned} \Psi : \quad \mathbb{D}^c &\to \mathbb{D}^c \\ re^{2i\pi\theta} &\mapsto re^{2i\pi(\Phi(\psi)*\eta_r)(\theta)}. \end{aligned}$$

Since ψ is continuous on the disk, the function $\Phi(\psi)$ is uniformly continuous. Thus $\Phi(\psi)*\eta_r$ converges uniformly to $\Phi(\psi)$ as r tends to 1. This implies that for any $x \in \partial\mathbb{D}$, $\lim_{y\to x} \Psi(y) = \psi(x)$. Besides, for any real $r > 1$, the convolution with η_r sends $Hom^{\mathbb{R}}_{\partial\mathbb{D}}$ on itself: this implies that Ψ is bijective. Since Ψ is differentiable, it remains to show that the Jacobian of Ψ is strictly positive. Yet, for any $x \in \mathbb{D}^c$, only the module of x is involved in the calculation of the module of $\Psi(x)$: the Jacobian matrix is triangular. Since η_r is even for any $r > 1$ and $\Phi(\psi)$ is strictly increasing (or decreasing), the derivative of $\Phi(\psi) * \eta_r$ is strictly positive (or negative). These two facts imply that the Jacobian matrix of Ψ is invertible, thus the function Ψ is a diffeomorphism. The last assertion about the orientation-preserving property is straightforward. □

3.3. Graphs and partial order. — The graphs with piecewise affine edges are interesting when one considers a special partial order on graphs studied in [**21**].

Definition 3.11. — *Let $\mathbb{G}$ and $\mathbb{G}'$ be two planar graphs. We say that $\mathbb{G}'$ is finer than $\mathbb{G}$ if $P(\mathbb{G}) \subset P(\mathbb{G}')$. We denote it by $\mathbb{G} \preccurlyeq \mathbb{G}'$.*

In fact, in Lemma 1.4.6. of [**21**], Lévy showed that this partial order is not directed. Yet, one can, by restricting it to a dense subspace of graphs, make it directed: for this, the edges of the graphs which we consider must be in a good subspace as defined below.

Definition 3.12. — *Let P be a subset of $P(M)$. A good subspace A of P is a dense subset of P for the convergence with fixed endpoints such that for any finite subset $\{c_1, \dots, c_n\}$ of A there exists a graph $\mathbb{G}$ such that $\{c_1, \dots, c_n\} \subset P(\mathbb{G})$.*

If A is a good subspace, $\mathcal{G}(A)$ endowed with $\preccurlyeq$ is directed. The following lemma is a reformulation of Proposition 1.4.8 of [**21**].

Lemma 3.2. — *For any Riemannian metric γ on M, the set $\mathsf{Aff}_\gamma(M)$ is a good subspace for $P(M)$.*

There are other natural examples of good subspaces of $P(M)$. For example, Baez in [**9**] used the good subspace of piecewise real-analytic paths in $P(\mathbb{R}^2)$ in order to define the Ashtekar and Lewandowski uniform measure. Another example of good subspace is used in the articles [**23**] and [**24**].

By definition, any path in M can be approximated by a sequence of paths in A if A is a good subspace. But $\mathsf{Aff}_\gamma(M)$ satisfies the stronger property which roughly asserts that $\mathcal{G}(\mathsf{Aff}_\gamma(M))$ is "dense" for a certain notion in the set of planar graphs. The next theorem is a direct consequence of Proposition 1.4.10. in [**21**]. It has to be noticed that, in the proof of Proposition 1.4.10. in [**21**], the measure of area does not have to be the measure of area associated with the chosen Riemannian metric. For the next theorem, let us suppose that M is an oriented compact surface with boundary.

Theorem 3.2. — *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ be a graph on M. Let γ be a Riemannian metric on M and let vol be a measure of area on M. There exists a sequence of finite planar graphs $\big(\mathbb{G}_n = (\mathbb{V}_n, \mathbb{E}_n, \mathbb{F}_n)\big)_{n\in\mathbb{N}}$ in $\mathcal{G}\big(\mathsf{Aff}_\gamma(M)\big)$ such that:*

1. *for any integer n, there exists ψ_n an orientation-preserving homeomorphism of M such that $\psi_n(\mathbb{G}) = \mathbb{G}_n$.*
2. *$\mathbb{V}_n = \mathbb{V}$,*
3. *for any edge $e \in \mathbb{E}$, $\psi_n(e)$ converges to e for the convergence with fixed endpoints,*
4. *for any face $F \in \mathbb{F}$, $vol(\psi_n(F)) \underset{n\to\infty}{\longrightarrow} vol(F)$.*

Another interesting property of $\mathcal{G}(\mathsf{Aff}_\gamma(\mathbb{R}^2))$ is the fact that any generic finite planar graph with piecewise affine edges can be sent by a piecewise smooth application on a subgraph of the $\mathbb{N}^2$ planar graph. We will prove this in Section 3.5, but before, we need to gather a few facts about graphs and triangulations.

3.4. Graphs and piecewise diffeomorphisms. — Since the plane can be endowed with the canonical Riemannian metric γ_0, it is natural to consider only the case where $\gamma = \gamma_0$ when one is working with the plane.

Definition 3.13. — *Let $\mathbb{G}$ be a finite planar graph in $\mathcal{G}\big(\mathsf{Aff}(\mathbb{R}^2)\big)$. It is simple if the boundary of any face of $\mathbb{G}$ is a simple loop. It is a triangulation if any bounded face is a non degenerate triangle.*

Definition 3.14. — *Let $\mathbb{G}$ be a finite planar graph in $\mathcal{G}\big(\mathsf{Aff}(\mathbb{R}^2)\big)$. A mesh of $\mathbb{G}$ is a simple graph $\mathbb{G}'$ in $\mathcal{G}\big(\mathsf{Aff}(\mathbb{R}^2)\big)$ such that $\mathbb{G} \preccurlyeq \mathbb{G}'$. A triangulation of $\mathbb{G}$ is a triangulation $\mathbb{T}$ such that $\mathbb{G} \preccurlyeq \mathbb{T}$ and the unbounded face of $\mathbb{T}$ is the unbounded face of $\mathbb{G}$.*

Two triangulations are homeomorphic if they are homeomorphic as finite planar graphs.

Definition 3.15. — *Let $\mathbb{G}$ and $\mathbb{G}'$ be two finite planar graphs in $\mathcal{G}\big(\mathsf{Aff}(\mathbb{R}^2)\big)$. A homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a $\mathbb{G} - \mathbb{G}'$ piecewise diffeomorphism if the three following assertions hold:*

1. *$\phi(\mathbb{G}) = \mathbb{G}'$,*
2. *there exists a mesh $\mathbb{G}_0$ of $\mathbb{G}$ (resp $\mathbb{G}'_0$ of $\mathbb{G}'$) such that $\phi(\mathbb{G}_0) = \mathbb{G}'_0$ and for any bounded face F of $\mathbb{G}_0$, $\phi_{|F} : F \to \phi(F)$ is a diffeomorphism whose Jacobian determinant is bounded below and above by some strictly positive real numbers and whose Jacobian determinant can also be extended on the boundary of F,*
3. *let F_∞ be the unbounded face of $\mathbb{G}_0$. The application $\phi_{|F_\infty} : F_\infty \to \phi(F_\infty)$ is a diffeomorphism.*

We will say that $\mathbb{G}_0$ is a good mesh for ϕ.

The piecewise diffeomorphisms we will construct will always be of the following form: they will be the extension (using Lemma 3.1 and the discussion before) of a piecewise affine homeomorphism from the interior of a piecewise affine Jordan curve to itself. Recall the definition of equivalence defined in Definition 3.8.

Proposition 3.2. — *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two homeomorphic simple finite planar graphs with piecewise affine edges. Let us choose an orientation-preserving homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(\mathbb{G}_1) = \mathbb{G}_2$. There exist two triangulations, $\mathbb{T}_1$ of $\mathbb{G}_1$, $\mathbb{T}_2$ of $\mathbb{G}_2$ and an orientation-preserving $\mathbb{G}_1 - \mathbb{G}_2$ piecewise-diffeomorphism ψ such that:*

1. *$\mathbb{T}_1$ is a good mesh for ψ,*
2. *ψ and ϕ are equivalent on $\mathbb{G}_1$,*
3. *$\psi(\mathbb{T}_1) = \mathbb{T}_2$.*

Consequently, the set of orientation-preserving $\mathbb{G}_1 - \mathbb{G}_2$ piecewise diffeomorphisms is not empty.

In order to prove this proposition, we will need the following result proved in the paper of Aronov-Seidel-Souvaine ([**6**]).

Theorem 3.3. — *Let Q_1 and Q_2 be two simple n-gons, seen as planar graphs with n vertices. Let us choose an orientation-preserving homeomorphism ψ which sends Q_1 on Q_2. Let T_1 (resp. T_2) be a triangulation of Q_1 (resp. Q_2). There exists $\hat{T}_1$ (resp. $\hat{T}_2$) a triangulation of Q_1 (resp. Q_2), finer than T_1 (resp. T_2) and an orientation-preserving homeomorphism ψ' such that ψ and ψ' are equivalent on Q_1 and $\psi'(\hat{T}_1) = \hat{T}_2$.*

Let $\mathbb{G}_1$, respectively $\mathbb{G}_2$, be a simple graph in $\mathcal{G}\big(\mathsf{Aff}(\mathbb{R}^2)\big)$ with only one face denoted F_1, respectively F_2. Let ψ be an orientation-preserving homeomorphism which sends $\mathbb{G}_1$ on $\mathbb{G}_2$. Then there exists a positive integer n such that ∂F_1 and ∂F_2 can be seen as two n-gons such that, when one considers these n-gons as graphs, ψ sends ∂F_1 on ∂F_2: in order to do so, it is enough to add some vertices on the boundaries of F_1 and F_2. This

remark will allow us to apply Theorem 3.3 to the faces of simple planar graphs with piecewise affine edges.

Let us remark also that the homeomorphism ψ between $\hat{T}_1$ and $\hat{T}_2$ in Theorem 3.3 can be chosen so that it is affine on each bounded face of $\hat{T}_1$.

Lemma 3.3. — *Let T_1 and T_2 be two triangulations in the plane. If they are homeomorphic, there exists a function ψ defined on the union of the bounded faces of T_1 and affine on each bounded face of T_1 such that ψ is an orientation-preserving homeomorphism which sends T_1 on T_2.*

Proof. — Let T_1 and T_2 be two homeomorphic triangulations and let ϕ be an orientation-preserving homeomorphism of $\mathbb{R}^2$ which sends T_1 on T_2. For any bounded face F of T_1, we can find an orientation-preserving affine map $\psi_{|F}$, defined on F, such that $\psi_{|F}$ and ϕ are equivalent on the border ∂F seen as a graph with 3 vertices: this map is actually unique.

Let us remark that for any triangle T, any $x \in T$ and any affine map F, $F(x)$ depends only on the image by F of the edge which contains x. This allows us to glue the affine maps $\left(\psi_{|F}\right)_F$ and to get the desired ψ. □

We can now prove Proposition 3.2.

Proof of Proposition 3.2. — Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two simple homeomorphic finite planar graphs with piecewise affine edges. Let ϕ be an orientation-preserving homeomorphism such that $\phi(\mathbb{G}_1) = \mathbb{G}_2$. For any face F of $\mathbb{G}_1$, F and $\phi(F)$ are simple polygons. As any polygon can be triangulated, one consequence of Theorem 3.3 and Lemma 3.3 is that there exists $\mathbb{T}_{1,F}$ (resp. $\mathbb{T}_{2,F}$) a triangulation of F (resp. $\phi(F)$) and $\psi_{|F}$ a function defined on F, affine on each bounded face of $\mathbb{T}_{1,F}$, such that $\psi_{|F}$ is an orientation-preserving homeomorphism between $\mathbb{T}_{1,F}$ and $\mathbb{T}_{2,F}$ and such that $\psi_{|F}$ and ϕ are equivalent on ∂F. We define $\mathbb{T}_1$ (resp. $\mathbb{T}_2$) as the triangulation obtained by taking the union of all the triangulations $(\mathbb{T}_{1,F})_F$ $((\mathbb{T}_{2,F})_F)$. As in the proof of Lemma 3.3, we can glue the $\psi_{|F}$ together: this gives a function $\psi_{|F_\infty^c}$ defined on the complementary of the unbounded face F_∞ of $\mathbb{G}_1$. As $\mathbb{G}_1$ is simple, the boundary of F_∞ is a Jordan curve. Thus, according to the discussion we had before Lemma 3.1, we can extend $\psi_{|F_\infty^c}$ on F_∞ and the resulting homeomorphism, denoted by ψ, is such that $\psi_{|F_\infty}$ is a diffeomorphism. By construction, ψ is an orientation-preserving $\mathbb{G} - \mathbb{G}'$ piecewise diffeomorphism, ϕ and ψ are equivalent on $\mathbb{G}_1$ and $\psi(\mathbb{T}_1) = \mathbb{T}_2$. □

3.5. Universality of $\mathbb{N}^2$. — We have seen after Lemma 2.3 that the set of piecewise horizontal or vertical paths is not dense in $P(\mathbb{R}^2)$ for the convergence with fixed endpoints. Thus, when one works with a stochastically continuous random holonomy field, it would seem that it is not possible to consider only such paths. In the following, we show that, in some sense, we can always inject any graph in the $\mathbb{N}^2$ graph defined in Exemple 3.2. This property is crucial in the study of planar Markovian holonomy fields.

Definition 3.16. — *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ be a finite planar graph. We say that $\mathbb{G}$ is generic if for any vertex $v \in \mathbb{V}$, $\#\mathbb{E}_v \leq 4$, where we remind the reader that $\mathbb{E}_v$ is the set of edges $e \in \mathbb{E}$ such that $\underline{e} = v$.*

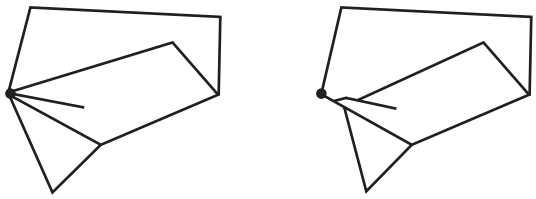


FIGURE 6. An approximation by a generic graph.

It is worth noticing that any finite planar graph can be approximated by a generic graph. This is illustrated in Figure 6.

Lemma 3.4. — *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ be a finite planar graph in $\mathcal{G}\big(\mathsf{Aff}\left(\mathbb{R}^2\right)\big)$. Let v be a vertex of $\mathbb{G}$. There exists a sequence of generic graphs $\mathbb{G}_n = (\mathbb{V}_n, \mathbb{E}_n, \mathbb{F}_n)$ in $\mathcal{G}\big(\mathsf{Aff}\left(\mathbb{R}^2\right)\big)$ such that:*

1. *$v \in \mathbb{G}_n$,*
2. *there exists an injective function $\mathcal{L}_n : L_v(\mathbb{G}) \to L_v(\mathbb{G}_n)$ such that for any loop $l \in L_v(\mathbb{G})$, $\mathcal{L}_n(l)$ converges with fixed endpoints to l.*

The notion of generic graphs was defined so that one could send any of such graph in $\mathbb{N}^2$.

Proposition 3.3. — *A finite planar graph $\mathbb{G}$ in $\mathcal{G}\big(\mathsf{Aff}\left(\mathbb{R}^2\right)\big)$ is generic if and only if there exists ψ an orientation-preserving homeomorphism of $\mathbb{R}^2$ such that $\psi\big(\mathbb{G}\big)$ is a subgraph of the $\mathbb{N}^2$ planar graph.*

Proof. — Since the planar graph $\mathbb{N}^2$ is generic, every subgraph of it is generic. This proves one implication and it remains to prove that one can send any generic finite planar graph in the planar graph $\mathbb{N}^2$.

Let us consider a generic finite planar graph $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ in $\mathcal{G}\big(\mathsf{Aff}\left(\mathbb{R}^2\right)\big)$. For each $v \in \mathbb{V}$, we choose a point $\tilde{v}$ of $\mathbb{N}^2$ such that the points $(\tilde{v})_{v\in\mathbb{V}}$ are all distinct. For each of these points, we choose a subset $\mathbb{E}_{\tilde{v}}$ of edges in $\frac{1}{3}\mathbb{N}^2$ going out of $\tilde{v}$ such that $\#\mathbb{E}_{\tilde{v}} = \#\mathbb{E}_v$. We consider the two pre-graphs:

1. $\mathbb{G}_p$ such that the set of edges $\mathbb{E}_p$ is equal to $\{e_p([0, \frac{1}{3}]), e_p \text{ represents } e \in \mathbb{E}\}$,
2. $\mathbb{G}'_p$ such that the set of edges $\mathbb{E}'_p$ is equal to $\cup_{v\in\mathbb{V}}\mathbb{E}_{\tilde{v}}$.

Let us define the application $\mathcal{S} : v \mapsto \tilde{v}$. Because $\#\mathbb{E}_{\tilde{v}} = \#\mathbb{E}_v$ and thanks to the shape of the graphs, we can choose a bijection $\mathcal{E} : \mathbb{E}_p \to \mathbb{E}'_p$ such that the conditions of Theorem 3.1 hold. Using this theorem, there exists an orientation-preserving homeomorphism ψ such that $\psi(\mathbb{G}_p) = \mathbb{G}'_p$ and ψ induces the two bijections $\mathcal{S}$ and $\mathcal{E}$. Let us define $\mathbb{G}' = \psi(\mathbb{G})$: we approximate $\mathbb{G}'$ in a $\frac{1}{3^k}\mathbb{N}^2$. For k big enough this approximation defines a graph $\tilde{\mathbb{G}}$ without new vertices. By construction, the assumptions 1. to 4. of Theorem 3.1 hold for $\mathbb{G}$ and $\tilde{\mathbb{G}}$. Using a dilation we can suppose that $k = 1$. Using Theorem 3.1, there exists an orientation-preserving homeomorphism ψ which sends $\mathbb{G}$ to the subgraph $\tilde{\mathbb{G}}$ of the $\mathbb{N}^2$ planar graph. □

Corollary 3.1. — *Let $\mathbb{G}$ be a finite generic planar graph in $\mathcal{G}\big(\mathsf{Aff}\,(\mathbb{R}^2)\big)$ which is simple. There exists a subgraph $\mathbb{G}'$ of the $\mathbb{N}^2$ graph such that the set of orientation-preserving $\mathbb{G}-\mathbb{G}'$ piecewise diffeomorphisms is not empty.*

Proof. — Let $\mathbb{G}$ be a finite generic planar graph in $\mathcal{G}\big(\mathsf{Aff}\,(\mathbb{R}^2)\big)$ which is simple. Due to Proposition 3.3 there exists a subgraph $\mathbb{G}'$ of the $\mathbb{N}^2$ planar graph such that $\mathbb{G}$ and $\mathbb{G}'$ are homeomorphic: the graph $\mathbb{G}'$ is also simple. By Proposition 3.2 the set of orientation-preserving $\mathbb{G}-\mathbb{G}'$ piecewise diffeomorphisms is not empty. □

4. Planar Markovian holonomy fields

We have now all the notions in order to define continuous and discrete planar Markovian holonomy fields: these are families of random holonomy fields on subsets of $P(\mathbb{R}^2)$ satisfying an area-preserving homeomorphism invariance and an independence property.

4.1. Definitions. — First, we define the strong and weak notions of (continuous) planar Markovian holonomy fields. We will use the following notation: if l is a simple loop in $\mathbb{R}^2$, $\mathsf{Int}(l)$ will stand for the bounded connected component of $\mathbb{R}^2 \setminus l$.

Definition 4.1. — *A G-valued strong (continuous) planar Markovian holonomy field is the data, for each measure of area vol on $\mathbb{R}^2$ of a gauge-invariant random holonomy field $\mathbb{E}_{vol}$ on $\mathbb{R}^2$ of weight $\mathbb{E}_{vol}(\mathbb{1}) = 1$, such that the three following axioms hold:*

$\mathbf{P_1}$: *Let vol and vol′ be two measures of area on $\mathbb{R}^2$. Let $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ be a locally bi-Lipschitz homeomorphism which preserves the orientation and which sends vol on vol′ (i.e. $vol' = vol \circ \psi^{-1}$). The mapping from $\mathcal{M}ult(P(\mathbb{R}^2), G)$ to itself induced by ψ, denoted also by ψ, satisfies:*

$$\mathbb{E}_{vol'} \circ \psi^{-1} = \mathbb{E}_{vol}.$$

Moreover, let $\mathbb{G}$ and $\mathbb{G}'$ be two finite planar graphs, let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism which preserves the orientation, which sends vol on vol′ and which sends $\mathbb{G}$ on $\mathbb{G}'$. The mapping from $\mathcal{M}ult(P(\mathbb{G}'), G)$ to $\mathcal{M}ult\big(P(\mathbb{G}), G\big)$ induced by ϕ, denoted also by ϕ, satisfies:

$$(\mathbb{E}_{vol'})_{|\mathcal{M}ult(P(\mathbb{G}'),G)} \circ \phi^{-1} = (\mathbb{E}_{vol})_{|\mathcal{M}ult(P(\mathbb{G}),G)}\,.$$

$\mathbf{P_2}$: *For any measure of area vol on $\mathbb{R}^2$, for any simple loops l_1 and l_2 such that $\mathsf{Int}(l_1)$ and $\mathsf{Int}(l_2)$ are disjoint, under $\mathbb{E}_{vol}$, the two families:*

$$\Big\{h(p), p \in P\big(\overline{\mathsf{Int}(l_1)}\big)\Big\} \text{ and } \Big\{h(p), p \in P\big(\overline{\mathsf{Int}(l_2)}\big)\Big\}$$

are $\mathcal{I}$-independent.

$\mathbf{P_3}$: *For any measures of area on $\mathbb{R}^2$, vol and vol′, if l is a simple loop such that vol and vol′ are equal when restricted to the interior of l, the following equality holds:*

$$\big(\mathbb{E}_{vol}\big)_{|\mathcal{M}ult(P(\overline{\mathsf{Int}(l)}),G)} = \big(\mathbb{E}_{vol'}\big)_{|\mathcal{M}ult(P(\overline{\mathsf{Int}(l)}),G)}.$$

When we will work with Markovian holonomy fields, it will be convenient to have the notion of weak (continuous) planar Markovian holonomy fields.

Definition 4.2. — *A G-valued weak (continuous) planar Markovian holonomy field is the data, for each measure of area vol on $\mathbb{R}^2$ of a gauge-invariant random holonomy field $\mathbb{E}_{vol}$ on $\mathsf{Aff}\left(\mathbb{R}^2\right)$ of weight $\mathbb{E}_{vol}(\mathbb{1}) = 1$, such that the three following axioms hold:*

$\mathbf{wP_1}$: *Let vol and vol′ be two measures of area on $\mathbb{R}^2$. Let $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism at infinity which preserves the orientation and which sends vol on vol′ (i.e. $vol' = vol \circ \psi^{-1}$). Let $p_1, \dots, p_n$ be paths in $\mathsf{Aff}\left(\mathbb{R}^2\right)$ such that for any $i \in \{1, \dots, n\}$, $p'_i = \psi(p_i)$ is in $\mathsf{Aff}\left(\mathbb{R}^2\right)$. Then for any continuous function $f : G^n \to \mathbb{R}$,*

$$\mathbb{E}_{vol}\Big[f\big(h(p_1), \dots, h(p_n)\big)\Big] = \mathbb{E}_{vol'}\Big[f\big(h(p'_1), \dots, h(p'_n)\big)\Big].$$

$\mathbf{wP_2}$: *For any measure of area vol on $\mathbb{R}^2$, for any simple loops l_1 and l_2 in $\mathsf{Aff}\left(\mathbb{R}^2\right)$ such that $\overline{\mathsf{Int}(l_1)}$ and $\overline{\mathsf{Int}(l_2)}$ are disjoint, under $\mathbb{E}_{vol}$, the two families:*

$$\Big\{h(p), p \in \mathsf{Aff}\left(\mathbb{R}^2\right) \cap P\big(\overline{\mathsf{Int}(l_1)}\big)\Big\} \text{ and } \Big\{h(p), p \in \mathsf{Aff}\left(\mathbb{R}^2\right) \cap P\big(\overline{\mathsf{Int}(l_2)}\big)\Big\}$$

are independent.

$\mathbf{wP_3}$: *For any measures of area on $\mathbb{R}^2$, vol and vol′, if l is a simple loop such that vol and vol′ are equal when restricted to the interior of l, the following equality holds:*

$$\big(\mathbb{E}_{vol}\big)_{|\mathcal{M}ult(\mathsf{Aff}(\overline{\mathsf{Int}(l)}),G)} = \big(\mathbb{E}_{vol'}\big)_{|\mathcal{M}ult(\mathsf{Aff}(\overline{\mathsf{Int}(l)}),G)}.$$

It can seem strange that we replaced the $\mathcal{I}$-independence by the usual independence in $\mathbf{wP_2}$, but this was precisely the point of Remark 2.6: in the setting of $\mathbf{wP_2}$, the endpoints set of any set of paths in $\overline{\mathsf{Int}(l_1)}$ and of any set of paths in $\overline{\mathsf{Int}(l_2)}$ are disjoint, thus the $\mathcal{I}$-independence notion is the same as the usual independence. As a consequence, any strong planar Markovian holonomy field defines, by restriction, a weak planar Markovian holonomy field. We will see later that the two notions are equivalent when we restrict them to stochastically continuous objects.

By G-valued (continuous) planar Markovian holonomy fields, we will denote the family of G-valued strong or weak (continuous) planar Markovian holonomy fields.

Definition 4.3. — *Let $\big(\mathbb{E}_{vol}\big)_{vol}$ be a G-valued planar Markovian holonomy field. We say that $\big(\mathbb{E}_{vol}\big)_{vol}$ is stochastically continuous if, for any measure of area vol on $\mathbb{R}^2$, $\mathbb{E}_{vol}$ is stochastically continuous.*

A discrete counterpart exists for planar Markovian holonomy fields.

Definition 4.4. — *A G-valued strong discrete planar Markovian holonomy field is the data, for each measure of area vol, for each finite planar graph $\mathbb{G}$, of a gauge-invariant random holonomy field $\mathbb{E}^{\mathbb{G}}_{vol}$ on $P(\mathbb{G})$ of weight $\mathbb{E}^{\mathbb{G}}_{vol}(\mathbb{1}) = 1$, such that the four following axioms hold:*

$\mathbf{DP_1}$: *Let vol and vol′ be two measures of area on $\mathbb{R}^2$, let $\mathbb{G}$ and $\mathbb{G}'$ be two finite planar graphs. Let ψ be a homeomorphism which preserves the orientation, satisfies $\psi(\mathbb{G}) = \mathbb{G}'$ and such that for any $F \in \mathbb{F}^b$, $vol(F) = vol'(\psi(F))$. The mapping from $\mathcal{M}ult(P(\mathbb{G}'), G)$ to $\mathcal{M}ult\big(P(\mathbb{G}), G\big)$ induced by ψ, denoted also by ψ, satisfies:*

$$\mathbb{E}^{\mathbb{G}'}_{vol'} \circ \psi^{-1} = \mathbb{E}^{\mathbb{G}}_{vol}.$$

DP$_2$: *For any measure of area vol on $\mathbb{R}^2$, for any finite planar graph $\mathbb{G}$, for any simple loops l_1 and l_2 in $P(\mathbb{G})$, such that $\mathsf{Int}(l_1) \cap \mathsf{Int}(l_2) = \emptyset$, under $\mathbb{E}^{\mathbb{G}}_{vol}$, the two families:*

$$\Big\{h(p), p \in P(\mathbb{G}) \cap P\big(\overline{\mathsf{Int}(l_1)}\big)\Big\} \text{ and } \Big\{h(p), p \in P(\mathbb{G}) \cap P\big(\overline{\mathsf{Int}(l_2)}\big)\Big\}$$

are $\mathcal{I}$-independent.

DP$_3$: *For any measures of area on $\mathbb{R}^2$, vol and vol$'$, if l is a simple loop such that vol and vol$'$ are equal when restricted to the interior of l, if $\mathbb{G}$ is included in $\overline{\mathsf{Int}(l)}$, then the following equality holds:*

$$\mathbb{E}^{\mathbb{G}}_{vol} = \mathbb{E}^{\mathbb{G}}_{vol'}.$$

DP$_4$: *For any measure of area vol on $\mathbb{R}^2$, for any finite planar graphs $\mathbb{G}_1$ and $\mathbb{G}_2$, such that $\mathbb{G}_1 \preccurlyeq \mathbb{G}_2$:*

$$\mathbb{E}^{\mathbb{G}_2}_{vol} \circ \rho^{-1}_{P(\mathbb{G}_1),P(\mathbb{G}_2)} = \mathbb{E}^{\mathbb{G}_1}_{vol},$$

where we remind the reader that

$$\rho_{P(\mathbb{G}_1),P(\mathbb{G}_2)} : \mathcal{M}ult\big(P(\mathbb{G}_2), G\big) \to \mathcal{M}ult\big(P(\mathbb{G}_1), G\big)$$

is the restriction map.

We will use also the following weak version of discrete planar Markovian holonomy fields.

Definition 4.5. — *A G-valued weak discrete planar Markovian holonomy field is the data, for each measure of area vol, for each finite graph $\mathbb{G}$ in $\mathcal{G}\big(\mathsf{Aff}\left(\mathbb{R}^2\right)\big)$, of a gauge-invariant random holonomy field $\mathbb{E}^{\mathbb{G}}_{vol}$ on $P(\mathbb{G})$ of weight $\mathbb{E}^{\mathbb{G}}_{vol}(\mathbb{1}) = 1$, such that the four following axioms hold:*

wDP$_1$: *Let vol and vol$'$ be two measures of area on $\mathbb{R}^2$, let $\mathbb{G}$ and $\mathbb{G}'$ be two* simple *finite planar graphs in $\mathcal{G}\big(\mathsf{Aff}\left(\mathbb{R}^2\right)\big)$. Let ψ be a $\mathbb{G} - \mathbb{G}'$ piecewise diffeomorphism which preserves the orientation. Suppose that for any bounded face F of $\mathbb{G}$, $vol(F) = vol'(\psi(F))$. Then the mapping from $\mathcal{M}ult(P(\mathbb{G}'), G)$ to $\mathcal{M}ult\big(P(\mathbb{G}), G\big)$ induced by ψ satisfies:*

$$\mathbb{E}^{\mathbb{G}'}_{vol'} \circ \psi^{-1} = \mathbb{E}^{\mathbb{G}}_{vol}.$$

wDP$_2$: *For any measure of area vol on $\mathbb{R}^2$, for any finite graph $\mathbb{G}$ in $\mathcal{G}\big(\mathsf{Aff}\left(\mathbb{R}^2\right)\big)$, for any simple loops l_1 and l_2 in $P(\mathbb{G})$, such that $\overline{\mathsf{Int}(l_1)} \cap \overline{\mathsf{Int}(l_2)} = \emptyset$, under $\mathbb{E}^{\mathbb{G}}_{vol}$, the two families:*

$$\Big\{h(p), p \in P(\mathbb{G}) \cap P\big(\overline{\mathsf{Int}(l_1)}\big)\Big\} \text{ and } \Big\{h(p), p \in P(\mathbb{G}) \cap P\big(\overline{\mathsf{Int}(l_2)}\big)\Big\}$$

are independent.

wDP$_3$: *For any measures of area on $\mathbb{R}^2$, vol and vol$'$, if l is a simple loop such that vol and vol$'$ are equal when restricted to the interior of l, if $\mathbb{G}$ is included in $\overline{\mathsf{Int}}(l)$, then the following equality holds:*

$$\mathbb{E}^{\mathbb{G}}_{vol} = \mathbb{E}^{\mathbb{G}}_{vol'}.$$

wDP$_4$: *For any measure of area vol on $\mathbb{R}^2$, for any finite planar graphs $\mathbb{G}_1$ and $\mathbb{G}_2$ in $\mathcal{G}\big(\mathsf{Aff}\left(\mathbb{R}^2\right)\big)$, such that $\mathbb{G}_1 \preccurlyeq \mathbb{G}_2$:*

$$\mathbb{E}^{\mathbb{G}_2}_{vol} \circ \rho^{-1}_{P(\mathbb{G}_1),P(\mathbb{G}_2)} = \mathbb{E}^{\mathbb{G}_1}_{vol},$$

where again $\rho_{P(\mathbb{G}_1),P(\mathbb{G}_2)} : \mathcal{M}ult\big(P(\mathbb{G}_2),G\big) \to \mathcal{M}ult\big(P(\mathbb{G}_1),G\big)$ is the restriction map.

Let us remark that the Axioms **DP$_3$** and **wDP$_3$** can be directly deduced respectively from **DP$_1$** and **wDP$_1$** by considering the identity function of the plane. Yet, in order to have a similar formulation for continuous and discrete objects we preferred to keep them in the definitions.

As for the continuous objects, any G-valued strong discrete planar Markovian holonomy field defines, by restriction, a weak discrete planar Markovian holonomy field. By G-valued discrete planar Markovian holonomy fields, we will denote the family of G-valued strong or weak discrete planar Markovian holonomy fields. In any assertion about G-valued discrete planar Markovian holonomy fields, the reader will have to understand that, in the case we are working with a weak discrete planar Markovian holonomy field, all the graphs must be in $\mathcal{G}\big(\mathsf{Aff}\left(\mathbb{R}^2\right)\big)$.

From now on, if not specified, all the planar Markovian holonomy fields will be G-valued, thus we will omit to specify it.

Remark 4.1. — *Let $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ be a discrete planar Markovian holonomy field. As an application of Proposition 2.1, the Axiom* **DP$_4$** *or* **wDP$_4$** *allows us to define for any measure of area vol and any possibly infinite planar graph $\mathbb{G}$, a unique gauge-invariant random holonomy field $\mathbb{E}^{\mathbb{G}}_{vol}$ on $P(\mathbb{G})$ whose weight $\mathbb{E}^{\mathbb{G}}_{vol}(\mathbb{1})$ is equal to 1, such that, for any finite planar graph $\mathbb{G}_f \preccurlyeq \mathbb{G}$, $\mathbb{E}^{\mathbb{G}}_{vol} \circ \rho^{-1}_{P(\mathbb{G}_f),P(\mathbb{G})} = \mathbb{E}^{\mathbb{G}_f}_{vol}$.*

Besides, Lemma 3.2 shows that the set of finite planar graphs $\mathcal{G}\big(\mathsf{Aff}(\mathbb{R}^2)\big)$ is directed: the family

$$\left\{\left(\mathcal{M}ult\big(P\left(\mathbb{G}\right),G\big),\mathcal{B},\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G}\in\mathcal{G}(\mathsf{Aff}(\mathbb{R}^2))},\left(\rho_{P(\mathbb{G}),P(\mathbb{G}')}\right)_{\mathbb{G},\mathbb{G}'\in\mathcal{G}(\mathsf{Aff}(\mathbb{R}^2)),\mathbb{G}\preccurlyeq\mathbb{G}'}\right\}$$

is a projective family. Hence we can define a unique gauge-invariant random holonomy field on $\mathsf{Aff}\left(\mathbb{R}^2\right)$, whose weight is equal to 1, which we denote by $\mathbb{E}^{\mathsf{Aff}}_{vol}$, such that for any finite planar graph $\mathbb{G}_f \in \mathcal{G}\big(\mathsf{Aff}(\mathbb{R}^2)\big)$, $\mathbb{E}^{\mathsf{Aff}}_{vol} \circ \rho^{-1}_{P(\mathbb{G}_f),\mathsf{Aff}(\mathbb{R}^2)} = \mathbb{E}^{\mathbb{G}_f}_{vol}$.

The notions of area-dependent continuity and locally stochastically $\frac{1}{2}$-Hölder continuity that we are going to define are similar to the notions explained in Definition 3.2.8 of [**21**] that Lévy used for discrete Markovian holonomy fields.

Definition 4.6. — *Let $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ be a family of random holonomy fields such that for any measure of area vol and any finite planar graph $\mathbb{G}$, $\mathbb{E}^{\mathbb{G}}_{vol}$ is a random holonomy field on $P(\mathbb{G})$. It is locally stochastically $\frac{1}{2}$-Hölder continuous if for any measure of area vol on $\mathbb{R}^2$, $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G}}$ is a uniformly locally $\frac{1}{2}$-Hölder continuous family of random holonomy fields.*

It is continuously area-dependent if, for any sequence of finite planar graphs $\mathbb{G}_n$ which are the images of a common graph $\mathbb{G}$ by a sequence of homeomorphisms ψ_n ($\psi_n(\mathbb{G}) = \mathbb{G}_n$)

and such that $vol(\psi_n(F))$ tends to $vol(F)$ as n tends to infinity for any bounded face F of $\mathbb{G}$, the following convergence holds:

$$\mathbb{E}^{\mathbb{G}_n}_{vol} \circ \psi_n^{-1} \underset{n\to\infty}{\longrightarrow} \mathbb{E}^{\mathbb{G}}_{vol},$$

where we denote by ψ_n the induced map from $\mathcal{M}ult\big(P(\mathbb{G}_n), G\big)$ to $\mathcal{M}ult\big(P(\mathbb{G}), G\big)$.

It is regular if it is locally stochastically $\frac{1}{2}$-Hölder continuous and continuously area-dependent.

We define also the notion of stochastic continuity in law.

Definition 4.7. — *Let $\big(\mathbb{E}^{\mathbb{G}}_{vol}\big)_{\mathbb{G},vol}$ be a family of random holonomy fields such that for any measure of area vol and any finite planar graph $\mathbb{G}$, $\mathbb{E}^{\mathbb{G}}_{vol}$ is a random holonomy field on $P(\mathbb{G})$. The family $\big(\mathbb{E}^{\mathbb{G}}_{vol}\big)_{\mathbb{G},vol}$ is stochastically continuous in law if for any measure of area vol, for any integer m, any finite planar graph $\mathbb{G}$, any sequence of finite planar graphs $(\mathbb{G}_n)_{n\geq 0}$ and any sequence of m-tuples of loops in $\mathbb{G}_n$, $((l^n_k)^m_{k=1})_{n\in\mathbb{N}}$, if there exists a m-tuple of loops in $\mathbb{G}$, $(l_k)^m_{k=1}$ such that for any $i \in \{1,..,k\}$, l^n_i converges with fixed endpoints to l_i when n goes to infinity, then the law of $\big(h(l^n_k)\big)^m_{k=1}$ under $\mathbb{E}^{\mathbb{G}_n}_{vol}$ converges to the law of $\big(h(l_k)\big)^m_{k=1}$ under $\mathbb{E}^{\mathbb{G}}_{vol}$ when n goes to infinity.*

Let us also remak that the Axioms $\mathbf{DP_1}$ and $\mathbf{wDP_1}$ are not just a discrete version of $\mathbf{P_1}$ and $\mathbf{wP_1}$ since in $\mathbf{DP_1}$ and $\mathbf{wDP_1}$ we do not require that vol' is the image of vol by ψ. Thus, it is not obvious that any planar Markovian holonomy field, when restricted to graphs, defines a discrete planar Markovian holonomy field. For now, we define the notion of constructibility but later, in Theorem 5.1 and Theorem 10.4, we will show that, under some regularity conditions, any planar Markovian holonomy field is constructible.

Definition 4.8. — *Let $\big(\mathbb{E}_{vol}\big)_{vol}$ be a weak (resp. strong) planar Markovian holonomy field. It is constructible if the family of measures:*

$$\Big(\big(\mathbb{E}_{vol}\big)_{|\mathcal{M}ult(P(\mathbb{G}),G)}\Big)_{\mathbb{G},vol}$$

is a weak (resp. strong) discrete planar Markovian holonomy field.

Remark 4.2. — *If $\big(\mathbb{E}_{vol}\big)_{vol}$ is a constructible stochastically continuous planar Markovian holonomy field, its restriction to graphs defines a stochastically continuous in law discrete planar Markovian holonomy field.*

We have seen, in Remark 4.1, that given a strong discrete planar Markovian holonomy field $\big(\mathbb{E}^{\mathbb{G}}_{vol}\big)_{\mathbb{G},vol}$, we could define a family of probability measures $\big(\mathbb{E}^{\mathsf{Aff}}_{vol}\big)_{vol}$. If $\big(\mathbb{E}^{\mathbb{G}}_{vol}\big)_{\mathbb{G},vol}$ is locally stochastically $\frac{1}{2}$-Hölder continuous, so is $\mathbb{E}^{\mathsf{Aff}}_{vol}$ for any measure of area vol. By Theorem 2.1, one can extend $\mathbb{E}^{\mathsf{Aff}}_{vol}$ as a stochastically continuous random holonomy field on $\mathbb{R}^2$, denoted by $\mathbb{E}_{vol}$. We have thus defined $\big(\mathbb{E}_{vol}\big)_{vol}$ a family of stochastically continuous random holonomy fields on $\mathbb{R}^2$.

If $\big(\mathbb{E}^{\mathbb{G}}_{vol}\big)_{\mathbb{G},vol}$ is continuously area-dependent, the family $\big(\mathbb{E}_{vol}\big)_{vol}$ is a stochastically continuous strong planar Markovian holonomy field: this is a slight modification of Theorem 3.2.9 in [**21**]. The only difficult point is to prove that the axiom $\mathbf{P_1}$ is valid for $\big(\mathbb{E}_{vol}\big)_{vol}$.

Using the same arguments as Lévy used in Proposition 3.4.1 of [**21**], if $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ is continuously area-dependent then for any finite planar graph $\mathbb{G}$, for any measure of area vol,

$$\mathbb{E}^{\mathbb{G}}_{vol} = \left(\mathbb{E}_{vol}\right)_{|\mathcal{M}ult(P(\mathbb{G}),G)}.$$

Let us remark that it is important, in order to prove this assertion, that we consider all the homeomorphisms in the Axiom $\mathbf{DP_1}$. Using this assertion, we see that $\left(\mathbb{E}_{vol}\right)_{vol}$ satisfies the second assertion in the Axiom $\mathbf{P_1}$. It remains to prove the first assertion. Let vol, vol' and ψ which satisfy the conditions of this first assertion. Let p_1, ..., p_n be paths on the plane and let f be a continuous function on G^n. We need to prove that:

$$\mathbb{E}_{vol}\left[f(h(p_1),...,h(p_n))\right] = \mathbb{E}_{vol'}\left[f(h(\psi(p_1)),...,h(\psi(p_n)))\right]. \tag{9}$$

Let us consider, for any $i \in \{1,...,n\}$, a sequence of piecewise affine paths $(p_i^j)_{j\in\mathbb{N}}$ which converges with fixed endpoints to p_i when j goes to infinity. Using Lemmas 2.1 and 2.2, for any $i \in \{1,...,n\}$, $\psi(p_i^j)$ converges with fixed endpoints to $\psi(p_i)$ when j goes to infinity. Since $\mathbb{E}_{vol}$ is stochastically continuous, it is enough to prove the Equation (9) when p_1, ..., p_n are piecewise affine paths. But in this case, there exists a graph $\mathbb{G}$ such that $\{p_1,...,p_n\} \subset P(\mathbb{G})$ and $\psi(\mathbb{G})$ is also a planar graph: the Equality (9) is a consequence of the second assertion in the Axiom $\mathbf{P_1}$ that we already proved. Thus, we proved the following theorem.

Theorem 4.1. — *Let $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ be a strong discrete planar Markovian holonomy field. If it is regular then there exists a unique stochastically continuous strong planar Markovian holonomy field $\left(\mathbb{E}_{vol}\right)_{vol}$ such that, for any finite planar graph $\mathbb{G}$ and any measure of area vol, $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ is the restriction to $\mathcal{M}ult(P(\mathbb{G}),G)$ of $\mathbb{E}_{vol}$:*

$$\mathbb{E}_{vol} \circ \rho^{-1}_{P(\mathbb{G}),P(M)} = \mathbb{E}^{\mathbb{G}}_{vol}$$

Remark 4.3. — *The proof of Lévy of the axiom $\mathbf{A_4}$ page* 123 *of* [**21**] *in the proof of Theorem* 3.2.9 *is problematic. In this article, we corrected this proof by considering the modified Axiom $\mathbf{P_1}$ in Definition 4.1 and Axiom $\mathbf{A_4}$ in Definition 12.5.*

The next assertion is a consequence of Theorem 4.1 and Remark 4.2 when one considers strong discrete planar Markovian holonomy fields. It is a direct consequence of Theorem 2.1 and Remark 4.2 for weak discrete planar Markovian holonomy fields.

Corollary 4.1. — *Any strong discrete planar Markovian holonomy field which is regular is stochastically continuous in law. Any weak discrete planar Markovian holonomy field which is locally stochastically $\frac{1}{2}$-Hölder continuous is stochastically continuous in law.*

In the rest of the paper, we will mostly work with stochastically continuous in law discrete planar Markovian holonomy fields.

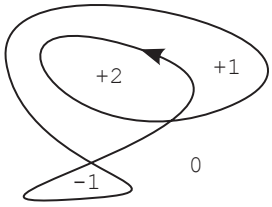


FIGURE 7. The index of a curve.

4.1.1. Example: the index field. — In this section, we define one of the simplest planar Markovian holonomy field. For any parametrized loop l, for any x in $\mathbb{R}^2 \setminus l\big([0,1]\big)$, the index of l with respect to x is defined as the integer:

$$\mathfrak{n}_l(x) = \frac{1}{2i\pi} \oint_l \frac{dz}{z-x}.$$

Actually, one needs to approximate uniformly l by piecewise smooth loops and take the limit. Given the Banchoff-Pohl's inequality proved in [**27**], the index of any rectifiable loop is square-integrable, thus integrable since it takes values in $\mathbb{N}$. Since any loop is bounded, $\mathfrak{n}_l$ vanishes outside a ball, so that $\mathfrak{n}_l$ is also integrable against any measure of area *vol*. Besides if l_1 and l_2 are based at the same point, using the additivity of the curve integral, we get:

$$\mathfrak{n}_{l_1 l_2} = \mathfrak{n}_{l_1} + \mathfrak{n}_{l_2}. \tag{10}$$

An other way to define the index field is by first constructing the L^2-functions valued non-random holonomy field on $\mathsf{Aff}(\mathbb{R}^2)$ which sends l on $\mathfrak{n}_l$: this can be defined as a combinatorial object. Using the L^2 norm on the set of $\mathbb{N}$-valued functions on the plane and considering the Lebesgue measure on the plane, it is easy to see that this holonomy field is locally stochastically $\frac{1}{2}$-Hölder continuous. Using Theorem 2.1, we can extend it in order to get a stochastically continuous non-random planar holonomy field on the plane.

Let D be an element of the Lie algebra $\mathfrak{g}$ of G. We can now define the index field driven by D.

Definition 4.9. — *The index field driven by D is the only planar Markovian holonomy field $\big(\mathbb{E}_{vol}\big)_{vol}$ such that for any measure of area vol, any loops $l_1, \dots, l_n$ based at the same point and any continuous function f from G^n to $\mathbb{R}$ invariant by diagonal conjugation, we have:*

$$\mathbb{E}_{vol}\Big[f\big(h(l_1), \dots, h(l_n)\big)\Big] = f\left(e^{D\int_{\mathbb{R}^2} \mathfrak{n}_{l_1}(x) vol(dx)}, \dots, e^{D\int_{\mathbb{R}^2} \mathfrak{n}_{l_n}(x) vol(dx)}\right).$$

The existence of such a planar Markovian holonomy field is due to the fact that one can consider for any finite family of loops $(l_1, ..., l_n)$ based at 0, the random holonomy field on $(l_1, ..., l_n)$ such that $(h(l_1), ..., h(l_n))$ has the law of $\left(Ue^{D\int_{\mathbb{R}^2} \mathfrak{n}_{l_1}(x) vol(dx)}U^{-1}, ..., Ue^{D\int_{\mathbb{R}^2} \mathfrak{n}_{l_n}(x) vol(dx)}U^{-1}\right)$, where U is a Haar random variable on G. It is a gauge-invariant random holonomy field due to the Equation (10) and it is actually a measure on $\mathcal{M}ult_{P(\mathbb{R}^2)}(\{l_1, ..., l_n\}, G)$. An application of Proposition 2.5 allows us to conclude. An interesting fact with this planar Markovian holonomy field

is that it is stochastically continuous and constructible. It can also be used in order to add a drift to any holonomy field on the plane.

Lemma 4.1. — *Let μ be a random holonomy field on the plane, let vol be a measure of area and let D be an element of the center of $\mathfrak{g}$. There exists a planar holonomy field $\mu^{D,vol}$ such that for any loops $l_1, \dots, l_n$ based at the same point and any continuous function f from G^n to $\mathbb{R}$ invariant by diagonal conjugation, we have:*

$$\mu^{D,vol}\Big[f\big(h(l_1),\dots,h(l_n)\big)\Big] = \mu\left[f\left(e^{D\int_{\mathbb{R}^2}\mathfrak{n}_{l_1}(x)vol(dx)}h(l_1),\dots,e^{D\int_{\mathbb{R}^2}\mathfrak{n}_{l_n}(x)vol(dx)}h(l_n)\right)\right].$$

Any regularity which holds for μ holds for $\mu^{D,vol}$. Besides, if $(\mathbb{E}_{vol})_{vol}$ is a planar Markovian holonomy field, so is $(\mathbb{E}^{D,vol}_{vol})_{vol}$.

4.2. Restriction and extension of the structure group. — We have seen in Section 2.2.5 how to restrict and extend the structure group of a gauge-invariant random holonomy field. We will explain how to apply this to discrete planar Markovian holonomy fields, but Propositions 4.1 and 4.2 can also be applied to planar Markovian holonomy fields. Let us consider H a closed subgroup of G.

4.2.1. Extension. — Let $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ be a H-valued (weak or strong) discrete planar Markovian holonomy field. Recall the notation $\mu\circ\hat{i}_P^{-1}$ defined in Notation 2.3. Following Section 2.2.5, in the gauge-invariant setting, for any finite planar graph $\mathbb{G}$ and any measure of area *vol*, we can see $\mathbb{E}^{\mathbb{G}}_{vol}$ as a G-valued gauge-invariant random holonomy field on $P(\mathbb{G})$ by considering $\mathbb{E}^{\mathbb{G}}_{vol}\circ\hat{i}^{-1}_{P(\mathbb{G})}$. It is not difficult to verify next proposition.

Proposition 4.1. — *The family $\left(\mathbb{E}^{\mathbb{G}}_{vol}\circ\hat{i}^{-1}_{P(\mathbb{G})}\right)_{\mathbb{G},vol}$ is a G-valued (weak or strong) discrete planar Markovian holonomy field. The regularities are the same for the H-valued and the G-valued random holonomy fields.*

4.2.2. Restriction. —

Proposition 4.2. — *Let $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ be a G-valued stochastically continuous in law (weak or strong) discrete planar Markovian holonomy field. Let us suppose that for any finite planar graph $\mathbb{G}$, any vertex v of $\mathbb{G}$, any measure of area vol and any simple loop $l \in L_v(\mathbb{G})$,*

$$h(l) \in H, \quad \mathbb{E}^{\mathbb{G}}_{vol} \text{ a.s.},$$

then there exists a H-valued stochastically continuous in law (weak or strong) discrete planar Markovian holonomy field $\left(\tilde{\mathbb{E}}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ such that:

$$\mathbb{E}^{\mathbb{G}}_{vol} = \tilde{\mathbb{E}}^{\mathbb{G}}_{vol}\circ\hat{i}^{-1}_{P(\mathbb{G})},$$

for any finite planar graph $\mathbb{G}$ and any measure of area vol.

The proof of Proposition 4.2 relies heavily on a theorem which will be proved later, namely Theorem 10.1, thus it will be given page 202. It is more difficult than one might think to prove this proposition because of the non-unicity of the random holonomy field μ_H in Proposition 2.7. In fact, one can show in general that the natural choice we made in Proposition 2.7 does not allow one to define a H-valued discrete Markovian holonomy field: we will give an exemple page 201 which illustrates this fact. For now, let us give

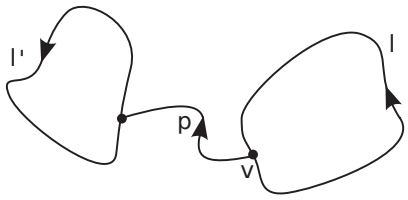


FIGURE 8. Two simple loops l, l' with a path p joining $\underline{l}$ to $\underline{l'}$.

a simple argument which shows the problem which appears when one wants to restrict the structure group of a discrete Markovian holonomy field.

Let $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ be a G-valued stochastically continuous in law discrete planar Markovian holonomy field which satisfies the condition of Proposition 4.2. Let $\mathbb{G}$ be a finite planar graph and let vol be a measure of area on the plane, it is natural to set:

$$\tilde{\mathbb{E}}^{\mathbb{G}}_{vol} = \left(\widehat{\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{|L_v(\mathbb{G})} \circ \iota^{-1}} \right)_{|\mathcal{I}_H}, \tag{11}$$

where v is any vertex of $\mathbb{G}$, $\iota : \mathcal{M}ult(L_v(\mathbb{G}), H) \to \mathcal{M}ult(P(\mathbb{G}), H)$ is any map given by Lemma 2.5, $\mathcal{I}_H$ is the H-invariant σ-field and $\widehat{\ }$ is the gauge-invariant extension (where the gauge group is now built on H) given by Proposition 2.2.

Let l and l' be two simple loops in $P(\mathbb{G})$, with $\underline{l} = v$, such that $\overline{\mathsf{Int}(l)} \cap \overline{\mathsf{Int}(l')} = \emptyset$ as shown in the Figure 8. If the family of measures $\left(\tilde{\mathbb{E}}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ just defined above were a discrete planar Markovian holonomy field, then $h(l)$ and $h(l')$ would be independent under $\tilde{\mathbb{E}}^{\mathbb{G}}_{vol}$. But if p is the path from v to $\underline{l'}$ used to define ι and if f_1, f_2 are two continuous functions on H invariant by conjugation by H, we have:

$$\tilde{\mathbb{E}}^{\mathbb{G}}_{vol}\Big(f_1\big(h\left(l\right)\big)f_2\big(h\left(l'\right)\big)\Big) = \mathbb{E}^{\mathbb{G}}_{vol}\Big(f_1\big(h(l)\big)f_2\big(h(pl'p^{-1})\big)\Big). \tag{12}$$

In the r.h.s. appear the two loops l and $pl'p^{-1}$ which are not of null intersection (as they share at least v) and only appear functions invariant by conjugation by H and not by G. This does not allow us to split the expectation into a product.

5. Weak constructibility and locality

In this section, we are going to prove that any weak continuous planar Markovian holonomy field is constructible. For this, we will need a proposition which is a direct consequence of an important theorem of Moser and Dacorogna in [**11**]. We denote by Leb the Lesbegue measure on $\mathbb{R}^2$.

Proposition 5.1. — *Let Q be a simple n-gon in $\mathbb{R}^2$. Let f and g be two strictly positive functions on Q which are in $C^1(Q) \cap C^0(\overline{Q})$. Suppose that:*

$$\int_Q f d\mathsf{Leb} = \int_Q g d\mathsf{Leb}.$$

Then there exists $\phi \in Diff^1(Q) \cap Diff^0(\overline{Q})$, a homeomorphism of $\overline{Q}$ which restricts to a diffeomorphism of Q, such that:

$$g.\mathsf{Leb}_{|Q} = \left(f.\mathsf{Leb}_{|Q}\right) \circ \phi^{-1}.$$

and $\phi(x) = x$ for any $x \in \partial Q$.

Proof. — In [**11**], page 15 the authors define for any integer $k \geq 1$, a property (H_k) for open subsets of $\mathbb{R}^n$. They show in Theorem 7 of the same paper, that for any integer $k \geq 1$, any open domain Ω which satisfies (H_k), any positive functions f and g in $C^k(\Omega)$ with $f + \frac{1}{f}$ and $g + \frac{1}{g}$ bounded and satisfying:

$$\int_\Omega f d\mathsf{Leb} = \int_\Omega g d\mathsf{Leb},$$

there exists $\phi \in Diff^1(\Omega) \cap Diff^0(\overline{\Omega})$ with $\phi(x) = x$ on $\partial\Omega$ such that:

$$g.d\mathsf{Leb}_{|\Omega} = \left(f.d\mathsf{Leb}_{|\Omega}\right) \circ \phi^{-1}.$$

Besides, Proposition $A.2$ of the same paper asserts that any domain with Lipschitz boundary satisfies (H_k) for every $k \geq 1$. The proposition follows from this discussion. □

Which this proposition, we can prove the following theorem which allows us to link the continuous and the discrete weak planar Markovian holonomy fields.

Theorem 5.1. — *Any weak planar Markovian holonomy field is constructible.*

Proof. — Let $(\mathbb{E}_{vol})_{vol}$ be a weak planar Markovian holonomy field. Let $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ be the family of random holonomy fields that we get by restricting $(\mathbb{E}_{vol})_{vol}$ on $\mathcal{M}ult(P(\mathbb{G}), G)$ for any finite planar graph $\mathbb{G}$ in $\mathcal{G}\left(\mathsf{Aff}\left(\mathbb{R}^2\right)\right)$. As explained before Definition 4.8, we only have to check that the Axiom $\mathbf{wDP_1}$ is satisfied by $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$. Let vol and vol' be two measures of area on $\mathbb{R}^2$. Consider $\mathbb{G}$ and $\mathbb{G}'$ two simple finite planar graphs in $\mathcal{G}\big(\mathsf{Aff}(\mathbb{R}^2)\big)$. Let ψ be an orientation-preserving $\mathbb{G} - \mathbb{G}'$ piecewise diffeomorphism. Let us suppose that for any bounded face F of $\mathbb{G}$, $vol(F) = vol'(\psi(F))$.

Let F'_∞ be the unbounded face of $\mathbb{G}'$. Let us suppose that we managed to construct an orientation-preserving diffeomorphism at infinity Φ on $\mathbb{R}^2$ such that:

$$vol'_{|(F'_\infty)^c} = \left(vol \circ (\Phi \circ \psi)^{-1}\right)_{|(F'_\infty)^c}, \tag{13}$$

$$\Phi_{|\mathbb{G}'} = Id_{|\mathbb{G}'}. \tag{14}$$

As $\mathbb{G}'$ is a simple graph, the boundary of F'_∞ is a simple loop. Applying the Axiom $\mathbf{wP_3}$ and using the condition (13):

$$\mathbb{E}^{\mathbb{G}'}_{vol'} = \mathbb{E}^{\mathbb{G}'}_{vol\circ(\Phi\circ\psi)^{-1}}.$$

Yet by condition (14), $\mathbb{G}' = \Phi(\mathbb{G}') = \Phi \circ \psi(\mathbb{G})$. Thus, as an application of Axiom $\mathbf{wP_1}$, we get:

$$\mathbb{E}^{\mathbb{G}'}_{vol'} = \mathbb{E}^{\mathbb{G}'}_{vol\circ(\Phi\circ\psi)^{-1}} = \mathbb{E}^{\Phi\circ\psi(\mathbb{G})}_{vol\circ(\Phi\circ\psi)^{-1}} = \mathbb{E}^{\mathbb{G}}_{vol},$$

which is the desired equality.

It remains to construct an orientation-preserving diffeomorphism at infinity Φ on $\mathbb{R}^2$ satisfying the two conditions (13) and (14). This will be done by applying twice the

Proposition 5.1. Let $\mathbb{G}_0$ be a good mesh for ψ. First we regularize the measure $vol \circ \psi^{-1}$, which does not have a smooth density, by applying the proposition for each face of the mesh $\mathbb{G}_0$. Then we transport the resulting measure of area on vol' by applying again Proposition 5.1 for each bounded face of $\mathbb{G}'$.

Let us fix a measure of area vol'' such that, for any bounded face F of $\mathbb{G}_0$, $vol''(\psi(F)) = vol(F)$. Let us consider any bounded face F_0 of $\mathbb{G}_0$ which is included in a bounded face of $\mathbb{G}$. By definition of a $\mathbb{G} - \mathbb{G}'$ piecewise diffeomorphism and the definition of a good mesh for ψ, $\psi_{|F_0}$ is a diffeomorphism from F_0 to $\psi(F_0)$ which are two simple n-gons. Thus, $\widetilde{vol}_{|\psi(F_0)} = vol_{|F_0} \circ (\psi_{|F_0})^{-1}$ defines a measure with smooth density on $\psi(F_0)$. Using the condition on the Jacobian determinant of ψ, this smooth density can be extended as a continuous function on $\overline{\psi(F_0)}$. Using Proposition 5.1, we can consider $\phi_{|\psi(F_0)} \in Diff^1(\psi(F_0)) \cap Diff^0(\overline{\psi(F_0)})$ with $\phi_{|\psi(F_0)}(x) = x$ for any $x \in \partial\psi(F_0)$ such that:

$$vol''_{|\psi(F_0)} = \widetilde{vol}_{|\psi(F_0)} \circ (\phi_{|\psi(F_0)})^{-1}.$$

Let us finally set $\phi_{|\psi(F_\infty)} = Id_{|\psi(F_\infty)}$, where F_∞ is the unbounded face of $\mathbb{G}$. Thanks to the boundary condition on $\phi_{|\psi(F)}$ for any face F of $\mathbb{G}_0$, we can glue together all the homeomorphisms $\phi_{|\psi(F)}$ constructed for each face F of $\mathbb{G}_0$. It defines an orientation-preserving diffeomorphism at infinity ϕ_1 on $\mathbb{R}^2$ such that $vol''_{|\psi(F_\infty)^c} = (vol \circ (\phi_1 \circ \psi)^{-1})_{|\psi(F_\infty)^c}$ and $(\phi_1)_{|\mathbb{G}'} = Id_{|\mathbb{G}'}$.

For any bounded face F of $\mathbb{G}'$, we have:

$$vol''(F) = vol(\psi^{-1}(F)) = vol'(F).$$

Besides, $\mathbb{G}'$ is a simple graph: we can apply Proposition 5.1 for any bounded face F of $\mathbb{G}'$ in order to transport $vol''_{|F}$ on $vol'_{|F}$. Applying the same arguments (gluing the homeomorphisms as we just did) allows us to construct an orientation-preserving homeomorphism ϕ_2 such that:

$$(vol')_{|(F'_\infty)^c} = (vol'' \circ \phi_2^{-1})_{|(F'_\infty)^c}$$
$$(\phi_2)_{|\mathbb{G}'} = Id_{|\mathbb{G}'},$$

where we recall that $F'_\infty = \psi(F_\infty)$ is the unbounded face of $\mathbb{G}'$. The orientation-preserving diffeomorphism at infinity $\Phi = \phi_2 \circ \phi_1$ satisfies the two conditions (13) and (14). □

6. Group of reduced loops

One of our goal is to construct planar Markovian holonomy fields. For this purpose, we will need the group of reduced loops. Indeed, we have seen that, in order to construct a gauge-invariant random holonomy field on P, it is enough to construct a measure on $\mathcal{M}ult_P(L, G)$ for any set L of loops of P: this was the loop paradigm explained in Lemma 2.4. If L is the set of loops $L_v(\mathbb{G})$, where $\mathbb{G}$ is a finite planar graph, v is a vertex of $\mathbb{G}$ and if P is equal to $P(\mathbb{G})$, then:

$$\mathcal{M}ult_P(L, G) = \mathsf{Hom}(\pi_1(\mathbb{G}, v), G^\vee),$$

where $G^\vee$ is the group based on the same set at G, endowed with the multiplication $._\vee$ such that $x._\vee y = yx$ for any $x, y \in G$ and $\pi_1(\mathbb{G}, v)$ is the fundamental group of $\mathbb{G}$ based at v. This shows the importance of the group $\pi_1(\mathbb{G}, v)$.

6.1. Definition and facts. — The group of based reduced loops $RL_v(\mathbb{G})$ is the fundamental group of $\mathbb{G}$ based at v: $RL_v(\mathbb{G}) = \pi_1(\mathbb{G}, v)$. For convenience we define it using a combinatorial point of view, as Lévy does in Section 1.3.4 of [**21**].

Let l be a loop in $P(\mathbb{G})$. Recall the definition of equivalence of paths explained in Definition 2.7. The equivalence class of l in $P(\mathbb{G})$, denoted by $[l]_\simeq$, contains a unique element of shortest combinatorial length, which is said to be reduced. Besides, if l_1 and l_2 are two loops in $P(\mathbb{G})$ based at v, $[l_1 l_2]_\simeq$ depends only on $[l_1]_\simeq$ and $[l_2]_\simeq$. Thus, it is equivalent to speak about equivalence classes or about reduced paths and the set of reduced paths is endowed with an internal operation.

Definition 6.1. — *The set of reduced loops in $P(\mathbb{G})$ based at v will be denoted by $RL_v(\mathbb{G})$. Let l_1 and l_2 be two loops in $RL_v(\mathbb{G})$, we define $l_1 \times l_2 = [l_1 l_2]_\simeq$.*

Endowed with this operation, $RL_v(\mathbb{G})$ is a group. The existence of the inverse of a loop l based at v is due to the fact that $[ll^{-1}]_\simeq = [1_v]_\simeq$, where 1_v is the trivial path constant to v. In the following, we will denote the reduced product of l_1 with l_2 by $l_1 l_2$ rather than $l_1 \times l_2$.

Let us state a simple, yet crucial lemma about lassos.

Lemma 6.1. — *Two lassos based at the same point and whose meanders represent the same cycle are conjugated in $RL_v(\mathbb{G})$.*

Proof. — Let l and l' be two lassos based at v. They can be written as $l = sms^{-1}$ and $l' = s'm's'^{-1}$, where s and s' are respectively the spoke of l and l'. As the loops m and m' are related, there exist c and d two paths such that $m = cd$ and $m' = dc$. Let us denote by p the loop $s'c^{-1}s^{-1}$, then $l' = plp^{-1}$. □

Definition 6.2. — *A loop in a planar graph $\mathbb{G}$ is called a facial lasso if it is a lasso and its meander represents a non-oriented facial cycle of $\mathbb{G}$.*

The exact definition of facial cycle, an oriented or non-oriented cycle which represents the boundary of a face, is given in [**21**], in Definition 1.3.13. For any face F of $\mathbb{G}$, we will denote by ∂F both the non-oriented and oriented facial cycles associated with F. If we specify that ∂F is oriented, we will consider the anti-clockwise orientation.

In the following, we address the problem of creating families of lassos which generate the whole group $RL_v(\mathbb{G})$. The well-known Lemma 6.2 provides a solution of this problem which is not adapted to our context, but will nevertheless be the departure point of our discussion. In order to state it, we need the definition of a spanning tree.

Definition 6.3. — *Let $\mathbb{G} = \big(\mathbb{V}, \mathbb{E}, \mathbb{F}\big)$ be a finite planar graph. A spanning tree T is a subset of $\mathbb{E}$ such that:*

- *if an edge e is in T, e^{-1} is also in T,*
- *the set of non degenerate loops in T is empty,*
- $\mathbb{V} = \big\{\underline{e}, e \in \mathbb{E}\big\}$.

If $\mathbb{G}$ is composed of a unique edge e which is a loop, the set $\{\underline{e}\}$ is considered as a spanning tree of $\mathbb{G}$.

In the following, we will often restrict a spanning tree T of $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ to a subgraph $\mathbb{G}' = (\mathbb{V}', \mathbb{E}', \mathbb{F}')$ of $\mathbb{G}$. The restriction is defined as following: if $\mathbb{E} = \{l, l^{-1}\}$ with l a loop, then $T' = \{\underline{l}\}$ and in the other cases we consider $T' = T \cap \mathbb{E}'$.

Lemma 6.2. — *Let $\mathbb{G}$ be a finite planar graph and v be a vertex of $\mathbb{G}$. Let $T \subset \mathbb{E}$ be a spanning tree of $\mathbb{G}$ rooted at v. If u and w are vertices of $\mathbb{G}$, we set $[u, w]_T$ to be the unique injective path in T joining u to w. Let $\mathbb{E}^+$ be an orientation of $\mathbb{G}$. The group $RL_v(\mathbb{G})$ is freely generated by the loops $\left\{l_{e,T} : e \in (\mathbb{E} \setminus T)^+\right\}$ where for any edge e, $l_{e,T}$ is equal to $[v, \underline{e}]_T e [\overline{e}, v]_T$.*

Proof. — We only have to prove that $\left(\left\{l_{e,T} : e \in (\mathbb{E} \setminus T)^+\right\}, RL_v(\mathbb{G})\right)$ satisfies the universal property of free groups: given any function f from $\left\{l_{e,T} : e \in (\mathbb{E} \setminus T)^+\right\}$ to a group G there exists a homomorphism $\phi : RL_v(\mathbb{G}) \to G$ such that

$$\phi(l_{e,T}) = f(l_{e,T}),$$

for any $e \in (\mathbb{E} \setminus T)^+$.

Let G be any group and let 1 be its neutral element. Let $\mathbb{E}^+$ be an orientation of $\mathbb{G}$. We recall the Equation (1), in Subsection 2.2, which shows that one can construct a multiplicative function from $P(\mathbb{G})$ to G by specifying the value on $\mathbb{E}^+$. In the definition of multiplicative functions, we asked that the function reverses the order of multiplication. Only for this proof, we will suppose that it preserves the order. This means that if $g \in \mathcal{M}ult(P(\mathbb{G}), G)$, then for any path p_1 and p_2 in P which can be concatenated, $g(p_1 p_2) = g(p_1) g(p_2)$. Let f be a function from $\left\{l_{e,T} : e \in (\mathbb{E} \setminus T)^+\right\}$ to G. We define the element ϕ in $G^{\mathbb{E}^+}$ by:

$$\phi(e) = \begin{cases} f(l_{e,T}), & \text{if } e \in (\mathbb{E} \setminus T)^+, \\ 1, & \text{otherwise.} \end{cases}$$

This defines an element of $\mathcal{M}ult(P(\mathbb{G}), G)$, called also ϕ, which restriction on $L_v(\mathbb{G})$ induces a homeomorphism from $RL_v(\mathbb{G})$ to G. Beside, for any path p in T, $\phi(p) = 1$. Let e be any element of $(\mathbb{E} \setminus T)^+$. Then, we have:

$$\phi(l_{e,T}) = \phi([v, \underline{e}]_T e [\overline{e}, v]_T) = \phi([v, \underline{e}]_T) \phi(e) \phi([\overline{e}, v]_T)) = f(l_{e,T}).$$

The universal property of free groups holds: $RL_v(\mathbb{G})$ is the free group generated by $\left\{l_{e,T} : e \in (\mathbb{E} \setminus T)^+\right\}$. □

Remark 6.1. — *The loops $l_{e,T}$ defined above are actually lassos. Besides, since $\mathbb{G}$ is a planar graph, $\#(\mathbb{E} \setminus T)^+ = \#\mathbb{E}^+ - \#T = \#\mathbb{E}^+ - \#\mathbb{V} + 1 = \#\mathbb{F}^b$.*

6.2. Example: $RL_0(\mathbb{N}^2)$. — We define in this section a family of facial lassos in $\mathbb{N}^2$. This family will be important in Section 10: even if this family can be studied with the help of Proposition 6.1, we give an elementary proof that it generates $RL_0(\mathbb{N}^2)$.

Notation 6.1. — *Let (i, j) and (k, l) be couples of reals such that $i = k$ or $j = l$. We denote by $(i, j) \to (k, l)$ the straight line from (i, j) to (k, l). If $j = l$ and $k = i + 1$ it will also be denoted by $e^r_{i,j}$; if $i = k$ and $l = j + 1$ it will also be denoted by $e^u_{i,j}$.*

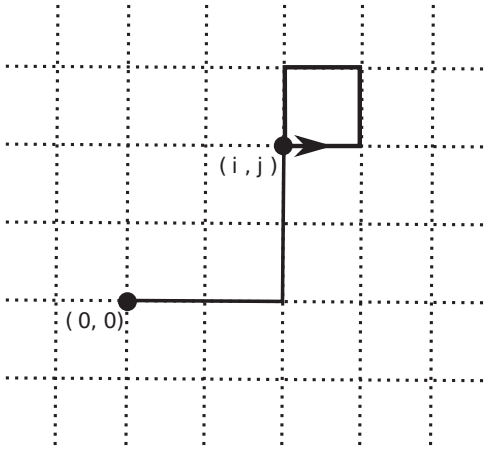


FIGURE 9. The lasso $L_{i,j}$.

Definition 6.4. — *Let i, j be two non negative integers. Let $\partial c_{i,j}$ be the loop in $L(\mathbb{N}^2)$ defined by:*

$$\begin{aligned}\partial c_{i,j} &= (i,j) \to (i+1,j) \to (i+1,j+1) \to (i,j+1) \to (i,j)\\ &= e^r_{i,j} e^u_{i+1,j} (e^r_{i,j+1})^{-1} (e^u_{i,j})^{-1}.\end{aligned}$$

Let $p_{i,j}$ be the path in $L_0(\mathbb{N}^2)$ defined by:

$$p_{i,j} = (0,0) \to (i,0) \to (i,j) = e^r_{0,0} ... e^r_{i-1,0} e^u_{i,0} ... e^u_{i,j-1}.$$

Let $L_{i,j}$ be the reduced loop based at 0:

$$L_{i,j} = \left[p_{i,j} \partial c_{i,j} p^{-1}_{i,j} \right]_{\simeq}.$$

One can refer to Figure 9 to have a clear representation of the lasso $L_{i,j}$.

Lemma 6.3. — *The family $\left(L_{i,j}\right)_{(i,j)\in\mathbb{N}^2}$ is a freely generating subset of $RL_0(\mathbb{N}^2)$.*

Proof. — We only have to work with the finite planar graph:

$$\mathbb{G} = \mathbb{N}^2 \cap \left\{(x,y), x \leq k, y \leq k'\right\},$$

where k and k' are any positive integers. We remind the reader that the intersection of a graph with a set was defined before Definition 3.5. The Lemma 6.2 implies that $RL_0(\mathbb{G})$ is a free group of rank $k \times k'$. Let l be a loop in $RL_0(\mathbb{G})$. We endow the graph $\mathbb{G}$ with the following orientation: from bottom to top, from left to right. Let T be the tree defined by:

$$T = \left\{ \left(e^u_{i,j}\right)^{\pm 1}, i \in \{0,\dots,k\}, j \in \{0,\dots,k'-1\} \right\} \cup \left\{ \left(e^r_{i,0}\right)^{\pm 1}, i \in \{0,\dots,k-1\} \right\}.$$

The root of T will be chosen to be $(0,0)$. One can look at Figure 10 to have a better idea of the graph and the tree we have just constructed. Applying Lemma 6.2 to this situation, l can be written as the reduced concatenation of some elements of $\left\{l^{\pm 1}_{e,T}\right\}_{e\in(\mathbb{E}\setminus T)^+}$, where

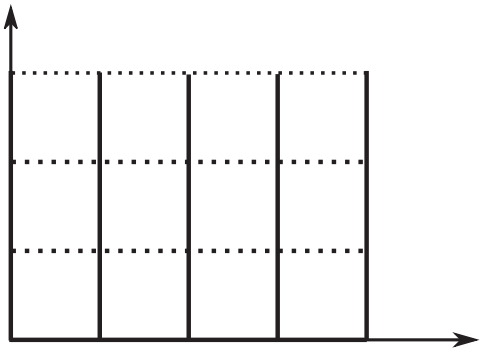


FIGURE 10. The spanning tree T.

$\mathbb{E}$ is the set of edges of $\mathbb{G}$. Moreover $(\mathbb{E} \setminus T)^+$ is equal to $\left\{e^r_{i,j}, i \in \{0, \dots, k-1\}, j \in \{1, \dots, k'\}\right\}$. Since

$$(l_{e^r_{i,j},T})^{-1} = L_{i,0}L_{i,1}...L_{i,j-1},$$

the family $\left(L_{i,j}\right)_{(i,j)\in\mathbb{N}^2}$ is a generating subset of $RL_0(\mathbb{G})$ whose cardinal is $k \times k'$: it is a freely generating subset of $RL_0(\mathbb{G})$. □

6.3. Family of generators of $RL_v(\mathbb{G})$. — As $RL_v(\mathbb{G})$ is a group, it is interesting to know some generating families: we have already seen an example in the last subsection. In this section, in the setting of planar graphs, we will generalize the Section 2.9 in [**21**] about tame generators. Our proofs do not use the ideas in [**21**] but rather uses a recursive decomposition of graphs. The next definition follows Definition 2.4.6. of [**21**].

Definition 6.5. — *Let T be a spanning tree rooted at v and $l_{e,T}$ be the loops defined in Lemma 6.2. Let F be a bounded face of $\mathbb{G}$ and let c_F be a simple loop representing the facial non-oriented cycle associated with F: it can be written as $c_F = e_1...e_n$. We define the reduced path $\mathbf{l}_{c_F,T} = l_{e_1,T} \dots l_{e_n,T}$ in $RL_v(\mathbb{G})$.*

For each rooted spanning tree T and each choice of loops $(c_F)_{F\in\mathbb{F}^b}$ such that c_F is a representative of thenon-oriented facial cycle associated with F, we have defined a new family of loops:

$$\left(\mathbf{l}_{c_F,T}\right)_{F\in\mathbb{F}^b}.$$

The family $(c_F)_{F\in\mathbb{F}^b}$ is called a family of facial loops of $\mathbb{G}$. The difference with Definition 2.4.7 in [**21**] is that the choice of c_F is not given by the choice of T. There is freedom to choose the base point of c_F.

A remark that we will often use is that, when one changes the root of T from v to v', this has the effect to conjugate the family $\left(\mathbf{l}_{c_F,T}\right)_{F\in\mathbb{F}^b}$ by $[v', v]_T$. This comes from the fact that, for any spanning tree T, any vertices v, v' and v'', we have the equality in the set of reduced paths, $[v, v'']_T = [v, v']_T[v', v'']_T$. Proposition 6.1 and Lemma 6.4 give the two most important properties of these new families of reduced loops.

Proposition 6.1. — *For any vertex v, for any spanning tree T rooted at v and any family of facial loops $(c_F)_{F\in\mathbb{F}^b}$, the following assertions hold:*

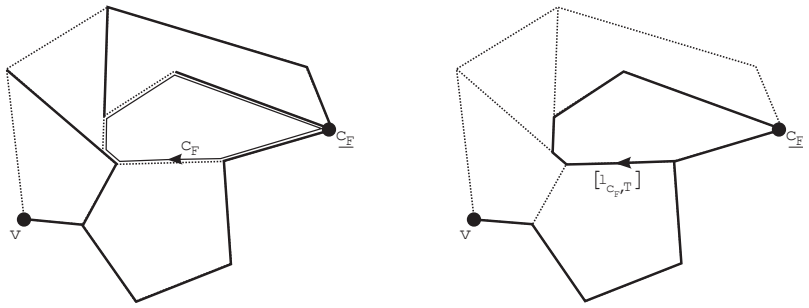


FIGURE 11. A graph, a spanning tree, a facial cycle: the associated reduced facial lasso.

1. *for any bounded face F, $\mathbf{l}_{c_F,T}$ is a facial lasso based at v whose meander represents the non-oriented facial cycle ∂F,*
2. *$(\mathbf{l}_{c_F,T})_{F\in\mathbb{F}^b}$ freely generates $RL_v(\mathbb{G})$.*

Proof. — 1. The equality $\mathbf{l}_{c_F,T} = [v,\underline{c}_F]_T c_F [v,\underline{c}_F]_T^{-1}$ allows us to see that $\mathbf{l}_{c_F,T}$ is a lasso of meander c_F and spoke $[v,\underline{c}_F]_T$.

2. As seen in Lemma 6.2 and Remark 6.1, $RL_v(\mathbb{G})$ is a free group of rank $\#\mathbb{F}^b$. Thus, we have only to show that $(\mathbf{l}_{c_F,T})_{F\in\mathbb{F}^b}$ generates $RL_v(\mathbb{G})$. Using Lemma 6.2, it is enough to show that for every $e \in \mathbb{E}\setminus T$, $l_{e,T}$ is a product of elements of the form $\mathbf{l}_{c_F,T}^{\pm 1}$. Let e be an edge which is not in T. As T is a tree, there exist c, p and p' three simple paths in T which do not intersect, except at the point $\overline{c} = \underline{p} = \underline{p'}$, such that:
   - $[v,\underline{e}]_T = c\,p$,
   - $[v,\overline{e}]_T = c\,p'$,
   - the meander m of $l_{e,T}$ is pep'^{-1}.

   Let v' be any point of $\mathbb{G}$ inside the meander of $l_{e,T}$. Since T is a tree, $[v,v']_T$ must begin with the path c. If not, it would create a non degenerate loop in T. Define $\mathbb{G}'$ (resp. T') the restriction of $\mathbb{G}$ (resp. T) to the closure of the inside of the meander m of $l_{e,T}$. We have drawn an example in Figure 12. We set $\overline{c}$ to be the root of T'. For every bounded face F of $\mathbb{G}$ inside m, $\mathbf{l}_{c_F,T} = c\,\mathbf{l}_{c_F,T'}c^{-1}$ where $\mathbf{l}_{c_F,T'}$ is the facial lasso based at $\overline{c}$ defined in $\mathbb{G}'$ thanks to T'.

   Applying Lemma 6.4 to $\mathbb{G}'$ endowed with T', m can be written as a product of lassos of the form $\mathbf{l}_{c_F,T'}^{\pm 1}$, thus $l_{e,T}$ can be written as a product of lassos of the form $\mathbf{l}_{c_F,T}^{\pm 1}$.

□

Lemma 6.4. — *Let $\mathbb{G}$ be a finite planar graph, let $\mathbb{F}^b$ be the set of bounded faces of $\mathbb{G}$. Let v be a vertex on the boundary of the unbound face and $(F_i)_{i=1}^{\#\mathbb{F}^b}$ be an enumeration of the bounded faces. Let l_∞ be the only loop in $P(\mathbb{G})$ based at v with anti-clockwise orientation which represents the non-oriented facial cycle of the unbounded face F_∞. For any spanning tree T rooted at v and any family of facial loops $(c_F)_{F\in\mathbb{F}^b}$, there exists a permutation σ of $\{1,...,\#\mathbb{F}^b\}$ and an application $\epsilon : \{1,...,n\} \to \{-1,1\}$ such that the*

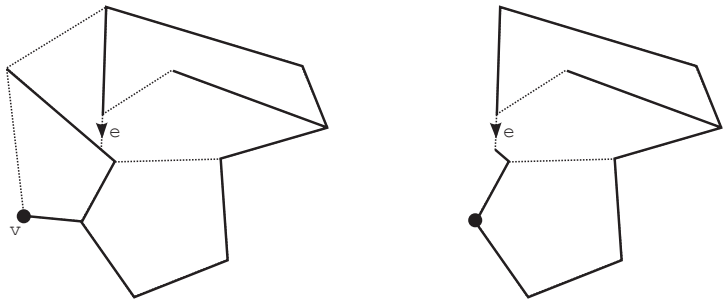


FIGURE 12. The restriction of $\mathbb{G}$ used in Proposition 6.1.

equality

$$\mathbf{l}^{\epsilon(n)}_{c_{F_{\sigma(n)}},T}\mathbf{l}^{\epsilon(n-1)}_{c_{F_{\sigma(n-1)}},T}...\mathbf{l}^{\epsilon(1)}_{c_{F_{\sigma(1)}},T} = l_\infty$$

holds in $RL_v(\mathbb{G})$. *Besides, for any integer* $k \in \{1,\dots,n\}$, $\epsilon(k)$ *is equal to* 1 *if and only if* $c_{F_{\sigma(k)}}$ *is oriented anti-clockwise.*

Proof. — In this proof, all the equalities will hold in $RL_v(\mathbb{G})$: from now on we will omit to specify this. The last assertion comes from a topological index argument. Let us suppose that there exists a permutation σ of $\{1,...,\#\mathbb{F}^b\}$ and an application $\epsilon : \{1,...,n\} \to \{-1,1\}$ such that

$$\mathbf{l}^{\epsilon(n)}_{c_{F_{\sigma(n)}},T}\mathbf{l}^{\epsilon(n-1)}_{c_{F_{\sigma(n-1)}},T}...\mathbf{l}^{\epsilon(1)}_{c_{F_{\sigma(1)}},T} = l_\infty.$$

We can compute the index of l_∞:

$$\mathfrak{n}_{l_\infty} = \mathfrak{n}_{\mathbf{l}_{c_{F_{\sigma(n)}},T}}\epsilon(n) + \dots + \mathfrak{n}_{\mathbf{l}_{c_{F_{\sigma(1)}},T}}\epsilon(1)$$

Let F be any bounded face of $\mathbb{G}$. We can evaluate the last equality for any $x \in F$. This implies that for any $i \in \{1,\dots,n\}$, $1 = \mathfrak{n}_{\mathbf{l}_{c_{F_{\sigma(i)}},T}}\epsilon(i)$, hence the second assertion.

Let us show the first part of Lemma 6.4. The proof goes by induction on the number $\#\mathbb{F}^b$ of bounded faces . For a graph with only one bounded face F the result is true since $l_\infty = l^\epsilon_{c_F}$, with ϵ being -1 of 1, depending on the orientation of c_F.

There exists a unique way to write l_∞ as $p_1e_1p_2e_2\dots e_np_n$ with p_i a path in T (which can be constant) and e_i an edge in $\mathbb{E}\setminus T$ bounding F_∞ for any i in $\{1,...,n\}$. Let us decompose the graph $\mathbb{G}$ in n subgraphs. The i-th subgraph $\mathbb{G}_i$ is the part of $\mathbb{G}$ which is inside the meander m_i of $l_{e_i,T}$. The vertex $v_i = \underline{m_i}$ will be the chosen point on the boundary of $\mathbb{G}_i$, then:

- the restriction T_i of T to $\mathbb{G}_i$ is still a spanning tree of $\mathbb{G}_i$,
- for any bounded face F in $\mathbb{G}_i$, $\mathbf{l}_{c_F,T} = [v,v_i]_T\mathbf{l}_{c_F,T_i}[v,v_i]_T^{-1}$, where $\mathbf{l}_{c_F,T_i}$ is the facial lasso based at v_i defined in the graph $\mathbb{G}_i$.

If $n > 1$, each of the graphs $\mathbb{G}_i$ has strictly less than $\#\mathbb{F}^b$ bounded faces. An example is drawn in Figure 13. By induction the result holds for $\mathbb{G}_i$, based at v_i and endowed with T_i. It follows that $l_{e_i,T}$, which is equal to $[v,v_i]_Tm_i[v,v_i]_T^{-1}$, is an ordered product of all the facial lasso (or their inverse) associated with the faces F in $\mathbb{G}_i$. But the family

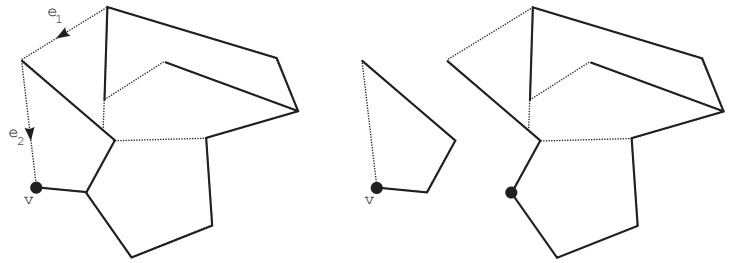


FIGURE 13. Decomposition when $n > 1$.

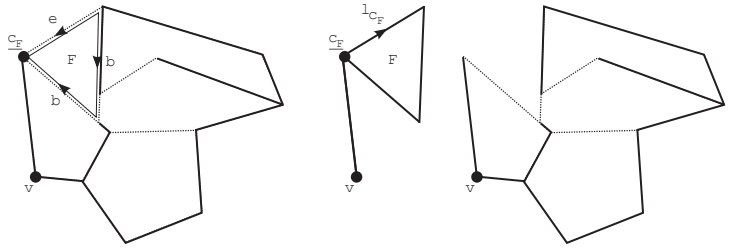


FIGURE 14. Decomposition when $\mathbb{G}_l$ has as many faces as $\mathbb{G}$.

$(\mathbb{G}_i)_i$ induces a partition of the set of bounded faces of $\mathbb{G}$. As $l_\infty = l_{e_1,T}...l_{e_n,T}$ it is now clear that the result holds.

It remains the case where $n = 1$. In this case, $l_\infty = pep'$, with p and p' two simple paths in T and e an edge in $\mathbb{E} \setminus T$ bounding F_∞. We have to find a new way of decomposing $\mathbb{G}$ in order to apply the induction hypothesis. Let F be the only bounded face which is surrounded by e. We can suppose c_F turning clockwise thus it can be decomposed as $c_F = ae^{-1}b$. Consider the loop: $l = [v, \underline{c}_F]_T ae^{-1}[\underline{e}, v]_T$. This is a lasso and as before we consider $\mathbb{G}_l$, which is the graph $\mathbb{G}$ restricted to the closure of the interior of the meander m_l of l. We base this graph at $v_l = \underline{m}_l$.

First of all, if $\mathbb{G}_l$ has the same number of faces than $\mathbb{G}$, as in Figure 14, then the equality $l = l_\infty^{-1}$ must hold and thus one has $l_\infty = ([v, \underline{e}]_T b[\underline{c}_F, v]_T)\mathbf{l}_{c_F}^{-1}$. The path $\tilde{l} = [v, \underline{e}]_T b[\underline{c}_F, v]_T$ is a loop based at v which represents the non-oriented facial cycle of the unbounded face of the graph obtained when one removes e to $\mathbb{G}$. On this graph, T is still a spanning tree and this graph has one less bounded face. The induction hypothesis allows us to conclude.

In the case where $\mathbb{G}_l$ has less faces than $\mathbb{G}$, as in Figure 15, the restriction of T in m_l is not a spanning tree. We will define T_l to be the restriction of T in $\overline{\mathsf{Int}(m_l)}$ to which one adds all the edges in the path a and we root is at v_l. With these modifications, T_l is a spanning tree of $\mathbb{G}_l$ and for any bounded face F of $\mathbb{G}_l$, we have $\mathbf{l}_{c_F,T} = s_l \mathbf{l}_{c_F,T_l} s_l^{-1}$, where $\mathbf{l}_{c_F,T_l}$ is the facial lasso based at v_l defined in $\mathbb{G}_l$ thanks to T_l and s_l is the spoke of l. We define also $l' = [v, \underline{c}_F]_T a[\overline{e}, v]_T$ and $\mathbb{G}_{l'}$ the part of $\mathbb{G}$ inside $\overline{\mathsf{Int}(m_{l'})}$ of l'. The restriction of T to $\mathbb{G}_{l'}$ is denoted $T_{l'}$. In this case $T_{l'}$ is a spanning tree of $\mathbb{G}_{l'}$. Besides, for any bounded face F in $\mathbb{G}_{l'}$, $\mathbf{l}_{c_F,T} = s_{l'} \mathbf{l}_{c_F,T_{l'}} s_{l'}^{-1}$, where $\mathbf{l}_{c_F,T_{l'}}$ is the facial lasso based at $v_{l'} = \underline{m}_{l'}$ defined in $\mathbb{G}_{l'}$ thanks to $T_{l'}$ and $s_{l'}$ is the spoke of l'. In the case we are

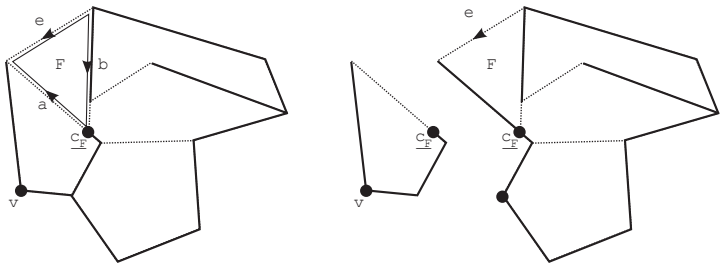


FIGURE 15. Decomposition when $\mathbb{G}_l$ has less faces than $\mathbb{G}$.

studying, $\mathbb{G}_l$ and $\mathbb{G}_{l'}$ have strictly less bounded faces than $\mathbb{G}$, thus we can apply the induction hypothesis. Using the link between facial lassos in $\mathbb{G}_l$ (resp. in $\mathbb{G}_{l'}$) and in $\mathbb{G}$, there exists an ordering on the bounded faces of $\mathbb{G}_l$ (resp. $\mathbb{G}_{l'}$) such that l (resp. l') is the ordered product of the facial lassos $(\mathbf{l}^{\pm 1}_{c_F,T})_{F\in\mathbb{F}^b_l}$ (resp. $(\mathbf{l}^{\pm 1}_{c_F,T})_{F\in\mathbb{F}^b_{l'}}$), where $\mathbb{F}^b_l$ (resp. $\mathbb{F}^{b'}_l$) is the set of bounded faces of $\mathbb{G}_l$ (resp. $\mathbb{G}_{l'}$). Since $l_\infty = [v,\underline{e}]_T e[\overline{e},v]_T = l^{-1}l'$ and as any bounded face F of $\mathbb{G}$ is either a bounded face of $\mathbb{G}_l$ or $\mathbb{G}_{l'}$, we can conclude that there exists a permutation σ of $\{1,...,\#\mathbb{F}^b\}$ and an application $\epsilon : \{1,...,n\} \to \{-1,1\}$ such that:

$$\mathbf{l}^{\epsilon(n)}_{c_{F_{\sigma(n)}},T}\mathbf{l}^{\epsilon(n-1)}_{c_{F_{\sigma(n-1)}},T}...\mathbf{l}^{\epsilon(1)}_{c_{F_{\sigma(1)}},T} = l_\infty.$$

This allows us to conclude. □

Let us finish with a proposition which will be needed in Proposition 8.4.

Proposition 6.2. — *Let $\mathbb{G} = (\mathbb{V},\mathbb{E},\mathbb{F})$ be a finite planar graph, let v be a vertex of $\mathbb{G}$. Let l_1 and l_2 be two simple loops in $\mathbb{G}$ such that $\overline{\mathsf{Int}(l_1)}$ and $\overline{\mathsf{Int}(l_2)}$ are disjoint. There exists a spanning tree T, rooted at v, such that for any family of facial loops $(c_F)_{F\in\mathbb{F}^b}$ the following assertions hold:*

1. *for every loop l in $P(\mathbb{G})$ included in $\overline{\mathsf{Int}(l_1)}$, $\left[[v,\underline{l}]_T l[v,\underline{l}]_T^{-1}\right]_\simeq$ is a product in $RL_v(\mathbb{G})$ of elements of $\left\{\mathbf{l}^{\pm 1}_{c_F,T}; F\in\mathbb{F}^b, F\subset\overline{\mathsf{Int}(l_1)}\right\}$,*
2. *for every loop l in $P(\mathbb{G})$ included in $\overline{\mathsf{Int}(l_2)}$, $\left[[v,\underline{l}]_T l[v,\underline{l}]_T^{-1}\right]_\simeq$ is a product in $RL_v(\mathbb{G})$ of elements of $\left\{\mathbf{l}^{\pm 1}_{c_F,T}; F\in\mathbb{F}^b, F\subset\overline{\mathsf{Int}(l_2)}\right\}$.*

Proof. — Consider $\mathbb{G}$ a finite planar graph, v a vertex of $\mathbb{G}$ and l_1, l_2 two simple loops in $\mathbb{G}$. Suppose that $\overline{\mathsf{Int}(l_1)}$ and $\overline{\mathsf{Int}(l_2)}$ are disjoint. We can decompose l_1 and l_2 as a concatenation of edges of $\mathbb{G}$:

$$l_1 = e_1^1 \dots e_1^n,$$
$$l_2 = e_2^1 \dots e_2^m.$$

The set $\left\{e_1^1,\dots,e_1^{n-1},e_2^1,\dots,e_2^{m-1}\right\}$ can be extended as a spanning tree T, rooted at v, of the graph $\mathbb{G}$. Thanks to the construction, the restriction T_1 of T to $\overline{\mathsf{Int}(l_1)}$ is a

spanning tree of the restriction $\mathbb{G}_1$ of $\mathbb{G}$ to $\overline{\mathsf{Int}(l_1)}$. We set v_1 to be equal to $\underline{e_1^1}$: this is the root of T_1. Applying Proposition 6.1, for any loop l inside l_1,

$$\Big[[v_1,\underline{l}]_{T_1} l [v_1,\underline{l}]_{T_1}^{-1}\Big]_{\simeq}$$

is a product of elements of $\Big\{\mathbf{l}_{c_F,T_1}^{\pm 1}; F\in\mathbb{F}^b, F\subset\overline{\mathsf{Int}(l_1)}\Big\}$. Besides for any vertex w in $\mathbb{G}_1$, $[v,w]_T=[v,v_1]_T[v_1,w]_{T_1}$ in $RL_v(\mathbb{G})$. Thus for any face $F\in\mathbb{F}^b$ such that $F\subset\overline{\mathsf{Int}(l_1)}$, $\mathbf{l}_{c_F,T}=[v,v_1]_T\mathbf{l}_{c_F,T_1}[v,v_1]_T^{-1}$ in $RL_v(\mathbb{G})$ and for any loop l in $P(\mathbb{G})$ included in $\overline{\mathsf{Int}(l_1)}$, $[v,\underline{l}]_T l [v,\underline{l}]_T^{-1}$ is equal in $RL_v(\mathbb{G})$ to $[v,v_1]_T[v_1,\underline{l}]_{T_1} l [v_1,\underline{l}]_{T_1}^{-1}[v,v_1]_T^{-1}$. Thus $[v,\underline{l}]_T l [v,\underline{l}]_T^{-1}$ is a product of elements of $\Big\{\mathbf{l}_{c_F,T}^{\pm 1}; F\in\mathbb{F}^b, F\subset\overline{\mathsf{Int}(l_1)}\Big\}$. The same holds for the loop l_2. □

6.4. Construction of random holonomy fields III: the planar case and the group of reduced loops. — Let us explain two applications of the group of reduced loops which concern the uniqueness and construction of random holonomy fields on the plane.

Proposition 6.3. — *Let μ and ν be two stochastically continuous measures on $\big(\mathcal{M}ult(P(\mathbb{R}^2),G),\mathcal{B}\big)$ which are invariant by gauge transformations. The two assertions are equivalent:*

1. *μ and ν are equal,*
2. *there exist $v\in\mathbb{R}^2$ and A_v a good subspace of $L_v(\mathbb{R}^2)$, such that for any finite planar graph $\mathbb{G}$ in $\mathcal{G}\big(A_v\big)$ which has v as a vertex, there exist a rooted spanning tree T and a family of facial loops $(c_F)_{F\in\mathbb{F}^b}$ of $\mathbb{G}$ such that the law of $(h(\mathbf{l}_{c_F,T}))_{F\in\mathbb{F}^b}$ is the same under μ and under ν.*

Proof. — It is an easy application of the multiplicative property of random holonomy fields, Proposition 2.4 and Proposition 6.1. □

Proposition 6.4. — *Let us suppose that for any finite planar graph $\mathbb{G}$ in $\mathcal{G}\big(\mathsf{Aff}(\mathbb{R}^2)\big)$, we are given a diagonal conjugation-invariant measure $\mu_{\mathbb{G}}$ on $G^{\#\mathbb{F}^b}$, a rooted spanning tree T and a family of facial loops $(c_F)_{F\in\mathbb{F}^b}$. For any finite planar graph $\mathbb{G}$, there is only one possibility to extend $\mu_{\mathbb{G}}$ as a gauge-invariant random field on $P(\mathbb{G})$, extension which will be also called $\mu_{\mathbb{G}}$, such that the law of $(h(\mathbf{l}_{c_F,T}))_{F\in\mathbb{F}^b}$ under $\mu_{\mathbb{G}}$ is the same as the law of the canonical projections under the measure $\mu_{\mathbb{G}}$ on $G^{\#\mathbb{F}^b}$.*

If $(\mu_{\mathbb{G}})_{\mathbb{G}\in\mathcal{G}(\mathsf{Aff}(\mathbb{R}^2))}$ is uniformly locally stochastically $\frac{1}{2}$-Hölder continuous, if for any finite planar graphs $\mathbb{G}$ and $\mathbb{G}'$ in $\mathcal{G}\big(\mathsf{Aff}(\mathbb{R}^2)\big)$ such that $\mathbb{G}\preccurlyeq\mathbb{G}'$ and for any family of facial loops $(c_F)_{F\in\mathbb{F}^b}$ of $\mathbb{G}$, $(h(\mathbf{l}_{c_F,T}))_{F\in\mathbb{F}^b}$ has the same law under $\mu_{\mathbb{G}}$ as under $\mu_{\mathbb{G}'}$, then there exists a unique stochastically continuous random holonomy field μ on the plane such that for any finite planar graph $\mathbb{G}$ in $\mathcal{G}\big(\mathsf{Aff}(\mathbb{R}^2)\big)$, for any rooted spanning tree T and for any family of facial loops $(c_F)_{F\in\mathbb{F}^b}$ of $\mathbb{G}$, the law of $(h(\mathbf{l}_{c_F,T}))_{F\in\mathbb{F}^b}$ is the same under μ as under $\mu_{\mathbb{G}}$.

Proof. — For any finite planar graph $\mathbb{G}$ in $\mathcal{G}\big(\mathsf{Aff}(\mathbb{R}^2)\big)$ and any vertex v of $\mathbb{G}$, there exists a natural measurable function from $\mathsf{Hom}\big(RL_v(\mathbb{G}),G^\vee\big)$ to $\mathcal{M}ult_{P(\mathbb{G})}(L_v(\mathbb{G}),G)$. Thus, we can transport any measure from the first space to the second. Using the freeness of

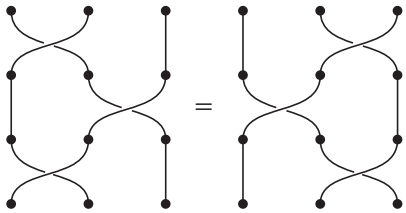


FIGURE 16. The braid relation

the generating families $\mathbf{l}_{c_F,T}$, the multiplicity property of random holonomy fields and Proposition 2.5, we can extend $\mu_{\mathbb{G}}$ as a gauge-invariant random field on $P(\mathbb{G})$. This gauge-invariant random field does not depend on the choice of v. Then an application of Proposition 2.6 and Lemma 3.2 allows us to construct the desired μ. The uniqueness of μ is a consequence of Proposition 6.3. □

7. Braids and probabilities I: an algebraic point of view and finite random sequences

For any finite planar graph $\mathbb{G}$, we have constructed in the last section a set of generating family of facial lassos of $\mathbb{G}$. It is natural to wonder what is the transformation which sends one generating family to an other. It has to be noticed that, as soon as the root of the spanning tree is chosen, for any generating family of lassos we have constructed in the last section, their product, up to some suitable permutation, is always equal to the same loop. This remark and Artin's theorem 7.1 motivate the study of the group of braids.

7.1. Generators, Relations, Actions. — We have seen a geometric definition of the braid group in the introduction of this paper. One can also define the braid group with a generator-relation presentation.

Definition 7.1. — *Let n be an integer greater than 2. The braid group with n strands $\mathcal{B}_n$ is the group with the following presentation:*

$$\left\langle (\beta_i)_{i=1}^{n-1} \;\middle|\; \forall i,j \in \{1,\dots,n-1\}, \begin{array}{l} |i-j|=1 \Longrightarrow \beta_i\beta_j\beta_i = \beta_j\beta_i\beta_j \\ |i-j|>1 \Longrightarrow \beta_i\beta_j = \beta_j\beta_i \end{array} \right\rangle.$$

The elements $(\beta_i)_{i=1}^{n-1}$ we defined in Figure 4 satisfy the braid group relations which now seem more natural. An example of the first relation between β_i and β_j when $|i-j|=1$ is given in Figure 16.

This presentation of the braid group is not intuitive, yet it allows us to recall some natural actions of the braid group $\mathcal{B}_n$: one on the free group of rank n and one on G^n.

Definition 7.2. — *Let $\mathbb{F}_n$ be the free group of rank n generated by $e_1, ..., e_n$. We define the natural action of $\mathcal{B}_n$ on $\mathbb{F}_n$ by:*

$$\begin{aligned}\beta_i e_i &= e_{i+1},\\ \beta_i e_{i+1} &= e_{i+1} e_i e_{i+1}^{-1},\\ \beta_i e_j &= e_j, \text{ for any } j \notin \{i, i+1\}.\end{aligned}$$

One can verify easily that the braid group relations is satisfied in this last definition: the natural action of $\mathcal{B}_n$ on $\mathbb{F}_n$ is well defined. There is a diagrammatic way to compute the action: one puts $e_1, \ldots, e_n$ at the bottom of a diagram representing β, then we propagate these $e_1, \ldots, e_n$ in the diagram from the bottom to the top with the rule that, at each crossing, the value on the string which is behind does not change and the value on the upper string is conjugated by the value of the other so that the product from right to left remains unchanged. At the end we get a n-uple $(f_1, \ldots, f_n)$ at the top of the diagram: the braid sends e_i on f_i.

Definition 7.3. — *Consider G an arbitrary group, we call the natural action of $\mathcal{B}_n$ on G^n the action such that:*

$$\beta_i \bullet (x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n) = (x_1, ..., x_{i-1}, x_i x_{i+1} x_i^{-1}, x_i, \ldots, x_n), \tag{15}$$

for any integer $i \in \{1, \ldots, n-1\}$ and n-tuple $(x_i)_{i=1}^n$ in G^n.

One can verify easily that the braid group relations is satisfied in this last definition: the natural action of $\mathcal{B}_n$ on G^n is well defined. There is a diagrammatic way to compute the action: one puts $x_1, \ldots, x_n$ at the upper part of a diagram representing β, then we propagate these $x_1, \ldots, x_n$ in the diagram from the top to the bottom with the rule that, at each crossing, the value on the string which is behind does not change and the value on the upper string is conjugated by the value of the other so that the product from left to right remains unchanged. At the end we get a n-uple $(y_1, \ldots, y_n)$ at the top of the diagram: the braid sends $(x_1, \ldots, x_n)$ on $(y_1, \ldots, y_n)$.

Let h be a G-valued multiplicative function on the free group. This means that $h\left(x^{-1}\right) = h(x)^{-1}$ and $h(xy) = h(y)h(x)$ for any x and y in $\mathbb{F}_n$. For any n-uple $(f_1, \ldots, f_n)$ of elements of $\mathbb{F}_n$, we define $h(f_1, \ldots, f_n) = (h(f_1), \ldots, h(f_n))$. We have the following lemma which shows how both actions are linked and which is a consequence of the diagrammatic formulation of both actions.

Lemma 7.1. — *Let $\mathbb{F}_n$ be the free group of rank n generated by $e_1, ..., e_n$. For any braid $\beta \in \mathcal{B}_n$, we have:*

$$h\left(\beta \bullet (e_1, \ldots, e_n)\right) = \beta^{-1} \bullet h(e_1, \ldots, e_n).$$

With the n-diagrams picture in mind, it is obvious that the application, which sends a braid on the permutation obtained by erasing the information at each crossing, is a homomorphism: it is the one which sends β_i on the transposition $(i, i+1)$ for any integer $i \in \{1, ..., n-1\}$.

Lemma 7.2. — *The operation of erasing the information at each crossing induces a natural homomorphism from $\mathcal{B}_n$ to $\mathfrak{S}_n$. We will denote the image of β by σ_β.*

7.2. Artin theorem. — Let us consider $\mathbb{F}_n$ the free group of rank n generated by $e_1, ..., e_n$. For any braid β with n strands, we have seen how to associate an action a_β on $\mathbb{F}_n$ which is in fact an automorphism. Thus, there exists a morphism from $\mathcal{B}_n$ in $\mathbb{A}ut(\mathbb{F}_n)$ which is moreover injective. In [**8**] and [**7**], Artin gave a sufficient and necessary condition for an automorphism of $\mathbb{F}_n$ to be the induced action of a braid in $\mathcal{B}_n$.

Theorem 7.1. — *An automorphism* $\mathbf{a}$ *of* $\mathbb{F}_n$ *is the induced action of a braid in* $\mathcal{B}_n$ *if and only if the two following conditions hold:*

Conjugacy property: *for any* i *in* $\{1, ..., n\}$, $\mathbf{a}(e_i)$ *is in the same conjugacy class as one of the elements of* $(e_j)_{j=1}^n$.

Product invariance: $\mathbf{a}(e_n \ ... \ e_1) = e_n \ ... \ e_1$.

Remark 7.1. — *Let* β *be a braid in* $\mathcal{B}_n$ *and* a_β *the induced action on* $\mathbb{F}_n$. *For each* i *in* $\{1, ..., n\}$, $a_\beta(e_i)$ *is conjugated to* $e_{\sigma_\beta(i)}$ *and this property characterizes* σ_β.

7.3. Braids and the group of reduced loops. —

Proposition 7.1. — *Let* $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ *be a finite planar graph, let* v *be a vertex of* $\mathbb{G}$ *and let* T *and* T' *be two spanning trees of* $\mathbb{G}$ *rooted at* v. *Let us consider* $(c_F)_{F\in\mathbb{F}^b}$ *and* $(c'_F)_{F\in\mathbb{F}^b}$ *two families of facial loops turning anti-clockwise. There exists an enumeration of the bounded faces* $(F_i)_{i=1}^{\#\mathbb{F}^b}$ *and a braid* β *in* $\mathcal{B}_{\#\mathbb{F}^b}$ *such that:*

$$\beta \bullet \left(\mathbf{l}_{c_{F_i},T}\right)_{i=1}^{\#\mathbb{F}^b} = \left(\mathbf{l}_{c'_{F_{\sigma(i)}},T'}\right)_{i=1}^{\#\mathbb{F}^b},$$

where $\sigma = \sigma_\beta$ *and where* β *is seen as acting on the free group generated by* $\left(\mathbf{l}_{c_{F_i},T}\right)_{i=1}^{\#\mathbb{F}^b}$.

Proof. — For any bounded face F of $\mathbb{G}$, the first part of Proposition 6.1 asserts that $\mathbf{l}_{c_F,T}$ and $\mathbf{l}_{c'_F,T'}$ are facial lassos based at v whose meanders represent the facial cycle ∂F oriented anti-clockwise. By Lemma 6.1, we deduce that $\mathbf{l}_{c'_F,T'}$ is conjugated to $\mathbf{l}_{c_F,T}$ in $RL_v(\mathbb{G})$. Besides, thanks to Lemma 6.4, we can find an enumeration of the bounded faces $(F_i)_{i=1}^{\#\mathbb{F}^b}$ and a permutation σ of $\{1, ..., \#\mathbb{F}^b\}$ such that:

1. $\mathbf{l}_{c_{F_n},T}\ \mathbf{l}_{c_{F_{n-1}},T}... \ \mathbf{l}_{c_{F_1},T} = l_\infty$,
2. $\mathbf{l}_{c'_{F_{\sigma(n)}},T'}\ \mathbf{l}_{c'_{F_{\sigma(n-1)}},T'}... \ \mathbf{l}_{c'_{F_{\sigma(1)}},T'} = l_\infty$,

in $RL_v(\mathbb{G})$, where l_∞ is the facial loop based at v, turning anti-clockwise, representing the non-oriented facial cycle ∂F_∞. Besides, Proposition 6.1 tells us that both $(\mathbf{l}_{c_F})_{F\in\mathbb{F}^b}$ and $(\mathbf{l}_{c'_F})_{F\in\mathbb{F}^b}$ are free families of generators of the free group $RL_v(\mathbb{G})$. A natural automorphism of $RL_v(\mathbb{G})$ is defined by:

$$\forall i \in \{1, ..., \#\mathbb{F}^b\},\ \mathbf{a}(\mathbf{l}_{c_{F_i},T}) = \mathbf{l}_{c'_{F_{\sigma(i)}},T'}.$$

This automorphism of free group satisfies the conditions of Artin's theorem given in Theorem 7.1. There exists a braid β such that $\mathbf{a}$ is equal to $\mathbf{a}_\beta$, the action induced by β on the free group $RL_v(\mathbb{G})$ with free generators $(\mathbf{l}_{c_F,T})_{F\in\mathbb{F}^b}$. Using Remark 7.1, it is straightforward to see that σ is equal to σ_β. □

7.4. Braids and finite sequence of random variables. — In the last section we have understood the transformations between families of loops of the form $(\mathbf{l}_{c_F,T})_{F\in\mathbb{F}^b}$. In the context of random holonomy fields, a random variable is associated with any loop. Thus it is natural to study the action of the braid groups on finite sequence of random variables. When one has to deal with non-commutative random variables (i.e. random variables in a non-commutative group), this action is in some sense more appropriate than the symmetrical group action which is often studied in the mathematical literature. This leads to a theory of braidability which is more efficient than the exchangeability concept for sequences of random variables in a non-commutative group. Let n be an integer strictly greater than 1 and let us consider G an arbitrary topological group.

Notation 7.1. — *The braid group $\mathcal{B}_n$ acts on the set of n-tuple of G-valued random variables according to the formula:*

$$\beta_i \bullet (X_1, ..., X_{i-1}, X_i, X_{i+1}, ..., X_n) = (X_1, ..., X_{i-1}, X_i X_{i+1} X_i^{-1}, X_i, \dots, X_n) \tag{16}$$

for any $i \in \{1, ..., n-1\}$.

We remind the reader that the notation σ_β was defined in Lemma 7.2.

Definition 7.4. — *Let $(X_1, ..., X_n)$ be a finite sequence of G-valued random variables. It is* purely invariant by braids *if for any braid $\beta \in \mathcal{B}_n$ one has the equality in law:*

$$\beta \bullet (X_1, ..., X_n) = \sigma_\beta \bullet (X_1, ..., X_n),$$

where $\sigma \bullet (X_1, ..., X_n) = \left(X_{\sigma^{-1}(1)}, ..., X_{\sigma^{-1}(n)}\right)$ for any permutation $\sigma \in \mathfrak{S}_n$.
It is invariant by braids *if for any braid $\beta \in \mathcal{B}_n$, one has the equality in law:*

$$\beta \bullet (X_1, ..., X_n) = (X_1, ..., X_n).$$

Some of the main concepts we will need in the following are the notion of support of a measure and various notions of invariance by conjugation. If m is a probability measure on G, the support of m is the smallest closed subset of G of measure 1 for m. It will be denoted by $\mathsf{Supp}(m)$. The closure of the subgroup generated by the support of m is denoted by H_m. If X is a G-valued random variable and m its law, we define $\mathsf{Supp}(X) = \mathsf{Supp}(m)$ and $H_X = H_m$.

Definition 7.5. — *Let T be a finite index set of cardinal strictly greater than 1. Let $(X_t)_{t\in T}$ be a sequence of G-valued random variables. We say that $(X_t)_{t\in T}$ is* auto-invariant by conjugation *if for any different elements i and j in T and for any $g \in \mathsf{Supp}(X_j)$, we have the equality in law:*

$$g X_i g^{-1} = X_i. \tag{17}$$

This definition can be extended to collections of measures on G.

The first result on random sequences purely invariant by braids is the following proposition.

Proposition 7.2. — *A finite sequence of independent G-valued random variables is auto-invariant by conjugation if and only if it is purely invariant by braids.*

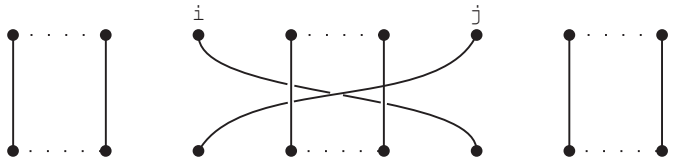


FIGURE 17. The braid $\beta_{(i,j)}$.

Proof. — Let $(X_1, \dots, X_n)$ be a finite sequence of G-valued random variables which are independent. Let us suppose that $(X_1, \dots, X_n)$ is auto-invariant by conjugation and let us prove that it is purely invariant by braids. Since $\beta \mapsto \sigma_\beta$ is a morphism and using the independence of the variables, we just have to show that $(X_2, X_2^{-1}X_1X_2)$ and (X_2, X_1) have the same law since the permutation associated with β_1 is the transposition $(1, 2)$. This result follows from the independence of the variables X_1 and X_2 and from the invariance by conjugation of the law of X_1 by any element g in the support of X_2.

Now, let us suppose that $(X_1, \dots, X_n)$ is purely invariant by braids and let us prove that it is auto-invariant by conjugation. Let i and j be two distinct integers in $\{1, \dots, n\}$. Let us suppose that $i < j$. Let $\beta_{(i,j)}$ be the braid defined by:

$$\beta_{(i,j)} = \beta_i^{-1} \dots \beta_{j-2}^{-1} \beta_{j-1} \dots \beta_i.$$

An example of such a braid is shown in Figure 17. By considering only the i^{th} and j^{th} positions in the equality in law $\beta_{(i,j)} \bullet (X_1, ..., X_n) = \sigma_{\beta_{i,j}} \bullet (X_1, ..., X_n)$, we get the following equality in law:

$$(X_i, X_j) = (X_i, X_iX_jX_i^{-1}).$$

By disintegration and using the independence of the variables, one gets the desired result. □

The proof of Proposition 7.2 is straightforward, but looking at the following equality in law:

$$(X_2^{-1}X_1X_2, X_2^{-1}X_1^{-1}X_2X_1X_2) = (X_1, X_2),$$

where (X_1, X_2) is an auto-invariant by conjugation couple of random variables, one can see that it gives identities which, at first glance, do not seem trivial. A last remark to be made about Proposition 7.2 is that there exist finite sequences of non-independent G-valued random variables which are purely invariant by braids.

8. Planar Yang-Mills fields

8.1. Construction of pure planar Yang-Mills fields. — First of all, let us recall the definition of Lévy processes that we introduced in Section 1.1.

Definition 8.1. — *A Lévy process $(Z_t)_{t\geq 0}$ is a random process from $\mathbb{R}^+$ to G, left continuous with right limit with independent and stationary right increments. This means:*

- $\forall\ 0 \leq t_0 < \dots < t_n$, $\left(Z_{t_{i-1}}^{-1} Z_{t_i}\right)_{i=1}^n$ *are independent,*
- $\forall\ 0 \leq s < t$, $Z_s^{-1}Z_t$ *has the same law as* Z_{t-s}.

We say that $(Z_t)_{t\geq 0}$ is invariant by conjugation by G, or conjugation-invariant, if and only if for any $g \in G$, the process $(g^{-1}Z_t g)_{t\geq 0}$ has the same law as $(Z_t)_{t\geq 0}$.

There is a correspondence between continuous semi-groups of convolution of probability measures starting from the Dirac measure on the neutral element of G and Lévy processes. The goal of this section is to construct a family of planar Markovian holonomy fields called the planar Yang Mills fields. In order to do this, given any Lévy process Y which is invariant by conjugation by G, we construct, in Proposition 8.1, for any finite planar graph $\mathbb{G}$, an random holonomy field on $P(\mathbb{G})$ associated to Y. In Propositions 8.2 and 8.4, we show that these random holonomy fields allow us to define a family of random holonomy fields on $\mathbb{R}^2$ which is a strong planar Markovian holonomy field. In Section 8.2, we weaken the condition on the Lévy process by using our results about the extension of the structure group.

Proposition 8.1. — *Let $\mathbb{G}$ be a finite planar graph, let vol be a measure of area and let $Y = (Y_t)_{t\geq 0}$ be a Lévy process on G invariant by conjugation. There exists a unique random holonomy field $\mathbb{E}^{Y,\mathbb{G}}_{vol}$ on $P(\mathbb{G})$, whose weight is equal to 1, such that for any rooted spanning tree T of $\mathbb{G}$, any family $(c_F)_{F\in\mathbb{F}^b}$ of facial loops of $\mathbb{G}$, each oriented anti-clockwise, under $\mathbb{E}^{Y,\mathbb{G}}_{vol}$:*

1. *the random variables $\left(h\left(\mathbf{l}_{c_F,T}\right)\right)_{F\in\mathbb{F}^b}$ are independent,*
2. *for any $F \in \mathbb{F}^b$, $h(\mathbf{l}_{c_F,T})$ has the same law as $Y_{vol(F)}$.*

The family $\left(\mathbb{E}^{Y,\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ is the discrete planar Yang-Mills field associated with $(Y_t)_{t\geq 0}$.

Proof. — Let $\mathbb{G}$ be a finite planar graph, let *vol* be a measure of area and let $Y = (Y_t)_{t\geq 0}$ be a Lévy process on G invariant by conjugation. For any positive real t, let us denote by m_t the law of Y_t. For any rooted spanning tree T and any family $(c_F)_{F\in\mathbb{F}^b}$ of facial loops oriented anti-clockwise, we define the measure $\mathbb{E}^{Y,\mathbb{G}}_{vol,T,(c_F)_{F\in\mathbb{F}^b}}$ on $\left(\mathcal{M}ult(P(\mathbb{G}),G),\mathcal{B}\right)$ as the unique gauge-invariant probability measure such that, under $\mathbb{E}^{Y,\mathbb{G}}_{vol,T,(c_F)_{F\in\mathbb{F}^b}}$:

1. the random variables $\left(h\left(\mathbf{l}_{c_F,T}\right)\right)_{F\in\mathbb{F}^b}$ are independent,
2. for any $F \in \mathbb{F}^b$, $h(\mathbf{l}_{c_F,T})$ has the same law as $Y_{vol(F)}$.

Since $\otimes_{F\in\mathbb{F}^b} m_{vol(F)}$ is invariant by diagonal conjugation, by applying the first part of Proposition 6.4 we see that the definition makes sense. We will show that the probability measure $\mathbb{E}^{Y,\mathbb{G}}_{vol,T,(c_F)_{F\in\mathbb{F}^b}}$ neither depends on the choice of T, nor on the choice of $(c_F)_{F\in\mathbb{F}^b}$. Thanks to the uniqueness property in this last definition, we have to prove that given another rooted spanning tree T' and another family of facial loops $(c'_F)_{F\in\mathbb{F}^b}$ oriented anti-clockwise, under $\mathbb{E}^{Y,\mathbb{G}}_{vol,T,(c_F)_{F\in\mathbb{F}^b}}$, $\left(h(\mathbf{l}_{c'_F,T'})\right)_{F\in\mathbb{F}^b}$ has the same law as $\left(h\left(\mathbf{l}_{c_F,T}\right)\right)_{F\in\mathbb{F}^b}$.

First of all, let us prove that one can suppose that T and T' are rooted at the same vertex v of $\mathbb{G}$. Let v be the root of T, let v' be a vertex of $\mathbb{G}$ and let us define the rooted spanning tree $\tilde{T}$ as the tree T rooted at v'. When we change the root of T from v to v' we conjugate every of the $\mathbf{l}_{c_F,T}$ by the same path $[v',v]_T$. By Remark 2.7, since

$\mathbb{E}^{Y,\mathbb{G}}_{vol,T,(c_F)_{F\in\mathbb{F}^b}}$ is gauge-invariant, under $\mathbb{E}^{Y,\mathbb{G}}_{vol,T,(c_F)_{F\in\mathbb{F}^b}}$, $\left(h(\mathbf{l}_{c_F,\tilde{T}})\right)_{F\in\mathbb{F}^b}$ has the same law as $\left(h(\mathbf{l}_{c_F,T})\right)_{F\in\mathbb{F}^b}$, namely $\otimes_{F\in\mathbb{F}^b} m_{vol(F)}$: $\mathbb{E}^{Y,\mathbb{G}}_{vol,T,(c_F)_{F\in\mathbb{F}^b}} = \mathbb{E}^{Y,\mathbb{G}}_{vol,\tilde{T},(c_F)_{F\in\mathbb{F}^b}}$.

Now let us assume that T and T' are rooted at the same vertex. By Proposition 7.1, there exists an enumeration $(F_i)_{i=1}^{\#\mathbb{F}^b}$ of the bounded faces of $\mathbb{G}$, a braid β in $\mathcal{B}_{\#\mathbb{F}^b}$ such that:

$$\beta \bullet \left(\mathbf{l}_{c_{F_i},T}\right)_{i=1}^{\#\mathbb{F}^b} = \left(\mathbf{l}_{c'_{F_{\sigma_\beta(i)}},T'}\right)_{i=1}^{\#\mathbb{F}^b}.$$

Using Lemma 7.1, $h\left(\beta \bullet \left(\mathbf{l}_{c_{F_i},T}\right)_{i=1}^{\#\mathbb{F}^b}\right) = \beta^{-1} \bullet \left(h(\mathbf{l}_{c_{F_i},T})\right)_{i=1}^{\#\mathbb{F}^b}$, and thus:

$$\beta^{-1} \bullet \left(h(\mathbf{l}_{c_{F_i},T})\right)_{i=1}^{\#\mathbb{F}^b} = \sigma_{\beta^{-1}} \bullet \left(h(\mathbf{l}_{c'_{F_i},T'})\right)_{i=1}^{\#\mathbb{F}^b}.$$

Applying the Proposition 7.2, under $\mathbb{E}^{Y,\mathbb{G}}_{vol,T,(c_F)_{F\in\mathbb{F}^b}}$, the following equality in law holds:

$$\beta^{-1} \bullet \left(h(\mathbf{l}_{c_{F_i},T})\right)_{i=1}^{\#\mathbb{F}^b} = \sigma_{\beta^{-1}} \bullet \left(h(\mathbf{l}_{c_{F_i},T})\right)_{i=1}^{\#\mathbb{F}^b}.$$

From this, we get the equality in law under $\mathbb{E}^{Y,\mathbb{G}}_{vol,T,(c_F)_{F\in\mathbb{F}^b}}$:

$$\left(h(\mathbf{l}_{c'_{F_i},T'})\right)_{i=1}^{\#\mathbb{F}^b} = \left(h(\mathbf{l}_{c_{F_i},T})\right)_{i=1}^{\#\mathbb{F}^b},$$

which is what we expected. □

This proposition allows us not to have to choose a special rooted tree for each graph in order to construct planar Yang-Mills fields. More importantly, it will allow us to show the independence property and the area-preserving homeomorphism invariance of the family of random holonomy fields which we will construct thanks to Proposition 6.4.

Proposition 8.2. — *Let $(Y_t)_{t\geq 0}$ be a G-valued Lévy process invariant by conjugation. There exists a unique family of gauge-invariant stochastically continuous random holonomy fields $\left(\mathbb{E}^Y_{vol}\right)_{vol}$, whose weight is equal to* 1*, such that for any measure of area vol, for any finite planar graph $\mathbb{G}$, for any rooted spanning tree T of $\mathbb{G}$ and any family of facial loops $(c_F)_{F\in\mathbb{F}^b}$ oriented anti-clockwise, under $\mathbb{E}^Y_{vol}$:*

1. *the random variables $\left(h\left(\mathbf{l}_{c_F,T}\right)\right)_{F\in\mathbb{F}^b}$ are independent,*
2. *for any $F\in\mathbb{F}^b$, $h(\mathbf{l}_{c_F,T})$ has the same law as $Y_{vol(F)}$.*

The family $\left(\mathbb{E}^Y_{vol}\right)_{vol}$ is the planar Yang-Mills field associated with $(Y_t)_{t\geq 0}$.

In order to prove this result, we will use the following statement, from [**21**], which allows us to bound the distance of a Lévy process to the neutral element.

Proposition 8.3. — *Let $(Y_t)_{t\geq 0}$ be a Lévy process on G. There exists a constant K such that:*

$$\forall t\geq 0, \mathbb{E}\left[d_G\left(1,Y_t\right)\right] \leq K\sqrt{t}.$$

Proof of Proposition 8.2. — Let *vol* be a measure of area on the plane. In order to prove the result, we will apply Proposition 6.4 to the family of measures $\left(\mathbb{E}_{vol}^{Y,\mathbb{G}}\right)_{\mathbb{G}\in\mathcal{G}(\mathsf{Aff}(\mathbb{R}^2))}$. Then we will study the restriction to general finite planar graphs of the random holonomy field that we will have defined. In order to do all this, we have to prove a compatibility condition and a uniform locally stochastically $\frac{1}{2}$-Hölder continuity property for the family $\left(\mathbb{E}_{vol}^{Y,\mathbb{G}}\right)_{\mathbb{G}\in\mathcal{G}(\mathsf{Aff}(\mathbb{R}^2))}$.

Compatibility condition : Let us consider $\mathbb{G}_1$ and $\mathbb{G}_2$ two graphs in $\mathcal{G}\big(\mathsf{Aff}(\mathbb{R}^2)\big)$ such that $\mathbb{G}_1 \preccurlyeq \mathbb{G}_2$. Let us consider m a vertex of $\mathbb{G}_1$ and $\mathbb{G}_2$. Using Proposition 8.1, it is enough to show that $\mathbb{G}_1$ satisfies the following property:

$$(\mathcal{H})\begin{cases}\text{there exists a family of facial loops } (c_F)_{F\in\mathbb{F}_1^b} \text{ oriented anti-clockwise and}\\ \text{a spanning tree } T_1 \text{ of } \mathbb{G}_1 \text{ rooted at } m, \text{ such that under } \mathbb{E}_{vol}^{Y,\mathbb{G}_2}:\\ \quad 1.\ \text{the random variables } \left(h\left(\mathbf{l}_{c_F,T}\right)\right)_{F\in\mathbb{F}_1^b} \text{ are independent,}\\ \quad 2.\ \text{for any } F\in\mathbb{F}_1^b,\ h\left(\mathbf{l}_{c_F,T}\right) \text{ has the same law as } Y_{vol(F)}.\end{cases}$$

We show this by a kind of induction argument on the finite set $\left[\mathbb{G}_1,\mathbb{G}_2\right] = \left\{\mathbb{G},\mathbb{G}_1 \preccurlyeq \mathbb{G} \preccurlyeq \mathbb{G}_2\right\}$ endowed with the partial order $\preccurlyeq$. It is clearly true that $(\mathcal{H})$ holds for $\mathbb{G} = \mathbb{G}_2$. Consider a finite planar graph $\mathbb{G}$ in $\left[\mathbb{G}_1,\mathbb{G}_2\right]$ satisfying $(\mathcal{H})$, we will show that there exists $\mathbb{G}' \in \left[\mathbb{G}_1,\mathbb{G}\right[$ for which $(\mathcal{H})$ is still valid. Thanks to Proposition 8.1, property $(\mathcal{H})$ holds for $\mathbb{G}$ for any family of facial loops $(c_F)_{F\in\mathbb{F}^b}$ oriented anti-clockwise and any choice of spanning tree T rooted at m. Since $\mathbb{G}_1 \preccurlyeq \mathbb{G}$, at least one of the following assertions is true:

1. there exist an edge of $\mathbb{G}_1$, e, and a vertex v of $\mathbb{G}$ of degree two such that $v \in e\big((0,1)\big)$,
2. there exists a face F_1 of $\mathbb{G}_1$, bounded or not, such that the restriction of $\mathbb{G}$ to F_1 has a unique face F_0 and ∂F_0, oriented anti-clockwise, contains a sequence of the form ee^{-1} with the interior of e included in F_1,
3. there exists a face F_1 of $\mathbb{G}_1$ which contains more than one face of $\mathbb{G}$.

Let us consider the three possibilities.

1. Consider a family of facial loops $(c_F)_{F\in\mathbb{F}^b}$ for $\mathbb{G}$, oriented anti-clockwise, none of which is based at v, and a choice of spanning tree T of $\mathbb{G}$ rooted at m. Let e_1 and e_2 be the two edges of $\mathbb{G}$ such that $e = e_1e_2$ and $\overline{e_1} = v$. We consider $\mathbb{G}'$, the graph defined by:

$$\left(\mathbb{V}',\mathbb{E}',\mathbb{F}'\right) = \left(\mathbb{V}\setminus\{v\},\mathbb{E}\setminus\left\{e_1^{\pm1},e_2^{\pm1}\right\}\cup\left\{(e_1e_2)^{\pm1}\right\},\mathbb{F}\right).$$

   By construction $\mathbb{G}' \in \left[\mathbb{G}_1,\mathbb{G}\right[$. Besides, $(c_F)_{F\in\mathbb{F}'^b}$ is still a family of facial loops for $\mathbb{G}'$ oriented anti-clockwise and $T' = \left(T\setminus\left\{e_1^{\pm1},e_2^{\pm1}\right\}\right)\cup\left\{(e_1e_2)^{\pm1}\right\}$ is a spanning tree of $\mathbb{G}'$ rooted at m. It is now obvious that $\mathbb{G}'$ satisfies property $(\mathcal{H})$ with the choices of $(c_F)_{F\in\mathbb{F}'^b}$ and T'.
2. We will consider that F_1 is bounded, the unbounded case is similar. In this case, let v be the vertex of e of degree 1 and define $F_0' = F_0\cup e\big((0,1)\big)\cup\{v\}$. Consider any family of facial loops for $\mathbb{G}$ oriented anti-clockwise, $(c_F)_{F\in\mathbb{F}^b}$, such that $\underline{c_{F_0}} \neq v$. Let us choose any spanning tree of $\mathbb{G}$ rooted at m, T. We

consider $\mathbb{G}'$, the graph defined by:

$$\left(\mathbb{V}', \mathbb{E}', \mathbb{F}'\right) = \left(\mathbb{V} \setminus v, \mathbb{E} \setminus \{e, e^{-1}\}, (\mathbb{F} \setminus F_0) \cup F_0'\right).$$

The spanning tree T of $\mathbb{G}$ must include the unoriented edge $\left\{e, e^{-1}\right\}$ in order to cover v, thus we can define $T' = T \setminus \{e, e^{-1}\}$. The facial loop c_{F_0} contains the sequence ee^{-1}. We define ${c'}_{F_0'}$ from c_{F_0} by removing this sequence. For any other face $F \in \mathbb{F}'$, we set ${c'}_F = c_F$. For any face $F \in \mathbb{F}'^b$, using the identification between F_0 and F_0', $\mathbf{l}_{c_F,T} = \mathbf{l}_{c'_F,T'}$ in $RL_m(\mathbb{G})$, and by Remark 2.1, $h(\mathbf{l}_{c_F,T}) = h(\mathbf{l}_{c'_F,T'})$. The graph $\mathbb{G}'$ satisfies property $(\mathcal{H})$ with the choices of $\left(c'_{F'}\right)_{F' \in \mathbb{F}'^b}$ and T'.

3. We will study this case under the hypothesis that F_1 is bounded, the unbounded case being easier. The key point will be the semigroup property satisfied by the marginal distributions of the Lévy process Y. Let F_r and F_l be two faces of $\mathbb{G}$ contained in F_1 and adjacent, sharing an edge e on their boundaries. We can find a facial loop oriented anti-clockwise representing the boundary of F_r (resp. F_l) of the form $c_{F_r} = e_1 \dots e_n e$ (resp. $c_{F_l} = e^{-1} e'_1 \dots e'_m$). We define $F_{r,l} = F_r \cup F_l \cup e\big((0,1)\big)$. We complete the family (c_{F_r}, c_{F_l}) in order to have a family of facial loops $(c_F)_{F \in \mathbb{F}^b}$ oriented anti-clockwise for $\mathbb{G}$. Let us consider $\mathbb{G}'$, the graph defined by:

$$\left(\mathbb{V}', \mathbb{E}', \mathbb{F}'\right) = \left(\mathbb{V}, \mathbb{E} \setminus \left\{e, e^{-1}\right\}, (\mathbb{F} \setminus \{F_r, F_l\}) \cup F_{r,l}\right).$$

It is still a finite planar graph. Let us consider T any spanning tree of $\mathbb{G}'$ rooted at m: it is also a spanning tree of $\mathbb{G}$ rooted at m. We define $c'_{F_{r,l}} = e_1 \dots e_n e'_1 \dots e'_m$. For any other face F' of $\mathbb{G}'$ different from $F_{r,l}$, F' is a face of $\mathbb{G}$ and we set $c'_{F'} = c_{F'}$. Once these choices made, it needs only a simple verification to check that the following equalities hold in $RL_m(\mathbb{G})$:

$$\mathbf{l}_{c'_{F_{r,l}},T} = \mathbf{l}_{c_{F_r},T} \mathbf{l}_{c_{F_l},T},$$
$$\mathbf{l}_{c'_{F'},T} = \mathbf{l}_{c_{F'},T}, \forall F' \in \mathbb{F}'^b, F' \neq F_{r,l}.$$

Using the multiplicativity of h:

$$h(\mathbf{l}_{c'_{F_{r,l}},T}) = h(\mathbf{l}_{c_{F_l},T}) h(\mathbf{l}_{c_{F_r},T}),$$
$$h(\mathbf{l}_{c'_{F'},T}) = h(\mathbf{l}_{c_{F'},T}), \forall F' \in \mathbb{F}'^b, F' \neq F_{r,l}.$$

Let us recall that under $\mathbb{E}_{vol}^{Y,\mathbb{G}_2}$,

(a) the random variables $\left(h\left(\mathbf{l}_{c_F,T}\right)\right)_{F \in \mathbb{F}^b}$ are independent,

(b) for any $F \in \mathbb{F}^b$, $h(\mathbf{l}_{c_F,T})$ has the same law as $Y_{vol(F)}$.

Using the semigroup property of the marginal distributions of the process Y, we can conclude that $\mathbb{G}'$ satisfies $(\mathcal{H})$ with the choices of $({c'}_{F'})_{F' \in \mathbb{F}'^b}$ and T.

By descending induction, it follows that $\mathbb{G}_1$ satisfies property $(\mathcal{H})$.

Uniform $\frac{1}{2}$-Hölder continuity : Let $\mathbb{G}$ be a finite planar graph with piecewise affine edges. Let l be a simple loop in $\mathbb{G}$ bounding a disk D. A consequence of what we have just seen is that the law of $h(l)$ under $\mathbb{E}_{vol}^{Y,\mathbb{G}}$ is the same as under

$\mathbb{E}^{Y,\mathbb{G}(l)}_{vol}$, where $\mathbb{G}(l)$ is the graph containing only the edge l (see Example 3.1). Thus:

$$\int_{\mathcal{M}ult(P(\mathbb{G}),G)} d_G\big(1,h(l)\big)\mathbb{E}^{Y,\mathbb{G}}_{vol}(dh) = \int_{\mathcal{M}ult(P(\mathbb{G}(l)),G)} d_G\big(1,h(l)\big)\mathbb{E}^{Y,\mathbb{G}(l)}_{vol}(dh)$$
$$= \mathbb{E}\left[d_G\left(1,Y_{vol(D)}\right)\right] \leq K\sqrt{vol(D)},$$

where the last inequality comes from Proposition 8.3 and where K depends only on G. The family $(\mathbb{E}^{Y,\mathbb{G}}_{vol})_{\mathbb{G}\in\mathcal{G}(\mathsf{Aff}(\mathbb{R}^2))}$ is uniformly locally stochastically $\frac{1}{2}$-Hölder continuous.

Thus, we can apply Proposition 6.4 in order to construct a stochastically continuous random holonomy field $\mathbb{E}^{Y}_{vol}$ such that for any finite planar graph $\mathbb{G} \in \mathcal{G}(\mathsf{Aff}\left(\mathbb{R}^2\right))$, for any rooted spanning tree T of $\mathbb{G}$ and any family of facial loops $(c_F)_{F\in\mathbb{F}^b}$ oriented anti-clockwise, under $\mathbb{E}^{Y}_{vol}$:

1. the random variables $\left(h\left(\mathbf{l}_{c_F,T}\right)\right)_{F\in\mathbb{F}^b}$ are independent,
2. for any $F \in \mathbb{F}^b$, $h\left(\mathbf{l}_{c_F,T}\right)$ has the same law as $Y_{vol(F)}$.

It remains to prove that this property is true for any finite planar graph, not necessarily with piecewise affine edges. Let $(m_t)_{t\in\mathbb{R}^+}$ be the continuous semi-group of convolution associated with $(Y_t)_{t\geq 0}$. Let $\mathbb{G} = (\mathbb{V},\mathbb{E},\mathbb{F})$ be a finite planar graph, let T be a rooted spanning tree and let $\left(c_F\right)_{F\in\mathbb{F}^b}$ be a family of facial loops oriented anti-clockwise. Let us consider a sequence of finite planar graphs $\left(\mathbb{G}_n = (\mathbb{V}_n,\mathbb{E}_n,\mathbb{F}_n)\right)_{n\in\mathbb{N}}$ in $\mathcal{G}\big(\mathsf{Aff}(\mathbb{R}^2)\big)$ and $(\psi_n)_{n\in\mathbb{N}}$ a sequence of orientation-preserving homeomorphisms which satisfy the conditions of Theorem 3.2. For any integer n, $(\psi_n(c_F))_{F\in\mathbb{F}^b}$ is a family of facial loops for $\mathbb{G}_n$ which is oriented anti-clockwise and $\psi_n(T)$ is a spanning tree of $\mathbb{G}_n$. Using the discussion we had before, the law of $\left(h(\mathbf{l}_{\psi_n(c_F),\psi_n(T)})\right)_{F\in\mathbb{F}^b}$ under $\mathbb{E}^{Y}_{vol}$ is $\bigotimes_{F\in\mathbb{F}^b} m_{vol(\psi_n(F))}$. As for any edge $e \in \mathbb{E}$, $(\psi_n(e))_{n\geq 0}$ converges to e for the convergence with fixed endpoints, for any face $F \in \mathbb{F}^b$, one has:

$$\mathbf{l}_{\psi_n(c_F),\psi_n(T)} \underset{n\to\infty}{\longrightarrow} \mathbf{l}_{c_F,T}$$

for the fixed endpoints convergence. Besides, using condition 4 of Theorem 3.2 and the continuity of $(m_t)_{t\in\mathbb{R}^+}$,

$$\bigotimes_{F\in\mathbb{F}^b} m_{vol(\psi_n(F))} \underset{n\to\infty}{\longrightarrow} \bigotimes_{F\in\mathbb{F}^b} m_{vol(F)}.$$

Since $\mathbb{E}^{Y}_{vol}$ is stochastically continuous, we get that under $\mathbb{E}^{Y}_{vol}$, the law of $\left(h(\mathbf{l}_{c_F,T})\right)_{F\in\mathbb{F}^b}$ is $\bigotimes_{F\in\mathbb{F}^b} m_{vol(F)}$. □

Let us remark that, in the latest argument, we actually proved that the family $\left(\mathbb{E}^{Y,\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ is continuously area-dependent.

For any Lévy process which is invariant by conjugation, we have constructed a family of gauge-invariant stochastically continuous random holonomy fields. In the following, we will show that this family is a strong planar Markovian holonomy field.

Proposition 8.4. — *Let $(Y_t)_{t\geq 0}$ be a G-valued Lévy process invariant by conjugation. The family of random holonomy fields $\left(\mathbb{E}^Y_{vol}\right)_{vol}$ is a constructible stochastically continuous strong planar Markovian holonomy field.*

Proof. — We have already shown that the family $\left(\mathbb{E}^{Y,\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ satisfies the Axiom $\mathbf{DP_4}$, is continuously area-dependent and locally stochastically $\frac{1}{2}$-Hölder continuous. By Theorem 4.1, it remains to check that $\left(\mathbb{E}^{Y,\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ satisfies the three Axioms $\mathbf{DP_1}$, $\mathbf{DP_2}$ and $\mathbf{DP_3}$ in Definition 4.4.

Besides, $\left(\mathbb{E}^{Y,\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ is stochastically continuous in law: using a continuity argument, it is enough to show that $\mathbf{wDP_2}$ holds instead of $\mathbf{DP_2}$. Let us give briefly the arguments which allow us to do so: first of all, using Theorem 3.2, we see that it is enough to consider graphs with piecewise affine edges. Let us suppose that the Axiom $\mathbf{wDP_2}$ holds, let us consider $\mathbb{G}$ a finite planar graph with piecewise affine edges and two loops l_1 and l_2 in $P(\mathbb{G})$ such that $\mathsf{Int}(l_1)\cap\mathsf{Int}(l_2)=\emptyset$. The only interesting case is when $\overline{\mathsf{Int}(l_1)}\cap\overline{\mathsf{Int}(l_2)}\neq\emptyset$: let us consider a point v in this intersection. Using Remark 2.7, it is enough to prove that for any family of loops $(l^1_i)_{i=1}^n$ (resp. $(l^2_i)_{i=1}^m$) in $\overline{\mathsf{Int}(l_1)}$ (resp. in $\overline{\mathsf{Int}(l_2)}$) based at v, $\left(h(l^1_i)\right)_{i=1}^n$ is $\mathcal{I}$-independent of $\left(h(l^2_i)\right)_{i=1}^m$ under the measure $\mathbb{E}^{Y,\mathbb{G}}_{vol}$. Let us consider such families of loops $(l^1_i)_{i=1}^n$ and $(l^2_i)_{i=1}^m$. One can always approximate $\mathbb{G}$ by a finite planar graph $\mathbb{G}'$ with piecewise affine edges such that there exist l'_1 and l'_2 two loops in $\mathbb{G}'$ and p a path in $\mathbb{G}'$ such that:

1. $\overline{\mathsf{Int}(l'_1)}\cap\overline{\mathsf{Int}(l'_2)}=\emptyset$,
2. for any loop l in $\overline{\mathsf{Int}(l_1)}$ based at v, there exists a loop in $\overline{\mathsf{Int}(l'_1)}$ which approximates l for the convergence with fixed endpoints,
3. for any loop l in $\overline{\mathsf{Int}(l_2)}$ based at v, there exist a loop in $\overline{\mathsf{Int}(l'_2)}$, denoted by l' such that $pl'p^{-1}$ approximates l for the convergence with fixed endpoints.

Let us consider the graph $\mathbb{G}'$, the two loops l'_1 and l'_2 and the path p given by the last assertion. We can approximate $(l^1_i)_{i=1}^n$ and $(l^2_i)_{i=1}^m$ by two families $(l'^1_i)_{i=1}^n$ and $(pl'^2_ip^{-1})_{i=1}^m$ such that the first one is in $\overline{\mathsf{Int}(l'_1)}$ and $(l'^2_i)_{i=1}^m$ is in $\overline{\mathsf{Int}(l'_2)}$. The $\mathcal{I}$-independence of $(l^1_i)_{i=1}^n$ and $(l^2_i)_{i=1}^m$ under the measure $\mathbb{E}^{Y,\mathbb{G}}_{vol}$ would be a consequence of the $\mathcal{I}$-independence of $(l'^1_i)_{i=1}^n$ and $(pl'^2_ip^{-1})_{i=1}^m$ under $\mathbb{E}^{Y,\mathbb{G}'}_{vol}$, which is equivalent to the the $\mathcal{I}$-independence of $(l'^1_i)_{i=1}^n$ and $(l'^2_i)_{i=1}^m$ under $\mathbb{E}^{Y,\mathbb{G}'}_{vol}$. Using Remark 2.6, this is equivalent to the independence of $(l'^1_i)_{i=1}^n$ and $(l'^2_i)_{i=1}^m$ under $\mathbb{E}^{Y,\mathbb{G}'}_{vol}$ which is granted since we supposed that the Axiom $\mathbf{wDP_2}$ holds.

Let us prove that that $\left(\mathbb{E}^{Y,\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ satisfies the three Axioms $\mathbf{DP_1}$, $\mathbf{wDP_2}$ and $\mathbf{DP_3}$.

$\mathbf{DP_1}$: Consider vol and vol' two measures of area on $\mathbb{R}^2$. Let $\mathbb{G}$ and $\mathbb{G}'$ be two finite planar graphs. Let ψ be a homeomorphism which preserves the orientation. Let us suppose that $\psi(\mathbb{G})=\mathbb{G}'$ and for any $F\in\mathbb{F}^b$, $vol(F)=vol'(\psi(F))$. Let $(c'_F)_{F\in\mathbb{F}'^b}$ be a family of facial loops oriented anti-clockwise for $\mathbb{G}'$ and let T' be a rooted spanning tree of $\mathbb{G}'$. We consider $\left(c_F=\psi^{-1}\left(c'_{\psi(F)}\right)\right)_{F\in\mathbb{F}^b}$ and $T=\psi^{-1}(T')$. The family $(c_F)_{F\in\mathbb{F}^b}$ is a family of facial loops for $\mathbb{G}$ which are oriented anti-clockwise and T is a rooted spanning tree of $\mathbb{G}$. Recall the notations used in the proof of

Proposition 8.1. By construction, we have the equality:

$$\mathbb{E}^{Y,\mathbb{G}'}_{vol',T',(c'_F)_{F\in\mathbb{F}'^b}} \circ \psi^{-1} = \mathbb{E}^{Y,\mathbb{G}}_{vol,T,(c_F)_{F\in\mathbb{F}^b}},$$

where we denoted also by ψ the induced application from $\mathcal{M}ult(P(\mathbb{G}'),G)$ to $\mathcal{M}ult(P(\mathbb{G}),G)$ induced by the homeomorphism ψ. Using the Proposition 8.1, we get:

$$\mathbb{E}^{Y,\mathbb{G}'}_{vol'} \circ \psi^{-1} = \mathbb{E}^{Y,\mathbb{G}}_{vol}.$$

wDP$_2$: Let vol be a measure of area on $\mathbb{R}^2$. Let $\mathbb{G} = (\mathbb{V},\mathbb{E},\mathbb{F})$ be a finite planar graph in $\mathcal{G}\big(\mathsf{Aff}\left(\mathbb{R}^2\right)\big)$ and let m be a vertex of $\mathbb{G}$. Let L_1 and L_2 be two simple loops in $P(\mathbb{G})$ whose closure of the interiors are disjoint. As an application of Proposition 6.2, we can consider T a spanning tree rooted at m, such that for any family of facial loops $(c_F)_{F\in\mathbb{F}^b}$ oriented anti-clockwise:

1. for every loop l in $P(\mathbb{G})$ inside $\overline{\mathsf{Int}(L_1)}$, $\left[[m,\underline{l}]_T l [m,\underline{l}]_T^{-1}\right]_{\simeq}$ is a product in $RL_m(\mathbb{G})$ of elements of $\left\{\mathbf{l}^{\pm 1}_{c_F,T}; F\in\mathbb{F}^b, F\subset\overline{\mathsf{Int}(L_1)}\right\}$,
2. for every loop l in $P(\mathbb{G})$ inside $\overline{\mathsf{Int}(L_2)}$, $\left[[m,\underline{l}]_T l [m,\underline{l}]_T^{-1}\right]_{\simeq}$ is a product in $RL_m(\mathbb{G})$ of elements of $\left\{\mathbf{l}^{\pm 1}_{c_F,T}; F\in\mathbb{F}^b, F\subset\overline{\mathsf{Int}(L_2)}\right\}$.

Let us consider $(p_i)_{i=1}^n$ (resp. $(p'_j)_{j=1}^{n'}$), some paths in $P(\mathbb{G})$ which are inside $\overline{\mathsf{Int}(L_1)}$ (resp. $\overline{\mathsf{Int}(L_2)}$). Recall the definition given by the Equality (6). For any continuous function f_1 (resp. f_2) defined on G^n (resp. on $G^{n'}$), using the gauge-invariance, we have that $\mathbb{E}^{Y,\mathbb{G}}_{vol}\left[f_1\left((h(p_i))_{i=1}^n\right) f_2\left(\left(h(p'_j)\right)_{j=1}^{n'}\right)\right]$ is equal to:

$$\mathbb{E}^{Y,\mathbb{G}}_{vol}\left[\hat{f}_{1\,J_{p_1,\dots,p_n}}\left(h(p_1),\dots,h(p_n)\right)\hat{f}_{2\,J_{p'_1,\dots,p'_{n'}}}\left(h(p'_1),\dots,h(p'_{n'})\right)\right].$$

Thus, it is also equal to:

$$\mathbb{E}^{Y,\mathbb{G}}_{vol}\left[\hat{f}_{1\,J_{p_1,\dots,p_n}}\left(\left(h(\tilde{l}_i)\right)_{i=1}^n\right)\hat{f}_{2\,J_{p'_1,\dots,p'_{n'}}}\left(\left(h(\tilde{l}'_i)\right)_{i=1}^{n'}\right)\right].$$

where, for any $i\in\{1,\dots,n\}$, $\tilde{l}_i = [m,\underline{p_i}]_T p_i [m,\overline{p_i}]_T^{-1}$ and for any $i\in\{1,\dots,n'\}$, $\tilde{l}'_i = [m,\underline{p'_i}]_T p'_i [m,\overline{p'_i}]_T^{-1}$. Recall the form of T given in the proof of Proposition 6.2: this implies that there exist $(l_i)_{i=1}^n$ some loops in $\overline{\mathsf{Int}(L_1)}$ and $(l'_i)_{i=1}^{n'}$ some loops in $\overline{\mathsf{Int}(L_2)}$ such that for any $i\in\{1,\dots,n\}$ and any $j\in\{1,\dots,n'\}$,

$$\tilde{l}_i = [m,\underline{l_i}]_T l_i [m,\underline{l_i}]_T^{-1},$$
$$\tilde{l}'_j = [m,\underline{l'_j}]_T l'_j [m,\underline{l'_j}]_T^{-1},$$

in $RL_m(\mathbb{G})$. Using the properties satisfied by T, the following assertions hold:

$$\sigma\left(\left(h(\tilde{l}_i)\right)_{i=1}^n\right) \subset \sigma\left(\left\{h(\mathbf{l}_{c_F,T}); F\in\mathbb{F}^b, F\subset\overline{\mathsf{Int}(L_1)}\right\}\right),$$
$$\sigma\left(\left(h(\tilde{l}'_i)\right)_{i=1}^{n'}\right) \subset \sigma\left(\left\{h(\mathbf{l}_{c_F,T}); F\in\mathbb{F}^b, F\subset\overline{\mathsf{Int}(L_2)}\right\}\right).$$

We recall that $\overline{\mathsf{Int}(L_1)} \cap \overline{\mathsf{Int}(L_2)} = \emptyset$, thus the two σ-fields:

$$\sigma\left(\left\{h(\mathbf{l}_{c_F,T}); F \in \mathbb{F}^b, F \subset \overline{\mathsf{Int}(L_1)}\right\}\right),$$
$$\sigma\left(\left\{h(\mathbf{l}_{c_F,T}); F \in \mathbb{F}^b, F \subset \overline{\mathsf{Int}(L_2)}\right\}\right)$$

are independent under $\mathbb{E}^{Y,\mathbb{G}}_{vol}$. Thus $\mathbb{E}^{Y,\mathbb{G}}_{vol}\left[f_1\left((h(p_i))_{i=1}^n\right) f_2\left(\left(h(p'_j)\right)_{j=1}^{n'}\right)\right]$ is equal to:

$$\mathbb{E}^{Y,\mathbb{G}}_{vol}\left[\hat{f}_{1\,J_{p_1,\ldots,p_n}}\left(\left(h(\tilde{l}_i)\right)_{i=1}^{n}\right)\right]\mathbb{E}^{Y,\mathbb{G}}_{vol}\left[\hat{f}_{2\,J_{p'_1,\ldots,p'_{n'}}}\left(\left(h(\tilde{l}'_i)\right)_{i=1}^{n'}\right)\right],$$

which is equal to $\mathbb{E}^{Y,\mathbb{G}}_{vol}\left[f_1\left((h(p_i))_{i=1}^n\right)\right]\mathbb{E}^{Y,\mathbb{G}}_{vol}\left[f_2\left((h(p'_j))_{j=1}^{n'}\right)\right]$: the axiom $\mathbf{wDP_2}$ is satisfied.

$\mathbf{DP_3}$: Let l be a simple loop, let vol and vol' be two measures of area on $\mathbb{R}^2$ which agree on the interior of l. Let $\mathbb{G}$ be a finite planar graph included in the closure of the interior of l. The bounded faces of $\mathbb{G}$ are in the interior of l thus for any bounded face F of $\mathbb{G}$, $vol(F) = vol'(F)$. By definition, it is clear that $\mathbb{E}^{Y,\mathbb{G}}_{vol} = \mathbb{E}^{Y,\mathbb{G}}_{vol'}$.

We have proved all the conditions on $\left(\mathbb{E}^{Y,\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ we needed in order to apply Theorem 4.1. The family of holonomy fields $\left(\mathbb{E}^{Y}_{vol}\right)_{vol}$ is a constructible stochastically continuous strong planar Markovian holonomy field. □

For this new construction, we used the loop paradigm which links the multiplicative functions on a set P and the pre-multiplicative functions on its set of loops. The edge paradigm given by the Equation (1) can be used to give an explicit formula for discrete planar Yang-Mills fields associated with a conjugation invariant Lévy process with density.

Proposition 8.5. — *Let $\left(Y_t\right)_{t\geq 0}$ be a G-valued Lévy process invariant by conjugation. Let us suppose that for any positive real t, Y_t has a density Q_t with respect to the Haar measure. Let $\left(\mathbb{E}^{Y,\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ be the discrete planar Yang-Mills field associated with $\left(Y_t\right)_{t\geq 0}$. For any finite planar graph $\mathbb{G}$ and for any measure of area vol:*

$$\mathbb{E}^{Y,\mathbb{G}}_{vol}(dh) = \prod_{F\in\mathbb{F}^b} Q_{vol(F)}\big(h(\partial F)\big) \bigotimes_{e\in\mathbb{E}^+} dh(e), \tag{18}$$

where ∂F is the anti-clockwise oriented facial cycle associated with F, the notation $Q_{vol(F)}\big(h(\partial F)\big)$ means that we consider $Q_{vol(F)}\big(h(c)\big)$ where c represents ∂F (this does not depend on the choice of c since $Q_{vol(F)}$ is invariant by conjugation) and $\bigotimes_{e\in\mathbb{E}^+} dh(e)$ is the push forward of $\bigotimes_{e\in\mathbb{E}^+} dg_e$ on $\mathcal{M}ult(P,G)$ by the edge paradigm identification. It is independent of the choice of orientation $\mathbb{E}^+$.

Recall the definition of $L_{i,j}$ in Definition 6.4. In order to make the proof simple, we will use the upcoming Theorem 10.7 which roughly asserts that a stochastically continuous planar Markovian holonomy field is characterized by the law of the random sequence $(h(L_{n,0}))_{n\in\mathbb{N}}$.

Proof. — A slight modification of Section 4.3 in [**21**] shows that:

$$\Big(\prod_{F\in\mathbb{F}^b} Q_{vol(F)}(h(\partial F))\bigotimes_{e\in\mathbb{E}^+} dh(e)\Big)_{\mathbb{G},vol} \tag{19}$$

is a stochastically continuous in law discrete planar Markovian holonomy field. Let $\mathbb{G}=(\mathbb{V},\mathbb{E},\mathbb{F})$ be the planar graph $\mathbb{N}^2\cap\big(\mathbb{R}^+\times[0,1]\big)$. Let us consider $\mathbb{E}^+$ an orientation of $\mathbb{E}$. As an application of Theorem 10.7, we only have to check that for any positive real α, $\big(h(L_{n,0})\big)_{n\in\mathbb{N}}$ has the same law under $\mathbb{E}^{Y,\mathbb{G}}_{\alpha dx}$ as under $\prod\limits_{F\in\mathbb{F}^b} Q_\alpha\big(h(\partial F)\big)\bigotimes\limits_{e\in\mathbb{E}^+} dh(e)$. The value of α will not matter, thus we set α equals to 1.

By Proposition 8.1, under $\mathbb{E}^{Y,\mathbb{G}}_{dx}$, $\big(h(L_{n,0})\big)_{n\in\mathbb{N}}$ are i.i.d. random variables which have the same law as Y_1. It remains to prove that this property is true under $\prod\limits_{F\in\mathbb{F}^b} Q_1\big(h(\partial F)\big)\bigotimes\limits_{e\in\mathbb{E}^+} dh(e)$.

Under the probability law $\bigotimes_{e\in\mathbb{E}^+} dh(e)$, $\big(h(e)\big)_{e\in\mathbb{E}^+}$ are i.i.d. and Haar distributed. Using the multiplicativity property of random holonomy fields, for any integer n, $h(L_{n,0})$ is a product of elements of $(h(e))_{e\in\mathbb{E}}=(h(e))_{e\in\mathbb{E}^+}\cup(h(e)^{-1})_{e\in\mathbb{E}^+}$. An important remark is that for any integer n, the edge $e^r_{n,0}$, defined in the Notation 6.1, appears only once in the reduced decomposition of $L_{n,0}$ and in no other reduced decomposition of $L_{m,0}$ with $m\neq n$. Applying Lemma 8.1, one has that under $\bigotimes_{e\in\mathbb{E}^+} dh(e)$, $\big(h(L_{n,0})\big)_{n\in\mathbb{N}}$ are independent and each of them is a Haar random variable. Recall the notation $\partial c_{i,j}$ defined in Definition 6.4. Since Y_1 is invariant by conjugation, for any bounded face F in $\mathbb{F}$, there exists an integer $n\in\mathbb{N}$ such that $Q_1\left(\partial F\right)=Q_1\big(h(\partial c_{n,0})\big)=Q_1\big(h(L_{n,0})\big)$. Let $f:G^{\mathbb{N}}\to\mathbb{R}$ be a measurable function, the following sequence of equality holds:

$$\begin{aligned}
&\int_{\mathcal{M}ult\left(P(\mathbb{G}),G\right)} f\Big(\big(h(L_{n,0})\big)_{n\in\mathbb{N}}\Big)\prod_{F\in\mathbb{F}^b} Q_1\big(h(\partial F)\big)\bigotimes_{e\in\mathbb{E}^+} dh(e)\\
&\qquad=\int_{\mathcal{M}ult\left(P(\mathbb{G}),G\right)} f\Big(\big(h(L_{n,0})\big)_{n\in\mathbb{N}}\Big)\prod_{n\in\mathbb{N}} Q_1\big(h(\partial c_{n,0})\big)\bigotimes_{e\in\mathbb{E}^+} dh(e)\\
&\qquad=\int_{\mathcal{M}ult\left(P(\mathbb{G}),G\right)} f\Big(\big(h(L_{n,0})\big)_{n\in\mathbb{N}}\Big)\prod_{n\in\mathbb{N}} Q_1\big(h(L_{n,0})\big)\bigotimes_{e\in\mathbb{E}^+} dh(e)\\
&\qquad=\int_{G^{\mathbb{N}}} f\Big(\big(g_n\big)_{n\in\mathbb{N}}\Big)\bigotimes_{n\in\mathbb{N}}\Big(Q_1(g_n)dg_n\Big),
\end{aligned}$$

which is the assertion we had to prove. □

Lemma 8.1. — *Let $(\alpha_i)_{i=1}^\infty$ be a sequence of independent G-valued random variables which are Haar distributed. Let $(\beta_j)_{j=1}^\infty$ be a sequence of G-valued random variables such that for every $j\in\mathbb{N}^*$, β_j is a product of elements of $\{\alpha_i,\alpha_i^{-1},i\in\mathbb{N}^*\}$:*

$$\beta_j=w_j\big((\alpha_i,\alpha_i^{-1})_{i\in\mathbb{N}^*}\big),$$

with w_j being a finite word.

Suppose that for any $j\in\mathbb{N}^*$, there exists an index i_j such that α_{i_j} appears exactly once in w_j and in no other word $(w_{j'})_{j'\neq j}$. Then $(\beta_j)_{j=1}^\infty$ is a family of independent Haar distributed random variables.

Proof. — Let k be any positive integer and let $(j, j_1, \dots, j_k)$ be a $k+1$-tuple of positive integers. Let $i_j \in \mathbb{N}^*$ such that α_{i_j} appears exactly once in w_j and in no other word $(w_{j'})_{j' \neq j}$. Let $F : G^k \to \mathbb{R}$ and $f : G \to \mathbb{R}$ be two continuous functions. There exist w_1 and w_2 two words in $(\alpha_i, \alpha_i^{-1})_{i \neq i_j}$, J a subset of $\mathbb{N} \setminus \{i_j\}$ and $\tilde{F}$ a continuous function from $G^{\#J}$ to $\mathbb{R}$ such that a.s.:

$$f(\beta_j) F(\beta_{j_1} \dots \beta_{j_k}) = f(w_1 \alpha_{i_j} w_2) \tilde{F}\left((\alpha_i)_{i \in J}\right).$$

Thus, the following sequence of equalities holds:

$$\begin{aligned}
\mathbb{E}\left[f(\beta_j) F(\beta_{j_1} \dots \beta_{j_k})\right] &= \mathbb{E}\left[f(w_1 \alpha_{i_j} w_2) \tilde{F}\left((\alpha_i)_{i \in J}\right)\right] \\
&= \int_G \mathbb{E}\left[f(w_1 x w_2) \tilde{F}\left((\alpha_i)_{i \in J}\right)\right] dx \\
&= \int_G \mathbb{E}\left[f(x) \tilde{F}\left((\alpha_i)_{i \in J}\right)\right] dx \\
&= \left(\int_G f(x) dx\right) \mathbb{E}\left[\tilde{F}\left((\alpha_i)_{i \in J}\right)\right],
\end{aligned}$$

where we used for the third equality the translation invariance of the Haar measure. Thus for any $j \in \mathbb{N}^*$, β_j is a Haar random variable which is independent of $(\beta_j, j \neq i)$. □

8.2. Construction of general planar Yang-Mills fields. — In the last subsection we didn't create all the possible planar Yang-Mills fields. Indeed, we considered only Lévy processes which were invariant by conjugation by G. We are going to see in this subsection, that for any G-valued self-invariant by conjugation Lévy process Y, one can construct a planar Markovian holonomy field associated to Y. Let us introduce the notion of support of a process.

Definition 8.2. — *Let $Y = \left(Y_t\right)_{t \in \mathbb{R}^+}$ be a random process. We define the support of Y by the following equality:*

$$H_Y = \overline{\langle \bigcup_{t \in \mathbb{R}^+} H_{Y_t} \rangle}.$$

An other formulation is to say that the support of a process, say Y, is the smallest closed group such that for any $t \in \mathbb{R}^+$, $\mathbb{P}(Y_t \in H_Y) = 1$. If Y is a Lévy process, we can consider Y as a process living in H_Y.

Remark 8.1. — *Let Y be a Lévy process and let us suppose that for any $t \geq 0$, e is in $\mathsf{Supp}(Y_t)$. Then for any $t \in \mathbb{R}^+$, $H_Y = H_{Y_t}$. Indeed, using the property that Y is a Lévy process, for any $0 \leq t < s$, $\mathsf{Supp}(Y_s) = \overline{\mathsf{Supp}(Y_t)\mathsf{Supp}(Y_{s-t})}$. Thus, using the condition on the support of Y_t, H_{Y_t} is increasing in t. Yet, using the same argument, we see that $H_{Y_{2t}} = H_{Y_t}$, thus H_{Y_t} does not depend on t. This remark explains why we impose that for any $t \geq 0$, $e \in \mathsf{Supp}(Y_t)$ in Proposition 9.9.*

We weaken the notion of invariance by conjugation for a random process.

Definition 8.3. — *Let $(Y_t)_{t \geq 0}$ be a G-valued process. It is self-invariant by conjugation if it is invariant by conjugation by H_Y.*

Let η be a finite Borel measure on G^n. For any g in G, the measure η^g on G^n is the unique measure such that for any continuous function $f : G^n \to \mathbb{R}$:

$$\eta^g(f) = \int_G f(g^{-1}g_1g, \dots, g^{-1}g_ng)\eta(dg_1, \dots, dg_n). \tag{20}$$

We can now construct a planar Yang-Mills field associated with any self-invariant by conjugation Lévy process.

Theorem 8.1. — *For every G-valued self-invariant by conjugation Lévy process Y, there exists a unique stochastically continuous strong planar Markovian holonomy field $\left(\mathbb{E}^Y_{vol}\right)_{vol}$, called the planar Yang-Mills field associated with Y, such that for any measure of area vol, for any finite planar graph $\mathbb{G}$, for any rooted spanning tree T of $\mathbb{G}$ and any family of facial loops $(c_F)_{F\in\mathbb{F}^b}$ oriented anti-clockwise, under $\mathbb{E}^Y_{vol}$, the law of $\left(h(\mathbf{l}_{c_F,T})\right)_{F\in\mathbb{F}^b}$ is:*

$$\int_G \left(\otimes_{F\in\mathbb{F}^b} m_{vol(F)}\right)^g dg,$$

where $(m_t)_{t\geq 0}$ is the semi-group of convolution of measures associated with Y.

Let us notice that on $\left(\mathbb{E}^Y_{vol}, \mathcal{B}\right)$, $\left(h(\mathbf{l}_{c_F,T})\right)_{F\in\mathbb{F}^b}$ is not, in general, a sequence of independent variables.

Proof. — The unicity part uses the same arguments as usual. Let us prove the existence of the stochastically continuous strong planar Markovian holonomy field $\left(\mathbb{E}^Y_{vol}\right)_{vol}$. Let $Y = \left(Y_t\right)_{t\geq 0}$ be a G-valued self-invariant by conjugation Lévy process. Let H_Y be the support of Y. Using the discussion after Definition 8.2, we can see the process Y as a H_Y-valued Lévy process which is invariant by conjugation (by H_Y). Thus, applying Propositions 8.2 and 8.4, there exists a H_Y-valued stochastically continuous strong planar Markovian holonomy field such that for any measure of area vol, any finite planar graph $\mathbb{G}$, for any rooted spanning tree T of $\mathbb{G}$ and any family of facial loops $(c_F)_{F\in\mathbb{F}^b}$ oriented anti-clockwise, under $\mathbb{E}^Y_{vol}$:

1. the random variables $\left(h\left(\mathbf{l}_{c_F,T}\right)\right)_{F\in\mathbb{F}^b}$ are independent,
2. for any $F \in \mathbb{F}^b$, $h(\mathbf{l}_{c_F,T})$ has the same law as $Y_{vol(F)}$.

Recall that Proposition 4.1 can also be applied to planar (continuous) Markovian holonomy fields. Thus we can extend the group on which $\mathbb{E}^Y_{vol}$ is defined, from H_Y to G: we will denote it $\mathbb{E}^Y_{vol}$. It is a G-valued stochastically continuous strong planar Markovian holonomy field and by definition, for any measure of area vol, for any finite planar graph $\mathbb{G}$, for any rooted spanning tree T of $\mathbb{G}$ and any family of facial loops $(c_F)_{F\in\mathbb{F}^b}$ oriented anti-clockwise, under $\mathbb{E}^Y_{vol}$, the law of $\left(h(\mathbf{l}_{c_F,T})\right)_{F\in\mathbb{F}^b}$ is:

$$\int_G \left(\otimes_{F\in\mathbb{F}^b} m_{vol(F)}\right)^g dg.$$

This ends the proof of the theorem. □

Definition 8.4. — *By construction, the planar Yang-Mills field associated with a self-invariant by conjugation Lévy process Y is constructible. Its restriction to multiplicative*

functions on finite planar graphs is called the discrete planar Yang-Mills field associated with Y, we will denote it $\left(\mathbb{E}^{Y,\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$.

We are led to classify the planar Yang-Mills fields according to their degree of symmetry. In Section 11, we will prove equivalent conditions in order to classify planar Yang-Mills fields.

Definition 8.5. — *Let $\left(\mathbb{E}^{Y}_{vol}\right)_{vol}$ be a planar Yang-Mills field associated with a G-valued self-invariant by conjugation Lévy process $Y = \left(Y_t\right)_{t\in\mathbb{R}^+}$. The planar Yang-Mills field $\left(\mathbb{E}^{Y}_{vol}\right)_{vol}$ and the Lévy process Y are* pure *if $\left(Y_t\right)_{t\geq 0}$ is invariant by conjugation by G and* mixed *if not pure. They are also* non-degenerate *if $H_Y = G$ and* degenerate *if $H_Y \neq G$. The same definition holds for the discrete planar Yang-Mills field associated with Y.*

According to this definition, any planar Yang-Mills field is either pure non-degenerate, pure degenerate or mixed degenerate.

9. Braids and probabilities II: a geometric point of view, infinite random sequences and random processes

We would like now to characterize the discrete planar Markovian holonomy fields. For this, we will use intensively the invariance by area-preserving homeomorphisms. The braid group will appear again in a geometric way as the diffeotopy group of the n-punctured disk. In the next subsection, we explain the general idea which is used in the proof of Theorem 10.1. In order to characterize the Markovian holonomy fields, one will have to understand the generalization of the next subsection to any surface and thus to study the mapping class groups of general surfaces.

9.1. Braids as the diffeotopy group of the n-punctured disk. — Let $\mathbb{D}$ be the disk of center 0 and radius 1 and $Q_n = \{q_k = \frac{2k-1-n}{n},\ 1 \leq k \leq n\}$. Let $\mathrm{Diff}(\mathbb{D}, Q_n, \partial\mathbb{D})$ be the group of diffeomorphisms of $\mathbb{D}$ which fix the set Q_n and fix pointwise a neighborhood of $\partial\mathbb{D}$. The class of isotopy of the identity mapping in $\mathrm{Diff}(\mathbb{D}, Q_n, \partial\mathbb{D})$ is a normal subgroup called $\mathrm{Diff}_0(\mathbb{D}, Q_n, \partial\mathbb{D})$. We define the diffeotopy group of the disk with n points by:

$$\mathcal{M}_n(\mathbb{D}) = \mathrm{Diff}(\mathbb{D}, Q_n, \partial\mathbb{D}) \Big/ \mathrm{Diff}_0(\mathbb{D}, Q_n, \partial\mathbb{D}).$$

One important theorem is that:

$$\mathcal{M}_n(\mathbb{D}) \simeq \mathcal{B}_n. \tag{21}$$

This isomorphism is constructed by sending some special elements, the half-twists, on the canonical free family of generators of the braid groups. A half-twist permutes the points q_k and q_{k+1} for some k and does not move the other points $(q_i)_{i\notin\{k,k+1\}}$. For a precise definition, one considers for $1 \leq k \leq n$, t_k the isotopy class of the diffeomorphism $\tilde{t}_k$ equals to identity outside the disk of radius $\frac{2}{n}$ centered at $\frac{q_k+q_{k+1}}{2}$ and defined by

$\tilde{t}_k(x) = \psi \circ t \circ \psi^{-1}$, where:

$$\psi : x \mapsto \frac{n}{2}\left(x - \frac{q_k + q_{k+1}}{2}\right),$$
$$t(re^{i\theta}) = re^{i2\pi\left(\theta+\alpha(r)\right)},$$

and α is a smooth function from $[0,1]$ to itself, which is equal to 0 on a neighborhood of 1 and to $\frac{1}{2}$ at $\frac{1}{2}$.

This geometric construction of the braid group allows us to recover the action of the braid group which was given by Definition 7.2. Indeed, the group $\mathcal{M}_n(\mathbb{D})$ acts on the fundamental group of $\mathbb{D}\setminus Q_n$ which is isomorphic to $\mathbb{F}_n$ the free group of rank n. We will take $-\frac{i}{2}$ as the base point for the fundamental group of $\mathbb{D}\setminus Q_n$. Let x_k be the homotopy class of the loop based at $-\frac{i}{2}$ which goes only around q_k anti-clockwise. One can verify that the action of $\mathcal{M}_n(\mathbb{D})$ on $\mathbb{F}_n$, with the identification given in (21), is the action given by Definition 7.2.

Besides, given a finite graph $\mathbb{G}$ on $\mathbb{R}^2$, the fundamental group of $\mathbb{G}$ is isomorphic to the fundamental group of the disk without one point in each of the bounded faces: $\pi_1(\mathbb{G}) \simeq \pi_1\left(\left(\mathbb{D}\setminus Q_{|\mathbb{F}^b|}\right)\right)$. Thus, we will have a natural action of a braid group on $\pi_1(\mathbb{G})$ which is isomorphic to the reduced group of loops on $\mathbb{G}$ defined in Section 6.1. One consequence of the existence of such action is Proposition 10.1.

9.2. Infinite sequence. — Using Proposition 7.2, every finite sequence of i.i.d. random variables which is auto-invariant by conjugation is invariant by braids. It is natural to wonder if one can characterize finite sequence of random variables which are invariant by braids. As for exchangeable sequences of random variables, it is easier to work with infinite sequence of random variables.

Definition 9.1. — *An infinite random sequence $\xi = \left(\xi_i\right)_{i\in\mathbb{N}^*}$ in G is braidable (or braid-invariant or invariant by braids) if for any integer n greater that* 1 *and any braid $\beta \in \mathcal{B}_n$, the following equality in law holds:*

$$\beta \bullet \left(\xi_i\right)_{1\le i\le n} = \left(\xi_i\right)_{1\le i\le n}.$$

We say that ξ is spreadable if for any increasing sequence of positive integers $(k_i)_{i\ge1}$, we have the equality in law:

$$\left(\xi_{k_i}\right)_{1\le i} = \left(\xi_i\right)_{1\le i}.$$

These properties seem to be quite different, yet we are going to prove that one condition is weaker than the other.

Lemma 9.1. — *Any braidable infinite family of random variables is spreadable.*

Proof. — Let $\mathbf{k} = (k_1 < k_2 < ... < k_n)$ be a finite strictly increasing sequence of integers. We define the braid $\beta_{\mathbf{k}}$ by:

$$\beta_{\mathbf{k}} = \beta_n^{-1}\dots\beta_{k_n-1}^{-1}\beta_2^{-1}\dots\beta_{k_2-1}^{-1}\beta_1^{-1}\dots\beta_{k_1-1}^{-1}$$

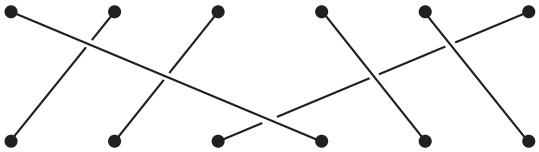


FIGURE 18. The braid $\beta_{\mathbf{k}}$, $\mathbf{k} = (2, 3, 6)$.

We have drawn in Figure 18 the braid $\beta_{\mathbf{k}}$ with $\mathbf{k} = (2, 3, 6)$. As the lines attaching $(i, 1)$ to $(k_i, 0)$ for $i = 1, ..., n$ are behind in the diagram, this braid verifies that for any element $(g_1, ..., g_{k_n})$ of G^{k_n}, for any integer i between 1 and n, $\big(\beta_{\mathbf{k}} \bullet (g_1, ..., g_{k_n})\big)_i = g_{k_i}$.

Let $\xi = (\xi_i)_{i\in\mathbb{N}^*}$ be a braidable random sequence, the following equality in law holds:

$$\beta_{\mathbf{k}} \bullet \big(\xi_i\big)_{1\leq i\leq k_n} = \big(\xi_i\big)_{1\leq i\leq k_n},$$

By restricting it for i between 0 and $n-1$, we get the desired equality in law:

$$(\xi_{k_1}, \dots \xi_{k_n}) = (\xi_1, \dots \xi_n),$$

from which one can conclude that ξ is spreadable. □

Let m be a probability measure on G. We denote by $m^{\otimes\infty}$ the measure on $G^{\mathbb{N}^*}$ such that the unidimensional projections are independent and identically distributed with law m.

Definition 9.2. — *Let ξ be an infinite random sequence in G and $\mathcal{A}$ be a σ-field. We say that ξ is i.i.d. conditionally to $\mathcal{A}$ if there exists a random measure η on G, $\mathcal{A}$-measurable, such that the conditional distribution of ξ given $\mathcal{A}$ is $\eta^{\otimes\infty}$: $\mathbb{P}\big[\xi \in . \mid \mathcal{A}\big] = \eta^{\otimes\infty}$. It is conditionally i.i.d. if there exists a σ-field $\mathcal{A}$ such that it is i.i.d. conditionally to $\mathcal{A}$.*

If ξ is i.i.d. conditionally to $\mathcal{A}$, its law is of the form:

$$\int_{\mathcal{M}_1(G)} m^{\otimes\infty} d\nu(m),$$

where ν is the law of η. If we just want to keep in mind the form of the law of ξ, we will say that ξ is a *mixture of i.i.d. random sequences.* Let us state an extension of de Finetti-Ryll-Nardzewski's theorem for the braid group.

Theorem 9.1. — *Let $\xi = (\xi_n)_{n\in\mathbb{N}^*}$ be a sequence of G-valued random variables. The following conditions are equivalent:*

1. *the sequence ξ is braidable,*
2. *the sequence ξ is i.i.d. conditionally to the tail σ-field $\mathcal{T}_\xi = \cap_{n\in\mathbb{N}^*}\sigma\left(\xi_k, k \geq n\right)$ and conditionally to $\mathcal{T}_\xi$, almost surely the law of ξ_1 is invariant by conjugation by its own support.*

If one of the two conditions holds, then the law of ξ is of the form $\int_{\mathcal{M}_1(G)} m^{\otimes\infty} d\nu(m)$, where ν-a.s., m is almost surely invariant by its own support.

Proof. — Let $\xi = (\xi_n)_{n\in\mathbb{N}^*}$ be a sequence of G-valued random variables. An application of Proposition 7.2 to any subsequence of the form $(\xi_n)_{n=1}^N$ shows that condition 2 implies condition 1.

Now, let us suppose that ξ is braidable. As a consequence of Lemma 9.1, the infinite sequence ξ is spreadable. Using the de Finetti-Ryll-Nardzewski's theorem (Theorem 1.1 of [**16**]), any spreadable infinite sequence of random variables in G is i.i.d. conditionally to the tail σ-field: ξ is conditionally i.i.d., conditionally to $\mathcal{T}_\xi$. Besides, conditionally to $\mathcal{T}_\xi$, ξ is still braidable: an application of Proposition 7.2 shows that conditionally to $\mathcal{T}_\xi$, $(\xi_n)_{n\in\mathbb{N}^*}$ is an i.i.d. sequence of random variables invariant by conjugation by their own support: the condition 2 holds. □

In the next theorem, we give a condition under which one can characterize the mixture which appears in the last theorem. In order to do so, we consider the diagonal conjugation of G on $G^{\mathbb{N}^*}$ defined for any $g \in G$ and $(x_n)_{n\in\mathbb{N}^*} \in G^{\mathbb{N}^*}$ by $g.(x_n)_{n\in\mathbb{N}^*} = (g^{-1}x_n g)_{n\in\mathbb{N}^*}$. We need also to define the property $(\mathcal{P})$:

$(\mathcal{P})$ For any integer $n \in \mathbb{N}^*$, $(\xi_k)_{k\leq n}$ and $(\xi_k)_{k>n}$ are $\mathcal{I}$-independent,

where we recall that $(\xi_k)_{k\leq n}$ and $(\xi_k)_{k>n}$ are $\mathcal{I}$-independent if for any positive integer m and any continuous functions $f : G^n \to \mathbb{R}$ and $g : G^m \to \mathbb{R}$ which are invariant by diagonal conjugation, $\mathbb{E}[f((\xi_k)_{k\leq n})g((\xi_k)_{k=n+1}^{n+m})] = \mathbb{E}[f((\xi_k)_{k\leq n}]\mathbb{E}[g((\xi_k)_{k=n+1}^{n+m})]$.

Theorem 9.2. — *Let $\xi = (\xi_n)_{n\in\mathbb{N}^*}$ be a braidable sequence of G-valued random variables. Suppose that ξ is invariant (in law) by diagonal conjugation: it means that for any $g \in G$, $g.\xi$ has the same law as ξ. If ξ satisfies the property $(\mathcal{P})$, there exists m_0 a probability measure on G, invariant by conjugation by its own support, such that the law of ξ is:*

$$\int_G \left(m_0^{\otimes\infty}\right)^g dg,$$

where $\left(m_0^{\otimes\infty}\right)^g$ is defined using a similar equation as (20).

Proof. — Let ξ be an infinite braidable sequence of G-valued random variables which is invariant by diagonal conjugation and satisfies the property $(\mathcal{P})$. As a consequence of Theorem 9.1, there exists a random measure η on G, which is almost surely invariant by conjugation by its own support, such that the conditional distribution of ξ given $\mathcal{T}_\xi$ is $\eta^{\otimes\infty}$. Let ν be the law of η, the law of ξ is

$$\int_{\mathcal{M}^1(G)} m^{\otimes\infty} d\nu(m).$$

As ξ is invariant by diagonal conjugation by G, we only have to show that there exists a probability measure m_0 such that the law of ξ on the invariant σ-field $\mathcal{I}$ is equal to $m_0^{\otimes\infty}$. Let us remark that this would imply also that m_0 is invariant by conjugation by any element of its own support. Let k be a positive integer. Let $f : G^k \to \mathbb{R}$ be a continuous function invariant by diagonal conjugation. As G^k is compact, f is bounded. As the sequence ξ satisfies the property $(\mathcal{P})$, $\left(f\left(\xi_{ik+1}, \ldots, \xi_{ik+k}\right)\right)_{i\geq 0}$ is an i.i.d. sequence

of bounded random variables. Thus, by the law of large numbers, there exists a real $l^k(f)$ such that:

$$\frac{1}{n}\sum_{0\leq i<n} f\left(\xi_{ik+1},\dots,\xi_{ik+k}\right) \underset{n\to\infty}{\to} l^k(f) \text{ a.s..}$$

Yet, by disintegration and the law of large numbers, we get also that:

$$\frac{1}{n}\sum_{0\leq i<n} f\left(\xi_{ik+1},\dots,\xi_{ik+k}\right) \underset{n\to\infty}{\to} \eta^{\otimes k}(f), \text{ a.s..}$$

The random variable $\eta^{\otimes k}(f)$ is thus almost surely constant. Let us define the set of measures:

$$\Omega_{f,k} = \left\{ m \in \mathcal{M}_1(G), m^{\otimes k}(f) = l^k(f) \right\}.$$

We just proved that for any positive integer k, for any continuous function $f : G^k \to \mathbb{R}$, invariant by diagonal conjugation,

$$\nu(\Omega_{f,k}) = 1.$$

Let us consider $\mathcal{F}_k$ a dense countable set of continuous functions which are invariant by diagonal conjugation of G on G^k. Our previous discussion allows us to write the following equality:

$$\nu\left(\bigcap_{k\in\mathbb{N}^*} \bigcap_{f\in\mathcal{F}_k} \Omega_{f,k} \right) = 1.$$

Let us take a measure $m_0 \in \bigcap_{k\in\mathbb{N}^*} \bigcap_{f\in\mathcal{F}_k} \Omega_{f,k}$: for any positive integer k and any continuous function f on G^k invariant by diagonal conjugation,

$$m^{\otimes k}(f) = m_0^{\otimes k}(f), \ \nu(dm) \text{ a.s.}$$

We have just managed to prove that on the invariant σ-field, the law of ξ is $m_0^{\otimes\infty}$: the law of ξ is thus $\int_G (m_0^{\otimes\infty})^g dg$. □

Let ξ be an infinite sequence of G-valued random variables which satisfies the properties in Theorem 9.2. Let m_0 be a probability measure on G, given by Theorem 9.2, such that the law of ξ is $\int_G \left(m_0^{\otimes\infty}\right)^g dg$. What we aim to do next is to give conditions which ensure the fact that ξ is actually a sequence of i.i.d. random variables: for this, it is enough to show that m_0 is invariant by conjugation by G.

9.3. Degeneracy of the mixture. — Since m_0 is invariant by conjugation by its own support, one particular possibility in order to prove that m_0 is invariant by conjugation by G is to prove that $\mathsf{Supp}(m_0) = G$. We will call this case the non-degeneracy case.

9.3.1. Non-degeneracy case. —

Proposition 9.1. — *Let G be a finite group. Let ξ be an infinite sequence of G-valued random variables which is braidable, invariant by diagonal conjugation, satisfies property $(\mathcal{P})$ and such that $e \in \mathsf{Supp}(\xi_1)$. The following assertions are equivalent:*

1. *ξ is a sequence of i.i.d. random variables which support generates G;*
2. *there exists an integer k such that $\mathsf{Supp}(\prod_{i=1}^k \xi_i) = G$.*

In order to prove this proposition, we will need the two following facts which hold only when G is finite. The first assertion is that for any measure m on G such that $e \in \mathsf{Supp}(m)$, there exists an integer k such that for any $k' \geq k$, $\mathsf{Supp}(m^{*k'}) = H_m$, where H_m was defined after Definition 7.4. The second asserts that no subgroup of G can intersect every conjugacy classes of G: this is the following Jordan's theorem.

Theorem 9.3. — *Let G be a finite group, let H be a subgroup of G. Let us suppose that:*

$$G = \bigcup_{g \in G} g^{-1} H g,$$

then $G = H$.

Proof. — The proof is taken from Serre's lecture [**25**] and can be summarized in a simple calculation:

$$\#G \leq \#\Big(\bigcup_{g \in G} g^{-1} H g \Big) = \#\Big(\bigcup_{g \in G} \big(g^{-1} H g \setminus e\big) \cup \{e\} \Big) \leq \frac{\#G}{\#H}(\#H - 1) + 1,$$

which can hold if and only if $H = G$. □

We can now handle the proof of Proposition 9.1.

Proof of Proposition 9.1. — It is quite obvious that the assertion 1 implies the assertion 2. It remains to prove the other implication. As a consequence of Theorem 9.2, there exists a probability measure m_0 invariant by conjugation by its own support such that the law of ξ is $\int_G (m_0^{\otimes \infty})^g dg$. Let m_0 be such a probability measure. As $e \in \mathsf{Supp}(\xi_1)$, e is in the support of m_0: the support of m_0^{*k} is increasing in k and so is the support of $\prod_{i=1}^k \xi_i$. Let k be an integer such that $\mathsf{Supp}\big(\prod_{i=1}^k \xi_i\big) = G$: for any $k' \geq k$, $\mathsf{Supp}\big(\prod_{i=1}^{k'} \xi_i\big) = G$. Let N be an integer greater than k such that $\mathsf{Supp}(m_0^{*N}) = H_{m_0^{*N}}$. As the law of $\prod_{i=1}^N \xi_i$ is $\int_G (m_0^{*N})^g dg$, its support is then equal to $\cup_{g \in G} g^{-1} H_{m_0^{*N}} g$. On the other side, since N is greater that k, $\mathsf{Supp}\big(\prod_{i=1}^N \xi_i\big) = G$. Thus, one has the equality:

$$G = \bigcup_{g \in G} g^{-1} H_{m_0^{*N}} g$$

which implies, thanks to Jordan's theorem (Theorem 9.3), that $H_{m_0^{*N}} = G$. This implies that $H_{m_0} = G$: the support of m_0 generates the group G. We recall that m_0 was invariant by conjugation by its support, hence by the sub-group generated by its support H_{m_0}. The probability m_0 is thus invariant by conjugation by G and the law of ξ is thus $m_0^{\otimes \infty}$. Thus, the assertion 2 implies the assertion 1. □

For an arbitrary compact Lie group, it is not true in general that for any measure m on G such that $e \in \mathsf{Supp}(m)$, there exists k such that $\mathsf{Supp}(m^{*k}) = H_m$. Thus, in order to deal with any compact Lie group, we will substitute this fact by the Ito-Kawada's theorem, Theorem 9.4.

As for Jordan's theorem, it does not hold when G is infinite, as in every compact Lie group, any maximal torus intersects all the conjugacy classes. Thus, we have to impose that the subgroup H intersects every conjugacy class "as much as" G does, which is the meaning of the condition imposed in the upcoming Proposition 9.3 or Proposition

9.4. Doing so, we will be able to prove the following proposition which holds for any arbitrary compact Lie group.

Proposition 9.2. — *Let G be a compact Lie group. Let ξ be an infinite sequence of G-valued random variables which is braidable, invariant by diagonal conjugation and satisfies property $(\mathcal{P})$. Let us suppose that $e \in \mathsf{Supp}(\xi_1)$. The following assertions are equivalent:*

1. *ξ is a sequence of i.i.d. random variables which support generates G.*
2. *the random variables $\prod_{k=1}^{n} \xi_k$ converge in law to a Haar random variable as n goes to infinity.*

Let us introduce Itô-Kawada's theorem and a measurable version of the theorem of Jordan which holds for any compact Lie group.

Definition 9.3. — *Let m be a probability measure on G. It is:*

- aperiodic *if its support $\mathsf{Supp}(m)$ is not contained in a left or right proper coset of a proper closed subgroup of G,*
- non-degenerate *if $H_m = G$.*

Remark 9.1. — *It is obvious that m is non-degenerate if it is seen as a measure on H_m. Besides, if $e \in \mathsf{Supp}(m)$ then m is aperiodic.*

Under the condition of aperiodicity and non-degeneracy, the Itô-Kawada's theorem (Theorem 3.3.5. of [**26**], first proved in [**17**]) explains the behavior of m^{*n} when n goes to infinity.

Theorem 9.4 **(Itô-Kawada's theorem)**. — *Let G be a compact topological Hausdorff group. Let μ be a non-degenerate and aperiodic probability measure on G. The sequence μ^{*n} converges in distribution to the normalized Haar measure on G as n goes to infinity.*

Let us state our generalization of Jordan's theorem, Theorem 9.3, valid for any compact Lie group. Recall that for any compact Lie group K, we denote by λ_K the normalized Haar measure on K.

Proposition 9.3. — *Let G be a compact Lie group, let H be a closed subgroup. If*

$$\int_G \lambda_{g^{-1}Hg} dg = \lambda_G,$$

then $G = H$.

Proof. — Let G be a compact Lie group, let H be a proper closed subgroup. We want to prove that $\int_G \lambda_{g^{-1}Hg} \neq \lambda_G$: it is enough to construct a continuous function $\phi : G \to \mathbb{R}$, invariant by conjugation, such that $\lambda_H(\phi) \neq \lambda_G(\phi)$. The space $H \setminus G$ of right cosets of H is a nice topological space: it is a differentiable manifold and there exists $\tilde{f}$ a non-constant real continuous function on $H \setminus G$. Let $p : G \to H \setminus G$ be the canonical projection, the function $f = \tilde{f} \circ p$ is a real non-constant square-integrable function f on G invariant by left multiplication by H: for any $g \in G$, for any $h \in H$, $f(g) = f(hg)$. One can also assume that f is of zero mean on G.

Let $E = \left\{\phi \in L^2(G),\ \int_G \phi(g) d\lambda_G(g) = 0\right\}$ be the space of square-integrable zero mean functions on G. The group G acts on E, by left multiplication on the argument and this representation has no non-zero fixed point. On the other hand, the restriction of this representation on H has at least one fixed point, namely f. We can decompose E as a sum of finite dimensional irreducible representations of G:

$$E = \bigoplus_{i=1}^{\infty} E_i.$$

None of the E_i is the trivial representation of G as we have restricted the action of G to zero mean functions. The action of H on E admits a fixed point f. We can decompose f on $\bigoplus_{i=1}^{\infty} E_i$. As, for any integer i, the space E_i is invariant under the action of H, there exists at least an integer i_0 such that E_{i_0} seen as a H-module is not irreducible. We denote by χ_{i_0} the character of the G-module E_{i_0}. By the classical theory of character,

$$\int_G \chi_{i_0} d\lambda_G = \dim(E_{i_0}^G) = 0,$$

whereas:

$$\int_G \chi_{i_0} d\lambda_H = \int_H \chi_{i_0} d\lambda_H = \dim(E_{i_0}^H) \geq 1,$$

where $E_{i_0}^G$ and $E_{i_0}^H$ are the vector spaces of fixed points in E_{i_0} under the actions of G and H. Thus, we just found a central function χ_{i_0} such that:

$$\int_G \chi_{i_0} d\lambda_G \neq \int_G \chi_{i_0} d\lambda_H.$$

This ends the proof. □

We give an other formulation of Proposition 9.3 which seems interesting to us.

Proposition 9.4. — *Let G be a compact Lie group, let H be a closed subgroup of G. If for any G-invariant function f the equality $\lambda_G(f) = \lambda_H(f)$ holds, then $G = H$.*

We have now all the tools in order to prove Proposition 9.2.

Proof of Proposition 9.2. — As a consequence of Theorem 9.4, Remark 9.1 and the fact that $e \in \mathsf{Supp}(\xi_1)$, it is easy to see that the condition 1 implies condition 2. Let us prove the other implication.

Let ξ be an infinite sequence of G-valued random variables which is braidable, invariant by diagonal conjugation and which satisfies property $(\mathcal{P})$. Let us suppose that e is in $\mathsf{Supp}(\xi_1)$. As a consequence of Theorem 9.2, there exists a probability measure m_0 invariant by conjugation by its own support such that the law of ξ is $\int_G (m_0^{\otimes\infty})^g dg$. Let m_0 be such a probability measure. Using the hypothesis on ξ_1, $e \in \mathsf{Supp}(m_0)$.

Let us suppose that $\prod_{k=1}^n \xi_k$ converges in law to a Haar random variable. As we have seen in Remark 9.1, the measure m_0 is aperiodic and non-degenerate if seen as a measure on H_{m_0}. Thanks to the Itô-Kawada's theorem, m_0^{*n} converges to the Haar probability measure on H_{m_0}, which we will denote by $\lambda_{H_{m_0}}$, when n goes to infinity. For any integer

n, the law of $\prod_{i=1}^{n} \xi_i$ is $\int_G (m_0^{*n})^g \, dg$ and thus, using the hypothesis on the law of $\prod_{i=1}^{n} \xi_i$ and our previous discussion, one gets the equality:

$$\lambda_G = \int_G \lambda_{g^{-1} H_{m_0} g} dg.$$

By Proposition 9.3, it follows that $H_{m_0} = G$. Since the measure m_0 is invariant by conjugation by H_{m_0}, it is invariant by conjugation by G. This implies that the law of ξ is $m_0^{\otimes \infty}$. □

9.3.2. General case. — Actually, one would like weaker conditions on ξ in order to understand the case when ξ is an infinite sequence of i.i.d. random variables such that $H_{\xi_1} \neq G$. The main result in the general case is Theorem 9.5.

Theorem 9.5. — *Let G be a compact Lie group. Let ξ be an infinite sequence of G-valued random variables which is braidable, invariant by diagonal conjugation and satisfies property $(\mathcal{P})$. The following assertions are equivalent:*

1. *the sequence ξ is a sequence of i.i.d. random variables invariant by conjugation by G,*
2. *there exists ν a probability measure on G such that for any positive integer n, the law of $\prod_{k=1}^{n} \xi_k$ is ν^{*n}.*

We recall basic, yet crucial, results about representations and integration. First of all, Peter-Weyl's theorem asserts that the set of matrix elements of irreducible representations

$$\left\{ g \mapsto v(\pi(g)w), \pi \in \hat{G}, v \in V_\pi^*, w \in V_\pi \right\},$$

where $\hat{G}$ is the set of irreducible representations of G, is dense for the uniform norm in the set of continuous functions on G. Thus, any measure m on G is fully characterized by its Fourier coefficients defined as:

$$\forall \pi \in \hat{G}, \, \pi(m) = \int_G \pi(g) m(dg).$$

Secondly, let $\pi : G \to Gl(V)$ be an irreducible representation of dimension d_π. Let A be a matrix acting on V. By the Schur's lemma,

$$\int_G \pi(g) A \pi(g)^{-1} dg = \frac{Tr(A)}{d_\pi} Id.$$

Definition 9.4. — *Let m be a probability measure on a compact Lie group G. It is quasi-invariant by conjugation if there exists ν a probability measure on G such that for any $n \in \mathbb{N}$*

$$\int_G (m^g)^{*n} dg = \nu^{*n}.$$

It is obvious that any conjugation-invariant probability measure is quasi-invariant by conjugation. We can state now the main characterization of quasi-invariant by conjugation probability measures.

Proposition 9.5. — *Let m be a probability measure on a compact Lie group G. It is quasi-invariant by conjugation if and only if for any irreducible representation π of G, the matrix $\pi(m)$ has only one eigenvalue.*

Proof. — Let m be a probability measure on a compact Lie group G. Let π be an irreducible representation of G and let n be a positive integer. Let us compute $\pi\left(\int_G (m^{*n})^g dg\right)$ and $\pi\left(\left(\int_G m^g dg\right)^{*n}\right)$:

$$\begin{aligned}
\pi\left(\int_G (m^{*n})^g dg\right) &= \int_{G^2} \pi(gg'g^{-1}) m^{*n}(dg') dg \\
&= \int_G \pi(g) \left(\int_G \pi(g') m^{*n}(dg')\right) \pi(g)^{-1} dg \\
&= \frac{1}{d_\pi} Tr\left(\pi(m)^n\right) Id, \\
\pi\left(\left(\int_G m^g dg\right)^{*n}\right) &= \pi\left(\left(\int_G m^g dg\right)\right)^n \\
&= \left(\frac{1}{d_\pi} Tr(\pi(m))\right)^n Id.
\end{aligned}$$

The Proposition 9.5 is equivalent to the following assertion: for any positive integer n,

$$\int_g (m^g)^{*n} dg = \left(\int_g m^g dg\right)^{*n} \tag{22}$$

if and only if for any irreducible representation π of G, the matrix $\pi(m)$ has only one eigenvalue. Yet, using the remark about Peter-Weyl's theorem, Equality (22) holds for any positive integer n if and only if for any irreducible representation π of G, for any positive integer n:

$$\pi\left(\int_g (m^g)^{*n} dg\right) = \pi\left(\left(\int_g m^g dg\right)^{*n}\right),$$

hence if and only if for any irreducible representation π of G, for any positive integer n:

$$Tr\left(\pi(m)^n\right) = Tr\left(\left(\frac{Tr(\pi(m))}{d_\pi} Id\right)^n\right).$$

The proposition is a consequence of the link between the traces of the positive powers of a finite matrix and the set of its eigenvalues and the fact that the matrix $\frac{Tr(\pi(\nu))}{d_\pi} Id$ has only one eigenvalue. □

It is natural to ask if a quasi-invariant by conjugation probability measure is invariant by conjugation. The answer is no and we will construct a counter-example in the symmetric group $\mathfrak{S}_3$.

Proposition 9.6. — *Let $(\mu_t)_{t\geq 0}$ (reps. $(\eta_t)_{t\geq 0}$) be the continuous semi-group of convolution of measures starting from δ_{id} on the symmetric group $\mathfrak{S}_3$, associated with the*

jump measure m (resp. m_0):

$$\begin{aligned} m((12)) &= 0, & m_0((12)) &= 1,\\ m((13)) &= 1, & m_0((13)) &= 1,\\ m((23)) &= 2, & m_0((23)) &= 1,\\ m((123)) &= 2, & m_0((123)) &= 1,\\ m((132)) &= 0, & m_0((132)) &= 1. \end{aligned}$$

The measure μ_1 is quasi-invariant by conjugation and for any $n \in \mathbb{N}$,

$$\int_G (\mu_1^g)^{*n} dg = \eta_1^{*n}.$$

Proof. — We have to check that the condition of Proposition 9.5 is fulfilled by μ_1. Actually we only have to show that for any $\pi \in \widehat{\mathfrak{S}_3}$, $\pi(m)$ has only one eigenvalue, which is equal to the one of $\pi(m_0)$. The group $\mathfrak{S}_3$ has only three irreducible representations two of which have dimension one. It remains to compute $\pi(m)$ where π is the representation of $\mathfrak{S}_3$ on $\{(a,b,c) \in \mathbb{R}^3, a+b+c=0\}$. We leave this calculation as an exercise. □

The quasi-invariance by conjugation property does not imply the invariance by conjugation property. Yet, we have the following theorem.

Theorem 9.6. — *Let m be a probability measure on G. Let us suppose that m is invariant by conjugation by its own support. Then, m is quasi-invariant by conjugation if and only if m is invariant by conjugation by G.*

Proof. — As we have already seen, a probability measure which is invariant by conjugation is quasi-invariant by conjugation. It remains to prove the "only if" part of the theorem. Let m be a quasi-invariant by conjugation probability measure. We know, using Proposition 9.5, that for any irreducible representation π of G, $\pi(m)$ has only one eigenvalue. Any irreducible representation $\pi \in \hat{G}$ determines by restriction a representation of H_m. Since H_m is a closed subgroup of G, it is a compact Lie group, thus we can apply Peter-Weyl's theorem which allows us to decompose any representation of H_m as a direct sum of irreducible representations:

$$\pi = \bigoplus_{i=1}^{n} \pi_i,$$

with $\pi_i \in \hat{H}_m$. As m is invariant by conjugation by H_m, thanks to Schur's lemma, for any $i \in \{1, \ldots, n\}$, $\pi_i(m)$ is a scalar matrix, hence $\pi(m)$ is diagonal. As it has only one eigenvalue, it is a multiple of the identity. Let $\pi \in \hat{G}$ acting on V, let $w \in V$, $v \in V^*$ and let $h \in G$:

$$\begin{aligned} \int_G v(\pi(hgh^{-1})w) m(dg) &= v\left(\pi(h)\left(\int_G \pi(g) m(dg)\right)\pi(h)^{-1} w\right)\\ &= v\left(\left(\int_G \pi(g) m(dg)\right) w\right)\\ &= \int_G v(\pi(g)w) m(dg). \end{aligned}$$

Thus, using Peter-Weyl's theorem, m is invariant by conjugation by G. □

We have now all the tools in order to prove Theorem 9.5.

Proof of Theorem 9.5. — Let ξ be an infinite sequence of G-valued random variables which is braidable, invariant by diagonal conjugation and which satisfies property $(\mathcal{P})$. As a consequence of Theorem 9.2, there exists a probability measure m_0 invariant by conjugation by its own support such that the law of ξ is $\int_G (m_0^{\otimes\infty})^g dg$.

Let us suppose that ξ is a sequence of i.i.d. random variables, then one can take as m_0 the law of ξ_1: the law of ξ is equal to $m_0^{\otimes\infty}$. Thus for any n, the law of $\prod_{k=1}^n \xi_k$ is m_0^{*n}.

Instead of assuming that ξ is a sequence of i.i.d. random variables, let us suppose that there exists ν a probability measure on G such that for any n, the law of $\prod_{k=1}^n \xi_k$ is ν^{*n}. The law of $\prod_{k=1}^n \xi_k$ is $\int_G (m_0^{*k})^g dg$: it shows that the probability measure m_0 is quasi-invariant by conjugation. Yet, it is also invariant by conjugation by its own support. By Theorem 9.6, m is invariant by conjugation by G and thus ξ is a sequence of i.i.d. random variables. □

9.4. Processes. — In this subsection, we apply the results we proved in Section 9.2 and 9.3 in order to prove similar results for G-valued processes indexed by $\mathbb{R}^+$. We define the increments of a process as Kallenberg does in [**16**]: what is interesting for us are the rational increments as defined in Definition 9.5.

9.4.1. Definitions. —

Definition 9.5. — *Let X be a G-valued random process indexed by $\mathbb{R}^+$. We define the (rational) increments of X for $n \in \mathbb{N}^* \cup (\mathbb{N}^*)^{-1}$ and $j \geq 1$ as:*

$$X_{n,j} = X_{\frac{j-1}{n}}^{-1} X_{\frac{j}{n}}. \tag{23}$$

It has spreadable (resp. braidable) increments if for every $n \in \mathbb{N}^* \cup (\mathbb{N}^*)^{-1}$, the sequence $(X_{n,i})_{0<i}$ is spreadable (resp. braidable).

Definition 9.6. — *Let $(X_t)_{t\in\mathbb{R}^+}$ be a Lévy process. It has auto-invariant by conjugation increments if for any $0 = t_0 \leq t_1 \leq t_2 \cdots \leq t_k$, the sequence of increments $(X_{t_{i-1}}^{-1} X_{t_i})_{i=1}^k$ is auto-invariant by conjugation .*

Recall the notion of self-invariance in Definition 8.3. In the next proposition, we link the different notions of invariance by conjugation that can be applied to a Lévy process.

Proposition 9.7. — *Let X be a G-valued Lévy process. The three following conditions are equivalent:*

1. *X has auto-invariant by conjugation increments,*
2. *for any $t \in \mathbb{R}^+$, X_t is invariant by conjugation by its own support,*
3. *X is invariant by conjugation by H_X, thus self-invariant by conjugation.*

Proof. — Let X be a G-valued Lévy process. We will show that 1 implies 2, 3 implies 1 and at last 2 implies 3.

Let us assume that X has auto-invariant by conjugation increments. Let $t \in \mathbb{R}^+$, then $(X_t, X_t^{-1} X_{2t})$ is auto-invariant by conjugation: X_t is invariant by conjugation by

the support of $X_t^{-1}X_{2t}$. As X is a Lévy process, $X_t^{-1}X_{2t}$ and X_t has the same law, thus the same support: X_t is invariant by conjugation by its own support.

Now, let us show that 3 implies 1. Let us assume that X is invariant by conjugation by H_X. By definition for any $t \in \mathbb{R}^+$, $\mathsf{Supp}(X_t) \subset H_X$. Let $t_1 < t_2$ and $t_3 < t_4$ be four non negative reals. As the process X is invariant by conjugation by H_X, $X_{t_1}^{-1}X_{t_2}$ is invariant by $\mathsf{Supp}(X_{t_4-t_3})$ and thus by $\mathsf{Supp}(X_{t_3}^{-1}X_{t_4})$. This implies easily that X has auto-invariant by conjugation increments.

It remains to prove that 2 implies 3. Let us assume that for any $t \in \mathbb{R}^+$, X_t is invariant by conjugation by its own support. Let us first remark that, if U and V are two random independent variables in G, $\mathsf{Supp}(UV) = \overline{\mathsf{Supp}(U).\mathsf{Supp}(V)}$. Besides, if they are both invariant in law by conjugation by a set S, UV is also invariant by conjugation by S. Moreover, if U is invariant by conjugation by any element of S, it is invariant by conjugation by any element of the closure of the semi-group generated by S: $\overline{\bigcup_{k=1}^{\infty} S^k}$, which, in the case where G is compact, is a group. Let n be an integer and let t be a positive real. Using the hypothesis on X, $X_{\frac{t}{n}}$ is invariant by conjugation by $\mathsf{Supp}(X_{\frac{t}{n}})$. Taking n independent copies of $X_{\frac{t}{n}}$ and applying the remarks above, we find that X_t is still invariant by conjugation by $\mathsf{Supp}(X_{\frac{t}{n}})$ and thus also for any integer $k \geq 1$, by $\overline{\mathsf{Supp}(X_{\frac{t}{n}})^k} = \mathsf{Supp}(X_{\frac{k}{n}t})$, or by the semi-group generated by $\mathsf{Supp}(X_{\frac{k}{n}t})$, which is nothing but $H_{X_{\frac{k}{n}t}}$. Thus, X_t is invariant by conjugation by:

$$\overline{\bigcup_{q\in\mathbb{Q}^+} H_{X_{qt}}}.$$

Since the laws of $\big(X_t\big)_{t\geq 0}$ form a continuous semi-group of convolution of measures, for any $s \geq 0$, $X_s \in \overline{\bigcup_{q\in\mathbb{Q}^+} H_{X_{q.t}}}$ a.s. and thus $H_{X_s} \subset \overline{\bigcup_{q\in\mathbb{Q}^+} H_{X_{qt}}}$, hence the equality:

$$H_X = \overline{\bigcup_{q\in\mathbb{Q}^+} H_{X_{qt}}}.$$

The conclusion of the discussion is that, for any positive real t, X_t is invariant by conjugation by H_X. Using the fact that X has independent and stationary increments, it implies that the Lévy process X is invariant by conjugation by H_X. □

In the following we will need a weak version of the notion of independence of increments.

Definition 9.7. — *Let $(X_t)_{t\in\mathbb{R}^+}$ be a G-valued process. It has $\mathcal{I}$-independent increments if for any increasing sequence of real $0 = t_0 < t_1 < \cdots < t_n < \ldots$ the sequence $(X_{t_{n-1}}^{-1}X_{t_n})_{n\in\mathbb{N}^*}$ satisfies the property $(\mathcal{P})$ defined page 187.*

9.4.2. Generalized Bülhmann's theorem. — We can now state the generalization of Bühlmann's theorem (Theorem 1.19 of [**16**]) for the braid group.

Theorem 9.7. — *Let X be a G-valued stochastically continuous process indexed by $\mathbb{R}_+$ with $X_0 = e$. The following conditions are equivalent:*

1. *X has braidable increments,*

2. *X is a mixture of self-invariant by conjugation Lévy processes.*

The σ-field which makes the rational increments, as defined in Definition 9.5, conditionally i.i.d. is the σ-field $\mathcal{T} = \cap_{t\in\mathbb{Q}^+}\sigma(X_t^{-1}X_s, s > t)$. Besides, the following conditions are equivalent:

1. *X is invariant by conjugation by G and has braidable and $\mathcal{I}$-independent increments,*
2. *there exists a self-invariant by conjugation Lévy process Y, such that the law of X is UYU^{-1}, where U is a Haar variable on G independent of Y.*

Proof. — Let us consider the first part of the theorem. Let us show that the condition 2 implies the first one: it is enough to show that any self-invariant by conjugation Lévy process has braidable increments. Let Z be any self-invariant by conjugation Lévy process. By Proposition 9.7, for any $n \in \mathbb{N}^* \cup (\mathbb{N}^*)^{-1}$, the sequence of increments $(Z_{n,j})_j$ defined in Definition 9.5 is a sequence of i.i.d. random variables which are invariant by conjugation by their own support. Hence, by Theorem 9.1, it is braidable: the process Z has braidable increments.

Now, let us consider X a G-valued stochastically continuous process indexed by $\mathbb{R}_+$ with $X_0 = e$. Let us suppose that X has braidable increments. Following the proof of Theorem 1.19 of [**16**], we introduce the processes:

$$Y_n^k(t) = X\left(\frac{k-1}{n}\right)^{-1} X\left(t + \frac{k-1}{n}\right), \quad t \in \mathbb{Q} \cap [0, n^{-1}],\ k \in \mathbb{N}^*,\ n \in \mathbb{N}^*$$

Let n be a positive integer. Using the same arguments used in Lemma 9.1, notice that the sequence $(Y_n^k)_{k\in\mathbb{N}^*}$ is spreadable. Applying to these sequences the deFinetti-Ryll-Nardzewski's theorem (Theorem 1.1 and Corollary 1.6 in [**16**]) which is valid for sequences in Polish spaces, we conclude that for any $n \in \mathbb{N}^*$, conditionally to the tail σ-field $\mathcal{T}_n$, the sequence $(Y_n^k)_{k\in\mathbb{N}^*}$ is a sequence of i.i.d. random variables. We considered $t \in \mathbb{Q}^+ \cap [0, n^{-1}]$ in the definition of Y_n^k as the product of a countable family of Polish spaces is still a Polish space. The σ-field $\mathcal{T}_n$ we are conditioning on is a.s. independent of n: we call it $\mathcal{T}$. Given $\mathcal{T}$, X has conditionally stationary independent (rational) increments. For any $t \in \mathbb{Q}^+$, let m_t be the law of X_t conditionally to $\mathcal{T}$: for any $t \in \mathbb{Q}^+$ and any $s \in \mathbb{Q}^+$, almost surely $m_t * m_s = m_{t+s}$. Besides, using the stochastic continuity of X and the fact that $X_0 = e$, one has that almost surely $(m_t)_{t\in\mathbb{Q}^+}$ is uniformly continuous. We can extend the semi-group $(m_t)_{t\in\mathbb{Q}^+}$ in order to get a semi-group $(m_t)_{t\in\mathbb{R}^+}$: by stochastic continuity the process X is then a mixture of Lévy processes.

Let q be in $\mathbb{Q}^+$. Using a similar argument as in the proof of Theorem 9.1, applied to the sequence $(X_{nq})_{n\in\mathbb{N}}$, conditionally on $\mathcal{T}$, the random variable X_q is invariant by conjugation by its own support. Using a continuity argument allows us to extend the result for any $q \in \mathbb{R}^+$. The result follows from Proposition 9.7: X is a mixture of self-invariant by conjugation Lévy process.

The second part of the theorem is deduced from Theorem 9.2 applied to the increments of X. □

9.4.3. Degeneracy of the mixture. — In this subsection, we generalize the Sections 9.3.1 and 9.3.2 in the setting of processes. The proofs will be omitted as the theorems follow directly from their counterpart in the setting of sequences and from Theorem 9.7. Recall

Definition 8.5, where we defined the notions of pure/mixed, degenerate/non-degenerate Lévy processes. Let us state the consequence of Proposition 9.1 for processes.

Proposition 9.8. — *Let G be a finite group. Let X be a G-valued stochastically continuous process invariant by conjugation by G such that $X_0 = e$ and which has braidable and $\mathcal{I}$-independent increments. The following assertions are equivalent:*

1. *X is a pure non-degenerate Lévy process,*
2. *there exists $t \in \mathbb{R}^+$ such that $\mathsf{Supp}(X_t) = G$.*

If one of the two conditions holds then for any $t \in \mathbb{R}^+$, $\mathsf{Supp}(X_t) = G$.

Let us remark that, in order to prove the last proposition, we have to replace the property of $\mathsf{Supp}(m^{*k})$ we used in the proof of Proposition 9.1 by the following straightforward lemma.

Lemma 9.2. — *Let G be a finite group. Let $\big(Y_t\big)_{t\geq 0}$ be a Lévy process on G, we have for any real $t \geq 0$, $\mathsf{Supp}(Y_t) = H_Y$.*

We now state the consequence of Proposition 9.2.

Proposition 9.9. — *Let G be a compact Lie group. Let X be a G-valued stochastically continuous process invariant by conjugation by G such that $X_0 = e$ and which has braidable and $\mathcal{I}$-independent increments. Let us suppose that $e \in \mathsf{Supp}(X_t)$ for any $t \in \mathbb{R}^+$. The following conditions are equivalent:*

1. *the process X is a pure non-degenerate Lévy process,*
2. *the random variable X_t converges in law to a Haar random variable on G when t goes to infinity.*

In order to conclude this section, it remains to state the consequence of Theorem 9.5.

Theorem 9.8. — *Let G be a compact Lie group. Let X be a G-valued stochastically continuous process invariant by conjugation by G such that $X_0 = e$ and which has braidable and $\mathcal{I}$-independent increments. The following assertions are equivalent:*

1. *the process X is a pure (i.e. invariant by conjugation by G) Lévy process,*
2. *there exists a G-valued Lévy process Z such that for any $t \in \mathbb{R}^+$, X_t has the same law as Z_t.*

Remark 9.2. — *Let us suppose that X satisfies the hypothesis of the last theorem. By Theorem 9.7 there exists a Lévy process Y which is self-invariant by conjugation and such that for any function f invariant by conjugation by G, $\mathbb{E}[f(X_t)] = \mathbb{E}[f(Y_t)]$. Let us suppose that the condition 2 of Theorem 9.8 holds: there exists Z, a G-valued Lévy process, invariant by conjugation by G, such that for any $t \in \mathbb{R}^+$, X_t has the same law as Z_t. This means that the Lévy process Y satisfies for any $t \geq 0$ and any invariant by conjugation function f on G:*

$$\mathbb{E}[f(Y_t)] = \mathbb{E}[f(Z_t)].$$

A Lévy process $Y = \big(Y_t\big)_{t\geq 0}$ is one marginal quasi-invariant by conjugation Lévy processes if there exists a conjugation-invariant Lévy process $Z = \big(Z_t\big)_{t\geq 0}$ such that for any $t \geq 0$ and any invariant by conjugation function f on G, $\mathbb{E}[f(Y_t)] = \mathbb{E}[f(Z_t)]$.

The last theorem is now equivalent to the fact that any one marginal quasi-invariant by conjugation Lévy process which is self-invariant by conjugation is a Lévy process invariant by conjugation by G. Let us remark that the condition of self-invariance by conjugation is important. Indeed, the Lévy process Y associated with $(\mu_t)_{t\geq 0}$ defined in Proposition 9.6 is a one marginal quasi-invariant by conjugation Lévy process. Indeed, if Z is the Lévy process associated with $(\eta_t)_{t\geq 0}$ defined in the same proposition, for any $t \geq 0$ and any invariant by conjugation function f on G, $\mathbb{E}[f(Y_t)] = \mathbb{E}[f(Z_t)]$. Yet the process Y is not invariant by conjugation by G.

We have put the emphasis on Theorem 9.8 as it has the simplest formulation. Yet, a more general theorem is the following.

Theorem 9.9. — *Let G be a compact Lie group. Let X be a G-valued stochastically continuous process invariant by conjugation by G such that $X_0 = e$ and which has braidable and $\mathcal{I}$-independent increments. The following assertions are equivalent:*

1. *the process X is a pure (i.e. invariant by conjugation by G) Lévy process,*
2. *there exists $(m_t)_{t\in\mathbb{R}^+}$ a family of measures on G invariant by conjugation by G such that for any $t \geq 0$ and any $k \in \mathbb{N}$, the law of X_{kt} is m_t^{*k}.*

Proof. — Let X be a G-valued process invariant by conjugation by G such that $X_0 = e$ and which has braidable and $\mathcal{I}$-independent increments. If it is a pure Lévy process, the condition 2 is satisfied by using, for any $t \in \mathbb{R}^+$, the law of X_t for m_t.

Now let us suppose condition 2. Let $(m_t)_{t\in\mathbb{R}^+}$ be a family of measures on G invariant by conjugation by G such that for any positive integer k and any $t \geq 0$, the law of X_{kt} is m_t^{*k}. Let $t \in \mathbb{R}^+$ and let us consider the sequence $\xi = \left(X_{(k-1)t}^{-1} X_{kt}\right)_{k\geq 1}$. Using the stochastic continuity, this is a sequence of G-valued random variables which is braidable, invariant by diagonal conjugation and satisfies property $(\mathcal{P})$. Besides, for any integer k, the law of $\prod_{i=k}^{1} \xi_i$ is the law of X_{kt} and thus is equal to m_t^{*k}. By Theorem 9.5, ξ is a sequence of i.i.d. random variables invariant by conjugation by G. Thus, by stochastic continuity, the process X is a Lévy process which is also invariant by conjugation by G. □

10. Characterization of stochastically continuous in law weak discrete planar Markovian holonomy fields

In this section we will show the following theorem.

Theorem 10.1. — *Let $\left(\mathbb{E}_{vol}^{\mathbb{G}}\right)_{\mathbb{G},vol}$ be a G-valued stochastically continuous in law weak discrete planar Markovian holonomy field. There exists a G-valued Lévy process, $\left(Y_t\right)_{t\geq 0}$, self-invariant by conjugation such that $\left(\mathbb{E}_{vol}^{\mathbb{G}}\right)_{\mathbb{G},vol}$ is the weak discrete planar Yang-Mills field associated with $\left(Y_t\right)_{t\geq 0}$. This means that for any measure of area vol and any graph $\mathbb{G} \in \mathcal{G}\left(\mathsf{Aff}(\mathbb{R}^2)\right)$, $\mathbb{E}_{vol}^{\mathbb{G}}$ is equal to $\mathbb{E}_{vol}^{Y,\mathbb{G}}$, where $\left(\mathbb{E}_{vol}^{Y,\mathbb{G}}\right)_{\mathbb{G},vol}$ is the discrete planar Yang-Mills field associated with $\left(Y_t\right)_{t\geq 0}$.*

If G is Abelian, the Lévy process is unique and is characterized by the fact that for any simple loop l in $\mathsf{Aff}(\mathbb{R}^2)$, under $\mathbb{E}^{\mathbb{G}(l)}_{vol}$ (see Example 3.1), $h(l)$ has the same law as $Y_{vol(\mathsf{Int}(l))}$.

It is important to note that the Lévy process $\left(Y_t\right)_{t\geq 0}$ is not unique if G is non-Abelian: it is unique up to an equivalence. We say that $(Y_t)_{t\geq 0}$ and $(Y'_t)_{t\geq 0}$ are equivalent if they have the same law when we restrict their law to the invariant σ-field on $G^{\mathbb{R}^+}$. We believe, but did not prove it, that two Lévy processes are equivalent if and only if there exists a non-random element $g \in G$ such that the law of $(Y_t)_{t\geq 0}$ and the law of $(g^{-1}Y'_t g)_{t\geq 0}$ are equal. The Theorem 10.1 can be now summarized as:

Theorem 10.2. — *There exists a one-to-one correspondence between the set of equivalence classes of G-valued self-invariant by conjugation Lévy processes and the set of G-valued stochastically continuous in law weak discrete planar Markovian holonomy fields.*

Before proving Theorem 10.1, let us consider its consequences.

Theorem 10.3. — *For a discrete planar Markovian holonomy field, the following conditions are equivalent:*

- *it is stochastically continuous in law,*
- *it is regular.*

If the discrete planar Markovian holonomy field is a weak one, then one can replace the regularity condition by the locally stochastically $\frac{1}{2}$-Hölder continuity.

Proof. — We already saw in Corollary 4.1 that, depending if we are working with weak or strong discrete planar Markovian holonomy fields, the regularity or the locally stochastically $\frac{1}{2}$-Hölder continuity implies the stochastically continuity in law of the discrete planar Markovian holonomy field.

Besides if a discrete planar Markovian holonomy field $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ is stochastically continuous in law, its restriction to the piecewise affine graphs is a stochastically continuous in law weak discrete planar Markovian holonomy field. By Theorem 10.1, there exists a planar Yang-Mills field $\left(\mathbb{E}^{Y}_{vol}\right)_{vol}$ such that $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G}\in\mathcal{G}(\mathsf{Aff}(\mathbb{R}^2)),vol} = \left((\mathbb{E}^{Y}_{vol})_{|\mathcal{M}ult(P(\mathbb{G}),G)}\right)_{\mathbb{G}\in\mathcal{G}(\mathsf{Aff}(\mathbb{R}^2)),vol}$. Using the stochastic continuity in law of both of the fields, this equality holds without the restriction on the graphs. Using the proof of Proposition 8.2, up to a slight modification since Y is only self-invariant by conjugation, $\left((\mathbb{E}^{Y}_{vol})_{|\mathcal{M}ult(P(\mathbb{G}),G)}\right)_{\mathbb{G},vol}$ is locally stochastically $\frac{1}{2}$-Hölder continuous and continuously area-dependent, thus $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ is also locally stochastically $\frac{1}{2}$-Hölder continuous and continuously area-dependent. □

In Section 4 we have defined four different notions of planar Markovian holonomy fields. By now, we know that, by restriction, a strong planar Markovian holonomy field defines a weak continuous one. Using the results of Section 5, a weak planar Markovian holonomy field defines, when restricted, a discrete planar Markovian holonomy field. Theorem 10.1 now allows us to show that the four different notions are in some sense equivalent when one considers stochastically continuous objects. Indeed, a stochastically continuous in law weak discrete planar Markovian holonomy field is the restriction of a

planar Yang-Mills field, which by the results of Section 8 was shown to be a stochastically continuous strong planar Markovian holonomy field. Besides, by construction, any planar Yang-Mills field is constructible. This discussion allows us to state the following theorems.

Theorem 10.4. — *Any stochastically continuous strong planar Markovian holonomy field is a planar Yang-Mills field: as a consequence it is constructible.*

Theorem 10.5. — *Let $(E_{vol})_{vol}$ be a family of stochastically continuous random holonomy fields. We have equivalence between:*

1. *$(E_{vol})_{vol}$ is a strong planar Markovian holonomy field,*
2. *$\left((E_{vol})_{|\mathcal{M}ult(\mathsf{Aff}(\mathbb{R}^2),G)}\right)_{vol}$ is a weak planar Markovian holonomy field,*
3. *$\left((E_{vol})_{|\mathcal{M}ult(P(\mathbb{G}),G)}\right)_{\mathbb{G},vol}$ is a strong discrete planar Markovian holonomy field,*
4. *$\left((E_{vol})_{|\mathcal{M}ult(P(\mathbb{G}),G)}\right)_{\mathbb{G}\in\mathcal{G}(\mathsf{Aff}(\mathbb{R}^2)),vol}$ is a weak discrete planar Markovian holonomy field,*
5. *$(E_{vol})_{vol}$ is a planar Yang-Mills field associated with a self-invariant by conjugation Lévy process.*

Theorem 10.6. — *Any G-valued stochastically continuous in law weak discrete planar Markovian holonomy field is the restriction of a unique G-valued stochastically continuous strong planar Markovian holonomy field.*

The last theorem is different from Theorem 4.1 which only allowed us to extend a stochastically continuous in law strong discrete planar Markovian holonomy field. We encourage the reader to have a look at the diagram page 227 where we drawn the different links between all the notions introduced or used in this paper.

The last consequence of Theorem 10.1 is the Proposition 4.2. Before giving the proof of this proposition, we will construct an explicit G-valued stochastically continuous in law discrete planar Markovian holonomy field $(\mathbb{E}^{\mathbb{G}}_{vol})_{\mathbb{G},vol}$ which satisfies the hypothesis of Proposition 4.2 but for which the natural restriction defined by the Equation (11) is not a discrete planar Markovian holonomy field.

For this, we consider the symmetrical group $G = \mathfrak{S}_3$. Let H be the subgroup of G isomorphic to $\mathbb{Z}/3\mathbb{Z}$ which contains the neutral element e and the two 3-cycles c and c^2. Let X be a H-valued Lévy process which jumps only by multiplication by c. As H is abelian, X is a self-invariant by conjugation G-valued Lévy process and, because of the condition on the jumps, for any positive time t:

$$\mathbb{P}[X_t = c] \neq \mathbb{P}\left[X_t = c^2\right]. \tag{24}$$

Let $\left(\mathbb{E}^X_{vol}\right)_{vol}$ be the G-valued planar Yang-Mills field associated with X and let us consider the restriction $\left(\mathbb{E}^{X,\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ of $\left(\mathbb{E}^X_{vol}\right)_{vol}$ to the finite planar graphs: it is a G-valued stochastically continuous in law discrete planar Markovian holonomy field. Since H is normal in G, $\left(\mathbb{E}^{X,\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ satisfies the conditions stated in Proposition 4.2.

The natural restriction of $\left(\mathbb{E}^{X,\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$, as defined by the Equation (11), and denoted by $\left(\tilde{\mathbb{E}}^{X,\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$, is neither a strong nor a weak H-valued discrete planar Markovian holonomy

field since it does not satisfy the weak independence property. Indeed, let us consider two loops l, l' and a path p as drawn in Figure 8 and let us suppose that they are drawn on a finite planar graph $\mathbb{G}$. Let us suppose that $vol(\mathsf{Int}(l)) = vol(\mathsf{Int}(l')) = 1$. If $\left(\tilde{\mathbb{E}}^{X,\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ satisfied the weak independence property, under $\tilde{\mathbb{E}}^{X,\mathbb{G}}_{vol}$, $h(l)$ and $h(l')$ would be independent. Recall the Equation (12), where f_1 and f_2 are, in this case, any functions on H since H is abelian. Under $\mathbb{E}^{X,\mathbb{G}}_{vol}$, the random couple $(h(l), h(pl'p^{-1}))$ has the same law as (UAU^{-1}, UBU^{-1}) where A, B and U are three independent random variables such that U is a Haar variable on G and the two random variables A and B have the same law as X_1. The two random variables UAU^{-1} and UBU^{-1} are not independent since the Equality (24) implies that:

$$\mathbb{P}\big[(UAU^{-1}, UBU^{-1}) = (c,c)\big] \neq \mathbb{P}[UAU^{-1} = c]\mathbb{P}[UBU^{-1} = c].$$

This proves that under $\tilde{\mathbb{E}}^{X,\mathbb{G}}_{vol}$, $h(l)$ and $h(l')$ are not independent.

Using Theorem 10.1, we can now prove Proposition 4.2.

Proof of Proposition 4.2. — As a consequence of Theorem 10.1, $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ is a discrete planar Yang-Mills field associated with a self-invariant by conjugation Lévy process $(Y_t)_{t\geq 0}$. Besides, the law of a simple loop l under $\mathbb{E}^{\mathbb{G}(l)}_{vol}$ is equal to $\int_G m^g_{vol(\mathsf{Int}(l))}dg$ where $m_{vol(\mathsf{Int}(l))}$ is the law of $Y_{vol(\mathsf{Int}(l))}$. Hence, under the hypothesis of Proposition 4.2, for any positive real t, Y_t is almost surely in H: we can find a modification of $(Y_t)_{t\geq 0}$ which is H-valued. Using Theorem 8.1, we can define a H-valued planar Yang-Mills field, associated with the H-valued Lévy process $(Y_t)_{t\geq 0}$, whose restriction to planar graphs is a H-valued stochastically continuous in law discrete planar Markovian holonomy field. It is easy to see that it satisfies the required conditions. □

In the following, we will prove Theorem 10.1. In Section 10.1, we show that the *two-dimensional time* objects which are the planar Markovian holonomy fields are characterized by a *one-dimensional time* process. Then in Section 10.2, it is shown that this one-dimensional time process has *$\mathcal{I}$-independent increments*. This allows us to prove Theorem 10.1 when the group G is abelian. In general, the result follows from the *braidability* of the one-dimensional time process which is proved in Section 10.3.

10.1. First correspondence for stochastically continuous in law weak discrete planar Markovian holonomy fields.— We can go further than Proposition 6.3 in the characterization of stochastically continuous in law weak discrete planar Markovian holonomy fields. We will need for that a continuous version of the loops $L_{i,j}$ which were defined in Definition 6.4.

Definition 10.1. — *For any $0 \leq s \leq t$ we define $\partial c^t_s = (s,0) \to (t,0) \to (t,1) \to (s,1) \to (s,0)$, and $p_s = (0,0) \to (s,0)$. In the following, we will focus on the reduced loop L^t_s defined by:*

$$L^t_s = [p_s \partial c^t_s p_s^{-1}]_{\simeq},$$

where the notion of reduced loops was defined in the beginning of Section 6.1. This notion can be applied here since p_s, ∂c^t_s, p_s^{-1} can be seen as paths on the same finite planar graph.

Remark 10.1. — *These loops satisfy the following equalities:*

$$\begin{aligned} L_r^t &= L_s^t L_r^s, \quad \forall\ 0 \leq r \leq s \leq t, \\ L_i^{i+1} &= L_{i,0}\ \ , \quad \forall\ i \in \mathbb{N}, \end{aligned}$$

where we considered the reduced product in the first equality and where $(L_{i,j})_{i,j}$ *were defined in Definition 6.4.*

In Theorem 10.7, we will show that the process $\left(h(L_0^t)\right)_{t\in\mathbb{R}^+}$ gives all the information one needs in order to understand a stochastically continuous in law weak discrete planar Markovian holonomy field. This theorem is motivated by the following lemma which is a straightforward application of Theorem 8.1.

Lemma 10.1. — *Let* Y *be a* G*-valued self-invariant by conjugation Lévy process. Let* U *be a Haar variable on* G *which is independent from* Y*. Let* $\left(\mathbb{E}^Y_{vol}\right)_{vol}$ *be the planar Yang-Mills field associated with* Y*. Under the measure* $\mathbb{E}^Y_{dx}$*,* $\left(h(L_0^t)\right)_{t\in\mathbb{R}^+}$ *has the same law as* $\left(UY_tU^{-1}\right)_{t\in\mathbb{R}^+}$*.*

Theorem 10.7. — *If* $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ *and* $\left(\tilde{\mathbb{E}}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ *are two stochastically continuous in law weak discrete planar Markovian holonomy fields, the three following assertions are equivalent:*

1. $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ *and* $\left(\tilde{\mathbb{E}}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ *are equal,*
2. $\left(h(L_0^t)\right)_{t\in\mathbb{R}^+}$ *has the same law under* $\mathbb{E}^{\mathsf{Aff}}_{dx}$ *as under* $\tilde{\mathbb{E}}^{\mathsf{Aff}}_{dx}$*,*
3. *for any positive real* α*,* $\left(h(L_{n,0})\right)_{n\in\mathbb{N}}$ *has the same law under* $\mathbb{E}^{\mathbb{N}^2}_{\alpha dx}$ *as under* $\tilde{\mathbb{E}}^{\mathbb{N}^2}_{\alpha dx}$*.*

We remind the reader that $\mathbb{E}^{\mathsf{Aff}}_{dx}$ and $\mathbb{E}^{\mathbb{N}^2}_{dx}$ were defined in Remark 4.1. The proof will consist in proving the equivalence between the conditions 1 and 2, then between 2 and 3.

Proof. — Let $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ and $\left(\tilde{\mathbb{E}}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ be two stochastically continuous in law weak discrete planar Markovian holonomy fields.

Since the condition 1 clearly implies condition 2, let us show that 2 implies 1. Let us suppose that $\left(h(L_0^t)\right)_{t\in\mathbb{R}^+}$ has the same law under $\mathbb{E}^{\mathsf{Aff}}_{dx}$ as under $\tilde{\mathbb{E}}^{\mathsf{Aff}}_{dx}$. Let *vol* be a measure of area and let $\mathbb{G}$ be a finite planar graph in $\mathcal{G}\left(\mathsf{Aff}\left(\mathbb{R}^2\right)\right)$. We have to show that $\mathbb{E}^{\mathbb{G}}_{vol} = \tilde{\mathbb{E}}^{\mathbb{G}}_{vol}$. The proof will consist in a sequence of simplifications, changing the graph and the measure of area little by little. By Proposition 2.4, these measures are characterized by the way they integrate functions of the form: $h \mapsto f(h(l_1), \ldots, h(l_m))$, where f is a continuous function invariant by conjugation and $l_1, \ldots, l_n$ are elements of $L_v(\mathbb{G})$, where v is any given vertex of $\mathbb{G}$. Thus, we have to show that:

$$\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{|L_v(\mathbb{G})} = \left(\tilde{\mathbb{E}}^{\mathbb{G}}_{vol}\right)_{|L_v(\mathbb{G})}.$$

Since these two measures are defined on multiplicative functions on loops, we can suppose that $\mathbb{G}$ is simple. Let us consider a sequence of generic graphs $(\mathbb{G}_n)_{n\geq 0}$, which will be simple and which approximate the graph $\mathbb{G}$ in the sense of Lemma 3.4. Using the stochastic continuity in law of $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ (resp. $\left(\tilde{\mathbb{E}}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$), the measure $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{|L_v(\mathbb{G})}$

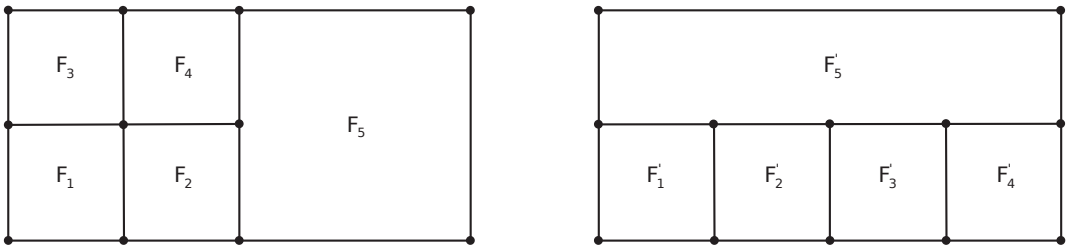


FIGURE 19. Graphs $\mathbb{G}$ and $\tilde{\mathbb{G}}$.

$\left(\text{resp. } \left(\tilde{\mathbb{E}}^{\mathbb{G}}_{vol}\right)_{|L_v(\mathbb{G})}\right)$ can be recovered using the sequence of measures $\left(\left(\mathbb{E}^{\mathbb{G}_n}_{vol}\right)_{|L_v(\mathbb{G}_n)}\right)_{n\in\mathbb{N}}$ $\left(\text{resp. } \left(\left(\tilde{\mathbb{E}}^{\mathbb{G}_n}_{vol}\right)_{|L_v(\mathbb{G}_n)}\right)_{n\in\mathbb{N}}\right)$. This allows us to suppose that $\mathbb{G}$ is also generic.

Using Corollary 3.1, there exists $\mathbb{G}'$ a subgraph of the $\mathbb{N}^2$ graph such that the set of $\mathbb{G}-\mathbb{G}'$ piecewise diffeomorphisms is not empty: let ψ be such a homeomorphism. Let vol' be a measure of area on $\mathbb{R}^2$ such that for any bounded face F of $\mathbb{G}$, $vol'(\psi(F)) = vol(F)$. Using the Axiom $\mathbf{wDP_1}$, we know that:

$$\mathbb{E}^{\mathbb{G}'}_{vol'} \circ \psi^{-1} = \mathbb{E}^{\mathbb{G}}_{vol}.$$

By definition of $\mathbb{E}^{\mathbb{N}^2}_{vol'}$, the measure $\mathbb{E}^{\mathbb{G}'}_{vol'}$ is equal to $\left(\mathbb{E}^{\mathbb{N}^2}_{vol'}\right)_{|\mathcal{M}ult(P(\mathbb{G}'),G)}$. The same discussion holds for $\left(\tilde{\mathbb{E}}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$. Thus, if we show that, for any measure of area vol',

$$\mathbb{E}^{\mathbb{N}^2}_{vol'} = \tilde{\mathbb{E}}^{\mathbb{N}^2}_{vol'},$$

we will get the conclusion that $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ and $\left(\tilde{\mathbb{E}}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ are equal.

Let vol' be a measure of area on $\mathbb{R}^2$. Since $\{L_{i,j}, i,j \in \mathbb{N}^2\}$ is a family which generates $RL_0(\mathbb{N}^2)$ and since we are considering gauge-invariant measures, we only have to prove that $(h(L_{i,j}))_{i,j}$ has the same law under $\mathbb{E}^{\mathbb{N}^2}_{vol'}$ as under $\tilde{\mathbb{E}}^{\mathbb{N}^2}_{vol'}$. Let us show that it is actually enough to know that $(h(L_{n,0}))_{n\in\mathbb{N}}$ has the same law under $\mathbb{E}^{\mathbb{N}^2}_{vol'}$ as under $\tilde{\mathbb{E}}^{\mathbb{N}^2}_{vol'}$.

Let us consider the two finite planar graphs $\mathbb{G}$ and $\tilde{\mathbb{G}}$ drawn in Figure 19. Let vol'' be a measure of area such that $vol''_{|F_\infty} = vol'_{|\tilde{F}_\infty}$, where F_∞ (resp. $\tilde{F}_\infty$) is the unbounded face of $\mathbb{G}$ (resp. $\tilde{\mathbb{G}}$). Besides, we impose that the following condition holds for vol'':

$$\forall i \in \{1,\dots,5\}, vol''(F'_i) = vol'(F_i). \tag{25}$$

The loops $(L_{i,j})_{i,j\in\{0,1\}}$ belong to $L_{(0,0)}(\mathbb{G})$ and the loops $(L_{i,0})_{i\in[|0,3|]}$ are in $L_{(0,0)}(\tilde{\mathbb{G}})$. Let us approximate the loops $(L_{i,j})_{i,j\in\{0,1\}}$ by loops whose intersection is reduced to the base point. Such loops are drawn in bold in the left part of Figure 20. The two graphs $\mathbb{G}_1$ and $\mathbb{G}_2$ drawn in Figure 20 satisfy the hypothesis of Theorem 3.1 and are in $\mathcal{G}\big(\mathsf{Aff}(\mathbb{R}^2)\big)$: they are homeomorphic. Thus, by Proposition 3.2, there exists an orientation-preserving $\mathbb{G}_1 - \mathbb{G}_2$ piecewise diffeomorphism which we denote by ψ. We can suppose, up to a modification of $\mathbb{G}_1$ and $\mathbb{G}_2$ which will not change the general form of both graphs and thus will not invalidate the discussion, that $vol'(F) = vol''(\psi(F))$ for any bounded face F of $\mathbb{G}_1$. This last assertion is essentially due to the condition (25) on

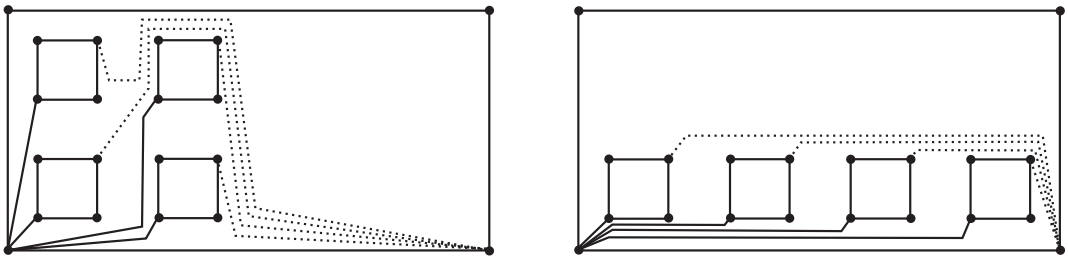


FIGURE 20. Graphs $\mathbb{G}_1$ and $\mathbb{G}_2$.

vol''. Using Axiom $\mathbf{wDP_1}$ and using the stochastic continuity in law property, we conclude that under $\mathbb{E}^{\mathbb{N}^2}_{vol'}$ (resp. $\tilde{\mathbb{E}}^{\mathbb{N}^2}_{vol'}$), $(h(L_{0,1}), h(L_{0,0}), h(L_{1,1}), h(L_{1,0}))$ has the same law as $(h(L_{i,0}))_{i\in[|0,3|]}$ under $\mathbb{E}^{\mathbb{N}^2}_{vol''}$ (resp. $\tilde{\mathbb{E}}^{\mathbb{N}^2}_{vol''}$). A slight generalization of these arguments allows us to show that for any integer n there exists a measure of area vol'' such that under $\mathbb{E}^{\mathbb{N}^2}_{vol}$ (resp. $\tilde{\mathbb{E}}^{\mathbb{N}^2}_{vol}$), $(h(L_{0,n-1}), \dots, h(L_{0,0}), \dots, h(L_{n-1,n-1}), \dots, h(L_{n-1,0}))$ has the same law as $(h(L_{i,0}))_{i\in[|0,n^2-1|]}$ under $\mathbb{E}^{\mathbb{N}^2}_{vol''}$ (resp. $\tilde{\mathbb{E}}^{\mathbb{N}^2}_{vol''}$).

Thus it is now enough to show that for any measure of area vol'', $(h(L_{i,0}))_{i\in\mathbb{N}}$ has the same law under $\mathbb{E}^{\mathbb{N}^2}_{vol''}$ as under $\tilde{\mathbb{E}}^{\mathbb{N}^2}_{vol''}$. Let $n \in \mathbb{N}$, let $\mathbb{S}$ be the graph defined as the intersection of the $\mathbb{N}^2$ graph and $[0, n+1] \times [0, 1]$. Let us consider an orientation-preserving homeomorphism ψ such that its restiction on $\mathbb{R}^+ \times \mathbb{R}$ is given by:

$$\begin{aligned}\psi : \quad & \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2 \\ & (x, y) \mapsto \left(vol''\left([0, x] \times [0, 1]\right), y\right).\end{aligned}$$

The image of $\mathbb{S}$ by ψ, $\psi(\mathbb{S})$ is a simple graph in $\mathcal{G}(\mathsf{Aff}(\mathbb{R}^2))$ and for any bounded face F of $\mathbb{S}$, $vol''(F) = dx(\psi(F))$. Let us define for any $i \in \{0, \dots, n+1\}$, $t_i = vol''(L^i_0)$. We can apply the Axiom $\mathbf{wDP_1}$ to the two graphs $\mathbb{S}$ and $\psi(\mathbb{S})$, to the two measures of area vol'' and dx and to ψ. It shows that under $\mathbb{E}^{\mathbb{N}^2}_{vol''}$ (resp. $\tilde{\mathbb{E}}^{\mathbb{N}^2}_{vol''}$), $(h(L_{i,0}))^n_{i=0}$ has the same law as $(h(L^{t_{i+1}}_{t_i}))^n_{i=0}$ under $\mathbb{E}^{\mathsf{Aff}}_{dx}$ (resp. $\tilde{\mathbb{E}}^{\mathsf{Aff}}_{dx}$). It remains to show that for any integer n, any sequence of positive reals $t_0 < \dots < t_n$, $(h(L^{t_{i+1}}_{t_i}))^n_{i=0}$ has the same law under $\mathbb{E}^{\mathsf{Aff}}_{dx}$ as under $\tilde{\mathbb{E}}^{\mathsf{Aff}}_{dx}$. Yet, we started with the fact that $(h(L^t_0))_{t\in\mathbb{R}^+}$ has the same law under $\mathbb{E}^{\mathsf{Aff}}_{dx}$ as under $\tilde{\mathbb{E}}^{\mathsf{Aff}}_{dx}$. This allows us to conclude.

Let us prove the equivalence between the conditions 2. and 3. Suppose that for any positive real α, $(h(L_{n,0}))_{n\in\mathbb{N}}$ has the same law under $\mathbb{E}^{\mathbb{N}^2}_{\alpha dx}$ as under $\tilde{\mathbb{E}}^{\mathbb{N}^2}_{\alpha dx}$. By the Axiom $\mathbf{wDP_4}$, we can change $\mathbb{E}^{\mathbb{N}^2}_{\alpha dx}$ (resp. $\tilde{\mathbb{E}}^{\mathbb{N}^2}_{\alpha dx}$) by $\mathbb{E}^{\mathsf{Aff}}_{\alpha dx}$ (resp. $\tilde{\mathbb{E}}^{\mathsf{Aff}}_{\alpha dx}$). As an application of the Axiom $\mathbf{wDP_1}$, with ψ given by:

$$\psi : (x, y) \mapsto (\alpha x, y),$$

the random vector $\left(h(L^{\alpha(n+1)}_{\alpha.n})\right)_{n\in\mathbb{N}}$ has the same law under $\mathbb{E}^{\mathsf{Aff}}_{dx}$ as under $\tilde{\mathbb{E}}^{\mathsf{Aff}}_{dx}$. Using the fact that for any positive integers p and q:

$$h\left(L^{\frac{p}{q}}_0\right) = \prod_{i=0}^{p-1} h\left(L^{\frac{i+1}{q}}_{\frac{i}{q}}\right),$$

we can conclude that $(h(L_t))_{t\in\mathbb{Q}^+}$ has the same law under $\mathbb{E}^{\mathsf{Aff}}_{dx}$ as under $\tilde{\mathbb{E}}^{\mathsf{Aff}}_{dx}$ and by stochastically continuity the same assertion holds for $(h(L_t))_{t\in\mathbb{R}^+}$. The conditions 3. implies the second one. The other implication can be proved using the same arguments: this shows the equivalence between condition 2. and condition 3. □

10.2. Law of the conjugacy classes and the Abelian case. — Let $\left(\mathbb{E}^{\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$ be a stochastically continuous in law weak discrete planar Markovian holonomy field and let $\mathbb{E}^{\mathsf{Aff}}_{dx}$ be the usual expectation associated with it.

Definition 10.2. — *Until the end of this section, we define for any* $0 \leq s \leq t$,

$$Z^t_s = h(L^t_s),$$

and $Z_t = Z^t_0$.

Remark 10.2. — *Thanks to the multiplicativity property of random holonomy fields and Remark 10.1, for any* $0 \leq r \leq s \leq t$, $Z^t_r = Z^s_r Z^t_s$, *hence for any* $0 \leq s \leq t$,

$$Z^t_s = (Z_s)^{-1} Z_t.$$

Since $Z^t_s = h\big(p_s \partial c^t_s p_s^{-1}\big)$, *by Remark 2.7, under* $\mathbb{E}^{\mathsf{Aff}}_{dx}$, Z^t_s *has the same law as* $h(\partial c^t_s)$. *Besides the left translation by* s *sends* dx *on itself and* ∂c^t_s *on* ∂c^{t-s}_0: *applying the Axiom* $\mathbf{wDP_1}$ *for weak discrete planar Markovian holonomy fields (Definition 4.4), we get that under* $\mathbb{E}^{\mathsf{Aff}}_{dx}$,

$$Z^t_s \text{ has the same law as } Z_{t-s}.$$

Moreover, using the stochastic continuity property, under $\mathbb{E}^{\mathsf{Aff}}_{dx}$, *the process* $(Z_t)_{t\geq 0}$ *is stochastically continuous and* Z_0 *is equal to the neutral element of* G.

A simple but important lemma is the following.

Lemma 10.2. — *Under* $\mathbb{E}^{\mathsf{Aff}}_{dx}$, *for any* $t_0 > 0$, *for any finite subset* T *of* $[0, t_0]$ *and any finite subset* T' *of* $[t_0, \infty[$, $(Z_t)_{t\in T}$ *and* $(Z^{-1}_{t_0} Z_t)_{t\in T'}$ *are* $\mathcal{I}$*-independent. This means that for any continuous functions* $f : G^T \to \mathbb{R}$ *and* $f' : G^{T'} \to \mathbb{R}$ *invariant by diagonal conjugation by* G,

$$\mathbb{E}^{\mathsf{Aff}}_{dx}\Big[f\left((Z_t)_{t\in T}\right) f'\left(\left(Z^{-1}_{t_0} Z_t\right)_{t\in T'}\right)\Big] = \mathbb{E}^{\mathsf{Aff}}_{dx}\left[f\left((Z_t)_{t\in T}\right)\right] \mathbb{E}^{\mathsf{Aff}}_{dx}\Big[f'\left(\left(Z^{-1}_{t_0} Z_t\right)_{t\in T'}\right)\Big].$$

Proof. — Let $t_0 > 0$, let T be a finite subset of $[0, t_0]$ and let T' be a finite subset of $[t_0, \infty[$. Obviously we can suppose that $T' \subset]t_0, \infty[$. Let t'_0 be any real strictly greater that t_0 such that $T' \subset [t'_0, \infty[$. We remind the reader that for any $t \in T'$,

$$Z^{-1}_{t'_0} Z_t = h\big(p_{t'_0} \partial c^t_{t'_0} p^{-1}_{t'_0}\big),$$

thus for any continuous functions $f : G^T \to \mathbb{R}$ and $f' : G^{T'} \to \mathbb{R}$ invariant by diagonal conjugation by G:

$$\mathbb{E}^{\mathsf{Aff}}_{dx}\Big[f\big((Z_t)_{t\in T}\big) f'\left(\left(Z^{-1}_{t'_0} Z_t\right)_{t\in T'}\right)\Big] = \mathbb{E}^{\mathsf{Aff}}_{dx}\Big[f\big((h(L^t_0))_{t\in T}\big) f'\big((h(\partial c^t_{t'_0}))_{t\in T'}\big)\Big].$$

Let us denote by t_1 the maximum of T'. The loop L_0^t is in $\overline{\mathsf{Int}(L_0^{t_0})}$ for any $t \in T$ and the loop $\partial c_{t'_0}^t$ is in $\overline{\mathsf{Int}(\partial c_{t'_0}^{t_1})}$ for any $t \in T'$. Besides we have $\mathsf{Int}(L_0^{t_0}) \cap \overline{\mathsf{Int}(\partial c_{t'_0}^{t_1})} = \emptyset$. We can thus apply $\mathbf{wDP_2}$:

$$\mathbb{E}_{dx}^{\mathsf{Aff}}\left[f\left((Z_t)_{t\in T}\right)f'\left(\left(Z_{t'_0}^{-1}Z_t\right)_{t\in T'}\right)\right]=\mathbb{E}_{dx}^{\mathsf{Aff}}\left[f\left((Z_t)_{t\in T}\right)\right]\mathbb{E}_{dx}^{\mathsf{Aff}}\left[f'\left(\left(Z_{t'_0}^{-1}Z_t\right)_{t\in T'}\right)\right].$$

Using the stochastic continuity of $\mathbb{E}_{dx}^{\mathsf{Aff}}$ and taking the limit $t'_0 \to t_0$, we get the Lemma 10.2. □

When G is Abelian, for any n-tuple $(g_1, \dots, g_n)$ of elements of G, the diagonal conjugacy class of $(g_1, \dots, g_n)$ is reduced to $\{(g_1, \dots, g_n)\}$. Thus, the last lemma asserts that $\sigma(\{Z_t, t \le t_0\})$ is independent of $\sigma(\{Z_{t_0}^{-1}Z_t, t \ge t_0\})$. Using the Remark 10.2, this implies that $(Z_t)_{t\in\mathbb{R}^+}$ is a Lévy process. Applying Theorem 10.7 and Lemma 10.1, we deduce that $\left(\mathbb{E}_{vol}^{\mathbb{G}}\right)_{\mathbb{G},vol}$ is the planar Yang-Mills field associated with the Lévy process $(Z_t)_{t\ge 0}$. The Abelian part of Theorem 10.1 is thus proved.

As we have seen in Remark 2.4, it is not enough to know the independence of the conjugacy classes of a couple of random variables in order to understand the law of this couple. Thus, when G is not Abelian, we have to get rid of the conjugacy classes in Lemma 10.2: it is what we intend to do in the following subsection.

10.3. Braidability, the non-commutative case. — The following property satisfied by the process $(Z_t)_{t\in\mathbb{R}^+}$ under $\mathbb{E}_{dx}^{\mathsf{Aff}}$ will be the key point for the proof of Theorem 10.1. Recall the notions and notations set in Definition 9.5.

Proposition 10.1. — *Under $\mathbb{E}_{dx}^{\mathsf{Aff}}$, the process $(Z_t)_{t\in\mathbb{R}^+}$ has braidable increments.*

Proof. — The proof will be essentially graphical. The braid group with m strands is generated by the elementary braids $(\beta_i)_{i=1}^{m-1}$ defined in Section 7. This allows us to reduce the braidability condition to the fact that for any $n \in \mathbb{N}^* \cup (\mathbb{N}^*)^{-1}$, any positive integers m and i such that $i < m$, the following equality in law holds:

$$\beta_i \bullet \left(Z_{n,1}, \dots, Z_{n,m}\right) = \left(Z_{n,1}, \dots, Z_{n,m}\right).$$

The proof does not depend on the value of n, we will suppose it is equal to 1. We remind the reader that $Z_{1,i} = h\left(p_{i-1}\partial c_{i-1}^i p_{i-1}^{-1}\right)$: we have to understand the law of the random variables associated with m lassos. Using the stochastic continuity of $\mathbb{E}_{dx}^{\mathsf{Aff}}$, we can "shrink" the meander of these lassos and we can suppose that their intersection is reduced to the base point as we did in the proof of Theorem 10.7. Let i be a positive integer, we will focus only on what happens in the interior of ∂c_{i-1}^{i+1}. Let us consider the graphs $\mathbb{G}_1$ and $\mathbb{G}_2$ drawn in Figure 21. They represent what is happening in ∂c_{i-1}^{i+1}: the loops in bold represent the part of the i^{th} and $i+1^{th}$ lassos inside ∂c_{i-1}^{i+1} and we added to it two paths in dots in order to consider simple graphs. The two graphs satisfy the hypothesis of Theorem 3.1 thus, they are homeomorphic. Let us consider an orientation-preserving homeomorphism ϕ between $\mathbb{G}_1$ and $\mathbb{G}_2$ which sends F_i on F'_i for any $i \in \{1, \dots, 5\}$. By Proposition 3.2, there exists an orientation-preserving $\mathbb{G}_1 - \mathbb{G}_2$ piecewise diffeomorphism ψ which is equivalent to ϕ on $\mathbb{G}_1$: it sends F_i on F'_i for any $i \in \{1, \dots, 5\}$. It is possible to take it such that ψ is the identity on the unbounded face of $\mathbb{G}_1$. Besides, one can remark that $\mathbb{G}_2$ is the horizontal flip of $\mathbb{G}_1$: for any integer $i \in \{1, \dots, 5\}$,

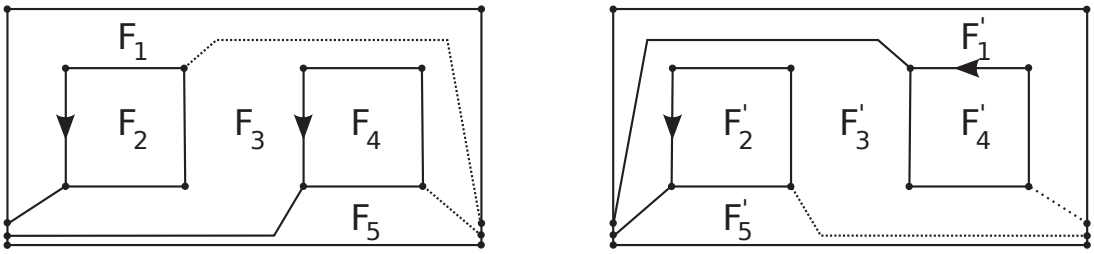


FIGURE 21. The graphs $\mathbb{G}_1$ and $\mathbb{G}_2$.

$dx(F_i) = dx(F_i')$. Thus for any bounded face F of $\mathbb{G}_1$, $dx(\psi(F)) = dx(F)$. Using the area-preserving homeomorphism invariance, namely Axiom $\mathbf{wDP_1}$, $\mathbb{E}_{dx}^{\mathbb{G}_2} \circ \psi^{-1} = \mathbb{E}_{dx}^{\mathbb{G}_1}$. Letting the shrinking parameter to zero in this equality allows us to recover the following equality in law: under $\mathbb{E}_{dx}^{\mathsf{Aff}}$, $\beta_i \bullet \big(Z_{1,1}, \dots, Z_{1,m}\big) = \big(Z_{1,1}, \dots, Z_{1,m}\big)$. □

Recall that we are working under $\mathbb{E}_{dx}^{\mathsf{Aff}}$. Using the results of Section 10.2 and 10.3, we already know that the process $Z = (Z_t)_{t\in\mathbb{R}^+}$ is invariant by conjugation by G and has braidable and $\mathcal{I}$-independent increments. By Theorem 9.7, there exists a self-invariant by conjugation Lévy process Y, such that the law of Z is UYU^{-1}, where U is a Haar variable on G independent of Y. Lemma 10.1, combined with Theorem 10.7 allows us to finish the proof of Theorem 10.1.

11. Classification of stochastically continuous strong planar Markovian holonomy fields

Let $\big(\mathbb{E}_{vol}\big)_{vol}$ be a stochastically continuous strong planar Markovian holonomy field. We just saw that it is a planar Yang-Mills field to which we can associate a Lévy process $Y = (Y_t)_{t\geq 0}$. In Definition 8.5, we defined the notions of pure non-degenerate/pure degenerate/mixed degenerate planar Yang-Mills field, according to the degree of symmetry and the support of Y. In this section, we will see equivalent conditions for $\big(\mathbb{E}_{vol}\big)_{vol}$ to be in each of these categories. The theorems explained below will be straightforward applications of Theorem 10.1 and Section 9.4.3. Indeed, by definition, $\big(\mathbb{E}_{vol}\big)_{vol}$ is a pure non-degenerate (resp. pure) planar Yang-Mills field if and only if Y is pure non-degenerate (resp. pure). Applying Proposition 9.8 (resp. Proposition 9.9, resp. Theorem 9.8) to the process Z_t defined in Definition 10.2 allows us to prove Theorem 11.1 (resp. Theorem 11.2, resp. Theorem 11.3). The first theorem gives an equivalent condition, when G is a finite group, for $\big(\mathbb{E}_{vol}\big)_{vol}$ to be a pure non-degenerate planar Yang-Mills field.

Theorem 11.1. — *Let G be a finite group, let $\big(\mathbb{E}_{vol}\big)_{vol}$ be a stochastically continuous strong planar Markovian holonomy field. It is a pure non-degenerate planar Yang-Mills field if and only if for any simple loop l, for any measure of area vol, the support of $h(l)$ under $\mathbb{E}_{vol}$ is G.*

The second theorem gives an equivalent condition, without any restriction on the compact Lie group G, for $\big(\mathbb{E}_{vol}\big)_{vol}$ to be a pure non-degenerate planar Yang-Mills field.

Theorem 11.2. — *Let $\big(\mathbb{E}_{vol}\big)_{vol}$ be a stochastically continuous strong planar Markovian holonomy field. Let us suppose that for any loop l and any measure of area vol, under $\big(\mathbb{E}_{vol}\big)_{vol}$, e is in the support of $h(l)$. The Markovian holonomy field $\big(\mathbb{E}_{vol}\big)_{vol}$ is a pure non-degenerate planar Yang-Mills field if and only if for any sequence of simple loops $\big(l_n\big)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ satisfying $vol\big(\mathsf{Int}(l_n)\big) \underset{n\to\infty}{\longrightarrow} \infty$, one has:*

$$\mathbb{E}_{vol} \circ h(l_n)^{-1} \underset{n\to\infty}{\longrightarrow} \lambda_G,$$

where λ_G is the Haar measure on G.

One could remove the condition on the support of $h(l)$ if one could understand the support of any Lévy process which is invariant by conjugation. The third theorem gives an equivalent condition, for any compact Lie group G, for $\big(\mathbb{E}_{vol}\big)_{vol}$ to be a pure planar Yang-Mills field.

Theorem 11.3. — *Let $\big(\mathbb{E}_{vol}\big)_{vol}$ be a stochastically continuous strong planar Markovian holonomy field. It is a pure planar Yang-Mills field if and only there exists a Lévy process $(Z_t)_{t\geq 0}$ such that for any simple loop l, for any measure of area vol, the law of $h(l)$ under $\big(\mathbb{E}_{vol}\big)_{vol}$ is the law of $Z_{vol(\mathsf{Int}(l))}$. If this condition holds, then $(Z_t)_{t\geq 0}$ is invariant by conjugation and it is the Lévy process associated with $\big(\mathbb{E}_{vol}\big)_{vol}$.*

12. Markovian holonomy fields

We will apply the results we got to the theory of Markovian holonomy fields defined in [**21**] by Lévy. The main result of this section is Theorem 12.3. After some basic definitions, taken from [**21**], and a summary of the results obtained in the same book, we will define the free boundary expectation on the plane associated with any Markovian holonomy field. By doing so, we obtain a planar Markovian holonomy field: this will allow us to apply the results obtain previously in the theory of planar Markovian holonomy fields.

12.1. Measured marked surfaces with G-constraints. — Until the end of the paper, M will be an *oriented smooth compact surface with boundary* and, from now on, as we only consider such surfaces, we will call them simply surfaces.

Definition 12.1. — *To any connected component of the boundary of M one can associate a non-oriented cycle (Definition 2.9). The union of these non-oriented cycles is denoted by $\mathcal{B}(M)$.*

A collection of marks $\mathcal{C}$ on M is a finite union of disjoint simple smooth non-oriented cycles in the interior of M. The couple $\big(M,\mathcal{C}\big)$ is called a marked surface and any element of $\mathcal{C}$ is called a mark.

The orientation of M induces an orientation on each connected component of the boundary: we denote by $\mathcal{B}^+(M)$ the subset of $\mathcal{B}(M)$ of positively oriented representative of each non-oriented cycle. The non-oriented cycles included in $\mathcal{C}$ does not carry a canonical orientation.

Let us recall that a non-oriented cycle is a set $\{c, c^{-1}\}$ where c and c^{-1} are oriented cycle, thus by definition a mark is an oriented cycle. Besides if M has only one boundary, we will denote by ∂M the positively oriented cycle associated with the unique boundary of M.

Definition 12.2. — *Let $\big(M, \mathcal{C}\big)$ be a marked surface. A graph on $\big(M, \mathcal{C}\big)$ is a graph on M such that each oriented cycle in $\mathcal{C}$ is represented by a loop in $L(\mathbb{G})$.*

The Proposition 1.3.10 in [**21**] asserts also that for any graph $\mathbb{G}$ on M, any cycle of $\mathcal{B}(M)$ is represented by a loop in $L(\mathbb{G})$.

Definition 12.3. — *Let $(M, \mathcal{C})$ be a marked surface. Let $\mathsf{Conj}(G)$ be the set of conjugacy classes of G. A set of G-constraints on $\big(M, \mathcal{C}\big)$ is a mapping C from $\mathcal{C} \cup \mathcal{B}(M)$ to $\mathsf{Conj}(G)$ such that $C(l^{-1}) = C(l)^{-1}$ for any $l \in \mathcal{C} \cup \mathcal{B}(M)$. The family of sets of G-constraints on $\big(M, \mathcal{C}\big)$ is denoted by $\mathsf{Conj}_G\big(M, \mathcal{C}\big)$.*

Notation 12.1. — *Let C be a set of G-constraints, let c be an oriented cycle in $\mathcal{C} \cup \mathcal{B}(M)$ and let x be an element of G. We will denote by $C_{c \to x}$ the unique set of G-constraints such that:*

1. *for any oriented cycle $c' \in \mathcal{C} \cup \mathcal{B}(M)$ different of c and c^{-1}, $C_{l \to x}(c') = C(c')$,*
2. *$C_{c \to x}(c) = \big[x\big]$ and $C_{c \to x}(c^{-1}) = \big[x^{-1}\big]$,*

where we recall that $[x]$ is the conjugacy class of x. Besides, we will denote by $c \to [x]$ the set of G-constraints defined on $\{c, c^{-1}\}$ which sends c on $[x]$ and c^{-1} on $[x^{-1}]$.

Definition 12.4. — *A measured marked surface with G-constraints is a quadruple $\big(M, vol, \mathcal{C}, C\big)$ where $\big(M, \mathcal{C}\big)$ is a marked surface, vol is a measure of area on M and C is a set of G-constraints on $\big(M, \mathcal{C}\big)$.*

The isomorphism notion on the set of measured marked surfaces with G-constraints is the following: $\big(M, vol, \mathcal{C}, C\big)$ and $\big(M', vol', \mathcal{C}', C'\big)$ are isomorphic if and only if there exists a diffeomorphism $\psi : M \to M'$ such that:

- *$vol \circ \psi^{-1} = vol'$,*
- *ψ sends $\mathcal{C}$ on $\mathcal{C}'$,*
- *$\forall\ l \in \mathcal{C} \cup \mathcal{B}(M), C'(\psi(l)) = C(l)$.*

12.2. Splitting of a surface. — An important notion for the definition of Markovian holonomy fields is the operation of splitting. We will not define this notion properly in this paper but instead we refer the reader to Section 1.1.2 in [**21**] for a rigorous definition.

Let M be a surface, a splitting of M is the data of a surface M' and a gluing: $M' \to M$, which is an application which glues two boundary components of M'. The set which consists of the image of the boundary glued and its inverse is the joint of the gluing: we split according to this non-oriented cycle. Thus, we will consider a splitting as the inverse of the gluing: a splitting is the action to split a surface according to a non-oriented cycle drawn on it. We will also say that we split a surface according to a mark l and by this, we mean that we split according to the non-oriented cycle $\{l, l^{-1}\}$. There is uniqueness (modulo isomorphism) of the splitting according to a mark on a surface M: the split surface of M according to l is denoted by $Spl_l(M)$.

Let $\big(M,\mathcal{C},vol,C\big)$ be a measured marked surface with G-constraints. Let l be a mark in $\mathcal{C}$ and $f_l : Spl_l(M) \to M$ be the gluing associated with the splitting $Spl_l(M)$. Thanks to the empty intersection of the marks on M, we can transport the marks on $Spl_l(M)$. We will denote them $Spl_l(\mathcal{C})$. Since outside a negligible subset, a gluing is a diffeomorphism, it is possible to transport the measure of area on $Spl_l(M)$ by setting $Spl_l(vol) = vol \circ f_l$. In order to transport the G-constraints on $Spl_l(M)$, we set $Spl_l(C)(l') = C(f_l(l'))$ for any $l' \in Spl_l(\mathcal{C}) \cup \mathcal{B}(Spl_l(M))$.

12.3. Markovian holonomy fields. — The definition of a Markovian holonomy field was first stated in Definition 3.1.2 of [**21**]. In this paper, we only consider oriented Markovian holonomy fields: for sake of simplicity, we will call them Markovian holonomy fields. In the following definition, we add the condition that the measures are non-degenerate, which means that their weight is strictly positive. Besides, we changed the Axiom $\mathbf{A_4}$: in order to undestand why, one can read Remark 4.3.

Definition 12.5. — *A G-valued Markovian holonomy field, HF, is the data, for each measured marked surface with G-constraints $(M,vol,\mathcal{C},C)$ of a non-degenerate finite measure $\mathsf{HF}_{(M,vol,\mathcal{C},C)}$ on $\big(\mathcal{M}ult\big(P(M),G\big),\mathcal{I}\big)$ such that:*

- $\mathbf{A_1}$ **:** *For any $(M,vol,\mathcal{C},C)$, $\mathsf{HF}_{(M,vol,\mathcal{C},C)}\big(\exists l \in \mathcal{C} \cup \mathcal{B}(M), h(l) \notin C(l)\big) = 0$.*
- $\mathbf{A_2}$ **:** *For any $\big(M,vol,\mathcal{C}\big)$ and any event Γ in $\mathcal{I}$, the function $C \mapsto \mathsf{HF}_{(M,vol,\mathcal{C},C)}(\Gamma)$ is a measurable function on $\mathsf{Conj}_G\big(M,\mathcal{C}\big)$.*
- $\mathbf{A_3}$ **:** *For any $(M,vol,\mathcal{C},C)$ and any $l \in \mathcal{C}$,*

$$\mathsf{HF}_{\left(M,vol,\mathcal{C}\setminus\{l,l^{-1}\},C_{|\mathcal{B}(M)\cup\mathcal{C}\setminus\{l,l^{-1}\}}\right)} = \int_G \mathsf{HF}_{\left(M,vol,\mathcal{C},C_{l\to[x]}\right)} dx,$$

where $C_{l\to[x]}$ is defined in Notation 12.1.

- $\mathbf{A_4}$ **:** *Let $\psi : (M,vol,\mathcal{C},C) \to (M',vol',\mathcal{C}',C')$ be a bi-Lipschitz homeomorphism which preserves the orientation such that $vol \circ \psi^{-1} = vol'$, $\psi(\mathcal{C}) = \mathcal{C}'$ and $C = C' \circ \psi$. The mapping from $\mathcal{M}ult(P(M'),G)$ to $\mathcal{M}ult\big(P(M),G\big)$ induced by ψ, also denoted ψ, satisfies:*

$$\mathsf{HF}_{(M',vol',\mathcal{C}',C')} \circ \psi^{-1} = \mathsf{HF}_{(M,vol,\mathcal{C},C)}.$$

Moreover, let $\mathbb{G}$ (resp. $\mathbb{G}'$) be a graph on $(M,\mathcal{C})$ (resp. on $(M,\mathcal{C}')$), let ϕ be a homeomorphism from $(M,vol,\mathcal{C},C)$ to $(M',vol',\mathcal{C}',C')$ which sends $\mathbb{G}$ on $\mathbb{G}'$, which preserves the orientation and such that $vol \circ \phi^{-1} = vol'$, $\phi(\mathcal{C}) = \mathcal{C}'$ and $C = C' \circ \phi$. The mapping from $\mathcal{M}ult(P(\mathbb{G}'),G)$ to $\mathcal{M}ult\big(P(\mathbb{G}),G\big)$ induced by ϕ, also denoted ϕ, satisfies:

$$\big(\mathsf{HF}_{(M',vol',\mathcal{C}',C')}\big)_{|\mathcal{M}ult(P(\mathbb{G}'),G)} \circ \phi^{-1} = \big(\mathsf{HF}_{(M,vol,\mathcal{C},C)}\big)_{|\mathcal{M}ult(P(\mathbb{G}),G)}.$$

- $\mathbf{A_5}$ **:** *For any $(M_1,vol_1,\mathcal{C}_1,C_1)$ and $(M_2,vol_2,\mathcal{C}_2,C_2)$, one has the identity:*

$$\mathsf{HF}_{\left(M_1\sqcup M_2,vol_1\sqcup vol_2,\mathcal{C}_1\sqcup\mathcal{C}_2,C_1\sqcup C_2\right)} = \mathsf{HF}_{\left(M_1,vol_1,\mathcal{C}_1,C_1\right)} \otimes \mathsf{HF}_{\left(M_2,vol_2,\mathcal{C}_2,C_2\right)}.$$

- $\mathbf{A_6}$ **:** *For any $(M,vol,\mathcal{C},C)$, any $l \in \mathcal{C}$ and any gluing along l, $\psi : Spl_l(M) \to M$, one has:*

$$\mathsf{HF}_{\left(Spl_l(M),Spl_l(vol),Spl_l(\mathcal{C}),Spl_l(C)\right)} = \mathsf{HF}_{\left(M,vol,\mathcal{C},C\right)} \circ \psi^{-1}.$$

$\mathbf{A_7}$: *For any $(M, vol, \emptyset, C)$ and for any l in $\mathcal{B}(M)$,*

$$\int_G \mathsf{HF}_{(M,vol,\emptyset,C_{l\to x})}(\mathbb{1})dx = 1.$$

The Markovian holonomy fields seem to be quite complicated objects. Actually it is easier to understand them when they are exposed in a less formal way. A Markovian holonomy field is a *family of measures*. For each surface M with marks, set of G-constraints and measure of area, we are given a *gauge-invariant random holonomy field on M* which *satisfies the set of G-constraints* ($\mathbf{A_1}$). Moreover, the family of measures given by a Markovian holonomy field is *invariant under a class of area-preserving homeomorphisms*, $\mathbf{A_4}$, and satisfies a kind of *Markov property*, $\mathbf{A_5}$ and $\mathbf{A_6}$. The measures associated with $\big(M, vol, \mathcal{C}, C\big)$, seen as a function of the G-constraints, provide a *regular disintegration of* $\mathsf{HF}_{\big(M,vol,\emptyset,C_{|\mathcal{B}(M)}\big)}$ (Axioms $\mathbf{A_1}$, $\mathbf{A_2}$ and $\mathbf{A_3}$). The last assumption is a *normalization* axiom.

As for the planar Markovian holonomy fields, if not specified, all the Markovian holonomy fields will be G-valued, thus we will omit to specify it. In the definition of a Markovian holonomy field, we didn't specify any regularity condition on the field. In what follows, we will focus only on regular Markovian holonomy field in the following sense:

Definition 12.6. — *Let HF be a Markovian holonomy field.*

1. *We say that HF is stochastically continuous if, for any $(M, vol, \mathcal{C}, C)$, $\mathsf{HF}_{(M,vol,\mathcal{C},C)}$ is stochastically continuous (Definition 2.14).*
2. *We say that HF is Fellerian if, for any $\big(M, vol, \mathcal{C}\big)$, the function*

$$(t, C) \mapsto \mathsf{HF}_{(M,vol,\mathcal{C},C)}(\mathbb{1}),$$

   defined on $\mathbb{R}_+^* \times \mathsf{Conj}_G\big(M, \mathcal{C}\big)$ is continuous.
3. *We say that HF is regular if it is both stochastically continuous and Fellerian.*

12.4. Partition functions for oriented surfaces. — Given an even positive integer g and p a positive integer, let $\Sigma_{p,g}^+$ be the connected sum of $\frac{g}{2}$ tori with p holes. For $g = 0$ we define $\Sigma_{p,0}^+$ to be the sphere with p holes.

The classification of surfaces asserts that any connected oriented compact surface is diffeomorphic to one and exactly one of $\big\{\Sigma_{p,g}^+, p \in \mathbb{N}, g \in 2\mathbb{N}\big\}$. Besides, as a consequence of a theorem of Moser and as explained in Proposition 4.1.1 of [**21**], if M and M' are oriented, if $(M, vol, \emptyset, C)$ and $(M', vol', \emptyset, C')$ are two measured marked surfaces with G-constraints, then they are isomorphic if and only if:

1. M and M' are diffeomorphic,
2. $vol(M) = vol'(M')$,
3. there exists a bijection $\psi = \mathcal{B}^+(M) \to \mathcal{B}^+(M')$ such that $C = C' \circ \psi$ on $\mathcal{B}^+(M)$.

Let HF be a Markovian holonomy field, we define the partition functions of HF.

Definition 12.7. — *Let g be an even positive integer, p be a positive integer and t be a positive real. Let vol be a measure of area on $\Sigma_{p,g}^+$ of total mass t. Let $\big\{b_1, b_2, ..., b_p\big\}$*

be an enumeration of $\mathcal{B}^+(\Sigma^+_{p,g})$. *We define the mapping:*

$$\begin{aligned} Z^+_{p,g,t}(x_1,...,x_p): \quad G^p \quad &\longrightarrow \mathbb{R}^*_+ \\ (x_1,...,x_p) \;&\mapsto\; Z^+_{p,g,t}(x_1,...,x_p) \;= \mathsf{HF}_{\left(\Sigma^+_{p,g},vol,\emptyset,(b_i\mapsto[x_i])_{i=1}^p\right)}(\mathbb{1}), \end{aligned}$$

It is called the partition function of the surface of genus g *with* p *holes. Using the diffeomorphism invariance given by Axiom* $\mathbf{A_4}$ *and using Moser's theorem, it depends neither on the choice of vol nor on the choice of the enumeration:* $Z^+_{p,g,t}$ *is a symmetric function.*

Remark 12.1. — *If* $p = 0$, *we define* $Z^+_{0,g,t}$ *as the positive number which is equal to the mass of* $\mathsf{HF}_{\left(\Sigma^+_{0,g},vol,\emptyset,\emptyset\right)}$.

The discussion on the notion of isomorphism between measured marked surfaces with G-constraints implies that if $(M, vol, \emptyset, C)$ is a measured marked surface with G-constraints then there exist p and g such that M is diffeomorphic to $\Sigma^+_{p,g}$. This implies:

$$\mathsf{HF}_{(M,vol,\emptyset,C)}(\mathbb{1}) = Z^+_{p,g,vol(M)}(x_1,...,x_p),$$

where x_1, ..., x_p are representatives of the p constraints put on $\mathcal{B}^+(M)$.

The Fellerian condition satisfied by regular Markovian holonomy fields implies that their partition functions are continuous in $(t, x_1, ..., x_p)$. Besides we can reformulate the Axiom of normalization $\mathbf{A_7}$ (Definition 12.5) in terms of partition functions. If HF is a Markovian holonomy field, for any $t > 0$,

$$\int_G Z^+_{1,0,t}(g)dg = 1,$$

that is to say: $Z^+_{1,0,t}dg$ is a probability measure on G. In one of the main theorems proved in Chapter 4 of [**21**], Lévy characterized the family of probability measures $\left(Z^+_{1,0,t}dg\right)_{t>0}$.

Theorem 12.1. — *Let* HF *be a regular Markovian holonomy field. The probability measures* $\left(Z^+_{1,0,t}dg\right)_{t>0}$ *on* G *are the one dimensional distributions of a unique conjugation-invariant Lévy process* $(Y_t)_{t\geq 0}$ *issued from the neutral element. Moreover, this Lévy process determines completely the partition functions of* HF.

We say that $Y = (Y_t)_{t\geq 0}$ (resp. HF) is the Lévy process (resp. a regular Markovian holonomy field) associated with HF (resp. to Y). Given this theorem, it is natural to ask if every Lévy process which is conjugation-invariant is associated with a regular Markovian holonomy field. Of course, some other conditions must hold such as the existence of a conjugation-invariant square-integrable density. Indeed, as the constraints on the boundary are given by specifying a conjugacy class, $Z^+_{1,0,t}(x)$ is a function of $\left[x\right]$. Besides, by the definition of regularity, it must be continuous in x thus square-integrable. To finish, let us remark that $Z^+_{1,0,t}(x)$ is strictly positive since we supposed that the measures $\left(\mathsf{H}_{(M,vol,\mathcal{C},C)}\right)_{(M,vol,\mathcal{C},C)}$ are non-degenerate. Hence the natural following definition:

Definition 12.8. — *Let* $(Y_t)_{t\geq 0}$ *be a Lévy process on* G *issued from the neutral element. It is admissible if:*

– *it is invariant by conjugation by* G,

– *the distribution of Y_t admits a strictly positive square-integrable density Q_t with respect to the Haar measure on G for any $t > 0$.*

The discussion we just had allows us to write the following proposition.

Proposition 12.1. — *Let* HF *be a regular Markovian holonomy field, the Lévy process $(Y_t)_{t\geq 0}$ associated with* HF *is an admissible Lévy process.*

In fact, we get all the admissible Lévy processes by studying the Lévy processes which are associated with regular Markovian holonomy fields: this is given by Theorem 4.3.1 in the book [**21**].

Theorem 12.2. — *Every G-valued admissible Lévy process Y is associated with a regular Markovian holonomy field.*

The proof of this assertion consists in constructing, just as we did for planar Markovian holonomy fields, for every G-valued admissible Lévy, a special Markovian holonomy field YM which will be called a Yang-Mills field. For this, Lévy used the edge paradigm for random holonomy fields. A Yang-Mills field is a kind of deformation of a uniform measure.

12.5. Uniform measure and Yang-Mills fields. — Let $(M, vol, \mathcal{C}, C)$ be a measured marked surface with G-constraints, endowed with a graph $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$. The uniform measure on $\mathcal{M}ult\big(P(\mathbb{G}), G\big)$ is almost a product of Haar measures as for any orientation $\mathbb{E}^+$ of $\mathbb{G}$, $\mathcal{M}ult\big(P(\mathbb{G}), G\big) \simeq G^{\mathbb{E}^+}$. But one has to be careful: since $(M, \mathcal{C}, C)$ is an oriented marked surface with G-constraints, the elements in $\mathcal{M}ult\big(P(\mathbb{G}), G\big)$ that we have to consider have to obey the constraints.

Notation 12.2. — *For any conjugacy class $\mathcal{O} \subset G$ and any integer $n \geq 1$, we denote by $\delta_{\mathcal{O}(n)}$ the natural extension to G^n of the unique G^n-invariant probability measure on*

$$\mathcal{O}(n) \ = \big\{(x_1, ..., x_n) \in G^n : x_1...x_n \in \mathcal{O}\big\}$$

under the G^n action $(g_1, ..., g_n) \bullet (x_1, ..., x_n) = (g_1 x_1 g_2^{-1}, ..., g_n x_n g_1^{-1})$.

Let $l_1, ..., l_q$ be q disjoint simple loops in $L(\mathbb{G})$ such that $\mathcal{C} \cup \mathcal{B}(M) = \{l_1, l_1^{-1}, ..., l_q, l_q^{-1}\}$. For any $i \in \{1, ..., q\}$, we can decompose $l_i = e_{i,1}...e_{i,n_i}$ with $e_{i,j} \in \mathbb{E}$ for any i and j. We can consider $\mathbb{E}^+$, an orientation of $\mathbb{G}$, such that for any $i \in \{1, ..., q\}$ and $j \in \{1, ..., n_i\}$, $e_{i,j} \in \mathbb{E}^+$. We label e_1, ..., e_m the other edges of $\mathbb{E}^+$. Any measure constructed on $G^{\mathbb{E}^+}$ defines canonically a unique measure on $\mathcal{M}ult\big(P(\mathbb{G}), G\big)$. The uniform measure $\mathsf{U}^{\mathbb{G}}_{M,\mathcal{C},C}$ is the measure provided by the following measure on $G^{\mathbb{E}^+}$:

$$dg_1 \otimes ... \otimes dg_m \otimes \delta_{C(l_1)(n_1)}(dg_{1,n_1}...dg_{1,1}) \otimes ... \otimes \delta_{C(l_q)(n_q)}(dg_{q,n_q}...dg_{q,1}).$$

This probability measure on $\mathcal{M}ult\big(P(\mathbb{G}), G\big)$ does not depend on any of the choices we made.

Notation 12.3. — *We define also a similar measure without constraints for any surface M (resp. $\mathbb{R}^2$) endowed with a graph (resp. a planar graph) $\mathbb{G}$. Let $\mathbb{E}^+$ be an orientation of $\mathbb{G}$. The measure on $\mathcal{M}ult\big(P(\mathbb{G}),G\big)$ seen on $G^{\mathbb{E}^+}$ as:*

$$\bigotimes_{e\in\mathbb{E}^+} dg_e,$$

is denoted by $\mathsf{U}^{\mathbb{G}}$.

We can give the definition of Yang-Mills field.

Definition 12.9. — *Let $(Y_t)_{t\geq 0}$ be an admissible Lévy process on G. For any positive real t, let Q_t be the density of Y_t. A regular Markovian holonomy field YM is called a Yang-Mills field (or sometimes a Yang-Mills measure) associated with $(Y_t)_{t\geq 0}$ if for any measured marked surface with G-constraints $(M, vol, \mathcal{C}, C)$ and any graph $\mathbb{G}$ on $\big(M,\mathcal{C}\big)$,*

$$\Big(\mathsf{YM}_{(M,vol,\mathcal{C},C)}\Big)_{|\mathcal{M}ult(P(\mathbb{G}),G)} = \prod_{F\in\mathbb{F}} Q_{vol(F)}\big(h(\partial F)\big)\mathsf{U}^{\mathbb{G}}_{M,\mathcal{C},C}(dh),$$

where ∂F is the oriented facial cycle associated with F, defined in Definition 1.3.13 *of* [**21**] *and the notation $Q_{vol(F)}\big(h(\partial F)\big)$ means that we consider $Q_{vol(F)}\big(h(c)\big)$ where c represents ∂F: this does not depend on the choice of c since $Q_{vol(F)}$ is invariant by conjugation.*

A Yang-Mills field associated with $(Y_t)_{t\geq 0}$ is a regular Markovian holonomy field associated with $(Y_t)_{t\geq 0}$. Thus, in order to prove Theorem 12.2, it is enough to prove the following.

Proposition 12.2. — *For any G-valued admissible Lévy process $(Y_t)_{t\geq 0}$ there exists a unique Yang-Mills field YM associated with $(Y_t)_{t\geq 0}$.*

For this proposition one has to introduce, as we did for planar Markovian holonomy fields, a discrete analog of Markovian holonomy fields: the discrete Markovian holonomy fields. The definition of discrete Markovian holonomy fields can be found in Section 3.2 of [**21**]. Then one can show that the family of measures:

$$\Big(\big(\mathsf{YM}_{(M,vol,\mathcal{C},C)}\big)_{|\mathcal{M}ult(P(\mathbb{G}),G)}\Big)_{\big(M,vol,\mathcal{C},C,\mathbb{G}\big)}$$

is a Fellerian continuously area-dependent (Proposition 4.3.11 in [**21**]) and locally stochastically $\frac{1}{2}$-Hölder continuous (Proposition 4.3.15 in [**21**]) discrete Markovian holonomy field (Proposition 4.3.10 in [**21**]) associated with Y. Then it is shown, in Theorem 3.2.9 of [**21**], that under these regularity conditions, every discrete Markovian holonomy field can be extended to a regular Markovian holonomy field. It has to be noticed that we changed the definition of Markovian holonomy fields (Axiom $\mathbf{A}_4$): this allows us to correct the arguments used in the proof of Axiom $\mathbf{A}_4$ in Theorem 3.2.9 of [**21**] by using the one explained before Theorem 4.1.

The definition of discrete Markovian holonomy field follows closely the definition of a Markovian holonomy field except for the invariance by homeomorphisms which becomes almost a combinatorial condition. It is the same difference between the Axioms $\mathbf{P_1}$ and $\mathbf{DP_1}$ of Definitions 4.1 and 4.4 of Section 4.1.

Remark 12.2. — *The difference between the assumption* $\mathbf{A_4}$ *in Definition 3.1.2 in* [**21**] *and* $\mathbf{D_4}$ *in Definition 3.2.1 in* [**21**] *makes the proof of Lemma 3.2.2. in the same book incomplete. Thus, it is not clear that any Markovian holonomy field determines a discrete Markovian holonomy field.*

This remark leads us to the following definition.

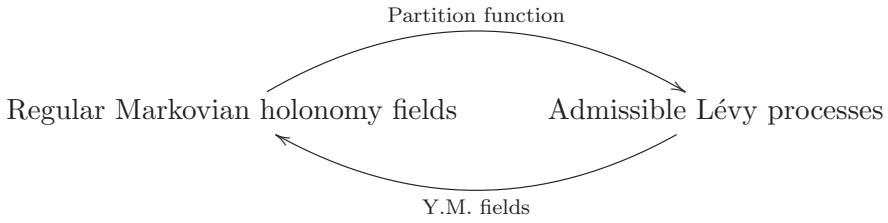
Definition 12.10. — *Let* HF *be a Markovian holonomy field. It is constructible if the family of measures:*

$$\Big(\big(\mathsf{HF}_{(M,vol,\mathcal{C},C)}\big)_{|\mathcal{M}ult(P(\mathbb{G}),G)}\Big)_{\big(M,vol,\mathcal{C},C,\mathbb{G}\big)}$$

is a discrete Markovian holonomy field.

It is still an open question to know if any Markovian holonomy field is constructible.

12.6. Conjecture and main theorem. — We can summarize the results of Proposition 12.1 and Theorem 12.2 by the following diagram.



Besides, it was shown that the left arrow goes into the constructible regular Markovian holonomy fields and the composition of the two arrows is equal to the identity on the set of admissible Lévy processes. It is natural to wonder if the two arrows are each other inverse: this leads us to the following conjecture.

Conjecture 12.1. — *Every regular Markovian holonomy field is a Yang-Mills field.*

From this conjecture, it would be true that every regular Markovian holonomy field is constructible.

In order to state the main result of the section, we need the notion of planar mark. Let M be an oriented smooth compact surface with boundary.

Definition 12.11. — *A planar mark is a mark* l *on* M *such that* l *cuts* M *in two parts, one of which is of genus* 0*.*

We can now state the main result of the section.

Theorem 12.3. — *Let* $\big(\mathsf{HF}_{(M,vol,\mathcal{C},C)}\big)_{(M,vol,\mathcal{C},C)}$ *be a regular Markovian holonomy field and* $(Y_t)_{t\in\mathbb{R}^+}$ *its associated* G*-valued Lévy process. Let* $\big(\mathsf{YM}_{(M,vol,\mathcal{C},C)}\big)_{M,vol,\mathcal{C},C}$ *be the Yang-Mills field associated with* $(Y_t)_{t\in\mathbb{R}^+}$*.*

Let $\big(M, vol, \emptyset, C\big)$ be a measured marked surface with G-constraints, let l be a planar mark on M, let M_1 be a part of M of genius 0 determined by l and let m be a point in M_1. The following equality holds:

$$\Big(\mathsf{HF}_{(M,vol,\emptyset,C)}\Big)_{|\mathcal{M}ult(L_m(M_1),G)} = \Big(\mathsf{YM}_{(M,vol,\emptyset,C)}\Big)_{|\mathcal{M}ult(L_m(M_1),G)}.$$

Let $\mathcal{C}$ be a collection of marks on M which do not intersect the mark l. Let us choose an orientation of $\mathcal{C}$ denoted by $\mathcal{C}^+$. Let C be a set of G-constraints on $\mathcal{B}(M)$. We endow the set of G-constraints on $\mathcal{C} \cup \mathcal{B}(M)$ with the measure $d\lambda_{C_{|\mathcal{B}(M)}}$ coming from:

$$\bigotimes_{c\in\mathcal{C}^+} dg_c \otimes \bigotimes_{b\in\mathcal{B}(M)^+} \delta_{C(b)}.$$

By disintegration, for any set of constraints on $\mathcal{B}(M)$, $d\lambda_{C_{|\mathcal{B}(M)}}$ almost surely:

$$\Big(\mathsf{HF}_{(M,vol,\mathcal{C},C)}\Big)_{|\mathcal{M}ult(L_m(M_1),G)} = \Big(\mathsf{YM}_{(M,vol,\mathcal{C},C)}\Big)_{|\mathcal{M}ult(L_m(M_1),G)}.$$

Remark 12.3. — *In order to prove the Conjecture 12.1, one would have to generalize Theorem 12.3 in order to include all the remaining loops, including the generators of the fundamental group of the surface.*

12.7. Markovian holonomy fields and free boundary condition on the plane. — Given a Markovian holonomy field HF, we have noticed that the measures $\mathsf{HF}_{(M,vol,\mathcal{C},C)}$ is not in general a probability measure. One way to deal with probability measures would be to normalize them by their mass. Yet, a better way to get a probability measure in our case is to consider the free boundary condition measure.

12.7.1. Free boundary condition on a surface. —

Definition 12.12. — *Let HF be a regular Markovian holonomy field. For any surface M homomorphic to a disk $\Sigma^+_{0,1}$ endowed with a measure of area vol, we define the free boundary condition expectation on M associated with HF as the probability measure on $\big(\mathcal{M}ult\big(P(M),G\big),\mathcal{B}\big)$ such that for any positive integer n, any measurable function $f : G^n \to \mathbb{R}^+$ and any finite family c_1, ... c_n of elements of $P(M)$:*

$$\mathbb{E}^{\mathsf{HF}}_{M,vol}\Big(f\big(h(c_1),...,h(c_n)\big)\Big) = \int_G \widehat{\mathsf{HF}}_{(M,vol,\emptyset,\partial M\to[x])}\Big(f\big(h(c_1),...,h(c_n)\big)\Big)dx,$$

where $\widehat{\mathsf{HF}}_{M,vol,\emptyset,\partial M\to[x]}$ is the extension of $\mathsf{HF}_{M,vol,\emptyset,\partial M\to[x]}$ to the Borel σ-field given by Proposition 2.2.

Remark 12.4. — *In this definition we have extended the σ-field to the Borel σ-field, in a way such that the new measure becomes invariant by the gauge group. In order for the definition of $\mathbb{E}^{\mathsf{HF}}_{M,vol}$ to be consistent with the way we named it, one has to verify that it is indeed a probability measure. Since the constant function $\mathbb{1}$ is gauge-invariant, $\hat{\mathbb{1}}_{J_{c_1,...,c_n}} = \mathbb{1}$, thus:*

$$\mathbb{E}^{\mathsf{HF}}_{M,vol}(\mathbb{1}) = \int_G \mathsf{HF}_{(M,vol,\emptyset,\partial M\to[x])}(\mathbb{1})dx = 1,$$

the last equality coming from the normalization Axiom $\mathbf{A_7}$ in Definition 12.5.

We can calculate the free boundary condition expectation on M of a Yang-Mills field.

Lemma 12.1. — *Let* YM *be the Yang-Mills field associated with the Lévy process* $(Y_t)_{t\in\mathbb{R}^+}$. *For any positive real* t, *let* Q_t *be the density of* Y_t. *For any surface* M *homeomorphic to a disk endowed with a measure of area vol and a graph* $\mathbb{G}$,

$$\left(\mathbb{E}^{\mathsf{YM}}_{M,vol}\right)_{|\mathcal{M}ult(P(\mathbb{G}),G)} = \prod_{F\in\mathbb{F}} Q_{vol(F)}\left(h(\partial F)\right)\mathsf{U}^{\mathbb{G}}(dh),$$

where $\mathsf{U}^{\mathbb{G}}$ *was defined in Notation 12.3 and where we used the notation* $Q_{vol(F)}\left(h(\partial F)\right)$ *already used in Proposition 8.5.*

Proof. — This follows from the fact that $\int_G \mathsf{U}^{\mathbb{G}}_{M,\emptyset,\partial M\to x}dx = \mathsf{U}^{\mathbb{G}}$ which is a consequence of:

$$\int_G \left[\int_{G^n} f\delta_{[y](n)}(dx_1,...,dx_n)\right]dy = \int_{G^n} f dx_1...dx_n,$$

given by the Equality (26) of Lemma 2.3.4 in [**21**]. □

12.7.2. Free boundary condition on the plane. — In this section, we define the free boundary expectation on the plane. This measure allows us to define a discrete planar Markovian holonomy field.

Proposition 12.3. — *Let* HF *be a regular Markovian holonomy field. Let* (M, vol) *and* (M', vol') *be two measured compact two-dimensional sub-manifolds of* $\mathbb{R}^2$ *which are homeomorphic to the unit disk. Let us suppose that* M *is included in the interior of* M', *denoted by* $\mathsf{Int}(M')$. *Let us assume that* $vol'_{|M} = vol$. *The free boundary condition expectations on* M *and* M' *are related by:*

$$\mathbb{E}^{\mathsf{HF}}_{M,vol} = \mathbb{E}^{\mathsf{HF}}_{M',vol'} \circ \rho^{-1}_{M,M'},$$

where we remind the reader that $\rho_{M,M'}$ *was defined in Notation 2.2. Thus, for any measure of area vol on* $\mathbb{R}^2$, *the family:*

$$\left\{\left(\mathcal{M}ult\left(P\left(M\right),G\right),\mathcal{B},\mathbb{E}^{\mathsf{HF}}_{M,vol_{|M}}\right)_{M\subset\mathbb{R}^2},\left(\rho_{M,M'}\right)_{M\subset\mathsf{Int}(M')}\right\},$$

is a projective family of probability spaces.

Proof. — Let (M, vol) and (M', vol') be two measured compact sub-manifolds of $\mathbb{R}^2$ which are homeomorphic to the unit disk and such that $M \subset \mathsf{Int}(M')$ and $vol'_{|M} = vol$. Let m be a point of M. Since $\mathbb{E}^{\mathsf{HF}}_{M,vol}$ and $\mathbb{E}^{\mathsf{HF}}_{M',vol'} \circ \rho^{-1}_{M,M'}$ are gauge-invariant, it is enough, by Proposition 2.4, to show that, for any positive integer n, for any continuous conjugation-invariant function f on G^n and any n-tuple of loops $l_1, ..., l_n$ in M based at m:

$$\mathbb{E}^{\mathsf{HF}}_{M',vol'}\left[f\left(h\left(l_1\right),...,h\left(l_n\right)\right)\right] = \mathbb{E}^{\mathsf{HF}}_{M,vol}\left[f\left(h\left(l_1\right),...,h\left(l_n\right)\right)\right].$$

The following sequence of equalities gives the desired result:

$$
\begin{aligned}
&\mathbb{E}^{\mathsf{HF}}_{M',vol'}\Big[f\big(h\left(l_1\right),...,h\left(l_n\right)\big)\Big]\\
&=\int_G\int_{\mathcal{M}(P(M'),G)} f\big(h(l_1),...,h(l_n)\big)\mathsf{HF}_{(M',vol',\emptyset,\partial M'\to[x])}(dh)dx\\
&=\int_G\int_G\int_{\mathcal{M}(P(M'),G)} f\big(h(l_1),...,h(l_n)\big)\mathsf{HF}_{(M',vol',\partial M,\{\partial M'\to[x],\partial M\to[y]\})}(dh)dydx \qquad \mathbf{A_3}\\
&=\int_G\int_G\int_{\mathcal{M}(P((M'\setminus\mathsf{Int}(M))\sqcup M),G)} f\big(h(l_1),...,h(l_n)\big)\\
&\qquad\qquad \mathsf{HF}_{\left((M'\setminus\mathsf{Int}(M))\sqcup M,vol'_{|(M'\setminus\mathsf{Int}(M))}\sqcup vol,\emptyset,\{\partial M'\to[x],\partial M\to[y]\}\right)}(dh)dydx \qquad \mathbf{A_6}\\
&=\int_G\int_G\int_{\mathcal{M}(P(M'\setminus\mathsf{Int}(M)),G)}\int_{\mathcal{M}(P(M),G)} f\big(h(l_1),...,h(l_n)\big)\mathsf{HF}_{(M,vol,\emptyset,\{\partial M\to[y]\})}(dh)\\
&\qquad\qquad \mathsf{HF}_{\left(M'\setminus\mathsf{Int}(M),vol'_{|M'\setminus\mathsf{Int}(M)},\emptyset,\{\partial M'\to[x],\partial M\to[y]\}\right)}(dh')dydx \qquad \mathbf{A_5}\\
&=\int_G\int_G\int_{\mathcal{M}(P(M),G)} f\big(h(l_1),...,h(l_n)\big)\mathsf{HF}_{(M,vol,\emptyset,\{\partial M\to[y]\})}(dh)\\
&\qquad\qquad \mathsf{HF}_{\left(M'\setminus\mathsf{Int}(M),vol'_{|M'\setminus\mathsf{Int}(M)},\emptyset,\{\partial M'\to[x],\partial M\to[y]\}\right)}(\mathbb{1})dydx\\
&=\int_G\int_{\mathcal{M}(P(M),G)} f\big(h(l_1),...,h(l_n)\big)\mathsf{HF}_{(M,vol,\emptyset,\{\partial M\to[y]\})}(dh)\\
&\qquad\qquad \Big(\int_G \mathsf{HF}_{\left(M'\setminus\mathsf{Int}(M),vol'_{|M'\setminus\mathsf{Int}(M)},\emptyset,\{\partial M'\to[x],\partial M\to[y]\}\right)}(\mathbb{1})dx\Big)dy\\
&=\int_G\int_{\mathcal{M}(P(M),G)} f\big(h(l_1),...,h(l_n)\big)\mathsf{HF}_{(M,vol,\emptyset,\{\partial M\to[y]\})}(dh)dy \qquad \mathbf{A_7}\\
&=\mathbb{E}^{\mathsf{HF}}_{M,vol}\Big[f\big(h(l_1),...,h(l_n)\big)\Big],
\end{aligned}
$$

where we applied successively the definition of $\mathbb{E}^{\mathsf{HF}}_{M',vol'}$, the Axioms $\mathbf{A_3}$, $\mathbf{A_6}$ and $\mathbf{A_5}$. Then after a change of notation and a Fubini exchange of integrals, the normalization Axiom $\mathbf{A_7}$ with the definition of $\mathbb{E}^{\mathsf{HF}}_{M,vol}$ lead us to the result. □

The free boundary expectation on the plane is the projective limit of this family of measured spaces.

Definition 12.13. — *Let* HF *be a regular Markovian holonomy field. Let vol be a measure of area on* $\mathbb{R}^2$. *The free boundary condition expectation on* $\mathbb{R}^2$ *defined on* $\big(\mathcal{M}ult(P(\mathbb{R}^2),G),\mathcal{B}\big)$ *is the projective limit of:*

$$\left\{\left(\mathcal{M}ult\left(P\left(M\right),G\right),\mathcal{B},\mathbb{E}^{\mathsf{HF}}_{M,vol_{|M}}\right)_{M\subset\mathbb{R}^2},\left(\rho_{M,M'}\right)_{M\subset\mathsf{Int}(M')}\right\}.$$

We will denote it by $\mathbb{E}^{\mathsf{HF}}_{vol}$. *This random holonomy field is gauge-invariant.*

Using the Lemma 12.1, for any embedded planar graph $\mathbb{G}$, we have an explicit formula for the restriction on $\mathcal{M}ult(P(\mathbb{G}),G)$ of the the free boundary condition expectation on the plane associated with a Yang-Mills field. Yet, we saw in Proposition 3.1, that any

finite planar graph $\mathbb{G}'$ can be seen as a subgraph of an embedded planar graph. It is thus possible to give an explicit formula for the restriction on $\mathcal{M}ult(P(\mathbb{G}'),G)$ of the free boundary condition expectation on the plane associated with a Yang-Mills field.

Proposition 12.4. — *Suppose that $\mathbb{R}^2$ is endowed with a measure of area vol. Let $\mathbb{G}=(\mathbb{V},\mathbb{E},\mathbb{F})$ be a finite planar graph. Let $Y=(Y_t)_{t\geq 0}$ be a G-valued admissible Lévy process with associated semigroup of densities $(Q_t)_{t\geq 0}$. Let YM be the Yang-Mills field associated with Y. The free boundary condition expectation on $\mathbb{R}^2$ satisfies:*

$$\big(\mathbb{E}^{\mathsf{YM}}_{vol}\big)_{|\mathcal{M}ult(P(\mathbb{G}),G)}(dh)=\prod_{F\in\mathbb{F}^b}Q_{vol(F)}\big(h(\partial F)\big)\mathsf{U}^{\mathbb{G}}(dh),$$

where ∂F is the anti-clockwise oriented facial cycle associated with F and where we used the same convention as before for $Q_{vol(F)}\big(h(\partial F)\big)$.

In order to simplify the proof, we will use the upcoming Theorem 12.4.

Proof. — We have already seen in the proof of Proposition 8.5 that

$$\left(\prod_{F\in\mathbb{F}^b}Q_{vol(F)}\big(h(\partial F)\big)\mathsf{U}^{\mathbb{G}}(dh)\right)_{\mathbb{G},vol}$$

is a stochastically continuous in law weak discrete planar Markovian holonomy field. Using Theorem 12.4 and the constructibility result of Section 5, $\Big(\big(\mathbb{E}^{\mathsf{YM}}_{vol}\big)_{|\mathcal{M}ult(P(\mathbb{G}),G)}\Big)_{\mathbb{G},vol}$ is a stochastically continuous in law weak discrete planar Markovian holonomy field. Using Theorem 10.7, we only have to check that for any positive real α, $\big(h(L_{n,0})\big)_{n\in\mathbb{N}}$ has the same law under $\mathbb{E}^{\mathsf{YM}}_{\alpha dx}$ as under $\prod_{F\in\mathbb{F}^b}Q_\alpha\big(h(\partial F)\big)\mathsf{U}^{\mathbb{N}^2}(dh)$, where $\mathbb{F}^b$ is the set of bounded faces of the $\mathbb{N}^2$ graph. The value of α will not be important, so we will suppose that $\alpha=1$. In fact, we will prove that for any positive integer n, $\big(\mathbb{E}^{\mathsf{YM}}_{dx}\big)_{|\mathcal{M}ult(P(\mathbb{G}_n),G)}(dh)=\prod_{F\in\mathbb{F}^b_n}Q_1(h(\partial F))\mathsf{U}^{\mathbb{G}_n}(dh)$, where $\mathbb{G}_n=(\mathbb{V}_n,\mathbb{E}_n,\mathbb{F}_n)=\mathbb{N}^2\cap\big([0,n]\times[0,1]\big)$.

Let $\partial D(0,n+1)$ be the loop based at $(n+1,0)$ turning anti-clockwise, representing the cycle bounding the disk of radius $n+1$ centered at $(0,0)$. We define the graph $\mathbb{G}'_n=(\mathbb{V}'_n,\mathbb{E}'_n,\mathbb{F}'_n)$ by:

- $\mathbb{E}'_n=\mathbb{E}_n\cup\big\{\partial D(0,n+1),\partial D(0,n+1)^{-1},e^r_{n,0},(e^r_{n,0})^{-1}\big\}$,
- $\mathbb{V}'_n=\mathbb{V}_n\cup\big\{(n+1,0)\big\}$.

The finite planar graph $\mathbb{G}'_n$ is an embedded graph and $\mathbb{G}_n$ is a subgraph of $\mathbb{G}'_n$. Using Lemma 12.1, we know that:

$$\big(\mathbb{E}^{\mathsf{YM}}_{dx}\big)_{|\mathcal{M}ult(P(\mathbb{G}'_n),G)}(dh)=\prod_{F\in\mathbb{F}'^b_n}Q_{dx(F)}\big(h(\partial F)\big)\mathsf{U}^{\mathbb{G}'_n}(dh).$$

Since $\big(\prod_{F\in\mathbb{F}^b}Q_{vol(F)}\big(h(\partial F)\big)\mathsf{U}^{\mathbb{G}}(dh)\big)_{\mathbb{G},vol}$ is a weak discrete planar Markovian holonomy field, the restriction of $\prod_{F\in\mathbb{F}'^b_n}Q_{dx(F)}\big(h(\partial F)\big)\mathsf{U}^{\mathbb{G}'_n}(dh)$ to $\mathcal{M}ult(P(\mathbb{G}_n),G)$ is equal

to $\prod_{F\in\mathbb{F}_n^b} Q_1(h(\partial F))\mathsf{U}^{\mathbb{G}_n}(dh)$. This implies the equality $\left(\mathbb{E}^{\mathsf{YM}}_{dx}\right)_{|\mathcal{M}ult(P(\mathbb{G}_n),G)}(dh) = \prod_{F\in\mathbb{F}_n^b} Q_1(h(\partial F))\mathsf{U}^{\mathbb{G}_n}(dh)$. □

In particular, this last proposition and Proposition 8.5 show that the free boundary condition expectation on $\mathbb{R}^2$ of a Yang-Mills field associated with an admissible Lévy process Y is the planar Yang-Mills field associated with Y. This implies the following result.

Proposition 12.5. — *Let* YM *be the Yang-Mills field associated with an admissible Lévy process* $Y = (Y_t)_{t\in\mathbb{R}^+}$. *For any planar graph* $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$, *any measure of area vol, any family of facial loops* $(c_F)_{F\in\mathbb{F}^b}$ *oriented anti-clockwise and any rooted spanning tree* T *of* $\mathbb{G}$, *under the free boundary condition on the plane* $\mathbb{E}^{\mathsf{YM}}_{vol}$, *the random variables* $\left(h\left(\mathbf{l}_{c_F,T}\right)\right)_{F\in\mathbb{F}^b}$ *are independent and for any* $F\in\mathbb{F}^b$, $h(\mathbf{l}_{c_F,T})$ *has the same law as* $Y_{vol(F)}$.

12.7.3. Free boundary condition expectation on $\mathbb{R}^2$ *as a planar Markovian holonomy field.* — The free boundary condition expectation on $\mathbb{R}^2$ is interesting as it allows us to link the theory of Markovian holonomy fields with the one of planar Markovian holonomy fields. Consider HF a regular Markovian holonomy field and let $\left(\mathbb{E}^{\mathsf{HF}}_{vol}\right)_{vol}$ be the free boundary condition expectation on the plane associated with HF.

Theorem 12.4. — *The family* $\left(\mathbb{E}^{\mathsf{HF}}_{vol}\right)_{vol}$ *is a stochastically continuous strong planar Markovian holonomy field.*

Using the theory of planar Markovian holonomy fields we developed, it is enough to show that for any *vol*, $\mathbb{E}^{\mathsf{HF}}_{vol}$ is stochastically continuous and that its restriction to $P(\mathsf{Aff}(\mathbb{R}^2))$ is a stochastically continuous weak planar Markovian holonomy field. As we have already checked the weight condition and as we have noticed the gauge-invariance of the free boundary condition expectation in Definition 12.13, it is enough to show that it is stochastically continuous and that the Axioms $\mathbf{wP_1}$, $\mathbf{wP_2}$ and $\mathbf{wP_3}$ in Definition 4.2 hold. These are proved in Lemmas 12.2, 12.3, 12.4 and 12.5.

Lemma 12.2. — *For any measure of area vol,* $\mathbb{E}^{\mathsf{HF}}_{vol}$ *is a stochastically continuous random holonomy field.*

Proof. — Let *vol* be a measure of area, let p_n be a sequence of paths which converges, as n goes to infinity, to a path p for the convergence with fixed endpoints. Let $\mathbb{D}$ be a disk centered at $(0,0)$ such that for any integer n, $p_n\in\mathbb{D}$. We remind the reader that $\widehat{\mathsf{HF}}_{\left(\mathbb{D},vol_{|\mathbb{D}},\emptyset,\partial\mathbb{D}\to[x]\right)}$ is the extension given by Proposition 2.2 of $\mathsf{HF}_{\left(\mathbb{D},vol_{|\mathbb{D}},\emptyset,\partial\mathbb{D}\to[x]\right)}$ on the Borel σ-field. By definition,

$$\begin{aligned}\mathbb{E}^{\mathsf{HF}}_{vol}\left[d_G(h(p_n),h(p)\right] &= \int_G \widehat{\mathsf{HF}}_{\left(\mathbb{D},vol_{|\mathbb{D}},\emptyset,\partial\mathbb{D}\to[x]\right)}\left[d_G(h(p_n),h(p))\right]dx\\ &= \int_G \mathsf{HF}_{\left(\mathbb{D},vol_{|\mathbb{D}},\emptyset,\partial\mathbb{D}\to[x]\right)}\left[\widehat{(d_G)}_{J_{\{p_n,p\}}}(h(p_n),h(p))\right]dx.\end{aligned}$$

Since p_n and p have the same endpoints, $J_{\{p_n,p\}}$ is equal to G^2 and its action on G^2 is given by: $(k_1,k_2)\bullet(g_1,g_2) = (k_2^{-1}g_1k_1, k_2^{-1}g_2k_1)$. The invariance of d_G, by right and left

translations, implies that $\widehat{(d_G)}_{J_{\{p_n,p\}}} = d_G$. This leads to:

$$\mathbb{E}^{\mathsf{HF}}_{vol}\big[d_G(h(p_n),h(p)\big] = \int_G \mathsf{HF}_{\left(\mathbb{D},vol_{|\mathbb{D}},\emptyset,\partial\mathbb{D}\to[x]\right)}\big[d_G(h(p_n),h(p))\big]dx.$$

Since HF is regular, it is stochastically continuous, thus we have:

$$\mathsf{HF}_{\left(\mathbb{D},vol_{|\mathbb{D}},\emptyset,\partial\mathbb{D}\to[x]\right)}\big[d_G(h(p_n),h(p))\big] \underset{n\to\infty}{\longrightarrow} 0.$$

Thus, with an argument of dominated convergence, $\mathbb{E}^{\mathsf{HF}}_{vol}\big[d_G(h(p_n),h(p))\big]$ converges to zero as n goes to infinity. □

Lemma 12.3. — *The family of random holonomy fields* $\left(\mathbb{E}^{\mathsf{HF}}_{vol}\right)_{vol}$ *satisfies the area-preserving diffeomorphisms at infinity invariance* $\mathbf{wP_1}$.

Proof. — Consider *vol* and *vol*′ two measures of area on $\mathbb{R}^2$. Let ψ be a diffeomorphism at infinity which preserves the orientation and let R be a positive real such that:

1. $vol' = vol \circ \psi^{-1}$,
2. $\psi : \mathbb{D}(0,R)^c \to \psi\big(\mathbb{D}(0,R)^c\big)$ is a diffeomorphism.

Using the gauge-invariance of $\mathbb{E}^{\mathsf{HF}}_{vol}$, it is enough to consider loops based at the same point. Let $l_1,\dots,l_n$ be loops in $\mathsf{Aff}\left(\mathbb{R}^2\right)$ based at the same point such that for any $i \in \{1,\dots,n\}$, $l'_i = \psi(l_i)$ is in $\mathsf{Aff}\left(\mathbb{R}^2\right)$. Let R' be a positive real such that R' is greater than R and such that for any $i \in \{1,\dots,n\}$, l_i is in $M_{R'} = \mathbb{D}(0,R')$. The set $M' = \psi\left(M_{R'}\right)$ is a connected compact two-dimensional sub-manifold of $\mathbb{R}^2$. Let us consider $f : G^n \to \mathbb{R}$, a continuous function invariant by diagonal conjugation. We have the following equalities:

$$\begin{aligned}
\mathbb{E}^{\mathsf{HF}}_{vol}\Big[f\big((h(l_i)^n_{i=1})\big)\Big] &= \mathbb{E}^{\mathsf{HF}}_{M_{R'},vol_{|M_{R'}}}\Big[f\big((h(l_i)^n_{i=1})\big)\Big] \\
&= \int_G \int_{\mathcal{M}ult(P(M_R),G)} f\Big(\big(h(l_i)^n_{i=1}\big)\Big)\mathsf{HF}_{\left(M_{R'},vol_{|M_{R'}},\emptyset,\partial M_R\to[x]\right)}(dh)dx \\
&= \int_G \int_{\mathcal{M}ult(P(M'),G)} f\Big(\big(h(l'_i)^n_{i=1}\big)\Big)\mathsf{HF}_{\left(M',vol'_{|M'},\emptyset,\partial M'\to[x]\right)}(dh)dx \quad \mathbf{A}_4 \\
&= \mathbb{E}^{\mathsf{HF}}_{M',vol'_{|M'}}\Big[f\big((h(l'_i)^n_{i=1})\big)\Big] = \mathbb{E}^{\mathsf{HF}}_{vol'}\Big[f\big((h(l'_i)^n_{i=1})\big)\Big].
\end{aligned}$$

Thus the Axiom $\mathbf{wP_1}$ is satisfied by $\left(\mathbb{E}^{\mathsf{HF}}_{vol}\right)_{vol}$. □

Lemma 12.4. — *The family of random holonomy fields* $\left(\mathbb{E}^{\mathsf{HF}}_{vol}\right)_{vol}$ *satisfies the weak independence property* $\mathbf{wP_2}$.

Proof. — Let *vol* be a measure of area on $\mathbb{R}^2$. Let l and l' be two loops in $\mathsf{Aff}(\mathbb{R}^2)$ such that $\overline{\mathsf{Int}(l)} \cap \overline{\mathsf{Int}(l')} = \emptyset$. We can always consider $\tilde{l}$ and $\tilde{l}'$ two smooth simple loops in $\mathbb{R}^2$ such that the closure of their interiors are also disjoint and such that $l \subset \mathsf{Int}(\tilde{l})$ and $l' \subset \mathsf{Int}(\tilde{l}')$. Using this remark, we can suppose instead that l and l' are smooth. Using the gauge-invariance of $\mathbb{E}^{\mathsf{HF}}_{vol}$, as we did in order to show the Axiom $\mathbf{wDP_2}$ in the proof of Proposition 8.4, we can work with loops. Let us consider $l_1,\dots,l_n$ some loops

in $\overline{\mathsf{Int}(l)}$ and $l'_1, \dots, l'_m$ some loops in $\overline{\mathsf{Int}(l')}$. The aim is to prove that for any continuous functions f and g, from G^n, respectively G^m, to $\mathbb{R}$, we have:

$$\mathbb{E}^{\mathsf{HF}}_{vol}\Big[f\big((h(l_i))_{i=1}^n\big)g\big((h(l'_i))_{i=1}^m\big)\Big] = \mathbb{E}^{\mathsf{HF}}_{vol}\Big[f\big((h(l_i))_{i=1}^n\big)\Big]\mathbb{E}^{\mathsf{HF}}_{vol}\Big[g\big((h(l'_i))_{i=1}^m\big)\Big].$$

We will use the notations and result stated in Remark 2.6. Let $\mathcal{L}_0$ be a smooth loop such that $\mathcal{L}_0$ surrounds l and l'. We will make the following abuse of notation: depending on the context $\mathcal{L}_0$ we either stand for $\overline{\mathsf{Int}(\mathcal{L}_0)}$ or for the oriented cycle represented by $\mathcal{L}_0$. Besides, we will suppose that the orientation of $\mathcal{L}_0$ was chosen such that $\mathcal{L}_0 = \partial\overline{\mathsf{Int}(\mathcal{L}_0)}$. The same will hold for l and l'. Using the different axioms in Definition 12.5, we have:

$$\begin{aligned}
&\mathbb{E}^{\mathsf{HF}}_{dx}\Big[f\big((h(l_i))_{i=1}^n\big)g\big((h(l'_i))_{i=1}^m\big)\Big] = \mathbb{E}^{\mathsf{HF}}_{\mathcal{L}_0,dx}\Big[f\big((h(l_i))_{i=1}^n\big)g((h(l'_i))_{i=1}^m)\Big]\\
&= \int_G \int_{\mathcal{M}ult(P(\mathcal{L}_0),G)} \widehat{(f\otimes g)}_{J_{(l_i)_{i=1}^n,(l'_i)_{i=1}^m}}\big((h(l_i))_{i=1}^n,(h(l'_i))_{i=1}^m\big)\mathsf{HF}_{(\mathcal{L}_0,dx,\emptyset,\mathcal{L}_0\to[y])}(dh)dy\\
&= \int_G \int_{\mathcal{M}ult(P(\mathcal{L}_0),G)} \widehat{f}_{J_{(l_i)_{i=1}^n}}\big((h(l_i))_{i=1}^n\big)\ \widehat{g}_{J_{(l'_i)_{i=1}^m}}\big((h(l'_i))_{i=1}^m\big)\mathsf{HF}_{(\mathcal{L}_0,dx,\emptyset,\mathcal{L}_0\to[y])}(dh)dy\\
&= \int_{G^3} \int_{\mathcal{M}ult(P(\mathcal{L}_0),G)} \widehat{f}_{J_{(l_i)_{i=1}^n}}\big((h(l_i))_{i=1}^n\big)\ \widehat{g}_{J_{(l'_i)_{i=1}^m}}\big((h(l'_i))_{i=1}^m\big) \qquad \mathbf{A_3}\\
&\qquad\qquad \mathsf{HF}_{(\mathcal{L}_0,dx,\emptyset,\{\mathcal{L}_0\to[y],l'\to[z],l\to[w]\})}(dh)dydzdw\\
&= \int_{G^3} \int_{\mathcal{M}ult(P(l),G)} \widehat{f}_{J_{(l_i)_{i=1}^n}}\big((h(l_i))_{i=1}^n\big)\mathsf{HF}_{(l,dx,\emptyset,l\to[w])}(dh) \qquad \mathbf{A_6}\\
&\qquad\qquad \int_{\mathcal{M}ult(P(l'),G)} \widehat{g}_{J_{(l'_i)_{i=1}^m}}\big((h(l'_i))_{i=1}^n\big)\mathsf{HF}_{(l',dx,\emptyset,l'\to[z])}(dh) \qquad \mathbf{A_5}\\
&\qquad\qquad\qquad \int_{\mathcal{M}ult(P(\overline{\mathcal{L}_0\setminus(l\cup l')}),G)} \mathsf{HF}_{(\mathcal{L}_0,dx,\emptyset,\{\mathcal{L}_0\to[y],l'\to[z],l\to[w]\})}(dh)dydzdw.
\end{aligned}$$

Using the Axiom $\mathbf{A_7}$, $\int_G \int_{\mathcal{M}ult(P(\overline{\mathcal{L}_0\setminus(l\cup l')}),G)} \mathsf{HF}_{(\mathcal{L}_0,dx,\emptyset,\{\mathcal{L}_0\to[y],l'\to[z],l\to[w]\})}(dh)dy = 1$. Thus $\mathbb{E}^{\mathsf{HF}}_{dx}\Big[f\big((h(l_i))_{i=1}^n\big)g\big((h(l'_i))_{i=1}^m\big)\Big]$ is equal to

$$\begin{aligned}
&\int_G \int_{\mathcal{M}ult(P(l),G)} \hat{f}_{J_{(l_i)_{i=1}^n}}\big((h(l_i))_{i=1}^n\big)\mathsf{HF}_{(l,dx,\emptyset,l\to[w])}(dh)dw\\
&\qquad\qquad \int_G \int_{\mathcal{M}ult(P(l'),G)} \hat{g}_{J_{(l'_i)_{i=1}^m}}\big((h(l'_i))_{i=1}^m\big)\mathsf{HF}_{(l',dx,\emptyset,l'\to[z])}(dh)dz,
\end{aligned}$$

which is equal to $\mathbb{E}^{\mathsf{HF}}_{dx}\Big[f\big((h(l_i))_{i=1}^n\big)\Big]\mathbb{E}^{\mathsf{HF}}_{dx}\Big[g\big((h(l'_i))_{i=1}^m\big)\Big]$. □

Lemma 12.5. — *The family of random holonomy fields* $\big(\mathbb{E}^{\mathsf{HF}}_{vol}\big)_{vol}$ *satisfies the locality property* $\mathbf{wP_3}$.

Proof. — Let l be a simple loop, let vol and vol' be two measures of area whose restrictions to the closure of the interior of l are equal. The random holonomy fields $\mathbb{E}^{\mathsf{HF}}_{vol}$ and $\mathbb{E}^{\mathsf{HF}}_{vol'}$ being gauge invariant and stochastically continuous, by Proposition 2.4, we only have to prove, for any loops $l_1, \dots, l_n$ in $\mathsf{Int}(l)$ and for any continuous function

$f : G^n \to \mathbb{R}$ invariant under the diagonal action of G, that:

$$\mathbb{E}^{\mathsf{HF}}_{vol}\Big[f\big(h(l_1),\dots,h(l_n)\big)\Big] = \mathbb{E}^{\mathsf{HF}}_{vol'}\Big[f\big(h(l_1),\dots,h(l_n)\big)\Big].$$

Using Riemann's uniformization theorem, we can find a smooth curve $\tilde{l}$ in the interior of l such that $l_1,\dots,l_n$ are in the interior of $\tilde{l}$. Let M be the closure of the interior of $\tilde{l}$, we can write the following equalities:

$$\begin{aligned}\mathbb{E}^{\mathsf{HF}}_{vol}\Big[f\big(h(l_1),\dots,h(l_n)\big)\Big] &= \mathbb{E}^{\mathsf{HF}}_{M,vol_{|M}}\Big[f\big(h(l_1),\dots,h(l_n)\big)\Big]\\ &= \mathbb{E}^{\mathsf{HF}}_{M,vol'_{|M}}\Big[f\big(h(l_1),\dots,h(l_n)\big)\Big]\\ &= \mathbb{E}^{\mathsf{HF}}_{vol'}\Big[f\big(h(l_1),\dots,h(l_n)\big)\Big].\end{aligned}$$

This allows us to conclude. □

Remark 12.5. — *Using the same kind of calculations as the one explained in this subsection and using Theorem 12.1, it is easy to see that for any simple loop l, the law of $h(l)$ under $\mathbb{E}^{\mathsf{HF}}_{vol}$ is the law of $Y_{vol(\overline{\mathsf{Int}(l)})}$ where $(Y_t)_{t\in\mathbb{R}^+}$ is the Lévy process associated with* HF.

12.8. Proof of Theorem 12.3. — We have now all the tools in order to prove Theorem 12.3.

Proof of Theorem 12.3. — Let us remark that the second part about marks is a consequence of the first part by conditioning: we will prove the first assertion. Let $\big(\mathsf{HF}_{(M,vol,\mathcal{C},C)}\big)_{(M,vol,\mathcal{C},C)}$ be a regular Markovian holonomy field and $(Y_t)_{t\in\mathbb{R}^+}$ its associated G-valued Lévy process. Let $\big(\mathsf{YM}_{(M,vol,\mathcal{C},C)}\big)_{M,vol,\mathcal{C},C}$ be the Yang-Mills field associated with $(Y_t)_{t\in\mathbb{R}^+}$.

Let $\big(M, vol, \emptyset, C\big)$ be a measured marked surface with G-constraints, let l be a planar mark on M, let M_1 be a part of M of genius 0 determined by l and let m be a point in M_1. We want to prove that:

$$\Big(\mathsf{HF}_{(M,vol,\emptyset,C)}\Big)_{|\mathcal{M}ult(L_m(M_1),G)} = \Big(\mathsf{YM}_{(M,vol,\emptyset,C)}\Big)_{|\mathcal{M}ult(L_m(M_1),G)}.$$

For any loops $l_1,...,l_n$ in M_1 based at m and any continuous function f invariant by diagonal conjugation, $\int f(h(l_1),...,h(l_n))\mathsf{HF}_{(M,vol,\emptyset,C)}(dh)$ is equal to:

$$\int\int f(h(l_1),...,h(l_n))\mathsf{HF}_{(M_1,vol_{|M_1},\emptyset,C_{|\partial M_1\setminus\{l,l^{-1}\}}\cup\{l\to[x]\})}(dh)\\ \mathsf{HF}_{M_2,vol_{|M_2},\emptyset,C_{|\partial M_2\setminus\{l,l^{-1}\}}\cup\{l\to[x]\}}(\mathbb{1})dx$$

where M_2 is the second part of M determined by l. Using Theorem 12.1, HF and YM have the same partition functions. Thus, since l is a planar mark, it is enough to show that for any measure marked surface with G-constraints $(M, vol, \emptyset, C)$ such that M is homeomorphic to a sphere with a positive number p of holes,

$$\mathsf{HF}_{(M,vol,\emptyset,C)} = \mathsf{YM}_{(M,vol,\emptyset,C)}.$$

The proof can be made by induction on the number of holes: we will only prove the case where $p = 1$ since the arguments for the induction are similar. Let $\left(\mathbb{E}^{\mathsf{HF}}_{vol}\right)_{vol}$ be the free boundary condition expectation on the plane, defined in Definition 12.13, associated with HF. It is a stochastically continuous strong planar Markovian holonomy field as shown in Theorem 12.4. Hence, by Theorem 5.1, it induces a stochastically continuous in law weak discrete planar Markovian holonomy field $\left(\mathbb{E}^{\mathsf{HF},\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$. The Remark 12.5 ensures that the condition in order to apply Theorem 11.3 is satisfied by $\left(\mathbb{E}^{\mathsf{HF},\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$. It is equal to the pure discrete planar Yang-Mills field, denoted by $\left(\mathbb{E}^{Y,\mathbb{G}}_{vol}\right)_{\mathbb{G},vol}$, associated with the Lévy process $(Y_t)_{t\in\mathbb{R}^+}$. By stochastic continuity, for any measure of area vol, $\mathbb{E}^{\mathsf{HF}}_{vol} = \mathbb{E}^{Y}_{vol}$, where $\mathbb{E}^{Y}_{vol}$ is the pure continuous planar Yang-Mills field associated with $(Y_t)_{t\in\mathbb{R}^+}$.

Let $\left(\mathbb{E}^{\mathsf{YM}}_{vol}\right)_{vol}$ be the associated free boundary condition expectation on the plane associated with YM. Using Proposition 12.5, for any measure of area vol, $\mathbb{E}^{\mathsf{YM}}_{vol} = \mathbb{E}^{Y}_{vol}$. Recall the notation for the free boundary condition on a surface and let us consider a disk-shaped suface M endowed with a measure of area vol. The last two equalities imply that:

$$\mathbb{E}^{\mathsf{HF}}_{M,vol} = \mathbb{E}^{\mathsf{YM}}_{M,vol}.$$

Let t be equal to $vol(M)$. Using Definition 12.12:

$$\mathbb{E}^{\mathsf{HF}}_{M,vol} = \int_G \widehat{\mathsf{HF}}_{(M,vol,\emptyset,\{\partial M\to[x]\})}dx,$$

and a similar equation holds for YM. Thus, if we set $Z_t(x) = \mathsf{HF}_{(M,vol,\emptyset,\{\partial M\to[x]\})}(\mathbb{1})$, which is also equal to $\mathsf{YM}_{(M,vol,\emptyset,\{\partial M\to[x]\})}(\mathbb{1})$ and which is strictly positive, we can write:

$$\mathbb{E}^{\mathsf{HF}}_{M,vol} = \int_G \frac{\widehat{\mathsf{HF}}_{(M,vol,\emptyset,\{\partial M\to[x]\})}}{Z_t(x)} Z_t(x)dx.$$

Besides, the law of $h(\partial M)$ is $Z_t(g)dg$: it implies that $\frac{\widehat{\mathsf{HF}}_{(M,vol,\emptyset,\{\partial M\to[x]\})}}{Z_t(x)}$ is a disintegration of $\mathbb{E}^{\mathsf{HF}}_{M,vol}$ with respect to $h(\partial M)$. The same discussion holds for YM. By almost sure uniqueness of the disintegration we have:

$$\frac{\widehat{\mathsf{HF}}_{(M,vol,\emptyset,\{\partial M\to[x]\})}}{Z_t(x)} = \frac{\widehat{\mathsf{YM}}_{(M,vol,\emptyset,\{\partial M\to[x]\})}}{Z_t(x)}, \text{ a.s. in } x,$$

thus:

$$\widehat{\mathsf{HF}}_{(M,vol,\emptyset,\{\partial M\to[x]\})} = \widehat{\mathsf{YM}}_{(M,vol,\emptyset,\{\partial M\to[x]\})}, \text{ a.s. in } x. \tag{26}$$

It remains to remove the *a.s.* part. Using Proposition 2.4 and Lemma 2.3, we need to show that, for any continuous function f invariant by diagonal conjugation from G^n to G and any piecewise affine loops $l_1, ..., l_n$ in the interior of M (for any Riemannian metric) based at the same point:

$$\widehat{\mathsf{HF}}_{(M,vol,\emptyset,\{\partial M\to[x]\})}\big(f\big(h(l_1), ..., h(l_n)\big)\big) = \widehat{\mathsf{YM}}_{(M,vol,\emptyset,\{\partial M\to[x]\})}\big(f\big(h(l_1), ..., h(l_n)\big)\big).$$

Yet, given such n-tuple, we can always find a mark l such that $l_1, ..., l_n$ is in the interior of l. Thus, it is enough to show that for any mark l, for any $x \in G$, once we restrain the

measures on $\mathcal{M}ult(P(\overline{\mathsf{Int}(l)}), G)$, we have the equality:

$$\widehat{\mathsf{HF}}_{(M,vol,\emptyset,\{\partial M\to[x]\})} = \widehat{\mathsf{YM}}_{(M,vol,\emptyset,\{\partial M\to[x]\})}.$$

Let l be a mark on M and let us denote by M' the closure of the interior of l. Let us suppose that the orientation of l is such that $l = \partial M'$. Let us recall that $Z^+_{2,0,s}$ was the notation for the partition function of the regular Markovian holonomy field HF associated to the planar annulus of total volume which is equal to s. Applying the Axioms $\mathbf{A_3}$, $\mathbf{A_6}$ and $\mathbf{A_5}$, we get that for any continuous function f invariant by diagonal conjugation from G^n to G and any loops $l_1, ..., l_n$ in M':

$$\begin{aligned}
&\widehat{\mathsf{HF}}_{(M,vol,\emptyset,\{\partial M\to[x]\})}\Big(f\big(h(l_1),...,h(l_n)\big)\Big)\\
&\quad= \int f\big(h(l_1),...,h(l_n)\big)\mathsf{HF}_{(M,vol,\emptyset,\{\partial M\to[x]\})}(dh)\\
&\quad= \int_G\int f\big(h(l_1),...,h(l_n)\big)\mathsf{HF}_{(M,vol,l,\{\partial M\to[x],l\to[y]\})}(dh)dy && \mathbf{A_3}\\
&\quad= \int_G\int f\big(h(l_1),...,h(l_n)\big) && \mathbf{A_6},\mathbf{A_5}\\
&\qquad\qquad \mathsf{HF}_{\left(M',vol_{|M'},\emptyset,\{\partial M'\to[y]\}\right)}\otimes\mathsf{HF}_{\left(M\setminus\mathsf{Int}(M'),vol_{|M\setminus\mathsf{Int}(M')},\emptyset,\{\partial M\to[x],\partial M'\to[y]\}\right)}(dh)dy\\
&\quad= \int_G\int f\big(h(l_1),...,h(l_n)\big)\mathsf{HF}_{\left(M',vol_{|M'},\emptyset,\{\partial M'\to[y]\}\right)}(dh)Z^+_{2,0,vol(M\setminus\mathsf{Int}(M'))}(x,y^{-1})dy.
\end{aligned}$$

Thus, if we only consider the restriction on $\mathcal{M}ult\big(P(M'),G\big)$:

$$\widehat{\mathsf{HF}}_{(M,vol,\emptyset,\{\partial M\to[x]\})} = \int_G \widehat{\mathsf{HF}}_{\left(M',vol_{|M'},\emptyset,\{\partial M'\to[y]\}\right)}Z^+_{2,0,vol(M\setminus\mathsf{Int}(M'))}(x,y^{-1})dy.$$

Recall that the partition functions of HF and YM are equal. Thus:

$$\widehat{\mathsf{YM}}_{(M,vol,\emptyset,\{\partial M\to[x]\})} = \int_G \widehat{\mathsf{YM}}_{\left(M',vol_{|M'},\emptyset,\{\partial M'\to[y]\}\right)}Z^+_{2,0,vol(M\setminus\mathsf{Int}(M'))}(x,y^{-1})dy,$$

Once we restrain the measures on $\mathcal{M}ult\big(P(M'),G\big)$, using Equation (26), for any $x\in G$:

$$\begin{aligned}
\widehat{\mathsf{HF}}_{(M,vol,\emptyset,\{\partial M\to[x]\})} &= \int_G \widehat{\mathsf{HF}}_{\left(M',vol_{|M'},\emptyset,\{\partial M'\to[y]\}\right)}Z^+_{2,0,vol(M\setminus\mathsf{Int}(M'))}(x,y^{-1})dy\\
&= \int_G \widehat{\mathsf{YM}}_{\left(M',vol_{|M'},\emptyset,\{\partial M'\to[y]\}\right)}Z^+_{2,0,vol(M\setminus\mathsf{Int}(M'))}(x,y^{-1})dy\\
&= \widehat{\mathsf{YM}}_{(M,vol,\emptyset,\{\partial M\to[x]\})}.
\end{aligned}$$

This proves that the Equality (26) now holds for any x in G. □

Planar Objects
General Objects
Restriction
Strong
Weak
Free boundary expectation
Regular Markovian H.F.
Thm. 5.1
Thm. 12.4
Def. 12.13
Thm. 4.1
Weak discrete
Def. 12.9
Conj. 12.1
Partition function, Thm. 12.1
Thm. 10.1
Strong discrete
Planar Y.M.
Y.M. measures
Thm. 8.1
Thm. 8.1
Prop. 12.2
Lévy processes
Self-invariant by conjugation
Admissible
is

References

[1] S. Albeverio, R. Høegh-Krohn, and H. Holden. Random fields with values in Lie groups and Higgs fields. *Lecture notes in Physics, Editors: S. Albeverio, G. Casati, D. Merlini*, 262:1–13, 1986.

[2] S. Albeverio, R. Høegh-Krohn, and H. Holden. Stochastic Lie group-valued measures and their relations to stochastic curve integrals, gauge fields and Markov cosurfaces. *Stochastic processes—mathematics and physics, Editors: S. Albeverio, Ph. Blanchard, L. Streit*, 1986.

[3] S. Albeverio, R. Høegh-Krohn, and H. Holden. Stochastic Multiplicative Measures, Generalized Markov Semigroups, and Group-Valued Stochastic Processes and Fields. *Journal of function analysis*, 78:154–184, 1988.

[4] S. Albeverio, R. Høegh-Krohn, and H. Holden. Stochastic multiplicative measures, generalized Markov semigroups, and group-valued stochastic processes and fields. *J. Funct. Anal.*, 78(1):154–184, 1988.

[5] M. Anshelevich and A.N. Sengupta. Quantum free Yang-Mills on the plane. *Journal of Geometry and Physics*, 62:330–343, 2012.

[6] B. Aronov, R. Seidel, and D. Souvaine. On compatible triangulations of simple polygons. *CGTA: Computational Geometry: Theory and Applications*, 3:27–35, 1993.

[7] E. Artin. Theorie der Zöpfe. *Abh. Math. Sem. Hamburgischen Univ.*, 4:47–72, 1925.

[8] E. Artin. Theory of braids. *Annals of Mathematics*, 48(1):101–126, 1947.

[9] J.C. Baez. Generalized measures in gauge theory. *Letters in Mathematical Physics*, 31(3):213–223, 1994.

[10] G. Cébron, A. Dahlqvist, and F. Gabriel. The generalized master fields. *arxiv.org/abs/1601.00214*.

[11] B. Dacorogna and J. Moser. On a partial differential equation involving the Jacobian determinant. *Annales de l'I.H.P., section C*, 7(1):1–26, 1990.

[12] B.K. Driver. YM_2 : continuum expectations, lattice convergence, and lassos. *Comm. Math. Phys.*, 123(4):575–616, 1989.

[13] B.K. Driver. Two-dimensional Euclidean quantized Yang-Mills fields. In *Probability models in mathematical physics*, pages 21–36. World Sci. Publ., 1991.

[14] L. Gross. A Poincaré lemma for connection forms. *J. Funct. Anal.*, 63(1):1–46, 1985.

[15] L. Gross. The Maxwell equations for Yang-Mills theory. In *CMS Conf. Proc., Editors: J.S. Feldman and L.M. Rosen*, volume 9, pages 193–203. Amer. Math. Soc., 1988.

[16] O. Kallenberg. *Probabilistic symmetries and invariance principles*. Probability and its Applications. Springer, 2005.

[17] Y. Kawada and K. Itô. On the probability distribution on a compact group. *Proc. Phys.-Math. Soc. Japan*, 22(3):977–998, 1940.

[18] T. Lévy. The master field on the plane. *arxiv.org/abs/1112.2452*.

[19] T. Lévy. Construction et étude à l'échelle microscopique de la mesure de Yang-Mills sur les surfaces compactes. *C. R. Acad. Sci. Paris Ser. I Math.*, 330(11):1019–1024, 2000.

[20] T. Lévy. *Yang-Mills Measure on compact surfaces*. Number 790. American Mathematical Society, 2003.

[21] T. Lévy. *Two-dimensional Markovian holonomy fields*. Number 329. Astérisque, 2010.

[22] B. Mohar and C. Thomassen. *Graphs on surfaces*. Johns Hopkins University Press, 2001.

[23] A.N. Sengupta. The Yang-Mills measure for S^2. *J. Funct. Anal.*, 108(2):231–273, 1992.

[24] A.N. Sengupta. Gauge theory on compact surfaces. *Mem. Amer. Math. Soc.*, 126(600):viii+85, 1997.

[25] J.P. Serre, M. Buhler, and C. Goldstein. *Groupes finis : cours à l'E.N.S.J.F, 1978/1979*. Montrouge : Ecole normale supérieure de jeunes filles, ENSJF, 1979.

[26] K. Stromberg. Probabilities on a compact group. *Transactions of the American Mathematical Society*, 94(2):295–309, 1960.

[27] A. Vogt. The isoperimetric inequality for curves with self-intersections. *Canad. Math. Bull.*, 24:161–167, 1981.

[28] E. Witten. On quantum gauge theories in two dimensions. *Comm. Math. Phys.*, 1(1991):153–209, 141.

[29] E. Witten. Two-dimensional gauge theories revisited. *J. Geom. Phys.*, 9(4):303–368, 1992.

[30] C.N. Yang and R.L. Mills. Conservation of isotopic spin and isotopic gauge invariance. *Physical Rev.*, 96(2):191–195, 1954.

"It has long been an axiom of mine that the little things are infinitely the most important."

Sir Arthur Conan Doyle

PARTITIONS ET GÉOMÉTRIE

Abstract. — In this article, we define and study a new geometry and a new order on the set of partitions of an even number of objects. One of the definitions involves the partition algebra, a structure of algebra on the set of such partitions depending on an integer parameter N. Then we emulate the theory of random matrices in a combinatorial framework: for any parameter N, we introduce a family of linear forms on the partition algebras which allows us to define a notion of weak convergence similar to the convergence in moments in random matrices theory.

A renormalization of the partition algebras allows us to consider the weak convergence as a simple convergence in a fixed space. This leads us to the definition of a deformed partition algebra for any integer parameter N and to the definition of two transforms: the cumulants transform and the exclusive moments transform. Using an improved triangle inequality for the distance defined on partitions, we prove that the deformed partition algebras, endowed with a deformation of the linear forms converge as N go to infinity. This result allows us to prove combinatorial properties about geodesics and a convergence theorem for semi-groups of functions on partitions.

At the end we study a sub-algebra of functions on infinite partitions with finite support: a new addition operation and a notion of $\mathcal{R}$-transform are defined. We introduce the set of multiplicative functions which becomes a Lie group for the new addition and multiplication operations. For each of them, the Lie algebra is studied.

The appropriate tools are developed in order to understand the algebraic fluctuations of the moments and cumulants for converging sequences. This allows us to extend all the results we got for the zero order of fluctuations to any order.

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1. Introduction

This article is the first of a self-contained set of three articles [**8**], [**9**] and [**10**] on a combinatorial method in random matrices theory based on a geometry on partitions and a new point of view on usual/free cumulants based on dualities between groups and sub-algebras of partitions. This general method allows us to work with random matrices which are invariant by conjugation by the symmetric group instead of the unitary or orthogonal group, besides, no more assumption about the factorization of moments is needed. The first article is about the combinatorial framework based on the partition algebra. In the second article we will apply this framework to random matrices, and the third one will put the emphasis on the random walks on the symmetric group and the link with the $\mathfrak{S}_\infty$-Yang-Mills measure.

This set of articles has to be considered as the continuation of what could be called the Gauge Theory School in random matrices. The article of F. Xu [**19**] is one of the pioneer work about this point of view on random matrices. Later, this point of view was developed by A. Sengupta [**18**], then highly improved by T. Lévy [**13**], [**14**], then it was used by two students of T. Lévy: A. Dahlqvist in [**1**] and G. Cébron [**6**], [**5**].

We wrote these articles as a lecture for graduate students with the intention that no special requirement is needed to understand them. The reader will find a new presentation and introduction to the random matrices theory. To achieve this, we only used the Gauge Theory School's papers, the seminal article for partition algebras [**11**], and the book [**17**] which, in some sense, we tried partially to generalize. Another point of view on random matrices which are invariant by conjugation by the symmetric group was given first by C. Male in his paper on traffics [**15**]. Yet, the goal was to develop the ideas of the Gauge Theory School and thus we did not use this article. In the forthcoming article [**7**], the author and his coauthors build connections between the notions developed here and the notions developed in [**15**]. In some sense, these articles can be seen also a bridge to go from the book [**17**] to the traffic interpretation of [**15**], traffics which have shown their importance in the study of random graphs [**16**]. At the moment the author was finishing these articles, he was informed of M. Capitaine and M. Casalis's work, [**4**],

on their Schur-Weyl's interpretation of non-commutative free cumulants for unitary and orthogonal invariant random matrices.

The point of view developed in the three articles [**8**], [**9**] and [**10**] allows us to recover in a very simple way some famous theorems. The reader will also find in these articles a simple tool box in order to prove convergences of random matrices (for example random walks on the symmetric group). He will also find the tools in order to understand the algebraic fluctuations of moments of random matrices. Besides, this point of view allows us to define a general notion of freeness for matrices which are invariant by conjugation by the symmetric group and we construct the first non-commutative multiplicative Lévy processes for this notion of freeness. We will formulate two equivalent definitions of this freeness: one based on cumulants, and the other on moments. This freeness notion is linked with a new $\mathcal{R}$-transform which generalizes the old known $\mathcal{R}$-transform. A Kreweras complement is defined for partitions: this generalizes the notion already set for permutations. Amongst others, we will state a matricial Wick's theorem, which allows us to recover the Wick law for Gaussian Hermitian or symmetric matrices. We will also recover theorems about convergence of Hermitian Lévy processes proved in [**3**], [**2**] and unitary Lévy processes proved in [**5**]: we extend them to the symmetric and the orthogonal case. A new central limit theorem will be stated, which generalizes the non-commutative and the commutative central limit theorem. In the article [**9**], convergences of random walks on the symmetric group will be proved, and will be used in order to show that the Wilson loops of the $\mathfrak{S}_N$-Yang-Mills measure converge in probability when N goes to infinity. This will imply a result about some convergence of ramified coverings on the disk. We will also see how to inject the usual theory of probabilities in this framework. This last assertion shows that one could, in this framework, study the probabilistic fluctuations.

1.1. Renormalization and a physical point of view. — In this article, we emulate the theory of random matrices in a combinatorial framework. Given a partition p of a number of points, and an integer N, we consider (p, N) as a physical system involving N particles. When the number of points is even, by polarizing the points in two sets, we can consider (p, N) as a discrete time transformation operation. A partition p can be seen as an elementary evolution of a system of size N: we can define the composition of two partitions. Later in the paper, we consider these discrete-time transformations also as the Hamiltonian of continuous time transformations.

An evolution of a system of size N is a linear combination of elementary evolutions of size N. Thus, every transformation is uniquely characterized by a size N and by a finite number of coefficients which, as we will see in the article, are bare quantities. Two questions arise: how to describe a system of infinite size and how to renormalize the bare quantities. As one does for perturbative renormalization, the important idea is to consider observables: we define some observables, one for each partition. In Theorem 4.1, we show how the bare coefficient must be renormalized in order to have finite observables at the limit $N = \infty$.

Then, we show that, by using the same renormalization, the composition of two evolutions converge also: this is proved in Theorems 6.1 and 7.1. In Theorem 7.2, we consider continuous-time evolution transformations: we show that if the Hamiltonian is renormalized as we did for discrete time transformations, then the evolution converges.

In Theorem 10.2, we characterize the Hamiltonian so that the factorization property of large N holds.

We study also the development in powers of $1/N$ of systems of size N which converge to a continuous system.

The main novelty is to show that, even if one knows how to renormalize the bare constants, it does not seem interesting to define a vector space of infinite systems since all systems considered are defined in the same vector space whose basis is the set of partitions of $2k$ elements. In order to have an interesting space of infinite systems, one has to consider a renormalization of the algebras in which are defined the N-dimensional systems: the limit defines a non-trivial algebra in which one can study continuous evolutions of continuous systems.

Let us remark that a consequence of our results is that, in our toy-model, given a continuous system, one has canonically a sequence of approximations by systems involving N particles.

1.2. Layout of the article. — Using the set $\mathcal{P}_k$ of partitions of $2k$ elements as basis, one can define an algebra known as the partition algebra which definition depends on a parameter $N \in \mathbb{N}$: the partition algebra $\mathbb{C}[\mathcal{P}_k(N)]$. For a comprehensive study of this algebra, we recommend the article [**11**]. The main definitions are set in Section 2.

In Section 3, we define a geometry on the set of partitions of $2k$ elements which generalizes a well-known geometry on the symmetric group $\mathfrak{S}_k$. This geometry is defined by constructing a kind of Cayley graph for $\mathcal{P}_k$. This geometry allows to define a new order on $\mathcal{P}_k$ for which we construct the Hasse diagram and we compute the Möbius function. Using this new geometry, in Section 4 we define two notions of convergence of sequences which are shown to be equivalent. We define the notion of coordinate numbers, normalized moments, exclusive coordinate numbers and exclusive normalized moments. One of the results that we prove is that exclusive coordinate numbers and exclusive normalized moments are equal. In Section 5, a new deformed partition algebra is defined: $\mathbb{C}[\mathcal{P}_k(N, N)]$. These algebras are shown to converge to a new algebra: this is obtained by an improvement of the triangle inequality proved in Section 6 for the distance defined on the set of partitions of $2k$ elements. Let us remark that we define in the same section a Kreweras complement for partitions which generalizes the notion for permutations. We use these results in Section 7 in order to show that the multiplication is continuous for the notion of convergence of elements of $\prod_{N\in\mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$. We also study the convergence of semi-groups in $\prod_{N\in\mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$. In Section 8, using the convergence of sequences defined in Section 4, we show how one can prove combinatorial results, for example, a new proof of the improved triangle inequality is given.

In Section 9, we develop the notion of algebraic fluctuations, and extend the results already proved for the zero order of fluctuations to any order.

In Section 10, we construct an algebra $\mathfrak{E}[\mathcal{P}]$ which elements are functions on $\cup_{k\in\mathbb{N}}\mathcal{P}_k$. This algebra can be endowed with two special laws: $\boxplus$ and $\boxtimes$. We study two subgroups of $\mathfrak{E}[\mathcal{P}]$ associated with the operations $\boxplus$ and $\boxtimes$, the group of multiplicative invertible elements. These groups are Lie groups, the Lie algebras of these groups are studied. We also define the $\mathcal{R}_A$-transform, which generalizes the usual $\mathcal{R}$-transform and we define

two other transformations linked with the notion of exclusive moments. To finish the article, we extend these definitions to the setting of higher order fluctuations.

2. Partition algebra

2.1. First definitions. — Let k and N be two positive integers. We will consider three different algebras $\mathbb{C}[\mathfrak{S}_k], \mathbb{C}[\mathcal{B}_k(N)], \mathbb{C}[\mathcal{P}_k(N)]$: respectively the symmetric algebra, the Brauer algebra, and the partition algebra. These algebras satisfy the inclusions:

$$\mathbb{C}[\mathfrak{S}_k] \subset \mathbb{C}[\mathcal{B}_k(N)] \subset \mathbb{C}[\mathcal{P}_k(N)].$$

Thus, we will first construct $\mathbb{C}[\mathcal{P}_k(N)]$ and we will see the two other algebras as subalgebras of $\mathbb{C}[\mathcal{P}_k(N)]$. The reference article for the partition algebra is the article [**11**] of T.Halverson and A.Ram.

Let us consider $2k$ elements which we denote by: $1, \dots, k$ and $1', \dots, k'$. We define $\mathcal{P}_k$ as the set of set partitions of $\{1, \dots, k\} \cup \{1', \dots, k'\}$. If $k = 0$, we consider $\mathcal{P}_k = \{\emptyset\}$. Let p be an element of $\mathcal{P}_k$. We will denote by $p^1, \dots, p^r$ the blocks in p. The number of connected components $\mathsf{nc}(p)$, the propagating number $\mathsf{pn}(p)$ and the length $\mathsf{l}(p)$ of p are defined respectively by:

$$\begin{aligned} \mathsf{nc}(p) &= r, \\ \mathsf{pn}(p) &= \#\left\{i, p^i \text{ contains both an element of } \{1, \dots, k\} \text{ and one of } \{1', \dots, k'\}\right\}, \\ \mathsf{l}(p) &= k. \end{aligned}$$

Any partition $p \in \mathcal{P}_k$ can be represented by a graph. For this we consider two rows: k vertices are in the top row, labeled by 1 to k from left to right and k vertices are in the bottom row, labeled from $1'$ to k' from left to right. Any edge between two vertices means that the labels of the two vertices are in the same block of the partition p. Examples are given in Figure 1 and 2.

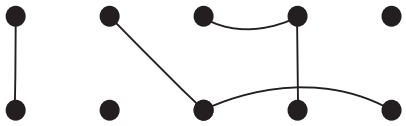

FIGURE 1. Partition $p_1 = \{\{1', 1\}\{2'\}\{2, 3', 5'\}\{3, 4, 4'\}\{5\}\}$.

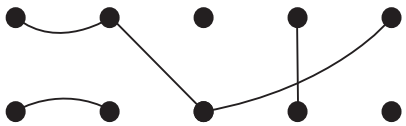

FIGURE 2. Partition $p_2 = \{\{1', 2'\}\{1, 2, 3', 5\}\{3\}\{4', 4\}\{5'\}\}$.

The notion of tensor product of partitions will be also very useful.

Definition 2.1. — *Let k and l be two positive integers. Let p be an element of $\mathcal{P}_k$ and let p' be an element of $\mathcal{P}_l$. Let us consider two diagrams: one associated with p, another with p'. Let $p \otimes p'$ be the partition in $\mathcal{P}_{k+l}$ associated with the diagram where one has put the diagram associated with p on the left of the diagram associated with p'.*

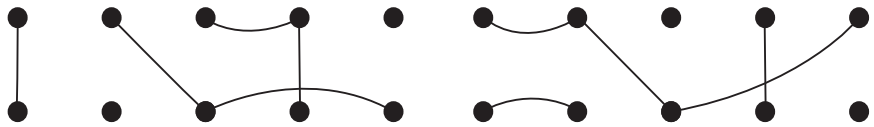

FIGURE 3. Partition $p_1 \otimes p_2$.

Let p_1 and p_2 be two elements of $\mathcal{P}_k$. We say that p_1 is *coarser* than p_2 if any two elements which are in the same block of p_2 are also in the same block of p_1. This order is directed: for any partitions p_1 and p_2 in $\mathcal{P}_k$ there exists a third partition p_3 which is coarser than p_1 and p_2. For example, one can consider the partition $p_1 \vee p_2$ defined as follows.

Definition 2.2. — *We define $p_1 \vee p_2$ as the partition in $\mathcal{P}_k$ such that for any $i, j \in \{1, \dots, k\} \cup \{1', \dots, k'\}$, i and j are in the same block of $p_1 \vee p_2$ if and only if there exists $i = x_0, x_1, \dots, x_l = j$ such that for any $n \in \{0, \dots, l-1\}$, $x_n \in \{1, \dots, k\} \cup \{1', \dots, k'\}$ and the two elements x_n and x_{n+1} are in the same block of either p_1 or p_2.*

It is always interesting to have a graphical representation for the operations defined on partitions. One can recover a diagram representing $p_1 \vee p_2$ by putting a diagram representing p_2 over one representing p_1.

FIGURE 4. Two diagrams which represent $p_1 \vee p_2 = \{\{1, 1', 2, 2', 3', 5, 5'\}, \{3, 4, 4'\}\}$.

We will need also later to take the infimum of p and p'

Definition 2.3. — *Let p_1 and p_2 be two elements of $\mathcal{P}_k$. We define $p_1 \wedge p_2$ as the partition in $\mathcal{P}_k$ such that for any $i, j \in \{1, ..., k\} \cup \{1', ..., k'\}$, i and j are in the same block of $p_1 \wedge p_2$ if and only if they are in the same block of p_1 and in the same block of p_2.*

Let us play a little with the graphical representation of p_1 and p_2 in order to define other natural operations on the set of partitions.

We will use later the transposition of a partition: it is the partition obtained by permuting the role of $\{1, \dots, k\}$ and $\{1', \dots, k'\}$. For example if $k = 3$, let $p = \big\{\{1, 1', 3'\}, \{2, 3\}, \{2'\}\big\}$, then ${}^t p = \big\{\{1', 1, 3\}, \{2', 3'\}, \{2\}\big\}$. For every diagram associated with p, the diagram obtained by flipping it according to a horizontal axis is a diagram associated with ${}^t p$. One can find an example in Figure 5

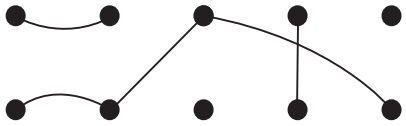

FIGURE 5. Partition ${}^t p_2$

An other thing we can do is to put one diagram representing p_2 above one diagram representing p_1. Let us identify the lower vertices of p_2 with the upper vertices of p_1. We obtain a graph with vertices on three levels, then erase the vertices in the middle row, keeping the edges obtained by concatenation of edges passing through the deleted vertices. Any connected component entirely included in the middle row is then removed. Let us denote by $\kappa(p_1, p_2)$ the number of such connected components. We obtain an other diagram associated with a partition denoted by $p_1 \circ p_2$. For any elements p_1 and p_2 of $\mathcal{P}_k$, the partition $p_1 \circ p_2$ does not depend on the choice of diagrams representing the partitions p_1 and p_2.

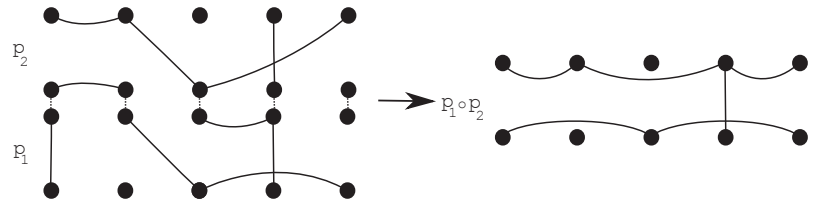

FIGURE 6. Partition $p_1 \circ p_2$.

The set of Brauer elements and the set of permutations will be stable by this operation of concatenation.

Definition 2.4. — *The set of Brauer elements $\mathcal{B}_k$ is the set of pair partitions in $\mathcal{P}_k$. The set of permutation $\mathfrak{S}_k$ is the set of pair partitions in $\mathcal{P}_k$ whose propagating number is equal to k.*

For any p_1 and p_2 in $\mathcal{B}_k$ (resp. $\mathfrak{S}_k$), $p_1 \circ p_2 \in \mathcal{B}_k$ (resp. $\mathfrak{S}_k$). Let us define the three algebras $\mathbb{C}[\mathfrak{S}_k], \mathbb{C}[\mathcal{B}_k(N)]$ and $\mathbb{C}[\mathcal{P}_k(N)]$.

Definition 2.5. — *The partition algebra $\mathbb{C}[\mathcal{P}_k(N)]$ is the associative algebra over $\mathbb{C}$ with basis $\mathcal{P}_k$ endowed with the multiplication defined by:*

$$\forall p_1, p_2 \in \mathcal{P}_k,\ p_1 p_2 = N^{\kappa(p_1,p_2)}(p_1 \circ p_2).$$

The Brauer algebra $\mathbb{C}[\mathcal{B}_k(N)]$ (resp. symmetric algebra $\mathbb{C}[\mathfrak{S}_k]$) is the sub-algebra of $\mathbb{C}[\mathcal{P}_k(N)]$ generated by the elements of $\mathcal{B}_k$ (resp. the elements of $\mathfrak{S}_k$).

Notation 2.1. — *In all the paper, A_k will stand either for $\mathcal{P}_k$ or $\mathcal{B}_k$ or $\mathfrak{S}_k$. Thus for any $N \in \mathbb{N}$, $\mathbb{C}[A_k(N)]$ will stand for $\mathbb{C}[\mathcal{P}_k(N)]$, $\mathbb{C}[\mathcal{B}_k(N)]$ or $\mathbb{C}[\mathfrak{S}_k(N)]$.*

Let us remark that actually, the algebra $\mathbb{C}[\mathfrak{S}_k(N)]$ does not depend on N. We can see any permutation $\sigma \in \mathfrak{S}_k$ as a bijection from $\{1, \ldots, k\}$ to itself: for any $i \in \{1, \ldots, k\}$ there exists a unique $j \in \{1', \ldots, k'\}$ such that $\{i, j'\} \in \sigma$, we set $\sigma(i) = j$. For any

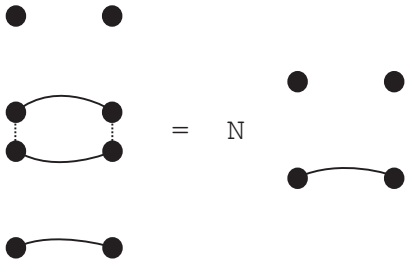

FIGURE 7. Example of a product which involves the counting of loops.

permutations σ_1 and σ_2, the bijection associated with $\sigma_1\sigma_2$ is the composition of the two bijections associated with σ_1 and σ_2.

We can extend the operations of transposition, tensor product and multiplication on the partition algebra, by linearity or bi-linearity.

The sub-algebra $\mathbb{C}[\mathfrak{S}_k]$ is not only stable for the $\circ$ operation. It also satisfies the following property which can be proved by looking at the propagating number.

Lemma 2.1. — *Let $p, p' \in \mathcal{P}_k$, if $p \circ p' \in \mathfrak{S}_k$ then p and p' are in $\mathfrak{S}_k$.*

Besides, for any partition $\sigma \in \mathfrak{S}_k$ and any $p \in A_k$, $\kappa(\sigma, p) = \kappa(p, \sigma) = 0$. Let us remark that, for any integer N, the algebras $\mathbb{C}[A_k(N)]$ have the same neutral element, denoted by id_k or id, for the product operation:

$$id_k = \big\{\{i, i'\}, i \in \{1, \dots, k\}\big\},$$

whose diagram for $k = 5$ is drawn in Figure 8. A consequence of Lemma 2.1 is that, as $id_k \in \mathfrak{S}_k$, the only invertible elements of $A_k(N)$, for the multiplication operation, are the permutations. The inverse of a permutation σ is $\sigma^{-1} = {}^t\sigma$.

FIGURE 8. The neutral element id_5.

We will later need some special permutations.

Definition 2.6. — *Let $I \subset \{1, \dots, k\}$: $I = \{i_1, \dots, i_l\}$ with $i_1 < \cdots < i_l$. We define σ_I the permutation which sends i_j on j for any $j \in \{1, \dots, l\}$ and $i \notin I$ on $l+i-\#\{n, i_n < i\}$. This is the partition:*

$$\sigma_I = \big\{\{i_j, j'\}, j \in \{1, \dots, l\}\big\} \cup \big\{\{i, (l + i - \#\{n, i_n < i\})'\}, i \notin I\big\}.$$

Definition 2.7. — *The transposition $(1, 2)$ is the partition $\sigma_{\{2\}}$ in $\mathcal{P}_2$ defined by:*

$$(1, 2) = \big\{\{1, 2'\}, \{2, 1'\}\big\}.$$

The Weyl contraction is the Brauer element in $\mathcal{P}_2$ defined by:

$$[1, 2] = \big\{\{1, 2\}, \{1', 2'\}\big\}.$$

These partitions are drawn in Figure 9.

FIGURE 9. The transposition $(1,2)$ and the Weyl contraction $[1,2]$.

Definition 2.8. — *Let i,j be two distinct integers in $\{1,\dots,k\}$. The transposition (i,j) in $\mathfrak{S}_k$ is:*

$$(i,j) = \sigma_{\{i,j\}}^{-1}\big((1,2)\otimes Id_{k-2}\big)\sigma_{\{i,j\}} = \{\{i',j\},\{i,j'\}\}\cup\{\{l,l'\},l\notin\{i,j\}\}.$$

The set of transpositions on k elements is:

$$\mathcal{T}_k = \big\{(i,j), i,j\in\{1,\dots,k\}, i\neq j\big\}.$$

The Weyl contraction $[i,j]$ in $\mathcal{B}_k$ is:

$$[i,j] = \sigma_{\{i,j\}}^{-1}\big([1,2]\otimes Id_{k-2}\big)\sigma_{\{i,j\}} = \{\{i,j\},\{i',j'\}\}\cup\{\{l,l'\},l\notin\{i,j\}\}.$$

Due to the remark we made after Lemma 2.1, the product does not depend on which $\mathbb{C}[\mathcal{B}_k(N)]$ one considers to define the product. We denote by $\mathcal{W}_k$ the set of Weyl contractions in $\mathcal{B}_k$:

$$\mathcal{W}_k = \big\{[i,j], i,j\in\{1,\dots,k\}, i\neq j\big\}.$$

A notion linked with the tensor operation, which will be central in the asymptotic freeness results in the article [**9**], is the notion of irreducibility of partitions. Let p be in $\mathcal{P}_k$.

Definition 2.9. — *A cycle of p is a block of $p\vee id$. The set of cycles of p is denoted by $\mathsf{C}(p)$. The number of cycles of p is denoted by $\mathsf{c}(p)$. The partition p is composed if $\mathsf{c}(p)>1$. The partition p is irreducible if it is not composed. By convention, the empty partition is irreducible.*

Let us consider the set of irreducible partitions.

Definition 2.10. — *We will denote by $A_k^{(i)}$ the set of irreducible partitions of A_k.*

It has to be noted that for any integer k:

$$\mathfrak{S}_k^{(i)} = \{\sigma^{-1}(1,\dots,k)\sigma, \sigma\in\mathfrak{S}_k\},$$

where $(1,...,k)\in\mathcal{S}_k$ is the k-cycle equal to $\sigma_{\{2,3,\dots,k\}}$.

The partition p is composed if and only if there exist p_1 and p_2 two partitions non equal to the empty partition, and I a subset of $\{1,\dots,k\}$ such that $\#I = \mathsf{l}(p_1)$, $\mathsf{l}(p_2) = k-\#I$ and:

$$\sigma_I^{-1}(p_1\otimes p_2)\sigma_I = p.$$

Let us define the decomposition of p into two partitions.

Definition 2.11. — *The set of decompositions of p into two partitions is:*

$$\mathfrak{F}_2(p) = \left\{(p_1, p_2, I), \sigma_I^{-1}(p_1 \otimes p_2)\sigma_I = p\right\}.$$

Let us remark that for any partition, even the irreducible partitions, $\mathfrak{F}_2(p) \neq \emptyset$. For example, if p is irreducible:

$$\mathfrak{F}_2(p) = \{(p, \emptyset, \{1, \ldots, k\}), (\emptyset, p, \emptyset)\}.$$

Let also remark that $\mathfrak{F}_2(\emptyset) = \{(\emptyset, \emptyset, \emptyset)\}$.

We will need a notion of weak irreducibility later: this is based on the notions of extraction and restriction. For any partition p we have a lot of choice in order to represent p as a graph: the complete graph which represents p is the graph such that i and j, two elements of $\{1, ..., k\} \cup \{1', ..., k'\}$ are linked if and only if i and j are in the same block of p.

Definition 2.12. — *Let J be a subset of $\{1, \ldots, k\} \cup \{1', \ldots, k'\}$. Let us denote by J^s the symmetrization of J:*

$$J^s = J \cup \{j \in \{1', \ldots, k'\}, \exists i \in J \cap \{1, \ldots, k\}, j = i'\} \cup \{i \in \{1, \ldots, k\}, i' \in J\}.$$

We define:

- *The* extraction *of p to J, denoted p_J. Let us take the complete graph which represents p, let us erase all the vertices which are not in J^s and all the edges which are not between two vertices in J^s and at last let us label the remaining vertices from left to right. This is the graph of p_J.*
- *The* restriction *of p to J, denoted $p_{|J}$. Let us take the complete graph which represents p, let us erase all the edges which are not between two vertices in J and let us connect each $i \notin J^s$ with i'. This is the graph of $p_{|J}$.*

By convention, if $J^s = \{1, \ldots, k\} \cup \{1', \ldots, k'\}$, then $p_J = \emptyset$ and $p_{|J} = id$.

Definition 2.13. — *The support of p is:*

$$\mathsf{S}(p) = \{1, \ldots, k\} \setminus \{i \in \{1, \ldots, k\}, \{i, i'\} \in p\}.$$

The partition p is weakly irreducible if $p_{\mathsf{S}(p)}$ is irreducible. In particular the permutation id_k is weakly irreducible.

2.2. Partitions and representation. — In this section, we define a natural action of the partition algebra (and by restriction of the Brauer and of the symmetric algebra) on $\left(\mathbb{C}^N\right)^{\otimes k}$. This action will be useful in order to translate combinatorial properties into linear algebraic properties.

Let N and k be two positive integers.

Definition 2.14. — *For any $p \in \mathcal{P}_k$ and any k-uples $(i_1, \ldots, i_k)$ and $(i_{1'}, \ldots, i_{k'})$ of elements of $\{1, \ldots, N\}$, we set:*

$$p_{i_{1'}, \ldots, i_{k'}}^{i_1, \ldots, i_k} = \begin{cases} 1, & \text{if for any two elements } r \text{ and } s \in \{1, \ldots, k\} \cup \{1', \ldots, k'\} \text{ which} \\ & \text{are in the same block of } p, \text{ one has } i_r = i_s, \\ 0, & \text{otherwise.} \end{cases}$$

We can now define the action of the partition algebra $\mathbb{C}[\mathcal{P}_k(N)]$ on $\left(\mathbb{C}^N\right)^{\otimes k}$. Let $(e_1, \ldots, e_N)$ be the canonical basis of $\mathbb{C}^N$.

Definition 2.15. — *The action of the partition algebra $\mathbb{C}[\mathcal{P}_k(N)]$ on $\left(\mathbb{C}^N\right)^{\otimes k}$ is defined by the fact that for any $p \in \mathcal{P}_k$, for any $(i_1, \ldots, i_k) \in \{1, \ldots, N\}^k$:*

$$p.(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{(i_{1'}, \ldots, i_{k'}) \in \{1, \ldots, N\}^k} p^{i_1, \ldots, i_k}_{i_{1'}, \ldots, i_{k'}} \; e_{i_{1'}} \otimes \cdots \otimes e_{i_{k'}}.$$

This action defines a representation of the partition algebra $\mathbb{C}[\mathcal{P}_k(N)]$ on $\left(\mathbb{C}^N\right)^{\otimes k}$ which we denote by $\rho_N^{\mathcal{P}_k}$:

$$\rho_N^{\mathcal{P}_k} : \mathbb{C}[\mathcal{P}_k(N)] \mapsto \mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right).$$

Let us define E_i^j be the matrix which sends e_j on e_i and any other element of the canonical basis on 0. Let p be a partition in $\mathcal{P}_k$. We can write the matrix of $\rho_N^{\mathcal{P}_k}(p)$ in the basis $(e_{i_1} \otimes \cdots \otimes e_{i_k})_{(i_l)_{l=1}^k \in \{1, \ldots, N\}^k}$:

$$\rho_N^{\mathcal{P}_k}(p) = \sum_{(i_1, \ldots, i_k, i_{1'}, \ldots, i_{k'}) / p^{i_1, \ldots, i_k}_{i_{1'}, \ldots, i_{k'}} = 1} E^{i_1}_{i_{1'}} \otimes \ldots \otimes E^{i_k}_{i_{k'}}. \tag{1}$$

For example, if p is the transposition $(1, 2)$, then:

$$\rho_N^{\mathcal{P}_2}((1,2)) = \sum_{a,b=1}^{N} E^b_a \otimes E^a_b.$$

We think that this presentation allows us to understand, in an easier way, the representation $\rho_N^{\mathcal{P}_k}$. We illustrate in Figure 10, how to find the partition which representation is given by a sum of the form (1). The partition p_1 used in Figure 10 is the partition drawn in Figure 1.

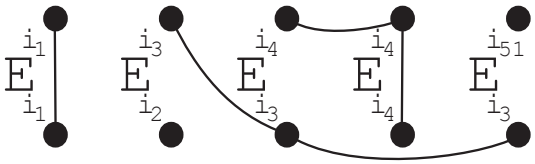

FIGURE 10. $\sum_{i_1, i_2, i_3, i_4, i_5} E^{i_1}_{i_1} \otimes E^{i_3}_{i_2} \otimes E^{i_4}_{i_3} \otimes E^{i_4}_{i_4} \otimes E^{i_5}_{i_3} = \rho_N^{\mathcal{P}_5}(p_1)$.

Let us suppose that $N \geq 2k$. Using Theorem 3.6 in [**11**], the application $\rho_N^{\mathcal{P}_k}$ is injective. Actually, if one considers only its restriction to the symmetric algebra or the Brauer algebra, it is enough to ask for $N \geq k$. For $N = k - 1$ this result does not hold, this is a consequence of the Mandelstam's identity which asserts that:

$$\sum_{\sigma \in \mathfrak{S}_k} (-1)^{\epsilon(\sigma)} \rho_{k-1}^{\mathcal{P}_k}(\sigma) = 0,$$

where $\epsilon(\sigma)$ is the signature of σ.

Let us remark that the natural action of $\mathbb{C}[\mathcal{P}_k(N)]$ on $\left(\mathbb{C}^N\right)^{\otimes k}$ behaves well under the operation of product tensor.

Lemma 2.2. — *Let k and k' be two positive integers. Let $p \in \mathbb{C}[\mathcal{P}_k]$ and $p' \in \mathbb{C}[\mathcal{P}_{k'}]$. We have for any integer N:*

$$\rho_N^{\mathcal{P}_{k+k'}}(p \otimes p') = \rho_N^{\mathcal{P}_k}(p) \otimes \rho_N^{\mathcal{P}_k}(p').$$

2.3. The exclusive basis of $\mathbb{C}[\mathcal{P}_k]$. — The basis used to define the partition algebra is quite natural, yet, it is not always very easy to work with. Indeed, if we look at the representation $\rho_N^{\mathcal{P}_k}$ of a partition, we see that the condition we used to define the delta function is not exclusive. It means that we did not use the following exclusive delta function:

$$(p_{i_{1'},\dots,i_{k'}}^{i_1,\dots,i_k})^{ex} = \begin{cases} 1, & \text{if for any two elements } r \text{ and } s \in \{1,\dots,k\} \cup \{1',\dots,k'\}, \\ & i_r = i_s \text{ if and only if } r \text{ and } s \text{ are in the same block of } p, \\ 0, & \text{otherwise.} \end{cases}$$

By changing in Definition 2.15 the delta function defined in Definition 2.14 by this new exclusive delta function, we define a new function:

$$\tilde{\rho}_N^{\mathcal{P}_k} : \mathbb{C}[\mathcal{P}_k(N)] \to \mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right).$$

Does it exist, for any partition $p \in \mathcal{P}_k$ an element $p^c \in \mathbb{C}[\mathcal{P}_k]$ such that for any integer N, $\rho_N^{\mathcal{P}_k}(p^c) = \tilde{\rho}_N^{\mathcal{P}_k}(p)$? The answer is given by the following definition, as explained by Equation (2.3) of [**11**].

Definition 2.16. — *We define the family $(p^c)_{p\in\mathcal{P}_k}$ as the only family of elements in $\mathbb{C}[\mathcal{P}_k]$ defined by the relation:*

$$p = \sum_{p' \text{ coarser than } p} p'^c.$$

The notion of being coarser defines a partial order on $\mathcal{P}_k$: the relation can be inverted. The family $(p^c)_{p\in\mathcal{P}_k}$ is well defined and it is a basis of the partition algebra $\mathbb{C}[\mathcal{P}_k]$. We will call $(p^c)_{p\in\mathcal{P}_k}$ the exclusive partition basis, it satisfies the following proposition.

Proposition 2.1. — *For any positive integers k and N, for any partition $p \in \mathcal{P}_k$,*

$$\rho_N^{\mathcal{P}_k}(p^c) = \tilde{\rho}_N^{\mathcal{P}_k}(p).$$

3. Geometry on the set of partitions

In this section, we define a new geometry on the set of partitions $\mathcal{P}_k$ which generalizes some well-known geometry on the symmetric group. We will see three ways to construct a distance on $\mathcal{P}_k$: one will allow us to work with linear algebra, another to compute the distance in a combinatorial way, and the last one will use a graph which we will consider as the generalized Cayley graph of $\mathcal{P}_k$.

Depending on the context, we will consider a partition either as an element of $\mathcal{P}_k$ or as an element of $\mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right)$ via the action defined in Definition 2.15. We remind the reader that $(e_1,\dots,e_N)$ is the canonical base of $\mathbb{C}^N$. The family $\{e_{i_1} \otimes \cdots \otimes$

$e_{i_k}, (i_1, \ldots, i_k) \in \{1, \ldots, N\}^k\}$ is a basis of $\left(\mathbb{C}^N\right)^{\otimes k}$: let Tr^k be the trace with respect to this canonical basis. We do not renormalize it, thus $Tr^k\left(Id_{(\mathbb{C}^N)^{\otimes k}}\right) = N^k$. We can define the trace of a partition.

Definition 3.1. — *Let k and N be two positive integers, let p be a partition in $\mathcal{P}_k$. We define:*

$$Tr_N(p) = Tr^k\left(\rho_N^{\mathcal{P}_k}(p)\right).$$

For any integer N, we extend Tr_N by linearity to $\mathbb{C}[\mathcal{P}_k(N)]$.

Let us remark that, if one does not want to use the representation $\rho_N^{\mathcal{P}_k}$, one could have also define the trace by defining for any partition $p \in \mathcal{P}_k$,

$$Tr_N(p) = N^{\mathsf{nc}(p \vee id)}. \tag{2}$$

3.1. Definition of the geodesic order. — We can now define a distance on $\mathcal{P}_k$.

Proposition 3.1. — *Let N be a positive integer, let p and p' be two elements of $\mathcal{P}_k$. The number:*

$$d(p, p') = -log_N\left(\frac{Tr_N({}^t pp')}{\sqrt{Tr_N({}^t pp)Tr_N({}^t p'p')}}\right)$$

does not depend on N: it is called the distance between p and p'.

The fact that $d(p, p')$ does not depend on N is a consequence of Lemma 3.1. Actually we have not proven yet that it is a distance, even if it is fairly easy to see that it satisfies the strict positivity property: it is a consequence of the Cauchy-Schwarz's inequality.

The easiest way to prove that $d(p, p')$ does not depend on N is to show that it is a combinatorial object.

Lemma 3.1. — *For any p and p' in $\mathcal{P}_k$:*

$$d(p, p') = \frac{1}{2}\big(\mathsf{nc}(p) + \mathsf{nc}(p')\big) - \mathsf{nc}(p \vee p').$$

Proof. — This is a consequence of the following equality which holds for any p and p' in $\mathcal{P}_k$ and any positive integer N:

$$Tr_N({}^t pp') = N^{\mathsf{nc}({}^t p \circ p' \vee id) + \kappa({}^t p, p')} = N^{\mathsf{nc}(p \vee p')}, \tag{3}$$

which is a consequence of Equality (2) and the combinatorial equality:

$$\mathsf{nc}({}^t p \circ p' \vee id) + \kappa({}^t p, p') = \mathsf{nc}(p \vee p').$$

This latter equality can be understood by flipping the diagram of ${}^t p$ over the one of p': the flip transposes ${}^t p$ thus we get the two diagrams of p and p' one over the other. By definition, the diagram constructed by putting a diagram representing p' over one representing p is associated with $p \vee p'$. □

It remains to show that d satisfies the triangle inequality on the set of partitions $\mathcal{P}_k$. For that we will show that it is a geodesic distance on a graph.

Definition 3.2. — *We define the weighted graph $\mathbb{G}_k = (\mathbb{V}_k, \mathbb{E}_k, w_k)$ such that:*

– *the set of vertices* $\mathbb{V}_k$ *is* $\mathcal{P}_k$,
– *there exists an edge* e *in* $\mathbb{E}_k$ *between* p *and* p' *two elements of* $\mathcal{P}_k$ *if and only if:*
  • *one can go from one to the other by gluing two blocks. Let us suppose that we can go from* p *to* p'*. If* p *is the partition* $\{p^1, \dots, p^r\}$ *then there exist* i *and* j*, distinct, such that* $p' = \{p^s, s \in \{1, \dots, r\} \setminus \{i, j\}\} \cup \{p^i \cup p^j\}$*. The weight of the edge* e *is set to* 0.5*:* $w_k(e) = 0.5$*.*
  • *one can go from one to the other by permuting two elements of* $\{1, \dots, k\} \cup \{1', \dots, k'\}$ *which are in distinct blocks. Let us suppose that we can go from* p *to* p' *by permuting two elements. In this case, if* p *is the partition* $\{p^1, \dots, p^r\}$*, there exist* $s, t \in \{1, \dots, k, 1', \dots, k'\}$ *distinct and* $i, j \in \{1, \dots, r\}$ *distinct, such that* $s \in p^i$*,* $t \in p^j$ *and* $p' = \{p^s, s \in \{1, \dots, r\} \setminus \{i, j\}\} \cup \{(p^i \setminus \{s\}) \cup \{t\}, (p^j \setminus \{t\}) \cup \{s\}\}$*. The weight of the edge* e *is set to* 1*:* $w_k(e) = 1$*.*

Remark 3.1. — *The graph* $\mathbb{G}_k$ *plays the role of the Cayley graph of* $\mathcal{P}_k$*. Actually, if one considers the subgraph* $\mathbb{S}_k$ *obtained by restraining it to the vertices which are permutations, one really obtains the Cayley graph of the symmetric group* $\mathfrak{S}_k$*. The Cayley graph* $\mathbb{B}_k$ *of* $\mathcal{B}_k$ *is defined as the restriction of* $\mathbb{G}_k$ *to the vertices which are in* $\mathcal{B}_k$*.*

We gave this definition so that the reader can understand easily why this graph is a generalization of the usual Cayley graph. Yet, there is an other graph which will be used in Proposition 3.2. Let us define $\mathbb{G}'_k = (\mathbb{V}'_k, \mathbb{E}'_k, w'_k)$ such that:

– the set of vertices $\mathbb{V}'_k$ is $\mathcal{P}_k$,
– there exists an edge in $\mathbb{E}'_k$ between p and p' two elements of $\mathcal{P}_k$ if and only if one can go from one to the other by gluing two blocks,
– the weight function w'_k is constant equal to $1/2$.

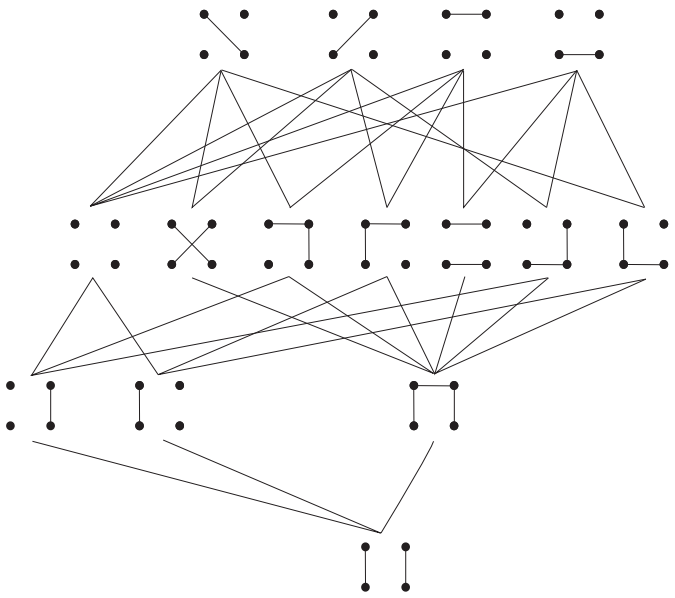

FIGURE 11. The graph $\mathbb{G}'_2$.

From now on, when we will consider the Cayley graph for $\mathcal{P}_k$, we will consider this graph $\mathbb{G}'_k$. The graphs $\mathbb{G}_k$ and $\mathbb{G}'_k$ are interesting as they allow us to better understand the distance d.

Proposition 3.2. — *Let p and p' be two elements of $\mathcal{P}_k$. Let us define $C_{\mathbb{G}_k}(p,p')$ (resp. $C_{\mathbb{G}'_k}(p,p')$) the set of paths π in $\mathbb{G}_k$ (resp. $\mathbb{G}'_k$) which begin in p and finish in p'. Let us define the geodesic distance on $\mathbb{G}_k$ and on $\mathbb{G}'_k$ between p and p' by:*

$$d_{\mathbb{G}_k}(p,p') = \min_{\pi\in C_{\mathbb{G}_k}(p,p'),\pi=e_1\dots e_l} w(e_1)+\cdots+w(e_l),$$
$$d_{\mathbb{G}'_k}(p,p') = \min_{\pi\in C_{\mathbb{G}'_k}(p,p'),\pi=e_1\dots e_l} w(e_1)+\cdots+w(e_l).$$

We have the equalities:

$$d(p,p') = d_{\mathbb{G}_k}(p,p') = d_{\mathbb{G}'_k}(p,p').$$

Proof. — Let p and p' be two elements of $\mathcal{P}_k$. It is enough to prove that $d_{\mathbb{G}_k}(p,p') = d_{\mathbb{G}'_k}(p,p')$ and $d(p,p') = d_{\mathbb{G}'_k}(p,p')$.

First, let us show that $d_{\mathbb{G}_k}(p,p') = d_{\mathbb{G}'_k}(p,p')$. This assertion comes from the fact that one can permute two elements of $\{1,\dots,k\}\cup\{1',\dots k'\}$ in the partition p by gluing two blocks of p and then splitting one block of the resulting partition. Indeed, let us suppose that $p=\{p^1,\dots,p^r\}$. Let $s,t\in\{1,\dots,k,1',\dots,k'\}$, distinct, and let $i,j\in\{1,\dots,r\}$, distinct, such that $s\in p^i$ and $t\in p^j$. Then:

$$p' = \{p^s, s\in\{1,\dots,r\}\setminus\{i,j\}\}\cup\{(p^i\setminus\{s\})\cup\{t\},(p^j\setminus\{t\})\cup\{s\}\}$$

can be obtained by:

1. gluing p_i and p_j,
2. splitting $p_i\cup p_j$ in two: $(p^i\setminus\{s\})\cup\{t\}$ and $(p^j\setminus\{t\})\cup\{s\}$.

The weight of this path is equal to $0.5+0.5=1$. Thus, to compute the distance $d_{\mathbb{G}_k}(p,p')$, it is enough to look only at paths in $\mathbb{G}'_k$: $d_{\mathbb{G}_k}(p,p') = d_{\mathbb{G}'_k}(p,p')$.

Then, let us show that $d(p,p') = d_{\mathbb{G}'_k}(p,p')$. For this, let us see what happens to the distance $d(p,p')$ between p and p' when one moves from p' to one neighborhood of p' in $\mathbb{G}'_k$. Suppose that we glue two blocks of p', then $\mathsf{nc}(p)$ is constant, $\mathsf{nc}(p')$ decreases by 1 and $\mathsf{nc}(p\vee p')$ stays constant or decreases by 1. In this case $d(p,p')$ will increase or decrease by 0.5. Suppose now that we cut one block of p', then $\mathsf{nc}(p)$ is constant, $\mathsf{nc}(p')$ increases by 1 and $\mathsf{nc}(p\vee p')$ stays constant or increases by 1. In this case $d(p,p')$ will also increase or decrease by 0.5.

Thus a gluing/cutting can at most increase the value of $d(p,p')$ by 0.5. It implies that $d(p,p')\leq d_{\mathbb{G}'_k}(p,p')$. We have to show now that $d_{\mathbb{G}'_k}(p,p')\leq d(p,p')$. Let us remark that $p\vee p'$ is coarser than p: we can go from p to $p\vee p'$ by doing $\mathsf{nc}(p)-\mathsf{nc}(p\vee p')$ gluing of blocks. The same holds for p': we can go from p' to $p\vee p'$ by doing $\mathsf{nc}(p')-\mathsf{nc}(p\vee p')$ gluing of blocks. Thus one can go from p to $p\vee p'$ and then from $p\vee p'$ to p' in $\mathsf{nc}(p)+\mathsf{nc}(p')-2\mathsf{nc}(p\vee p')$ steps in $\mathbb{G}'_k$. Thus $d_{\mathbb{G}'_k}(p,p')\leq\frac{1}{2}\left[\mathsf{nc}(p')+\mathsf{nc}(p')-2\mathsf{nc}(p\vee p')\right] = d(p,p')$. □

The function $d_{\mathbb{G}'_k}$ is a geodesic distance on a graph: it is thus a distance. As we have just shown that $d = d_{\mathbb{G}'_k}$, the next corollary is immediately proved.

Corollary 3.1. — *The function $d : \mathcal{P}_k \times \mathcal{P}_k \to \mathbb{R}^+$ is a distance.*

Lemma 3.2. — *The restriction of d to the permutation group is quite usual:*

$$d(\sigma, \sigma') = k - \mathsf{nc}(\sigma^{-1}\sigma'),$$

for any $\sigma, \sigma' \in \mathfrak{S}_k$. This distance is in fact the geodesic distance on the Cayley graph $\mathcal{S}_k$ of $\mathbb{S}_k$. By Lemma 6.26 *of* [**13**]*, the restriction of the distance d to $\mathcal{B}_k$ is also the geodesic distance on the Cayley graph $\mathbb{B}_k$ of $\mathcal{B}_k$.*

Using this distance, we can define a notion of set-geodesic for any of the three sets of partitions we are interested in. We remind the reader that the notation A_k was settled in Notation 2.1.

Definition 3.3. — *Let $p \in A_k$, the set-geodesic $[id, p]_{A_k}$ is defined by:*

$$[id, p]_{A_k} = \left\{p' \in A_k, d(id, p) = d(id, p') + d(p', p)\right\}.$$

A geodesic in a graph between two vertices p and p' is a path in this graph which length is equal to the geodesic distance. Using Proposition 3.2 and Lemma 3.2, one shows that for any $p \in A_k$, the set-geodesic $[id, p]_{A_k}$ is the union of the geodesics between id and p in the Cayley graph of A_k.

The distance on A_k allows us to define a new partial order on A_k.

Definition 3.4. — *Let p and p' be elements of A_k, we write that $p' \leq p$ if and only if $d(id, p) = d(id, p') + d(p', p)$.*

This is a partial order as the restriction of d to $A_k \times A_k$ is a distance.

3.2. Characterization of the order. — Let us define a notion of defect. Let p and p' in $\mathcal{P}_k$.

Definition 3.5. — *We define the defect of p' from not being on the set-geodesic $[id, p]_{\mathcal{P}_k}$ by:*

$$\mathsf{df}(p', p) = d(id, p') + d(p', p) - d(id, p) = \mathsf{nc}(p') - \mathsf{nc}(p' \vee id) - \mathsf{nc}(p \vee p') + \mathsf{nc}(p \vee id).$$

A simple but very useful lemma is the following.

Lemma 3.3. — *If p is coarser than p', then:*

$$\mathsf{df}(p', p) = \mathsf{nc}(p') - \mathsf{nc}(p' \vee id) - \mathsf{nc}(p) + \mathsf{nc}(p \vee id).$$

If p is finer than p' then:

$$\mathsf{df}(p', p) = \mathsf{nc}(p \vee id) - \mathsf{nc}(p' \vee id).$$

Proof. — This is a simple calculation, where one has to use the fact that $\mathsf{nc}(p \vee p') = \mathsf{nc}(p)$ if p is coarser than p' and $\mathsf{nc}(p \vee p') = \mathsf{nc}(p')$ if p is finer than p'. □

We will now characterize the order by constructing the Hasse diagram of $(\mathcal{P}_k, \leq)$. For this, we define the notion of pivotal block for a partition $p \in \mathcal{P}_k$.

Definition 3.6. — *We define the set of pivotal blocks for p as the set of blocks b of p such that there exists a way to cut b into two blocks in order to cut a cycle of p into two cycles. We denote by $\mathsf{Piv}(p)$ the set of pivotal blocks for p.*

For example, if $p = \{\{1, 2, 1', 2'\}\} \in \mathcal{P}_2$, the block $\{1, 2, 1', 2'\}$ is a pivotal block for p since we can cut it in order to get the new partition $\{\{1, 1'\}, \{2, 2'\}\}$ which has one more cycle.

Definition 3.7. — *We denote by $\Delta(p)$ the set of all partitions p' which are obtained by cutting in p a pivotal block for p into two blocks in such way that p' has one more cycle than p. This defines a function Δ from $\mathcal{P}_k$ to the subsets of $\mathcal{P}_k$, namely $\mathcal{P}(\mathcal{P}_k)$. The admissible splits of p are $\mathsf{Sp}(p) = \bigcup_{k=0}^{\infty} \Delta^k(p)$.*

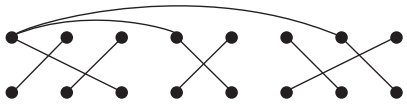

FIGURE 12. A partition p such that $(1, 2, 3)(4, 5)(6, 7, 8) \in \mathsf{Sp}(p)$.

Lemma 3.4. — *If p' is finer than p, the three following assertions are equivalent:*

1. $p' \in [id, p]_{\mathcal{P}_k}$,
2. $\mathsf{nc}(p') - \mathsf{nc}(p' \vee id) = \mathsf{nc}(p) - \mathsf{nc}(p \vee id)$,
3. $p' \in \mathsf{Sp}(p)$.

Proof. — The fact that the two first assertions are equivalent is a consequence of Lemma 3.3. Let us prove that the second and the third assertions are equivalent. If p' is finer than p then one can go from p to p' only by cutting blocks. At each step the number of blocks goes up by one and the number of cycle is either constant or goes up by one. Thus, if we want that the number of blocks minus the number of cycle is constant at the beginning and at the end it must be constant during all the path from p to p'. This means that at each step the number of cycles must go up by one, and thus we are cutting a pivotal block in a way to create one more cycle. This proves that the second condition is equivalent to $p' \in \mathsf{Sp}(p)$. □

Definition 3.8. — *For any $p \in \mathcal{P}_k$, let us define $\mathsf{Gl}_c(p)$ the set of partitions p' in $\mathcal{P}_k$ such that p' is obtained by gluing blocks of p which are in the same cycle of p. The set $\mathsf{Gl}_c(p)$ is only:*

$$\mathsf{Gl}_c(p) = \{p' \in \mathcal{P}_k \mid \mathsf{nc}(p \vee id) = \mathsf{nc}(p' \vee id) \text{ and } p' \text{ is coarser than } p\}.$$

The following proposition is straightforward.

Lemma 3.5. — *If p' is coarser than p, the two following assertions are equivalent:*

1. $p' \in [id, p]_{\mathcal{P}_k}$,
2. $\mathsf{nc}(p' \vee id) = \mathsf{nc}(p \vee id)$,
3. $p' \in \mathsf{Gl}_c(p)$.

Lemmas 3.4 and 3.5 allow us to prove the following characterization of the geodesic order.

Theorem 3.1. — *The partition p' is in $[id,p]_{\mathcal{P}_k}$ if and only there exists $p'' \in \mathcal{P}_k$ such that the two following conditions hold:*

1. $p'' \in \mathsf{Gl}_c(p)$,
2. $p' \in \mathsf{Sp}(p'')$.

If so, then $p'' = p \vee p'$.

Thus, if one defines the set of admissible gluings $\mathsf{Sp}^{-1}(p)$ as the set of all partitions p' such that $p \in \mathsf{Sp}(p')$, then $p' \in [id,p]_{\mathcal{P}_k}$ if and only if $\mathsf{Gl}_c(p) \cap \mathsf{Sp}^{-1}(p') \neq \emptyset$, and if so $\mathsf{Gl}_c(p) \cap \mathsf{Sp}^{-1}(p') = \{p \vee p'\}$.

An other formulation in order to state this theorem is the following theorem which allows us to construct the Hasse diagram for the geodesic order.

Theorem 3.2. — *We define the graph $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$ such that:*

- *the set of vertices $\mathcal{V}_k$ is $\mathcal{P}_k$,*
- *there exists an oriented edge from p to p' if one can go from p to p' by gluing two blocks of p which are not in the same cycle of p or if one can go from p' to p by gluing two blocks which are in the same cycle of p'.*

The graph $\mathcal{G}_k$ is the Hasse diagram of the geodesic order on $\mathcal{P}_k$: $p' \in [id,p]_{\mathcal{P}_k}$ if and only if there exists a path from p' to p in $\mathcal{G}_k$.

Remark 3.2. — *Let $\{p,p'\}$ be an edge in $\mathbb{G}'_k$, the Cayley graph of $\mathcal{P}_k$. As the Cayley graph is not oriented, we can always suppose that one can go from p to p' by gluing two blocks of p. If the two blocks we glue are not in the same cycle of p, then the oriented edge (p,p') is in the Hasse diagram for the geodesic order. If not, then it means that (p',p) is in the Hasse diagram for the geodesic order.*

Thus, for any non-oriented edge e of $\mathbb{G}'_k$, there exists an orientation e^+ of e such that e^+ is an edge of the Hasse diagram for the geodesic order and $(e^+)^{-1}$ is not: the Hasse diagram is obtained by chosing a good orientation on the edges of the Cayley graph $\mathbb{G}'_k$. For $\mathcal{P}_2$, the Hasse diagram is obtained by considering the graph drawn in Figure 3.1 and orienting the edges from down to top.

Proof of Theorem 3.1. — Using Lemma 3.8, we see that $p' \in [id,p]_{\mathcal{P}_k}$ if and only if $\mathsf{nc}(p \vee p' \vee id) = \mathsf{nc}(p \vee id)$ and $p' \in [id,p]_{\mathcal{P}_k}$, thus, if and only if $\mathsf{nc}(p \vee p' \vee id) = \mathsf{nc}(p \vee id)$ and:

$$\mathsf{df}(p',p) = \mathsf{nc}(p') - \mathsf{nc}(p' \vee id) - \mathsf{nc}(p \vee p') + \mathsf{nc}(p \vee id) = 0,$$

which is equivalent to $\mathsf{nc}(p \vee p' \vee id) = \mathsf{nc}(p \vee id)$ and:

$$\mathsf{df}(p', p \vee p') = \mathsf{nc}(p') - \mathsf{nc}(p' \vee id) - \mathsf{nc}(p \vee p') + \mathsf{nc}(p \vee p' \vee id) = 0,$$

which is again equivalent to $\mathsf{nc}(p \vee p' \vee id) = \mathsf{nc}(p \vee id)$ and $p' \in [id, p \vee p']_{\mathcal{P}_k}$. Using Lemmas 3.4 and 3.5, we get that $p' \in [id,p]_{\mathcal{P}_k}$ if and only if:

$$p \vee p' \in \mathsf{Gl}_c(p) \cap \mathsf{Sf}^{-1}(p').$$

Let us prove that $\#\left(\mathsf{Gl}_c(p) \cap \mathsf{Sf}^{-1}(p')\right) \leq 1$. Let $p'' \in \mathsf{Gl}_c(p) \cap \mathsf{Sf}^{-1}(p')$: this implies that:

$$\mathsf{nc}(p'' \vee id) = \mathsf{nc}(p \vee id)$$
$$\mathsf{nc}(p'') - \mathsf{nc}(p'' \vee id) = \mathsf{nc}(p') - \mathsf{nc}(p' \vee id).$$

Thus $\mathsf{nc}(p'') - \mathsf{nc}(p \vee id) = \mathsf{nc}(p') - \mathsf{nc}(p' \vee id)$. Yet the fact that $\mathsf{df}(p', p) \geq 0$ tells us that:

$$\mathsf{nc}(p \vee p') - \mathsf{nc}(p \vee id) \leq \mathsf{nc}(p') - \mathsf{nc}(p' \vee id).$$

Since p'' is coarser than p and than p', $\mathsf{nc}(p \vee p') \geq \mathsf{nc}(p'')$. Thus:

$$\mathsf{nc}(p'') - \mathsf{nc}(p'' \vee id) \leq \mathsf{nc}(p \vee p') - \mathsf{nc}(p'' \vee id) \leq \mathsf{nc}(p') - \mathsf{nc}(p' \vee id).$$

Since the left and right hand sides are equal, we have that $\mathsf{nc}(p \vee p') = \mathsf{nc}(p'')$ and since p'' is coarser than $p \vee p'$, we obtain that $p'' = p \vee p'$. □

3.2.1. Properties of the admissible splitting. — Using the definition of the admissible splittings, the following proposition is straightforward.

Proposition 3.3. — *Let $p \in \mathcal{P}_k$, the set $\mathsf{Sp}(p) \cap \mathcal{B}_k$ is either empty or has one element.*

This last proposition leads us to the following definition.

Definition 3.9. — *For any positive integer k, we define:*

$$\overline{\mathfrak{S}_k} = \{p \in \mathcal{P}_k \mid \#\left(\mathsf{Sp}(p) \cap \mathfrak{S}_k\right) = 1\},$$
$$\overline{\mathcal{B}_k} = \{p \in \mathcal{P}_k \mid \#\left(\mathsf{Sp}(p) \cap \mathcal{B}_k\right) = 1\}.$$

For any $p \in \overline{B}_k$, we denote by $\mathsf{Mb}(p)$ the unique element in $\mathsf{Sp}(p) \cap \mathcal{B}_k$.

An other important lemma is the following.

Lemma 3.6. — *Let p be a partition in $\mathcal{P}_k$ which does not have any pivotal block, then $\mathsf{Sp}(p) = \{p\}$. In particular, if $p \in \mathcal{B}_k$, p does not have any pivotal block, thus:*

$$\mathsf{Sp}(p) = \{p\}.$$

Proof. — The first assertion is a direct consequence of the definitions. It remains to show that if $p \in \mathcal{B}_k$, p does not have any pivotal block. We can suppose that p is irreducible. Let us suppose that p has a pivotal block that we will denote by c. We can always suppose that c is of the form $\{i, (i+1)'\}$ or $\{i, i+1\}$ since we can shuffle the columns of p and take the transpose of p. With the same argument, we can suppose that when one cuts the block c, the new partition we get has the form $p_1 \otimes p_2$ with $p_1 \in \mathcal{P}_i$. Here is the contradiction: the partition p_1 must be composed of blocks of size two except one block which is equal to $\{i\}$. This is not possible since p_1 must be a partition of $2i$ elements. □

3.3. Computation of the Möbius function for the geodesic order. — Since we have defined an order on $\mathcal{P}_k$, namely the geodesic order, it would be interesting to compute the Möbius function. In order to do so, we need to define two orders.

Definition 3.10. — *Let p and p' be in $\mathcal{P}_k$. We say that:*
- *p' is coarser-compatible than p if and only if $p' \in \mathsf{Gl}_c(p)$. We denote this by $p' \dashv p$.*
- *p' is finer-admissible than p if and only if $p' \in \mathsf{Sp}(p)$. We denote this by $p' \sqsupset p$.*

Both notions define a partial order on $\mathcal{P}_k$.

We used the symbols $\dashv$ since when one reads it from right to left, one sees two segments which become one segment ; we are gluing two blocks of p. For $\sqsupset$, when one reads it from right to left, one sees a segment which splits into two parts: we are spliting one block of p.

We can define the matrices of the partial order $\leq, \dashv, \sqsupset$.

Definition 3.11. — *The matrices of the partial orders $\leq, \dashv, \sqsupset$ are:*
- *for the geodesic order $\leq$:* $G_{p,p'} = \delta_{p' \leq p}$,
- *for the coarser-compatible order $\dashv$:* $C_{p,p'} = \delta_{p' \dashv p}$,
- *for the finer-admissible order $\sqsupset$:* $S_{p,p'} = \delta_{p' \sqsupset p}$.

Using these definitions, we can translate the Theorem 3.1.

Theorem 3.3. — *The partial orders $\leq, \dashv$, and $\sqsupset$ are linked by the following equality:*

$$G = CS.$$

Proof. — Let us consider p and p' in $\mathcal{P}_k$. Using Theorem 3.1, we have:

$$G_{p,p'} = \delta_{p' \leq p} = \sum_{p'' \in \mathcal{P}_k | p'' \in \mathsf{Gl}_c(p), p' \in \mathsf{Sp}(p'')} 1.$$

Thus:

$$G_{p,p'} = \sum_{p'' \in \mathcal{P}_k} \delta_{p'' \in \mathsf{Gl}_c(p)} \delta_{p' \in \mathsf{Sp}(p'')} = \sum_{p'' \in \mathcal{P}_k} \delta_{p'' \dashv p} \delta_{p' \sqsupset p''} = \sum_{p'' \in \mathcal{P}_k} C_{p,p''} S_{p'',p'}.$$

This allows us to finish the proof. □

The notion of Möbius function is linked with the inverse of the matrix of the order.

Definition 3.12. — *Let T be a finite set endowed with a partial order. Let M be the matrix of the order as we defined in Definition 3.11. The Möbius function is the function such that for any a and b in T,*

$$\mu(a, b) = (M^{-1})_{b,a}.$$

Our goal is to compute the Möbius function for $(\mathcal{P}_k, \leq)$. In order to compute the inverse of G, we need the following lemma.

Lemma 3.7. — *Let p, p' and p'' be in $\mathcal{P}_k$. Let us suppose that $p' \dashv p''$ and $p'' \sqsupset p$, then $p'' = p \wedge p'$.*

Let us remark that it is the "infimum version" of the last assertion of Theorem 3.1 which asserts that $p'' \dashv p$ and $p' \sqsupset p''$ if and only if $p'' = p \vee p'$.

Proof of Lemma 3.7. — Let p, p' and p'' be three partitions in $\mathcal{P}_k$ which satisfy the hypotheses of Lemma 3.7. Using Lemmas 3.4 and 3.5, we know that:

$$\mathsf{nc}(p' \vee id) = \mathsf{nc}(p'' \vee id)$$
$$\mathsf{nc}(p'') - \mathsf{nc}(p'' \vee id) = \mathsf{nc}(p) - \mathsf{nc}(p \vee id).$$

Thus by using the two last equalities:

$$\mathsf{nc}(p'') - \mathsf{nc}(p' \vee id) = \mathsf{nc}(p) - \mathsf{nc}(p \vee id).$$

Besides, $p' \leq p''$ and $p'' \leq p$ thus $p' \leq p$: this implies that $\mathsf{df}(p, p') = 0$, thus:

$$\mathsf{nc}(p') - \mathsf{nc}(p' \vee id) + \mathsf{nc}(p \vee id) - \mathsf{nc}(p \vee p') = 0.$$

Using the two lattest equations, we get:

$$\mathsf{nc}(p'') + \mathsf{nc}(p \vee p') - \mathsf{nc}(p') - \mathsf{nc}(p) = 0. \tag{4}$$

Using the triangle inequality, we know that $d(p, p \wedge p') + d(p \wedge p', p'') - d(p, p') \geq 0$, thus:

$$\mathsf{nc}(p \wedge p') + \mathsf{nc}(p \vee p') - \mathsf{nc}(p) - \mathsf{nc}(p') \geq 0. \tag{5}$$

Since p'' is finer than p' and than p, p'' is finer than $p \wedge p'$: this implies that $\mathsf{nc}(p \wedge p') \leq \mathsf{nc}(p'')$. Using Equations (4) and (5), we get that $\mathsf{nc}(p'') = \mathsf{nc}(p \wedge p')$. Thus $p'' = p \wedge p'$. □

Theorem 3.4. — *Let p and p' in $\mathcal{P}_k$. We have:*

$$(G^{-1})_{p,p'} = \delta_{p \wedge p' \sqsupset p} \delta_{p' \dashv p \wedge p'} \mu_f(p \wedge p', p) \mu_f(p \wedge p', p'),$$

where for any partition p_1 and p_2 such that p_1 is finer than p_2:

$$\mu_f(p_1, p_2) = (-1)^{\mathsf{nc}(p_1) - \mathsf{nc}(p_2)} \prod_{i=3}^{\mathsf{nc}(p_1)} ((i-1)!)^{r_i}$$

where r_i is the number of blocks of p_2 which contains exactly i blocks of p_1.

Proof. — Using Theorem 3.3, we know that $G = CS$. Thus $G^{-1} = S^{-1}C^{-1}$. Let p and p' be two partitions in $\mathcal{P}_k$, we know that:

$$(G^{-1})_{p,p'} = \sum_{p'' \in \mathcal{P}_k} (S^{-1})_{p,p''} (C^{-1})_{p'',p'}.$$

Since S, respectively C, is the matrix of the order $\sqsupset$, respectively $\dashv$, for any partitions p_1 and p_2 in $\mathcal{P}_k$:

$$(S^{-1})_{p_1,p_2} = \delta_{p_2 \sqsupset p_1} (S^{-1})_{p_1,p_2},$$
$$(C^{-1})_{p_1,p_2} = \delta_{p_2 \dashv p_1} (C^{-1})_{p_1,p_2}.$$

This is due to the fact that the inverse of an upper triangular invertible matrix is still upper triangular. Thus:

$$(G^{-1})_{p,p'} = \sum_{p'' \in \mathcal{P}_k | p'' \sqsupset p, p' \dashv p''} (S^{-1})_{p,p''} (C^{-1})_{p'',p'}.$$

Using Lemma 3.7, we get that:

$$(G^{-1})_{p,p'} = (S^{-1})_{p,p \wedge p'} (C^{-1})_{p \wedge p',p'}.$$

It remains to compute $(S^{-1})_{p,p \wedge p'}$ and $(C^{-1})_{p \wedge p',p'}$.

Each time that one considers a partial order on a finite set, its matrix O can be written as $Id + \tilde{O}$, with $\tilde{O}$ being a nilpotent matrice. Thus:

$$O^{-1} = \sum_{i=0}^{\infty} (-1)^i \tilde{O}^i,$$

this is the Rota's Formula for the Möbius inversion. Thus, for any p and p' in $\mathcal{P}_k$:

$$\begin{aligned}\left(S^{-1}\right)_{p,p\wedge p'} &= \sum_{i=0}^{\infty} (-1)^i \sum_{(p_0,\dots,p_i)\in\mathcal{P}_k | p=p_0\neq p_1\neq\dots\neq p_i=p\wedge p'} \left[\prod_{l=0}^{i-1} S_{p_l,p_{l+1}}\right] \\ &= \sum_{i=0}^{\infty} (-1)^i \sum_{(p_0,\dots,p_i)\in\mathcal{P}_k | p=p_0\neq p_1\neq\dots\neq p_i=p\wedge p'} \left[\prod_{l=0}^{i-1} \delta_{p_{l+1}\sqsupset p_l}\right].\end{aligned}$$

Yet, if $p \wedge p' \sqsupset p$, then for any positive integer i, for any $i+1$-tuple $(p_0, ..., p_i)$:

$$\prod_{l=0}^{i-1} \delta_{p_{l+1}\sqsupset p_l} = \prod_{l=0}^{i-1} \delta_{p_{l+1} \text{ finer than } p_l}.$$

Thus:

$$(S^{-1})_{p,p\wedge p'} = \delta_{p\wedge p'\sqsupset p} \sum_{i=0}^{\infty} (-1)^i \sum_{(p_0,\dots,p_i)\in\mathcal{P}_k | p=p_0\neq p_1\neq\dots\neq p_i=p\wedge p'} \left[\prod_{l=0}^{i-1} \delta_{p_{l+1} \text{ finer than } p_l}\right].$$

This implies that:

$$\left(S^{-1}\right)_{p,p\wedge p'} = \delta_{p\wedge p'\sqsupset p} (F^{-1})_{p,p\wedge p'},$$

with F being the matrix such that for any $p_1, p_2 \in \mathcal{P}_k$, $F_{p_1,p_2} = \delta_{p_2 \text{ finer than } p_1}$. The inverse of this matrix is well known, for any $p_1, p_2 \in \mathcal{P}_k$ such that p_2 is finer than p_1:

$$(F^{-1})_{p_1,p_2} = \mu_f(p_2, p_1),$$

where μ_f is the Möbius function for the order of being finer and is given in the statement of Theorem 3.4. Similar arguments allow us to compute the inverse of C and to obtain that:

$$(C^{-1})_{p\wedge p',p'} = \delta_{p'\dashv p\wedge p'} \mu_f(p \wedge p', p').$$

This allows us to obtain the desired formula for G^{-1}. □

Theorem 3.5. — *The Möbius function for $(\mathcal{P}_k, \leq)$, denoted by $\mu_{\leq}$, is given by:*

$$\forall p_1, p_2 \in \mathcal{P}_k, \mu_{\leq}(p_1, p_2) = \delta_{p_1\dashv p_1\wedge p_2} \delta_{p_1\wedge p_2\sqsupset p_2} \mu_f(p_1 \wedge p_2, p_1) \mu_f(p_1 \wedge p_2, p_2).$$

Let us remark that if $p_1 \dashv p_1 \wedge p_2$ then $p_1 \leq p_1 \wedge p_2$ and if $p_1 \wedge p_2 \sqsupset p_2$ then $p_1 \wedge p_2 \leq p_2$. Thus if both conditions hold, $p_1 \leq p_2$ by transitivity of the geodesic order. This is why we do not add the condition that $p_1 \leq p_2$ in the formula for the Möbius function $\mu_{\leq}$.

Remark 3.3. — *Let us remark that, as a by-product of the proof of Theorem 3.4, we computed the matrices C^{-1} and S^{-1}, thus we know the Möbius functions for the orders $\dashv$, $\sqsupset$ and $\leq$.*

3.4. Some properties of the geodesic order. —

3.4.1. Factorization of the geodesics. — The following property, known for $\mathfrak{S}_k$ and $\mathcal{B}_k$, is still true for $\mathcal{P}_k$: a geodesic between id and $p_1 \otimes p_2$ must be the tensor product of a geodesic between id and p_1 and a geodesic between id and p_2.

Lemma 3.8. — *Let $p \in A_k$, we have:*

$$[id, p]_{A_k} \simeq \prod_{C \in \mathsf{C}(p)} \left[id_{\frac{\#C}{2}}, p_C \right]_{A_{\frac{\#C}{2}}}.$$

In particular if p_1 and p_2 are two partitions, $p_1 \in A_{k_1}$ and $p_2 \in A_{k_2}$, then $p' \in [id, p_1 \otimes p_2]_{A_{k_1+k_2}}$ if and only if there exist $p'_1 \in A_{k_1}$ and $p'_2 \in A_{k_2}$ such that $p' = p'_1 \otimes p'_2$, $p'_1 \in [id, p_1]_{A_k}$ and $p'_2 \in [id, p_2]_{A_k}$.

3.4.2. No Brauer element is smaller than a permutation. — In the following lemma, we show that the geodesic in the Cayley graph of $\mathcal{P}_k$ between two permutations either stay in the set of permutations or intersect $\mathcal{P}_k \setminus \mathcal{B}_k$. Using the fact that $[id, p]_{A_k}$ is the union of the geodesics between id and p in the Cayley graph of A_k, we get an equality between $[id, \sigma]_{\mathcal{B}_k}$ and $[id, \sigma]_{\mathfrak{S}_k}$.

Lemma 3.9. — *Let $\sigma \in \mathfrak{S}_k$, then $[id, \sigma]_{\mathcal{B}_k} = [id, \sigma]_{\mathfrak{S}_k}$.*

Proof. — We will do a proof by contradiction. Let $S \subset \mathfrak{S}_k$ be the set of permutations such that $[id, \sigma]_{\mathcal{B}_k} \neq [id, \sigma]_{\mathfrak{S}_k}$. Let $\sigma \in S$ be a permutation such that $d(id, \sigma) = \min_{\sigma' \in S} d(id, \sigma')$. Let us consider b an element of $\mathcal{B}_k \setminus \mathfrak{S}_k$ such that $b \in [id, \sigma]_{\mathcal{B}_k}$. There exists a geodesic in $\mathcal{B}_k$ which goes through b and goes from id to σ. Let $b' \in \mathcal{B}_k$ be the unique element on this geodesic such that $d(id, b') = 1$. Let us remark that $b \in [b', \sigma]_{\mathcal{B}_k}$: this implies that b' can not be a permutation. Indeed, if b' was a permutation, then $[b', \sigma]_{\mathcal{B}_k} \neq [b', \sigma]_{\mathfrak{S}_k}$ and thus, $[id, b'^{-1}\sigma]_{\mathcal{B}_k} \neq [id, b'^{-1}\sigma]_{\mathfrak{S}_k}$. Yet $d(id, b'^{-1}\sigma) = d(b', \sigma) = d(id, \sigma) - 1$. This would contradict the fact that $d(id, \sigma) = \min_{\sigma' \in S} d(id, \sigma')$. Thus b' must be an element of $\mathcal{B}_k \setminus \mathfrak{S}_k$. Since $d(id, b') = 1$, there exist i and j in $\{1, \ldots, k\}$ such that b' is equal to the Weyl contraction $[i, j]$ in $\mathcal{B}_k$. Thus there exist i and j in $\{1, \ldots, k\}$ such that $[i, j] \in [id, \sigma]_{\mathcal{B}_k}$. Using Theorem 3.1, this means that $[i, j] \vee \sigma \in \mathsf{Gl}_c(\sigma)$ and $[i, j] \in \mathsf{Sp}([i, j] \vee \sigma)$. The first condition tells us that i and j must be in the same cycle of σ, let us suppose so. The second condition implies that σ has only one cycle which is not trivial and it implies that:

$$\mathsf{nc}(\sigma \vee [i, j]) - \mathsf{nc}(\sigma \vee [i, j] \vee id) = \mathsf{nc}([i, j]) - \mathsf{nc}([i, j] \vee id) = 1.$$

Yet $\mathsf{nc}(\sigma \vee [i, j]) - \mathsf{nc}(\sigma \vee [i, j] \vee id) = 0$ thus $[i, j]$ can not be in $[id, \sigma]_{\mathcal{B}_k}$. This yields the contradiction. □

This lemma is the key point which will allow us to explain in the second article [**9**] why processes on $U(N)$ and $O(N)$ have the same limit when one only considers usual moments.

3.4.3. Geodesics and tensor product. — For the last geometric proposition, we need to define the left and right parts of a partition p.

Definition 3.13. — *Let k and l be two positive integers. Let $p \in A_{k+l}$, we denote by p_k^g the extraction of p to $\{1, ..., k\}$ and p_k^d the extraction of p to $\{k+1, ..., k+l\}$. The partition p_k^g is in $\mathcal{P}_k$ and p_k^d is in $\mathcal{P}_l$.*

Proposition 3.4. — *Let k_1 and k_2 be two positive integers and let $k = k_1 + k_2$. Let p be an element of $\mathcal{P}_k$. Let p_1 and p_2 be respectively in $\mathcal{P}_{k_1}$ and $\mathcal{P}_{k_2}$. We have equivalence between:*

1. $p_1 \otimes p_2 \sqsupset p$,
2. $p_1 \sqsupset p_{k_1}^g$, $p_2 \sqsupset p_{k_1}^d$ *and* $p_{k_1}^g \otimes p_{k_1}^d \sqsupset p$.

Proof. — First of all, it is not difficult to see that the second condition implies the first one. Indeed, if $p_1 \sqsupset p_{k_1}^g$ and $p_2 \sqsupset p_{k_1}^d$ then $p_1 \otimes p_2 \sqsupset p_{k_1}^g \otimes p_{k_1}^d$. Thus, by transitivity of the order, if $p_{k_1}^g \otimes p_{k_1}^d \sqsupset p$, we get that $p_1 \otimes p_2 \sqsupset p$.

It remains to show that the first condition implies the second one. Let us remark that p is coarser than $p_1 \otimes p_2$ if and only if $p_{k_1}^g$ is coarser than p_1 and $p_{k_1}^d$ is coarser than p_2. Thus it remains to prove that under the condition that p is coarser than $p_1 \otimes p_2$, $p_1 \otimes p_2 \in [id, p]_{A_k}$ implies that $p_1 \in [id, p_{k_1}^g]_{\mathcal{P}_{k_1}}$, $p_2 \in [id, p_{k_1}^d]_{\mathcal{P}_{k_2}}$ and $p_k^g \otimes p_k^d \in [id, p]_{A_k}$. Since for any partitions the defect between two partitions is always positive, the result would be a consequence of the following equality:

$$\mathsf{df}(p_1 \otimes p_2, p) = \mathsf{df}(p_1, p_k^g) + \mathsf{df}(p_2, p_k^d) + \mathsf{df}(p_k^g \otimes p_k^d, p).$$

Let us prove this equality: using Lemma 3.3:

$$\begin{aligned}
&\mathsf{df}(p_1 \otimes p_2, p) - \mathsf{df}(p_1, p_k^g) - \mathsf{df}(p_2, p_k^d) - \mathsf{df}(p_k^g \otimes p_k^d, p) \\
&\quad = \mathsf{nc}(p_1 \otimes p_2) - \mathsf{nc}((p_1 \otimes p_2) \vee id) - \mathsf{nc}(p) + \mathsf{nc}(p \vee id) \\
&\qquad - \mathsf{nc}(p_1) + \mathsf{nc}(p_1 \vee id) + \mathsf{nc}(p_k^l) - \mathsf{nc}(p_k^g \vee id) \\
&\qquad - \mathsf{nc}(p_2) + \mathsf{nc}(p_2 \vee id) + \mathsf{nc}(p_k^r) - \mathsf{nc}(p_k^d \vee id) \\
&\qquad - \mathsf{nc}(p_k^g \otimes p_k^d) + \mathsf{nc}((p_k^g \otimes p_k^d) \vee id) + \mathsf{nc}(p) - \mathsf{nc}(p \vee id) \\
&\quad = 0,
\end{aligned}$$

since:

$$\begin{aligned}
\mathsf{nc}(p_1 \otimes p_2) &= \mathsf{nc}(p_1) + \mathsf{nc}(p_2), \\
\mathsf{nc}((p_1 \otimes p_2) \vee id) &= \mathsf{nc}(p_1 \vee id) + \mathsf{nc}(p_2 \vee id), \\
\mathsf{nc}(p_k^g \otimes p_k^d) &= \mathsf{nc}(p_k^g) + \mathsf{nc}(p_k^d), \\
\mathsf{nc}((p_k^g \otimes p_k^d) \vee id) &= \mathsf{nc}(p_k^g \vee id) + \mathsf{nc}(p_k^d \vee id).
\end{aligned}$$

This ends the proof. □

4. Convergence of elements of $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$

4.1. Coordinate numbers and moments. —

4.1.1. Definitions. — Let k be a positive integer, recall the notation A_k defined in Notation 2.1. For each integer N, we have defined an algebra $\mathbb{C}[A_k(N)]$. Let $(E_N)_{N\in\mathbb{N}}$ be a sequence such that for any positive integer N, $E_N \in \mathbb{C}[A_k(N)]$. For each integer N, the algebra $\mathbb{C}[A_k(N)]$, seen as a vector space has the same basis A_k. Thus, we could study the convergence of $(E_N)_{N\in\mathbb{N}}$ only from the vector space point of view by saying that the sequence $(E_N)_{N\in\mathbb{N}}$ converges if and only if the coordinates of E_N in the basis A_k converge. Actually, this convergence forgets the fact that $\mathbb{C}[A_k(N)]$ is an algebra which depends on an integer N. In order to define a better definition of convergence, we have to define the coordinate numbers of E in $\mathbb{C}[A_k(N)]$.

Definition 4.1. — *Let N be an integer. Let E be an element of $\mathbb{C}[A_k(N)]$. We define the numbers $\left(\kappa^p(E)\right)_{p\in A_k}$ as the only numbers such that:*

$$E = \sum_{p\in A_k} \frac{\kappa^p(E)}{N^{\frac{-k+\mathsf{nc}(p)}{2}+d(id,p)}}\ p.$$

The family $(\kappa^p(E))_{p\in A_k(N)}$ is called the coordinate numbers *of E.*

After Definition 4.4, we will explain how we get this definition, and why this definition is in fact the most natural thing one can do. We will need to use the following equality: for any integer k, for any $p \in A_k$,

$$\frac{-k+\mathsf{nc}(p)}{2} + d(id,p) = \mathsf{nc}(p) - \mathsf{nc}(p\vee id). \tag{6}$$

This implies the following remark.

Remark 4.1. — *For any integer k, any integer N, for any $E \in \mathbb{C}[\mathcal{P}_k(N)]$:*

$$E = \sum_{p\in A_k} \frac{\kappa^p(E)}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}}p.$$

We will consider the coordinate numbers as linear applications from $\mathbb{C}[A_k(N)]$ to $\mathbb{R}$:

$$\begin{aligned}\kappa^p : \mathbb{C}[A_k(N)] &\to \mathbb{R}\\ E &\mapsto \kappa^p(E).\end{aligned}$$

The notion of coordinate numbers allows us to define a strong convergence for any sequence $(E_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$.

Definition 4.2. — *Let $(E_N)_{N\in\mathbb{N}}$ be an element of $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$. The sequence $(E_N)_{N\in\mathbb{N}}$ converges strongly if the coordinate numbers of E_N converges when N goes to infinity: for any $p \in A_k$, $\kappa^p(E_N)$ converges when N goes to infinity.*

The goal now is to give a dual definition of convergence. We have seen in Definition 2.15 that any element of $\mathbb{C}[A_k(N)]$ can be seen as an element of $\mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right)$ and we defined in Definition 3.1 the trace of any element $\mathbb{C}[A_k(N)]$. Using this trace and the structure of algebra of $\mathbb{C}[A_k(N)]$, we define, for any element of $\mathbb{C}[A_k(N)]$ and any element $p \in A_k$, the p-normalized moment of E.

Definition 4.3. — *Let $N \in \mathbb{N}$, let $p \in A_k$ and $E \in \mathbb{C}[A_k(N)]$. The p-normalized moment of E is:*

$$m_p(E) = \frac{1}{Tr_N(p)} Tr_N(E\, {}^t p).$$

Using these normalized moments, we can define a weak notion of convergence for any sequence $(E_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$.

Definition 4.4. — *The sequence $(E_N)_{N\in\mathbb{N}}$ converges in moments if the normalized moments of E_N converges when N goes to infinity: for any $p \in A_k$, $m_p(E_N)$ converges when N goes to infinity.*

4.1.2. Coordinate numbers-moments transformation. — We can now explain how we ended up with Definition 4.1 and how we had the idea to define the distance on the set of partitions. The idea behind these definitions is that we want to know, given a sequence $(E_N) \in \prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$, how the usual coordinates of E_N in the basis A_k must scale so that for any $p \in A_k$, $m_p(E_N)$ converges when N goes to infinity. Let N be an integer, we have $E_N = \sum_{p\in A_k} a_N^p p$. Thus

$$m_{p_0}(E_N) = \sum_{p\in A_k} \frac{Tr_N(p\ {}^t p_0)}{Tr_N(p_0)} a_N^p.$$

Thus the vector $m_N = (m_{p_0}(E_N))_{p_0}$ and $a_N = (a_N^p)_p$ are linked by the relation $m_N = M_N a_N$ where $M_N = \left(\frac{Tr_N(p\ {}^t p_0)}{Tr_N(p_0)}\right)_{p_0,p}$.

There are then two possible possibilities: to invert M_N for N big enough. This is the usual way, which leads to the Weingarten function. Or, one can make the following Ansatz: if we write the system, we see that for any p, $(a_N)_p$ is going to be multiplied by $(M_N)_{p_0,p}$ for any $p_0 \in A_k$. Thus we make the assumption that $(a_N)_p$ must decrease as the inverse of the maximum of $(M_N)_{p_0,p}$ over p_0. That is $a_N^p \sim a^p N^{-\eta_p}$, where η_p is given by:

$$\eta_p = \sup_{p_0} \lim_{N\to\infty} \log_N \left(\frac{Tr_N(p\ {}^t p_0)}{Tr_N(p_0)}\right).$$

The goal now is to know in which p_0 the supremum is obtained. It is more than tempting, seeing the scalar product $Tr_N(p_0\ {}^t p)$ to write what is inside the log_N as:

$$\begin{aligned}\frac{Tr_N(p\ {}^t p_0)}{Tr_N(p_0)} &= \frac{Tr_N(p\ {}^t p_0)}{\sqrt{Tr_N(p\ {}^t p) Tr_N(p_0\ {}^t p_0)}} \frac{\sqrt{Tr_N(p_0\ {}^t p_0) Tr_N(p\ {}^t p)}}{Tr_N(p_0)} \\ &= \frac{Tr_N(p\ {}^t p_0)}{\sqrt{Tr_N(p\ {}^t p) Tr_N(p_0\ {}^t p_0)}} \frac{\sqrt{Tr_N(p_0\ {}^t p_0) Tr_N(id_k\ {}^t id_k)}}{Tr_N(p_0\ {}^t id_k)} \sqrt{\frac{Tr_N(p\ {}^t p)}{Tr_N(id_k\ {}^t id_k)}}.\end{aligned}$$

We recognize thus the distance that we defined. In fact the intuition that is should be a distance comes from the fact that one can write:

$$\eta_p = \sup_{p_0} \left[-d(p, p_0) + d(p_0, id_k)\right] + \frac{1}{2}(-k + \mathsf{nc}(p)).$$

If d were a distance, then by the triangle inequality, for any p_0,

$$d(p_0, id_k) - d(p_0, p) \leq d(p, id_k).$$

This shows that the supremum is obtained at $p_0 = p$, and thus the Ansatz tells us that:

$$a_N^p \sim a^p N^{-\left[\frac{1}{2}(-k+\mathsf{nc}(p))+d(id,p)\right]},$$

to be compared with the Definition 4.1.

The first main result is given by Theorem 4.1 which shows the equivalence between strong and weak convergence.

Theorem 4.1. — *Let $(E_N)_{N\in\mathbb{N}}$ be a sequence such that for any $N \in \mathbb{N}$, E_N is an element of $\mathbb{C}[A_k(N)]$. The sequence $(E_N)_{N\in\mathbb{N}}$ converges strongly if and only if it converges in moments. Let us suppose that $(E_N)_{N\in\mathbb{N}}$ converges in moments or strongly, for any $p \in A_k$:*

$$\lim_{N\to\infty} m_p(E_N) = \sum_{p'\in[id,p]_{A_k}} \lim_{N\to\infty} \kappa^{p'}(E_N). \tag{7}$$

Proof. — Let $(E_N)_{N\in\mathbb{N}}$ be an element of $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$, let $p \in A_k$ and let N be a positive integer. Using the coordinate numbers of E_N, we can calculate the p-normalized moment of E_N:

$$\begin{aligned} m_p(E_N) = \frac{1}{Tr_N(p)} Tr_N(E_N\, {}^t p) &= \frac{1}{Tr_N(p)} Tr_N\left(\sum_{p'\in A_k} \frac{\kappa^{p'}(E_N)}{N^{\mathsf{nc}(p')-\mathsf{nc}(p'\vee id)}} p'\, {}^t p\right) \\ &= \sum_{p'\in A_k} \kappa^{p'}(E_N) \frac{Tr_N(p'\, {}^t p)}{Tr_N(p) N^{\mathsf{nc}(p')-\mathsf{nc}(p'\vee id)}}. \end{aligned}$$

Using the Equality (3):

$$m_p(E_N) = \sum_{p'\in A_k} \kappa^{p'}(E_N) N^{\mathsf{nc}(p\vee p')-\mathsf{nc}(p\vee id)-\mathsf{nc}(p')+\mathsf{nc}(p'\vee id)}.$$

Hence, using Definition 3.5:

$$m_p(E_N) = \sum_{p'\in A_k} \kappa^{p'}(E_N) N^{-\mathsf{df}(p',p)}. \tag{8}$$

Let us suppose that $(E_N)_{N\in\mathbb{N}}$ converges strongly. The triangle inequality for d shows that for any $p \in A_k$, $m_p(E_N)$ converges when N goes to infinity and:

$$\lim_{N\to\infty} m_p(E_N) = \sum_{p'\in[id,p]_{A_k}} \lim_{N\to\infty} \kappa^{p'}(E_N).$$

Now, let us suppose that it converges in moments. We can write (8) as:

$$m^N = G_N \kappa^N,$$

where $m^N = (m_p(E_N))_{p\in A_k(N)}$, $\kappa^N = (\kappa^p(E_N))_{p\in A_k(N)}$, and $G_N = \left(N^{-\mathsf{df}(p',p)}\right)_{p,p'\in A_k(N)}$. The sequence $(G_N)_{N\in\mathbb{N}}$ converges to the matrix G defined in Definition 3.11, and since G is invertible, $\kappa^N = G_N^{-1}m^N$ converges to $G^{-1}m$ where $m = \left(\lim_{N\to\infty} m_p(E_N)\right)_{p\in A_k}$. □

Let us take some notations in order to simplify our up-coming discussions.

Notation 4.1. — *Let $(E_N)_{N\in\mathbb{N}}$ be an element of $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$. From now on, we will say that $(E_N)_{N\in\mathbb{N}}$ converges if and only if it converges either strongly or in moments. Besides, let suppose that $(E_N)_{N\in\mathbb{N}}$ converges, then we will set, for any partition $p \in A_k$ and any $P \subset A_k$:*

$$\begin{aligned} m_p(E) &= \lim_{N\to\infty} m_p(E_N), \\ \kappa^p(E) &= \lim_{N\to\infty} \kappa^p(E_N) \\ \kappa^P(E) &= \sum_{p\in P} \kappa^p(E). \end{aligned}$$

4.2. Consequences of Theorem 4.1.— We have already an interesting corollary of Theorem 4.1.

Theorem 4.2. — *For this theorem, let us suppose that A is equal either to $\mathfrak{S}$ or $\mathcal{B}$. Let $(E_N)_{N\in\mathbb{N}}$ be an element of $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$ which converges in moments, then for any $p \in \mathcal{P}_k$, the limit of $m_p(E_N)$ exists. Besides, for any $p \in \mathcal{P}_k$, the following equality holds:*

$$m_p(E) = \sum_{p'\in A_k, p'\leq p} \kappa^{p'}(E).$$

Proof. — If $(E_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$ converges in moments then it converges strongly. Thus seens as an element of $\prod_{N\in\mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$ it converges also strongly and thus in moments. The Equation (7) allows to conclude. □

In the case where $A = \mathcal{B}$, one can also prove that, under some hypotheses, the convergence of the $\mathfrak{S}$-moments is equivalent to the convergence of the $\mathfrak{S}$-coordinate numbers.

Theorem 4.3. — *Let $(E_N)_{N\in\mathbb{N}}$ be an element of $\prod_{N\in\mathbb{N}} \mathbb{C}[\mathcal{B}_k(N)]$ and let us suppose that for any $p \in \mathcal{B}_k$, $(m_p(E_N))_{N\in\mathbb{N}}$ is bounded. The following assertions are equivalent:*

- *for any $\sigma \in \mathfrak{S}_k$, $\kappa^\sigma(E_N)$ converges when N goes to infinity,*
- *for any $\sigma \in \mathfrak{S}_k$, $m_\sigma(E_N)$ converges when N goes to infinity.*

and if one of the condition is satisfied, then for any $\sigma \in \mathfrak{S}_N$,

$$m_\sigma(E) = \sum_{\sigma'\in[id,\sigma]_{\mathfrak{S}_k}} \kappa^{\sigma'}(E).$$

Proof. — Let $(E_N)_{N\in\mathbb{N}}$ be an element of $\prod_{N\in\mathbb{N}} \mathbb{C}[\mathcal{B}_k(N)]$ which satisfies the hypothesis of the theorem. First of all, using the same notations of the proof of Theorem 4.1, we know that, for N big enough $\kappa^N = G_N^{-1} m^N$. As the sequence $(m^N)_{N\in\mathbb{N}}$ is bounded and as G_N^{-1} converges to G^{-1} when N goes to infinity, we deduce that $(\kappa^N)_{N\in\mathbb{N}}$ is also bounded.

Let $\sigma \in \mathfrak{S}_k$, using Equation (8), for any integer N,

$$m_\sigma(E_N) = \sum_{p'\in\mathcal{B}_k} \kappa^{p'}(E_N) N^{-\mathsf{df}(p',\sigma)}.$$

Yet, if $p' \in \mathcal{B}_k \setminus \mathfrak{S}_k$, using Lemma 3.9, $\mathsf{df}(p',\sigma) < 0$.

Let us suppose that for any $\sigma' \in \mathfrak{S}_k$, $\kappa^{\sigma'}(E_N)$ converges, then $m_\sigma(E_N)$ converges as N goes to infinity, and:

$$\lim_{N\to\infty} m_\sigma(E_N) = \sum_{\sigma'\in[id,\sigma]_{\mathfrak{S}_k}} \kappa^{\sigma'}(E_N).$$

Let us suppose now that for any $\sigma \in \mathfrak{S}_k$, $m_\sigma(E_N)$ converges when N goes to infinity, then for any increasing sequence $(i_N)_{N\in\mathbb{N}}$ of integers such that for any $\sigma' \in \mathfrak{S}_k$, $\kappa^{\sigma'}(E_{i_N})$ converges, we have:

$$\lim_{N\to\infty} m_\sigma(E_N) = \sum_{\sigma'\in[id,\sigma]_{\mathfrak{S}_k}} \lim_{N\to\infty} \kappa^{\sigma'}(E_{i_N}).$$

Hence, for any $\sigma' \in \mathfrak{S}_k$, $\lim_{N\to\infty} \kappa^{\sigma'}(E_{i_N})$ does not depend on the sequence $(i_N)_{N\in\mathbb{N}}$: this shows that for any $\sigma' \in \mathfrak{S}_k$, $\kappa^{\sigma'}(E_N)$ converges when N goes to infinity. Again we get also:

$$\lim_{N\to\infty} m_\sigma(E_N) = \sum_{\sigma'\in[id,\sigma]_{\mathfrak{S}_k}} \kappa^{\sigma'}(E_N).$$

This finishes the proof. □

4.3. Exclusive coordinate numbers and moments. —

4.3.1. Exclusive coordinate numbers. — In Section 2.3, we defined an other basis of $\mathbb{C}[\mathcal{P}_k]$, namely the exclusive basis. In the case we are working with an element $E \in \mathbb{C}[A_k(N)]$ we can also define the exclusive coordinate numbers.

Definition 4.5. — *Let k and N be two positive integers. Let E be an element of $\mathbb{C}[A_k(N)]$. We define the numbers $\left(\kappa_c^p(E)\right)_{p\in\mathcal{P}_k}$ as the only numbers such that:*

$$E = \sum_{p\in\mathcal{P}_k} \frac{\kappa_c^p(E)}{N^{d(id,p)+\frac{-k+\mathsf{nc}(p)}{2}}}\ p^c = \sum_{p\in\mathcal{P}_k} \frac{\kappa_c^p(E)}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}}\ p^c.$$

The family $(\kappa_c^p(E))_{p\in\mathcal{P}_k}$ is called the exclusive coordinate numbers *of E.*

The next proposition shows that one can choose to work either with the exclusive basis or with the usual basis of $\mathbb{C}[A_k]$ in order to study the convergence of $(E_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$.

Theorem 4.4. — *Let $(E_N)_{N\in\mathbb{N}}$ be an element of $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$. The exclusive coordinate numbers $(\kappa_c^p(E_N))_{p\in\mathcal{P}_k}$ converge as N goes to infinity if and only if $(E_N)_{N\in\mathbb{N}}$ converges. Besides, if $(E_N)_{N\in\mathbb{N}}$ converges then for any $p\in\mathcal{P}_k$, $\kappa_c^p(E_N)$ converges as N goes to infinity, and for any $p\in\mathcal{P}_k$:*

$$\lim_{N\to\infty}\kappa_c^p(E_N)=\sum_{p'\in A_k, p'\in \mathsf{Sp}(p)}\lim_{N\to\infty}\kappa_{p'}(E_N)$$

Proof. — Let $(E_N)_{N\in\mathbb{N}}$ be an element of $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$. Then for any positive integer N:

$$E_N=\sum_{p\in A_k}\frac{\kappa^p(E_N)}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}}p=\sum_{p\in A_k}\frac{\kappa^p(E_N)}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}}\sum_{p'\in\mathcal{P}_k|p' \text{ coarser than } p}p'^c$$

$$=\sum_{p\in A_k, p'\in\mathcal{P}_k|p' \text{ coarser than } p}\kappa^p(E_N)N^{-\mathsf{nc}(p)+\mathsf{nc}(p\vee id)+\mathsf{nc}(p')-\mathsf{nc}(p'\vee id)}\frac{p'^c}{N^{\mathsf{nc}(p')-\mathsf{nc}(p'\vee id)}},$$

and using Lemma 3.3:

$$E_N=\sum_{p'\in\mathcal{P}_k}\left(\sum_{p\in A_k, p \text{ finer than } p'}\kappa^p(E_N)N^{-\mathsf{df}(p,p')}\right)\frac{p'^c}{N^{\mathsf{nc}(p')-\mathsf{nc}(p'\vee id)}}.$$

Thus, for any integer N, for any $p'\in\mathcal{P}_k$

$$\kappa_c^{p'}(E_N)=\sum_{p\in A_k, p \text{ finer than } p'}\kappa^p(E_N)N^{-\mathsf{df}(p,p')}. \tag{9}$$

The result follows from this equality, the usual arguments already explained in Theorem 4.1 and Lemma 3.4. □

Let us remark that, using Equality (9), one has the following proposition.

Proposition 4.1. — *Let A be either $\mathfrak{S}$ or $\mathcal{B}$. Let N be an integer, let $E\in\mathbb{C}[A_k(N)]$, for any $p\in A_k$:*

$$\kappa_c^p(E)=\kappa^p(E).$$

Proof. — This is a consequence of Equality (9) and the fact that p' in $\mathcal{P}_k$ is finer than $p'\in A_k$ implies that $p'\notin A_k$. □

4.3.2. Exclusive moments. — As we did for the coordinate numbers, one can define exclusive normalized moments.

Definition 4.6. — *Let $N\in\mathbb{N}$, let $p\in\mathcal{P}_k$ and $E\in\mathbb{C}[A_k(N)]$. The p-exclusive normalized moment of E is:*

$$m_{p^c}(E)=\frac{1}{Tr_N(p)}Tr_N(E\ {}^t(p^c)).$$

One can also give a combinatorial definition of the p-exclusive normalized moment.

Lemma 4.1. — *Let p and p' be in $\mathcal{P}_k$, then:*

$$Tr_N(p\ {}^t(p'^c)) = \delta_{p' \text{ coarser than } p} \frac{N!}{(N-\mathsf{nc}(p'))!}.$$

The easiest way to prove this lemma is to do it graphicaly: we see that p' must be coarser than p, if not the trace is equal to zero, and if p' is coarser than p, it is equal to $\frac{N!}{(N-\mathsf{nc}(p'))!}$.

Recall the Definition 3.8. Similarly to what we proved for coordinate numbers, we prove the following proposition. Let us consider $(E_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$.

Proposition 4.2. — *The sequence $(E_N)_{N\in\mathbb{N}}$ converges in normalized moments if and only if for any $p \in \mathcal{P}_k$, $(m_{p^c}(E_N))_{N\in\mathbb{N}}$ converges. Besides, if $(E_N)_{N\in\mathbb{N}}$ converges in normalized moments then for any $p \in \mathcal{P}_k$:*

$$\lim_{N\to\infty} m_p(E_N) = \sum_{p'\in\mathcal{P}_k, p'\in\mathsf{Gl}_c(p)} \lim_{N\to\infty} m_{p'^c}(E_N).$$

Proof. — Because of Theorem 4.2, it is enough to consider $(E_N)_{N\in\mathbb{N}}$ an element of $\prod_{N\in\mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$. By computation:

$$m_p(E_N) = \sum_{p' \text{coarser than } p} N^{\mathsf{nc}(p'\vee id)-\mathsf{nc}(p\vee id)} m_{p'^c}(E_N).$$

We are in the same setting as for the proof of Theorem 4.1: we can write this equality as:

$$m_N = C_N m_{c,N},$$

where $(m_N)_p = m_p(E_N)$, $(m_{c,N})_p = m_{p^c}(E_N)$ and C_N converges to the matrix C defined in Definition 3.11. With the same arguments than in the proof of Theorem 4.1, we get that m_N converges to infinity if and only if $m_{c,N}$ converges to infinity: the sequence $(E_N)_{N\in\mathbb{N}}$ converges in $\mathcal{P}_k-$exclusive normalized moments if and only if it converges in normalized moments and in this case:

$$\lim_{N\to\infty} m_p(E_N) = \sum_{p'\in\mathsf{Gl}_c(p)} \lim_{N\to\infty} m_{p'^c}(E_N).$$

This finishes the proof. □

4.3.3. In the exclusive world, coordinate numbers and moments are equal. — We will prove that the limit of exclusive normalized moments are in fact equal to the limit of the exclusive coordinate numbers. Let $(E_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$. Let us suppose that $(E_N)_{N\in\mathbb{N}}$ converges in normalized moments.

Theorem 4.5. — *For any $p \in \mathcal{P}_k$,*

$$\lim_{N\to\infty} m_{p^c}(E_N) = \lim_{N\to\infty} \kappa_c^p(E_N).$$

Proof. — We will prove that for any integer N, any $p \in A_k$, seen as an element of $\mathbb{C}[A_k(N)]$, for any $p' \in \mathcal{P}_k$,

$$\kappa_c^{p'}(p) = \left(\prod_{i=0}^{\mathsf{nc}(p')-1} \left(\frac{N}{N-k} \right) \right) m_{p'^c}(p).$$

Indeed by the Equality (9), we get that for any $p' \in \mathcal{P}_k$:

$$\kappa_c^{p'}(p) = \delta_{p' \text{ coarser than } p} N^{\mathsf{nc}(p')-\mathsf{nc}(p' \vee id)}. \tag{10}$$

Let $p' \in \mathcal{P}_k$, by Lemma 4.1:

$$m_{p'^c}(p) = \frac{1}{N^{\mathsf{nc}(p' \vee id)}} Tr_N(p\ {}^t(p'^c)) = \delta_{p' \text{ coarser than } p} \frac{N!}{(N-\mathsf{nc}(p'))!} N^{-\mathsf{nc}(p' \vee id)}.$$

The theorem is now a simple consequence of a linearity argument and taking N going to infinity. □

Let us remark that one can prove Theorem 4.5 also by a purely combinatorial argument using Theorem 3.3. Indeed, using similar notations as the one explained in Notation 4.1, the Theorem 4.4 shows that:

$$\left(\kappa_c^p(E)\right)_{p \in \mathcal{P}_k} = S \left(\kappa^p(E)\right)_{p \in \mathcal{P}_k}.$$

The Equation (7) asserts that:

$$\left(m_p(E)\right)_{p \in \mathcal{P}_k} = G \left(\kappa^p(E)\right)_{p \in \mathcal{P}_k} = CS \left(\kappa^p(E)\right)_{p \in \mathcal{P}_k},$$

where the last equation is a consequence of Theorem 3.3. The Proposition 4.2 proves that:

$$\left(m_{p^c}(E)\right)_{p \in \mathcal{P}_k} = C^{-1} \left(m_p(E)\right)_{p \in \mathcal{P}_k}.$$

Thus:

$$\left(m_{p^c}(E)\right)_{p \in \mathcal{P}_k} = S \left(\kappa^p(E)\right)_{p \in \mathcal{P}_k} = \left(\kappa_c^p(E)\right)_{p \in \mathcal{P}_k}.$$

Using this discussion, we see that one can give an expression of the exclusive moments which involves the coordinate numbers.

Theorem 4.6. — *For any $p \in \mathcal{P}_k$,*

$$\lim_{N \to \infty} m_{p^c}(E_N) = \sum_{p' \in A_k, p' \in \mathsf{Sp}(p)} \lim_{N \to \infty} \kappa^{p'}(E_N).$$

Using Lemma 3.6, one gets the following corollary.

Corollary 4.1. — *Let p be a partition in $\mathcal{P}_k$ which does not have any pivotal block, then:*

$$\lim_{N \to \infty} m_{p^c}(E_N) = \delta_{p \in A_k} \lim_{N \to \infty} \kappa^p(E_N). \tag{11}$$

In particular, for any $p \in \mathcal{B}_k$, the Equality (11) is valid.

Recall the Definition 3.9, using Proposition 3.3, one gets also the following corollary.

Corollary 4.2. — *Let us suppose that A_k is equal either to $\mathfrak{S}_k$ or $\mathcal{B}_k$. For any $p \in \mathcal{P}_k$:*

$$\lim_{N\to\infty} m_{p^c}(E_N) = \delta_{p\in\overline{A_k}} \lim_{N\to\infty} \kappa^{\mathsf{Mb}(p)}(E_N).$$

Let us remark that some simple equalities hold also for finite N, since, for example, as a consequence of Equation 9 and the proof of Theorem 4.5, if A_k is equal either to $\mathfrak{S}_k$ or $\mathcal{B}_k$, for any $N \in \mathbb{N}$ and any $p \in A_k$:

$$\kappa^p(E_N) = \left(\prod_{i=0}^{\mathsf{nc}(p')-1} \left(\frac{N}{N-k} \right) \right) m_{p^c}(E_N).$$

At the beginning of this section, we have argued that the simplest notion of convergence of elements of $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$ was not interesting as it did not take into account the fact that $\mathbb{C}[A_k(N)]$ is an algebra which depends on the parameter N. In the Section 5, we will slightly modify the product defined on $\mathbb{C}[A_k(N)]$ in order to define a new algebra $\mathbb{C}[A_k(N,N)]$. In this new algebra the strong convergence will be the usual notion of convergence in vector spaces.

4.4. Projections and conjugation. — Let A_k be either $\mathcal{B}_k$ or $\mathfrak{S}_k$. Recall the Definition 3.11 where we defined the matrices G, C and S. There exist two natural projections that one can define on $\mathbb{C}[\mathcal{P}_k]$. In order to define them, we need to define the restriction and extension applications.

Definition 4.7. — *The restriction on A_k is defined by:*

$$\begin{aligned} \mathsf{R}_{|A_k} : \quad & \mathbb{C}[\mathcal{P}_k] \to \mathbb{C}[A_k] \\ & (x_p)_{p\in\mathcal{P}_k} \mapsto (x_p)_{p\in A_k}. \end{aligned}$$

The extension from A_k, denoted by E^{A_k}, is the unique application such that $\mathsf{R}_{|A_k} \circ \mathsf{E}^{A_k} = \mathsf{Id}_{A_k}$ and $\left(\mathsf{E}^{A_k}\left((x_p)_{p\in A_k}\right)\right)_{p_0} = 0$ for any $p_0 \notin A_k$.

We define the restriction of G to A_k as:

$$G_{|A_k} = \mathsf{R}_{|A_k} G \mathsf{E}^{A_k}.$$

This is the matrix of the order $\leq$ restrained to A_k.

Definition 4.8. — *The cumulant-projection on A is defined as the application:*

$$\mathcal{C}_\kappa^A : \mathbb{C}[\mathcal{P}_k] \to \mathbb{C}[\mathcal{P}_k]$$

such that:

$$\mathcal{C}_\kappa^A = \mathsf{E}^{A_k} \circ (G_{|A_k})^{-1} \circ \mathsf{R}_{|A_k} \circ G.$$

One can remark that $\mathsf{R}_{|A} \circ G \circ \mathsf{E}^{A_k} \circ (G_{|A_k})^{-1} = \mathsf{Id}_{A_k}$.

Lemma 4.2. — *The cumulant-projection is a projection and $Im(\mathcal{C}_\kappa^A) = \mathsf{E}^{A_k}(\mathbb{C}[A_k])$.*

Proof. — The first assertion is a a consequence of the following computation:

$$\begin{aligned}\mathcal{C}_\kappa^A \circ \mathcal{C}_\kappa^A &= \mathsf{E}^{A_k} \circ (G_{|A_k})^{-1} \circ \mathsf{R}_{|A_k} \circ G \circ \mathsf{E}^{A_k} \circ (G_{|A_k})^{-1} \circ \mathsf{R}_{|A_k} \circ G \\ &= \mathsf{E}^{A_k} \circ (G_{|A_k})^{-1} \circ \big(\mathsf{R}_{|A_k} \circ G \circ \mathsf{E}^{A_k} \circ (G_{|A_k})^{-1}\big) \circ \mathsf{R}_{|A_k} \circ G \\ &= \mathsf{E}^{A_k} \circ (G_{|A_k})^{-1} \circ \mathsf{R}_{|A_k} \circ G = \mathcal{C}_\kappa^A.\end{aligned}$$

Besides, if $x \in \mathsf{E}^{A_k}(\mathbb{C}[A_k])$, there exists $y \in \mathbb{C}[A_k]$ such that $x = \mathsf{E}^{A_k}(y)$. This implies that $\mathcal{C}_\kappa^A(x) = \mathcal{C}_\kappa^A(\mathsf{E}^{A_k}(y)) = \mathsf{E}^{A_k} \circ (G_{|A_k})^{-1} \circ \mathsf{R}_{|A_k} \circ G \circ \mathsf{E}^{A_k}(y) = \mathsf{E}^{A_k} \circ (G_{|A_k})^{-1} \circ G_{|A_k}(y) = \mathsf{E}^{A_k}(y) = x$. Thus $\mathsf{E}^{A_k}(\mathbb{C}[A_k]) \subset Im(\mathcal{C}_\kappa^A)$. It is clear also that $Im(\mathcal{C}_\kappa^A) \subset \mathsf{E}^{A_k}(\mathbb{C}[A_k])$, hence the equality between the r.h.s. and the l.h.s. □

Definition 4.9. — *The moment-projection on A is defined as the application:*

$$\mathcal{C}_m^A : \mathbb{C}[\mathcal{P}_k] \to \mathbb{C}[\mathcal{P}_k]$$

such that:

$$\mathcal{C}_m^A = G \circ \mathcal{C}_\kappa^A \circ G^{-1}.$$

Since $\mathcal{C}_\kappa^A$ is a projection, $\mathcal{C}_m^A$ is also a projection which image is given gy $G \circ \mathsf{E}^{A_k}(\mathbb{C}[A_k])$. It can also be interesting to define $\mathcal{C}_{m^c}^A$ by $\mathcal{C}_{m^c}^A = S \circ \mathcal{C}_\kappa^A \circ S^{-1}$. This is again a projection which image is given by $S \circ \mathsf{E}^{A_k}(\mathbb{C}[A_k])$. In Lemma 4.4, we characterize the set $S \circ \mathsf{E}^{A_k}(\mathbb{C}[A_k])$.

By looking at the definition of the cumulant and moment-projections, one can see that one needs to compute the inverse of $G_{|A_k}$: this amounts to compute the Möbius function for $(A_k, \leq)$. If A is equal to $\mathfrak{S}$, this Möbius functions is well-known since one can use the fact that $(A_k, \leq)$ is isomorphic to the poset of non-crossing partitions for which the Möbius function has been computed by Kreweras in [**12**].

Definition 4.10. — *Let $(m_p)_{p \in \mathcal{P}_k}$ be a element of $\mathbb{C}[\mathcal{P}_k]$. We say that $(m_p)_{p \in \mathcal{P}_k}$ is G-invariant if:*

$$(m_p)_{p \in \mathcal{P}_k} = \mathcal{C}_m^A\big((m_p)_{p \in \mathcal{P}_k}\big).$$

Using the fact that $\mathcal{C}_m^A$ is a projection and that $G = CS$, we get the following lemma.

Lemma 4.3. — *Let $(m_p)_{p \in \mathcal{P}_k}$ be a element of $\mathbb{C}[\mathcal{P}_k]$. It is G-invariant if and only if one of the following conditions is satisfied:*

- $(m_p)_{p \in \mathcal{P}_k} \in G \circ \mathsf{E}^{A_k}(\mathbb{C}[A_k])$,
- $G^{-1}(m_p)_{p \in \mathcal{P}_k} \in \mathsf{E}^{A_k}(\mathbb{C}[A_k])$,
- $C^{-1}(m_p)_{p \in \mathcal{P}_k} \in S \circ \mathsf{E}^{A_k}(\mathbb{C}[A_k])$.

The three conditions have to be understood as conditions on respectively the moments, the coordinate numbers, and the exclusive moments: this is why we wrote the three conditions even if they are obviously equivalent. Let us also remark that the sets $\mathsf{E}^{A_k}(\mathbb{C}[A_k])$ and $S \circ \mathsf{E}^{A_k}(\mathbb{C}[A_k)$ are easy to understand. Recall Definition 3.9 where we defined $\mathsf{Mb}(p)$.

Lemma 4.4. — *The set* $\mathsf{E}^{A_k}(\mathbb{C}[A_k])$ *is the set of elements* x *of* $\mathbb{C}[\mathcal{P}_k]$ *such that for any* $p \notin A_k$, $x_p = 0$.

The set $S \circ \mathsf{E}^{A_k}(\mathbb{C}[A_k])$ *is the set of elements* x *in* $\mathbb{C}[\mathcal{P}_k]$ *such that for any* $p \in \mathcal{P}_k$, $x_p = \delta_{p \in \overline{A_k}} x_{\mathsf{Mb}(p)}$.

Proof. — The first assertion is straightforward. The second is a direct consequence of Proposition 3.3 and Lemma 3.6. Indeed, if $x \in S \circ \mathsf{E}^{A_k}(\mathbb{C}[A_k])$, there exists $y \in \mathbb{C}[A_k]$ such that for any $p \in \mathcal{P}_k$:

$$x_p = \sum_{p' \in A_k | p' \sqsupset p} y_{p'}.$$

If $p \in A_k$, using Lemma 3.6, $x_p = y_p$ and if $p \notin A_k$, using Proposition 3.3, $x_p = y_{\mathsf{Mb}(p)} = x_{\mathsf{Mb}(p)}$. Using the same arguments, one can show that if x in $\mathbb{C}[\mathcal{P}_k]$ satisfies that for any $p \in \mathcal{P}_k$, $x_p = \delta_{p \in \overline{A_k}} x_{\mathsf{Mb}(p)}$, then $x \in S \circ \mathsf{E}^{A_k}(\mathbb{C}[A_k])$. □

Let us consider (A, G) an element of $\{(\mathfrak{S}, U), (\mathcal{B}, O), (\mathcal{P}, \mathfrak{S})\}$. For any positive integer N, $U(N)$, $O(N)$ and $\mathfrak{S}(N)$ are respectively the unitary group of size N, the orthogonal group of size N and the group of permutation matrices of size N. The notion of G-invariant element of $\mathbb{C}[\mathcal{P}_k]$ in Definition 4.10 is motivated by the following proposition.

Proposition 4.3. — *Let* $(E_N)_{N \in \mathbb{N}}$ *be an element of* $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$ *which converges when* N *goes to infinity. For any positive integer* N*, we define*

$$F_N = \int_{G(N)} g^{\otimes k} \rho_N^{\mathcal{P}_k}(E_N)(g^*)^{\otimes k} dg,$$

where dg *is the Haar probability measure on* $G(N)$*. There exists a sequence* $(\tilde{E}_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ *such that for any positive integer* N*:* $\rho_N^{\mathcal{P}_k}(\tilde{E}_n) = F_N$*. The sequence* $(\tilde{E}_N)_{N \in \mathbb{N}}$ *converges as* N *goes to infinity and:*

$$\left(\kappa^p(\tilde{E})\right)_{p \in \mathcal{P}_k} = \mathcal{C}_\kappa^A \left(\left(\kappa^p(E)\right)_{p \in \mathcal{P}_k} \right),$$
$$\left(m_p(\tilde{E})\right)_{p \in \mathcal{P}_k} = \mathcal{C}_m^A \left(\left(m_p(E)\right)_{p \in \mathcal{P}_k} \right),$$
$$\left(m_{p^c}(\tilde{E})\right)_{p \in \mathcal{P}_k} = \mathcal{C}_{m^c}^A \left(\left(m_{p^c}(E)\right)_{p \in \mathcal{P}_k} \right).$$

Proof. — Let us consider $(E_N)_{N \in \mathbb{N}}$ and $(F_N)_{N \in \mathbb{N}}$ two sequences which satisfy the asusmptions of the proposition. The existence of a sequence $(\tilde{E}_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ such that for any positive integer N: $\rho_N^{\mathcal{P}_k}(\tilde{E}_n) = F_N$ is a consequence of the Schur-Weyl-Jones dualities which are explained in the Section 3 of [**9**]. Let us prove that $(\tilde{E}_N)_{N \in \mathbb{N}}$ converges as N goes to infinity and that the two equalities stated in the proposition are satisfied.

In order to do so, it is enough to show that the normalized moments of $(\tilde{E}_N)_{N \in \mathbb{N}}$ converge as N goes to infinity. Let p be in A_k, we have to prove that $m_p(\tilde{E}_N)$ converges

as N goes to infinity. Yet, for any positive integer N:

$$\begin{aligned} m_p(\tilde{E}_N) &= \frac{1}{N^{\mathsf{nc}(p\vee id)}} Tr^k\left(\rho_N^{\mathcal{P}_k}(\tilde{E}_n)\rho_N^{\mathcal{P}_k}({}^tp)\right) \\ &= \frac{1}{N^{\mathsf{nc}(p\vee id)}} Tr^k\left(\left(\int_{G(N)} g^{\otimes k}\rho_N^{\mathcal{P}_k}(E_N)(g^*)^{\otimes k}dg\right)\rho_N^{\mathcal{P}_k}({}^tp)\right) \\ &= \frac{1}{N^{\mathsf{nc}(p\vee id)}}\int_{G(N)} Tr^k\left(\rho_N^{\mathcal{P}_k}(E_N)(g^*)^{\otimes k}\rho_N^{\mathcal{P}_k}({}^tp)g^{\otimes k}\right)dg \end{aligned}$$

By the Schur-Weyl-Jones dualities, for any $g \in G_N$, $(g^*)^{\otimes k}\rho_N^{\mathcal{P}_k}({}^tp)g^{\otimes k} = p$, thus:

$$m_p(\tilde{E}_N) = \frac{1}{N^{\mathsf{nc}(p\vee id)}} Tr^k\left(\rho_N^{\mathcal{P}_k}(E_N)\rho_N^{\mathcal{P}_k}({}^tp)\right) = m_p(E_N).$$

This implies that $m_p(\tilde{E}_N)$ converges as N goes to infinity, and:

$$\left(m_p(\tilde{E})\right)_{p\in A_k} = \mathsf{R}_{|A_k}\left((m_p(E))_{p\in\mathcal{P}_k}\right).$$

Considering this last equality with the following equalities:

$$\begin{aligned} \left(\kappa^p(\tilde{E})\right)_{p\in\mathcal{P}_k} &= \mathsf{E}^{A_k}\left((\kappa^p(\tilde{E}))_{p\in A_k}\right), \\ \left(\kappa^p(\tilde{E})\right)_{p\in A_k} &= (G_{|A_k})^{-1}\left((m_p(\tilde{E}))_{p\in A_k}\right), \\ \left(m_p(E)\right)_{p\in\mathcal{P}_k} &= G\left((\kappa^p(E))_{p\in\mathcal{P}_k}\right), \end{aligned}$$

one gets:

$$\left(\kappa^p(\tilde{E})\right)_{p\in\mathcal{P}_k} = \mathsf{E}^{A_k}\circ(G_{|A_k})^{-1}\circ\mathsf{R}_{|A_k}\circ G\left((\kappa^p(E))_{p\in\mathcal{P}_k}\right),$$

which is nothing but the first equality we had to prove. In order to prove the second one, one can do the following computations:

$$\begin{aligned} \left((m_p(\tilde{E}))_{p\in\mathcal{P}_k}\right) = G\left((\kappa^p(\tilde{E}))_{p\in\mathcal{P}_k}\right) &= G\circ\mathcal{C}_\kappa^A\left((\kappa^p(E))_{p\in\mathcal{P}_k}\right) \\ &= G\circ\mathcal{C}_\kappa^A\circ G^{-1}\left((m_p(E))_{p\in\mathcal{P}_k}\right) \\ &= \mathcal{C}_m^A\left((m_p(E))_{p\in\mathcal{P}_k}\right). \end{aligned}$$

The third one is a consequence of the fact that $\left((m_{p^c}(\tilde{E}))_{p\in\mathcal{P}_k}\right) = S\left((\kappa^p(\tilde{E}))_{p\in\mathcal{P}_k}\right)$. □

5. The deformed partition algebra

Let us define a deformation of the partition algebra by modifying the multiplication which was set in Definition 2.5. Let k and N be two positive integers.

Definition 5.1. — *We define the application:*

$$M_k^N : A_k \to A_k$$
$$p \mapsto \frac{1}{N^{d(id,p)+\frac{-k+\mathsf{nc}(p)}{2}}} p = \frac{1}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}} p.$$

This application can be extended as an isomorphism of vector spaces from $\mathbb{C}[A_k]$ *to itself.*

Seen as a vector space, the algebra $\mathbb{C}[A_k(N)]$ is isomorphic to $\mathbb{C}[A_k]$. Thus, we can see M_k^N as an isomorphism of vector spaces from $\mathbb{C}[A_k]$ to $\mathbb{C}[A_k(N)]$. Let us endow $\mathbb{C}[A_k]$ with a structure of associative algebra by taking the pullback of the structure of algebra of $\mathbb{C}[A_k(N)]$ by M_k^N: for any p_1, p_2 in A_k the new product of p_1 with p_2 is given by:

$$p_1 \cdot_N p_2 = \left(M_k^N\right)^{-1} \left[M_k^N(p_1) M_k^N(p_2)\right].$$

This is the deformed algebra $\mathbb{C}[A_k(N,N)]$. Using the definition of M_k^N, one gets the following proposition.

Proposition 5.1. — *The deformed algebra* $\mathbb{C}[A_k(N,N)]$ *is the associative algebra over* $\mathbb{C}$ *with basis* $\mathcal{P}_k$*, endowed with the multiplication defined by the fact that for any* $p_1, p_2 \in A_k$*:*

$$p_1 \cdot_N p_2 = N^{\kappa(p_1,p_2)} N^{d(id,p_1\circ p_2)-d(id,p_1)-d(id,p_2)+\frac{k+\mathsf{nc}(p_1\circ p_2)-\mathsf{nc}(p_1)-\mathsf{nc}(p_2)}{2}} (p_1 \circ p_2).$$

One can write the exponent in an other form so that it looks like a triangle inequality.

Lemma 5.1. — *Let* p *and* p' *in* A_k*. We have the equality:*

$$d(id, p\circ p') - d(id,p) - d(id,p') + \frac{k + \mathsf{nc}(p\circ p') - \mathsf{nc}(p) - \mathsf{nc}(p')}{2} + \kappa(p,p')$$
$$= d({}^t p', p) - d(id,p) - d(id,p') + k + \mathsf{nc}(p\circ p') - \mathsf{nc}(p) - \mathsf{nc}(p') + 2\kappa(p,p').$$

Proof. — In order to prove this equality, for exemple one can consider N to the power to the r.h.s and the l.h.s. and one can use the following equality:

$$N^{-d({}^t p', p)} = \frac{Tr(pp')}{N^{\frac{\mathsf{nc}(p)+n p(p')}{2}}} = N^{\kappa(p,p')} \frac{Tr(p\circ p')}{N^{\frac{\mathsf{nc}(p)+n p(p')}{2}}} = N^{\kappa(p,p')} \frac{N^{-d(id,p\circ p')+\frac{k+\mathsf{nc}(p\circ p')}{2}}}{N^{\frac{\mathsf{nc}(p)+n p(p')}{2}}}.$$

This allows us to prove Lemma 5.1. □

Using the definition of the deformed algebra $\mathbb{C}[A_k(N,N)]$, we have the straightforward proposition.

Proposition 5.2. — *The application* M_k^N *can be extended as an isomorphism of algebra from* $\mathbb{C}[A_k(N,N)]$ *to* $\mathbb{C}[A_k(N)]$*. Its extension will be also denoted by* M_k^N*.*

For any integer N*, the deformed algebra* $\mathbb{C}[A_k(N,N)]$ *is isomorphic to* $\mathbb{C}[A_k(N)]$*.*

Actually, the application M_k^N is not only compatible with the multiplication, but also with the $\otimes$ operation defined in Definition 2.1.

Lemma 5.2. — *Let k and k' be two positive integers. Let $p \in A_k$ and $p' \in A_{k'}$. The following equality holds:*

$$M_{k+k'}^N(p \otimes p') = M_k^N(p) \otimes M_{k'}^N(p'). \tag{12}$$

The definition of the morphism M_k^N was not chosen randomly: it was set so that the following lemma holds.

Lemma 5.3. — *Let $E \in \mathbb{C}[A_k(N)]$, we have:*

$$\left(M_k^N\right)^{-1}(E) = \sum_{p \in A_k} \kappa^p(E)p.$$

Thus, one can see that we will be able to formulate the strong convergence in $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$ by using the morphisms $\left(M_k^N\right)_{N\in\mathbb{N}}$ and the usual notion of convergence in vector spaces. Indeed, for any integers N and k, any element in $\mathbb{C}[A_k(N,N)]$ can be considered as an element of $\mathbb{C}[A_k]$. This allows us to state the following lemma.

Lemma 5.4. — *Let $(E_N)_{N\in\mathbb{N}}$ be an element of $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$. The sequence $(E_N)_{N\in\mathbb{N}}$ converges strongly if and only if:*

$$(M_k^N)^{-1}(E_N)$$

converges when N goes to infinity in $\mathbb{C}[A_k]$ for the usual convergence in finite dimensional vector spaces.

6. Refined geometry of the partition algebra

In the last section, we defined the deformed algebra $\mathbb{C}[A_k(N,N)]$ and we explained that the strong convergence can be seen as the natural notion of convergence in finite dimensional vector space as soon as one works in the deformed algebra. In this section, we will study the convergence of the algebras $\mathbb{C}[A_k(N,N)]$.

The core of Section 3 was to prove the triangle inequality for the function d defined on A_k in Definition 3.1. The study of the convergence of the algebras $\mathbb{C}[A_k(N,N)]$ will use intensively the following improved triangle inequality for A_k.

Proposition 6.1. — *Let p and p' be two elements of $\mathcal{P}_k$, the following improved triangle inequality holds:*

$$d(p',p) \leq d(p',id) + d(p,id) - k - \mathsf{nc}(p \circ {}^t p') + \mathsf{nc}(p) + \mathsf{nc}(p') - 2\kappa(p, {}^t p').$$

The restriction of the improved triangle inequality to the permutations is obvious as it is a consequence of the usual triangle inequality. Indeed, for any permutations σ and σ', $\mathsf{nc}(\sigma) = 0$ and $\kappa(p,p') = 0$. Yet, this is indeed an improved triangle inequality as soon as one considers elements on $\mathcal{B}_k$: let us suppose that p and p' are equal to the Weyl contraction $[1,2]$. The triangle inequality asserts that $0 \leq 2$, since $d(id,[1,2]) = 1$. Yet, in this case:

$$d(p',id) + d(p,id) - k - \mathsf{nc}(p \circ {}^t p') + \mathsf{nc}(p) + \mathsf{nc}(p') - 2\kappa(p, {}^t p') = 0.$$

The improved triangle inequality asserts thus the stronger fact that $0 \leq 0$.

In fact, we can see this improved triangle inequality as a consequence of the usual triangle inequality and an inequality between $d(p, p \circ p')$ and $d(id, p')$. If we consider p and p' in the symmetric group, then we know that $d(p, p \circ p') = d(p, pp') = d(id, p')$. Yet, this equality does not hold any more in the general case, we only get the following inequality.

Proposition 6.2. — *Let p and p' in $\mathcal{P}_k$, we have the following inequality:*

$$d(p, p \circ p') \leq d(id, p') - \frac{k + \mathsf{nc}(p \circ p') - \mathsf{nc}(p) - \mathsf{nc}(p')}{2} - \kappa(p, p').$$

Proof. — Let p and p' in $\mathcal{P}_k$ and let us define $\tau \in \mathfrak{S}_{2k}$:

$$\tau_k = (1, k+1)(2, k+2) \dots (k, 2k).$$

Let us apply the triangle inequality:

$$d\left(p \otimes id_k, \left((p \circ p') \otimes id_k\right) \tau\right) \leq d(p \otimes id_k, p \otimes p') + d\left(p \otimes p', \left((p \circ p') \otimes id_k\right) \tau\right). \tag{13}$$

The goal is to understand each of these three terms. The term $d(p \otimes id_k, p \otimes p')$ is simple:

$$d(p \otimes id_k, p \otimes p') = d(id, p').$$

Let us study $d\left(p \otimes id_k, ((p \circ p') \otimes id_k)\, \tau\right)$. Using the definition of the distance in Proposition 3.1, and the Equality (3):

$$N^{-d(p \otimes id_k, ((p \circ p') \otimes id_k)\tau)} = \frac{Tr_N\left[(p \otimes id_k)\ {}^t(((p \circ p') \otimes id_k)\, \tau)\right]}{N^{\frac{\mathsf{nc}(p)+k}{2}} N^{\frac{\mathsf{nc}(p \circ p')+k}{2}}},$$

since $\mathsf{nc}(p \otimes id_k) = \mathsf{nc}(p) + k$ and $\mathsf{nc}(((p \circ p') \otimes id_k)\, \tau) = \mathsf{nc}((p \circ p') \otimes id_k) = \mathsf{nc}((p \circ p') + k$. Yet:

$$Tr_N\left[(p \otimes id_k)\ {}^t\left(\left((p \circ p') \otimes id_k\right) \tau\right)\right] = Tr_N\left[p\ {}^t(p \circ p')\right].$$

Thus, using again Proposition 3.1:

$$d\left(p \otimes id_k, \left((p \circ p') \otimes id_k\right) \tau\right) = d(p, p \circ p') + k.$$

Let us consider $d\left(p \otimes p', ((p \circ p') \otimes id_k)\, \tau\right)$. Using the same arguments:

$$N^{-d(p \otimes p', ((p \circ p') \otimes id_k)\tau)} = \frac{Tr_N\left(pp'\ {}^t(p \circ p')\right)}{N^{\frac{\mathsf{nc}(p)+\mathsf{nc}(p')}{2}} N^{\frac{\mathsf{nc}(p \circ p')+k}{2}}}.$$

Using the definition of $\kappa(p, p')$ and the Equality (3):

$$\begin{aligned} N^{-d(p \otimes p', ((p \circ p') \otimes id_k)\tau)} &= N^{\kappa(p,p')} \frac{Tr_N\left(p \circ p'\ {}^t(p \circ p')\right)}{N^{\frac{\mathsf{nc}(p)+\mathsf{nc}(p')}{2}} N^{\frac{\mathsf{nc}(p \circ p')+k}{2}}} \\ &= N^{\kappa(p,p') + \frac{1}{2}[\mathsf{nc}(p \circ p') - \mathsf{nc}(p) - \mathsf{nc}(p') - k]}. \end{aligned}$$

Thus:

$$d\left(p \otimes p', \left((p \circ p') \otimes id_k\right) \tau\right) = -\kappa(p, p') - \frac{1}{2}[\mathsf{nc}(p \circ p') - \mathsf{nc}(p) - \mathsf{nc}(p') - k].$$

Let us come back to the triangle inequality (13). This shows that:

$$d(p, p \circ p') + k \leq d(id, p') - \kappa(p, p') - \frac{1}{2}[\mathsf{nc}(p \circ p') - \mathsf{nc}(p) - \mathsf{nc}(p') - k],$$

and thus:

$$d(p, p\circ p') \leq d(id, p') - \frac{\mathsf{nc}(\mathsf{p}\circ\mathsf{p}') + k - \mathsf{nc}(p) - \mathsf{nc}(p')}{2} - \kappa(p, p').$$

This is the inequality we wanted to prove. □

Proof of Proposition 6.1. — Let p and p' be two elements of A_k. Using the triangle inequality:

$$d(id, p\circ p') \leq d(id, p) + d(p, p\circ p').$$

And an application of Proposition 6.2 implies that:

$$d(id, p\circ p') \leq d(id, p) + d(id, p') - \frac{k + \mathsf{nc}(p\circ p') - \mathsf{nc}(p) - \mathsf{nc}(p')}{2} - \kappa(p, p'). \tag{14}$$

And using Lemma 5.1:

$$d({}^t p', p) \leq d(id, p) - d(id, p') + k + \mathsf{nc}(p\circ p') - \mathsf{nc}(p) - \mathsf{nc}(p') + 2\kappa(p, p').$$

The result follows then from the fact that $\mathsf{nc}({}^t p') = \mathsf{nc}(p')$. □

We can generalize the inequality (14) to a n-uple of elements of A_k.

Lemma 6.1. — *For any positive integer n, for any $(p_i)_{i=1}^n \in (A_k)^n$:*

$$d(id, \circ_{i=1}^n p_i) \leq \sum_{i=1}^n d(id, p_i) - \frac{1}{2}\left[(n-1)k + \mathsf{nc}(\circ_{i=1}^n p_i) - \sum_{i=1}^n \mathsf{nc}(p_i)\right] - \sum_{i=1}^{n-1} \kappa(p_i, p_{i+1}),$$

where we have used the notation $\circ_{i=1}^n p_i = p_1 \circ \ldots \circ p_n$.

In fact, the best way to understand the improved triangle inequality is to work with the equivalent inequality (14). This formulation of the improved triangle inequality leads us to the next notion.

Definition 6.1. — *Let p and p' be two elements of A_k. We will say that $p \prec p\circ p'$ if and only if:*

$$d(id, p\circ p') - d(id, p) - d(id, p') + \frac{k + \mathsf{nc}(p\circ p') - \mathsf{nc}(p) - \mathsf{nc}(p')}{2} + \kappa(p, p') = 0.$$

Let $p_0 \in A_k$. We will write that $p \prec p_0$ if there exists $p' \in A_k$ such that $p_0 = p\circ p'$ and $p \prec p\circ p'$.

Definition 6.2. — *Let us suppose that $p \prec p_0$. We define for any $p \prec p_0$:*

$$K_{p_0}(p) = \{p' \in A_k, p\circ p' = p_0\}.$$

Let us suppose that $p \prec p\circ p'$. We recall that:

$$\begin{aligned} d(id, p\circ p') &\leq d(id, p) + d(p, p\circ p') \\ &\leq d(id, p) + d(id, p') - \frac{k + \mathsf{nc}(p\circ p') - \mathsf{nc}(p) - \mathsf{nc}(p')}{2} - \kappa(p, p'). \end{aligned}$$

Thus, if the first term and the third term are equal, then $p \in [id, p\circ p']_{A_k}$. We have shown the following lemma.

Lemma 6.2. — *Let p and p_0 in A_k, if $p \prec p_0$ then there exists $p' \in A_k$ such that $p_0 = p \circ p'$ and*

$$p \in [id, p_0]_{A_k}.$$

Let us remark that for any $\sigma \in \mathfrak{S}_k$, $\{\sigma' \in \mathfrak{S}_k, \sigma' \prec \sigma\} = [id, \sigma]_{\mathfrak{S}_k}$. This is due to the fact that $\kappa(\sigma, \sigma') = 0$ for any couple of permutations, the fact that nc is constant on the set of permutations and the fact that any permutation is invertible. Using similar arguments and Lemma 2.1, one can have the better result.

Lemma 6.3. — *Let k be an integer. Let $\sigma \in \mathfrak{S}_k$, then:*

$$\{p \in \mathcal{P}_k, p \prec \sigma\} = [id, \sigma]_{\mathfrak{S}_k}.$$

Besides, for any $p \in \mathcal{P}_k$, $\{\sigma \in \mathfrak{S}_k, \sigma \prec p\} = [id, p]_{\mathcal{P}_k} \cap \mathfrak{S}_k$.

Let us state a consequence of Lemma 6.2: the factorization property for $\prec$.

Lemma 6.4. — *Let k and l be two positive integers. Let $a \in \mathcal{P}_k$ and $b \in \mathcal{P}_l$. For any $p \in \mathcal{P}_{k+l}$ such that $p \prec a \otimes b$, there exist $p_1 \prec a$ and $p_2 \prec b$ such that $p = p_1 \otimes p_2$.*

This lemma is a consequence of Lemma 6.2 and the factorization property for the geodesics stated in Lemma 3.8.

Let p and p_0 in A_k such that $p \prec p_0$. Let us have a little discussion on $K_{p_0}(p)$: by definition this is not empty but it is not reduced to a unique partition. For example, one can show that if $p = \{\{1, 2, 1', 2'\}\}$ and $p_0 = \{\{1', 2'\}, \{1\}, \{2\}\}$ then:

$$K_{p_0}(p) = \left\{\left\{\{1\}, \{2\}, \{1'\}, \{2'\}\right\}, \left\{\{1\}, \{2\}, \{1', 2'\}\right\}\right\}.$$

Let $(1, \ldots, k)$ be the k-cycle in $\mathfrak{S}_k$ that we already defined in Definition 2.10. It is well-known that the poset of non-crossing partition over $\{1, \ldots, k\}$ is isomorphic to $([id, (1, \ldots, k)]_{\mathfrak{S}_k}, \leq)$. From now on, we will consider any non-crossing partition over $\{1, \ldots, k\}$ as an element of $[id, (1, .., k)]_{\mathfrak{S}_k}$. The following lemma is straightforward.

Lemma 6.5. — *If $p_0 = (1, ..., k)$ and $p \in [id, (1, ..., k)]_{\mathfrak{S}_k}$, then $K_{p_0}(p)$ is the Kreweras complement of the non-crossing partition corresponding to p.*

We are going now to see one of the main results of the paper, namely the fact that the improved triangle inequality implies the convergence of the deformed algebras $\left(\mathbb{C}[A_k(N, N)])\right)_{N \in \mathbb{N}}$ stated in the forthcoming Theorem 6.1. Before doing so, we need to define the notion of convergence of algebras.

Definition 6.3. — *Let C be a finite set of elements. For any $N \in \mathbb{N} \cup \{\infty\}$, let L_N be an algebra such that C is a linear basis of L_N. For any elements x and y of C, for each $N \in \mathbb{N} \cup \{\infty\}$, we denote the product of x with y in L_N by $x._N y$.*

We say that L_N converges to the algebra L_∞ when N goes to infinity if for any x and y in C,

$$x._N y \underset{N \to \infty}{\longrightarrow} x._\infty y \text{ in } \mathbb{C}[C],$$

for the usual notion of convergence in finite dimensional linear spaces.

Let us state the convergence of the deformed algebras $\left(\mathbb{C}[A_k(N, N)])\right)_{N \in \mathbb{N}}$.

Theorem 6.1. — *As N goes to infinity, the deformed algebra $\mathbb{C}[A_k(N,N)]$ converges to the deformed algebra $\mathbb{C}[A_k(\infty,\infty)]$ which is the associative algebra over $\mathbb{C}$ with basis A_k endowed with the multiplication defined by:*

$$\forall p,p' \in \mathcal{P}_k,\ p._\infty p' = \delta_{p\prec p\circ p'}\ p\circ p'.$$

Proof. — For any $N \in \mathbb{N}\cup\{\infty\}$, A_k is a linear basis of $\mathbb{C}[A_k(N,N)]$. By bi-linearity of the product, it is enough to prove that for any p and p' in A_k, $p._N p'$ converges to $\delta_{p\prec p\circ p'}p\circ p'$. Let p and p' be two elements of $\mathcal{P}$. We have:

$$p._N p' = N^{d(id,p\circ p')-d(id,p)-d(id,p')+\frac{k+\mathsf{nc}(p\circ p')-\mathsf{nc}(p)-\mathsf{nc}(p')}{2}+\kappa(p,p')}(p\circ p').$$

By the version of the improved triangle inequality stated in Proposition 6.1 or in the inequality (14), we have:

$$d(id,p\circ p')-d(id,p)-d(id,p')+\frac{k+\mathsf{nc}(p\circ p')-\mathsf{nc}(p)-\mathsf{nc}(p')}{2}+\kappa(p,p')\leq 0.$$

According to Definition 6.1, we have $p._N p' \underset{N\to\infty}{\longrightarrow} \delta_{p\prec p\circ p'}\ p\circ p'$. □

To conclude this section, let us remark that for any integer k, we have the inclusion of algebras:

$$\mathbb{C}[\mathfrak{S}_k(\infty,\infty)]\subset\mathbb{C}[\mathcal{B}_k(\infty,\infty)]\subset\mathbb{C}[\mathcal{P}_k(\infty,\infty)].$$

7. Consequences of the convergence of the deformed algebras.

7.1. Convergence of a product. — Let k be an integer. As usual, let A_k be $\mathfrak{S}_k$, $\mathcal{B}_k$ or $\mathcal{P}_k$. Let us give the first consequence of Theorem 6.1 for the product of two elements of $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$. Recall the Notation 4.1.

Theorem 7.1. — *Let $(E_N)_{N\in\mathbb{N}}$ and $(F_N)_{N\in\mathbb{N}}$ be two elements of $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$. Let us suppose that $(E_N)_{N\in\mathbb{N}}$ and $(F_N)_{N\in\mathbb{N}}$ converge, then the sequence $\left(E_NF_N\right)_{N\in\mathbb{N}}$ converges. Besides,*

– *for any $p_0\in A_k$:*

$$\kappa^{p_0}(EF)=\sum_{p\in A_k,p\prec p_0}\kappa^p(E)\kappa^{K_{p_0}(p)}(F), \tag{15}$$

– *for any $p_0\in\mathcal{P}_k$:*

$$m_{p_0}(EF)=\sum_{p\in A_k,p\leq p_0}\kappa^p(E)m_{\iota_{p\circ p_0}}(F). \tag{16}$$

Proof. — Let $(E_N)_{N\in\mathbb{N}}$ and $(F_N)_{N\in\mathbb{N}}$ elements of $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$. Let us suppose that $(E_N)_{N\in\mathbb{N}}$ and $(F_N)_{N\in\mathbb{N}}$ converge. We have by definition:

$$(M_k^N)^{-1}(E_NF_N)=\left(M_k^N\right)^{-1}(E_N)._N\left(M_k^N\right)^{-1}(F_N).$$

We know, by Lemma 5.4, that $\left(M_k^N\right)^{-1}(E_N)$ and $\left(M_k^N\right)^{-1}(F_N)$, seen as elements of $\mathbb{C}[A_k]$, converge when N goes to infinity. Besides, the algebra $\mathbb{C}[A_k(N,N)]$ converges to

$\mathbb{C}[A_k(\infty,\infty)]$, as it was proved in Theorem 6.1. Thus $(M_k^N)^{-1}(E_N F_N)$ converges when N goes to infinity. Again, by Lemma 5.4 and Theorem 4.1, this shows that $(E_N F_N)_{N\in\mathbb{N}}$ converges. Besides, using Lemma 5.3, we have:

$$(M_k^N)^{-1}(E_N F_N) = \sum_{p\in A_k} \kappa^{p_0}(E_N F_N) p_0,$$

$$\left(M_k^N\right)^{-1}(E_N).{}_N \left(M_k^N\right)^{-1}(F_N) = \sum_{p\in A_k, p'\in A_k} \kappa^{p}(E_N)\kappa^{p'}(F_N) p.{}_N p'.$$

Using the formula for the limit of $._N$ shown in Theorem 6.1, for any $p_0 \in \mathcal{P}_k$:

$$\kappa^{p_0}(EF) = \sum_{p\in A_k, p\prec p_0} \kappa^{p}(E)\kappa^{K_{p_0}(p)}(F).$$

For the second equality, one could use the link, between A_k-moments and coordinate numbers when $N\to\infty$ given by Equality (7). Yet, this happens to be more difficult than a direct proof. Indeed, by bi-linearity, we have only to show that the equality (16) holds when, for any integer N:

$$E_N = \frac{1}{N^{\frac{-k+\mathsf{nc}(p)}{2}+d(id,p)}} p,$$

with $p\in A_k$. Let N be an integer, let us suppose that E_N is of this form. Let $p_0\in\mathcal{P}_k$, we have:

$$\begin{aligned} m_{p_0}(E_N F_N) &= \frac{1}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}} \frac{1}{N^{\mathsf{nc}(p_0\vee id)}} Tr(F_N\ {}^t p_0 p) \\ &= N^{-\mathsf{nc}(p)+\mathsf{nc}(p\vee id)-\mathsf{nc}(p_0\vee id)} N^{\kappa({}^t p_0, p)} Tr(F_N\, {}^t p_0 \circ p) \\ &= N^{-\mathsf{nc}(p)+\mathsf{nc}(p\vee id)-\mathsf{nc}(p_0\vee id)} N^{\kappa({}^t p_0, p)+\mathsf{nc}({}^t p\circ p_0\vee id)} m_{{}^t p\circ p_0}(F_N) \\ &= N^{-\mathsf{nc}(p)+\mathsf{nc}(p\vee id)-\mathsf{nc}(p_0\vee id)} N^{\kappa({}^t p_0, p)+\mathsf{nc}({}^t p_0\circ p\vee id)} m_{{}^t p\circ p_0}(F_N). \end{aligned}$$

Yet, we have seen in the proof of Lemma 3.1, that:

$$\mathsf{nc}({}^t p_0 \circ p \vee id) + \kappa({}^t p_0, p) = \mathsf{nc}(p_0 \vee p),$$

thus:

$$\begin{aligned} m_{p_0}(E_N F_N) &= N^{-\mathsf{nc}(p)+\mathsf{nc}(p\vee id)-\mathsf{nc}(p_0\vee id)+\mathsf{nc}(p_0\vee p)} m_{{}^t p\circ p_0}(F_N) \\ &= N^{-\mathsf{df}(p,p_0)} m_{{}^t p\circ p_0}(F_N). \end{aligned}$$

One gets that $m_{p_0}(E_N F_N)$ converges when N goes to infinity to $\delta_{p\le p_0} m_{{}^t p\circ p_0}(F_N)$. □

Remark 7.1. — *Under the same assumptions, one can prove that for any $p_0\in\mathcal{P}_k$:*

$$m_{p_0}(EF) = \sum_{p\in A_k, p\le p_0} m_{p_0\circ {}^t p}(E)\kappa^p(F). \tag{17}$$

7.2. Semi-groups. — In this subsection, we will study convergence of sequences of semi-groups in $\mathbb{C}[A_k(N)]$. Semi-groups in different algebras will appear in the paper: for this paper, a family $(a_t)_{t\geq 0}$ is a semi-group if there exists h, called the generator, such that for any $t_0 \geq 0$:

$$\frac{d}{dt}_{|t=t_0} a_t = h a_{t_0}.$$

If we consider the algebra $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$, we are led to the next definition.

Definition 7.1. — *The family* $\left((E_t^N)_N\right)_{t\geq 0}$ *is a semi-group in* $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$ *if there exists* $(H_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$*, called the generator, such that for any* $t \geq 0$*, for any integer* N*:*

$$\frac{d}{dt}_{|t=t_0} E_t^N = H_N E_{t_0}^N.$$

From now on, let us suppose that $\left((E_t^N)_N\right)_{t\geq 0}$ is a semi-group in $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$ whose generator is $(H_N)_{N\in\mathbb{N}}$. Let us define the convergence for semi-groups in $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$.

Definition 7.2. — *The semi-group* $\left((E_t^N)_N\right)_{t\geq 0}$ *converges if and only if for any* $t \geq 0$*,* E_t^N *converges as* N *goes to infinity.*

The next theorem shows that a semi-group in $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$ converges if the initial condition and the generator converge. Recall the Notation 4.1.

Theorem 7.2. — *The semi-group* $\left((E_t^N)_N\right)_{t\geq 0}$ *converges if the sequences* $(E_0^N)_{N\in\mathbb{N}}$ *and* $(H_N)_{N\in\mathbb{N}}$ *converge as* N *goes to infinity.*

Besides, we have the two differential systems of equations:

$$\forall p \in A_k, \forall t_0 \geq 0, \quad \frac{d}{dt}_{|t=t_0} \kappa^p(E_t) = \sum_{p_1\in A_k, p_1 \prec p} \kappa^{p_1}(H)\kappa^{K_p(p_1)}(E_{t_0}). \tag{18}$$

$$\forall p \in \mathcal{P}_k, \forall t_0 \geq 0, \quad \frac{d}{dt}_{|t=t_0} m_p(E_t) = \sum_{p_1\in A_k, p_1 \leq p} \kappa^{p_1}(H) m_{{}^t p_1 \circ p}(E_{t_0}). \tag{19}$$

Proof. — Let us suppose that $(H_N)_{N\in\mathbb{N}}$ converges. For any integer N and any $t \geq 0$, we define:

$$\tilde{E}_t^N = (M_k^N)^{-1}(E_t^N),$$
$$\tilde{H}_N = (M_k^N)^{-1}(H_N).$$

As for any integer N, M_k^N is a morphism of algebra, the family $\left((\tilde{E}_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a semi-group in $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N,N)]$ and its generator is $\left(\tilde{H}_N\right)_{N\in\mathbb{N}}$. An application of Lemma 5.3 allows us to write the condition of semi-group in the basis A_k ; for any $t_0 \geq 0$:

$$\frac{d}{dt}_{|t=t_0} \sum_{p_0\in A_k} \kappa^{p_0}(E_t^N)p_0 = \left(\sum_{p\in A_k} \kappa^p(H_N)p\right).{}_N\left(\sum_{p'\in A_k} \kappa^{p'}(E_{t_0}^N)p'\right).$$

Then the following equality must hold for any positive integer N, any $t_0 \geq 0$ and any $p_0 \in A_k$:

$$\frac{d}{dt}_{|t=t_0} \kappa^{p_0}(E_t^N)$$
$$= \sum_{p,p' \in A_k, p \circ p' = p_0} \kappa^p(H_N)\kappa^{p'}(E_t^N) N^{d(id,p\circ p') - d(id,p) - d(id,p') + \frac{k + \mathsf{nc}(p\circ p') - \mathsf{nc}(p) - \mathsf{nc}(p')}{2} + \kappa(p,p')}.$$

Let us take N going to infinity. Because of the hypotheses and because of the improved triangle inequality, this differential system converges: $\kappa^p(E_t^N)$ must converge for any $p \in A_k$ and any real $t \geq 0$. Besides, we get for any $t_0 \geq 0$:

$$\forall p \in A_k, \frac{d}{dt}_{|t=t_0} \kappa^p(E_t) = \sum_{p_1 \in A_k, p_1 \prec p} \kappa^{p_1}(H)\kappa^{K_p(p_1)}(E_{t_0}).$$

Since the semi-group converges, using the usual notations, we can write that for any $p \in \mathcal{P}_k$ and any $t_0 \geq 0$:

$$\frac{d}{dt}_{|t=t_0} m_p(E_t) = m_p\left(HE_{t_0}\right),$$

and using Equality (16), one has:

$$\lim_{N\to\infty} m_p(H_N E_{t_0}^N) = \sum_{p_1 \in A_k, p_1 \leq p} \kappa^{p_1}(H) m_{{}^t p_1 \circ p}(E_{t_0}).$$

Hence we recover the second system of differential equations. □

Of course one also has, using Equality (17) instead of (16), that for any $p \in \mathcal{P}_k$ and any $t_0 \geq 0$:

$$\frac{d}{dt}_{|t=t_0} m_p(E_t) = \sum_{p_1 \in A_k, p_1 \leq p} m_{p \circ {}^t p_1}(H)\kappa^{p_1}(E_{t_0}).$$

Moreover, Theorem 7.2 can be very easily generalized for any semi-group with time dependent generator. In order to finish the section, let us prove a consequence of Lemma 2.1.

Theorem 7.3. — *Let $\left((E_t^N)_N\right)_{t\geq 0}$ be a semi-group in $\prod_{N\in\mathbb{N}} \mathbb{C}[\mathcal{B}_k(N)]$. Let us suppose that the sequence $(E_0^N)_{N\in\mathbb{N}}$ converges as N goes to infinity. Let us suppose that for any $\sigma \in \mathfrak{S}_k$, $\kappa^\sigma(H_N)$ converges when N goes to infinity. Then for any $\sigma \in \mathfrak{S}_k$, for any positive real t, $\kappa^\sigma(E_t^N)$ converges as N goes to infinity. Besides for any $\sigma \in \mathfrak{S}_k$ and any $t_0 \geq 0$:*

$$\frac{d}{dt}_{|t=t_0} \kappa^\sigma(E_t) = \sum_{\sigma_1 \in \mathfrak{S}_k, \sigma_1 \in [id,\sigma]_{\mathfrak{S}_k}} \kappa^{\sigma_1}(H)\kappa^{\sigma_1^{-1}\sigma}(E_{t_0}). \tag{20}$$

Proof. — Let $\left((E_t^N)_N\right)_{t\geq 0}$ be a semi-group in $\prod_{N\in\mathbb{N}} \mathbb{C}[\mathcal{B}_k(N)]$ which satisfies the hypotheses of the theorem. Let $\sigma \in \mathfrak{S}_k$ and let N be a positive integer. We have seen in

the last proof that for any $t_0 \geq 0$:

$$\frac{d}{dt}_{|t=t_0} \kappa^\sigma(E_t^N)$$
$$= \sum_{p,p' \in \mathcal{B}_k, p \circ p' = \sigma} \kappa^p(H_N)\kappa^{p'}(E_{t_0}^N) N^{d(id,p\circ p') - d(id,p) - d(id,p') + \frac{k + \mathsf{nc}(p\circ p') - \mathsf{nc}(p) - \mathsf{nc}(p')}{2} + \kappa(p,p')}.$$

Yet, by Lemma 2.1, if $p \circ p' = \sigma$, then p and p' are in $\mathfrak{S}_k$. Thus,

$$\frac{d}{dt}_{|t=t_0} \kappa^\sigma(E_t^N) = \sum_{p,p' \in \mathfrak{S}_k, p\circ p' = \sigma} \kappa^p(H_N)\kappa^{p'}(E_{t_0}^N) N^{-\mathsf{df}(p,\sigma)}.$$

Thus, we see that $\left((\kappa^\sigma(E_t^N))_{\sigma \in \mathfrak{S}_k}\right)_{t \geq 0}$ satisfies a linear differential system whose coefficients converge by hypothesis. Thus, for any $\sigma \in \mathfrak{S}_k$, for any positive real t, $\kappa^\sigma(E_t^N)$ converges as N goes to infinity. The Equation (20) is obtained by taking N going to infinity in the last equation. □

8. Geometric and combinatorial consequences of Theorem 4.1

In Section 6, we showed new inequalities on the set of partitions $\mathcal{P}_k$. The proofs were quite combinatorial, and used only the notion of distance. In this section, we want to show that one can prove new inequalities or equalities, by using Theorem 4.1 as a black box.

8.1. Geometric consequences of Theorem 4.1. — First, let us give a new proof of the improved triangle inequality.

Proof of Proposition 6.1. — Let k be an integer. Let p and p' be two elements of A_k. Let us consider $(\mathbf{p}_N)_{N\in\mathbb{N}}$ and $(\mathbf{p}'_N)_{N\in\mathbb{N}}$ such that for any integer N:

$$\mathbf{p}_N = M_k^N(p),$$
$$\mathbf{p}'_N = M_k^N(p').$$

Using Lemma 5.4, $(\mathbf{p}_N)_{N\in\mathbb{N}}$ and $(\mathbf{p}'_N)_{N\in\mathbb{N}}$ converge strongly. Let N be an integer. Applying the Equality (12), we have:

$$\mathbf{p}_N \otimes \mathbf{p}'_N = M_{2k}^N(p \otimes p').$$

Thus, using Lemma 5.4, $\mathbf{p}_N \otimes \mathbf{p}'_N$ converges strongly when N goes to infinity. An application of Theorem 4.1 shows that it converges in moments: for any $\tilde{p} \in A_{2k}$,

$$m_{\tilde{p}}(\mathbf{p}_N \otimes \mathbf{p}'_N) \text{ converges when } N \to \infty.$$

For any partition $\tilde{p} \in A_k$, we define $P(\tilde{p})$ be the partition in A_{2k}:

$$P(\tilde{p}) = (\tilde{p} \otimes id_k)(1, k+1)(2, k+2)\dots(k, 2k).$$

Then for any $E \in \mathbb{C}[A_k(N)]$ and $F \in \mathbb{C}[A_k(N)]$, and any $p_0 \in A_k$, we have:

$$m_{P(p_0)}(E \otimes F) = m_{p_0}(EF).$$

Thus for any $p_0 \in A_k$, $m_{p_0}(\mathbf{p}_N\mathbf{p}'_N)$ which is equal to $m_{P(p_0)}(\mathbf{p}_N \otimes \mathbf{p}'_N)$ converges as N goes to infinity. Using again the Theorem 4.1, we have that $\mathbf{p}_N\mathbf{p}'_N$ converges strongly

as N goes to infinity. It implies, because of Lemma 5.4 that $(M_k^N)^{-1}(\mathbf{p}_N\mathbf{p}'_N)$ converges in $\mathbb{C}[A_k]$ when N goes of infinity. We can calculate this last expression:

$$\begin{aligned}(M_k^N)^{-1}(\mathbf{p}_N\mathbf{p}'_N) &= (M_k^N)^{-1}(M_k^N(p)M_k^N(p')) \\ &= p._N p' \\ &= N^{d({}^tp',p)-d(id,p)-d(id,p')+k+\mathsf{nc}(p\circ p')-\mathsf{nc}(p)-\mathsf{nc}(p')+2\kappa(p,p')}(p\circ p'),\end{aligned}$$

where we used Lemma 5.1. Thus we must have that for any p and p' in A_k:

$$d({}^tp',p) \leq d(id,p)+d(id,p')-k-\mathsf{nc}(p\circ p')+\mathsf{nc}(p)+\mathsf{nc}(p')-2\kappa(p,p').$$

The improved inequality is just a consequence of the last inequality as soon as we see that for any $p \in A_k$, $\mathsf{nc}({}^tp) = \mathsf{nc}(p)$, and $d(id,p) = d(id,{}^tp)$. □

Again, using the same ideas, one can show the following interesting property, which will be important in order to compute in Theorem 7.6 in [**9**] the law of the product of two $\mathcal{P}$-free families.

Proposition 8.1. — *Let p_0, p_1 and p_2 be three partitions in A_k. Let τ be the partition in A_{2k} defined by:*

$$\tau = (1,k+1)(2,k+2)\dots(k,2k).$$

We have:

$$\delta_{p_1\otimes p_2\in[id,(p_0\otimes id_k)\tau]_{A_{2k}}} = \delta_{p_1\circ p_2\in[id,p_0]_{A_k}}\delta_{p_1\prec p_1\circ p_2}.$$

Proof. — Let p_0, p_1 and p_2 be three partitions in A_k. Let us consider $(\mathbf{p}_N^1)_{N\in\mathbb{N}}$ and $(\mathbf{p}_N^2)_{N\in\mathbb{N}}$ such that for any integer N, $\mathbf{p}_N^1 = M_k^N(p_1)$ and $\mathbf{p}_N^2 = M_k^N(p_2)$. Using Lemma 5.4, $(\mathbf{p}_N^1)_{N\in\mathbb{N}}$ and $(\mathbf{p}_N^2)_{N\in\mathbb{N}}$ converge strongly. Thus, $(\mathbf{p}_N^1\otimes\mathbf{p}_N^2)_{N\in\mathbb{N}}$ converges strongly, and by Theorem 4.1 it converges in moments. Let us calculate, using two ways, the limit of $m_{(p_0\otimes id_k)\tau}\left[\mathbf{p}_N^1\otimes\mathbf{p}_N^2\right]$, where $\tau = (1,k+1)(2,k+2)\dots(k,2k)$.

First, using Theorem 4.1 and the Equation (7), we get that:

$$\lim_{N\to\infty} m_{(p_0\otimes id_k)\tau}\left[\mathbf{p}_N^1\otimes\mathbf{p}_N^2\right] = \sum_{p\in[(p_0\otimes id_k)\tau]_{A_{2k}}}\lim_{N\to\infty}\kappa^p\left[\mathbf{p}_N^1\otimes\mathbf{p}_N^2\right].$$

Yet, for any $p\in A_{2k}$, $\kappa^p\left[\mathbf{p}_N^1\otimes\mathbf{p}_N^2\right] = \delta_{p=p_1\otimes p_2}$, thus:

$$\lim_{N\to\infty} m_{(p_0\otimes id_k)\tau}\left[\mathbf{p}_N^1\otimes\mathbf{p}_N^2\right] = \delta_{p_1\otimes p_2\in[id,(p_0\otimes id_k)\tau]_{A_{2k}}}.$$

Then, using the fact that $m_{(p_0\otimes id_k)\tau}\left[\mathbf{p}_N^1\otimes\mathbf{p}_N^2\right] = m_{p_0}\left[\mathbf{p}_N^1\mathbf{p}_N^2\right]$, and using again Theorem 4.1 and the Equation (7):

$$\lim_{N\to\infty} m_{(p_0\otimes id_k)\tau}\left[\mathbf{p}_N^1\otimes\mathbf{p}_N^2\right] = \sum_{p\in[id,p_0]_{A_k}}\lim_{N\to\infty}\kappa^p\left[\mathbf{p}_N^1\mathbf{p}_N^2\right].$$

Let $p\in A_k$, $\kappa^p\left[\mathbf{p}_N^1\mathbf{p}_N^2\right]$ is the coefficient of p in the expression $\left(M_k^N\right)^{-1}(\mathbf{p}_N^1\mathbf{p}_N^2)$. Let us remark that $\left(M_k^N\right)^{-1}(\mathbf{p}_N^1\mathbf{p}_N^2) = \left(M_k^N\right)^{-1}(M_k^N(p_1)M_k^N(p_2)) = p_1._N p_2$ which converges in $\mathbb{C}[A_k]$ to $\delta_{p_1\prec p_1\circ p_2}p_1\circ p_2$. Thus,

$$\lim_{N\to\infty}\kappa^p\left[\mathbf{p}_N^1\mathbf{p}_N^2\right] = \delta_{p_1\prec p_1\circ p_2}\delta_{p=p_1\circ p_2}.$$

This implies that:

$$\lim_{N\to\infty} m_{(p_0\otimes id_k)\tau}\left[\mathbf{p}_N^1\otimes\mathbf{p}_N^2\right]=\delta_{p_1\prec p_1\circ p_2}\delta_{p_1\circ p_2\in[id,p_0]_{A_k}}.$$

Using the two ways to compute $\lim_{N\to\infty} m_{(p_0\otimes id_k)\tau}\left[\mathbf{p}_N^1\otimes\mathbf{p}_N^2\right]$, we get:

$$\delta_{p_1\otimes p_2\in[id,(p_0\otimes id_k)\tau]_{A_{2k}}}=\delta_{p_1\circ p_2\in[id,p_0]_{A_k}}\delta_{p_1\prec p_1\circ p_2}.$$

which was the desired equality. □

In fact, one can always prove the results by a combinatorial argument: the ideas we present are more an automatic way to get combinatorial results that one can prove after by combinatorial means. For example, let us consider Definition 9.1 ; using Proposition 8.1, one can now expect that $\mathsf{df}(p_1\otimes p_2,(p_0\otimes id_k)\tau)=\mathsf{df}(p_1\circ p_2,p_0)+\eta(p_1,p_2)$. Indeed, we have the following proposition.

Proposition 8.2. — *Let p_0, p_1 and p_2 be three partitions in A_k. Let τ be the partition in A_{2k} defined by:*

$$\tau=(1,k+1)(2,k+2)\dots(k,2k).$$

We have:

$$\mathsf{df}(p_1\otimes p_2,(p_0\otimes id_k)\tau)=\mathsf{df}(p_1\circ p_2,p_0)+\eta(p_1,p_2).$$

Proof. — The proof is only based on calculations. Let p and p' be two partitions in A_k, then:

$$\mathsf{df}(p',p)=\mathsf{nc}(p')-\mathsf{nc}(p'\vee id)-\mathsf{nc}(p'\vee p)+\mathsf{nc}(p\vee id),$$

and $\eta(p,p')$ is equal to:

$$\mathsf{nc}(p)+\mathsf{nc}(p')-\mathsf{nc}(p\circ p')-\mathsf{nc}(p\vee id)-\mathsf{nc}(p'\vee id)+\mathsf{nc}(p\circ p'\vee id)-\kappa(p,p').$$

Thus:

$$\begin{aligned}
&\mathsf{df}(p_1\circ p_2,p_0)+\eta(p_1,p_2)-\mathsf{df}(p_1\otimes p_2,(p_0\otimes id_k)\tau)\\
&=\mathsf{nc}(p_1\circ p_2)-\mathsf{nc}((p_1\circ p_2)\vee id)-\mathsf{nc}((p_1\circ p_2)\vee p_0)+\mathsf{nc}(p_0\vee id)+\mathsf{nc}(p_1)+\mathsf{nc}(p_2)\\
&\quad-\mathsf{nc}(p_1\circ p_2)-\mathsf{nc}(p_1\vee id)-\mathsf{nc}(p_2\vee id)+\mathsf{nc}((p_1\circ p_2)\vee id)-\kappa(p_1,p_2)\\
&\quad-\mathsf{nc}(p_1\otimes p_2)+\mathsf{nc}((p_1\otimes p_2)\vee id)+\mathsf{nc}([(p_0\otimes id_k)\tau]\vee[p_1\otimes p_2])\\
&\quad-\mathsf{nc}([(p_0\otimes id_k)\tau]\vee id_{2k}).
\end{aligned}$$

Using the following equalities:

$$\begin{aligned}
\mathsf{nc}(p_1\otimes p_2)&=\mathsf{nc}(p_1)+\mathsf{nc}(p_2),\\
\mathsf{nc}([p_1\otimes p_2]\vee id)&=\mathsf{nc}(p_1\vee id)+\mathsf{nc}(p_2\vee id),
\end{aligned}$$

we get:

$$\begin{aligned}
&\mathsf{df}(p_1\circ p_2,p_0)+\eta(p_1,p_2)-\mathsf{df}(p_1\otimes p_2,(p_0\otimes id_k)\tau)\\
&\quad=-\mathsf{nc}((p_1\circ p_2)\vee p_0)+\mathsf{nc}(p_0\vee id)-\kappa(p_1,p_2)\\
&\quad+\mathsf{nc}([(p_0\otimes id_k)\tau]\vee[p_1\otimes p_2])-\mathsf{nc}([(p_0\otimes id_k)\tau]\vee id_{2k}).
\end{aligned}$$

The equalities:

$$Tr_N(p_0) = Tr_N((p_0 \otimes id_k)\tau),$$
$$N^{\kappa(p_1,p_2)} Tr_N((p_1 \circ p_2)^t p_0) = Tr_N((p_1 p_2)^t p_0) = Tr_N((p_1 \otimes p_2)^t [(p_0 \otimes id_k)\tau]),$$

allow us to prove, as an application of Equations (2) and (3), that:

$$\mathsf{nc}(p_0 \vee id) = \mathsf{nc}([(p_0 \otimes id_k)\tau] \vee id_{2k}),$$
$$\mathsf{nc}((p_1 \circ p_2) \vee p_0) + \kappa(p_1, p_2) = \mathsf{nc}([(p_0 \otimes id_k)\tau] \vee [p_1 \otimes p_2]).$$

Thus $\mathsf{df}(p_1 \circ p_2, p_0) + \eta(p_1, p_2) - \mathsf{df}(p_1 \otimes p_2, (p_0 \otimes id_k)\tau) = 0$. □

8.2. Combinatorial consequences of Theorem 4.1. — Let us remark the following important, yet simple theorem.

Theorem 8.1. — *Let $(m_p)_{p \in A_k}$ be a family of complex numbers. There exists a sequence $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ which converges and such that:*

$$\lim_{N \to \infty} m_p(E_N) = m_p.$$

Proof. — Let us consider $(m_p)_{p \in A_k}$ a family of complex numbers. Let us consider $(\kappa^p)_{p \in A_k}$, the unique family of real such that for any $p \in A_k$:

$$m_p = \sum_{p' \in [id,p]_{A_k}} \kappa^{p'}.$$

Let us consider then:

$$E_N = M_k^N \left(\sum_{p \in A_k} \kappa^p p \right).$$

According to Lemma 5.4, $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ converges strongly. Thus, by Theorem 4.1 it converges in moments and for any $p \in A_k$:

$$\lim_{N \to \infty} m_p(E_N) = \sum_{p' \in [id,p]_{A_k}} \lim_{N \to \infty} \kappa^{p'}(E_N).$$

Yet, using Lemma 5.3, $\kappa^{p'}(E_N)$ is equal to κ^p. Thus:

$$\lim_{N \to \infty} m_p(E_N) = \sum_{p' \in [id,p]_{A_k}} \kappa^{p'} = m_p.$$

This concludes the proof. □

This theorem shows that, in order to understand the transformation between moments and coordinate numbers, we have an approximation setting in which one can work with: the space of convergent sequences in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. Let us show some examples of propositions that one can get using this point of view. For this, we need the notion of cumulants and exclusive moments. Let us consider $(m_p)_{p \in A_k}$ a family of complex numbers.

Definition 8.1. — *The cumulants of* $(m_p)_{p\in A_k}$ *is the unique family of complex numbers* $(\kappa^p)_{p\in A_k}$ *such that for any* $p \in A_k$*:*

$$m_p = \sum_{p'\in[id,p]_{A_k}} \kappa^{p'}.$$

The exclusive moments of $(m_p)_{p\in A_k}$ *is the only family* $(m_{p^c})_{p\in\mathcal{P}_k}$ *of complex numbers such that:*

$$m_{p^c} = \sum_{p'\in \mathsf{Sp}(p)\cap A_k} \kappa_{p'}.$$

Let us consider the cumulants $(\kappa^p)_{p\in A_k}$ and the exclusive moments $(m_{p^c})_{p\in\mathcal{P}_k}$ of $(m_p)_{p\in A_k}$.

Proposition 8.3. — *Let* p *and* p_0 *be two elements of* A_k*. Then:*

$$\delta_{p\in[id,p_0]_{A_k}} m_{{}^t p\circ p_0} = \sum_{p'\in[id,p_0]_{A_k}} \delta_{p\prec p'} \kappa^{K_{p'}(p)}.$$

where for any $P \subset A_k$, $\kappa^P = \sum_{p\in P} \kappa^p$.

By specifying $p = id$ in Proposition 8.4, we get back the Equation (7). Besides, one can get a similar formula for $m_{p_0\circ {}^t p}(E)$ by using Equation (17). Using Theorem 8.1, the last proposition is a consequence of Proposition 8.4.

Proposition 8.4. — *For any integer* N*, let us consider* E_N *an element of* $\mathbb{C}[A_k(N)]$*. Let us suppose that* $(E_N)_{N\in\mathbb{N}}$ *converges. Let* p *and* p_0 *be two elements of* A_k*. Then:*

$$\delta_{p\in[id,p_0]_{A_k}} m_{{}^t p\circ p_0}(E) = \sum_{p'\in[id,p_0]_{A_k}} \delta_{p\prec p'} \kappa^{K_{p'}(p)}(E).$$

Proof. — Let p and p_0 be two elements of A_k. Let us consider for any N, $M_k^N(p) \in \mathbb{C}[A_k(N)]$. The sequence $\left(M_k^N(p)\right)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$ converges by Lemma 5.4. Let us apply the Theorem 7.1 to the product $M_k^N(p)E_N$. We remind the reader that $\kappa^{p'}(M_k^N(p)) = \delta_{p=p'}$ for any $p' \in A_k$. Using the Equation (15) in Theorem 7.1, we know that

$$\lim_{N\to\infty} \kappa^{p'}(M_k^N(p)E_N) = \delta_{p\prec p'} \kappa^{K_{p'}(p)}(E).$$

Let us use Equation (7):

$$\lim_{N\to\infty} m_{p_0}(M_k^N(p)E_N) = \sum_{p'\in[id,p_0]_{A_k}} \delta_{p\prec p'} \kappa^{K_{p'}(p)}(E).$$

Yet, according to Equation (16),

$$\lim_{N\to\infty} m_{p_0}(M_k^N(p)E_N) = \delta_{p\in[id,p_0]_{A_k}} m_{{}^t p\circ p_0}(E),$$

hence the equality stated in Proposition 8.4. □

8.3. Convergence of the modified observables. — In Section 5, we have defined a deformed partition algebra, by deforming the multiplication. Yet, we have not defined any deformed linear form m_p on the algebra $\mathbb{C}[A_k(N,N)]$. In fact, on $\mathbb{C}[A_k(N,N)]$, for any $p \in \mathcal{P}_k$ we define:

$$m_p^N : \mathbb{C}[A_k(N,N)] \to \mathbb{C}$$
$$E \mapsto m_p\left(M_k^N(E)\right).$$

A consequence of Theorem 4.1 is that for any $E \in A_k$, for any $p \in \mathcal{P}_k$, $m_p^N(E)$ converges as N goes to infinity: let us denote the limit by $m_p^\infty(E)$. We already know that the algebra $\mathbb{C}[A_k(N,N)]$ converges to $\mathbb{C}[A_k(\infty,\infty)]$ when N goes to infinity. Thus, we have that:

Theorem 8.2. — *For any integer k, $\left(\mathbb{C}[A_k(N,N)], (m_p^N)_{p\in\mathcal{P}_k}\right)$ converges to*

$$\left(\mathbb{C}[A_k(\infty,\infty)], (m_p^\infty)_{p\in\mathcal{P}_k}\right)$$

as N goes to infinity. This means that:

1. *the algebra $\mathbb{C}[A_k(N,N)]$ converges to $\mathbb{C}[A_k(\infty,\infty)]$ as N goes to infinity,*
2. *for any $E \in \mathbb{C}[A_k(N,N)]$, for any $p \in \mathcal{P}_k$, $m_p^N(E)$ converges to $m_p^\infty(E)$ as N goes to infinity, where $m_p^\infty(E)$ is defined below.*

Besides, let $E = \sum_{p\in A_k} E_p p$ and $F = \sum_{p\in A_k} F_p p$ in $\mathbb{C}[A_k(\infty,\infty)]$, then

$$EF = \sum_{p_1,p_2\in A_k} E_{p_1}F_{p_2}\delta_{p_1\prec p_1\circ p_2} p_1 \circ p_2.$$

And if $p_0 \in A_k$:

$$m_{p_0}^\infty(E) = \sum_{p\in A_k} \delta_{p\in[id,p_0]_{A_k}} E_p.$$

In fact, this theorem has to be read in the other way: given the algebra with linear forms $(\mathbb{C}[A_k(\infty,\infty)], (m_p^\infty)_{p\in\mathcal{P}_k})$, one can find an approximation given by $\left(\mathbb{C}[A_k(N,N)], (m_p^N)_{p\in\mathcal{P}_k}\right)$.

9. Algebraic fluctuations

In this section, we generalize Sections 4, 5 and 7 in order to study the asymptotic developments of the coordinate numbers and normalized moments. The proofs will be either omitted or simplified as they will use the same arguments as we have seen in Sections 4, 5 and 7.

In order to study the asymptotic developments, we need to introduce two notions of defect. One already seen is linked with the triangle inequality and the other to the improved triangle inequality. Let k be a positive integer, let p and p' be two elements of A_k. We recall that we defined in Definition 3.5 the defect of p' from not being on the set-geodesic $[id,p]_{\mathcal{P}_k}$ by:

$$\mathsf{df}(p',p) = d(id,p') + d(p',p) - d(id,p) = \mathsf{nc}(p') - \mathsf{nc}(p'\vee id) - \mathsf{nc}(p\vee p') + \mathsf{nc}(p\vee id).$$

Definition 9.1. — *The defect $\eta(p,p')$ that $p \prec p \circ p'$ is not satisfied by:*

$$d(id,p)+d(id,p')-d(id,p\circ p')-\frac{k+\mathsf{nc}(p\circ p')-\mathsf{nc}(p)-\mathsf{nc}(p')}{2}-\kappa(p,p').$$

We warn the reader that, in general:

$$\mathsf{df}(p,p\circ p')\neq\eta(p,p'),$$

even if this equality is true when one considers $p,p'\in\mathfrak{S}_k$. Let us remark that if p and p_0 are elements of A_k, $p\prec p_0$ holds if and only if there exists p' such that $p_0=p\circ p'$ and $\eta(p,p')=0$.

Let us define the N-development algebra of order m of A_k. This algebra is the good setting in order to study fluctuations of the coordinate numbers and moments.

Definition 9.2. — *Let N,k and m be integers, let X be a formal variable. The N-development algebra of order m of A_k, $\mathbb{C}_{(m)}[A_k(N)]$, is the associative algebra generated by the elements of the form:*

$$\frac{p}{X^i},$$

where $p\in A_k$ and $i\in\{0,\dots,m\}$. The product is defined such that, for any p and p' in A_k, and any i and j in $\{0,\dots,m\}$:

$$\frac{p}{X^i}.\frac{p'}{X^j}=\frac{1}{N^{\max(i+j+\eta(p,p')-m,0)}}\frac{p\circ p'}{X^{\min(i+j+\eta(p,p'),m)}}.$$

This product is well defined: indeed the improved triangle inequality, Proposition 6.1 or Lemma 6.1, assert that for any $p,p'\in A_k$, $\eta(p,p')\geq 0$, thus, for any $i,j\in\{0,\dots,m\}$, any $p,p'\in A_k$, we have $min(i+j+\eta(p,p'),m)\geq 0$. This implies that:

$$\frac{p\circ p'}{X^{\min(i+j+\eta(p,p'),m)}}$$

is an element of the canonical basis of the N-development algebra of order m of A_k. Let us remark that for any positive integers k, N and m, $\mathbb{C}_{(m)}[\mathfrak{S}_k(N)]\subset\mathbb{C}_{(m)}[\mathcal{B}_k(N)]\subset\mathbb{C}_{(m)}[\mathcal{P}_k(N)]$, where these inclusions are inclusions of algebras.

9.1. Coordinate numbers. — Let N be a positive integer, the N-development algebra of order 0 of A_k is canonically isomorphic to $\mathbb{C}[A_k(N,N)]$.

Lemma 9.1. — *Let N be an integer, the application:*

$$L_k^N : A_k\to\mathbb{C}_{(0)}[A_k(N)]$$
$$p\mapsto\frac{p}{X^0}$$

can be extended as an isomorphism of algebra between $\mathbb{C}[A_k(N,N)]$ and $\mathbb{C}_{(0)}[A_k(N)]$.

Proof. — Let us show that for any p, p' in A_k, $L_k^N(p._N p')=L_k^N(p)L_k^N(p')$. As for any $p,p'\in A_k$, $\eta(p,p')\geq 0$, $L_k^N(p)L_k^N(p')$ is equal to:

$$\frac{p}{X^0}\frac{p'}{X^0}=\frac{1}{N^{\eta(p,p')}}\frac{p\circ p'}{X^0}=\frac{1}{N^{\eta(p,p')}}L_k^N(p\circ p').$$

Yet, looking at the definition of $\eta(p,p')$ given in Definition 9.1, for any integer N the following equation holds in $\mathbb{C}[A_k(N,N)]$:

$$p._N p' = \frac{1}{N^{\eta(p,p')}} p \circ p'.$$

This allows us to conclude. □

Let m be a positive integer. We define, for any $i \leq m$, the coordinate numbers of order i of any element of $\mathbb{C}_{(m)}[A_k(N)]$.

Definition 9.3. — *Let $E \in \mathbb{C}_{(m)}[A_k(N)]$. The coordinate numbers of E up to the order m are the elements $(\kappa_i^p(E))_{i\in\{0,\dots,m\},p\in\mathcal{P}_k}$ such that:*

$$E = \sum_{p\in A_k} \sum_{i=0}^{m} \kappa_i^p(E) \frac{p}{X^i}.$$

Let $p \in A_k$ and $i \leq m$. The number $\kappa_i^p(E)$ is the coordinate number of E on p of order i.

We define also a notion of convergence. In order to do so, we must not forget that, when $m = 0$, $\mathbb{C}_{(m)}[A_k(N)]$ is isomorphic to the deformed algebra $\mathbb{C}[A_k(N,N)]$ and not the algebra $\mathbb{C}[A_k(N)]$. Let $(E_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$.

Definition 9.4. — *The sequence $(E_N)_{N\in\mathbb{N}}$ converges if and only if for any $i \in \{0,\dots,m-1\}$, and any $p \in A_k$, $\kappa_i^p(E_N)$ is independent of N, and for any $p \in A_k$, $\kappa_m^p(E_N)$ converges when N goes to infinity.*

Notation 9.1. — *Let us suppose that $(E_N)_{N\in\mathbb{N}}$ converges as N goes to infinity. We denote, for any $i \in \{0,\dots,m\}$ and any $p \in A_k$: $\kappa_i^p(E) = \lim_{N\to\infty} \kappa_i^p(E_N)$.*

9.2. Convergences: $\mathbb{C}_{(m)}[A_k(N)]$ and multiplication. — Using Lemma 9.1 and Theorem 6.1, since the algebra $\mathbb{C}[A_k(N,N)]$ is isomorphic to $\mathbb{C}_{(0)}[A_k(N)]$ by an isomorphism which sends the canonical base of the first algebra on the canonical base of the second algebra, we know that the algebra $\mathbb{C}_{(0)}[A_k(N)]$ converges as N tends to infinity. In fact, the result holds for any $m \in \mathbb{N}$. Let k and m be two integers. Let X be a formal variable.

Definition 9.5. — *The ∞-development algebra of order m of A_k, denoted by $\mathbb{C}_{(m)}[A_k(\infty)]$ is the associative algebra generated by the elements of the form:*

$$\frac{p}{X^i},$$

where $p \in A_k$ and $i \in \{0,\dots,m\}$. The product is defined such that, for any p and p' in A_k, and any i and j in $\{0,\dots,m\}$,

$$\frac{p}{X^i}\frac{p}{X^j} = \delta_{i+j+\eta(p,p')\leq m} \frac{p\circ p'}{X^{i+j+\eta(p,p')}}.$$

Let us recall Definition 6.3, where we defined the convergence of algebras. We then have the following proposition.

Proposition 9.1. — *When N goes to infinity, the N-development algebra of order m of A_k, $\mathbb{C}_{(m)}[A_k(N)]$ converges to the ∞-development algebra of order m of A_k, namely $\mathbb{C}_{(m)}[A_k(\infty)]$.*

Proof. — The algebras $\mathbb{C}_{(m)}[A_k(N)]$ have, for any integer N, the same linear basis $\left(\frac{p}{X^i}\right)_{i\in\{0,\dots,m\},p\in A_k}$. Since for any $p,p'\in A_k$, any $i,j\in\mathbb{N}$:

$$\frac{p}{X^i}\frac{p'}{X^{i'}}=\frac{1}{N^{\max(i+j+\eta(p,p')-m,0)}}\frac{p\circ p'}{X^{\min(i+j+\eta(p,p'),m)}}\underset{N\to\infty}{\longrightarrow}\delta_{i+j+\eta(p,p')\leq m}\frac{p\circ p'}{X^{i+j+\eta(p,p')}},$$

where the first product is seen in $\mathbb{C}_{(m)}[A_k(N)]$, the algebra $\mathbb{C}_{(m)}[A_k(N)]$ converges to $\mathbb{C}_{(m)}[A_k(\infty)]$ as N goes to infinity. □

Let us write the first easiest consequence of the Proposition 9.1, which can be proved by using a bi-linearity argument, Proposition 9.1 and Definition 9.4.

Proposition 9.2. — *Let $(E_N)_{N\in\mathbb{N}}$ and $(F_N)_{N\in\mathbb{N}}$ be elements of $\prod_{N\in\mathbb{N}}\mathbb{C}_{(m)}[A_k(N)]$. Let us suppose that the two sequences $(E_N)_{N\in\mathbb{N}}$ and $(F_N)_{N\in\mathbb{N}}$ converge. The sequence $(E_NF_N)_{N\in\mathbb{N}}$ converges and, using Notations 9.1, for any $i_0\in\{0,\dots,m\}$ and for any $p_0\in A_k$:*

$$\kappa_{i_0}^{p_0}(EF)=\sum_{p,p'\in A_k,\eta(p,p')\leq i_0,p\circ p'=p_0}\ \sum_{i\in\{0,\dots,i_0-\eta(p,p')\}}\kappa_i^p(E)\kappa_{i_0-\eta(p,p')-i}^{p'}(F).$$

As for Section 7.2, the good behavior of the product, given by Proposition 9.2, implies a criterion for the convergence of semi-groups in $\prod_{N\in\mathbb{N}}\mathbb{C}_{(m)}[A_k(N)]$.

Definition 9.6. — *Let $\left((E_t^N)_N\right)_{t\geq 0}$ be a semi-group in $\prod_{N\in\mathbb{N}}\mathbb{C}_{(m)}[A_k(N)]$. The semi-group $\left((E_t^N)_N\right)_{t\geq 0}$ converges if and only if for any $t\geq 0$, $\left(E_t^N\right)_{N\in\mathbb{N}}$ converges.*

Proposition 9.3. — *Let $m\in\mathbb{N}$. Let us consider $\left(\left(E_t^N\right)_N\right)_{t\geq 0}$ a semi-group in $\prod_{N\in\mathbb{N}}\mathbb{C}_{(m)}[A_k(N)]$ which generator is denoted by $(H_N)_{N\in\mathbb{N}}$. It converges if the sequences $(E_0^N)_{N\in\mathbb{N}}$ and $(H_N)_{N\in\mathbb{N}}$ converge. Besides, using Notation 9.1, for any $p\in A_k$, for any $t_0\geq 0$ and any $i\in\{0,\dots,m\}$,*

$$\frac{d}{dt}_{|t=t_0}\kappa_{i_0}^{p_0}(E_t)=\sum_{p,p'\in A_k,\eta(p,p')\leq i_0,p\circ p'=p_0}\ \sum_{i\in\{0,\dots,i_0-\eta(p,p')\}}\kappa_i^p(H)\kappa_{i_0-\eta(p,p')-i}^{p'}(E_{t_0}).$$

In order to finish this section, let us introduce the evaluation morphism: it is a morphism which allows us to inject an element from $\mathbb{C}_{(m)}[A_k(N)]$ in $\mathbb{C}[A_k(N)]$. Let N be a positive integer, the function $eval_N$ is defined by:

$$\begin{array}{rccc} eval_{(m)}^N: & \mathbb{C}_{(m)}[A_k(N)] & \to & \mathbb{C}[A_k(N)] \\ & \displaystyle\sum_{p\in A_k}\sum_{i=0}^m\kappa_i^p(E)\frac{p}{X^i} & \mapsto & \displaystyle\sum_{p\in A_k}\sum_{i=0}^m\kappa_i^p(E)\frac{1}{N^i}\frac{p}{N^{-\frac{k}{2}+\frac{\mathrm{nc}(p)}{2}+d(id,p)}}. \end{array}$$

Lemma 9.2. — *For any positive integer N, $eval_{(m)}^N$ is a morphism of algebra.*

Proof. — Let N be a positive integer, let $i,j \in \{0,\dots,m\}$ and $p,p' \in A_k$. Then:

$$\begin{aligned} eval^N_{(m)}\left(\frac{p}{X^i}\frac{p'}{X^j}\right) &= eval^N_{(m)}\left(\frac{1}{N^{\max(i+j+\eta(p,p')-m,0)}}\frac{p\circ p'}{X^{\min(i+j+\eta(p,p'),m)}}\right)\\ &= \frac{1}{N^{i+j+\eta(p,p')-\frac{k}{2}+\frac{\mathsf{nc}(p\circ p')}{2}+d(id,p\circ p')}}p\circ p'\\ &= \left(\frac{1}{N^i}\frac{p}{N^{-\frac{k}{2}+\frac{\mathsf{nc}(p)}{2}+d(id,p)}}\right)\left(\frac{1}{N^j}\frac{p'}{N^{-\frac{k}{2}+\frac{\mathsf{nc}(p')}{2}+d(id,p')}}\right)\\ &= eval^N_{(m)}\left(\frac{p}{X^i}\right)eval^N_{(m)}\left(\frac{p}{X^j}\right). \end{aligned}$$

The other properties can be easily verified. □

The function $eval^N_{(m)}$ has an inverse if and only if $m=0$. This will motivate us in order to define a notion of convergence up to order m of fluctuations for sequences in $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$. Then, given a linear or multiplicative problem in $\mathbb{C}[A_k(N)]$, one can try to find a similar problem in $\mathbb{C}_{(m)}[A_k(N)]$, solve this last problem, and push by $eval^N_{(m)}$ the solution on a solution of the first problem.

9.3. Convergence at any order of fluctuations in $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$. — We are interested in elements in $\mathbb{C}[A_k(N)]$ and we want to define a notion of strong convergence up to the m^{th} order of fluctuations. Let m be an integer, let $(E_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$.

Definition 9.7. — *The sequence $(E_N)_{N\in\mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations if and only if there exist two families of real $(\kappa_i^p)_{i\in\{0,\dots,m-1\},p\in A_k}$ and $(\kappa^p_{m,N})_{p\in A_k,N\in\mathbb{N}}$ such that:*

- $\forall p\in A_k,\ \kappa^p(E_N)=\sum_{i=0}^{m-1}\frac{\kappa_i^p}{N^i}+\frac{\kappa^p_{m,N}}{N^m}$,
- $\forall p\in A_k$, $\kappa^p_{m,N}$ *converges as N goes to infinity.*

The families $(\kappa_i^p)_{i\in\{0,\dots,m-1\},p\in A_k}$ and $(\kappa^p_{m,N})_{p\in A_k}$ are uniquely defined.

For any $p\in A_k$, any integer N and any $i\in\{0,\dots,m-1\}$, κ_i^p is the coordinate number of E_N on p of order i, and $\kappa^p_{m,N}$ is the coordinate number of E_N on p of order m.

Notation 9.2. — *Let us suppose that $(E_N)_{N\in\mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations. From now on, the coordinate numbers of E_N on p of order i will be denoted by $\kappa_i^p(E_N)$. For any $p\in A_k$ and any $i\in\{0,\dots,m\}$, we will define:*

$$\kappa_i^p(E)=\lim_{N\to\infty}\kappa_i^p(E_N).$$

When one works in $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$, one has to be aware that the coordinate numbers of higher order of fluctuations are only defined for a sequence $(E_N)_{N\in\mathbb{N}}$ which converges strongly. Thus, one must not forgot that the notation $\kappa_i^p(E_N)$ means that we are looking at the coordinate numbers of E_N seen as an element of the sequence $(E_N)_{N\geq 0}$.

The Definition 9.7 might seem strange as it only uses once the notion of convergence. Yet, it is easy to see that an equivalent definition is the following one. The sequence

$(E_N)_{N\in\mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations if and only if there exists a family $(\kappa_i^p)_{i\in\{0,\dots,m\},p\in A_k}$ of real numbers such that for any $i\in\{0,\dots,m\}$,

$$N^i\left(\kappa^p(E_N)-\sum_{j=0}^{i-1}\frac{\kappa_j^p}{N^j}\right)\underset{N\to\infty}{\longrightarrow}\kappa_i^p,$$

with the convention $\sum_{j=0}^{-1}\frac{\kappa_j^p}{N^j}=0$. This equivalent definition explains why the families $(\kappa_i^p)_{i\in\{0,\dots,m-1\},p\in A_k}$ and $(\kappa_{m,N}^p)_{N\in\mathbb{N},p\in A_k}$ defined in Definition 9.7 are uniquely defined.

The next lemma allows to make a link between the convergence of elements of $\prod_{N\in\mathbb{N}}\mathbb{C}_{(m)}[A_k(N)]$ and the convergence up to the m^{th} order of fluctuations of elements of $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$.

Lemma 9.3. — *Let us suppose that $(E_N)_{N\in\mathbb{N}}$ converges. Then $\left(eval_{(m)}^N(E_N)\right)_{N\in\mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations.*

The notion of strong convergence to the m^{th} order of fluctuations allows us to inject canonically an element of $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$ which converges strongly up to the m^{th} order of fluctuations into $\prod_{N\in\mathbb{N}}\mathbb{C}_{(m)}[A_k(N)]$.

Definition 9.8. — *Let us suppose that $(E_N)_{N\in\mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations. For any $p\in A_k$, any integer N, let $(\kappa_i^p)_{i\in\{0,\dots,m-1\}}$ and $\kappa_{m,N}^p$ be the coordinate numbers of E_N on p. We define the lift of the sequence $(E_N)_{N\in\mathbb{N}}$ as $(\tilde{E}_N)_{N\in\mathbb{N}}\in\prod_{N\in\mathbb{N}}\mathbb{C}_{(m)}[A_k(N)]$ defined by:*

$$\tilde{E}_N=\sum_{p\in A_k}\left(\left(\sum_{i=0}^{m-1}\kappa_i^p\frac{p}{X^i}\right)+\kappa_{m,N}^p\frac{p}{X^m}\right).$$

The following lemma is then straightforward.

Lemma 9.4. — *Let us suppose that $(E_N)_{N\in\mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations. Let $(\tilde{E}_N)_{N\in\mathbb{N}}$ be its canonical lift in $\prod_{N\in\mathbb{N}}\mathbb{C}_{(m)}[A_k(N)]$. Then $(\tilde{E}_N)_{N\in\mathbb{N}}$ converges as N goes to infinity and for any $N\in\mathbb{N}$, one has $eval_{(m)}^N(\tilde{E}_N)=E_N$.*

We are going to define a weak notion of convergence up to the m^{th} order of fluctuations and we will show that this notion is equivalent to the strong convergence notion we defined in Definition 9.7.

Definition 9.9. — *The sequence $(E_N)_{N\in\mathbb{N}}$ converges in moments up to the m^{th} order of fluctuations if and only if there exist two families $(m_p^i)_{i\in\{0,\dots,m-1\},p\in A_k}$ and $(m_{p,N}^m)_{N\in\mathbb{N},p\in A_k}$ such that:*

- $\forall p\in A_k,\ m_p(E_N)=\sum_{i=0}^{m-1}\frac{m_p^i}{N^i}+\frac{m_p^m}{N^m}$,
- $\forall p\in A_k,\ m_p^{m,N}$ *converges as N goes to infinity.*

The families $(m_p^i)_{i\in\{0,\dots,m-1\},p\in A_k}$ *and* $(m_{p,N}^m)_{N\in\mathbb{N},p\in A_k}$ *are uniquely defined.*

For any $p \in A_k$*, any integer* N*, and any* $i \in \{0,\dots,m-1\}$*,* m_p^i *is the* i^{th}*-order fluctuations of the* p*-normalized moment of* E_N*, and* $m_{p,N}^m$ *is the* m^{th}*-order fluctuations of the* p*-normalized moment of* E_N*.*

Notation 9.3. — *Let us suppose that* $(E_N)_{N\in\mathbb{N}}$ *converges in moments up to the* m^{th} *order of fluctuations. From now on, the* i^{th}*-order fluctuations of the* p*-normalized moment of* E_N *will be denoted by* $m_p^i(E_N)$*. For any* $p \in A_k$ *and any* $i \in \{0,\dots,m\}$*, we define:*

$$m_p^i(E) = \lim_{N\to\infty} m_p^i(E_N).$$

The same remark about the coordinate numbers of E_N on p of order i, explained just after Notation 9.2, can be made for the fluctuations of the p-normalized moments of E_N. The next theorem shows that the strong convergence up to the m^{th} order of fluctuations is equivalent to the convergence in moments up to the m^{th} order of fluctuations. We recall that $m \in \mathbb{N}$ and $(E_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$.

Theorem 9.1. — *The sequence* $(E_N)_{N\in\mathbb{N}}$ *converges strongly up to the* m^{th} *order of fluctuations if and only if it converges in moments up to the* m^{th} *order of fluctuations. We will say that* $(E_N)_{N\in\mathbb{N}}$ *converges up to the* m^{th} *order of fluctuations.*

Let us suppose that $(E_N)_{N\in\mathbb{N}}$ *converges up to the* m^{th} *order of fluctuations, then, seen as an element of* $\prod_{N\in\mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$*, it converges also up to the* m^{th} *order of fluctuations and for any* $i_0 \in \{0,\dots,m\}$ *and any* $p \in \mathcal{P}_k$*:*

$$\lim_{N\to\infty} m_p^{i_0}(E_N) = \sum_{p'\in A_k, \mathsf{df}(p',p)\leq i_0} \kappa_{i_0-\mathsf{df}(p',p)}^{p'}(E). \tag{21}$$

Proof. — We have already seen the arguments in order to prove the second assertion. Let us prove that $(E_N)_{N\in\mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations if and only if it converges in moments up to the m^{th} order of fluctuations. Let us consider p in A_k.

Let us suppose that $(E_N)_{N\in\mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations. The coordinate numbers of E_N are defined up to order m of fluctuations and:

$$E_N = \sum_{p\in A_k} \sum_{i=0}^{m} \frac{\kappa_i^p(E_N)}{N^i} \frac{p}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}}.$$

Besides, for any $p \in A_k$ and any $i \leq m-1$, $\kappa_i^p(E_N)$ does not depend on N and $\kappa_m^p(E_N)$ converges when N goes to infinity. We can compute the p-normalized moments of E_N,

using the same arguments as for the proof of Theorem 4.1:

$$m_p(E_N) = \frac{1}{Tr_N(p)} Tr_N(E_N\, {}^t p) = \sum_{p' \in A_k} \sum_{i=0}^{m} \kappa_i^{p'}(E_N) \frac{1}{N^{i+\mathsf{df}(p',p)}}$$
$$= \sum_{j=0}^{m-1} \left(\sum_{(p',i) \in A_k \times \{0,\dots,m-1\}, i+\mathsf{df}(p',p)=j} \kappa_i^{p'}(E_N) \right) \frac{1}{N^j}$$
$$+ \left(\sum_{(p',i) \in A_k \times \{0,\dots,m\}, i+\mathsf{df}(p',p) \geq m} \frac{\kappa_i^{p'}(E_N)}{N^{i+\mathsf{df}(p',p)-m}} \right) \frac{1}{N^m}.$$

Let us define for any $N \in \mathbb{N}$, any $j \in \{0, \dots, m-1\}$ and any $p \in A_k$:

$$m_p^j(E_N) = \sum_{(p',i) \in A_k \times \{0,\dots,m-1\}, i+\mathsf{df}(p',p)=j} \kappa_i^{p'}(E_N)$$

and

$$m_p^m(E_N) = \sum_{(p',i) \in A_k \times \{0,\dots,m\}, i+\mathsf{df}(p',p) \geq m} \frac{\kappa_i^{p'}(E_N)}{N^{i+\mathsf{df}(p',p)-m}},$$

so that, for any $p \in A_k$ and any $N \in \mathbb{N}$:

$$m_p(E_N) = \sum_{j=0}^{m-1} \frac{m_p^i(E_N)}{N^j} + \frac{m_p^m(E_N)}{N^m}.$$

For any $p \in A_k$ and any $i \leq m-1$, $m_p^i(E_N)$ does not depend on N and for any $p \in A_k$, $\kappa_m^p(E_N)$ converges when N goes to infinity. Thus $m_p^m(E_N)$ converges when N goes to infinity to

$$\sum_{p' \in A_k, \mathsf{df}(p',p) \leq m} \kappa_{m-\mathsf{df}(p',p)}^{p'}(E).$$

By Definition 9.9, this shows that $(E_N)_{N \in \mathbb{N}}$ converges in moments up to the m^{th} order of fluctuations and the Equation (21) holds.

Let us suppose now that $(E_N)_{N \in \mathbb{N}}$ converges in moments up to the m^{th} order of fluctuations. Then, by Theorem 4.1, it converges strongly up to order 0 of fluctuation. Let us suppose that $(E_N)_{N \in \mathbb{N}}$ converges strongly up to order l of fluctuations with $l < m$. Thus, the coordinate numbers of E_N up to order l of fluctuations are well defined and we can write:

$$E_N = \sum_{p \in \mathcal{P}_k} \left(\sum_{j=0}^{l-1} \frac{\kappa_j^p(E)}{N^j} + \frac{\kappa_l^p(E_N)}{N^l} \right) p,$$

where, for any $p \in \mathcal{P}_k$, $\kappa_l^p(E_N)$ is converging when N goes to infinity to a number $\kappa_l^p(E)$. We can use the computation of the normalized moments of E_N that we already did in order to show that for any partition $p \in A_k$:

$$m_p(E_N) = \sum_{j=0}^{l-1} \left(\sum_{(p',i)\in A_k\times\{0,\dots,l-1\}, i+\mathsf{df}(p',p)=j} \kappa_i^{p'}(E) \right) \frac{1}{N^j} + \left(\sum_{(p',i)\in A_k\times\{0,\dots,l\}, i+\mathsf{df}(p',p)\geq l} \frac{\kappa_i^{p'}(E_N)}{N^{i+\mathsf{df}(p',p)-l}} \right) \frac{1}{N^l}.$$

Thus, using the same notations than those used in the first part of the proof, we get:

$$m_p(E_N) = \sum_{j=0}^{l} \frac{m_p^j(E)}{N^j} + \sum_{p'\in A_k, \mathsf{df}(p',p)=0} \frac{\kappa_l^{p'}(E_N) - \kappa_l^{p'}(E)}{N^l} + \sum_{(p',i)\in A_k\times\{0,\dots,l\}, i+\mathsf{df}(p',p)-l=1} \frac{\kappa_i^{p'}(E_N)}{N^{l+1}} + o\left(\frac{1}{N^{l+1}}\right).$$

Let us use the fact that $(E_N)_{N\in\mathbb{N}}$ converges in moments up to the order $l+1$ of fluctuations: for any $p' \in A_k$,

$$N^{l+1}\left(m_p(E_N) - \sum_{j=0}^{l} \frac{m_p^j(E)}{N^j} \right)$$

converges as N goes to infinity. This implies that for any $p \in A_k$,

$$\sum_{p'\in[id,p]_{A_k}} N(\kappa_l^{p'}(E_N) - \kappa_l^{p'}(E))$$

converges as N goes to infinity. We are thus in the same setting as for the order 0 of fluctuations: for any $p \in A_k$,

$$N(\kappa_l^{p'}(E_N) - \kappa_l^{p'}(E))$$

converges as N goes to infinity: this is equivalent to say that $(E_N)_{N\in\mathbb{N}}$ converges strongly up to order $l+1$ of fluctuations. This implies by recurrence that $(E_N)_{N\in\mathbb{N}}$ converges strongly up to order m of fluctuations. □

9.4. Multiplication and convergence of fluctuations in $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$. — The results in Section 9.3 were only algebraic: we will now give the similar results for elements in $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$. The main ingredients used in order to do so are Lemma 9.2, Lemma 9.3 and Lemma 9.4 which respectively assert that $eval_{(m)}^N$ is a morphism of algebra, compatible with the strong convergence notion and, in some sense, can be inverted.

Theorem 9.2. — *Let $(E_N)_{N\in\mathbb{N}}$ and $(F_N)_{N\in\mathbb{N}}$ be elements of $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$. Let us suppose that the sequences $(E_N)_{N\in\mathbb{N}}$ and $(F_N)_{N\in\mathbb{N}}$ converge up to the m^{th} order of fluctuations. Then, the sequence $(E_N F_N)_{N\in\mathbb{N}}$ converges up to the m^{th} order of fluctuations.*

Besides, using Notations 9.2 and 9.3, for any $i_0 \in \{0,\dots,m\}$:

– *for any* $p_0 \in A_k$*:*

$$\kappa_{i_0}^{p_0}(EF) = \sum_{p,p' \in A_k, \eta(p,p') \leq i_0, p \circ p' = p_0} \sum_{i=0}^{i_0 - \eta(p,p')} \kappa_i^p(E) \kappa_{i_0 - \eta(p,p') - i}^{p'}(F). \tag{22}$$

– *for any* $p_0 \in \mathcal{P}_k$*:*

$$m_{p_0}^{i_0}(EF) = \sum_{p_1 \in A_k} \sum_{i+j+\mathsf{df}(p_1,p_0)=i_0} \kappa_i^{p_1}(E) m_{i_{p_1 \circ p_0}}^j(F). \tag{23}$$

Proof. — Let $(E_N)_{N\in\mathbb{N}}$ and $(F_N)_{N\in\mathbb{N}}$ be elements of $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$. Let us suppose that the sequences $(E_N)_{N\in\mathbb{N}}$ and $(F_N)_{N\in\mathbb{N}}$ converge strongly or in moments up to the m^{th} order of fluctuations. By Lemma 9.4, let us consider the canonical lifts of $(E_N)_{N\in\mathbb{N}}$ (resp. $(F_N)_{N\in\mathbb{N}}$) in $\prod_{N\in\mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$: $(\tilde{E}_N)_{N\in\mathbb{N}}$ (resp. $(\tilde{F}_N)_{N\in\mathbb{N}}$). The two sequences $(\tilde{E}_N)_{N\in\mathbb{N}}$ and $(\tilde{F}_N)_{N\in\mathbb{N}}$ converge. According to Proposition 9.2, the sequence $(\tilde{E}_N \tilde{F}_N)_{N\in\mathbb{N}}$ converges. For any $i_0 \in \{0, \ldots, m\}$ and for any $p_0 \in A_k$:

$$\kappa_{i_0}^{p_0}(\tilde{E}\tilde{F}) = \sum_{p,p' \in A_k, \eta(p,p') \leq i_0, p \circ p' = p_0} \sum_{i \in \{0,\ldots,i_0 - \eta(p,p')\}} \kappa_i^p(\tilde{E}) \kappa_{i_0 - \eta(p,p') - i}^{p'}(\tilde{F}). \tag{24}$$

An application of Lemma 9.3 shows that the sequence $\left(eval_{(m)}^N(\tilde{E}_N \tilde{F}_N) \right)_{N\in\mathbb{N}}$ converges up to the m^{th} order of fluctuations. As $eval_{(m)}^N$ is a morphism of algebra, Lemma 9.2, for any $N \in \mathbb{N}$,

$$eval_{(m)}^N(\tilde{E}_N \tilde{F}_N) = eval_{(m)}^N(\tilde{E}_N) eval_{(m)}^N(\tilde{F}_N) = E_N F_N.$$

We deduce that $(E_N F_N)_{N\in\mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations. The equality (22) is deduced from (24).

In order to prove the equality (23), the best way is to come back to the definitions, and do a proof similar to the one for (16) in Theorem 7.1. □

Let us consider the implication of Proposition 9.3 for semi-groups in $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$. From now on, let us suppose that $\left(\left(E_t^N \right)_N \right)_{t\geq 0}$ is a semi-group in $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$ whose generator is $(H_N)_{N\in\mathbb{N}}$. We would like to state a theorem for the fluctuations of $\left(\left(E_t^N \right)_N \right)_{t\geq 0}$ similar to Theorem 7.2. For this, we need the following definition.

Definition 9.10. — *The semi-group* $\left(\left(E_t^N \right)_N \right)_{t\geq 0}$ *converges to the* m^{th} *order of fluctuations if and only if for any* $t \geq 0$, $\left(E_t^N \right)_{N\in\mathbb{N}}$ *converges up to the* m^{th} *order of fluctuations.*

Now we can state the theorem about the convergence to the m^{th} order of fluctuations of a semi-group in $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$. The proof will not be given, as it is a direct consequence of Proposition 9.3 with a lift-argument as for the last proof.

Theorem 9.3. — *The semi-group* $\left(\left(E_t^N \right)_N \right)_{t\geq 0}$ *converges to the* m^{th} *order of fluctuations if the sequences* $(E_0^N)_{N\in\mathbb{N}}$ *and* $(H_N)_{N\in\mathbb{N}}$ *converge up to the* m^{th} *order of fluctuations. Besides, we have the two differential systems of equations:*

– *for any $p_0 \in A_k$, for any $t_0 \geq 0$, for any $i_0 \in \{0, \dots, m\}$:*

$$\frac{d}{dt}_{|t=t_0} \kappa_{i_0}^{p_0}(E_t) = \sum_{p,p' \in A_k, \eta(p,p') \leq i_0, p \circ p' = p_0} \quad \sum_{i \in \{0,\dots,i_0 - \eta(p,p')\}} \kappa_i^p(H) \kappa_{i_0 - \eta(p,p') - i}^{p'}(E_{t_0}).$$

– *for any $p_0 \in \mathcal{P}_k$, for any $t_0 \geq 0$, for any $i \in \{0, ..., m\}$:*

$$\frac{d}{dt}_{|t=t_0} m_{p_0}^{i_0}(E_t) = \sum_{p_1 \in A_k} \sum_{i+j+\mathsf{df}(p_1,p_0)=i_0} \kappa_i^{p_1}(H_t) m_{t_{p_1 \circ p}}^j(E_{t_0}).$$

10. An introduction to the general $\mathcal{R}$-transform

10.1. The zero order. — Up to now, we only worked with partitions which have a fixed length: we worked in A_k for a fixed integer k. Yet, we could have worked with $A_\infty = \cup_{k \in \mathbb{N}} A_k$ endowed with the product: $pp' = \delta_{\mathsf{l}(p)=\mathsf{l}(p')} pp'$ where we recall that $\mathsf{l}(p)$ is the length of p. With this definition, we see that all the results hold when one changes k by $k = \infty$. For example $\mathbb{C}[A_\infty(N, N)]$ converges when N goes to infinity to an algebra $\mathbb{C}[A_\infty(\infty, \infty)]$.

10.1.1. Order zero: general definitions and Lie algebras. — Recall that A is either $\mathfrak{S}$, $\mathcal{B}$ or $\mathcal{P}$.

Definition 10.1*. — Let us define the algebra $\mathfrak{E}_g[A] = \prod_{k=0}^\infty \mathbb{C}\left[A_k(\infty, \infty)\right]$. Two subspaces of $\mathfrak{E}_g[A]$ will be interesting for us:*

$$\mathfrak{E}[A] = \{E \in \mathfrak{E}_g[A], E_\emptyset = 1\} \text{ and } \mathfrak{e}[A] = \{E \in \mathfrak{E}_g[A], E_\emptyset = 0\}.$$

Any element $E \in \mathfrak{E}_g[A]$ is of the form $E = \left(\sum_{p \in A_k} (E_k)_p p\right)_{k \in \mathbb{N}}$. In order to simplify the notations, we will use the following conventions: for any integer k, for any $p \in A_k$,

$$E_p = E(p) = (E_k)_p,$$

and for any positive integer k:

$$E_k = \sum_{p \in A_k} E_p p.$$

The algebra $\mathfrak{E}_g[A]$ is naturally endowed with a natural addition and multiplication given, for any $E, F \in \mathfrak{E}_g[A]$ and any $k \in \mathbb{N}^*$ by:

$$(E + F)_k = E_k + F_k$$
$$(E \boxtimes F)_k = E_k F_k.$$

By convention $(E \boxtimes F)_\emptyset = E_\emptyset F_\emptyset$. Besides, one can construct an other law on $\mathfrak{E}_g[A]$.

Definition 10.2*. — Let E and F be two elements of $\mathfrak{E}_g[A]$. We denote by $E \boxplus F$ the element of $\mathfrak{E}_g[A]$ such that for any $p \in A_{\mathsf{l}(p)}$:*

$$(E \boxplus F)_p = \sum_{(p_1,p_2,I) \in \mathfrak{F}_2(p)} E(p_1) F(p_2),$$

where $\mathfrak{F}_2(p)$ was defined in Definition 2.11.

In fact, the two operations $\boxtimes$ and $\boxplus$ are convolution operations.

Remark 10.1. — *The sets* $\mathfrak{E}[A]$ *and* $\mathfrak{e}[A]$ *are stable by the* $\boxplus$ *and* $\boxtimes$ *operations. Besides,* $\mathfrak{E}[A]$ *is an affine space whose underlying vector space is* $\mathfrak{e}[A]$.

The operation $\boxplus$ on $\mathfrak{E}[A]$ is commutative, it defines a structure of group on $\mathfrak{E}[A]$. The neutral element $0_{\mathfrak{E}}$ is the only element in $\mathfrak{E}[A]$ such that for any $k \in \mathbb{N}^*$, any $p \in A_k$, $(0_{\mathfrak{E}})_p = 0$.

The operation $\boxtimes$ is not commutative and the set of invertible elements in $\mathfrak{E}[A]$ is the set of elements E such that $E_{id_k} \neq 0$ for any $k \geq 1$, we denote it by $G\mathfrak{E}[A]$. We denote by $1_{\mathfrak{E}}$ the neutral element for $\boxtimes$ which is the only element such that for any $k \geq 1$, $(1_{\mathfrak{E}})_k = id_k$.

Let us consider an interesting sub-vector space of $\mathfrak{E}_g[A]$. Recall the notation $A_k^{(i)}$ defined in Definition 2.10.

Definition 10.3. — *We define* $\mathfrak{E}_g^{(i)}[A] = \prod_{k=0}^{\infty} \mathbb{C}\left[A_k^{(i)}\right]$. *Two subspaces of* $\mathfrak{E}_g[A]$ *will be interesting for us:*

$$\mathfrak{E}^{(i)}[A] = \mathfrak{E}_g^{(i)}[A] \cap \mathfrak{E}[A] \text{ and } \mathfrak{e}^{(i)}[A] = \mathfrak{E}_g^{(i)}[A] \cap \mathfrak{e}[A].$$

When $A = \mathfrak{S}$, we have already seen after Definition 2.10 that

$$A_k^{(i)} = \{\sigma^{-1}(1, \ldots, k)\sigma, \sigma \in \mathfrak{S}_k\}.$$

Let us consider $(\mathfrak{E}^{(i)}[A])^{\mathfrak{S}}$ the set of elements of $\mathfrak{E}^{(i)}[A]$ which are invariant by conjugation by any permutation: this means that for any positive integer k and any $\sigma \in \mathfrak{S}_k$, the following equality holds in $\mathbb{C}[A_k(N)]$ for any positive integer N:

$$\sigma E_k \sigma^{-1} = E_k,$$

Proposition 10.1. — *The affine space* $(\mathfrak{E}^{(i)}[\mathfrak{S}])^{\mathfrak{S}}$ *can ben identified with the affine space* $\mathbb{C}_1[[z]]$ *of formal power series which constant term is equal to* 1 *by the following isomorphism :*

$$\begin{aligned}(\mathfrak{E}^{(i)}[\mathfrak{S}])^{\mathfrak{S}} &\to \mathbb{C}_1[[z]] \\ E &\mapsto \sum_{k \in \mathbb{N}} E_{(1,\ldots,k)} z^k.\end{aligned}$$

Any element E in $\mathfrak{E}_g[A]$ can be restricted in order to obtain an element of $\mathfrak{E}_g^{(i)}[A]$ that we denote by $E_{|\mathfrak{E}^i[A]}$. Conversely, given an element of $\mathfrak{E}_g^{(i)}[A]$, one can inject it non-trivially in $\mathfrak{E}_g[A]$ in a natural way. Recall the definition of the extraction of p in Definition 2.12, and the definition of cycles given in Definition 2.9. We only consider the injection of an element of $\mathfrak{E}^{(i)}[A]$ in $\mathfrak{E}[A]$.

Definition 10.4. — *For any* $E \in \mathfrak{E}^{(i)}[A]$, *we denote by* $\mathfrak{M}(E)$ *the unique element of* $\mathfrak{E}[A]$ *such that for any integer* k, *any* $p \in A_k$,

$$(\mathfrak{M}(E))_p = \prod_{C \in \mathsf{C}(p)} E_{p_C}.$$

Any element of the image of the application:

$$\begin{aligned}\mathfrak{M} : \mathfrak{E}^{(i)}[A] &\to \mathfrak{E}[A] \\ E &\mapsto \mathfrak{M}(E)\end{aligned}$$

is called multiplicative and we denote $\mathfrak{ME}[A] = \mathfrak{M}\left[\mathfrak{E}[A]\right]$.

Let us remark that $0_{\mathfrak{E}}$ and $1_{\mathfrak{E}}$ are multiplicative elements. This is not the only property satisfied by $\mathfrak{ME}[A]$.

Theorem 10.1. — *The set* $\mathfrak{ME}[A]$ *is stable by the operations* $\boxplus$ *and* $\boxtimes$.

Proof. — Let E and F be two elements of $\mathfrak{ME}[A]$. Let us show that $E \boxplus F$ is multiplicative. Let p_1 and p_2 be two partitions, we have to show that:

$$(E \boxplus F)_{p_1 \otimes p_2} = (E \boxplus F)_{p_1} (E \boxplus F)_{p_2}. \tag{25}$$

Yet, by definition:

$$(E \boxplus F)_{p_1 \otimes p_2} = \sum_{(a_1, a_2, I) \in \mathfrak{F}_2(p_1 \otimes p_2)} E_{a_1} F_{a_2},$$

and:

$$(E \boxplus F)_{p_1} (E \boxplus F)_{p_2} = \sum_{(a_1^1, a_2^1, I^1) \in \mathfrak{F}_2(p_1), (a_1^2, a_2^2, I^2) \in \mathfrak{F}_2(p_2)} E_{a_1^1} E_{a_1^2} F_{a_2^1} F_{a_2^2}.$$

Using the fact that E and F are multiplicative, that $E_\emptyset = 1 = F_\emptyset$ and using the fact that for any $(a_1, a_2, I) \in \mathfrak{F}_2(p_1 \otimes p_2)$, a_1 and a_2 can be decomposed uniquely into two parts in order to get two 3-tuples $(a_1^1, a_2^1, I^1) \in \mathfrak{F}_2(p_1)$ and $(a_1^2, a_2^2, I^2) \in \mathfrak{F}_2(p_2)$, one gets the Equality (25).

Let us show that $E \boxtimes F$ is multiplicative. Let p_1 and p_2 be two partitions, we have to show that:

$$(E \boxtimes F)_{p_1 \otimes p_2} = (E \boxtimes F)_{p_1} (E \boxtimes F)_{p_2}.$$

By definition:

$$(E \boxtimes F)_{p_1 \otimes p_2} = \sum_{a, b / a \circ b = p_1 \otimes p_2, a \prec p_1 \otimes p_2} E_a F_b.$$

Yet, using Lemma 6.4, any partition a such that $a \prec p_1 \otimes p_2$ can be decomposed uniquely as $a_1 \otimes a_2$ such that $a_1 \prec p_1$ and $a_2 \prec p_2$. Then if b is a partition such that $a_1 \otimes a_2 \circ b = p_1 \otimes p_2$, b can be also decomposed uniquely as $b = b_1 \otimes b_2$ with $a_1 \otimes b_1 = p_1$ and $a_2 \otimes b_2 = p_2$. Using the multiplicative property of E and F, one gets:

$$\begin{aligned}(E \boxtimes F)_{p_1 \otimes p_2} &= \sum_{a_1, a_2, b_1, b_2 / a_1 \circ b_1 = p_1, a_2 \circ b_2 = p_2, a_1 \prec p_1, a_2 \prec p_2} E_{a_1} E_{a_2} F_{b_1} F_{b_2} \\ &= \sum_{a_1, b_1 / a_1 \circ b_1 = p_1, a_1 \prec p_1} E_{a_1} F_{b_1} \sum_{a_2, b_2 / a_2 \circ b_2 = p_2, a_2 \prec p_2} E_{a_2} F_{b_2} \\ &= (E \boxtimes F)_{p_1} (E \boxtimes F)_{p_2}.\end{aligned}$$

This ends the proof. □

Let us justify our notation $\boxplus$. If we consider the pull-back of the $\boxplus$ operation from $\mathfrak{ME}[A]$ to $\mathfrak{E}^{(i)}[A]$ and if one consider only the coefficients for the non-empty partitions, one simply obtains the usual additive law on $\mathfrak{E}^{(i)}[A]$. We will also see in the article [**9**] that $\boxplus$ is the natural operation which appears when one is working with sum of free elements.

We believe that the inverse of a multiplicative element for the $\boxplus$ and $\boxtimes$ is still multiplicative, but we have not yet written the proof. It is natural to wonder, as we have two semi-groups $(\mathfrak{ME}[A], \boxplus)$ and $(\mathfrak{ME}[A] \cap G\mathfrak{E}_A, \boxtimes)$ on which one can define differentiable one-parameter semi-groups, what are the "Lie algebras" of these two semi-groups. Let us remark that $\mathfrak{ME}[A] \cap G\mathfrak{E}_A$ is only the set of elements E of $\mathfrak{ME}[A]$ such that $E_{id_1} \neq 0$. We need to define two ways to inject $\mathfrak{e}^{(i)}[A]$ in $\mathfrak{e}[A]$, the first of which is the natural injection.

Definition 10.5. — *For any $E \in \mathfrak{e}^{(i)}[A]$, we denote by $\mathfrak{I}(E)$ the unique element of $\mathfrak{e}[A]$ such that, for any positive integer k, any irreducible $p \in A_k$,*

$$\left(\mathfrak{I}(E)\right)_p = E_p,$$

and for any non-irreducible $p \in A_k$, $\left(\mathfrak{I}(E)\right)_p = 0$. We define $\mathfrak{me}_{\boxplus}[A] = \mathfrak{I}(\mathfrak{e}^{(i)}[A])$.

The second injection uses the notion of support of a partition and the notion of weakly irreducible partitions defined in Definition 2.13. Recall also the notion of extraction defined in Definition 2.12.

Definition 10.6. — *For any $E \in \mathfrak{e}^{(i)}[A]$, we denote by $\mathfrak{J}(E)$ the unique element of $\mathfrak{e}[A]$ such that, for any integer k, any weakly irreducible $p \neq id_k$ in A_k:*

$$\left(\mathfrak{J}(E)\right)_p = E(p_{\mathsf{S}(p)}),$$

and $\left(\mathfrak{J}(E)\right)_{id_k} = k\left(\mathfrak{J}(E)\right)_{id_1}$ and for any other $p \in A_k$, $\left(\mathfrak{J}(E)\right)_p = 0$. We define $\mathfrak{me}_{\boxtimes}[A] = \mathfrak{J}(\mathfrak{e}^{(i)}[A])$.

Due to the definitions, it is obvious that the sets $\mathfrak{me}_{\boxtimes}[A]$ and $\mathfrak{me}_{\boxplus}[A]$ are vector spaces. Let us define the exponentiation of any element of $\mathfrak{e}[A]$ associated with the operation $\boxplus$.

Definition 10.7. — *Let $E \in \mathfrak{e}[A]$. The $\boxplus$-semi group associated with E is the family $(e_{\boxplus}^{tE})_{t\geq 0}$ of elements of $\mathfrak{E}[A]$ such that for any $t_0 \geq 0$:*

$$\frac{d}{dt}_{|t=t_0} e_{\boxplus}^{tE} = E \boxplus e_{\boxplus}^{t_0 E},$$
$$e_{\boxplus}^{0E} = 0_{\mathfrak{E}}.$$

Due to the commutativity of $\boxplus$, one has that for any $E, F \in \mathfrak{e}[A]$, $e_{\boxplus}^{E} \boxplus e_{\boxplus}^{F} = e_{\boxplus}^{E \boxplus F}$. Let us define the exponentiation associated with the operation $\boxtimes$.

Definition 10.8. — *Let $E \in \mathfrak{e}[A]$. The $\boxtimes$-semi group associated with E is the family $(e_{\boxtimes}^{tE})_{t\geq 0}$ of elements of $\mathfrak{E}[A]$ such that for any $t_0 \geq 0$:*

$$\frac{d}{dt}_{|t=t_0} e_{\boxtimes}^{tE} = E \boxtimes e_{\boxtimes}^{t_0 E},$$
$$e_{\boxtimes}^{0E} = 1_{\mathfrak{E}}.$$

We defined $e_{\boxplus}^{tE}$ and $e_{\boxtimes}^{tE}$ as a one-parameter semi-group for two reasons: it will appear later in this formulation, and it allows us to have a Lie group/Lie algebra formalism. As noticed by G. Cébron, an equivalent definition is given by the next proposition.

Proposition 10.2. — *Let $E \in \mathfrak{e}[A]$, for any $t \in \mathbb{R}^+$,*

$$e_{\boxplus}^{tE} = \sum_{n=0}^{\infty} \frac{t^n}{n!} E^{\boxplus n} \text{ and } e_{\boxtimes}^{tE} = \sum_{n=0}^{\infty} \frac{t^n}{n!} E^{\boxtimes n},$$

where $E^{\boxplus 0} = 0_{\mathfrak{E}}$ and $E^{\boxtimes 0} = 1_{\mathfrak{E}}$.

Actually, we will use implicitely this fact when we will have to compute a element of the form $e_{\boxplus}^{tE}$ in the article [**9**]. Besides, if one wants to make everything explicits, for example this implies that for any $t \in \mathbb{R}^+$, any positive integer k, any $p \in A_k$ and any $E \in \mathfrak{e}[A]$,

$$\left(e_{\boxtimes}^{tE}\right)_p = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{(p_1,\ldots,p_n)\in A_k, p_1 \prec p_1\circ p_2 \prec \ldots \prec p_1\circ\ldots\circ p_n / p_1\circ\ldots\circ p_n = p} E_{p_1} E_{p_2} \ldots E_{p_n}.$$

The next theorem shows that $\mathfrak{me}_{\boxplus}[A]$ and $\mathfrak{me}_{\boxtimes}[A]$ are the Lie algebras of respectively $(\mathfrak{ME}[A], \boxplus)$ and $(\mathfrak{ME}[A] \cap G\mathfrak{E}_A, \boxtimes)$.

Theorem 10.2. — *Let $E \in \mathfrak{me}_{\boxplus}[A]$, for any $t \geq 0$,*

$$e_{\boxplus}^{tE} \in \mathfrak{ME}[A].$$

Besides for any differentiable one-parameter semi-group $(E^t)_{t\geq 0}$ in $(\mathfrak{ME}[A], \boxplus)$ such that $E^0 = 0_{\mathfrak{E}}$, there exists $E \in \mathfrak{me}_{\boxplus}[A]$ such that for any $t \geq 0$,

$$e_{\boxplus}^{tE} = E^t.$$

Let $E \in \mathfrak{me}_{\boxtimes}[A]$. For any $t \geq 0$,

$$e_{\boxtimes}^{tE} \in \mathfrak{ME}[A].$$

Besides for any differentiable one-parameter semi-group $(E^t)_{t\geq 0}$ in $(\mathfrak{ME}[A], \boxtimes)$ such that $E^0 = 1_{\mathfrak{E}}$, there exists $E \in \mathfrak{me}_{\boxtimes}[A]$ such that for any $t \geq 0$,

$$e_{\boxtimes}^{tE} = E^t.$$

Proof. — Before doing the proof, let us give the two general ideas that we will use.

1. Let $(E^t)_{t\geq 0}$ be a differentiable family of elements of $\mathfrak{E}[A]$ such that $E^0 = 0_{\mathfrak{E}}$ or $E^0 = 1_{\mathfrak{E}}$. In order to prove that $E^t \in \mathfrak{ME}[A]$ for any real $t \geq 0$, it is enough to show that $(E^t)_{t\geq 0}$ and $\left(\mathfrak{M}\left(E^t_{|\mathfrak{E}^{(i)}[A]}\right)\right)_{t\geq 0}$ satisfy the same differential linear equations.
2. Let $(E^t)_{t\geq 0}$ be a differentiable one-parameter semi-group for the $\boxplus$ operation (resp. $\boxtimes$ operation), which is in $\mathfrak{ME}[A]$ and which starts at $0_{\mathfrak{E}}$ (resp. $1_{\mathfrak{E}}$). In order to

prove that there exists $E \in \mathfrak{me}_{\boxplus}[A]$ (resp. $E \in \mathfrak{me}_{\boxtimes}[A]$) such that for any $t \geq 0$, $e_{\boxplus}^{tE} = E^t$ (resp. $e_{\boxtimes}^{tE} = E^t$), it is enough to show that:

$$(E^t)_{t\geq 0} \text{ and } \left(e_{\boxplus}^{t\mathfrak{J}\left(\left(\frac{d}{dt}_{|t=0} E_t\right)_{|\mathfrak{e}^{(i)}[A]}\right)} \right)_{t\geq 0}$$

$$\left(\text{resp. } (E^t)_{t\geq 0} \text{ and } \left(e_{\boxtimes}^{t\mathfrak{J}\left(\left(\frac{d}{dt}_{|t=0} E_t\right)_{|\mathfrak{e}^{(i)}[A]}\right)} \right)_{t\geq 0} \right)$$

satisfy the same differential linear equations.

Let us prove Theorem 10.2. Let $E \in \mathfrak{me}_{\boxplus}[A]$. For any $t \geq 0$ we consider $E^t = e_{\boxplus}^{tE}$. Let n be a positive integer and let us consider n irreducible partitions $p_1, \ldots, p_n$ in $\cup_{k\in\mathbb{N}^*} A_k^{(i)}$. For any real $t_0 \geq 0$, we have:

$$\begin{aligned}\frac{d}{dt}_{|t=t_0} E^t_{p_1\otimes\cdots\otimes p_n} &= \left(E \boxplus E^{t_0}\right)_{p_1\otimes\ldots\otimes p_n} \\ &= \sum_{(p_1',p_2',I)\in\mathfrak{F}_2(p_1\otimes\ldots\otimes p_n)} E(p_1')E^{t_0}(p_2').\end{aligned}$$

Yet, we must not forget that E is in $\mathfrak{me}_{\boxplus}[A]$: for any integer k, any $p \in \mathcal{P}_k$, if p is not irreducible or if $p = \emptyset$, then $E(p) = 0$. Thus the sum we are considering can be taken over the $(p_1', p_2', I) \in \mathfrak{F}_2(p_1 \otimes \ldots \otimes p_n)$ such that p_1' is irreducible and not equal to $\emptyset$: this means in particular that $p'_{1\mathsf{S}(p_1')}$ is one of the $(p_i)_{i=1}^n$. Thus:

$$\frac{d}{dt}_{|t=t_0} E^t_{p_1\otimes\ldots\otimes p_n} = \sum_{i=1}^n E(p_i) E^{t_0}_{p_1\otimes\cdots\otimes p_{i-1}\otimes p_{i+1}\otimes\cdots\otimes p_n}.$$

On the other hand,

$$\begin{aligned}\frac{d}{dt}_{|t=t_0} \left(E^t_{p_1}\ldots E^t_{p_n}\right) &= \sum_{i=1}^n \left(\frac{d}{dt}_{|t=t_0} E^t_{p_i}\right) \prod_{j\neq i} E^{t_0}_{p_j} \\ &= \sum_{i=1}^n E(p_i) \prod_{j\neq i} E^{t_0}_{p_j}.\end{aligned}$$

This allows us to conclude that $E^t \in \mathfrak{ME}[A]$ for any real $t \geq 0$.

Let $(E^t)_{t\geq 0}$ be a differentiable one-parameter semi-group for the $\boxplus$ operation which is in $\mathfrak{ME}[A]$ and such that $E^0 = 0_{\mathfrak{E}}$. Then using the same calculation that we did, for any integer n and any irreducible partitions $p_1, \ldots, p_n$ in $\cup_{k\in\mathbb{N}^*} A_k^{(i)}$, for any real $t_0 \geq 0$, we have:

$$\frac{d}{dt}_{|t=t_0} (E^t_{p_1\otimes\ldots\otimes p_n}) = \frac{d}{dt}_{|t=t_0} (E^t_{p_1}\ldots E^t_{p_n}) = \sum_{i=1}^n \left(\frac{d}{dt}_{|t=t_0} E^t_{p_i}\right) \prod_{j\neq i} E^{t_0}_{p_j}.$$

Yet p_i is irreducible, thus using the fact that $\frac{d}{dt}_{|t=0} E^t_{\emptyset} = 0$, we get that $\frac{d}{dt}_{|t=t_0} E^t_{p_i} = \left(\left(\frac{d}{dt}_{|t=0} E^t\right) \boxplus E^{t_0}\right)_{p_i} = \frac{d}{dt}_{|t=0} E^t_{p_i}$, and thus:

$$\begin{aligned}\frac{d}{dt}_{|t=t_0}(E^t_{p_1\otimes\ldots\otimes p_n}) &= \sum_{i=1}^{n}\left(\frac{d}{dt}_{|t=0} E^t_{p_i}\right)\prod_{j\neq i} E^{t_0}_{p_j}\\ &= \left(\mathfrak{I}\left(\left(\frac{d}{dt}_{|t=0} E_t\right)_{|\mathfrak{E}^{(i)}[A]}\right)\boxplus E^{t_0}\right)_{p_1\otimes\ldots\otimes p_n}\end{aligned}$$

and thus there exists $E \in \mathfrak{me}_{\boxplus}[A]$ such that for any $t \geq 0$, $e^{tE}_{\boxplus} = E^t$.

Now, let $E \in \mathfrak{me}_{\boxtimes}[A]$. For any $t \geq 0$ we consider $E^t = e^{tE}_{\boxtimes}$. Let n be an integer and let us consider n irreducible partitions $p_1, \ldots, p_n$ in $\cup_{k\in\mathbb{N}^*} A^{(i)}_k$. For any real $t_0 \geq 0$, we have:

$$\begin{aligned}\frac{d}{dt}_{|t=t_0} E^t_{p_1\otimes\ldots\otimes p_n} &= \left(E \boxtimes E^{t_0}\right)_{p_1\otimes\ldots\otimes p_n}\\ &= \sum_{a,b/a\circ b=p_1\otimes\cdots\otimes p_n, a\prec p_1\otimes\cdots\otimes p_n} E_a E^{t_0}_b.\end{aligned}$$

Yet, we must not forget that E is in $\mathfrak{me}_{\boxtimes}[A]$: for any integer k, any $p \in \mathcal{P}_k$, if p is not weakly irreducible then $E(p) = 0$. Thus the sum we are considering can be taken over the couples (a, b) such that a is weakly irreducible. Besides, $E_{id_l} = lE_{id_1}$ for any integer l. Thus:

$$E_{id_{\sum_{i=1}^n l(p_i)}} E^{t_0}_{p_1\otimes\cdots\otimes p_n} = \sum_{i=1}^{n} E_{id_{l(p_i)}} E^{t_0}_{p_1\otimes\cdots\otimes p_n}.$$

Thus, we get:

$$\frac{d}{dt}_{|t=t_0} E^t_{p_1\otimes\ldots\otimes p_n} = \sum_{i=1}^{n}\sum_{a,b|a\circ b=p_i, a\prec p_i} E_a E^{t_0}_{p_1\otimes\cdots\otimes p_{i-1}\otimes b\otimes p_{i+1}\otimes\cdots\otimes p_n}.$$

On the other hand,

$$\begin{aligned}\frac{d}{dt}_{|t=t_0}\left(E^t_{p_1}\ldots E^t_{p_n}\right) &= \sum_{i=1}^{n}\left(\frac{d}{dt}_{|t=t_0} E^t_{p_i}\right)\prod_{j\neq i} E^{t_0}_{p_j}\\ &= \sum_{i=1}^{n}\sum_{a,b|a\circ b=p_i, a\prec p_i} E_a E^{t_0}_b \prod_{j\neq i} E^{t_0}_{p_j}.\end{aligned}$$

This allows us to conclude that $E^t \in \mathfrak{ME}[A]$ for any real $t \geq 0$.

Let $(E^t)_{t\geq 0}$ be a differentiable one-parameter semi-group for the $\boxtimes$ operation which is in $\mathfrak{ME}[A]$ and such that $E^0 = 1_{\mathfrak{E}}$. Then using the same calculation that we did, for any integer n and any irreducible partitions $p_1, \ldots, p_n$ in $\cup_{k\in\mathbb{N}^*} A^{(i)}_k$, for any real $t_0 \geq 0$, we have:

$$\frac{d}{dt}_{|t=t_0}(E^t_{p_1\otimes\ldots\otimes p_n}) = \frac{d}{dt}_{t=t_0}(E^t_{p_1}\ldots E^t_{p_n}) = \sum_{i=1}^{n}\left(\frac{d}{dt}_{|t=t_0} E^t_{p_i}\right)\prod_{j\neq i} E^{t_0}_{p_j}.$$

Yet for any $i \in \{1, ..., n\}$, $\frac{d}{dt}_{|t=t_0} E^t_{p_i} = \sum_{a,b|a\circ b=p_i, a\prec p_i} \frac{d}{dt}_{|t=0} E^t_{p_i} E^{t_0}_b$, and thus:

$$\frac{d}{dt}_{|t=t_0}(E^t_{p_1\otimes...\otimes p_n}) = \sum_{i=1}^{n} \left(\sum_{a,b/a\circ b=p_i, a\prec p_i} \frac{d}{dt}_{|t=0} E^t_{p_i} E^{t_0}_b \right) \prod_{j\neq i} E^{t_0}_{p_j}$$
$$= \left(\mathfrak{J} \left(\left(\frac{d}{dt}_{|t=0} E_t \right)_{|\mathfrak{e}^{(i)}[A]} \right) \boxtimes E^{t_0} \right)_{p_1\otimes...\otimes p_n}$$

and thus there exists $E \in \mathfrak{me}_{\boxtimes}[A]$ such that for any $t \geq 0$, $e^{tE}_{\boxtimes} = E^t$. □

Remark 10.2. — *In fact, $\mathfrak{e}[A]$ is endowed with two structures of Lie algebras. Indeed, it is a vector space for the addition and multiplication by a scalar, and we can define two Lie brackets on it, one named $[.,.]_{\boxplus}$ which comes from the $\boxplus$ operation and the other named $[.,.]_{\boxtimes}$ which comes from the $\boxtimes$ operation. In order to know which bracket is considered on $\mathfrak{e}[A]$, we will denote it either by $\mathfrak{e}_{\boxplus}[A]$ or by $\mathfrak{e}_{\boxtimes}[A]$.*

Since the operation $\boxplus$ is commutative, the bracket $[.,.]_{\boxplus}$ is trivial. Thus $\mathfrak{me}_{\boxplus}$ is a sub-Lie algebra of $\mathfrak{e}_{\boxplus}$. Since the operation $\boxtimes$ is not commutative, the bracket $[.,.]_{\boxtimes}$ is not trivial and for any E and F in $\mathfrak{e}_{\boxtimes}[A]$,

$$[E, F]_{\boxtimes} = E \boxtimes F - F \boxtimes E.$$

Then, it is not difficult to see directly that $\mathfrak{me}_{\boxtimes}[A]$ is a sub-Lie algebra of $\mathfrak{e}_{\boxtimes}[A]$.

An element E belongs to $\mathfrak{me}_{\boxplus}[A]$ or $\mathfrak{me}_{\boxtimes}[A]$ if some conditions on its coordinates are satisfied. It would be interesting to know that are the conditions on the moments or on the exclusive moments of E which allows us to know if E is in $\mathfrak{me}_{\boxplus}[A]$ or $\mathfrak{me}_{\boxtimes}[A]$. In the Section 10.2 of [**9**], we prove the following theorem.

Theorem 10.3. — *Let E be in $\mathfrak{e}[A]$. Following the notations of Theorem 8.2, for any positive integer k, any $p \in \mathcal{P}_k$, we define:*

$$m^{\infty}_p(E) = \sum_{p'\in A_k} \delta_{p'\in[id,p_0]_{A_k}} E_{p'}.$$

We have the following characterization of $\mathfrak{me}_{\boxplus}[A]$ and $\mathfrak{me}_{\boxtimes}[A]$.

1. *$E \in \mathfrak{me}_{\boxplus}[A]$ if and only if for any positive integer k and l, for any $p_1 \in A_k$ and any $p_2 \in A_l$:*
$$m^{\infty}_{p_1\otimes p_2}(E) = 0.$$
2. *$E \in \mathfrak{me}_{\boxtimes}[A]$ if and only if for any positive integer k and l, for any $p_1 \in A_k$ and any $p_2 \in A_l$:*
$$m^{\infty}_{p_1\otimes p_2}(E) = m^{\infty}_{p_1}(E) + m^{\infty}_{p_2}(E).$$

In order to state a similar theorem for the exclusive moments, we need the definition of exclusive-irreducible partitions.

Definition 10.9. — *For any positive integer k, we define $\mathsf{0}_k = \{1, ..., k, 1', ..., k'\}$. We say that a partition p is exclusive-irreducible if there exists a cycle c_0 of p such that for any cycle c of p which is different from c_0, the extraction of p on c is equal to $\mathsf{0}_{\mathsf{l}(p_c)}$. If the cycle c_0 is unique it is called the exclusive-support of p: it is denoted by $\mathsf{Supp}^c(p)$.*

If it is not unique, then up to a permutation of the columns, p is a tensor product of elements of $(\mathsf{0}_k)_k$.

Recall the definition of the extraction of a partition p on a subset of $\{1,...,k,1',...,k'\}$ (Definition 2.12). We only state the theorem for $\mathfrak{me}_{\boxtimes}[\mathcal{P}]$ but one could state a similar result for $\mathfrak{me}_{\boxplus}[\mathcal{P}]$.

Theorem 10.4. — *Let E be an element of $\mathfrak{me}_{\boxtimes}[\mathcal{P}]$. Let us denote by E^c the element such that for any positive integer k and any $p \in \mathcal{P}_k$, $E^c_p = \sum_{p' \sqsupset p} E_{p'}$. The element E is in $\mathfrak{me}_{\boxtimes}[\mathcal{P}]$ if and only if for any positive integer k, for any $p \in \mathcal{P}_k$,*

1. *if p is not exclusive-irreducible, $E^c_p = 0$,*
2. *if p is exclusive-irreducible and is a tensor product of elements of $(\mathsf{0}_k)_k$ up to a permutation of the columns, then $E^c_p = \sum_{b \text{ cycle of } p} E^c_{p_b}$,*
3. *if p is exclusive-irreducible and the exclusive-support of p is defined then $E^c_p = E^c_{p_{\mathsf{Supp}^c(p)}}$.*

Proof. — Let $E \in \mathfrak{me}_{\boxtimes}[\mathcal{P}]$, let k be a positive integer and let p be a partition in $\mathcal{P}_k$. By definition,

$$E^c_p = \sum_{p' \sqsupset p} E_{p'}.$$

Yet, if $p' \sqsupset p$ and p is not exclusive-irreducible, then p' is not weakly irreducible. This implies that $E^c_p = 0$ if p is not exclusive-irreducible. Let us suppose that p is exclusive-irreducible. Let us suppose that it is a tensor product of elements of $(\mathsf{0}_l)_l$. We will only consider the case where it is of the form $p = \mathsf{0}_a \otimes \mathsf{0}_b$ with $a + b = k$. The partition p' which are weakly irreducible and such that $p' \sqsupset p$ are the partitions of the form $p'_1 \otimes id_b$ with p'_1 weakly irreducible and $p'_1 \sqsupset \mathsf{0}_a$ or $id_a \otimes p'_2$ with p'_2 weakly irreducible and $p'_2 \sqsupset \mathsf{0}_b$. These two sets are not disjoint, but one must not forget that $E_{id_a \otimes id_b} = E_{id_a} + E_{id_b}$ since $E \in \mathfrak{me}_{\boxtimes}[\mathcal{P}]$. Thus:

$$\begin{aligned} E^c_p &= \sum_{p'_1 \neq id_a, p'_1 \sqsupset \mathsf{0}_a} E_{p'_1 \otimes id_b} + \sum_{p'_2 \neq id_b, p'_2 \sqsupset \mathsf{0}_b} E_{id_a \otimes p'_2} + E_{id_a \otimes id_b} \\ &= \sum_{p'_1 \neq id_a, p'_1 \sqsupset \mathsf{0}_a} E_{p'_1} + \sum_{p'_2 \neq id_b, p'_2 \sqsupset \mathsf{0}_b} E_{p'_2} + E_{id_a} + E_{id_b} = E^c_{\mathsf{0}_a} + E^c_{\mathsf{0}_b}. \end{aligned}$$

The argument can be easily generalized for any tensor product of elements of $(\mathsf{0}_l)_l$. Using similar argument, one prove that if p is exclusive-irreducible and the exclusive-support of p is defined, then $E^c_p = E^c_{p_{\mathsf{Supp}^c(p)}}$. This implies that E^c satisfies the three conditions stated in the theorem.

Now, let us consider $E \in \mathfrak{e}[\mathcal{P}]$ such that for any positive integer k, for any $p \in \mathcal{P}_k$, if one defines E^c as in the theorem, one has:

1. if p is not exclusive-irreducible, $E^c_p = 0$,
2. if p is exclusive-irreducible and is a tensor product of elements of $(\mathsf{0}_k)_k$ up to a permutation of the columns, then $E^c_p = \sum_{b \text{ cycle of } p} E^c_{p_b}$,

3. if p is exclusive-irreducible and the exclusive-support of p is defined then $E^c_p = E^c_{p_{\mathsf{Supp}^c(p)}}$.

Let us prove that $E \in \mathfrak{me}_{\boxtimes}[\mathcal{P}]$. Let k and l be two positive integers. Let p_1 and p_2 be two irreducible partitions in $\mathcal{P}_k$ and $\mathcal{P}_l$. Then, using the notations of Theorem 10.3, and using Theorem 3.3:

$$m^{\infty}_{p_1\otimes p_2}(E) = \sum_{p\dashv p_1\otimes p_2} (E^c)_p = \sum_{p'_1\dashv p_1, p'_2\dashv p_2} (E^c)_{p'_1\otimes p'_2}\,.$$

Using the fact that if $p'_1 \otimes p'_2$ is not exclusive-irreducible then $E^c_p = 0$, we get:

$$\begin{aligned} m^{\infty}_{p_1\otimes p_2}(E) &= \sum_{p'_1\dashv p_1, p'_1\neq 0_{\mathsf{l}(p_1)}} (E^c)_{p'_1\otimes 0_{\mathsf{l}(p_2)}} + \sum_{p'_2\dashv p_2, p'_2\neq 0_{\mathsf{l}(p_2)}} (E^c)_{0_{\mathsf{l}(p_1)}\otimes p'_2} + E^c_{0_{\mathsf{l}(p_1)}\otimes 0_{\mathsf{l}(p_2)}} \\ &= \sum_{p'_1\dashv p_1, p'_1\neq 0_{\mathsf{l}(p_1)}} (E^c)_{p'_1} + \sum_{p'_2\dashv p_2, p'_2\neq 0_{\mathsf{l}(p_2)}} (E^c)_{p'_2} + E^c_{0_{\mathsf{l}(p_1)}} + E^c_{0_{\mathsf{l}(p_2)}} \\ &= m^{\infty}_{p_1}(E) + m^{\infty}_{p_2}(E). \end{aligned}$$

The argument can be easily generalized for more that two irreducible partitions. Using Theorem 10.3, this implies that $E \in \mathfrak{me}_{\boxtimes}[A]$. □

10.1.2. The $\mathcal{R}_A$-transform. — We will define the notion of $\mathcal{R}_A$-transform. This application will be defined as the inverse of the $\mathcal{M}_A$-transform whose definition lies on the Equation (7).

Definition 10.10. — *The $\mathcal{M}_A$-transform is the application:*

$$\begin{aligned} \mathcal{M}_A : \mathfrak{E}[A] &\to \mathfrak{E}[A] \\ E &\to \mathcal{M}_A(E) \end{aligned}$$

such that for any $E \in \mathfrak{E}[A]$, for any integer k, any $p \in A_k$:

$$(\mathcal{M}_A(E))_p = \sum_{p'\in[id,p]_{A_k}} E_{p'}.$$

This application is a bijection. Thus we can consider its inverse.

Definition 10.11. — *The $\mathcal{R}_A$-transform is the inverse of the $\mathcal{M}_A$-transform:*

$$\mathcal{R}_A = \mathcal{M}_A^{-1}.$$

We will often forget about the indices A when we will work with the $\mathcal{R}$-transforms. One can show that the $\mathcal{R}_A$-transform is a bijection from $\mathfrak{ME}[A]$ to itself.

Proposition 10.3. — *The $\mathcal{R}_A$-transform is a bijection from $\mathfrak{ME}[A]$ to itself.*

Proof. — We recall that the $\mathcal{R}_A$-transform is, by definition, a bijection from $\mathfrak{E}[A]$ to itself. Let $E \in \mathfrak{ME}[A]$, we have to show that $\mathcal{M}_A[E] \in \mathfrak{ME}[A]$ and $\mathcal{R}_A[E] \in \mathfrak{ME}[A]$. Let k and l be two positive integers. Let $p_1 \in \mathcal{P}_k$ and $p_2 \in \mathcal{P}_l$.

Let us show that $\mathcal{M}_A[E] \in \mathfrak{ME}[A]$. Using Lemma 3.8 and the multiplicative property of E, we have:

$$\begin{aligned}(\mathcal{M}_A[E])_{p_1\otimes p_2} = \sum_{p'\in[id,p_1\otimes p_2]_{A_k}} E_{p'} &= \sum_{p'_1\in[id,p_1]_{A_k},p'_2\in[id,p_2]_{A_k}} E_{p'_1\otimes p'_2}\\ &= \sum_{p'_1\in[id,p_1]_{A_k},p'_2\in[id,p_2]_{A_k}} E_{p'_1}E_{p'_2}\\ &= \sum_{p'_1\in[id,p_1]_{A_k}} E_{p'_1} \sum_{p'_2\in[id,p_2]_{A_k}} E_{p'_2}\\ &= (\mathcal{M}_A[E])_{p_1}\,(\mathcal{M}_A[E])_{p_2}\,.\end{aligned}$$

Now, let us show that $\mathcal{R}_A[E] \in \mathfrak{ME}[A]$. Let us consider $(\tilde{E}_p)_{p\in \bigcup_{k\in\mathbb{N}^*} A_k^{(i)}}$ such that for any positive integer k, any $p \in A_k^{(i)}$,

$$E_p = \sum_{p'\in[id,p]_{A_k}} \prod_{c\in\mathsf{C}(p')} \tilde{E}_{p_c}.$$

Using the multiplicativity of E, and Lemma 3.8, we see that E being in $\mathfrak{ME}[A]$, the family $(\tilde{E}_p)_{p\in \bigcup_{k\in\mathbb{N}} A_k^{(i)}}$ satisfies in fact that for any integer k, any $p \in A_k$:

$$E_p = \sum_{p'\in[id,p]_{A_k}} \prod_{c\in\mathsf{C}(p')} \tilde{E}_{p_c}.$$

Thus $\prod_{c\in\mathsf{C}(p')} \tilde{E}_{p_c}$ is equal to $(\mathcal{R}_A[E])_{p'}$ and thus $\mathcal{R}_A[E] \in \mathfrak{ME}[A]$. □

We can also translate the Lemma 3.9 in terms of $\mathcal{R}$-transform. Recall Definition 4.7. We denote by $\mathsf{R}_{|\mathfrak{S}}$ the restriction function from $\mathfrak{E}[\mathcal{B}]$ to $\mathfrak{E}[\mathfrak{S}]$ such that for any $E \in \mathfrak{E}[\mathcal{B}]$, for any $k \in \mathbb{N}$,

$$\big(\mathsf{R}_{|\mathfrak{S}}(E)\big)_k = (\mathsf{R}_{|\mathfrak{S}_k} \circ \mathsf{E}^{\mathcal{B}_k})(E_k).$$

Proposition 10.4. — *The following diagram is commutative:*

$$\begin{array}{ccc}\mathfrak{E}[\mathcal{B}] & \xrightarrow{\mathcal{R}_{\mathcal{B}}} & \mathfrak{E}[\mathcal{B}]\\ \downarrow{\scriptstyle \mathsf{R}_{|\mathfrak{S}}} & & \downarrow{\scriptstyle \mathsf{R}_{|\mathfrak{S}}}\\ \mathfrak{E}[\mathfrak{S}] & \xrightarrow{\mathcal{R}_{\mathfrak{S}}} & \mathfrak{E}[\mathfrak{S}]\end{array}$$

Proof. — It is only a consequence of the fact that:

$$\begin{array}{ccc}\mathfrak{E}[\mathcal{B}] & \xleftarrow{\mathcal{M}_{\mathcal{B}}} & \mathfrak{E}[\mathcal{B}]\\ \downarrow{\scriptstyle \mathsf{R}_{|\mathfrak{S}}} & & \downarrow{\scriptstyle \mathsf{R}_{|\mathfrak{S}}}\\ \mathfrak{E}[\mathfrak{S}] & \xleftarrow{\mathcal{M}_{\mathfrak{S}}} & \mathfrak{E}[\mathfrak{S}]\end{array}$$

is commutative. Indeed, using Lemma 3.9, if $E \in \mathfrak{E}[\mathcal{B}]$, and if $\sigma \in \mathfrak{S}_k$:

$$\mathsf{R}_{|\mathfrak{S}}\left[\mathcal{M}_{\mathcal{B}}(E)\right](\sigma) = \left(\mathcal{M}_{\mathcal{B}}(E)\right)(\sigma) = \sum_{p \in [id,\sigma]_{\mathcal{B}_k}} E_p = \sum_{p \in [id,\sigma]_{\mathfrak{S}_k}} E_p = \left[\mathcal{M}_{\mathfrak{S}} \circ \mathsf{R}_{|\mathfrak{S}}(E)\right](\sigma).$$

This concludes the proof. □

It is well-known in the literature that there exists a notion of $\mathcal{R}$-transform on $\mathbb{C}_1[[z]]$ which we will call the $\mathcal{R}_u$-transform. In order to finish this section, we make the link between the $\mathcal{R}_A$-transform and the $\mathcal{R}_u$-transform.

Definition 10.12. — *Let $M(z)$ be a formal power serie in $\mathbb{C}_1[[z]]$, that is a formal power serie of the form:*

$$M(z) = 1 + \sum_{n=1}^{\infty} a_n z^n.$$

Let $C(z)$ be the formal power serie $C(z) = 1 + \sum_{n=1}^{\infty} k_n z^n$ such that $C[zM(z)] = M(z)$. The $\mathcal{R}_u$-transform of M is C.

The $\mathcal{R}_A$-transform is a generalization of the usual $\mathcal{R}_u$-transform. Indeed, we have the following theorem.

Theorem 10.5. — *Using the identification $(\mathfrak{E}^{(i)}[\mathfrak{S}])^{\mathfrak{S}} \simeq \mathbb{C}_1[[z]]$ explained in Proposition 10.1, the following diagram is commutative:*

$$\begin{array}{ccc} (\mathfrak{E}^{(i)}[\mathfrak{S}])^{\mathfrak{S}} & \xrightarrow{\mathcal{R}_u} & (\mathfrak{E}^{(i)}[\mathfrak{S}])^{\mathfrak{S}} \\ \Big\downarrow \mathfrak{M} & & \Big\downarrow \mathfrak{M} \\ \mathfrak{E}[\mathfrak{S}] & \xrightarrow{\mathcal{R}_{\mathfrak{S}}} & \mathfrak{E}[\mathfrak{S}] \end{array}$$

Proof. — Let E be an element of $\mathfrak{E}^{(i)}[\mathfrak{S}] \simeq \mathbb{C}_1[[z]]$. Using Theorem 2.7 of [**17**], and using the bijection between non-crossing partitions of k elements and the set $[id, (1, \ldots, k)]_{\mathfrak{S}_k}$, we know that $\mathcal{R}_u(E)$ is characterized by the fact that for any integer $k > 0$:

$$E_{(1,\ldots,k)} = \sum_{p \in [id,(1,\ldots,k)]_{\mathfrak{S}_k}} \prod_{c \text{ cycle of } p} \mathcal{R}_u(E)(1, \ldots, \#c).$$

Or, with our notations:

$$E_{(1,\ldots,k)} = \sum_{p \in [id,(1,\ldots,k)]_{\mathfrak{S}_k}} \mathfrak{M}\left[\mathcal{R}_u(E)\right].$$

By the factorization property of the geodesics, Lemma 3.8, for any $\sigma \in \mathfrak{S}_k$:

$$\left[\mathfrak{M}(E)\right](\sigma) = \sum_{\sigma' \in [id,\sigma]_{\mathfrak{S}_k}} \mathfrak{M}[\mathcal{R}_u(E)](\sigma).$$

This is equivalent to the fact that $\mathcal{R}_{\mathfrak{S}}\left[\mathfrak{M}(E)\right] = \mathfrak{M}[\mathcal{R}_u(E)]$. □

10.1.3. Transformations linked with the exclusive moments. — For the sake of clarity, we only consider the case where $A = \mathcal{P}$.

Definition 10.13. — *The $\mathcal{M}^{c\rightarrow}$-transform is the application:*

$$\begin{aligned}\mathcal{M}^{c\rightarrow} : \mathfrak{E}[\mathcal{P}] &\rightarrow \mathfrak{E}[\mathcal{P}] \\ E &\mapsto \mathcal{M}^{c\rightarrow}(E),\end{aligned}$$

such that for any $E \in \mathfrak{E}[\mathcal{P}]$, for any integer k, any $p \in \mathcal{P}_k$:

$$\left(\mathcal{M}^{c\rightarrow}(E)\right)_p = \sum_{p' \in \mathsf{Gl}_c(p)} E_{p'}.$$

This application is a bijection, it is the application which transforms exclusive moments in moments. Thus we can consider its inverse. The $\mathcal{M}^c$-transform is the inverse of the $\mathcal{M}^{c\rightarrow}$-transform: $\mathcal{M}^c = (\mathcal{M}^{c\rightarrow})^{-1}$. Using the same arguments than Proposition 10.3, one can proof that the $\mathcal{M}^c$ is a bijection from $\mathfrak{ME}[\mathcal{P}]$ to itself.

Let us remark that this last proposition holds since, if p' is coarser-compatible than $p_1 \otimes p_2$ this means that there exists p'_1 and p'_2 such that $p' = p'_1 \otimes p'_2$ and such that p'_1 (resp. p'_2) is coarser-compatible than p_1 (resp. p_2). Thus, if one has defined $\mathcal{M}^{c\rightarrow}(E)$ by replacing the coarser-compatibility order by the coarser order then this proposition (and other good properties) would not have hold.

Let us define a last transformation on $\mathfrak{E}[\mathcal{P}]$.

Definition 10.14. — *The $\mathcal{M}^{\rightarrow c}$-transform is the application:*

$$\begin{aligned}\mathcal{M}^{\rightarrow c} : \mathfrak{E}[\mathcal{P}] &\rightarrow \mathfrak{E}[\mathcal{P}] \\ E &\mapsto \mathcal{M}^{\rightarrow c}(E),\end{aligned}$$

such that for any $E \in \mathfrak{E}[\mathcal{P}]$, for any integer k, any $p \in \mathcal{P}_k$:

$$\left(\mathcal{M}^{\rightarrow c}(E)\right)_p = \sum_{p' \in \mathsf{Sp}(p)} E_{p'}$$

This is again a bijection. The applications defined above are actually linked.

Theorem 10.6. — *The following diagram is commutative.*

$$\begin{array}{ccc} \mathfrak{E}[\mathcal{P}] & \xrightarrow{\mathcal{M}^{\rightarrow c}} & \mathfrak{E}[\mathcal{P}] \\ & {\scriptstyle \mathcal{M}_{\mathcal{P}}} \searrow & \downarrow {\scriptstyle \mathcal{M}^{c\rightarrow}} \\ & & \mathfrak{E}[\mathcal{P}] \end{array}$$

Proof. — This is a straithforward application of Theorem 3.3. □

10.2. Higher order. — In Definition 9.5, we defined the ∞-development algebra of order m of A_k. Thus, one can also define a higher order $\mathcal{R}$-transform: we will only give definitions in this section. Let $m \in \mathbb{N}$ be the higher order of fluctuations which we are working with.

Definition 10.15. — *Let us define the algebra* $\mathfrak{E}_{g,(m)}[A] = \prod_{k=0}^{\infty} \mathbb{C}_{(m)}[A_k(\infty)]$. *We also consider the subspace of* $\mathfrak{E}_{g,(m)}[A]$ *defined by:*

$$\mathfrak{E}_{(m)}[A] = \{E \in \mathfrak{E}_{g,(m)}[A], E_{\emptyset,0} = 1, E_{\emptyset,i} = 0, \forall i \geq 1\}.$$

Let us remark that $\mathfrak{E}_{(0)}[A] = \mathfrak{E}[A]$. Any element $E \in \mathfrak{E}[A]$ is of the form:

$$\left(\sum_{p \in A_k, i \in \{0,\dots,m\}} (E_k)_{p,i} \frac{p}{X^i} \right)_{k \in \mathbb{N}}.$$

Again, in order to simplify the notations, we will use the following conventions: for any $p \in \cup_{k=0}^{\infty} A_k$ and any $i \in \{0, \dots, m\}$:

$$E_{p,i} = E^i(p) = (E_{\mathsf{l}(p)})_{p,i},$$

and for any integer k:

$$E_k = \sum_{p \in A_k, i \in \{0,\dots,m\}} E_{p,i} \frac{p}{X^i}.$$

As for $\mathfrak{E}_g[A]$, the algebra $\mathfrak{E}_{g,(m)}[A]$ is naturally endowed with a natural addition and multiplication given, for any $E, F \in \mathfrak{E}_{g,(m)}[A]$, by:

$$(E + F)_k = E_k + F_k,$$
$$(E \boxtimes F)_k = E_k F_k.$$

Besides, we can also construct an other law on $\mathfrak{E}_{g,(m)}[A]$.

Definition 10.16. — *Let* E *and* F *be two elements of* $\mathfrak{E}_{g,(m)}[A]$. *We denote by* $E \boxplus F$ *the element of* $\mathfrak{E}_{g,(m)}[A]$ *such that for any positive integer* k, *any* $p \in A_k$ *and any* $i \in \{0, \dots, m\}$:

$$(E \boxplus F)^i(p) = \sum_{(p_1,p_2,I) \in \mathfrak{F}_2(p)} \sum_{i_1=1}^{i} E^{i_1}(p_1) F^{i-i_1}(p_2),$$

where $\mathfrak{F}_2(p)$ *was defined in Definition 2.11.*

Again, the subset $\mathfrak{E}_{(m)}[A]$ is stable by the $\boxtimes$ and $\boxplus$ operations. Besides, $\mathfrak{E}_{(m)}[A]$ is an affine space.

The operation $\boxplus$ is commutative, it defines a structure of group on $\mathfrak{E}_{(m)}[A]$. The neutral element is the element $0_{\mathfrak{E}_{(m)}} \in \mathfrak{E}_{(m)}[A]$ such that, for any positive integer k, any $p \in A_k$ and any $i \in \{0, \dots, m\}$, $(E)_{p,i} = 0$.

The operation $\boxtimes$ is not commutative and the set of invertible elements in $\mathfrak{E}_{(m)}[A]$ is the set of elements E such that $E_{id_k,0} \neq 0$ for any $k \geq 1$. We denote by $1_{\mathfrak{E}_{(m)}}$ the neutral element for $\boxtimes$ which is the only element in $\mathfrak{E}_{(m)}[A]$ such that for any $k \geq 1$, $(E_k)_k = \frac{id_k}{X^0}$.

We can also define a $\mathcal{R}_A^{(m)}$-transform. For this, we need to define the $\mathcal{M}_A^{(m)}$-transform whose definition lies on Equality (21).

Definition 10.17. — *The general $\mathcal{M}_A^{(m)}$ transform is the application:*

$$\mathcal{M}_A^{(m)} : \mathfrak{E}_{(m)}[A] \to \mathfrak{E}_{(m)}[A]$$
$$E \to \mathcal{M}_A^{(m)}(E)$$

such that for any $E \in \mathfrak{E}_{(m)}[A]$, for any positive integer k, any $p \in A_k$ and any $i \in \{0, \dots, m\}$:

$$\left(\mathcal{M}_A^{(m)}(E)\right)_{p,i} = \sum_{p' \in A_k, \mathsf{df}(p',p) \leq i} E_{p',i-\mathsf{df}(p',p)}.$$

This application is a bijection: we can consider its inverse.

Definition 10.18. — *The $\mathcal{R}_A^{(m)}$-transform is the inverse of the $\mathcal{M}_A$-transform:*

$$\mathcal{R}_A^{(m)} = \left(\mathcal{M}_A^{(m)}\right)^{-1}.$$

11. Conclusion

We have defined a geometry on partitions, and new notions of convergence for elements of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. Using Schur-Weyl's duality and similar results, we will link the study of random matrices with the study of elements in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ and in $\mathfrak{E}[A]$. In the article [**9**], we apply the results proved in this article to the theory of random matrices invariant in law by conjugation by the symmetric group. We also study additive and multiplicative unitary or orthogonal invariant Lévy processes. In the article [**10**], we apply the results of the first two articles to the study of random walks on the symmetric group and the study of the $\mathfrak{S}_\infty$-Yang-Mills theory.

References

[1] A. Dahlqvist. Integration formula for Brownian motion on classical compact Lie groups. *http://arxiv.org/abs/1212.5107*.

[2] F. Benaych-Georges. Classical and free infinitely divisible distributions and random matrices. *The Annals of Probability*, 33, no. 3:1134–1170, 2005.

[3] T. Cabanal-Duvillard. A matrix representation of the Bercovici-Pata bijection. *Electronic Journal of Probability*, 10:632–661, 2005.

[4] M. Capitaine and M. Casalis. Geometric interpretation of the cumulants for random matrices previously defined as convolutions on the symmetric group. *Lecture Notes in Mathematics, Séminaire de Probabilités XLI*, 1934:93–119, 2008.

[5] G. Cébron. Matricial model for the free multiplicative convolution. *To appear in Annals of Probability*.

[6] G. Cébron. Free convolution operators and free Hall transform. *J. Funct. Anal.*, 265:2645–2708, 2013.

[7] G. Cébron and A. Dahlqvist and F. Gabriel and C. Male. Traffic-partition correspondence. *Work in progress*.

[8] F. Gabriel. Combinatorial theory of permutation-invariant random matrices I: Partitions, geometry and renormalization. *http://arxiv.org/abs/1503.02792*, 2015.

[9] F. Gabriel. Combinatorial theory of permutation-invariant random matrices II: Freeness and Lévy processes. *http://arxiv.org/abs/1507.02465*, 2015.

[10] F. Gabriel. Combinatorial theory of permutation-invariant random matrices III: Random walks on $\mathfrak{S}_N$, ramified coverings and the $\mathfrak{S}_\infty$ Yang-Mills measure. *http://arxiv.org/abs/1510.01046*, 2015.

[11] T. Halverson and A. Ram. Partition algebras. *European Journal of Combinatorics*, 26:869–921, 2005.

[12] G. Kreweras. Sur les partitions non-croisées d'un cycle. *Discrete Math.*, 1:333–350, 1972.

[13] T. Lévy. The master field on the plane. *arxiv.org/abs/1112.2452*.

[14] T. Lévy. Schur-Weyl duality and the heat kernel measure on the unitary group. *Advances in Mathematics*, 218, no. 2:537–575, 2008.

[15] C. Male. The distributions of traffics and their free product. *arxiv.org/abs/1111.4662*, 2011.

[16] C. Male and S. Péché. Uniform regular weighted graphs with large degree: Wigner's law, asymptotic freeness and graphons limit. *http://arxiv.org/abs/1410.8126*, 2014.

[17] A. Nica and R. Speicher. *Lectures on the Combinatorics of Free Probability*. Number 335. Lecture Note Series, London Mathematical Society, Cambridge University Press, 2006.

[18] A.N. Sengupta. Traces in Two-Dimensional QCD: The Large-N Limit. *Traces in geometry, number theory and quantum fields*, 218, no. 2:193–212, 2008.

[19] F. Xu. A random matrix model from two-dimensional Yang-Mills theory. *Comm. Math. Phys.*, 190, no. 2:287–307, 1997.

"The present is filled with past experience ringing in various ways and now is colored by this symphony of resonance."

Paul Lansky

MATRICES ALÉATOIRES INVARIANTES PAR LE GROUPE SYMÉTRIQUE

Abstract. — In this article, we study random matrices in a framework based on the geometric study of partitions and some dualities as the Schur-Weyl's duality. This gives a unified and simple framework in order to understand families of random matrices which are invariant by conjugation in law by any group whose associated tensor category is spanned by partitions. This includes the unitary groups, the unitary reflection groups, the orthogonal groups, the bistochastic groups, the hyperoctahedral groups and the symmetric groups. For each choice of symmetry one can associate a subset of partitions A which allows us to define a notion of A-free cumulants. Besides, we introduce some observables on random matrices, namely the $\mathcal{P}$-moments, which generalize the normalized moments. One of the various by-products we get is that for any family of random matrices which is invariant by the unitary group, if it converges in non-commutative distribution then the $\mathcal{P}$-moments of this family converge in expectation. This implies a simple formula which allows us to compute the asymptotic of any product of the entries of a family of random matrices which is invariant in law by conjugation by the unitary group and which converges in non-commutative distribution. We prove similar results when the family is invariant in law by conjugation by the orthogonal group.

This setting leads to a unified way in order to define and study new notions of asymptotic freeness associated to each symmetry. As a by-product, we prove that independence and invariance in law by conjugation by the bistochastic group implies asymptotic Voiculescu's freeness. We show that there exist two formulations for each notion of asymptotic freeness: one uses some modified moments, and the other uses cumulants.

In this setting, a non-commutative central limit theorem is proved and the notion of asymptotic factorization is also studied. We also show how to inject the theory of classical probabilities in this new framework: as a consequence, the classical cumulants can be seen as a special case of the new free cumulants we defined.

At last, we give general theorems about convergence of matrix-valued additive and multiplicative Lévy processes which are invariant in law by conjugation by the symmetric group. Using these results, we give a unified point of view on the study of matricial Lévy processes on some Lie algebras and some Lie groups.

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1. Introduction

This article is the second one of a set of three, namely [**15**], [**16**] and [**17**]. Here we will not restate the results of [**15**], we will only refer to them. Thus we encourage the reader to read the first article before this one. Besides, the reader must not be surprised by the absence of introduction in this article since the introduction of [**15**] was a general introduction for the three articles, [**15**], [**16**] and [**17**], which are actually a unique article cut into three. In this article we apply the combinatorial framework developed in [**15**] to the study of random matrices. It has to be noticed that some results about the convergence of generalized observables for unitarily invariant random matrices were also proved independently in the paper [**13**]. They used the theory of traffics developped by C. Male [**26**], which is a generalization of the theory of free probability, and proved an equivalent result in this setting. We will explain the links between this article and the theory of traffics of C. Male in the forthcoming article [**14**].

1.1. Layout of the article. — The core of this article is the existence of dualities for some groups of matrices. This idea was first used in order to study random matrices in [**2**]. Let G be a group whose associated tensor category is spanned by partitions: to G is associated a subset of the partitions $\mathcal{P} = \bigcup_{k\in\mathbb{N}} \mathcal{P}_k$ that we will denote in the layout by A. An introduction to the dualities for subgroups of $O(N)$ can be found in [**3**]. For the unitary reflection group, the duality was proved by Tanabe in [**31**].

In Section 2, we recall some basic notions in random matrices theory such as the notion of empirical eigenvalues distribution. In order to prove convergence of this distribution, one can study observables. We generalize these observables to a family wich is called the A-moments. Then we begin our study of convergence of families of random matrices using these observables. A family of matrices is said to converge in expectation in A-moments if the expectation of any A-moments converges.

The link with the article [**15**] is explained in Section 3, where dualities like the Schur-Weyl's duality are explained. In the article [**3**], T. Banica and R. Speicher caracterized the easy orthogonal groups, which are roughly the groups G such that $\mathfrak{S}(N) \subset G \subset$

$O(N)$ and which satisfy a duality with some sub-algebra of the partition algebra. These groups are interesting for our setting based on partitions. We focus in this article mainly on the unitary, orthogonal and symmetric groups.

We define, in Section 4, the notions of A-non-commutative (or free) cumulants for any family of random matrices which converges in expectation in A-moments. Using the results of Section 3, we also define non-commutative cumulants for any family of random matrices with a given size: these cumulants are shown to converge to the first free cumulants we defined when the family of matrices converges in expectation in A-moments. This allows to prove in a simple way that for any family or random matrices which is invariant by conjugation by G, if it converges in expectation in A-moments then it converges in $\mathcal{P}$-moments (Theorem 4.5). For example this shows that a unitarily invariant family of random matrices converges in expectation in $\mathcal{P}$-moments if it converges in non-commutative distribution and satisfies an asymptotic factorization of its moments: for example Hadamard products between matrices of this family also converge in non-commutative distribution. We also study the development in $\frac{1}{N}$ of the observables, which is sometimes called the topological expansion.

We introduce the exclusive moments in Section 5: the study of the exclusive moments allows us to compute the asymptotic of any product of the entries of a family of random matrices which converges in non-commutative distribution and which is unitarily invariant (Theorem 5.4).

We generalize the notion of non-commutative law and $\mathcal{R}$-transform in Section 6. Then we define and study the notion of A-freeness in Section 7. We study the links between the different notions of A-freeness and the Voiculescu's notion of freeness: some of the links are summarized in Section 7.1.3. We prove for example, in Theorem 7.3, that asymptotic $\mathcal{P}$-freeness and a condition of asymptotic G-invariance imply A-freeness. Besides asymptotic $\mathcal{B}$-freeness implies asymptotic $\mathfrak{S}$-freeness which rhougly implies asymptotic Voiculescu's freeness. We prove Theorem 7.10 which is a generalization of a well-known result: independence and G-invariance implies asymptotic A-freeness. This is also generalized in Section 7.1.9, with Theorem 7.11, which allows us to suppose a weaker condition than the G-invariance. We recover in this setting a result from [**28**]: a unitarily invariant family of random matrices which converges in distribution is free from its transpose family. We also compute the $\mathcal{R}$-transform of the sum and the multiplication of two asymptotically A-free families of random matrices. This result extends the well-known formulas which use the Kreweras complement for non-crossing partitions. At the end, we define a notion of A-freeness of higher order in order to deal with the expansion in $\frac{1}{N}$ of the observables, and prove similar theorems for A-freeness of higher order.

Using the notion of asymptotically A-freeness, in Section 8 is proved a A-non-commutative central limit theorem, generalization of the non-commutative central limit theorem, which is shown in Section 9 to generalize also the classical central limit theorem.

In Section 9, we gather some simple and known facts about approximation of probability measures using the empirical eigenvalues distribution of random matrices. This gives two ways to inject the classical probabilities in the setting developed in the article: one of them allows us to express classical cumulants as non-commutative cumulants.

This shows that probabilistic fluctuations of random matrices invariant by conjugation by the easy orthogonal groups could also be handled in this new setting.

In the last section, Section 10, we study Lévy processes on $\mathcal{M}_N(\mathbb{C})$ and their asymptotic properties when the size N goes to infinity. We prove a new general result on the convergence in probability of the observables and the existence of expansion in series in $\frac{1}{N}$ in Section 10.2. In order to illustrate these theorems, we apply them to additive and multiplicative Brownian motions: for example we recover well-known results about GUE and brownian motion on $U(N)$. A matricial Wick theorem is explained in order to illustrate the notion of N-finite dimension A-non-commutative cumulants we defined earlier. Then we apply the results of Section 10.2 to Lévy processes with an application to free infinitely divisible measures: we extend known results ([**5**], [**10**]) to the real setting where one considers $\mathcal{M}_N(\mathbb{R})$-valued Lévy processes.

2. Random matrices and observables

2.1. Introduction to random matrices. — Random matrices are random variables which take values in a set of matrices $\mathcal{M}_{n,m}(\mathbb{C})$. In this article we will only consider square random matrices, we will say that M is of size N if M is a square matrix of size $N \times N$ and we will write $M \in \mathcal{M}_N(\mathbb{C})$.

Actually any random matrix M of size N that we will consider in this article is automatically supposed to be in $L^{\infty^-}(\Omega) \otimes \mathcal{M}_N(\mathbb{C})$. This means that for any i, j in $\{1, ..., N\}$ and for any positive integer k, $\mathbb{E}[\mid M_{i,j} \mid^k] < \infty$. Besides, all the random matrices we consider are defined on the same probability space. In fact, one can only suppose that for each integer N, all the random matrices of size N that we consider are defined on the same probability space.

In this article, we are interested in the convergence of some observables when the size of the random matrices goes to infinity. For example, let $(M_N)_{N\in\mathbb{N}}$ be a sequence of random matrices such that $M_N \in \mathcal{M}_N(\mathbb{C})$. In the usual method of moments, one is interested in the convergence of:

$$\frac{1}{N} Tr\left((M_N)^k\right), \tag{1}$$

or the convergence of the mean moments:

$$\mathbb{E}\left[\frac{1}{N} Tr\left((M_N)^k\right)\right], \tag{2}$$

when $k \in \mathbb{N}$.

The method of moments is justified by the fact that a $N \times N$ random matrix M_N has N random eigenvalues: $\lambda_1(M_N), ..., \lambda_N(M_N)$ and for any integer $k \in \mathbb{N}$:

$$\frac{1}{N} Tr\left((M_N)^k\right) = \frac{1}{N} \sum_{i=1}^{N} \lambda_i(M_N)^k.$$

Thus, if we define the empirical eigenvalues distribution of M_N by:

$$\eta_{M_N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(M_N)},$$

we get that:

$$\frac{1}{N}Tr\left((M_N)^k\right) = \int_{\mathbb{C}} z^k \eta_{M_N}(dz).$$

Let us suppose that M_N is symmetric or Hermitian for any integer N: η_{M_N} is a measure supported by the real line. Using the Carleman's continuity theorem, Theorem 2.2.9 in [**32**], one can use the convergence of (1) or (2) to prove that almost surely or in expectation the random measures $(\eta_{M_N})_{N\in\mathbb{N}}$ converge when N goes to infinity. Likewise, we can do the same for unitary or orthogonal matrices.

Theorem 2.1. — *Let $(M_N)_{N\in\mathbb{N}}$ be a sequence of random matrices such that for any integer N, M_N is of size N. We suppose that for any positive integer k,*

$$\mathbb{E}\left[\frac{1}{N}Tr\left((M_N)^k\right)\right]$$

converges when N goes to infinity.

1. *If for any integer N, M_N is a unitary or orthogonal matrix then there exists μ a probability measure on the circle such that the mean empirical eigenvalues distribution $\mathbb{E}[\eta_{M_N}]$ of M_N converges to μ as N tends to infinity.*
2. *If for any integer N, M_N is symmetric or Hermitian (resp. skew-symmetric or skew-hermitian), under a condition of uniform subgaussianity on $(\mathbb{E}[\eta_{M_N}])_{N\in\mathbb{N}}$, the measure $\mathbb{E}[\eta_{M_N}]$ converges to a probability measure, named μ, supported by the real line (resp. the imaginary line) as N goes to infinity.*

Besides, for any integer k:

$$\int_{\mathbb{C}} z^k \mu(dz) = \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}Tr\left((M_N)^k\right)\right].$$

Proof. — We will only remind the reader the proof for the unitary case, the symmetric case being deduced from a tightness argument and Theorem 2.2.9 of [**32**]. Let $(M_N)_{N\in\mathbb{N}}$ be a sequence of random matrices such that for any integer N, M_N is of size N. For any integer N, the measure $\mathbb{E}[\eta_{M_N}]$ is supported by the unit circle since the eigenvalues of a unitary matrix are on the unit circle $\mathbb{U}$. Thus the sequence $(\mathbb{E}[\eta_{M_N}])_{N\in\mathbb{N}}$ is tight: we only have to show that any limit of a subsequence is determined by its positive moments. Let μ and μ' be two probability measures on the circle such that for any integer k:

$$\int_{\mathbb{U}} z^k \mu(dz) = \int_{\mathbb{U}} z^k \mu'(dz). \tag{3}$$

Let us show that $\mu = \mu'$.

The unital algebra generated by the functions $z \mapsto z$ and $z \mapsto \overline{z}$ is dense in the set of continuous complex functions on $\mathbb{U}$. Indeed, it is stable by complex conjugation, and it separates points, thus dense by the complex Stone-Weierstrass's theorem. Yet, on $\mathbb{U}$, it is also the algebra of functions which are linear combination of the functions $z \mapsto z^k$ and $z \mapsto \overline{z}^k$ where $k \in \mathbb{N}$.

Yet, because of (3), for any integer k, $\int_{\mathbb{U}} \overline{z^k}\mu(dz) = \overline{\int_{\mathbb{U}} z^k \mu(dz)} = \overline{\int_{\mathbb{U}} z^k \mu'(dz)} = \int_{\mathbb{U}} \overline{z^k}\mu'(dz)$. By linearity and density, we see that the two measures μ and μ' are equal. □

Actually, the author's motivation in order to study the convergence of (2) does not come from the convergence of the empirical eigenvalues distributions. Indeed, the motivation comes from Yang-Mills theory and random holonomy fields ([**18**]) where one of the main problem is to prove convergence of the expectation of generalizations of observables of the form (2). Thus, in this article, we will mainly focus on the observables.

2.2. Observables. —

2.2.1. Definitions: the zero order case. — Let N and k be two positive integers. Let $M_1, ..., M_k$ be k random matrices of size N defined on the same probability space. We would like to measure the system $(M_1, ..., M_k)$. Let Tr^k be the trace on $\left(\mathbb{C}^N\right)^{\otimes k}$ defined according to the canonical basis $\{e_{i_1} \otimes ... \otimes e_{i_k}, (i_1, ..., i_k) \in N^k\}$ such that $Tr^k(Id^{\otimes k}) = N^k$. Recall the definition of the set of partitions $\mathcal{P}_k$ and Definition 2.15 in [**15**] where we defined the representation $\rho_N^{\mathcal{P}_k}$ of $\mathbb{C}[\mathcal{P}_k(N)]$. Let p be in $\mathcal{P}_k$.

Definition 2.1. — *The mean p-normalized moment of $(M_1, ..., M_k)$ is:*

$$\mathbb{E}m_p(M_1, ..., M_k) = \frac{1}{Tr^k(\rho_N^{\mathcal{P}_k}(p))}\mathbb{E}\left[Tr^k\left((M_1 \otimes ... \otimes M_k)\rho_N^{\mathcal{P}_k}({}^t p)\right)\right].$$

We will denote $Tr_N(p) = Tr^k(\rho_N^{\mathcal{P}_k}(p))$ and from now on, we will forget about the $\rho_N^{\mathcal{P}_k}$: any partition in $\mathcal{P}_k$ will be seen as an endomorphism of $\left(\mathbb{C}^N\right)^{\otimes k}$, where the parameter N has to be deduced from the formulas.

For any k-tuple $(M_1, ..., M_k)$ of matrices of size N:

$$Tr^k\left[(M_1 \otimes ... \otimes M_k)\ {}^t p\right] = \sum_{i_1,...,i_k,i_{1'},...,i_{k'} \in \{1,...,N\}/p^{i_1,...,i_k}_{i_{1'},...,i_{k'}}=1} (M_1)^{i_1}_{i_{1'}} ... (M_k)^{i_k}_{i_{k'}}$$

where for any $l \in \{1, ..., k\}$, and any $(i, j) \in \{1, ..., N\}^2$, $M_l(e_j) = (M_l)^j_i e_i$ and the notation $p^{i_1,...,i_k}_{i_{1'},...,i_{k'}}$ was defined in Definition 2.14 of [**15**]. This implies the following lemma which shows that the notion of mean p-normalized moments generalizes the observables (2), as explained in [**23**].

Lemma 2.1. — *Let $\sigma \in \mathfrak{S}_k$, let M be a random of size N. Then:*

$$\mathbb{E}m_\sigma(M, ..., M) = \mathbb{E}\left[\prod_{c\ cycle\ of\ \sigma} \frac{1}{N} Tr\left(M^{\#c}\right)\right],$$

where $\#c$ stands for the length of the cycle c.

When p is an element of the Brauer algebra, then $\mathbb{E}m_p(M, ..., M)$ is the expectation of a product of normalized traces of products in M and ${}^t M$. In general, when $p \in \mathcal{P}_k$, the calculation of $\mathbb{E}m_p(M, ..., M)$ has to do with the computation of the graph-test observables of [**26**], where one also considers non-connected graph-tests. This link will be developed in the forthcoming article [**14**].

Let I be a set of indexation. We will study the convergence of observables of some family $(M_i^N)_{i \in I}$ of random matrices of size N when N goes to infinity. In this setting,

we consider each sequence of random matrices $\left(M^N\right)_{N\in\mathbb{N}}$, such that for any integer N the matrix M^N is of size N, as an element of the algebra

$$\prod_{N=1}^{\infty}\left(L^{\infty^-}(\Omega)\otimes\mathcal{M}_N(\mathbb{C})\right)$$

which we will denote by $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$.

For any family of elements $(a_i)_{i\in I}$ in $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$, the algebra generated by $(a_i)_{i\in I}$ is simply:

$$\mathcal{A}\left(a_i\right)_{i\in I}=\left\{P(a_{i_1},...,a_{i_k})\mid(i_1,...,i_k)\in I^k,P\in\mathbb{C}\{X_1,...,X_k\},k\in\mathbb{N}\right\},\tag{4}$$

where $\mathbb{C}\{X_1,...,X_k\}$ is the algebra of non-commutative polynomials. Recall the notation that we set in [**15**]: the letter A stands either for $\mathcal{P}$, $\mathcal{B}$ or $\mathfrak{S}$ and thus for any integer k, A_k stands either for $\mathcal{P}_k$, $\mathcal{B}_k$ or $\mathfrak{S}_k$. For any $i\in I$, let $(M_i^N)_{N\in\mathbb{N}}$ be an element of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$.

Definition 2.2. — *Let k be a positive integer, let $(i_1,...,i_k)\in I^k$. The sequence $\left(M_{i_1}^N\otimes...\otimes M_{i_k}^N\right)_{N\in\mathbb{N}}$ converges in expectation in A_k-moments if for any $p\in A_k$,*

$$\mathbb{E}m_p\left(M_{i_1}^N,...,M_{i_k}^N\right)$$

converges when N goes to infinity.

The family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in A-moments if for any positive integer k, for any element $(B_1^N)_{N\in\mathbb{N}}$,..., $(B_k^N)_{N\in\mathbb{N}}$ in $\mathcal{A}\left(\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\right)$, the sequence $\left(B_1^N\otimes...\otimes B_k^N\right)_{N\in\mathbb{N}}$ converges in expectation in A_k-moments.

Notation 2.1. — *Let us suppose that the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in A-moments. For any positive integer k, for any $(i_1,...,i_k)\in I^k$, for any $p\in A_k$, we set:*

$$\mathbb{E}m_p(M_{i_1},...,M_{i_k})=\lim_{N\to\infty}\mathbb{E}m_p(M_{i_1}^N,...,M_{i_k}^N).$$

In fact, it is an easy exercise to prove the following lemma.

Lemma 2.2. — *The family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in A-moments if and only if for any integer k, for any $(i_1,...,i_k)\in I^k$, the sequence $\left(M_{i_1}^N\otimes...\otimes M_{i_k}^N\right)_{N\in\mathbb{N}}$ converges in expectation in A_k-moments.*

Remark 2.1. — *If the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in:*

- *expectation in $\mathfrak{S}$-moments then it converges in non-commutative distribution,*
- *expectation in $\mathcal{B}$-moments if and only if $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\cup\left(({}^tM_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$, the family and the transpose of the family, converges in $\mathfrak{S}$-moments,*
- *expectation in $\mathcal{B}$-moments then $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\cup\left(({}^tM_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in non-commutative distribution,*
- *expectation in $\mathcal{P}$-moments then $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in distribution of traffics (see* [**26**] *and the correspondence explained in* [**14**]*).*

Remark 2.2. — *Let us remark also that the convergence in expectation in $\mathcal{P}$-moments implies the convergence in non-commutative distribution of the algebra of traffics generated by $\left(M_i^N\right)_{i\in I}$. This is the point of view developed in the paper in preparation* [**13**] *; in this paper, we do not take this point of view. Yet let us make a similar remark than C. Male in* [**26**]*, if $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in $\mathcal{P}$-moments, then for any i and j in I, the Hadamard product $(M_i^N \circ M_j^N)_{N\in\mathbb{N}}$ converges in $\mathcal{S}$-moments, and in fact also in $\mathcal{P}$-moments.*

Let $(M_N)_{N\in\mathbb{N}}$ be in $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. Using Lemma 2.1, we see that $(M_N)_{N\in\mathbb{N}}$ converges in expectation in $\mathfrak{S}$-moments if for any positive integer k, any k-tuple $(n_1,...,n_k)$ of integers,

$$\mathbb{E}\left[\prod_{i=1}^{k}\frac{1}{N}Tr(M_N^{n_i})\right],$$

converges when N goes to infinity. This is slightly more demanding than the usual convergence of moments, where one only considers the convergence of the mean moments (2). This is due to the fact that usually one works with sequences of random matrices such that some concentration of measure happens and implies that the following asymptotic factorization property holds.

2.2.2. The A-factorization property. — Recall the operation $\otimes$ on partitions defined in Definition 2.1 of [**15**]. Let $(M_i)_{i\in I}$ be a family of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$.

Definition 2.3. — *Let us suppose that the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in A-moments. We say that $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ satisfies the asymptotic A-factorization property if for any integer k_1 and k_2, for any $p_1\in A_{k_1}$ and $p_2\in A_{k_2}$, for any element $(B_1^N)_{N\in\mathbb{N}},...,\ (B_{k_1+k_2}^N)_{N\in\mathbb{N}}$ in $\mathcal{A}\left(\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\right)$,*

$$\mathbb{E}m_{p_1\otimes p_2}(B_1,...,B_{k_1+k_2}) = \mathbb{E}m_{p_1}(B_1,...,B_{k_1})\mathbb{E}m_{p_2}(B_{k_1+1},...,B_{k_2}).$$

Let us explain the importance of the factorization property.

Theorem 2.2. — *Let us suppose that $(M_i)_{i\in I}$ is in $L^{\infty^-}\otimes\mathcal{M}(\mathbb{R})$, converges in expectation in A-moments and satisfies the asymptotic A-factorization property, then the A-moments of $(M_i)_{i\in I}$ converge in probability. This means that for any integer k, any $p\in A_k$, any $i_1,...,i_k$ in I:*

$$m_p(M_{i_1}^N\otimes...\otimes M_{i_k}^N) = \frac{1}{N^{\mathsf{nc}(p\vee id)}}Tr^k\left((M_{i_1}^N\otimes...\otimes M_{i_k}^N)^t p\right)$$

converges in probability as N goes to infinity to $\mathbb{E}m_p(M_{i_1},...,M_{i_k})$.

If $(M_i)_{i\in I}$ is a family of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$, the same results holds if we suppose that either $(M_i,\overline{M}_i)_{i\in I}$ or $(M_i,(M_i)^*)_{i\in I}$ satisfy the asymptotic A-factorization.

Proof. — Let $(M_i)_{i\in I}$ be a family of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{R})$ which converge in expectation in A-moments and which satisfies the asymptotic A-factorization property. We

use an argument from T. Lévy [**24**]. Let k be an integer, let $p \in A_k$, and let $i_1, ..., i_k$ be in I. The variance $\mathbb{V}\mathsf{ar}\big[m_p\left(M_{i_1}^N \otimes ... \otimes M_{i_k}^N\right)\big]$ is equal to:

$$\mathbb{E}\left[m_{p\otimes p}\big(M_{i_1}^N \otimes ... \otimes M_{i_k}^N \otimes M_{i_1}^N \otimes ... \otimes M_{i_k}^N\big)\right] - \mathbb{E}\left[m_p\big(M_{i_1}^N \otimes ... \otimes M_{i_k}^N\big)\right]^2,$$

which, by definition, is equal to:

$$\mathbb{E}m_{p\otimes p}\big(M_{i_1}^N \otimes ... \otimes M_{i_k}^N \otimes M_{i_1}^N \otimes ... \otimes M_{i_k}^N\big) - \big[\mathbb{E}m_p\big(M_{i_1}^N \otimes ... \otimes M_{i_k}^N\big)\big]^2.$$

Thus the variance has a limit which is given by:

$$\mathbb{E}m_{p\otimes p}(M_{i_1} \otimes ... \otimes M_{i_k} \otimes M_{i_1} \otimes ... \otimes M_{i_k}) - [\mathbb{E}m_p(M_{i_1} \otimes ... \otimes M_{i_k})]^2,$$

which is equal to zero since $(M_i)_{i\in I}$ satisfies the asymptotic A-factorization property.

The same argument can be applied for a family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$, by computing $\mathbb{V}\mathsf{ar}\big[m_p\left(M_{i_1}^N \otimes ... \otimes M_{i_k}^N\right)\big]$, which is equal to:

$$\mathbb{E}\Big[m_{p\otimes p}\Big(M_{i_1}^N \otimes ... \otimes M_{i_k}^N \otimes \overline{M_{i_1}^N} \otimes ... \otimes \overline{M_{i_k}^N}\Big)\Big] - \mathbb{E}\big[m_p\big(M_{i_1}^N \otimes ... \otimes M_{i_k}^N\big)\big]\,\mathbb{E}\Big[m_p\Big(\overline{M_{i_1}^N} \otimes ... \otimes \overline{M_{i_k}^N}\Big)\Big],$$

or:

$$\mathbb{E}\Big[m_{p\otimes {}^t p}\Big(M_{i_1}^N \otimes ... \otimes M_{i_k}^N \otimes \big(M_{i_1}^N\big)^* \otimes ... \otimes \big(M_{i_k}^N\big)^*\Big)\Big] - \mathbb{E}\big[m_p\big(M_{i_1}^N \otimes ... \otimes M_{i_k}^N\big)\big]\,\mathbb{E}\Big[m\,{}^t p\Big(\big(M_{i_1}^N\big)^* \otimes ... \otimes \big(M_{i_k}^N\big)^*\Big)\Big],$$

and by using the same arguments than before. □

As a consequence, for example, when one considers orthogonal (resp. unitary) matrices, the factorization property of $(M_i)_{i\in I}$ (resp. $(M_i, \overline{M_i})_{i\in I}$ or $(M_i, (M_i)^*)_{i\in I}$) implies the convergence of probability of the empirical eigenvalues distributions. Actually, one can go further than the convergence in probability using combinatorial arguments: indeed, in [**15**] and [**16**] we developed the tools in order to compute algebraic fluctuations: let us suppose that the algebraic fluctuations are in fact in powers of $\frac{1}{N^2}$, then the variance of the moments would be in $\frac{1}{N^2}$, and it would imply that the moments converge almost surely.

2.2.3. Definitions: the higher order case. — We can also define the notion of convergence in expectation in A-moments up to any order of fluctuations for $\big((M_i^N)_{N\in\mathbb{N}}\big)_{i\in I}$. Let m be a non-negative integer.

Definition 2.4. — *Let k be a positive integer and let $(i_1, ..., i_k) \in I^k$. The sequence $\left(M_{i_1}^N \otimes ... \otimes M_{i_k}^N\right)_{N\in\mathbb{N}}$ converges in expectation in A_k-moments up to order m of fluctuations if there exists a family of complex numbers*

$$\left(\mathbb{E}m_p^i\left(M_1^N, ..., M_k^N\right)\right)_{p\in A_k, i\in\{0,...,m\}, N\in\mathbb{N}},$$

such that:

- $\forall p \in A_k, \forall N \in \mathbb{N}^*, \mathbb{E}m_p(M_1^N, ..., M_k^N) = \sum_{i=0}^m \frac{\mathbb{E}m_p^i(M_1^N,...,M_k^N)}{N^i}$,
- $\forall p \in A_k, \forall i \in \{0, ..., m-1\}, \mathbb{E}m_p^i(M_1^N, ..., M_k^N)$ *does not depend on* N,
- $\forall p \in A_k, \mathbb{E}m_p^m(M_1^N, ..., M_k^N)$ *converges as* N *goes to infinity.*

The family $\left(\mathbb{E}m_p^i(M_1^N,...,M_k^N)\right)_{p\in A_k,i\in\{0,...,m\},N\in\mathbb{N}}$ *is uniquely defined and for any* $i\in\{0,...,m\}$*, any* $p\in A_k$ *and any* $N\in\mathbb{N}$, $\mathbb{E}m_p^i(M_1^N,...,M_k^N)$ *is called the* i^{th}*-order fluctuations of the mean* p*-normalized moment.*

The family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in expectation in* A*-moments up to order* m *of fluctuations if for any positive integer* k*, for any element* $(B_1^N)_{N\in\mathbb{N}}$,..., $(B_k^N)_{N\in\mathbb{N}}$ *in* $\mathcal{A}\left(\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\right)$*, the sequence* $(B_1^N\otimes...\otimes B_k^N)_{N\in\mathbb{N}}$ *converges in expectation in* A_k*-moments up to order* m *of fluctuations.*

Notation 2.2. — *Let us suppose that the family* $\left(\left(M_i^N\right)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in expectation in* A*-moments up to order* m *of fluctuations. For any positive integer* k*, for any* k*-tuple* $(i_1,...,i_k)\in I^k$*, for any* $j\in\{0,...,m\}$ *and any* $p\in A_k$*, we set:*

$$\mathbb{E}m_p^j(M_{i_1},...,M_{i_k})=\lim_{N\to\infty}\mathbb{E}m_p^j(M_{i_1}^N,...,M_{i_k}^N).$$

2.2.4. Convergence and transposition. — In fact the convergence in expectation in $\mathcal{B}$ (resp. $\mathcal{P}$)-moments implies the convergence, of the family and its transpose family, in expectation in $\mathcal{B}$ (resp. $\mathcal{P}$)-moments. In order to prove so, we need the following definition. Let k be a positive integer.

Definition 2.5. — *Let* p *be a partition in* $\mathcal{P}_k$*, let* $k_1<k$*. The partition* $S_{k_1}(p)$ *is the partition obtained by permuting* i *with* i' *for any* $i\in\{k_1+1,...,k\}$ *in the definition of* p*.*

Let us give an example. Let $k=4$, $k_1=2$ and let $p=\{\{1,3,2'\},\{2,4',1'\},\{4,3'\}\}$, then $S_{k_1}(p)=\{\{1,3',2'\},\{2,4,1'\},\{4',3\}\}$.

Lemma 2.3. — *Let* k_1 *and* k_2 *be non-negative integers such that* $k_1+k_2=k$*. Let* $E\in\mathsf{End}\left((\mathbb{C}^N)^{\otimes k_1}\right)$ *and* $F\in\mathsf{End}\left((\mathbb{C}^N)^{\otimes k_2}\right)$*, for any* $p\in\mathcal{P}_k$*:*

$$m_p(E\otimes{}^tF)=m_{S_{k_1}(p)}(E\otimes F).$$

For any non-negative integer k_1 such that $k_1<k$, the two sets $\mathcal{B}_k$ and $\mathcal{P}_k$ are stable by the operation S_{k_1}. This remark and Lemma 2.3 imply the following proposition.

Proposition 2.1. — *Let us suppose that* A *stands either for* $\mathcal{B}$ *or* $\mathcal{P}$*. Let* m *be a non-negative integer. Let us suppose that the family* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in expectation in* A*-moments up to order* m *of fluctuations then* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\cup\left(({}^tM_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in expectation in* A*-moments up to order* m *of fluctuations.*

For example, this implies that for a family of random orthogonal matrices $\left((O_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$, in order to prove that the family $\left((O_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\cup\left(((O_i^N)^{-1})_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in $\mathfrak{S}$-moments, we only have to prove that $\left((O_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in $\mathcal{B}$-moments.

2.2.5. Some special cases. — We will see along this paper, that in some special cases, the theory becomes even easier. Let m be a non-negative integer. Using the fact that for any symmetric of skew-symmetric matrix M, the matrix ${}^tM$ is equal either to M or $-M$, one has the following result.

Proposition 2.2. — *Let $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ be a family of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{R})$. If for any positive integer N and any $i\in I$, M_N is symmetric or skew-symmetric, the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in $\mathcal{B}$-moments up to order m of fluctuations as N goes to infinity if and only if it converges in expectation in $\mathfrak{S}$-moments up to order m of fluctuations as N goes to infinity.*

For orthogonal matrices, one has a similar result when considering only one element $(M^N)_{N\in\mathbb{N}}$ in $L^{\infty^-}\otimes\mathcal{M}(\mathbb{R})$. From now on, we denote by $\mathfrak{S}(N)$ the set of permutation matrices of size N.

Proposition 2.3. — *Let $(M^N)_{N\in\mathbb{N}}$ be an element of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{R})$. If for any positive integer N, M^N is orthogonal, the matrix M^N converges in expectation in $\mathcal{B}$-moments up to order m of fluctuations as N goes to infinity if and only if it converges in expectation in $\mathfrak{S}$-moments up to order m of fluctuations as N goes to infinity.*

Besides if for any positive integer N, M^N is almost surely in $\mathfrak{S}(N)$, the matrix M^N converges in expectation in $\mathcal{P}$-moments up to order m of fluctuations as N goes to infinity if and only if it converges in expectation in $\mathfrak{S}$-moments up to order m of fluctuations as N goes to infinity.

The first part of this proposition comes from the fact that for any orthogonal matrix M, $M\ {}^tM = Id$. The second part is a consequence of the following general proposition. Recall the notion of convergence in moments that we defined in Definition 4.4 of [**15**].

Proposition 2.4. — *Let k be a positive integer. For each integer N, let E_N be in the algebra generated by the elements $S^{\otimes k}$ with $S\in\mathfrak{S}(N)$ and denoted by $\mathbb{C}\left[\rho^k_{\mathfrak{S}(N)}\right]$. Then $(E_N)_{N\in\mathbb{N}}$ converges in $\mathcal{P}_k$-moments if and only if it converges in $\mathfrak{S}_k$-moments.*

Again, this is a simple consequence of the following lemma.

Lemma 2.4. — *Let k and N be two positive integers. Let p be in $\mathcal{P}_k$, there exist an integer $l\leq k$ and $\sigma\in\mathfrak{S}_l$ such that for any $S\in\mathfrak{S}(N)$:*

$$m_p\left(S^{\otimes k}\right)=m_\sigma\left(S^{\otimes l}\right),$$

with the convention that $m_\emptyset\left(S^{\otimes 0}\right)=1$.

Proof. — The proof uses a recurence argument. Let k and N be two positive integers, let p be in $\mathcal{P}_k$ and let S be a permutation in $\mathfrak{S}(N)$. If $k=1$, then $m^{\{\{1\},\{1'\}\}}(S)$ is equal to one and thus equal to $m_\emptyset(S^{\otimes 0})$. Let us suppose that $k\neq 1$, then, with a slight abuse of notation:

$$m_p(S^{\otimes k})=\frac{1}{Tr(p)}Tr(S^{\otimes k}\ {}^tp)=\frac{1}{N^{\mathsf{nc}(p\vee id)}}\sum_{i_1,\ldots,i_k,i_{1'},\ldots,i_{k'}\in\{1,\ldots,N\},p^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}}=1}S^{i_1}_{i_{1'}}\ldots S^{i_k}_{i_{k'}}.$$

Let us suppose that $p\notin\mathfrak{S}_k$, one of the three assertions is true:

- there exist r, s in $\{1,\ldots,k\}$ such that r and s are in the same block of p,
- there exist r', s' in $\{1',\ldots,k'\}$ such that r' and s' are in the same block of p,
- there exist r in $\{1,\ldots,k\}\cup\{1',\ldots,k'\}$ such that $\{r\}$ is a block of p.

Let us remark that for any $i, j, l \in \{1, ..., N\}$,

$$S_i^j S_i^l = S_i^j \delta_{l=j} \text{ and } S_j^i S_l^i = S_j^i \delta_{l=j}.$$

Besides, for any $i_1, ..., i_k$ and $i_{1'}, ..., i_{k'}$ in $\{1, ..., N\}$, we have:

$$\sum_{i=1}^{N} S_{i_1'}^i S_{i_2'}^{i_2} ... S_{i_k'}^{i_k} = S_{i_2'}^{i_2} ... S_{i_k'}^{i_k} \text{ and } \sum_{i=1}^{N} S_i^{i_1} S_{i_2'}^{i_2} ... S_{i_k'}^{i_k} = S_{i_2'}^{i_2} ... S_{i_k'}^{i_k}.$$

The first two equations assert that, when there exists an horizontal line in p, which means that there exist r, s (or r', s') in $\{1, ..., k\}$ (resp. $\{1', ..., k'\}$) such that r and s are in the same block of p, then we can glue the two blocks r' and s' (resp. r and s) belong to and then remove the column of r or s (resp. r' or s') without changing the value of the sum we need to compute. We get a new partition $p' \in \mathcal{P}_{k-1}$ with the same number of cycles than p: we get then the equality $m_p(S^{\otimes k}) = m_{p'}(S^{\otimes k-1})$.

The last two equations assert that, each time there exists a block of p which is composed of one unique element, let say r in $\{1, ..., k\}$, one can delete the column of this element and the sum over i_r without changing the value of the sum we need to compute. Let us denote by p' the new partition. If $\{r'\}$ is not a block of p then the new partition has as many cycles as p. If $\{r'\}$ is a block of p, we lose one block by doing so, but, one must not forget that we still have to sum over $i_{r'}$ even if it does not appear any more. Thus, at the end we still get the equality $m_p(S^{\otimes k}) = m_{p'}(S^{\otimes k-1})$. □

3. Schur-Weyl's duality

In the last section, we introduced observables proportional to:

$$\mathbb{E}\left[Tr^k\left((M_1 \otimes ... \otimes M_k)\ {}^t p\right)\right].$$

It is not difficult to see that for any $S \in \mathfrak{S}(N)$ and any $p \in \mathcal{P}_k$, $S^{\otimes k} p (S^{-1})^{\otimes k} = p$. Thus, using the tracial property, $\mathbb{E}\left[Tr^k\left((M_1 \otimes ... \otimes M_k)\ {}^t p\right)\right]$ is equal to:

$$Tr^k\left(\left(\frac{1}{N!}\sum_{S \in \mathfrak{S}(N)} (S^{-1})^{\otimes k} \mathbb{E}[M_1 \otimes ... \otimes M_k] S^{\otimes k}\right) {}^t p\right).$$

Let us remark that $\frac{1}{N!}\sum_{S \in \mathfrak{S}(N)} (S^{-1})^{\otimes k} \mathbb{E}[M_1 \otimes ... \otimes M_k] S^{\otimes k}$ commutes with $S^{\otimes k}$ for any $S \in \mathfrak{S}(N)$. The same discussion could have been made by replacing $\mathcal{P}_k$ by $\mathcal{B}_k$ (resp. $\mathfrak{S}_k$), $\mathfrak{S}(N)$ by $O(N)$ (resp. $U(N)$) and the sum $\frac{1}{N!}\sum_{S \in \mathfrak{S}(N)}$ by an integration against the Haar measure of $O(N)$ (resp. of $U(N)$).

This motivates the study of endomorphisms of $\left(\mathbb{C}^N\right)^{\otimes k}$ which commute with $M^{\otimes k}$, where M is either in $\mathfrak{S}(N)$, $O(N)$ or $U(N)$. We introduce for this the Schur-Weyl's duality and some similar statements. We will focus on the duality between:

- the unitary group $U(N)$ and the symmetric group $\mathfrak{S}_k$,
- the orthonormal group $O(N)$ and the Brauer algebra $\mathbb{C}[\mathcal{B}_k(N)]$,
- the symmetric group $\mathfrak{S}(N)$ and the partition algebra $\mathbb{C}[\mathcal{P}_k(N)]$.

One can find more informations about these dualities in [**19**], and for the last duality one can look at [**22**], [**27**] and [**20**].

Remark 3.1. — *Actually, all the paper can be applied not only to the three groups we considered but also to the unitary reflection groups and to the easy orthogonal groups. We will not explain the duality for the imprimitive unitary reflection group $G(m,p,n)$, defined in Section* 2.1 *of* [**31**] *since the partition subalgebra that one get is not simple to define, and we will never use it in the article. For a definition of easy orthogonal groups, one can have a look to* [**3**] *and there exists a duality between:*

- *the group $\mathfrak{S}(N)\times\mathbb{Z}/2\mathbb{Z}$ (group of permutation matrices multiplied by ± 1) and the partition algebra $\mathbb{C}[\mathcal{P}_k(N)]$,*
- *the hyperoctahedral group $H(N)$, which consists of matrices which have exactly one nonzero enty in each row and each column which is equal to ± 1, and the algebra $\mathbb{C}[\mathcal{H}_k(N)]$, where $\mathcal{H}_k$ is the subset of $\mathcal{P}_k$ of partitions which have blocks of even size,*
- *the bistochastic group $B(N)$, which consists of orthogonal matrices having sum* 1 *in each row and each column and the algebra $\mathbb{C}[\mathcal{B}s_k(N)]$, where $\mathcal{B}s_k$ is the subset of $\mathcal{P}_k$ of partitions which have blocks of size less of equal to two.*
- *the group $B(N)\times\mathbb{Z}/2\mathbb{Z}$ (group of bistochastic matrices multiplied by ± 1) and the algebra $\mathbb{C}[\mathcal{B}s_k(N)]$.*

Thus, in the following Notation 3.1, one could also consider (G,A) being one of the five couples:

$$(\mathfrak{S},\mathcal{P}),(O,\mathcal{B}),(U,\mathfrak{S}),(H(N),\mathcal{H}),(B(N),\mathcal{B}s).$$

We ommited in the list the two groups $\mathfrak{S}(N)\times\mathbb{Z}/2\mathbb{Z}$ and $B(N)\times\mathbb{Z}/2\mathbb{Z}$ since the invariance in law by conjugation by $\mathfrak{S}(N)\times\mathbb{Z}/2\mathbb{Z}$ (resp. $B(N)\times\mathbb{Z}/2\mathbb{Z}$) is equivalent to the invariance in law by $\mathfrak{S}(N)$ (resp. $B(N)$). We decided to focus on the three first couples since $(U,\mathfrak{S})$ is the usual setting for asymptotic freeness, $(O,\mathcal{B})$ is interesting as soon as we consider matrices and their transpose and $(\mathfrak{S},\mathcal{P})$ is the most general setting. Yet, it has to be noticed that in Section 7.1.4, we prove a new assertion using the bistochastic group stating that in fact, in order to get asymptotic freeness, it is enought to have independence and invariance by conjugation by the bistochastic group.

We have made a distinction between $\mathfrak{S}_k$ and $\mathfrak{S}(N)$. Indeed, $\mathfrak{S}_k$ will stand for the symmetric group, seen as a group of permutations, and $\mathfrak{S}(N)$ will be seen as the subgroup of $GL(N)$ of permutation matrices.

3.1. Tensor action of a subgroup of $GL(N)$. — Let k and N be two positive integers. Let G be any subgroup of $GL(N)$, for example $U(N)$, $O(N)$ or $\mathfrak{S}(N)$. We can define a natural action of G on the vector space $\left(\mathbb{C}^N\right)^{\otimes k}$.

Definition 3.1. — *The tensor action of G on $\left(\mathbb{C}^N\right)^{\otimes k}$ is the application:*

$$\rho_G^k : G \to \mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right)$$
$$U \mapsto \rho_G^k(U),$$

defined such that for any $U\in G$, any $x_1,...,x_k\in\mathbb{C}^N$,

$$\rho_G^k(U)(x_1\otimes ...\otimes x_n) = Ux_1\otimes ...\otimes Ux_n.$$

In representation theory, it is common to study the set of endomorphisms which commute with the action of the group. Let us define the commutant and some algebras associated with the representations we have just defined.

Definition 3.2. — *Let C be a subalgebra of $\mathsf{End}\left((\mathbb{C}^N)^{\otimes k}\right)$. We denote by C' the commutant of C defined by:*

$$C' = \left\{ E \in \mathsf{End}\left((\mathbb{C}^N)^{\otimes k}\right), \forall F \in C, EF = FE \right\}.$$

Definition 3.3. — *For any subgroup $G(N)$ of $GL(N)$, we define:*

$$\mathbb{C}\left[\rho^k_{G(N)}\right] = \mathbb{C}\left[\left\{\rho^k_G(g), g \in G\right\}\right] \subset \mathsf{End}\left((\mathbb{C}^N)^{\otimes k}\right).$$

For A_k being either $\mathcal{P}_k$, $\mathcal{B}_k$ or $\mathfrak{S}_k$, we define:

$$\mathbb{C}\left[\rho^{A_k}_N\right] = \mathbb{C}\left[\left\{\rho^{A_k}_N(p), p \in A_k\right\}\right] \subset \mathsf{End}\left((\mathbb{C}^N)^{\otimes k}\right).$$

The Schur-Weyl's duality and some similar dualities (as the partition-symmetric group duality proved in [**22**] and [**27**]) assert that the algebras we have just defined are mutual commutants.

Theorem 3.1. — *We have the set of equalities:*

1. *$U(N) - \mathfrak{S}_k$-duality:*

$$\left(\mathbb{C}\left[\rho^k_{U(N)}\right]\right)' = \mathbb{C}\left[\rho^{\mathfrak{S}_k}_N\right] \text{ and } \left(\mathbb{C}\left[\rho^{\mathfrak{S}_k}_N\right]\right)' = \mathbb{C}\left[\rho^k_{U(N)}\right].$$

2. *$O(N) - \mathcal{B}_k$-duality:*

$$\left(\mathbb{C}\left[\rho^k_{O(N)}\right]\right)' = \mathbb{C}\left[\rho^{\mathcal{B}_k}_N\right] \text{ and } \left(\mathbb{C}\left[\rho^{\mathcal{B}_k}_N\right]\right)' = \mathbb{C}\left[\rho^k_{O(N)}\right].$$

3. *$\mathfrak{S}(N) - \mathcal{P}_k$-duality:*

$$\left(\mathbb{C}\left[\rho^k_{\mathfrak{S}(N)}\right]\right)' = \mathbb{C}\left[\rho^{\mathcal{P}_k}_N\right] \text{ and } \left(\mathbb{C}\left[\rho^{\mathcal{P}_k}_N\right]\right)' = \mathbb{C}\left[\rho^k_{\mathfrak{S}(N)}\right].$$

The last duality is the simplest to show, one can look at the proof in [**20**], and it uses the exclusive basis defined in Definition 2.16 of [**15**].

The main implication of this theorem for this article is that we can extend the definition of coordinate numbers for any endomorphism of $\left(\mathbb{C}^N\right)^{\otimes k}$ which commutes with the action of $\mathfrak{S}(N)$. As we have seen in [**15**], the representation $\rho^{\mathcal{P}_k}_N$ is injective as soon as $N \geq 2k$, and the representations $\rho^{\mathcal{B}_k}_N$ and $\rho^{\mathfrak{S}_k}_N$ are injective as soon as $N \geq k$. Let us suppose that $N \geq 2k$, let E be an endomorphism of $\left(\mathbb{C}^N\right)^{\otimes k}$ which commutes with the action of $\mathfrak{S}(N)$ on $\left(\mathbb{C}^N\right)^{\otimes k}$. There exists a unique family of reals $\left(\kappa^p(E)\right)_{p \in \mathcal{P}_k}$ such that:

$$E = \sum_{p \in \mathcal{P}_k} \frac{\kappa^p(E)}{N^{\mathsf{nc}(p) - \mathsf{nc}(p \vee id)}} \rho^{\mathcal{P}_k}_N(p).$$

This family is the coordinate numbers of E.

The $U(N)-\mathfrak{S}_k$ (resp. $O(N)-\mathcal{B}_k$) duality asserts that if E commutes with the action of $U(N)$ on $\left(\mathbb{C}^N\right)^{\otimes k}$, then we can define the coordinate numbers as soon as $N \geq k$ and in this case:

$$\forall p \notin \mathfrak{S}_k (\text{resp. } \forall p \notin \mathcal{B}_k), \kappa^p(E) = 0.$$

Using this discussion, we can extend all the results of the article [**15**] to endomorphisms of $(\mathbb{C}^N)^{\otimes k}$ which commute with the action of either $\mathfrak{S}(N)$, $O(N)$ or $U(N)$ on $(\mathbb{C}^N)^{\otimes k}$. Let us define a notation which will be used all along the article.

Notation 3.1. — *The letter G will denote either $\mathfrak{S}$, O or U. Thus for any integer N, $G(N)$ stands either for $\mathfrak{S}(N)$, $O(N)$ or $U(N)$. Besides, by (G, A) we denote one of the three couples $(\mathfrak{S}, \mathcal{P})$, $(O, \mathcal{B})$ or $(U, \mathfrak{S})$.*

Let us finish with some definitions about endomorphisms of $\left(\mathbb{C}^N\right)^{\otimes k}$ which will be useful in Section 10. We remind the reader that the permutation σ_I was defined in Definition 2.6 of [**15**]. Let l and k be two integers, such that $l \leq k$.

Definition 3.4. — *Let $A \in \mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes l}\right)$, let $B \in \mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k-l}\right)$ and let $I \subset \{0, ..., k\}$ such that $\#I = l$. The endomorphism $\mathcal{I}_I(A, B)$ is the endomorphism in $\mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right)$ defined by:*

$$\mathcal{I}_I(A, B) = \rho_N^{\mathfrak{S}_k}\left(\sigma_I^{-1}\right)(A \otimes B)\rho_N^{\mathfrak{S}_k}(\sigma_I).$$

Let us state a simple lemma that will be often used without referring to it.

Lemma 3.1. — *The application:*

$$\begin{array}{rcl} \mathcal{I}_I : \mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes l}\right) \times \mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k-l}\right) & \to & \mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right) \\ (A, B) & \mapsto & \mathcal{I}_I(A, B), \end{array}$$

is bi-linear and for any $(p, p') \in \mathcal{P}_l \times \mathcal{P}_{k-l}$,

$$\mathcal{I}_I(\rho_N^{\mathcal{P}_l}(p) \otimes \rho_N^{\mathcal{P}_{k-l}}(p')) = \rho_N^{\mathcal{P}_k}\left(\sigma_I^{-1}(p \otimes p')\sigma_I\right).$$

Proof. — Let $(p, p') \in \mathcal{P}_l \times \mathcal{P}_{k-l}$. Then:

$$\mathcal{I}_I(\rho_N^{\mathcal{P}_l}(p) \otimes \rho_N^{\mathcal{P}_{k-l}}(p')) = \rho_N^{\mathfrak{S}_k}\left(\sigma_I^{-1}\right)\left(\rho_N^{\mathcal{P}_l}(p) \otimes \rho_N^{\mathcal{P}_{k-l}}(p')\right)\rho_N^{\mathfrak{S}_k}(\sigma_I).$$

Using Lemma 2.2 of [**15**] which asserts that the representations are compatible with the operation $\otimes$:

$$\mathcal{I}_I(\rho_N^{\mathcal{P}_l}(p) \otimes \rho_N^{\mathcal{P}_{k-l}}(p')) = \rho_N^{\mathcal{P}_k}\left(\sigma_I^{-1}\right)\left(\rho_N^{\mathcal{P}_l}(p \otimes p')\right)\rho_N^{\mathcal{P}_k}(\sigma_I) = \rho_N^{\mathcal{P}_k}\left(\sigma_I^{-1}(p \otimes p')\sigma_I\right),$$

hence the lemma. □

4. Random Matrices and cumulants

4.1. Cumulants. — Let k be a positive integer, let $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ be a k-tuple of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converges in expectation in A-moments. We will define three notions of cumulants up to any order of fluctuations associated with $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$: the $\mathfrak{S}$, the $\mathcal{B}$ and the $\mathcal{P}$-cumulants. Recall the results about the distance on $\mathcal{P}_k$, the geodesic order denoted by $\leq$ and the set-geodesic $[id,p]_{A_k}$ which were the core of the article [**15**].

4.1.1. Zero order. —

Definition 4.1. — *The family of A-cumulants*

$$\left(\mathbb{E}\kappa_A^p\left[\left(M_1^N\right)_{N\in\mathbb{N}},...,\left(M_k^N\right)_{N\in\mathbb{N}}\right]\right)_{p\in A_k}$$

is the unique family of complex numbers such that for any $p\in A_k$:

$$\mathbb{E}m_p\left(M_1,...,M_k\right)=\sum_{p'\in A_k,p'\leq p}\mathbb{E}\kappa_A^{p'}\left[\left(M_1^N\right)_{N\in\mathbb{N}},...,\left(M_k^N\right)_{N\in\mathbb{N}}\right].$$

We will denote them by $\left(\mathbb{E}\kappa_A^p\left[M_1,...,M_k\right]\right)_{p\in A_k}$.

These numbers are well defined since $\leq$ is a partial order. Besides, since we computed the Mobius function for the geodesic order in Theorem 3.5 of [**15**], one can invert the moment-cumulant formula.

Let us also remark that if the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ converges in expectation in $\mathcal{P}$-moments, the restriction of the $\mathcal{P}$-cumulants to $\mathfrak{S}_k$ is not in general equal to the $\mathfrak{S}$-cumulants. This is due to the fact that a geodesic in $\mathcal{P}_k$ between two permutations can go through $\mathcal{P}_k\setminus\mathfrak{S}_k$. Yet, if we suppose that the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ converges in expectation in $\mathcal{B}$-moments the following assertion is true.

Lemma 4.1. — *For any $\sigma\in\mathfrak{S}_k$, $\mathbb{E}\kappa_{\mathcal{B}}^{\sigma}\left[M_1,...,M_k\right]=\mathbb{E}\kappa_{\mathfrak{S}}^{\sigma}\left[M_1,...,M_k\right]$.*

This lemma is a simple consequence of Lemma 3.9 in [**15**] which asserts that for any $\sigma\in\mathfrak{S}_k$, $[id_k,\sigma]_{\mathcal{B}_k}=[id_k,\sigma]_{\mathfrak{S}_k}$.

Remark 4.1. — *Let us recall the definitions in Section 4.4 of [**15**]. The cumulants that we defined are linked by the fact that, if A stands either for $\mathfrak{S}$ or $\mathcal{B}$:*

$$\mathsf{E}^{A_k}\left[\left(\mathbb{E}\kappa_A^p[M_1,...,M_k]\right)_{p\in A_k}\right]=\mathcal{C}_\kappa^A\left[\left(\mathbb{E}\kappa_{\mathcal{P}}^p[M_1,...,M_k]\right)_{p\in\mathcal{P}_k}\right],$$

and

$$\mathsf{E}^{\mathfrak{S}_k}\left[\left(\mathbb{E}\kappa_A^p[M_1,...,M_k]\right)_{p\in\mathfrak{S}_k}\right]=\mathcal{C}_\kappa^{\mathfrak{S}}\circ\mathsf{E}^{\mathcal{B}_k}\left[\left(\mathbb{E}\kappa_{\mathcal{B}}^p[M_1,...,M_k]\right)_{p\in\mathcal{B}_k}\right].$$

We can also give a new characterization of the asymptotic A-factorization property whose proof is similar to the proof of the fact that the $\mathcal{R}_A$-transform defined in [**15**] is a bijection between multiplicative elements. Thus, we will not provide a proof for it.

Proposition 4.1. — *The family* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ *satisfies the asymptotic A-factorization property if and only if for any integer* k_1 *and* k_2*, for any* $p_1 \in A_{k_1}$ *and* $p_2 \in A_{k_2}$*, for any element* $(B_1^N)_{N\in\mathbb{N}},...,\ (B_{k_1+k_2}^N)_{N\in\mathbb{N}}$ *in* $\mathcal{A}\left(\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k\right)$*,*

$$\mathbb{E}\kappa_A^{p_1\otimes p_2}(B_1,...,B_{k_1+k_2}) = \mathbb{E}\kappa_A^{p_1}(B_1,...,B_{k_1})\mathbb{E}\kappa_A^{p_1}(B_{k_1+1},...,B_{k_2}).$$

We stated the result for a finite family, yet it implies a similar statement for infinite families. For any integer k, let $(1,...,k)$ be the permutation in $\mathfrak{S}_k$ which, for any $i \in \{1,...,k\}$, sends i on $i+1$ modulo k.

Proposition 4.2. — *Using the bijection between the poset of non-crossing partitions and elements of* $([id_k,(1,...,k)]_{\mathfrak{S}_k},\leq)$*, we see that this definition of cumulants is a generalization of the usual free cumulants. Indeed, if the family* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ *converges in expectation in* $\mathfrak{S}$*-moments and satisfies the asymptotic* $\mathfrak{S}$*-factorization, then for any* $\sigma \in \mathfrak{S}_k$*:*

$$\mathbb{E}\kappa_{\mathfrak{S}}^{\sigma}[M_1,...,M_k] = \prod_{c\in\mathsf{C}(\sigma)} \kappa^{\#c}\left[(M_i)_{i\in c}\right],$$

where $\mathsf{C}(\sigma)$ *is the set of cycles of* σ *and for any positive integer* n*,* κ^n *stands for the usual free cumulant.*

For a definition of the usual free cumulants, one can read Chapter 2.5 of [**21**].

Proposition 4.3. — *Let us suppose that the* $\mathfrak{S}$*-factorization property asymptotically holds for* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ *then* $\mathbb{E}\kappa_{\mathfrak{S}}^{(1,...,k)}(M_1,...,M_k)$ *is the usual non-commutative cumulant* $\kappa^k(M_1,...,M_k)$*.*

4.1.2. Higher order. — Let m be an integer. Let us suppose that $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ converges in expectation in A-moments up to order m of fluctuations. Recall the definition of the defect $\mathsf{df}(p',p)$ defined in Definition 3.5 in [**15**].

Definition 4.2. — *The family of A-cumulants up to order* m *of fluctuations*

$$\left(\mathbb{E}\kappa_{i,A}^p\left[\left(M_1^N\right)_{N\in\mathbb{N}},...,\left(M_k^N\right)_{N\in\mathbb{N}}\right]\right)_{p\in A_k, i\in\{0,...,m\}}$$

is the unique family of complex numbers such that for any $i \in \{0,...,m\}$ *and any* $p \in A_k$*:*

$$\mathbb{E}m_p^i(M_1,...,M_k) = \sum_{p'\in A_k, \mathsf{df}(p',p)\leq i} \mathbb{E}\kappa_{i-\mathsf{df}(p',p),A}^{p'}\left[\left(M_1^N\right)_{N\in\mathbb{N}},...,\left(M_k^N\right)_{N\in\mathbb{N}}\right].$$

We will denote them by $\left(\mathbb{E}\kappa_{i,A}^p[M_1,...,M_k]\right)_{p\in A_k, i\in\{0,...,m\}}$*.*

We recall that for any p and p' in A_k, $\mathsf{df}(p',p) = 0$ if and only if $p' \in [id_k,p]_{A_k}$. Thus we get the following lemma.

Lemma 4.2. — *For any* $p \in A_k$*,* $\mathbb{E}\kappa_A^p[M_1,...,M_k] = \mathbb{E}\kappa_{0,A}^p[M_1,...,M_k]$*.*

Again, the restriction to $\mathfrak{S}_k$ or $\mathcal{B}_k$ of the $\mathcal{P}$-cumulants of higher order is not equal to the $\mathfrak{S}_k$ or $\mathcal{B}_k$-cumulants of higher order. Besides, the Lemma 4.1 does not hold anymore for $m > 0$.

The Definitions 4.1 and 4.2 only allow us to define cumulants at the limit $N = \infty$: we can define cumulants only for k-tuple of elements in $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which converges in expectation in A-moments. In the following, we define a notion of N-dimensional cumulants, defined for any element of $L^{\infty^-}(\Omega) \otimes \mathcal{M}_N(\mathbb{C})$ and we show that these N-dimensional cumulants converge to the cumulants we have just defined.

4.2. N-dimensional cumulants. —

4.2.1. Zero order. — Let N and k be two positive integers. Let $M_1, ..., M_k$ be k random matrices in $L^{\infty^-}(\Omega) \otimes \mathcal{M}_N(\mathbb{C})$.

Let us choose a possibility for (G, A) as explained in Notation 3.1. Let dg be the Haar measure on $G(N)$. As explained in the beginning of Section 3, we are interested in the element:

$$\int_{G(N)} g^{\otimes k} \mathbb{E}[M_1 \otimes ... \otimes M_k](g^{-1})^{\otimes k} dg, \tag{5}$$

which is an element of $\left(\mathbb{C}\left[\rho^k_{G(N)}\right]\right)'$. Using Theorem 3.1, this endomorphism is an element of $\mathbb{C}\left[\rho_N^{A_k}\right]$. Let us remind that if A is equal to $\mathcal{B}$ or $\mathfrak{S}$, then $\rho_N^{A_k}$ is injective as soon as $N \geq k$, and if $A = \mathcal{P}$, then $\rho_N^{A_k}$ is injective as soon as $N \geq 2k$. This allows us to define the N-dimensional cumulants as the coordinate numbers of (5).

Definition 4.3. — *Let us suppose that $N \geq k$ if $A \in \{\mathcal{B}, \mathfrak{S}\}$ or that $N \geq 2k$ if $A = \mathcal{P}$. The family of N-dimensional A-cumulants of $(M_1, ..., M_k)$ is the family $\left(\mathbb{E}\kappa_A^p(M_1, ..., M_k)\right)_{p \in A_k}$ such that, for any $p \in A_k$,*

$$\mathbb{E}\kappa_A^p(M_1, ..., M_k) = \kappa^p\left(\int_{G(N)} g^{\otimes k} \mathbb{E}[M_1 \otimes ... \otimes M_k](g^{-1})^{\otimes k} dg\right).$$

We warn the reader that the three notions of cumulants are really distinct: for any $\sigma \in \mathfrak{S}_k$ in general $\mathbb{E}\kappa_{\mathcal{P}}^{\sigma}(M_1, ..., M_k)$, $\mathbb{E}\kappa_{\mathcal{B}}^{\sigma}(M_1, ..., M_k)$ and $\mathbb{E}\kappa_{\mathfrak{S}}^{\sigma}(M_1, ..., M_k)$ are distinct. Thus there is no similar result to Lemma 4.1 for the N-dimensional A-cumulants.

In the following we define the convergence in expectation in A-cumulants. For any $i \in I$, let $(M_i^N)_{N \in \mathbb{N}}$ be an element of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$.

Definition 4.4. — *The family $\left((M_i^N)_{N \in \mathbb{N}}\right)_{i \in I}$ converges in expectation in A-cumulants if for any positive integer k, for any element $(B_1^N)_{N \in \mathbb{N}}$,..., $(B_k^N)_{N \in \mathbb{N}}$ in $\mathcal{A}\left(\left((M_i^N)_{N \in \mathbb{N}}\right)_{i \in I}\right)$, for any $p \in A_k$, the sequence of cumulants:*

$$\left(\mathbb{E}\kappa_A^p(B_1^N, ..., B_k^N)\right)_{N \geq 2k}$$

converges when N goes to infinity.

The first main and direct consequence of Theorem 4.1 of [**15**] is the following result which links the two notions of convergence.

Theorem 4.1. — *The family* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in expectation in* A*-moments if and only if it converges in expectation in* A*-cumulants. If so, we say that* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in* A*-expectation.*

Let us suppose that $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in* A*-expectation. For any positive integer* k*, for any* $(i_1,...,i_k)\in I^k$*, for any* $p\in A_k$*:*

$$\mathbb{E}m_p(M_{i_1},...,M_{i_k})=\sum_{p'\in[id,p]_{A_k}}\lim_{N\to\infty}\mathbb{E}\kappa_A^{p'}\left(M_{i_1}^N,...,M_{i_k}^N\right). \tag{6}$$

Remark 4.2. — *Let us remark that the convergence in* $\mathcal{P}$*-expectation implies the convergence in* $\mathcal{B}$*-expectation which implies the convergence in* $\mathfrak{S}$*-expectation. The easiest way to see this, is to consider the convergence in expectation in moments.*

Looking at Theorem 4.1 and Definition 4.1, we see that the N-dimensional A-cumulants of $(M_i^N)_{i\in I}$ converge to the A-cumulants of $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$

Theorem 4.2. — *Let us suppose that the family* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in* A*-expectation. For any integer* k*, for any* $(i_1,...,i_k)\in I^k$*, for any* $p\in A_k$*,*

$$\lim_{N\to\infty}\mathbb{E}\kappa_A^p(M_{i_1}^N,...,M_{i_k}^N)=\mathbb{E}\kappa_A^p\left[\left(M_{i_1}^N\right)_{N\in\mathbb{N}},...,\left(M_{i_k}^N\right)_{N\in\mathbb{N}}\right].$$

Let us remark that this theorem allows us to define in a more natural way the A-cumulants $\left(\mathbb{E}\kappa_A^p\left[\left(M_{i_1}^N\right)_{N\in\mathbb{N}},...,\left(M_{i_k}^N\right)_{N\in\mathbb{N}}\right]\right)_{p\in A_k}$.

4.2.2. Higher order. — One can define also the N-dimensional A-cumulants of higher order. Actually, one can not define them just for a family of random matrices of size N as we did for the zero order: the higher order N-dimensional A-cumulants are defined for families of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. For each $i\in I$, let $(M_i^N)_{N\in\mathbb{N}}$ be an element of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. The definition of higher order N-dimensional A-cumulants can not be disjoint from the definition of convergence in expectation in A-cumulants up to a given order of fluctuations. Let m be a non-negative integer.

Definition 4.5. — *Let* k *be a positive integer and let* $(i_1,...,i_k)\in I^k$*. The sequence* $\left(M_{i_1}^N\otimes...\otimes M_{i_k}^N\right)_{N\in\mathbb{N}}$ *converges in expectation in* A_k*-cumulants up to order* m *of fluctuations if there exists a family of complex numbers*

$$\left(\mathbb{E}\kappa_{i,A}^p(M_{i_1}^N,...,M_{i_k}^N)\right)_{p\in A_k,i\in\{0,...,m\},N\geq 2k},$$

such that:

- $\forall p\in A_k,\mathbb{E}\kappa_A^p(M_{i_1}^N,...,M_{i_k}^N)=\sum_{i=0}^m\frac{\mathbb{E}\kappa_{i,A}^p(M_{i_1}^N,...,M_{i_k}^N)}{N^i}$,
- $\forall p\in A_k,\forall i\in\{0,...,m-1\},\mathbb{E}\kappa_{i,A}^p(M_{i_1}^N,...,M_{i_k}^N)$ *does not depend on* N,
- $\forall p\in A_k,\mathbb{E}\kappa_{m,A}^p(M_{i_1}^N,...,M_{i_k}^N)$ *converges as* N *goes to infinity.*

The family $\left(\mathbb{E}\kappa_{i,A}^p(M_{i_1}^N,...,M_{i_k}^N)\right)_{p\in A_k,i\in\{0,...,m\},N\geq 2k}$ *is uniquely defined.*

The family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in expectation in* A*-cumulants up to order* m *of fluctuations if for any positive integer* k*, for any element* $(B_1^N)_{N\in\mathbb{N}}$*,...,* $(B_k^N)_{N\in\mathbb{N}}$

in $\mathcal{A}\left(\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\right)$, *the sequence* $(B_1^N \otimes ... \otimes B_k^N)_{N\in\mathbb{N}}$ *converges in expectation in* A_k*-cumulants up to order* m *of fluctuations.*

Actually, there is a slight abuse of notation as for any $p \in A_k$, for any $i \in \{0, ..., m\}$ and any $N \in \mathbb{N}$, $\mathbb{E}\kappa_{i,A}^p(M_1^N, ..., M_k^N)$ does not depend only on $(M_1^N, ..., M_k^N)$, it depends on all the sequences $(M_1^N)_{N\in\mathbb{N}}$,..., $(M_k^N)_{N\in\mathbb{N}}$. In particular, we warn the reader that $\mathbb{E}\kappa_{0,A}^p(M_1^N, ..., M_k^N)$ is not equal to $\mathbb{E}\kappa_A^p(M_1^N, ..., M_k^N)$.

Let us focus on a generalization of Theorem 4.1, which is a consequence of Theorem 9.1 of the article [**15**].

Theorem 4.3. — *The family* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in expectation in* A*-moments up to order* m *of fluctuations if and only if it converges in expectation in* A*-cumulants up to order* m *of fluctuations. If so, we say that* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in* A*-expectation up to order* m *of fluctuations.*

Let us suppose that $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in* A*-expectation up to order* m *of fluctuations. For any positive integer* k*, for any* $(i_1, ..., i_k) \in I^k$*, for any* $i \in \{0, ..., m\}$ *and any* $p \in A_k$ *:*

$$\lim_{N\to\infty} \mathbb{E}m_p^i\left(M_{i_1}^N, ..., M_{i_k}^N\right) = \sum_{p'\in A_k, \mathsf{df}(p',p)\leq i} \lim_{N\to\infty} \mathbb{E}\kappa_{i-\mathsf{df}(p',p),A}^{p'}\left(M_{i_1}^N, ..., M_{i_k}^N\right). \tag{7}$$

Remark 4.3. — *Let us remark that again the convergence in* $\mathcal{P}$*-expectation up to order* m *of fluctuations implies the convergence in* $\mathcal{B}$*-expectation up to order* m *of fluctuations which implies the convergence in* $\mathfrak{S}$*-expectation up to order* m *of fluctuations.*

Looking at Theorem 4.3 and Definition 4.2, we get the following theorem.

Theorem 4.4. — *Let us suppose that the family* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in* A*-expectation up to order* m *of fluctuations. For any positive integer* k*, for any* $(i_1, ..., i_k) \in I^k$*, for any* $i \in \{0, ..., m\}$ *and any* $p \in A_k$*,*

$$\lim_{N\to\infty} \mathbb{E}\kappa_{i,A}^p\left[M_{i_1}^N, ..., M_{i_k}^N\right] = \mathbb{E}\kappa_{i,A}^p\left[(M_{i_1}^N)_{N\in\mathbb{N}}, ..., (M_{i_k}^N)_{N\in\mathbb{N}}\right].$$

4.3. Special case: G-invariant families. — Let N and k be two positive integers.

Definition 4.6. — *Let* $M_1, ..., M_k$ *be* k *random matrices in* $L^{\infty^-}(\Omega) \otimes \mathcal{M}_N(\mathbb{C})$*. The* k*-tuple* $(M_1, ..., M_k)$ *is* $G(N)$*-invariant if for any* $g \in G(N)$*, we have the equality in law:*

$$\left(gM_1g^{-1}, ..., gM_kg^{-1}\right) = (M_1, ..., M_k).$$

In general, if $(M_i)_{i\in I}$ *is a family of random matrices of size* N*, then* $(M_i)_{i\in I}$ *is* $G(N)$*-invariant if for any integer* k*, for any* $(i_1, ..., i_k) \in I^k$*,* $(M_{i_1}, ..., M_{i_k})$ *is* $G(N)$*-invariant.*

Let $((M_i^N)_{N\in\mathbb{N}})_{i\in I}$ *be a family of elements of* $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$*. We say that* $((M_i^N)_{N\in\mathbb{N}})_{i\in I}$ *is* G*-invariant if for any integer* N*,* $(M_i^N)_{i\in I}$ *is* $G(N)$*-invariant.*

Let us remark that the invariance in law by conjugation by $U(N)$ implies the invariance in law by conjugation by $O(N)$ which implies the invariance in law by conjugation by $\mathfrak{S}(N)$. Recall the notation $p_{i_1,...,i_k}^{i_{1'},...,i_{k'}}$ defined in Section 2.3 of [**15**]. In order to define some micro-observables, we need to define the kernel of a partition.

Definition 4.7. — *Let $(i_1, ..., i_k, i_{1'}, ..., i_{k'})$ be a $2k$-tuple of $\{1, ..., N\}$. We denote by $\mathsf{Ker}\,((i_1, ..., i_k, i_{1'}, ..., i_{k'}))$ the unique partition $p \in \mathcal{P}_k$ such that $(p^{i_1,...,i_k}_{i_{1'},...,i_{k'}})^{ex} = 1$: for any r and s in $\{1, ..., k, 1', ..., k'\}$, r and s are in the same block if and only if $i_r = i_s$.*

This definition allows to define some observables $(\delta_p)_{p\in\mathcal{P}_k}$ for family of random matrices which is invariant in law by conjugation by the symmetric group.

Definition 4.8. — *Let $(M_1, ..., M_k)$ be a k-tuple of random matrices of size N. Let us suppose that $(M_1, ..., M_k)$ is invariant in law by conjugation by the symmetric group $\mathfrak{S}(N)$. Let p be a partition in $\mathcal{P}_k$, we define:*

$$\delta_p\,(M_1, ..., M_k) = \mathbb{E}\big[\,(M_1)^{i_1}_{i_{1'}} ...(M_k)^{i_k}_{i_{k'}}\big],$$

where $(i_1, ..., i_k, i_{1'}, ..., i_{k'})$ is any $2k$-tuple of $\{1, ..., N\}$ such that $\mathsf{Ker}\,((i_1, ..., i_k, i_{1'}, ..., i_{k'})) = p$.

We will study the asymptotics of these observables in Section 5 when $(M_i)_{i=1}^k$ is a family which converges in A-expectation.

4.3.1. Zero order. — Let us choose a possibility for (G, A) as explained in Notation 3.1. Let k and N be two positive integers such that $N \geq k$ if $A \in \{\mathcal{B}, \mathfrak{S}\}$ or such that $N \geq 2k$ if $A = \mathcal{P}$. Let $M_1, ..., M_k$ be k random matrices in $L^{\infty^-}(\Omega) \otimes \mathcal{M}_N(\mathbb{C})$. Let us suppose that $(M_1, ..., M_k)$ is $G(N)$-invariant. The notion of N-dimensional A-cumulants is simpler as one can remove the integration $\int_{G(N)} dg$ in (5). Indeed, by definition, we have the following lemma.

Lemma 4.3. — *Let us suppose that $(M_1, ..., M_k)$ is $G(N)$-invariant. Then:*

$$\mathbb{E}[M_1 \otimes ... \otimes M_k] = \sum_{p\in A_k} \mathbb{E}\kappa^p_A(M_1, ..., M_k)\frac{p}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}}.$$

This lemma shows that the three notions of N-dimensional cumulants are linked for such k-tuple.

Proposition 4.4. — *If $G = U$, then:*

1. *for any $p \notin \mathfrak{S}_k$, $\mathbb{E}\kappa^p_{\mathcal{P}}(M_1, ..., M_k) = \mathbb{E}\kappa^p_{\mathcal{B}}(M_1, ..., M_k) = 0$,*
2. *for any $p \in \mathfrak{S}_k$, $\mathbb{E}\kappa^p_{\mathfrak{S}}(M_1, ..., M_k) = \mathbb{E}\kappa^p_{\mathcal{B}}(M_1, ..., M_k) = \mathbb{E}\kappa^p_{\mathcal{P}}(M_1, ..., M_k)$.*

If $G = O$, then:

1. *for any $p \notin \mathcal{B}_k$, $\mathbb{E}\kappa^p_{\mathcal{P}}(M_1, ..., M_k) = 0$,*
2. *for any $p \in \mathcal{B}_k$, $\mathbb{E}\kappa^p_{\mathcal{B}}(M_1, ..., M_k) = \mathbb{E}\kappa^p_{\mathcal{P}}(M_1, ..., M_k)$.*

Let $((M^N_i)_{N\in\mathbb{N}})^k_{i\in1}$ be a G-invariant family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$. Using Theorem 4.2, the Proposition 4.4 implies the following result on the $\mathfrak{S}$, $\mathcal{B}$ and $\mathcal{P}$-cumulants of $((M^N_i)_{N\in\mathbb{N}})^k_{i\in1}$. Let us remark that we will use Lemma 4.1 for the third assertion when G is equal to O. Recall the notation that we explained at the end of Definition 4.1.

Theorem 4.5. — *Let us suppose that the family* $((M_i^N)_{N\in\mathbb{N}})_{i\in 1}^k$ *converges in* A*-expectation, then it converges in* $\mathcal{P}$*-expectation. Besides, for any* $p \in \mathcal{P}_k$*:*

$$\lim_{N\to\infty} \mathbb{E}m_p(M_1^N, ..., M_k^N) = \sum_{p'\in A_k, p'\leq p} \mathbb{E}\kappa_A^{p'}(M_1, ..., M_k). \tag{8}$$

Moreover, if $G = U$*, then:*

1. *for any* $p \notin \mathfrak{S}_k$*:*
   - *if* $p \in \mathcal{P}_k$, $\mathbb{E}\kappa_{\mathcal{P}}^p[M_1, ..., M_k] = 0$,
   - *if* $p \in \mathcal{B}_k$, $\mathbb{E}\kappa_{\mathcal{B}}^p[M_1, ..., M_k] = 0$,
2. *for any* $p \in \mathfrak{S}_k$,

$$\begin{aligned}\mathbb{E}\kappa_{\mathfrak{S}}^p[M_1, ..., M_k] &= \mathbb{E}\kappa_{\mathcal{B}}^p[M_1, ..., M_k] \\ &= \mathbb{E}\kappa_{\mathcal{P}}^p[M_1, ..., M_k].\end{aligned}$$

If $G = O$*, then:*

1. *for any* $p \in \mathcal{P}_k \setminus \mathcal{B}_k$, $\mathbb{E}\kappa_{\mathcal{P}}^p\left[(M_1^N)_{N\in\mathbb{N}}, ..., (M_k^N)_{N\in\mathbb{N}}\right] = 0$,
2. *for any* $p \in \mathcal{B}_k$,

$$\mathbb{E}\kappa_{\mathcal{B}}^p[M_1, ..., M_k] = \mathbb{E}\kappa_{\mathcal{P}}^p[M_1, ..., M_k],$$

3. *for any* $p \in \mathfrak{S}_k$,

$$\mathbb{E}\kappa_{\mathfrak{S}}^p\left[(M_1^N)_{N\in\mathbb{N}}, ..., (M_k^N)_{N\in\mathbb{N}}\right] = \mathbb{E}\kappa_{\mathcal{B}}^p\left[(M_1^N)_{N\in\mathbb{N}}, ..., (M_k^N)_{N\in\mathbb{N}}\right].$$

Proof. — If $A = \mathcal{P}$, there is nothing to prove. Let us suppose that A is equal to $\mathcal{B}$ or $\mathfrak{S}$. Let us suppose that the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ converges in A-expectation. Using the Theorem 4.1, it converges in expectation in A-cumulants. Yet Proposition 4.4 implies that it converges in expectation in $\mathcal{P}$-cumulants. Indeed, for any positive integer l, for any element B_1,..., B_l of $\mathcal{A}\left(\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\right)$, for any positive integer N, and for any $p \in \mathcal{P}_l$:

- If $p \in A_k$, $\mathbb{E}\kappa_{\mathcal{P}}^p[B_1^N, ..., B_l^N] = \mathbb{E}\kappa_A^p[B_1^N, ..., B_l^N]$.
- If $p \notin A_k$, $\mathbb{E}\kappa_{\mathcal{P}}^p[B_1^N, ..., B_l^N] = 0$.

Thus using Theorem 4.1, the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ converges in $\mathcal{P}$-expectation.

We have already noticed that the convergence in $\mathcal{P}$-expectation implies the convergence in $\mathcal{B}$-expectation: thus we can consider, for any $p \in \mathcal{B}_k$, the free cumulant $\mathbb{E}\kappa_{\mathcal{B}}^p[M_1, ..., M_k]$. Taking the limit, when N goes to infinity, of the equalities in Proposition 4.4 implies the second part of the assertions. The Equality (8) is a consequence of our discussions and Theorem 4.1 of [**15**]. □

Let us suppose that the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ converges in expectation in A-moments. If it satisfies the asymptotic A-factorization property, then due to Proposition 4.1, for any integer k_1 and k_2, for any $p_1 \in A_{k_1}$ and $p_2 \in A_{k_2}$, for any element $(B_1^N)_{N\in\mathbb{N}}$,..., $(B_{k_1+k_2}^N)_{N\in\mathbb{N}}$ in $\mathcal{A}\left(\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k\right)$

$$\mathbb{E}\kappa_A^{p_1\otimes p_2}(B_1, ..., B_{k_1+k_2}) = \mathbb{E}\kappa_A^{p_1}(B_1, ..., B_{k_1})\mathbb{E}\kappa_A^{p_2}(B_{k_1+1}, ..., B_{k_2}).$$

Using Theorem 4.5, we see that this last equality holds for p_1 and p_2 in respectively $\mathcal{P}_{k_1}$ and $\mathcal{P}_{k_2}$. Using again Proposition 4.1, we deduce that the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ satisfies the asymptotic $\mathcal{P}$-factorization property.

Proposition 4.5. — *The family* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ *satisfies the asymptotic A-factorization property if and only if it satisfies the asymptotic* $\mathcal{P}$*-factorization property.*

This proposition can be extended to any family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$, with I being any indexation set. Using the Theorem 4.5 and our discussion on the asymptotic A-factorization property, one has for example the following corollary.

Corollary 4.1. — *For any positive integer* N*, let* $(M_i^N)_{i\in I}$ *be a family of random matrices of size* N *which is invariant by conjugation by* $U(N)$*. If* $(M_i^N)_{i\in I}$ *converges in non-commutative distribution as* N *goes to infinity and satisfies the asymptotically* $\mathfrak{S}$*-factorization property, then* $(M_i^N)_{i\in I}\cup\left({}^tM_i^N\right)_{i\in I}$ *converges in non-commutative distribution. Besides, for any positive integer* k*, for any* $p\in\mathcal{P}_k$*, for any* $(i_1,...,i_k)\in I^k$*,* $\mathbb{E}m_p(M_{i_1}^N,...,M_{i_k}^N)$ *converges as* N *goes to infinity to:*

$$\sum_{\sigma\in\mathfrak{S}_k|\sigma\leq p}\prod_{c=(j_1,...,j_r)\ cycle\ of\ \sigma}\kappa\left(\left(M_{i_{j_1}}^N\right)_{N\in\mathbb{N}},...,\left(M_{i_{j_r}}^N\right)_{N\in\mathbb{N}}\right)$$

where the κ *stand for the usual non-commutative cumulants in free probability.*

A similar corollary can be written for families of random matrices which are invariant in law by conjugation by the orthogonal group. Recall the notion of cycles in Definition 2.9 of [**15**] and the notion of extraction in Definition 2.12 of the same article.

Corollary 4.2. — *Let* $(M_i)_{i\in I}$ *be a family of elements of* $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ *which is* O*-invariant. If* $(M_i)_{i\in I}\cup({}^tM_i)_{i\in I}$ *converges in non-commutative distribution and if the* $\mathfrak{S}$*-factorization property holds for this family, then for any positive integer* k*, for any* $p\in\mathcal{P}_k$*, for any* $(i_1,...,i_k)\in I^k$*,* $\mathbb{E}m_p\left(M_{i_1}^N,...,M_{i_k}^N\right)$ *converges as* N *goes to infinity to:*

$$\sum_{b\in\mathcal{B}_k|b\leq p}\prod_{c=\{j_1,...,j_r\}\cup\{(j_1)',...,(j_r)'\}\ cycle\ of\ b,j_1<...<j_r}\mathbb{E}\kappa_{\mathcal{B}}^{b_c}\left((M_{i_{j_1}}^N)_{N\in\mathbb{N}},...,(M_{i_{j_r}}^N)_{N\in\mathbb{N}}\right).$$

4.3.2. Higher order. — Let m be a non negative integer, let us choose a possibility for (G,A) as explained in Notation 3.1. Let k be a positive integer, let $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in1}^k$ be a G-invariant family of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. The following proposition is a consequence of Proposition 4.4 and Theorem 4.3. The proof which is similar to the one of Theorem 4.5 will be omitted. Recall the notation that we explained in Definition 4.2.

Proposition 4.6. — *Let us suppose that the family* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ *converges in expectation in A-moments up to order* m *of fluctuations, then it converges in expectation in* $\mathcal{P}$*-moments up to order* m *of fluctuations. Besides, for any* $i\in\{0,...,m\}$ *and any* $p\in\mathcal{P}_k$*:*

$$\lim_{N\to\infty}\mathbb{E}m_p^i(M_1^N,...,M_k^N)=\sum_{p'\in A_k,\mathsf{df}(p',p)\leq i}\mathbb{E}\kappa_{i-\mathsf{df}(\mathsf{p}',\mathsf{p}),A}^{p'}(M_1,...,M_k).$$

Moreover, if $G=U$*, then:*

1. *For any* $p\notin\mathfrak{S}_k$*, for any* $i\in\{0,...,m\}$*:*

$$\mathbb{E}\kappa_{i,\mathcal{P}}^p[M_1,...,M_k]=\mathbb{E}\kappa_{i,\mathcal{B}}^p[M_1,...,M_k]=0.$$

2. *For any $p \in \mathfrak{S}_k$, for any $i \in \{0, ..., m\}$:*

$$\begin{aligned}\mathbb{E}\kappa^p_{i,\mathfrak{S}}[M_1, ..., M_k] &= \mathbb{E}\kappa^p_{i,\mathcal{B}}[M_1, ..., M_k] \\ &= \mathbb{E}\kappa^p_{i,\mathcal{P}}[M_1, ..., M_k].\end{aligned}$$

If $G = O$, then:

1. *For any $p \notin \mathcal{B}_k$, for any $i \in \{0, ..., m\}$:*

$$\mathbb{E}\kappa^p_{i,\mathcal{P}}[M_1, ..., M_k] = 0.$$

2. *For any $p \in \mathcal{B}_k$, for any $i \in \{0, ..., m\}$:*

$$\mathbb{E}\kappa^p_{i,\mathcal{B}}[M_1, ..., M_k] = \mathbb{E}^p_{i,\mathcal{P}}[M_1, ..., M_k].$$

3. *For any $p \in \mathfrak{S}_k$:*

$$\mathbb{E}\kappa^p_{0,\mathfrak{S}}[M_1, ..., M_k] = \mathbb{E}\kappa^p_{0,\mathcal{B}}[M_1, ..., M_k].$$

5. Exclusive moments and cumulants

5.1. Exclusive moments. — Let N and k be positive integers. Let $M_1, ..., M_k$ be k random matrices of size N defined on the same probability space, we are going to define the mean exclusive normalized moments. Let p in $\mathcal{P}_k$.

Definition 5.1. — *The mean exclusive p-normalized moment of the k-tuple $(M_1, ..., M_k)$ is:*

$$\mathbb{E}m_{p^c}(M_1, ..., M_k) = \frac{1}{Tr^k(\rho_N^{\mathcal{P}_k}(p))}\mathbb{E}\left[Tr^k\left((M_1 \otimes ... \otimes M_k)\rho_N^{\mathcal{P}_k}({}^t p^c)\right)\right].$$

We will define a second notion of mean exclusive normalized moments. Let us consider a choice of (G, A) as explained in Notation 3.1.

Definition 5.2. — *Let p in $\mathcal{P}_k$, the mean exclusive p-normalized A-moment of the k-tuple $(M_1, ..., M_k)$, denoted by $\mathbb{E}m^A_{p^c}(M_1, ..., M_k)$, is:*

$$\frac{1}{Tr^k(\rho_N^{\mathcal{P}_k}(p))}\mathbb{E}\left[Tr^k\left(\left(\int_{G(N)} g^{\otimes k}(M_1 \otimes ... \otimes M_k)(g^{-1})^{\otimes k}dg\right)\rho_N^{\mathcal{P}_k}({}^t p^c)\right)\right].$$

Let us remark that for any $p \in \mathcal{P}_k$, the mean exclusive p-normalized moment is equal to the mean exclusive p-normalized $\mathcal{P}$-moment.

Lemma 5.1. — *For any $p \in \mathcal{P}_k$:*

$$\mathbb{E}m_{p^c}(M_1, ..., M_k) = \mathbb{E}m^{\mathcal{P}}_{p^c}(M_1, ..., M_k).$$

One can define the convergence in expectation in exclusive A-moments, almost as we did in Definition 2.2, except that we ask for the convergence of the observables for any $p \in \mathcal{P}_k$. Let $(M_i)_{i\in I}$ be a family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$.

Definition 5.3. — *Let k be a positive integer, let $(i_1, ..., i_k) \in I^k$. The sequence $\left(M_{i_1}^N \otimes ... \otimes M_{i_k}^N\right)_{N\in\mathbb{N}}$ converges in expectation in exclusive A-moments if for any $p \in \mathcal{P}_k$,*

$$\mathbb{E}m_{p^c}^A(M_{i_1}^N, ..., M_{i_k}^N)$$

converges when N goes to infinity.

The family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in exclusive A-moments if for any positive integer k, for any element $(B_1^N)_{N\in\mathbb{N}}$,..., $(B_k^N)_{N\in\mathbb{N}}$ in $\mathcal{A}\left(\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\right)$, the sequence $(B_1^N \otimes ... \otimes B_k^N)_{N\in\mathbb{N}}$ converges in expectation in exclusive A_k-moments.

Again, if $(M_1^N, ..., M_k^N)$ converges in expectation in exclusive A-moments, we denote by $\mathbb{E}m_{p^c}^A(M_1, ..., M_k)$ the limit of $\mathbb{E}m_{p^c}^A(M_1^N, ..., M_k^N)$.

Let us recall the notations and definitions in the Sections 3.2 and 3.3 of [**15**]. In these sections, we defined the set of admissible splittings of a partition p, denoted by $\mathsf{Sp}(p)$, the set of admissible gluings of p, denoted by $\mathsf{Gl}_c(p)$, and two partial orders $\sqsupset$ and $\dashv$. Then Proposition 4.2 of [**15**] implies the following proposition.

Proposition 5.1. — *The family $(M_i)_{i\in I}$ converges in expectation in A-moments if and only if it converges in expectation in A-exclusive moments. Besides if one condition holds then for any positive integer k, for any $p \in A_k$, for any $i_1, ..., i_k$ in I:*

$$\mathbb{E}m_p(M_{i_1}, ..., M_{i_k}) = \sum_{p'\in\mathcal{P}_k,\ p'\dashv p} \mathbb{E}m_{p'^c}^A(M_{i_1}, ..., M_{i_k}).$$

Proof. — This is due to the fact that, using Theorem 3.1, if p is in A_k, then:

$$\mathbb{E}m_p(M_{i_1}, ..., M_{i_k}) = m_p\left(\int_{G(N)} g^{\otimes}\mathbb{E}[M_{i_1} \otimes ... \otimes M_{i_k}](g^{-1})^{\otimes k}dg\right),$$

and with a slight abuse of notation:

$$\int_{G(N)} g^{\otimes}\mathbb{E}[M_{i_1} \otimes ... \otimes M_{i_k}](g^{-1})^{\otimes k}dg \in \mathbb{C}[A_k(N)].$$

The proposition now is a consequence of Proposition 4.2 of [**15**]. □

By looking at the proof, one can understand why the equality in Proposition 5.1 does not hold for any $p \in \mathcal{P}_k$. If $p \in \mathcal{P}_k \setminus A_k$, the same equality would hold if one changes the left hand side by $\mathbb{E}m_p(HM_{i_1}H^{-1}, ..., HM_{i_k}H^{-1})$ with H being a sequence of Haar random variable on the groups of symmetry, namely $(G(N))_{N\in\mathbb{N}}$, associated with A with the condition that for every positive integer N, H^N is independent of $(M_i^N)_{i\in I}$. This discussion implies that if the family $(M_i)_{i\in I}$ is G-invariant, then Proposition 5.1 does hold for any $p \in \mathcal{P}_k$.

Let $(M_i)_{i\in I}$ be a family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ and let us suppose until the end of this section that $(M_i)_{i\in I}$ converges in expectation in A-moments. Using the Theorem 4.6 of [**15**], we get the following statement.

Theorem 5.1. — *For any positive integer k, any $i_1, ..., i_k$ in I and any $p \in \mathcal{P}_k$,*

$$\mathbb{E}m_{p^c}^A(M_{i_1}, ..., M_{i_k}) = \sum_{p'\in A_k, p'\sqsupset p} \mathbb{E}\kappa_A^{p'}(M_{i_1}, ..., M_{i_k}).$$

Besides, as a consequence of Corollary 4.1 of [**15**], we get that the limits of exclusive moments and finite-dimensional cumulants are equal for some special partitions.

Theorem 5.2. — *Let k be a positive integer, let p be a partition in $\mathcal{P}_k$ which does not have any pivotal block. For any $i_1, ..., i_k$ in I,*

$$\mathbb{E}m^A_{p^c}(M_{i_1}, ..., M_{i_k}) = \delta_{p\in A_k}\mathbb{E}\kappa^p_A(M_{i_1}, ..., M_{i_k}) \tag{9}$$

In particular, for any $p \in \mathcal{B}_k$, the Equality (9) is valid.

This last corollary is very crucial in order to study the asymptotic of the random walk on the symmetric group. The Corollary 4.2 of [**15**] allows us to assert the following proposition. Recall the definition of $\mathsf{Mb}(p)$ in Definition 3.9 of [**15**].

Theorem 5.3. — *Let us suppose that A is equal either to $\mathfrak{S}$ or $\mathcal{B}$. For any positive integer k, for any $i_1, ..., i_k$ in I, for any partition p in $\mathcal{P}_k$:*

$$\mathbb{E}m^A_{p^c}(M_{i_1}, ..., M_{i_k}) = \delta_{p\in\overline{A_k}}\mathbb{E}\kappa^{\mathsf{Mb}(p)}_A(M_{i_1}, ..., M_{i_k}).$$

5.1.1. The G-invariant case. — As always, when one imposes some invariance, the theory becomes simpler. Let us choose a possibility for (G, A) as explained in Notation 3.1. Let $(M_i)_{i\in I}$ be a family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which is G-invariant.

The Lemma 5.1 does not hold for general random matrices if one substitutes $\mathcal{P}$ in the r.h.s. by any other choice of A. Yet, in the setting where $(M_i)_{i\in I}$ is G-invariant we get the following lemma.

Lemma 5.2. — *For any positive integer k and N, for any k-tuple $(i_1, ..., i_k)$ of elements of I, for any $p \in \mathcal{P}_k$:*

$$\mathbb{E}m_{p^c}(M_{i_1}, ..., M_{i_k}) = \mathbb{E}m^A_{p^c}(M_{i_1}, ..., M_{i_k}).$$

Using this lemma, and using the discussion after Proposition 5.1, one can see that $(M_i)_{i\in I}$ converges in expectation in A-moments if and only if for any positive integer k, for any $(i_1, ..., i_k) \in I^k$, for any $p \in \mathcal{P}_k$, $\mathbb{E}m_{p^c}(M^N_{i_1}, ..., M^N_{i_k})$ converges as N goes to infinity. If so, then for any $p \in \mathcal{P}_k$:

$$\mathbb{E}m_p(M_{i_1}, ..., M_{i_k}) = \sum_{p'\in\mathcal{P}_k,\ p'\dashv p} \mathbb{E}m_{p'^c}(M_{i_1}, ..., M_{i_k}),$$

$$\mathbb{E}m_{p^c}(M_{i_1}, ..., M_{i_k}) = \sum_{p'\in A_k, p'\sqsupset p} \mathbb{E}\kappa^{p'}_A(M_{i_1}, ..., M_{i_k}).$$

If G is equal either to O of U, this last equality becomes much simpler, since it asserts that:

$$\mathbb{E}m_{p^c}(M_{i_1}, ..., M_{i_k}) = \delta_{p\in\overline{A_k}}\mathbb{E}\kappa^{\mathsf{Mb}(p)}_A(M_{i_1}, ..., M_{i_k}),$$

moreover, if G equal to U, and if the asymptotic $\mathfrak{S}$-factorization holds for $(M_i)_{i\in I}$, this last equality asserts that:

$$\mathbb{E}m_{p^c}(M_{i_1}, ..., M_{i_k}) = \delta_{p\in\overline{\mathfrak{S}_k}} \prod_{c=(j_1,...,j_r) \text{ cycle of } \mathsf{Mb}(p)} \kappa(M_{i_{j_1}}, ..., M_{i_{j_r}}), \tag{10}$$

where κ is the usual non commutative cumulant in free probability theory.

We will see in the following that the symmetry by conjugation by G allows also to link the exclusive moments with the moments of the entries of the matrices. This will allow to see the Equation (10) as a kind of generalization of Theorem 2.6 of [**12**].

As noticed by C. Male in [**26**] we have the following fact.

Lemma 5.3. — *For any positive integer k and N, for any k-tuple $(i_1, ..., i_k)$ of elements of I, for any $p \in \mathcal{P}_k$, one has:*

$$\delta_p(M_{i_1}^N, ..., M_{i_k}^N) = \frac{(N - \mathsf{nc}(p))!}{N!} N^{\mathsf{nc}(p \vee id)} \mathbb{E} m_{p^c}(M_{i_1}^N, ..., M_{i_k}^N).$$

Proof. — Let N and k be two positive integers, let us consider $(i_1, ..., i_k)$ a k-tuple of elements of I, and let p be a partition in $\mathcal{P}_k$. Since $(M_{i_1}^N, ..., M_{i_k}^N)$ is invariant in law by conjugation by $G(N)$, it is invariant in law by conjugation by the symmetric group $\mathfrak{S}(N)$. We have already seen that $\mathbb{E}\big[\big(M_{i_1}^N\big)_{j_{1'}}^{j_1} ...(M_{i_k}^N)_{j_{k'}}^{j_k} \big]$ does not depend on the value of $(j_1, ..., j_k, j_{1'}, ...j_{k'})$ as long as $\mathsf{Ker}\left((j_1, ..., j_k, j_{1'}, ...j_{k'})\right) = p$. Since there exist exactly $\frac{N!}{(N-\mathsf{nc}(p))!}$ of $2k$-tuples $(j_1, ..., j_k, j_{1'}, ...j_{k'})$ such that $\mathsf{Ker}\left((j_1, ..., j_k, j_{1'}, ...j_{k'})\right) = p$, we get the following equalities:

$$\begin{aligned}
&\frac{(N - \mathsf{nc}(p))!}{N!} N^{\mathsf{nc}(p \vee id)} \mathbb{E} m_{p^c}(M_{i_1}^N, ..., M_{i_k}^N) \\
&= \frac{(N - \mathsf{nc}(p))!}{N!} \sum_{(j_1,...,j_k,j_{1'},...j_{k'}) \in \{1,...,N\}^{2k} | \mathsf{Ker}((j_1,...,j_k,j_{1'},...j_{k'}))=p} \mathbb{E}\big[\big(M_{i_1}^N\big)_{j_{1'}}^{j_1} ...(M_{i_k}^N)_{j_{k'}}^{j_k} \big] \\
&= \delta_p(M_{i_1}^N, ..., M_{i_k}^N).
\end{aligned}$$

□

We can state now a corollary of Theorem 5.3 and the discussion we had.

Theorem 5.4. — *Let k be a positive integer, let $(M_1, ...M_k)$ be a k-tuple of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which is U-invariant, converges in $\mathfrak{S}$-expectation and satisfies the asymptotic $\mathfrak{S}$-factorization property. Then for any $2k$-tuple, $\mathbb{I} = (i_1, ..., i_k, i_{1'}, ..., i_{k'}) \in \{1, ..., N\}^{2k}$, one has that for any positive integer N:*

$$\mathbb{E}\left[\big(M_1^N\big)_{i_{1'}}^{i_1} ...(M_k^N)_{i_{k'}}^{i_k} \right] = N^{\mathsf{nc}(\mathsf{Ker}(\mathbb{I}) \vee id) - \mathsf{nc}(\mathsf{Ker}(\mathbb{I}))} \Big(\delta_{\mathsf{Ker}(\mathbb{I}) \in \overline{\mathfrak{S}_k}} \kappa_{\mathsf{Mb}(\mathsf{Ker}(\mathbb{I}))}(M_1, ..., M_k) + o(1) \Big).$$

where, as usual, for any $\sigma \in \mathfrak{S}_k$, $\kappa_\sigma\big(M_1, ..., M_k\big)$ is equal to

$$\prod_{c \text{ cycle of } \sigma, c=(i_1,...,i_r)} \kappa(M_{i_1}, ..., M_{i_r}),$$

and κ stands for the free cumulants.

One can state also a similar theorem when G is equal to O or $\mathfrak{S}$ but it would not use the usual free cumulants but the $\mathcal{B}$ and $\mathcal{P}$-cumulants that we defined. Yet, it would show that one can compute cumulants as limits of some normalized moments of the entries of the matrices or one can compute the asymptotics of any moments of the entries of the matrices by using the cumulants.

5.2. Exclusive cumulants. — Recall the notion of exclusive coordinate numbers defined in Definition 4.5 of [**15**]. We can define the N-dimensional mean A-exclusive cumulants.

Definition 5.4. — *Let k and N be two positive integers such that $N \geq 2k$. Let $M_1, ..., M_k$ be k random matrices in $L^{\infty^-}(\Omega) \otimes \mathcal{M}_N(\mathbb{C})$. The family of N-dimensional mean A-exclusive cumulants of $(M_1, ..., M_k)$ is the family $\left(\mathbb{E}\kappa_A^{p^c}(M_1, ..., M_k)\right)_{p \in \mathcal{P}_k}$ such that, for any $p \in \mathcal{P}_k$,*

$$\mathbb{E}\kappa_A^{p^c}(M_1, ..., M_k) = \kappa_c^p \left(\int_{G(N)} g^{\otimes k} \mathbb{E}[M_1 \otimes ... \otimes M_k](g^{-1})^{\otimes k} dg \right).$$

Let us remark that the A-exclusive cumulants are defined for any $p \in \mathcal{P}_k$. A consequence of Theorem 4.5 of [**15**] is the following theorem.

Theorem 5.5. — *Let $(M_i)_{i \in I}$ be a family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which converges in A-moments. Then, for any positive integer k, any $p \in \mathcal{P}_k$, any $i_1, ..., i_k$ in I:*

$$\mathbb{E}m_{p^c}^A(M_{i_1}, ..., M_{i_k}) = \mathbb{E}\kappa_A^{p^c}(M_{i_1}, ..., M_{i_k}).$$

6. The $\mathcal{R}$-transform and the non-commutative law

6.1. The $\mathcal{R}$-transform. —

6.1.1. Zero order. — Let $(M_N)_{N \in \mathbb{N}}$ be an element of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which converges in A-expectation. We define the A-law and the $\mathcal{R}_A$-functional of $(M_N)_{N \in \mathbb{N}}$ as following. Recall the notation $\mathfrak{E}[A]$ defined in Definition 10.1 of [**15**].

Definition 6.1. — *The A-law of $(M_N)_{N \in \mathbb{N}}$, denoted by $\mathcal{M}_A((M_N)_{N \in \mathbb{N}})$ or $\mathcal{M}_A(M)$, is the element of $\mathfrak{E}[A]$ such that for any $k \geq 1$, for any $p \in A_k$:*

$$(\mathcal{M}_A(M))_p = \mathbb{E}m_p[M, ..., M].$$

The $\mathcal{R}_A$-functional of $(M_N)_{N \in \mathbb{N}}$, denoted by $\mathcal{R}_A((M_N)_{N \in \mathbb{N}})$ or $\mathcal{R}_A(M)$, is the element of $\mathfrak{E}[A]$ such that for any $k \geq 1$, for any $p \in A_k$:

$$(\mathcal{R}_A(M))_p = \mathbb{E}\kappa^p[M, ..., M].$$

Recall the definition of the $\mathcal{M}_A$ and $\mathcal{R}_A$-transforms, transformations defined on $\mathfrak{E}[A]$, given in Definitions 10.9 and 10.10 of [**15**]. Using all the definitions we have just made and Theorem 4.1, we get the following theorem. By definition, the following equalities hold:

$$\mathcal{R}_A(M) = \mathcal{R}_A[\mathcal{M}_A(M)], \ \mathcal{M}_A(M) = \mathcal{M}_A[\mathcal{R}_A(M)].$$

With these definitions, one can reformulate the definition of the asymptotic A-factorization property for $(M_N)_{N \in \mathbb{N}}$. Recall the definition of $\mathfrak{ME}[A]$ given in Definition 10.4 in [**15**].

Proposition 6.1. — *The asymptotic A-factorization property holds for $(M_N)_{N \in \mathbb{N}}$ if and only if $\mathcal{M}_A(M)$ or $\mathcal{R}_A(M)$ is in $\mathfrak{ME}[A]$.*

6.1.2. Higher order. — Let m be a non-negative integer, let us suppose that the sequence $(M_N)_{N\in\mathbb{N}}$ converges in A-expectation up to order m of fluctuations. We can define the A-law and the $\mathcal{R}_A^{(m)}$-functional of $(M_N)_{N\in\mathbb{N}}$ up to order m of fluctuations. Recall Definition 10.14 of [**15**] where we defined the algebra $\mathfrak{E}_{(m)}[A]$.

Definition 6.2. — *We define the A-law of $(M_N)_{N\in\mathbb{N}}$ up to order m of fluctuations, denoted by $\mathcal{M}_A^{(m)}\left((M_N)_{N\in\mathbb{N}}\right)$ or $\mathcal{M}_A^{(m)}(M)$, as the element of $\mathfrak{E}_{(m)}[A]$ such that for any positive integer k, any $p\in A_k$, any $i\in\{0,...,m\}$:*

$$\left(\mathcal{M}_A^{(m)}(M)\right)_{p,i}=\mathbb{E}m_p^i[M,...,M].$$

We define the $\mathcal{R}_A^{(m)}$-functional of $(M_N)_{N\in\mathbb{N}}$ up to order m of fluctuations, denoted by $\mathcal{R}_A^{(m)}\left((M_N)_{N\in\mathbb{N}}\right)$ or $\mathcal{R}_A^{(m)}(M)$ as the element of $\mathfrak{E}_{(m)}[A]$ such that for any integer k, any $p\in A_k$, any $i\in\{0,...,m\}$:

$$\left(\mathcal{R}_A^{(m)}(M)\right)_{p,i}=\mathbb{E}\kappa_{i,A}^p[M,...,M].$$

Using these new definitions, Definitions 10.16 and 10.17 of [**15**], the following equalities hold:

$$\mathcal{R}_A^{(m)}(M)=\mathcal{R}_A^{(m)}\left[\mathcal{M}^{(m)}(M)\right],\ \mathcal{M}_A^{(m)}(M)=\mathcal{M}_A^{(m)}\left[\mathcal{R}^{(m)}(M)\right].$$

6.2. The non-commutative law. —

6.2.1. Zero order. — We have defined in the last section the law of an element $(M_N)_{N\in\mathbb{N}}$ in $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converges in A-expectation. We generalize this for a family of elements in $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converges in A-expectation. Let $(M_i)_{i\in I}$ be a family of elements in $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converges in A-expectation.

Definition 6.3. — *The A-law of $(M_i)_{i\in I}$ is the application:*

$$\bigcup_{l\in\mathbb{N}}\left(\left(\bigcup_{k=1}^{\infty}\left(\mathbb{C}\{X_1,...,X_k\}\times I^k\right)\right)^l\times A_l\right)\to\mathbb{C},$$
$$\left(\left(P_n,\{i_1^n,...,i_{k_n}^n\}\right)_{n=1}^l,p\right)\mapsto\lim_{N\to\infty}\mathbb{E}m_p\left[\left(P_n\left[M_{i_1}^N,...,M_{i_{k_n}}^N\right]\right)_{n=1}^l\right].$$

One can also define the $\mathcal{R}_A$-functional of $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$.

Definition 6.4. — *The $\mathcal{R}_A$-functional of $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ is the application:*

$$\bigcup_{l\in\mathbb{N}}\left(\left(\bigcup_{k=1}^{\infty}\left(\mathbb{C}\{X_1,...,X_k\}\times I^k\right)\right)^l\times A_l\right)\to\mathbb{C},$$
$$\left(\left(P_n,\{i_1^n,...,i_{k_n}^n\}\right)_{n=1}^l,p\right)\mapsto\mathbb{E}\kappa_A^p\left[\left(\left(P_n\left[M_{i_1}^N,...,M_{i_{k_n}}^N\right]\right)_{N\in\mathbb{N}}\right)_{n=1}^l\right].$$

6.2.2. Higher order. — One can also define the A-law and the $\mathcal{R}_A$-functional of the fluctuations up to order m. Let m be an integer, let us suppose that $(M_i)_{i\in I}$ converges in A-expectation up to order m of fluctuations.

Definition 6.5. — *The A-law up to order m of fluctuations of $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ is the application:*

$$\bigcup_{l\in\mathbb{N}}\left(\left(\bigcup_{k=1}^{\infty}\left(\mathbb{C}\{X_1,...,X_k\}\times I^k\right)\right)^l\times A_l\right)\times(\{0,...,m\})\to\mathbb{C}$$

which sends $\left(\left(P_n,\{i_1^n,...,i_{k_n}^n\}\right)_{n=1}^l,p,i_0\right)$ on $\mathbb{E}m_p^{i_0}\left[\left(P_n\left[M_{i_1},...,M_{i_{k_n}}\right]\right)_{n=1}^l\right]$.

Definition 6.6. — *The $\mathcal{R}_A$-functional up to order m of fluctuations of $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ is the application:*

$$\bigcup_{l\in\mathbb{N}}\left(\left(\bigcup_{k=1}^{\infty}\left(\mathbb{C}\{X_1,...,X_k\}\times I^k\right)\right)^l\times A_l\right)\times(\{0,...,m\})\to\mathbb{C}$$

which sends $\left(\left(P_n,\{i_1^n,...,i_{k_n}^n\}\right)_{n=1}^l,p,i_0\right)$ on $\mathbb{E}\kappa_{i_0,A}^p\left[\left(\left(P_n\left[M_{i_1}^N,...,M_{i_{k_n}}^N\right]\right)_{N\in\mathbb{N}}\right)_{n=1}^l\right]$.

7. A-Freeness

Let $\left(M_1^N\right)_{N\in\mathbb{N}}$ and $\left(M_2^N\right)_{N\in\mathbb{N}}$ be two elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converge in A-expectation. Can we recover the A-law of the couple $\left(\left(M_1^N\right)_{N\in\mathbb{N}},\left(M_2^N\right)\right)_{N\in\mathbb{N}}$ knowing the A-law of $\left(M_1^N\right)_{N\in\mathbb{N}}$ and $\left(M_2^N\right)_{N\in\mathbb{N}}$? Actually, there is no general formula to do so, except in some particular cases. For example, this is possible if $\left(M_1^N\right)_{N\in\mathbb{N}}$ and $\left(M_2^N\right)_{N\in\mathbb{N}}$ are asymptotically A-free. In this section, we define the notion of asymptotic A-freeness.

7.1. Zero order. —

7.1.1. Definitions and general results. — Let I and J be two indexation sets. Recall Definition 2.9 in [**15**] where the notion of irreducible partitions is defined. Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. Let us suppose that the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ converge in A-expectation. We will need some notations in order to define the notion of asymptotic A-freeness.

Notation 7.1. — *Let k be a positive integer, let $(B_i)_{i=1}^k$ be a k-tuple of elements of*

$$\mathcal{A}\left((L_i)_{i\in I}\right)\cup\mathcal{A}\left((M_j)_{j\in J}\right).$$

We denote by $L_{(B_i)_{i=1}^k}$ the set of $i\in\{1,...,k\}$ such that $\left(B_i^N\right)_{N\in\mathbb{N}}\in\mathcal{A}\left((L_i)_{i\in I}\right)$. We denote by $M_{(B_i)_{i=1}^k}$ the set of $i\in\{1,...,k\}$ such that $\left(B_i^N\right)_{N\in\mathbb{N}}\in\mathcal{A}\left((M_j)_{j\in J}\right)$. We say that $(B_i)_{i=1}^k$ is mixed *if $L_{(B_i)_{i=1}^k}$ and $M_{(B_i)_{i=1}^k}$ are not empty.*

Let p be in A_k. The decomposition of $\{1,...,k\}\cup\{1',...,k'\}$ by the cycles of p induces a partition π of $\{1,...,k\}$. The partition p is compatible with $(B_i)_{i=1}^k$ if any block of π is included in $L_{(B_i)_{i=1}^k}$ or in $M_{(B_i)_{i=1}^k}$.

Recall the definition of the extraction of a partition, defined in Definition 2.12 of [**15**]. Let us define the notion of A-freeness.

Definition 7.1. — *The families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free if and only if for any positive integer k, for any partition $p\in A_k$, for any mixed k-tuple $(B_i)_{i=1}^k$ of elements of $\mathcal{A}\left((L_i)_{i\in I}\right)\cup\mathcal{A}\left((M_j)_{j\in J}\right)$, the sequence $\left(\mathbb{E}\kappa_A^p(B_1^N,...,B_k^N)\right)_N$ converges as N goes to infinity and the two following conditions hold:*

1-Compatibility condition : *If p is not compatible with $(B_i)_{i=1}^k$:*

$$\lim_{N\to\infty}\mathbb{E}\kappa_A^p(B_1^N,...,B_k^N)=0.$$

2-Compatible factorization property : *If p is compatible with $(B_i)_{i=1}^k$:*

$$\lim_{N\to\infty}\mathbb{E}\kappa_A^p\left(B_1^N,...,B_k^N\right)=\mathbb{E}\kappa_A^{p_{L_{(B_i)_{i=1}^k}}}\left((B_i)_{i\in L_{(B_i)_{i=1}^k}}\right)\mathbb{E}\kappa_A^{p_{M_{(B_i)_{i=1}^k}}}\left((B_i)_{i\in M_{(B_i)_{i=1}^k}}\right).$$

In particular, if the families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free, for any integers k and l, any $(i_1,...,i_k)\in I^k$, any $(j_1,...,j_l)\in I^l$ and any $p\in A_{k+l}$,

$$\mathbb{E}\kappa_A^p(M_{i_1}^N,...,M_{i_k}^N,L_{j_1}^N,...,L_{j_l}^N)$$

converges when N goes to infinity. By Theorem 4.1 of [**15**], for any positive integers k and l, any $(i_1,...,i_k)\in I^k$, any $(j_1,...,j_l)\in I^l$, $M_{i_1}^N\otimes...\otimes M_{i_k}^N\otimes L_{j_1}^N\otimes...\otimes L_{j_l}^N$ converges in expectation in A_{k+l}-moments as N goes to infinity. Using Lemma 2.2, the family $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ converges in expectation in A-moments. We just proved the first part of the following theorem.

Theorem 7.1. — *Let us suppose that the families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free. The family $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ converges in A-expectation. Besides, the A-law of $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ only depends on the A-law of $(L_i)_{i\in I}$ and the A-law of $(M_j)_{j\in J}$.*

The key idea in order to prove the second part of this theorem is the following lemma. In this article, by monomial, we understand a product of powers of the formal variables with nonnegative integer exponents: the coefficient is equal to one.

Lemma 7.1. — *Let k be a positive integer, let $p\in A_k$ and let $\left(P_1,(i_1^1,...,i_{n_1}^1)\right)$,..., $\left(P_k,(i_1^k,...,i_{n_k}^k)\right)$ be elements of $\bigcup_{m\in\mathbb{N}}(\mathbb{C}\{X_1,...,X_m\}\times I^m)$ such that for any $j\in\{1,...,k\}$, P_j is a monomial.*

There exist a positive integer l, a partition $p'\in A_l$ and a l-tuple $(j_1,...,j_l)\in I^l$ such that for any positive integer N, any family of random matrices $(M_i)_{i\in I}$ of size N:

$$\mathbb{E}m_p[P_1(M_{i_1},...,M_{i_{n_1}}),...,P_k(M_{i_1},...,M_{i_{n_k}})]=\mathbb{E}m_{p'}[M_{j_1},...,M_{j_l}].$$

In order to illustrate the last lemma, let us give a simple example: for any positive integer N, for any matrices B and C of size N:

$$\mathbb{E}m_{id_2}[BCBB,BB]=\mathbb{E}m_{(1,2,3,4)(5,6)}[B,C,B,B,B,B].$$

Proof of Theorem 7.1. — It remains to prove that the A-law of $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ only depends on the A-laws of $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$. Let us denote by $(N_i)_{i\in I\sqcup J}$ the family such that for any $i\in I$, $N_i=L_i$ and for any $j\in J$, $N_j=M_j$. Let k be a positive integer, and let $\left(P_1,(i_1^1,...,i_{n_1}^1)\right)$,..., $\left(P_k,(i_1^k,...,i_{n_k}^k)\right)$ be elements of $\cup_{m\in\mathbb{N}}\mathbb{C}\{X_1,...,X_m\}\times(I\sqcup J)^m$, let $p\in A_k$. By multi-linearity, we can suppose that each polynomial is a monomial. Let us use Lemma 7.1, let l be an integer and let $(j_1,...,j_l)\in(I\sqcup J)^l$ be a l-tuple such that for any integer N, any family of random matrices $(C_i)_{i\in I\sqcup J}$ of size N:

$$\mathbb{E}m_p[P_1(C_{i_1},...,C_{i_{n_1}}),...,P_k(C_{i_1},...,C_{i_{n_k}})]=\mathbb{E}m_{p'}[C_{j_1},...,C_{j_l}].$$

Using Theorem 4.1:

$$\mathbb{E}m_p[P_1(N_{i_1},...,N_{i_{n_1}}),...,P_k(N_{i_1},...,N_{i_{n_k}})]=\sum_{p''\in[id_l,p']_{A_k}}\mathbb{E}\kappa_A^{p''}[N_{j_1},...,N_{j_l}].$$

By definition of A-freeness, for any $p''\in A_k$, $\mathbb{E}\kappa_A^{p''}[N_{j_1},...,N_{j_l}]$ only depends on the $\mathcal{R}_A$-functionals of $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$, and thus it only depends on the A-law of $(L_i)_{i\in I}$ and the A-law of $(M_j)_{j\in J}$. □

The dependance is actually polynomial.

7.1.2. The special case of G-invariant random matrices. — As seen already, the case where one considers G-invariant random matrices is quite special, and often easier. Let us choose a possibility for (G,A) as explained in Notation 3.1. Let us consider two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. Let us suppose that both families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ converge in A-expectation. Using Theorem 4.5, we already know that if $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are G-invariant then the two families converge in $\mathcal{P}$-expectation. The following theorem is a straightforward consequence of our discussions in Section 4.3.

Theorem 7.2. — *Let us suppose that the family $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ is G-invariant. The two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free if and only if they are asymptotically $\mathcal{P}$-free.*

We can prove an other version of this theorem which use the notion of asymptotically G-invariance.

Definition 7.2. — *Let us suppose that $(M_j)_{j\in J}$ converges in $\mathcal{P}$-expectation. We say that $(M_j)_{j\in J}$ is asymptotically G-invariant if for any integer k, for any $p\in\mathcal{P}_k$, for any $(j_1,...,j_k)\in J^k$,*

$$\mathbb{E}m_p\left(M_{j_1},...,M_{j_k}\right)=\lim_{N\to\infty}\mathbb{E}m_p\left(U^NM_{j_1}^N(U^N)^{-1},...,U^NM_{j_k}^N(U^N)^{-1}\right),$$

where, for any integer N, U^N is a Haar random variable on $G(N)$ which is independent from $(M_j^N)_{j\in J}$.

We could have also written this definition as a condition on the $\mathcal{P}$-law of $(M_i)_{i\in I}$, since we have the following lemma which is a consequence of Proposition 4.3, Lemma 4.3 and 4.4 of [**15**]. Recall the Definition 3.9 of [**15**] where we defined the notation Mb.

Lemma 7.2. — *Let us suppose that G is equal to O or U. The family $(M_j)_{j\in J}$ is asymptotically G-invariant if and only if for any positive integer k, for any $(j_1,...,j_k)\in J^k$ one the following equivalent condition holds:*

- *for any $p\in\mathcal{P}_k\setminus A_k$, $\mathbb{E}\kappa_{\mathcal{P}}^p(M_{j_1},...,M_{j_k})=0$,*
- *$(\mathbb{E}m_p(M_{j_1},...,M_{j_k}))_{p\in\mathcal{P}_k}$ is G-invariant (see Definition 4.10 of [**15**]),*
- *for any $p\in\mathcal{P}_k$, $\mathbb{E}m_{p^c}(M_{j_1},...,M_{j_k})=\delta_{p\in\overline{A_k}}\mathbb{E}m_{(\mathsf{Mb}(p))^c}(M_{j_1},...,M_{j_k})$.*

If $(M_j)_{j\in J}$ is asymptotically G-invariant then for any $p\in A_k$:

$$\mathbb{E}\kappa_{\mathcal{P}}^p(M_{j_1},...,M_{j_k})=\mathbb{E}\kappa_A^p(M_{j_1},...,M_{j_k}).$$

As noticed by C. Male, the following theorem is a generalization of the "rigidity of freeness" theorem of [**26**]. A consequence of the Theorem 7.3 is that if two families are $\mathcal{P}$-free, satisfy the asymptotic $\mathfrak{S}$-factorization property and and that one of the two families is U, O or B-asymptotically invariant, then the two families are free in the meaning of Voiculescu.

Theorem 7.3. — *Let us suppose that the family $(M_j)_{j\in J}$ is asymptotically G-invariant, if the families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically $\mathcal{P}$-free then they are asymptotically A-free.*

Proof. — Let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ satisfy the conditions of the theorem. It is enough to prove that for any positive integers k_1 and k_2, for any $b\in A_{k_1+k_2}$, any $(i_1,...,i_{k_2})\in I^{k_1}$ and $(j_1,...,j_{k_1})\in J^{k_2}$, $\mathbb{E}\kappa_A^b(L_{i_1}\otimes...\otimes L_{i_{k_1}}\otimes M_{j_1}\otimes...\otimes M_{j_{k_2}})$ is equal to:

$$\delta_{\exists(b_1,b_2)\in A_{k_1}\times A_{k_2},b=b_1\otimes b_2}\mathbb{E}\kappa_A^{b_1}(L_{i_1}\otimes...\otimes L_{i_{k_1}})\mathbb{E}\kappa_A^{b_2}(M_{j_1}\otimes...\otimes M_{j_{k_2}}).$$

Let us denote by k the integer k_1+k_2. In order to compute $\mathbb{E}\kappa_A^b(L_{i_1}\otimes...\otimes L_{i_{k_1}}\otimes M_{j_1}\otimes...\otimes M_{j_{k_2}})$, we need to compute $\int_{G(N)}g^{\otimes k}\mathbb{E}\left[L_{i_1}^N\otimes...\otimes L_{i_{k_1}}^N\otimes M_{j_1}^N\otimes...\otimes M_{j_{k_2}}^N\right](g^{-1})^{\otimes k}dg$ with N greater than $2k$. This is equal to:

$$\int_{G(N)}g^{\otimes k}\left(\frac{1}{N!}\sum_{S\in\mathfrak{S}(N)}S^{\otimes k}\mathbb{E}\Big[L_{i_1}^N\otimes...\otimes L_{i_{k_1}}^N\otimes M_{j_1}^N\otimes...\otimes M_{j_{k_2}}^N\Big](S^{-1})^{\otimes k}\right)(g^{-1})^{\otimes k}dg,$$

and thus this is equal to:

$$\sum_{p\in\mathcal{P}_k}\mathbb{E}\kappa_{\mathcal{P}}^p(L_{i_1}^N,...,L_{i_{k_1}}^N,M_{j_1}^N,...,M_{j_{k_2}}^N)\frac{1}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}}\int_{G(N)}g^{\otimes k}p(g^{-1})^{\otimes k}dg. \tag{11}$$

Using Proposition 4.3 of [**15**], for any $p\in\mathcal{P}_k$, the sequence:

$$\left(E_N^p=\frac{1}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}}\int_{G(N)}g^{\otimes k}p(g^{-1})^{\otimes k}dg\right)_{N\in\mathbb{N}}\in\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$$

converges: we can neglect in the sum 11 any element of the form $o(1)\frac{p}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}}$. Using the asymptotic $\mathcal{P}$-freeness of $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$, this implies that we can study the

following expression instead of the expression (11):

$$\sum_{(p_1,p_2)\in\mathcal{P}_{k_1}\times\mathcal{P}_{k_2}} \mathbb{E}\kappa_{\mathcal{P}}^{p_1}\left(L_{i_1}^N,...,L_{i_{k_1}}^N\right)\mathbb{E}\kappa_{\mathcal{P}}^{p_2}\left(M_{j_1}^N,...,M_{j_{k_2}}^N\right)$$
$$\frac{1}{N^{\mathsf{nc}(p_1\otimes p_2)-\mathsf{nc}((p_1\otimes p_2)\vee id)}}\int_{G(N)} g^{\otimes k}(p_1\otimes p_2)(g^{-1})^{\otimes k}dg,$$

and using the fact that the family $(M_j)_{j\in J}$ is asymptotically G-invariant, which implies that for any $p\in\mathcal{P}\setminus A$, $\mathbb{E}\kappa_{\mathcal{P}}^{p}(M_{i_1},...,M_{i_k})=0$ and for any $p\in A$, $\mathbb{E}\kappa_{A}^{p}(M_{i_1},...,M_{i_k})=\mathbb{E}\kappa_{\mathcal{P}}^{p}(M_{i_1},...,M_{i_k})$, we can study in fact:

$$\sum_{(p_1,p_2)\in\mathcal{P}_{k_1}\times A_{k_2}} \mathbb{E}\kappa_{\mathcal{P}}^{p_1}\left(L_{i_1}^N,...,L_{i_{k_1}}^N\right)\mathbb{E}\kappa_{A}^{p_2}\left(M_{j_1}^N,...,M_{j_{k_2}}^N\right)$$
$$\frac{1}{N^{\mathsf{nc}(p_1\otimes p_2)-\mathsf{nc}((p_1\otimes p_2)\vee id)}}\int_{G(N)} g^{\otimes k}(p_1\otimes p_2)(g^{-1})^{\otimes k}dg.$$

Yet, we remind that for any $(p_1,p_2)\in\mathcal{P}_{k_1}\times A_{k_2}$,

$$\begin{aligned}\int_{G(N)} g^{\otimes k}(p_1\otimes p_2)(g^{-1})^{\otimes k}dg &= \int_{G(N)}\left(g^{\otimes k_1}p_1(g^{-1})^{\otimes k_1}\right)\otimes\left(g^{\otimes k_2}p_2(g^{-1})^{\otimes k_2}\right)dg\\ &= \left(\int_{G(N)} g^{\otimes k_1}p_1(g^{-1})^{\otimes k_1}dg\right)\otimes p_2.\end{aligned}$$

Thus we have to study:

$$\sum_{p_1\in\mathcal{P}_{k_1}} \mathbb{E}\kappa_{\mathcal{P}}^{p_1}\left(L_{i_1}^N,...,L_{i_{k_1}}^N\right)\frac{\int_{G(N)} g^{\otimes k_1}p_1(g^{-1})^{\otimes k_1}dg}{N^{\mathsf{nc}(p_1)-\mathsf{nc}(p_1\vee id)}}$$
$$\otimes\sum_{p_2\in A_{k_2}}\mathbb{E}\kappa_{A}^{p_2}\left(M_{j_1}^N,...,M_{j_{k_2}}^N\right)\frac{p_2}{N^{\mathsf{nc}(p_2)-\mathsf{nc}(p_2\vee id)}}.$$

At last, let us remark that $\sum_{p_1\in\mathcal{P}_{k_1}}\mathbb{E}\kappa_{\mathcal{P}}^{p_1}\left(L_{i_1}^N,...,L_{i_{k_1}}^N\right)\frac{\int_{G(N)} g^{\otimes k_1+k_2}p_1(g^{-1})^{\otimes k_1}dg}{N^{\mathsf{nc}(p_1)-\mathsf{nc}(p_1\vee id)}}$ is in fact equal to:

$$\sum_{p_1\in A_k}\mathbb{E}\kappa_{A}^{p_1}\left(L_{i_1}^N,...,L_{i_{k_1}}^N\right)\frac{p_1}{N^{\mathsf{nc}(p_1)-\mathsf{nc}(p_1\vee id)}},$$

and thus we get that the limit, as N goes to infinity, of the coordinate numbers of $\int_{G(N)} g^{\otimes k}\mathbb{E}\left[L_{i_1}^N\otimes...\otimes L_{i_{k_1}}^N\otimes M_{j_1}^N\otimes...\otimes M_{j_{k_2}}^N\right](g^{-1})^{\otimes k}dg$ is equal to the limit, as N goes to infinity, of the coordinate numbers of:

$$\sum_{(p_1,p_2)\in A_{k_1}\times A_{k_2}}\mathbb{E}\kappa_{A}^{p_1}\left(L_{i_1}^N,...,L_{i_{k_1}}^N\right)\mathbb{E}\kappa_{A}^{p_2}\left(M_{j_1}^N,...,M_{j_{k_2}}^N\right)\frac{p_1\otimes p_2}{N^{\mathsf{nc}(p_1\otimes p_2)-\mathsf{nc}((p_1\otimes p_2)\vee id)}}.$$

This implies that for any $b\in A_{k_1+k_2}$,

$$\begin{aligned}&\mathbb{E}\kappa_A^b(L_{i_1}\otimes...\otimes L_{i_{k_1}}\otimes M_{j_1}\otimes...\otimes M_{j_{k_2}})\\ &\qquad=\delta_{\exists(b_1,b_2)\in A_{k_1}\times A_{k_2},b=b_1\otimes b_2}\mathbb{E}\kappa_A^{b_1}(L_{i_1}\otimes...\otimes L_{i_{k_1}})\mathbb{E}\kappa_A^{b_2}(M_{j_1}\otimes...\otimes M_{j_{k_2}}),\end{aligned}$$

and thus $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free. □

Using the Theorem 4.5 and the Definition 7.1 of asymptotic freeness, we can already see that a unitarily invariant family of random matrices which converges in expectation in $\mathfrak{S}$-moments is asymptotically $\mathfrak{S}$-free from its transpose family: this gives us the first example of families of random matrices which are $\mathfrak{S}$-free. This result was proved in [**28**] as a consequence of a more general result on second-order freeness ; this result is equivalent to the following theorem.

Theorem 7.4. — *Let us suppose that the family $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ is U-invariant, converges in expectation in $\mathfrak{S}$-moments and satisfies the asymptotic $\mathfrak{S}$-factorization property. Then the families $(L_i)_{i\in I}$ and $({}^tM_j)_{j\in J}$ are asymptotically $\mathfrak{S}$-free.*

Proof. — Since $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ is U-invariant, by Corollary 4.1, $(L_i)_{i\in I}\cup({}^tM_j)_{j\in J}$ converges in expectation in $\mathcal{P}$-moments. Let us prove that for any positive integer k and l, any $\sigma\in\mathfrak{S}_{k+l}$ and any $(i_1,...,i_k)\in I^k$ and any $(j_1,...,j_l)\in J^l$, $\mathbb{E}\kappa^\sigma_{\mathfrak{S}}(L_{i_1},...,L_{i_k},{}^tM_{j_1},...,{}^tM_{j_l})$ is equal to:

$$\delta_{\exists(\sigma_1,\sigma_2)\in\mathfrak{S}_k\times\mathfrak{S}_l|\sigma=\sigma_1\otimes\sigma_2}\mathbb{E}\kappa^{\sigma_1}_{\mathfrak{S}}(L_{i_1},...,L_{i_k})\mathbb{E}\kappa^{\sigma_2}_{\mathfrak{S}}({}^tM_{j_1},...,{}^tM_{j_l}).$$

It has to be noticed that the family $(L_i)_{i\in I}\cup({}^tM_j)_{j\in J}$ is O-invariant: using the point 3. of Theorem 4.5 in the case where $G=O$, we know that:

$$\mathbb{E}\kappa^\sigma_{\mathfrak{S}}(L_{i_1},...,L_{i_k},{}^tM_{j_1},...,{}^tM_{j_l})=\mathbb{E}\kappa^\sigma_{\mathcal{B}}(L_{i_1},...,L_{i_k},{}^tM_{j_1},...,{}^tM_{j_l})$$

Besides, using the definition of S_k given in Definition 2.5, the fact that the family $(L_{i_1},...,L_{i_k},{}^tM_{j_1},{}^tM_{j_l})$ is O-invariant, and using Lemma 4.3, we get:

$$\mathbb{E}\kappa^\sigma_{\mathcal{B}}(L_{i_1},...,L_{i_k},{}^tM_{j_1},...,{}^tM_{j_l})=\mathbb{E}\kappa^{S_k(\sigma)}_{\mathcal{B}}(L_{i_1},...,L_{i_k},M_{j_1},...,M_{j_l}).$$

Since $(L_{i_1},...,L_{i_k},M_{j_1},...,M_{j_l})$ is U-invariant, because of Theorem 4.5, we know that the right-hand-side is equal to zero if $S_k(\sigma)$ is not in $\mathfrak{S}_{k+l}$. Let us remark that $S_k(\sigma)$ is in $\mathfrak{S}_{k+l}$ if and only if there exists $(\sigma_1,\sigma_2)\in\mathfrak{S}_k\times\mathfrak{S}_l$ such that $\sigma=\sigma_1\otimes\sigma_2$. This proves that if σ can not be decomposed as $\sigma_1\otimes\sigma_2$ with $(\sigma_1,\sigma_2)\in\mathfrak{S}_k\times\mathfrak{S}_l$, $\mathbb{E}\kappa^\sigma_{\mathfrak{S}}(L_{i_1},...,L_{i_k},{}^tM_{j_1},...,{}^tM_{j_l})=0$.

It remains to consider the case when σ can be decomposed as $\sigma_1\otimes\sigma_2$ with $(\sigma_1,\sigma_2)\in\mathfrak{S}_k\times\mathfrak{S}_l$. In this case, using the asymptotic $\mathfrak{S}$-factorization property, we get:

$$\begin{aligned}\mathbb{E}\kappa^{\sigma_1\otimes\sigma_2}_{\mathfrak{S}}(L_{i_1},...,L_{i_k},{}^tM_{j_1},...,{}^tM_{j_l})&=\mathbb{E}\kappa^{\sigma_1\otimes{}^t\sigma_2}_{\mathfrak{S}}(L_{i_1},...,L_{i_k},M_{j_1},...,M_{j_l})\\&=\mathbb{E}\kappa^{\sigma_1}_{\mathfrak{S}}(L_{i_1},...,L_{i_k})\mathbb{E}\kappa^{{}^t\sigma_2}_{\mathfrak{S}}(M_{j_1},...,M_{j_l})\\&=\mathbb{E}\kappa^{\sigma_1}_{\mathfrak{S}}(L_{i_1},...,L_{i_k})\mathbb{E}\kappa^{\sigma_2}_{\mathfrak{S}}({}^tM_{j_1},...,{}^tM_{j_l}).\end{aligned}$$

This ends the proof. □

Let us remark that the proof has nothing special to do with random matrices and would work in an abstract setting. Also, we could have use a weaker condition on $(M_i)_{i\in I}$, namely the fact that $(M_i)_{i\in I}$ is asymptotically U-invariant.

7.1.3. Links between the different notion of asymptotic freeness. — We defined three notions of asymptotic freeness, namely the $\mathfrak{S}$, $\mathcal{B}$ and $\mathcal{P}$-asymptotic freeness. Besides, since the work of Voiculescu, a notion of asymptotic freeness was defined. In this section, we will explain the links between these notions using results that one can find in this paper, and we will explain the following diagram. In order to shorten the notations, we will ommit to specify that the notions of freeness that we consider in this section are "asymptotic" notions.

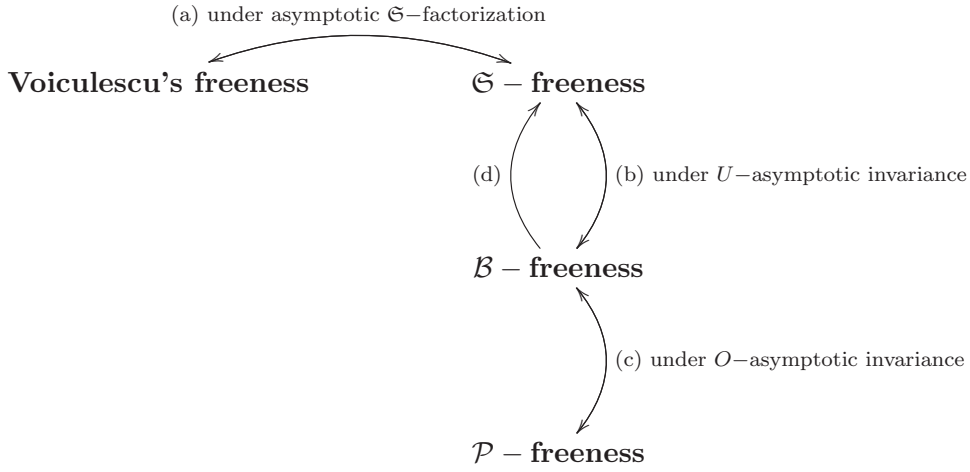

Diagram 1. The different notions of freeness.

Let us focus on the implications which are true. The equivalence denoted by (a) in Diagram 1 is a consequence of Proposition 4.3, and the fact that the freeness in free probability theory can be stated as a property of vanishing cumulants (Speicher's condition, Theorem 11.16 in [**29**]). The equivalences denoted by (b) and (c) in Diagram 1 are consequences of Theorems 7.3. The implication denoted by (d) is a direct consequence of Lemma 4.1.

Now, let us focus on the implications which are in general not true:

1. the $\mathcal{P}$-freeness does not imply the $\mathfrak{S}$-freeness. Let us considers two diagonal matrices which are independent and which have i.i.d. diagonal entries, then they are $\mathcal{P}$-free, but they are not $\mathfrak{S}$-free. Indeed, if they were, the limit of the eigenvalues distribution of their sum would be the free convolution of the limiting eigenvalues distributions of the two matrices, yet it is clear that it is the usual convolution of the limiting eigenvalues distributions of the two matrices. Since free convolution and usual convolution are not equivalent, one can conclude. More generaly, one can see the fact that $\mathcal{P}$-freeness does not imply the $\mathfrak{S}$-freeness as a consequence of Proposition 7.1.
2. The $\mathcal{P}$-freeness does not imply the $\mathcal{B}$-freeness: if it was the case, since $\mathcal{B}$-freeness implies $\mathfrak{S}$-freeness the first point would be true.
3. the $\mathfrak{S}$-freness does not implies the $\mathcal{B}$-freeness. By Theorem 7.4, if one considers $M \in L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which is U invariant and which converges in $\mathfrak{S}$-moments, then M and M^t are asymptotically $\mathfrak{S}$-free. It M and M^t were $\mathcal{B}$-free, this would imply that M is asymptotically $\mathfrak{S}$-free with itself: this is usually not true.

For the moment, we do not have a good argument which would explain why the asymptotic $\mathcal{B}$-freeness does not imply the $\mathcal{P}$-freeness. Besides, as explained in Remark 3.1, one can define other notions of asymptotic freeness, for exemple the $\mathcal{B}s$ and $\mathcal{H}$ notions of asymptotic freeness. It would be interesting to know the links between all of them.

Let us finish with a proposition which is totally inspired by a result of C. Male, namely the first point of Corollary 3.5 of [**26**]. The proof that we give here differs from the one of C. Male as it is based on the cumulants we introduced in this article and the new Theorem 7.8.

Proposition 7.1. — *Let* $\mathsf{0}_2$ *be the partition* $\{\{1,2,1',2'\}\}$. *Let* $\mathbf{M} = (M_1^N, M_2^N)_{N\in\mathbb{N}}$ *and* $\mathbf{L} = (L_1^N, L_2^N)_{N\in\mathbb{N}}$ *be two families of elements of* $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ *which converges in* $\mathcal{P}$*-expectation. Let us suppose that* $\mathbf{M}$ *and* $\mathbf{L}$ *are asymptotically* $\mathcal{P}$*-free. If*

$$\mathbb{E}\kappa_{\mathcal{P}}^{\mathsf{0}_2}(M_1 \otimes M_2)\mathbb{E}\kappa_{\mathcal{P}}^{\mathsf{0}_2}(L_1 \otimes L_2) \neq 0,$$

then $\mathbf{M}$ *and* $\mathbf{L}$ *are not asymptotically* $\mathfrak{S}$*-free.*

Proof. — Let $\mathbf{M} = (M_1^N, M_2^N)_{N\in\mathbb{N}}$ and $\mathbf{L} = (L_1^N, L_2^N)_{N\in\mathbb{N}}$ be two families of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which converge in $\mathcal{P}$-expectation and which are asymptotically $\mathcal{P}$-free. We will apply a theorem that we will prove in a more general setting, namely Theorem 7.8. It shows that:

$$\begin{aligned}\mathbb{E}m_{(1,2,3,4)}[M_1, L_1, M_2, L_2] &= \mathbb{E}m_{(1,2)}[M_1L_1, M_2L_2] \\ &= \sum_{p\in\mathcal{P}_k, p\in[id,(1,2)]_{\mathcal{P}_2}} \mathbb{E}\kappa_{\mathcal{P}}^{p}[M_1, M_2]\mathbb{E}m_{\iota_{p\circ(1,2)}}[L_1, L_2].\end{aligned}$$

Yet, $[id,(1,2)]_{\mathcal{P}_2} = \{id, \mathsf{0}_2, (1,2)\}$, thus $\mathbb{E}m_{(1,2,3,4)}[M_1, L_1, M_2, L_2]$ is equal to:

$$\mathbb{E}\kappa_{\mathcal{P}}^{id}[M_1, M_2]\mathbb{E}m_{(1,2)}[L_1, L_2] + \mathbb{E}\kappa_{\mathcal{P}}^{\mathsf{0}_2}[M_1, M_2]\mathbb{E}m_{\mathsf{0}_2}[L_1, L_2] + \mathbb{E}\kappa_{\mathcal{P}}^{(1,2)}[M_1, M_2]\mathbb{E}m_{id}[L_1, L_2].$$

Using the links between cumulants and moments, we can write again this expression as:

$$\mathbb{E}m_{id}[M_1, M_2]\mathbb{E}m_{(1,2)}[L_1, L_2] + \mathbb{E}m_{(1,2)}[M_1, M_2]\mathbb{E}m_{id}[L_1, L_2] - \mathbb{E}m_{id}[M_1, M_2]\mathbb{E}m_{id}[L_1, L_2] + \mathbb{E}\kappa_{\mathcal{P}}^{\mathsf{0}_2}[M_1, M_2]\mathbb{E}\kappa_{\mathcal{P}}^{\mathsf{0}_2}[L_1, L_2].$$

If $\mathbf{M}$ and $\mathbf{L}$ were asymptotically $\mathfrak{S}$-free, using the same arguments but using the set-geodesic in $\mathfrak{S}_k$ would lead to the following expression for $\mathbb{E}m_{(1,2,3,4)}[M_1, L_1, M_2, L_2]$:

$$\mathbb{E}m_{id}[M_1,M_2]\mathbb{E}m_{(1,2)}[L_1, L_2] + \mathbb{E}m_{(1,2)}[M_1,M_2]\mathbb{E}m_{id}[L_1, L_2] - \mathbb{E}m_{id}[M_1,M_2]\mathbb{E}m_{id}[L_1, L_2].$$

Thus, we see that if $\mathbb{E}\kappa_{\mathcal{P}}^{\mathsf{0}_2}(M_1, M_2)\mathbb{E}\kappa_{\mathcal{P}}^{\mathsf{0}_2}(L_1, L_2) \neq 0$, then $\mathbf{M}$ and $\mathbf{L}$ are not asymptotically $\mathfrak{S}$-free. □

Remark 7.1. — *Recall that in Remark 3.1, we defined for any positive integer* k, $\mathcal{H}_k$ *as the subset of* $\mathcal{P}_k$ *of partitions which have blocks of even size. Let us remark that in Proposition 7.1 one can replace the condition of* $\mathcal{P}$*-free by* $\mathcal{H}$*-free since* $\mathsf{0}_2 \in [id,(1,2)]_{\mathcal{H}_2}$.

Using this result, we will show in the article [**17**] that the $\mathcal{P}$-free Lévy process which is the limit of random walks on the symmetric group is not a $\mathfrak{S}$-free Lévy process: its increments are not $\mathfrak{S}$-free, and thus since the $\mathfrak{S}$-asymptotic factorization will hold, this is not a free Lévy process.

7.1.4. The $\mathcal{B}s$-asymptotic freeness implies $\mathfrak{S}$-asymptotic. — In Section 3.1, in Remak 3.1, we explained that the paper could be generalized to any of the easy orthogonal groups and the category of partitions associated to them. We also explained by we decided to focus on the three groups $U(N)$, $O(N)$, $\mathfrak{S}(N)$. Yet, we would like to give an example of the use of the other groups in order to explain that a deeper study of the other notions of asymptotic freeness, namely the $\mathcal{B}s$ and the $\mathcal{H}$-asymptotic freeness, could be interesting.

Lemma 7.3. — *Let k be a positive integer. Let $\sigma \in \mathfrak{S}_k$, then we have the equality:*

$$[id, \sigma]_{\mathfrak{S}_k} = [id, \sigma]_{\mathcal{B}s_k}$$

Proof. — The proof is similar to the proof of Lemma 3.9 of [**15**]. One has just to check that for any permutation $\sigma \in \mathfrak{S}_k$ and any $p \in \mathcal{B}s_k$ such that $d(id, p) = 1$, p is not in the set-geodesic $[id, \sigma]_{\mathcal{P}_k}$. □

A consequence of this is that for any family $(M_i)_{i\in I}$ of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which converges in $\mathcal{B}s$-expectation, for any positive integer k, for any $\sigma \in \mathfrak{S}_k$ and any $(i_1, ..., i_k) \in I^k$:

$$\mathbb{E}\kappa^{\sigma}_{\mathcal{B}s}\left[M_{i_1}, ..., M_{i_k}\right] = \mathbb{E}\kappa^{\sigma}_{\mathfrak{S}}\left[M_{i_1}, ..., M_{i_k}\right].$$

This fact implies the following theorem.

Theorem 7.5. — *Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which converge in $\mathcal{B}s$-expectation. Let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically $\mathcal{B}s$-free, then they are $\mathfrak{S}$-free. In particular, if each family satisfy the asymptotic $\mathfrak{S}$-factorization property, then $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are free in the meaning of Voiculescu.*

Using this theorem and Theorem 7.10, one gets the following theorem.

Theorem 7.6. — *Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which converge in $\mathcal{B}s$-expectation and which satisfy the asymptotic $\mathfrak{S}$-factorization. Let us suppose that L_i is B-invariant and let us suppose that for every positive integer N, the two families $\left(L_i^N\right)_{i\in I}$ and $(M_j^N)_{j\in J}$ are independent. Then the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically free in the meaning of Voiculescu.*

7.1.5. Definition of freeness using exclusive moments. — Let us give an other definition of asymptotically A-freeness. Recall Definition 3.13 of [**15**], where we defined for any positive integer $l \leq k$ and $p \in \mathcal{P}_k$, the left and right parts p_l^g and p_l^d.

Theorem 7.7. — *The families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free if and only if for any positive integer k, for any partition $p \in \mathcal{P}_k$, for any integer $l \in]0, ..., k[$, for any k-tuple $(B_i)_{i=1}^k$ such that $(B_i)_{i=1}^l$ is in $\mathcal{A}\left((L_i)_{i\in I}\right)$ and $(B_i)_{i=l+1}^k$ is in $\mathcal{A}\left((M_j)_{j\in J}\right)$, $\mathbb{E}m^A_{p^c}\left((B_i^N)_{i=1}^k\right)$ converges as N goes to infinity, and:*

$$\lim_{N\to\infty} \mathbb{E}m^A_{p^c}\left((B_i^N)_{i=1}^k\right) = \delta_{p_l^g \otimes p_l^d \sqsupset p} \mathbb{E}m^A_{(p_l^g)^c}\left[(B_i)_{i=1}^l\right] \mathbb{E}m^A_{(p_l^d)^c}\left[(B_i)_{i=l+1}^k\right].$$

For sake of clarity, we will say that the families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free in exclusive moments if the conditions on the exclusive moments hold. In the upcoming paper [**14**], the author and his coauthors explain the link between this definition of asymptotic A-freeness in exclusive moments and the definition of C. Male based on traffics and free-product of traffic defined in [**26**].

Proof of Theorem 7.7. — Let k be a positive integer, let l be an integer in $]0,...,k[$ and let $p \in \mathcal{P}_k$. Let $(B_i)_{i=1}^{l}$ be a l-tuple of elements of $\mathcal{A}((L_i)_{i\in I})$ and let $(B_i)_{i=l+1}^{k}$ be a $(k-l)$-tuple of elements of $\mathcal{A}((M_j)_{j\in J})$. Let us prove that:

$$\sum_{p_1'\in A_l, p_2'\in A_{k-l}, p_1'\otimes p_2'\sqsupset p} \mathbb{E}\kappa_A^{p_1'}\left((B_i)_{i=1}^{l}\right)\mathbb{E}\kappa_A^{p_2'}\left((B_i)_{i=l+1}^{k}\right) \tag{12}$$

$$= \delta_{p_l^g\otimes p_l^d\sqsupset p}\mathbb{E}m^A_{(p_l^g)^c}\left[(B_i)_{i=1}^{l}\right]\mathbb{E}m^A_{(p_l^d)^c}\left[(B_i)_{i=l+1}^{k}\right]. \tag{13}$$

Let us consider the left hand side: the Proposition 3.4 of [**15**] shows that this is also equal to:

$$\delta_{p_l^g\otimes p_l^d\sqsupset p}\sum_{p_1\in A_l, p_1\sqsupset p_l^g}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^{l}\right)\sum_{p_2\in A_{k-l}, p_2\sqsupset p_l^d}\mathbb{E}\kappa_A^{p_2}\left((B_i)_{i=l+1}^{k}\right),$$

and using Theorem 5.1, it is thus equal to:

$$\mathbb{E}m^A_{p^c}\left((B_i)_{i=1}^{k}\right) = \delta_{p_l^g\otimes p_l^d\sqsupset p}\mathbb{E}m^A_{(p_l^g)^c}\left[(B_i)_{i=1}^{l}\right]\mathbb{E}m^A_{(p_l^d)^c}\left[(B_i)_{i=l+1}^{k}\right].$$

Let us suppose that the families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free. Then the left hand side (12) is equal to:

$$\sum_{p'\in A_k, p'\sqsupset p}\mathbb{E}\kappa_A^{p'}\left((B_i)_{i=1}^{k}\right).$$

which, by Theorem 5.1 is equal to $\mathbb{E}m^A_{p^c}\left((B_i)_{i=1}^{k}\right)$. This implies that

$$\mathbb{E}m^A_{p^c}\left((B_i)_{i=1}^{k}\right) = \delta_{p_l^g\otimes p_l^d\sqsupset p}\mathbb{E}m^A_{(p_l^g)^c}\left[(B_i)_{i=1}^{l}\right]\mathbb{E}m^A_{(p_l^d)^c}\left[(B_i)_{i=l+1}^{k}\right],$$

and thus $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free in exclusive moments.

Now, let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free in exclusive moments. Then the right hand side (13) is equal to $\mathbb{E}m^A_{p^c}\left((B_i)_{i=1}^{k}\right)$. Thus:

$$\sum_{p_1'\in A_l, p_2'\in A_{k-l}, p_1'\otimes p_2'\sqsupset p}\mathbb{E}\kappa_A^{p_1'}\left((B_i)_{i=1}^{l}\right)\mathbb{E}\kappa_A^{p_2'}\left((B_i)_{i=l+1}^{k}\right) = \mathbb{E}m^A_{p^c}\left((B_i)_{i=1}^{k}\right).$$

This equality is valid for any partition $p \in \mathcal{P}_k$, yet for any $p \in \mathcal{P}_k$, we know that:

$$\sum_{p'\in A_k, p'\sqsupset p}\mathbb{E}\kappa_A^{p'}\left((B_i)_{i=1}^{k}\right) = \mathbb{E}m^A_{p^c}\left((B_i)_{i=1}^{k}\right).$$

By inversion of the matrix order or $\sqsupset$, we get that for any $p \in A_k$:

$$\mathbb{E}\kappa_A^{p}\left((B_i)_{i=1}^{k}\right) = \delta_{\exists(p_1,p_2)\in A_l\times A_{k-l}, p=p_1\otimes p_2}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^{l}\right)\mathbb{E}\kappa_A^{p_2}\left((B_i)_{i=l+1}^{k}\right).$$

This shows that the families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free. □

Remark 7.2. — *We proved Theorem 7.7 under the assumption that we were working with sequences of random matrices. Yet, it has to be noticed that we did use only a combinatorial argument: this implies that the proof would work exactly the same for general A-free non-commutative random variables.*

7.1.6. Factorization and freeness. — Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. Let us suppose that both families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ converge in A-expectation and satisfy the asymptotic A-factorization property.

Proposition 7.2. — *Let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free, then the asymptotic A-factorization property holds for $(L_i)_{i\in I}\cup(M_j)_{j\in J}$.*

Proof. — This is a direct consequence of Proposition 4.1 and the definition of asymptotic A-freeness. □

7.1.7. Sum and product of families of A-free elements. — In this new setting, we can generalize some well-known theorems about the sum and the multiplication of asymptotic free elements. Recall the Definitions 6.1 and 6.2 of [**15**] about the notion $\prec$ and the generalized Kreweras complement. Recall the notation $\mathfrak{F}_2(p)$ defined in Definition 2.11 in [**15**]. We need an other notation.

Notation 7.2. — *Let k be a positive integer. Let $(C_i)_{i=1}^k$ be a k-tuple of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converges in expectation in A-moments. Let K be a subset of $\mathcal{P}_k$, we set:*

$$\mathbb{E}\kappa_A^K\left[(C_i)_{i=1}^k\right]=\sum_{p\in K\cap A_k}\mathbb{E}\kappa_A^p\left[(C_i)_{i=1}^k\right].$$

Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$.

Theorem 7.8. — *Let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free. Let k be a positive integer. Let us consider $\left((B_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ and $\left((C_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ such that for any $i\in\{1,...,k\}$, $\left(B_i^N\right)_{N\in\mathbb{N}}\in\mathcal{A}\left((L_i)_{i\in I}\right)$ and $\left(C_i^N\right)_{N\in\mathbb{N}}\in\mathcal{A}\left((M_j)_{j\in J}\right)$.*

Then, for any $p\in A_k$:

$$\mathbb{E}\kappa_A^p\left[B_1+C_1,...,B_k+C_k\right]=\sum_{(p_1,p_2,I)\in\mathfrak{F}_2(p)}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i\in I}\right)\mathbb{E}\kappa_A^{p_2}\left((C_j)_{j\in\{1,...,k\}\setminus I}\right),$$

$$\mathbb{E}\kappa_A^p\left[B_1C_1,...,B_kC_k\right]=\sum_{p_1\in A_k,p_1\prec p}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{K_p(p_1)}\left((C_i)_{i=1}^k\right),$$

$$\mathbb{E}m_p\left[B_1C_1,...,B_kC_k\right]=\sum_{p_1\in A_k,p_1\in[id,p]_{A_k}}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}m_{t_{p_1}\circ p}\left((C_i)_{i=1}^k\right).$$

Proof. — Let k be a positive integer, let us consider $\left((B_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ and $\left((C_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ such that for any $i\in\{1,...,k\}$, $\left(B_i^N\right)_{N\in\mathbb{N}}\in\mathcal{A}\left((L_i)_{i\in I}\right)$, and $\left(C_i^N\right)_{N\in\mathbb{N}}\in\mathcal{A}\left((M_j)_{j\in J}\right)$. Let p be in A_k. Let us prove first that:

$$\mathbb{E}\kappa_A^p\left[B_1+C_1,...,B_k+C_k\right]=\sum_{(p_1,p_2,I)\in\mathfrak{F}_2(p)}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i\in I}\right)\mathbb{E}\kappa_A^{p_2}\left((C_j)_{j\in\{1,...,k\}\setminus I}\right).$$

By multilinearity, we have:

$$\mathbb{E}\kappa_A^p\left[B_1+C_1,...,B_k+C_k\right]=\sum_{(U^{(1)},...,U^{(k)})\in\{B,C\}^k}\mathbb{E}\kappa_A^p\left[U_1^{(1)},...,U_k^{(k)}\right].$$

Using the asymptotic A-freeness of $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$, we see that every term is equal to zero except for the choices of $\left(U^{(1)},...,U^{(k)}\right)\in\{B,C\}^k$ such that for any cycle of p and for any i and j in this cycle, $U^{(i)}=U^{(j)}$. Such a choice gives a decomposition of p in two partitions. Recall the notation $\mathfrak{F}_2(p)$ defined in Definition 2.11 in [**15**]. We get:

$$\mathbb{E}\kappa_A^p\left[B_1+C_1,...,B_k+C_k\right]=\sum_{(p_1,p_2,I)\in\mathfrak{F}_2(p)}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i\in I}\right)\mathbb{E}\kappa_A^{p_2}\left((C_j)_{j\in\{1,...,k\}\setminus I}\right).$$

Let us prove now that:

$$\mathbb{E}\kappa_A^p\left[B_1C_1,...,B_kC_k\right]=\sum_{p_1\in A_k,p_1\prec p}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{K_p(p_1)}\left((C_i)_{i=1}^k\right).$$

For this, we only have to prove that for any $p\in A_k$,

$$\sum_{p'\in[id,p]_{A_k}}\sum_{p_1\in A_k,p_1\prec p'}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{K_{p'}(p_1)}\left((C_i)_{i=1}^k\right)=\mathbb{E}m_p\left[B_1C_1,...,B_kC_k\right].$$

Let p be in A_k, we have:

$$\begin{aligned}\sum_{p'\in[id,p]_{A_k}}\sum_{p_1\in A_k,p_1\prec p'}&\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{K_{p'}(p_1)}\left((C_i)_{i=1}^k\right)\\&=\sum_{p'\in[id,p]_{A_k}}\sum_{p_1,p_2\in A_k,p_1\prec p_1\circ p_2=p'}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{p_2}\left((C_i)_{i=1}^k\right)\\&=\sum_{p_1,p_2\in A_k,p_1\prec p_1\circ p_2,p_1\circ p_2\in[id,p]_{A_k}}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{p_2}\left((C_i)_{i=1}^k\right).\end{aligned}$$

We recall that we proved in Proposition 8.1 of [**15**], that for any p,p_1,p_2 in A_k:

$$\delta_{p_1\otimes p_2\in[id,(p\otimes id_k)\tau]_{A_{2k}}}=\delta_{p_1\circ p_2\in[id,p]_{A_k}}\delta_{p_1\prec p_1\circ p_2}.$$

where $\tau=(1,k+1)(2,k+2)...(k,2k)$. Thus,

$$\begin{aligned}\sum_{p'\in[id,p]_{A_k}}\sum_{p_1\in A_k,p_1\prec p'}&\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{K_{p'}(p_1)}\left((C_i)_{i=1}^k\right)\\&=\sum_{p_1,p_2\in A_k,p_1\otimes p_2\in[id,(p\otimes id_k)\tau]_{A_{2k}}}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{p_2}\left((C_i)_{i=1}^k\right).\end{aligned}$$

Using the asymptotic A-freeness of $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$, we know that for any p_1 and p_2 in A_k,

$$\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{p_2}\left((C_i)_{i=1}^k\right)=\mathbb{E}\kappa_A^{p_1\otimes p_2}\left(B_1,...,B_k,C_1,...,C_k\right).$$

Besides, if $p' \in A_{2k}$ is not compatible with $(B_1, ..., B_k, C_1, ..., C_k)$, which means that there exist no partitions p_1 and p_2 in A_k such that $p_1 \otimes p_2 = p'$ then:

$$\mathbb{E}\kappa_A^{p'}(B_1, ..., B_k, C_1, ..., C_k) = 0.$$

This implies that:

$$\begin{aligned}
&\sum_{p' \in [id,p]_{A_k}} \sum_{p_1 \in A_k, p_1 \prec p'} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right) \mathbb{E}\kappa_A^{K_{p'}(p_1)}\left((C_i)_{i=1}^k\right) \\
&\qquad = \sum_{p_1, p_2 \in A_k, p_1 \otimes p_2 \in [id, (p \otimes id_k)\tau]_{A_{2k}}} \mathbb{E}\kappa_A^{p_1 \otimes p_2}(B_1, ..., B_k, C_1, ..., C_k) \\
&\qquad = \sum_{p' \in A_k, p' \in [id, (p \otimes id_k)\tau]_{A_{2k}}} \mathbb{E}\kappa_A^{p'}(B_1, ..., B_k, C_1, ..., C_k) \\
&\qquad = \mathbb{E}m_{(p \otimes id_k)\tau}(B_1, ..., B_k, C_1, ..., C_k) \\
&\qquad = \mathbb{E}m_p(B_1C_1, ..., B_kC_k).
\end{aligned}$$

Let us prove now that:

$$\mathbb{E}m_p[B_1C_1, ..., B_kC_k] = \sum_{p_1 \in A_k, p_1 \in [id,p]_{A_k}} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right) \mathbb{E}m_{t_{p_1} \circ p}\left((C_i)_{i=1}^k\right).$$

For this, we use the equality we just proved, the Definition 4.1 and the Proposition 8.3 of [**15**]. Indeed,

$$\begin{aligned}
\mathbb{E}m_p[B_1C_1, ..., B_kC_k] &= \sum_{p' \in A_k, p' \in [id,p]_{A_k}} \mathbb{E}\kappa_A^p[B_1C_1, ..., B_kC_k] \\
&= \sum_{p' \in A_k, p' \in [id,p]_{A_k}} \sum_{p_1 \in A_k, p_1 \prec p'} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right) \mathbb{E}\kappa_A^{K_{p'}(p_1)}\left((C_i)_{i=1}^k\right) \\
&= \sum_{p_1 \in A_k} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right) \sum_{p' \in A_k, p' \in [id,p]_{A_k}} \delta_{p_1 \prec p'} \mathbb{E}\kappa_A^{K_{p'}(p_1)}\left((C_i)_{i=1}^k\right) \\
&= \sum_{p_1 \in A_k} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right) \delta_{p_1 \in [id,p]_{A_k}} \mathbb{E}m_{t_{p_1} \circ p}\left((C_i)_{i=1}^k\right) \\
&= \sum_{p_1 \in A_k, p_1 \in [id,p]_{A_k}} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right) \mathbb{E}m_{t_{p_1} \circ p}\left((C_i)_{i=1}^k\right).
\end{aligned}$$

This finishes the proof. □

Let us remark that if p is the cycle $(1, ..., k)$, using the fact that the set-geodesic $([id, (1, ..., k)]_{\mathfrak{S}_k}, \leq)$ is isomorphic to the poset of non-crossing partitions of $\{1, ..., k\}$, we recover the classical formula, given in the Appendix of [**23**], which asserts that

$$\lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} Tr\left(B_1^N C_1^N ... B_k^N C_k^N\right)\right] = \sum_{\pi \in NC_k} \tau_\pi(B_1, ..., B_k) \kappa_{\pi^\vee}(C_1, ..., C_k),$$

where we used the notation NC_k for the non-crossing partitions of k elements, the notation τ_π for the non-commutative moments and $\pi^\vee$ for the Kreweras complement of a non-crossing partition π.

Recall the definition of the operations $\boxplus$ and $\boxtimes$ defined in Section 10.1 of [**15**]. As a consequence of Theorem 7.8, we get the following theorem.

Theorem 7.9. — *Let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free. Let us consider two sequences $(B_N)_{N\in\mathbb{N}}$ and $(C_N)_{N\in\mathbb{N}}$ such that $(B_N)_{N\in\mathbb{N}} \in \mathcal{A}\big((L_i)_{i\in I}\big)$, $(C_N)_{N\in\mathbb{N}} \in \mathcal{A}\big((M_j)_{j\in J}\big)$.*

The $\mathcal{R}_A$-transform of $(B_N + C_N)_{N\in\mathbb{N}}$ and $(B_N C_N)_{N\in\mathbb{N}}$ are given by:

$$\mathcal{R}_A\left[(B_N + C_N)_{N\in\mathbb{N}}\right] = \mathcal{R}_A\left[(B_N)_{N\in\mathbb{N}}\right] \boxplus \mathcal{R}_A\left[(C_N)_{N\in\mathbb{N}}\right],$$
$$\mathcal{R}_A\left[(B_N C_N)_{N\in\mathbb{N}}\right] = \mathcal{R}_A\left[(B_N)_{N\in\mathbb{N}}\right] \boxtimes \mathcal{R}_A\left[(C_N)_{N\in\mathbb{N}}\right].$$

Actually, by using the up-coming Theorem 7.10, one could apply directly the results from the article [**15**], instead of the proof given in this section. We will explain more after Therorem 7.10. Yet the resulting proof could not be applied abstractly to two families which are A-free: both families should be families in $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$. Instead, the proof that we gave for Theorem 7.8 is general enough to be applied in an abstract setting.

7.1.8. G-invariance and independence imply A-asymptotic freeness. — The following theorem generalizes the well-known fact that U and O invariance and independence implies freeness in the meaning of Voiculescu. Let us choose a possibility for (G, A) as explained in Notation 3.1. Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$. Let us suppose that the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ converge in A-expectation.

Theorem 7.10. — *Let us suppose that $(L_i)_{i\in I}$ is G-invariant and let us suppose that for every positive integer N, the two families $(L_i^N)_{i\in I}$ and $(M_j^N)_{j\in I}$ are independent. Then the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free.*

Actually, the compatibility condition and the compatible factorization property hold for any integer N big enough so that the N-dimensional A-cumulants are defined: for any positive integer k, for any partition $p \in A_k$, for any positive integer $N \geq 2k$, for any mixed k-tuple $\left(B_i^N\right)_{i=1}^k$ of elements of $\mathcal{A}\left((L_i^N)_{i\in I}\right) \cup \mathcal{A}\left((M_j^N)_{j\in J}\right)$,

1-Compatibility condition : *If p is not compatible with $(B_i^N)_{i=1}^k$:*

$$\mathbb{E}\kappa_A^p\left(B_1^N, ..., B_k^N\right) = 0.$$

2-Compatible factorization property : *If p is compatible with $(B_i^N)_{i=1}^k$:*

$$\mathbb{E}\kappa_A^p\left(B_1^N, ..., B_k^N\right) = \mathbb{E}\kappa_A^{p_{L_{(B_i)_{i=1}^k}}}\left(\left(B_i^N\right)_{i\in L_{(B_i)_{i=1}^k}}\right) \mathbb{E}\kappa_A^{p_{M_{(B_i)_{i=1}^k}}}\left(\left(B_i^N\right)_{i\in M_{(B_i)_{i=1}^k}}\right).$$

Proof. — Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which satisfy the hypotheses stated in Theorem 7.10. Let k be a positive integer, let p be in A_k, let $(B_i)_{i=1}^k$ be any mixed k-tuple of elements of $\mathcal{A}\left((L_i)_{i\in I}\right) \cup \mathcal{A}\left((M_j)_{j\in J}\right)$.

We have to show that $\mathbb{E}\kappa^p\left[B_1^N,...,B_k^N\right]$ converges as N goes to infinity, and we have to compute its limit. For this, we need to compute, for every positive integer N, $\int_{G(N)} g^{\otimes k}\mathbb{E}\left[B_1^N\otimes...\otimes B_k^N\right](g^{-1})^{\otimes k}dg$. Using the independence for each positive integer N of the families $\left\{\left(B_i^N\right)/i\in L_{(B_i)_{i=1}^k}\right\}$ and $\left\{\left(B_i^N\right)/i\in M_{(B_i)_{i=1}^k}\right\}$, one can cut the expectation $\mathbb{E}\left[B_1^N\otimes...\otimes B_k^N\right]$ in two. Thus, $\mathbb{E}\left[B_1^N\otimes...\otimes B_k^N\right]$ is equal to:

$$\mathbb{E}\left[\mathcal{I}_{L_{(B_i)_{i=1}^k}}\left[(B_i^N)_{i\in L_{(B_i)_{i=1}^k}}\otimes Id^{\otimes k-\#L_{(B_i)_{i=1}^k}}\right]\right]$$
$$\mathbb{E}\left[\mathcal{I}_{M_{(B_i)_{i=1}^k}}\left[(B_i^N)_{i\in M_{(B_i)_{i=1}^k}}\otimes Id^{\otimes k-\#M_{(B_i)_{i=1}^k}}\right]\right].$$

For example, if X and Y are two independent random matrices:

$$\mathbb{E}[X\otimes Y\otimes X]=\mathbb{E}[X\otimes Id\otimes X]\mathbb{E}[Id\otimes Y\otimes Id].$$

Since for any positive integer N, $(L_i^N)_{i\in I}$ is invariant by conjugation by $G(N)$, the family $(B_i^N)_{i\in L_{(B_i)_{i=1}^k}}$ is invariant by conjugation by $G(N)$. Thus

$$\mathbb{E}\left[\mathcal{I}_{L_{(B_i)_{i=1}^k}}\left[(B_i^N)_{i\in L_{(B_i)_{i=1}^k}}\otimes Id^{\otimes k-\#L_{(B_i)_{i=1}^k}}\right]\right]$$

commutes with the action of G on $\left(\mathbb{C}^N\right)^{\otimes k}$.

This implies that $\int_{G(N)} g^{\otimes k}\mathbb{E}\left[B_1^N\otimes...\otimes B_k^N\right](g^{-1})^{\otimes k}dg$ is equal to:

$$\int_{G(N)}(g')^{\otimes k}\mathbb{E}\left[\mathcal{I}_{L_{(B_i)_{i=1}^k}}\left[(B_i^N)_{i\in L_{(B_i)_{i=1}^k}}\otimes Id^{\otimes k-\#L_{(B_i)_{i=1}^k}}\right]\right](g'^{-1})^{\otimes k}dg'$$
$$\int_{G(N)} g^{\otimes k}\mathbb{E}\left[\mathcal{I}_{M_{(B_i)_{i=1}^k}}\left[(B_i^N)_{i\in M_{(B_i)_{i=1}^k}}\otimes Id^{\otimes k-\#M_{(B_i)_{i=1}^k}}\right]\right](g^{-1})^{\otimes k}dg.$$

It is now obvious that when one writes everything in terms of N-dimensional cumulants and partitions, the only partitions which will appear will be compatible with $(B_i)_{i=1}^k$, and if p is compatible, the compatible factorization property will hold for any $N\geq 2k$ if $A=\mathcal{P}$ and for any $N\geq k$ if $A=\mathfrak{S}$ or $A=\mathcal{B}$. □

Remark 7.3. — *Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families which satisfy the hypotheses of Theorem 7.10. Then for any integer N, the family $(L_i^N)_{i\in I}\cup((L_i^N)^*)_{i\in I}$ is also invariant by conjugation by $G(N)$, and the two families $(L_i^N)_{i\in I}\cup((L_i^N)^*)_{i\in I}$ and $(M_j^N)_{j\in J}$ are independent. Thus, if $(L_i)_{i\in I}\cup(L_i^*)_{i\in I}$ converges in A-expectation, then $(L_i^N)_{i\in I}\cup(L_i^*)_{i\in I}$ and $(M_j)_{j\in J}$ are also asymptotically A-free.*

Now we can recover the Theorem 7.8 as a simple consequence of the results in [**15**]. Indeed, let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families which are A-free. We have seen that the A-law of $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ only depends on the A-law of $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$. Thus it remains unchanged if we suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are G-invariant. Besides because of Theorem 7.10, we can always suppose that, for any positive integer N, $(L_i^N)_{i\in I}$ and $(M_j^N)_{j\in J}$ are independent. Let k be a positive integer. Let us consider $\left((B_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ and $\left((C_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ such that for any $i\in\{1,...,k\}$,

$(B_i^N)_{N\in\mathbb{N}} \in \mathcal{A}\big((L_i)_{i\in I}\big)$, $(C_i^N)_{N\in\mathbb{N}} \in \mathcal{A}\big((M_j)_{j\in J}\big)$. Then, for any $p \in A_k$, for any positive integer N,

$$\begin{aligned}\mathbb{E}\left[B_1^N C_1^N \otimes ... \otimes B_k^N C_k^N\right] &= \mathbb{E}\left[(B_1^N \otimes ... \otimes B_k^N)(C_1^N \otimes ... \otimes C_k^N)\right] \\ &= \mathbb{E}\left[B_1^N \otimes ... \otimes B_k^N\right]\mathbb{E}\left[C_1^N \otimes ... \otimes C_k^N\right].\end{aligned}$$

Recall Lemma 4.3 and Theorem 4.2: an application of the Equation (15) of Theorem 7.1 in [**15**] gives:

$$\mathbb{E}\kappa_A^p\left[B_1C_1, ..., B_kC_k\right] = \sum_{p_1\in A_k, p_1\prec p} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{K_p(p_1)}\left((C_i)_{i=1}^k\right).$$

Besides, for any positive integer N,

$$\mathbb{E}m_p\left[B_1^N C_1^N, ..., B_k^N C_k^N\right] = m_p\left[\mathbb{E}\left[B_1^N \otimes ... \otimes B_k^N\right]\mathbb{E}\left[C_1^N \otimes ... \otimes C_k^N\right]\right].$$

The Equation (16) of Theorem 7.1 in [**15**] gives:

$$\mathbb{E}m_p\left[B_1^N C_1^N, ..., B_k^N C_k^N\right] \sum_{p_1\in A_k, p_1\leq p} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}m_{t_{p_1}\circ p}\left((C_i)_{i=1}^k\right).$$

7.1.9. Strong asymptotically G-invariance. — Let $(L_i)_{i\in I} \cup (M_j)_{j\in J}$ be a family of elements in $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$. Let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ converge in $\mathcal{P}$-expectation. Seeing that in Theorem 7.3 we only need the asymptotic G-invariance of one of the two families, one can wonder if it is possible to state a version of Theorem 7.10 where one could replace the condition of G-invariance imposed for $(L_i)_{i\in I}$ by the condition that $(L_i)_{i\in I}$ is asymptotically G-invariant.

One can not hope that such generalization is true, at least with the definition of asymptotically G-invariant that we gave. Indeed, let us suppose that $(A, G) = (\mathcal{P}, \mathfrak{S})$: $(L_i)_{i\in I}$ is asymptotically $\mathfrak{S}$-invariant. Thus, if the generalization of Theorem 7.3 was true, this would imply that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically $\mathcal{P}$-free. Let us consider the special case where for any positive integer N, M_N is the diagonal matrix with $\lfloor\frac{N}{2}\rfloor$ zeros followed by $\lceil\frac{N}{2}\rceil$ ones and L_N is the diagonal matrix with $\lfloor\frac{N}{2}\rfloor$ ones follows by $\lceil\frac{N}{2}\rceil$ zeros. The sequences of random matrices $(M_N)_{N\in\mathbb{N}}$ and $(L_N)_{N\in\mathbb{N}}$ converges in $\mathcal{P}$-expectation, have the same $\mathcal{P}$-law, and it is easy to see that:

$$m_{id_1}(M) = m_{id_1}(L) = \frac{1}{2}.$$

Let us suppose that $(M_N)_{N\in\mathbb{N}}$ and $(L_N)_{N\in\mathbb{N}}$ are asymptotically $\mathcal{P}$-free. Let us denote by $\mathbb{1}_2$ the partition in $\mathcal{P}_2$ equal to $\{\{1, 2, 1', 2'\}\}$: one can see that $id_1 \otimes id_1 \sqsupset \mathbb{1}_2$. Thus, we should have the following equality:

$$m_{\mathbb{1}_2}(M, L) = m_{id_1}(M)m_{id_1}(L),$$

yet the left hand-side is equal to zero, and the right hand side is equal to $\frac{1}{4}$. Thus $(M_N)_{N\in\mathbb{N}}$ and $(L_N)_{N\in\mathbb{N}}$ are not $\mathcal{P}$-free.

Actually, one can state a generalization of Theorem 7.10 but one has to give a stronger definition of the asymptotic G-invariance.

Notation 7.3. — *Let N and k be two positive integers, let $\mathbb{I} = (i_1, ...i_k)$ be a k-tuple of elements of I. Let $\mathbb{J} = (j_1, j_{1'}, ..., j_k, j_{k'})$ be a $2k$-tuple of $\{1, ..., N\}^{2k}$. We denote by $L^N_{\mathbb{I},\mathbb{J}}$ the product:*

$$(L^N_{i_1})^{j_1}_{j_{1'}} ... (L^N_{i_k})^{j_k}_{j_{k'}}.$$

Definition 7.3. — *The family $(L_i)_{i\in I}$ is asymptotically strongly G-invariant if it is asymptotically G-invariant and for any positive integer k, for any k-tuple $\mathbb{I} = (i_1, ...i_k)$ of elements of I:*

$$\sup_{\mathbb{J},\mathbb{J}'\in\{1,...,N\}^{2k}|\mathsf{Ker}(\mathbb{J})=\mathsf{Ker}(\mathbb{J}')} N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)} \mid \mathbb{E}\left[M^N_{\mathbb{I},\mathbb{J}} - M^N_{\mathbb{I},\mathbb{J}'}\right] \mid \underset{N\to\infty}{\longrightarrow} 0.$$

Let us choose a possibility for (G, A) as explained in Notation 3.1. We recall that we supposed that the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ converge in $\mathcal{P}$-expectation. We can state a generalization of Theorem 7.10 which proof is similar to the proofs in [**26**].

Theorem 7.11. — *Let us suppose that $(L_i)_{i\in I}$ is asymptotically strongly G-invariant and that for every positive integer N, the two families $(L^N_i)_{i\in I}$ and $(M^N_j)_{j\in I}$ are independent. Then the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free.*

Proof. — Let us suppose that the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ satisfy the hypotheses stated in the theorem. Let us suppose that we proved that the two families are asymptotically $\mathcal{P}$-free then, by definition, $(L_i)_{i\in I}$ is asymptotically G-invariant, and thus, by Theorem 7.3, we get that the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free. It remains to prove that the two families are asymptotically $\mathcal{P}$-free. Let k be a positive integer, let l be an integer in $]0, ..., k[$ and let $p \in \mathcal{P}_k$. Let $(B_i)_{i=1}^l$ be a l-tuple of elements of $(L_i)_{i\in I}$ and let $(B_i)_{i=l+1}^k$ be a $(k-l)$-tuple of elements of $(M_j)_{j\in J}$. It is enough to prove that:

$$\lim_{N\to\infty} \mathbb{E}m_{p^c}\left((B^N_i)_{i=1}^k\right) = \delta_{p^g_l\otimes p^d_l \sqsupset p} \mathbb{E}m_{(p^g_l)^c}\left[(B_i)_{i=1}^l\right] \mathbb{E}m_{(p^d_l)^c}\left[(B_i)_{i=l+1}^k\right].$$

Let N be a positive integer, we have:

$$\mathbb{E}m_{p^c}\left((B^N_i)_{i=1}^k\right) = \frac{1}{N^{\mathsf{nc}(p\vee id)}} \sum_{\mathbb{J}\in\{1,...,N\}^{2k}|\mathsf{Ker}(\mathbb{J})=p} \mathbb{E}\left[B^N_{(1,...,k),\mathbb{J}}\right].$$

Any $\mathbb{J} \in \{1, ..., N\}^{2k}$ can be writen as the concatenation of $\mathbb{J}_1 \in \{1, ..., N\}^{2l}$ and $\mathbb{J}_2 \in \{1, ..., N\}^{2(k-l)}$, that we denote by $\mathbb{J}_1\mathbb{J}_2$.The right hand side can be written as:

$$\frac{1}{N^{\mathsf{nc}(p\vee id)}} \sum_{\mathbb{J}_1\in\{1,...,N\}^{2l},\mathbb{J}_2\in\{1,...,N\}^{2(k-l)}|\mathsf{Ker}(\mathbb{J}_1\mathbb{J}_2)=p} \mathbb{E}\left[B^N_{(1,...,l),\mathbb{J}_1}\right] \mathbb{E}\left[B^N_{(l+1,...,k),\mathbb{J}_2}\right].$$

Let $\mathbb{J}_1$ be in $\{1, ..., N\}^{2l}$. Since $(L_i)_{i\in I}$ is asymptotically G-invariant, using the same arguments as for Lemma 5.3, one gets:

$$\mathbb{E}\left[B^N_{(1,...,l),\mathbb{J}_1}\right] = N^{\mathsf{nc}(\mathsf{Ker}(\mathbb{J}_1)\vee id)-\mathsf{nc}(\mathsf{Ker}(\mathbb{J}_1))}\left[\mathbb{E}m_{\mathsf{Ker}(\mathbb{J}_1)^c}\left[B^N_1, ..., B^N_l\right] + o(1)\right]$$

where the $o(1)$ is uniform in $\mathbb{J}_1$. Let us remark also that if $\mathbb{J}_1 \in \{1,...,N\}^{2l}$ and $\mathbb{J}_2 \in \{1,...,N\}^{2(k-l)}$ satisfy that $\mathsf{Ker}(\mathbb{J}_1\mathbb{J}_2) = p$, then $\mathsf{Ker}(\mathbb{J}_1) = p_l^g$ and $\mathsf{Ker}(\mathbb{J}_2) = p_l^d$. This implies that $\mathbb{E}m_{p^c}\left((B_i^N)_{i=1}^k\right)$ is equal to:

$$\frac{1}{N^{\mathsf{nc}(p\vee id)}} \sum_{\mathbb{J}_1,\mathbb{J}_2|\mathsf{Ker}(\mathbb{J}_1\mathbb{J}_2)=p} N^{\mathsf{nc}(p_l^g\vee id)-\mathsf{nc}(p_l^g)} \left[\mathbb{E}m_{(p_l^g)^c}\left[B_1^N,...,B_l^N\right] + o(1)\right] \mathbb{E}\left[B^N_{(l+1,...,k),\mathbb{J}_2}\right]$$

or to:

$$N^{-\mathsf{nc}(p\vee id)+\mathsf{nc}(p_l^g\vee id)-\mathsf{nc}(p_l^g)} \sum_{\mathbb{J}_2|\mathsf{Ker}(\mathbb{J}_2)=p_l^d} \mathbb{E}\left[B^N_{(l+1,...,k),\mathbb{J}_2}\right] \sum_{\mathbb{J}_1|\mathsf{Ker}(\mathbb{J}_1)=p_l^g,\mathsf{Ker}(\mathbb{J}_1\mathbb{J}_2)=p} \left[\mathbb{E}m_{(p_l^g)^c}\left[B_1^N,...,B_l^N\right] + o(1)\right].$$

When $\mathbb{J}_2 \in \{1,...,N\}^{2(k-l)}$ is fixed then there exists exactly $N^{\mathsf{nc}(p)-\mathsf{nc}(p_l^d)}$ elements $\mathbb{J}_1$ in $\{1,...,N\}^{2l}$ such that $\mathsf{Ker}(\mathbb{J}_1) = p_l^g$ and $\mathsf{Ker}(\mathbb{J}_1\mathbb{J}_2) = p$, thus we can go on our calculations, and $\mathbb{E}m_{p^c}\left((B_i^N)_{i=1}^k\right)$ is equal to:

$$N^{-\mathsf{nc}(p\vee id)+\mathsf{nc}(p_l^g\vee id)-\mathsf{nc}(p_l^g)+\mathsf{nc}(p)-\mathsf{nc}(p_l^d)} \left[\mathbb{E}m_{(p_l^g)^c}\left[B_1^N,...,B_l^N\right] + o(1)\right] \sum_{\mathbb{J}_2|\mathsf{Ker}(\mathbb{J}_2)=p_l^g} \mathbb{E}\left[B^N_{(l+1,...,k),\mathbb{J}_2}\right]$$

or:

$$N^{(\mathsf{nc}(p)-\mathsf{nc}(p\vee id))-\left(\mathsf{nc}(p_l^g\otimes p_l^d)-\mathsf{nc}((p_l^g\otimes p_l^d)\vee id)\right)} \left[\mathbb{E}m_{(p_l^g)^c}\left[B_1^N,...,B_l^N\right] + o(1)\right] \mathbb{E}m_{(p_l^d)^c}\left[B_{l+1}^N,...,B_k^N\right].$$

Since $p_l^g \otimes p_l^d$ is finer than p, using Lemma 3.4 of [**15**], one gets that:

$$N^{(\mathsf{nc}(p)-\mathsf{nc}(p\vee id))-\left(\mathsf{nc}(p_l^g\otimes p_l^d)-\mathsf{nc}((p_l^g\otimes p_l^d)\vee id)\right)} \underset{N\to\infty}{\longrightarrow} \delta_{p_l^g\otimes p_l^d \sqsupset p},$$

this allows to conclude the proof of the theorem. □

Remark 7.4. — *As a corollary of the Theorem 7.11, we see that if one can prove that $(M_i)_{i\in I}$ and $(L_i)_{i\in I}$ converges in $\mathcal{P}$-moments, that the asymptotic $\mathfrak{S}$-factorization holds for both family and that one of the family is asymptotically strongly U or O-invariant, then $(M_i)_{i\in I}$ and $(L_i)_{i\in I}$ are asymptotically free in the meaning of Voiculescu.*

In particular, this is general enough to be used for the general Wigner real and compex matrices, where the matrices are symmetric or Hermitian, with the variance's entries equal to $1/N$ and with a condition of bounded moments on the entries of the matrix, but where one does not ask for an identically distribution property.

7.1.10. Matricial A-free Lévy processes. — Let $(M_t)_{t\geq 0}$ be a family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$. We suppose that for any positive integer N, $M_0^N = Id$. Let us suppose that the family $(M_t)_{t\geq 0}$ converges in A-expectation.

Definition 7.4. — *The family $(M_t)_{t\geq 0}$ is a matricial A-free additive Lévy process if both following assertions hold:*

A-freeness of the additive increments : *for any $0 \leq t < s$, the two families $\left(M_s^N - M_t^N\right)_{N\in\mathbb{N}}$ and $\left((M_u^N)_{u\leq t}\right)_{N\in\mathbb{N}}$ are asymptotically A-free,*

Translation invariance : *for any $0 \leq t < s$, $\left(M_s^N - M_t^N\right)_{N\in\mathbb{N}}$ has the same A-law as $\left(M_{s-t}^N\right)_{N\in\mathbb{N}}$.*

We also give a multiplicative version.

Definition 7.5. — *The family $(M_t)_{t\geq 0}$ is a matricial A-free multiplicative Lévy process if both following assertions hold:*

A-freeness of the multiplicative increments : *for any $0 \leq t < s$, the two families $\left(M_s^N (M_t^N)^{-1}\right)_{N\in\mathbb{N}}$ and $\left((M_u^N)_{u\leq t}\right)_{N\in\mathbb{N}}$ are asymptotically A-free,*

Translation invariance : *for any $0 \leq t < s$, $\left(M_s^N (M_t^N)^{-1}\right)_{N\in\mathbb{N}}$ has the same A-law as $\left(M_{t-s}^N\right)_{N\in\mathbb{N}}$*

7.2. Higher order. — We can define a notion of freeness up to higher order of fluctuations for some families of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. Let m be a non-negative integer. Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. Let us suppose that the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ converge in A-expectation up to order m of fluctuations.

Definition 7.6. — *The families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free up to order m of fluctuations if and only if for any positive integer k, for any partition $p \in \mathcal{P}_k$, for any mixed k-tuple $(B_i)_{i=1}^k$ of elements of $\mathcal{A}\left((L_i)_{i\in I}\right) \cup \mathcal{A}\left((M_j)_{j\in J}\right)$, the sequence $\left(B_1^N, ..., B_k^N\right)_N$ converges in expectation in A_k-cumulants up to order m of fluctuations as N goes to infinity and for any $i_0 \in \{0, ..., m\}$, the two conditions hold:*

Compatibility condition : *If p is not compatible with $(B_i)_{i=1}^k$:*

$$\lim_{N\to\infty} \mathbb{E}\kappa_{i_0,A}^p\left(B_1^N, ..., B_k^N\right) = 0.$$

Compatible factorization property : *If p is compatible with $(B_i)_{i=1}^k$:*

$$\lim_{N\to\infty} \mathbb{E}\kappa_{i_0,A}^p\Big(B_1^N, ..., B_k^N\Big) = \sum_{i_1=0}^{i_0} \mathbb{E}\kappa_{i_1,A}^{p_{|L_{(B_j)_{j=1}^k}}}\left((B_i)_{i\in L_{(B_j)_{j=1}^k}}\right) \mathbb{E}\kappa_{i_0-i_1,A}^{p_{|M_{(B_j)_{j=1}^k}}}\left((B_i)_{i\in M_{(B_j)_{j=1}^k}}\right).$$

Two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free if and only if they are asymptotically A-free up to order 0 of fluctuations. All the theorems we proved in the zero order case can be easily generalized for the notion of A-freeness up to order m of fluctuations.

Theorem 7.12. — *Let us suppose that the families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free up to order m of fluctuations. The family $(L_i)_{i\in I} \cup (M_j)_{j\in J}$ converges in A-expectation up to order m of fluctuations. Besides, the A-law of $(L_i)_{i\in I} \cup (M_j)_{j\in J}$ up to order m of fluctuations only depends on the A-law of $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ up to order m of fluctuations.*

The dependence is again polynomial. As for the zero order of fluctuations, we can state a theorem for higher order of fluctuations which is similar to Theorem 7.8.

Theorem 7.13. — *Let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free up to order m of fluctuations. Let k be a positive integer. Let us consider $\big((B_i^N)_{N\in\mathbb{N}}\big)_{i=1}^k$ and $\big((C_i^N)_{N\in\mathbb{N}}\big)_{i=1}^k$ such that for any $i\in\{1,...,k\}$, $\big(B_i^N\big)_{N\in\mathbb{N}}\in\mathcal{A}\big((L_i)_{i\in I}\big)$, $\big(C_i^N\big)_{N\in\mathbb{N}}\in\mathcal{A}\big((M_j)_{j\in J}\big)$. Then, for any $p\in A_k$ and any $i_0\in\{0,...,m\}$, one has:*

$$\mathbb{E}\kappa_{i_0,A}^{p}[B_1+C_1,...,B_k+C_k]=$$
$$\sum_{(p_1,p_2,I)\in\mathfrak{F}_2(p)}\sum_{i=0}^{i_0}\mathbb{E}\kappa_{i,A}^{p_1}(B_1,...,B_k)\mathbb{E}\kappa_{i_0-i,A}^{p_2}(C_1,...,C_k),$$

$$\mathbb{E}\kappa_{i_0,A}^{p}[B_1C_1,...,B_kC_k]=$$
$$\sum_{p',p''\in A_k,\eta(p',p'')\leq i_0,p'\circ p''=p}\ \sum_{i=0}^{i_0-\eta(p',p'')}\mathbb{E}\kappa_{i,A}^{p'}(B_1,...,B_k)\mathbb{E}\kappa_{i_0-\eta(p',p'')-i,A}^{p''}(C_1,...,C_k),$$

$$\mathbb{E}m_{p}^{i_0}[B_1C_1,...,B_kC_k]=$$
$$\sum_{p'\in A_k}\ \sum_{i,j,i+j+\mathsf{df}(p',p)=i_0}\mathbb{E}\kappa_{i,A}^{p'}(B_1,...,B_k)\mathbb{E}m_{i_{p'\circ p}}^{j}(C_1,...,C_k).$$

Proof. — One could do a combinatorial proof as we did for Theorem 7.8, using results such that Proposition 8.2 in [**15**]. In order to give a shorter proof, one can use the same arguments as explained after Remark 7.3. Using Theorem 7.12 and the up-coming Theorem 7.15, one can suppose that the families we consider satisfy the hypotheses of Theorem 7.15. In this case the theorem is only a consequence of a simple calculation for the first equality and a consequence of Theorem 9.2 of [**15**] for the others equalities. □

Recall the definition of the operations $\boxplus$ and $\boxtimes$ defined in Section 10.1 of [**15**]. As a consequence of this theorem, we have the following theorem.

Theorem 7.14. — *Let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free up to order m of fluctuations. Let us consider two sequences $(B_N)_{N\in\mathbb{N}}$ and $(C_N)_{N\in\mathbb{N}}$ such that $(B_N)_{N\in\mathbb{N}}\in\mathcal{A}\big((L_i)_{i\in I}\big)$, $(C_N)_{N\in\mathbb{N}}\in\mathcal{A}\big((M_j)_{j\in J}\big)$.*
The $\mathcal{R}_A^{(m)}$-transform of $(B_N+C_N)_{N\in\mathbb{N}}$ and $(B_NC_N)_{N\in\mathbb{N}}$ are given by:

$$\mathcal{R}_A^{(m)}\left[(B_N+C_N)_{N\in\mathbb{N}}\right]=\mathcal{R}_A^{(m)}\left[(B_N)_{N\in\mathbb{N}}\right]\boxplus\mathcal{R}_A^{(m)}\left[(C_N)_{N\in\mathbb{N}}\right],$$
$$\mathcal{R}_A^{(m)}\left[(B_NC_N)_{N\in\mathbb{N}}\right]=\mathcal{R}_A^{(m)}\left[(B_N)_{N\in\mathbb{N}}\right]\boxtimes\mathcal{R}_A^{(m)}\left[(C_N)_{N\in\mathbb{N}}\right].$$

Besides, the following theorem allows us to construct examples of families of sequences of matrices which are asymptotically A-free up to order m of fluctuations.

Theorem 7.15. — *Let us choose a possibility for (G,A) as explained in Notation 3.1. Let us suppose that the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ converge in A-expectation up to order m of fluctuations.*

Let us suppose that $(L_i)_{i\in I}$ is G-invariant. Besides let us suppose that for every integer N, the two families $(L_i^N)_{i\in I}$ and $(M_j^N)_{j\in I}$ are independent. Then the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free up to order m of fluctuations.

Actually, the compatibility condition and the compatible factorization property hold again for any integer N big enough so that the N-dimensional A-cumulants are defined. The Theorem 7.15 is general enough so that one can apply it to families of random matrices of the form $(L_i,(L_i)^*)_{i\in I}$ and $(M_j,(M_j)^*)_{j\in J}$ as long as one can prove that $(L_i,(L_i)^*)_{i\in I}$ and $(M_j,(M_j)^*)_{j\in J}$ converge in expectation in A-expectation.

Moreover, we can also define the notion of matricial A-free Lévy process up to order m of fluctuations by changing, in Definitions 7.4 and 7.5, the notion of asymptotic A-freeness, by the notion of asymptotic A-freeness up to order m of fluctuations, and using the A-law up to order m of fluctuations in order to define the translation invariance property.

8. The A-non-commutative central limit theorem

In this section, we prove a A-non-commutative central limit theorem. Recall the definition of $e_{\boxplus}^E$ defined in Definition 10.7 of [**15**]. Recall also the notation E_k, for $E\in\mathfrak{E}[A]$ and $k\in\mathbb{N}$, set in Section 10 of the same article.

Theorem 8.1. — *Let $(L_N)_{N\in\mathbb{N}}$ be an element of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converges in A-expectation and such that $(\mathcal{R}_A[L])_1=0$. For any integer $i\in\mathbb{N}$, let $(L_N^i)_{N\in\mathbb{N}}\in L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. Let us suppose that:*

- *for any $i\in\mathbb{N}$, $(L_N^i)_{N\in\mathbb{N}}$ has the same A-law as $(L_N)_{N\in\mathbb{N}}$,*
- *for any $i\in\mathbb{N}$, $(L_N^i)_{N\in\mathbb{N}}$ is asymptotically A-free from $\left((L_N^j)_{N\in\mathbb{N}}\right)_{j\neq i}$.*

Then, as n goes to infinity, the $\mathcal{R}_A$-transform of $\frac{1}{\sqrt{n}}\sum_{i=1}^n L^i$ converges to:

$$e_{\boxplus}^{(\mathcal{R}_A[L])_2}. \tag{14}$$

Proof. — Let us consider $L^i=(L_N^i)_{N\in\mathbb{N}}$ as in the theorem. By multi-linearity, for any integer $n,k\in\mathbb{N}$, and any $p\in A_k$:

$$\left(\mathcal{R}_A\left[\frac{1}{\sqrt{n}}\sum_{i=1}^n L^i\right]\right)_p=\frac{1}{n^{\frac{2}{k}}}\left(\mathcal{R}_A\left[\sum_{i=1}^n L^i\right]\right)_p.$$

Recall the notion of cycle defined in Definition 2.9 of [**15**] and the notation $\mathsf{C}(p)$ for the set of cycles of p. Using the asymptotic A-freeness property of the sequences $(L_N^i)_{N\in\mathbb{N}}$ and using the fact that for any positive integer i, $\left(L_N^i\right)$ has the same A-law as $(L_N)_{N\in\mathbb{N}}$, one has:

$$\left(\mathcal{R}_A\left[\frac{1}{\sqrt{n}}\sum_{i=1}^n L^i\right]\right)_p=\frac{1}{n^{\frac{k}{2}}}\sum_{f:\mathsf{C}(p)\to\{1,\ldots,n\}}\prod_{i=1}^n(\mathcal{R}_A[L])_{p_{\left(\bigcup_{c\in f^{-1}(i)}c\right)}}$$

where we recall that for any subset J, p_J is the extraction of p to J. We recall the convention $(\mathcal{R}_A[L])_\emptyset=1$. Let us consider a function $f:\mathsf{C}(p)\to\{1,\ldots,n\}$. This defines

a partition of the cycles of p by considering $\{f^{-1}(i), i \in \{1,...,n\}\}$. We say that $f \sim f'$ if the underlying partitions of $\mathsf{C}(p)$ are equal. It is obvious that, if $f \sim f'$, then:

$$\prod_{i=1}^{n} (\mathcal{R}_A[L])_{p\left(\bigcup_{c\in f^{-1}(i)} c\right)} = \prod_{i=1}^{n} (\mathcal{R}_A[L])_{p\left(\bigcup_{c\in f'^{-1}(i)} c\right)}.$$

Let π be a partition of $\mathsf{C}(p)$, we denote by $\mathsf{nc}(\pi)$ the number of blocks of π. By a counting argument, there exist approximatively $n^{\mathsf{nc}(\pi)}$ functions $f : C(p) \to \{1,...,n\}$ such that the underlying partition of f is π. Thus we can write:

$$\left(\mathcal{R}_A\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} L^i\right]\right)_p \simeq \frac{1}{n^{\frac{k}{2}}} \sum_{\pi \text{ partition of } \mathsf{C}(p)} n^{\mathsf{nc}(\pi)} \prod_{t \text{ block of } \pi} (\mathcal{R}_A[L])_{p_{\bigcup_{c\in t} c}}. \tag{15}$$

The condition $(\mathcal{R}[A])_1 = 0$ allows us to show that for any partition π of $\mathsf{C}(p)$, if $\mathsf{nc}(\pi) > \frac{k}{2}$ then $\prod_{t \text{ block of } \pi} (\mathcal{R}_A[L])_{p_{\bigcup_{c\in t} c}} = 0$. Indeed let us suppose that $\mathsf{nc}(\pi) > \frac{k}{2}$, then one block of π must be a cycle of size 1 of p. Thus in the considered product, one would have an element of the form $\mathcal{R}_A[L]_{p'}$ with $p' \in \mathcal{P}_1$: the product is then equal to zero. Thus, in the r.h.s., in the sum, one can impose the condition $\mathsf{nc}(\pi) \leq \frac{k}{2}$. This shows that $\left(\mathcal{R}_A\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} L^i\right]\right)_p$ converges when n goes to infinity.

Let us understand what the limit is: if $\mathsf{nc}(\pi) < \frac{k}{2}$, then the term associated with π disappears when n goes to infinity. Only remain the partitions π of $\mathsf{C}(p)$ such that $\mathsf{nc}(\pi) = \frac{k}{2}$ and such that none of the block of π is a cycle of length 1 of p. Thus, in order that the limit is not equal to zero, p must consist in some cycles of size 2 and an even number of cycles of length 1 ; the blocks of π are either two cycles of size 1 of p or a cycle of size 2 of p. This implies, by sending n to infinity in (15), that:

$$\lim_{n\to\infty} \mathcal{R}_A\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} L^i\right] = e_{\boxplus}^{(\mathcal{R}_A[L])_2},$$

which is the equality we wanted to prove. □

Using the intuition we will develop in Section 10.3, one can show easily the following theorem which proof will be given in a later version of the article.

Theorem 8.2. — *Let $E \in (\mathbb{C}[A_2])^{\mathfrak{S}}$ be an element of $\mathbb{C}[A_2]$ which is invariant by conjugation by $\mathfrak{S}_2$. This means that for any permutation $\sigma \in \mathfrak{S}_2$, for any positive integer N, $\sigma E \sigma^{-1} = E$ in $\mathbb{C}[A_k(N)]$.*

There exists a sequence of random matrices $(M_N)_{N\in\mathbb{N}}$, such that for any positive integer N, $M_N \in \mathcal{M}_N(\mathbb{C})$ and such that:

$$\mathcal{R}_A[M] = e_{\boxplus}^{E}.$$

Using Theorem 8.2, the Theorem 8.1 asserts now that the A-law of $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} L^i$ converges to the A-law of a sequence of random matrices which $\mathcal{R}_A$-transform is given by (14). The Theorem 8.1 is a generalization of the usual non-commutative central limit theorem, proved in [**33**], that one recovers when $A = \mathfrak{S}$ and when the $\mathfrak{S}$-factorization is assumed: this is due to the link between the usual $\mathcal{R}_u$-transform and the $\mathcal{R}_{\mathfrak{S}}$-transform

explained in Theorem 10.3 of [**15**]. This Theorem is also closely linked with the traffic free central limit theorem proved by C. Male by [**26**]. We will see that it also gives the usual central limit theorem given in Theorem 9.1. Yet, we warn the reader that Theorem 8.1 is not just an interpolation between the usual and the non-commutative central limit theorem since the space of possible limits is, in some sense, of dimension $\mathsf{dim}\left(\mathbb{C}^{\mathfrak{S}}[A_2]\right)$ equal to eleven if $A = \mathcal{P}$.

One could easily extend Theorem 8.1 to families of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$.

9. Classical probabilities

The goal of this section is to show that the classical probabilities are in some sense included in the theory we developed: for example, one can recover in a new way the usual cumulants as a particular case of $\mathcal{P}$-cumulants. This is partly based on easy and common knowledge, shared by researchers, that we found interesting to gather here.

9.1. Approximation of a measure by random matrices. — We consider the set $P^{\infty}(\mathbb{C})$ of probability measures such that for any integer $k \in \mathbb{N}$,

$$\int_{\mathbb{C}} |z|^k \, d\mu(z) < \infty.$$

Let μ be in $P^{\infty}(\mathbb{C})$. Let us see how to construct a sequence of random matrices $(M_N)_{N\in\mathbb{N}}$ in $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ such that for any positive integer n,

$$\mathbb{E}m_{(1,\dots,n)}\left[M_N^{\otimes n}\right] \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{C}} z^n d\mu(z).$$

If $(M_N)_{N\in\mathbb{N}}$ is such a sequence of random matrices, we say that $(M_N)_{N\in\mathbb{N}}$ is a weak approximation of μ. If, besides, the factorization property holds for $(M_N)_{N\in\mathbb{N}}$, we say that $(M_N)_{N\in\mathbb{N}}$ is a strong approximation of μ. In this last case, for any non-negative integer n and any $\sigma \in \mathfrak{S}_n$,

$$\mathbb{E}m_{\sigma}\left[M_N^{\otimes n}\right] \underset{N\to\infty}{\longrightarrow} \prod_{c\in\mathsf{C}(\sigma)} \int_{\mathbb{C}} z^{\#c} d\mu(z),$$

where, here, $\mathsf{C}(\sigma)$ is the set of cycles of σ as defined in the usual way. Let us give two simple approximations of μ.

Lemma 9.1. — *Let X be a random variable which law is μ. The sequence $(XId_N)_{N\in\mathbb{N}}$ is a weak approximation of μ. It will be called the natural weak approximation of μ.*

Proof. — Indeed, for any non negative integer n, $\mathbb{E}m_{(1,\dots,n)}[(XId_N)^{\otimes n}] = \mathbb{E}[X^n] = \int_{\mathbb{C}} z^n d\mu(z)$. Thus, the equality holds for any integer N and not only at the limit $N = \infty$. □

The last example does not provide a strong approximation of μ: for any non-negative integer n, for any $\sigma \in \mathfrak{S}_n$:

$$\mathbb{E}m_{\sigma}\left[(XId_N)^{\otimes n}\right] = \mathbb{E}[X^n] = \int_{\mathbb{C}} z^k d\mu(z) \neq \prod_{c\in\mathsf{C}(\sigma)} \int_{\mathbb{C}} z^{\#c} d\mu(z).$$

Thus $\mathcal{R}_{\mathfrak{S}}(XId)$ is not multiplicative. Yet, this is the simplest way to inject the classical probabilities in the theory we developed. For example, if μ and ν are two probability measures in $P^{\infty}(\mathbb{C})$, let us consider $(XId_N)_{N\in\mathbb{N}}$ and $(YId_N)_{N\in\mathbb{N}}$ two independent natural weak approximations associated respectively with μ and ν. Then $(XId_N)_{N\in\mathbb{N}}$ and $(YId_N)_{N\in\mathbb{N}}$ are asymptotically $\mathfrak{S}$-free, besides the sum $((X+Y)Id_N)_{N\in\mathbb{N}}$ is a natural weak approximation of the classical convolution $\mu * \nu$: this implies the usual central limit theorem for random variables with moments of any order.

Theorem 9.1. — *Let $(X_i)_{i=1}^{\infty}$ be a sequence of i.i.d. real random variables such that, for any integer k, $\mathbb{E}\left[|X_1|^k\right] < \infty$, and such that $\mathbb{E}[X_1] = 0$. The moments of $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i$ converge to the moments of a Gaussian of law $\mathcal{N}(0, \mathbb{E}\left[X_1^2\right])$.*

Proof. — Let $(X_i)_{i=1}^{\infty}$ be a sequence of i.i.d. random variables whose law admits moments up to any order and such that $\mathbb{E}[X_1] = 0$. Let us consider for any integer i, the sequences of matrices $(X_i Id_N)_{N=0}^{\infty}$. Using Theorem 7.10, these sequences of matrices are asymptotically $\mathfrak{S}$-free, as they are independent and each of them is invariant by conjugation by the unitary group. Applying the central limit theorem proved in Theorem 8.1, one has that the $\mathcal{R}$-transform of:

$$\left(\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right) Id_N\right)_{N\in\mathbb{N}}$$

converges when n goes to infinity. Yet, using the N-dimensional cumulants, we see that its $\mathcal{R}$-transform is equal to:

$$\left(\mathbb{E}\left[\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right)^k\right] id_k\right)_{k\in\mathbb{N}}.$$

Thus for any positive integer k, when n goes to infinity, $\mathbb{E}\left[\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right)^k\right]$ converges to:

$$\left(e_{\boxplus}^{\mathbb{E}[X_1^2]id_2}\right)_{id_k} = \sum_{\text{pairings of } \{1,\ldots,k\}} \mathbb{E}[X_1^2] = \mathbb{E}[X_1^2].\#\{\text{pairings of } \{1,\ldots,k\}\}.$$

Hence, for any positive integer k, $\mathbb{E}\left[\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right)^k\right]$ converges to the moments of a random variable of law $\mathcal{N}(0, \mathbb{E}\left[X_1^2\right])$. □

It would be interesting to have also a strong approximation of μ. For this, we need the following notation.

Notation 9.1. — *Let N be a positive integer, let $(\alpha_i)_{i=1}^{N}$ be N complex numbers. We denote by* $\mathsf{Diag}\left[(\alpha_i)_{i=1}^{N}\right]$ *the matrix $\sum_i \alpha_i E_i^i$, where for any $i \in \{1,\ldots,N\}$, E_i^i is the elementary matrix whose elements are equal to zero except in position (i,i) where it is equal to one.*

Proposition 9.1. — *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of random variables which are independent with same law μ. The sequence $\left(\mathsf{Diag}\left[(X_i)_{i=1}^N\right]\right)_{N\in\mathbb{N}}$ is a strong approximation of μ. It will be called the natural strong approximation of μ.*

Proof. — Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of random variables which are independent with same law μ. Let us denote by $(M_N)_{N\in\mathbb{N}}$ the sequence $\left(\mathsf{Diag}\left[(X_i)_{i=1}^N\right]\right)_{N\in\mathbb{N}}$. For any positive integer k, for any permutation $\sigma\in\mathfrak{S}_k$:

$$m^\sigma\left[M_N\otimes ...\otimes M_N\right]=\prod_{c\in\mathsf{C}(\sigma)}\frac{1}{N}Tr\left(M_N^{\#c}\right)=\prod_{c\in\mathsf{C}(\sigma)}\left[\frac{1}{N}\sum_{i=1}^N X_i^{\#c}\right].$$

By the law of large numbers $\mathbb{E}m_\sigma\left[M_N^{\otimes n}\right]\underset{N\to\infty}{\longrightarrow}\prod_{c\in\mathsf{C}(\sigma)}\int_{\mathbb{C}}z^{\#c}d\mu(z)$. This concludes the proof. □

9.2. A Schur-Weyl interpretation of classical cumulants. —

9.2.1. A simple duality. — Let N and k be positive integers. Let $\mathcal{D}_N(\mathbb{C})$ be the set of matrices $M\in\mathcal{M}_N(\mathbb{C})$ which are diagonal. We will study elements of $(\mathcal{D}_N(\mathbb{C}))^{\otimes k}$ which commute with the action of $\mathfrak{S}(N)$ on $\left(\mathbb{C}^N\right)^{\otimes k}$.

Definition 9.1. — *Let J be a finite subset, we denote by P_J the set of partitions of J. For any positive integer k, we simply denote by P_k the set of partitions of $\{1,...,k\}$.*

Definition 9.2. — *Let $\pi=\{\pi^1,...,\pi^r\}$ be in P_k. Let $\pi'^i=\{j'\in\{1',...,k'\},j\in\pi_i\}$. We define f_π in $\mathcal{P}_k$ equal to $\{\{\pi_i\cup\pi_i'\},i\in\{1,...,n\}\}$. We set:*

$$\mathcal{D}_k=\{f_\pi,\pi\in\mathsf{P}_k\}.$$

One example is given in Figure 1.

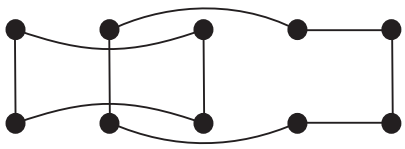

FIGURE 1. Partition f_π with $\pi=\{\{1,3\},\{2,4,5\}\}$.

One of the main feature of this new set is that for any f in $\mathcal{D}_k$, any f' which is coarser than f is in $\mathcal{D}_k$; besides f_π is coarser than $f_{\pi'}$ if and only if π is coarser than π'. Let us recall the notion of exclusive basis defined in Definition 2.16 of [**15**]. From the discussion we just had, we obtain the following lemma.

Lemma 9.2. — *For any partition π of $\{1,...,k\}$,*

$$f_\pi=\sum_{\pi'\ coarser\ than\ \pi}(f_{\pi'})^c.$$

A consequence is that for any $\pi\in\mathsf{P}_k$, $(f_{\pi'})^c\in\mathbb{C}[\mathcal{D}_k]$.

In the following, we denote by $f_{\pi'}^c$ the element $(f_{\pi'})^c$. Using this lemma, it is easy to prove the following result.

Lemma 9.3. — *Let E be an element of $\mathcal{D}_N(\mathbb{C})^{\otimes k}$ seen as an endomorphism of $\left(\mathbb{C}^N\right)^{\otimes k}$. Let us suppose that E commutes with the action $\rho^k_{\mathfrak{S}(N)}$ of $\mathfrak{S}(N)$ on $\left(\mathbb{C}^N\right)^{\otimes k}$. Then:*

$$E \in \rho_N^{\mathcal{P}_k}\left[\mathbb{C}\left[\mathcal{D}_k\right]\right].$$

Besides, if $N \geq k$, the restriction of $\rho_N^{\mathcal{P}_k}$ to $\mathbb{C}\left[\mathcal{D}_k\right]$ is a bijection.

Proof. — Using the Lemma 9.2, it is enough to show that E can be written as a linear combination of elements of the form $\rho_N^{\mathcal{P}_k}(f_\pi^c)$ with π being any partition of $\{1,...,k\}$. Any element of $\mathcal{D}_N(\mathbb{C})^{\otimes k}$ is a linear combination of elements of the form $E_{i_1}^{i_1} \otimes ... \otimes E_{i_k}^{i_k}$. Let $E \in \mathcal{D}_N(\mathbb{C})^{\otimes k}$:

$$E = \sum_{(i_1,...,i_k)\in\{1,...,N\}^k} c_{i_1,...,i_k} E_{i_1}^{i_1} \otimes ... \otimes E_{i_k}^{i_k}.$$

Any choice $(i_1,...,i_k) \in \{1,...,N\}^k$ defines a partition of $\{1,...,k\}$ denoted also by $\mathsf{Ker}((i_1,...,i_k))$ which is the unique partition of $\{1,...,k\}$ such that two elements u and v of $\{1,...,k\}$ are in the same block if and only if $i_u = i_v$.

If E commutes with the action $\rho^k_{\mathfrak{S}(N)}$ of $\mathfrak{S}(N)$ on $\left(\mathbb{C}^N\right)^{\otimes k}$, then $c_{i_1,...,i_k}$ only depends on the partition of k which is induced by $(i_1,...,i_k)$. Thus:

$$\begin{aligned} E &= \sum_{\pi\in\mathsf{P}_k} c_\pi \sum_{(i_1,...,i_k)|\mathsf{Ker}((i_1,...,i_k))=\pi} E_{i_1}^{i_1} \otimes ... \otimes E_{i_k}^{i_k} \\ &= \sum_{\pi\in\mathsf{P}_k} c_\pi \rho_N^{\mathcal{P}_k}\left[f_\pi^c\right]. \end{aligned}$$

Let us suppose that $N \geq k$, let $(c_\pi)_{\pi\in\mathsf{P}_k} \in \mathbb{C}^{\mathsf{P}_k}$ such that $\sum_{\pi\in\mathsf{P}_k} c_\pi \rho_N^{\mathcal{P}_k}\left[f_\pi^c\right] = 0$. Let $\pi_0 \in \mathsf{P}_k$, and let us consider $(i_1,...,i_k) \in \{1,...,N\}^k$ such that $\mathsf{Ker}((i_1,...,i_k)) = \pi_0$. Such k-tuple exists because $N \geq k$. Then:

$$0 = \left(\sum_{\pi\in\mathsf{P}_k} c_\pi \rho_N^{\mathcal{P}_k}\left[f_\pi^c\right]\right)(e_{i_1} \otimes ... \otimes e_{i_k}) = c_{\pi_0}\left(e_{i_1} \otimes ... \otimes e_{i_k}\right),$$

and thus $c_\pi = 0$ for any $\pi \in \mathsf{P}_k$. □

On the set $\mathcal{D}_k$, the geodesic order becomes much simpler than on $\mathcal{P}_k$.

Proposition 9.2. — *Let π and π' be in P_k. The following assertions are equivalent:*

- *$f_{\pi'} \in [id, f_\pi]_{\mathcal{P}_k}$,*
- *f_π is coarser than $f_{\pi'}$,*
- *π is coarser than π'.*

As a consequence, the ordered set $(\mathcal{D}_k, \leq)$ is isomorphic to the set P_k endowed with the order that we will also denote by $\leq$ which is defined by the fact that for any π and π' in P_k, $\pi' \leq \pi$ if and only if π' is finer than π

Proof. — Let π and π' be in P_k. We have already noticed that f_π is coarser than $f_{\pi'}$ if and only if π is coarser than π'. Besides it is easy to see that $\mathsf{Gl}_c(f_\pi) = f_\pi$ and $f_{\pi'}$ is always in $\mathsf{Sp}(f_\pi \vee f_{\pi'})$. Using Theorem 3.2 of [**15**], we see that $f_{\pi'} \in [id, f_\pi]_{\mathcal{P}_k}$ if and only if $f_\pi \vee f_{\pi'} = f_\pi$, thus if and only if f_π is coarser than $f_{\pi'}$. □

We will give a kind of Schur-Weyl duality interpretation of classical cumulants. For any $\pi \in \mathsf{P}_k$, let us denote by $\mathsf{C}(\pi)$ the blocks of π. Let us recall the classical notion of cumulants.

Definition 9.3. — *Let $m = (m_I)_{I\subset\{1,...,k\}}$ be a family of complex numbers. The sequence of cumulants of $(\kappa_I)_{I\subset\{1,...,k\}}$ associated with m is the only sequence of complex numbers such that for any $J \subset \{1, ..., k\}$:*

$$m_J = \sum_{\pi\in\mathsf{P}_J} \prod_{c\in\mathsf{C}(\pi)} \kappa_c.$$

In classical probabilities, for any $I \subset \{1, ..., k\}$, we consider m_I equal to $\mathbb{E}\left[\prod_{i\in I} X_i\right]$ where $(X_i)_1^k$ is a family of random variables which has mixed moments of any order. In this case, we will denote by $\kappa_J\left((X_i)_{i=1}^k\right)$ the cumulant κ_J associated with $\left(\mathbb{E}\left[\prod_{i\in I} X_i\right]\right)_{I\subset\{1,...,k\}}$. If $J = \{1, ..., k\}$, we also denote the cumulant $\kappa_J\left((X_i)_{i=1}^k\right)$ by $\kappa_k\left((X_i)_{i=1}^k\right)$. In our framework, we prefer to work with partitions and not with sets: this leads us to the following lemma.

Lemma 9.4. — *Let $m = (m_I)_{I\subset\{1,...,k\}}$ be a family of complex numbers. For any $\pi \in \mathsf{P}_k$ we define $m_\pi = \prod_{c\in\mathsf{C}(\pi)} m_c$. Let $(\eta_\pi)_{\pi\in\mathsf{P}_k}$ be a family of complex numbers such that $m_\pi = \sum_{\pi'\leq\pi} \eta_{\pi'}$. Then for any $\pi \in \mathsf{P}_k$,*

$$\eta_\pi = \prod_{c\in\mathsf{C}(\pi)} \kappa_c,$$

where $(\kappa_I)_{I\subset\{1,...,k\}}$ is the family of cumulants associated with m.

Proof. — The proof relies on the fact that there exists a unique family $(\eta_\pi)_{\pi\in\mathsf{P}_k}$ such that for any $\pi \in \mathsf{P}_k$, $m_\pi = \sum_{\pi'\leq\pi} \eta_{\pi'}$. If $(\kappa_I)_{I\subset\{1,...,k\}}$ is the family of cumulants associated with m, then $\left(\prod_{c\in\mathsf{C}(\pi)} \kappa_c\right)_{\pi\in\mathsf{P}_k}$ satisfies these equalities. □

The Schur-Weyl duality interpretation of classical cumulants is given by the following theorem.

Theorem 9.2. — *Let $(X_1, ..., X_k)$ be a k-tuple of random variables. Let $l \geq k$, and let us consider $(X_1^n, ..., X_k^n)_{n\in\{1,...,l\}}$ a family of l i.i.d. k-tuple of random variables which have the same law as $(X_1, ..., X_k)$. Let us consider, for any $i \in \{1, ..., k\}$:*

$$M_i = \mathsf{Diag}\left[(X_i^n)_{n=1}^l\right].$$

We have the following formula:

$$\kappa_k(X_1, ..., X_k) = \mathbb{E}\kappa^{f_{\{\{1,...,k\}\}}}\left[M_1, ..., M_k\right].$$

Proof. — The endomorphism $\mathbb{E}\left[\bigotimes_{i=1}^{k} M_i\right]$ is in $(\mathcal{D}_N)^{\otimes k}$ and commutes with the action $\rho^k_{\mathfrak{S}(N)}$ of $\mathfrak{S}(N)$ on $\left(\mathbb{C}^N\right)^{\otimes k}$: by Lemma 9.3, it belongs to $\rho_N^{\mathcal{P}_k}\left[\mathbb{C}\left[\mathcal{D}_k\right]\right]$ and, since $l \geq k$, the numbers $\mathbb{E}\kappa^{f_{\{\{1,\ldots,k\}\}}}\left[M_1, ..., M_k\right]$ are well defined.

Following the calculation in the proof of Lemma 9.3, and using the independence of the $(X_1^n, ..., X_k^n)_{n\in\{1,\ldots,k\}}$, one gets:

$$\mathbb{E}\left[\bigotimes_{i=1}^{k} M_i\right] = \sum_{\pi\in\mathsf{P}_k}\left(\prod_{c\in\mathsf{C}(\pi)}\mathbb{E}\left[\prod_{i\in c} X_i\right]\right) f_\pi^c.$$

Using the definition of finite-dimensional cumulants, Lemma 9.3, Lemma 9.2 and Proposition 9.2, one gets:

$$\begin{aligned}\mathbb{E}\left[\bigotimes_{i=1}^{k} M_i\right] &= \sum_{\pi\in\mathsf{P}_k}\mathbb{E}\kappa^{f_\pi}\left[M_1, ..., M_k\right] f_\pi \\ &= \sum_{\pi\in\mathsf{P}_k}\mathbb{E}\kappa^{f_\pi}\left[M_1, ..., M_k\right]\left(\sum_{\pi\leq\pi'} f_{\pi'}^c\right) \\ &= \sum_{\pi\in\mathsf{P}_k}\left(\sum_{\pi'\leq\pi}\mathbb{E}\kappa^{f_{\pi'}}\left[M_1, ..., M_k\right]\right) f_\pi^c.\end{aligned}$$

Using the second part of Lemma 9.3, we obtain the fact that for any $\pi \in \mathsf{P}_k$,

$$\sum_{\pi'\leq\pi}\mathbb{E}\kappa^{f_{\pi'}}\left[M_1, ..., M_k\right] = \prod_{c\in\mathsf{C}(\pi)}\mathbb{E}\left[\prod_{i\in\pi} X_i\right].$$

An application of Lemma 9.4 and considering the partition $\pi = \{\{1, ..., k\}\}$ allow us to conclude. □

In fact from the last proof, using the same notations, one has the more general formula:

$$\mathbb{E}\kappa^{f_\pi}[M_1, ..., M_k] = \prod_{c\in\mathsf{c}(\pi)}\kappa_{\#c}\left((X_i)_{i\in c}\right).$$

The theorem 9.2 shows that one could be able to study the probabilities fluctuations of random matrices in the framework we developed. Let us remark that one can recover the classical central limit theorem using Theorem 9.2.

An other proof of Theorem 9.1. — Let $(X_i)_{i=1}^{\infty}$ be a sequence of i.i.d. random variables whose law admits moments up to any order and such that $\mathbb{E}[X_1] = 0$. Let k be a positive integer. We have to understand $\kappa_k\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i, ..., \frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\right)$ when n goes to infinity. Let us consider $\left(Y_i^l\right)_{i,l\in\mathbb{N}}$ a family of independent identically distributed random variables which have the same law as X_1. Let us consider for any positive integers i and N:

$$M_i^N = \mathsf{Diag}\left[\left(Y_i^l\right)_{l=1}^N\right].$$

Using the $\mathfrak{S}$-invariance, the independence property and Theorem 7.10, for any $i \geq 1$, $\left(M_i^N\right)_{N\in\mathbb{N}}$ is asymptotically $\mathcal{P}$-free from $\left((M_j^N)_{j\neq i}\right)_{N\in\mathbb{N}}$ and for any positive integer n, $\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n M_i^N\right)_{N\in\mathbb{N}}$ is a strong approximation of the law of $\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i$. Let us apply the Theorem 8.1: using the fact that $\mathbb{E}[X_1] = 0$, the $\mathcal{R}$-transform of $\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n M_i^N\right)_{N\in\mathbb{N}}$ converges, when n goes to infinity, to $e_{\boxplus}^{\mathbb{E}[X_1^2]f_{\{\{1,2\}\}}}$. Yet, using Theorem 9.2, $\kappa_k\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i, ..., \frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\right)$ is equal to:

$$\lim_{N\to\infty} \mathbb{E}\kappa^{f_{\{\{1,...,k\}\}}}\left[\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n M_i^N\right)^{\otimes k}\right] = \left[\mathcal{R}_{\mathcal{P}}\left(\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n M_i^N\right)_{N\in\mathbb{N}}\right)\right]_{f_{\{\{1,...,k\}\}}}.$$

Thus $\kappa_k\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i, ..., \frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\right)$ converges, as n goes to infinity, to:

$$\left(e_{\boxplus}^{\mathbb{E}[X_1^2]f_{\{\{1,2\}\}}}\right)_{f_{\{\{1,...,k\}\}}}$$

which is equal to 0 if $k = 1$, $\mathbb{E}[X_1^2]$ if $k = 2$ and 0 if $k \geq 3$. □

10. Lévy processes

10.1. Generalities about Lévy processes. — Let us recall the notion of Lévy processes.

Definition 10.1. — *Let E be a topological group. Let $(X_t)_{t\geq 0}$ be a cadlàg process in E which is stochastically continuous such that X_0 is the neutral element of E. The process $(X_t)_{t\geq 0}$ is a Lévy process if for any $0 < t < s$, $X_s X_t^{-1}$ is independent of $(X_u)_{u\leq t}$ and $X_s X_t^{-1}$ has the same law as X_{s-t}.*

Let us remark that we consider actually right Lévy processes as defined by M. Liao in [**25**]. From now on, we will only consider groups of matrices. If E is a subgroup of $(\mathcal{M}_N(\mathbb{C}), +)$, the Lévy processes are called additive Lévy processes. If E is a subgroup of $GL(N)$, the Lévy processes are called multiplicative Lévy processes.

Recall that the notation G stands either for O, U, or $\mathfrak{S}$. Following Definition 4.6, we define the notion of G-invariant Lévy processes. Let N be a positive integer.

Definition 10.2. — *Let $(X_t)_{t\geq 0}$ be an additive or multiplicative Lévy process in $\mathcal{M}_N(\mathbb{C})$, $(X_t)_{t\geq 0}$ is a G-invariant Lévy process if for any $g \in G(N)$, we have the following equality in law: $\left(gX_t g^{-1}\right)_{t\geq 0} = (X_t)_{t\geq 0}$.*

In the following, we only consider Lévy processes such that for any $t \geq 0$, X_t is in $L^{\infty^-} \otimes \mathcal{M}_N(\mathbb{C})$. In this case, for any $k \in \mathbb{N}$,

$$G_k = \frac{d}{dt}_{|t=0} \mathbb{E}\left[X_t^{\otimes k}\right]$$

is defined. In the next lemma, we show that this is the only data one needs in order to compute $\mathbb{E}[X_t^{\otimes k}]$ for any $t \in \mathbb{R}^+$. Recall the Definition 3.4.

Lemma 10.1. — *Let $(X_t)_{t\geq 0}$ be a matrix-valued Lévy process such that for any positive real t, $X_t \in L^{\infty^-} \otimes \mathcal{M}_N(\mathbb{C})$. Let us define for any positive integer k:*

$$G_k = \frac{d}{dt}_{|t=0} \mathbb{E}\left[X_t^{\otimes k}\right].$$

For any positive integer k, for any positive real t_0,

– *if $(X_t)_{t\geq 0}$ is an additive Lévy process:*

$$\frac{d}{dt}_{|t=t_0} \mathbb{E}\left[X_t^{\otimes k}\right] = \sum_{l=0}^{k-1} \sum_{I\subset\{1,\ldots,k\},\#I=l} \mathcal{I}_I\left[\mathbb{E}\left[X_{t_0}^{\otimes l}\right], G_{k-l}\right],$$

– *if $(X_t)_{t\geq 0}$ is a multiplicative Lévy process:*

$$\frac{d}{dt}_{|t=t_0} \mathbb{E}\left[X_t^{\otimes k}\right] = G_k \mathbb{E}\left[X_{t_0}^{\otimes k}\right].$$

Proof. — Let $(H_t)_{t\geq 0}$ and $(U_t)_{t\geq 0}$ be two matrix-valued Lévy processes which are respectively additive and multiplicative. Let us define for any integer k:

$$G_k^H = \frac{d}{dt}_{|t=0} \mathbb{E}\left[H_t^{\otimes k}\right],\ G_k^U = \frac{d}{dt}_{|t=0} \mathbb{E}\left[U_t^{\otimes k}\right].$$

Let t_0 be a non negative real. Using the independence and the stationarity properties of the processes $(H_t)_{t\geq 0}$ and $(U_t)_{t\geq 0}$ we have the two following sets of equalities.
For the additive Lévy process $(H_t)_{t\geq 0}$:

$$\begin{aligned}
\frac{d}{dt}_{|t=t_0} \mathbb{E}\left[H_t^{\otimes k}\right] &= \lim_{s\to 0} \frac{\mathbb{E}\left[H_{t_0+s}^{\otimes k}\right] - \mathbb{E}\left[H_{t_0}^{\otimes k}\right]}{s} \\
&= \lim_{s\to 0} \frac{\mathbb{E}\left[(H_{t_0} + (H_{t_0+s} - H_{t_0}))^{\otimes k}\right] - \mathbb{E}\left[H_{t_0}^{\otimes k}\right]}{s} \\
&= \lim_{s\to 0} \frac{\sum_{l=0}^{k-1} \sum_{I\subset\{1,\ldots,k\},\#I=l} \mathcal{I}_I\left[\mathbb{E}\left[H_{t_0}^{\otimes l}\right], \mathbb{E}\left[H_s^{\otimes k-l}\right]\right]}{s} \\
&= \sum_{l=0}^{k-1} \sum_{I\subset\{1,\ldots,k\},\#I=l} \mathcal{I}_I\left[\mathbb{E}\left[H_{t_0}^{\otimes l}\right], \lim_{s\to 0} \frac{\mathbb{E}\left[H_s^{\otimes k-l}\right]}{s}\right] \\
&= \sum_{l=0}^{k-1} \sum_{I\subset\{1,\ldots,k\},\#I=l} \mathcal{I}_I\left[\mathbb{E}\left[H_{t_0}^{\otimes l}\right], G_{k-l}^H\right].
\end{aligned}$$

For the multiplicative Lévy process $(U_t)_{t\geq 0}$, let us consider $(U'_t)_{t\geq 0}$ a process which has the same law as $(U_t)_{t\geq 0}$ and which is independent of $(U_t)_{t\geq 0}$. Then we have:

$$\begin{aligned}\frac{d}{dt}_{|t=t_0}\mathbb{E}\left[U_t^{\otimes k}\right] = \lim_{s\to 0}\frac{\mathbb{E}\left[U_{t_0+s}^{\otimes k}\right]-\mathbb{E}\left[U_{t_0}^{\otimes k}\right]}{s} &= \lim_{s\to 0}\frac{\mathbb{E}\left[(U'_sU_{t_0})^{\otimes k}\right]-\mathbb{E}\left[\left(U_{t_0}^{\otimes k}\right)\right]}{s}\\ &= \lim_{s\to 0}\frac{\mathbb{E}\left[(U'_s)^{\otimes k}U_{t_0}^{\otimes k}\right]-\mathbb{E}\left[\left(U_{t_0}^{\otimes k}\right)\right]}{s}\\ &= \lim_{s\to 0}\frac{\mathbb{E}\left[(U'_s)^{\otimes k}\right]-Id^{\otimes k}}{s}\mathbb{E}\left[U_{t_0}^{\otimes k}\right]\\ &= G_k^U\mathbb{E}\left[U_{t_0}^{\otimes k}\right].\end{aligned}$$

This allows us to conclude the proof. □

We will also use the more general fact that for any multiplicative matrix-valued Lévy process $(U_t)_{t\geq 0}$, for any integers k and l, the family $\left(\mathbb{E}\left[U^{\otimes k}\otimes\overline{U}^{\otimes l}\right]\right)_{t\geq 0}$ is a semi-group. Indeed, using the same arguments, one can see that for any $t_0\geq 0$,

$$\frac{d}{dt}_{|t=t_0}\mathbb{E}\left[U_t^{\otimes k}\otimes\overline{U_t}^{\otimes l}\right] = \left(\frac{d}{dt}_{|t=0}\mathbb{E}\left[U_t^{\otimes k}\otimes\overline{U_t}^{\otimes l}\right]\right)\mathbb{E}\left[U_{t_0}^{\otimes k}\otimes\overline{U_{t_0}}^{\otimes l}\right]. \tag{16}$$

10.2. Convergence of Lévy processes. —

10.2.1. Zero order. — In this section, we state and prove the main theorems for the convergence of Lévy processes. Actually, we have decided to cut it in four parts for the sake of clarity. Yet, these theorems have to be read one after the other as if it was only one theorem.

Let us choose a possibility for (G,A) as explained in Notation 3.1. Let $\left((X_t^N)_{t\geq 0}\right)_{N\geq 0}$ be a sequence of G-invariant Lévy processes which are either all additive or multiplicative. Let us suppose that for any positive integer N, and any $t\geq 0$, $X_t^N\in L^{\infty^-}\otimes\mathcal{M}_N(\mathbb{C})$. We define, for any positive integer k and N :

$$G_k^N = \frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\right].$$

Let k and N be two positive integers. The endomorphism G_k^N commutes with the tensor action of G on $\left(\mathbb{C}^N\right)^{\otimes k}$. We can apply Section 3 to G_k^N: with a slight abuse of notation G_k^N belongs to $\mathbb{C}[A_k(N)]$. Thus, we can define the coordinate numbers $\left(\kappa^p\left(G_k^N\right)\right)_{p\in A_k}$ of G_k^N. Recall the notion of convergence for elements of $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$ defined in [**15**]. Every time we will consider the sequence $\left(G_k^N\right)_{N\geq 0}$, we will consider it as an element of $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$.

Theorem 10.1. — *If for any positive integer k, the sequence $\left(G_k^N\right)_{N\geq 0}$ converges, then the process $(X_t^N)_{t\geq 0}$ converges in $\mathcal{P}$-expectation as N goes to infinity. Besides, $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial A- and $\mathcal{P}$-non-commutative Lévy process.*

Proof. — Let us prove that for any real $t \geq 0$, $\left(X_t^N\right)_{N\in\mathbb{N}}$ converges in A-expectation. Using Lemma 2.2, it is enough to show that for anypositive integer k, $\left(X_t^N\right)^{\otimes k}$ converges in expectation in A_k-moments as N tends to infinity. Let k be a positive integer. Recall that, since $\left(X_t^N\right)_{t\geq 0}$ is G-invariant, using the dualities in Theorem 3.1, $\mathbb{E}\left[(X_t^N)^{\otimes k}\right]$ and G_k^N are in $\mathbb{C}\left[\rho_N^{A_k}\right]$.

In the additive case, by Lemma 10.1, for any $t_0 \geq 0$:

$$\frac{d}{dt}_{|t=t_0} \mathbb{E}\left[(X_t^N)^{\otimes k}\right] = \sum_{l=0}^{k-1} \sum_{I\subset\{0,...,k\},\#I=l} \mathcal{I}_I\left[\mathbb{E}\left[(X_{t_0}^N)^{\otimes l}\right], G_{k-l}^N\right]. \tag{17}$$

Thus, for any $p \in \mathcal{P}_k$ and any $t_0 \geq 0$:

$$\frac{d}{dt}_{|t=t_0} \kappa^p\left[\mathbb{E}\left[(X_t^N)^{\otimes k}\right]\right] = \sum_{(p_1,p_2,I)\in\mathfrak{S}_2(p)/p_1\neq p} \kappa^{p_1}\left[\mathbb{E}\left[(X_t^N)^{\otimes \mathsf{l}(p_1)}\right]\right] \kappa^{p_2}\left[G_{\mathsf{l}(p_2)}^N\right],$$

where for any partition p, $\mathsf{l}(p)$ is the lenght of p defined as the unique integer such that $p \in \mathcal{P}_{\mathsf{l}(p)}$. Since for any positive integer k, G_k^N converges when N goes to infinity, for any partition $p_2 \in \cup_{k\in\mathbb{N}} A_k$, the sequence $\left(\kappa^{p_2}\left[G_{\mathsf{l}(p_2)}^N\right]\right)_{N\in\mathbb{N}}$ converges when N goes to infinity. By recurrence on the length of p, we see that for any positive integer k, any $p \in \mathcal{P}_k$ and any $t \geq 0$, $\kappa^p\left[\mathbb{E}\left[X_t^{\otimes k}\right]\right]$ converges as N goes to infinity.

In the multiplicative case, by Lemma 10.1, for any $t_0 \geq 0$:

$$\frac{d}{dt}_{|t=t_0} \mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\right] = G_k^N \mathbb{E}\left[\left(X_{t_0}^N\right)^{\otimes k}\right].$$

Thus, $\mathbb{E}\left[(X_t^N)^{\otimes k}\right]$ is a semi-group in $\mathbb{C}\left[\rho_N^{A_k}\right]$. The Theorem 7.2 of [**15**] allows us to conclude.

We must now prove that the family $\left(X_t^N\right)_{t\geq 0}$ converges in A-expectation. We will only explain the multiplicative case as the additive case is similar. We have to prove that for any $0 \leq t_1 \leq t_2 ... \leq t_n$, $\left(X_{t_i}^N\right)_{i=1}^n$ converges in A-expectation as N tends to infinity. Let us define $t_0 = 0$. Using the definition of Lévy processes, we know that for any positive integer N, $(X_{t_i}^N (X_{t_{i-1}}^N)^{-1})_{i=1}^n$ is a family of independent random matrices, each of them is invariant in law by conjugation by $G(N)$. Besides for any $i \in \{1, ..., n\}$, $X_{t_i}^N (X_{t_{i-1}}^N)^{-1}$ has the same law as $X_{t_i - t_{i-1}}^N$. Using the first part of the proof, we know that for any $i \in \{1, ..., n\}$, $X_{t_i}^N (X_{t_{i-1}}^N)^{-1}$ converges in A-expectation as N tends to infinity. Using Theorem 7.10, one concludes that $(X_{t_i}^N)_{i=1}^n$ converges in A-expectation as N tends to infinity. Because of the discussion we just had, we also get that $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a A-non-commutative matricial Lévy process.

An application of Theorem 4.5 allows us to prove that $\left(\left(X_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation. By definition, we have proved that it is a matricial A-free Lévy process, and since $\left(X_t^N\right)_{t\geq 0}$ is G-invariant for any positive integer N, as a consequence of Theorem 7.2, it is also a $\mathcal{P}$-non-commutative Lévy process. □

We define the $\mathcal{R}_A$-functional of the family $\left(\left(G_k^N\right)_{N\in\mathbb{N}}\right)_{k\in\mathbb{N}}$. Recall Definition 10.1 of [**15**].

Definition 10.3. — *Let us suppose that for any positive integer k, $\left(G_k^N\right)_{N\in\mathbb{N}}$ converges as N goes to infinity. We define $\mathcal{R}_A(G)$ as the only element in $\mathfrak{e}[A]$ such that for any positive integer k and any $p \in A_k$,*

$$\left(\mathcal{R}_A(G)\right)_p = \lim_{N\to\infty} \kappa^p\left(G_k^N\right).$$

From now on, we suppose that for any positive integer k, $(G_k^N)_{N\in\mathbb{N}}$, seen as an element of $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$, converges. Recall the notions of $\mathfrak{me}_{\boxplus}[A]$ and $\mathfrak{me}_{\boxtimes}[A]$ defined in Definition 10.5 and 10.6 in [**15**].

Definition 10.4. — *We say that $\left(G_k^N\right)_{k,N}$ condensates (resp. weakly condensates) if $\mathcal{R}_A(G) \in \mathfrak{me}_{\boxplus}[A]$ (resp. $\mathcal{R}_A(G) \in \mathfrak{me}_{\boxtimes}[A]$).*

Let us compute the $\mathcal{R}_A$-functional of $\left(X_t^N\right)_{t\geq 0}$. Recall the notation $\mathfrak{E}^{(i)}[A]$ defined in Definition 10.3 of [**15**]. If E is an element of $\mathfrak{E}[A]$, we denote by $E_{|\mathfrak{E}^{(i)}[A]}$ the restriction of E to $\bigcup_{k\in\mathbb{N}^*} A_k^{(i)}$. Recall also Definitions 10.7 and 10.8 in [**15**].

Theorem 10.2. — *Let us suppose that for any postive integer N, $\left(X_t^N\right)_{t\geq 0}$ is an additive Lévy process. For any real $t_0 \geq 0$,*

$$\frac{d}{dt}_{|t=t_0} \mathcal{R}_A[X_t] = \mathcal{R}_A[G] \boxplus \mathcal{R}_A[X_{t_0}].$$

This equation can also be written in the following way, for any $t_0 \geq 0$:

$$\mathcal{R}_A[X_{t_0}] = e_{\boxplus}^{t_0\mathcal{R}_A[G]}.$$

A consequence is that, for any real $t_0 \geq 0$:

$$\left(\mathcal{R}_A\left[X_{t_0}\right]\right)_{|\mathfrak{E}^{(i)}[A]} = t_0\left(\mathcal{R}_A\left[G\right]\right)_{|\mathfrak{E}^{(i)}[A]}.$$

Let us suppose that for any positive integer N, $\left(X_t^N\right)_{t\geq 0}$ is a multiplicative Lévy process. For any real $t_0 \geq 0$,

$$\frac{d}{dt}_{|t=t_0} \mathcal{R}_A[X_t] = \mathcal{R}_A[G] \boxtimes \mathcal{R}_A[X_{t_0}].$$

This equation can also be written in the following way, for any $t_0 \geq 0$:

$$\mathcal{R}_A[X_{t_0}] = e_{\boxtimes}^{t_0\mathcal{R}_A[G]}.$$

Besides, for any positive integer k, any $p \in \mathcal{P}_k$ and any $t_0 \geq 0$:

$$\frac{d}{dt}_{|t=t_0} \mathbb{E}m_p(X_t, ..., X_t) = \sum_{p_1\in[id,p]_{A_k}} \kappa_A^{p_1}(G_k)\mathbb{E}m_{t_{p_1}\circ p}(X_{t_0}, ..., X_{t_0}).$$

Proof. — The first equation appears when one takes the limit of Equation (17) when N goes to infinity. Let k be a positive integer, let p be an irreducible partition in A_k. Using the first equality and using the fact that $(\mathcal{R}_A[G])_\emptyset = 0$, we get:

$$\frac{d}{dt}_{|t=t_0} (\mathcal{R}_A[X_t])_p = (\mathcal{R}_A[G])_p \,,$$

hence $(\mathcal{R}_A[X_t])_p = t\,(\mathcal{R}_A[G])_p$ for any $t \geq 0$.

The equations in the multiplicative case are consequences of Equations (18) and (19) in Theorem 7.2 of [**15**]. □

Theorem 10.3. — *In the additive case, if* $\left(G_k^N\right)_{k,N}$ *condensates, the asymptotic* $\mathcal{P}$*-factorization property holds for* $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$.

In the multiplicative case, if $\left(G_k^N\right)_{k,N}$ *weakly condensates, the asymptotic* $\mathcal{P}$*-factorization property holds for* $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$.

Proof. — Let us prove first that for any positive real t, the A-factorization property holds for $(X_t^N)_{N\in\mathbb{N}}$. Let t be a positive real. Using Theorem 10.2, in the additive case $\mathcal{R}_A[X_t] = e_{\boxplus}^{t\mathcal{R}[G]}$ and in the multiplicative case $\mathcal{R}_A[X_t] = e_{\boxtimes}^{t\mathcal{R}[G]}$. By Definition 10.4, $(G_k^N)_{k,N}$ condensates if $\mathcal{R}_A(G) \in \mathfrak{me}_{\boxplus}[A]$, and $(G_k^N)_{k,N}$ weakly condensates if $\mathcal{R}_A(G) \in \mathfrak{me}_{\boxtimes}[A]$. By Theorem 10.2 of [**15**], we see that under the condensation property, in the additive case as for the multiplicative case, $\mathcal{R}_A[X_t] \in \mathfrak{ME}[A]$, which is equivalent, by Proposition 6.1, to say that the asymptotic A-factorization property holds for $(X_t^N)_{N\in\mathbb{N}}$.

As a consequence of Proposition 7.2, the asymptotic A-factorization property holds for $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ since the increments are asymptotically A-free. Then using the fact that the Lévy processes we considered are G-invariant, an application of Proposition 4.5 allows us to assert that the asymptotic $\mathcal{P}$-factorization property holds for $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$. □

In order to show that $(G_k^N)_{N\in\mathbb{N}}$ converges for any positive integer k, it is often easier to compute the normalized moments of G_k^N. Thus it would be interesting to have a criterion about normalized moments in order to know if $(G_k^N)_{k,N\in\mathbb{N}}$ condensates or weakly condensates.

Lemma 10.2. — *The family* $\left(G_k^N\right)_{k,N}$ *condensates if and only if for any positive integer* k *and* l*, for any* $p_1 \in A_k$ *and any* $p_2 \in A_l$*,*

$$m_{p_1\otimes p_2}\left(\left(G_{k+l}^N\right)_{N\in\mathbb{N}}\right) = 0.$$

The family $\left((G_k^N)_{N\in\mathbb{N}}\right)_{k\in\mathbb{N}}$ *weakly condensates if and only if for any positive integer* k *and* l*, for any* $p_1 \in A_k$ *and any* $p_2 \in A_l$*,*

$$m_{p_1\otimes p_2}\left(\left(G_{k+l}^N\right)_{N\in\mathbb{N}}\right) = m_{p_1}\left(\left(G_k^N\right)_{N\in\mathbb{N}}\right) + m_{p_2}\left(\left(G_l^N\right)_{N\in\mathbb{N}}\right).$$

We will only give the proof for the weak condensation, the first assertion can be proved with a similar argument.

Proof. — Let us recall that we defined an application $\mathfrak{J} : \mathfrak{e}^{(i)}[A] \to \mathfrak{e}[A]$ in Definition 10.6 of [**15**]. Besides, we defined an application $\mathcal{M}_A$ in Definition 10.9 of [**15**] which can be restrained to $\mathfrak{e}[A]$. In this proof, we will consider thus $\mathcal{M}_A : \mathfrak{e}[A] \to \mathfrak{e}[A]$. Let

us define two others applications. The first application allows us to extend a function defined on the irreducible partitions to all the partitions:

$$\mathfrak{A}_A : \mathfrak{e}^{(i)}[A] \to \mathfrak{e}[A]$$

such that for any $E \in \mathfrak{e}^{(i)}[A]$, for any positive integer k, for any $p \in A_k$,

$$\left(\mathfrak{A}_A(E)\right)_p = \sum_{C \in \mathsf{C}(p)} E_{p_c}.$$

The second one $\mathcal{M}_A^{(i)}$ is defined as the application:

$$\mathcal{M}_A^{(i)} : \mathfrak{e}^{(i)}[A] \to \mathfrak{e}^{(i)}[A]$$

such that for any $E \in \mathfrak{e}^{(i)}[A]$, for any positive integer k, for any $p \in A_k$ which is irreducible:

$$\left(\mathcal{M}_A^{(i)}(E)\right)_p = \left(\left(\mathcal{M}_A \circ \mathfrak{J}\right)(E)\right)_p$$

Let us remark that $\mathcal{M}_A^{(i)}$ is a bijection. Let us prove that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{e}^{(i)}[A] & \xrightarrow{\mathfrak{J}} & \mathfrak{e}[A] \\ {\scriptstyle \mathcal{M}_A^{(i)}}\downarrow & & \downarrow{\scriptstyle \mathcal{M}_A} \\ \mathfrak{e}^{(i)}[A] & \xrightarrow{\mathfrak{A}_A} & \mathfrak{e}[A] \end{array}$$

Let us consider $E \in \mathfrak{e}^{(i)}[A]$, let k be a positive integer, let $p \in A_k$, let p_1, ..., p_r be the extraction of p on its cycles. We can suppose that $p = p_1 \otimes ... \otimes p_r$ with p_1, ..., p_r some irreducible partitions. Using the factorization of the geodesics and using the fact that for any $p' \in A_k$, $\left(\mathfrak{J}(E)\right)_{p'}$ is equal to zero if p' is not weakly irreducible, we get:

$$\begin{aligned} \left(\mathcal{M}_A\left(\mathfrak{J}(E)\right)\right)_p = \sum_{p' \in [id, p_1 \otimes ... \otimes p_r]_{A_k}} \left(\mathfrak{J}(E)\right)_{p'} &= \sum_{i=1}^{r} \sum_{p'_i \in [id, p_i]_{A_k}} \left(\mathfrak{J}(E)\right)_{p'_i} \\ &= \sum_{i=1}^{r} \left(\mathcal{M}_A^{(i)}(E)\right)_{p_i} = \left(\left(\mathfrak{A}_A \circ \mathcal{M}_A^{(i)}\right)(E)\right)_p. \end{aligned}$$

Using the commutative diagram, one gets the following equalities:

$$\mathfrak{A}_A\left[\mathfrak{e}^i[A]\right] = \mathfrak{A}_A \circ \mathcal{M}_A^{(i)}\left[\left(\mathcal{M}_A^{(i)}\right)^{-1}\left[\mathfrak{e}^i[A]\right]\right] = \mathcal{M}_A \circ \mathfrak{J}\left[\mathfrak{e}^i[A]\right] = \mathcal{M}_A\left[\mathfrak{me}_{\boxtimes}[A]\right].$$

Thus any element $E \in \mathfrak{e}[A]$ belongs to $\mathfrak{me}_{\boxtimes}[A]$ if and only if $\mathcal{M}_A(E) \in \mathfrak{A}_A\left[\mathfrak{e}^i[A]\right]$. By applying this to $\mathcal{R}_A(G)$ one gets the expected result. □

10.2.2. The $*$-convergence. — Let us consider the same sequence $\left(\left(X_t^N\right)_{t\geq 0}\right)_{N\geq 0}$ of G-invariant Lévy processes used in the last subsection. If the matrices X_t^N are complex-valued, one can not deduce any convergence in probability of normalized A-moments using the Theorem 10.3: one has to show that $\left(\left(X_t^N, \left(X_t^N\right)^*\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ or $\left(\left(X_t^N, \overline{X_t^N}\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ converges as N goes to infinity and satisfies the A-factorization property in order to be able to apply the second part of Theorem 2.2. This is what we do in this section: we generalize Theorems 10.1, 10.2 and 10.3.

We will suppose that for any integer N, $(X_t^N)_{t\geq 0}$ is a multiplicative Lévy process, since often the additive Lévy processes considered are in the Hermitian matrices space for which the convergence of $\left(\left(X_t^N, \left(X_t^N\right)^*\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ is a direct consequence of the convergence of $\left(\left(X_t^N\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$.

Let us recall that according to the Equation (16), $\left(\mathbb{E}\left[(X_t^N)^{\otimes k}\otimes\left(\overline{X_t^N}\right)^{\otimes l}\right]\right)_{t\geq 0}$ is a semi-group of endomorphisms. We define, for any integers k, l and N:

$$G_{k,l}^N = \frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\otimes\left(\overline{X_t^N}\right)^{\otimes l}\right].$$

Recall the operation S_k defined in Definition 2.5. For any positive integer k and l, for any real $t\geq 0$, the endomorphism $\mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\otimes\left((X_t^N)^*\right)^{\otimes l}\right]$ commutes with the tensor action of G on $\left(\mathbb{C}^N\right)^{\otimes(k+l)}$: according to Theorem 3.1, it is an element of $\mathbb{C}\left[\rho_N^{A_{k+l}}\right]$, or with a slight abuse of notation it is an element of $\mathbb{C}[A_{k+l}(N)]$. This implies that $\mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\otimes\left(\overline{X_t^N}\right)^{\otimes l}\right]$ which is equal to $S_k\left[\mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\otimes\left((X_t^N)^*\right)^{\otimes l}\right]\right]$ is in $\mathbb{C}[\mathcal{P}_{k+l}(N)]$ if $G=\mathfrak{S}$, in $\mathbb{C}[\mathcal{B}_{k+l}(N)]$ if $G=O$ and in $S_k\left[\mathbb{C}[\mathfrak{S}_k]\right]\subset\mathcal{B}_{k+l}$ if $G=U$. The same conclusion holds for $G_{k,l}^N$. The proof of the following theorem is similar to 10.1, thus we will omit it.

Theorem 10.4. — *If for any positive integer k and l, the sequence $\left(G_{k,l}^N\right)_{N\geq 0}$, seen as an element of $\prod_{N\in\mathbb{N}}\mathbb{C}\left[\mathcal{P}_{k+l}(N)\right]$ converges, then the family $\left(\left(X_t^N, (X_t^N)^*\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ converges in $\mathcal{P}$-expectation as N goes to infinity.*

Let us suppose, from now on that for any positive integer k and l, the sequence $\left(G_{k,l}^N\right)_{N\geq 0}$ converges. We can define for any positive integer k and l,

$$(\mathcal{R}(G))_{k,l} = \sum_{p\in\mathcal{P}_{k+l}}\kappa^p(G_{k,l})p\in\mathbb{C}\left[\mathcal{P}_{k+l}(\infty,\infty)\right].$$

Since $\left(\left(X_t^N,(X_t^N)^*\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ converges in $\mathcal{P}$-expectation as N goes to infinity, we can define for any positive integer k and l and any real $t\geq 0$:

$$(\mathcal{R}(X_t))_{k,l}=\sum_{p\in\mathcal{P}_{k+l}}\kappa^p\left(\left(X_t^N\right)_{N\in\mathbb{N}},...,\left(X_t^N\right)_{N\in\mathbb{N}},\left(\overline{X_t^N}\right)_{N\in\mathbb{N}},...,\left(\overline{X_t^N}\right)_{N\in\mathbb{N}}\right)p,$$

where we wrote k times $(X_t^N)_{N\in\mathbb{N}}$ and l times $\left(\overline{X_t^N}\right)_{N\in\mathbb{N}}$.

Definition 10.5. — *We say that $\left(G_{k,l}^N\right)_{k,l,N}$ weakly condensates if for any positive integer k and l, for any $p\in\mathcal{P}_{k+l}$, if p is not weakly irreducible,*

$$\kappa^p(G_{k,l})=0,$$

and

$$\kappa^{id_{k+l}}(G_{k,l})=k\kappa^{id_1}(G_{1,0})+l\kappa^{id_1}(G_{0,1}).$$

This definition allows us to state the following theorem which is a generalization of Theorems 10.2 and 10.3.

Theorem 10.5. — *For any real $t_0\geq 0$, the following equality holds in $\mathbb{C}[\mathcal{P}_{k+l}(\infty,\infty)]$:*

$$\frac{d}{dt}_{|t=t_0}\left(\mathcal{R}(X_t)\right)_{k,l}=(\mathcal{R}(G))_{k,l}\left(\mathcal{R}(X_{t_0})\right)_{k,l}.$$

Moreover, for any integer k and l, any $p\in\mathcal{P}_k$ and any $t_0\geq 0$:

$$\frac{d}{dt}_{|t=t_0}m_p\left[\mathbb{E}\left(X_t^{\otimes k}\otimes\overline{X_t}^{\otimes l}\right)\right]=\sum_{p_1\in[id,p]_{A_k}}\kappa^{p_1}(G_{k,l})m_{t_{p_1\circ p}}\left[\mathbb{E}\left(X_{t_0}^{\otimes k}\otimes\overline{X_{t_0}}^{\otimes l}\right)\right],$$

where $m_p\left[\mathbb{E}\left(X_{t_0}^{\otimes k}\otimes\overline{X_{t_0}}^{\otimes l}\right)\right]$ stands for the limit as N goes to infinity of the mean moment $m_p\left[\mathbb{E}\left(X_{t_0}^{N\otimes k}\otimes\left(\overline{X_{t_0}^N}\right)^{\otimes l}\right)\right]$.

Besides, if $\left(G_{k,l}^N\right)_{k,l,N}$ weakly condensates, the asymptotic $\mathcal{P}$-factorization property holds for $\left(\left(X_t^N,(X_t^N)^*\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ and thus the normalized $\mathcal{P}$-moments of $\left(X_t^N\right)_{t\geq 0}$ converge in probability.

10.2.3. Higher order. — More generally, one can state similar results for the fluctuations than those we get in Section 10.2.1. The proofs are similar to the one for the zero order, and all the tools that one needs in order to prove them are included either in [**15**], namely Theorem 9.3, or in this article, thus we will omit them. Let m be a non negative integer.

Theorem 10.6. — *Let us suppose that for anypositive integer k, $(G_k^N)_{N\in\mathbb{N}}$, seen as an element of $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$, converges up to order m of fluctuations. Then $(X_t^N)_{t\geq 0}$ converges in $\mathcal{P}$-expectation up to order m of fluctuations as N goes to infinity. Besides the family $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial A- and $\mathcal{P}$-non-commutative Lévy process up to any order of fluctuations.*

Let us define the $\mathcal{R}_A^{(m)}$-functional of the family $\left(\left(G_k^N\right)_{N\in\mathbb{N}}\right)_{k\in\mathbb{N}}$.

Definition 10.6. — *Let us suppose that* $\left(G_k^N\right)_{N\in\mathbb{N}}$*, seen in* $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$*, converges up to order* m *of fluctuations. We define* $\mathcal{R}_A^{(m)}(G)$ *as the only element in* $\mathfrak{E}^{(m)}[A]$ *such that for any positive integer* k*, any* $p\in A_k$ *and any* $i\in\{0,...,m\}$*:*

$$\left(\mathcal{R}_A(G)\right)_{p,i}=\lim_{N\to\infty}\kappa_i^p\left(G_k^N\right),$$

and for any $i\in\{0,...,m\}$*,* $\left(\mathcal{R}_A(G)\right)_{\emptyset,i}=0$*.*

From now on, we suppose that $\left(G_k^N\right)_{N\in\mathbb{N}}$ converges up to order m of fluctuations. Let us recall the two operations $\boxplus$ and $\boxtimes$ defined in Section 10.1 of [**15**].

Theorem 10.7. — *Let us suppose that for any positive integer* N*,* $(X_t^N)_{t\geq 0}$ *is an additive Lévy process. For any real* $t_0\geq 0$*,*

$$\frac{d}{dt}_{|t=t_0}\mathcal{R}_A^{(m)}[X_t]=\mathcal{R}_A^{(m)}[G]\boxplus\mathcal{R}_A^{(m)}[X_{t_0}].$$

Let us suppose that for any integer N*,* $(X_t^N)_{t\geq 0}$ *is a multiplicative Lévy process. For any real* $t_0\geq 0$*,*

$$\frac{d}{dt}_{|t=t_0}\mathcal{R}_A^{(m)}[X_t]=\mathcal{R}_A^{(m)}[G]\boxtimes\mathcal{R}_A^{(m)}[X_{t_0}].$$

Besides, for any integer k*, any* $p\in\mathcal{P}_k$*, any* $t_0\geq 0$ *and any* $i_0\in\{0,...,m\}$*:*

$$\frac{d}{dt}_{|t=t_0}\mathbb{E}m_p^{i_0}(X_t,...,X_t)=\sum_{p_1\in A_k}\sum_{i+j+\mathsf{df}(p_1,p)=i_0}\kappa_i^{p_1}(G_k)\mathbb{E}m_{i_{p_1\circ p}}^j(X_{t_0},...,X_{t_0}).$$

10.2.4. Special cases. — Using Lemma 3.9 in [**15**] which states that any geodesic in the set of Brauer elements between permutations is included in the set of permutations, we can state a slightly stronger theorem for sequence of O-invariant Lévy processes. Let $\left((X_t^N)_{t\geq 0}\right)_{N\geq 0}$ be a sequence of O-invariant Lévy processes which are either all multiplicative or additive. Let us suppose that for any positive integer N, for any positive real t, $X_t^N\in L^{\infty^-}\otimes\mathcal{M}_N(\mathbb{C})$. As usual, let us define for any positive integer k and N:

$$G_k^N=\frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\right].$$

Let us recall, with a slight abuse of notation, that using the dualities in Theorem 3.1, for any positive integers k and N, $G_k^N\in\mathbb{C}[\mathcal{B}_k]$. As a consequence of Proposition 2.3 in this article and Theorem 4.3, Theorem 7.3, Equations (18) and (20) in the article [**15**] one can prove the following theorem.

Theorem 10.8. — *Let us suppose that for any positive integer* k*, the* $\mathfrak{S}_k$*-normalized moments of* $\left(G_k^N\right)$ *converge as* N *goes to infinity and that the* $\mathcal{B}_k$*-normalized moments of* $\left(G_k^N\right)$ *are bounded, then* $(X_t^N)_{t\geq 0}$ *converges in* $\mathfrak{S}$*-expectation as* N *goes to infinity.*

As a special case, let us suppose that for any positive integer N*,* $\left(X_t^N\right)_{t\geq 0}$ *is a* O*-invariant Lévy process which is defined on the set of orthogonal or symmetric matrices, then the condition of boundeness of the* $\mathcal{B}_k$*-normalized moments of* $\left(G_k^N\right)$ *is not necessary:*

the $\mathcal{B}$-normalized moments of $\left(G_k^N\right)_{N\in\mathbb{N}}$ will converge and because of Theorem 10.1, $\left(\left(X_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation.

Besides let $\left((Y_t^N)_{t\geq 0}\right)_{N\geq 0}$ be a sequence of U-invariant Lévy processes which is of the same type than $\left((X_t^N)_{t\geq 0}\right)_{N\geq 0}$ (i.e. multiplicative of additive). Let us define for any positive integer k and N:

$$G_k'^N = \frac{d}{dt}_{|t=0} \mathbb{E}\left[\left(Y_t^N\right)^{\otimes k}\right].$$

Let us suppose that $\left(G_k'^N\right)_{N\in\mathbb{N}}$ seen as an element of $\prod_{N\in\mathbb{N}}\mathbb{C}[\mathfrak{S}(N)]$ converges. Let us suppose that for any positive integer k and any $\sigma\in\mathfrak{S}_k$:

$$\lim_{N\to\infty} m_\sigma\left(G_k'^N\right) = \lim_{N\to\infty} m_\sigma\left(G_k^N\right) \text{ or } \lim_{N\to\infty}\kappa^\sigma\left(G_k'^N\right) = \lim_{N\to\infty}\kappa^\sigma\left(G_k^N\right),$$

then $\left(\left(X_t^N\right)_{t\geq 0}\right)_{N\geq 0}$ has the same $\mathfrak{S}$-law as $\left(\left(Y_t^N\right)_{t\geq 0}\right)_{N\geq 0}$.

The reader must be aware that Theorem 10.8 holds only for O-invariant Lévy processes. It does not hold for a general $\mathfrak{S}$-invariant Lévy process, due to the fact that the geodesics from a permutation to the identity might go through an element of $\mathcal{P}_k\setminus\mathfrak{S}_k$. Yet, a similar result holds for $\mathfrak{S}(N)$-invariant Lévy processes which are defined on the set of permutation matrices: this is a consequence of Lemma 2.4.

Theorem 10.9. — *Let us suppose that for any integer N, $\left(X_t^N\right)_{t\geq 0}$ is a $\mathfrak{S}$-invariant Lévy process which is defined on the set of permutation matrices.*

Let us suppose that for any integer k, the $\mathfrak{S}$-normalized moments of $\left(G_k^N\right)_{N\in\mathbb{N}}$ converge, then the $\mathcal{P}$-normalized moments of $\left(G_k^N\right)_{N\in\mathbb{N}}$ converge. As a consequence, because of Theorem 10.1, $\left(\left(X_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation.

10.3. Brownian motions, convergence and matricial Wick formula. — In this section, we study the convergence of Brownian motions on some Lie algebras and Lie groups. We decided to write a whole section about Brownian motions since we will explain different ideas than the ones which will be presented in the general setting of Lévy processes in Section 10. Besides, the intuitions that we will develop in this section will allow us to prove Theorem 8.2.

10.3.1. General definitions. — Let us define some useful space of matrices. We will follow the presentation of Lévy in [**23**]. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, let N be a positive integer.

Definition 10.7. — *The spaces of skew-symmetric and symmetric real matrices of size N are respectively:*

$$\mathfrak{a}_N = \{M\in\mathcal{M}_N(\mathbb{R}),\ {}^tM = -M\},$$
$$\mathfrak{s}_N = \{M\in\mathcal{M}_N(\mathbb{R}),\ {}^tM = M\}.$$

The space of skew-Hermitian matrices of size N is:

$$\mathfrak{u}_N = \{M \in \mathcal{M}_N(\mathbb{C}), M^* = -M\}.$$

It is easy to see that the space of Hermitian matrices of size N is $i\mathfrak{u}_N$. We will use the conventions:

$$\mathfrak{u}(N,\mathbb{K}) = \begin{cases} \mathfrak{a}_N & \text{if } \mathbb{K} = \mathbb{R}, \\ \mathfrak{u}_N & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

and:

$$\mathfrak{h}(N,\mathbb{K}) = \begin{cases} \mathfrak{s}_N & \text{if } \mathbb{K} = \mathbb{R}, \\ i\mathfrak{u}_N & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

Even more generally, we will consider:

$$\mathfrak{g}^\epsilon(N,\mathbb{K}) = \begin{cases} \mathfrak{u}(N,\mathbb{K}) & \text{if } \epsilon = -1, \\ \mathfrak{h}(N,\mathbb{K}) & \text{if } \epsilon = 1. \end{cases}$$

Besides, we will also use the following notation:

$$\beta_{\mathbb{K}} = \begin{cases} 1 & \text{if } \mathbb{K} = \mathbb{R}, \\ 2 & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

Let us consider $\epsilon \in \{-1, 1\}$.

Definition 10.8. — *We will always consider $\mathfrak{g}^\epsilon(N,\mathbb{K})$ as a Lie algebra endowed with the scalar product:*

$$\forall X, Y \in \mathfrak{g}^\epsilon(N,\mathbb{K}), \quad <X,Y>= \frac{\beta_{\mathbb{K}} N}{2} Tr(X^*Y).$$

Let us remark that the scalar product on $\mathbb{R} = \mathfrak{g}^1(1,\mathbb{R})$ is $<x,y>= \frac{1}{2}xy$ for any reals x and y.

Definition 10.9. — *Let d be the dimension of $\mathfrak{g}^\epsilon(N,\mathbb{K})$. Let $(X_i)_{i=1}^d$ be an orthonormal basis of $\mathfrak{g}^\epsilon(N,\mathbb{K})$. We define the Casimir of $\mathfrak{g}^\epsilon(N,\mathbb{K})$ as:*

$$C_{\mathfrak{g}^\epsilon(N,G)} = \sum_{i=1}^{d} X_i \otimes X_i.$$

This is an easy and well-known result that the definition of the Casimir does not depend on the orthonormal basis we have chosen. We remind the reader that we defined the transpositions and the Weyl contractions in Definition 2.8 of [**15**]. Using the results of Lévy in [**23**], we know the value of the Casimir operator.

Lemma 10.3. — *The Casimir of $\mathfrak{g}^\epsilon(N,\mathbb{K})$ is equal to:*

$$\frac{1}{N}\rho_N^{\mathcal{B}_2}\big(\epsilon(1,2) + (2-\beta_{\mathbb{K}})[1,2]\big).$$

This lemma implies the following result.

Lemma 10.4. — *Let d be the dimension of $\mathfrak{g}^\epsilon(N,\mathbb{K})$, and let $(X_i)_{i=1}^d$ be an orthonormal basis of $\mathfrak{g}^\epsilon(N,\mathbb{K})$. Then:*

$$\sum_{i=1}^{d} X_i X_i = c_{\mathfrak{g}^\epsilon(N,\mathbb{K})} Id,$$

with $c_{\mathfrak{g}^\epsilon(N,\mathbb{K})} = \epsilon + \frac{2-\beta_{\mathbb{K}}}{N}$.

In the following, we will use the following notation:

$$U(N,\mathbb{K}) = \begin{cases} O(N) & \text{if } \mathbb{K}=\mathbb{R}, \\ U(N) & \text{if } \mathbb{K}=\mathbb{C}. \end{cases}$$

The group N, $U(N,\mathbb{K})$ is a Lie group, whose Lie algebra is $\mathfrak{u}(N,\mathbb{K})$.

10.3.2. Brownian motion on $\mathfrak{g}^\epsilon(N,\mathbb{K})$. — Let N be an integer and let ϵ be in $\{-1,1\}$. As Lévy did in [**23**], we now define the notion of Brownian motion on $\mathfrak{g}^\epsilon(N,\mathbb{K})$. Let d be the dimension of $\mathfrak{g}^\epsilon(N,\mathbb{K})$, and let $(X_i)_{i=1}^d$ be an orthonormal basis of $\mathfrak{g}^\epsilon(N,\mathbb{K})$.

Definition 10.10. — *Let $\left((B_t^i)_{t\geq 0}\right)_{i=1}^d$ be a d-tuple of independent real Brownian motions. The process $(H_t)_{t\geq 0}$ where for any $t\geq 0$:*

$$H_t = \sum_{i=1}^{d} B_t^i X_i.$$

is a Brownian motion on $\mathfrak{g}^\epsilon(N,\mathbb{K})$. Any process which has the same law as $(H_t)_{t\geq 0}$ is called a Brownian motion on $g^\epsilon(N,\mathbb{K})$.

Remark 10.1. — *The law of the process $(H_t)_{t\geq 0}$ does not depend on the choice of the orthonormal basis of $\mathfrak{g}^\epsilon(N,\mathbb{K})$: this allows us to show that $(H_t)_{t\geq 0}$ is a $U(N,\mathbb{K})$-invariant process.*

A consequence of Definition 10.10 is that any Brownian motion in $\mathfrak{g}^\epsilon(N,\mathbb{K})$ is an additive Lévy process. Besides, we can compute the bracket of H_t with itself:

$$dH_t \otimes dH_t = \sum_{i,j=1}^{d} dB_t^i dB_t^j X_i \otimes X_j = \left(\sum_{i=1}^{d} X_i \otimes X_i\right) dt = C_{\mathfrak{g}^\epsilon(N,\mathbb{K})} dt, \tag{18}$$

due to the fact that for any $i,j \in \{0,...,d\}$, $dB_t^i dB_t^j = \delta_{i,j} dt$. Let us remark that one can easily compute the bracket of H_t with ${}^tH_t$ and ${}^tH_t$ with itself since:

$$dH_t \otimes {}^t dH_t = S_1\left[dH_t \otimes dH_t\right]$$
$${}^t dH_t \otimes {}^t dH_t = S_0\left[dH_t \otimes dH_t\right],$$

where S_k is the operation defined in Definition 2.5. We will be interested in the case where $\mathbb{K}=\mathbb{C}$ and $\epsilon=-1$: let us suppose so just for this discussion. Then, we get the following equalities:

$$dH_t \otimes {}^t dH_t = -\frac{1}{N}\rho_N^{\mathcal{B}_k}\left([1,2]\right) dt, \tag{19}$$

$${}^t dH_t \otimes {}^t dH_t = -\frac{1}{N}\rho_N^{\mathfrak{S}_k}(1,2) dt. \tag{20}$$

We recall that we defined the insertion operator $\mathcal{I}_I$ in Definition 3.4. Using the Itô's formula, one has the following lemma.

Lemma 10.5. — *Let us consider $(H_t)_{t\geq 0}$ a Brownian motion on $\mathfrak{g}^\epsilon(N,\mathbb{K})$. For any integer $k \in \mathbb{N}\setminus\{2\}$, $G_k^N = \frac{d}{dt}_{|t=0}\mathbb{E}\left[H_t^{\otimes k}\right] = 0$ and:*

$$G_2^N = \frac{d}{dt}_{|t=0}\mathbb{E}\left[H_t^{\otimes 2}\right] = C_{\mathfrak{g}^\epsilon(N,\mathbb{K})}.$$

10.3.3. Brownian motion on $U(N,\mathbb{K})$. — We can now define the multiplicative notion of Brownian Motion.

Definition 10.11. — *Let $(H_t)_{t\geq 0}$ be a Brownian motion on $\mathfrak{u}(N,\mathbb{K})$. The solution of the Stratonovich stochastic equation:*

$$\left\{\begin{array}{ll} dU_t & = dH_t \circ U_t, \\ U_0 & = I_N, \end{array}\right.$$

is a Brownian motion on $U(N,\mathbb{K})$. We can reformulate this in terms of Itô integral. The solution of the Itô stochastic equation:

$$\left\{\begin{array}{ll} dU_t & = dH_t U_t + \frac{c_{\mathfrak{u}(N,\mathbb{K})}}{2} U_t dt, \\ U_0 & = I_N, \end{array}\right.$$

is a Brownian motion on $U(N,\mathbb{K})$. Any process which has the same law as $(U_t)_{t\geq 0}$ is a Brownian motion on $U(N,\mathbb{K})$.

It is an important result that any Brownian motion on $U(N,\mathbb{K})$ is a multiplicative Lévy process which is invariant by conjugation by $U(N,\mathbb{K})$: this is a consequence of the fact that the linear Brownian motion is also invariant by conjugation by $U(N,\mathbb{K})$.

We can also compute the bracket of U_t with itself. Let $(H_t)_{t\geq 0}$ be an additive Brownian motion on $\mathfrak{u}(N,\mathbb{K})$ and let $(U_t)_{t\geq 0}$ be the Brownian motion on $U(N,\mathbb{K})$ associated with $(H_t)_{t\geq 0}$. Then:

$$dU_t \otimes dU_t = dH_t U_t \otimes dH_t U_t = (dH_t \otimes dH_t)(U_t \otimes U_t) = C_{\mathfrak{u}(N,\mathbb{K})}(U_t \otimes U_t)dt,$$

where the last equality is a consequence of the additive case.

This remark allows us to compute the action of the infinitesimal generator on the tensor product. Using the Itô's formula and Lemma 10.4, we get the following formula given in [**23**] :

Lemma 10.6. — *Let us consider $(U_t)_{t\geq 0}$ a Brownian motion on $U(N,\mathbb{K})$. For any positive integer k:*

$$G_k^N = \frac{d}{dt}_{|t=0}\mathbb{E}\left[U_t^{\otimes k}\right] = \frac{k c_{\mathfrak{u}_{N,\mathbb{K}}}}{2} Id^{\otimes k} + \sum_{1\leq i<j\leq k} \mathcal{I}_{\{i,j\}}\left[C_{\mathfrak{u}(N,\mathbb{K})}, Id^{\otimes k-2}\right].$$

We can also generalize this lemma in order to understand $\frac{d}{dt}_{|t=0}\mathbb{E}\left[U_t^{\otimes k} \otimes \overline{U_t}^{\otimes l}\right]$ when $\mathbb{K} = \mathbb{C}$. Since H_t is skew-Hermitian, $\overline{dH_t} = -{}^t dH_t$. Besides, by Definition 10.11, we

know that $dU_t = dH_t U_t - \frac{1}{2} U_t dt$, thus:

$$d\overline{U_t} = -\,{}^t dH_t \overline{U_t} - \frac{1}{2}\overline{U_t} dt.$$

This implies that:

$$dU_t \otimes d\overline{U_t} = -dH_t U_t \otimes {}^t dH_t \overline{U_t} = \frac{1}{N}\rho_N^{\mathcal{B}_k}\left([1,2]\right)\left(U_t \otimes \overline{U_t}\right) dt,$$
$$d\overline{U_t} \otimes d\overline{U_t} = {}^t dH_t \overline{U_t} \otimes {}^t dH_t \overline{U_t} = -\frac{1}{N}\rho_N^{\mathfrak{S}_k}\left((1,2)\right)\left(\overline{U_t} \otimes \overline{U_t}\right) dt.$$

This remark, the Equations (19) and (20) allow us to use the Itô's formula in order to show the following result which was already proved by T. Lévy in [**23**], and by A. Dahlqvist in [**1**].

Lemma 10.7. — *Let us consider $(U_t)_{t\geq 0}$ a Brownian motion on $U(N,\mathbb{C})$. For any positive integers k and k', we have, with a slight abuse of notations:*

$$G_{k,k'}^N = \frac{d}{dt}_{|t=0} \mathbb{E}\left[U_t^{\otimes k} \otimes \overline{U_t}^{\otimes k'}\right]$$
$$= -\frac{k+k'}{2} Id^{\otimes k} - \frac{1}{N}\left[\sum_{1\leq i<j\leq k+k'} (i,j) + \sum_{1\leq i\leq k<j\leq k+k'} ([i,j]-(i,j))\right].$$

10.3.4. Convergence of additive Brownian motion. — Let us state the results about convergence of additive Brownian motions.Let $\epsilon \in \{-1,1\}$. Let A be equal to $\mathfrak{S}$ if $\mathbb{K} = \mathbb{C}$ and equal to $\mathcal{B}$ if $\mathbb{K} = \mathbb{R}$. For any integer N, let $\left(H_t^N\right)_{t\geq 0}$ be a Brownian motion on $\mathfrak{g}^\epsilon(N,\mathbb{K})$.

Theorem 10.10. — *The process $\left(H_t^N\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation up to any order of fluctuations as N tends to infinity. The family $\left((H_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial A- and $\mathcal{P}$-non-commutative additive Lévy process up to any order of fluctuations.*

Besides, the asymptotic $\mathcal{P}$-factorization property holds for $\left(\left(H_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$: this implies that the $\mathcal{P}$-moments of $\left(H_t^N\right)_{t\geq 0}$ converges in probability to a non-random constant as N goes to infinity. Moreover, for any positive real t:

$$\mathcal{R}_A\left((H_t^N)_{N\in\mathbb{N}}\right) = e_{\boxplus}^{t\mathcal{R}_A(G)},$$

with:

$$\mathcal{R}_A(G) = \epsilon(1,2) + (2-\beta_{\mathbb{K}})[1,2].$$

Proof of Theorem 10.10. — We have seen that $\left(H_t^N\right)_{t\geq 0}$ is invariant by conjugation by $U(N,\mathbb{K})$. As an application of Theorems 10.2, 10.3 and 10.6, using the notations of Lemma 10.5, it remains to show that for any integer k, $\left(G_k^N\right)_{N\in\mathbb{N}}$, seen as an element of $\prod_{N\in\mathbb{N}} \mathbb{C}\left[A_k(N)\right]$, converges up to any order of fluctuations, and that:

$$\mathcal{R}_A(G) = \epsilon(1,2) + (2-\beta_{\mathbb{K}})[1,2].$$

Indeed if so, by Theorem 10.6 the process $\left(H_t^N\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation up to any order of fluctuations as N tends to infinity. Since $(1,2)$ and $[1,2]$ are

irreducible, $\mathcal{R}_A[G] \in \mathfrak{me}_{\boxplus}[A]$: by Theorem 10.3, the asymptotic $\mathcal{P}$-factorization holds for $\left(\left(H_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$. Using the fact that we are considering matrices which are in $\mathfrak{g}^{\epsilon}(N,\mathbb{K})$, this implies that the asymptotic $\mathcal{P}$-factorization also holds for $\left(\left(H_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0} \cup \left(\left(\overline{H_t^N}\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$: by Theorem 2.2, the assertion on the convergence in probability of the $\mathcal{P}$-moments is proved. The remaining assertion is a consequence of Theorem 10.2.

By Lemma 10.5, $G_k^N = 0$ if $k \neq 2$ and $G_2^N = C_{\mathfrak{u}(N,\mathbb{K})}$. Recall the notation M_k^N defined in Definition 5.1 of [**15**]. Using Lemma 10.3 and Lemma 5.4 of [**15**], it is enough to show that:

$$(M_2^N)^{-1}\left(\frac{1}{N}\left(\epsilon(1,2)+(2-\beta_{\mathbb{K}})[1,2]\right)\right)$$

converges in $\mathbb{C}[A_2]$ and has an asymptotic development as an infinite power series in $\frac{1}{N}$. For any integer N :

$$(M_2^N)^{-1}\left(\frac{1}{N}\left(\epsilon(1,2)+(2-\beta_{\mathbb{K}})[1,2]\right)\right) = \epsilon(1,2)+(2-\beta_{\mathbb{K}})[1,2].$$

Thus for any positive integer k, $\left(G_k^N\right)_{N\in\mathbb{N}}$, seen as an element of $\prod_{N\in\mathbb{N}}\mathbb{C}\left[A_k(N)\right]$, converges up to any order of fluctuations, and using the discussion we just had:

$$\mathcal{R}_A[G] = \epsilon(1,2)+(2-\beta_{\mathbb{K}})[1,2].$$

This concludes the proof. □

When $\epsilon = 1$, it is well-known that the limit of the mean empirical eigenvalues distribution of H_1^N is the Wigner semicircular distribution.

Theorem 10.11. — *For any positive integer N, let $\left(H_t^N\right)_{t\geq 0}$ be a Brownian motion on $\mathfrak{g}^1(N,\mathbb{K})$. The mean empirical eigenvalues distribution of H_1^N converges in probability to the Wigner semicircle distribution:*

$$\mu_{sc} = \frac{1}{2\pi}\sqrt{4-|x|^2}\mathbb{1}_{[-2,2]}dx.$$

Proof. — Using the usual arguments we explained in the beginning of Section 2.1, namely Theorem 2.1, we only have to prove that for any positive integer k, $\frac{1}{N}\mathbb{E}\left[Tr\left(\left(H_1^{\mathfrak{g}^1(N,\mathbb{K})}\right)^k\right)\right]$ converges when N goes to infinity to:

$$\int_{\mathbb{R}} x^k \mu_{sc}(dx) = \#\Big\{\text{Non-crossing pair-partitions of } \{1,...,k\}\Big\}.$$

Actually, one needs to prove some uniformity sub-Gaussian property for the mean empirical eigenvalues distribution, this can be proved using concentration of measure for the operator norm of a random matrix. We refer to the discussion in [**32**] at the beginning of Section 2.4.2.

Let us remark that, using the bijection between non-crossing pair-partitions of $\{1,...,k\}$ and involutions without fixed points in $[id,(1,...,k)]_{\mathfrak{S}_k}$, we see that:

$$\#\Big\{\text{Non-crossing pair-partitions of } \{1,...,k\}\Big\} = \sum_{\sigma \text{ involution in } \mathfrak{S}_k, \#\mathsf{C}(\sigma)=\frac{k}{2}, \sigma\in[id,(1,...,k)]_{\mathfrak{S}_k}} 1.$$

Thus:

$$\#\Big\{\text{Non-crossing pair-partitions of } \{1,...,k\}\Big\} = \left(\mathcal{M}_{\mathfrak{S}}\left[e_{\boxplus}^{(1,2)}\right]\right)_{(1,...,k)},$$

where the application $\mathcal{M}_{\mathfrak{S}}$ was defined in Definition 10.9 of [**15**]. Let A be equal to $\mathfrak{S}$ if $\mathbb{K}=\mathbb{C}$ or $\mathcal{B}$ if $\mathbb{K}=\mathbb{R}$. By Theorem 10.10, $\mathcal{R}_A\left[(H_1^N)_{N\in\mathbb{N}}\right] = e_{\boxplus}^{(1,2)+(2-\beta_{\mathbb{K}})[1,2]}$, thus:

$$\begin{aligned}\mathcal{M}_A\left[\left(H_1^N\right)_{N\in\mathbb{N}}\right] &= \mathcal{M}_A\left[\mathcal{R}_A\left[\left(H_1^N\right)_{N\in\mathbb{N}}\right]\right]\\ &= \mathcal{M}_A\left[e_{\boxplus}^{(1,2)+(2-\beta_{\mathbb{K}})[1,2]}\right].\end{aligned}$$

As an application of Proposition 10.4 of [**15**] and using the fact that $\left(e_{\boxplus}^{E}\right)_{|\mathfrak{E}[\mathfrak{S}]} = e_{\boxplus}^{E_{|\mathfrak{S}}}$, one gets, for any positive integer k:

$$\begin{aligned}\left(\mathcal{M}_A\left[\left(H_1^N\right)_{N\in\mathbb{N}}\right]\right)_{(1,...,k)} &= \left(\mathcal{M}_{\mathfrak{S}}\left[\left(e_{\boxplus}^{(1,2)+(2-\beta_{\mathbb{K}})[1,2]}\right)_{|\mathfrak{E}[\mathfrak{S}]}\right]\right)_{(1,...,k)}\\ &= \left(\mathcal{M}_{\mathfrak{S}}\left[e_{\boxplus}^{(1,2)}\right]\right)_{(1,...,k)}\\ &= \#\Big\{\text{Non-crossing pair-partitions of } \{1,...,k\}\Big\}\\ &= \int_{\mathbb{R}} x^k \mu_{sc}(dx),\end{aligned}$$

which is the equality we wanted to prove. □

10.3.5. Matricial Wick Formula. — In this section, we only write in a more usual way the equation stated in Theorem 10.10: one can recover a matricial Wick formula. In order to do so, we need the following definition. Let n be a positive integer.

Definition 10.12. — *A one-specie pairing of n is a partition of $\{1,...,n\}$ into pairs. A two-species pairing of n is a partition of $\{1,...,n\}$ into pairs, with a partition in two sets of these pairs: (T_m, W_m).*

We denote by $\mathcal{F}_n(1)$ the set of one-specie pairings of n and $\mathcal{F}_n(2)$ the set of two-species parings of n. By convention $\#W_m = 0$ for any one-specie pairing m.

Any one-specie pairing of n can be written as $\big\{\{i,j_i\}, i\in I\big\}$ where I is a subset of $\{1,...,n\}$ of size $n/2$. For example, the partition $\{\{1,4\},\{2,3\},\{5,6\}\}$ is a one-specie pairing of 6 and $\{\{1,4\},\{2,3\},\{5,6\}\}$ with the partition $T_m = \{\{1,4\},\{5,6\}\}$, $W_m = \{\{2,3\}\}$ is a two-species pairing of 6.

There is an obvious bijection between one-specie pairings of n and involutions in $\mathfrak{S}_n$, thus we will often consider a one-specie pairing as a permutation of $\{1,...,n\}$. We can also inject the set of two-species pairings of n in $\mathcal{B}_n$. Recall the definitions of the transposition (i,j) and the Weyl contraction $[i,j]$ stated in Definition 2.8 of [**15**].

Definition 10.13. — *Let $m = \left(\left\{\{i, j_i\}, i \in I\right\}, T_m, W_m\right)$ be a two-species pairing of n. We consider $b_m \in \mathcal{B}_n$ equal to:*

$$b_m = \prod_{i \in I, \{i,j_i\} \in T_m} (i, j_i) \prod_{i \in I, \{i,j_i\} \in W_m} [i, j_i].$$

The elements in the product commute, thus the order is not important.

Often we will consider a two-species pairing m of n either as a colored partition of $\{1, ..., n\}$ or as an element of $\mathcal{B}_n$.

Definition 10.14. — *Let M be a random matrix in $\mathcal{M}_N(\mathbb{K})$. It is a standard Gaussian $\mathfrak{g}^\epsilon(N, \mathbb{K})$ matrix if the equality in law holds:*

$$M = \sqrt{N} H_1,$$

where $(H_t)_{t \geq 0}$ is a Brownian motion on $\mathfrak{g}^\epsilon(N, K)$.

When $\epsilon = 1$ this is an equivalent definition for the G.O.E and G.U.E ensembles.

Theorem 10.12. — *For any integer N, let M_N be a standard Gaussian $\mathfrak{g}^1(N, \mathbb{K})$ random matrix. The mean empirical eigenvalues distribution of $\frac{1}{\sqrt{N}} M_N$ converges in probability to the Wigner semicircular distribution μ_{sc} as N goes to infinity.*

Besides, one has a matricial Wick theorem.

Theorem 10.13. — *Let $(M_1, ..., M_k)$ be a random vector of matrices such that there exist a $k \times k'$ complex matrix A and a k'-tuple of independent standard Gaussian $\mathfrak{g}^\epsilon(N, \mathbb{K})$ matrices $(\tilde{M}_1, ..., \tilde{M}_{k'})$ such that the following equality in law holds:*

$$(M_i)_{i=1}^k = \left(\sum_{j=1}^{k'} A_{i,j} \tilde{M}_j \right)_{i=1}^k,$$

or in a shorter way:

$${}^t(M_1, ..., M_k) = A.{}^t(\tilde{M}_1, ..., \tilde{M}_{k'}).$$

Let us define for any i, j in {1,...,k}, $C(M_i, M_j) = (A\,{}^t A)_{i,j}$. Then:

$$\mathbb{E}[M_1 \otimes ... \otimes M_k] = \sum_{m \in \mathcal{F}_k(3 - \beta_{\mathbb{K}})} \left(\epsilon^{\#T_m} \prod_{\{i,j\} \in m} C(M_i, M_j) \right) b_m,$$

where we have made a slight abuse of notation by writing b_m instead of $\rho_N^{\mathcal{B}_k}(b_m)$.

Proof. — First of all, the result holds if one considers only one matrix M. Indeed, if M is a standard Gaussian $\mathfrak{g}^\epsilon(N, \mathbb{K})$ matrix, then:

$$\mathbb{E}\left[M^{\otimes k}\right] = \sum_{m \in \mathcal{F}_k(3 - \beta_{\mathbb{K}})} \epsilon^{\#T_m} b_m.$$

Indeed, M has the same law than $\sqrt{N}H_1$ where $(H_t)_{t\geq 0}$ is a Brownian motion on $\mathfrak{g}^\epsilon(N,K)$. Yet, we have seen in the proof of Theorem 10.10 that for any positive integer k:

$$G_k = \frac{d}{dt}_{|t=0} \mathbb{E}[H_t^{\otimes k}] = \delta_{k=2}\left[M_2^N\left(\epsilon(1,2) + (2-\beta_{\mathbb{K}})[1,2]\right)\right].$$

Using Lemma 10.1, for any positive real t_0:

$$\frac{d}{dt}_{|t=t_0} \mathbb{E}\left[H_t^{\otimes k}\right] = \sum_{I\subset\{1,...,k\},\#I=k-2} \mathcal{I}_I\left[\mathbb{E}\left[H_{t_0}^{\otimes(k-2)}\right], G_2\right].$$

Let $\left(E_t^k\right)_{k\in\mathbb{N},t\geq 0}$ be such that for any $k\in\mathbb{N}$, $(E_t^k)_{t\geq 0}\in\mathbb{C}[\mathcal{P}_k]^{\mathbb{R}^+}$ and such that $\left(E_t^k\right)_{k\in\mathbb{N},t\geq 0}$ is the solution of: $\forall k\in\mathbb{N}$:

$$\frac{d}{dt}_{|t=t_0} E_t^k = \sum_{I\subset\{0,...,k\},\#I=k-2} \mathcal{I}_I\left[E_t^{k-2}, \epsilon(1,2) + (2-\beta_{\mathbb{K}})[1,2]\right], \tag{21}$$

with the initial condition: $\forall k\in\mathbb{N}, E_0^k = id_k$. Then for any positive integer k, for any positive real t,

$$\mathbb{E}\left[H_t^{\otimes k}\right] = \rho_N^{\mathcal{B}_k}\left[M_2^N\left(E_t^k\right)\right].$$

Thus, we only have to find a solution of Equation (21): this is given for any positive real t and any $k\in\mathbb{N}$ by:

$$E_t^k = \sum_{m\in\mathcal{F}_k(3-\beta_{\mathbb{K}})} \epsilon^{\#T_m} t^{\frac{k}{2}} b_m.$$

Thus for any integer k:

$$\begin{aligned}\mathbb{E}\left[M^{\otimes k}\right] = N^{\frac{k}{2}}\mathbb{E}[H_1^{\otimes k}] = N^{\frac{k}{2}}\rho_N^{\mathcal{B}_k}\left[M_2^N\left(\sum_{m\in\mathcal{F}_k(3-\beta_{\mathbb{K}})} \epsilon^{\#T_m} b_m\right)\right]\\ = \sum_{m\in\mathcal{F}_k(3-\beta_{\mathbb{K}})} \epsilon^{\#T_m} b_m.\end{aligned}$$

Now let $(M_1,...,M_k)$ be a random vector of matrices such that there exist a $k\times k'$ complex matrix A and a k'-tuple of independent standard Gaussian $\mathfrak{g}^\epsilon(N,\mathbb{K})$ matrices $(\tilde{M}_1,...,\tilde{M}_k)$ such that the following equality in law holds:

$${}^t(M_1,...,M_k) = A.{}^t(\tilde{M}_1,...,\tilde{M}_{k'}).$$

Let us define for any i,j in {1,...,k}, $C(M_i,M_j) = (A\,{}^tA)_{i,j}$. Then:

$$\begin{aligned}\mathbb{E}\left[M_1\otimes...\otimes M_k\right] &= \sum_{i_1,...,i_k=1}^{k'} A_1^{i_1}...A_k^{i_k}\mathbb{E}[\tilde{M}_{i_1}\otimes...\otimes\tilde{M}_{i_k}]\\ &= \sum_{\pi\in\mathsf{P}_k}\sum_{\mathsf{Ker}((i_1,...,i_k))=\pi} A_1^{i_1}...A_k^{i_k}\mathbb{E}[\tilde{M}_{i_1}\otimes...\otimes\tilde{M}_{i_k}].\end{aligned}$$

Using the independence of the $\left(\tilde{M}_i\right)_{i=1}^{k'}$, and using the result already proved for one matrix, one gets:

$$\begin{aligned}&\mathbb{E}\left[M_1 \otimes ... \otimes M_k\right]\\ &= \sum_{\pi \in \mathsf{P}_k} \left(\sum_{\mathsf{Ker}((i_1,...,i_k))=\pi} A_1^{i_1}...A_k^{i_k} \right) \sum_{m \in \mathcal{F}_k(3-\beta_{\mathbb{K}}), m \leq \pi} \epsilon^{\#T_m} b_m.\end{aligned}$$

We can go on the calculation:

$$\begin{aligned}&\mathbb{E}\left[M_1 \otimes ... \otimes M_k\right]\\ &= \sum_{m \in \mathcal{F}_k(3-\beta_{\mathbb{K}})} \left(\sum_{\pi \in \mathsf{P}_k, m \leq \pi} \quad \sum_{\mathsf{Ker}((i_1,...,i_k))=\pi} A_1^{i_1}...A_k^{i_k} \right) \epsilon^{\#T_m} b_m,\end{aligned}$$

and the result follow from the easy equality which holds for any $m \in \mathcal{F}_k(3-\beta_{\mathbb{K}})$:

$$\begin{aligned}\sum_{\pi \in \mathsf{P}_k, m \leq \pi} \quad \sum_{\mathsf{Ker}((i_1,...,i_k))=\pi} A_1^{i_1}...A_k^{i_k} &= \prod_{\{i,j\} \in m} \sum_{l=1}^{k'} A_i^l A_j^l\\ &= \prod_{\{i,j\} \in m} C(M_i, M_j).\end{aligned}$$

Hence the matricial Wick formula. □

We stated the result for the calculation of $\mathbb{E}[M_1 \otimes ... \otimes M_k]$, yet using Theorem 10.13, one can see that for any $l \in \{1, ..., k-1\}$, one can calculate the value of $\mathbb{E}\left[M_1 \otimes ... \otimes M_l \otimes M_l \otimes \overline{M_{l+1}} \otimes ... \otimes \overline{M_k}\right]$. One can use two ways to do it: one can either decompose any random matrices using its real part and imaginary part, and apply Theorem 10.13 for the real matrices, or one can use the fact that M^* is either equal to M or $-M$. Let us illustrate the second possibility: for example, let us consider $\mathbb{E}\left[M^{\otimes k} \otimes \overline{M}^{\otimes k}\right]$, M being a standard Gaussian $\mathfrak{g}^1(N, \mathbb{C})$. Using Theorem 10.13, we know that:

$$\mathbb{E}\left[M^{\otimes k} \otimes (M^*)^{\otimes k}\right] = \mathbb{E}\left[M^{\otimes k} \otimes M^{\otimes k}\right] = \sum_{m \in \mathcal{F}_k(1)} \epsilon^{\#T_m} b_m.$$

And thus:

$$\mathbb{E}\left[M^{\otimes k} \otimes \overline{M}^{\otimes k}\right] = S_k\left(\mathbb{E}[M^{\otimes k} \otimes (M^*)^{\otimes k}]\right) = \sum_{m \in \mathcal{F}_k(1)} \epsilon^{\#T_m} S_k\left(b_m\right),$$

where the operation S_k was defined in Definition 2.5.

10.3.6. Convergence of multiplicative Brownian motion. — Let us now state the results about convergence of multiplicative Brownian motions. Let A be equal to $\mathfrak{S}$ if $\mathbb{K} = \mathbb{C}$ or equal to $\mathcal{B}$ if $\mathbb{K} = \mathbb{R}$. For any integer N, let $\left(U_t^N\right)_{t \geq 0}$ be a Brownian motion on $U(N, \mathbb{K})$. The next theorem is inspired by the results of T. Lévy in [**23**]: we extend slightly his results to our setting.

Theorem 10.14. — *The process* $\left(U_t^N\right)_{t\geq 0}$ *converges in* $\mathcal{P}$*-expectation up to any order of fluctuations as* N *tends to infinity. The family* $\left(\left(U_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ *is a matricial* A *and* $\mathcal{P}$*-non-commutative multiplicative Lévy process up to any order of fluctuations.*

The asymptotic $\mathcal{P}$*-factorization property holds for* $\left(\left(U_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$*. For any* $t\in\mathbb{R}^+$*:*

$$\mathcal{R}_A\left(\left(U_t^N\right)_{N\in\mathbb{N}}\right)=e_{\boxtimes}^{t\mathcal{R}_A(G)},$$

where, for any positive integer k*,*

$$\left(\mathcal{R}_A(G)\right)_k=-\frac{k}{2}id_k+\sum_{1\leq i<j\leq k}\left(-(i,j)+(2-\beta_{\mathbb{K}})\,[i,j]\right). \tag{22}$$

And for any $t_0\geq 0$*, any positive integer* k*, and for any* $p\in\mathcal{P}_k$*:*

$$\frac{d}{dt}_{|t=t_0}\mathbb{E}m_p(U_t,...,U_t)=\sum_{p_1\in A_k,p_1\leq p}\left(\mathcal{R}_A(G)\right)_{p_1}\mathbb{E}m_{t_{p_1\circ p}}(U_{t_0},...,U_{t_0}). \tag{23}$$

Let us suppose that $\mathbb{K}=\mathbb{C}$*. Then the family* $\left(U_t^N,\left(U_t^N\right)^*\right)_{t\geq 0}$ *converges in* $\mathcal{P}$*-expectation up to any order of fluctuations as* N *tends to infinity. Besides, the asymptotic* $\mathcal{P}$*-factorization property holds for* $\left(\left(U_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}\cup\left(\left(\left(U_t^N\right)^*\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$*. In particular, the normalized* $\mathcal{P}$*-moments of the family* $\left(U_t^N,\left(U_t^N\right)^*\right)_{t\geq 0}$ *converges in probability to the limit of their expectation as* N *goes to infinity.*

Let us use the same notations as for Theorem 10.5. Then for any real $t_0\geq 0$*, for any positive integer* k *and* l*, the following equality holds in* $\mathbb{C}[\mathcal{P}_{k+l}(\infty,\infty)]$*:*

$$\frac{d}{dt}_{|t=t_0}\left(\mathcal{R}(U_t)\right)_{k,l}=\left(\mathcal{R}(G)\right)_{k,l}\left(\mathcal{R}(U_{t_0})\right)_{k,l},$$

where

$$\left(\mathcal{R}(G)\right)_{k,l}=-\frac{k+l}{2}id_{k+l}-\sum_{1\leq i<j\leq k+k'}(i,j)-\sum_{1\leq i\leq k<j\leq k+k'}\left([i,j]-(i,j)\right).$$

Moreover, for any positive integer k *and* l*, any* $p\in\mathcal{P}_{k+l}$ *and any* $t_0\geq 0$*:*

$$\frac{d}{dt}_{|t=t_0}m_p\left[\mathbb{E}\left(U_t^{\otimes k}\otimes\overline{U_t}^{\otimes l}\right)\right]=\sum_{p_1\in A_k,p_1\leq p}\left(\left(\mathcal{R}(G)\right)_{k,l}\right)_{p_1}m_{t_{p_1\circ p}}\left[\mathbb{E}\left(U_t^{\otimes k}\otimes\overline{U_t}^{\otimes l}\right)\right],$$

where $m_p\left[\mathbb{E}\left(U_t^{\otimes k}\otimes\overline{U_t}^{\otimes l}\right)\right]$ *stands for the limit as* N *goes to infinity of the mean normalized moment* $m_p\left[\mathbb{E}\left(U_t^{N\otimes k}\otimes\left(\overline{U_t^N}\right)^{\otimes l}\right)\right]$*.*

Proof of Theorem 10.14. — For any integer N, let $(U_t^N)_{t\geq 0}$ be a Brownian motion on $U(N,\mathbb{K})$. Let us prove first all the statements about $\left((U_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$. We have seen that $\left(U_t^N\right)_{t\geq 0}$ is invariant by conjugation by $U(N,\mathbb{K})$. As an application of Theorems 10.6, 10.2 and 10.3, using the notations of Lemma 10.6, it remains to show that for any

positive integer k, $\left(G_k^N\right)_{N\in\mathbb{N}}$, seen as an element of $\prod_{N\in\mathbb{N}}\mathbb{C}\left[A_k(N)\right]$, converges up to any order of fluctuations and that for any positive integer k,

$$\left(\mathcal{R}_A[G]\right)_k = -\frac{k}{2}id_k + \sum_{1\leq i<j\leq k}\left(-(i,j)+(2-\beta_{\mathbb{K}})[i,j]\right).$$

Indeed if so, by Theorem 10.6 the process $\left(U_t^N\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation up to any order of fluctuations as N tends to infinity. Since (i,j) and $[i,j]$ are weakly irreducible, $\mathcal{R}_A[G]\in\mathfrak{me}_{\boxtimes}[A]$: by Theorem 10.3, the asymptotic $\mathcal{P}$-factorization holds for $\left((U_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$. The remaining assertion is a consequence of Theorem 10.2.

Using Lemma 10.6, Lemma 10.3 of this article and Lemma 5.4 of [**15**], it is enough to show that for any positive integer k:

$$\left(M_k^N\right)^{-1}\left(\frac{kc_{\mathfrak{u}(N,\mathbb{K})}}{2}id_k + \frac{1}{N}\sum_{1\leq i<j\leq k}\left(-(i,j)+(2-\beta_{\mathbb{K}})[i,j]\right)\right)$$

converges in $\mathbb{C}[A_k]$ and has an asymptotic development as an infinite power series in $\frac{1}{N}$. Using Lemma 10.4, for any positive integer N, this is equal to:

$$\frac{k}{2}\left(-1+\frac{2-\beta_{\mathbb{K}}}{N}\right)id_k + \sum_{1\leq i<j\leq k}\left(-(i,j)+(2-\beta_{\mathbb{K}})[i,j]\right).$$

Thus for any positive integer k, $\left(G_k^N\right)_{N\in\mathbb{N}}$, seen as an element of $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$, converges up to any order of fluctuations, and using the discussion we just had:

$$\mathcal{R}_A(G) = \mathfrak{J}\left(-\frac{1}{2}id_1 - (1,2) + (2-\beta_{\mathbb{K}})[1,2]\right)$$

where $\mathfrak{J}$ was defined in Definition 10.6 of [**15**], or equivalently, for any integer k,

$$\left(\mathcal{R}_A(G)\right)_k = -\frac{k}{2}id_k + \sum_{1\leq i<j\leq k}\left(-(i,j)+(2-\beta_{\mathbb{K}})[i,j]\right).$$

This finishes the first part of the theorem.

Let us suppose that $\mathbb{K}=\mathbb{C}$. We can prove the assertions about $\left(\left(U_t^N,(U_t^N)^*\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ and $\left(\left(U_t^N,\overline{U_t^N}\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ by using similar arguments to the ones we just used and using Lemma 10.7, Theorem 10.4, Theorem 10.5 and Theorem 2.2. □

One can also calculate for any positive real t, the limit of the moments of the mean empirical eigenvalues distribution of U_t^N. The first proof of the next theorem was given by Biane in [**8**], and then by Lévy in [**24**]. The proof we give here is just a reformulation in our framework of the latter proof. For this, let us define, for any real $t\geq 0$ and any positive integer k,

$$m_k(t) = e^{-\frac{kt}{2}}\sum_{l=0}^{k-1}\frac{(-t)^l}{l!}k^{l-1}\binom{k}{l+1}.$$

Theorem 10.15. — *For any integer N, let $\left(U_t^N\right)_{t\geq 0}$ be a Brownian motion on $U(N,\mathbb{K})$. For any $t\geq 0$, the empirical eigenvalues distribution of U_t^N converges in probability to a measure ν_t as N goes to infinity and for any positive integer k:*

$$\int_{\mathbb{U}} z^k \nu_t(dz) = \int_{\mathbb{U}} z^{-k}\nu_t(dz) = m_k(t).$$

Thus for any integer $r\geq 1$, any integers $l_1,...,l_r$ in $\mathbb{Z}$, for any real $t\geq 0$,

$$\mathbb{E}\left[\frac{1}{N}Tr\left((U_t^N)^{l_1}\right)...\frac{1}{N}Tr\left((U_t^N)^{l_r}\right)\right] \underset{N\to\infty}{\longrightarrow} \prod_{i=1}^{r} m_{l_i}(t).$$

Proof. — For any positive integer N, let $\left(U_t^{N,\mathbb{K}}\right)_{t\geq 0}$ be a Brownian motion on $U(N,\mathbb{K})$. Let $\mathcal{R}_A\left[G^{\mathbb{K}}\right]$ be the $\mathcal{R}_A$-functional of the family $\left(\left(G_k^{N,\mathbb{K}}\right)\right)_{k\in\mathbb{N}}$, where as usual,

$$G_k^{N,\mathbb{K}} = \frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(U_t^{N,\mathbb{K}}\right)^{\otimes k}\right].$$

Using Theorem 10.14, $\left(\left(U_t^{N,\mathbb{K}}\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation and each normalized $\mathfrak{S}$-moment of $\left(\left(U_t^{N,\mathbb{K}}\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ converges in probability to the limit of their expectation. Using a similar argument to the one used in Theorem 2.1, for any positive real t, the empirical eigenvalues distribution of $U_t^{N,\mathbb{K}}$ converges in probability to a probability measure called $\nu_{\mathbb{K},t}$. Again using Theorem 10.14, the restriction on $\mathfrak{E}[\mathfrak{S}]$ of $\mathcal{R}_{\mathcal{B}}\left[G^{\mathbb{R}}\right]$ is equal to $\mathcal{R}_{\mathfrak{S}}\left[G^{\mathbb{C}}\right]$. Thus, by Theorem 10.8, $\left(\left(U_t^{N,\mathbb{R}}\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ and $\left(\left(U_t^{N,\mathbb{C}}\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ have the same $\mathfrak{S}$-law: in particular, for any positive real t,

$$\nu_{\mathbb{C},t} = \nu_{\mathbb{R},t}.$$

We denote for any positive real t, $\nu_t = \nu_{\mathbb{C},t}$. We have to show now that for any $k\geq 1$:

$$\int_{\mathbb{U}} z^k \nu_t(dz) = m_k(t).$$

For any integer $k\geq 1$, let us denote by $m_{\nu,k}(t)$ the moment $\int_{\mathbb{U}} z^k\nu_t(dz)$. Using the Equality (22) in Theorem 10.14, the family $(m_{\nu,k}(t))_{t\geq 0,k\in\mathbb{N}}$ satisfies the following system of differential equations:

$$\forall k\geq 1, \forall t_0\geq 0, \frac{d}{dt}_{|t=t_0} m_{\nu,k}(t) = -\frac{k}{2}m_{\nu,k}(t_0) - \frac{k}{2}\sum_{l=1}^{k-1} m_{\nu,l}(t_0)m_{\nu,k-l}(t_0).$$

In [**23**], Lévy proved in Lemma 2.4 that $(m_k(t))_{t\geq 0,k\in\mathbb{N}}$ also satisfies the same system of linear differential equations with the same initial conditions. By unicity we get that:

$$\int_{\mathbb{U}} z^k \nu_t(dz) = m_k(t).$$

This allows us to conclude. □

10.4. Lévy processes. —

10.4.1. The generator. — The main step in order to show the convergence of the Brownian motions in Section 10.3, was to compute the action of the generator at time $t = 0$ on the application $M \to M^{\otimes k}$. The Itô's formula allowed us to do so in Lemma 10.5 and Lemma 10.6. Fortunately, there exists an automatic way to compute the generator given by Theorem 31.5 in [**30**] and Hunt's Theorem, Theorem 1.1 in [**25**].

Theorem 10.16. — *Let E be a finite dimensional vector space of dimension d, let $(Y_i)_{i=1}^d$ be a basis of E. Let $(X_t)_{t\geq 0}$ be an additive Lévy process in E. There exist:*

1. *$Y_0 \in E$,*
2. *a symmetric positive semidefinite matrix $(y_{i,j})_{i,j=1}^d$,*
3. *a Lévy measure Π on E, that is a measure on E such that $\Pi(\{0\}) = 0$ and such that, if B is the ball of center 0 and radius 1 in E:*

$$\int_B || x ||_E^2 \, \Pi(dx) \leq \infty \text{ and } \Pi(B^c) < \infty,$$

such that the generator G of $(X_t)_{t\geq 0}$ is given for any $f \in C_0^2(E)$ and any $y \in E$ by $Gf(y) = \frac{d}{dt}_{|t=0}\mathbb{E}[f(y + X_t)]$ which is equal to:

$$\partial_{X_0} f(y) + \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}\partial_{Y_i}\partial_{Y_j} f(y) + \int_{\mathcal{H}_N} [f(y + x) - f(y) - \mathbb{1}_B(x)\partial_x f(y)]\,\Pi(dx).$$

Conversely, every operator of this form is the generator of a unique Lévy process $(X_t)_{t\geq 0}$.

Besides, let us suppose that E is equal to the Lie algebra $\mathfrak{g}$ of a compact Lie group G. Let H be a Lie subgroup of G. Let us suppose that Y_0, the operator $\sum_{i,j=1}^{N^2} y_{i,j}\partial_{Y_i}\partial_{Y_j}$ and Π are invariant by conjugation by any element of H, then the Lévy process associated is invariant by conjugation by H.

Remark 10.2. — *One can change $\mathbb{1}_B$ in the form of the generator given by the last theorem by anything of the form $\mathbb{1}_V$ where V is a neighborhood of 0. This operation only changes the drift X_0. This remark will be important latter as we will work with a Lévy measure which is compactly supported: it is then easier to suppose that $\mathsf{Supp}(\Pi) \subset B$.*

A similar result exists for compact Lie groups. Let G be a compact Lie group and let $\mathfrak{g}$ be the Lie algebra of G. Let $A : G \to \mathfrak{g}$ be a smooth mapping such that:

$$A(Id) = 0 \text{ and } d_{Id} = id_{\mathfrak{g}}.$$

As explained by G. Cébron in [**10**] for the case of $U(N)$-valued Lévy processes, one can use on $U(N, \mathbb{K})$ the mapping:

$$A : M \mapsto \frac{M - M^*}{2}.$$

From now on, we will denote this application $i\Im$, even when we are working on $O(N)$. Besides, it is also invariant by conjugation by $U(N, \mathbb{K})$: for any $U \in U(N, \mathbb{K})$, for any $M \in \mathcal{M}(N, \mathbb{K})$:

$$UA(M)U^{-1} = A(UMU^{-1}).$$

For any matrix M, we denote by $\Re(M)$ the matrix $\frac{M+M^*}{2}$. We recall also that any element X in the Lie algebra $\mathfrak{g}$ induces a right invariant vector field which is defined

for any g in $\mathfrak{g}$, by $X^r(g) = DR_g(X)$, with DR_g being the diffential map of the right multiplication operation $R_g : h \mapsto gh$. Hunt's theorem, see [**25**], allows to compute the generator of a Lévy process on a compact Lie group.

Theorem 10.17. — *Let $(X_t)_{t\geq 0}$ be a Lévy process on a compact Lie group G. Let d be the dimension of the Lie algebra $\mathfrak{g}$ of G. Let $(Y_i)_{i=1}^d$ be a basis of $\mathfrak{g}$. There exist:*

1. *$Y_0 \in \mathfrak{g}_N$,*
2. *a symmetric positive semidefinite matrix $(y_{i,j})_{i,j=1}^d$,*
3. *a Lévy measure Π on G, that is a measure on G such that $\Pi(\{Id\}) = 0$ and for any neighborhood V of Id in G, we have:*

$$\int_V ||\, A(x)\,||_{\mathfrak{g}}^2\, \Pi(dx) \leq \infty \text{ and } \Pi(V^c) < \infty,$$

such that the generator G of $(X_t)_{t\geq 0}$ is given for any $f \in C^2(G)$ and any $h \in U_N$ by $Gf(h) = \frac{d}{dt}_{|t=0}\mathbb{E}[f(X_t h)]$ which is equal to:

$$Y_0^r f(h) + \frac{1}{2}\sum_{i,j=1}^d y_{i,j} Y_i^r Y_j^r f(h) + \int_G [f(gh) - f(h) - A(g)^r f(h)]\, \Pi(dg).$$

Conversely, every operator of this form is the generator of a unique Lévy process $(X_t)_{t\geq 0}$.

Let H be a Lie subgroup of G. Let us suppose that Y_0, the measure Π and the operator $\sum_{i,j=1}^d y_{i,j} Y_i^l Y_j^l$ are invariant by conjugation by any element of H, then the Lévy process associated is invariant by conjugation by H.

Remark 10.3. — *We will apply Theorem 10.16 and Theorem 10.17 to the case where E is one of the Lie algebra defined in Section 10.3.1 and where G is one of the Lie group defined in the same section. In Definition 10.8, we defined a scalar product on these Lie algebras: from now on, we will always assume that the basis $(Y_i)_{i=1}^d$ used in both theorems is an orthonormal basis for this scalar product.*

10.4.2. Free additive and multiplicative divisible measures. — One of our goal is to study the convergence of some natural Lévy processes using the new setting we developed and to characterize the limit of the mean eigenvalues distribution. In order to do so, we need to introduce the well-known notions of free additive and multiplicative infinitely divisible measures. First, let us recall the notions of free additive and multiplicative convolutions.

We denote for any topological set K, $\mathcal{P}(K)$ the set of probability measure on K and $\mathcal{P}^c(K)$ the elements in $\mathcal{P}(K)$ which are compactly supported. Let $\mathbb{U}$ be the unit circle. Recall all the definitions in Section 10 of [**15**].

Definition 10.15. — *Let μ be in $\mathcal{P}^c(\mathbb{C})$. For any positive integer n, let $[(1,...,n)]_{\mathfrak{S}_n}$ be the conjugacy class of $(1,...,n)$ in $\mathfrak{S}_n$. Let $\mathcal{M}^{(i)}(\mu)$ be the element of $\mathfrak{E}^{(i)}[\mathfrak{S}]$ such that for any integer n, for any $\sigma \in [(1,...,n)]_{\mathfrak{S}_n}$, $\left(\mathcal{M}^{(i)}(\mu)\right)_\sigma = \int_{\mathbb{C}} z^n d\mu(z)$. The moment invariant of μ is:*

$$\mathcal{M}(\mu) = \mathfrak{M}\left(\mathcal{M}^{(i)}(\mu)\right).$$

The cumulant invariant of μ is:

$$\mathcal{R}(\mu) = \mathcal{R}_{\mathfrak{S}}\left[\mathcal{M}(\mu)\right].$$

The moment invariant and the cumulant invariant characterize the measures which are compactly supported either by the real line or by the unit circle, this is a well-known consequence of the Stone-Weierstrass's theorem.

Lemma 10.8. — *Let μ_1, μ_2 be two measures either both in $\mathcal{P}^c(\mathbb{R})$ or in $\mathcal{P}(\mathbb{U})$. If either $\mathcal{M}(\mu_1) = \mathcal{M}(\mu_2)$ or $\mathcal{R}(\mu_1) = \mathcal{R}(\mu_2)$ then $\mu_1 = \mu_2$.*

Recall the notion of strong approximation of a measure μ defined in Section 9.1. The following lemma is straightforward.

Lemma 10.9. — *Let $\mu \in \mathcal{P}^c(\mathbb{C})$. Let $(M_N)_{N\in\mathbb{N}}$ be an element of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which is a strong approximation of μ, then $\mathcal{M}_{\mathfrak{S}}[(M_N)_{N\in\mathbb{N}}] = \mathcal{M}[\mu]$.*

In order to define the free additive convolution, we need the following lemma.

Lemma 10.10. — *For any Hermitian matrix M of size N, we denote by $\lambda_1(M) \geq ... \geq \lambda_N(M)$ the eigenvalues of M. Let A and B be two Hermitian matrices, for any $i \in \{1, ..., N\}$:*

$$\lambda_N(A) + \lambda_N(B) \leq \lambda_i(A+B) \leq \lambda_1(A) + \lambda_1(B).$$

Now we can define the free additive convolution of two probability measures supported by the real line.

Proposition 10.1. — *Let μ and ν be in $\mathcal{P}^c(\mathbb{R})$. There exists a unique probability measure in $\mathcal{P}^c(\mathbb{R})$, denoted $\mu \boxplus \nu$, such that the equation:*

$$\mathcal{R}(\mu \boxplus \nu) = \mathcal{R}(\mu) \boxplus \mathcal{R}(\nu)$$

holds in $\mathfrak{E}[\mathfrak{S}]$. Moreover, the application $\boxplus : \mathcal{P}^c_{\mathbb{R}} \times \mathcal{P}^c_{\mathbb{R}} \to \mathcal{P}^c_{\mathbb{R}}$ can be extended by continuity from $\mathcal{P}^c_{\mathbb{R}}$ to $\mathcal{P}_{\mathbb{R}}$.

Proof. — We will only give a proof of the first assertion. The second result is a consequence of Proposition 4.13. of [**7**].

Let μ and ν be in $\mathcal{P}^c(\mathbb{R})$. Their moments are finite: let us consider two independent natural strong approximations of μ and ν, as defined in Proposition 9.1, respectively $\left(M_\mu^N\right)_{N\in\mathbb{N}}$ and $\left(M_\nu^N\right)_{N\in\mathbb{N}}$. For each positive integer N, let us consider U_N a matrix such that $\left(M_\mu^N, M_\nu^N, U_N\right)$ are independent, and U_N is Haar distributed on $U(N)$. Then for any integer N, M_μ^N is independent of $U_N M_\nu^N U_N^*$ which is invariant by conjugation by $U(N)$. By Theorem 7.10, $\left(M_\mu^N\right)_{N\in\mathbb{N}}$ and $\left(U_N M_\nu^N U_N^*\right)_{N\in\mathbb{N}}$ are asymptotically $\mathfrak{S}$-free. By Theorem 7.1,

$$M^N = M_\mu^N + U_N M_\nu^N U_N^*$$

converges in $\mathfrak{S}$-expectation when N goes to infinity. Besides, by Lemma 10.10, the mean eigenvalues distributions $\mathbb{E}\left[\eta_{M^N}\right]$ of M^N are uniformly compactly supported. By Theorem 2.1, there exists a measure that we denote $\mu \boxplus \nu$ such that:

$$\mathbb{E}\left[\eta_{M^N}\right] \underset{N\to\infty}{\longrightarrow} \mu \boxplus \nu.$$

By Theorem 7.9, we also get that:

$$\mathcal{R}_{\mathfrak{S}}\left(\left(M^N\right)_{N\in\mathbb{N}}\right) = \mathcal{R}_{\mathfrak{S}}\left(\left(M_\mu^N\right)_{N\in\mathbb{N}}\right) \boxplus \mathcal{R}_{\mathfrak{S}}\left(\left(M_\nu^N\right)_{N\in\mathbb{N}}\right).$$

Let us remark that $\left(M^N\right)_{N\in\mathbb{N}}$ is a strong approximation of $\mu\boxplus\nu$. Indeed, as $(M_\mu^N)_{N\in\mathbb{N}}$ and $(M_\nu^N)_{N\in\mathbb{N}}$ are strong approximations, $\mathcal{R}_{\mathfrak{S}}\left(\left(M_\mu^N\right)_{N\in\mathbb{N}}\right)$ and $\mathcal{R}_{\mathfrak{S}}\left(\left(M_\nu^N\right)_{N\in\mathbb{N}}\right)$ are multiplicative. By Theorem 10.1 of [**15**],

$$\mathcal{R}_{\mathfrak{S}}\left(\left(M_\mu^N\right)_{N\in\mathbb{N}}\right) \boxplus \mathcal{R}_{\mathfrak{S}}\left(\left(M_\nu^N\right)_{N\in\mathbb{N}}\right)$$

is also multiplicative. This implies that $\mathcal{R}_{\mathfrak{S}}\left(\left(M^N\right)_{N\in\mathbb{N}}\right)$ is multiplicative, and thus $\mathcal{M}_{\mathfrak{S}}\left(\left(M^N\right)_{N\in\mathbb{N}}\right)$ is multiplicative. Using Lemma 10.9, the measure $\mu\boxplus\nu$ satisfies:

$$\mathcal{R}(\mu\boxplus\nu) = \mathcal{R}(\mu)\boxplus\mathcal{R}(\nu),$$

which is the equality we wanted to prove. □

Proposition 10.2. — *Let μ and ν be in $\mathcal{P}(\mathbb{U})$. There exists a unique probability measure in $\mathcal{P}(\mathbb{U})$, denoted $\mu\boxtimes\nu$, such that the equation:*

$$\mathcal{R}(\mu\boxtimes\nu) = \mathcal{R}(\mu)\boxtimes\mathcal{R}(\nu)$$

holds in $\mathfrak{E}[\mathfrak{S}]$.

Proof. — The proof is similar to the proof of Proposition 10.1. Let μ and ν be in $\mathcal{P}(\mathbb{U})$. We can consider two independent natural strong matricial approximations of μ and ν, respectively $\left(M_\mu^N\right)_{N\in\mathbb{N}}$ and $\left(M_\nu^N\right)_{N\in\mathbb{N}}$. For each positive integer N, let us consider U_N a matrix such that $\left(M_\mu^N, M_\nu^N, U_N\right)$ are independent, and U_N is Haar distributed on $U(N)$. By the Theorems 7.1 and 7.10, the matrix:

$$M^N = M_\mu^N U_N M_\nu^N U_N^*$$

converges in $\mathfrak{S}$-expectation when N goes to infinity. For any integer N, M^N is a unitary matrix: the mean eigenvalues distribution $\mathbb{E}\left[\eta_{M^N}\right]$ of M^N is supported by $\mathbb{U}$. By Theorem 2.1, there exists a measure that we denote $\mu\boxtimes\nu$ such that:

$$\mathbb{E}\left[\eta_{M^N}\right] \underset{N\to\infty}{\longrightarrow} \mu\boxtimes\nu.$$

Besides, by Theorem 7.9, we also get that:

$$\mathcal{R}\left(\left(M^N\right)_{N\in\mathbb{N}}\right) = \mathcal{R}\left(\left(M_\mu^N\right)_{N\in\mathbb{N}}\right) \boxtimes \mathcal{R}\left(\left(M_\nu^N\right)_{N\in\mathbb{N}}\right),$$

in $\mathfrak{E}[\mathfrak{S}]$. Using similar arguments to those used in Proposition 10.1, $\left(M^N\right)_{N\in\mathbb{N}}$ is a strong approximation of $\mu\boxtimes\nu$. Using Lemma 10.9, the measure $\mu\boxtimes\nu$ satisfies:

$$\mathcal{R}(\mu\boxtimes\nu) = \mathcal{R}(\mu)\boxtimes\mathcal{R}(\nu),$$

which is the equality we wanted to prove. □

As one can do for the usual convolution, we can define the notion of free additive and multiplicative infinitely divisible measures.

Definition 10.16. — *Let μ be in $\mathcal{P}(\mathbb{R})$ and ν be in $\mathcal{P}(\mathbb{U})$. The measure μ is a free additive infinitely divisible measure if for any integer $n \geq 1$ there exists $\mu_{\frac{1}{n}} \in \mathcal{P}(\mathbb{R})$ such that $\mu = \mu_{\frac{1}{n}}^{\boxplus n}$. The measure ν is a free multiplicative infinitely divisible measure if for any integer $n \geq 1$ there exists $\nu_{\frac{1}{n}} \in \mathcal{P}(\mathbb{U})$ such that $\nu = \nu_{\frac{1}{n}}^{\boxtimes n}$.*

Let $\lambda_{\mathbb{U}}$ be the uniform probability measure on $\mathbb{U}$. Let μ be any measure in $\mathcal{P}(\mathbb{R})$ or $\mathcal{P}(\mathbb{U}) \setminus \{\lambda_{\mathbb{U}}\}$ which is a free additive (resp. multiplicative) infinitely divisible measure. There exists a continuous one parameter semi-group of measures $(\mu_t)_{t\geq 0}$ for the free additive (resp. multiplicative) convolution such that $\mu = \mu_1$ and $\mu_0 = \delta_0$ (resp. $\mu_0 = \delta_1$). This semi-group is unique in the additive case and in the multiplicative case there is a canonical way to associate a free multiplicative convolution semi-group. We will not explain in details how to construct these continuous one parameter semi-group of measures. Just let us explain briefly what is the canonical choice for the free multiplicative convolution.

Let ν be a free multiplicative infinitely divisible measure in $\mathcal{P}(\mathbb{U}) \setminus \{\lambda_{\mathbb{U}}\}$. From the explanations given in Section 2.6 of [**10**], there exist a unique $\omega \in \mathbb{U}$ and a unique measure ρ on $\mathbb{U}$ such that $\rho(\mathbb{U}) < \infty$, such that the S-transform of ν is given by:

$$S(z) = \omega \exp\left(\int_{\mathbb{U}} \frac{1+z+\zeta z}{1+z-\zeta z} d\rho(\zeta)\right).$$

Conversely, any function of the form given above is the S-transform of a unique free multiplicative infinitely divisible measure in $\mathcal{P}(\mathbb{U})$. We did not define in the article what was the S-transform but one can read [**10**] for an explanation. The only thing to be known for this article is that it characterizes the measures defined on $\mathbb{U}$ and it transforms the $\boxtimes$ operation in a multiplication in the space of formal power series. Let $\mathsf{arg}(\omega)$ be the argument of ω defined in $]-\pi, \pi]$. Let us define for any $t \geq 0$,

$$S_t(z) = e^{it\mathsf{arg}(\omega)} \exp\left(\int_{\mathbb{U}} \frac{1+z+\zeta z}{1+z-\zeta z} d(t\rho)(\zeta)\right),$$

then for any positive real t there exists ν_t such that the S-transform of ν_t is S_t. Yet, for any positive real t_1 and t_2, $S_{t_1} S_{t_2} = S_{t_1+t_2}$ thus $\nu_{t_1} \boxtimes \nu_{t_2} = \nu_{t_1+t_2}$. The family $(\nu_t)_{t\geq 0}$ is a one-parameter semi-group of measures for the free multiplicative convolution which is in fact continuous and which satisfies $\nu_1 = \nu$. We see here that one could have chosen any $\mathsf{arg}(\omega) + 2k\pi$, with $k \in \mathbb{Z}$, instead of $\mathsf{arg}(\omega)$ in the definition of S_t: this would have given a different one parameter continuous semi-group hence the non-unicity property stated above. Each time we will consider the canonical semi-group associated with ν, the reader will have to understand that we consider the one constructed just above with the use of $\mathsf{arg}(\omega)$.

Let μ be in $\mathcal{P}(\mathbb{R})$ and let ν be in $\mathcal{P}(\mathbb{U}) \setminus \{\lambda_{\mathbb{U}}\}$. Let us suppose from now on that μ is a free additive infinitely divisible measure and that ν is a free multiplicative infinitely divisible measure. Let $(\mu_t)_{t\geq 0}$ and $(\nu_t)_{t\geq 0}$ be the canonical continuous semi-group associated respectively with μ and ν. By definition, $(\mathcal{R}(\mu_t))_{t\geq 0}$ and $(\mathcal{R}(\nu_t))_{t\geq 0}$ are continuous one parameter semi-groups in respectively $(\mathfrak{E}[\mathfrak{S}], \boxplus)$ and $(\mathfrak{E}[\mathfrak{S}], \boxtimes)$ which begin at the

neutral element. Thus there exist $\mathcal{G}(\mu)$ and $\mathcal{LR}(\nu)$ in $\mathfrak{E}[\mathfrak{S}]$ such that:

$$\frac{d}{dt}_{|t=t_0} \mathcal{R}(\mu_t) = \mathcal{G}(\mu) \boxplus \mathcal{R}(\mu_{t_0}), \tag{24}$$

$$\frac{d}{dt}_{|t=t_0} \mathcal{R}(\nu_t) = \mathcal{LR}(\nu) \boxtimes \mathcal{R}(\nu_{t_0}), \tag{25}$$

hold in $\mathfrak{E}[\mathfrak{S}]$. The element $\mathcal{G}(\mu)$ is easy to understand.

Lemma 10.11. — *The element $\mathcal{G}(\mu)$ is the unique element in $\mathfrak{me}_{\boxplus}[\mathfrak{S}]$ such that, for any positive integer k, for any $p \in \mathfrak{S}_k$ which is irreducible:*

$$(\mathcal{G}(\mu))_p = (\mathcal{R}(\mu))_p .$$

Definition 10.17. — *The log-cumulant invariant of ν is $\mathcal{LR}(\nu)$.*

This definition is equivalent to the one given by G. Cébron in [**11**] and [**10**]. Let us remark that in some sense, the S-transform is the exponentiation in the algebra of formal series of the log-cumulant. One can introduce the S-transform as a way to send the $\boxtimes$ operation on the simpler multiplication of formal series.

Using Lemma 10.8, we get the following lemma.

Lemma 10.12. — *Let ν_1 and ν_2 be two free multiplicative infinitely divisible measures in $\mathcal{P}(\mathbb{U})$ distinct from $\lambda_{\mathbb{U}}$. If $\mathcal{LR}(\nu_1)$ is equal to $\mathcal{LR}(\nu_2)$, then $\nu_1 = \nu_2$.*

It remains to characterize the cumulant invariant of free additive infinitely divisible measures and the log-cumulant invariant of free multiplicative infinitely divisible measures. This is given by the two following theorems. We warn the reader that these theorems are only reformulations of well-known results about the R- and S-transforms of free infinitely divisible measures that one can find in [**4**] and [**6**].

Let us recall that a measure ρ on $\mathbb{R}$ is a Lévy measure if $\rho(\{0\}) = 0$ and

$$\int_{\mathbb{R}} \min(x^2, 1)\rho(dx) < \infty.$$

Let us also recall that any multiplicative element E of $\mathfrak{E}[\mathfrak{S}]$ is determined by its restriction on the irreducible partitions: this gives an element of $\mathfrak{E}^{(i)}[\mathfrak{S}]$. Let us suppose that E is invariant by conjugation by any permutation: we recall that this means that for any positive integer k and any $\sigma \in \mathfrak{S}_k$, the following equality holds in $\mathbb{C}[A_k(N)]$ for any positive integer N:

$$\sigma E_k \sigma^{-1} = E_k.$$

The restriction of E on the irreducible partitions is an element of $\left(\mathfrak{E}^{(i)}[\mathfrak{S}]\right)^{\mathfrak{S}}$. Yet, $\mathfrak{E}^{(i)}[\mathfrak{S}]$ is isomorphic to the affine space of formal series of the form $\mathbb{C}_1[[z]]$ with constant coefficient equal to 1. Let us suppose that for a complex number z, the evaluation of this formal series converges, then we denote this evaluation by $E(z)$.

Theorem 10.18. — *Let μ be a free additive infinitely divisible measure. For any complex number z such that $Im(z) < 0$, $\mathcal{R}(\mu)(z)$ is defined, and there exist $\eta \in \mathbb{R}$, $a \in \mathbb{R}^+$*

and ρ a Lévy measure on $\mathbb{R}$ which are unique and such that:

$$\mathcal{R}(\mu)(z) = 1 + \eta z + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1-tz} - 1 - tz\mathbb{1}_{[-1,1]}(t) \right) \rho(dt). \tag{26}$$

Conversely, for any $\eta \in \mathbb{R}$, $a \in \mathbb{R}^+$ and any Lévy measure ρ on $\mathbb{R}$, there exists a free additive infinitely divisible measure $\mu \in \mathcal{P}(\mathbb{R})$ such that for any complex number z such that $Im(z) < 0$, the equality (26) is satisfied. In particular, if the Lévy measure of μ is compactly supported, then there exist $\eta \in \mathbb{R}$, $a \in \mathbb{R}^+$ and ρ a Lévy measure on $\mathbb{R}$ which are unique and such that:

$$\mathcal{R}(\mu) = \mathcal{M}\left(1 + \sum_{n \in \mathbb{N}^*} \kappa_n z^n\right),$$

where we used the bijection explained in Proposition 10.1 *of* [**15**]*, and where:*

- $\kappa_1 = \eta$,
- $\kappa_2 = a + \int_{\mathbb{R}} x^2 \rho(dx)$,
- $\forall n \geq 3$, $\kappa_n = \int_{\mathbb{R}} x^n \rho(dx)$.

The last triplet (η, a, ρ) will be called the $\boxplus$-characteristic triplet of μ.

A similar theorem exists for the free multiplicative infinitely divisible measures. Recall that ν, a measure on $\mathbb{U}$, is a Lévy measure if $\nu(\{1\}) = 0$ and

$$\int_{\mathbb{U}} (1 + \Re(\zeta)) d\nu(\zeta) < \infty,$$

where $\Re(\zeta)$ is the real part of ζ. Recall the definition of $\mathfrak{J}$ in Definition 10.6 of [**15**].

Theorem 10.19. — *Let η be a free multiplicative infinitely divisible measure. There exist $\omega \in \mathbb{U}$, $b \in \mathbb{R}^+$ and ν a Lévy measure on $\mathbb{U}$ which are unique and such that:*

$$\mathcal{LR}(\nu) = \mathfrak{J}\left[(L\kappa_n)_{n \in \mathbb{N}^*}\right] \tag{27}$$

where:

- $L\kappa_1 = i\mathsf{arg}(\omega) - \frac{b}{2} + \int_{\mathbb{U}} (\Re(\zeta) - 1) d\nu(\zeta)$,
- $L\kappa_2 = -b + \int_{\mathbb{U}} (\zeta - 1)^2 d\nu(\zeta)$,
- $\forall n \geq 3$, $L\kappa_n = \int_U (\zeta - 1)^n d\nu(\zeta)$.

Conversely, for any element $\mathcal{LR}$ of the form given by (27), there exists a free multiplicative infinitely divisible measure $\mu \in \mathcal{P}(\mathbb{U})$ such that $\mathcal{LR}(\mu) = \mathcal{LR}$. The triplet (ω, b, ν) is called the $\boxtimes$-characteristic triplet of ν.

Proof. — One could give a proof by using the characterization of the S-transform of a multiplicative infinitely divisible measure. Yet, G. Cébron showed already in [**10**] the forthcoming Theorem 10.26. In the proof of Theorem 10.26, we compute the log-cumulant invariant of μ given by Equation (29). This allows to conclude. □

In the following, we will give a new proof of results found in [**5**], [**9**] and [**10**], in the setting we have developed.

Theorem 10.20. — *Let $\mu \in \mathcal{P}(\mathbb{R})$ be a free additive infinitely divisible measure. Let us suppose that the Lévy measure ρ of μ is compactly supported. Let $(\mu_t)_{t\geq 0}$ be the continuous semi-group associated with μ. There exists $\left(\left(X_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ a matricial $\mathcal{P}$-non-commutative additive Lévy process such that:*

- *for any positive integer N, for any positive real t, X_t^N is hermitian,*
- *for any positive integer N, $(X_t^N)_{t\geq 0}$ is U-invariant,*
- *for any $t \geq 0$, $\mathcal{M}(\mu_t) = \mathcal{M}_{\mathfrak{S}}[X_t]$.*

In [**5**] it is proved that one can remove the condition on the measure μ. The Theorem 10.20 is a consequence of Theorem 10.24. In [**10**], a similar result for free multiplicative infinitely divisible probability measures is proved.

Theorem 10.21. — *Let $\nu \in \mathcal{P}(\mathbb{U}) \setminus \{\lambda_{\mathbb{U}}\}$ be a free multiplicative infinitely divisible measure. Let $(\nu_t)_{t\geq 0}$ be the canonical continuous semi-group associated with ν. There exists $\left(\left(X_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ a matricial $\mathcal{P}$-non-commutative multiplicative Lévy process such that:*

- *for any positive integer N, for any positive real t, X_t^N is unitary,*
- *for any positive integer N, $(X_t^N)_{t\geq 0}$ is U-invariant,*
- *for any $t \geq 0$: $\mathcal{M}(\nu_t) = \mathcal{M}_{\mathfrak{S}}[X_t]$.*

This theorem is a consequence of Theorem 10.26. We will also extend these theorems to the real setting.

Theorem 10.22. — *Let us use the same notations than those in Theorem 10.20. There exists $\left(\left(X_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ a matricial $\mathcal{P}$-non-commutative additive Lévy process such that:*

- *for any integer N, for any positive real t, X_t^N is symmetric,*
- *for any integer N, $(X_t^N)_{t\geq 0}$ is O-invariant,*
- *for any $t \geq 0$: $\mathcal{M}(\mu_t) = \mathcal{M}_{\mathfrak{S}}[X_t]$.*

This theorem is a consequence of Theorem 10.25. In order to state a similar theorem for multiplicative semi-groups of measures, we need to define the notion of being even according to the conjugation.

Definition 10.18. — *Let ν be a measure on $\mathbb{U}$. It is even according to the conjugation if for any continuous function $f : \mathbb{U} \to \mathbb{R}$, one has:*

$$\int_{\mathbb{U}} f(\overline{z}) d\nu(z) = \int_{\mathbb{U}} f(z) d\nu(z).$$

Theorem 10.23. — *Let us use the same notations than those in Theorem 10.21. Let (ω, b, η) be the characteristic triplet of ν. Let us suppose that η is even according to the conjugation and ω is equal to one. There exists $\left(\left(X_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ a matricial $\mathcal{P}$-non-commutative multiplicative Lévy process such that:*

- *for any integer N, for any positive real t, X_t^N is orthogonal,*
- *for any integer N, $(X_t^N)_{t\geq 0}$ is O-invariant,*
- *for any $t \geq 0$: $\mathcal{M}(\nu_t) = \mathcal{M}_{\mathfrak{S}}[X_t]$.*

This theorem is a consequence of Theorem 10.28.

10.4.3. Strong approximation in $\mathfrak{h}(N,\mathbb{C})$. —

Theorem 10.24. — *Let (η,a,ρ) be the characteristic triplet of μ, a free additive multiplicative infinitely divisible probability measure on $\mathbb{R}$. Let $(\mu_t)_{t\geq 0}$ be the one-parameter semi-group for the free additive convolution associated with μ. Let us suppose that the measure ρ has a compact support. Let us define for any integer N:*

$$a_N = aId_{N^2},$$

$$\rho_N(f) = N\int_{\mathbb{R}}\int_{U(N)} f\left(g\begin{pmatrix} x & 0 & \cdots & 0\\ 0 & 0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 0\end{pmatrix}g^*\right)dgd\rho(x).$$

For any positive integer N, let $(X_t^N)_{t\geq 0}$ be a Lévy process on $\mathfrak{h}(N,\mathbb{C})$ with characteristic triplet $\left(\eta I_N, a_N, \rho_N\right)$. The process $(X_t^N)_{t\geq 0}$ converges in $\mathcal{P}$-expectation as N goes to infinity. The family $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial $\mathcal{P}$-non-commutative additive Lévy process and the asymptotic $\mathcal{P}$-factorization property holds for $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$: this implies that the normalized $\mathcal{P}$-moments of $\left(X_t^N\right)_{t\geq 0}$ converges in probability when N goes to infinity to the limit of their expectation. Besides, for any $t\geq 0$ and any $k\in\mathbb{N}$,

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}Tr\left(\left(X_t^N\right)^k\right)\right]\underset{N\to\infty}{\longrightarrow}\int_{\mathbb{R}} z^k d\mu_t.$$

Proof. — For any positive integer N, let $(X_t^N)_{t\geq 0}$ be a Lévy process on $\mathfrak{h}(N,\mathbb{C})$ with characteristic triplet $\left(\eta I_N, a_N, \rho_N\right)$ defined in the theorem. Using the last assertion of Theorem 10.16, for any positive integer N, $(X_t^N)_{t\geq 0}$ is a Lévy process invariant by conjugation by $U(N)$.

Let us prove that for any positive integer k and N:

$$\left(G_k^N = \frac{d}{dt}_{t=0}\mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\right]\right)_{N\in\mathbb{N}}$$

converges in $\mathfrak{S}$-moments, and let us compute $\mathcal{R}_{\mathfrak{S}}(G)$. Let us suppose, without loss of generality, that for any $x\in\mathsf{Supp}(\rho)$, the matrix:

$$\begin{pmatrix} x & 0 & \cdots & 0\\ 0 & 0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 0\end{pmatrix}$$

is in the ball of center 0 and radius 1. We can compute the action of the generator using Theorem 10.16. Depending on the value of k, we have:

– if $k=1$:

$$\frac{d}{dt}_{|t=0}\mathbb{E}\left[X_t^N\right] = \eta I_N,$$

– if $k = 2$:

$$\frac{d}{dt}_{|t=0} \mathbb{E}\left[\left(X_t^N\right)^{\otimes 2}\right] = \frac{a}{2} C_{\mathcal{H}_N} + \int_{\mathcal{H}_N} g^{\otimes 2} \rho_N(dg),$$

– if $k \geq 3$:

$$\frac{d}{dt}_{|t=0} \mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\right] = \int_{\mathcal{H}_N} g^{\otimes k} \rho_N(dg).$$

The convergence of G_1^N is obvious, and we have already understood the convergence of $\frac{a}{2} C_{\mathcal{H}_N}$ in Section 10.3. Let k be an integer greater than two, it remains to study the sequence $\left(\int_{\mathcal{H}_N} g^{\otimes k} \rho_N(dg)\right)_{N \in \mathbb{N}}$. Let σ be a permutation in $\mathfrak{S}_k$, we have the following equalities:

$$m_\sigma\left(\int_{\mathcal{H}_N} g^{\otimes k} \rho_N(dg)\right) = \frac{1}{Tr_N(\sigma)} Tr^k\left[\int_{\mathcal{H}_N} g^{\otimes k} \rho_N(dg)\ {}^t\sigma\right] = \frac{N}{Tr_N(\sigma)} \int_{\mathbb{R}} x^k d\rho(x).$$

We recall that $Tr_N(\sigma) = N^{\mathsf{nc}(\sigma \vee id)}$ and $\mathsf{nc}(\sigma \vee id)$ is the number of cycles of σ, thus:

$$m_\sigma\left(\int_{\mathcal{H}_N} g^{\otimes k} \rho_N(dg)\right) \underset{N \to \infty}{\longrightarrow} \delta_{\sigma \in [(1,\ldots,k)]_{\mathfrak{S}_k}} \int_{\mathbb{R}} x^k d\rho(x), \tag{28}$$

where we recall that $[(1, ..., k)]_{\mathfrak{S}_k}$ is the conjugacy class of $(1, ..., k)$ in $\mathfrak{S}_k$. Thus for any $k \geq 1$, $\left(G_k^N\right)_{N \in \mathbb{N}}$ converges in $\mathfrak{S}$-moments.

We would like to compute $\mathcal{R}_{\mathfrak{S}}[G]$: we see in (28) that the moments of G_k^N concentrate on the k-cycles as N goes to infinity. It is easy to compute the limit of the coordinate numbers of G_k^N: using Theorem 4.1 in the article [**15**], we get:

$$\begin{aligned}
\left(\mathcal{R}[G]\right)_{(1)} &= \lim_{N \to \infty} \kappa^{(1)}(G_1^N) = \eta, \\
\left(\mathcal{R}[G]\right)_{(1,2)} &= \lim_{N \to \infty} \kappa^{(1,2)}(G_2^N) = \frac{a}{2} + \int_{\mathbb{R}} x^2 d\rho(dx), \\
\forall k \geq 3, \left(\mathcal{R}[G]\right)_{(1,\ldots,k)} &= \lim_{N \to \infty} \kappa^{(1,\ldots,k)}(G_k^N) = \int_{\mathbb{R}} x^k d\rho(dx),
\end{aligned}$$

and for any partition p which is not irreducible, $\left(\mathcal{R}[G]\right)_p = 0$.

By Theorem 10.1, the process $(X_t^N)_{t \geq 0}$ converges in $\mathcal{P}$-expectation as N goes to infinity. Besides the family $\left((X_t^N)_{N \in \mathbb{N}}\right)_{t \geq 0}$ is a matricial $\mathcal{P}$-non-commutative additive Lévy process. Using the calculation of $\mathcal{R}_{\mathfrak{S}}[G]$, and using Theorem 10.3, the asymptotic $\mathcal{P}$-factorization property holds for $\left((X_t^N)_{N \in \mathbb{N}}\right)_{t \geq 0}$. Finally, using Theorem 10.2, we know that for any positive real t_0:

$$\frac{d}{dt}_{|t=t_0} \mathcal{R}_{\mathfrak{S}}\left[X_t\right] = \mathcal{R}_{\mathfrak{S}}[G] \boxplus \mathcal{R}_{\mathfrak{S}}\left[X_{t_0}\right].$$

Yet, using Theorem 10.18, Lemma 10.11 and Equation (24), we see that $\left(\mathcal{R}(\mu_t)\right)_{t \geq 0}$ satisfies the same differential equation with same initial conditions. Thus, for any $t \geq 0$, $\mathcal{R}_{\mathfrak{S}}[X_t] = \mathcal{R}(\mu_t)$. Which is equivalent to the fact that $\mathcal{M}_{\mathfrak{S}}[X_t] = \mathcal{M}(\mu_t)$. If one considers

the evaluation of this equality to the cycles $(1, ..., k)$ with k being any positive integer, one gets that for any $t \geq 0$ and any $k \in \mathbb{N}$:

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N} Tr\left(\left(X_t^N\right)^k\right)\right] \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} z^k d\mu_t.$$

It remains to prove that the normalized $\mathcal{P}$-moments of $(X_t^N)_{t\geq 0}$ converges in probability when N goes to infinity to the limit of their expectation. Since for any non-negative real t and any positive integer N the random matrix X_t^N is Hermitian, the $\mathcal{P}$-convergence and the asymptotic $\mathcal{P}$-factorization of $(X_t^N)_{t\geq 0}$ implies the $\mathcal{P}$-convergence and the asymptotic $\mathcal{P}$-factorization of $(X_t^N)_{t\geq 0} \cup \left(\left(X_t^N\right)^*\right)_{t\geq 0}$. Using Theorem 2.2, we get that the normalized $\mathcal{P}$-moments of $(X_t^N)_{t\geq 0}$ converges in probability when N goes to infinity to the limit of their expectation. □

10.4.4. Convergence in $\mathfrak{s}_N$. —

Theorem 10.25. — *Let (η, a, ρ) be the characteristic triplet of μ, a free additive multiplicative infinitely divisible measure on $\mathbb{R}$. Let $(\mu_t)_{t\geq 0}$ be the one-parameter semi-group for the free additive convolution associated with μ. Let us suppose that the measure ρ has a compact support. Let us define for any integer N:*

$$a_N = aId_{\frac{N(N+1)}{2}},$$

$$\rho_N(f) = N \int_{\mathbb{R}} \int_{O(N)} f\left(g \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} {}^t g\right) dg d\rho(x).$$

For any positive integer N, let $(X_t^N)_{t\geq 0}$ be a Lévy process on $\mathfrak{h}(N, \mathbb{R})$ with characteristic triplet $\left(\eta I_N, a_N, \rho_N\right)$. The process $(X_t^N)_{t\geq 0}$ converges in expectation in $\mathcal{P}$-moments as N goes to infinity. The family $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial $\mathcal{P}$-non-commutative additive Lévy process and the asymptotic $\mathcal{P}$-factorization property holds for $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$: this implies that as N goes to infinity, the normalized $\mathcal{P}$-moments of $\left(X_t^N\right)_{t\geq 0}$ converge in probability to the limit of their expectation. Besides, for any $t \geq 0$ and any $k \in \mathbb{N}$,

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N} Tr\left(\left(X_t^N\right)^k\right)\right] \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} z^k d\mu_t.$$

Proof. — The proof is similar to the proof of Theorem 10.24. Let us show that for any positive integer k,

$$\left(G_k^N = \frac{d}{dt}_{|t=0} \mathbb{E}\left[(X_t^N)^{\otimes k}\right]\right)_{N\in\mathbb{N}}$$

converges in $\mathcal{B}$-moments. Yet, using Theorem 10.8 or Proposition 2.2, it is enough to show that for any positive integer k G_k^N converges in $\mathfrak{S}$-moments as N goes to infinity. As for the proof of Theorem 10.24:

– if $k = 1$:

$$\frac{d}{dt}_{t=0} \mathbb{E}\left[X_t^N\right] = \eta I_N,$$

– if $k = 2$:

$$\frac{d}{dt}_{t=0} \mathbb{E}\left[\left(X_t^N\right)^{\otimes 2}\right] = \frac{a}{2} C_{\mathfrak{s}_N} + \int_{\mathfrak{s}_N} g^{\otimes 2} \rho_N(dg),$$

– if $k \geq 3$:

$$\frac{d}{dt}_{t=0} \mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\right] = \int_{\mathfrak{s}_N} g^{\otimes k} \rho_N(dg).$$

We have already handled the convergence of ηI_N and $\frac{a}{2} C_{\mathfrak{s}_N}$ in Section 10.3. It remains to show that for any $k \geq 2$, $\left(\int_{\mathfrak{s}_N} g^{\otimes k} \rho_N(dg)\right)_{N \in \mathbb{N}}$ converges in $\mathfrak{S}$-moments. But this calculation has already been made in the proof of Theorem 10.24.

By Theorem 10.8 and 10.1, the process $(X_t^N)_{t \geq 0}$ converges in $\mathcal{P}$-expectation as N goes to infinity. The family $\left((X_t^N)_{N \in \mathbb{N}}\right)_{t \geq 0}$ is a matricial $\mathcal{P}$-non-commutative additive Lévy process.

Besides, for any positive integer k, for any $\sigma \in \mathfrak{S}_k$, the limit of $m_\sigma \left(\frac{d}{dt}_{|t=0} \mathbb{E}\left[(X_t^N)^{\otimes k}\right]\right)$ are equal in the Hermitian and symmetric case. Using Theorem 10.8, $\left((X_t^N)_{N \in \mathbb{N}}\right)_{t \geq 0}$ has the same $\mathfrak{S}$-law as the matricial $\mathcal{P}$-non-commutative additive Lévy process defined in the proof of Theorem 10.24. This proves that for any $t \geq 0$ and any $k \in \mathbb{N}$,

$$\lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} Tr\left(\left(X_t^N\right)^k\right)\right] \underset{N \to \infty}{\longrightarrow} \int_{\mathbb{R}} z^k d\mu_t.$$

Moreover, since the asymptotic $\mathfrak{S}$-factorization holds in the Hermitian case, the asymptotic $\mathfrak{S}$-factorization property holds for $(X_t^N)_{N \in \mathbb{N}}$ for any $t \geq 0$. But for any positive real t, for any positive integer N, X_t^N is symmetric: using a similar argument as for Proposition 2.2, the asymptotic $\mathcal{B}$-factorization property holds for $(X_t^N)_{N \in \mathbb{N}}$. By Proposition 4.5, it satisfies also the asymptotic $\mathcal{P}$-factorization property, and using the fact that the increments are independent and G-invariant, we get that the asymptotic $\mathcal{P}$-factorization property holds for $\left((X_t^N)_{N \in \mathbb{N}}\right)_{t \geq 0}$. An application of Theorem 2.2 allows to state that the normalized $\mathcal{P}$-moments of $\left(X_t^N\right)_{t \geq 0}$ converge in probability to the limit of their expectation as N goes to infinity. □

10.4.5. Convergence in $U(N)$. — Let us introduce the model of G. Cébron explained in [**10**].

Theorem 10.26. — *Let (ω, b, ν) be the characteristic triplet associated with μ, a free multiplicative infinitely divisible measure on the circle $\mathbb{U}$. Let $(\mu_t)_{t \geq 0}$ be the one-parameter semi-group for the free multiplicative convolution associated with μ. Let us*

define:

$$b_N = bId_{N^2},$$
$$\nu_N(f) = N\int_{\mathbb{U}}\int_{U(N)} f\left(g\begin{pmatrix}\zeta & 0 & \cdots & 0\\ 0 & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 1\end{pmatrix}g^*\right)dgd\nu(\zeta).$$

For any positive integer N, let $(Y_t^N)_{t\geq 0}$ be a Lévy process with characteristic triplet $(i\mathsf{arg}(\omega)I_N, b_N, \nu_N)$. The process $(Y_t^N)_{t\geq 0}$ converges in expectation in $\mathcal{P}$-moments as N goes to infinity. The family $\left((Y_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial $\mathcal{P}$-non-commutative multiplicative Lévy process. Besides the asymptotic $\mathcal{P}$-factorization property holds for the family $\left((Y_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$. For any $t\geq 0$ and any $k\in\mathbb{N}$,

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}Tr\left(\left(Y_t^N\right)^k\right)\right]\underset{N\to\infty}{\longrightarrow}\int_{\mathbb{U}} z^k d\mu_t.$$

Proof. — The proof follows exactly the one of Theorem 10.24. For any positive integer N, let $(Y_t^N)_{t\geq 0}$ be a $U(N)$-valued Lévy process with characteristic triplet $(i\mathsf{arg}(\omega)I_N, b_N, \nu_N)$. Using the last assertion of Theorem 10.17, for any positive integer N, $(Y_t^N)_{t\geq 0}$ is a Lévy process invariant by conjugation by $U(N)$. We compute the action of the generator at time $t=0$ on the application $U\mapsto U^{\otimes k}$. This computation follows the work of Cébron in Propositions 5.2 and 5.3 of [**10**]. Applying Theorem 10.17, for any positive integer k:

$$\begin{aligned}G_k^N &= \frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(Y_t^N\right)^{\otimes k}\right]\\ &= ki\mathsf{arg}(w)Id^{\otimes k} + \frac{b}{2}kc_{\mathfrak{g}}Id^{\otimes k} + \frac{1}{2}\sum_{i,j=1}^{k}\mathcal{I}_{\{i,j\}}(C_{\mathfrak{g}^1(N,\mathbb{K})}, Id^{\otimes k-2})\\ &+ \int_{U(N)}\left(M^{\otimes k} - Id^{\otimes k} - \sum_{i=1}^{k}\mathcal{I}_{\{i\}}(i\Im(M), Id^{\otimes k-1})\right)\nu_N(dM).\end{aligned}$$

Using in the last term the fact that:

$$M^{\otimes k} = (M - Id + Id)^{\otimes k} = Id^{\otimes k} + \sum_{m=1}^{k}\sum_{1\leq i_1<i_2<\ldots<i_m\leq k}\mathcal{I}_{\{i_1,\ldots,i_m\}}\left[(M-Id)^{\otimes m}, Id^{\otimes k-m}\right]$$

and the fact that $M - Id - i\Im(M) = \Re(M) - Id$, one gets:

$$\frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(Y_t^N\right)^{\otimes k}\right] = ki\mathsf{arg}(w)Id^{\otimes k} + \sum_{i=1}^{k}\mathcal{I}_{\{i\}}\left[\int_{U(N)}(\Re(M)-Id)\nu_N(dM), Id^{\otimes k-1}\right]$$
$$+ \frac{b}{2}kc_{\mathfrak{g}}Id^{\otimes k} + \frac{b}{2}\sum_{i,j=1}^{k}\mathcal{I}_{\{i,j\}}(C_{\mathfrak{g}^1(N,\mathbb{K})}, Id^{\otimes k-2})$$
$$+ \sum_{2<m\leq l}\sum_{1\leq i_1<\ldots<i_m\leq k}\mathcal{I}_{\{i_1,\ldots,i_m\}}\left[\int_{U(N)}(M-Id)^{\otimes m}\nu_N(dM), Id^{\otimes k-m}\right],$$

result obtained in [**10**], Proposition 5.3. The proof now differs from the one of C. Cébron. We want to understand the limit of $\frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(Y_t^N\right)^{\otimes k}\right]$ when N tends to infinity. Using our work on the Brownian motion and on the Casimir element, it remains to understand the limit of the $\mathfrak{S}$-cumulants or of the $\mathfrak{S}$-moments of:

$$\left(A_N = \sum_{i=1}^{k}\mathcal{I}_{\{i\}}\left[\int_{U(N)}(\Re(M)-Id)\nu_N(dM), Id^{\otimes k-1}\right]\right)_{N\in\mathbb{N}}$$

and of

$$\left(B_N = \mathcal{I}_{\{i_1,\ldots,i_m\}}\left[\int_{U(N)}(M-Id)^{\otimes m}\nu_N(dM), Id^{\otimes k-m}\right]\right)_{N\in\mathbb{N}}$$

for any $m \in \{2,...,k\}$ and for any $1 \leq i_1 < ... < i_m \leq k$.

It is easier to work with the $\mathfrak{S}$-moments. Let σ be in $\mathfrak{S}_k$, for any positive integer N:

$$m_\sigma(A_N) = k\int_{\mathbb{U}}(\Re(\zeta)-1)d\nu(\zeta).$$

Using the cumulant-moment equation:

$$\lim_{N\to\infty}\kappa^\sigma(A_N) = \left(k\int_{\mathbb{U}}(\Re(\zeta)-1)d\nu(\zeta)\right)\mathbb{1}_{\sigma=id_k},$$

or, written in a different way:

$$\lim_{N\to\infty}\left(M_N^k\right)^{-1}(A_N) = \left(k\int_{\mathbb{U}}(\Re(\zeta)-1)d\nu(\zeta)\right)id_k,$$

where we recall that M_N^k is the application defined in Definition 5.1 of [**15**].

Let m be a positive integer. Let us understand the sequence $(B_N)_{N\in\mathbb{N}}$. For this, it is enough to understand the sequence:

$$\left(\tilde{B}_N = \int_{U(N)}(M-Id)^{\otimes m}\nu_N(dM)\right)_{N\in\mathbb{N}}$$

For any $\sigma \in \mathfrak{S}_m$:

$$m_\sigma\left(\tilde{B}_N\right) = \frac{N}{N^{\mathsf{nc}(\sigma\vee id)}}\int_{\mathbb{U}}(\zeta-1)^m d\nu(\zeta).$$

Thus, as N goes to infinity, for any $\sigma \in \mathfrak{S}_k$:

$$m_\sigma\left(\tilde{B}_N\right) \underset{N\to\infty}{\longrightarrow} \delta_{\sigma\in[(1,\ldots,m)]_{\mathfrak{S}_m}} \int_{\mathbb{U}} (\zeta-1)^m d\nu(\zeta),$$

where we recall that $[(1,...,m)]_{\mathfrak{S}_m}$ is the conjugacy class in $\mathfrak{S}_m$ of the cycle $(1,...,m)$. We can compute the limit of the coordinate numbers of $\tilde{B}_N$ using Theorem 4.1 in the article [**15**]:

$$\kappa^\sigma\left(\tilde{B}_N\right) \underset{N\to\infty}{\longrightarrow} \delta_{\sigma\in[(1,\ldots,m)]_{\mathfrak{S}_m}} \int_{\mathbb{U}} (\zeta-1)^m d\nu(\zeta).$$

Hence:

$$\lim_{N\to\infty} \left(M_N^k\right)^{-1}(\tilde{B}_N) = \sum_{\sigma\in[(1,\ldots,m)]_{\mathfrak{S}_m}} \left(\int_{\mathbb{U}} (\zeta-1)^m d\nu(\zeta)\right)\sigma.$$

Using all the results we just had, we can conclude that

$$\left(M_N^k\right)^{-1}\left[\frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(Y_t^N\right)^{\otimes k}\right]\right]$$

converges as N goes to infinity and its limit is equal to:

$$k\left(i\mathsf{arg}(\omega)-\frac{b}{2}+\int_{\mathbb{U}}(\Re(\zeta)-1)d\nu(\zeta)\right)Id_k - b\sum_{1\le i<j\le k}(i,j) + \sum_{2<m\le k}\sum_{c\in\mathcal{C}_m}\left(\int_{\mathbb{U}}(\zeta-1)^m d\nu(\zeta)\right)c, \tag{29}$$

where $\mathcal{C}_m$ is the set of permutations in $\mathfrak{S}_k$ which have only one non-trivial cycle which is of length m.

By Theorem 10.1, the process $\left(Y_t^N\right)_{t\ge 0}$ converges in $\mathcal{P}$-expectation as N goes to infinity. Besides the family $\left((Y_t^N)_{N\in\mathbb{N}}\right)_{t\ge 0}$ is a matricial $\mathcal{P}$-non-commutative additive Lévy process. Using the calculations we just did and using Theorem 10.3, since any $c \in \mathcal{C}_m$ is weakly irreducible, the asymptotic $\mathcal{P}$-factorization property holds for $\left((Y_t^N)_{N\in\mathbb{N}}\right)_{t\ge 0}$. Finally, using Theorem 10.2, we know that for any positive real t_0:

$$\frac{d}{dt}_{|t=t_0}\mathcal{R}_{\mathfrak{S}}\left[Y_t\right] = \mathcal{R}_{\mathfrak{S}}[G]\boxtimes\mathcal{R}_{\mathfrak{S}}\left[X_{t_0}\right].$$

Yet, using Theorem 10.19 and Equation (25), we see that $(\mathcal{R}(\mu_t))_{t\ge 0}$ satisfies the same differential equation with same initial conditions. Thus, for any $t \ge 0$, $\mathcal{R}_{\mathfrak{S}}[X_t] = \mathcal{R}(\mu_t)$. Which is equivalent to the fact that $\mathcal{M}_{\mathfrak{S}}[X_t] = \mathcal{M}(\mu_t)$. If one considers the evaluation of this equality on the cycles $(1,...,k)$ with k being any positive integer, one gets that for any $t \ge 0$ and any $k \in \mathbb{N}$:

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}Tr\left(\left(Y_t^N\right)^k\right)\right] \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{U}} z^k d\mu_t.$$

This concludes the proof. □

In the theorem we just proved, we only proved the $\mathcal{P}$-asymptotic factorization for $(Y_t^N)_{t\geq 0}$, yet it would be interesting to have the convergence in $\mathcal{P}$-expectation and the asymptotic $\mathcal{P}$-asymptotic factorization of $\left(Y_t^N, \left(Y_t^N\right)^*\right)_{t\geq 0}$ as N goes to infinity. Indeed, this would allow to improve the convergence in expectation of the normalized moments into a convergence in probability. In [**10**], G. Cébron used some concentration of measure arguments in order to prove the almost sure convergence of the empirical measure. The goal of the following discussion is to show that actually, in our setting, no more work as to be done in order to get the weaker result of convergence in probability.

Let k and k' be two positive integers. In order to prove that the normalized $\mathcal{P}$-moments of $\left(Y_t^N\right)_{t\geq 0}$ converge in probability to the limit of their expectation, we need to understand $\left(Y_t^N, \left(Y_t^N\right)^*\right)_{t\geq 0}$. Yet it is easier to understand $\left(Y_t^N, \overline{Y_t^N}\right)_{t\geq 0}$ since $\mathbb{E}\left[(Y_t^N)^{\otimes k}\otimes(\overline{Y_t^N})^{\otimes k'}\right]$ is a semi-group. Let us define:

$$G_{k,k'} = \frac{d}{dt}_{|t=0}\mathbb{E}\left[(Y_t^N)^{\otimes k}\otimes(\overline{Y_t^N})^{\otimes k'}\right].$$

Using Theorem 10.17, we can compute $G_{k,k'}$ and we can see that it is composed of three parts: a drift and Brownian parts which were already studied in Section 10.3, and a third part which remains to be understood:

$$\int_{U(N,\mathbb{C})}\left(M^{\otimes k}\otimes\overline{M}^{\otimes k'} - Id^{\otimes k+k'} - \sum_{i=1}^{k}\mathcal{I}_{\{i\}}(i\Im(M), Id^{\otimes k+k'-1}) + \sum_{i=k+1}^{k+k'}\mathcal{I}_{(i)}(i^t\Im(M), Id^{\otimes k+k'-1})\right)\nu_N(dM),$$

where we used the fact that $\overline{i\Im(M)} = -{}^t(i\Im(M))$. Using the same argument as the one in the proof of Theorem 10.26, we know that $M^{\otimes k}\otimes\overline{M}^{\otimes k'} - Id^{\otimes k+k'}$ is equal to:

$$\sum_{m=1,l=1}^{k,k'}\sum_{1\leq i_1<\ldots<i_m\leq k<j_1\leq\ldots\leq j_l\leq k+k'}\mathcal{I}_{\{i_1,\ldots,i_m,j_1,\ldots,j_l\}}\left[(M-Id)^{\otimes m}\otimes(\overline{M}-Id)^{\otimes l}, Id^{\otimes k+k'-m-l}\right].$$

Let us remark also that $\overline{M} + {}^t(i\Im(M)) = {}^t\Re(M)$. As for the proof of Theorem 10.26, it remains to understand the limit of the $\mathcal{B}$-moments or of the $\mathcal{B}$-cumulants of:

$$\int_{U(N,\mathbb{C})}(\Re(M)-Id)\nu_N(dM), \int_{U(N,\mathbb{C})}({}^t\Re(M)-Id)\nu_N(dM),$$
$$\int_{U(N,\mathbb{C})}(M-Id)^{\otimes m}\otimes\left(\overline{M}-Id\right)^{\otimes l}\nu_N(dM),$$

for any $(m,l)\in\{1,...,k\}\times\{1,...,k'\}$. The two first sequences are easy to understand, only the last sequence is interesting, let (m,l) in $\{1,...,k\}\times\{1,...,k'\}$ and let us understand:

$$\left(C_N = \int_{U(N,\mathbb{C})}(M-Id)^{\otimes m}\otimes\left(\overline{M}-Id\right)^{\otimes l}\nu_N(dM)\right)_{N\in\mathbb{N}}.$$

For any positive integer N, C_N commutes with the tensor action of $O(N)$: this is an element of $\mathbb{C}[\mathcal{B}_{m+l}(N)]$. Let us recall the operation S_m defined in Definition 2.5. The element $S_m(C_N)$ is equal to:

$$\int_{U(N)} (M - Id)^{\otimes m} \otimes (M^* - Id)^{\otimes l} \nu_N(dM)$$

which commutes with the tensor action of $U(N)$: it is an element of $\mathbb{C}[\mathfrak{S}_{m+l}(N)]$, and the convergence in $\mathfrak{S}$-moments of $S_m(C_N)$ implies the convergence in $\mathbb{C}[\mathcal{B}_{m+l}(N)]$-moments of $(C_N)_{N\in\mathbb{N}}$. Yet, for any $\sigma \in \mathfrak{S}_{m+l}$:

$$\lim_{N\to\infty} m_\sigma(S_m(C_N)) = \delta_{[(1,...,m+l)]_{\mathfrak{S}_{m+l}}}(\sigma) \int_{\mathbb{U}} (\zeta - 1)^m (\overline{\zeta} - 1)^l d\nu(\zeta).$$

where we remind that $[(1, ..., m+l)]_{\mathfrak{S}_{m+l}}$ is the conjugacy class in $\mathfrak{S}_{m+l}$ of the $(m+l)$-cycle $(1, ..., m+l)$. Using the cumulant-moment relation, we get that for any $\sigma \in \mathfrak{S}_{m+l}$:

$$\lim_{N\to\infty} \kappa^\sigma(S_m(C_N)) = \delta_{[(1,...,m+l)]_{\mathfrak{S}_{m+l}}}(\sigma) \int_{\mathbb{U}} (\zeta - 1)^m (\overline{\zeta} - 1)^l d\nu(\zeta),$$

Hence:

$$\lim_{N\to\infty} \left(M_N^{m+l}\right)^{-1} (C_N) = \sum_{\sigma\in[(1,...,m+l)]_{\mathfrak{S}_{m+l}}} \left(\int_{\mathbb{U}} (\zeta - 1)^m (\overline{\zeta} - 1)^l d\nu(\zeta)\right) S_m(\sigma).$$

At the end, we get that $(G^N_{k,k'})_{N\in\mathbb{N}}$, seen as an element of $\prod_{N\in\mathbb{N}} \mathbb{C}[\mathcal{B}_{k+k'}(N)]$ converges as N goes to infinity, and $(G^N_{k,k'})_{k,k',N\in\mathbb{N}}$ weakly condensates.

Using Theorem 10.4 and 10.5, this implies that the family $\left(Y_t^N, (Y_t^N)^*\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation as N goes to infinity and the $\mathcal{P}$-asymptotic factorization property holds for this family. In particular for any positive integer m, any m-tuple $t_1, ..., t_m$ of non negative reals, for any m-tuple $(k_1, ..., k_m) \in \mathbb{Z}^m$:

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{i=1}^m \frac{1}{N} Tr\left((Y_{t_i}^N)^{k_i}\right)\right] = \prod_{i=1}^m \int_{\mathbb{U}} z^{k_i} d\mu_t.$$

Besides, this shows that for any positive integer k, any k-tuple $t_1, ..., t_k$ of non negative reals, for any k-tuple $(l_1, ..., l_k) \in \mathbb{Z}^k$ and any $p \in \mathcal{P}_k$, $m_p\left(\bigotimes_{i=1}^k \left(Y_{t_i}^N\right)^{l_i}\right)$ converges in probability as N goes to infinity. This discussion allows to state the following theorem.

Theorem 10.27. — *For any positive integer N, let $\left(Y_t^N\right)_{t\geq 0}$ be the Lévy process defined in Theorem 10.26. As N goes to infinity, the family $\left(\left(Y_t^N\right)_{t\geq 0}, \left((Y_t^N)^*\right)_{t\geq 0}\right)$ converges in expectation in $\mathcal{P}$-moments and satisfies the $\mathcal{P}$-factorization property.*

In particular the family $\left(\left(Y_t^N\right)_{t\geq 0}, \left((Y_t^N)^*\right)_{t\geq 0}\right)$ converges in probability in $\mathcal{P}$-moments. This means that for any $p \in \mathcal{P}$, for any positive integer k, for any k-tuple $(M_1, ..., M_k)$ of elements of $\mathcal{A}\left(\left(\left(Y_t^N\right)_{t\geq 0}, \left((Y_t^N)^*\right)_{t\geq 0}\right)\right)$,

$$m_p\left(M_1^N \otimes ... \otimes M_k^N\right)$$

converges in probability as N goes to infinity.

10.4.6. Convergence in $O(N)$. —

Theorem 10.28. — *Let μ be a free multiplicative infinitely divisible measure on the circle $\mathbb{U}$. Let (ω, b, ν) be the characteristic triplet of μ. Let $(\mu_t)_{t\geq 0}$ be the one-parameter semi-group for the free multiplicative convolution associated with μ. Let us suppose that ν is even according to the conjugation and that ω is equal to 1. Let us define:*

$$b_N = bId_{\frac{N(N-1)}{2}},$$

$$\nu_N(f) = N\int_{[-\pi,\pi]}\int_{O(N)} f\left(g\begin{pmatrix} cos\theta & -sin\theta & 0 & \cdots & 0\\ sin\theta & cos\theta & 0 & \cdots & 0\\ 0 & 0 & 1 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & 1\end{pmatrix}{}^t g\right) dg d\nu(\theta),$$

where in the last equality, we considered ν as a measure on $[-\pi, \pi]$.

For any positive integer N, let $(Y_t^N)_{t\geq 0}$ be a Lévy process in $O(N)$ with characteristic triplet $(0, b_N, \nu_N)$. The process $(Y_t^N)_{t\geq 0}$ converges in expectation in $\mathcal{P}$-moments as N goes to infinity and the family $\left((Y_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial $\mathcal{P}$-non-commutative multiplicative Lévy process.

Besides, the asymptotic $\mathcal{P}$-factorization property holds for $\left((Y_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$: this implies that the normalized $\mathcal{P}$-moments of $\left(Y_t^N\right)_{t\geq 0}$ converge in probability to the limit of their expectation as N goes to infinity. Moreover for any $t \geq 0$ and any $k \in \mathbb{N}$,

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\frac{1}{N}Tr\left(\left(Y_t^N\right)^k\right)\right] \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{U}} z^k d\mu_t.$$

Proof. — The proof is the same as the proof of Theorem 10.26 and uses the same arguments as for Theorem 10.25. □

References

[1] A. Dahlqvist. Integration formula for Brownian motion on classical compact Lie groups. *http://arxiv.org/abs/1212.5107*.

[2] B. Collins and P. Śniady. Integration with respect to the Haar measure on unitary, orthogonal and symplectic group. *Com. in Math. Phys.*, 264(3):773–795, 2004.

[3] T. Banica and R. Speicher. Liberation of orthogonal Lie groups. *arxiv.org/pdf/0808.2628v2.pdf*.

[4] O.E. Barndorff-Nielsen and S. Thorbjørnsen. A connection between free and classical infinite divisibility. *Infin. Dimens. Anal. Quantum. Probab. Relat. Top.*, 07:573–590, 2004.

[5] F. Benaych-Georges. Classical and free infinitely divisible distributions and random matrices. *The Annals of Probability*, 33, no. 3:1134–1170, 2005.

[6] H. Bercovici and D. Voiculescu. Lévy-Hinchin type theorems for multiplicative and additive free convolution. *Pacific Journal of Mathematics*, 153, no. 2:217–248, 1992.

[7] H. Bercovici and D. Voiculescu. Free convolution of measures with unbounded support. *Indiana Univ. Math. J.*, 42(3):733–773, 1993.

[8] P. Biane. Free Brownian motion, free stochastic calculus and random matrices. *Free probability theory (Waterloo, ON, 1995)*, 12 of Field Inst. Commun.:1–19, 1997.

[9] T. Cabanal-Duvillard. A matrix representation of the Bercovici-Pata bijection. *Electronic Journal of Probability*, 10:632–661, 2005.

[10] G. Cébron. Matricial model for the free multiplicative convolution. *To appear in Annals of Probability*.
[11] G. Cébron. Free convolution operators and free Hall transform. *J. Funct. Anal.*, 265:2645–2708, 2013.
[12] B. Collins, J. Mingo, P. Sniady, and R. Speicher. Second order freeness and fluctuations of random matrices: III. Higher order freeness and free cumulants. *Documenta Math.*, 12:1–70, 2007.
[13] G. Cébron and A. Dahlqvist and C. Male. On the limiting distribution of traffics of unitarily invariant random matrices. *Work in progress*.
[14] G. Cébron and A. Dahlqvist and F. Gabriel and C. Male. Traffic-partition correspondence.
[15] F. Gabriel. Combinatorial theory of permutation-invariant random matrices I: Partitions, geometry and renormalization. *http://arxiv.org/abs/1503.02792*, 2015.
[16] F. Gabriel. Combinatorial theory of permutation-invariant random matrices II: Freeness and Lévy processes. *http://arxiv.org/abs/1507.02465*, 2015.
[17] F. Gabriel. Combinatorial theory of permutation-invariant random matrices III: Random walks on $\mathfrak{S}_N$, ramified coverings and the $\mathfrak{S}_\infty$ Yang-Mills measure. *http://arxiv.org/abs/1510.01046*, 2015.
[18] F. Gabriel. Planar Markovian holonomy fields. A first step to the characterization of Markovian holonomy fields. *http://arxiv.org/abs/1501.05077*, 2015.
[19] R. Goodman and N.R. Wallach. *Representations and Invariants of the Classical Groups*. Number 68. Encyclopedia Math. Appl., Cambridge University Press, Cambridge, 1998.
[20] T. Halverson and A. Ram. Partition algebras. *European Journal of Combinatorics*, 26:869–921, 2005.
[21] F. Hiai and D. Petz. *The Semicircle Law, Free Random Variables and Entropy*. Number 77. Mathematical Surveys and Monographs, American Mathematical Society, 2000.
[22] V.F.R. Jones. The Potts model and the symmetric group. *Subfactors: Proceedings of the Taniguchi Symposium on Operator Algebras (Kyuzeso, 1993)*, pages 259–267, 1994.
[23] T. Lévy. The master field on the plane. *arxiv.org/abs/1112.2452*.
[24] T. Lévy. Schur-Weyl duality and the heat kernel measure on the unitary group. *Advances in Mathematics*, 218, no. 2:537–575, 2008.
[25] M. Liao. *Lévy processes in Lie groups*. Number 162. Cambridge Univ. Press, 2004.
[26] C. Male. The distributions of traffics and their free product. *arxiv.org/abs/1111.4662*, 2011.
[27] P. Martin. The Potts model and the symmetric group. *Series on Advances in Statistical Mechanics*, 1991.
[28] J.A. Mingo and M. Popa. Freeness and the transposes of unitarily invariant random matrices. *J. of Func. Anal.*, 271, no. 4:883–921, 2016.
[29] A. Nica and R. Speicher. *Lectures on the Combinatorics of Free Probability*. Number 335. Lecture Note Series, London Mathematical Society, Cambridge University Press, 2006.
[30] K. Sato. *Léévy processes and infinitely divisible distributions*. Number 68. Cambridge studies in advanced mathematics, 1999.
[31] K. Tanabe. On the centralizer algebra of the unitary reflection group G(m,p,n). *Nagoya Mathematical Journal*, 148:113–126, 1997.
[32] T. Tao. *Topics in Random matrix theory*. Number 132. Graduate Studies in Mathematics, American Mathematical Society, 2012.
[33] Dan Voiculescu. Limit laws for random matrices and free products. *Inv. math.*, 104:201–220, 1991.

“Le raisonnement achevé est alors comme une cathédrale gothique ; on y voit les fondements des temps anciens, des premiers balbutiements où la fragilité de la structure était vouée inévitablement à disparaître pour ne plus se manifester que par une empreinte fossile.”

A. Bouthier

REVÊTEMENTS ALÉATOIRES

Abstract. — The aim of this article is to study some asymptotics of a natural model of random ramified coverings on the disk of degree N. We prove that the monodromy field, called also the holonomy field, converges in probability to a non-random field as N goes to infinity. In order to do so, we use the fact that the monodromy field of random uniform labelled simple ramified coverings on the disk of degree N has the same law as the $\mathfrak{S}(N)$-Yang-Mills measure associated with the random walk by transposition on $\mathfrak{S}(N)$.

This allows us to restrict our study to random walks on $\mathfrak{S}(N)$: we prove theorems about asymptotics of random walks on $\mathfrak{S}(N)$ in a new framework based on the geometric study of partitions and the Schur-Weyl-Jones's dualities. In particular, given a sequence of conjugacy classes $(\lambda_N \subset \mathfrak{S}(N))_{N\in\mathbb{N}}$, we define a notion of convergence for $(\lambda_N)_{N\in\mathbb{N}}$ which implies the convergence in non-commutative distribution and in $\mathcal{P}$-expectation of the λ_N-random walk to a $\mathcal{P}$-free multiplicative Lévy process. This limiting process is shown not to be a free multiplicative Lévy process and we compute its log-cumulant functional. We give also a criterion on $(\lambda_N)_{N\in\mathbb{N}}$ in order to know if the limit is random or not.

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1. Intoduction

Yang-Mills theory was introduced by Yang and Mills, in 1954, in [**25**] as a theory of random connections on a principal bundle with gauge symmetry. In two dimensions, it has been defined by mathematicians ([**3**], [**4**], [**5**], [**7**], [**8**], [**13**], [**14**], [**16**], [**18**], [**23**], [**24**]) and it has become well understood that it was a theory of random multiplicative functions from the set of paths of a two dimentional surface to a compact group G. In [**12**], the author proved that an axiomatic formulation of planar Yang-Mills measures, similar to the axioms for Lévy processes, could be set: this allowed to prove a correspondance between Yang-Mills measures and a set of Lévy processes on G. In the following, by Yang-Mills measure, we consider the one given by chosing a Brownian motion on G.

When the structure group G is a discrete group, T. Lévy proved in [**18**] that the Yang-Mills measure could be seen as the random monodromy field of a random ramified G-bundle. Since ramified $\mathfrak{S}(N)$-bundles are in bijection with ramified coverings with N sheets, one recovers the link explained by A. D'Adda and P. Provero in [**1**] and [**2**] between $\mathfrak{S}(N)$-Yang Mills measure and random branched $\mathfrak{S}(N)$ coverings. It has to be noticed that this link is different from the $U(N)$-Yang Mills measure/ramified coverings partly explained in [**17**] and also known as the Yang-Mills/String duality. The theory of random ramified coverings has also some interesting and challenging links with quantum gravity ([**26**]).

In this article, we study the asymptotic of the theory of random ramified coverings coming from the $\mathfrak{S}(N)$-Yang Mills measure as N goes to infinity: we construct the $\mathfrak{S}(\infty)$-master field. The rigorous study of the asymptotics of Yang-Mills measures driven by the Brownian motion on the unitary group begun with M. Anshelevich and A.N. Sengupta in [**6**] where the convergence was proved for a weak Yang-Mills measure and T. Lévy in [**15**] where asymptotics and Makeenko-Migdal equations were proved for the full Yang-Mills measure. In this last article, the unitary, orthogonal and sympleptic groups were considered, and the convergence of the Yang-Mills measure driven by the different Brownian motions, as the dimension of the group goes to infinity, was proved by using estimates for the speed of convergence in non-commutative distribution of arbitrary words in independent Brownian motions. In the article [**9**] in preparation, the author and his co-author show how to prove asymptotics of Yang-Mills measures driven by Lévy processes on the unitary and orthogonal groups without using any estimates for the speed of convergence: the asymptotic of Yang-Mills measure is a consequence of the convergence in non-commutative distribution of the Lévy processes considered and a kind of two-dimensional Kolmogorov's continuity theorem proved by T. Lévy in [**18**].

Using similar arguments, we prove convergence of Yang-Mills measure driven by random walks on the symmetric groups by proving the convergence in non-commutative distribution of some continuous-time random walks on the symmetric groups. For sake of simplicity, for any integer N, we only consider random walks which jump by an element which is drawn uniformly from a conjugacy class $\lambda(N)$ of $\mathfrak{S}(N)$. If the conjugacy class $\lambda(N)$ converges in some sense, then the random walks will converge in non-commutative distribution. In particular, the eigenvalue distribution will converge when N goes to infinity. When λ_N is the set of transpositions, this result was shown using representation theory [**21**]. Besides, it seems possible that some of these results could be deduced from the proofs of articles like [**20**], [**19**] where the distance from the identity was proved to converge. The study of asymptotics linked with random walks was also one of the concern of the article [**22**]. In these last articles, the heuristic idea was to consider the symmetric group as a "Lie group" whose "Lie algebra" would be in some sense $\mathbb{Z}[C]$ where $C = \cup_{k=2}^{\infty} \left([1,N]^k/\sim\right)$ where $(i_1,...,i_k) \sim (j_1,...,j_k)$ if the second one is obtained by a cyclic permutation of the first one. In this picture, the exponential of $c \in C$ would just be the permutation which has c as a single non-trivial cycle. The interesting fact about this is that, one can link easily the Brownian motion on the "Lie algebra" which drives the Brownian motion on the symmetric group (the random walk by transposition) with some Erdös-Renyi random graph process. Using the natural coupling between the two processes, one can then transfer results from Erdös-Renyi random graph processes to the study of random walks on the symmetric graph.

In this article, we use a generalization of the non-commutative probability ideas, defined in [**10**] and [**11**], in order to prove asymptotics and phase transitions for the random walks on the symmetric group without using the coupling with the Erdös-Renyi random graph processes. This allows us to show that asymptotics of random walks on the symmetric groups can be studied with the same tools than the one used for the study of multiplicative unitary Lévy processes (Section 10 of [**11**]). This method is a generalization of the method used by T. Lévy in [**17**] or [**15**] in order to study the large N asymptotics of the Brownian motions on $U(N), O(N)$ and $Sp(N)$. In particular, we do not use any theory of representations, as opposed to [**21**] where some results where given for the random walk by transpositions. We prove results in a more general setting, in particular we do not ask that the elements of λ_N have bounded support when N goes to infinity. This allows us to show that there exist two behaviors for the eigenvalues empirical distribution: if the size of the support is $o(N)$ then it converges in probability to a non random probability measure we are able to compute explicitely, and if the support is growing like αN, the eigenvalues empirical distribution converges in law to a random measure. As an application of asymptotic $\mathcal{P}$-freeness of matrices which are invariant by the symmetric group, we get that the whole random walk converges in distribution in non-commutative distribution to a process whose increments are not free but $\mathcal{P}$-free. This gives the first non-trivial example of $\mathcal{P}$-free multiplicative Lévy process which log cumulant functional is computed.

1.1. Layout. — The results we present in this article are based on the study of the asymptotic of random walks on the symmetric group (Section 2). The theorems about convergence of random walks on the symmetric group $\mathfrak{S}(N)$ are presented in Section

2. After some preliminary results explained in Section 2.2, we give the proofs of these theorems in Section 2.3. The log-cumulant functional for the limit of random walks on the symmetric group is computed in Section 2.4.

A short presentation of Yang-Mills measure with $\mathfrak{S}(N)$-gauge group is explained in Section 3. In the same section, we prove that the Wilson loops in $\mathfrak{S}(N)$-Yang-Mills measure converge in probability as N goes to infinity to a non-random field: the $\mathfrak{S}(\infty)$-master field.

Based on the results of T. Lévy in [**18**], we explain in Section 4 how to link the study of random coverings of the disk and the study of $\mathfrak{S}(N)$-Yang-Mills measure. This allows us to prove that the monodromy field of a model of random simple ramified labelled covering of the unit disk with N sheets converges in probability to a non-random field as N goes to infinity.

2. Convergence of random walks on $\mathfrak{S}(N)$

2.1. General theorems of convergence. — In this section, we state the general theorems about convergence of random walks on the symmetric group. The proofs will be given in Section 2.3. Let N be a positive integer, let us consider λ_N a conjugacy class of $\mathfrak{S}(N)$, the symmetric group on N elements. We denote by $\#\lambda_N$ the size of the conjugacy class λ_N. Let σ be in λ_N and let i be in $\{1,...,N\}$. For any $k \in \{1,...,N\}$, the period of k in σ is the smallest positive integer n such that $\sigma^n(k) = k$. We denote by $\lambda_N(i)$ the number of elements in $\{1,...,N\}$ which period in σ is equal to i: this number does not depend on the choice of σ. Thus, we can see λ_N as a partition of N:

$$\lambda = (\lambda_N(i))_{i=1}^{\infty} \text{ and } \sum_{i=1}^{\infty} \lambda_N(i) = N.$$

Definition 2.1. — *We define the λ_N-random walk on $\mathfrak{S}(N)$, denoted by $\left(S_t^N\right)_{t\geq 0}$, as the Markov process on $\mathfrak{S}(N)$ such that $S_0^N = id_N$ and whose generator is given by:*

$$\forall f \in \mathbb{R}^{\mathfrak{S}(N)}, \forall \sigma_0 \in \mathfrak{S}(N), H_N f(\sigma_0) = \frac{N}{\lambda_N(1^c)} \frac{1}{\#\lambda_N} \sum_{\sigma \in \lambda_N} \left[f(\sigma\sigma_0) - f(\sigma_0)\right],$$

where we used the following notation:

$$\lambda_N(1^c) = N - \lambda_N(1).$$

This random walk is invariant by conjugation by $\mathfrak{S}(N)$.

Lemma 2.1. — *Let σ be in $\mathfrak{S}(N)$, let $\left(S_t^N\right)_{t\geq 0}$ be a λ_N-random walk on $\mathfrak{S}(N)$. We have the equality in law: $\left(\sigma S_t^N \sigma^{-1}\right)_{t\geq 0} = \left(S_t^N\right)_{t\geq 0}$.*

Proof. — It is a consequence of the fact that the generator of $\left(S_t^N\right)_{t\geq 0}$ is invariant by conjugation by $\mathfrak{S}(N)$. □

For any positive integer N, let λ_N be a conjugacy class of $\mathfrak{S}(N)$.

Definition 2.2. — *The sequence $(\lambda_N)_{N\in\mathbb{N}}$ converges if and only if there exists:*

$$(\lambda(i))_{i\geq 2}\in\left\{(a_i)_{i\in\mathbb{N}\setminus\{0,1\}}\mid \forall i\geq 2, a_i\geq 0, \sum_{i=2}^{\infty}a_i\leq 1\right\}$$

such that for any integer $i\geq 2$:

$$\frac{\lambda_N(i)}{\lambda_N(1^c)}\underset{N\to\infty}{\longrightarrow}\lambda(i),$$

and there exists $\alpha\in[0,1]$ such that:

$$\frac{\lambda_N(1)}{N}\underset{N\to\infty}{\longrightarrow}1-\alpha.$$

The sequence $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent if $\alpha=0$ and it is macroscopic if $\alpha>0$.

Moreover we say that $(\lambda_N)_{N\in\mathbb{N}}$ converges at any order of fluctuations if one can find $\left((\lambda^{(m)}(i))_{m\in\mathbb{N}}\right)_{i\geq 2}$ and $\left(\beta^{(m)}\right)_{m\in\mathbb{N}}$ such that for any $k\geq 0$

$$\frac{\lambda_N(i)}{\lambda_N(1^c)}-\sum_{m=0}^{k}\frac{\lambda^{(m)}(i)}{N^m}=o\left(\frac{1}{N^k}\right),$$

$$\frac{\lambda_N(1)}{N}-\sum_{m=0}^{k}\frac{\beta^{(m)}}{N^m}=o\left(\frac{1}{N^k}\right).$$

For any positive integer N, let us consider $\left(S_t^N\right)_{t\geq 0}$ a λ_N-random walk on $\mathfrak{S}(N)$. Recall Definition 5.1 of [**11**]. Let us suppose that the sequence $(\lambda_N)_{N\in\mathbb{N}}$ converges.

Theorem 2.1. — *For any $t\geq 0$, the mean empirical eigenvalues distribution of $\left(S_t^N\right)_{N\in\mathbb{N}}$ converges as N goes to infinity to a probability measure $\overline{\mu}_t^{\lambda}$ which has the form:*

$$\overline{\mu}_t^{\lambda}=\sum_{n\in\mathbb{N}^*}\sum_{k=0}^{n-1}\frac{m_{n^c}(t)}{n}\delta_{e^{\frac{2ik\pi}{n}}}+m_{\infty^c}(t)\lambda_{\mathbb{U}},$$

with $m_{\infty^c}(t)=1-\sum_{k=0}^{\infty}m_{n^c}(t)\geq 0$ and $m_{n^c}(t)\geq 0$ for any integer n.

Recall that $[\sigma]$ is the conjugacy class of σ which can be seen as a partition of N. Let us take the convention that for any positive integer k: $\frac{\left((1-0)^k-1\right)}{0}=-k$. Let us consider the unique solution $\left((m_{\sigma^c}(t))_{\sigma\in\cup_k\mathfrak{S}_k}\right)_{t\geq 0}$ of the system of differential equations: $\forall k\in\mathbb{N}^*$, $\forall\sigma_0\in\mathfrak{S}_k$, $\forall t_0\geq 0$:

$$\frac{d}{dt}_{|t=t_0}m_{\sigma_0^c}(t)=\frac{\left((1-\alpha)^k-1\right)}{\alpha}m_{\sigma_0^c}(t_0)$$
$$+\sum_{\sigma\in\mathfrak{S}_k\setminus\{id_k\},\sigma\leq\sigma_0}\alpha^{\mathsf{nc}(\sigma\vee id)-[\sigma](1)-1}\left(\prod_{i=2}^{k}(\lambda(i))^{\frac{[\sigma](i)}{i}}\right)(1-\alpha)^{[\sigma](1)}m_{(t_{\sigma\sigma_0})^c}(t_0),$$

with the initial conditions: $\forall k\in\mathbb{N}^*,\forall\sigma\in\mathfrak{S}_k$:

$$m_{\sigma^c}(0)=\delta_{\sigma=id_k}.$$

Then for any positive integer n, for any real $t \geq 0$:

$$m_{n^c}(t) = m_{(1,...,n)^c}(t),$$

where $(1, ..., n) \in \mathfrak{S}_n$ is a n cycle. Besides for any $t \geq 0$ and any $\sigma \in \mathfrak{S}_k$:

$$m_{\sigma^c}(t) = \lim_{N\to\infty} \mathbb{E} m_{\sigma^c}\left[\left(S_t^N\right)^{\otimes k}\right].$$

This theorem can be extended to the whole process $\left(S_t^N\right)_{t\geq 0}$. Recall the notion of convergence in $\mathcal{P}$-expectation defined in Definition 2.2 of [**11**]. Recall also the Definition 7.5 in [**11**].

Theorem 2.2. — *The family $\left(S_t^N\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation as N goes to infinity. The process $\left(\left(S_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial $\mathcal{P}$-free multiplicative Lévy process. It is not a $\mathfrak{S}$-free multiplicative Lévy process. In particular, if $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent, it is not a free multiplicative Lévy process in the sense of Voiculescu: the multiplicative increments are not asymptotically free.*

Besides, if $(\lambda_N)_{N\in\mathbb{N}}$ converges at any order of fluctuations, then the family $\left(S_t^N\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation up to any order of fluctuations as N goes to infinity.

In fact, the proof of Theorem 2.1 will show that the empirical eigenvalues distribution of $\left(S_t^N\right)_{N\in\mathbb{N}}$ converges in law as N goes to infinity to a random measure: depending on the behavior of $(\lambda_N)_{N\in\mathbb{N}}$, one can know if the limit is or is not non-random.

Theorem 2.3. — *Let t be a positive real. The empirical eigenvalues distribution $\mu_t^{\lambda_N}$ of S_t^N converges in law to a random measure on $\mathbb{U}$, denoted by μ_t^{λ}. Two behaviors are possible:*

1. *if the sequence $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent then the limiting measure is a non-random measure on $\mathbb{U}$, $\mu_t^{\lambda} = \overline{\mu}_t^{\lambda}$, and the convergence holds in probability. The family $\left(S_t^N\right)_{N\in\mathbb{N}}$ satisfies the asymptotic $\mathcal{P}$-factorization and thus it converges in probability in $\mathcal{P}$-normalized moments,*
2. *if the sequence $(\lambda_N)_{N\in\mathbb{N}}$ is macroscopic, then the limiting measure is not a non-random measure on $\mathbb{U}$ and the family $\left(S_t^N\right)_{N\in\mathbb{N}}$ does not satisfy the asymptotic $\mathcal{P}$-factorization.*

In the case where the sequence $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent, we can compute explicitly the measure μ_t^{λ}. Given Theorem 2.1, we only need to compute its moments or equivalently $m_{n^c}(t)$.

Theorem 2.4. — *Let us suppose that $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent. Let us use the same notations as in Theorem 2.1 and in Definition 2.2. Let n be a positive integer and let $t \geq 0$. We have:*

$$m_{n^c}(t) = e^{-nt} \sum_{k=0}^{n-1} t^k \frac{n^{k-1}}{k!} \sum_{(i_1,...,i_k)\in\mathbb{N}^*, \sum_{j=1}^k i_j = n-1} \prod_{j=1}^{k} \lambda(i_j + 1). \tag{1}$$

We used the usual conventions for the products and the sums, thus $m_{1^c}(t) = e^{-t}$. *In particular for any positive integer* n, *and any* $t \geq 0$:

$$m_n(t) := \int_{z\in\mathbb{U}} z^n d\overline{\mu}_t^{\lambda}(z) = \sum_{d|n} e^{-dt} \sum_{k=0}^{d-1} t^k \frac{d^{k-1}}{k!} \sum_{(i_1,\ldots,i_k)\in\mathbb{N}^*, \sum_{j=1}^k i_j = d-1} \prod_{j=1}^{k} \lambda(i_j+1).$$

In particular, let us consider a positive integer k, *let us suppose that* $\lambda(k) = 1$ *and for any positive integer* $l \neq k$, $\lambda(l) = 0$. *This means that we are considering a random walk which jumps by multiplication by a uniform* k*-cycle. Let* t *be a non-negative real. If there does not exist any positive integer* u *such that* $n = u(k-1)+1$, *then* $m_{n^c}(t) = 0$. *Besides, for any* $u \in \mathbb{N}$:

$$m_{(u(k-1)+1)^c}(t) = e^{-(u(k-1)+1)t} t^u \frac{(u(k-1)+1)^{u-1}}{u!}. \tag{2}$$

Let us suppose for this discussion that $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent. In Theorem 2.1, we saw that the measure μ_t^{λ} is the sum of an atomic part and $m_{\infty^c}(t)$ times the Lebesgue measure on the unit circle. In fact there exists a real $t_c^{\lambda} \geq 0$ such that μ_t^{λ} is purely atomic for $t < t_c^{\lambda}$ and for any $t > t_c$, μ_t^{λ} is a sum of a purely atomic and a multiple of the Lebesgue measure. This critical time is the same critical as found by N. Berestycki in [**19**] for the phase transition for the distance to the identity.

Theorem 2.5. — *Let us suppose that* $(\lambda_N)_{N\in\mathbb{N}}$ *is evanescent. The function* $t \mapsto m_{\infty^c}(t)$ *is continuous and converges to* 1 *as* t *goes to infinity. Besides, if we define:*

$$t_c^{\lambda} = \delta_{\sum_{j=2}^{\infty}\lambda(j)=1} \frac{1}{\sum_{j=2}^{\infty}(j-1)\lambda(j)}, \tag{3}$$

for any $0 \leq t \leq t_c^{\lambda}$, $m_{\infty^c}(t) = 0$ *and for any* $t > t_c^{\lambda}$, $m_{\infty^c}(t) > 0$.

Using the theorems already explained, we recover Theorem 3 of [**20**], Theorem 4 of [**21**]. We recommend the reader to have also a look at Theorem 3 of [**19**].

Corollary 2.1. — *Let us suppose that* $(\lambda_N)_{N\in\mathbb{N}}$ *converges and is evanescent. For any positive integer* N, *let* $\left(S_t^N\right)_{t\geq 0}$ *be a* λ_N*-random walk on the symmetric group. For any permutations* σ *and* σ' *in* $\mathfrak{S}(N)$, *let* $d_{\mathfrak{S}(N)}(\sigma,\sigma')$ *be the distance in* $\mathfrak{S}(N)$ *between* σ *and* σ' *defined as* $N - \mathsf{nc}(\sigma \vee \sigma')$, *where* nc *is the function which gives the number of cycles. Then for any* $t \geq 0$, $\frac{1}{N} d_{\mathfrak{S}(N)}\left(id_N, S_t^N\right)$ *converges in probability when* N *goes to infinity to the non-random continuous function:*

$$d^{\lambda}(t) = 1 - \sum_{k=1}^{\infty} \frac{1}{k} m_{k^c}(t),$$

where $m_{k^c}(t)$ *is given by Equation* (1).

Using the Equation (2), we recover Equation (5) of [**19**], yet this expression of $d^{\lambda}(t)$ for general λ seems to be new. The function $d^{\lambda}(t)$ was studied in [**19**], when $\left(\lambda_N\left(1^c\right)\right)_{N\in\mathbb{N}}$ is constant and equal to a positive integer a: using t_c^{λ} defined before, it was shown that $d^{\lambda}(t)$ is C^{∞} on a subset of the form $\mathbb{R}^+ \subset I$, with I a bounded interval of $]t_0, \infty[$, for any $t < t_c^{\lambda}$, $d^{\lambda}(t) = \frac{t}{a}$ and $\left(d^{\lambda}\right)''\left(\left(t^{\lambda_c}\right)^+\right) = -\infty$. Using the Stirling's formula, it is easy

to see that for the random walk which only jumps by multiplication by a k-cycle, the set I is empty.

In the second part of the article, in order to construct the Yang-Mills $\mathfrak{S}_\infty$-field, we will need the following result whose proof will not be given since it is an easy consequence of Theorems 2.2, 2.3, Lemma 2.1 of this article and the Theorem 7.8 and Proposition 2.1 of [**11**].

Let n be a positive integer and for any positive integer N, let us consider $\left(S_t^{1,N}\right)_{t\geq 0}$, $\ldots,\left(S_t^{n,N}\right)_{t\geq 0}$, n independent λ_N-random walks on $\mathfrak{S}(N)$. Recall the Equation (4) of [**11**] and Section 2.2 of [**10**].

Theorem 2.6. — *The family of random matrices:*

$$\mathbf{F}=\left(\left(S_t^{k,N}\right)_{N\in\mathbb{N}},\left(\left(S_t^{k,N}\right)^{-1}\right)_{N\in\mathbb{N}}\right)_{t\geq 0,k\in\{1,\ldots,n\}}$$

converges in $\mathcal{P}$-expectation as N goes to infinity. Besides, if $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent, then $\mathbf{F}$ satisfies the asymptotic $\mathcal{P}$-factorization property as N goes to infinity. This implies that $\mathbf{F}$ converges in probability in $\mathcal{P}$-normalized moment: for any positive integer k, for any $p\in\mathcal{P}_k$, for any k-uple $\left(\left(B_1^N\right)_{N\in\mathbb{N}},\ldots,\left(B_k^N\right)_{N\in\mathbb{N}}\right)$ of elements of $\mathcal{A}(\mathbf{F})$:

$$\frac{1}{N^{\mathsf{nc}(p\vee id)}}Tr^k\left(\left(B_1^N\otimes\ldots\otimes B_k^N\right)\rho_N^{\mathcal{P}_k}(p)\right)$$

converges in probability to the limit of its expectation as N goes to infinity.

Besides, for any $i\in\{1,\ldots,k\}$, $\left(S_t^{i,N}\right)_{t\geq 0}\cup\left(\left(S_t^{i,N}\right)^{-1}\right)_{t\geq 0}$ is asymptotically $\mathcal{P}$-free but not asymptotically free in the sense of Voiculescu from

$$\left(\left(S_t^{k,N}\right)_{N\in\mathbb{N}},\left(\left(S_t^{k,N}\right)^{-1}\right)_{N\in\mathbb{N}}\right)_{t\geq 0,k\in\{1,\ldots,n\}\setminus\{i\}}.$$

2.2. Preliminary results. — Before we prove the theorems we just stated, we need some preliminary results. We could have written them in [**11**], yet since most of them are specific to permutation matrices we prefered to explain them in this article.

2.2.1. Exclusive moments for permutation matrices. — Let k be a positive integer. Recall the notions of irreducible partition and of extraction defined in Definitions 2.9 and 2.12 of [**10**]. Let p be a partition, we recall that $\mathsf{l}(p)$ is the lengh of p: it is the unique integer such that $p\in\mathcal{P}_{\mathsf{l}(p)}$.

Definition 2.3. — *Let $I=\{b_1,\ldots,b_s\}$ be a partition of $\{1,\ldots,k\}$, let σ be an irreducible permutation of $\{1,\ldots,s\}$. For any integer $l\in\{1,\ldots,s\}$, we denote by b_l' the set $\{j\in\{1',\ldots,k'\},\exists i\in b_l, j=i'\}$. The partition:*

$$p=\left\{b_l\cup b'_{\sigma(l)},l\in\{1,\ldots,s\}\right\}$$

is called the necklace associated with (I,σ). The true-length of p, denoted $|\,p\,|$, is equal to s.

Let p be an irreducible partition in $\mathcal{P}_k$: it is a chain if and only if there exist two blocks s and r of p such that the partition $\tilde{p} = p \setminus \{r, s\} \cup \{r \cup s\}$ is a necklace. The true-length of p, denoted $|p|$, is equal to the true-length of $\tilde{p}$.

A partition p in $\mathcal{P}_k$ is a parure if for any cycle c of p the extraction of p on c is either a chain or a necklace. The true-length of p is defined by:

$$|p| = \sum_{c \text{ cycle of } p} |p_c| .$$

Let N be a positive integer, let S be a permutation in $\mathfrak{S}(N)$. Recall Sections 2.2 and 2.3 of [**10**]. For sake of clarity, in the following we will denote by $m_{p^c}(S)$ the number $\frac{1}{N^{\mathrm{nc}(p \vee id)}} Tr(S^{\otimes k} \circ \rho_N^{\mathcal{P}_k}(p^c))$.

Proposition 2.1. — *Let S be a permutation in $\mathfrak{S}(N)$, let p be a partition in $\mathcal{P}_k$. If p is not a parure then $m_{p^c}(S) = 0$. If p is a necklace, then $m_{p^c}(S) = m_{(1,\ldots,|p|)^c}(S)$, and if p is a chain, then:*

$$m_{p^c}(S) = 1 - \sum_{l=1}^{|p|} m_{(1,\ldots,l)^c}(S) .$$

Proof. — Let S be a permutation in $\mathfrak{S}(N)$. Let p be a partition in $\mathcal{P}_k$. Let us consider a block b of p. Let u and v (resp. u' and v') be in $\{1, ..., k\}$ (resp. $\{1', ..., k'\}$) which are in the same block of p. Then u' and v' (resp. u and v) must be in the same block of p if one wants $m_{p^c}(S)$ not to be equal to zero. This is a consequence of the fact that for any $i, j, l \in \{1, ..., N\}$,

$$S_i^j S_i^l = S_i^j \delta_{l=j} \text{ and } S_j^i S_l^i = S_j^i \delta_{l=j}. \tag{4}$$

Yet, if p is not a parure, these conditions on the blocks of p are not satisfied, thus $m_{p^c}(S) = 0$.

The assertion about the exclusive moments when p is a necklace is a consequence of the Equations 4. Let us suppose that p is a chain. Using the Equations (4), we get that $m_{p^c}(S)$ is the fraction of elements of $\{1, ..., N\}$ which period in S is strictly greater than $|p|$. Thus it is equal to one minus the fraction of elements of $\{1, ..., N\}$ which period in S is less than $|p|$. Since for any positive integer l, $m_{(1,\ldots,l)^c}(S)$ is the fraction of elements of $\{1, ..., N\}$ which period in S is equal to l, we get the following equality:

$$m_{p^c}(S) = 1 - \sum_{l \geq 1, l \leq |p|} m_{(1,\ldots,l)^c}(S),$$

which is the equality we had to prove. □

2.2.2. Link between moments and exclusive moments for permutation matrices. — When one considers permutation matrices, an interesting link occurs between normalized moments and normalized exclusive moments. Let N be a positive integer and let S be a permutation in $\mathfrak{S}(N)$. For any positive integer l, $(1, ..., l)$ is the l-cycle in $\mathfrak{S}_l$.

Proposition 2.2. — *For any positive integer k,*

$$Tr\left[S^{\otimes k}\circ\rho_N^{\mathcal{P}_k}\left((1,...,k)\right)\right]=\sum_{d\in\mathbb{N}^*,d|k}Tr\left[S^{\otimes d}\circ\rho_N^{\mathcal{P}_k}\left((1,...,d)^c\right)\right].$$

Proof. — This is due to the fact that for any positive integer k,

$$Tr\left[S^{\otimes k}\circ\rho_N^{\mathcal{P}_k}\left((1,...,k)\right)\right]=\frac{1}{N}Tr\left(S^k\right)$$

is equal to the fraction of elements $i\in\{1,...,N\}$ whose period divides k and $Tr\left[S^{\otimes d}\circ\rho_N^{\mathcal{P}_k}\left((1,...,d)^c\right)\right]$ is equal to the fraction of elements $i\in\{1,...,N\}$ whose period is equal to d. □

2.2.3. Measures. — The following lemma is a special and easy case of the problem of moments.

Lemma 2.2. — *Let $(\kappa_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{i=1}^{\infty}\kappa_i\leq\kappa_0$. There exists a unique measure μ on $\mathbb{U}$ whose weight is equal to κ_0 such that:*

$$\forall n\in\mathbb{N}^*,\int_{\mathbb{U}}z^nd\mu(z)=\sum_{d\in\mathbb{N}^*,d|n}\kappa_d.$$

Besides μ is equal to:

$$\sum_{n\in\mathbb{N}^*}\sum_{k=0}^{n-1}\frac{\kappa_n}{n}\delta_{e^{\frac{2ik\pi}{n}}}+\left[\kappa_0-\sum_{i=1}^{\infty}\kappa_i\right]\lambda_{\mathbb{U}}.$$

Proof. — Any measure on the unit circle $\mathbb{U}$ is characterized by its non-negative moments as we saw in the proof of Theorem 2.1 of [**11**]. Let $(\kappa_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{i=1}^{\infty}\kappa_i\leq\kappa_0$. It is enough to see that the moments of $\sum_{n\in\mathbb{N}^*}\sum_{k=0}^{n-1}\frac{\kappa_n(t)}{n}\delta_{e^{\frac{2ik\pi}{n}}}+\left[\kappa_0-\sum_{i=1}^{\infty}\kappa_i\right]\lambda_{\mathbb{U}}$ are equal to the ones expected. □

Let $(\kappa_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{i=1}^{\infty}\kappa_i\leq\kappa_0$. Let μ be the unique measure associated to $(\kappa_n)_{n\in\mathbb{N}}$ which is given by Lemma 2.2. By looking at the form of μ, one can see that the weight of the purely atomic part of μ is equal to $\sum_{n\geq1}\kappa_n$.

2.2.4. Criterion for non $\mathfrak{S}$-freeness. — In the article [**11**], we saw that the Voiculescu asymptotic freeness and the asymptotic $\mathcal{P}$-freeness are not the same notions. Let us state some consequence of Proposition 7.1 of [**11**] when one considers random matrices whose entries are equal either to zero or one. Recall that $\mathsf{0}_2$ is the partition $\{\{1,2,1',2'\}\}$ in $\mathcal{P}_2$. Recall that the notion of $\mathcal{L}^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ was defined at the beginning of Section 2.1 of [**11**]. At last, recall Definition 2.1 and Notation 2.1 of [**11**].

Lemma 2.3. — *Let $S=\left(S^N\right)_{N\in\mathbb{N}}$ be an element of $L^{\infty^-}\otimes\mathcal{M}(\{0,1\})$. Let us suppose that S converges in expectation in $\mathcal{P}$-moments then:*

$$\mathbb{E}\kappa_{\mathcal{P}}^{\mathsf{0}_2}[S,S]=\mathbb{E}m_{id_1}[S]-\mathbb{E}m_{id_2}[S,S].$$

If the asymptotic $\mathfrak{S}$-factorization property holds for S, then:

$$\mathbb{E}\kappa_{\mathcal{P}}^{0_2}[S,S] = \mathbb{E}m_{id_1}[S]\left(1 - \mathbb{E}m_{id_1}[S]\right).$$

Proof. — Indeed, we have $\mathbb{E}\kappa_{\mathcal{P}}^{0_2}[S,S] = \mathbb{E}m_{0_2}[S,S] - \mathbb{E}m_{id_2}[S,S]$. Yet, one does not forget that for any integer N, S^N is a matrix of zeros and ones, thus for any positive integer N, $\mathbb{E}m_{0_2}\left[S^N \otimes S^N\right] = \mathbb{E}m_{id_1}\left[S^N\right]$. This implies that:

$$\mathbb{E}\kappa_{\mathcal{P}}^{0_2}[S,S] = \mathbb{E}m_{id_1}[S] - \mathbb{E}m_{id_2}[S,S].$$

The second assertion is a direct consequence of the $\mathfrak{S}$-factorization property. □

Proposition 2.3. — *Let S_1 and S_2 be two elements of $L^{\infty^-}\otimes\mathcal{M}(\{0,1\})$ which converge in expectation in $\mathcal{P}$-moments and which are asymptotically $\mathcal{P}$-free. If for $i \in \{1,2\}$, $\mathbb{E}m_{id_2}[S_i,S_i] \neq \mathbb{E}m_{id_1}[S_i]$, which is equivalent to say that $\frac{1}{N}Tr\left(S_i^N\right)$ does not converge in law to a random variable which takes value in $\{0,1\}$, then S_1 and S_2 are not asymptotically $\mathfrak{S}$-free.*

In particular, if the asymptotic $\mathcal{P}$-factorization property holds for both of them, if $\mathbb{E}m_{id}[S_1] \notin \{0,1\}$ and $\mathbb{E}m_{id}[S_2] \notin \{0,1\}$, then S_1 and S_2 are not asymptotically free in the sense of Voiculescu.

Proof. — This is a consequence of Proposition 7.1 of [**11**], of Lemma 2.3 of this article and also of the fact that under the hypothesis of asymptotic factorization the $\mathfrak{S}$-freeness is equivalent to the usual freeness as explained in Section 7.1.3 of [**11**]. □

2.3. Proofs of the theorems of Section 2.1. — We can now prove the theorems of Section 2.1.

Proof of Theorem 2.1 and Theorem 2.2. — For any positive integer N, let us consi;der λ_N a conjugacy class of $\mathfrak{S}(N)$. Let us suppose that $(\lambda_N)_{N\in\mathbb{N}}$ converges as N goes to infinity. For any positive integer N, let us consider $\left(S_t^N\right)_{t\geq 0}$ a λ_N-random walk on $\mathfrak{S}(N)$. Let N and k be two positive integers and let us define:

$$G_k^N = \frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(S_t^N\right)^{\otimes k}\right] = \frac{N}{\lambda_N(1^c)}\frac{1}{\#\lambda_N}\sum_{\sigma\in\lambda_N}\left[\sigma^{\otimes k} - id^{\otimes k}\right].$$

Let p be a partition in $\mathcal{P}_k$. Let $\sigma_N \in \lambda_N$, let us remark that:

$$m_{p^c}\left(G_k^N\right) = \frac{N}{\lambda_N(1^c)}\left[m_{p^c}\left(\sigma_N^{\otimes k}\right) - m_{p^c}\left(id^{\otimes k}\right)\right],$$

thus, using Proposition 2.1, if p is not a parure, then $m_{p^c}\left(G_k^N\right) = 0$.

Let us suppose that p is an irreducible parure, then it is either a necklace or a chain. Yet, using Proposition 2.1, it is enough to consider the case when p is a necklace, and thus when it is a cycle. Let us suppose that $p = (1,...,k)$, then:

$$m_{p^c}\left(G_k^N\right) = \frac{N}{\lambda_N(1^c)}\left[m_{(1,...,k)^c}\left(\sigma_N^{\otimes k}\right) - m_{(1,...,k)^c}\left(id^{\otimes k}\right)\right].$$

If $k = 1$, then:

$$m_{p^c}\left(G_k^N\right) = \frac{N}{\lambda_N(1^c)}\left[\frac{\lambda_N(1)}{N} - \frac{N}{N}\right] = -1.$$

If $k \neq 1$, then:

$$m_{p^c}\left(G_k^N\right) = \frac{N}{\lambda_N(1^c)}\left[\frac{\lambda_N(k)}{N}\right] = \frac{\lambda_N(k)}{\lambda_N(1^c)} \underset{N\to\infty}{\longrightarrow} \lambda(k).$$

If p is a chain, let us remark that, using again Proposition 2.1:

$$\begin{aligned} m_{p^c}\left(G_k^N\right) &= \frac{N}{\lambda_N(1^c)} m_{p^c}\left(\sigma_N^{\otimes k}\right) \\ &= \frac{N}{\lambda_N(1^c)}\left[1 - \sum_{l=1}^{|p|} m_{(1,\ldots,l)^c}\left(\sigma_N^{\otimes l}\right)\right] \underset{N\to\infty}{\longrightarrow} 1 - \sum_{l=2}^{|p|}\lambda(l). \end{aligned}$$

Thus, for any irreducible partition, $m_{p^c}\left(G_k^N\right)$ converges as N goes to infinity. Yet, if p is irreducible, any partition p' which is coarser than p is also irreducible. This implies that, for any irreducible partition, $m_p\left(G_k^N\right)$ converges as N goes to infinity.

Let us remark that for any partition p in $\mathcal{P}_k$:

$$m_p\left(G_k^N\right) = \frac{N}{\lambda_N(1^c)}\left[m_p\left(\sigma_N^{\otimes k}\right) - 1\right]. \tag{5}$$

Thus, we have proved that for any irreducible partition p, $m_p(\sigma_N^{\otimes k})$ converges as N goes to infinity, and besides, if $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent, then $\lim_{N\to\infty} m_p(\sigma_N^{\otimes k}) = 1$. Let r be a positive integer, let us consider r irreducible partitions p_1, ..., p_r, we have:

$$\begin{aligned} m_{p_1\otimes\ldots\otimes p_r}\left(G_k^N\right) &= \frac{N}{\lambda_N(1^c)}\left[m_{p_1\otimes\ldots\otimes p_r}\left(\sigma_N^{\otimes k}\right) - 1\right] \\ &= \frac{N}{\lambda_N(1^c)}\left[\prod_{i=1}^{r} m_{p_i}\left(\sigma_N^{\otimes k}\right) - 1\right] \\ &= \sum_{i=1}^{r}\left(\frac{N}{\lambda_N(1^c)}\left[m_{p_i}\left(\sigma_N^{\otimes k}\right) - 1\right]\prod_{l=i+1}^{r} m_{p_l}\left(\sigma_N^{\otimes k}\right)\right) \\ &= \sum_{i=1}^{r}\left(m_{p_i}\left(G_k^N\right)\prod_{l=i+1}^{r} m_{p_l}\left(\sigma_N^{\otimes k}\right)\right). \end{aligned}$$

This proves that $m_{p_1\otimes\ldots\otimes p_r}\left(G_k^N\right)$ converges as N goes to infinity. Thus for any positive integer k, for any partition $p \in \mathcal{P}_k$, $m_p\left(G_k^N\right)$ converges as N goes to infinity.

Using Theorem 10.1 of [**11**], the family $\left(S_t^N\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation and $\left(\left(S_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a $\mathcal{P}$-matricial free Lévy process. It is easy to see that the last assertion of Theorem 2.2 holds.

In particular, for any $t \geq 0$, $\left(S_t^N\right)_{N\in\mathbb{N}}$ converges in $\mathcal{P}$-expectation as N goes to infinity. Using Theorem 2.1 of [**11**], we deduce that the mean empirical eigenvalues distribution of $\left(S_t^N\right)_{N\in\mathbb{N}}$ converges as N goes to infinity to a probability measure $\overline{\mu}_t^{\lambda}$ defined on the circle $\mathbb{U}$. Besides, the measure $\overline{\mu}_t^{\lambda}$ is characterized by the fact that for any positive integer

n:

$$\int_{\mathbb{U}} z^n d\overline{\mu}_t^{\lambda} = \lim_{N\to\infty} \mathbb{E} m_{(1,\ldots,n)} \left[\left(S_t^N \right)^{\otimes n} \right].$$

Using Proposition 2.2, we get that:

$$\int_{\mathbb{U}} z^n d\overline{\mu}_t^{\lambda} = \sum_{d\in\mathbb{N}^*, d|n} \lim_{N\to\infty} \mathbb{E} m_{(1,\ldots,d)^c} \left[\left(S_t^N \right)^{\otimes d} \right].$$

We are in the setting of Lemma 2.2 thus $\overline{\mu}_t^{\lambda}$ is equal to:

$$\overline{\mu}_t^{\lambda} = \sum_{n\in\mathbb{N}^*} \sum_{k=0}^{n-1} \frac{m_{n^c}(t)}{n} \delta_{e^{\frac{2ik\pi}{n}}} + m_{\infty^c}(t)\lambda_{\mathbb{U}},$$

where for any integer $n \in \mathbb{N}^*$, $m_{n^c}(t) = \lim_{N\to\infty} \mathbb{E} m_{(1,\ldots,n)^c} \left[\left(S_t^N \right)^{\otimes n} \right]$.

For any positive integer k, any $\sigma \in \mathfrak{S}_k$ and any $t \geq 0$, let us denote by $m_{\sigma^c}(t)$ the limit $\lim_{N\to\infty} \mathbb{E} m_{\sigma^c} \left[\left(S_t^N \right)^{\otimes k} \right]$. Using Theorem 5.2 of [**11**], we know that:

$$m_{\sigma^c}(t) = \mathbb{E}\kappa_{\mathcal{P}}^{\sigma}\left[S_t, ..., S_t\right].$$

Using Theorem 10.2 of [**11**], and using the same notations as for this theorem, we get that $m_{\sigma^c}(t)$ satisfies the system of equations, $\forall t_0 \geq 0$, $\forall k \in \mathbb{N}^*$, $\forall \sigma_0 \in \mathfrak{S}_k$:

$$\frac{d}{dt}_{|t=t_0} m_{\sigma_0^c}(t) = \sum_{p_1\in\mathcal{P}_k, p_2\in\mathcal{P}_k | p_1\circ p_2 = \sigma_0, p_1 \prec \sigma_0} \left(\mathcal{R}_{\mathcal{P}}[G]\right)_{p_1} \left(\mathcal{R}_{\mathcal{P}}[S_t]\right)_{p_2}.$$

Yet, we saw, in Lemma 6.3 of [**10**], that if p_1 and p_2 are two partitions such that $p_1 \circ p_2 = \sigma_0$ and $p_1 \prec \sigma_0$, then p_1 and p_2 are two permutations and $p_1 \in [id, \sigma_0]_{\mathfrak{S}_k}$. Thus $\forall t_0 \geq 0$, $\forall k \in \mathbb{N}^*$, $\forall \sigma_0 \in \mathfrak{S}_k$:

$$\frac{d}{dt}_{|t=t_0} m_{\sigma_0^c}(t) = \sum_{\sigma\in\mathfrak{S}_k | \sigma \leq \sigma_0} \left(\mathcal{R}_{\mathcal{P}}[G]\right)_{\sigma} m_{t\sigma\sigma_0}(t_0).$$

Using again Theorem 5.2 of [**11**] and thus the fact that for any permutation σ, $\left(\mathcal{R}_{\mathcal{P}}[G]\right)_{\sigma} = \lim_{N\to\infty} m_{\sigma^c}\left(G_k^N\right)$, we see that we only need to compute $\lim_{N\to\infty} m_{\sigma^c}\left(G_k^N\right)$ for any permutation in $\mathfrak{S}_k$ in order to finish the proof of Theorem 2.1.

Let σ_0 be a permutation in $\mathfrak{S}_k$ and let us compute $m_{\sigma_0^c}\left(G_k^N\right)$. If σ_0 is equal to id_k, then:

$$\begin{aligned} m_{id_k^c}\left(G_k^N\right) &= \frac{N}{\lambda_N(1^c)} \frac{1}{\#\lambda_N} \sum_{\sigma\in\lambda_N} \left[m_{id_k^c}\left(\sigma^{\otimes k}\right) - m_{id_k^c}\left(id^{\otimes k}\right) \right] \\ &= \frac{N}{\lambda_N(1^c)} \left[m_{id_k^c}\left(\sigma^{\otimes k}\right) - m_{id_k^c}\left(id^{\otimes k}\right) \right], \end{aligned}$$

where σ is any permutation in λ_N.

In the following, we will use the following convention: for any $n \in \mathbb{N}$ for any $m \in \mathbb{N}$:

$$\frac{n!}{(n-m)!} = \prod_{n-m+1\leq i\leq n} i,$$

with of course the fact that $\prod_{i\in\emptyset} i = 1$. This is not of course a convention if $n-m+1 > 0$.

Now, if σ is a permutation in λ_N:

$$m_{id_k^c}\left(\sigma^{\otimes k}\right) = \frac{1}{N^k}\frac{\lambda_N(1)!}{(\lambda_N(1)-k)!}.$$

Besides, $m_{id_k^c}\left(id^{\otimes k}\right) = \frac{N!}{(N-k)!}$. Thus:

$$\begin{aligned} m_{id_k^c}\left(G_k^N\right) &= \frac{N}{\lambda_N(1^c)}\frac{1}{N^k}\left[\frac{\lambda_N(1)!}{(\lambda_N(1)-k)!} - \frac{N!}{(N-k)!}\right] \\ &= \frac{N}{\lambda_N(1^c)}\left[\prod_{i=0}^{k-1}\left(1-\frac{\lambda_N(1^c)+i}{N}\right) - \prod_{i=0}^{k-1}\left(1-\frac{i}{N}\right)\right]. \end{aligned}$$

Let us denote by α the limit of $\frac{\lambda_N(1^c)}{N}$ as N goes to infinity. We get:

$$\lim_{N\to\infty} m_{id_k^c}\left(G_k^N\right) = \begin{cases} -k & \text{if } (\lambda_N)_{N\in\mathbb{N}} \text{ is evanescent,} \\ \frac{1}{\alpha}((1-\alpha)^k - 1) & \text{if } (\lambda_N)_{N\in\mathbb{N}} \text{ is macroscopic.} \end{cases}$$

Now, let us suppose that σ_0 is not equal to id_k. Let σ be in λ_N, since $m_{\sigma_0}\left(id^{\otimes k}\right) = 0$, we get that:

$$m_{\sigma_0^c}\left(G_k^N\right) = \frac{N}{\lambda_N(1^c)} m_{\sigma_0^c}\left(\sigma^{\otimes k}\right).$$

Yet, by denoting by $[\sigma_0]$ the conjugacy class of σ_0, it is easy to see that:

$$m_{\sigma_0^c}\left(\sigma^{\otimes k}\right) = \frac{1}{N^{\mathsf{nc}(\sigma_0\vee id)}}\prod_{i=1}^{k}\frac{\left(\frac{\lambda_N(i)}{i}\right)!}{\left(\frac{\lambda_N(i)}{i}-\frac{[\sigma_0](i)}{i}\right)!}i^{\frac{[\sigma_0](i)}{i}},$$

thus:

$$m_{\sigma_0^c}\left(G_k^N\right) = \frac{N}{\lambda_N(1^c)}\frac{1}{N^{\mathsf{nc}(\sigma_0\vee id)}}\prod_{i=1}^{k}\frac{\left(\frac{\lambda_N(i)}{i}\right)!}{\left(\frac{\lambda_N(i)}{i}-\frac{[\sigma_0](i)}{i}\right)!}i^{\frac{[\sigma_0](i)}{i}},$$

Let us notice that $\mathsf{nc}(\sigma_0 \vee id) = \sum_{i=1}^{k}\frac{[\sigma_0](i)}{i}$. Thus:

$$m_{\sigma_0^c}\left(G_k^N\right) = \frac{N}{\lambda_N(1^c)}\left(\prod_{i=2}^{k}\frac{1}{N^{\frac{[\sigma_0](i)}{i}}}\frac{\left(\frac{\lambda_N(i)}{i}\right)!}{\left(\frac{\lambda_N(i)}{i}-\frac{[\sigma_0](i)}{i}\right)!}i^{\frac{[\sigma_0](i)}{i}}\right)\frac{1}{N^{[\sigma_0](1)}}\frac{\lambda_N(1)!}{(\lambda_N(1)-[\sigma_0](1))!}.$$

We recall that for any $i \geq 2$, there exists $\lambda(i)$ such that $\frac{\lambda_N(i)}{\lambda_N(1^c)}$ converges to $\lambda(i)$ when N goes to infinity, and $\lim_{N\to\infty}\frac{\lambda_N(1)}{N} \to 1-\alpha$. Thus, when N goes to infinity, $m_{\sigma^c}\left(G_k^N\right)$ has the same limit as:

$$\frac{N}{\lambda_N(1^c)}\left(\prod_{i=2}^{k}\left(\frac{\lambda_N(i)}{\lambda_N(1^c)}\frac{\lambda_N(1^c)}{N}\right)^{\frac{[\sigma_0](i)}{i}}\right)\left(\frac{\lambda_N(1)}{N}\right)^{[\sigma_0](1)},$$

or the same limit as:

$$\left(\frac{\lambda_N(1^c)}{N}\right)^{\mathsf{nc}(\sigma_0\vee id)-[\sigma_0](1)-1}\prod_{i=2}^{k}\left(\frac{\lambda_N(i)}{\lambda_N(1^c)}\right)^{\frac{[\sigma_0](i)}{i}}\left(\frac{\lambda_N(1)}{N}\right)^{[\sigma_0](1)}.$$

This implies that:

$$\lim_{N\to\infty} m_{\sigma_0^c}\left(G_k^N\right)=\alpha^{\mathsf{nc}(\sigma_0\vee id)-[\sigma_0](1)-1}\left(\prod_{i=2}^{k}(\lambda(i))^{\frac{[\sigma_0](i)}{i}}\right)(1-\alpha)^{[\sigma_0](1)}.$$

Let us remark that, since $\sigma_0\neq id_k$, $\mathsf{nc}(\sigma_0\vee id)-[\sigma_0](1)-1$ is always non negative. So the following formula has a meaning even if $\alpha=0$. Using these calculations, we recover the system of differential equations stated in Theorem 2.1.

At last, let us prove that $\left(S_t^N\right)_{t\geq 0}$ is not a $\mathfrak{S}$-free multiplicative Lévy process. In order to do so, we will prove that the increments of $\left(S_t^N\right)_{t\geq 0}$ are not asymptotically $\mathfrak{S}$-free as N goes to infinity. Let t_1 and t_2 be two positive reals. For any positive integer N, let $S_{t_2}^{'N}$ be a random variable which has the same law as $S_{t_2}^N$ and which is independent with $S_{t_1}^N$. Since $\left(S_t^N\right)_{t\geq 0}$ is a Lévy process, it is enough to prove that $S_{t_1}^N$ and $S_{t_2}^{'N}$ are not asymptotically free as N goes to infinity. We already know that $S_{t_1}^N$ and $S_{t_2}^{'N}$ are asymptotically $\mathcal{P}$-free. Besides, using the relation between moments and exclusive moments, we know that for any real $t\geq 0$, $\mathbb{E}m_{id_1}[S_t]=\mathbb{E}m_{id_1^c}[S_t]$ and $\mathbb{E}m_{id_2}[S_t,S_t]=\mathbb{E}m_{id_2^c}[S_t,S_t]$. Using the differential system of equations proved in Theorem 2.1, we know that for any $t_0\geq 0$,

$$\frac{d}{dt}_{|t=t_0}\mathbb{E}m_{id_1}[S_t]=-\mathbb{E}m_{id_1}[S_{t_0}]$$
$$\frac{d}{dt}_{|t=t_0}\mathbb{E}m_{id_2}[S_t,S_t]=(-2+\alpha)\mathbb{E}m_{id_2}[S_{t_0},S_{t_0}],$$

and $\mathbb{E}m_{id_1}[S_0]=\mathbb{E}m_{id_2}[S_0,S_0]=1$. Since $\alpha\in[0,1]$, this implies that for any positive real t, $\mathbb{E}m_{id_1}[S_t]\neq\mathbb{E}m_{id_2}[S_t,S_t]$. An application of Proposition 2.3 allows to conclude that $S_{t_1}^N$ and $S_{t_2}^{'N}$ are not asymptotically $\mathfrak{S}$-free. Moreover, we will see that if $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent, then the asymptotic $\mathcal{P}$-factorization property holds for $\left(S_{t_1}^N\right)_{N\in\mathbb{N}}$ and $\left(S_{t_2}^{'N}\right)_{N\in\mathbb{N}}$. Using again Proposition 2.3, this shows that if $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent then the process $\left(\left(S_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is not a free multiplicative Lévy process in the sense of Voiculescu. □

Now we have proved the convergence in $\mathcal{P}$-expectation, let us understand when the convergence holds in probability or not, and let us consider the consequences for the empirical eigenvalues distribution.

Proof of Theorem 2.3. — Let us remark that if (λ_N) is macroscopic, we already saw that $m_{id_k^c}(G_k)$, which is also the limit of the coordinate numbers of G_k^N on id_k, is equal to $\frac{\left((1-\alpha)^k-1\right)}{\alpha}\neq k.m_{id_1^c}(G_k)$: the generator does not condensate weakly, thus the family $\left(S_t^N\right)_{N\in\mathbb{N}}$ does not satisfy the asymptotic $\mathcal{P}$-factorization property. Besides, by looking

at the system of differential equations satisfied by the limits of the observable, this implies actually that for any $t > 0$:

$$\lim_{N\to\infty} \mathbb{E}m_{id_2}\left[S_t^N \otimes S_t^N\right] \neq \left(\lim_{N\to\infty} \mathbb{E}m_{id_1}\left[S_t^N\right]\right)^2.$$

Thus, if (λ_N) is macroscopic, for any positive real t the $\mathfrak{S}$-asymptotic factorization property does not hold for $\left(S_t^N\right)_{N\in\mathbb{N}}$.

Let p be a partition in $\mathcal{P}_k$; we can suppose, up to a permutation of the columns, that there exist r irreducible partitions p_1, ..., p_r such that $p = p_1 \otimes ... \otimes p_r$. We saw in the proof of Theorem 2.1 that for any integer N:

$$m_p\left(G_k^N\right) = \sum_{i=1}^{r}\left(m_{p_i}\left(G_k^N\right)\prod_{l=i+1}^{r} m_{p_l}\left(\sigma_N^{\otimes k}\right)\right),$$

where $\sigma_N \in \lambda_N$. Besides, using Equation (5), $\lim_{N\to\infty} m_p\left(\sigma_N^{\otimes k}\right) = 1+\alpha\left(\lim_{N\to\infty} m_p\left(G_k^N\right)\right)$, where we recall that $\alpha = \lim_{N\to\infty}\frac{\lambda_N(1^c)}{N}$. Thus, denoting by $m_p\left(G_k\right)$ the limit of $m_p\left(G_k^N\right)$:

$$m_p\left(G_k\right) = \sum_{i=1}^{r}\left(m_{p_i}\left(G_k\right)\prod_{l=i+1}^{r}\left[1+\alpha m_{p_l}\left(G_k\right)\right]\right).$$

Recall Theorem 10.3 of [**10**]: this last equation implies that if (λ_N) is evanescent then $\left(\left(G_k^N\right)_k\right)_N$ weakly condensates. Thus, if (λ_N) is evanescent, then by Theorem 10.3 of [**11**], we know that $\left(\left(S_t^N\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ satisfies the asymptotic $\mathcal{P}$-factorization property. Using Theorem 2.2 of [**11**], this implies that, in this case, $\left(\left(S_t^N\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ converges in probability in $\mathcal{P}$-moments.

Let us translate these results for the empirical eigenvalues distribution. Let t be a positive real. Let us denote by $\mu_t^{\lambda_N}$ the random empirical eigenvalues distribution of S_t^N. Let r be a positive integer, let $(n_1, ..., n_r)$ be in $\mathbb{Z}^r$ and let σ be a permutation which has r cycles, the i^{th} having size $|\, n_i \,|$. Then using the definitions and the fact that any permutation matrice is real valued, we get:

$$\mathbb{E}\left[\prod_{i=1}^{r}\left(\int_{\mathbb{U}} z^{n_i}\mu_t^{\lambda_N}(dz)\right)\overline{\prod_{i=1}^{r}\left(\int_{\mathbb{U}} z^{n'_i}\mu_t^{\lambda_N}(dz)\right)}\right] = \mathbb{E}m_\sigma\left[\left(S_t^N\right)^{\otimes r}\right]. \tag{6}$$

Since $\left(S_t^N\right)_{N\in\mathbb{N}}$ converges in $\mathfrak{S}$-expectation, we get that the measures $\mu_t^{\lambda_N}$ converge in law to a random measure on $\mathbb{U}$, denoted by μ_t^{λ}.

The measure μ_t^{λ} is not random if and only if for any positive integer r, and any $(n_1, ..., n_r)$ in $\mathbb{Z}^r$:

$$\mathbb{E}\left[\prod_{i=1}^{r}\left(\int_{\mathbb{U}} z^{n_i}\mu_t^{\lambda}(dz)\right)\overline{\prod_{i=1}^{r}\left(\int_{\mathbb{U}} z^{n'_i}\mu_t^{\lambda}(dz)\right)}\right] = \prod_{i=1}^{r}\mathbb{E}\left[\int_{\mathbb{U}} z^{n_i}\mu_t^{\lambda}(dz)\right]\prod_{i=1}^{r}\overline{\mathbb{E}\left[\int_{\mathbb{U}} z^{n'_i}\mu_t^{\lambda}(dz)\right]}.$$

Using the Equation (6), this means that the measure μ_t^λ is not random if and only if the asymptotic $\mathfrak{S}$-factorization holds for $\left(S_t^N\right)_{N\in\mathbb{N}}$. Using the results previously obtained, we get that measure μ_t^λ is not random if and only if $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent. □

From now on, we will suppose that $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent: the limiting empirical eigenvalues distribution is not-random. Let us compute this limiting measure.

Proof of Theorem 2.4. — We recall that in the proof of Theorem 2.1, we used the following notation: for any positive integer k, any $\sigma\in\mathfrak{S}_k$ and any $t\geq 0$, we denote by $m_{\sigma^c}(t)$ the limit $\lim_{N\to\infty}\mathbb{E}m_{\sigma^c}\left[\left(S_t^N\right)^{\otimes k}\right]$. Besides, we proved that the family $(m_{\sigma^c}(t))_{t,\sigma}$ satisfies the system of differential equations stated in Theorem 2.1. Since we suppose that $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent, for any $t_0\geq 0$ and any $\sigma_0\in\mathfrak{S}_k$, one has:

$$\frac{d}{dt}_{|t=t_0} m_{\sigma_0^c(t)} = -km_{\sigma_0^c}(t_0) + \sum_{\sigma\in\mathfrak{S}_k\setminus\{id_k\},\sigma\leq\sigma_0} 0^{\mathsf{nc}(\sigma\vee id)-[\sigma](1)-1}\left(\prod_{i=2}^{k}(\lambda(i))^{\frac{[\sigma](i)}{i}}\right) m_{({}^t\sigma\sigma_0)^c}(t_0),$$

yet $\mathsf{nc}(\sigma\vee id)-[\sigma](1)-1=0$ if and only if σ is weakly irreducible, thus if and only if this is a cycle. Thus, if we set $m_{n^c}(t)=m_{(1,\ldots,n)^c}(t)$, for any $t_0\geq 0$ and any positive integer n:

$$\frac{d}{dt}_{|t=t_0} m_{n^c(t)} = -nm_{\sigma_0^c}(t_0) + \sum_{k=2}^{n}\sum_{\sigma\in\mathfrak{S}_n\setminus\{id_k\},\ \sigma\text{ is a k cycle },\sigma\leq(1,\ldots,n)} \lambda(k)m_{({}^t\sigma\sigma_0)^c}(t_0). \tag{7}$$

Yet, using Theorem 2.3, the $\mathfrak{S}$-asymptotic factorization holds when $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent. This implies that we can wite Equation (7) only in terms of $(m_{n^c}(t))_{n,t}$. For any positive integer n, any $t_0\geq 0$:

$$\frac{d}{dt}_{|t=t_0} m_{n^c}(t) = -nm_{n^c}(t) + \sum_{k=2}^{n}\lambda(k)\frac{n}{k}\sum_{(n_1,\ldots,n_k)\in(\mathbb{N}^*)^k|\sum_{i=1}^k n_i=n}\prod_{i=1}^{k} m_{n_i^c}. \tag{8}$$

Let us introduce the generating formal series of $\left(e^{nt}m_{n^c}(t)\right)_{n\geq 1}$:

$$\mathsf{R}(t,z)=\sum_{n\geq 1} e^{nt}m_{n^c}(t)z^n.$$

Let us remark that $\mathsf{R}(0,z)=z$. The Equation (8) can be written as:

$$\partial_t\mathsf{R}(t,z) = z\partial\mathsf{R}(t,z)\mathsf{LS}(\mathsf{R})(t,z), \tag{9}$$

where we defined:

$$\mathsf{LS}(z)=\sum_{n\geq 1}\lambda(n+1)z^n.$$

Let us define $\mathsf{S}(t,z)$ the reciprocal formal series such that for any $t\geq 0$:

$$\mathsf{S}(t,\mathsf{R}(t,z))=z.$$

Let us remark that $\mathsf{S}(0,z)=z$. The Equation (9) implies an equation on S:

$$\partial_t\mathsf{S}(t,z) = -\mathsf{LS}(z)\mathsf{S}(t,z).$$

Thus $\mathsf{S}(t,z)$ is given by $\mathsf{S}(t,z) = ze^{-t\mathsf{LS}(z)}$. Let $t \geq 0$ and let n be a positive integer. Using the usual notations, since $e^{nt}m_{n^c}(t) = [z^n]\mathsf{R}(t,\bullet)$, we can compute $e^{nt}m_{n^c}(t)$ by using the Lagrange inversion. This implies that:

$$[z^n]\mathsf{R}(t,\bullet) = \frac{1}{n}\left[z^{n-1}\right]e^{tn\mathsf{LS}(z)},$$

thus $m_{1^c}(t) = e^{-t}$ and for $n > 1$:

$$m_{n^c}(t) = e^{-nt}\sum_{k=1}^{n-1} t^k \frac{n^{k-1}}{k!} \sum_{(i_1,\ldots,i_k)\in(\mathbb{N}^*)^k, i_1+\ldots+i_k=n-1} \prod_{j=1}^{k}\lambda(i_j+1),$$

hence the assertions in Theorem 2.4. □

Let us prove the assertion on the existence of a phase transition for the random walks on the symmetric group.

Proof of Theorem 2.5. — Let us suppose that the sequence $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent. Let us show that the function $f(t) = \sum_{n=1}^{\infty} m_{n^c}(t)$, which is equal to $1-m_{\infty^c}(t)$ is continuous, converges to 0 as t goes to infinity. Indeed, we have:

$$f(t) = \sum_{k,n=0}^{\infty} \frac{1}{n}e^{-nt}t^k\frac{n^k}{k!}p(k,n),$$

where $p(k,n) = \sum_{(i_1,\ldots,i_k)\in\mathbb{N}^*, \sum_{j=1}^k i_j = n-1}\prod_{j=1}^k \lambda(i_j+1)$. For any k and n in $\mathbb{N}$, $f_{k,n}(t) = \frac{1}{n}e^{-nt}t^k\frac{n^k}{k!}p(k,n)$ is continuous and goes to zero as t goes to infinity, besides $f_{k,n}$ is non-negative and maximal at $t_{k,n} = \frac{k}{n}$ and using Stirling's formula, there exists a constant C such that $f_{k,n}(t_{k,n}) = \frac{1}{n}e^{-k}\frac{k^k}{k!}p(k,n) \leq C\frac{1}{k^{3/2}}p(k,n)$. In order to finish, one has to remark that:

$$\sum_{n\in\mathbb{N}} p(k,n) = \sum_{(i_1,\ldots,i_k)\in\mathbb{N}^*}\prod_{j=1}^{k}\lambda(i_j+1) = \left(\sum_{i\in\mathbb{N}^*}\lambda(i+1)\right)^k \leq 1,$$

thus $\sum_{k,n} f_{k,n}(t_{k,n}) < \infty$. This allows to apply the dominated convergence theorem, thus f is a continous function and converges to zero as t goes to infinity.

Recall the definition of t_c^λ given by Equation (3). Let us prove that $f(t) = 1$ for any $t \leq t_c^\lambda$ and $f(t) < 1$ for any $t > t_c^\lambda$. Using the generating function $\mathsf{R}(t,\bullet)$ of $e^{nt}m_{n^c}(t)$, we know that for any real $t \geq 0$:

$$f(t) = \mathsf{R}\left(t, e^{-t}\right).$$

Using the fact that $\mathsf{S}(t,\mathsf{R}(t,e^{-t})) = e^{-t}$, and given that $\mathsf{S}(t,z) = ze^{-t\mathsf{LS}(z)}$, we get that:

$$\mathsf{R}(t,e^{-t})e^{-t\mathsf{LS}(\mathsf{R}(t,e^{-t}))} = e^{-t}.$$

Thus for any $t \geq 0$, $f(t)$ is a solution in $[0,1]$ of $\Phi_t(z) = ze^{-t(\mathsf{LS}(z)-1)} = 1$. The function Φ_t is log-concave on $[0,1]$, $\Phi_t(0) = 0$ and $\Phi_t(1) = e^{-(t\mathsf{LS}(1)-1)}$. If $\mathsf{LS}(1) = \sum_{i=2}^{\infty}\lambda(i)$ is not equal to one it must be stricly smaller than 1, thus in this case for any $t > 0$, $\Phi_t(1) > 1$ and thus there exists a unique solution of $\Phi_t(z) = 1$ in $[0,1]$ which is in fact

in $]0,1[$. Thus we recover the delta function in Equation (3). Let us suppose now that $\sum_{i=2}^{\infty}\lambda(i)=1$. Then $\mathsf{LS}(1)=1$: thus, since Φ_t is log-concave, there exists a solution ν_t (which is unique) of $\Phi_t(z)=1$ on $]0,1[$ if and only if $\Phi_t'(1)<0$. Since $\Phi_t'(1)=1-t\mathsf{LS}'(1)$, we get that the critical time after which one observes a solution in $[0,1]$ which is different from the trivial solution 1 is equal to $\frac{1}{\mathsf{LS}'(1)}$ which is the value of t_c given by Equation (3). Since $f(t)$ is a continuous function which must converge to zero as t goes to infinity, it must be equal to 1 if $t\leq t_c$ and then it must be equal to $\nu(t)$ if $t>t_c$. □

Let us finish with the proof of Corollary 2.1.

Proof of Corollary 2.1. — Let t be a non-negative real number, let N be a positive integer, we have to understand:

$$\frac{1}{N}d_{\mathfrak{S}(N)}\left(id_N,S_t^N\right)=1-\frac{\mathsf{nc}\left(S_t^N\vee id_N\right)}{N}.$$

Recall that:

$$\frac{1}{N}\mathsf{nc}\left(S_t^N\vee id_N\right)=\sum_{k\geq 1}\frac{1}{k}m_{(1,\ldots,k)^c}\left(\left(S_t^N\right)^{\otimes k}\right)$$

since $m_{(1,\ldots,k)^c}\left(\left(S_t^N\right)^{\otimes k}\right)$ is the fraction of integers in $\{1,\ldots,N\}$ whose period in S_t^N is equal to k. It remains to see if one can interchange the limit and the sum. For any positive integer N, for any $\sigma\in\mathfrak{S}(N)$, if $c_k(\sigma)$ is the numbers of cycles of size k in σ, we have for any $K\in\mathbb{N}^*$:

$$\sum_{k\geq K}\frac{1}{k}m_{(1,\ldots,k)^c}\left(\sigma^{\otimes k}\right)=\frac{1}{N}\sum_{k\geq K}c_k(\sigma)\leq\frac{1}{N}\frac{N}{K}=\frac{1}{K},$$

since there can not be more than $\frac{N}{K}$ cycles in σ of size bigger than K. Thus:

$$\sup_N\sum_{k\geq K}\frac{1}{k}m_{(1,\ldots,k)^c}\left(\left(S_t^N\right)^{\otimes k}\right)\underset{N\to\infty}{\longrightarrow}0,$$

almost surely. Thus we can interchange limits and thus, since for any integer $k\geq 1$, $m_{(1,\ldots,k)^c}\left(\left(S_t^N\right)^{\otimes k}\right)$ converges in probability to $m_{k^c}(t)$, then we have the convergence in probability:

$$\lim_{N\to\infty}\frac{1}{N}\mathsf{nc}\left(S_t^N\vee id_N\right)=\sum_{k\geq 1}\lim_{N\to\infty}\frac{1}{k}m_{(1,\ldots,k)^c}\left(\left(S_t^N\right)^{\otimes k}\right)=\sum_{k=1}^{\infty}\frac{1}{k}m_{k^c}(t).$$

This allows to conclude the proof. □

2.4. Log-cumulant calculations. — In the article [**11**], we studied the log-cumulant invariant of a free multiplicative infinitely divisible measure. The log-cumulant was defined in Definition 10.18. We can generalize the definition of log-cumulant in the setting of $\mathcal{P}$-free multiplicative Lévy processes. Recall the notion of matricial $\mathcal{P}$-free multiplicative Lévy process defined in Definition 7.4 of [**11**].

Definition 2.4. — *Let $(S_t)_{t\geq 0}$ be a matricial $\mathcal{P}$-free multiplicative Lévy process. The log-cumulant invariant of $(S_t)_{t\geq 0}$ is the unique element $\mathcal{LR}\left((S_t)_{t\geq 0}\right) \in \mathfrak{E}[\mathcal{P}]$ such that:*

$$\forall t \geq 0,\ \frac{d}{dt}_{|t=t_0} \mathcal{R}(S_t) = \mathcal{LR}\left((S_t)_{t\geq 0}\right) \boxtimes \mathcal{R}(S_{t_0}).$$

Let $(S_t)_{t\geq 0} = \left(\left(S_t^N\right)_{N\geq 1}\right)_{t\geq 0}$ be a matricial $\mathcal{P}$-free multiplicative Lévy process. For any positive integer k and N, we consider $G_k^N = \frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(S_t^N\right)^{\otimes k}\right]$. Recall that, using the same notation as in Definition 10.3 of [**10**], $\mathcal{LR}\left((S_t)_{t\geq 0}\right) = \mathcal{R}_{\mathcal{P}}[G]$.

In the setting of $\mathcal{P}$-free multiplicative Lévy processes which are invariant by conjugation by the unitary group, the log-cumulant invariant is an important tool in order to caracterize them. We hope that this could also be the case for more general $\mathcal{P}$-free multiplicative Lévy processes. We are thus interested in computing the log-cumulant invariant for some examples of $\mathcal{P}$-free multiplicative Lévy processes: the following theorem is the first computation of the log-cumulants of $\mathcal{P}$-free multiplicative Lévy processes which are not free Lévy processes in the sense of Voiculescu.

For any positive integer N, let λ_N be a conjugacy of $\mathfrak{S}(N)$ and let us consider $\left(S_t^N\right)_{N\in\mathbb{N}}$ a λ_N-random walk on $\mathfrak{S}(N)$. For any positive integer t, let us denote by S_t^{λ} the family $\left(S_t^N\right)_{N\in\mathbb{N}}$. Let us suppose that $(\lambda_N)_{N\in\mathbb{N}}$ converges as N goes to infinity and that it is evanescent. As we already did, for any $i \geq 2$, we set $\lambda(i) = \lim_{N\to\infty} \frac{\lambda_N(i)}{\lambda_N(1^c)}$.

We need to define the notion of ears.

Definition 2.5. — *Let k be a positive integer, let i be an element of $\{1,...,k\}$ and let $p \in \mathcal{P}_k$. We say that $\{i,i'\}$ is an ear of p if $\{i,i'\}$ are in the same block of p. The set of ears of p is denoted by $\mathsf{E}(p)$. The head of p, denoted by $\mathsf{H}(p)$, is the extraction of p to $\{1,...,k,1',...,k'\} \setminus \cup_{i,i'\in\mathsf{E}(p)}\{i,i'\}$.*

Let us state the main result about the log-cumulant functional. Recall the notion of true-length that we defined in Definition 2.3.

Theorem 2.7. — *The log-cumulant invariant of $\left(\mathsf{S}_t^{\lambda}\right)_{t\geq 0}$, denoted by $\mathcal{LR}^{\lambda}$, is characterized by:*

1. *$\mathcal{LR}^{\lambda} \in \mathfrak{me}_{\boxtimes}[\mathcal{P}]$,*
2. *for any positive integer k, for any irreducible partition $p \in \mathcal{P}_k$, if $\mathsf{H}(p)$ is not a parure then $\left(\mathcal{LR}^{\lambda}\right)_p = 0$,*
3. *for any positive integer k, for any irreducible partition $p \in \mathcal{P}_k$, if $\mathsf{H}(p)$ is a necklace then:*

$$\left(\mathcal{LR}^{\lambda}\right)_p = (-1)^{\#\mathsf{E}(p)}\lambda(|\,\mathsf{H}(p)\,|),$$

with the convention that $|\,\emptyset\,| = 0$ and $\lambda(0) = 1$, and if $\mathsf{H}(p)$ is a chain then:

$$\left(\mathcal{LR}^{\lambda}\right)_p = (-1)^{\#\mathsf{E}(p)}\left(1 - \sum_{i=2}^{|\mathsf{H}(p)|}\lambda(i)\right).$$

Proof. — Let us denote $E = \mathcal{LR}^{\lambda}$. Let us consider $F \in \mathfrak{me}_{\boxtimes}[\mathcal{P}]$ which satisfies the conditions 2. and 3. of the theorem. Since we supposed in this section that $(\lambda_N)_{N\in\mathbb{N}}$ converges and is evanescent, by the proof of Theorem 2.3 we know that the generator of $\left(S_t^N\right)_{N\in\mathbb{N}}$ weakly condensates as N goes to infinity. By definition, this implies that $E \in \mathfrak{me}_{\boxtimes}[\mathcal{P}]$. Besides, we have computed the exclusive moments of the generator in Section 2.3. Using the notations of [**10**], we know $\mathcal{M}^{\to c}(E)$: the goal is to invert the transformation $\mathcal{M}^{\to c}$.

It remains to show that for any weakly irreducible partition p, $E_p = F_p$. Let us denote by E^c the element $\mathcal{M}^{\to c}(E)$. Recall that $|\,p\,|$ is the true-length of a partition p. Using the calculations in the proof of Theorem 2.1, we know for any irreducible partition p, if p is not a parure then $E_p^c = 0$ and if p is a necklace of true-length equal to 1 then $E_p^c = -1$, if it is a necklace of true-length greater than 1 then $E_p^c = \lambda(|\,p\,|)$, and if p is a chain then $E_p^c = 1 - \sum_{k=2}^{|p|}\lambda(k)$. Recall Definition 3.11 of [**10**]. Since $E \in \mathfrak{me}_{\boxtimes}[\mathcal{P}]$, E is characterized by the fact that for any irreducible partition p, $E_p^c = \sum_{p'\sqsupset p} E_{p'}$. Let p be an irreducible partition in $\mathcal{P}_k$, it remains to prove that:

$$E_p^c = \sum_{p'\sqsupset p} F_{p'}.$$

Let us recall that $F_{p'} = 0$ if p' is not weakly irreducible. Yet if $p' \sqsupset p$ and p' is weakly irreducible, this means that one can get p' by chosing a certain number of ears of p and by cutting each of them in p. Let us consider the two possible cases $p = \mathsf{0}_k$ or $p \neq \mathsf{0}_k$. If $p = \mathsf{0}_k$, then:

$$\begin{aligned}\sum_{p'\sqsupset \mathsf{0}_k} F_{p'} &= \left(\sum_{I\subset\{1,\dots,k\},\#I>1} (-1)^{\#I-1}\right)(-1) - k \\ &= -\left(\left(\sum_{l=0}^{k}(-1)^{l-1}\frac{k!}{l!(k-l)!}\right)+1\right) = -1 = E_{\mathsf{0}_k}^c,\end{aligned}$$

the last equality coming from the fact that $\mathsf{0}_k$ is a necklace of true-length equal to 1.

If $p \neq \mathsf{0}_k$, then:

$$\sum_{p'\sqsupset p} F_{p'} = \sum_{I\subset \mathsf{E}(p)} (-1)^{\#\mathsf{E}(p)-\#I} F_{\mathsf{T}(p)} = \delta_{\#\mathsf{O}(p)=0} F_{\mathsf{T}(p)} = \delta_{\#\mathsf{O}(p)=0} F_p = \delta_{\#\mathsf{O}(p)=0} E_{p^c} = E_{p^c},$$

since the only irreducible parure in $\mathcal{P}_k$ which has ears is $\mathsf{0}_k$. □

Let us remark that, since we saw that $\mathcal{LR}\left((S_t)_{t\geq 0}\right) = \mathcal{R}_{\mathcal{P}}[G]$, we could have try to prove the last theorem by computing the coordinate numbers of G_k^N for any positive integer k and N. For example, if one considers the random walk by transposition, if $\mathcal{T}_N$

is the set of transpositions in $\mathfrak{S}(N)$:

$$G_k^N = \frac{1}{N-1}\sum_{\tau\in\mathcal{T}_N}\left(\tau^{\otimes k} - Id_k\right)$$
$$= \frac{1}{2(N-1)}\sum_{i,j=1}^{N}\left(\left(Id_N - E_i^i - E_j^j + E_i^j + E_j^i\right)^{\otimes k} - Id^{\otimes k}\right),$$

where E_i^j is the usual elementary matrix which sends e_j on e_i and where $(e_1, ..., e_N)$ is the canonical basis of $\mathbb{C}^N$. Then one can develop the tensor product and compute the coordinate numbers and their limits. Yet, one can see that it becomes less tractable as soon as one considers general random walks on the symmetric group.

3. Convergence of $YM(\mathfrak{S}(N))$

We will not go into all the details of the theory of planar Yang-Mills fields, one can read [**12**] and [**18**] to have an introduction on planar Yang-Mills fields and planar Markovian holonomy fields. Yet, our presentation will be adequate so that the reader does not have to read other articles in order to understand the main result of this section, namely Theorem 3.2. The general ideas are all taken from the article [**9**] where asymptotics of unitary Yang-Mills measures are proved. In this article, the Yang-Mills measure with $\mathfrak{S}(N)$ gauge group will denote for us the planar Markovian holonomy field associated with the $\mathcal{T}_N$-random walk, where $\mathcal{T}_N$ is the set of transposition in $\mathfrak{S}(N)$. Yet, this section can easily be generalized to planar Markovian holonomy fields associated with any λ_N-random walk.

Definition 3.1. — *The set of paths P in the plane is the set of rectifiable oriented curves drawn in $\mathbb{R}^2$ up to increasing reparametrization. The set of loops based at 0, denoted by L_0, is the set of paths l such that the two endpoints of l are 0. A loop is simple if it does not intersect with itself, except at the endpoints.*

We will consider Aff *and* Aff_0 *respectively the set of piecewise affine paths in $\mathbb{R}^2$ and the set of piecewise affine loops based at 0.*

We can define two operations on P: the concatenation and the inversion. Given two paths p_1 and p_2 such that the starting point of p_2 is the arrival point of p_1, it is natural to concatenate p_1 and p_2 by gluing them at the arrival point of p_1: it defines a new path p_1p_2. The inversion of p_1, denoted by p_1^{-1}, is defined by changing the orientation of p_1. T. Lévy defined in [**18**], the notion of convergence with fixed endpoints. For any $p \in P$, $\underline{p}$ denotes the starting point of p and $\overline{p}$ denotes the arrival point of p. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of paths. The sequence $(p_n)_{n\in\mathbb{N}}$ converges with fixed endpoints if and only if there exists a path p such that for any integer n, p_n and p have the same endpoints and:

$$|\,\mathbf{l}(p_n) - \mathbf{l}(p)\,| + \inf\sup_{t\in[0,1]}|\,p_n(t) - p(t)\,| \underset{n\to\infty}{\longrightarrow} 0,$$

where the infimum is taken on the parametrization of the paths p_n and p and where $\mathbf{l}(p)$ is the length of p.

Let J be a subset of P, let G be a group. The set of multiplicative functions $\mathcal{M}ult(J,G)$ from J to G is the subset of functions f in G^J such that for any $p_1, p_2, p_3 \in J$ such that $p_1p_2 \in J$ and $p_3^{-1} \in J$, one has:

$$f(p_1p_2) = f(p_2)f(p_1),$$
$$f\left(p_3^{-1}\right) = f(p_3)^{-1}.$$

For any $p \in P$, we define h_p or $h(p)$ as the evaluation on p:

$$h_p : \mathcal{M}ult(J,G) \to G$$
$$h \mapsto h(p).$$

Let N be a positive integer. We are going to define a gauge-invariant measure on the set of multiplicative functions from P to $\mathfrak{S}(N)$. Thus we endow $\mathcal{M}ult(J,\mathfrak{S}(N))$ with the cylinder σ-field $\mathcal{B}$ which is the trace on $\mathcal{M}ult(J,G)$ of the cylinder σ-field on $\mathfrak{S}(N)^J$. Let us denote by V the set $\{x \in \mathbb{R}^2, \exists p \in J, x = \underline{p} \text{ or } x = \overline{p}\}$. For any function $j : V \to \mathfrak{S}(N)$ and any $h \in \mathcal{M}ult(J,\mathfrak{S}(N))$, we define $j \bullet h \in \mathcal{M}ult(J,\mathfrak{S}(N))$ such that:

$$\forall c \in J, (j \bullet h)(c) = j_{\overline{c}}^{-1} h(c) j_{\underline{c}}.$$

A measure μ on $\mathcal{M}ult\,(J,\mathfrak{S}(N))$ is gauge-invariant if for any continuous function f from $(\mathcal{M}ult(J,\mathfrak{S}(N)),\mathcal{B})$ to $\mathbb{R}$, for any function $j : V \to \mathfrak{S}(N)$:

$$\int_{\mathcal{M}ult(P,G)} f(j \bullet h) d\mu(h) = \int_{\mathcal{M}ult(P,G)} f(h) d\mu(h).$$

In the up-coming paper [**9**], the author and his co-authors proved the following theorem which is a slight generalization of Theorem 3.3.1 proved by T.Lévy in [**18**]. The original formulation by T. Lévy of this theorem is the first part of Theorem 3.1.

Theorem 3.1. — *Let us denote by dx the Lebesgue measure on $\mathbb{R}^2$. Let $(\Gamma_N, d_N)_{N\in\mathbb{N}}$ be a sequence of complete metric groups such that for any $N \in \mathbb{N}$, translations and inversion are isometries on Γ_N. For any integer N, let $H_N \in \mathcal{M}ult(\mathsf{Aff}, \Gamma_N)$ be a multiplicative function. Assume that there exists $K_N \geq 0$ such that for any $N \in \mathbb{N}$, for all simple loop $l \in \mathsf{Aff}$ bounding a disk D and such that $\mathbf{l}(l) \leq K_N^{-1}$, the inequality:*

$$d_N(1, H_N(l)) \leq K_N \sqrt{dx(D)} \tag{10}$$

holds.

Then for each integer N, the function H_N admits a unique extension as an element of $\mathcal{M}ult(P,G)$, also denoted by H_N, which is continuous for the convergence with fixed endpoints.

Let (E,d) be a metric space. For any integer N, let $\psi_N : \Gamma_N \to E$ be a Lipchitz function of Lipchitz norm $||\psi_N||_{\mathsf{Lip}}$. Let us assume that the three following conditions hold:

1. *for any $l \in \mathsf{Aff}_0$, $\psi_N(H_N(l))$ converges to a limit when N goes to infinity,*
2. $\sup_{N\in\mathbb{N}} ||\psi_N||_{\mathsf{Lip}} \leq \infty$,
3. $\sup_{N\in\mathbb{N}} K_N < \infty$,

then for any $l \in L_0$, $\left(\psi_N(H_N(l))\right)_{N\in\mathbb{N}}$ *converges to a limit* $\phi(l)$. *Besides, the function:*

$$\begin{aligned}\phi : L_0 &\to E\\ l &\mapsto \phi(l)\end{aligned}$$

is continuous for the convergence with fixed endpoints.

Recall that $\mathcal{T}_N$ is the set of transpositions in $\mathfrak{S}(N)$. Let $\left(S_t^N\right)_{t\geq 0}$ be the $\mathcal{T}_N$ random walk on $\mathfrak{S}(N)$. Let us explain how the first part of Theorem 3.1 allows us to construct the Yang-Mills field associated with $\left(S_t^N\right)_{t\geq 0}$. In order to do so, we need the notion of finite planar graph: it will be the usual notion, except that we ask that the bounded faces are homeomorphic to an open disk. Let $\mathbb{G}$ be a finite planar graph: the set of bounded of faces of $\mathbb{G}$ is denoted by $\mathbb{F}$. For any finite planar graph $\mathbb{G}$, we define $P(\mathbb{G})$ as the set of paths that one can draw by concatenating edges of $\mathbb{G}$. Let us define also $\mathcal{G}(\mathsf{Aff})$ the set of finite planar graphs $\mathbb{G}$ whose edges are piecewise affine.

In order to construct a measure on $(\mathcal{M}ult(P,\mathfrak{S}(N)),\mathcal{B})$, first we construct for any finite planar graph $\mathbb{G} \in \mathcal{G}(\mathsf{Aff})$ an associated measure $\mu_{\mathbb{G}}$ on $\left(\mathcal{M}ult(P(\mathbb{G}),\mathfrak{S}(N)),\mathcal{B}\right)$. We will give the construction given by the author in [**12**], but one can have a look at [**18**] where a different formulation is given.

We need to introduce the loop paradigm for two dimensional Yang-Mills fields. Let us consider a finite planar graph $\mathbb{G}$ in $\mathcal{G}(\mathsf{Aff})$, let us consider v_0 a vertex of $\mathbb{G}$ and T a covering tree of $\mathbb{G}$. Let us consider for any bounded face F of $\mathbb{G}$ a loop $c_F \in P(\mathbb{G})$ which represents ∂F. For any vertex v of $\mathbb{G}$, we denote by $[v_0,v]_T$ the unique injective path in T which goes from v_0 to v. Let $L_{v_0}(\mathbb{G})$ be the set of loops l in $P(\mathbb{G})$ such that $\underline{l} = v_0$. We define the facial lasso $\mathsf{l}_F \in L_{v_0}(\mathbb{G})$ by:

$$\mathsf{l}_F = [v_0,v]_T\, c_F\, [v_0,v]_T^{-1}.$$

It was proved in Proposition 6.1 that the application:

$$\begin{aligned}\Phi_{T,(c_{\mathbb{F}})_{F\in\mathbb{F}}} : \mathcal{M}ult\left(L_{v_0}(\mathbb{G}),\mathfrak{S}(N)\right) &\to \left(\mathfrak{S}(N)\right)^{\mathbb{F}}\\ h &\mapsto \left(h\left(\mathsf{l}_F\right)\right)_{F\in\mathbb{F}},\end{aligned}$$

is a bijection and for any loop $l \in L_{v_0}(\mathbb{G})$, there exists a word $w\left(\left(\mathsf{l}_F\right)_{F\in\mathbb{F}},\left(\mathsf{l}_F^{-1}\right)_{F\in\mathbb{F}}\right)$ in the letters $\left(\mathsf{l}_F\right)_{F\in\mathbb{F}}$ and $\left(\mathsf{l}_F^{-1}\right)_{F\in\mathbb{F}}$ such that $h_l = w\left(\left(h_{\mathsf{l}_F}\right)_{F\in\mathbb{F}},\left(h_{\mathsf{l}_F^{-1}}\right)_{F\in\mathbb{F}}\right)$.

Using the fact that for any $\sigma \in \mathfrak{S}(N)$, σ^{-1} is in the same conjugacy class as σ, we know that for any non-negative real t, S_t^N has the same law as $\left(S_t^N\right)^{-1}$. This implies, with Proposition 8.1 proved by the author in [**12**], the following proposition.

Proposition 3.1. — *There exists a unique gauge-invariant measure* $\mu_{\mathbb{G}}^{v_0,T,(c_F)_{F\in\mathbb{F}}}$ *on* $\mathcal{M}ult\left(P(\mathbb{G}),\mathfrak{S}(N)\right)$ *such that under this measure:*

1. *the random variables* $\left(h\left(\mathsf{l}_F\right)\right)_{F\in\mathbb{F}}$ *are independent,*
2. *for any* $F \in \mathbb{F}$, $h\left(\mathsf{l}_F\right)$ *has the same law as* $S^N_{dx(F)}$.

This measure does not depend neither on the choice of v_0 *nor* T *nor on the choice of* $(c_F)_{F\in\mathbb{F}}$, *we denote it by* $\mu_{\mathbb{G}}$.

Let $\mathbb{G}$ and $\mathbb{G}'$ be two finite planar graphs in $\mathcal{G}(\mathsf{Aff})$ such that $\mathbb{G}'$ is coarser than $\mathbb{G}$. Any function in $\mathcal{M}ult(P(\mathbb{G}),\mathfrak{S}(N))$ allows us to define, by restriction, an element of $\mathcal{M}ult(P(\mathbb{G}'),\mathfrak{S}(N))$. The measures $(\mu_{\mathbb{G}})_{\mathbb{G}}$ are compatible with the applications of restriction we have just described. The family $\left(\mathcal{M}ult(P(\mathbb{G}),\mathfrak{S}(N)),\mu_{\mathbb{G}}\right)_{\mathbb{G}\in\mathcal{G}(\mathsf{Aff})}$ is thus a projective system and, as explained in Proposition 2.1 of [**12**] and in [**18**], we can take the projective limit.

Definition 3.2. — *The affine Yang-Mills measure associated to $(S_{N,t})_{t\geq 0}$ is the projective limit of:*

$$\left(\mathcal{M}ult(P(\mathbb{G}),\mathfrak{S}(N)),\mu_{\mathbb{G}}\right)_{\mathbb{G}\in\mathcal{G}(\mathsf{Aff})}.$$

The affine Yang-Mills measure $YM_{\mathsf{Aff}}^{\mathfrak{S}(N)}$ is a gauge-invariant measure on $\mathcal{M}ult(\mathsf{Aff},\mathfrak{S}(\mathsf{N}))$.

Let us consider a simple loop l in Aff and let $\mathbb{G}^l$ be the finite planar graph in $\mathcal{G}(\mathsf{Aff})$ which has l as unique edge. In this case, $\mathcal{M}ult(P(\mathbb{G}^l,\mathfrak{S}(N)))\simeq\mathfrak{S}(N)$ and for any continuous function $f:\mathfrak{S}(N)\to\mathbb{R}$:

$$YM_{\mathsf{Aff}}^{\mathfrak{S}(N)}\left[f(h_l)\right]=\mathbb{E}\left[f\left(S^N_{dx(\mathsf{Int}(l))}\right)\right], \tag{11}$$

where $\mathsf{Int}(l)$ is the bounded component of $\mathbb{R}^2\setminus l$. This last equality shows that under $YM_{\mathsf{Aff}}^{\mathfrak{S}(N)}$, h_l has the same law as $S^N_{dx(\mathsf{Int}(l))}$. This will allow us to use the first part of Theorem 3.1 in order to construct the Yang-Mills measure, as it was done by T. Lévy in [**18**] and then by the author in [**12**]. Before doing so, we need some estimates on the walk $\left(S_t^N\right)_{t\geq 0}$: in order to do so, let us define a distance on $\mathfrak{S}(N)$.

Definition 3.3. — *We will consider any element of $\mathfrak{S}(N)$ as a permutation matrix of size N. Let us consider for any $\sigma,\sigma'\in\mathfrak{S}(N)$:*

$$d_N(\sigma,\sigma')=\left[2\left(1-\frac{1}{N}Tr\left(\sigma\sigma'^{-1}\right)\right)\right]^{\frac{1}{2}},$$

where we recall that Tr is the usual trace on $\mathcal{M}_N(\mathbb{C})$ which satisfies $Tr\left(Id_N\right)=N$.

Since the permutation matrices are orthogonal, for any σ and σ' in $\mathfrak{S}(N)$:

$$d_N\left(\sigma,\sigma'\right)=\left[\frac{1}{N}Tr\left(\left(\sigma-\sigma'\right).{}^t\left(\sigma-\sigma'\right)\right)\right]^{\frac{1}{2}}.$$

This shows that d_N is a distance on $\mathfrak{S}(N)$. Let us control the distance of $\left(S_t^N\right)_{t\geq 0}$ to the identity.

Lemma 3.1. — *For any real $t\geq 0$, $\mathbb{E}\left[d_N\left(id,S_t^N\right)\right]\leq\sqrt{2t}$.*

Proof. — Let t be a non-negative real. By definition:

$$\mathbb{E}\left[d_N\left(id,S_t^N\right)^2\right]=2\left[1-\mathbb{E}\left[\frac{1}{N}Tr\left(S_t^N\right)\right]\right].$$

A simple calculation allows us to write that:

$$\frac{1}{N}\sum_{\tau\in\mathcal{T}_N}(\tau - Id) = \rho_N^{\mathcal{P}_k}\left[\frac{1}{N}\mathsf{0}_1 - id\right],$$

where we recall that $\mathsf{0}_1$ is the partition $\{\{1\},\{1'\}\}$. This implies that for any $t_0 \geq 0$:

$$\frac{d}{dt}_{|t=t_0}\mathbb{E}\left[S_t^N\right] = \rho_N^{\mathcal{P}_k}\left(\frac{1}{N}\mathsf{0}_1 - Id\right)\mathbb{E}\left[S_{t_0}^N\right].$$

Thus, by linearity:

$$\frac{d}{dt}_{|t=t_0}\mathbb{E}\left[\frac{1}{N}Tr\left(S_t^N\right)\right] = \frac{1}{N}\mathbb{E}\left[\frac{1}{N}Tr\left(\rho_N^{\mathcal{P}_k}(\mathsf{0}_1)S_{t_0}^N\right)\right] - \mathbb{E}\left[\frac{1}{N}Tr\left(S_{t_0}^N\right)\right],$$

and, using the fact that $\frac{1}{N}Tr\left(\rho_N^{\mathcal{P}_k}(\mathsf{0}_1)\sigma\right) = 1$ for any $\sigma \in \mathfrak{S}(N)$, we get the differential equation:

$$\frac{d}{dt}_{|t=t_0}\mathbb{E}\left[\frac{1}{N}Tr\left(S_{t_0}^N\right)\right] = \frac{1}{N} - \mathbb{E}\left[\frac{1}{N}Tr\left(S_{t_0}^N\right)\right],$$
$$\mathbb{E}\left[\frac{1}{N}Tr\left(S_0^N\right)\right] = 1.$$

The solution is given by the function $t \mapsto \frac{1}{N} + \left(1 - \frac{1}{N}\right)e^{-t}$. Thus for any real $t \geq 0$:

$$\mathbb{E}\left[\frac{1}{N}Tr\left(S_t^N\right)\right] = \frac{1}{N} + \left(1 - \frac{1}{N}\right)e^{-t}, \tag{12}$$

and thus:

$$\mathbb{E}\left[d_N\left(id, S_t^N\right)^2\right] = 2\left[1 - \frac{1}{N}\right][1 - e^{-t}].$$

This implies that for any $t \geq 0$, and any positive integer N,

$$\left(\mathbb{E}\left[d_N\left(id, S_t^N\right)\right]\right)^2 \leq \mathbb{E}\left[d_N\left(id, S_t^N\right)^2\right] \leq 2t.$$

This allows us to finish the proof. □

Using Lemma 3.1, we can prove the following proposition.

Proposition 3.2. — *The measure $YM_{\mathsf{Aff}}^{\mathfrak{S}(N)}$ can be extended by continuity to a measure on $\mathcal{M}ult(P,\mathfrak{S}(N))$. This means that there exists a measure $YM^{\mathfrak{S}(N)}$ on $\mathcal{M}ult(P,\mathfrak{S}(N))$ such that:*

1. *the restriction of $YM^{\mathfrak{S}(N)}$ on $\mathcal{M}ult(\mathsf{Aff},\mathfrak{S}(\mathsf{N}))$ is equal to $YM_{\mathsf{Aff}}^{\mathfrak{S}(N)}$,*
2. *for any sequence of paths $(p_n)_{n\in\mathbb{N}}$ and any path $p \in P$ such that $(p_n)_{n\in\mathbb{N}}$ converges with fixed endpoints to p, we have:*

$$YM^{\mathfrak{S}(N)}\left[d_N\left(h_{p_n}, h_p\right)\right] \underset{n\to\infty}{\longrightarrow} 0.$$

We only recall the proof given in [**18**].

Proof. — Let $(\Omega, \mathcal{A}, \mathbb{P})$ be equal to $\left(\mathcal{M}ult(\mathsf{Aff}, \mathfrak{S}(N)), \mathcal{B}, YM_{\mathsf{Aff}}^{\mathfrak{S}(N)}\right)$. For any $p \in \mathsf{Aff}$, h_p is a function on $\mathcal{M}ult(\mathsf{Aff}, \mathfrak{S}(N))$ thus it can be seen as a G-valued random variable on $(\Omega, \mathcal{A}, \mathbb{P})$. For any positive integer N, let $\Gamma_N = L(\Omega, \mathcal{A}, \mathbb{P}; \mathfrak{S}(N))$ be the set of $\mathfrak{S}(N)$-valued random variables defined on Ω: this is a group for the pointwise multiplication of random variables. We endow Γ_N with the distance:

$$d_N(X, Y) = \mathbb{E}\left[d_N(X, Y)\right].$$

It is a distance which is invariant by translations and inversion. Let us consider the mapping:

$$\begin{aligned} H_N : \mathsf{Aff} &\to \Gamma_N \\ l &\to h_l. \end{aligned}$$

This is a multiplicative function. Besides, using Lemma 3.1 and Equality 11, we get that for any simple loop l: $d_N(1, h_l) \leq \sqrt{2}\sqrt{dx(\mathsf{Int}(l))}$. We can apply Theorem 3.1: there exists an extension:

$$\begin{aligned} H_N : P &\to \Gamma_N \\ p &\mapsto H(p) \end{aligned}$$

which is continuous for the convergence with fixed endpoints. For any sequence of paths $(p_n)_{n\in\mathbb{N}}$ and any path $p \in P$ such that $(p_n)_{n\in\mathbb{N}}$ converges with fixed endpoints to p, we get:

$$d_N\left(H\left(p_n\right), H\left(p\right)\right) = YM_{\mathsf{Aff}}^{\mathfrak{S}(N)}\left[d_N(H(p_n), H(p))\right] \underset{n\to\infty}{\longrightarrow} 0.$$

Thus we have constructed a $\mathfrak{S}(N)$-valued process $\left(H(p)\right)_{p\in P}$ on Ω such that for any p and p' in P such that $\overline{p} = \underline{p'}$, almost surely $H(pp') = H(p')H(p)$, $H(p^{-1}) = H(p)^{-1}$. Using Proposition 2.1 in [**12**] this allows us to construct a measure on $\mathcal{M}ul(P, \mathfrak{S}(N))$ called $YM^{\mathfrak{S}(N)}$, such that the process $(h_p)_{p\in P}$ has the same law under $YM^{\mathfrak{S}(N)}$ as the process $(H(p))_{p\in P}$ under $YM_{\mathsf{Aff}}^{\mathfrak{S}(N)}$. The measure $YM^{\mathfrak{S}(N)}$ satisfies the desired properties. □

Now that we have defined the Yang-Mills measure $YM^{\mathfrak{S}(N)}$ for any positive integer N, we are interested in the convergence of these measures as N goes to infinity. Let us define the notion of Wilson loops. Using the gauge-invariance of the Yang-Mills measure, we can restrict ourself to the study of Wilson loops based at 0 instead of studying all the Wilson loops.

Definition 3.4. — *Let l_0 be a loop based at 0, the Wilson loop on l_0 is the function:*

$$\begin{aligned} W_{l_0}^N : \mathcal{M}ult(P, \mathfrak{S}(N)) &\to \mathbb{R} \\ (h_p)_{p\in P} &\mapsto \frac{1}{N} Tr(h_{l_0}). \end{aligned}$$

The main result about the limit of Yang-Mills measure on the symmetric group is given by the following result.

Theorem 3.2. — *For any loop l based at 0 the Wilson loop on l converges in probability to a constant denoted by $\phi(l)$ when N goes to infinity. The function:*

$$\begin{aligned}\phi : L_0 &\to \mathbb{R}\\ l &\mapsto \phi_l,\end{aligned}$$

is continuous for the convergence with fixed endpoints.

The asymptotic factorization property holds: for any positive integer k, any k-tuple of loops l_1, ..., l_k in L_0:

$$YM^{\mathfrak{S}(N)}\left[W_{l_1}^N ... W_{l_k}^N\right] \underset{N\to\infty}{\longrightarrow} \phi(l_1)...\phi(l_k).$$

The function ϕ in Theorem 3.2 is called the $\mathfrak{S}(\infty)$-master field. Let us prove Theorem 3.2 when one considers only piecewise affine loops.

Proposition 3.3. — *For any loop l in $\mathsf{Aff_0}$ the Wilson loop W_l^N converges in expectation and in probability as N tends to infinity to a constant $\phi(l)$.*

Proof. — Let l_0 be a loop in $\mathsf{Aff_0}$. Let $\mathbb{G}$ be a graph in $\mathcal{G}(\mathsf{Aff})$ such that l_0 is a loop in $\mathbb{G}$. Let us consider T a covering tree of $\mathbb{G}$, let us consider for any bounded face F of $\mathbb{G}$ a loop $c_F \in P(\mathbb{G})$ which represents ∂F and let us consider the facial lassos l_F associated with these choices of tree and loops. Let $w\left((\mathsf{l}_F)_{F\in\mathbb{F}}, \left(\mathsf{l}_F^{-1}\right)_{F\in\mathbb{F}}\right)$ be a word in the letters $(\mathsf{l}_F)_{F\in\mathbb{F}}$ and $\left(\mathsf{l}_F^{-1}\right)_{F\in\mathbb{F}}$ such that $h_{l_0} = w\left((h_{\mathsf{l}_F})_{F\in\mathbb{F}}, \left(h_{\mathsf{l}_F^{-1}}\right)_{F\in\mathbb{F}}\right)$: the random variable h_{l_0} is a product of random variables of the form h_{l_F} or $h_{\mathsf{l}_F}^{-1}$, with $F \in \mathbb{F}$.

Using Proposition 3.1, the random variables $(h_{\mathsf{l}_F})_{F\in\mathbb{F}}$ on $\left(\mathcal{M}ult(L_0,\mathbb{G}), \mathcal{B}, YM_{L_0}^{\mathfrak{S}(N)}\right)$ are independent and for any $F \in \mathbb{F}$, h_{l_F} has the same law as $S^N_{dx(F)}$. For all positive integer N, let $\left(S_{t,N}^{(1)}\right)_{t\geq 0}$, ..., $\left(S_{t,N}^{(\#\mathbb{F})}\right)_{t\geq 0}$ be $\#\mathbb{F}$ independent random walks identically distributed as $(S_t^N)_{t\geq 0}$. The discussion we just had implies that there exist two sequences of non-negative integers $(k_i)_{i=1}^{\#\mathbb{F}}$ and $(k_i')_{i=1}^{\#\mathbb{F}}$ such that the Wilson loop W_l^N is equal to:

$$\frac{1}{N^{\mathsf{nc}(\sigma\vee id)}} Tr^K\left(\left(S_{t,N}^{(1)}\right)^{\otimes k_1} \otimes \left(\left(S_{t,N}^{(1)}\right)^{-1}\right)^{\otimes k_1'} \otimes ... \otimes \left(S_{t,N}^{(\#\mathbb{F})}\right)^{\otimes k_{\#\mathbb{F}}} \otimes \left(\left(S_{t,N}^{(\#\mathbb{F})}\right)^{-1}\right)^{\otimes k'_{\#\mathbb{F}}} \circ \sigma\right),$$

where $K = \sum_{i=1}^{\#\mathbb{F}}(k_i + k_i')$ and σ is a K-cycle. An application of Theorem 2.6, applied to the evanescent family $((N-2, 2, 0, ..., 0))_{N\in\mathbb{N}}$ allows us to conclude. □

In order to generalize Proposition 3.3 and to prove Theorem 3.2, we need to have an estimate on the Lipschitz norm of the function which gives the expectation of the fraction of fixed points of a random permutation. Recall Definition 3.3 where we defined a distance d_N on $\mathfrak{S}(N)$.

Lemma 3.2. — *For any positive integer N and any $\mathfrak{S}(N)$-valued random variables S and S', one has:*

$$\left|\mathbb{E}\left[\frac{1}{N}Tr(S)\right] - \mathbb{E}\left[\frac{1}{N}Tr\left(S'\right)\right]\right| \leq \mathbb{E}\left[d_N\left(S, S'\right)\right].$$

Proof. — It is a consequence of the Cauchy-Schwarz's inequality. Indeed, for any positive integer N and any σ, $\sigma' \in \mathfrak{S}(N)$:

$$\frac{1}{N}\big|Tr(\sigma) - Tr(\sigma')\big| = \frac{1}{N}\left|Tr(\sigma - \sigma')\right| \leq \left[\frac{1}{N} Tr\big((\sigma-\sigma')^t(\sigma-\sigma')\big)\right]^{\frac{1}{2}} = d_N(\sigma, \sigma'),$$

hence the result by taking the expectation. □

We can finish the proof of Theorem 3.2.

Proof of Theorem 3.2. — We will use the second part of Theorem 3.1. For this, we consider $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space on which is defined for each positive integer N a process $\left(h_p^N\right)_{p\in P}$ whose law is the law of the canonical process $(h_p)_{p\in P}$ under the $\mathfrak{S}(N)$-Yang-Mills measure associated with the $\mathcal{T}_N$-random walk on $\mathfrak{S}(N)$. Recall the notations defined in the proof of Proposition 3.2: we consider $\Gamma_N = L\left(\Omega, \mathcal{A}, \mathbb{P} : \mathfrak{S}(N)\right)$ endowed with the distance d_N and we consider the mappings H_N defined from Aff to Γ_N.

Let us denote by E the space $L\left(\Omega, \mathcal{A}, \mathbb{P} : \mathbb{R}\right)$ of real valued random variables defined on Ω. Let us endow E with the distance $d(X, Y) = \mathbb{E}\left[\left|\, X - Y \,\right|\right]$. For any positive integer, let:

$$\begin{aligned} \psi_N : \Gamma_N &\to E \\ S &\mapsto \frac{1}{N} Tr\left(S\right). \end{aligned}$$

Using Lemma 3.2, for any positive integer N, ψ_N is Lipschitz and $\sup_{N\in\mathbb{N}} ||\psi_N||_{\mathsf{Lip}} \leq 1$. Besides, using Proposition 3.3, and using the dominated convergence theorem, we know that for any $l \in \mathsf{Aff}_0$, $\psi_N(H_N(l))$ converges in E to a limit which is the non-random variable $\phi(l)$. At last, the Lemma 3.1 shows that the constant K_N in (10) can be taken equal to $\sqrt{2}$ for any positive integer N. Thus we can apply the second part of Theorem 3.1: for any $l \in L_0$, $\psi_N\left(H_N(l)\right) = W_l^N$ converges to a limit $\phi(l)$ and the function:

$$\begin{aligned} \phi : L_0 &\to E \\ l &\mapsto \phi(l) \end{aligned}$$

is continuous for the convergence with fixed endpoints. Let l be a loop based at 0. One can approximate the loop $l \in L_0$ by a sequence of loops $(l_n)_{n\in\mathbb{N}}$ in Aff_0. Since for any positive integer n, $\phi(l_n)$ is almost surely constant then $\phi(l)$ is almost surely constant. The convergence in probability holds since we proved that:

$$\mathbb{E}\left[\left|\, W_l^N - \phi(l) \,\right|\right] \underset{N\to\infty}{\longrightarrow} 0.$$

The asymptotic factorization property is a simple consequence of the dominated convergence theorem. □

Remark 3.1. — *Theorem 3.2 is also true for more general sequence of Yang-Mills measures, the proof follows exactly the same steps. For any integer positive N, let us consider a conjugacy class λ_N of $\mathfrak{S}(N)$, and let $(S_t^N)_{t\geq 0}$ be a λ_N-random walk on $\mathfrak{S}(N)$. One can define for any positive integer N, a Yang-Mills measure $YM_{\lambda_N}^{\mathfrak{S}(N)}$ associated with $(S_t^N)_{t\geq 0}$. If the sequence $(\lambda_N)_{N\in\mathbb{N}}$ is evanescent, then Theorem 3.2 holds without any modification.*

If the sequence $(\lambda_N)_{N\in\mathbb{N}}$ is macroscopic, then Proposition 3.3 already does not hold: since the random walk does not satisfy the asymtotic $\mathcal{P}$-factorization property, for any loop $l \in \mathsf{Aff}_0$, the Wilson loop W_l^N converges in law to a random variable $\phi(l)$ which is not almost surely constant. A slight modification of Theorem 3.1 in order to consider functions of several loops and not only one (in order to consider the functions $(l_1,..,l_k) \mapsto \mathbb{E}\left[W_{l_1}^N ... W_{l_k}^N\right]$) allows to prove in a similar way that the process $(h(l))_{l\in L_0}$ under $YM_{\lambda_N}^{\mathfrak{S}(N)}$ converges in law to a real valued random process indexed by L_0.

4. Random ramified coverings

In this section, we present a natural model of random ramified coverings on the unit disk $\mathbb{D}$. This model was first defined in [**18**], in Chapter 5 in the general setting of ramified G-bundles when G is a finite group. We translate the results for random ramified coverings without any conditions on the monodromy on the boundary. This needs some simple verifications which will not be further discussed here.

Let Y be a finite subset of $\mathbb{D} \setminus \partial\mathbb{D}$.

Definition 4.1. — *A ramified covering of the disk with ramification locus Y is a continuous mapping $\pi : R \to \mathbb{D}$ from a surface R such that the following conditions hold:*

1. *the restriction of π to $\pi^{-1}(\mathbb{D} \setminus Y)$ is a covering,*
2. *for all $y \in Y$ and any $p \in \pi^{-1}(y)$, one can find a neighborhood U of p and an integer $n \geq 1$ such that the mapping:*

$$\begin{aligned} \pi_{|U} : (U, p) &\to (\pi(U), y) \\ x &\mapsto \pi(x) \end{aligned}$$

is conjugated to the mapping $z \mapsto z^n$: $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$.

The integer n is the order of ramification of p and will be denoted by $\mathsf{or}(p)$.

Let N be a positive integer. A ramified covering $\pi : R \to \mathbb{D}$ with ramification locus Y has degree N if the restriction π to $\pi^{-1}(\mathbb{D} \setminus Y)$ is a covering of degree N.

For sake of simplicity, in this paper, we will only consider simple ramified coverings but it is easy to extend the results to general ramified covering by using Chapter 5 of [**18**].

Definition 4.2. — *Let R be a ramified covering of the disk with ramification locus Y. Let $x \in Y$ be a ramification point of R. It is a simple ramification point if there exists $p_0 \in \pi^{-1}(x)$ such that $\mathsf{or}(p) = 2$, and for any other $p \in \pi^{-1}(x)$, $\mathsf{or}(p) = 1$. The ramified covering R is simple if for any $x \in Y$, x is a simple ramification point.*

Often we will denote the covering $\pi : R \to \mathbb{D}$ just by R. The set of simple ramified covering of the disk is too big to be interesting. As one does for the theory of random maps, we will only work with the isomorphism classes of simple ramified coverings.

Definition 4.3. — *Let $\pi : R \to \mathbb{D}$ and $\pi' : R' \to \mathbb{D}$ be two simple ramified coverings. They are isomorphic if there exists a homeomorphism $h : R \to R'$ such that $\pi' \circ h = \pi$.*

Let N be a positive integer. We denote by $\mathcal{R}^N(Y)$ the set of isomorphism classes of simple ramified coverings of degree N of $\mathbb{D}$ with ramification locus equal to Y. In fact, it is even easier to work with labelled simple ramified coverings since the set of automorphism of a labelled ramified covering is trivial.

Definition 4.4. — *Let $\pi : R \to \mathbb{D}$ be a simple ramified covering of the disk of degree N with ramification locus equal to Y. Let x be in $\mathbb{D} \setminus Y$. A labelling l of R at the point x is a bijection from $\{1, \dots, n\}$ to $\pi^{-1}(x)$. The pair (R, l) is a labelled simple ramified covering based at x.*

Let (R, l), (R, l') be two labelled simple ramified coverings based at x. They are isomorphic if there exists an isomorphism of simple ramified coverings $h : R \to R'$ such that $h \circ l = l'$.

Let x be a point of $\mathbb{D} \setminus Y$. The set of isomorphism classes of labelled simple ramified coverings of $\mathbb{D}$ with ramification locus Y based at x and with degree N is denoted by $\mathcal{R}_x^N(Y)$. In order to define a measure on $\mathcal{R}^N(Y)$ or $\mathcal{R}_x^N(Y)$, we need to define a σ-field. The σ-field we will consider will be a Borel σ-field.

Definition 4.5. — *We consider on $\mathcal{R}_x^N(Y)$ the topology generated by:*

$$\mathcal{V}((R, l), U) = \left\{R' \in \mathcal{R}_x^N(Y) \mid R_{|M\setminus U} \simeq R'_{|M\setminus U}\right\},$$

where U is any open subset such that $Y \subset U \subset \mathbb{D} \setminus x$.

Also, we consider on $\mathcal{R}^N(Y)$ the topology generated by:

$$\mathcal{V}(R, U) = \left\{R' \in \mathcal{R}^N(Y) \mid R_{|M\setminus U} \simeq R'_{|M\setminus U}\right\},$$

where U is any open subset such that $Y \subset U \subset \mathbb{D}$.

Let $\mathcal{T}_N$ be the set of transpositions in $\mathfrak{S}(N)$. The set $\mathcal{R}_x^N(Y)$ is in bijection with $(\mathcal{T}_N)^{\#Y}$: this is a finite set, and thus we can consider the uniform measure on $\mathcal{R}_x^N(Y)$. When one wants to define a measure on a finite set of objects, it is common to take into account the size of the automorphism group: in case of labelled ramified coverings, the uniform measure is the natural one.

Definition 4.6. — *The uniform measure on $\mathcal{R}_x^N(Y)$ is:*

$$\mathbf{U}_{x,Y}^N = \frac{1}{(\#\mathcal{T}_n)^{\#Y}} \sum_{(R,l)\in\mathcal{R}_x^N(Y)} \delta_{(R,l)}.$$

The natural measure on $\mathcal{R}^N(Y)$ is:

$$\mathbf{U}_Y^N = \frac{1}{(\#\mathcal{T}_n)^{\#Y}} \sum_{(R,l)\in\mathcal{R}^N(Y)} \frac{n!}{\#\mathsf{Aut}(R)} \delta_R.$$

Using the Equation (63) of [**18**], one gets the following lemma.

Lemma 4.1. — *Let $\mathcal{F} : \mathcal{R}_x^N(Y) \to \mathcal{R}^N(Y)$ be the application where one forgets about the labelling. We have $\mathbf{U}_Y^N = \mathbf{U}_{x,Y}^N \circ \mathcal{F}^{-1}$.*

Let $\mathcal{P}_N(dY)$ be a Poisson point process on $\mathbb{D}$ of intensity equal to $\frac{N}{2}dx$. On the set of finite subsets of $\mathbb{D}$, $F(\mathbb{D})$, we will consider the topology which makes the bijection $F(\mathbb{D}) \simeq \cup_{k\geq 0}(\mathbb{D}^k \setminus \Delta_k)/\mathfrak{S}_k$ continuous. In [**18**], Proposition 5.3.3, Lévy showed that:

Lemma 4.2. — *The application which sends Y, a finite subset of $\mathbb{D}$, on $\mathbf{U}_Y^N$ and the one which sends Y, a finite subset of $\mathbb{D}\setminus\{x\}$, on $\mathbf{U}_{x,Y}^N$ are continuous.*

Thus we can define the following measures on simple ramified coverings on the disk (labelled or not).

Definition 4.7. — *We consider on $\mathcal{R}^N(Y)$ and $\mathcal{R}_x^N(Y)$ respectively the Borel measures:*

$$\mathbf{U}^N = \int \mathbf{U}_Y^N \mathcal{P}_N(dY) \text{ and } \mathbf{U}_x^N = \int \mathbf{U}_{x,Y}^N \mathcal{P}_N(dY).$$

The main result in this article is that, in some sense, the measures $\mathbf{U}^N$ or $\mathbf{U}_x^N$ converge when N goes to infinity. This assertion has to be taken non-rigorously as the measures are not supported by the same space and the limiting object is not defined. What we will show instead is that the monodromies of the ramified coverings converge in probability. From now on, we will only consider the measure $\mathbf{U}_x^N$ on labelled ramified coverings. The case of non labelled ramified coverings could be also studied, yet it would be necessary to be a little more careful on how we define the associated holonomy process thus, for sake of clarity, we prefered to present the results in the setting of labelled ramified coverings.

Let R be a ramified covering in $\mathcal{R}_x^N(Y)$ and let l be the labelling of the sheets of R at x. Let c be a rectifiable loop in $\mathbb{D}$ based at x. We can transport the labelling l along the path c: it gives us an other labelling l' of the sheets above x. The unique element $\sigma \in \mathfrak{S}(N)$ such that $l' = l\sigma$ is called the monodromy of R along c with respect to l and is denoted by $\mathsf{mon}_{R,l}(c)$. Suppose that we label R at x with $l \circ \eta$ where $\eta \in \mathfrak{S}(N)$, then c transports the labelling $l\eta$ on $l\mathsf{mon}_{R,l}(c)\eta$: it shows that for any curve c_1 and c_2 based at x,

$$\mathsf{mon}_{R,l}(c_1) = \eta^{-1}\mathsf{mon}_{R,l\circ\eta}(c_1)\eta, \tag{13}$$

$$\mathsf{mon}_{R,l}(c_1^{-1}) = (\mathsf{mon}_{R,l}(c_1))^{-1}, \tag{14}$$

$$\mathsf{mon}_{R,l}(c_1c_2) = \mathsf{mon}_{R,l}(c_2)\mathsf{mon}_{R,l}(c_1), \tag{15}$$

where c_1c_2 is the concatenation of c_1 with c_2 and c_1^{-1} is the curve c_1 with reversed orientation.

If c is a rectifiable curve, $\mathcal{P}_N(dY)$-a.s. the range of c is inside $\mathbb{D}\setminus Y$. We can thus define the holonomy process associated with $\mathbf{U}_x^N$ by using the monodromy along any rectifiable loop based at x. The set of rectifiable loops in $\mathbb{D}$ based at x is denoted by $L_x(\mathbb{D})$.

Definition 4.8. — *The random holonomy field on $L_x(\mathbb{D})$ associated with $\mathbf{U}_x^N$ is the process $(\mathsf{m}(c))_{c\in L_x(\mathbb{D})}$ defined on $\left(\mathcal{R}_x^N, \mathbf{U}_x^N\right)$ where:*

$$\begin{aligned}\mathsf{m}_N(c)\colon \quad \mathcal{R}_x^N &\to \mathfrak{S}(N)\\ (R,l) &\mapsto \mathsf{mon}_{R,l}(c).\end{aligned}$$

For any c_1 and c_2 in $L_x(\mathbb{D})$, $\mathbf{U}_x^N$-a.s,

$$\mathsf{m}_N(c_1c_2) = \mathsf{m}_N(c_2)\mathsf{m}_N(c_1),$$
$$\mathsf{m}_N\left(c_1^{-1}\right) = \mathsf{m}_N(c_1)^{-1}.$$

It is quite natural to wonder how a change of the base point x changes the random holonomy field: in order to do so, we need to consider the same index set for the random processes.

Definition 4.9. — *Let $c_{0\to x}$ be a path from 0 to x. The random holonomy field on $L_0(\mathbb{D})$ associated with $\mathbf{U}_x^N$ is the process $\left(\mathsf{m}_N\left(c_{0\to x}^{-1}\ c\ c_{0\to x}\right)\right)_{c\in L_0(\mathbb{D})}$ defined on $\left(\mathcal{R}_x^N, \mathbf{U}_x^N\right)$.*

Its law does not depend on the choice of $c_{0\to x}$ and it was also proved by Lévy that the laws of the random holonomy field on $L_0(\mathbb{D})$ associated with $\mathbf{U}_x^N$ do not depend on the choice of x. From now on, we will only consider the random holonomy field on $L_0(\mathbb{D})$ associated with $\mathbf{U}_0^N$. Let us state a theorem which is a direct consequence of Proposition 5.4.4 of [**18**]. One can see also this theorem as a consequence of the results of [**12**].

Theorem 4.1. — *Let $\left(S_t^N\right)_{t\geq 0}$ be a $\mathcal{T}_N$-random walk on $\mathfrak{S}(N)$. The holonomy field on $L_0(\mathbb{D})$ associated with $\mathbf{U}_0^N$ has the same law as the holonomy process $(h(l))_{l\in L_0((D)}$ under the $\mathfrak{S}(N)$-valued Yang-Mills measure associated with $\left(S_t^N\right)_{t\geq 0}$.*

As already said, one can easily generalize this theorem in order to study any λ_N-random walk. In a nutshell we have the following "equality".

$$\mathsf{Monodromy\ of\ random\ ramified\ coverings} = \mathsf{Yang-Mills\ measure\ with}\ \mathfrak{S}(\mathsf{N})\ \mathsf{gauge\ group}.$$

Using this equality and Theorem 3.2, we have proved in this article that the traces of the monodromies of random ramified coverings of the disk of degree N converge in probability when N goes to infinity.

Theorem 4.2. — *There exists an application:*

$$\phi : L_0(\mathbb{D}) \to \mathbb{R}$$
$$l \mapsto \phi(l),$$

which is continuous for convergence with fixed endpoints such that for any $l \in L_0(\mathbb{D})$, $\frac{1}{N}Tr\left(\mathsf{m}_N(l)\right)$ converges in probability to $\phi(l)$ as N goes to infinity.

References

[1] A. D'Adda and P.Provero. Two-dimensional gauge theories of the symmetric group $\mathfrak{S}(n)$ and branched n-coverings of Riemann surfaces in the large-n limit. *Nucl.Phys.Proc.Suppl.*, 108:79–83, 2002.

[2] A. D'Adda and P.Provero. Two-dimensional gauge theories of the symmetric group $\mathfrak{S}_n$ in the large-n limit. *Commun.Math.Phys.*, 245:1–25, 2004.

[3] S. Albeverio, R. Høegh-Krohn, and H. Holden. Random fields with values in Lie groups and Higgs fields. *Lecture notes in Physics, Editors: S. Albeverio, G. Casati, D. Merlini*, 262:1–13, 1986.

[4] S. Albeverio, R. Høegh-Krohn, and H. Holden. Stochastic Lie group-valued measures and their relations to stochastic curve integrals, gauge fields and Markov cosurfaces. *Stochastic processes—mathematics and physics, Editors: S. Albeverio, Ph. Blanchard, L. Streit*, 1986.

[5] S. Albeverio, R. Høegh-Krohn, and H. Holden. Stochastic multiplicative measures, generalized Markov semigroups, and group-valued stochastic processes and fields. *J. Funct. Anal.*, 78(1):154–184, 1988.

[6] M. Anshelevich and A.N. Sengupta. Quantum free Yang–Mills on the plane. *Journal of Geometry and Physics*, 62:330–343, 2012.

[7] B.K. Driver. YM_2 : continuum expectations, lattice convergence, and lassos. *Comm. Math. Phys.*, 123(4):575–616, 1989.

[8] B.K. Driver. Two-dimensional Euclidean quantized Yang-Mills fields. In *Probability models in mathematical physics*, pages 21–36. World Sci. Publ., 1991.

[9] G. Cébron and A. Dahlqvist and F. Gabriel. The general Lévy Master Field. *http://arxiv.org/abs/1601.00214*.

[10] F. Gabriel. Combinatorial theory of permutation-invariant random matrices I: Partitions, geometry and renormalization. *http://arxiv.org/abs/1503.02792*, 2015.

[11] F. Gabriel. Combinatorial theory of permutation-invariant random matrices II: Freeness and Lévy processes. *http://arxiv.org/abs/1507.02465*, 2015.

[12] F. Gabriel. Planar Markovian holonomy fields. A first step to the characterization of Markovian holonomy fields. *http://arxiv.org/abs/1501.05077*, 2015.

[13] L. Gross. A Poincaré lemma for connection forms. *J. Funct. Anal.*, 63(1):1–46, 1985.

[14] L. Gross. The Maxwell equations for Yang-Mills theory. In *CMS Conf. Proc., Editors: J.S. Feldman and L.M. Rosen*, volume 9, pages 193–203. Amer. Math. Soc., 1988.

[15] T. Lévy. The master field on the plane. *arxiv.org/abs/1112.2452*.

[16] T. Lévy. *Yang-Mills Measure on compact surfaces*. Number 790. American Mathematical Society, 2003.

[17] T. Lévy. Schur-Weyl duality and the heat kernel measure on the unitary group. *Advances in Mathematics*, 218, no. 2:537–575, 2008.

[18] T. Lévy. *Two-dimensional Markovian holonomy fields*. Number 329. Astérisque, 2010.

[19] N. Berestycki. Emergence of giant cycles and slowdown transition in random transpositions and k-cycles. *Electr. J. Probab.*, 16:152–173, 2011.

[20] N. Berestycki and R. Durrett. Phase transition in the random transposition random walk. *Probab. Theor. Rel. Fields*, 136:203–233, 2006.

[21] Nathanael Berestycki and Gady Kozma. Cycle structure of the interchange process. *Bull. Soc. math. France.*, 143(2):265–280, 2015.

[22] Oded Schramm. Composition of random transpositions. *Israel J. Math.*, 147:221?243, 2005.

[23] A.N. Sengupta. The Yang-Mills measure for S^2. *J. Funct. Anal.*, 108(2):231–273, 1992.

[24] A.N. Sengupta. Gauge theory on compact surfaces. *Mem. Amer. Math. Soc.*, 126(600):viii+85, 1997.

[25] C.N. Yang and R.L. Mills. Conservation of isotopic spin and isotopic gauge invariance. *Physical Rev.*, 96(2):191–195, 1954.

[26] D. Zvonkine. Enumeration of ramified coverings of the sphere and 2-dimensional gravity. 2005.

“Patrick, c’est vert !”

Marie-Françoise Gabriel

ANNEXE: $det(A - A) = 0$ D'OÙ LE THÉORÈME DE CAYLEY-HAMILTON.

Abstract. — The goal of this note is to explain a proof of Cayley-Hamilton's theorem based on the new generalized polynomial algebra and on diagram calculations. By defining a universal characteristic generalized polynomial, we explain a proof Cayley-Hamilton's theorem similar to the "$P_M(M) = \det(M - M) = \det(0) = 0''$ argument. A generalization of Cayley-Hamilton's theorem is given for a family of matrices.

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1. Introduction.

For any complex matrix M of size n, let $P_M \in \mathbb{C}[X]$ be the characteristic polynomial defined by:

$$P_M(X) = \det(M - X\mathsf{Id}_n),$$

where Id_n is the identity matrix of size n. The Cayley-Hamilton's theorem asserts that:

Theorem 1.1. — *For any square complex matrix M,*

$$P_M(M) = 0.$$

A frequent error which arises in an undergraduate student's mind is to replace X by M in the definition of the characteristic polynomial, argument which leads to the following equalities:

$$P_M(M) = \det(M - M) = \det(0) = 0.$$

This proof is false, yet the proof we are going to explain, which uses the new generalized polynomial algebra introduced in [**1**], in order to study random Brownian motion in the set of invertible matrices, can be considered as a way to give meaning to such an intuition. Aimed to be explained to undergraduate students in first or second year, the use of exterior algebra is avoided. It has to be noticed that links between Cayley-Hamilton's theorem and Mandelstam's identity appeared in the physic community in [**5**].

In some sense, this note can also be seen as an introduction to some important tools used in random matrices: we already explained the history of the generalized polynomial algebra, besides, the use of representations of the symmetric group is one of the main tools in [**4**], [**2**] and in [**3**] in which a generalization of permutations is used.

2. Permutations, algebra of generalized polynomials.

The characteristic polynomial can be computed in a possibly new and easy graphical way: this uses the group of permutations and the algebra of generalized polynomials defined in [**1**].

2.1. Permutations.— Any permutation in $\mathfrak{S}_k$ can be seen as an endomorphism on $(\mathbb{C}^n)^{\otimes k}$.

Definition 2.1. — *The representation ρ_k^n of $\mathfrak{S}_k$ on $(\mathbb{C}^n)^{\otimes k}$ is given, for any $\sigma \in \mathfrak{S}_k$, by:*

$$\rho_k^n(\sigma) : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes k}$$

such that, for any $x_1 \otimes ... \otimes x_k \in (\mathbb{C}^n)^{\otimes k}$,

$$\rho_k^n(\sigma)\,(x_1 \otimes ... \otimes x_k) = x_{\sigma^{-1}(1)} \otimes ... \otimes x_{\sigma^{-1}(k)}.$$

Since we will use a lot the following fact, let us prove that for any permutations σ_1 and σ_2, $\rho_k^n(\sigma_1\sigma_2) = \rho_k^n(\sigma_1)\rho_k^n(\sigma_2)$. It is enough to consider $x_1 \otimes ... \otimes x_k \in (\mathbb{C}^n)^{\otimes k}$:

$$\begin{aligned}
(\rho_k^n(\sigma_1)\rho_k^n(\sigma_2))\,(x_1 \otimes ... \otimes x_k) &= \rho_k^n(\sigma_1)\left(x_{\sigma_2^{-1}(1)} \otimes ... \otimes x_{\sigma_2^{-1}(k)}\right) \\
&= x_{\sigma_2^{-1}(\sigma_1^{-1}(1))} \otimes ... \otimes x_{\sigma_2^{-1}(\sigma_1^{-1}(k))} \\
&= x_{(\sigma_1\sigma_2)^{-1}(1)} \otimes ... \otimes x_{(\sigma_1\sigma_2)^{-1}(k)} \\
&= \rho_k^n(\sigma_1\sigma_2)(x_1 \otimes ... \otimes x_k).
\end{aligned}$$

Let $\{e_i, i \in \{1, ..., n\}\}$ be the canonical basis of $\mathbb{C}^n$. Let σ be a permutation in $\mathfrak{S}_k$. It is useful to define the endomorphism $\rho_k^n(\sigma^c)$ on $(\mathbb{C}^n)^{\otimes k}$ such that for any $i_1, ..., i_k$ in $\{1, ..., n\}$,

$$\rho_k^n(\sigma^c)\,[e_{i_1} \otimes ... \otimes e_{i_k}] = \delta_{\#\{i_1,...,i_k\}=k}\left(e_{i_{\sigma^{-1}(1)}} \otimes ... \otimes e_{i_{\sigma^{-1}(k)}}\right).$$

Indeed, using these definitions, one gets the following result.

Lemma 2.1. — *For any positive integers k and n, we have the equality:*

$$\sum_{\sigma\in\mathfrak{S}_k} \epsilon(\sigma)\rho_k^n(\sigma) = \sum_{\sigma\in\mathfrak{S}_k} \epsilon(\sigma)\rho_k^n(\sigma^c),$$

where for any permutation $\sigma \in \mathfrak{S}_k$, $\epsilon(\sigma)$ is the signature of σ.

Proof. — It is enough to prove that for any integers $i_1, ..., i_k$ in $\{1, ..., n\}$, such that at least two elements are equal, $\left[\sum_{\sigma\in\mathfrak{S}_k} \epsilon(\sigma)\rho_k^n(\sigma)\right](e_{i_1} \otimes ... \otimes e_{i_k}) = 0$. Let us suppose that $i_1 = i_2$, and let $(1, 2)$ be the transposition in $\mathfrak{S}_k$ which exchanges 1 and 2, then:

$$\rho_k^n\left((1, 2)\right)(e_{i_1} \otimes ... \otimes e_{i_k}) = e_{i_1} \otimes ... \otimes e_{i_k}.$$

Let us denote by x the vector $e_{i_1} \otimes ... \otimes e_{i_k}$. We have the following sequence of equalities:

$$\left[\sum_{\sigma\in\mathfrak{S}_k}\epsilon(\sigma)\rho_k^n(\sigma)\right]x = \left[\sum_{\sigma\in\mathfrak{S}_k}\epsilon(\sigma)\rho_k^n(\sigma)\right]\left(\rho_k^n\big((1,2)\big)x\right)$$
$$= \left[\sum_{\sigma\in\mathfrak{S}_k}\epsilon(\sigma)\rho_k^n(\sigma\circ(1,2))\right]x$$
$$= -\left[\sum_{\sigma\in\mathfrak{S}_k}\epsilon(\sigma\circ(1,2))\rho_k^n(\sigma\circ(1,2))\right]x$$
$$= -\left[\sum_{\sigma\in\mathfrak{S}_k}\epsilon(\sigma)\rho_k^n(\sigma)\right]x.$$

Thus $\left[\sum_{\sigma\in\mathfrak{S}_k}\epsilon(\sigma)\rho_k^n(\sigma)\right](e_{i_1}\otimes ...\otimes e_{i_k}) = 0$. □

Let us consider on $\mathbb{C}^n$ the scalar product defined such that for any integers i and j in $\{1,...,n\}$, $\langle e_i, e_j\rangle = \delta_{i=j}$. Let M be a square matrix of size n, let i and j be two integers in $\{1,...,n\}$. We define:

$$M_i^j = \langle Me_j, e_i\rangle.$$

One interesting fact about the representation ρ_k^n is the following lemma, already stated in Definition 2.1 of [**4**].

Lemma 2.2. — *Let M be a square matrix of size n, let $\sigma\in\mathfrak{S}_k$. Let $Tr^{\otimes k}$ be the trace on $(\mathbb{C}^n)^{\otimes k}$:*

$$Tr^{\otimes k}\left(M^{\otimes k}\circ\rho_k^n(\sigma)\right) = \prod_{c\in\mathsf{C}(\sigma)} Tr(M^{\#c}),$$

where $\mathsf{C}(\sigma)$ is the set of cycles of σ. Besides:

$$Tr^{\otimes k}\left(M^{\otimes k}\circ\rho_k^n(\sigma^c)\right) = \sum_{(i_1,...,i_k)\in\{1,...,n\}^k,\#\{i_1,...,i_k\}=k}\ \prod_{j=1}^{k} M_{i_j}^{i_{\sigma^{-1}(j)}}.$$

For example, if $\sigma=(1,3)(2,4,5)$, then:

$$Tr^{\otimes 5}\left(M^{\otimes 5}\circ\rho_5^n(\sigma)\right) = Tr(M^2)Tr(M^3).$$

Proof. — Let M be a square matrix of size n, let $\sigma\in\mathfrak{S}_k$. A basis of $(\mathbb{C}^n)^{\otimes k}$ is given by $\mathcal{B} = \{(e_{i_1}\otimes ...\otimes e_{i_k}), (i_1,...,i_k)\in\{1,...,n\}^k$. On $(\mathbb{C}^n)^{\otimes k}$, let us consider the scalar product such that $\mathcal{B}$ is an orthonormal basis: the trace $Tr^{\otimes k}$ is given by:

$$Tr^{\otimes k}(A) = \sum_{i_1,...,i_k=1}^{n}\left\langle e_{i_1}\otimes ...\otimes e_{i_k}, A\left(e_{i_1}\otimes ...\otimes e_{i_k}\right)\right\rangle,$$

for any $A \in \mathsf{End}\left((\mathbb{C}^n)^{\otimes k}\right)$. This implies the following sequence of equalities:

$$\begin{aligned}
Tr^{\otimes k}\left(M^{\otimes k} \circ \rho_k^n(\sigma)\right) &= \sum_{i_1,\dots,i_k=1}^{n} \left\langle e_{i_1} \otimes \dots \otimes e_{i_k}, M^{\otimes k} \circ \rho_k^n(\sigma)\left(e_{i_1} \otimes \dots \otimes e_{i_k}\right)\right\rangle \\
&= \sum_{i_1,\dots,i_k=1}^{n} \left\langle e_{i_1} \otimes \dots \otimes e_{i_k}, M^{\otimes k}\left(e_{i_{\sigma^{-1}(1)}} \otimes \dots \otimes e_{i_{\sigma^{-1}(k)}}\right)\right\rangle \\
&= \sum_{i_1,\dots,i_k=1}^{n} \prod_{j=1}^{k} \left\langle e_{i_j}, M e_{i_{\sigma^{-1}(j)}}\right\rangle \\
&= \sum_{i_1,\dots,i_k=1}^{n} \prod_{j=1}^{k} M_{i_j}^{i_{\sigma^{-1}(j)}} \\
&= \sum_{\left(i_1,\dots,i_{\ell(c)}\right)_{c\in\mathsf{C}(\sigma)} \in \prod\limits_{c\in\mathsf{C}(\sigma)} \{1,\dots,n\}^{\ell(c)}} \left[\prod_{c\in\mathsf{C}(\sigma)} \left(\prod_{k=1}^{\ell(c)-1} M_{i_k}^{i_{k+1}}\right) M_{i_l}^{i_1}\right] \\
&= \prod_{c=(j_1,\dots,j_l)\in\mathsf{C}(\sigma)} \sum_{i_1,\dots,i_l=1}^{n} \left(\prod_{k=1}^{l-1} M_{i_k}^{i_{k+1}}\right) M_{i_l}^{i_1} \\
&= \prod_{c=(j_1,\dots,j_l)\in\mathsf{C}(\sigma)} Tr\left(M^l\right),
\end{aligned}$$

where for any $c \in \mathsf{C}(\sigma)$ we denoted by $\ell(c)$ is the length of the cycle c. The proof of the second equality follows the same calculations. □

2.2. Algebra of generalized polynomials.— The algebra of generalized polynomials is a generalization of the algebra of polynomials. Let us remind the reader the definition of the algebra of polynomials. It is the vector space which base is denoted by $\left(X^k\right)_{k=0}^{\infty}$, endowed with the following product: for any non negative integers i and j, $X^i X^j = X^{i+j}$. Our interest in this algebra is the existence of the evaluation morphism e_M. Let M be a square matrix of size n.

Definition 2.2. — *The evaluation in M, defined on the algebra of polynomials, is the unique morphism of algebra e_M between $\mathbb{C}[X]$ and $M_n(\mathbb{C})$ such that $e_M(X) = M$. If P is a polynomial, we will often denote by $P(M)$ the matrix $e_M(P)$.*

Definition 2.3. — *The algebra of generalized polynomials, denoted by $\mathbb{C}\{X\}$, is the complex associative and commutative algebra generated by $\left(X^k\right)_{k=0}^{\infty}$ and $\left(\tau(X^k)\right)_{k=0}^{\infty}$ with neutral element for the multiplication is given by X^0 and with the relations:*

$$\forall k, k' \in \mathbb{N},\ X^k X^{k'} = X^{k+k'}.$$

The algebra $\mathbb{C}\{X\}$ is thus a complex vector space, and one of its basis is given by:

$$\left(X^k \tau(X^{l_1})\tau(X^{l_2})\dots\tau(X^{l_m})\right)_{(k,m)\in\mathbb{N}\times\mathbb{N}, 0\le l_1\le\dots\le l_m}.$$

Let us remark that any element of the form $\tau(X^{l_1})...\tau(X^{l_m})$ is written in this basis as $X^0\tau(X^{l_1})...\tau(X^{l_m})$. Later this remark will be important. Let us give an example of computation in $\mathbb{C}\{X\}$:

$$\left(X^2\tau(X^1)\tau(X^3)\right)\left(X^4\tau(X^0)\right) = X^6\tau(X^0)\tau(X^1)\tau(X^3).$$

Readers who are interested in the universal property satisfied by $\mathbb{C}\{X\}$ can have a look at the article [**1**]. An interesting subalgebra of $\mathbb{C}\{X\}$ is the trace algebra.

Definition 2.4. — *The trace algebra is the unital sub-algebra of* $\mathbb{C}\{X\}$ *generated by* $\left(\tau(X^k)\right)_{k\in\mathbb{N}}$.

There exists a unique morphism of vector spaces denoted by τ, from $\mathbb{C}\{X\}$ onto the trace algebra, such that for any non-negative integers k and k', any $0 \leq l_1 \leq ... \leq l_{k'}$:

$$\tau\left(X^k\tau(X^{l_1})...\tau(X^{l_{k'}})\right) = \tau(X^k)\tau(X^{l_1})...\tau(X^{l_{k'}}).$$

It becomes natural to denote the trace algebra by $\tau(\mathbb{C}\{X\})$.

For every positive integer n, we will also define the morphism of vector spaces τ_n such that for any non-negative integers k and k', any $0 \leq l_1 \leq ... \leq l_{k'}$:

$$\begin{aligned}\tau_n\left(X^k\tau(X^{l_1})...\tau(X^{l_{k'}})\right) &= \tau(X^k)\tau(X^{l_1})...\tau(X^{l_{k'}}),\ \text{if } k \neq 0,\\ &= n\tau(X^{l_1})...\tau(X^{l_{k'}}),\ \text{if } k = 0.\end{aligned}$$

Definition 2.5 **(Partial evaluation)**. — *The partial evaluation in* M*, denoted by* tr_M*, is the unique morphism of algebra from* $\mathbb{C}\{X\}$ *to* $\mathbb{C}[X]$ *such that:*

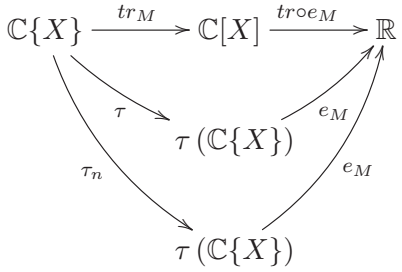
$$\begin{aligned}tr_M\left(X^k\right) &= X^k \qquad \text{, for any integer } k \in \mathbb{N},\\ tr_M\left(\tau\left(X^l\right)\right) &= tr\left(M^l\right), \text{ for any integer } l \in \mathbb{N}.\end{aligned}$$

Definition 2.6 **(Trace evaluation)**. — *The trace evaluation in* M*, defined on* $\tau\left(\mathbb{C}\{X\}\right)$ *and also denoted by* e_M*, is the unique morphism of algebras from* $\tau(\mathbb{C}\{X\})$ *to* $\mathbb{C}$ *such that:*

$$e_M\left(\tau(X^k)\right) = tr(M^k).$$

The morphisms τ, τ_n, the evaluation, the partial evaluation and the trace evaluation are actually tightly linked: this is the meaning of the following lemma whose proof is straightforward by checking it on the basis of $\mathbb{C}\{X\}$.

Lemma 2.3. — *Let M be a complex square matrix of size n. The following diagram is commutative:*



Let us illustrate this lemma with an example. Let us consider $P = X^2\tau(X^3)$, then:

$$tr \circ e_M \circ tr_M(P) = tr \circ e_M(X^2 tr(M^3)) = tr\left(M^2 tr(M^3)\right) = tr(M^2)tr(M^3),$$
$$e_M \circ \tau_2(P) = e_M(\tau(X^2)\tau(X^3)) = tr(M^2)tr(M^3),$$

hence $tr \circ e_M \circ tr_M(P) = e_M \circ \tau_2(P)$.

3. The generalized characteristic polynomial.

The characteristic polynomial is a polynomial which strongly depends on the matrix. Using the algebra of generalized polynomials, we can define a universal generalized characteristic polynomial which only depends on the size of the matrix: the characteristic polynomial is then the partial evaluation of this universal generalized characteristic polynomial.

Let us consider A a unital associative algebra and let k be a positive integer.

Definition 3.1 **(Indexed permutations)**. — *An indexed permutation of size k is an element of $\mathfrak{S}_k \times A^k$. The set of indexed permutations of size k is denoted by $A\mathfrak{S}_k$.*

Any indexed permutation $(\sigma, (a_1, ..., a_k))$ can be graphically represented by drawing the diagram which represents σ, and writing below it the elements a_1 to a_k as shown in Figure 1. Let us remark that there exists a canonical bijection between $A\mathfrak{S}_1$ and A, and a canonical bijection between $A\mathfrak{S}_0$ and $\{1\}$, where 1 is the neutral element of A. From now on, we will always identify $A\mathfrak{S}_1$ with A and $A\mathfrak{S}_0$ with the set $\{1\}$.

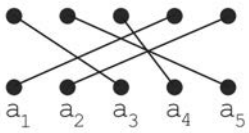


FIGURE 1. Graphical representation of $(\sigma_1, \mathbf{a}_1)$ with $\sigma_1 = (1,3,4)(2,5)$ and $\mathbf{a}_1 = (a_1, a_2, a_3, a_4, a_5)$.

Let us suppose that A is endowed with an application τ from A to the center of A, $\mathcal{Z}(A)$, such that:

$$\forall (a,b) \in A^2, \tau(ab) = \tau(ba),$$
$$\forall (a,b) \in \mathcal{Z}(A) \times A, \tau(ab) = a\tau(b).$$

We are going to define, for any subset $I \subset \{1,...,k\}$, an application:

$$tr_I : A\mathfrak{S}_k \to \mathcal{Z}(A) \times A\mathfrak{S}_{k-\#I}.$$

Let us consider I a subset of $\{1,...,k\}$ and let σ be a permutation in $\mathfrak{S}_k$. We consider $\mathcal{C}_\sigma(I)$ the set of cycles c of σ which are included in I. For any $c = (i_1,...,i_r)$ in $\mathcal{C}_\sigma(I)$, we define for any $(a_1,...,a_k) \in A^k$:

$$\tau_c(a_1,...,a_k) = \tau\left(\prod_{j=1}^{r} a_{i_j}\right),$$

which does not depend on how we wrote $c = (i_1,...,i_r)$ since τ is tracial.

Let us consider $i \in \{1,...,k\} \setminus I$, we consider the bigger integer r such that for any $j \in \{1,...,r\}$, $\sigma^j(i) \in I$, and we define for any $(a_1,...,a_k) \in A^k$:

$$\tau_I^{i,\sigma}(a_1,...,a_k) = \prod_{j=1}^{r+1} a_{\sigma^j(i)},$$

and we define $\tilde{\sigma}_{\sigma,I}(i) = \sigma^{r+1}(i)$. Let us consider ϕ the unique increasing bijection between $\{1,...,k-\#I\}$ and $\{1,...,k\} \setminus I$.

Definition 3.2. — *The application*

$$tr_I : A\mathfrak{S}_k \to \mathcal{Z}(A) \times A\mathfrak{S}_{k-\#I}$$

is defined such that for any $(\sigma, a) \in A\mathfrak{S}_k$*:*

$$tr_I((\sigma,a)) = \left(\prod_{c\in\mathcal{C}_\sigma(I)} \tau_c(a), \left(\phi^{-1}\tilde{\sigma}_{\sigma,I}\phi, \left(\tau_I^{\tilde{\sigma}_{\sigma,I}^{-1}(\phi(i)),\sigma}(a)\right)_{i=1}^{k-\#I}\right)\right).$$

One can look at Figure 2 in order to see that this application is actually easy to understand and to compute graphically. Indeed, it is enought to add the vertical lines joining the two levels corresponding to points in I, as we did in Figure 2, to follow the lines, to multiply the elements of A that stand on the way and to apply τ when resulting path is a loop.

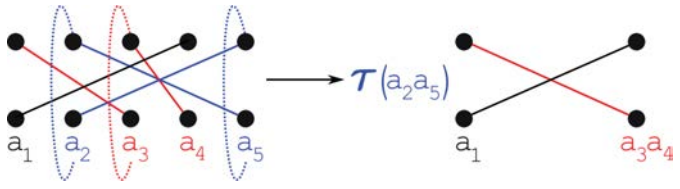


FIGURE 2. Graphical computation of $tr_{\{2,3,5\}}(\sigma_1, \mathbf{a}_1)$.

Let us denote by mult the multiplication from $A \times A$ to A. Let us remark that for any $(\sigma, a) \in A\mathfrak{S}_k$,

$$\tau\left(\mathsf{mult}\left(tr_{\{1,\dots,k-1\}}(\sigma,a)\right)\right) = \mathsf{mult}\left(tr_{\{1,\dots,k\}}(\sigma,a)\right). \tag{1}$$

Let us consider the set of $\mathbb{C}\{X\}$-indexed partitions, where $\mathbb{C}\{X\}$ is endowed with the application τ_n defined in Definition 2.4.

Definition 3.3. — *The generalized characteristic polynomial of degree n is:*

$$P_n^g = \sum_{\sigma\in\mathfrak{S}_{n+1}} \epsilon(\sigma)\mathsf{mult}\left(tr_{\{1,\dots,n\}}\left[\left(\sigma,(X^1,\dots,X^1,X^0)\right)\right]\right).$$

Remark 3.1. — *Let us compute some examples of generalized characteristic polynomials.*

- ***n=1****: There is only two permutations in $\mathfrak{S}_2$, the identity and the transposition. Thus:*

$$P_1^g = -X^1 + \tau(X).$$

- ***n=2****: We have to compute a sum on the permutations of $\mathfrak{S}_3$. The identity permutation gives a term equal to $\tau(X)^2$, the $(1,2)$ transposition gives $-\tau(X^2)X^0$, the $(2,3)$ and $(1,3)$ transpositions give $-X^1\tau(X)$, the two 3-cycles give X^2. Thus:*

$$P_2^g = 2X^2 - 2X^1\tau(X) + \tau(X)^2 - \tau(X^2).$$

This generalized characteristic polynomial of degree n is the universal generalized polynomial from which one can recover easily the characteristic polynomial P_M for any square matrix M of size n.

Theorem 3.1. — *The characteristic polynomial of M is given by:*

$$P_M = \frac{1}{n!} tr_M(P_n^g),$$

where the partial evaluation tr_M was defined in Definition 2.5.

Let us illustrate it by giving an example: let us suppose that M is a square matrix of size 2. The theorem asserts that the characteristic polynomial of M is equal to $\frac{1}{2}\left[tr_M\left(2X^2 - 2X\tau(X) + \tau(X)^2 - \tau(X^2)\right)\right]$, and thus to:

$$X^2 - tr(M)X + \frac{1}{2}(tr(M)^2 - tr(M^2)).$$

Using the equality $\det(M) = \frac{1}{2}(tr(M)^2 - tr(M^2))$, valid for any square matrix of size 2, we recover the well-known result that $P_M(X) = X^2 - tr(M)X + \det(M)$.

Proof of Theorem 3.1.. — Let P_n^g be the generalized characteristic polynomial of degree n, let M be a square matrix of size n and let l be an integer in $\{0,\dots,n\}$.

By a simple calculation, using the definition of the determinant, we get that the coefficient of X^l in P_M, denoted by $[P_M]_l$, is equal to:

$$[P_M]_l = (-1)^l \sum_{I\subset\{1,\dots,n\},\#I=l} \det\left[M_{I^c}\right],$$

where M_{I^c} is the matrix restricted to $I^c \times I^c$ and I^c is the complement of I in $\{1,\dots,n\}$.

Let us study the coefficient of X^l in $tr_M(P_n^g)$, namely $[tr_M(P_n^g)]_l$. We need to understand what are the permutations $\sigma \in \mathfrak{S}_{n+1}$ for which the monomial:

$$tr_M\left[\mathsf{mult}\left(tr_{\{1,...,n\}}\left[\left(\sigma,(X^1,...,X^1,X^0)\right)\right]\right)\right]$$

is a scalar multiple of X^l. Let σ be such a permutation: this means that $n+1$ is in a cycle of size $l+1$ of σ. The set of partitions such that $n+1$ is in a cycle of size $l+1$ is in $\frac{n!}{(n-l)!}$-to-one correspondence with $\mathfrak{S}_{n-l}$. This correspondence and the first part of Lemma 2.2 allow us to compute $[tr_M\left(P_n^g\right)]_l$:

$$[tr_M(P_n^g)]_l = \frac{n!}{(n-l)!}\sum_{\sigma\in\mathfrak{S}_{n-l}}\epsilon(\sigma)(-1)^l Tr^{\otimes n-l}\left(M^{\otimes n-l}\circ\rho_{n-l}^n(\sigma)\right).$$

Using Lemma 2.1, it remains to prove that:

$$\frac{1}{(n-l)!}\sum_{\sigma\in\mathfrak{S}_{n-l}}\epsilon(\sigma)Tr^{\otimes n-l}\left(M^{\otimes n-l}\circ\rho_{n-l}^n(\sigma^c)\right) = \sum_{I\subset\{1,...,n\},\#I=l}\det\left[M_{I^c}\right].$$

Using the second part of Lemma 2.2:

$$\begin{aligned}
\text{l.h.s.} &= \frac{1}{(n-l)!}\sum_{\sigma\in\mathfrak{S}_{n-l}}\epsilon(\sigma)\sum_{(j_1,...,j_{n-l})\in\{1,...,n\},\#\{j_1,...,j_{n-l}\}=n-l}\prod_{k=1}^{n-l}M_{j_k}^{j_{\sigma^{-1}(k)}}\\
&= \frac{1}{(n-l)!}\sum_{(j_1,...,j_{n-l})\in\{1,...,n\},\#\{j_1,...,j_{n-l}\}=n-l}\sum_{\sigma\in\mathfrak{S}_{n-l}}\epsilon(\sigma)\prod_{k=1}^{n-l}M_{j_k}^{j_{\sigma(k)}}\\
&= \sum_{J\subset\{1,...,n\},\#J=n-l}\sum_{\sigma\in\mathfrak{S}_{n-l}}\epsilon(\sigma)\prod_{k=1}^{n-l}M_{J_k}^{J_{\sigma(k)}},
\end{aligned}$$

where in the last equality we used the following notation: if J is a subset of $\{1,...,n\}$ of cardinal $n-l$, the family $(J_1,...,J_{n-l})$ is the increasing family of elements of J. At the end, $\frac{1}{(n-l)!}\sum_{\sigma\in\mathfrak{S}_{n-l}}\epsilon(\sigma)Tr^{\otimes n-l}\left(M^{\otimes n-l}\circ\rho_{n-l}^n(\sigma^c)\right)$ is equal to:

$$\sum_{I\subset\{1,...,n\},\#I=l}\sum_{\sigma\in\mathfrak{S}_{n-l}}\epsilon(\sigma)\prod_{k=1}^{n-l}M_{(I^c)_k}^{(I^c)_{\sigma(k)}} = \sum_{I\subset\{1,...,n\},\#I=l}\det[M_{I^c}],$$

equality which allows us to finish the proof. □

4. The Mandelstam's identities.

In order to prove Cayley-Hamilton's theorem, we will use two identities that are tightly linked with the Mandelstam's identities. Usualy, Mandelstam's identities assert that there exists some relations between elements of the family $(tr(M^k))_{k\in\mathbb{N}}$ for any square matrix M of finite size. The first new version of Mandelstam's identity is given by the following equality in the trace algebra.

Lemma 4.1. — *Let P_n^g be the generalized characteristic polynomial of degree n. The following equality holds:*

$$\tau_n(P_n^g) = 0.$$

Before giving a proof of this lemma, let us give an example. When $n = 2$:

$$\tau_2(P_2^g) = 2\tau(X^2) - 2\tau(X)^2 + 2(\tau(X)^2 - \tau(X^2)) = 0.$$

Proof of Lemma 4.1. — Let n be a positive integer, we have to compute $\tau_n(P_n^g)$. Using the Equality (1):

$$\begin{aligned}\tau_n(P_n^g) &= \sum_{\sigma\in\mathfrak{S}_{n+1}} \epsilon(\sigma)\tau_n\left[\mathsf{mult}\left(tr_{\{1,...,n\}}\left[\left(\sigma,(X^1,...,X^1,X^0)\right)\right]\right)\right] \\ &= \sum_{\sigma\in\mathfrak{S}_{n+1}} \epsilon(\sigma)\mathsf{mult}\left(tr_{\{1,...,n+1\}}\left[\left(\sigma,(X^1,...,X^1,X^0)\right)\right]\right).\end{aligned}$$

Let us consider $\mathfrak{S}_{n+1}^f$ the set of permutations $\sigma \in \mathfrak{S}_{n+1}$ such that $\sigma(n+1) = n+1$: $\mathfrak{S}_{n+1}^f$ is in bijection with $\mathfrak{S}_n$. Let us define the application:

$$\begin{aligned}\Phi : \mathfrak{S}_{n+1} \setminus \mathfrak{S}_{n+1}^f &\to \mathfrak{S}_{n+1}^f \\ \sigma &\mapsto \sigma(n+1, \sigma^{-1}(n+1)),\end{aligned}$$

where for any distinct integers i, j in $\{1, ..., n+1\}$, (i, j) is the transposition which exchanges i and j. The application Φ is a n-to-one application. Besides for any $\sigma \in \mathfrak{S}_{n+1} \setminus \mathfrak{S}_{n+1}^f$, $\epsilon(\Phi(\sigma)) = -\epsilon(\sigma)$ and $\mathsf{mult}\left(tr_{\{1,...,n+1\}}\left[\left(\sigma,(X^1,...,X^1,X^0)\right)\right]\right)$ is equal to:

$$\frac{1}{n}\mathsf{mult}\left(tr_{\{1,...,n+1\}}\left[\left(\Phi(\sigma),(X^1,...,X^1,X^0)\right)\right]\right),$$

since we recall that $\tau_n(X^0) = n$. These remarks imply that $\tau_n(P_n^g) = 0$. □

The second identity that we present, asserts that for any positive integer n, the representation ρ_{n+1}^n is not injective.

Lemma 4.2. — *For any positive integer n, the Mandelstam's identity holds:*

$$\sum_{\sigma\in\mathfrak{S}_{n+1}} \epsilon(\sigma)\rho_{n+1}^n(\sigma) = 0.$$

Proof. — By definition, for any $k > n$, for any $\sigma \in \mathfrak{S}_k$, $\rho_k^n(\sigma^c) = 0$. The result follows from Lemma 2.1. □

5. Proof of Cayley-Hamilton's theorem.

Let M be a square matrix of size n. Let P_n^g and P_M be respectively the generalized characteristic polynomial of degree n and the characteristic polynomial of M. In order to prove that $P_M(M) = 0$, it is enough to show that for any square matrix L of size n, the equality

$$tr\left(P_M(M)L\right) = 0 \tag{2}$$

holds. In the following proof, we show at first that the first version of Mandelstam's identity, namely Lemma 4.1, implies that Equality (2) holds for $L = \mathsf{Id}_n$. Then we use the second version of Mandelstam's identity, namely Lemma 4.2, in order to prove the general case. We do so since the case where $L = \mathsf{Id}_n$ looks like the false replacement trick argument which consists of the following sequence of arguments.

"The polynomial $\det(X - X)$ is the null polynomial. Thus making an evaluation in M we get $\det(M - M) = 0$. But $\det(M - M)$ is the evaluation in M of the partial evaluation of $\det(X - X)$, namely $\det(X - M)$, which is the characteristic polynomial of M. "

One of the problems in this sequence of arguments is that the partial evaluation is not well defined. In Section 2.2, we managed to define such a partial evaluation which allows us to use a correct similar argument.

Proof of Theorem 1.1. — The first version of Mandelstam's identity, Lemma 4.1, asserts that $\tau_n(P_n^g) = 0$. Taking the partial evaluation, this implies that $e_M \circ \tau_n(P_n^g) = 0$. Using the commutative diagram of Lemma 2.3, we get:

$$tr \circ e_M \circ tr_M(P_n^g) = 0.$$

Yet, by Theorem 3.1, $tr_M(P_n^g) = n! P_M$ and we remind the reader that for any polynomial $P \in \mathbb{C}[X]$, $e_M(P) = P(M)$. Thus $tr(P_M(M)) = 0$ which is the Equality (2) when $L = \mathsf{Id}_n$.

Let us consider the general case. Let L be a square matrix of size n. Then $tr\left(P_M(M)L\right) = \frac{1}{n!} tr\left(tr_M(P_n^g)L\right)$. Using the graphical computation of P_n^g, and Lemma 2.2, it is easy to see that:

$$tr\left(tr_M(P_n^g)L\right) = tr\left(\left(\sum_{\sigma \in \mathfrak{S}_{n+1}} \epsilon(\sigma)\rho_{n+1}^n(\sigma)\right) \circ \left(M^{\otimes n} \otimes L\right)\right).$$

Using Lemma 4.2, this implies that $tr\left(P_M(M)L\right) = 0$. □

6. Cayley-Hamilton for a family of matrices.

Let us consider the generalized non-commutative polynomial algebra $\mathbb{C}\{X_1, ..., X_n\}$, whose definition can be easily guessed from Definition 2.3, adding the fact that X_1, ..., X_n does not commute, but keeping the fact that $(\tau(X_i))_{i=1}^k$ is in the center. On the algebra $\mathbb{C}\{X_1, ..., X_n\}$, we can consider a tracial state τ_n similar to the one considered on $\mathbb{C}\{X\}$. Besides, for any square matrices $M_1, ..., M_n$ of size n, let $tr_{(X_i \to M_i)_{i=1}^n}$ be the unique morphism of algebra from $\mathbb{C}\{X_1, ..., X_n\}$ to the algebra of non-commutative polynomials $\mathbb{C}\langle X_1, ..., X_n\rangle$ such that for any non-commutative monomial P in $X_1, ..., X_n$, $tr_{(X_i \to M_i)_{i=1}^n}(P(X_1, ..., X_n)) = P(X_1, ..., X_n)$ and $tr_{(X_i \to M_i)_{i=1}^n}\left(\tau\left(P(X_1, ..., X_n)\right)\right) = tr\left(P(M_1, ..., M_n)\right)$.

One can generalize the definitions of characteristic polynomial and generalized characteristic polynomial.

Definition 6.1. — *The generalized characteristic polynomial in n variables is:*

$$\overline{P}_n^g = \sum_{\sigma \in \mathfrak{S}_{n+1}} \epsilon(\sigma) \mathsf{mult}\left(tr_{\{1,...,n\}}\left[(\sigma, (X_1, ..., X_n, 1))\right]\right),$$

where 1 *is the identity in* $\mathbb{C}\{X_1, ..., X_n\}$ *and where we considered* $\mathbb{C}\{X_1, ..., X_n\}$*-indexed partitions with* $\mathbb{C}\{X_1, ..., X_n\}$ *endowed with the tracial state* τ_n*.*

For example the generalized characteristic polynomial in 2 variables is equal to:

$$X_1X_2 + X_2X_1 - \tau(X_1)X_2 - \tau(X_2)X_1 + [\tau(X_1)\tau(X_2) - \tau(X_1X_2)].$$

This generalized characteristic polynomial allows use to define a characteristic polynomial for n square matrices.

Definition 6.2. — *For any square matrices $M_1, ..., M_n$ of size n, the characteristic polynomial of $(M_1, ..., M_n)$ is the non-commutative polynomial:*

$$P_{M_1,...,M_n} = tr_{(X_i \to M_i)_{i=1}^n}\left[\overline{P}_n^g\right] \in \mathbb{C}\langle X_1, ..., X_n\rangle.$$

Let us remark that if M is a matrix of size n, the characteristic polynomial of M, $P_M(X)$, is equal to $P_{M,...,M}(X, ..., X)$. Let M_1 and M_2 be two square matrices of size 2. The non-commutative polynomial $P_{M_1,M_2}(X_1, X_2)$ is equal to:

$$X_1X_2 + X_2X_1 - Tr(M_1)X_1 - Tr(M_2)X_2 + [Tr(M_1)Tr(M_2) - Tr(M_1M_2)].$$

Using the same arguments than the ones used in order to prove Theorem 1.1, we get the following theorem.

Theorem 6.1. — *For any n-tuple of square matrices of size n, the following Cayley-Hamilton identity holds:*

$$P_{M_1,...,M_n}(M_1, ..., M_n) = 0.$$

Remark 6.1. — *Let us extend the notion of determinant of a matrix. Let M_1, ..., M_n be a n-tuple of square matrices of size n, the determinant* $\mathsf{det}(M_1, ..., M_n)$ *is defined as:*

$$\mathsf{det}(M_1, ..., M_n) = \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma)(M_1)_1^{\sigma(1)}...(M_n)_n^{\sigma(n)}.$$

This is a generalization of the determinant, since for any matrix M of size n,

$$\det(M) = \mathsf{det}(M, ..., M).$$

The polynomial $P_{M_1,...,M_n}$ can be also defined as:

$$P_{M_1,...,M_n} = \mathsf{det}\left(M_1 - X_1\mathsf{Id}_n, ..., M_n - X_n\mathsf{Id}_n\right).$$

References

[1] G. Cébron, Free convolution operators and free Hall transform *J. Funct. Anal.* **265** (2013) 2645-2708.

[2] F. Gabriel, Combinatorial theory of permutation-invariant random matrices I: Partitions, geometry and renormalization, *arxiv* (2015).

[3] F. Gabriel, Combinatorial theory of permutation-invariant random matrices II: Freeness and Lévy processes, *http://arxiv.org/pdf/1503.02792.pdf* (2015).

[4] T. Lévy, Schur-Weyl duality and the heat kernel measure on the unitary group, *Advances in Mathematics*, **218, no. 2** (2008) 537-575.

[5] D. E. Berenstein and L. F. Urrutia, The equivalence between the Mandelstam and the Cayley-Hamilton identities, *J. Math. Phys*, **35** (1994).