On the ampleness of the cotangent bundles of complete intersections

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PRÉSENTATION DES RÉSULTATS

En géométrie algébrique, la notion d'amplitude des fibrés en droites apparaît naturellement pour caractériser les variétés projectives.

Definition 0.1. Soient X un schéma, et $\mathscr{L}$ un fibré en droites sur X. Alors on dit que $\mathscr{L}$ est:

- *très ample*, s'il existe un plongement quelconque $i\colon X \hookrightarrow \mathbb{P}^N$ à valeurs dans un espace projectif $\mathbb{P}^N$ tel que $\mathscr{L} = i^*\mathscr{O}_{\mathbb{P}^N}(1)$;
- *ample*, s'il existe un nombre entier $k \geqslant 1$ tel que le fibré en droites $\mathscr{L}^k$ est très ample.

Il est un fait étonnant que l'amplitude des fibrés en droites bénéficie de critères divers, par exemple, ceux cohomologiques via les théorèmes d'annulations, ou celui numérique via le théorème de Nakai-Moishezon-Kleiman. De plus, sur le corps des nombres complexes, le célèbre théorème de plongement de Kodaira affirme que, sur une variété complexe compacte lisse, un fibré en droites holomorphe $\mathscr{L}$ est ample si et seulement s'il existe une métrique hermitienne de $\mathscr{L}$ dont la forme de courbure est strictement positive.

Pour des fibrés vectoriels, on peut aussi introduire la notion d'amplitude comme suit. Soit E un fibre vectoriels sur une variété X. Notons $\pi\colon \mathbf{P}(E) \longrightarrow X$ le fibré projectif des quotients de E de dimension 1, et notons $\mathscr{O}_{\mathbf{P}(E)}(1)$ le fibré en droites de Serre sur $\mathbf{P}(E)$, i.e. le quotient tautologique de π^*E.

Definition 0.2. Le fibré vectoriel E est (très) ample sur X si et seulement si le fibré en droites de Serre $\mathscr{O}_{\mathbf{P}(E)}(1)$ est (très) ample sur $\mathbf{P}(E)$.

Sur une variété lisse X, le fibré vectoriel le plus naturel serait, soit le fibré cotangent Ω_X, soit le fibré tangent $T_X = \Omega_X^\vee$. Rappelons que, dans sa solution célèbre de la Conjecture d'Hartshorne, Mori a établi que les seules variétés lisses ayant un fibré tangent ample sont les espaces projectifs. Par conséquent, un problème naturel surgit: déterminer les variétés lisses dont le fibré cotangent est ample.

Les variétés X à fibré cotangent ample jouissent de plusieurs propriétés intéressantes:

- toutes les sous-variétés de X sont de type général;
- il y a nombre fini d'applications rationnelles non constantes d'une variété projective fixée vers X;
- si X est définie sur $\mathbb{C}$, alors X est hyperbolique complexe au sens de Kobayashi, i.e. chaque application holomorphe $\mathbb{C} \longrightarrow X$ doit être constante.

De plus, si X est définie sur un corps de nombres K, alors l'ensemble des points K-rationnels de X est conjecturé par Lang avoir un cardinal fini.

Dans le cas de la dimension 1, nous avons une compréhension complète des variétés à fibré cotangent ample: ce sont les courbes algébriques lisses de genre $\geqslant 2$. Cependant, en dimension supérieure, peu d'exemples étaient connus jusqu'à très récemment, même s'ils étaient censés être raisonnablement abondants.

Dans cet objectif, Debarre a conjecturé que l'intersection de $c \geqslant N/2$ hypersurfaces génériques de degrés élevés dans $\mathbb{P}^N_{\mathbb{C}}$ devrait avoir un fibré cotangent ample.

Inspiré par les travaux de Debarre et Brotbek, dans cette thèse, nous établissons la Conjecture d'amplitude de Debarre en toute généralité, et notamment, nous obtenons une borne effective inférieure sur les degrés des hypersurfaces.

Théorème Principal. *Le fibré cotangent Ω_X d'une intersection $X := H_1 \cap \cdots \cap H_c \subset \mathbb{P}^N_{\mathbb{C}}$ de $c \geqslant N/2$ hypersurfaces génériques $H_1, \ldots, H_c$ de degrés élevés $d_1, \ldots, d_c \gg N^{N^2}$ est ample.*

En fait, au lieu de nous contenter de , nous allons montrer que ce résultat est valide pour chaque corps algébriquement clos, sans restriction sur sa caractéristique.

Esquissons maintenant la preuve.

Tout d'abord, nous élaborons une interprétation géométrique des différentielles symétriques sur les espaces projectifs. De cette manière, nous reconstruisons les différentielles symétriques de Brotbek sur X, lorsque les équations définissantes des hypersurfaces $H_1, \ldots, H_c$ sont de type Fermat généralisé. De plus, nous dévoilons des familles nouvelles de différentielles symétriques de degré inférieur sur toutes les intersections possibles de X avec des hyperplans de coordonnées.

Ensuite, nous introduisons ce que nous appelons la '*Méthode des Coefficients Mobiles*' ainsi que le '*Coup du Produit*' afin d'accomplir une démonstration de la conjecture d'amplitude de Debarre. De plus, nous obtenons une borne effective inférieure sur les degrés: $d_1, \ldots, d_c \geqslant N^{N^2}$. Enfin, grâce à des résultats connus au sujet de la conjecture de Fujita, nous établissons que $\mathsf{Sym}^\kappa \Omega_X$ est très ample pour tout $\kappa \geqslant 64 \left(\sum_{i=1}^{c} d_i \right)^2$.

Summary

In the first part of this thesis, we establish the *Debarre Ampleness Conjecture: The cotangent bundle* Ω_X *of the intersection* $X = H_1 \cap \cdots \cap H_c$ *of* $c \geqslant N/2$ *generic hypersurfaces* $H_i \subset \mathbb{P}^N_{\mathbb{C}}$ *of high degrees* $d_1, \ldots, d_c \gg 1$ *is ample*.

First of all, we provide a geometric interpretation of symmetric differential forms in projective spaces. Thereby, we reconstruct Brotbek's symmetric differential forms on X, where the defining hypersurfaces $H_1, \ldots, H_c$ are generalized Fermat-type. Moreover, we exhibit unveiled families of lower degree symmetric differential forms on all possible intersections of X with coordinate hyperplanes.

Thereafter, we introduce what we call the '*moving coefficients method*' and the '*product coup*' to settle the Debarre Ampleness Conjecture. In addition, we obtain an effective lower degree bound: $d_1, \ldots, d_c \geqslant N^{N^2}$. Lastly, thanks to known results about the Fujita Conjecture, we establish the very-ampleness of $\mathsf{Sym}^{\kappa}\, \Omega_X$ for all $\kappa \geqslant 64 \left(\sum_{i=1}^{c} d_i \right)^2$.

In the second part, we study the *General Debarre Ampleness Conjecture*, which stipulates that, over an algebraically closed field $\mathbb{K}$ with any characteristic, on an N-dimensional smooth projective $\mathbb{K}$-variety $\mathbf{P}$ equipped with $c \geqslant N/2$ very ample line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_c$, for all large degrees $d_1, \ldots, d_c \geqslant \mathsf{d} \gg 1$, for generic c hypersurfaces $H_1 \in \left| \mathscr{L}_1^{\otimes d_1} \right|, \ldots, H_c \in \left| \mathscr{L}_c^{\otimes d_c} \right|$, the complete intersection $X := H_1 \cap \cdots \cap H_c$ has ample cotangent bundle Ω_X.

On such an intersection variety X, we construct what we call '*generalized Brotbek's symmetric differential forms*', and we establish that, if $\mathscr{L}_1, \ldots, \mathscr{L}_c$ are *almost proportional* mutually, then the General Debarre Ampleness Conjecture holds true. Our method is effective, and in the case where $\mathscr{L}_1 = \cdots = \mathscr{L}_c$, we obtain the same lower degree bound $\mathsf{d} = N^{N^2}$ as in the first part.

These two works have been posted on arxiv.org.

Keywords. Debarre Ampleness Conjecture, Complete intersection, Cotangent bundle, Symmetric differential form, Moving Coefficients Method, Product coup, Cramer's rule, Base loci, Fibre dimension estimate, Core Lemma, Gaussian elimination, Formal matrices, Determinants, Complex hyperbolicity, Fujita Conjecture.

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In the end, I would like to dedicate this thesis to my parents for their endless love, support, and encouragement. Also, I would like to thank my girl (friend) Tongnuo for her kindness. It was in her pleasant company that I suddenly caught the inspiration of 'hidden' symmetric differential forms central to my thesis, and a wealth of other precious ideas.

ON THE AMPLENESS OF THE COTANGENT BUNDLES OF COMPLETE INTERSECTIONS

SONG-YAN XIE

Abstract. Based on a geometric interpretation of Brotbek's symmetric differential forms, for the intersection family $\mathscr{X}$ of general Fermat-type hypersurfaces in $\mathbb{P}^N_{\mathbb{K}}$ defined over any field $\mathbb{K}$, we construct/reconstruct explicit symmetric differential forms by applying Cramer's rule, skipping cohomology arguments, and we further exhibit unveiled families of lower degree symmetric differential forms on all possible intersections of $\mathscr{X}$ with coordinate hyperplanes.

Thereafter, we develop what we call the '*moving coefficients method*' to prove a conjecture made by Olivier Debarre: *for a generic choice of* $c \geqslant N/2$ *hypersurfaces* $H_1, \ldots, H_c \subset \mathbb{P}^N_{\mathbb{C}}$ *of degrees* $d_1, \ldots, d_c$ *sufficiently large, the intersection* $X := H_1 \cap \cdots \cap H_c$ *has ample cotangent bundle* Ω_X, and concerning effectiveness, the lower bound $d_1, \ldots, d_c \geqslant N^{N^2}$ works.

Lastly, thanks to known results about the Fujita Conjecture, we establish the very-ampleness of $\mathsf{Sym}^\kappa\, \Omega_X$ for all $\kappa \geqslant \kappa_0$, with a uniform lower bound $\kappa_0 = 64\left(\sum_{i=1}^c d_i\right)^2$ independent of X.

1. Introduction

In 2005, Debarre established that, in a complex abelian variety of dimension N, for $c \geqslant N/2$ sufficiently ample generic hypersurfaces $H_1, \ldots, H_c$, their intersection $X := H_1 \cap \cdots \cap H_c$ has ample cotangent bundle Ω_X, thereby answering a question of Lazarsfeld (cf. [18]). Then naturally, by thoughtful analogies between geometry of Abelian varieties and geometry of projective spaces, Debarre proposed the following conjecture in Section 3 of [18], extending in fact an older question raised by Schneider [58] in the surface case:

Conjecture 1.1. [Debarre Ampleness Conjecture] For all integers $N \geqslant 2$, for every integer $N/2 \leqslant c < N$, there exists a positive lower bound:

$$d \gg 1$$

such that, for all positive integers:

$$d_1, \ldots, d_c \geqslant d,$$

for generic choices of c hypersurfaces:

$$H_i \subset \mathbb{P}^N_{\mathbb{C}} \qquad (i = 1 \cdots c)$$

with degrees:

$$\deg H_i = d_i,$$

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Key words and phrases. Complex hyperbolicity, Fujita Conjecture, Debarre Ampleness Conjecture, Generic, Complete intersection, Cotangent bundle, Cramer's rule, Symmetric differential form, Moving Coefficients Method, Base loci, Fibre dimension estimate, Core Lemma, Gaussian elimination.

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the intersection:

$$X := H_1 \cap \cdots \cap H_c$$

has ample cotangent bundle Ω_X.

Precisely, according to a ground conceptualization due to Hartshorne [36], the expected ampleness is that, for all large degrees $k \geqslant \mathtt{k}_0 \gg 1$, the global symmetric k-differentials on X:

$$\Gamma\,(X,\,\mathsf{Sym}^k\,\Omega_X)$$

are so abundant and diverse, that firstly, at every point $x \in X$, the first-order jet evaluation map:

$$\Gamma\,(X,\,\mathsf{Sym}^k\,\Omega_X)\;\twoheadrightarrow\;\mathsf{Jet}_1\,\mathsf{Sym}^k\,\Omega_X\big|_x$$

is surjective, where for every vector bundle $E \to X$ the first-order jet of E at x is defined by:

$$\mathsf{Jet}_1\,E\big|_x\;:=\;\mathscr{O}_x(E)\Big/(\mathfrak{m}_x)^2\,\mathscr{O}_x(E),$$

and that secondly, at every pair of distinct points $x_1 \neq x_2$ in X, the simultaneous evaluation map:

$$\Gamma\,(X,\mathsf{Sym}^k\,\Omega_X)\;\twoheadrightarrow\;\mathsf{Sym}^k\,\Omega_X\big|_{x_1}\;\oplus\;\mathsf{Sym}^k\,\Omega_X\big|_{x_2}$$

is also surjective.

The hypothesis:

$$c\;\geqslant\;n$$

appears *optimal*, for otherwise when $c < n$, there are no nonzero global sections for all degrees $k \geqslant 1$:

$$\Gamma\,(X,\mathsf{Sym}^k\,\Omega_X)\;=\;0,$$

according to Brückmann-Rackwitz [8] and Schneider [58], whereas, in the threshold case $c = n$, nonzero global sections are known to exist.

As highlighted in [18], projective varieties X having ample cotangent bundles enjoy several fascinating properties, for instance the following ones.

- All subvarieties of X are all of general type.
- There are finitely many nonconstant rational maps from any fixed projective variety to X ([51]).
- If X is defined over $\mathbb{C}$, then X is Kobayashi-hyperbolic, i.e. every holomorphic map $\mathbb{C} \to X$ must be constant ([20, p. 16, Proposition 3.1], [27, p. 52, Proposition 4.2.1]).
- If X is defined over a number field K, the set of K-rational points of X is expected to be finite (Lang's conjecture, cf. [38], [48]).

Since ampleness of cotangent bundles potentially bridges Analytic Geometry and Arithmetic Geometry in a deep way, it is interesting to ask examples of such projective varieties. In one-dimensional case, they are in fact our familar Riemann surfaces/algebraic curves with genus $\geqslant 2$. However, in higher dimensional case, not many examples were known, even though they were expected to be reasonable abundant.

In this aspect, we would like to mention the following nice construction of Bogomolov, which is written down in the last section of [18]. If $X_1, \ldots, X_\ell$ are smooth complex projective varieties having positive dimensions:

$$\dim X_i\;\geqslant\;d\;\geqslant\;1\qquad\qquad{\scriptstyle(i\,=\,1\,\cdots\,\ell)},$$

all of whose Serre line bundles $\mathscr{O}_{\mathbb{P}(\mathrm{T}_{X_i})}(1) \to \mathbb{P}(\mathrm{T}_{X_i})$ enjoy bigness:

$$\mathsf{dim}\,\Gamma\big(\mathbb{P}(\mathrm{T}_{X_i}),\, \mathscr{O}_{\mathbb{P}(\mathrm{T}_{X_i})}(k)\big) \,=\, \mathsf{dim}\,\Gamma\big(X_i,\, \mathsf{Sym}^k\Omega_{X_i}\big) \underset{k\to\infty}{\geqslant} \underbrace{\mathsf{constant}}_{>0} \cdot k^{2\,\dim X_i - 1},$$

then a generic complete intersection:

$$Y \,\subset\, X_1 \times \cdots \times X_\ell$$

having dimension:

$$\mathsf{dim}\, Y \,\leqslant\, \frac{d\,(\ell+1)+1}{2\,(d+1)}$$

has ample cotangent bundle Ω_Y.

In his Ph.D. thesis under the direction of Mourougane, Brotbek [5] reached an elegant proof of the Debarre Ampleness Conjecture in dimension $n = 2$, in all codimensions $c \geqslant 2$, for generic complete intersections $X^2 \subset \mathbb{P}^{2+c}(\mathbb{C})$ having degrees:

$$d_1, \ldots, d_c \,\geqslant\, \frac{8\,(n+c)+2}{n+c-1},$$

by extending the techniques of Siu [62, 63, 64], Demailly [20, 23, 22], Rousseau [56], Păun [52, 53], Merker [43], Diverio-Merker-Rousseau [26], Mourougane [50], and by employing the concept of *ampleness modulo a subvariety* introduced by Miyaoka in [47]. Also, for smooth complete intersections $X^n \subset \mathbb{P}^{n+c}(\mathbb{C})$ with $c \geqslant n \geqslant 2$, Brotbek showed using holomorphic Morse inequalities that when:

$$d_1, \ldots, d_c \,\geqslant\, \left[2^{n-1}\,(2n-2)\,\frac{n^2}{n+c+1}\binom{2n-1}{n}+1\right]\binom{n}{\lfloor\frac{n}{2}\rfloor}\frac{(2n+c)!}{(n+c)!}\,\frac{(c-n)!}{c!},$$

bigness of the Serre line bundle $\mathscr{O}_{\mathbb{P}(\mathrm{T}_X)}(1) \to \mathbb{P}(\mathrm{T}_X)$ holds:

$$\begin{aligned}\mathsf{dim}\,\Gamma\big(\mathbb{P}(\mathrm{T}_X),\, \mathscr{O}_{\mathbb{P}(\mathrm{T}_X)}(k)\big) \,=\, \mathsf{dim}\,\Gamma\big(X,\, \mathsf{Sym}^k\,\Omega_X\big) &\underset{k\to\infty}{\geqslant} \tfrac{1}{2}\chi_{\mathsf{Euler}}(X,\, \mathsf{Sym}^k\,\Omega_X)\\ &\underset{k\to\infty}{\geqslant} \underbrace{\mathsf{constant}}_{>0}\cdot k^{2\,n-1},\end{aligned}$$

whereas a desirable control of the base locus of the inexplicitly given nonzero holomorphic sections seems impossible by means of currently available techniques.

To find an alternative approach, a key breakthrough happened in 2014, when Brotbek [7] obtained explicit global symmetric differential forms in coordinates by an intensive cohomological approach. More specifically, under the assumption that the ambient field $\mathbb{K}$ has characteristic zero, using exact sequences and the snake lemma, Brotbek firstly provided a key series of long injective cohomology sequences, whose left initial ends consist of the most general global twisted symmetric differential forms, and whose right target ends consist of huge dimensional linear spaces well understood. Secondly, Brotbek proved that the image of each left end, going through the full injections sequence, is exactly the kernel of a certain linear system at the right end. Thirdly, by focusing on pure Fermat-type hypersurface equations ([7, p. 26]):

$$F_j \,=\, \sum_{i=0}^{N} s_i^j\, Z_i^e \qquad (j = 1 \cdots c), \tag{1}$$

with integers $c \geqslant N/2$, $e \geqslant 1$, where s_i^j are some homogeneous polynomials of the same degree $\epsilon \geqslant 0$, Brotbek step-by-step traced back some kernel elements from each right end all the way to

the left end, every middle step being an application of Cramer's rule, and hence he constructed global twisted symmetric differential forms with neat determinantal structures ([7, p. 27–31]).

Thereafter, by employing the standard method of counting base-locus-dimension in two ways in algebraic geometry (see e.g. Lemma 8.15 below), Brotbek established that the Debarre Ampleness Conjecture holds when:

$$4c \geqslant 3N - 2,$$

for equal degrees:

$$d_1 = \cdots = d_c \geqslant 2N + 3, \tag{2}$$

the constructions being flexible enough to embrace 'approximately equal degrees', in the same sense as Theorem 5.2 below.

Inspired much by Brotbek's works, we propose the following answer to the Debarre Ampleness Conjecture.

Theorem 1.2. *The cotangent bundle of the intersection in $\mathbb{P}^N_{\mathbb{C}}$ of at least $N/2$ generic hypersurfaces with degrees $\geqslant N^{N^2}$ is ample.*

In fact, we will prove the following main theorem, which coincides with the above theorem for $r = 0$ and $\mathbb{K} = \mathbb{C}$, and whose effective bound $\mathfrak{d}_0 = N^{N^2}$ will be obtained in Theorem 11.2.

Theorem 1.3 (Ampleness). *Over any field $\mathbb{K}$ which is not finite, for all positive integers $N \geqslant 1$, for any nonnegative integers $c, r \geqslant 0$ with:*

$$2c + r \geqslant N,$$

there exists a lower bound $\mathfrak{d}_0 \gg 1$ such that, for all positive integers:

$$d_1, \ldots, d_c, d_{c+1}, \ldots, d_{c+r} \geqslant \mathfrak{d}_0,$$

for generic choices of $c + r$ hypersurfaces:

$$H_i \subset \mathbb{P}^N_{\mathbb{K}} \qquad (i = 1 \cdots c+r)$$

with degrees:

$$\deg H_i = d_i,$$

the cotangent bundle Ω_V of the intersection of the first c hypersurfaces:

$$V := H_1 \cap \cdots \cap H_c$$

restricted to the intersection of all the $c + r$ hypersurfaces:

$$X := H_1 \cap \cdots \cap H_c \cap H_{c+1} \cap \cdots \cap H_{c+r}$$

is ample.

First of all, remembering that *ampleness (or not) is preserved under any base change obtained by ambient field extension*, one only needs to prove the Ampleness Theorem 1.3 for algebraically closed fields $\mathbb{K}$.

Of course, we would like to have $\mathfrak{d}_0 = \mathfrak{d}_0(N, c, r)$ as small as possible, yet the optimal one is at present far beyond our reach, and we can only get exponential ones like:

$$\mathfrak{d}_0 = N^{N^2} \qquad \text{(Theorem 11.2)},$$

which confirms the large degree phenomena in Kobayashi hyperbolicity related problems ([26, 2, 22, 16, 64]). When $2\,(2c + r) \geqslant 3N - 2$, we obtain linear bounds for equal degrees:

$$d_1 = \cdots = d_{c+r} \ \geqslant \ 2N + 3,$$

hence we recover the lower bounds (2) in the case $r = 0$, and we also obtain quadratic bounds for all large degrees:

$$d_1, \ldots, d_{c+r} \ \geqslant \ (3N + 2)(3N + 3).$$

Better estimates of the lower bound $\mathfrak{d}_0$ will be explained in Section 12.

Concerning the proof, primarily, as anticipated/emphasized by Brotbek and Merker ([7, 44]), it is essentially based on constructing sufficiently many global negatively twisted symmetric differential forms, and then inevitably, one has to struggle with the overwhelming difficulty of clearing out their base loci, which seems, at the best of our knowledge, to be an incredible mission.

In order to bypass the complexity in these two aspects, the following seven ingredients are indispensable in our approach:

① generalized Brotbek's symmetric differential forms (Subsection 6.10);

② global *moving coefficients method* (MCM) (Subsection 7.2);

③ 'hidden' symmetric forms on intersections with coordinate hyperplanes (Subsection 6.4);

④ MCM on intersections with coordinate hyperplanes (Subsection 7.3);

⑤ Algorithm of MCM (Subsection 7.1);

⑥ Core Lemma of MCM (Section 10);

⑦ product coup (Subsection 5.3).

In fact, ① is based on a geometric interpretation of Brotbek's symmetric differential forms ([7, Lemma 4.5]), and has the advantage of producing symmetric differential forms by directly copying hypersurface equations and their differentials. Facilitated by ②, which is of certain combinatorial interest, ① amazingly cooks a series of global negatively twisted symmetric differential forms, which are of nice uniform structures. However, unfortunately, one still has the difficulty that all these obtained global symmetric forms happen to coincide with each other on the intersections with any two coordinate hyperplanes, so that their base locus stably keeps positive (large) dimension, which is an annoying obstacle to ampleness.

Then, to overcome this difficulty enters ③, which is arguably the most critical ingredient in harmony with MCM, and whose importance is much greater than its appearance as somehow a corollary of ①. Thus, to compensate the defect of ①-②, it is natural to design ④ which completes the framework of MCM. And then, ⑤ is smooth to be devised, and it provides suitable hypersurface equations for MCM. Now, the last obstacle to amplness is about narrowing the base loci, an ultimate difficulty solved by ⑥. Thus, the Debarre Conjecture is settled in the central cases of almost equal degrees. Finally, the magical coup ⑦ thereby embraces all large degrees for the Debarre Conjecture, and naturally shapes the formulation of the Ampleness Theorem.

Lastly, taking account of known results about the Fujita Conjecture in Complex Geometry (cf. survey [21]), we will prove in Section 13 the following

Theorem 1.4 (Effective Very Ampleness). *Under the same assumption and notation as in the Ampleness Theorem 1.3, if in addition the ambient field $\mathbb{K}$ has characteristic zero, then for generic*

choices of $H_1, \ldots, H_{c+r}$, the restricted cotangent bundle $\mathsf{Sym}^\kappa \Omega_V\big|_X$ is very ample on X, for every $\kappa \geqslant \kappa_0$, with the uniform lower bound:

$$\kappa_0 \,=\, 16\Big(\sum_{i=1}^{c} d_i + \sum_{i=1}^{c+r} d_i\Big)^2.$$

In the end, we would like to propose the following

Conjecture 1.5. **(i)** Over an algebraically closed field $\mathbb{K}$, for any smooth projective $\mathbb{K}$-variety $\mathbf{P}$ with dimension N, for any integers $c, r \geqslant 0$ with $2c + r \geqslant N$, for any very ample line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_{c+r}$ on $\mathbf{P}$, there exists a lower bound:

$$\mathsf{d}_0 \,=\, \mathsf{d}_0\,(\mathbf{P}, \mathscr{L}_\bullet) \,\gg\, 1$$

such that, for all positive integers:

$$d_1, \ldots, d_c, d_{c+1}, \ldots, d_{c+r} \,\geqslant\, \mathsf{d}_0,$$

for generic choices of $c + r$ hypersurfaces:

$$H_i \,\subset\, \mathbf{P} \qquad {\scriptstyle (i\,=\,1\,\cdots\,c+r)}$$

defined by global sections:

$$F_i \,\in\, \mathsf{H}^0\,(\mathbf{P},\, \mathscr{L}_i^{\otimes d_i}),$$

the cotangent bundle Ω_V of the intersection of the first c hypersurfaces:

$$V := H_1 \cap \cdots \cap H_c$$

restricted to the intersection of all the $c + r$ hypersurfaces:

$$X := H_1 \cap \cdots \cap H_c \cap H_{c+1} \cap \cdots \cap H_{c+r}$$

is ample.

(ii) There exists a uniform lower bound:

$$\mathsf{d}_0 \,=\, \mathsf{d}_0\,(\mathbf{P}) \,\gg\, 1$$

independent of the chosen very ample line bundles $\mathscr{L}_\bullet$.

(iii) There exists a uniform lower bound:

$$\kappa_0 \,=\, \kappa_0\,(\mathbf{P}) \,\gg\, 1$$

independent of $d_1, \ldots, d_{c+r}$, such that for generic choices of $H_1, \ldots, H_{c+r}$, the restricted cotangent bundle $\mathsf{Sym}^\kappa \Omega_V\big|_X$ is very ample on X, for every $\kappa \geqslant \kappa_0$.

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2. **Preliminaries and Restatements of the Ampleness Theorem 1.3**

2.1. **Two families of hypersurface intersections in $\mathbb{P}^N_{\mathbb{K}}$.** Fix an arbitrary algebraically closed field $\mathbb{K}$. Now, we introduce the fundamental object of this paper: the intersection family of $c+r$ hypersurfaces with degrees $d_1, \ldots, d_{c+r} \geqslant 1$ in the $\mathbb{K}$-projective space $\mathbb{P}^N_{\mathbb{K}}$ of dimension N, equipped with homogeneous coordinates $[z_0 : z_1 : \cdots : z_N]$.

Recalling that the projective parameter space of degree $d \geqslant 1$ hypersurfaces in $\mathbb{P}^N_{\mathbb{K}}$ is:

$$\mathbb{P}\Big(\underbrace{H^0(\mathbb{P}^N_{\mathbb{K}}, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(d))}_{\dim_{\mathbb{K}}=\binom{N+d}{N}}\Big) \;=\; \mathbb{P}\Big\{\sum_{|\alpha|=d} A_\alpha\, z^\alpha \colon A_\alpha \in \mathbb{K}\Big\},$$

we may denote by:

$$\mathbb{P}\Big(\underbrace{\oplus_{i=1}^{c+r} H^0(\mathbb{P}^N_{\mathbb{K}}, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(d_i))}_{\dim_{\mathbb{K}}=\sum_{i=1}^{c+r}\binom{N+d_i}{N}}\Big) = \mathbb{P}\Big\{\oplus_{i=1}^{c+r} \sum_{|\alpha|=d_i} A^i_\alpha\, z^\alpha \;:\; A^i_\alpha \in \mathbb{K}\Big\}$$

the projective parameter space of $c+r$ hypersurfaces with degrees $d_1, \ldots, d_{c+r}$. This $\mathbb{K}$-projective space has dimension:

$$\diamond := \sum_{i=1}^{c+r} \binom{N+d_i}{N} - 1, \tag{3}$$

hence we write it as:

$$\mathbb{P}^\diamond_{\mathbb{K}} = \operatorname{Proj} \mathbb{K}\big[\{A^i_\alpha\}_{\substack{1\leqslant i\leqslant c+r\\ |\alpha|=d_i}}\big], \tag{4}$$

where, as shown above, A^i_α are the homogeneous coordinates indexed by the serial number i of each hypersurface and by all multi-indices α with the weight $|\alpha| = d_i$ associated to the degree d_i monomials $z^\alpha \in \mathbb{K}[z_0, \ldots, z_N]$.

Now, we introduce the two subschemes:

$$\mathscr{X} \;\subset\; \mathscr{V} \;\subset\; \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}},$$

where $\mathscr{X}$ is defined by 'all' the $c+r$ bihomogeneous polynomials:

$$\mathscr{X} := \mathrm{V}\Big(\sum_{|\alpha|=d_1} A^1_\alpha\, z^\alpha, \ldots, \sum_{|\alpha|=d_c} A^c_\alpha\, z^\alpha, \sum_{|\alpha|=d_{c+1}} A^{c+1}_\alpha\, z^\alpha, \ldots, \sum_{|\alpha|=d_{c+r}} A^{c+r}_\alpha\, z^\alpha\Big), \tag{5}$$

and where $\mathscr{V}$ is defined by the 'first' c bihomogeneous polynomials:

$$\mathscr{V} := \mathrm{V}\Big(\sum_{|\alpha|=d_1} A^1_\alpha\, z^\alpha, \ldots, \sum_{|\alpha|=d_c} A^c_\alpha\, z^\alpha\Big). \tag{6}$$

Then we view $\mathscr{X}, \mathscr{V} \subset \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}}$ as two families of closed subschemes of $\mathbb{P}^N_{\mathbb{K}}$ parametrized by the projective parameter space $\mathbb{P}^\diamond_{\mathbb{K}}$.

2.2. **The relative cotangent sheaves family of $\mathscr{V}$.** A comprehensive reference on sheaves of relative differentials is [42, Section 6.1.2].

Let $\mathrm{pr}_1, \mathrm{pr}_2$ be the two canonical projections:

$$\begin{array}{ccc} & \mathbb{P}^{\diamond}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^{N}_{\mathbb{K}} & \\ {\scriptstyle \mathrm{pr}_1}\swarrow & & \searrow{\scriptstyle \mathrm{pr}_2} \\ \mathbb{P}^{\diamond}_{\mathbb{K}} & & \mathbb{P}^{N}_{\mathbb{K}}. \end{array} \tag{7}$$

Then, by composing with the subscheme inclusion:

$$i\colon \quad \mathscr{V} \hookrightarrow \mathbb{P}^{\diamond}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^{N}_{\mathbb{K}},$$

we receive a morphism:

$$\mathrm{pr}_1 \circ i\colon \quad \mathscr{V} \longrightarrow \mathbb{P}^{\diamond}_{\mathbb{K}},$$

together with a sheaf $\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamond}_{\mathbb{K}}}$ of relative differentials of degree 1 of $\mathscr{V}$ over $\mathbb{P}^{\diamond}_{\mathbb{K}}$.

Since pr_1 is of finite type and $\mathbb{P}^{\diamond}_{\mathbb{K}}$ is noetherian, a standard theorem ([42, p. 216, Proposition 1.20]) shows that the sheaf $\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamond}_{\mathbb{K}}}$ is coherent.

We may view $\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamond}_{\mathbb{K}}}$ as the family of the cotangent bundles for the intersection family $\mathscr{V}$, since the coherent sheaf $\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamond}_{\mathbb{K}}}$ is indeed locally free on the Zariski open set that consists of smooth complete intersections.

2.3. **The projectivizations and the Serre line bundles.** We refer the reader to [37, pp. 160-162] for the considerations in this subsection.

Starting with the noetherian scheme $\mathscr{V}$ and the coherent degree 1 relative differential sheaf $\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamond}_{\mathbb{K}}}$ on it, we consider the sheaf of relative $\mathscr{O}_{\mathscr{V}}$-symmetric differential algebras:

$$\mathbf{Sym}^{\bullet}\, \Omega^1_{\mathscr{V}/\mathbb{P}^{\diamond}_{\mathbb{K}}} := \bigoplus_{i \geqslant 0} \mathsf{Sym}^{i}\, \Omega^1_{\mathscr{V}/\mathbb{P}^{\diamond}_{\mathbb{K}}}.$$

According to the construction of [37, p. 160], noting that this sheaf has a natural structure of graded $\mathscr{O}_{\mathscr{V}}$-algebras, and moreover that it satisfies the condition (†) there, we receive the projectivization of $\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamond}_{\mathbb{K}}}$:

$$\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamond}_{\mathbb{K}}}) := \mathbf{Proj}\,(\mathbf{Sym}^{\bullet}\, \Omega^1_{\mathscr{V}/\mathbb{P}^{\diamond}_{\mathbb{K}}}). \tag{8}$$

As described in [37, p. 160], $\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamond}_{\mathbb{K}}})$ is naturally equipped with the so-called *Serre line bundle* $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamond}_{\mathbb{K}}})}(1)$ on it.

Similarly, replacing $\mathscr{V}$ by $\mathbb{P}^{\diamond}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^{N}_{\mathbb{K}}$, we obtain the relative differentials sheaf of $\mathbb{P}^{\diamond}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^{N}_{\mathbb{K}}$ with respect to pr_1 in (7):

$$\Omega^1_{\mathbb{P}^{\diamond}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^{N}_{\mathbb{K}}/\mathbb{P}^{\diamond}_{\mathbb{K}}} \;\cong\; \mathrm{pr}_2^{*}\, \Omega^1_{\mathbb{P}^{N}_{\mathbb{K}}},$$

and we thus obtain its projectivization:

$$\mathbf{P}(\Omega^1_{\mathbb{P}^{\diamond}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^{N}_{\mathbb{K}}/\mathbb{P}^{\diamond}_{\mathbb{K}}}) \;:=\; \mathbf{Proj}\,(\mathbf{Sym}^{\bullet}\, \Omega^1_{\mathbb{P}^{\diamond}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^{N}_{\mathbb{K}}/\mathbb{P}^{\diamond}_{\mathbb{K}}}) \;\cong\; \mathbb{P}^{\diamond}_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{Proj}\,(\mathbf{Sym}^{\bullet}\, \Omega^1_{\mathbb{P}^{N}_{\mathbb{K}}}). \tag{9}$$

We will abbreviate $\mathbf{Proj}\,(\mathbf{Sym}^{\bullet}\, \Omega^1_{\mathbb{P}^{N}_{\mathbb{K}}})$ as $\mathbf{P}(\Omega^1_{\mathbb{P}^{N}_{\mathbb{K}}})$, and denote its Serre line bundle by $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^{N}_{\mathbb{K}}})}(1)$. Then, the Serre line bundle $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^{\diamond}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^{N}_{\mathbb{K}}/\mathbb{P}^{\diamond}_{\mathbb{K}}})}(1)$ on the left hand side of (9) is nothing but the line

bundle $\widetilde{\pi}_2^*\,\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ on the right hand side, where $\widetilde{\pi}_2$ is the canonical projection:

$$\widetilde{\pi}_2\colon\quad \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) \to \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}). \tag{10}$$

Now, note that the commutative diagram:

$$\begin{array}{ccc} \mathscr{V} & \xhookrightarrow{\ i\ } & \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}} \\ & \searrow^{\mathrm{pr}_1\circ i} & \downarrow{\scriptstyle \mathrm{pr}_1} \\ & & \mathbb{P}^\diamond_{\mathbb{K}} \end{array}$$

induces the surjection (cf. [37, p. 176, Proposition 8.12]):

$$i^*\,\Omega^1_{\mathbb{P}^\diamond_{\mathbb{K}}\times_{\mathbb{K}}\mathbb{P}^N_{\mathbb{K}}/\mathbb{P}^\diamond_{\mathbb{K}}} \twoheadrightarrow \Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}},$$

and hence yields the surjection:

$$i^*\,\mathbf{Sym}^\bullet\,\Omega^1_{\mathbb{P}^\diamond_{\mathbb{K}}\times_{\mathbb{K}}\mathbb{P}^N_{\mathbb{K}}/\mathbb{P}^\diamond_{\mathbb{K}}} \twoheadrightarrow \mathbf{Sym}^\bullet\,\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}}.$$

Taking '**Proj**', thanks to (9), we obtain the commutative diagram:

$$\begin{array}{ccc} \mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}}) & \xhookrightarrow{\ \widetilde{i}\ } & \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) \\ \downarrow & & \downarrow \\ \mathscr{V} & \xhookrightarrow{\ i\ } & \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}}. \end{array} \tag{11}$$

Thus, the Serre line bundle $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}})}(1)$ becomes exactly the pull back of 'the Serre line bundle' $\widetilde{\pi}_2^*\,\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ under the inclusion $\widetilde{i}$:

$$\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}})}(1) \;=\; \widetilde{i}^*\big(\widetilde{\pi}_2^*\,\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)\big) \;=\; (\widetilde{\pi}_2\circ\widetilde{i})^*\,\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1). \tag{12}$$

2.4. **Restatement of Theorem 1.3.** Let $\widetilde{\pi}$ be the canonical projection:

$$\widetilde{\pi}\colon\quad \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) \to \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}},$$

and let π_1, π_2 be the compositions of $\widetilde{\pi}$ with $\mathrm{pr}_1, \mathrm{pr}_2$:

$$\begin{array}{ccccc} & & \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) & & \\ & \swarrow^{\pi_1:=\mathrm{pr}_1\circ\widetilde{\pi}} & \downarrow{\scriptstyle \widetilde{\pi}} & \searrow^{\pi_2:=\mathrm{pr}_2\circ\widetilde{\pi}} & \\ & & \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}} & & \\ & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \\ \mathbb{P}^\diamond_{\mathbb{K}} & & & & \mathbb{P}^N_{\mathbb{K}}. \end{array} \tag{13}$$

Let:

$$\mathbf{P} \;:=\; \widetilde{\pi}^{-1}(\mathscr{X}) \cap \mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}}) \;\subset\; \mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}}) \;\subset\; \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) \tag{14}$$

be 'the pullback' of:

$$\mathscr{X} \;\subset\; \mathscr{V} \;\subset\; \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}}$$

under the map $\widetilde{\pi}$, and let:

$$\mathscr{O}_{\mathbf{P}}(1) \;:=\; \mathscr{O}_{\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}})}(1)\Big|_{\mathbf{P}} \;=\; \widetilde{\pi}_2^*\,\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)\Big|_{\mathbf{P}} \qquad \text{[see (12)]} \tag{15}$$

be the restricted Serre line bundle.

Now, we may view $\mathbf{P}$ as a family of subschemes of $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$ parametrized by the projective parameter space $\mathbb{P}^\diamond_{\mathbb{K}}$ under the restricted map:

$$\pi_1 :\quad \mathbf{P} \longrightarrow \mathbb{P}^\diamond_{\mathbb{K}}. \tag{16}$$

Thus Theorem 1.3 can be reformulated as below, with the assumption that the hypersurface degrees $d_1, \ldots, d_{c+r}$ are sufficiently large:

$$d_1, \ldots, d_{c+r} \gg 1.$$

Theorem 1.3 (Version A). *For a generic point $t \in \mathbb{P}^\diamond_{\mathbb{K}}$, over the fibre:*

$$\mathbf{P}_t := \pi_1^{-1}(t) \cap \mathbf{P},$$

the restricted Serre line bundle:

$$\mathscr{O}_{\mathbf{P}_t}(1) := \mathscr{O}_{\mathbf{P}}(1)\Big|_{\mathbf{P}_t} \tag{17}$$

is ample.

From now on, every closed point:

$$t = \Big[\{A^i_\alpha\}_{\substack{1\leqslant i\leqslant c+r\\ |\alpha|=d}}\Big] \;\in\; \mathbb{P}^\diamond_{\mathbb{K}}$$

will be abbreviated as:

$$t \;=\; [F_1 \colon \cdots \colon F_{c+r}],$$

where:

$$F_i := \sum_{|\alpha|=d_i} A^i_\alpha\, z^\alpha \qquad (i = 1 \cdots c+r).$$

Then we have:

$$\mathbf{P}_t \;=\; \{t\} \times_{\mathbb{K}} {}_{F_{c+1},\ldots,F_{c+r}}\mathbf{P}_{F_1,\ldots,F_c},$$

for a uniquely defined subscheme:

$${}_{F_{c+1},\ldots,F_{c+r}}\mathbf{P}_{F_1,\ldots,F_c} \;\subset\; \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}). \tag{18}$$

Theorem 1.3 (Version B). *For a generic closed point:*

$$[F_1 \colon \cdots \colon F_{c+r}] \;\in\; \mathbb{P}^\diamond_{\mathbb{K}},$$

the Serre line bundle $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ is ample on ${}_{F_{c+1},\ldots,F_{c+r}}\mathbf{P}_{F_1,\ldots,F_c}$.

To have a better understanding of the above statements, we now investigate the geometry behind.

3. The background geometry

Since $\mathbb{K}$ is an algebraically closed field, throughout this section, we view each scheme in the classical sense (cf. [37, Chapter 1]), i.e. its underlying topological space ($\mathbb{K}$-variety) consists of all the closed points.

3.1. **The geometry of $\mathbb{P}^N_{\mathbb{K}}$ and $\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(1)$.** Recall that, the projective N-space $\mathbb{P}^N_{\mathbb{K}}$ is obtained by projectivizing the Euclidian $(N+1)$-space $\mathbb{K}^{N+1}$, i.e. is defined as the set of lines passing through the origin:

$$\mathbb{P}^N_{\mathbb{K}} := \mathbb{P}(\mathbb{K}^{N+1}) := \mathbb{K}^{N+1}\backslash\{0\} \ / \ \sim, \tag{19}$$

where the quotient relation $\sim$ for $z \in \mathbb{K}^{N+1}\backslash\{0\}$ is:

$$z \sim \lambda z \qquad (\forall\, \lambda \in \mathbb{K}^\times).$$

On $\mathbb{P}^N_{\mathbb{K}}$, there is the so-called *tautological line bundle* $\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-1)$, which at every point $[z] \in \mathbb{P}^N_{\mathbb{K}}$ has fibre:

$$\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-1)\Big|_{[z]} \ := \ \mathbb{K}\cdot z \ \subset \ \mathbb{K}^{N+1}.$$

Its dual line bundle is the well known:

$$\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(1) := \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-1)^\vee.$$

3.2. **The geometry of $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$ and $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)$.** For every point $[z] \in \mathbb{P}^N_{\mathbb{K}}$, the tangent space of $\mathbb{P}^N_{\mathbb{K}}$ at $[z]$ is:

$$\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}\Big|_{[z]} = \mathbb{K}^{N+1}/\mathbb{K}\cdot z,$$

and the total tangent space of $\mathbb{P}^N_{\mathbb{K}}$:

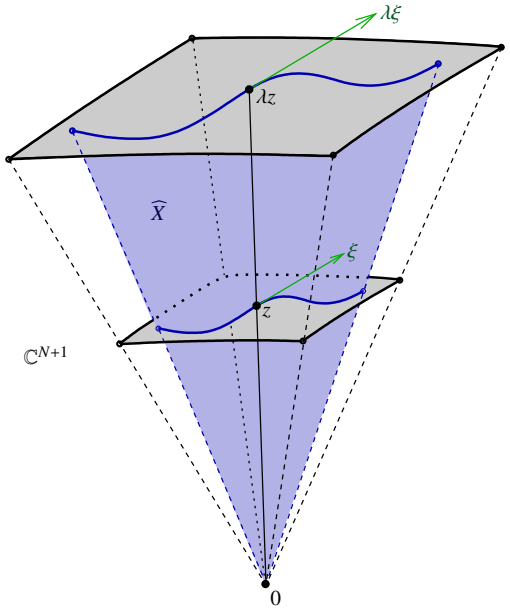
$$\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}} := \mathrm{T}_{\mathrm{hor}}\mathbb{K}^{N+1}/\sim, \tag{20}$$

is the quotient space of the horizontal tangent space of $\mathbb{K}^{N+1}\setminus\{0\}$:

$$\mathrm{T}_{\mathrm{hor}}\mathbb{K}^{N+1} := \Big\{(z,[\xi]) : \ z \in \mathbb{K}^{N+1}\setminus\{0\} \text{ and } [\xi] \in \mathbb{K}^{N+1}/\mathbb{K}\cdot z\Big\}, \tag{21}$$

by the quotient relation $\sim$:

$$(z,[\xi]) \sim (\lambda z, [\lambda\xi]) \qquad (\forall\, \lambda \in \mathbb{K}^\times).$$



Now, the $\mathbb{K}$-variety associated to $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$ is just the projectivized tangent space $\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$, which is obtained by projectivizing each tangent space $\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}\big|_{[z]}$ at every point $[z] \in \mathbb{P}^N_{\mathbb{K}}$:

$$\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})\big|_{[z]} := \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}\big|_{[z]}).$$

And the Serre line bundle $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ on $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$ corresponds to the 'Serre line bundle' $\mathscr{O}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ on $\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$, which after restricting on $\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})\big|_{[z]}$ becomes $\mathscr{O}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}|_{[z]}})}(1)$. In other words, the Serre line bundle $\mathscr{O}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ is the dual of the tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})}(-1)$, where the latter one, at every point $([z],[\xi]) \in \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$, has fibre:

$$\mathscr{O}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})}(-1)\big|_{([z],[\xi])} \;:=\; \mathbb{K}\cdot[\xi] \;\subset\; \mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}\big|_{[z]} = \mathbb{K}^{N+1}\,/\,\mathbb{K}\cdot z.$$

3.3. **The geometry of $\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}})$, $\mathbf{P}$ and $\mathbf{P}_t$.** Recalling (6), the $\mathbb{K}$-variety $\mathbb{V}$ associated to $\mathscr{V} \subset \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}}$ is:

$$\mathbb{V} \;:=\; \Big\{([F_1,\ldots,F_{c+r}],[z]) \in \mathbb{P}^\diamond_{\mathbb{K}} \times \mathbb{P}^N_{\mathbb{K}} \;:\; F_i(z)=0,\ \forall\, i=1\cdots c\Big\}.$$

Moreover, recalling (11), the $\mathbb{K}$-variety:

$$\mathbb{P}(\mathrm{T}_{\mathbb{V}/\mathbb{P}^\diamond_{\mathbb{K}}}) \subset \mathbb{P}^\diamond_{\mathbb{K}} \times \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$$

associated to $\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}}) \subset \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$ is:

$$\mathbb{P}(\mathrm{T}_{\mathbb{V}/\mathbb{P}^\diamond_{\mathbb{K}}}) \;:=\; \Big\{([F_1,\ldots,F_{c+r}],([z],[\xi])) \;:\; F_i(z)=0, dF_i\big|_z(\xi)=0, \forall\, i=1\cdots c\Big\}.$$

Similarly, the $\mathbb{K}$-variety:

$$\mathbb{P} \;\subset\; \mathbb{P}(\mathrm{T}_{\mathbb{V}/\mathbb{P}^\diamond_{\mathbb{K}}})$$

associated to $\mathbf{P} \subset \mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}})$ is:

$$\mathbb{P} \;:=\; \Big\{([F_1,\ldots,F_{c+r}],([z],[\xi])) \;:\; F_i(z)=0, dF_j\big|_z(\xi)=0, \forall\, i=1\cdots c+r, \forall\, j=1\cdots c\Big\},$$

and the $\mathbb{K}$-variety:

$$_{F_{c+1},\ldots,F_{c+r}}\mathbb{P}_{F_1,\ldots,F_c} \;\subset\; \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$$

associated to (18) is:

$$_{F_{c+1},\ldots,F_{c+r}}\mathbb{P}_{F_1,\ldots,F_c} \;:=\; \Big\{([z],[\xi]) \;:\; F_i(z)=0, dF_j\big|_z(\xi)=0, \forall\, i=1\cdots c+r, \forall\, j=1\cdots c\Big\}. \tag{22}$$

Now, the $\mathbb{K}$-variety $\mathbb{P}_t$ of $\mathbf{P}_t$ is:

$$\mathbb{P}_t \;:=\; \{t\} \times {}_{F_{c+1},\ldots,F_{c+r}}\mathbb{P}_{F_1,\ldots,F_c}.$$

4. **Some hints on the Ampleness Theorem 1.3**

The first three Subsections 4.1–4.3 consist of some standard knowledge in algebraic geometry, and the last Subsection 4.4 presents a helpful nefness criterion which suits our moving coefficients method.

4.1. **Ampleness is Zariski open.** The foundation of our approach is the following classical theorem of Grothendieck (cf. [33, III.4.7.1] or [40, p. 29, Theorem 1.2.17]).

Theorem 4.1. [Amplitude in families] *Let $f : X \to T$ be a proper morphism of schemes, and let $\mathscr{L}$ be a line bundle on X. For every point $t \in T$, denote by:*

$$X_t := f^{-1}(t), \quad \mathscr{L}_t := \mathscr{L}\big|_{X_t}.$$

Assume that, for some point $0 \in T$, $\mathscr{L}_0$ is ample on X_0 . Then in T, there is a Zariski open set U containing 0 such that $\mathscr{L}_t$ is ample on X_t, for all $t \in U$.

Note that in (13), $\pi_1 = \mathrm{pr}_1 \circ \pi$ is a composition of two proper morphisms, hence is proper, and so is (16). Therefore, by virtue of the above theorem, we only need to find one (closed) point $t \in \mathbb{P}^{\diamond}_{\mathbb{K}}$ such that:

$$\mathscr{O}_{\mathbf{P}_t}(1) \text{ is ample on } \mathbf{P}_t. \tag{23}$$

4.2. **Largely twisted Serre line bundle is ample.** Let:

$$\pi_0 \colon \quad \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) \to \mathbb{P}^N_{\mathbb{K}}$$

be the canonical projection. [37, p. 161, Proposition 7.10] yields that, for all sufficiently large integer ℓ [1], the twisted line bundle below is ample on $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$:

$$\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1) \otimes \pi_0^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell). \tag{24}$$

Recalling (10) and (13), and noting that:

$$\pi_2 \;=\; \pi_0 \circ \widetilde{\pi}_2, \tag{25}$$

for the following ample line bundle $\mathscr{H}$ on the scheme $\mathbb{P}^{\diamond}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}}$:

$$\mathscr{H} := \mathrm{pr}_1^* \mathscr{O}_{\mathbb{P}^{\diamond}_{\mathbb{K}}}(1) \otimes \mathrm{pr}_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(1),$$

the twisted line bundle below is ample on $\mathbb{P}^{\diamond}_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$:

$$\begin{aligned}
\widetilde{\pi}_2^* \mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1) \otimes \widetilde{\pi}^* \mathscr{H}^{\ell} &= \widetilde{\pi}_2^* \mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1) \otimes \widetilde{\pi}^* \Big({\mathrm{pr}_1}^* \mathscr{O}_{\mathbb{P}^{\diamond}_{\mathbb{K}}}(\ell) \otimes {\mathrm{pr}_2}^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell) \Big) \\
&= \widetilde{\pi}_2^* \mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1) \otimes (\mathrm{pr}_1 \circ \widetilde{\pi})^* \mathscr{O}_{\mathbb{P}^{\diamond}_{\mathbb{K}}}(\ell) \otimes (\mathrm{pr}_2 \circ \widetilde{\pi})^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell) \\
\text{[use (13)]} \qquad &= \widetilde{\pi}_2^* \mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1) \otimes \pi_1^* \mathscr{O}_{\mathbb{P}^{\diamond}_{\mathbb{K}}}(\ell) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell) \\
\text{[use (25)]} \qquad &= \widetilde{\pi}_2^* \mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1) \otimes \pi_1^* \mathscr{O}_{\mathbb{P}^{\diamond}_{\mathbb{K}}}(\ell) \otimes \widetilde{\pi}_2^* \left(\pi_0^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell) \right) \\
&= \pi_1^* \underbrace{\mathscr{O}_{\mathbb{P}^{\diamond}_{\mathbb{K}}}(\ell)}_{\text{ample}} \otimes \widetilde{\pi}_2^* \Big(\underbrace{\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1) \otimes \pi_0^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell)}_{\text{ample on } \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})} \Big).
\end{aligned} \tag{26}$$

In particular, for every point $t \in \mathbb{P}^{\diamond}_{\mathbb{K}}$, recalling (15), (17), restricting (26) to the subscheme:

$$\mathbf{P}_t \;=\; \pi_1^{-1}(t) \cap \mathbf{P},$$

[1] In fact, $\ell \geqslant 3$ is enough, see (222) below.

we receive an ample line bundle:

$$\mathscr{O}_{\mathbf{P}_t}(1) \otimes \underbrace{\pi_1^* \mathscr{O}_{\mathbb{P}^\diamond_{\mathbb{K}}}(\ell)}_{\text{trivial line bundle}} \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell) = \mathscr{O}_{\mathbf{P}_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell). \tag{27}$$

4.3. **Nefness of negatively twisted cotangent sheaf suffices.** As we mentioned at the end of Subsection 4.1, our goal is to show the existence of one such (closed) point $t \in \mathbb{P}^\diamond_{\mathbb{K}}$ satisfying (23). In fact, we can relax this requirement thanks to the following theorem.

Theorem 4.2. *For every point $t \in \mathbb{P}^\diamond_{\mathbb{K}}$, the following properties are equivalent.*

- **(i)** *$\mathscr{O}_{\mathbf{P}_t}(1)$ is ample on $\mathbf{P}_t$.*
- **(ii)** *There exist two positive integers $a, b \geqslant 1$ such that $\mathscr{O}_{\mathbf{P}_t}(a) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-b)$ is ample on $\mathbf{P}_t$.*
- **(iii)** *There exist two positive integers $a, b \geqslant 1$ such that $\mathscr{O}_{\mathbf{P}_t}(a) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-b)$ is nef on $\mathbf{P}_t$.*

Proof. It is clear that **(i)** $\Longrightarrow$ **(ii)** $\Longrightarrow$ **(iii)**, and we now show that **(iii)** $\Longrightarrow$ **(i)**.

In fact, the nefness of the negatively twisted Serre line bundle:

$$\mathscr{S}_t^a(-b) := \mathscr{O}_{\mathbf{P}_t}(a) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-b) \tag{28}$$

implies that:

$$\underbrace{(27)^{\otimes b}}_{\text{ample}} \otimes \underbrace{(28)^{\otimes \ell}}_{\text{nef}} = \underbrace{\mathscr{O}_{\mathbf{P}_t}(b + a\,\ell)}_{\text{ample !}} = \underbrace{\mathscr{O}_{\mathbf{P}_t}(1)}_{\text{ample}}{}^{\otimes (b + a\,\ell)}$$

is also ample, because of the well known fact that "ample $\otimes$ nef = ample" (cf. [40, p. 53, Corollary 1.4.10]). □

By definition, the nefness of (28) means that for every irreducible curve $C \subset \mathbf{P}_t$, the intersection number $C \cdot \mathscr{S}_t^a(-b)$ is $\geqslant 0$. Recalling now the classical result [37, p. 295, Lemma 1.2], we only need to show that the line bundle $\mathscr{S}_t^a(-b)$ has a nonzero section on the curve C:

$$H^0(C, \mathscr{S}_t^a(-b)) \neq \{0\}. \tag{29}$$

To this end, of course we like to construct sufficiently many global sections:

$$s_1, \ldots, s_m \in H^0(\mathbf{P}_t, \mathscr{S}_t^a(-b))$$

such that their base locus is empty or discrete, whence one of $s_1\big|_C, \ldots, s_m\big|_C$ suffices to conclude (29).

More flexibly, we have:

Theorem 4.3. *Suppose that there exist $m \geqslant 1$ nonzero sections of certain negatively twisted Serre line bundles:*

$$s_i \in H^0(\mathbf{P}_t, \mathscr{S}_t^{a_i}(-b_i)) \qquad (i = 1 \cdots m; a_i, b_i \geqslant 1)$$

such that their base locus is discrete or empty:

$$\dim \cap_{i=1}^m \mathsf{BS}(s_i) \leqslant 0,$$

then for all positive integers a, b with:

$$\frac{a}{b} \geqslant \max\left\{\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m}\right\},$$

the twisted Serre line bundle $\mathscr{S}_t^a(-b)$ is nef.

Proof. For every irreducible curve $C \subset \mathbf{P}_t$, noting that:

$$\underbrace{C}_{\dim = 1} \not\subset \underbrace{\cap_{i=1}^{m} \mathsf{BS}(s_i)}_{\dim \leqslant 0},$$

there exists some integer $1 \leqslant i \leqslant m$ such that:

$$C \not\subset \mathsf{BS}(s_i).$$

Therefore $s_i\big|_C$ is a nonzero section of $\mathscr{S}_t^{a_i}(-b_i)$ on the curve C:

$$s_i \in H^0(C, \mathscr{S}_t^{a_i}(-b_i)) \setminus \{0\},$$

and hence:

$$C \cdot \mathscr{S}_t^{a_i}(-b_i) \geqslant 0.$$

Thus we have the estimate:

$$\begin{aligned}
0 &\leqslant C \cdot (\mathscr{S}_t^{a_i}(-b_i))^{\otimes a} && [= a\, C \cdot \mathscr{S}_t^{a_i}(-b_i)\,] \\
&= C \cdot \mathscr{O}_{\mathbf{P_t}}(a_i\, a) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-b_i\, a) && [\text{see (28)}] \\
&= a_i\, C \cdot (\mathscr{O}_{\mathbf{P_t}}(a) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-b)) - (b_i\, a - a_i\, b)\; C \cdot \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(1) \\
&= a_i\; C \cdot (\mathscr{S}_t^{a}(-b)) - b\, b_i \underbrace{(a/b - a_i/b_i)}_{\geqslant 0}\; C \cdot \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(1).
\end{aligned}$$

Noting that $\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(1)$ is nef and hence is $\pi_2^*\, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(1)$ (cf. [40, p. 51, Example 1.4.4]), the above estimate immediately yields:

$$C \cdot (\mathscr{S}_t^{a}(-b)) \geqslant \frac{b\, b_i}{a_i} (a/b - a_i/b_i)\; \underbrace{C \cdot \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(1)}_{\geqslant 0} \geqslant 0. \qquad \square$$

Repeating the same reasoning as in the above two theorems, we obtain:

Proposition 4.4. *For every point $t \in \mathbb{P}^{\diamond}_{\mathbb{K}}$, if $\mathscr{O}_{\mathbf{P_t}}(\ell_1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-\ell_2)$ is nef on $\mathbf{P}_t$ for some positive integers $\ell_1, \ell_2 \geqslant 1$, then for any positive integers $\ell_1', \ell_2' \geqslant 1$ with $\ell_2'/\ell_1' < \ell_2/\ell_1$, the twisted line bundle $\mathscr{O}_{\mathbf{P_t}}(\ell_1') \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-\ell_2')$ is ample on $\mathbf{P}_t$.* $\square$

4.4. **A practical nefness criterion.** However, in practice, it is often difficult to gather enough global sections (with discrete base locus) to guarantee nefness of a line bundle. We need to be more clever to improve such a coarse nefness criterion with the help of nonzero sections of the same bundle restricted to proper subvarieties. First, let us introduce the theoretical reason behind.

Definition 4.5. Let X be a variety, and let $Y \subset X$ be a subvariety. A line bundle $\mathscr{L}$ on X is said to be *nef outside Y* if, for every irreducible curve $C \subset X$ with $C \not\subset Y$, the intersection number $C \cdot \mathscr{L} \geqslant 0$.

Of course, $\mathscr{L}$ is nef on X if and only if $\mathscr{L}$ is nef outside the empty set $\emptyset \subset X$.

Theorem 4.6 (Nefness Criterion). *Let X be a noetherian variety, and let $\mathscr{L}$ be a line bundle on X. Assume that there exists a set $\mathscr{V}$ of closed subvarieties of X satisfying:*

- **(i)** *$\emptyset \in \mathscr{V}$ and $X \in \mathscr{V}$;*
- **(ii)** *for every element $Y \in \mathscr{V}$ with $Y \neq \emptyset$, there exist finitely many elements $Z_1, \ldots, Z_\flat \in \mathscr{V}$ with $Z_1, \ldots, Z_\flat \subsetneq Y$ such that the restricted line bundle $\mathscr{L}\big|_Y$ is nef outside the union $Z_1 \cup \cdots \cup Z_\flat$.*

Then $\mathscr{L}$ is nef on X.

Proof. For every irreducible curve $C \subset X$, we have to show that $C \cdot \mathscr{L} \geqslant 0$.

Assume on the contrary that $C \cdot \mathscr{L} < 0$. Then introduce the subset $\mathscr{N} \subset \mathscr{V}$ consisting of all subvarieties $Y \in \mathscr{V}$ which contain the curve C. Clearly, $\mathscr{N} \ni X$, so $\mathscr{N}$ is nonempty. Note that there is a natural partial order '$<$' on $\mathscr{N}$ given by the strict inclusion relation '$\subsetneqq$'. Since X is noetherian, $\mathscr{N}$ has a minimum element $M \supset C$. We now show a contradiction.

In fact, according to **(ii)**, there exist some elements $\mathscr{V} \ni Z_1, \ldots, Z_\flat \subsetneqq M$ such that $\mathscr{L}\big|_M$ is nef outside $Z_1 \cup \cdots \cup Z_\flat$. Rembering that:

$$0 > C \cdot \mathscr{L} = C \cdot \mathscr{L}\big|_M,$$

the curve C is forced to lie in the union $Z_1 \cup \cdots \cup Z_\flat$, and thanks to irreducibility, it is furthermore contained in one certain:

$$\underbrace{Z_i}_{\subsetneqq M} \in \mathscr{N},$$

which contradicts the minimality of M! □

Now, using the same idea as around (29), we may realize **(ii)** above with the help of sections over proper subvarieties.

Corollary 4.7. *Let X be a noetherian variety, and let $\mathscr{L}$ be a line bundle on X. Assume that there exists a set $\mathscr{V}$ of closed subvarieties of X satisfying:*

- **(i)** *$\emptyset \in \mathscr{V}$ and $X \in \mathscr{V}$;*
- **(ii')** *every element $\emptyset \neq Y \in \mathscr{V}$ is a union of some elements $Y_1, \ldots, Y_\natural \in \mathscr{V}$ such that the union of base loci:*

$$\cup_{\bullet=1}^{\natural} \Big(\cap_{s \in H^0(Y_\bullet, \mathscr{L}|_{Y_\bullet})} \{s = 0\} \Big)$$

is contained in a union of some elements $\mathscr{V} \ni Z_1, \ldots, Z_\flat \subsetneqq Y$, except discrete points.

Then $\mathscr{L}$ is nef on X. □

5. A proof blueprint of the Ampleness Theorem 1.3

5.1. **Main Nefness Theorem.** Recalling Theorem 4.2, the Ampleness Theorem 1.3 is a consequence of the theorem below, whose effective bound $\mathsf{d}_0(\heartsuit)$ for $\heartsuit = 1$ will be given in Theorem 11.2.

Theorem 5.1. *Given any positive integer $\heartsuit \geqslant 1$, there exists a lower degree bound $\mathsf{d}_0(\heartsuit) \gg 1$ such that, for all degrees $d_1, \ldots, d_{c+r} \geqslant \mathsf{d}_0(\heartsuit)$, for a very generic*[2] *$t \in \mathbb{P}^\diamond_{\mathbb{K}}$, the negatively twisted Serre line bundle $\mathscr{O}_{\mathbf{P}_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-\heartsuit)$ is nef on $\mathbf{P}_t$.*

It suffices to find one such $t \in \mathbb{P}^\diamond_{\mathbb{K}}$ to guarantee'very generic' (cf. [40, p. 56, Proposition 1.4.14]).

We will prove Theorem 5.1 in two steps. At first, in Subsection 5.2, we sketch the proof in the central cases when all $c + r$ hypersurfaces are approximately of the same large degrees. Then, in Subsection 5.3, we play a *product coup* to embrace all large degrees.

[2] $t \in \mathbb{P}^\diamond_{\mathbb{K}} \setminus \cup_{i=1}^{\infty} Z_i$ for some countable proper subvarieties $Z_i \subsetneqq \mathbb{P}^\diamond_{\mathbb{K}}$.

5.2. **The central cases of relatively the same large degrees.**

Theorem 5.2. *For any fixed $c+r$ positive integers $\epsilon_1, \dots, \epsilon_{c+r} \geqslant 1$, for every sufficiently large integer $d \gg 1$, Theorem 5.1 holds with $d_i = d + \epsilon_i$, $i = 1 \cdots c+r$.*

When $c + r \geqslant N$, generically X is discrete or empty, so there is nothing to prove. Assuming $c + r \leqslant N - 1$, we now outline the proof.

Step 1. In the entire family of $c + r$ hypersurfaces with degrees $d + \lambda_1, \dots, d + \lambda_{c+r}$, whose projective parameter space is $\mathbb{P}^{\diamond}_{\mathbb{K}}$ (see (3)), we select a specific subfamily which best suits our moving coefficients method, whose projective parameter space is a subvariety:

$$\mathbb{P}^{\blacklozenge}_{\mathbb{K}} \ \subset \ \mathbb{P}^{\diamond}_{\mathbb{K}} \qquad \text{[see (140)]}.$$

For the details of this subfamily, see Subsection 7.1.

Recalling (5) and (7), we then consider the subfamily of intersections $\mathscr{Y} \subset \mathscr{X}$:

$$\mathrm{pr}_1^{-1}\left(\mathbb{P}^{\blacklozenge}_{\mathbb{K}}\right) \cap \mathscr{X} \ =: \ \underline{\mathscr{Y} \ \subset \ \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^{N}_{\mathbb{K}}} \ = \ \mathrm{pr}_1^{-1}\left(\mathbb{P}^{\blacklozenge}_{\mathbb{K}}\right).$$

Recalling (13), (14), we introduce the subscheme of $\mathbf{P}$:

$$\mathbf{P}' \ := \ \widetilde{\pi}^{-1}(\mathscr{Y}) \cap \mathbf{P} \ \subset \ \mathbb{P}^{\diamond}_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}),$$

which is parametrized by $\mathscr{Y}$. By restriction, (13) yields the commutative diagram:

$$\begin{array}{ccccc} & & \mathbf{P}' & & \\ & \swarrow^{\pi_1 = \mathrm{pr}_1 \circ \widetilde{\pi}} & \downarrow{\scriptstyle \widetilde{\pi}} & \searrow^{\pi_2 = \mathrm{pr}_2 \circ \widetilde{\pi}} & \\ & & \mathscr{Y} & & \\ & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \\ \mathbb{P}^{\blacklozenge}_{\mathbb{K}} & & & & \mathbb{P}^{N}_{\mathbb{K}}. \end{array} \tag{30}$$

Introducing the restricted Serre line bundle $\mathscr{O}_{\mathbf{P}'}(1) := \mathscr{O}_{\mathbf{P}}(1)\big|_{\mathbf{P}'}$ over $\mathbf{P}'$, in order to establish Theorem 5.2, it suffices to provide one such example. In fact, we will prove

Theorem 5.3. *For a generic closed point $t \in \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$, the bundle $\mathscr{O}_{\mathbf{P}'_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-\heartsuit)$ is nef on $\mathbf{P}'_t := \mathbf{P}_t$.*

Step 2. The central objects now are the *universal negatively twisted Serre line bundles*:

$$\mathscr{O}_{\mathbf{P}'}(a, b, -c) \ := \ \mathscr{O}_{\mathbf{P}'}(a) \otimes \pi_1^* \, \mathscr{O}_{\mathbb{P}^{\blacklozenge}_{\mathbb{K}}}(b) \otimes \pi_2^* \, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-c),$$

where a, c are positive integers such that $c/a \geqslant \heartsuit$, and where b are any integers.

Taking advantage of the moving coefficients method, firstly, we construct a series of global universal negatively twisted symmetric differential n-forms:

$$S_\ell \ \in \ \Gamma(\mathbf{P}', \, \mathscr{O}_{\mathbf{P}'}(n, N, -\heartsuit_\ell)) \qquad (\ell = 1 \cdots \maltese), \tag{31}$$

where $n := N - (c + r) \geqslant 1$ and all $\heartsuit_\ell / n \geqslant \heartsuit$, and where *we always use* the symbol '$\maltese$' to denote auxiliary positive integers, which vary according to the context.

Secondly, for every integer $1 \leqslant \eta \leqslant n - 1$, for every sequence of ascending indices :

$$0 \leqslant \nu_1 < \cdots < \nu_\eta \leqslant N,$$

considering the vanishing part of the corresponding η coordinates:

$$_{v_1,\dots,v_\eta}\mathbf{P}' \;:=\; \mathbf{P}' \;\cap\; \pi_2^{-1}\underbrace{\{z_{v_1}=\cdots=z_{v_\eta}=0\}}_{=:\,_{v_1,\dots,v_\eta}\mathbb{P}^N},$$

we construct a series of universal negatively twisted symmetric differential $(n-\eta)$-forms on it:

$$_{v_1,\dots,v_\eta}S_\ell \;\in\; \Gamma(_{v_1,\dots,v_\eta}\mathbf{P}',\,\mathscr{O}_{\mathbf{P}'}(n-\eta,N-\eta,-\,_{v_1,\dots,v_\eta}\heartsuit_\ell)) \qquad (\ell=1\cdots\maltese). \tag{32}$$

where all $_{v_1,\dots,v_\eta}\heartsuit_\ell/(n-\eta)\geqslant\blacktriangledown$.

This step will be accomplished in Sections 6 and 7.

Step 3. From now on, we view every scheme as its $\mathbb{K}$-variety.

Firstly, we control the base locus of all the global sections obtained in (31):

$$\mathsf{BS} := \text{Base Locus of } \{S_\ell\}_{1\leqslant\ell\leqslant\maltese} \;\subset\; \mathbf{P}'.$$

In fact, on the coordinates nonvanishing part of $\mathbf{P}'$:

$$\mathring{\mathbf{P}}' \;:=\; \mathbf{P}' \;\cap\; \pi_2^{-1}\{z_0\cdots z_N\neq 0\}\,,$$

we prove that:

$$\dim\,\mathsf{BS}\cap\mathring{\mathbf{P}}'\;\leqslant\;\dim\,\mathbb{P}^\blacklozenge_{\mathbb{K}}. \tag{33}$$

Secondly, we control the base locus of all the sections obtained in (32):

$$_{v_1,\dots,v_\eta}\mathsf{BS} := \text{Base Locus of } \{_{v_1,\dots,v_\eta}S_\ell\}_{1\leqslant\ell\leqslant\maltese} \;\subset\; _{v_1,\dots,v_\eta}\mathbf{P}'.$$

In fact, on the corresponding 'coordinates nonvanishing part' of $_{v_1,\dots,v_\eta}\mathbf{P}'$:

$$_{v_1,\dots,v_\eta}\mathring{\mathbf{P}}' \;:=\; _{v_1,\dots,v_\eta}\mathbf{P}' \;\cap\; \pi_2^{-1}\{z_{r_0}\cdots z_{r_{N-\eta}}\neq 0\}\,,$$

where:

$$\{r_0,\dots,r_{N-\eta}\} \;:=\; \{0,\dots,N\}\;\setminus\;\{v_1,\dots,v_\eta\}\,, \tag{34}$$

we prove that:

$$\dim\,{}_{v_1,\dots,v_\eta}\mathsf{BS}\cap\,{}_{v_1,\dots,v_\eta}\mathring{\mathbf{P}}'\;\leqslant\;\dim\,\mathbb{P}^\blacklozenge_{\mathbb{K}}. \tag{35}$$

This crucial step will be accomplished in Sections 9 and 10. Anticipating, we would like to emphasize that, in order to lower down dimensions of base loci for global symmetric differential forms (or for higher order jet differential forms in Kobayashi hyperbolicity conjecture), a substantial amount of algebraic geometry work is required, mainly because some already known/constructed sections have the annoying tendency to proliferate by multiplying each other without shrinking their base loci ($0\times\text{anything}=0$). Hence the first main difficulty is to devise a wealth of independent symmetric differential forms, which the *Moving Coefficients Method is designed for*, and the second main difficulty is to establish the emptiness/discreteness of their base loci, an ultimate difficulty that will be settled in the *Core Lemma* 9.5.

Step 4. Firstly, for the regular map:

$$\pi_1\colon\quad \mathbf{P}' \;\longrightarrow\; \mathbb{P}^\blacklozenge_{\mathbb{K}},$$

noting the dimension estimates (33), (35) of the base loci, applying now a classical theorem [35, p. 132, Theorem 11.12], we know that there exists a proper closed algebraic subvariety:

$$\Sigma \;\subsetneqq\; \mathbb{P}^\blacklozenge_{\mathbb{K}}$$

such that, for every closed point t outside Σ:

$$t \in \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus \Sigma,$$

(i) the base locus of the restricted symmetric differential n-forms:

$$\mathsf{BS}_t := \text{Base Locus of } \{S_\ell(t) := S_\ell\big|_{\mathbf{P}'_t}\}_{1\leqslant\ell\leqslant\maltese} \subset \mathbf{P}'_t$$

is discrete or empty over the coordinates nonvanishing part:

$$\dim \mathsf{BS}_t \cap \mathring{\mathbf{P}}'_t \leqslant 0, \tag{36}$$

where:

$$\mathring{\mathbf{P}}'_t := \mathring{\mathbf{P}}' \cap \pi_1^{-1}(t);$$

(ii) the base locus of the restricted symmetric differential $(n-\eta)$-forms:

$${}_{v_1,\ldots,v_\eta}\mathsf{BS}_t := \text{Base Locus of } \left\{{}_{v_1,\ldots,v_\eta}S_\ell(t) := {}_{v_1,\ldots,v_\eta}S_\ell\Big|_{{}_{v_1,\ldots,v_\eta}\mathbf{P}'_t}\right\}_{1\leqslant\ell\leqslant\maltese} \subset {}_{v_1,\ldots,v_\eta}\mathbf{P}'_t$$

is discrete or empty over the corresponding 'coordinates nonvanishing part':

$$\dim {}_{v_1,\ldots,v_\eta}\mathsf{BS}_t \cap {}_{v_1,\ldots,v_\eta}\mathring{\mathbf{P}}'_t \leqslant 0\,, \tag{37}$$

where:

$${}_{v_1,\ldots,v_\eta}\mathring{\mathbf{P}}'_t := {}_{v_1,\ldots,v_\eta}\mathring{\mathbf{P}}' \cap \pi_1^{-1}(t).$$

Secondly, there exists a proper closed algebraic subvariety:

$$\Sigma' \subsetneqq \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$$

such that, for every closed point t outside Σ':

$$t \in \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus \Sigma',$$

the fibre:

$$\mathscr{Y}_t := \mathscr{Y} \cap \mathrm{pr}_1^{-1}(t)$$

is smooth and of dimension $n = N - (c + r)$, and it satisfies:

$$\dim \mathscr{Y}_t \cap \mathrm{pr}_2^{-1}({}_{v_1,\ldots,v_n}\mathbb{P}^N) = 0 \qquad (0\leqslant v_1<\cdots<v_n\leqslant N), \tag{38}$$

i.e. the intersection of $\mathscr{Y}_t$ — (under the regular map pr_2) viewed as a dimension n subvariety in $\mathbb{P}^N$ — with every n coordinate hyperplanes:

$${}_{v_1,\ldots,v_n}\mathbb{P}^N := \{z_{v_1} = \cdots = z_{v_n} = 0\}$$

is just finitely many points, which we denote by:

$$\underbrace{{}_{v_1,\ldots,v_n}\mathscr{Y}_t}_{\#<\infty} \subset {}_{v_1,\ldots,v_n}\mathbb{P}^N. \tag{39}$$

Now, we shall conclude Theorem 5.3 for every closed point $t \in \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus (\Sigma \cup \Sigma')$.

Proof of Theorem 5.3. For the line bundle $\mathscr{L} = \mathscr{O}_{\mathbf{P}'_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-\heartsuit)$ over the variety $\mathbf{P}'_t$, we claim that the set of subvarieties:

$$\mathscr{V} := \left\{ \emptyset,\ \mathbf{P}'_t,\ {}_{\nu_1,\dots,\nu_\eta}\mathbf{P}'_t \right\}_{\substack{1 \leqslant \eta \leqslant n \\ 0 \leqslant \nu_1 < \cdots < \nu_\eta \leqslant N}}$$

satisfies the conditions of Theorem 4.6.

Indeed, firstly, recalling (36), the sections $\{S_\ell(t)\}_{\ell=1\cdots\clubsuit}$ have empty/discrete base locus over the coordinates nonvanishing part, i.e. outside $\cup_{j=0}^N {}_j\mathbf{P}'_t$. Hence, using an adaptation of Theorem 4.3, remembering $\heartsuit/1 \leqslant \min\{\diamondsuit_\ell/n\}_{1\leqslant\ell\leqslant\clubsuit}$, the line bundle $\mathscr{O}_{\mathbf{P}'_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N}(-\heartsuit)$ is nef outside $\cup_{j=0}^N {}_j\mathbf{P}'_t$.

Secondly, for every integer $\eta = 1 \cdots n-1$, recalling the dimension estimate (37), again by Theorem 4.3, remembering $\heartsuit/1 \leqslant \min\{{}_{\nu_1,\dots,\nu_\eta}\diamondsuit_\ell/(n-\eta)\}_{1\leqslant\ell\leqslant\clubsuit}$, the line bundle $\mathscr{O}_{\mathbf{P}'_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N}(-\heartsuit)$ is nef on ${}_{\nu_1,\dots,\nu_\eta}\mathbf{P}'_t$ outside $\cup_{j=0}^{N-\eta} {}_{\nu_1,\dots,\nu_\eta,r_j}\mathbf{P}'_t$ (see (34)).

Lastly, for $\eta = n$, noting that under the projection $\pi\colon \mathbf{P}'_t \to \mathscr{Y}_t$, thanks to (38), every subvariety ${}_{\nu_1,\dots,\nu_n}\mathbf{P}'_t$ contracts to discrete points ${}_{\nu_1,\dots,\nu_n}\mathscr{Y}_t$, we see that on ${}_{\nu_1,\dots,\nu_n}\mathbf{P}'_t$, the line bundle $\mathscr{O}_{\mathbf{P}'_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-\heartsuit) \cong \mathscr{O}_{\mathbf{P}'_t}(1)$ is not only nef, but also ample!

Summarizing the above three parts, by Theorem 4.6, we conclude the proof. □

5.3. **Product Coup.** We will use in an essential way Theorem 5.2 with all ϵ_i equal to either 1 or 2. To begin with, we need an elementary

Observation 5.4. *For all positive integers $d \geqslant 1$, every integer $d_0 \geqslant d^2 + d$ is a sum of nonnegative multiples of $d+1$ and $d+2$.*

Proof. According to the Euclidian division, we can write d_0 as:

$$d_0 = p\,(d+1) + q$$

for some positive integer $p \geqslant 1$ and residue number $0 \leqslant q \leqslant d$. We claim that $p \geqslant q$.

Otherwise, we would have:

$$p \leqslant q - 1 \leqslant d - 1,$$

which would imply the estimate:

$$d = p\,(d+1) + q \leqslant (d-1)\,(d+1) + d = d^2 + d - 1,$$

contradicting our assumption.

Therefore, we can write d_0 as:

$$d_0 = \underbrace{(p-q)}_{\geqslant 0}(d+1) + q\,(d+2),$$

which concludes the proof. □

Proof of Theorem 5.1. Take one sufficiently large integer d such that Theorem 5.2 holds for any integers $\epsilon_i \in \{1,2\}$, $i = 1 \cdots c+r$. Now, the above observation says that all large degrees $d_1, \dots, d_{c+r} \geqslant d^2 + d$ can be written as $d_i = p_i\,(d+1) + q_i\,(d+2)$, with some nonnegative integers $p_i, q_i \geqslant 0$, $i = 1 \cdots c+r$. Let $F_i := f_1^i \cdots f_{p_i}^i f_{p_i+1}^i \cdots f_{p_i+q_i}^i$ be a product of some p_i homogeneous polynomials $f_1^i, \dots, f_{p_i}^i$ each of degree $d+1$ and of some q_i homogeneous polynomials $f_{p_i+1}^i, \dots, f_{p_i+q_i}^i$ each of degree $d+2$, so that F_i has degree d_i.

Recalling (22), a point $([z],[\xi]) \in \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$ lies in ${}_{F_{c+1},\dots,F_{c+r}}\mathbb{P}_{F_1,\dots,F_c}$ if and only if:

$$F_i(z) = 0,\ dF_j\big|_z(\xi) = 0 \qquad (\forall\, i = 1 \cdots c+r, \forall\, j = 1 \cdots c).$$

Note that, for every $j = 1 \cdots c$, the pair of equations:

$$F_j(z) = 0,\ dF_j\big|_z(\xi) = 0 \tag{40}$$

is equivalent to either:

$$\exists\ 1 \leqslant v_j \leqslant p_j + q_j \quad s.t. \quad f^j_{v_j}(z) = 0,\ df^j_{v_j}\big|_z(\xi) = 0, \tag{41}$$

or to:

$$\exists\ 1 \leqslant w^1_j < w^2_j \leqslant p_j + q_j \quad s.t. \quad f^j_{w^1_j}(z) = 0,\ f^j_{w^2_j}(z) = 0. \tag{42}$$

Therefore, $([z], [\xi]) \in {}_{F_{c+1},\ldots,F_{c+r}}\mathbb{P}_{F_1,\ldots,F_c}$ is equivalent to say that there exists a subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, c\}$ of cardinality k ($k = 0$ for $\emptyset$) such that, firstly, for every index $j \in \{i_1 \cdots i_k\}$, (z, ξ) is a solution of (40) of type (41), secondly, for every index $j \in \{1, \ldots, c\} \setminus \{i_1 \cdots i_k\}$, (z, ξ) is a solution of (40) of type (42), and lastly, for every $j = c+1 \cdots c+r$, one of $f^j_1, \ldots, f^j_{p_j+q_j}$ vanishes at z. Thus, we see that the variety ${}_{F_{c+1},\ldots,F_{c+r}}\mathbb{P}_{F_1,\ldots,F_c}$ actually decomposes into a union of subvarieties:

$$\begin{aligned}{}_{F_{c+1},\ldots,F_{c+r}}\mathbb{P}_{F_1,\ldots,F_c} \ = \ \cup_{k=0\cdots c} \cup_{1\leqslant i_1<\cdots<i_k\leqslant c} \cup_{\substack{1\leqslant v_{i_j}\leqslant p_{v_j}+q_{v_j}\\ j=1\cdots k}} \cup_{\substack{\{r_1,\ldots,r_{c-k}\}=\{1,\ldots,c\}\setminus\{i_1,\ldots,i_k\}\\ 1\leqslant w^1_{r_l}<w^2_{r_l}\leqslant p_{r_l}+q_{r_l}\\ l=1\cdots c-k}} \cup_{\substack{1\leqslant u_j\leqslant p_j+q_j\\ j=c+1\cdots c+r}}\\ {}_{f^{r_1}_{w^1_{r_1}},f^{r_1}_{w^2_{r_1}},\ldots,f^{r_{c-k}}_{w^1_{r_{c-k}}},f^{r_{c-k}}_{w^2_{r_{c-k}}},f^{c+1}_{u_{c+1}},\ldots,f^{c+r}_{u_{c+r}}}\mathbb{P}_{f^{i_1}_{v_{i_1}},\ldots,f^{i_k}_{v_{i_k}}}.\end{aligned}$$

Similarly, we can show that the scheme ${}_{F_{c+1},\ldots,F_{c+r}}\mathbf{P}_{F_1,\ldots,F_c}$ also decomposes into a union of subschemes:

$$\begin{aligned}{}_{F_{c+1},\ldots,F_{c+r}}\mathbf{P}_{F_1,\ldots,F_c} \ = \ \cup_{k=0\cdots c} \cup_{1\leqslant i_1<\cdots<i_k\leqslant c} \cup_{\substack{1\leqslant v_{i_j}\leqslant p_{v_j}+q_{v_j}\\ j=1\cdots k}} \cup_{\substack{\{r_1,\ldots,r_{c-k}\}=\{1,\ldots,c\}\setminus\{i_1,\ldots,i_k\}\\ 1\leqslant w^1_{r_l}<w^2_{r_l}\leqslant p_{r_l}+q_{r_l}\\ l=1\cdots c-k}} \cup_{\substack{1\leqslant u_j\leqslant p_j+q_j\\ j=c+1\cdots c+r}}\\ {}_{f^{r_1}_{w^1_{r_1}},f^{r_1}_{w^2_{r_1}},\ldots,f^{r_{c-k}}_{w^1_{r_{c-k}}},f^{r_{c-k}}_{w^2_{r_{c-k}}},f^{c+1}_{u_{c+1}},\ldots,f^{c+r}_{u_{c+r}}}\mathbf{P}_{f^{i_1}_{v_{i_1}},\ldots,f^{i_k}_{v_{i_k}}}.\end{aligned} \tag{43}$$

Note that, for each subscheme on the right hand side, the number of polynomials on the lower-left of '$\mathbf{P}$' is $\#_L = 2(c-k)+r$, and the number of polynomials on the lower-right is $\#_R = k$, whence $2\#_R + \#_L = 2c+r \geqslant N$. Now, applying Theorem 5.2, we can choose one $\{f^\bullet_\bullet\}_{\bullet,\bullet}$ such that the twisted Serre line bundle $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)\otimes\pi_0^*\,\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-\heartsuit)$ is nef on each subscheme ${}_{f^{r_1}_{w^1_{r_1}},f^{r_1}_{w^2_{r_1}},\ldots,f^{r_{c-k}}_{w^1_{r_{c-k}}},f^{r_{c-k}}_{w^2_{r_{c-k}}},f^{c+1}_{u_{c+1}},\ldots,f^{c+r}_{u_{c+r}}}\mathbf{P}_{f^{i_1}_{v_{i_1}},\ldots,f^{i_k}_{v_{i_k}}}$, and therefore is also nef on their union ${}_{F_{c+1},\ldots,F_{c+r}}\mathbf{P}_{F_1,\ldots,F_c}$. Since nefness is a very generic property in family, we conclude the proof. □

6. Generalization of Brotbek's symmetric differentials forms

6.1. **Preliminaries on symmetric differential forms in projective space.** For a fixed algebraically closed field $\mathbb{K}$, for three fixed integers $N, c, r \geqslant 0$ such that $N \geqslant 2$, $2c + r \geqslant N$ and $c + r \leqslant N - 1$, for $c + r$ positive integers $d_1, \ldots, d_{c+r}$, let:

$$H_i \ \subset \ \mathbb{P}^N_{\mathbb{K}} \qquad (i = 1 \cdots c+r)$$

be $c + r$ hypersurfaces defined by some degree d_i homogeneous polynomials:

$$F_i \ \in \ \mathbb{K}[z_0, \ldots, z_N],$$

let V be the intersection of the first c hypersurfaces:

$$\begin{aligned} V \ &:= \ H_1 \cap \cdots \cap H_c \\ &= \{[z] \in \mathbb{P}^N_{\mathbb{K}} : \ F_i(z) = 0, \forall\, i = 1 \cdots c\}, \end{aligned} \tag{44}$$

and let X be the intersection of all the $c + r$ hypersurfaces:

$$\begin{aligned} X &:= \underbrace{H_1 \cap \cdots \cap H_c}_{=V} \cap \underbrace{H_{c+1} \cap \cdots \cap H_{c+r}}_{r \text{ more hypersurfaces}} \\ &= \{[z] \in \mathbb{P}^N_{\mathbb{K}} : \ F_i(z) = 0, \forall\, i = 1 \cdots c + r\}. \end{aligned} \tag{45}$$

It is well known that, for generic choices of $\{F_i\}_{i=1}^{c+r}$, the intersection $V = \cap_{i=1}^{c} H_i$ and $X = \cap_{i=1}^{c+r} H_i$ are both smooth complete, and we shall assume this henceforth. In Subsections 6.1–6.4, we focus on smooth $\mathbb{K}$-varieties to provide a geometric approach to generalize Brotbek's symmetric differential forms, where the ambient field $\mathbb{K}$ is assumed to be algebraically closed. In addition, in Subsection 6.5, we will give another quick algebraic approach, without any assumption on the ambient field $\mathbb{K}$.

Recalling (19), let us denote by:

$$\pi: \quad \mathbb{K}^{N+1} \setminus \{0\} \longrightarrow \mathbb{P}^N_{\mathbb{K}}$$

the canonical projection.

For every integer k, the standard twisted regular function sheaf $\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(k)$, geometrically, can be defined as, for all Zariski open subset U in $\mathbb{P}^N_{\mathbb{K}}$, the corresponding section set $\Gamma(U,\, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(k))$ consists of all the regular functions $\widehat{f}$ on $\pi^{-1}(U)$ satisfying:

$$\widehat{f}(\lambda z) = \lambda^k\, \widehat{f}(z) \qquad {\scriptstyle (\forall\, z \in \pi^{-1}(U),\, \lambda \in \mathbb{K}^\times).} \tag{46}$$

For the cone $\widehat{V} := \pi^{-1}(V)$ of V:

$$\widehat{V} \ = \ \{z \in \mathbb{K}^{N+1} \setminus \{0\} : \ F_i(z) = 0, \ \forall\, i = 1 \cdots c\},$$

recalling (20), (21), we can similarly define its horizontal tangent bundle $\mathrm{T}_{\mathrm{hor}}\widehat{V}$ which has fibre at any point $z \in \widehat{V}$:

$$\mathrm{T}_{\mathrm{hor}}\widehat{V}\Big|_z = \{[\xi] \in \mathbb{K}^{N+1}\Big/\mathbb{K}\cdot z : \ dF_i\Big|_z(\xi) = 0, \ \forall\, i = 1 \cdots c\}.$$

Its total space is:

$$\mathrm{T}_{\mathrm{hor}}\widehat{V} := \{(z, [\xi]) : \ z \in \widehat{V},\ [\xi] \in \mathbb{K}^{N+1}\Big/\mathbb{K}\cdot z, dF_i\Big|_z(\xi) = 0, \ \forall\, i = 1 \cdots c\}. \tag{47}$$

Then similarly we receive the total tangent bundle T_V of V as:

$$\mathrm{T}_V = \mathrm{T}_{\mathrm{hor}}\widehat{V}/\sim, \qquad \text{where } (z, [\xi]) \sim (\lambda z, [\lambda\xi]),\ \forall\, \lambda \in \mathbb{K}^\times.$$

Let Ω_V be the dual bundle of T_V, i.e. the cotangent bundle of V, and let $\Omega_{\mathrm{hor}}\widehat{V}$ be the dual bundle of $\mathrm{T}_{\mathrm{hor}}\widehat{V}$. For all positive integers $l \geqslant 1$ and all integers $\heartsuit \in \mathbb{Z}$, we use the standard notation $\mathsf{Sym}^l\, \Omega_V$ to denote the symmetric l-tensor-power of the vector bundle Ω_V, and we use $\mathsf{Sym}^l\, \Omega_V(\heartsuit)$ to denote the twisted vector bundle $\mathrm{Sym}^l\, \Omega_V \otimes \mathscr{O}_V(\heartsuit)$.

Proposition 6.1. *For two fixed integers $l \geqslant 1$, $\heartsuit \in \mathbb{Z}$, and for every Zariski open set $U \subset V$ together with its cone $\widehat{U} := \pi^{-1}(U)$, there is a canonical injection:*

$$\Gamma(U, \mathsf{Sym}^l\, \Omega_V(\heartsuit)) \hookrightarrow \Gamma(\widehat{U}, \mathsf{Sym}^l\, \Omega_{\mathrm{hor}}\widehat{V}),$$

whose image is the set of sections Φ enjoying:

$$\Phi(\lambda z, [\lambda\xi]) = \lambda^{\heartsuit}\, \Phi(z, [\xi]), \tag{48}$$

for all $z \in \widehat{U}$, for all $[\xi] \in \mathrm{T}_{\mathrm{hor}}\widehat{V}\Big|_z$ and for all $\lambda \in \mathbb{K}^\times$.

Proof. Note that we have two canonical injections of vector bundles:

$$\pi^*\mathsf{Sym}^l\,\Omega_V \hookrightarrow \mathsf{Sym}^l\Omega_{\mathrm{hor}}\widehat{V},$$
$$\pi^*\mathscr{O}_V(\heartsuit) \hookrightarrow \mathscr{O}_{\widehat{V}},$$

since the tensor functor is left exact (torsion free) in the category of $\mathbb{K}$-vector bundles, the tensoring of the above two injections remains an injection:

$$\pi^*\mathsf{Sym}^l\,\Omega_V \otimes \pi^*\mathscr{O}_V(\heartsuit) \hookrightarrow \mathsf{Sym}^l\Omega_{\mathrm{hor}}\widehat{V} \otimes \mathscr{O}_{\widehat{V}}.$$

Recalling that:

$$\mathsf{Sym}^l\,\Omega_V(\heartsuit) = \mathsf{Sym}^l\,\Omega_V \otimes \mathscr{O}_V(\heartsuit),$$

we can rewrite the above injection as:

$$\pi^*\mathsf{Sym}^l\,\Omega_V(\heartsuit) \hookrightarrow \mathsf{Sym}^l\Omega_{\mathrm{hor}}\widehat{V}.$$

With $U \subset X$ Zariski open, applying the global section functor $\Gamma(\widehat{U}, \cdot)$, which is left exact, we receive:

$$\Gamma(\widehat{U}, \pi^*\mathsf{Sym}^l\,\Omega_V(\heartsuit)) \hookrightarrow \Gamma(\widehat{U}, \mathsf{Sym}^l\Omega_{\mathrm{hor}}\widehat{V}).$$

Lastly, we have an injection:

$$\Gamma(U, \mathsf{Sym}^l\,\Omega_V(\heartsuit)) \hookrightarrow \Gamma(\widehat{U}, \pi^*\mathsf{Sym}^l\,\Omega_V(\heartsuit)),$$

whence, by composing the previous two injections, we conclude:

$$\Gamma(U, \mathsf{Sym}^l\,\Omega_V(\heartsuit)) \hookrightarrow \Gamma(\widehat{U}, \mathsf{Sym}^l\Omega_{\mathrm{hor}}\widehat{V}).$$

To view explicitly the image of this injection, notice that in the case $l = 0$, it is the standard injection:

$$\Gamma(U, \mathscr{O}_V(\heartsuit)) \hookrightarrow \Gamma(\widehat{U}, \mathscr{O}_{\widehat{V}})$$
$$f \mapsto \pi^* f,$$

whose image consists of, as a consequence of the definition (46) above, all functions $\widehat{f}$ on $\widehat{U}$ satisfying $\widehat{f}(\lambda z) = \lambda^{\heartsuit}\widehat{f}(z)$, for all $z \in \widehat{U}$ and for all $\lambda \in \mathbb{K}^\times$.

Furthermore, in the case $\heartsuit = 0$, the image of the injection:

$$\Gamma(U, \mathsf{Sym}^l\,\Omega_V) \hookrightarrow \Gamma(\widehat{U}, \mathsf{Sym}^l\Omega_{\mathrm{hor}}\widehat{V})$$
$$\omega \mapsto \pi^*\omega,$$

consists of sections $\widehat{\omega}$ on $\widehat{U}$ satisfying:

$$\widehat{\omega}\,(z, [\xi]) = \widehat{\omega}\,(\lambda z, [\lambda\xi]),$$

for all $z \in \widehat{U}$, all $[\xi] \in \mathrm{T}_{\mathrm{hor}}\widehat{V}\big|_z$ and all $\lambda \in \mathbb{K}^\times$.

As $\mathsf{Sym}^l\,\Omega_V(\heartsuit) = \mathsf{Sym}^l\,\Omega_V \otimes \mathscr{O}_V(\heartsuit)$, composing the above two observations by tensoring the corresponding two injections, we see that any element Φ in the image of the injection:

$$\Gamma(U, \mathsf{Sym}^l\,\Omega_V(\heartsuit)) \hookrightarrow \Gamma(\widehat{U}, \mathsf{Sym}^l\,\Omega_{\mathrm{hor}}\widehat{V}), \tag{49}$$

automatically satisfies (48). On the other hand, for every element Φ in $\Gamma(\widehat{U}, \mathsf{Sym}^l\,\Omega_{\mathrm{hor}}\widehat{V})$ satisfying (48), we can construct the corresponding element ϕ in $\Gamma(U, \mathsf{Sym}^l\,\Omega_V(\heartsuit))$, which maps to Φ under the injection (49). □

Let $Y \subset V$ be a regular subvariety. Replacing the underground variety V by Y, in much the same way we can show:

Proposition 6.2. *For two fixed integers $l \geqslant 1$, $\heartsuit \in \mathbb{Z}$, and for every Zariski open set $U \subset Y$ together with its cone $\widehat{U} := \pi^{-1}(U)$, there is a canonical injection:*

$$\Gamma(U, \mathsf{Sym}^l \Omega_V(\heartsuit)) \hookrightarrow \Gamma(\widehat{U}, \mathsf{Sym}^l \Omega_{\mathrm{hor}} \widehat{V}),$$

whose image is the set of sections Φ enjoying:

$$\Phi(\lambda z, [\lambda \xi]) = \lambda^{\heartsuit}\, \Phi(z, [\xi]), \tag{50}$$

for all $z \in \widehat{U}$, for all $[\xi] \in \mathrm{T}_{\mathrm{hor}} \widehat{V}\big|_z$ and for all $\lambda \in \mathbb{K}^{\times}$. □

In future applications, we will mainly interest in the sections:

$$\Gamma(Y, \mathsf{Sym}^l \Omega_V(\heartsuit)),$$

where $Y = X$ or $Y = X \cap \{z_{\nu_1} = 0\} \cap \cdots \cap \{z_{\nu_\eta} = 0\}$ for some vanishing coordinate indices $0 \leqslant \nu_1 < \cdots < \nu_\eta \leqslant N$.

6.2. **Global regular symmetric horizontal differential forms.** In our coming applications, we will be mainly concerned with *Fermat-type hypersursurfaces* H_i defined by some homogeneous polynomials F_i of the form:

$$F_i = \sum_{j=0}^{N} A_i^j\, z_j^{\lambda_j} \qquad (i = 1 \cdots c+r), \tag{51}$$

where $\lambda_0, \ldots, \lambda_N$ are some positive integers and where $A_i^j \in \mathbb{K}[z_0, z_1, \ldots, z_N]$ are some homogeneous polynomials, with all terms of F_i having the same degree:

$$\deg A_i^j + \lambda_j = \deg F_i =: d_i \qquad (i = 1 \cdots c+r;\, j = 0 \cdots N). \tag{52}$$

Differentiating F_i, we receive:

$$dF_i = \sum_{j=0}^{N} \mathsf{B}_i^j\, z_j^{\lambda_j - 1}, \tag{53}$$

where:

$$\mathsf{B}_i^j := z_j\, dA_i^j + \lambda_j\, A_i^j\, dz_j \qquad (i = 1 \cdots c+r;\, j = 0 \cdots N). \tag{54}$$

To make the terms of F_i have the same structure as that of dF_i, let us denote:

$$\widetilde{\mathsf{A}}_i^j := A_i^j\, z_j, \tag{55}$$

so that:

$$F_i = \sum_{j=0}^{N} \widetilde{\mathsf{A}}_i^j\, z_j^{\lambda_j - 1}.$$

Recalling (45), we denote the cone of X by:

$$\widehat{X} = \{z \in \mathbb{K}^{N+1} \setminus \{0\} :\ F_i(z) = 0,\ \forall\, i = 1 \cdots c + r\}.$$

For all $z \in \widehat{X}$ and $[\xi] \in \mathrm{T}_{\mathrm{hor}} \widehat{V}\big|_z$, by the very definition (47) of $\mathrm{T}_{\mathrm{hor}} \widehat{V}$, we have:

$$\begin{cases} \sum_{j=0}^{N} \widetilde{\mathsf{A}}_i^j\, z_j^{\lambda_j - 1}\,(z) = 0 & (i = 1 \cdots c+r), \\ \sum_{j=0}^{N} \mathsf{B}_i^j(z, \xi)\, z_j^{\lambda_j - 1}(z) = 0 & (i = 1 \cdots c). \end{cases} \tag{56}$$

For convenience, dropping z, ξ, we rewrite the above equations as:

$$\begin{cases} \sum_{j=0}^{N} \widetilde{\mathsf{A}}_i^j \, z_j^{\lambda_j - 1} = 0 & (i = 1 \cdots c+r), \\ \sum_{j=0}^{N} \mathsf{B}_i^j \, z_j^{\lambda_j - 1} = 0 & (i = 1 \cdots c), \end{cases}$$

and formally, we view them as a system of linear equations with respect to the unknown variables $z_0^{\lambda_0 - 1}, \ldots, z_N^{\lambda_N - 1}$, of which the associated coefficient matrix, of size $(c + r + c) \times (N + 1)$, is:

$$\mathsf{C} := \begin{pmatrix} \widetilde{\mathsf{A}}_1^0 & \cdots & \widetilde{\mathsf{A}}_1^N \\ \vdots & & \vdots \\ \widetilde{\mathsf{A}}_{c+r}^0 & \cdots & \widetilde{\mathsf{A}}_{c+r}^N \\ \mathsf{B}_1^0 & \cdots & \mathsf{B}_1^N \\ \vdots & & \vdots \\ \mathsf{B}_c^0 & \cdots & \mathsf{B}_c^N \end{pmatrix}, \tag{57}$$

so that the system reads as:

$$\mathsf{C} \begin{pmatrix} z_0^{\lambda_0 - 1} \\ \vdots \\ z_N^{\lambda_N - 1} \end{pmatrix} = 0. \tag{58}$$

Recalling our assumption:

$$n = \underbrace{N - (c + r)}_{= \dim X} \geqslant 1,$$

since $2c + r \geqslant N$, we have $1 \leqslant n \leqslant c$.

Let now D be the upper $(\underbrace{c + r + n}_{=N}) \times (N + 1)$ submatrix of C, i.e. consisting of the first $(c + r + n)$ rows of C:

$$\mathsf{D} := \begin{pmatrix} \widetilde{\mathsf{A}}_1^0 & \cdots & \widetilde{\mathsf{A}}_1^N \\ \vdots & & \vdots \\ \widetilde{\mathsf{A}}_{c+r}^0 & \cdots & \widetilde{\mathsf{A}}_{c+r}^N \\ \mathsf{B}_1^0 & \cdots & \mathsf{B}_1^N \\ \vdots & & \vdots \\ \mathsf{B}_n^0 & \cdots & \mathsf{B}_n^N \end{pmatrix}. \tag{59}$$

For $j = 0 \cdots N$, let $\widehat{\mathsf{D}}_j$ denote the submatrix of D obtained by omitting the $(j + 1)$-th column:

$$\widehat{\mathsf{D}}_j := \begin{pmatrix} \widetilde{\mathsf{A}}_1^0 & \cdots & \widehat{\widetilde{\mathsf{A}}_1^j} & \cdots & \widetilde{\mathsf{A}}_1^N \\ \vdots & & \vdots & & \\ \widetilde{\mathsf{A}}_{c+r}^0 & \cdots & \widehat{\widetilde{\mathsf{A}}_{c+r}^j} & \cdots & \widetilde{\mathsf{A}}_{c+r}^N \\ \mathsf{B}_1^0 & \cdots & \widehat{\mathsf{B}_1^j} & \cdots & \mathsf{B}_1^N \\ \vdots & & \vdots & & \\ \mathsf{B}_n^0 & \cdots & \widehat{\mathsf{B}_n^j} & \cdots & \mathsf{B}_n^N \end{pmatrix}, \tag{60}$$

and let D_j denote the $(j + 1)$-th column of D.

Denote:

$$W_j := \{z_j \neq 0\} \ \subset \ \mathbb{P}^N \qquad\qquad (j = 0 \cdots N) \tag{61}$$

the canonical affine open subsets, whose cones are:

$$\widehat{W}_j := \pi^{-1}(W_j) \ \subset \ \mathbb{K}^{N+1} \setminus \{0\}. \tag{62}$$

Denote also:

$$U_j := W_j \cap X \tag{63}$$

the open subsets of X, whose cones are:

$$\widehat{U}_j := \pi^{-1}(U_j) \ \subset \ \widehat{X}. \tag{64}$$

Recalling the horizontal tangent bundle of $\mathbb{K}^{N+1}$:

$$\mathrm{T}_{\mathrm{hor}}\mathbb{K}^{N+1} = \{(z, [\xi]) \ : \ z \in \mathbb{K}^{N+1} \setminus \{0\} \text{ and } [\xi] \in \mathbb{K}^{N+1}/\mathbb{K} \cdot z\},$$

now let $\Omega_{\mathrm{hor}}\mathbb{K}^{N+1}$ be its dual bundle.

Proposition 6.3. *For every $j = 0 \cdots N$, on the affine set:*

$$\widehat{W}_j = \{z_j \neq 0\} \ \subset \ \mathbb{K}^{N+1} \setminus \{0\},$$

the following affine symmetric horizontal differential n-form is well defined:

$$\widehat{\omega}_j := \frac{(-1)^j}{z_j^{\lambda_j - 1}} \det\left(\widehat{\mathsf{D}}_j\right) \ \in \ \Gamma\Big(\widehat{W}_j, \, \mathsf{Sym}^n \, \Omega_{\mathrm{hor}}\mathbb{K}^{N+1}\Big). \tag{65}$$

The essence of this proposition lies in the famous *Euler's Identity*.

Lemma 6.4. [Euler's Identity] *For every homogeneous polynomial $P \in \mathbb{K}[z_0, \ldots, z_N]$, one has:*

$$\sum_{j=0}^{N} \frac{\partial F}{\partial z_j} \cdot z_j = \deg F \cdot F,$$

where using differential writes as:

$$dF\Big|_z(z) =: dF(z, z) = \deg F \cdot F(z), \tag{66}$$

at all points $z = (z_0, \ldots, z_N) \in \mathbb{K}^{N+1}$. □

Proof of Proposition 6.3. Without loss of generality, we only prove the case $j = 0$.

Recalling the notation (55) and (54) where all $\widetilde{\mathsf{A}}_i^j$ are regular functions and all B_i^j are regular 1-forms on $\mathbb{K}^{N+1}$, we can see without difficulty that:

$$\begin{aligned}
\widehat{\omega}_0 &= \frac{1}{z_0^{\lambda_0 - 1}} \det\left(\widehat{\mathsf{D}}_0\right) \\
&= \frac{1}{z_0^{\lambda_0 - 1}} \ \det \begin{pmatrix} \widetilde{\mathsf{A}}_1^1 & \cdots & \widetilde{\mathsf{A}}_1^N \\ \vdots & & \vdots \\ \widetilde{\mathsf{A}}_{c+r}^1 & \cdots & \widetilde{\mathsf{A}}_{c+r}^N \\ \mathsf{B}_1^1 & \cdots & \mathsf{B}_1^N \\ \vdots & & \vdots \\ \mathsf{B}_n^1 & \cdots & \mathsf{B}_n^N \end{pmatrix} \ \in \ \Gamma\Big(\widehat{V}_0, \, \mathsf{Sym}^n \, \Omega \, \mathbb{K}^{N+1}\Big)
\end{aligned}$$

is a well defined regular symmetric differential n-form. Now we need an:

Observation 6.5. *Let $N \geqslant 1$ be a positive integer, let L be a field with* $\mathsf{Card}\, L = \infty$*, and let F be a polynomial:*

$$F \in L[z_0, \ldots, z_N].$$

Then F is a polynomial without the variable z_0:

$$F \in L[z_1, \ldots, z_N] \subset L[z_0, \ldots, z_N]$$

if and only if the evaluation map:

$$\begin{aligned} \mathsf{ev}_F : \quad L^{N+1} &\longrightarrow L \\ (x_0, \ldots, x_N) &\longmapsto F(x_0, \ldots, x_N) \end{aligned}$$

is independent of the first variable $x_0 \in L$. □

For the same reason as the above Observation, in order to show that $\widehat{\omega}_0$ descends to a regular symmetric horizontal differential n-form in $\Gamma(\widehat{V}_0, \mathsf{Sym}^n \Omega_{\mathrm{hor}} \mathbb{K}^{N+1})$, we only have to show, at every point $z \in \widehat{V}_0$, for all $\xi \in \mathrm{T}_z \mathbb{K}^{N+1} \cong \mathbb{K}^{N+1}$, $\lambda \in \mathbb{K}^\times$, that:

$$\widehat{\omega}_0(z, \xi + \lambda\, z) = \widehat{\omega}_0(z, \xi). \tag{67}$$

In fact, applying Euler's Identity (66) to the above formula (54), we receive:

$$\begin{aligned} \mathsf{B}_i^j(z, z) &= \lambda_j\, A_i^j(z)\, dz_j(z, z) + dA_i^j(z, z)\, z_j(z) \\ &= \lambda_j\, A_i^j(z)\, z_j(z) + \deg A_i^j \cdot A_i^j(z)\, z_j(z) \\ &= (\lambda_j + \deg A_i^j)\, \widetilde{\mathsf{A}}_i^j(z). \end{aligned}$$

Since B_i^j are 1-forms, we obtain:

$$\begin{aligned} \mathsf{B}_i^j(z, \xi + \lambda\, z) &= \mathsf{B}_i^j(z, \xi) + \lambda\, \mathsf{B}_i^j(z, z) \\ &= \mathsf{B}_i^j(z, \xi) + \underbrace{\lambda\, (\lambda_j + \deg A_i^j)}_{\text{`constant'}}\, \widetilde{\mathsf{A}}_i^j(z). \end{aligned}$$

Therefore, the matrix:

$$\begin{pmatrix} \widetilde{\mathsf{A}}_1^1 & \cdots & \widetilde{\mathsf{A}}_1^N \\ \vdots & & \vdots \\ \widetilde{\mathsf{A}}_{c+r}^1 & \cdots & \widetilde{\mathsf{A}}_{c+r}^N \\ \mathsf{B}_1^1 & \cdots & \mathsf{B}_1^N \\ \vdots & & \vdots \\ \mathsf{B}_n^1 & \cdots & \mathsf{B}_n^N \end{pmatrix} (z, \xi + \lambda\, z)$$

not only has the same first $c + r$ rows as the matrix:

$$\begin{pmatrix} \widetilde{\mathsf{A}}_1^1 & \cdots & \widetilde{\mathsf{A}}_1^N \\ \vdots & & \vdots \\ \widetilde{\mathsf{A}}_{c+r}^1 & \cdots & \widetilde{\mathsf{A}}_{c+r}^N \\ \mathsf{B}_1^1 & \cdots & \mathsf{B}_1^N \\ \vdots & & \vdots \\ \mathsf{B}_n^1 & \cdots & \mathsf{B}_n^N \end{pmatrix} (z, \xi),$$

but also for $\ell = 1 \cdots n$, the $(c + r + \ell)$-th row of the former one equals to the $(c + r + \ell)$-th row of the latter one plus a multiple of the ℓ-th row. Therefore both matrices have the same determinant, which verifies (67). □

Inspired by the explicit global symmetric differential forms in Lemma 4.5 of Brotbek's paper [7], we carry out a simple proposition employing the above notation. First, let us recall the well known Cramer's rule in a less familiar formulation (cf. [39, p. 513, Theorem 4.4]).

Theorem 6.6. [Cramer's rule] *In a commutative ring R, for all positive integers $N \geqslant 1$, let:*

$$A^0, A^1, \ldots, A^N \in R^N$$

be $(N + 1)$ column vectors, and suppose that $z_0, z_1, \ldots, z_N \in R$ satisfy:

$$A^0 z_0 + A^1 z_1 + \cdots + A^N z_N = \mathbf{0}. \tag{68}$$

Then for all index pairs $0 \leqslant i < j \leqslant N$, there holds the identity:

$$(-1)^j \det\,(A^0, \ldots, \widehat{A^j}, \ldots, A^N)\, z_i = (-1)^i \det\,(A^0, \ldots, \widehat{A^i}, \ldots, A^N)\, z_j. \tag{69}$$

Proof. By permuting the indices, without loss of generality, we may assume $i = 0$.

First, note that (68) yields:

$$A^0 z_0 = -\sum_{\ell=1}^{N} A^\ell z_\ell. \tag{70}$$

Hence we may compute the left hand side of (69) as:

$$\begin{aligned}
(-1)^j \det\,(A^0, A^1, \ldots, \widehat{A^j}, \ldots, A^N)\, z_0 \;&=\; (-1)^j \det\,(A^0 z_0, A^1, \ldots, \widehat{A^j}, \ldots, A^N) \\
\text{[substitute (70)]} \qquad &=\; (-1)^j \det\Big(-\sum_{\ell=1}^{N} A^\ell z_\ell, A^1, \ldots, \widehat{A^j}, \ldots, A^N\Big) \\
&=\; (-1)^{j+1} \sum_{\ell=1}^{N} \det\,(A^\ell, A^1, \ldots, \widehat{A^j}, \ldots, A^N)\, z^\ell \\
\text{[only } \ell = j \text{ is nonzero]} \qquad &=\; (-1)^{j+1} \det\,(A^j, A^1, \ldots, \widehat{A^j}, \ldots, A^N)\, z^j \\
&=\; (-1)^0 \det\,(\widehat{A^0}, A^1, \ldots, A^N)\, z^j,
\end{aligned}$$

which is exactly the right hand side. □

Proposition 6.7. *The following $(N + 1)$ affine regular symmetric horizontal differential n-forms:*

$$\widehat{\omega}_j := \frac{(-1)^j}{z_j^{\lambda_j - 1}} \det(\widehat{\mathsf{D}}_j) \quad \in \Gamma\Big(\widehat{U}_j,\, \mathsf{Sym}^n\, \Omega_{\mathrm{hor}} \widehat{V}\Big) \qquad (j = 0 \cdots N)$$

glue together to make a regular symmetric horizontal differential n-form on $\widehat{X}$:

$$\omega \in \Gamma\Big(\widehat{X},\, \mathsf{Sym}^n\, \Omega_{\mathrm{hor}} \widehat{V}\Big).$$

Proof. Our proof divides into two parts.

Part 1: To show that these affine regular symmetric horizontal differential n-forms $\widehat{\omega}_0, \ldots, \widehat{\omega}_N$ are well defined.

Part 2: To show that any two different affine regular symmetric horizontal differential n-forms $\widehat{\omega}_{j_1}$ and $\widehat{\omega}_{j_2}$ glue together along the intersection set $\widehat{U}_{j_1} \cap \widehat{U}_{j_2}$.

Proof of Part 1. The Proposition 6.3 above shows that the:

$$\widehat{\omega}_j := \frac{(-1)^j}{z_j^{\lambda_j - 1}} \det\big(\widehat{\mathsf{D}}_j\big) \quad \in \Gamma\Big(\widehat{W}_j,\, \mathsf{Sym}^n\, \Omega_{\mathrm{hor}}\mathbb{K}^{N+1}\Big) \qquad (j = 0 \cdots N),$$

are well defined, where:

$$\widehat{W}_j = \{z_j \neq 0\} \subset \mathbb{K}^{N+1} \setminus \{0\}.$$

Thanks to the canonical inclusion embedding of vector bundles:

$$\Big(\widehat{U}_j,\, \mathrm{T}_{\mathrm{hor}}\widehat{V}\Big) \hookrightarrow \Big(\widehat{W}_j,\, \mathrm{T}_{\mathrm{hor}}\mathbb{K}^{N+1}\Big),$$

a pullback of $\widehat{\omega}_j$ concludes the first part.

Proof of Part 2. Recalling the equations (58), in particular, granted that D consists of the first $(c + r + n)$ rows of C, we have:

$$\mathsf{D} \begin{pmatrix} z_0^{\lambda_0 - 1} \\ \vdots \\ z_N^{\lambda_N - 1} \end{pmatrix} = 0.$$

Now applying the above Cramer's rule to all the $(N + 1)$ columns of D and the $(N + 1)$ values $z_0^{\lambda_0 - 1}, \ldots, z_N^{\lambda_N - 1}$, for every index pair $0 \leqslant j_1 < j_2 \leqslant N$, we receive:

$$(-1)^{j_2} \det\big(\widehat{\mathsf{D}}_{j_2}\big)\, z_{j_1}^{\lambda_{j_1} - 1} = (-1)^{j_1} \det\big(\widehat{\mathsf{D}}_{j_1}\big)\, z_{j_2}^{\lambda_{j_2} - 1}.$$

When $z_{j_1} \neq 0, z_{j_2} \neq 0$, this immediately yields:

$$\underbrace{\frac{(-1)^{j_1}}{z_{j_1}^{\lambda_{j_1} - 1}} \det\big(\widehat{\mathsf{D}}_{j_1}\big)}_{= \widehat{\omega}_{j_1}} = \underbrace{\frac{(-1)^{j_2}}{z_{j_2}^{\lambda_{j_2} - 1}} \det\big(\widehat{\mathsf{D}}_{j_2}\big)}_{= \widehat{\omega}_{j_2}},$$

thus the two affine symmetric horizontal differential n-forms $\widehat{\omega}_{j_1}$ and $\widehat{\omega}_{j_2}$ glue together along their overlap set $\widehat{U}_{j_1} \cap \widehat{U}_{j_2}$. □

By permuting the indices, the above Proposition 6.7 can be trivially generalized to, instead of the particular upper $(c + r + n) \times (N + 1)$ submatrix D, all $(c + r + n) \times (N + 1)$ submatrices of C containing the upper $c + r$ rows, as follows.

For all n ascending positive integers $1 \leqslant j_1 < \cdots < j_n \leqslant c$, denote $\mathsf{C}_{j_1, \ldots, j_n}$ the $(c + r + n) \times (N + 1)$ submatrix of C consisting of the first upper $c + r$ rows and the rows $c + r + j_1, \ldots, c + r + j_n$. Also, for $j = 0 \cdots N$, let $\widehat{\mathsf{C}}_{j_1, \ldots, j_n;\, j}$ denote the submatrix of $\mathsf{C}_{j_1, \ldots, j_n}$ obtained by omitting the $(j + 1)$-th column.

Proposition 6.8. *The following $(N + 1)$ affine regular symmetric horizontal differential n-forms:*

$$\widehat{\omega}_{j_1, \ldots, j_n;\, j} := \frac{(-1)^j}{z_j^{\lambda_j - 1}} \det\big(\widehat{\mathsf{C}}_{j_1, \ldots, j_n;\, j}\big) \quad \in \Gamma\Big(\widehat{U}_j,\, \mathsf{Sym}^n\, \Omega_{\mathrm{hor}}\widehat{V}\Big) \qquad (j = 0 \cdots N)$$

glue together to make a global regular symmetric horizontal differential n-form on $\widehat{X}$. □

One step further, the above Proposition 6.8 can be generalized to a larger class of $(c + r + n) \times (N + 1)$ submatrices of C, as follows.

For any two positive integers $l_1 \geqslant l_2$ with:

$$l_1 + l_2 = c + r + n,$$

for any two ascending sequences of positive indices:

$$1 \leqslant i_1 < \cdots < i_{l_1} \leqslant c + r,$$
$$1 \leqslant j_1 < \cdots < j_{l_2} \leqslant c$$

satisfying:

$$\{j_1, \ldots, j_{l_2}\} \subset \{i_1, \ldots, i_{l_1}\},$$

denote $\mathsf{C}^{i_1,\ldots,i_{l_1}}_{j_1,\ldots,j_{l_2}}$ the $(c + r + n) \times (N + 1)$ submatrix of C consisting of the rows $i_1, \ldots, i_{l_1}$ and the rows $c + r + j_1, \ldots, c + r + j_{l_2}$. Also, for $j = 0, \ldots, N$, let $\widehat{\mathsf{C}}^{i_1,\ldots,i_{l_1}}_{j_1,\ldots,j_{l_2};\, j}$ denote the submatrix of $\mathsf{C}^{i_1,\ldots,i_{l_1}}_{j_1,\ldots,j_{l_2}}$ obtained by omitting the $(j + 1)$-th column.

By much the same proof of Proposition 6.7, we obtain:

Proposition 6.9. *The following $N + 1$ affine regular symmetric horizontal differential l_2-forms:*

$$\widehat{\omega}^{i_1,\ldots,i_{l_1}}_{j_1,\ldots,j_{l_2};\, j} := \frac{(-1)^j}{z_j^{\lambda_j - 1}} \det\Big(\widehat{\mathsf{C}}^{i_1,\ldots,i_{l_1}}_{j_1,\ldots,j_{l_2};\, j}\Big) \in \Gamma\Big(\widehat{U}_j,\, \mathsf{Sym}^{l_2}\, \Omega_{\mathrm{hor}}\widehat{V}\Big) \qquad (j = 0 \cdots N)$$

glue together to make a global regular symmetric horizontal differential l_2-form:

$$\widehat{\omega}^{i_1,\ldots,i_{l_1}}_{j_1,\ldots,j_{l_2}} \in \Gamma\Big(\widehat{X},\, \mathsf{Sym}^{l_2}\, \Omega_{\mathrm{hor}}\widehat{V}\Big). \qquad \square$$

6.3. **Global twisted regular symmetric differential forms.** Now, using the structure of the above explicit global forms, and applying the previous Proposition 6.2, we receive a result which, in the case of pure Fermat-type hypersurfaces (1) where all $\lambda_0 = \cdots = \lambda_N = \epsilon$ are equal, with also equal $\deg F_1 = \cdots = \deg F_c = e + \epsilon$, coincides with Brotbek's Lemma 4.5 in [7]; Brotbek also implicitly obtained such symmetric differential forms by his cohomological approach.

Proposition 6.10. *Under the assumptions and notation of Proposition 6.9, the global regular symmetric horizontal differential l_2-form $\widehat{\omega}^{i_1,\ldots,i_{l_1}}_{j_1,\ldots,j_{l_2}}$ is the image of a global twisted regular symmetric differential l_2-form:*

$$\omega^{i_1,\ldots,i_{l_1}}_{j_1,\ldots,j_{l_2}} \in \Gamma(X, \mathsf{Sym}^{l_2}\, \Omega_V(\heartsuit))$$

under the canonical injection as a particular case of Proposition 6.2:

$$\Gamma(X, \mathsf{Sym}^{l_2}\, \Omega_V(\heartsuit)) \hookrightarrow \Gamma(\widehat{X}, \mathsf{Sym}^{l_2}\, \Omega_{\mathrm{hor}}\widehat{V}),$$

where the degree:

$$\heartsuit := \sum_{p=1}^{l_1} \deg F_{i_p} + \sum_{q=1}^{l_2} \deg F_{j_q} - \sum_{j=0}^{N} \lambda_j + N + 1. \tag{71}$$

For all homogeneous polynomials $P \in \Gamma(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\deg P))$, by multiplication, one receives more global twisted regular symmetric differential l_2-forms:

$$P\, \omega^{i_1,\ldots,i_{l_1}}_{j_1,\ldots,j_{l_2}} \in \Gamma(X, \mathsf{Sym}^{l_2}\, \Omega_V(\deg P + \heartsuit)). \qquad \square$$

It is worth to mention that, again by applying Cramer's rule in linear algebra, one can construct determinantal shape sections concerning higher-order jet bundles on Fermat type hypersurfaces, as well as on their intersections.

Proof. According to the criterion (50) of Proposition 6.2, it is necessary and sufficient to show, for all $z \in \widehat{X}$, for all $[\xi] \in \mathrm{T}_{\mathrm{hor}}\widehat{V}\big|_z$ and for all $\lambda \in \mathbb{K}^\times$, that:

$$\widehat{\omega}^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2}}(\lambda z, [\lambda\xi]) = \lambda^{\heartsuit}\, \widehat{\omega}^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2}}(z, [\xi]). \tag{72}$$

We may assume $z \in \widehat{U}_0 = \{z_0 \neq 0\}$ for instance. Now, applying Proposition 6.9, we receive:

$$\begin{aligned}\widehat{\omega}^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2}}(\lambda z, [\lambda\xi]) &= \widehat{\omega}^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2};0}(\lambda z, [\lambda\xi]) \\ &= \frac{1}{z_0^{\lambda_0-1}} \begin{pmatrix} \widetilde{\mathsf{A}}^1_{i_1} & \cdots & \widetilde{\mathsf{A}}^N_{i_1} \\ \vdots & & \vdots \\ \widetilde{\mathsf{A}}^1_{i_{l_1}} & \cdots & \widetilde{\mathsf{A}}^N_{i_{l_1}} \\ \mathsf{B}^1_{j_1} & \cdots & \mathsf{B}^N_{j_1} \\ \vdots & & \vdots \\ \mathsf{B}^1_{j_{l_2}} & \cdots & \mathsf{B}^N_{j_{l_2}} \end{pmatrix}(\lambda z, \lambda\xi).\end{aligned} \tag{73}$$

For all $j = 0 \cdots N$, for all $p = 1 \cdots l_1$, and for all $q = 1 \cdots l_2$, recall the degree identity (52) which shows that the entry $\widetilde{\mathsf{A}}^j_{i_p} = z_j A^j_i$ is a homogeneous polynomial of degree:

$$\deg A^j_{i_p} + 1 = \deg F_{i_p} - \lambda_j + 1,$$

and therefore satisfies:

$$\widetilde{\mathsf{A}}^j_{i_p}(\lambda z) = \lambda^{\deg F_{i_p} - \lambda_j + 1} \widetilde{\mathsf{A}}^j_{i_p}(z). \tag{74}$$

Recalling also the notation (54), the entry $\mathsf{B}^j_{j_q}$ is a 1-form satisfying:

$$\begin{aligned}\mathsf{B}^j_{j_q}(\lambda z, \lambda[\xi]) &= \lambda^{\deg A^j_{j_q}+1}\, \mathsf{B}^j_{j_q}(z, [\xi]) \\ &= \lambda^{\deg F_{j_q} - \lambda_j + 1}\, \mathsf{B}^j_{j_q}(z, [\xi]).\end{aligned} \tag{75}$$

Now, let us continue to compute (73), starting by expanding the determinant:

$$\begin{pmatrix} \widetilde{\mathsf{A}}^1_{i_1} & \cdots & \widetilde{\mathsf{A}}^N_{i_1} \\ \vdots & & \vdots \\ \widetilde{\mathsf{A}}^1_{i_{l_1}} & \cdots & \widetilde{\mathsf{A}}^N_{i_{l_1}} \\ \mathsf{B}^1_{j_1} & \cdots & \mathsf{B}^N_{j_1} \\ \vdots & & \vdots \\ \mathsf{B}^1_{j_{l_2}} & \cdots & \mathsf{B}^N_{j_{l_2}} \end{pmatrix}(\lambda z, \lambda\xi) = \sum_{\sigma \in \mathfrak{S}_N} \mathrm{sign}(\sigma)\, \widetilde{\mathsf{A}}^{\sigma(1)}_{i_1} \cdots \widetilde{\mathsf{A}}^{\sigma(l_1)}_{i_{l_1}} \mathsf{B}^{\sigma(l_1+1)}_{j_1} \cdots \mathsf{B}^{\sigma(l_1+l_2)}_{j_{l_2}}(\lambda z, \lambda\xi). \tag{76}$$

With the help of the above two entry identities (74) and (75), each term in the above sum equals to:

$$\mathrm{sign}(\sigma)\, \widetilde{\mathsf{A}}^{\sigma(1)}_{i_1} \cdots \widetilde{\mathsf{A}}^{\sigma(l_1)}_{i_{l_1}} \mathsf{B}^{\sigma(l_1+1)}_{j_1} \cdots \mathsf{B}^{\sigma(l_1+l_2)}_{j_{l_2}}(z, \xi)$$

multiplied by $\lambda^{?}$, where:

$$\begin{aligned}
? &= \sum_{p=1}^{l_1} (\deg F_{i_p} - \lambda_{\sigma(p)} + 1) + \sum_{q=1}^{l_2} (\deg F_{j_q} - \lambda_{\sigma(l_1+q)} + 1) \\
&= \sum_{p=1}^{l_1} \deg F_{i_p} + \sum_{q=1}^{l_2} \deg F_{j_q} - \underbrace{\Big(\sum_{p=1}^{l_1} \lambda_{\sigma(p)} + \sum_{q=1}^{l_2} \lambda_{\sigma(l_1+q)} \Big)}_{= \sum_{j=1}^{l_1+l_2} \lambda_{\sigma(j)}} + \underbrace{l_1 + l_2}_{=N} \\
&= \sum_{p=1}^{l_1} \deg F_{i_p} + \sum_{q=1}^{l_2} \deg F_{j_q} - \sum_{j=1}^{N} \lambda_j + N \\
\text{[Use (71)]} \qquad &= \heartsuit + \lambda_0 - 1,
\end{aligned}$$

therefore (76) factors as:

$$\lambda^{\heartsuit+\lambda_0-1} \sum_{\sigma\in\mathbb{S}_N} \operatorname{sign}(\sigma)\, \widetilde{\mathsf{A}}^{\sigma(1)}_{i_1} \cdots \widetilde{\mathsf{A}}^{\sigma(l_1)}_{i_{l_1}} \mathsf{B}^{\sigma(l_1+1)}_{j_1} \cdots \mathsf{B}^{\sigma(l_1+l_2)}_{j_{l_2}}(z,\xi) \;=\; \lambda^{\heartsuit+\lambda_0-1} \begin{pmatrix} \widetilde{\mathsf{A}}^1_{i_1} & \cdots & \widetilde{\mathsf{A}}^N_{i_1} \\ \vdots & & \vdots \\ \widetilde{\mathsf{A}}^1_{i_{l_1}} & \cdots & \widetilde{\mathsf{A}}^N_{i_{l_1}} \\ \mathsf{B}^1_{j_1} & \cdots & \mathsf{B}^N_{j_1} \\ \vdots & & \vdots \\ \mathsf{B}^1_{j_{l_2}} & \cdots & \mathsf{B}^N_{j_{l_2}} \end{pmatrix}(z,\xi),$$

and thus (73) becomes:

$$\begin{aligned}
\frac{1}{(\lambda z_0)^{\lambda_0-1}} \lambda^{\heartsuit+\lambda_0-1} \begin{pmatrix} \widetilde{\mathsf{A}}^1_{i_1} & \cdots & \widetilde{\mathsf{A}}^N_{i_1} \\ \vdots & & \vdots \\ \widetilde{\mathsf{A}}^1_{i_{l_1}} & \cdots & \widetilde{\mathsf{A}}^N_{i_{l_1}} \\ \mathsf{B}^1_{j_1} & \cdots & \mathsf{B}^N_{j_1} \\ \vdots & & \vdots \\ \mathsf{B}^1_{j_{l_2}} & \cdots & \mathsf{B}^N_{j_{l_2}} \end{pmatrix}(z,\xi) &= \lambda^{\heartsuit} \frac{1}{z_0^{\lambda_0-1}} \begin{pmatrix} \widetilde{\mathsf{A}}^1_{i_1} & \cdots & \widetilde{\mathsf{A}}^N_{i_1} \\ \vdots & & \vdots \\ \widetilde{\mathsf{A}}^1_{i_{l_1}} & \cdots & \widetilde{\mathsf{A}}^N_{i_{l_1}} \\ \mathsf{B}^1_{j_1} & \cdots & \mathsf{B}^N_{j_1} \\ \vdots & & \vdots \\ \mathsf{B}^1_{j_{l_2}} & \cdots & \mathsf{B}^N_{j_{l_2}} \end{pmatrix}(z,\xi) \\
&= \lambda^{\heartsuit}\, \widehat{\omega}^{i_1,\ldots,i_{l_1}}_{j_1,\ldots,j_{l_2};0}(z,[\xi]) \\
&= \lambda^{\heartsuit}\, \widehat{\omega}^{i_1,\ldots,i_{l_1}}_{j_1,\ldots,j_{l_2}}(z,[\xi]),
\end{aligned}$$

which is exactly our desired equality (72). □

Now, let K be the $(c+r+c)\times(N+1)$ matrix whose first $c+r$ rows consist of all $(N+1)$ terms in the expressions of $F_1,\ldots,F_{c+r}$ in the exact order, i.e. the (i,j)-th entry of K is:

$$\mathsf{K}_{i,j} := A_i^{j-1}\, z_{j-1}^{\lambda_{j-1}} \qquad (i=1\cdots c+r;\, j=1\cdots N+1), \tag{77}$$

and whose last c rows consist of all $(N+1)$ terms in the expressions of $dF_1,\ldots,dF_c$ in the exact order, i.e. the $(c+r+i,j)$-th entry of K is:

$$\mathsf{K}_{c+r+i,j} := d(A_i^{j-1}\, z_{j-1}^{\lambda_{j-1}}) \qquad (i=1\cdots c;\, j=1\cdots N+1).$$

The j-th column K_j of K and the j-th column C_j of C are proportional:

$$\mathsf{K}_j = \mathsf{C}_j\, z_{j-1}^{\lambda_{j-1}-1} \qquad (j=1\cdots N+1). \tag{78}$$

In later applications, we will use Proposition 6.10 in the case:

$$l_1 = c + r, \quad l_2 = n,$$

and in abbreviation, dropping the upper indices, we will write these global symmetric differential forms $\omega^{1,\dots,c+r}_{j_1,\dots,j_n}$ as $\omega_{j_1,\dots,j_n}$. Since we will mainly consider the case where all coordinates are nonvanishing:

$$z_0 \neq 0, \dots, z_N \neq 0,$$

the corresponding symmetric horizontal differential n-forms $\widehat{\omega}_{j_1,\dots,j_n;\,j}$ of Proposition 6.8 read, in the set $\{z_0 \cdots z_N \neq 0\}$, as:

$$\begin{aligned}
\widehat{\omega}_{j_1,\dots,j_n;\,j} &= \frac{(-1)^j}{z_j^{\lambda_j - 1}} \det\big(\widehat{\mathsf{C}}_{j_1,\dots,j_n;\,j}\big) \\
\text{[Use (78)]} \qquad &= \frac{(-1)^j}{z_j^{\lambda_j - 1}} \Big(\prod_{0 \leqslant i \leqslant N, i \neq j} \frac{1}{z_i^{\lambda_i - 1}} \Big) \det\big(\widehat{\mathsf{K}}_{j_1,\dots,j_n;\,j}\big) \qquad (79) \\
&= \frac{(-1)^j}{z_0^{\lambda_0 - 1} \cdots z_N^{\lambda_N - 1}} \det\big(\widehat{\mathsf{K}}_{j_1,\dots,j_n;\,j}\big) \qquad (j = 0 \cdots N),
\end{aligned}$$

where $\widehat{\mathsf{K}}_{j_1,\dots,j_n;\,j}$ is defined as an analog of $\widehat{\mathsf{C}}_{j_1,\dots,j_n;\,j}$ in the obvious way.

6.4. **Regular twisted symmetric differential forms with some vanishing coordinates.** Investigating further the construction of symmetric differential forms via Cramer's rule, for every integer $1 \leqslant \eta \leqslant n - 1$, for every sequence of ascending indices:

$$0 \leqslant v_1 < \cdots < v_\eta \leqslant N,$$

by focusing on the intersection of X with the η coordinate hyperplanes:

$${}_{v_1,\dots,v_\eta}X := X \cap \{z_{v_1} = 0\} \cap \cdots \cap \{z_{v_\eta} = 0\},$$

we can also construct several twisted symmetric differential $(n - \eta)$-forms:

$$\Gamma\Big({}_{v_1,\dots,v_\eta}X,\, \mathrm{Sym}^{n-\eta}\, \Omega_V(?)\Big) \qquad (?\text{ are twisted degrees})$$

as follows, which will be essential ingredients towards the solution of the Debarre Ampleness Conjecture.

For every two positive integers $l_1 \geqslant l_2$ with:

$$l_1 + l_2 = c + r + n - \eta = N - \eta,$$

for any two sequences of ascending positive integers:

$$\begin{aligned}
1 \leqslant i_1 < \cdots < i_{l_1} \leqslant c + r \\
1 \leqslant j_1 < \cdots < j_{l_2} \leqslant c
\end{aligned}$$

such that the second one is a subsequence of the first one:

$$\{j_1, \dots, j_{l_2}\} \;\subset\; \{i_1, \dots, i_{l_1}\},$$

let us denote by ${}_{v_1,\dots,v_\eta}\mathsf{C}^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2}}$ the $(N - \eta) \times (N - \eta + 1)$ submatrix of C determined by the $(N - \eta)$ rows $i_1, \dots, i_{l_1}, c + r + j_1, \dots, c + r + j_{l_2}$ and the $(N - \eta + 1)$ columns which are complement to the columns $v_1 + 1, \dots, v_\eta + 1$. Also, for every index $j \in \{0, \dots, N\} \setminus \{v_1, \dots, v_\eta\}$, let ${}_{v_1,\dots,v_\eta}\widehat{\mathsf{C}}^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2};\,j}$

denote the submatrix of ${}_{v_1,\dots,v_\eta}\mathsf{C}^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2}}$ obtained by deleting the column which is originally contained in the $(j+1)$-th column of C. Analogously to (61)-(64), we denote:

$$ {}_{v_1,\dots,v_\eta}W_j := \{z_{v_1} = 0\} \cap \dots \cap \{z_{v_\eta} = 0\} \cap \{z_j \neq 0\} \subset \mathbb{P}^N, $$

whose cone is:

$$ {}_{v_1,\dots,v_\eta}\widehat{W}_j := \pi^{-1}({}_{v_1,\dots,v_\eta}W_j) \subset \mathbb{K}^{N+1} \setminus \{0\}, $$

and we denote also:

$$ {}_{v_1,\dots,v_\eta}U_j := {}_{v_1,\dots,v_\eta}W_j \cap X \subset {}_{v_1,\dots,v_\eta}X, $$

whose cone is:

$$ {}_{v_1,\dots,v_\eta}\widehat{U}_j := \pi^{-1}({}_{v_1,\dots,v_\eta}U_j) \subset {}_{v_1,\dots,v_\eta}\widehat{X} := \pi^{-1}({}_{v_1,\dots,v_\eta}X). $$

Now we have two very analogs of Propositions 6.9 and 6.10.

First, write the $(N-\eta+1)$ remaining numbers of the set-minus:

$$ \{0,\dots,N\} \setminus \{v_1,\dots,v_\eta\} $$

in the ascending order:

$$ r_0 < r_1 < \dots < r_{N-\eta}. \tag{80} $$

It is necessary to assume that $\lambda_0,\dots,\lambda_N \geqslant 2$.

Proposition 6.11. *For all $j = 0 \cdots N-\eta$, the following $(N+1-\eta)$ affine regular symmetric horizontal differential l_2-forms:*

$$ {}_{v_1,\dots,v_\eta}\widehat{\omega}^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2};\,r_j} := \frac{(-1)^j}{z_{r_j}^{\lambda_{r_j}-1}} \det\left({}_{v_1,\dots,v_\eta}\widehat{\mathsf{C}}^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2};\,r_j}\right) \in \Gamma\Big({}_{v_1,\dots,v_\eta}\widehat{U}_{r_j},\, \mathsf{Sym}^{l_2}\,\Omega_{\mathrm{hor}}\widehat{V}\Big) $$

glue together to make a regular symmetric horizontal differential l_2-form on ${}_{v_1,\dots,v_\eta}\widehat{X}$:

$$ {}_{v_1,\dots,v_\eta}\widehat{\omega}^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2}} \in \Gamma\Big({}_{v_1,\dots,v_\eta}\widehat{X},\, \mathsf{Sym}^{l_2}\,\Omega_{\mathrm{hor}}\widehat{V}\Big). $$

Proposition 6.12. *Under the assumptions and notation of the above proposition, the regular symmetric horizontal differential l_2-form ${}_{v_1,\dots,v_\eta}\widehat{\omega}^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2}}$ on ${}_{v_1,\dots,v_\eta}\widehat{X}$ is the image of a twisted regular symmetric differential l_2-form on ${}_{v_1,\dots,v_\eta}X$:*

$$ {}_{v_1,\dots,v_\eta}\omega^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2}} \in \Gamma\left({}_{v_1,\dots,v_\eta}X, \mathsf{Sym}^{l_2}\,\Omega_V({}_{v_1,\dots,v_\eta}\heartsuit^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2}})\right) $$

under the canonical injection:

$$ \Gamma\left({}_{v_1,\dots,v_\eta}X, \mathsf{Sym}^{l_2}\,\Omega_V({}_{v_1,\dots,v_\eta}\heartsuit^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2}})\right) \hookrightarrow \Gamma\left({}_{v_1,\dots,v_\eta}\widehat{X}, \mathsf{Sym}^{l_2}\,\Omega_{\mathrm{hor}}\widehat{V}\right), $$

where the twisted degree is:

$$ {}_{v_1,\dots,v_\eta}\heartsuit^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2}} := \sum_{p=1}^{l_1} \deg F_{i_p} + \sum_{q=1}^{l_2} \deg F_{j_q} - \Big(\sum_{j=0}^{N} \lambda_j - \sum_{\mu=1}^{\eta} \lambda_{v_\mu}\Big) + (N-\eta) + 1. \tag{81} $$

Furthermore, for all homogeneous polynomials $P \in \Gamma(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\deg P))$, by multiplication, one receives more twisted regular symmetric differential l_2-forms:

$$ P\,{}_{v_1,\dots,v_\eta}\omega^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2}} \in \Gamma\left({}_{v_1,\dots,v_\eta}X, \mathsf{Sym}^{l_2}\,\Omega_V(\deg P + {}_{v_1,\dots,v_\eta}\heartsuit^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2}})\right). $$ □

In our coming applications, we will use Proposition 6.12 in the case:

$$l_1 = c + r, \quad l_2 = n - \eta,$$

and in abbreviation we write these symmetric differential forms ${}_{v_1,\dots,v_\eta}\omega^{1,\dots,c+r}_{j_1,\dots,j_{n-\eta}}$ as ${}_{v_1,\dots,v_\eta}\omega_{j_1,\dots,j_{n-\eta}}$. Since we will mainly consider the case when all coordinates but $z_{v_1}, \dots, z_{v_\eta}$ are nonvanishinig:

$$z_{r_0} \neq 0, \dots, z_{r_{N-\eta}} \neq 0,$$

the corresponding symmetric horizontal differential $(n-\eta)$-forms ${}_{v_1,\dots,v_\eta}\omega_{j_1,\dots,j_{n-\eta}}$ of Proposition 6.11 read, in the set $\{z_{r_0} \cdots z_{r_{N-\eta}} \neq 0\}$, as:

$$\begin{aligned}
{}_{v_1,\dots,v_\eta}\widehat{\omega}_{j_1,\dots,j_{l_2};\,r_j} &:= \frac{(-1)^j}{z_{r_j}^{\lambda_{r_j}-1}} \det\left({}_{v_1,\dots,v_\eta}\widehat{\mathsf{C}}^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2};\,r_j}\right) \\
\text{[Use (78)]} \qquad &= \frac{(-1)^j}{z_{r_j}^{\lambda_{r_j}-1}} \Big(\prod_{0\leqslant i\leqslant N-\eta, i\neq j} \frac{1}{z_{r_i}^{\lambda_{r_i}-1}} \Big) \det\left({}_{v_1,\dots,v_\eta}\widehat{\mathsf{K}}_{j_1,\dots,j_{n-\eta};\,r_j}\right) \qquad (82) \\
&= \frac{(-1)^j}{z_{r_0}^{\lambda_{r_0}-1} \cdots z_{r_{N-\eta}}^{\lambda_{r_{N-\eta}}-1}} \det\left({}_{v_1,\dots,v_\eta}\widehat{\mathsf{K}}_{j_1,\dots,j_{n-\eta};\,r_j}\right) \qquad (j = 0 \cdots N-\eta),
\end{aligned}$$

where ${}_{v_1,\dots,v_\eta}\widehat{\mathsf{K}}_{j_1,\dots,j_{n-\eta};\,r_j}$ is defined as an analog of ${}_{v_1,\dots,v_\eta}\widehat{\mathsf{C}}_{j_1,\dots,j_{n-\eta};\,r_j}$ in the obvious way.

The two formulas (79), (82) will enable us to efficiently narrow the base loci of the obtained symmetric differential forms, as the matrix K directly copies the original equations/differentials of the hypersurface polymonials $F_1, \dots, F_{c+r}$. We will heartily appreciate such a formalism when a wealth of moving coefficient terms happen to tangle together.

6.5. **A scheme-theoretic point of view.** In future applications, we will only consider symmetric forms *in coordinates*. Nevertheless, in this subsection, let us reconsider the obtained symmetric forms in an algebraic way, dropping the assumption 'algebraically-closed' on the ambient field $\mathbb{K}$.

Recalling (3), (4), we may denote the projective parameter space of the $c + r$ hypersurfaces in (51) by:

$$\mathbb{P}^{\blacklozenge}_{\mathbb{K}} = \operatorname{Proj} \mathbb{K}\left[\left\{A^j_{i,\alpha}\right\}_{\substack{i=1\cdots c+r \\ j=0\cdots N \\ |\alpha|=d_i-\lambda_j}}\right],$$

so that the hypersurface coefficient polynomials A^j_i are written as:

$$A^j_i := \sum_{|\alpha|=d_i-\lambda_j} A^j_{i,\alpha}\, z^\alpha \qquad (i = 1 \cdots c+r,\ j = 0 \cdots N). \qquad (83)$$

Now, we give a scheme-theoretic explanation of Proposition 6.3, firstly by expressing $\widehat{\omega}_j$ in terms of affine coordinates.

For every index $j = 0 \cdots N$, in each affine set:

$$\widehat{W}_j = \{z_j \neq 0\} \subset \mathbb{K}^{N+1} \setminus \{0\},$$

the $c + r$ homogeneous hypersurface equations (51) in affine coordinates:

$$\left(\frac{z_0}{z_j}, \dots, \frac{\widehat{z_j}}{z_j}, \dots, \frac{z_N}{z_j}\right)$$

become:

$$\begin{aligned}(F_i)_j &= \sum_{k=0}^{N} (A_i^k)_j \left(\frac{z_k}{z_j}\right)^{\lambda_k} \\ \text{[see (55)]} \qquad &= \sum_{k=0}^{N} (\widetilde{\mathsf{A}}_i^k)_j \left(\frac{z_k}{z_j}\right)^{\lambda_k - 1},\end{aligned} \tag{84}$$

where for any homogeneous polynomial P, we dehomogenize:

$$(P)_j := \frac{P}{z_j^{\deg P}}.$$

Differentiating (84) for $i = 1 \cdots c$, we receive:

$$d\,(F_i)_j = \sum_{k=0}^{N} \mathsf{B}_{i,j}^k \left(\frac{z_k}{z_j}\right)^{\lambda_k - 1},$$

where:

$$\mathsf{B}_{i,j}^k := \frac{z_k}{z_j}\, d\,(A_i^k)_j + \lambda_k\, (A_i^k)_j\, d\left(\frac{z_k}{z_j}\right) \qquad (j,k = 0 \cdots N). \tag{85}$$

Computing $z_k\, d\,(A_i^k)_j$, we receive:

$$\begin{aligned} z_k\, d\,(A_i^k)_j &= z_k\, d\left(\frac{A_i^k}{z_j^{d_i - \lambda_k}}\right) \qquad \text{[use (52)]} \\ \text{[Leibniz's rule]} \qquad &= \frac{z_k\, d\, A_i^k}{z_j^{d_i - \lambda_k}} - (d_i - \lambda_k) \frac{z_k\, A_i^k}{z_j^{d_i - \lambda_k + 1}}\, dz_j \\ \text{[use (53)]} \qquad &= (\mathsf{B}_i^k - \lambda_k\, A_i^k\, dz_k) \frac{1}{z_j^{d_i - \lambda_j}} - (d_i - \lambda_k) \frac{z_k\, A_i^k}{z_j^{d_i - \lambda_j + 1}}\, dz_j, \end{aligned}$$

therefore (85) become:

$$\begin{aligned} \mathsf{B}_{i,j}^k &= (\mathsf{B}_i^k - \lambda_k\, A_i^k\, dz_k) \frac{1}{z_j^{d_i - \lambda_j + 1}} - (d_i - \lambda_k) \frac{z_k\, A_i^k}{z_j^{d_i - \lambda_j + 2}}\, dz_j + \lambda_k\, (A_i^k)_j\, d\left(\frac{z_k}{z_j}\right) \\ &= \frac{\mathsf{B}_i^k}{z_j^{d_i - \lambda_j + 1}} - \lambda_k\, (A_i^k)_j \frac{dz_k}{z_j} - (d_i - \lambda_k)\, (A_i^k)_j \frac{z_k}{z_j^2}\, dz_j + \lambda_k\, (A_i^k)_j \left(\frac{dz_k}{z_j} - \frac{z_k}{z_j^2}\, dz_j\right) \\ &= \frac{1}{z_j^{d_i - \lambda_j + 1}} \mathsf{B}_i^k - d_i \frac{z_k}{z_j^2}\, dz_j\, (A_i^k)_j \\ &= \frac{1}{z_j^{d_i - \lambda_j + 1}} \mathsf{B}_i^k - \frac{d_i}{z_j}\, dz_j\, (\widetilde{\mathsf{A}}_i^k)_j. \end{aligned} \tag{86}$$

Recalling the matrix C in (57), which is obtained by copying the homogeneous hypersurface equations $F_1, \dots, F_{c+r}$ and the differentials $dF_1, \dots, dF_c$, we define the matrix:

$$(\mathsf{C})_j := \begin{pmatrix} (\widetilde{\mathsf{A}}_1^0)_j & \cdots & (\widetilde{\mathsf{A}}_1^N)_j \\ \vdots & & \vdots \\ (\widetilde{\mathsf{A}}_{c+r}^0)_j & \cdots & (\widetilde{\mathsf{A}}_{c+r}^N)_j \\ \mathsf{B}_{1,j}^0 & \cdots & \mathsf{B}_{1,j}^N \\ \vdots & & \vdots \\ \mathsf{B}_{c,j}^0 & \cdots & \mathsf{B}_{c,j}^N \end{pmatrix}, \tag{87}$$

which is obtained by copying the dehomogenized hypersurface equations $(F_1)_j, \dots, (F_{c+r})_j$ and the differentials $d(F_1)_j, \dots, d(F_c)_j$. Recalling the matrices (59), (60), in the obvious way we also define $(\mathsf{D})_j$, $(\widehat{\mathsf{D}}_k)_j$ as:

$$(\mathsf{D})_{\mathsf{j}} := \begin{pmatrix} (\widetilde{\mathsf{A}}_1^0)_j & \cdots & (\widetilde{\mathsf{A}}_1^N)_j \\ \vdots & & \vdots \\ (\widetilde{\mathsf{A}}_{c+r}^0)_j & \cdots & (\widetilde{\mathsf{A}}_{c+r}^N)_j \\ \mathsf{B}_{1,j}^0 & \cdots & \mathsf{B}_{1,j}^N \\ \vdots & & \vdots \\ \mathsf{B}_{n,j}^0 & \cdots & \mathsf{B}_{n,j}^N \end{pmatrix}, \tag{88}$$

and:

$$(\widehat{\mathsf{D}}_k)_j := \begin{pmatrix} (\widetilde{\mathsf{A}}_1^0)_j & \cdots & \widehat{(\widetilde{\mathsf{A}}_1^k)_j} & \dots & (\widetilde{\mathsf{A}}_1^N)_j \\ \vdots & & \vdots & & \\ (\widetilde{\mathsf{A}}_{c+r}^0)_j & \cdots & \widehat{(\widetilde{\mathsf{A}}_{c+r}^k)_j} & \dots & (\widetilde{\mathsf{A}}_{c+r}^N)_j \\ \mathsf{B}_{1,j}^0 & \cdots & \widehat{\mathsf{B}_{1,j}^k} & \dots & \mathsf{B}_{1,j}^N \\ \vdots & & \vdots & & \\ \mathsf{B}_{n,j}^0 & \cdots & \widehat{\mathsf{B}_{n,j}^k} & \dots & \mathsf{B}_{n,j}^N \end{pmatrix} \qquad (k = 0 \cdots N). \tag{89}$$

Recalling $\widehat{\omega}_j$ of Proposition 6.3, now thanks to (86), we have the following nice:

Observation 6.13. *For every $j = 0 \cdots N$, one has the identity:*

$$\widehat{\omega}_j \;=\; \frac{(-1)^j}{z_j^{\lambda_j - 1}} \det(\widehat{\mathsf{D}}_j) \;=\; \frac{(-1)^j}{z_j^{-\heartsuit}} \det\Big((\widehat{\mathsf{D}}_j)_j\Big), \tag{90}$$

where for the moment $\heartsuit$ is defined in (71) for $\omega_{1,\dots,c}^{1,\dots,c+r}$:

$$\heartsuit := \sum_{p=1}^{c+r} d_p + \sum_{q=1}^{n} d_q - \sum_{j=0}^{N} \lambda_j + N + 1.$$

The proof is much the same as that of Proposition 6.10, hence we omit it here. □

Now, let $\mathrm{pr}_1, \mathrm{pr}_2$ be the two canonical projections:

$$\begin{array}{ccccc} & & \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^{N}_{\mathbb{K}} & & \\ & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \\ \mathbb{P}^{\blacklozenge}_{\mathbb{K}} & & & & \mathbb{P}^{N}_{\mathbb{K}}. \end{array}$$

Then thanks to the formula (90), we may view $\widehat{\omega}_j$ as a section of the twisted sheaf:

$$\mathsf{Sym}^n \, \Omega^1_{\mathbb{P}^{\blacklozenge}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}} / \mathbb{P}^{\blacklozenge}_{\mathbb{K}}} \otimes \mathrm{pr}_1^* \, \mathscr{O}_{\mathbb{P}^{\blacklozenge}_{\mathbb{K}}}(N) \otimes \mathrm{pr}_2^* \, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\heartsuit)$$

over the pullback:

$$\mathrm{pr}_2^{-1}(\mathsf{W}_j) \subset \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}}$$

of the canonical affine scheme

$$\mathsf{W}_j := \mathsf{D}(z_j) \subset \mathbb{P}^N_{\mathbb{K}}.$$

Using the same notation as (5), (6), recalling (51), (83), we now introduce the two subschemes:

$$\mathscr{X} \subset \mathscr{V} \subset \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}},$$

where $\mathscr{X}$ is defined by 'all' the $c + r$ bihomogeneous polynomials:

$$\mathscr{X} := \mathrm{V}\Big(\sum_{j=0}^{N} A_1^j \, z_j^{\lambda_j}, \ldots, \sum_{j=0}^{N} A_c^j \, z_j^{\lambda_j}, \sum_{j=0}^{N} A_{c+1}^j \, z_j^{\lambda_j}, \ldots, \sum_{j=0}^{N} A_{c+r}^j \, z_j^{\lambda_j} \Big),$$

and where $\mathscr{V}$ is defined by the 'first' c bihomogeneous polynomials:

$$\mathscr{V} := \mathrm{V}\Big(\sum_{j=0}^{N} A_1^j \, z_j^{\lambda_j}, \ldots, \sum_{j=0}^{N} A_c^j \, z_j^{\lambda_j} \Big).$$

Now, we may view each entry of the matrix (87) as a section in:

$$\Gamma \, \Big(\mathscr{X} \cap \mathrm{pr}_2^{-1}(\mathsf{W}_j), \, \mathsf{Sym}^{\bullet} \, \Omega^1_{\mathscr{V} / \mathbb{P}^{\blacklozenge}_{\mathbb{K}}} \otimes \mathrm{pr}_1^* \, \mathscr{O}_{\mathbb{P}^{\blacklozenge}_{\mathbb{K}}}(1) \Big),$$

where the symmetric degrees are 0 for the first $c + r$ rows and 1 for the last n rows. Noting that the $N + 1$ columns $C^0, \cdots, C^N$ of this matrix satisfy the relation:

$$\sum_{k=0}^{N} C^k \, \frac{z_k^{\lambda_k - 1}}{z_j^{\lambda_k - 1}} \; = \; \mathbf{0},$$

in particular, so do the columns of the submatrix (88). Now, recalling the submatrices (89) of (88), an application of Cramer's rule (Theorem 6.6) yields:

$$\begin{aligned} (-1)^{k_1} \det\Big((\widehat{\mathsf{D}}_{k_1})_j \Big) \, \frac{z_{k_2}^{\lambda_{k_2} - 1}}{z_j^{\lambda_{k_2} - 1}} \; &= \; (-1)^{k_2} \det\Big((\widehat{\mathsf{D}}_{k_2})_j \Big) \, \frac{z_{k_1}^{\lambda_{k_1} - 1}}{z_j^{\lambda_{k_1} - 1}} \\ &\in \; \Gamma \, \Big(\mathscr{X} \cap \mathrm{pr}_2^{-1}(\mathsf{W}_j), \, \mathsf{Sym}^n \, \Omega^1_{\mathscr{V} / \mathbb{P}^{\blacklozenge}_{\mathbb{K}}} \otimes \mathrm{pr}_1^* \, \mathscr{O}_{\mathbb{P}^{\blacklozenge}_{\mathbb{K}}}(N) \Big) \qquad (k = 0 \cdots N). \end{aligned} \tag{91}$$

Now, recalling (90), we may interpret Proposition 6.7 as follows. First, for $j = 0 \cdots N$, we view each:

$$\widehat{\omega}_j = \frac{(-1)^j}{z_j^{-\heartsuit}} \det\Big((\widehat{\mathsf{D}}_j)_j \Big)$$

as a section in:

$$\Gamma\,\Big(\mathscr{X}\cap \mathrm{pr}_2^{-1}(\mathsf{W}_j),\ \mathsf{Sym}^n\,\Omega^1_{\mathscr{V}/\mathbb{P}^\blacklozenge_{\mathbb{K}}}\otimes \mathrm{pr}_1^*\,\mathscr{O}_{\mathbb{P}^\blacklozenge_{\mathbb{K}}}(N)\otimes \mathrm{pr}_2^*\,\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\heartsuit)\Big).$$

Then, thanks to an observation below, for every different indices $j_1<j_2$, over the open set:

$$\mathscr{X}\cap \mathrm{pr}_2^{-1}(\mathsf{W}_{j_1}\cap \mathsf{W}_{j_2})\ \subset\ \mathscr{X},$$

the twisted sheaf:

$$\mathsf{Sym}^n\,\Omega^1_{\mathscr{V}/\mathbb{P}^\blacklozenge_{\mathbb{K}}}\otimes \mathrm{pr}_1^*\,\mathscr{O}_{\mathbb{P}^\blacklozenge_{\mathbb{K}}}(N)\otimes \mathrm{pr}_2^*\,\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\heartsuit)$$

has the two coinciding sections:

$$\begin{aligned}
\widehat{\omega}_{j_1} &= \frac{(-1)^{j_1}}{z_{j_1}^{-\heartsuit}}\det\Big((\widehat{\mathsf{D}}_{j_1})_{j_1}\Big) && \text{[Observation 6.13]}\\
\text{[use (91)]}\qquad &= \frac{(-1)^{j_2}}{z_{j_1}^{-\heartsuit}}\,\frac{z_{j_1}^{\lambda_{j_2}-1}}{z_{j_2}^{\lambda_{j_2}-1}}\det\Big((\widehat{\mathsf{D}}_{j_2})_{j_1}\Big)\\
\text{[Observation 6.14 below]}\qquad &= \frac{(-1)^{j_2}}{z_{j_1}^{-\heartsuit}}\,\frac{z_{j_1}^{\lambda_{j_2}-1}}{z_{j_2}^{\lambda_{j_2}-1}}\,\frac{z_{j_2}^{\heartsuit+\lambda_{j_2}-1}}{z_{j_1}^{\heartsuit+\lambda_{j_2}-1}}\det\Big((\widehat{\mathsf{D}}_{j_2})_{j_2}\Big)\\
&= \frac{(-1)^{j_2}}{z_{j_2}^{-\heartsuit}}\det\Big((\widehat{\mathsf{D}}_{j_2})_{j_2}\Big)\\
&= \widehat{\omega}_{j_2}.
\end{aligned}$$

Thus, the $N+1$ sections $\widehat{\omega}_0,\ldots,\widehat{\omega}_N$ glue together to make a global section:

$$\widehat{\omega}\ \in\ \Gamma\,\Big(\mathscr{X},\ \mathsf{Sym}^n\,\Omega^1_{\mathscr{V}/\mathbb{P}^\blacklozenge_{\mathbb{K}}}\otimes \mathrm{pr}_1^*\,\mathscr{O}_{\mathbb{P}^\blacklozenge_{\mathbb{K}}}(N)\otimes \mathrm{pr}_2^*\,\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\heartsuit)\Big).$$

Observation 6.14. *For all distinct indices $0\leqslant j_1,j_2\leqslant N$, one has the transition identities:*

$$\det\Big((\widehat{\mathsf{D}}_{j_2})_{j_1}\Big)\ =\ \frac{z_{j_2}^{\heartsuit+\lambda_{j_2}-1}}{z_{j_1}^{\heartsuit+\lambda_{j_2}-1}}\det\Big((\widehat{\mathsf{D}}_{j_2})_{j_2}\Big).$$

The proof is but elementary computations, so we omit it here. □

Next, repeating the same reasoning, using the obvious notation, we interpret Propositions 6.9 and 6.10 as:

Proposition 6.15. *Each of the following $N+1$ symmetric forms:*

$$\widehat{\omega}^{i_1,\ldots,i_{l_1}}_{j_1,\ldots,j_{l_2};\,j}\ =\ \frac{(-1)^j}{z_j^{\lambda_j-1}}\det\Big(\widehat{\mathsf{C}}^{i_1,\ldots,i_{l_1}}_{j_1,\ldots,j_{l_2};\,j}\Big)\ =\ \frac{(-1)^j}{z_j^{-\heartsuit}}\det\Big(\underbrace{\Big(\widehat{\mathsf{C}}^{i_1,\ldots,i_{l_1}}_{j_1,\ldots,j_{l_2};\,j}\Big)_j}_{\textit{guess what?}}\Big)\qquad (j=0\cdots N)$$

can be viewed as a section of:

$$\Gamma\Big(\mathrm{pr}_2^{-1}(\mathsf{W}_j),\ \mathsf{Sym}^n\,\Omega^1_{\mathbb{P}^\blacklozenge_{\mathbb{K}}\times_{\mathbb{K}}\mathbb{P}^N_{\mathbb{K}}/\mathbb{P}^\blacklozenge_{\mathbb{K}}}\otimes \mathrm{pr}_1^*\,\mathscr{O}_{\mathbb{P}^\blacklozenge_{\mathbb{K}}}(N)\otimes \mathrm{pr}_2^*\,\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\heartsuit)\Big),$$

with the twisted degree:

$$\heartsuit:=\sum_{p=1}^{l_1}\deg F_{i_p}+\sum_{q=1}^{l_2}\deg F_{j_q}-\sum_{j=0}^{N}\lambda_j+N+1.$$

Moreover, restricting on $\mathscr{X}$, they glue together to make a global section:

$$\widehat{\omega}^{i_1,\dots,i_{l_1}}_{j_1,\dots,j_{l_2}} \in \Gamma\Big(\mathscr{X}, \mathsf{Sym}^{l_2}\,\Omega^1_{\mathscr{Y}/\mathbb{P}^\blacklozenge_{\mathbb{K}}} \otimes \mathrm{pr}_1^*\,\mathscr{O}_{\mathbb{P}^\blacklozenge_{\mathbb{K}}}(N) \otimes \mathrm{pr}_2^*\,\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\heartsuit)\Big). \qquad \square$$

We may view Propositions 6.11 and 6.12 in a similar way.

7. Moving Coefficients Method

7.1. **Algorithm.** As explained in Subsection 4.3, we wish to construct sufficiently many *negatively twisted* symmetric differential forms, and for this purpose we investigate the *moving coefficients method* as follows. We will be concerned only with the central cases that all $c+r$ hypersurfaces are of the approximating big degrees $d+\epsilon_1,\dots,d+\epsilon_{c+r}$, where $\epsilon_1,\dots,\epsilon_{c+r}$ are some given positive integers negligible compared with the large integer $d \gg 1$ to be specified.

To understand the essence of the moving coefficients method while avoiding unnecessary complexity (see Section 12), we first consider the following $c+r$ cumbersome homogeneous polynomials $F_1,\dots,F_{c+r}$, each being the sum of a *dominant* Fermat-type polynomial plus an 'army' of *moving coefficient* terms:

$$F_i = \sum_{j=0}^{N} A_i^j\, z_j^d + \sum_{l=c+r+1}^{N} \sum_{0\leqslant j_0<\cdots<j_l\leqslant N} \sum_{k=0}^{l} M_i^{j_0,\dots,j_l;j_k}\, z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}} z_{j_k}^{d-l\mu_{l,k}}, \tag{92}$$

where all coefficients $A_i^\bullet, M_i^{\bullet;\bullet} \in \mathbb{K}[z_0,\dots,z_N]$ are some degree $\epsilon_i \geqslant 1$ homogeneous polynomials, and all integers $\mu^{l,k} \geqslant 2$, $d \gg 1$ are chosen subsequently by the following *Algorithm*, which is designed to make all the twisted symmetric differential forms obtained later have *negative twisted degrees*.

The procedure is to first construct $\mu^{l,k}$ in a lexicographic order with respect to indices (l,k), for $l = c+r+1\cdots N, k = 0\cdots l$, along with a set of positive integers δ_l.

Recall the integer $\blacktriangledown \geqslant 1$ in Theorem 5.1. We start by setting:

$$\delta_{c+r+1} \geqslant \max\{\epsilon_1,\dots,\epsilon_{c+r}\}. \tag{93}$$

For every integer $l = c+r+1\cdots N$, in this step, we begin with choosing $\mu_{l,0}$ that satisfies:

$$\mu_{l,0} \geqslant l\,\delta_l + l\,(\delta_{c+r+1}+1) + 1 + (l-c-r)\,\blacktriangledown, \tag{94}$$

then inductively we choose $\mu_{l,k}$ with:

$$\mu_{l,k} \geqslant \sum_{j=0}^{k-1} l\,\mu_{l,j} + (l-k)\,\delta_l + l\,(\delta_{c+r+1}+1) + 1 + (l-c-r)\,\blacktriangledown \qquad (k=1\cdots l). \tag{95}$$

If $l < N$, we end this step by setting:

$$\delta_{l+1} := l\,\mu_{l,l} \tag{96}$$

as the starting point for the next step $l+1$. At the end $l = N$, we demand the integer $d \gg 1$ to be big enough:

$$d \geqslant (N+1)\,\mu_{N,N}. \tag{97}$$

Roughly speaking, the Algorithm above is designed for the following three properties.

(i) For every integer $l = c+r+1\cdots N$, in this step, $\mu_{l,\bullet}$ $(\bullet = 0\cdots l)$ grows so drastically that the former ones are negligible compared with the later ones:

$$\mu_{l,0} \ll \mu_{l,1} \ll \cdots \ll \mu_{l,l}. \tag{98}$$

(ii) For all integer pairs (l_1, l_2) with $c + r + 1 \leqslant l_1 < l_2 \leqslant N$, all the integers $\mu_{l_1,\bullet_1}$ chosen in the former step l_1 are negligible compared with the integers $\mu_{l_2,\bullet_2}$ chosen in the later step l_2:

$$\mu_{l_1,\bullet_1} \ll \mu_{l_2,\bullet_2} \qquad (\forall\, 0 \leqslant \bullet_1 \leqslant l_1; 0 \leqslant \bullet_2 \leqslant l_2). \tag{99}$$

(iii) All integers $\mu_{l,k}$ are negligible compared with the integer d:

$$\mu_{l,k} \ll d \qquad (\forall\, c+r+1 \leqslant l \leqslant N; 0 \leqslant k \leqslant l). \tag{100}$$

7.2. **Global moving coefficients method.** First, for all $i = 1 \cdots c + r$, we write the polynomial F_i by extracting the terms for which $l = N$:

$$\begin{aligned} F_i = \sum_{j=0}^{N} A_i^j\, z_j^d + \sum_{l=c+r+1}^{N-1} \sum_{0 \leqslant j_0 < \cdots < j_l \leqslant N} \sum_{k=0}^{l} M_i^{j_0,\ldots,j_l;j_k}\, z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}} z_{j_k}^{d-l\mu_{l,k}} + \\ + \sum_{k=0}^{N} M_i^{0,\ldots,N;k}\, z_0^{\mu_{N,k}} \cdots \widehat{z_k^{\mu_{N,k}}} \cdots z_N^{\mu_{N,k}} z_k^{d-N\mu_{N,k}}, \end{aligned} \tag{101}$$

and now this second line consists of exactly all the moving coefficient terms which associate to all variables $z_0, \ldots, z_N$, namely of the form $M_i^{\bullet;\bullet}\, z_0^{\bullet} \cdots z_N^{\bullet}$.

To simplify the structure of the first line, associating each term in the second sums:

$$M_i^{j_0,\ldots,j_l;j_k}\, z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}} z_{j_k}^{d-l\mu_{l,k}}$$

with the 'corresponding' term in the first sum:

$$A_i^{j_k}\, z_{j_k}^d,$$

and noting a priori the inequalities guaranteed by the Algorithm:

$$d - l\,\mu_{l,k} \geqslant d - \underbrace{(N-1)\,\mu_{N-1,N-1}}_{=\,\delta_N \quad [\text{By } (96)]} \qquad (\forall\, c+r+1 \leqslant l \leqslant N-1; 0 \leqslant k \leqslant l),$$

we rewrite the F_i as:

$$F_i \;=\; \sum_{j=0}^{N} C_i^j\, z_j^{d-\delta_N} + \sum_{k=0}^{N} M_i^{0,\ldots,N;k}\, z_0^{\mu_{N,k}} \cdots \widehat{z_k^{\mu_{N,k}}} \cdots z_N^{\mu_{N,k}} z_k^{d-N\mu_{N,k}}, \tag{102}$$

where the homogeneous polynomials C_i^j are uniquely determined by gathering:

$$C_i^j\, z_j^{d-\delta_N} = A_i^j\, z_j^d + \sum_{l=c+r+1}^{N-1} \sum_{\substack{0 \leqslant j_0 < \cdots < j_l \leqslant N \\ j_k = j \text{ for some } 0 \leqslant k \leqslant l}} M_i^{j_0,\ldots,j_l;j_k}\, z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}} z_{j_k}^{d-l\mu_{l,k}}, \tag{103}$$

namely, after dividing out the common factor $z_j^{d-\delta_N}$ of both sides above:

$$C_i^j := A_i^j\, z_j^{\delta_N} + \sum_{l=c+r+1}^{N-1} \sum_{\substack{0 \leqslant j_0 < \cdots < j_l \leqslant N \\ j_k = j \text{ for some } 0 \leqslant k \leqslant l}} M_i^{j_0,\ldots,j_l;j_k}\, z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}} z_{j_k}^{\delta_N - l\mu_{l,k}}. \tag{104}$$

Next, we have two ways to manipulate the $(N+1)$ remaining moving coefficient terms in (102):

$$M_i^{0,\ldots,N;k}\, z_0^{\mu_{N,k}} \cdots \widehat{z_k^{\mu_{N,k}}} \cdots z_N^{\mu_{N,k}} z_k^{d-N\mu_{N,k}} \qquad (k = 0 \cdots N),$$

in order to ensure the negativity of the symmetric differential forms to be obtained later.

The first kind of manipulations are, for every chosen index $\nu = 0 \cdots N$, to associate all these $(N+1)$ moving coefficient terms:

$$\sum_{k=0}^{N} M_i^{0,\dots,N;k}\, z_0^{\mu_{N,k}} \cdots \widehat{z_k^{\mu_{N,k}}} \cdots z_N^{\mu_{N,k}}\, z_k^{d-N\mu_{N,k}}$$

with the term $C_i^\nu\, z_\nu^{d-\delta_N}$ by rewriting F_i in (102) as:

$$F_i = \sum_{\substack{j=0 \\ j\neq\nu}}^{N} C_i^j\, z_j^{d-\delta_N} + T_i^\nu\, z_\nu^{\mu_{N,0}}, \tag{105}$$

where T_i^ν is the homogeneous polynomial uniquely determined by solving:

$$T_i^\nu\, z_\nu^{\mu_{N,0}} = C_i^\nu\, z_\nu^{d-\delta_N} + \sum_{k=0}^{N} M_i^{0,\dots,N;k}\, z_0^{\mu_{N,k}} \cdots \widehat{z_k^{\mu_{N,k}}} \cdots z_N^{\mu_{N,k}}\, z_k^{d-N\mu_{N,k}}; \tag{106}$$

in fact, guided by properties (98), (99), (100), our algorithm a priori implies:

$$\mu_{N,0} \leqslant d-\delta_N,\ \ \mu_{N,k},\ \ d-N\mu_{N,k} \qquad (k=0\cdots N),$$

thus the right hand side of (106) has a common factor $z_\nu^{\mu_{N,0}}$.

The second kind of manipulations are, for every chosen integer $\tau = 0 \cdots N-1$, for every chosen index $\rho = \tau+1 \cdots N$, to associate each of the first $(\tau+1)$ moving coefficient terms:

$$M_i^{0,\dots,N;k}\, z_0^{\mu_{N,k}} \cdots \widehat{z_k^{\mu_{N,k}}} \cdots z_N^{\mu_{N,k}}\, z_k^{d-N\mu_{N,k}} \qquad (k=0\cdots\tau)$$

with the corresponding terms $C_i^k\, z_k^{d-\delta_N}$ and to associate the remaining $(N-\tau)$ moving coefficient terms:

$$\sum_{j=\tau+1}^{N} M_i^{0,\dots,N;j}\, z_0^{\mu_{N,j}} \cdots \widehat{z_j^{\mu_{N,j}}} \cdots z_N^{\mu_{N,j}}\, z_j^{d-N\mu_{N,j}}$$

with the term $C_i^\rho\, z_\rho^{d-\delta_N}$ by rewriting F_i as:

$$F_i = \sum_{k=0}^{\tau} E_i^k\, z_k^{d-N\mu_{N,k}} + \sum_{\substack{j=\tau+1 \\ j\neq\rho}}^{N} C_i^j\, z_j^{d-\delta_N} + P_i^{\tau,\rho}\, z_\rho^{\mu_{N,\tau+1}}, \tag{107}$$

where E_i^k and $P_i^{\tau,\rho}$ are the homogeneous polynomials uniquely determined by solving:

$$\begin{aligned} E_i^k\, z_k^{d-N\mu_{N,k}} &= C_i^k\, z_k^{d-\delta_N} + M_i^{0,\dots,N;k}\, z_0^{\mu_{N,k}} \cdots \widehat{z_k^{\mu_{N,k}}} \cdots z_N^{\mu_{N,k}}\, z_k^{d-N\mu_{N,k}} \qquad (k=0\cdots\tau), \\ P_i^{\tau,\rho}\, z_\rho^{\mu_{N,\tau+1}} &= C_i^\rho\, z_\rho^{d-\delta_N} + \sum_{j=\tau+1}^{N} M_i^{0,\dots,N;j}\, z_0^{\mu_{N,j}} \cdots \widehat{z_j^{\mu_{N,j}}} \cdots z_N^{\mu_{N,j}}\, z_j^{d-N\mu_{N,j}}, \end{aligned} \tag{108}$$

which is direct by the inequalities listed below granted by the Algorithm:

$$\begin{aligned} d-N\mu_{N,k} &\leqslant d-\delta_N \qquad (k=0\cdots\tau), \\ \mu_{N,\tau+1} &\leqslant \mu_{N,j}, \\ \mu_{N,\tau+1} &\leqslant d-N\mu_{N,j} \qquad (j=\tau+1\cdots N). \end{aligned} \tag{109}$$

Now thanks to the above two kinds of manipulations (105), (107), applying Proposition 6.8, 6.10, we receive the corresponding twisted symmetric differential forms with *negative degrees* as follows.

Firstly, for every index $\nu = 0 \cdots N$, applying Proposition 6.8, 6.10 with respect to the first kind of manipulation (105) on the hypersurface polynomial equations $F_1, \ldots, F_{c+r}$, for every n-tuple $1 \leqslant j_1 < \cdots < j_n \leqslant c$, we receive a twisted symmetric differential n-form:

$$\phi^{\nu}_{j_1,\ldots,j_n} \in \Gamma(X, \mathsf{Sym}^n \Omega_V(\heartsuit^{\nu}_{j_1,\ldots,j_n})), \tag{110}$$

whose twisted degree $\heartsuit^{\nu}_{j_1,\ldots,j_n}$, according to the formula (71), is negative:

$$\begin{aligned}
[\text{Use } \deg F_i = d + \epsilon_i \leqslant d + \delta_{c+r+1}] \qquad \heartsuit^{\nu}_{j_1,\ldots,j_n} &\leqslant N\,(d + \delta_{c+r+1}) - \left[N\,(d - \delta_N) + \mu_{N,0}\right] + N + 1 \\
&= N\,\delta_N + N\,(\delta_{c+r+1} + 1) + 1 - \mu_{N,0} \\
[\text{Use (94) for } l = N] \qquad &\leqslant -n \heartsuit.
\end{aligned}$$

Secondly, for every integer $\tau = 0 \cdots N-1$, for every index $\rho = \tau + 1 \cdots N$, applying Proposition 6.8, 6.10 with respect to the second kind of manipulation (107) on the hypersurface polynomial equations $F_1, \ldots, F_{c+r}$, for every n-tuple $1 \leqslant j_1 < \cdots < j_n \leqslant c$, we receive a twisted symmetric differential n-form:

$$\psi^{\tau,\rho}_{j_1,\ldots,j_n} \in \Gamma(X, \mathsf{Sym}^n \Omega_V(\heartsuit^{\tau,\rho}_{j_1,\ldots,j_n})), \tag{111}$$

whose twisted degree $\heartsuit^{\tau,\rho}_{j_1,\ldots,j_n}$, according to the formula (71), is negative too:

$$\begin{aligned}
\heartsuit^{\tau,\rho}_{j_1,\ldots,j_n} &\leqslant N\,(d + \delta_{c+r+1}) - \sum_{k=0}^{\tau} (d - N\,\mu_{N,k}) - (N - \tau - 1)\,(d - \delta_N) - \mu_{N,\tau+1} + N + 1 \\
&= \sum_{k=0}^{\tau} N\,\mu_{N,k} + (N - \tau - 1)\,\delta_N + N\,(\delta_{c+r+1} + 1) + 1 - \mu_{N,\tau+1} \\
&\leqslant -n \heartsuit \qquad [\text{use (95) for } l = N,\ k = \tau + 1].
\end{aligned}$$

7.3. **Moving coefficients method with some vanishing coordinates.** To investigate further the moving coefficients method, for all integers $1 \leqslant \eta \leqslant n-1$, for every sequence of ascending indices :

$$0 \leqslant \nu_1 < \cdots < \nu_\eta \leqslant N,$$

take the intersection of X with the η coordinate hyperplanes:

$$_{\nu_1,\ldots,\nu_\eta}X := X \cap \{z_{\nu_1} = 0\} \cap \cdots \cap \{z_{\nu_\eta} = 0\}.$$

Applying Proposition 6.12, in order to obtain more symmetric differential $(n-\eta)$-forms having negative twisted degree, we carry on manipulations as follows, which are much the same as before.

First, write the $(N - \eta + 1)$ remaining numbers of the set-minus:

$$\{0, \ldots, N\} \setminus \{\nu_1, \ldots, \nu_\eta\}$$

in the ascending order:

$$r_0 < \cdots < r_{N-\eta}.$$

Note that in Proposition 6.12, the coefficient terms associated with the vanishing variables $z_{\nu_1}, \ldots, z_{\nu_\eta}$ play no role, therefore we decompose F_i into two parts. The first part (the first two lines below)

is a very analog of (101) involving only the variables $z_{r_0},\dots,z_{r_{N-\eta}}$, while the second part (the third line) collects all the residue terms involving at least one of the vanishing coordinates $z_{v_1},\dots,z_{v_\eta}$:

$$\begin{aligned} F_i = &\sum_{j=0}^{N-\eta} A_i^{r_j}\, z_{r_j}^d + \sum_{l=c+r+1}^{N-\eta-1} \sum_{0\leqslant j_0<\cdots<j_l\leqslant N-\eta} \sum_{k=0}^{l} M_i^{r_{j_0},\dots,r_{j_l};r_{j_k}}\, z_{r_{j_0}}^{\mu_{l,k}} \cdots \widehat{z_{r_{j_k}}^{\mu_{l,k}}} \cdots z_{r_{j_l}}^{\mu_{l,k}} z_{r_{j_k}}^{d-l\mu_{l,k}} + \\ &+ \sum_{k=0}^{N-\eta} M_i^{r_0,\dots,r_{N-\eta};r_k}\, z_{r_0}^{\mu_{N-\eta,k}} \cdots \widehat{z_{r_k}^{\mu_{N-\eta,k}}} \cdots z_{r_{N-\eta}}^{\mu_{N-\eta,k}} z_{r_k}^{d-(N-\eta)\,\mu_{N-\eta,k}} + \\ &+ \underbrace{(\text{Residue Terms})_i^{v_1,\dots,v_\eta}}_{\text{negligible in the coming applications}} . \end{aligned} \tag{112}$$

Since every power associated with the vanishing variables $z_{v_1},\dots,z_{v_{v_\eta}}$ is $\geqslant 2$ thanks to the Algorithm in subsection 7.1, all the $(\text{Residue Terms})_i^{v_1,\dots,v_\eta}$ lie in the ideal:

$$(z_{v_1}^2,\dots,z_{v_\eta}^2) \subset \mathbb{K}[z_0,\dots,z_N].$$

Moreover, using for instance the lexicographic order, we can write them as:

$$(\text{Residue Terms})_i^{v_1,\dots,v_\eta} = \sum_{j=1}^{\eta} R_i^{v_1,\dots,v_\eta;v_j}\, z_{v_j}^2, \tag{113}$$

where $R_i^{v_1,\dots,v_\eta;v_j}$ are the homogeneous polynomials uniquely determined by solving:

$$R_i^{v_1,\dots,v_\eta;v_j}\, z_{v_j}^2 = A_i^{v_j}\, z_{v_j}^d + \sum_{l=c+r+1}^{N} \sum_{\substack{0\leqslant j_0<\cdots<j_l\leqslant N \\ \min\left(\{j_0,\dots,j_l\}\setminus\{v_1,\dots,v_\eta\}\right)=v_j}} \sum_{k=0}^{l} M_i^{j_0,\dots,j_l;j_k}\, z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}} z_{j_k}^{d-l\mu_{l,k}}.$$

Observing that the first two lines of (112) have exactly the same structure as (101), by mimicking the manipulation of rewriting (101) as (102), we can rewrite the first two lines of (112) as:

$$\sum_{j=0}^{N-\eta} {}_{v_1,\dots,v_\eta}C_i^{r_j}\, z_{r_j}^{d-\delta_{N-\eta}} + \sum_{k=0}^{N-\eta} M_i^{r_0,\dots,r_{N-\eta};r_k}\, z_{r_0}^{\mu_{N-\eta,k}} \cdots \widehat{z_{r_k}^{\mu_{N-\eta,k}}} \cdots z_{r_{N-\eta}}^{\mu_{N-\eta,k}} z_{r_k}^{d-(N-\eta)\,\mu_{N-\eta,k}}, \tag{114}$$

where the integer $\delta_{N-\eta}$ was defined in (96) for $l = N-\eta-1$:

$$\delta_{N-\eta} = (N-\eta-1)\,\mu_{N-\eta-1,N-\eta-1},$$

and where the homogeneous polynomials ${}_{v_1,\dots,v_\eta}C_i^{r_j}$ are obtained in the same way as C_i^j in (104):

$${}_{v_1,\dots,v_\eta}C_i^{r_j} := A_i^{r_j}\, z_j^{\delta_{N-\eta}} + \sum_{l=c+r+1}^{N-\eta-1} \sum_{\substack{0\leqslant j_0<\cdots<j_l\leqslant N-\eta \\ j_k=j \text{ for some } 0\leqslant k\leqslant l}} M_i^{r_{j_0},\dots,r_{j_l};r_{j_k}}\, z_{r_{j_0}}^{\mu_{l,k}} \cdots \widehat{z_{r_{j_k}}^{\mu_{l,k}}} \cdots z_{r_{j_l}}^{\mu_{l,k}} z_{r_{j_k}}^{(N-\eta-1)\mu_{N-\eta-1,N-\eta-1}-l\mu_{l,k}}.$$

Now substituting (114), (113) into the equation (112), we rewrite F_i as:

$$\begin{aligned} F_i = &\sum_{j=0}^{N-\eta} {}_{v_1,\dots,v_\eta}C_i^{r_j}\, z_{r_j}^{d-\delta_{N-\eta}} + \sum_{k=0}^{N-\eta} M_i^{r_0,\dots,r_{N-\eta};r_k}\, z_{r_0}^{\mu_{N-\eta,k}} \cdots \widehat{z_{r_k}^{\mu_{N-\eta,k}}} \cdots z_{r_{N-\eta}}^{\mu_{N-\eta,k}} z_{r_k}^{d-(N-\eta)\,\mu_{N-\eta,k}} + \\ &+ \underbrace{\sum_{j=1}^{\eta} R_i^{v_1,\dots,v_\eta;v_j}\, z_{v_j}^2}_{\text{negligible in the coming applications}}. \end{aligned} \tag{115}$$

Now, noting that the first line of F_i in (115) has exactly the same structure as (102), we repeat the two kinds of manipulations, as briefly summarized below.

The first kind of manipulations are, for every chosen index $\nu = 0 \cdots N-\eta$, to associate all these $(N+1-\eta)$ moving coefficient terms:

$$\sum_{k=0}^{N-\eta} M_i^{r_0,\ldots,r_{N-\eta};r_k}\, z_{r_0}^{\mu_{N-\eta,k}} \cdots \widehat{z_{r_k}^{\mu_{N-\eta,k}}} \cdots z_{r_{N-\eta}}^{\mu_{N-\eta,k}}\, z_{r_k}^{d-(N-\eta)\,\mu_{N-\eta,k}}$$

with the term ${}_{v_1,\ldots,v_\eta}C_i^{r_\nu}\, z_{r_\nu}^{d-\delta_{N-\eta}}$ by rewriting (114) as:

$$\sum_{\substack{j=0\\ j\neq\nu}}^{N-\eta} {}_{v_1,\ldots,v_\eta}C_i^{r_j}\, z_{r_j}^{d-\delta_{N-\eta}} + {}_{v_1,\ldots,v_\eta}T_i^{r_\nu}\, z_{r_\nu}^{\mu_{N-\eta,0}}, \tag{116}$$

where ${}_{v_1,\ldots,v_\eta}T_i^{r_\nu}$ is the homogeneous polynomial uniquely determined by solving:

$${}_{v_1,\ldots,v_\eta}T_i^{r_\nu}\, z_{r_\nu}^{\mu_{N-\eta,0}} = {}_{v_1,\ldots,v_\eta}C_i^{r_\nu}\, z_{r_\nu}^{d-\delta_{N-\eta}} + \sum_{k=0}^{N-\eta} M_i^{r_0,\ldots,r_{N-\eta};r_k}\, z_{r_0}^{\mu_{N-\eta,k}} \cdots \widehat{z_{r_k}^{\mu_{N-\eta,k}}} \cdots z_{r_{N-\eta}}^{\mu_{N-\eta,k}}\, z_{r_k}^{d-(N-\eta)\,\mu_{N-\eta,k}}. \tag{117}$$

The second kind of manipulations are, for every integer $\tau = 0 \cdots N-\eta-1$, for every index $\rho = \tau+1 \cdots N-\eta$, to associate each of the first $(\tau+1)$ moving coefficient terms:

$$M_i^{r_0,\ldots,r_{N-\eta};r_k}\, z_{r_0}^{\mu_{N-\eta,k}} \cdots \widehat{z_{r_k}^{\mu_{N-\eta,k}}} \cdots z_{r_{N-\eta}}^{\mu_{N-\eta,k}}\, z_{r_k}^{d-(N-\eta)\,\mu_{N-\eta,k}} \qquad (k = 0 \cdots \tau)$$

with the corresponding term ${}_{v_1,\ldots,v_\eta}C_i^{r_k}\, z_{r_k}^{d-\delta_{N-\eta}}$ and to associate the remaining $(N-\eta-\tau)$ moving coefficient terms:

$$\sum_{j=\tau+1}^{N-\eta} M_i^{r_0,\ldots,r_{N-\eta};r_j}\, z_{r_0}^{\mu_{N-\eta,j}} \cdots \widehat{z_{r_j}^{\mu_{N-\eta,j}}} \cdots z_{r_{N-\eta}}^{\mu_{N-\eta,j}}\, z_{r_j}^{d-(N-\eta)\,\mu_{N-\eta,j}},$$

with the term ${}_{v_1,\ldots,v_\eta}C_i^{r_\rho}\, z_{r_\rho}^{d-\delta_{N-\eta}}$ by rewriting (114) as:

$$\sum_{k=0}^{\tau} {}_{v_1,\ldots,v_\eta}E_i^{r_k}\, z_{r_k}^{d-(N-\eta)\,\mu_{N-\eta,k}} + \sum_{\substack{j=\tau+1\\ j\neq\rho}}^{N-\eta} {}_{v_1,\ldots,v_\eta}C_i^{r_j}\, z_{r_j}^{d-\delta_{N-\eta}} + {}_{v_1,\ldots,v_\eta}P_i^{r_\tau,r_\rho}\, z_{r_\rho}^{\mu_{N-\eta,\tau+1}}, \tag{118}$$

where ${}_{v_1,\ldots,v_\eta}E_i^{r_k}$ and ${}_{v_1,\ldots,v_\eta}P_i^{r_\tau,r_\rho}$ are the homogeneous polynomials uniquely determined by solving:

$$\begin{aligned}
{}_{v_1,\ldots,v_\eta}E_i^{r_k}\, z_{r_k}^{d-(N-\eta)\,\mu_{N-\eta,k}} &= {}_{v_1,\ldots,v_\eta}C_i^{r_k}\, z_{r_k}^{d-\delta_{N-\eta}} + M_i^{r_0,\ldots,r_{N-\eta};r_k}\, z_{r_0}^{\mu_{N-\eta,k}} \cdots \widehat{z_{r_k}^{\mu_{N-\eta,k}}} \cdots z_{r_{N-\eta}}^{\mu_{N-\eta,k}}\, z_{r_k}^{d-(N-\eta)\,\mu_{N-\eta,k}},\\
{}_{v_1,\ldots,v_\eta}P_i^{r_\tau,r_\rho}\, z_{r_\rho}^{\mu_{N-\eta,\tau+1}} &= {}_{v_1,\ldots,v_\eta}C_i^{r_\rho}\, z_{r_\rho}^{d-\delta_{N-\eta}} + \sum_{j=\tau+1}^{N-\eta} M_i^{r_0,\ldots,r_{N-\eta};r_j}\, z_{r_0}^{\mu_{N-\eta,j}} \cdots \widehat{z_{r_j}^{\mu_{N-\eta,j}}} \cdots z_{r_{N-\eta}}^{\mu_{N-\eta,j}}\, z_{r_j}^{d-(N-\eta)\,\mu_{N-\eta,j}},
\end{aligned} \tag{119}$$

which is possible by the Algorithm in subsection 7.1.

To summarize, taking the two forms (116), (118) of the first line of (115) into account, we can rewrite F_i in the following two ways. The first one is:

$$F_i = \sum_{\substack{j=0\\ j\neq\nu}}^{N-\eta} {}_{v_1,\ldots,v_\eta}C_i^{r_j}\, z_{r_j}^{d-\delta_{N-\eta}} + {}_{v_1,\ldots,v_\eta}T_i^{r_\nu}\, z_{r_\nu}^{\mu_{N-\eta,0}} + \underbrace{\sum_{j=1}^{\eta} R_i^{v_1,\ldots,v_\eta;v_j}\, z_{v_j}^2}_{\text{negligible in our coming applications}}, \tag{120}$$

and the second one is:

$$F_i = \sum_{k=0}^{\tau} {}_{v_1,\dots,v_\eta}E_i^{r_k}\, z_{r_k}^{d-(N-\eta)\,\mu_{N-\eta,k}} + \sum_{\substack{j=\tau+1\\ j\neq\rho}}^{N-\eta} {}_{v_1,\dots,v_\eta}C_i^{r_j}\, z_{r_j}^{d-\delta_{N-\eta}} + {}_{v_1,\dots,v_\eta}P_i^{r_\tau,r_\rho}\, z_{r_\rho}^{\mu_{N-\eta,\tau+1}} + \underbrace{\sum_{j=1}^{\eta} R_i^{v_1,\dots,v_\eta;v_j}\, z_{v_j}^2}_{\text{negligible in our coming applications}}. \tag{121}$$

Firstly, applying Proposition 6.12 to (120), for every $(n-\eta)$-tuple:

$$1 \leqslant j_1 < \cdots < j_{n-\eta} \leqslant c,$$

we receive a twisted symmetric differential $(n-\eta)$-form:

$${}_{v_1,\dots,v_\eta}\phi^{\nu}_{j_1,\dots,j_{n-\eta}} \in \Gamma\big({}_{v_1,\dots,v_\eta}X, \mathsf{Sym}^{n-\eta}\,\Omega_V\,({}_{v_1,\dots,v_\eta}\heartsuit^{\nu}_{j_1,\dots,j_{n-\eta}})\big), \tag{122}$$

whose twisted degree ${}_{v_1,\dots,v_\eta}\heartsuit^{\nu}_{j_1,\dots,j_{n-\eta}}$, according to the formula (81), is negative:

$$\begin{aligned} {}_{v_1,\dots,v_\eta}\heartsuit^{\nu}_{j_1,\dots,j_{n-\eta}} &\leqslant (N-\eta)\,(d+\delta_{c+r+1}) - \big[(N-\eta)\,(d-\delta_{N-\eta}) + \mu_{N-\eta,0}\big] + (N-\eta) + 1\\ &= (N-\eta)\,\delta_{N-\eta} + (N-\eta)\,(\delta_{c+r+1}+1) + 1 - \mu_{N-\eta,0}\\ \text{[use (94) for } l = N-\eta\text{]} \qquad &\leqslant -(n-\eta)\,\heartsuit. \end{aligned}$$

Secondly, applying Proposition 6.12 to (121), for every $(n-\eta)$-tuple:

$$1 \leqslant j_1 < \cdots < j_{n-\eta} \leqslant c,$$

we receive a twisted symmetric differential $(n-\eta)$-form:

$${}_{v_1,\dots,v_\eta}\psi^{\tau,\rho}_{j_1,\dots,j_{n-\eta}} \in \Gamma\big({}_{v_1,\dots,v_\eta}X, \mathsf{Sym}^{n-\eta}\,\Omega_V({}_{v_1,\dots,v_\eta}\heartsuit^{\tau,\rho}_{j_1,\dots,j_{n-\eta}})\big), \tag{123}$$

whose twisted degree ${}_{v_1,\dots,v_\eta}\heartsuit^{\tau,\rho}_{j_1,\dots,j_{n-\eta}}$, according to the formula (81), is negative also:

$$\begin{aligned} {}_{v_1,\dots,v_\eta}\heartsuit^{\tau,\rho}_{j_1,\dots,j_{n-\eta}} &\leqslant (N-\eta)\,(d+\delta_{c+r+1}) - \sum_{k=0}^{\tau}(d-(N-\eta)\,\mu_{N-\eta,k}) - (N-\eta-\tau-1)\,(d-\delta_{N-\eta}) -\\ &\quad - \mu_{N-\eta,\tau+1} + N - \eta + 1\\ &= \sum_{k=0}^{\tau}(N-\eta)\,\mu_{N-\eta,k} + (N-\eta-\tau-1)\,\delta_{N-\eta} + (N-\eta)\,(\delta_{c+r+1}+1) + 1 - \mu_{N-\eta,\tau+1}\\ &\leqslant -(n-\eta)\,\heartsuit \qquad \text{[use (95) for } l = N-\eta,\ k = \tau+1\text{]}. \end{aligned}$$

8. **Basic technical preparations**

8.1. **Fibre dimension estimates.** The first theorem below coincides with our geometric intuition, and one possible proof is mainly based on Complex Implicit Function Theorem. Here, we give a short proof by the same method as [59, p. 76, Theorem 7].

Theorem 8.1. (Analytic Fibre Dimension Estimate) *Let X, Y be two complex spaces and let $f\colon X \to Y$ be a regular map. Then the maximum fibre dimension is bounded from below by the dimension of the source space X minus the dimension of the target space Y:*

$$\mathsf{max}_{y\in Y}\,\mathsf{dim}_{\mathbb{C}}\, f^{-1}(y) \geqslant \mathsf{dim}_{\mathbb{C}}\, X - \mathsf{dim}_{\mathbb{C}}\, Y.$$

Equivalently:

$$\mathsf{dim}_{\mathbb{C}}\, X \leqslant \mathsf{dim}_{\mathbb{C}}\, Y + \mathsf{max}_{y\in Y}\,\mathsf{dim}_{\mathbb{C}}\, f^{-1}(y). \tag{124}$$

Proof. For every point $x \in X$, let $f(x) =: z \in Y$, and denote the germ dimension of Y at this point by:

$$d_z := \dim_{\mathbb{C}} (Y, z).$$

Then we can find holomorphic function germs $g_1, \ldots, g_{d_z} \in \mathscr{O}_{Y,z}$ vanishing at z such that:

$$(Y, z) \cap \{g_1 = \cdots = g_{d_z} = 0\} = \{z\}.$$

Pulling back by the holomorphic map f, we therefore realize:

$$(X, x) \cap \{g_1 \circ f = \cdots = g_{d_z} \circ f = 0\} = (f^{-1}(z), x).$$

Now, counting the germ dimension, we receive the estimate:

$$\dim_{\mathbb{C}} (f^{-1}(z), x) \geqslant \dim_{\mathbb{C}} (X, x) - d_z = \dim_{\mathbb{C}} (X, x) - \dim_{\mathbb{C}} (Y, z),$$

hence:

$$\begin{aligned} \dim_{\mathbb{C}} (X, x) &\leqslant \dim_{\mathbb{C}} (Y, z) + \dim_{\mathbb{C}} (f^{-1}(z), x) \\ &\leqslant \dim_{\mathbb{C}} Y + \max_{y \in Y} \dim_{\mathbb{C}} f^{-1}(y). \end{aligned}$$

Finally, let $x \in X$ vary in the above estimate, thanks to:

$$\dim_{\mathbb{C}} X = \max_{x \in X} \dim_{\mathbb{C}} (X, x),$$

we receive the desired estimate (124). □

With the same proof (cf. [54, p. 169, Proposition 12.30; p. 140, Corollary 10.27]), here is an algebraic version of the analytic fibre dimension estimate above, for every algebraically closed field $\mathbb{K}$ and for the category of $\mathbb{K}$-varieties in the classical sense ([37, §1.3, p. 15]), where dimension is defined to be the Krull dimension ([37, §1.1, p. 6]).

Theorem 8.2 (**Algebraic Fibre Dimension Estimate**). *Let X, Y be two $\mathbb{K}$-varieties, and let $f \colon X \to Y$ be a morphism. Then the dimension of the source variety X is bounded from above by the sum of the dimension of the target variety Y plus the maximum fibre dimension:*

$$\dim X \leqslant \dim Y + \max_{y \in Y} \dim f^{-1}(y). \tag{125}$$

In our future applications, f will always be surjective, so one may also refer to [59, p. 76, Theorem 7]. The above theorem will prove fundamental in estimating every base locus involved in this paper.

Corollary 8.3. *Let X, Y be two $\mathbb{K}$-varieties, and let $f \colon X \to Y$ be a morphism such that every fibre satisfies the dimension estimate:*

$$\dim f^{-1}(y) \leqslant \dim X - \dim Y \qquad (\forall\, y \in Y).$$

Then for every subvariety $Z \subset Y$, its inverse image:

$$f^{-1}(Z) \subset X$$

satisfies the transferred codimension estimate:

$$\operatorname{codim} f^{-1}(Z) \geqslant \operatorname{codim} Z.$$ □

8.2. **Matrix-rank estimates.** This subsection recalls some elementary rank estimates in linear algebra.

Lemma 8.4. *Let $\mathbb{K}$ be a field and let W be a finite-dimensional $\mathbb{K}$-vector space generated by a set of vectors $\mathscr{B}$. Then every subset $\mathscr{B}_1 \subset \mathscr{B}$ that consists of $\mathbb{K}$-linearly independent vectors can be extended to a bigger subset $\mathscr{B}_2 \subset \mathscr{B}$ which forms a basis of W.* □

Lemma 8.5. *Let $\mathbb{K}$ be a field, and let V be a $\mathbb{K}$-vector space. For all positive integers $e, k, l \geqslant 1$ with $k \geqslant l$, let $v_1, \ldots, v_e, v_{e+1}, \ldots, v_{e+k}$ be $(e+k)$ vectors such that:*

(i) *$v_1, \ldots, v_e$ are $\mathbb{K}$-linearly independent;*

(ii) *for every sequence of l ascending indices between $e+1$ and $e+k$:*

$$e + 1 \leqslant i_1 < \cdots < i_l \leqslant e + k,$$

there holds the rank inequality:

$$\mathsf{rank}_{\mathbb{K}} \{v_1, \ldots, v_e, v_{i_1}, \ldots, v_{i_l}\} \leqslant e + l - 1.$$

Then there holds the rank estimate:

$$\mathsf{rank}_{\mathbb{K}} \{v_1, \ldots, v_e, v_{e+1}, \ldots, v_{e+k}\} \leqslant e + l - 1.$$

Proof. Assume on the contrary that:

$$\mathsf{rank}_{\mathbb{K}} \{v_1, \ldots, v_e, v_{e+1}, \ldots, v_{e+k}\} =: e + l_0 \geqslant e + l,$$

that is, $l_0 \geqslant l$.

Now applying the above lemma to:

$$W = \mathsf{Span}_{\mathbb{K}} \underbrace{\{v_1, \ldots, v_e, v_{e+1}, \ldots, v_{e+k}\}}_{=\mathscr{B}},$$

$$\mathscr{B}_1 = \{v_1, \ldots, v_e\},$$

we receive a certain basis of V:

$$\mathscr{B}_2 = \{v_1, \ldots, v_e, v_{i_1}, \ldots, v_{i_{l_0}}\}.$$

In particular, as $l_0 \geqslant l$, the first $(e+l)$ vectors in $\mathscr{B}_2$ are $\mathbb{K}$-linearly independent:

$$\mathsf{rank}_{\mathbb{K}} \{v_1, \ldots, v_e, v_{i_1}, \ldots, v_{i_l}\} = e + l,$$

which contradicts condition **(ii)**. □

Let $\mathbb{K}$ be a field, and let p, q, e, l be positive integers with:

$$\mathsf{min}\{p, q\} \geqslant e + l.$$

Let $\mathsf{M} \in \mathsf{Mat}_{p\times q}(\mathbb{K})$ be a $p \times q$ matrix. For all sequences of ascending indices :

$$1 \leqslant i_1 < \cdots < i_k \leqslant p,$$

let us denote by $\mathsf{M}_{i_1,\ldots,i_k}$ the $k \times q$ submatrix of M that consists of the rows $i_1, \ldots, i_k$, and for all sequences of ascending indices:

$$1 \leqslant j_1 < \cdots < j_l \leqslant q,$$

let us denote by $\mathsf{M}^{j_1,\ldots,j_l}_{i_1,\ldots,i_k}$ the $k \times l$ submatrix of $\mathsf{M}_{i_1,\ldots,i_k}$ that consists of the columns $j_1, \ldots, j_l$.

Lemma 8.6. *If the first e rows of the matrix M are of full rank:*

$$\mathsf{rank}_{\mathbb{K}}\, \mathsf{M}_{1,\dots,e} = e,$$

and if all the $(e+l)\times(e+l)$ submatrices always selecting the first e rows of M are degenerate:

$$\mathsf{rank}_{\mathbb{K}}\, \mathsf{M}^{j_1,\dots,j_{e+l}}_{1,\dots,e,i_1,\dots,i_l} \leqslant e+l-1 \qquad (\forall\, e+1\leqslant i_1<\cdots<i_l\leqslant p;\, 1\leqslant j_1<\cdots<j_{e+l}\leqslant q), \tag{126}$$

then there holds the rank estimate:

$$\mathsf{rank}_{\mathbb{K}}\, \mathsf{M} \leqslant e+l-1$$

Proof. For every fixed sequence of ascending indices:

$$e+1\leqslant i_1<\cdots<i_l\leqslant p,$$

the rank inequalities (126) yields:

$$\mathsf{rank}_{\mathbb{K}}\, \mathsf{M}_{1,\dots,e,i_1,\dots,i_l} \leqslant e+l-1$$

Now applying the previous lemma to the rows of the matrix M, we conclude the desired rank estimate. □

Lemma 8.7. *Let $\mathbb{K}$ be a field and let e, m be positive integers. Let $\mathsf{H}\in \mathsf{Mat}_{e\times m}(\mathbb{K})$ be an $e\times m$ matrix with entries in $\mathbb{K}$ such that the sum of all m columns of H vanishes:*

$$\mathsf{H}_1+\cdots+\mathsf{H}_m = \mathbf{0}, \tag{127}$$

where we denote by H_i the i-th column of H. Then for every integer $j=1\cdots m$, the $e\times m$ submatrix $\widehat{\mathsf{H}}_j$ of H obtained by omitting the j-th column still has the same rank:

$$\mathsf{rank}_{\mathbb{K}}\, \widehat{\mathsf{H}}_j = \mathsf{rank}_{\mathbb{K}}\, \mathsf{H}.$$

Proof. Note that (127) yields:

$$\mathsf{H}_j = -(\mathsf{H}_1+\cdots+\mathsf{H}_{j-1}+\mathsf{H}_{j+1}+\cdots+\mathsf{H}_m),$$

therefore H_j lies in the $\mathbb{K}$-linear space generated by the columns of the matrix $\widehat{\mathsf{H}}_j$, thus we receive:

$$\mathsf{Span}_{\mathbb{K}}\{\mathsf{H}_1,\dots,\widehat{\mathsf{H}_j},\dots,\mathsf{H}_m\} = \mathsf{Span}_{\mathbb{K}}\{\mathsf{H}_1,\dots,\mathsf{H}_m\}.$$

Taking the dimension of both sides, we receive the desired rank equality. □

8.3. **Classical codimension formulas.** In an algebraically closed field $\mathbb{K}$, for all positive integers $p, q\geqslant 1$, denote by:

$$\mathsf{Mat}_{p\times q}(\mathbb{K}) = \mathbb{K}^{p\times q}$$

the space of all $p\times q$ matrices with entries in $\mathbb{K}$. For every integer $0\leqslant \ell\leqslant \max\{p,q\}$, we have a classical formula (cf. [31, p. 247 exercise 10.10, and the proof in p. 733]) for the codimension of the subvariety:

$$\Sigma^{p,q}_{\ell}\subset \mathsf{Mat}_{p\times q}(\mathbb{K})$$

which consists of all matrices with rank $\leqslant \ell$.

Lemma 8.8. *There holds the codimension formula:*

$$\mathsf{codim}\, \Sigma^{p,q}_{\ell} = \max\{(p-\ell)(q-\ell),\, 0\}.$$ □

In applications, we will use the following two direct consequences.

Corollary 8.9. *For every integer* $0 \leqslant \ell \leqslant \max\{p, q-1\}$*, the codimension of the subvariety:*

$$ {}^0\Sigma_\ell^{p,q} \subset \Sigma_\ell^{p,q}, $$

which consists of matrices whose sum of all the columns vanish, is:

$$ \operatorname{codim} {}^0\Sigma_\ell^{p,q} = \max\{(p-\ell)\,(q-1-\ell),\, 0\} + p. $$ □

Proof. Since every matrix in ${}^0\Sigma_\ell^{p,q}$ is uniquely determined by the first $(q-1)$ columns, thanks to Lemma 8.7, the projection morphism into the first $(q-1)$ columns:

$$ \pi\colon \quad {}^0\Sigma_\ell^{p,q} \longrightarrow \Sigma_\ell^{p,q-1} $$

is an isomorphism. Remembering that:

$$ \dim \Sigma_\ell^{p,q} = \dim \Sigma_\ell^{p,q-1} + p, $$

now a direct application of the preceding lemma finishes the proof. □

8.4. **Surjectivity of evaluation maps.** Given a field $\mathbb{K}$, for all positive integers $N \geqslant 1$, denote the affine coordinate ring of $\mathbb{K}^{N+1}$ by:

$$ \mathscr{A}(\mathbb{K}^{N+1}) := \mathbb{K}[z_0, \ldots, z_N]. $$

For all positive integers $\lambda \geqslant 1$, also denote by:

$$ \mathscr{A}_\lambda(\mathbb{K}^{N+1}) \subset \mathscr{A}(\mathbb{K}^{N+1}) $$

the $\mathbb{K}$-linear space spanned by all the degree λ homogeneous polynomials:

$$ \mathscr{A}_\lambda(\mathbb{K}^{N+1}) := \oplus_{\substack{\alpha_0+\cdots+\alpha_N=\lambda \\ \alpha_0,\ldots,\alpha_N\geqslant 0}} \mathbb{K} \cdot z_0^{\lambda_0} \cdots z_N^{\lambda_N} \cong \mathbb{K}^{\binom{N+\lambda}{N}}. $$

For every point $z \in \mathbb{K}^{N+1}$, denote by v_z the $\mathbb{K}$-linear evaluation map:

$$ \begin{aligned} v_z : \mathscr{A}(\mathbb{K}^{N+1}) &\longrightarrow \mathbb{K} \\ f &\longmapsto f(z), \end{aligned} $$

and for every tangent vector $\xi \in \mathrm{T}_z\mathbb{K}^{N+1} \cong \mathbb{K}^{N+1}$, denote by $d_z(\xi)$ the $\mathbb{K}$-linear differential evaluation map:

$$ \begin{aligned} d_z(\xi) : \mathscr{A}(\mathbb{K}^{N+1}) &\longrightarrow \mathbb{K} \\ f &\longmapsto d f\big|_z(\xi). \end{aligned} $$

For every polynomial $g \in \mathscr{A}(\mathbb{K}^{N+1})$, for every point $z \in \mathbb{K}^{N+1}$, denote by $(g \cdot v)_z$ the $\mathbb{K}$-linear evaluation map:

$$ \begin{aligned} (g \cdot v)_z : \mathscr{A}(\mathbb{K}^{N+1}) &\longrightarrow \mathbb{K} \\ f &\longmapsto (gf)(z), \end{aligned} $$

and for every tangent vector $\xi \in \mathrm{T}_z\mathbb{K}^{N+1} \cong \mathbb{K}^{N+1}$, denote by $d_z(g\cdot)(\xi)$ the $\mathbb{K}$-linear differential evaluation map:

$$ \begin{aligned} d_z(g\cdot)(\xi) : \mathscr{A}(\mathbb{K}^{N+1}) &\longrightarrow \mathbb{K} \\ f &\longmapsto d(gf)\big|_z(\xi). \end{aligned} $$

The following Lemma was obtained by Brotbek in another affine coordinates version [7, p. 36, Proof of Claim 3].

Lemma 8.10. *For all positive integers $\lambda \geqslant 1$, at every nonzero point $z \in \mathbb{K}^{N+1} \setminus \{0\}$, for every tangent vector $\xi \in \mathrm{T}_z\mathbb{K}^{N+1} \cong \mathbb{K}^{N+1}$ which does not lie in the line of z:*

$$\xi \notin \mathbb{K} \cdot z,$$

restricting on the subspace:

$$\mathscr{A}_\lambda(\mathbb{K}^{N+1}) \subset \mathscr{A}(\mathbb{K}^{N+1}),$$

the evaluation maps v_z and $d_z(\xi)$ are $\mathbb{K}$-linearly independent. In other words, the map:

$$\begin{pmatrix} v_z \\ d_z(\xi) \end{pmatrix} : \mathscr{A}_\lambda(\mathbb{K}^{N+1}) \longrightarrow \mathbb{K}^2$$

is surjective.

Proof. Step 1. For the case $\lambda = 1$, this lemma is evident. In fact, now every polynomials $\ell \in \mathscr{A}_1(\mathbb{K}^{N+1})$ can be viewed as, by evaluating $\ell(z)$ at every point $z \in \mathbb{K}^{N+1}$, a $\mathbb{K}$-linear form:

$$\ell \in (\mathbb{K}^{N+1})^\vee,$$

thus there is a canonical $\mathbb{K}$-linear isomorphism:

$$\mathscr{A}_1(\mathbb{K}^{N+1}) \cong (\mathbb{K}^{N+1})^\vee.$$

Moreover, it is easy to see:

$$d\ell\Big|_z(\xi) = \ell(\xi).$$

Since $z, \xi \in \mathbb{K}^{N+1}$ are $\mathbb{K}$-linearly independent, now recalling the Riesz Representation Theorem in linear algebra:

$$\mathbb{K}^{N+1} \cong \Big((\mathbb{K}^{N+1})^\vee\Big)^\vee, \tag{128}$$

we conclude the claim.

Step 2. For the general case $\lambda \geqslant 2$, first, we choose a degree $(\lambda - 1)$ homogeneous polynomial $g \in \mathscr{A}_{\lambda-1}(\mathbb{K}^{N+1})$ with $g(z) \neq 0$ (for instance, one of $z_0^{\lambda-1}, \ldots, z_N^{\lambda-1}$ succeeds), and then we claim, restricting on the $\mathbb{K}$-linear subspace obtained by multiplying $\mathscr{A}_1(\mathbb{K}^{N+1})$ with g:

$$g \cdot \mathscr{A}_1(\mathbb{K}^{N+1}) \subset \mathscr{A}_\lambda(\mathbb{K}^{N+1}),$$

that the evaluation maps v_z and $d_z(\xi)$ are $\mathbb{K}$-linearly independent.

In fact, for all $f \in \mathscr{A}(\mathbb{K}^{N+1})$, we have:

$$\begin{aligned} (g \cdot v)_z(f) &= (gf)(z) \\ &= g(z)\, f(z) \\ &= g(z)\, v_z(f), \end{aligned}$$

and by Leibniz's rule:

$$\begin{aligned} d_z(g\cdot)(\xi)(f) &= d(gf)\Big|_z(\xi) \\ &= g(z)\, df\Big|_z(\xi) + f(z)\, dg\Big|_z(\xi) \\ &= g(z)\, d_z(\xi)(f) + dg\Big|_z(\xi)\, v_z(f), \end{aligned}$$

in other words:

$$\begin{pmatrix} (g \cdot v)_z \\ d_z(g\cdot)(\xi) \end{pmatrix} = \underbrace{\begin{pmatrix} g(z) & 0 \\ dg\Big|_z(\xi) & g(z) \end{pmatrix}}_{\text{invertible, since } g(z) \neq 0} \begin{pmatrix} v_z \\ d_z(\xi) \end{pmatrix}. \tag{129}$$

Now, restricting (129) on the $\mathbb{K}$-linear subspace:

$$\mathscr{A}_1(\mathbb{K}^{N+1}) \subset \mathscr{A}(\mathbb{K}^{N+1}),$$

and recalling the result of Step 1 that the evaluation maps $v_z, d_z(\xi)$ are $\mathbb{K}$-linearly independent, we immediately see that the evaluation maps $(g \cdot v)_z, d_z(g\cdot)(\xi)$ are $\mathbb{K}$-linearly independent too. In other words, restricting on the $\mathbb{K}$-linear subspace:

$$g \cdot \mathscr{A}_1(\mathbb{K}^{N+1}) \subset \mathscr{A}_\lambda(\mathbb{K}^{N+1}),$$

the evaluation maps $v_z, d_z(\xi)$ are $\mathbb{K}$-linearly independent. □

Lemma 8.11. *For all positive integers $\lambda \geqslant 1$, for all polynomials $g \in \mathscr{A}(\mathbb{K}^{N+1})$, at every nonzero point $z \in \mathbb{K}^{N+1} \setminus \{0\}$ where g does not vanish:*

$$g(z) \neq 0,$$

and for every tangent vector $\xi \in \mathrm{T}_z\mathbb{K}^{N+1} \cong \mathbb{K}^{N+1}$ which does not lie in the line of z:

$$\xi \notin \mathbb{K} \cdot z,$$

restricting on the subspace:

$$\mathscr{A}_\lambda(\mathbb{K}^{N+1}) \subset \mathscr{A}(\mathbb{K}^{N+1}),$$

the evaluation maps $(g \cdot v)_z$ and $d_z(g\cdot)(\xi)$ are $\mathbb{K}$-linearly independent. In other words, the map:

$$\begin{pmatrix} (g \cdot v)_z \\ d_z(g\cdot)(\xi) \end{pmatrix} : \mathscr{A}_\lambda(\mathbb{K}^{N+1}) \longrightarrow \mathbb{K}^2$$

is surjective.

Proof. This is a direct consequence of formula (129) and of the preceding lemma. □

8.5. **Codimensions of affine cones.** Usually, it is more convenient to count dimension in an Euclidian space rather than in a projective space. Therefore we carry out the following lemma (cf. [37, p. 12, exercise 2.10]), which is geometrically obvious, as one point ($\dim_{\mathbb{K}} = 0$) in the projective space $\mathbb{P}^N_{\mathbb{K}}$ corresponds to one $\mathbb{K}$-line ($\dim_{\mathbb{K}} = 1$) in $\mathbb{K}^{N+1}$.

Lemma 8.12. *In an algebraically closed field $\mathbb{K}$, let $\pi\colon \mathbb{K}^{N+1} \to \mathbb{P}^N_{\mathbb{K}}$ be the canonical projection, and let:*

$$Y \subset \mathbb{P}^N_{\mathbb{K}}$$

be a nonempty algebraic set defined by a homogeneous ideal:

$$I \subset \mathbb{K}[z_0, \dots, z_N].$$

Denote by $C(Y)$ the affine cone over Y:

$$C(Y) := \pi^{-1}(Y) \cup \{\mathbf{0}\} \subset \mathbb{K}^{N+1}.$$

Then $C(Y)$ is an algebraic set in $\mathbb{K}^{N+1}$ which is also defined by the ideal I (considered as an ordinary ideal in $\mathbb{K}[z_0, \dots, z_N]$), and it has dimension one more than Y:

$$\dim C(Y) = \dim Y + 1.$$

In other words, they have the same codimension:

$$\operatorname{codim} C(Y) = \operatorname{codim} Y.$$ □

The essence of the above geometric lemma is the following theorem in commutative algebra (cf. [42, p. 73, Cor. 5.21]):

Theorem 8.13. *Let B be a homogeneous algebra over a field $\mathbb{K}$, then:*

$$\dim \operatorname{Spec} B = \dim \operatorname{Proj} B + 1. \qquad \square$$

8.6. **Full rank of hypersurface equation matrices.** In an algebraically closed field $\mathbb{K}$, for all positive integers $N \geqslant 2$, for all integers $e = 1 \cdots N$, for all positive integers $\epsilon_1, \ldots, \epsilon_e \geqslant 1$ and $d \geqslant 1$, consider the following e hypersurfaces:

$$H_1, \ldots, H_e \subset \mathbb{P}^N_{\mathbb{K}},$$

each being defined as the zero set of a degree $(d + \epsilon_i)$ Fermat-type homogeneous polynomial:

$$F_i = \sum_{j=0}^{N} A_i^j \, z_j^d \qquad (i = 1 \cdots e), \tag{130}$$

where all $A_i^j \in \mathscr{A}_{\epsilon_i}(\mathbb{K}^{N+1})$ are some degree ϵ_i homogeneous polynomials.

Now, denote by H the $e \times (N+1)$ matrix whose i-th row copies the $(N+1)$ terms of F_i in the exact order, i.e. the (i, j)-th entries of H are:

$$\mathsf{H}_{i,j} = A_i^{j-1} \, z_{j-1}^d \qquad (i = 1 \cdots e;\, j = 1 \cdots N+1),$$

so H writes as:

$$\mathsf{H} := \begin{pmatrix} A_1^0 \, z_0^d & \cdots & A_1^N \, z_N^d \\ \vdots & \ddots & \vdots \\ A_e^0 \, z_0^d & \cdots & A_e^N \, z_N^d \end{pmatrix}, \tag{131}$$

which we call *the hypersurface equation matrix* of $F_1, \ldots, F_e$. *Passim*, remark that by (130), the sum of all columns of H vanishes at every point $[z] \in X := H_1 \cap \cdots \cap H_e$.

Also introduce:

$$\mathbb{P}(\mathscr{M}) := \mathbb{P}\Big(\underbrace{\oplus_{\substack{1 \leqslant i \leqslant e \\ 0 \leqslant j \leqslant N}} \mathscr{A}_{\epsilon_i}(\mathbb{K}^{N+1})}_{=: \mathscr{M}} \Big)$$

the projectivized parameter space of $(A_i^j)_{\substack{1 \leqslant i \leqslant e \\ 0 \leqslant j \leqslant N}} \in \mathscr{M}$.

First, let us recall a classical theorem (cf. [59, p. 57, Theorem 2]) that somehow foreshadows Remmert's proper mapping theorem.

Theorem 8.14. *The image of a projective variety under a regular map is closed.* $\square$

The following lemma was proved by Brotbek in another version [7, p. 36, Proof of Claim 1], and the proof there is in affine coordinates $(\frac{z_0}{z_j}, \ldots, \widehat{\frac{z_j}{z_j}}, \ldots, \frac{z_N}{z_j})$:

$$\mathbb{K}^N \cong \{z_j \neq 0\} \subset \mathbb{P}^N_{\mathbb{K}} \qquad (j = 0 \cdots N).$$

Here, we may present a proof by much the same arguments in ambient coordinates $(z_0, \ldots, z_N)$:

$$\begin{aligned} \mathbb{K}^{N+1} \setminus \{0\} &\longrightarrow \mathbb{P}^N_{\mathbb{K}} \\ (z_0, \ldots, z_N) &\longmapsto [z_0 : \cdots : z_N]. \end{aligned}$$

Lemma 8.15. *In $\mathbb{P}(\mathscr{M})$, there exists a proper algebraic subset:*

$$\Sigma \subsetneq \mathbb{P}(\mathscr{M})$$

such that, for every choice of parameter outside Σ*:*

$$\left[(A_i^j)_{\substack{1\leqslant i\leqslant e\\ 0\leqslant j\leqslant N}}\right] \in \mathbb{P}(\mathscr{M}) \setminus \Sigma,$$

on the corresponding intersection:

$$X = H_1 \cap \cdots \cap H_e \subset \mathbb{P}^N_{\mathbb{K}},$$

the matrix H *has full rank* e *everywhere:*

$$\mathsf{rank}_{\mathbb{K}}\, \mathsf{H}(z) = e \qquad (\forall\, [z] \in X).$$

Sharing the same spirit as the famous *Fubini principle* in combinatorics, the essence of the proof below is to count dimension in two ways, which is a standard method in algebraic geometry having various forms (e.g. the proof of Bertini's Theorem in [37, p. 179], main arguments in [18, 7], etc).

Proof. Now, introduce the universal family $\mathscr{X} \hookrightarrow \mathbb{P}(\mathscr{M}) \times \mathbb{P}^N_{\mathbb{K}}$ of the intersections of such e Fermat-type hypersurfaces:

$$\mathscr{X} := \Big\{([A_i^j], [z]) \in \mathbb{P}(\mathscr{M}) \times \mathbb{P}^N_{\mathbb{K}} : \sum_{j=0}^N A_i^j z_j^d = 0, \text{ for } i = 1 \cdots e\Big\},$$

and then consider the subvariety $\mathscr{B} \subset \mathscr{X}$ that consists of all 'bad points' defined by:

$$\mathsf{rank}_{\mathbb{K}}\, \mathsf{H} \leqslant e - 1. \tag{132}$$

Let π_1, π_2 below be the two canonical projections:

$$\begin{array}{ccc} & \mathbb{P}(\mathscr{M}) \times \mathbb{P}^N_{\mathbb{K}} & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ \mathbb{P}(\mathscr{M}) & & \mathbb{P}^N_{\mathbb{K}}. \end{array}$$

Since $\mathbb{P}(\mathscr{M}) \times \mathbb{P}^N_{\mathbb{K}} \supset \mathscr{B}$ is a projective variety and π_1 is a regular map, now applying Theorem 8.14, we see that:

$$\pi_1(\mathscr{B}) \subset \mathbb{P}(\mathscr{M})$$

is an algebraic subvariety. Hence it is necessary and sufficient to show that:

$$\pi_1(\mathscr{B}) \neq \mathbb{P}(\mathscr{M}). \tag{133}$$

Our strategy is as follows.

Step 1. To decompose $\mathbb{P}^N_{\mathbb{K}}$ into a union of quasi-subvarieties:

$$\mathbb{P}^N_{\mathbb{K}} = \cup_{k=0}^N {}_k\mathring{\mathbb{P}}^N_{\mathbb{K}}, \tag{134}$$

where ${}_k\mathring{\mathbb{P}}^N_{\mathbb{K}}$ consists of points $[z] = [z_0 : z_1 : \cdots : z_N] \in \mathbb{P}^N_{\mathbb{K}}$ with exactly k vanishing homogeneous coordinates, the other ones being nonzero.

Step 2. For every integer $k = 0 \cdots N$, for every point $[z] \in {}_k\mathring{\mathbb{P}}^N_{\mathbb{K}}$, to establish the fibre dimension identity:

$$\dim \pi_2^{-1}([z]) \cap \mathscr{B} = \dim \mathbb{P}(\mathscr{M}) - \big(\max\{N - k - e + 1, 0\} + e\big). \tag{135}$$

Proof of Step 2. Without loss of generality, we may assume that the last k homogeneous coordinates of $[z]$ vanish:

$$z_{N-k+1} = \cdots = z_N = 0, \tag{136}$$

and then by the definition of ${}_k\mathring{\mathbb{P}}^N_{\mathbb{K}}$, none of the first $(N-k+1)$ coordinates $z_0, \ldots, z_{N-k}$ vanish.

Noting that:

$$\pi_2^{-1}([z]) \cap \mathscr{B} \;=\; \underbrace{\pi_1\Big(\pi_2^{-1}([z]) \cap \mathscr{B}\Big)}_{\text{by Theorem 8.14 is an algebraic set}} \;\times\; \underbrace{\{[z]\}}_{\text{one point set}},$$

and considering the canonical projection:

$$\widehat{\pi}\colon \quad \mathscr{M} \setminus \{0\} \longrightarrow \mathbb{P}(\mathscr{M}),$$

we receive:

$$\begin{aligned} \dim \pi_2^{-1}([z]) \cap \mathscr{B} &= \dim \pi_1\Big(\pi_2^{-1}([z]) \cap \mathscr{B}\Big) \\ \text{[use Lemma 8.12]} \qquad &= \dim \widehat{\pi}^{-1}\Big(\pi_1(\pi_2^{-1}([z]) \cap \mathscr{B})\Big) \cup \{0\} - 1. \end{aligned} \tag{137}$$

Now, observe that whatever choice of parameters:

$$(A_i^j)_{\substack{1 \leqslant i \leqslant e \\ 0 \leqslant j \leqslant N}} \in \mathscr{M},$$

the vanishing of the last k coordinates of $[z]$ in (136) makes the last k columns of $\mathsf{H}(z)$ in (131) vanish. It is therefore natural to introduce the submatrix ${}^{N+1-k}\mathsf{H}$ of H that consists of the remaining nonvanishing columns, i.e. the first $(N+1-k)$ ones. Since the sum of all columns of $\mathsf{H}(z)$ vanishes by (130), the sum of all columns of ${}^{N+1-k}\mathsf{H}(z)$ also vanishes.

Observe that the set:

$$\begin{aligned} \mathscr{M} \;&\supset \widehat{\pi}^{-1}\Big(\pi_1(\pi_2^{-1}([z]) \cap \mathscr{B})\Big) \cup \{0\} \\ &= \Big\{(A_i^j)_{\substack{1 \leqslant i \leqslant e \\ 0 \leqslant j \leqslant N}} \in \mathscr{M} \;:\; \text{sum of all the columns of } {}^{N+1-k}\mathsf{H}(z) \text{ vanishes,} \\ &\qquad\qquad \text{and } \underbrace{\operatorname{rank}_{\mathbb{K}} {}^{N+1-k}\mathsf{H}(z) \leqslant e-1}_{\text{from (132)}}\Big\} \end{aligned}$$

is nothing but the inverse image of:

$${}^0\Sigma^{e,N+1-k}_{e-1} \subset \mathsf{Mat}_{e\times(N+1-k)}(\mathbb{K}) \qquad \text{[use notation of Lemma 8.9]}$$

under the $\mathbb{K}$-linear map:

$$\begin{aligned} {}^{N-k+1}\mathsf{H}_z \;:\; \mathscr{M} &\longrightarrow \mathsf{Mat}_{e\times(N+1-k)}(\mathbb{K}) \\ (A_i^j)_{i,j} &\longmapsto \quad {}^{N-k+1}\mathsf{H}(z), \end{aligned}$$

which is surjective by Lemma 8.11.

Therefore we have the codimension identity:

$$\begin{aligned} \operatorname{codim} \widehat{\pi}^{-1}\Big(\pi_1(\pi_2^{-1}([z]) \cap \mathscr{B})\Big) \cup \{0\} &= \operatorname{codim} {}^0\Sigma^{e,N+1-k}_{e-1} \qquad [{}^{N-k+1}\mathsf{H}_z \text{ is linear and surjective}] \\ \text{[use Lemma (8.9)]} \qquad &= \max\{N-k-e+1, 0\} + e, \end{aligned} \tag{138}$$

and thereby we receive:

$$\begin{aligned}
\dim \pi_2^{-1}([z]) \cap \mathscr{B} &= \dim \widehat{\pi}^{-1}\Big(\pi_1(\pi_2^{-1}([z]) \cap \mathscr{B})\Big) \cup \{0\} - 1 && \text{[use (137)]}\\
\text{[by definition of codimension]} \qquad &= \dim \mathscr{M} - \operatorname{codim} \widehat{\pi}^{-1}\Big(\pi_1(\pi_2^{-1}([z]) \cap \mathscr{B})\Big) \cup \{0\} - 1\\
\text{[exercise]} \qquad &= \dim \mathbb{P}(\mathscr{M}) - \operatorname{codim} \widehat{\pi}^{-1}\Big(\pi_1(\pi_2^{-1}([z]) \cap \mathscr{B})\Big) \cup \{0\}\\
\text{[use (138)]} \qquad &= \dim \mathbb{P}(\mathscr{M}) - \big(\max\{N-k-e+1, 0\} + e\big),
\end{aligned}$$

which is exactly our claimed fibre dimension identity (135). □

Step 3. Applying Lemma 8.2 to the regular map:

$$\pi_2 : \quad \pi_2^{-1}({}_k\mathring{\mathbb{P}}^N_{\mathbb{K}}) \cap \mathscr{B} \longrightarrow {}_k\mathring{\mathbb{P}}^N_{\mathbb{K}},$$

remembering:

$$\dim {}_k\mathring{\mathbb{P}}^N_{\mathbb{K}} = N - k \qquad (k = 0 \cdots N),$$

together with the identity (135), we receive the dimension estimate:

$$\begin{aligned}
\dim \pi_2^{-1}({}_k\mathring{\mathbb{P}}^N_{\mathbb{K}}) \cap \mathscr{B} &\leqslant \dim {}_k\mathring{\mathbb{P}}^N_{\mathbb{K}} + \dim \mathbb{P}(\mathscr{M}) - \max\{N-k-e+1, 0\} - e\\
&\leqslant (N-k) + \dim \mathbb{P}(\mathscr{M}) - (N-k-e+1) - e\\
&= \dim \mathbb{P}(\mathscr{M}) - 1.
\end{aligned} \tag{139}$$

Note that $\mathscr{B}$ can be written as the union of $(N+1)$ quasi-subvarieties:

$$\begin{aligned}
\mathscr{B} &= \pi_2^{-1}\left(\mathbb{P}^N_{\mathbb{K}}\right) \cap \mathscr{B}\\
&= \pi_2^{-1}\left(\cup_{k=0}^N {}_k\mathring{\mathbb{P}}^N_{\mathbb{K}}\right) \cap \mathscr{B}\\
&= \left(\cup_{k=0}^N \pi_2^{-1}\left({}_k\mathring{\mathbb{P}}^N_{\mathbb{K}}\right)\right) \cap \mathscr{B}\\
&= \cup_{k=0}^N \left(\pi_2^{-1}\left({}_k\mathring{\mathbb{P}}^N_{\mathbb{K}}\right) \cap \mathscr{B}\right),
\end{aligned}$$

each one being, thanks to (139), of dimension less than or equal to:

$$\dim \mathbb{P}(\mathscr{M}) - 1,$$

and therefore we have the dimension estimate:

$$\dim \mathscr{B} \leqslant \dim \mathbb{P}(\mathscr{M}) - 1.$$

Finally, (133) follows from the dimensional comparison:

$$\dim \pi_1(\mathscr{B}) \leqslant \dim \mathscr{B} \leqslant \dim \mathbb{P}(\mathscr{M}) - 1.$$ □

In the more general context of our moving coefficients method, we now want to have an everywhere full-rank property analogous to Lemma 8.15 just obtained.

Observing that in (92), the number of terms in each polynomial F_i is:

$$(N+1) + \sum_{\ell=c+r+1}^{N} \binom{N+1}{\ell+1}(\ell+1),$$

and recalling that the $\mathbb{K}$-linear subspace $\mathscr{A}_{\epsilon_i}(\mathbb{K}^{N+1}) \subset \mathbb{K}[z_0, \ldots, z_N]$ spanned by all degree ϵ_i homogeneous polynomials is of dimension:

$$\dim_{\mathbb{K}} \mathscr{A}_{\epsilon_i}(\mathbb{K}^{N+1}) = \binom{N+\epsilon_i}{N},$$

we may denote by $\mathbb{P}^{\blacklozenge}_{\mathbb{K}}$ the projectivized parameter space of such $c+r$ hypersurfaces, with the integer:

$$\blacklozenge := \left[(N+1) + \sum_{\ell=c+r+1}^{N} \binom{N+1}{\ell+1}(\ell+1)\right] \sum_{i=1}^{c+r} \binom{N+\epsilon_i}{N}. \tag{140}$$

Now, by mimicking the construction of the matrix H in (131), employing the notation in Subsection 7.2, for every integer $\nu = 0 \cdots N$, let us denote by H^{ν} the $(c+r) \times (N+1)$ matrix whose i-th row copies the $(N+1)$ terms of F_i in (105). Also, for every integer $\tau = 0 \cdots N-1$, for every index $\rho = \tau+1 \cdots N$, let us denote by $\mathsf{H}^{\tau,\rho}$ the $(c+r) \times (N+1)$ matrix whose i-th row copies the $(N+1)$ terms of F_i in (107).

Lemma 8.16. *In $\mathbb{P}^{\blacklozenge}_{\mathbb{K}}$, there exists a proper algebraic subset:*

$$\Sigma \subsetneqq \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$$

such that, for every choice of parameter outside Σ:

$$[A^{\bullet}_{\bullet}, M^{\bullet}_{\bullet}] \quad \in \quad \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus \Sigma,$$

on the corresponding intersection:

$$X = H_1 \cap \cdots \cap H_{c+r} \subset \mathbb{P}^N_{\mathbb{K}},$$

all the matrices $\mathsf{H}^{\nu}, \mathsf{H}^{\tau,\rho}$ have full rank c:

$$\mathrm{rank}_{\mathbb{K}}\, \mathsf{H}^{\nu}(z) = c+r, \quad \mathrm{rank}_{\mathbb{K}}\, \mathsf{H}^{\tau,\rho}(z) = c+r \qquad (\forall\, [z] \in X).$$

We can copy the proof of Lemma 8.15 without much modification and thus everything works smoothly. Alternatively, we may present a short proof by applying Lemma 8.15.

Proof. Observation 1. We need only prove this lemma separately for each matrix H^{ν} (resp. $\mathsf{H}^{\tau,\rho}$), i.e. to show that there exists a proper algebraic subset:

$$\Sigma^{\nu} \text{ (resp. } \Sigma^{\tau,\rho}) \subsetneqq \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$$

outside of which every choice of parameter succeeds. Then the union of all these proper algebraic subsets works:

$$\Sigma := \cup_{\nu=0}^{N} \Sigma^{\nu} \quad \cup \cup_{\substack{\tau=0\cdots N-1 \\ \rho=\tau+1\cdots N}} \Sigma^{\tau,\rho} \qquad \subsetneqq \qquad \mathbb{P}^{\blacklozenge}_{\mathbb{K}}.$$

Observation 2. For each matrix H^{ν} (resp. $\mathsf{H}^{\tau,\rho}$), inspired by the beginning arguments in the proof of Lemma 8.15, especially (133), we only need to find one parameter:

$$[A^{\bullet}_{\bullet}, M^{\bullet}_{\bullet}] \quad \in \quad \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus \Sigma$$

with the desired property.

Observation 3. Now, setting all the moving coefficients zero:

$$M^{\bullet}_{\bullet} := 0,$$

thanks to (103), (106), the equations (105) become exactly the equations (130), and therefore all the matrices H^{ν} become the same matrix H of Lemma 8.15 (with $e = c + r$). Similarly, so do all the matrices $\mathsf{H}^{\tau,\rho}$.

Observation 4. Now, a direct application of Lemma 8.15 clearly yields more than one parameter, an infinity! □

Once again, by mimicking the construction of the matrix H in Lemma 8.15, employing the notation in subsection 7.3, let us denote by ${}_{v_1,\dots,v_\eta}\mathsf{H}^{\nu}$ (resp. ${}_{v_1,\dots,v_\eta}\mathsf{H}^{\tau,\rho}$) the $c \times (N+1)$ matrix whose i-th row copies the $(N+1)$ terms of F_i in (120) (resp. (121)).

Lemma 8.17. *In $\mathbb{P}^{\blacklozenge}_{\mathbb{K}}$, there exists a proper algebraic subset:*

$$ {}_{v_1,\dots,v_\eta}\Sigma \subsetneq \mathbb{P}^{\blacklozenge}_{\mathbb{K}} $$

such that, for every choice of parameter outside ${}_{v_1,\dots,v_\eta}\Sigma$:

$$ [A^{\bullet}_{\bullet}, M^{\bullet}_{\bullet}] \ \in \ \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus {}_{v_1,\dots,v_\eta}\Sigma $$

on the corresponding intersection:

$$ X = H_1 \cap \cdots \cap H_{c+r} \subset \mathbb{P}^N_{\mathbb{K}}, $$

all the matrices ${}_{v_1,\dots,v_\eta}\mathsf{H}^{\nu}$ and ${}_{v_1,\dots,v_\eta}\mathsf{H}^{\tau,\rho}$ have full rank $c + r$:

$$ \mathsf{rank}_{\mathbb{K}\ v_1,\dots,v_\eta}\mathsf{H}^{\nu}(z) = c + r, \quad \mathsf{rank}_{\mathbb{K}\ v_1,\dots,v_\eta}\mathsf{H}^{\tau,\rho}(z) = c + r \qquad (\forall\, [z] \in X). $$ □

The proof goes exactly the same way as in the preceding lemma.

9. **Controlling the base locus**

9.1. **Characterization of the base locus.** Now, we are in a position to characterize the base locus of all the obtained global twisted symmetric differential n-forms in (110), (111):

$$ \mathsf{BS} := \text{Base Locus of } \{\phi^{\nu}_{j_1,\dots,j_n}, \psi^{\tau,\rho}_{j_1,\dots,j_n}\}^{\nu,\tau,\rho}_{1 \leqslant j_1 < \cdots < j_n \leqslant c} \ \subset \ \mathbb{P}(\mathrm{T}_V\big|_X), \tag{141} $$

where $\mathbb{P}(\mathrm{T}_V\big|_X) \subset \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$ is given by:

$$ \mathbb{P}(\mathrm{T}_V\big|_X) := \Big\{([z],[\xi]) : \ F_i(z) = 0, dF_j\big|_z(\xi) = 0, \forall\, i = 1 \cdots c + r, \forall\, j = 1 \cdots c\Big\} $$

To begin with, for every $\nu = 0 \cdots N$, let us study the specific base locus:

$$ \mathsf{BS}^{\nu} := \text{Base Locus of } \{\phi^{\nu}_{j_1,\dots,j_n}\}_{1 \leqslant j_1 < \cdots < j_n \leqslant c} \ \subset \ \mathbb{P}(\mathrm{T}_V\big|_X) $$

associated with only the twisted symmetric differential forms obtained in (110).

For each sequence of ascending indices:

$$ 1 \leqslant j_1 < \cdots < j_n \leqslant c, $$

by mimicking the construction of the matrices $\mathsf{K}, \widehat{\mathsf{K}}_{j_1,\dots,j_n;\,j}$ at the end of Subsection 6.3, in accordance with the first kind of manipulation (105), we construct the $(c + r + c) \times (N + 1)$ matrix K^{ν} in the obvious way, i.e. by copying terms, differentials, and then we define the analogous $\widehat{\mathsf{K}}^{\nu}_{j_1,\dots,j_n;\,j}$.

First, let us look at points $([z],[\xi]) \in \mathsf{BS}^{\nu}$ having all coordinates nonvanishing:

$$ z_0 \cdots z_N \neq 0. \tag{142} $$

For each symmetric horizontal differential n-form $\widehat{\phi}^{\nu}_{j_1,\dots,j_n}$ which corresponds to $\phi^{\nu}_{j_1,\dots,j_n}$ in the sense of Propositions 6.10, 6.9, for every $j = 0 \cdots N$, we receive:

$$\begin{aligned} 0 &= \widehat{\phi}^{\nu}_{j_1,\dots,j_n;\,j}\,(z,\xi) && \text{[since } ([z],[\xi]) \in \mathsf{BS}^{\nu}\text{]} \\ &= \underbrace{\frac{(-1)^j}{z_0^{\star}\cdots z_N^{\star}}}_{\neq 0}\,\det\big(\widehat{\mathsf{K}}^{\nu}_{j_1,\dots,j_n;\,j}\big)\,(z,\xi), && \text{[use (79)]} \end{aligned}$$

where all integers $\star$ are of no importance here. Indeed, we can drop the nonzero factor $\frac{(-1)^j}{z_0^{\star}\cdots z_N^{\star}}$ and obtain:

$$\det\big(\underbrace{\widehat{\mathsf{K}}^{\nu}_{j_1,\dots,j_n;\,j}}_{N\times N\text{ matrix}}\big)\,(z,\xi) = 0.$$

In other words:

$$\mathsf{rank}_{\mathbb{K}}\,\widehat{\mathsf{K}}^{\nu}_{j_1,\dots,j_n;\,j}\,(z,\xi) \leqslant N-1.$$

Now, letting the index j run from 0 to N, we receive:

$$\mathsf{rank}_{\mathbb{K}}\,\underbrace{\mathsf{K}^{\nu}_{j_1,\dots,j_n}\,(z,\xi)}_{N\times(N+1)\text{ matrix}} \leqslant N-1, \tag{143}$$

where $\mathsf{K}^{\nu}_{j_1,\dots,j_n}$ is defined analogously to the matrix $\mathsf{C}^{\nu}_{j_1,\dots,j_n}$ before Proposition 6.8 in the obvious way.

Note that the first $c+r$ rows of $\mathsf{K}^{\nu}_{j_1,\dots,j_n}$ constitute the matrix H^{ν} in Lemma 8.16, which asserts that for a generic choice of parameter:

$$\mathsf{rank}_{\mathbb{K}}\,\mathsf{H}^{\nu}(z) = c + r.$$

Now, in (143), letting $1 \leqslant j_1 < \cdots < j_n \leqslant c$ vary, and applying Lemma 8.5, we immediately receive:

$$\mathsf{rank}_{\mathbb{K}}\,\mathsf{K}^{\nu}\,(z,\xi) \leqslant N-1.$$

Conversely, it is direct to see that any point $([z],[\xi]) \in {}_{X}\mathbb{P}(\mathrm{T}_{\mathrm{V}})$ satisfying this rank inequality lies in the base locus BS^{ν}.

Note that a point $([z],[\xi]) \in \mathbb{P}(\mathrm{T}_{\mathbb{P}^N})$ lies in $\mathbb{P}(\mathrm{T}_V\big|_X)$ if and only if the sum of all columns of $\mathsf{K}^{\nu}\,(z,\xi)$ vanishes. Summarizing the above analysis, restricting to the coordinates nonvanishing part of $\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$:

$$\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}) \;:=\; \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}) \cap \{z_0 \cdots z_N \neq 0\},$$

we conclude the following generic characterization of:

$$\mathsf{BS}^{\nu} \;\cap\; \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}),$$

where the exceptional locus Σ just below is defined in Lemma 8.16.

Proposition 9.1. *For every choice of parameter outside* Σ*:*

$$[A^{\bullet}_{\bullet}, M^{\bullet}_{\bullet}] \;\in\; \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus \Sigma$$

a point:

$$([z],[\xi]) \;\in\; \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$$

lies in the base locus:

$$([z],[\xi]) \;\in\; \mathsf{BS}^{\nu}$$

if and only if:

$$\mathrm{rank}_{\mathbb{K}}\, \mathsf{K}^{\nu}\,(z,\xi) \leqslant N-1, \ \textit{and the sum of all columns vanishes.} \qquad \square$$

Now, for every integer $\tau = 0 \cdots N-1$ and for every index $\rho = \tau+1 \cdots N$, the base locus:

$$\mathsf{BS}^{\tau,\rho} := \text{Base Locus of } \{\psi^{\tau,\rho}_{j_1,\ldots,j_n}\}_{1\leqslant j_1<\cdots<j_n\leqslant c} \ \subset \ \mathbb{P}(\mathrm{T}_V\big|_X)$$

associated with the twisted symmetric differential forms obtained in (111) enjoys the following generic characterization on the coordinates nonvanishing set $\{z_0 \cdots z_N \neq 0\}$. Of course, the matrix $\mathsf{K}^{\tau,\rho}$ is defined analogously to the matrix K^{ν} in the obvious way. A repetition of the preceding arguments yields:

Proposition 9.2. *For every choice of parameter outside* Σ*:*

$$[A^{\bullet}_{\bullet}, M^{\bullet}_{\bullet}] \ \in \ \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus \Sigma$$

a point:

$$([z],[\xi]) \ \in \ \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$$

lies in the base locus:

$$([z],[\xi]) \ \in \ \mathsf{BS}^{\tau,\rho}$$

if and only if:

$$\mathrm{rank}_{\mathbb{K}}\, \mathsf{K}^{\tau,\rho}\,(z,\xi) \leqslant N-1, \ \textit{and the sum of all columns vanishes.} \qquad \square$$

It is now time to clarify the (uniform) structures of the matrices K^{ν}, $\mathsf{K}^{\tau,\rho}$.

Thanks to the above two Propositions 9.1, 9.2, we may now receive a generic characterization of:

$$\mathsf{BS} \ \cap \ \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}).$$

Firstly, we construct the $(c+r+c)\times(2N+2)$ matrix M such that, for $i = 1\cdots c+r$, $j = 1\cdots c$, its i-row copies the $(2N+2)$ terms of F_i in (102) in the exact order, and its $(c+r+j)$-th row is the differential of the j-th row. In order to distinguish the first $(N+1)$ 'dominant' columns from the last $(N+1)$ columns of moving coefficient terms, we write M as:

$$\mathsf{M} = \Big(\mathsf{A}_0 \,|\, \cdots \,|\, \mathsf{A}_N \,|\, \mathsf{B}_0 \,|\, \cdots \,|\, \mathsf{B}_N\Big).$$

For every index $\nu = 0\cdots N$, comparing (105), (106) with (102), the matrix K^{ν} is nothing but:

$$\mathsf{K}^{\nu} = \Big(\mathsf{A}_0 \,|\, \cdots \,|\, \widehat{\mathsf{A}_{\nu}} \,|\, \cdots \,|\, \mathsf{A}_N \,|\, \mathsf{A}_{\nu} + \sum_{j=0}^{N} \mathsf{B}_j\Big). \tag{144}$$

Similarly, for every integer $\tau = 0\cdots N-1$ and for every index $\rho = \tau+1\cdots N$, comparing (107), (108) with (102), the matrix $\mathsf{K}^{\tau,\rho}$ is nothing but:

$$\mathsf{K}^{\tau,\rho} = \Big(\mathsf{A}_0+\mathsf{B}_0 \,|\, \cdots \,|\, \mathsf{A}_{\tau}+\mathsf{B}_{\tau} \,|\, \mathsf{A}_{\tau+1} \,|\, \cdots \,|\, \widehat{\mathsf{A}_{\rho}} \,|\, \cdots \,|\, \mathsf{A}_N \,|\, \mathsf{A}_{\rho} + \sum_{j=\tau+1}^{N} \mathsf{B}_j\Big). \tag{145}$$

Secondly, we introduce the algebraic subvariety:

$$\mathscr{M}^{N}_{2c+r} \ \subset \ \mathsf{Mat}_{(2c+r)\times 2(N+1)}(\mathbb{K}) \tag{146}$$

consisting of all $(c+r+c)\times 2(N+1)$ matrices $(\alpha_0 \,|\, \alpha_1 \,|\, \cdots \,|\, \alpha_N \,|\, \beta_0 \,|\, \beta_1 \,|\, \cdots \,|\, \beta_N)$ such that:

(i) the sum of these $(2N+2)$ colums is zero:

$$\alpha_0 + \alpha_1 + \cdots + \alpha_N + \beta_0 + \beta_1 + \cdots + \beta_N = \mathbf{0}; \tag{147}$$

(ii) for every index $\nu = 0 \cdots N$, replacing α_ν with $\alpha_\nu + (\beta_0 + \beta_1 + \cdots + \beta_N)$ in the collection of column vectors $\{\alpha_0, \alpha_1, \ldots, \alpha_N\}$, there holds the rank inequality:

$$\mathsf{rank}_{\mathbb{K}} \{\alpha_0, \ldots, \widehat{\alpha_\nu}, \ldots, \alpha_N, \alpha_\nu + (\beta_0 + \beta_1 + \cdots + \beta_N)\} \leqslant N - 1; \tag{148}$$

(iii) for every integer $\tau = 0 \cdots N-1$, for every index $\rho = \tau+1 \cdots N$, replacing α_ρ with $\alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_N)$ in the collection of column vectors $\{\alpha_0+\beta_0, \ldots, \alpha_\tau+\beta_\tau, \alpha_{\tau+1}, \ldots, \alpha_\rho, \ldots, \alpha_N\}$, there holds the rank inequality:

$$\mathsf{rank}_{\mathbb{K}} \{\alpha_0 + \beta_0, \alpha_1 + \beta_1, \ldots, \alpha_\tau + \beta_\tau, \alpha_{\tau+1}, \ldots, \widehat{\alpha_\rho}, \ldots, \alpha_N, \alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_N)\} \leqslant N - 1. \tag{149}$$

Proposition 9.3. *For every choice of parameter outside* Σ*:*

$$[A^{\bullet}_{\bullet}, M^{\bullet}_{\bullet}] \ \in \ \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus \Sigma$$

a point:

$$([z], [\xi]) \ \in \ \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N})$$

lies in the base locus:

$$([z], [\xi]) \ \in \ \mathsf{BS}$$

if and only if:

$$\mathsf{M}(z, \xi) \in \mathscr{M}^N_{2c+r}. \qquad \square$$

Furthermore, for all integers $1 \leqslant \eta \leqslant n-1$, for every sequence of ascending indices:

$$0 \leqslant \nu_1 < \cdots < \nu_\eta \leqslant N,$$

we also have to analyze the base locus of the twisted symmetric differential forms (122), (123):

$${}_{\nu_1,\ldots,\nu_\eta}\mathsf{BS} := \text{Base Locus of } \{{}_{\nu_1,\ldots,\nu_\eta}\phi^{\nu}_{j_1,\ldots,j_{n-\eta}},\ {}_{\nu_1,\ldots,\nu_\eta}\psi^{\tau,\rho}_{j_1,\ldots,j_{n-\eta}}\}^{\nu,\tau,\rho}_{1\leqslant j_1<\cdots<j_{n-\eta}\leqslant c} \tag{150}$$

in the intersection of the η hyperplanes:

$${}_{\nu_1,\ldots,\nu_\eta}\mathbb{P}(\mathrm{T}_{\mathbb{P}^N}) := \mathbb{P}(\mathrm{T}_{\mathbb{P}^N}) \cap \{z_{\nu_1} = \cdots = z_{\nu_\eta} = 0\},$$

and more specifically, we focus on the 'interior part':

$${}_{\nu_1,\ldots,\nu_\eta}\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) \ := \ {}_{\nu_1,\ldots,\nu_\eta}\mathbb{P}(\mathrm{T}_{\mathbb{P}^N}) \cap \{z_{r_0} \cdots z_{r_{N-\eta}} \neq 0\} \qquad \text{[see (80) for the indices } r_0, \ldots, r_{N-\eta}\text{]}.$$

Firstly, we construct the $(c+r+c) \times (2N+2-2\eta)$ matrix ${}_{\nu_1,\ldots,\nu_\eta}\mathsf{M}$, which will play the same role as the matrix M, whose i-row $(i = 1 \cdots c+r)$ copies the $(2N+2-2\eta)$ terms of (114) in the exact order, and whose $(c+r+j)$-th row $(j = 1 \cdots c)$ is the differential of the j-th row.

Secondly, in correspondence with $\mathscr{M}^N_{2c+r}$, by replacing plainly N with $N-\eta$, we introduce the algebraic variety:

$$\mathscr{M}^{N-\eta}_{2c+r} \ \subset \ \mathsf{Mat}_{(2c+r)\times 2(N-\eta+1)}(\mathbb{K}). \tag{151}$$

Thirdly, let us recall the exceptional subvatiety:

$${}_{\nu_1,\ldots,\nu_\eta}\Sigma \subsetneq \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$$

defined in Proposition 8.17.

By performing the same reasoning as in the preceding proposition, we get:

Proposition 9.4. *For every choice of parameter outside* ${}_{v_1,\dots,v_\eta}\Sigma$*:*

$$[A_\bullet^\bullet, M_\bullet^\bullet] \ \in \ \mathbb{P}_{\mathbb{K}}^{\blacklozenge} \setminus {}_{v_1,\dots,v_\eta}\Sigma$$

a point:

$$([z],[\xi]) \in {}_{v_1,\dots,v_\eta}\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N})$$

lies in the base locus (150)*:*

$$([z],[\xi]) \ \in \ {}_{v_1,\dots,v_\eta}\mathsf{BS}$$

if and only if:

$${}_{v_1,\dots,v_\eta}\mathsf{M}\,(z,\xi) \ \in \ \mathscr{M}_{2c+r}^{N-\eta}. \qquad \square$$

9.2. **Emptiness of the base loci.** First, for the algebraic varieties (146), (151), we claim the following codimension estimates, which serve as the engine of the moving coefficients method. However, we will not present it here but in the next section.

Lemma 9.5 (Core Lemma of MCM). (i) *For every positive integers* $N \geqslant 1$*, for every integers* $c, r \geqslant 0$ *with* $2c + r \geqslant N$*, there holds the codimension estimate:*

$$\mathsf{codim}\,\mathscr{M}_{2c+r}^{N} \ \geqslant \ \mathsf{dim}\,\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) \ = \ 2N-1.$$

(ii) *For every positive integer* $\eta = 1 \cdots N - (c+r) - 1$*, for every sequence of ascending indices:*

$$0 \leqslant v_1 < \cdots < v_\eta \leqslant N,$$

there holds the codimension estimate:

$$\mathsf{codim}\,\mathscr{M}_{2c+r}^{N-\eta} \ \geqslant \ \mathsf{dim}\,{}_{v_1,\dots,v_\eta}\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) \ = \ 2N-\eta-1. \qquad \square$$

Now, let us show the power of this Core Lemma.

Bearing Proposition 9.3 in mind, by mimicking the proof of Proposition 8.15, it is natural to introduce the subvariety:

$$M_{2c+r}^{N} \hookrightarrow \mathbb{P}_{\mathbb{K}}^{\blacklozenge} \times \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}),$$

which is defined 'in family' by:

$$M_{2c+r}^{N} \ := \ \Big\{([A_\bullet^\bullet, M_\bullet^\bullet];\, [z],[\xi]) \in \mathbb{P}_{\mathbb{K}}^{\blacklozenge} \times \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) \colon \ \mathsf{M}(z,\xi) \in \mathscr{M}_{2c+r}^{N}\Big\}.$$

Proposition 9.6. *There holds the dimension estimate:*

$$\mathsf{dim}\,M_{2c+r}^{N} \leqslant \mathsf{dim}\,\mathbb{P}_{\mathbb{K}}^{\blacklozenge}.$$

Proof. Let π_1, π_2 be the two canonical projections:

$$\begin{array}{ccc} & \mathbb{P}_{\mathbb{K}}^{\blacklozenge} \times \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ \mathbb{P}_{\mathbb{K}}^{\blacklozenge} & & \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}). \end{array}$$

By mimicking Step 2 in Lemma 8.15, for every point $([z],[\xi]) \in \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N})$, we claim the fibre dimension estimate:

$$\mathsf{dim}\,\pi_2^{-1}([z],[\xi]) \cap M_{2c+r}^{N} = \mathsf{dim}\,\mathbb{P}_{\mathbb{K}}^{\blacklozenge} - \mathsf{codim}\,\mathscr{M}_{2c+r}^{N} \tag{152}$$

Proof. Noting that:

$$\pi_2^{-1}([z],[\xi]) \cap M^N_{2c+r} = \underbrace{\pi_1\Big(\pi_2^{-1}([z],[\xi]) \cap M^N_{2c+r}\Big)}_{\text{by Theorem 8.14 is an algebraic set}} \times \underbrace{\{([z],[\xi])\}}_{\text{one point set}},$$

and considering the canonical projection:

$$\widehat{\pi}\colon \quad \mathbb{K}^{\blacklozenge} \setminus \{0\} \longrightarrow \mathbb{P}^{\blacklozenge}_{\mathbb{K}},$$

we receive:

$$\begin{aligned} \dim \pi_2^{-1}([z],[\xi]) \cap M^N_{2c+r} &= \dim \pi_1\Big(\pi_2^{-1}([z],[\xi]) \cap M^N_{2c+r}\Big) \\ \text{[use Lemma 8.12]} \qquad &= \dim \widehat{\pi}^{-1}\Big(\pi_1(\pi_2^{-1}([z],[\xi]) \cap M^N_{2c+r})\Big) \cup \{0\} - 1. \end{aligned} \tag{153}$$

Now, notice that the set:

$$\begin{aligned} \mathbb{K}^{\blacklozenge} \supset \widehat{\pi}^{-1}\Big(\pi_1(\pi_2^{-1}([z],[\xi]) \cap M^N_{2c+r})\Big) \cup \{0\} \\ = \Big\{(A^{\bullet}_{\bullet}, M^{\bullet}_{\bullet}) \in \mathbb{K}^{\blacklozenge} : \ \mathsf{M}(z,\xi) \in \mathscr{M}^N_{2c+r}\Big\} \end{aligned}$$

is nothing but the inverse image of:

$$\mathscr{M}^N_{2c+r} \subset \mathsf{Mat}_{(2c+r)\times 2(N+1)}(\mathbb{K})$$

under the $\mathbb{K}$-linear map:

$$\begin{aligned} \mathsf{M}_{z,\xi}\colon \qquad \mathbb{K}^{\blacklozenge} &\longrightarrow \mathsf{Mat}_{(2c+r)\times 2(N+1)}(\mathbb{K}) \\ (A^{\bullet}_{\bullet}, M^{\bullet}_{\bullet}) &\longmapsto \quad \mathsf{M}(z,\xi), \end{aligned}$$

which is surjective by the construction of M — see (102), (103), and by applying Lemma 8.11 — since $z_0 \neq 0, \ldots, z_N \neq 0$ and $\xi \notin \mathbb{K} \cdot z$.

Therefore, we have the codimension identity:

$$\operatorname{codim} \widehat{\pi}^{-1}\Big(\pi_1(\pi_2^{-1}([z],[\xi]) \cap M^N_{2c+r})\Big) \cup \{0\} = \operatorname{codim} \mathscr{M}^N_{2c+r} \qquad \text{[}\mathsf{M}_{z,\xi}\text{ is linear and surjective]}, \tag{154}$$

and thereby we receive:

$$\begin{aligned} \dim \pi_2^{-1}([z],[\xi]) \cap M^N_{2c+r} &= \dim \widehat{\pi}^{-1}\Big(\pi_1(\pi_2^{-1}([z],[\xi]) \cap M^N_{2c+r})\Big) \cup \{0\} - 1 && \text{[use (153)]} \\ \text{[by definition of codimension]} \qquad &= \dim \mathbb{K}^{\blacklozenge} - \operatorname{codim} \widehat{\pi}^{-1}\Big(\pi_1(\pi_2^{-1}([z],[\xi]) \cap M^N_{2c+r})\Big) \cup \{0\} - 1 \\ \text{[why?]} \qquad &= \dim \mathbb{P}^{\blacklozenge}_{\mathbb{K}} - \operatorname{codim} \widehat{\pi}^{-1}\Big(\pi_1(\pi_2^{-1}([z],[\xi]) \cap M^N_{2c+r})\Big) \cup \{0\} \\ \text{[use (154)]} \qquad &= \dim \mathbb{P}^{\blacklozenge}_{\mathbb{K}} - \operatorname{codim} \mathscr{M}^N_{2c+r}, \end{aligned}$$

which is exactly our claimed fibre dimension identity. □

Lastly, by applying the Fibre Dimension Estimate 8.2, we receive:

$$\begin{aligned} \dim M^N_{2c+r} &\leqslant \dim \overset{\circ}{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) + \dim \mathbb{P}^{\blacklozenge}_{\mathbb{K}} - \operatorname{codim} \mathscr{M}^N_{2c+r} && \text{[use (152)]} \\ \text{[use Core Lemma 9.5]} \qquad &\leqslant \dim \overset{\circ}{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) + \dim \mathbb{P}^{\blacklozenge}_{\mathbb{K}} - \dim \overset{\circ}{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) \\ &= \dim \mathbb{P}^{\blacklozenge}_{\mathbb{K}}, \end{aligned}$$

which is our claimed dimension estimate. □

Now, restricting the canonical projection π_1 to M^N_{2c+r}:

$$\pi_1 : \quad M^N_{2c+r} \longrightarrow \mathbb{P}^{\blacklozenge}_{\mathbb{K}},$$

according to the dimension inequality just obtained, we gain:

Proposition 9.7. *There exists a proper algebraic subset $\Sigma' \subsetneqq \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$ such that, for every choice of parameter outside Σ':*

$$P = [A^{\bullet}_{\bullet}, M^{\bullet}_{\bullet}] \quad \in \quad \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus \Sigma',$$

the intersection of the fibre $\pi_1^{-1}(P)$ with M^N_{2c+r} is discrete or empty:

$$\dim \pi_1^{-1}(P) \cap M^N_{2c+r} \leqslant 0. \qquad \square$$

Combining Propositions 9.3 and 9.7, we receive:

Proposition 9.8. *Outside the proper algebraic subset:*

$$\Sigma \cup \Sigma' \subsetneqq \mathbb{P}^{\blacklozenge}_{\mathbb{K}},$$

for every choice of parameter:

$$[A^{\bullet}_{\bullet}, M^{\bullet}_{\bullet}] \quad \in \quad \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus (\Sigma \cup \Sigma'),$$

the base locus in the coordinates nonvanishing set:

$$\mathsf{BS} \cap \{z_0 \cdots z_N \neq 0\}$$

is discrete or empty. $\square$

Moreover, bearing in mind Proposition 9.4, by repeating the same reasoning as in the preceding proposition, consider the subvariety:

$${}_{v_1,\ldots,v_\eta}M^{N-\eta}_{2c+r} \hookrightarrow \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \times {}_{v_1,\ldots,v_\eta}\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N})$$

which is defined 'in family' by:

$${}_{v_1,\ldots,v_\eta}M^{N}_{2c+r} := \Big\{([A^{\bullet}_{\bullet}, B^{\bullet}_{\bullet}];\, [z], [\xi]) \in \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \times {}_{v_1,\ldots,v_\eta}\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) :\ {}_{v_1,\ldots,v_\eta}\mathsf{M}(z,\xi) \in \mathscr{M}^{N-\eta}_{2c+r}\Big\},$$

and hence receive a very analog of Proposition 9.6.

Proposition 9.9. *There holds the dimension estimate:*

$$\dim {}_{v_1,\ldots,v_\eta}M^{N-\eta}_{2c+r} \leqslant \dim \mathbb{P}^{\blacklozenge}_{\mathbb{K}}. \qquad \square$$

Again, restricting the canonical projection π_1 to ${}_{v_1,\ldots,v_\eta}M^{N-\eta}_{2c+r}$:

$$\pi_1 : {}_{v_1,\ldots,v_\eta}M^{N-\eta}_{2c+r} \longrightarrow \mathbb{P}^{\blacklozenge}_{\mathbb{K}},$$

according to the dimension inequality above, we receive:

Proposition 9.10. *There exists a proper algebraic subset ${}_{v_1,\ldots,v_\eta}\Sigma' \subsetneqq \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$ such that, for every choice of parameter outside ${}_{v_1,\ldots,v_\eta}\Sigma'$:*

$$P = [A^{\bullet}_{\bullet}, M^{\bullet}_{\bullet}] \quad \in \quad \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus {}_{v_1,\ldots,v_\eta}\Sigma',$$

the intersection of the fibre $\pi_1^{-1}(P)$ with $M^{N-\eta}_{2c+r}$ is discrete or empty:

$$\dim \pi_1^{-1}(P) \cap M^{N-\eta}_{2c+r} \leqslant 0. \qquad \square$$

Combining now Propositions 9.4 and 9.10, we receive:

Proposition 9.11. *Outside the proper algebraic subset:*

$$_{v_1,\dots,v_\eta}\Sigma \cup {}_{v_1,\dots,v_\eta}\Sigma' \subsetneqq \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$$

for every choice of parameter:

$$[A^{\bullet}_{\bullet}, M^{\bullet}_{\bullet}] \in \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus ({}_{v_1,\dots,v_\eta}\Sigma \cup {}_{v_1,\dots,v_\eta}\Sigma'),$$

the base locus in the corresponding 'coordinates nonvanishing' set:

$${}_{v_1,\dots,v_\eta}\mathsf{BS} \cap \{z_{r_0} \cdots z_{r_{N-\eta}} \neq 0\}$$

is discrete or empty. □

10. The Engine of MCM

10.1. **Core Codimension Formulas.** Our motivation of this section is to prove the Core Lemma 9.5, which will succeed in Subsection 10.6.

As an essential step, by induction on positive integers $p \geqslant 2$ and $0 \leqslant \ell \leqslant p$, we first estimate the codimension ${}_{\ell}C_p$ of the algebraic variety:

$${}_{\ell}\mathbf{X}_p \subset \mathsf{Mat}_{p\times 2p}(\mathbb{K}) \tag{155}$$

which consists of $p \times 2p$ matrices $\mathsf{X}_p = (\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_p)$ such that:

(i) the first p column vectors have rank:

$$\mathsf{rank}_{\mathbb{K}}\{\alpha_1, \dots, \alpha_p\} \leqslant \ell; \tag{156}$$

(ii) for every index $\nu = 1 \cdots p$, replacing α_ν with $\alpha_\nu + (\beta_1 + \cdots + \beta_p)$ in the collection of column vectors $\{\alpha_1, \dots, \alpha_p\}$, there holds the rank inequality:

$$\mathsf{rank}_{\mathbb{K}}\{\alpha_1, \dots, \widehat{\alpha_\nu}, \dots, \alpha_p, \alpha_\nu + (\beta_1 + \cdots + \beta_p)\} \leqslant p - 1; \tag{157}$$

(iii) for every integer $\tau = 1 \cdots p-1$, for every index $\rho = \tau+1 \cdots p$, replacing α_ρ with $\alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_p)$ in the collection of column vectors $\{\alpha_1 + \beta_1, \dots, \alpha_\tau + \beta_\tau, \alpha_{\tau+1}, \dots, \alpha_\rho, \dots, \alpha_p\}$, there holds the rank inequality:

$$\mathsf{rank}_{\mathbb{K}}\{\alpha_1 + \beta_1, \dots, \alpha_\tau + \beta_\tau, \alpha_{\tau+1}, \dots, \widehat{\alpha_\rho}, \dots, \alpha_p, \alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_p)\} \leqslant p - 1. \tag{158}$$

Let us start with the easy case $\ell = 0$.

Proposition 10.1. *For every integer $p \geqslant 2$, the codimension value ${}_{\ell}C_p$ for $\ell = 0$ is:*

$${}_0C_p = p^2 + 1. \tag{159}$$

Proof. Now, **(i)** is equivalent to:

$$\underbrace{\alpha_1 = \cdots = \alpha_p = \mathbf{0}}_{\mathsf{codim} = p^2} .$$

Thus **(ii)** holds trivially, and the only nontrivial inequality in **(iii)** is:

$$\underbrace{\mathsf{rank}_{\mathbb{K}}\{\mathbf{0} + \beta_1, \dots, \mathbf{0} + \beta_p\} \leqslant p - 1}_{\mathsf{codim} = 1 \text{ by Lemma } 8.8} ,$$

which contributes one more codimension. □

For the general case $\ell = 1 \cdots p$, we will use Gaussian eliminations and do inductions on p, ℓ. First, let us count the codimension of the exceptional locus of Gaussian eliminations.

Proposition 10.2. *For every integer $p \geqslant 2$, the codimensions ${}_{\ell}C_p^0$ of the algebraic varieties:*

$$\{\alpha_1 + \beta_1 = \mathbf{0}\} \cap {}_{\ell}\mathbf{X}_p \subset \mathsf{Mat}_{p\times 2p}(\mathbb{K})$$

read according to the values of ℓ as:

$${}_{\ell}C_p^0 = \begin{cases} p+2 & (\ell = p-1, p), \\ p + (p-\ell)^2 & (\ell = 0 \cdots p-2). \end{cases}$$

The following lemma is the key ingredient for the proof.

Lemma 10.3. *In a field $\mathbb{K}$, let W be a $\mathbb{K}$-vector space. Let $p \geqslant 1$ be a positive integer. For any $(p+1)$ vectors:*

$$\alpha_1, \ldots, \alpha_p, \beta \in W,$$

the rank restriction:

$$\mathsf{rank}_{\mathbb{K}}\{\alpha_1, \ldots, \widehat{\alpha_\nu}, \ldots, \alpha_p, \alpha_\nu + \beta\} \leqslant p-1 \qquad (\nu = 1 \cdots p), \tag{160}$$

is equivalent to either:

$$\mathsf{rank}_{\mathbb{K}}\{\alpha_1, \ldots, \alpha_p, \beta\} \leqslant p-1,$$

or:

$$\mathsf{rank}_{\mathbb{K}}\{\alpha_1, \ldots, \alpha_p\} = p, \qquad (\alpha_1 + \cdots + \alpha_p) + \beta = \mathbf{0}.$$

Proof. Since '$\Longleftarrow$' is clear, we only prove the direction '$\Longrightarrow$'.

We divide the proof according to the rank of $\{\alpha_1, \ldots, \alpha_p\}$ into two cases.

Case 1: $\mathsf{rank}_{\mathbb{K}}\{\alpha_1, \ldots, \alpha_p\} \leqslant p-1$. Assume on the contrary that:

$$\mathsf{rank}_{\mathbb{K}}\{\alpha_1, \ldots, \alpha_p, \beta\} \geqslant p. \tag{161}$$

Since we have the elementary estimate:

$$\begin{aligned} \mathsf{rank}_{\mathbb{K}}\{\alpha_1, \ldots, \alpha_p, \beta\} &\leqslant \mathsf{rank}_{\mathbb{K}}\{\alpha_1, \ldots, \alpha_p\} + \mathsf{rank}_{\mathbb{K}}\{\beta\} \\ &\leqslant (p-1) + 1 \\ &= p, \end{aligned} \tag{162}$$

the inequalities '$\geqslant$' or '$\leqslant$' in (161) and (162) are exactly equalities '=', and thus we have:

$$\begin{gathered} \beta \notin \mathsf{Span}_{\mathbb{K}}\{\alpha_1, \ldots, \alpha_p\}, \\ \mathsf{rank}_{\mathbb{K}}\{\alpha_1, \ldots, \alpha_p\} = p-1. \end{gathered} \tag{163}$$

Consequently, it is clear that we can find a certain index $\nu \in \{1, \ldots, p\}$ such that:

$$\mathsf{rank}_{\mathbb{K}}\{\alpha_1, \ldots, \widehat{\alpha_\nu}, \ldots, \alpha_p\} = p-1,$$

whence the above rank inequality (160) implies:

$$\alpha_\nu + \beta \in \mathsf{Span}_{\mathbb{K}}\{\alpha_1, \ldots, \widehat{\alpha_\nu}, \ldots, \alpha_p\}, \tag{164}$$

which contradicts the formula (163).

Case 2: $\mathsf{rank}_{\mathbb{K}}\{\alpha_1, \ldots, \alpha_p\} = p$. Here, inequalities (160) also yield (164) for every ν, whence:

$$\beta + (\alpha_1 + \cdots + \alpha_p) \in \mathsf{Span}_{\mathbb{K}}\{\alpha_1, \ldots, \widehat{\alpha_\nu}, \ldots, \alpha_p\}.$$

Now, letting ν run from 1 to p, and noting that:

$$\cap_{\nu=1}^{p} \mathsf{Span}_{\mathbb{K}}\{\alpha_1, \ldots, \widehat{\alpha_\nu}, \ldots, \alpha_p\} = \{\mathbf{0}\},$$

we immediately conclude the proof. □

Proof of Proposition 10.2. For every matrix $\mathsf{X}_p = (\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_p)$ such that:

$$\underbrace{\alpha_1 + \beta_1 = \mathbf{0}}_{\mathsf{codim} = p}, \tag{165}$$

the conditions (158) in **(iii)** is trivial, and the restriction (157), thanks to the lemma just obtained, is equivalent either to:

$$\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \alpha_p, \beta_1 + \cdots + \beta_p\} \leqslant p - 1, \tag{166}$$

or to:

$$\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \alpha_p\} = p, \qquad \beta_1 + \cdots + \beta_p = -(\alpha_1 + \cdots + \alpha_p). \tag{167}$$

Now, since $\alpha_1 + \beta_1 = 0$, adding the first column vector of (166) to the last one, we get:

$$\underbrace{\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \alpha_p, \beta_2 + \cdots + \beta_p\} \leqslant p - 1}_{\mathsf{codim} = 2 \text{ by Lemma } 8.8},$$

and similarly, (167) is equivalent to:

$$\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \alpha_p\} = p, \qquad \underbrace{(\alpha_2 + \cdots + \alpha_p) + (\beta_2 + \cdots + \beta_p) = \mathbf{0}}_{\mathsf{codim} = p}.$$

Therefore, when $\ell = p - 1$ or $\ell = p$, we obtain the codimension formulas:

$$\begin{aligned} {}_{p-1}C^0_p &= p + 2, \\ {}_{p}C^0_p &= \min\{p + 2,\ p + p\} = p + 2. \end{aligned}$$

When $\ell = 0 \cdots p - 2$, the restriction **(ii)** is a consequence of **(i)**:

$$\begin{aligned} &\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \widehat{\alpha_\nu}, \dots, \alpha_p, \alpha_\nu + (\beta_1 + \cdots + \beta_p)\} \\ \leqslant\ &\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \widehat{\alpha_\nu}, \dots, \alpha_p\} + \mathsf{rank}_{\mathbb{K}} \{\alpha_\nu + (\beta_1 + \cdots + \beta_p)\} \\ \leqslant\ &\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \alpha_p\} + 1 \\ \leqslant\ &\ell + 1 \\ \leqslant\ &p - 1. \end{aligned}$$

Lastly, applying Lemma 8.8, restriction **(i)** contributes codimension $(p - \ell)^2$. Together with (165), this finishes the proof. □

Now, we claim the following *Codimension Induction Formulas*, the proof of which will appear in Subsection 10.5. In order to make sense of ${}_{\ell-2}C_{p-1}$ in (170) when $\ell = 1$, we henceforth make a *convention:*

$${}_{-1}C_{p-1} := \infty.$$

Proposition 10.4 (Codimension Induction Formulas). (i) *For every positive integer $p \geqslant 2$, for $\ell = p$, the codimension value ${}_pC_p$ satisfies:*

$${}_pC_p = \min\{p,\ \ {}_{p-1}C_p\}. \tag{168}$$

(ii) *For every positive integer $p \geqslant 3$, for $\ell = p - 1$, the codimension value ${}_\ell C_p$ satisfies:*

$${}_{p-1}C_p \geqslant \min\{{}_{p-1}C^0_p,\ \ {}_{p-1}C_{p-1} + 2,\ \ {}_{p-2}C_{p-1} + 1,\ \ {}_{p-3}C_{p-1}\}. \tag{169}$$

(iii) *For all integers $\ell = 1 \cdots p - 2$, the codimension values ${}_\ell C_p$ satisfy:*

$${}_\ell C_p \geqslant \min\{{}_\ell C^0_p,\ \ {}_\ell C_{p-1} + 2(p - \ell) - 1,\ \ {}_{\ell-1}C_{p-1} + (p - \ell),\ \ {}_{\ell-2}C_{p-1}\}. \tag{170}$$

In fact, all the above inequalities '$\geqslant$' should be exactly equalities '='. Nevertheless, '$\geqslant$' are already adequate for our purpose.

Now, let us establish the initial data for the induction process.

Proposition 10.5. *For the initial case $p = 2$, there hold the codimension values:*

$$ {}_0C_2 = 5, \qquad {}_1C_2 = 3, \qquad {}_2C_2 = 2. $$

Proof. Recalling formulas (159) and (168), we only need to prove ${}_1C_2 = 3$.

For every matrix:

$$ (\alpha_1, \alpha_2, \beta_1, \beta_2) \ \in \ {}_1\mathbf{X}_2 \setminus {}_0\mathbf{X}_2, $$

we have:

$$ \mathsf{rank}_{\mathbb{K}} \{\alpha_1, \alpha_2\} \ = \ 1, \tag{171} $$

$$ \mathsf{rank}_{\mathbb{K}} \{\alpha_1 + (\beta_1 + \beta_2), \alpha_2\} \ \leqslant \ 1, \tag{172} $$

$$ \mathsf{rank}_{\mathbb{K}} \{\alpha_1, \alpha_2 + (\beta_1 + \beta_2)\} \ \leqslant \ 1, \tag{173} $$

$$ \mathsf{rank}_{\mathbb{K}} \{\alpha_1 + \beta_1, \alpha_2 + \beta_2\} \ \leqslant \ 1. \tag{174} $$

Either α_1 or α_2 is nonzero. Firstly, assume $\alpha_1 \neq \mathbf{0}$. Then (171) yields:

$$ \alpha_2 \ \in \ \mathbb{K} \cdot \alpha_1, \tag{175} $$

and (173) yields:

$$ \alpha_2 + (\beta_1 + \beta_2) \ \in \ \mathbb{K} \cdot \alpha_1, $$

whence by subtracting we receive:

$$ \beta_1 + \beta_2 \ \in \ \mathbb{K} \cdot \alpha_1. \tag{176} $$

Next, adding the second column vector of (174) to the first one, we see:

$$ \mathsf{rank}_{\mathbb{K}} \{\alpha_1 + \alpha_2 + (\beta_1 + \beta_2), \alpha_2 + \beta_2\} \ \leqslant \ 1. \tag{177} $$

By (175) and (176):

$$ \alpha_1 + \alpha_2 + (\beta_1 + \beta_2) \ \in \ \mathbb{K} \cdot \alpha_1, $$

therefore (177) yields two possible situations, the first one is:

$$ \alpha_1 + \alpha_2 + \beta_1 + \beta_2 = \mathbf{0}, \tag{178} $$

and the second one is $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 \neq \mathbf{0}$ plus:

$$ \alpha_2 + \beta_2 \ \in \ \mathbb{K} \cdot \alpha_1. $$

Recalling (175), the latter case immediately yields:

$$ \beta_2 \ \in \ \mathbb{K} \cdot \alpha_1, $$

and then (176) implies:

$$ \beta_1 \ \in \ \mathbb{K} \cdot \alpha_1, $$

thus:

$$ \mathsf{rank}_{\mathbb{K}} \{\alpha_1, \alpha_2, \beta_1, \beta_2\} \ = \ 1. \tag{179} $$

Summarizing, the set:

$$ ({}_1\mathbf{X}_2 \setminus {}_0\mathbf{X}_2) \cap \{\alpha_1 \neq 0\} $$

is contained in the union of two algebraic varieties, the first one is defined by (175), (176), (178), and the second one is defined by (179). Since both of the two varieties are of codimension 3, we get:

$$\operatorname{codim}\left({}_1\mathbf{X}_2 \setminus {}_0\mathbf{X}_2\right) \cap \{\alpha_1 \neq 0\} \geqslant 3.$$

Secondly, by symmetry, we also have:

$$\operatorname{codim}\left({}_1\mathbf{X}_2 \setminus {}_0\mathbf{X}_2\right) \cap \{\alpha_2 \neq 0\} \geqslant 3.$$

Hence the union of the above two sets satisfies:

$$\operatorname{codim} {}_1\mathbf{X}_2 \setminus {}_0\mathbf{X}_2 \geqslant 3.$$

Now, recalling (159):

$$\operatorname{codim} {}_0\mathbf{X}_2 = 5 > 3,$$

we immediately receive:

$$\operatorname{codim} {}_1\mathbf{X}_2 \geqslant 3.$$

Finally, noting that ${}_1\mathbf{X}_2$ contains the subvariety:

$$\underbrace{\left\{\operatorname{rank}\{\alpha_1,\alpha_2,\beta_1,\beta_2\} \leqslant 1\right\}}_{\operatorname{codim}=3 \text{ by Lemma } 8.8} \subset \operatorname{Mat}_{2\times 4}(\mathbb{K}),$$

it follows:

$$\operatorname{codim} {}_1\mathbf{X}_2 \leqslant 3.$$

In conclusion, the above two estimates squeeze out the desired codimension identity. □

Admitting temporally Proposition 10.4, it is now time to deduce the crucial

Proposition 10.6 (Core Codimension Formulas). *For all integers $p \geqslant 2$, there hold the codimension estimates:*

$$ {}_\ell C_p \geqslant \ell + (p-\ell)^2 + 1 \qquad {}_{(\ell=0\cdots p-1)}, \tag{180}$$

and the codimension identity:

$$ {}_p C_p = p. \tag{180$'$}$$

Proof. The case $p = 2$ is already done by the previous proposition.

Reasoning by induction, assume the formulas (180) and (180′) hold for some integer $p - 1 \geqslant 2$, and prove them for the integer p.

Firstly, formula (159) yields the case $\ell = 0$.

Secondly, for the case $\ell = p - 1$, thanks to Proposition 10.2 and to the induction hypothesis, formula (169) immediately yields:

$$\begin{aligned} {}_{p-1}C_p &\geqslant \min\left\{{}_{p-1}C_p^0,\ {}_{p-1}C_{p-1}+2,\ {}_{p-2}C_{p-1}+1,\ {}_{p-3}C_{p-1}\right\} \\ &\geqslant \min\left\{p+2,\ (p-1)+2,\ (p-2)+1^2+1+1,\ (p-3)+2^2+1\right\} \\ &= p+1 \\ &= (p-1)+1^2+1. \end{aligned} \tag{181}$$

Similarly, for $\ell = 1\cdots p-2$, recalling formula (170):

$$ {}_\ell C_p \geqslant \min\left\{{}_\ell C_p^0,\ {}_\ell C_{p-1}+2(p-\ell)-1,\ {}_{\ell-1}C_{p-1}+(p-\ell),\ {}_{\ell-2}C_{p-1}\right\},$$

and computing:

$$\begin{aligned}
{}_{\ell}C^{0}_{p} &= p + (p-\ell)^2 \\
&= \ell + (p-\ell)^2 + \underbrace{(p-\ell)}_{\geqslant 2}, \\
{}_{\ell}C_{p-1} + 2(p-\ell) - 1 &\geqslant [\ell + \underline{(p-1-\ell)^2} + 1] + 2(p-\ell) - 1 \\
&= \ell + \underline{(p-\ell)^2 - 2(p-\ell) + 1} + 1 + 2(p-\ell) - 1 \\
&= \underbrace{\ell + (p-\ell)^2 + 1}_{\text{the desired lower bound!}}, \\
{}_{\ell-1}C_{p-1} + (p-\ell) &\geqslant [(\ell-1) + (p-\ell)^2 + 1] + (p-\ell) \\
&= \ell + (p-\ell)^2 + \underbrace{(p-\ell)}_{\geqslant 2}, \\
{}_{\ell-2}C_{p-1} &\geqslant (\ell-2) + (p-\ell+1)^2 + 1 \\
&= (\ell-2) + [(p-\ell)^2 + 2(p-\ell) + 1] + 1 \\
&= \ell + (p-\ell)^2 + \underbrace{2(p-\ell)}_{\geqslant 4},
\end{aligned}$$

we distinguish the desired lower bound without difficulty.

Lastly, the formula (168) and (181) immediately yield (180′):

$${}_{p}C_{p} = p,$$

which concludes the proof. □

Remark 10.7. In fact, the above estimates "⩾" in (180) are exactly identities "=". By the same reasoning, in Section 12, we will generalize the Core Codimension Formulas to cases of less number of moving coefficients terms, and thus receive better lower bounds on the hypersurfaces degrees.

10.2. **Gaussian eliminations.** Following the notation in (155), we denote by:

$$\mathsf{X}_p = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p)$$

the coordinate columns of $\mathsf{Mat}_{p\times 2p}(\mathbb{K})$, where each of the first p columns explicitly writes as:

$$\alpha_i = (\alpha_{1,i}, \ldots, \alpha_{p,i})^{\mathrm{T}},$$

and where each of the last p columns explicitly writes as:

$$\beta_i = (\beta_{1,i}, \ldots, \beta_{p,i})^{\mathrm{T}}.$$

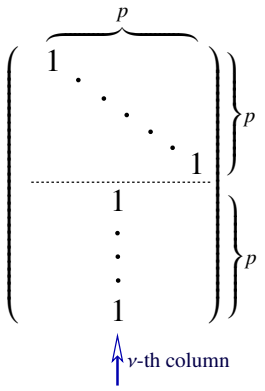
First, observing the structures of the matrices in (157), (158):

$$\begin{aligned}
\mathsf{X}^{0,\nu}_{p} &:= (\alpha_1 \mid \cdots \mid \underline{\widehat{\alpha_\nu}} \mid \cdots \mid \alpha_p \mid \underline{\alpha_\nu + (\beta_1 + \cdots + \beta_p)}), \\
\mathsf{X}^{\tau,\rho}_{p} &:= (\alpha_1 + \beta_1 \mid \cdots \mid \alpha_\tau + \beta_\tau \mid \alpha_{\tau+1} \mid \cdots \mid \underline{\widehat{\alpha_\rho}} \mid \cdots \mid \alpha_p \mid \underline{\alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_p)}),
\end{aligned}$$

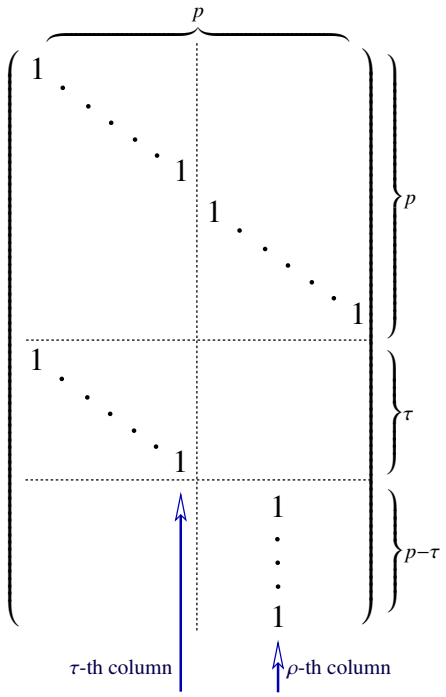
where, slightly differently, the second underlined columns are understood to appear in the first underlined removed places, we realize that they have the uniform shapes:

$$\begin{aligned}
\mathsf{X}^{0,\nu}_{p} &= \mathsf{X}_p\, \mathsf{I}^{0,\nu}_{p}, \\
\mathsf{X}^{\tau,\rho}_{p} &= \mathsf{X}_p\, \mathsf{I}^{\tau,\rho}_{p},
\end{aligned} \tag{182}$$

where the $2p \times p$ matrices $\mathsf{I}_p^{0,\nu}$ explicitly read as:



the upper $p \times p$ submatrix being the identity, the lower $p \times p$ submatrix being zero except its ν-th column being a column of 1, and where lastly, the $2p \times p$ matrices $\mathsf{I}_p^{\tau,\rho}$ explicitly read as:



the upper $p \times p$ submatrix being the identity, the lower $p \times p$ submatrix being zero except τ copies of 1 in the beginning diagonal and $p - \tau$ copies of 1 at the end of the ρ-th column.

Observation 10.8. *For all $p \geqslant 3$, $\tau = 1 \cdots p-1$, $\rho = \tau + 1 \cdots p$, the matrices $\mathsf{I}_p^{\tau,\rho}$ transform to $\mathsf{I}_{p-1}^{\tau-1,\rho-1}$ after deleting the first column and the rows $1, p+1$.* □

Next, observe that all matrices $\mathsf{X}^{\tau,\rho}$ have the same first column:

$$\alpha_1 + \beta_1 = (\,\alpha_{1,1} + \beta_{1,1} \mid \cdots \mid \alpha_{p,1} + \beta_{p,1})^{\mathrm{T}}.$$

Therefore, when $\alpha_{1,1}+\beta_{1,1}\neq 0$, operating Gaussian eliminations by means of the matrix:

$$\mathsf{G}:=\begin{pmatrix} 1 & & & \\ -\frac{\alpha_{2,1}+\beta_{2,1}}{\alpha_{1,1}+\beta_{1,1}} & 1 & & \\ \vdots & & \ddots & \\ -\frac{\alpha_{p,1}+\beta_{p,1}}{\alpha_{1,1}+\beta_{1,1}} & & & 1 \end{pmatrix}, \tag{183}$$

these matrices $\mathsf{X}^{\tau,\rho}$ become simpler:

$$\mathsf{G}\,\mathsf{X}_p^{\tau,\rho}=\begin{pmatrix} \alpha_{1,1}+\beta_{1,1} & \bullet & \cdots & \bullet \\ 0 & \star & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \star & \cdots & \star \end{pmatrix}, \tag{184}$$

where the lower-right $(p-1)\times(p-1)$ star submatrices enjoy amazing structural properties. At first, we need an:

Observation 10.9. *Let $p\geqslant 1$ be a positive integer, let A be a $p\times 2p$ matrix, let B be a $2p\times p$ matrix such that both its 1-st, $(p+1)$-th rows are $(1,\underbrace{0,\ldots,0}_{zeros})$. Then there holds:*

$$(A\,B)'=A''\,B''',$$

where $(A\,B)'$ means the $(p-1)\times(p-1)$ matrix obtained by deleting the first row and column of $A\,B$, and where A'' means the $(p-1)\times 2(p-1)$ matrix obtained by deleting the first row and the columns $1,p+1$ of A, and where B''' means the $2(p-1)\times(p-1)$ matrix obtained by deleting the first column and the rows $1,p+1$ of B. □

Now, noting that:

$$\mathsf{G}\,\mathsf{X}_p^{\tau,\rho}=\mathsf{G}\,(\mathsf{X}_p\,\mathsf{I}_p^{\tau,\rho})=(\mathsf{G}\,\mathsf{X}_p)\,\mathsf{I}_p^{\tau,\rho},$$

thanks to the above two observations, the $(p-1)\times(p-1)$ star submatrices enjoy the forms:

$$\begin{pmatrix} \star & \cdots & \star \\ \vdots & \ddots & \vdots \\ \star & \cdots & \star \end{pmatrix}=\mathsf{X}_p^{\mathsf{G}}\,\mathsf{I}_{p-1}^{\tau-1,\rho-1}, \tag{185}$$

where $\mathsf{X}_p^{\mathsf{G}}$ is the $(p-1)\times 2(p-1)$ matrix obtained by deleting the first row and the columns $1,p+1$ of $\mathsf{G}\,\mathsf{X}_p$.

Comparing (185) and (182), we immediately see that the star submatrices have the same structures as $\mathsf{X}_p^{0,\nu},\mathsf{X}_p^{\tau,\rho}$, which is the cornerstone of our induction approach.

10.3. **Study the morphism of left-multiplying by G.** Let us denote by:

$$D(\alpha_{1,1}+\beta_{1,1})\ \subset\ \mathsf{Mat}_{p\times 2p}(\mathbb{K})$$

the Zariski open set where $\alpha_{1,1}+\beta_{1,1}\neq 0$. Now, consider the regular map of left-multiplying by the function matrix G:

$$\begin{aligned} L_{\mathsf{G}}\colon\quad D(\alpha_{1,1}+\beta_{1,1}) &\longrightarrow D(\alpha_{1,1}+\beta_{1,1}) \\ \mathsf{X}_p &\longmapsto \mathsf{G}\,\mathsf{X}_p. \end{aligned}$$

Of course, it is not surjective, as (184) shows that its image lies in the variety:

$$\cap_{i=2}^{p}\{\alpha_{i,1}+\beta_{i,1}=0\}.$$

In order to compensate this loss of surjectivity, combing with the regular map:

$$\begin{aligned} \in\colon \quad D(\alpha_{1,1}+\beta_{1,1}) &\longrightarrow \mathsf{Mat}_{(p-1)\times 1}(\mathbb{K}) \\ \mathsf{X}_p &\longmapsto (\alpha_{2,1}+\beta_{2,1} \mid \cdots \mid \alpha_{p,1}+\beta_{p,1})^{\mathsf{T}}, \end{aligned}$$

we construct a regular map:

$$L_{\mathsf{G}} \oplus \in\colon \quad D(\alpha_{1,1}+\beta_{1,1}) \longrightarrow \underbrace{\Big(\cap_{\ell=2}^{p} \{\alpha_{i,1}+\beta_{i,1}=0\} \cap D(\alpha_{1,1}+\beta_{1,1}) \Big) \oplus \mathsf{Mat}_{(p-1)\times 1}(\mathbb{K})}_{=:\maltese},$$

which turns out to be an isomorphism. In fact, it has the inverse morphism:

$$\begin{aligned} \underbrace{\Big(\cap_{\ell=2}^{p} \{\alpha_{i,1}+\beta_{i,1}=0\} \cap D(\alpha_{1,1}+\beta_{1,1}) \Big) \oplus \mathsf{Mat}_{(p-1)\times 1}(\mathbb{K})}_{=\maltese} &\longrightarrow D(\alpha_{1,1}+\beta_{1,1}) \\ \mathsf{Y} \oplus (s_2,\ldots,s_p)^{\mathsf{T}} &\longmapsto {}^{-1}\mathsf{G} \cdot \mathsf{Y}, \end{aligned}$$

where the matrix ${}^{-1}\mathsf{G}$ is the "inverse" of the regular function matrix G in (183):

$$\begin{pmatrix} 1 & & & \\ \frac{s_2}{\alpha_{1,1}+\beta_{1,1}} & 1 & & \\ \vdots & & \ddots & \\ \frac{s_p}{\alpha_{1,1}+\beta_{1,1}} & & & 1 \end{pmatrix}. \tag{186}$$

Now, let us denote by:

$$\pi_p\colon \quad \mathsf{Mat}_{p\times 2p}(\mathbb{K}) \longrightarrow \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K})$$

the projection map obtained by deleting the first row and the columns $1, p+1$. Let us denote also:

$$\mathscr{L}_{\mathsf{G}} := \pi_p \circ L_{\mathsf{G}}.$$

It is worth to mention that there is a natural isomorphism:

$$\begin{aligned} \mathsf{R}\colon \qquad \maltese &\xrightarrow{\sim} D(\alpha_{1,1}+\beta_{1,1}), \\ \mathsf{Y} \oplus (s_2,\ldots,s_p)^{\mathsf{T}} &\longmapsto \ ? \end{aligned}$$

where ? is Y but replacing $(b_{2,1},\ldots,b_{p,1})^{\mathsf{T}}$ by $(s_2,\ldots,s_p)^{\mathsf{T}}$, and thus we obtain a commutative diagram:

$$\begin{array}{ccccc} D(\alpha_{1,1}+\beta_{1,1}) & \xrightarrow{L_{\mathsf{G}}\oplus\in} & \maltese & \xrightarrow{\mathsf{R}} & D(\alpha_{1,1}+\beta_{1,1}) \\ & \searrow^{\mathscr{L}_{\mathsf{G}}} & \downarrow{\scriptstyle \pi_p\oplus\mathbf{0}} & \swarrow_{\pi_p} & \\ & & \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}), & & \end{array} \tag{187}$$

where the horizontal maps are isomorphisms, and where the right vertical map is surjective with fibre:

$$\underbrace{\ker \pi_p}_{\mathbb{K}\text{-linear space}} \ \cap \ D(\alpha_{1,1}+\beta_{1,1}).$$

Recalling the end of Subsection 10.2, we in fact received the following key observation.

Observation 10.10. *For every positive integer $p \geqslant 3$, for every integer $\ell = 0 \cdots p-1$, the image of the variety:*

$$ {}_{\ell}\mathbf{X}_p \cap D(\alpha_{1,1}+\beta_{1,1}) \subset D(\alpha_{1,1}+\beta_{1,1}) $$

under the map:

$$ \mathscr{L}_{\mathsf{G}}\colon \quad D(\alpha_{1,1}+\beta_{1,1}) \longrightarrow \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}) $$

is contained in the variety:

$$ {}_{\ell}\mathbf{X}_{p-1} \subset \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}). $$

□

10.4. **A technical lemma.** Now, we carry out one preliminary lemma for the final proof of Proposition 10.4.

For all positive integers $p \geqslant 3$, for every integer $\ell = 0 \cdots p-1$, for every fixed $(p-1)\times(p-1)$ matrix J of rank ℓ, denote the space which consists of all the $p \times p$ matrices of the form:

$$ \left(\begin{array}{c|ccc} z_{1,1} & z_{1,2} & \cdots & z_{1,p} \\ \hline z_{2,1} & & & \\ \vdots & & J & \\ z_{p,1} & & & \end{array}\right) $$

by ${}_{J}S_{p,\ell} \cong \mathbb{K}^{2p-1}$. For every integer $j = \ell, \ell+1$, denote by ${}_{J}S^{j}_{p,\ell} \subset {}_{J}S_{p,\ell}$ the subvariety that consists of all the matrices having rank $\leqslant j$.

Lemma 10.11. *The codimensions of ${}_{J}S^{j}_{p,\ell}$ are:*

$$ \mathsf{codim}\ {}_{J}S^{j}_{p,\ell} = \begin{cases} 2(p-1-\ell)+1 & (j=\ell), \\ p-1-\ell & (j=\ell+1). \end{cases} $$

Proof. Step 1. We claim that the codimensions of ${}_{J}S^{j}_{p,\ell}$ are independent of the matrix J.

Indeed, choose two invertible $(p-1)\times(p-1)$ matrices L and R, which normalize the matrix J by multiplications on both sides:

$$ LJR = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} =: J_0, $$

where all the entries of J_0 are zeros except the last ℓ copies of 1 in the diagonal. Therefore, we obtain an isomorphism:

$$ \begin{aligned} LR\colon\ {}_{J}S_{p,\ell} &\xrightarrow{\sim} {}_{J_0}S_{p,\ell} \\ S &\longmapsto \begin{pmatrix} 1 & \\ & L \end{pmatrix} S \begin{pmatrix} 1 & \\ & R \end{pmatrix} \end{aligned} $$

whose inverse is:

$$ \begin{aligned} L^{-1}R^{-1}\colon\ {}_{J_0}S_{p,\ell} &\xrightarrow{\sim} {}_{J}S_{p,\ell} \\ S &\longmapsto \begin{pmatrix} 1 & \\ & L^{-1} \end{pmatrix} S \begin{pmatrix} 1 & \\ & R^{-1} \end{pmatrix}. \end{aligned} $$

Since the map LR preserves the rank of matrices, it induces an isomorphism between ${}_JS^j_{p,\ell}$ and ${}_{J_0}S^j_{p,\ell}$, which concludes the claim.

Step 2. For J_0, doing elementary row and column operations, we get:

$$\begin{aligned}
&\mathrm{rank}_{\mathbb{K}}\begin{pmatrix} z_{1,1} & z_{1,2} & \cdots & z_{1,p} \\ z_{2,1} & & & \\ \vdots & & J_0 & \\ z_{p,1} & & & \end{pmatrix} \\
&= \mathrm{rank}_{\mathbb{K}}\begin{pmatrix} z_{1,1} & z_{1,2} & \cdots & z_{1,p-\ell} & z_{1,p-\ell+1} & \cdots & z_{1,p} \\ z_{2,1} & & & & & & \\ \vdots & & \mathbf{0} & & & \mathbf{0} & \\ z_{p-\ell,1} & & & & & & \\ z_{p-\ell+1,1} & & & & 1 & & \\ \vdots & & \mathbf{0} & & & \ddots & \\ z_{p,1} & & & & & & 1 \end{pmatrix} \\
&= \mathrm{rank}_{\mathbb{K}}\begin{pmatrix} z_{1,1} - \sum_{k=p-\ell+1}^{p} z_{k,1}\, z_{1,k} & z_{1,2} & \cdots & z_{1,p-\ell} & 0 & \cdots & 0 \\ z_{2,1} & & & & & & \\ \vdots & & \mathbf{0} & & & \mathbf{0} & \\ z_{p-\ell,1} & & & & & & \\ 0 & & & & 1 & & \\ \vdots & & \mathbf{0} & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix} \\
&= \mathrm{rank}_{\mathbb{K}}\begin{pmatrix} z_{1,1} - \sum_{k=p-\ell+1}^{p} z_{k,1}\, z_{1,k} & z_{1,2} & \cdots & z_{1,p-\ell} \\ z_{2,1} & & & \\ \vdots & & \mathbf{0} & \\ z_{p-\ell,1} & & & \end{pmatrix} + \ell.
\end{aligned}$$

Step 3. In the $\mathbb{K}$-Euclidian space $\mathbb{K}^{2N-1}$ with coordinates $(z_{1,1}, z_{1,2}, \ldots, z_{1,N}, z_{2,1}, \ldots, z_{N,1})$, the algebraic subvariety defined by the rank inequality:

$$\mathrm{rank}_{\mathbb{K}}\begin{pmatrix} z_{1,1} - \sum_{k=p-\ell+1}^{p} z_{k,1}\, z_{1,k} & z_{1,2} & \cdots & z_{1,p-\ell} \\ z_{2,1} & & & \\ \vdots & & \mathbf{0} & \\ z_{p-\ell,1} & & & \end{pmatrix} \leqslant 0 \qquad (\text{resp. } \leqslant 1)$$

has codimension $2(p-1-\ell)+1$ (resp. $p-1-\ell$). □

10.5. **Proof of Proposition 10.4.** Recalling the definition (155), and applying Lemma 10.3, we receive:

Corollary 10.12. *For every integers $p \geqslant 1$, the difference of the varieties:*

$$ {}_p\mathbf{X}_p \;\setminus\; {}_{p-1}\mathbf{X}_p \;\subset\; \mathrm{Mat}_{p\times 2p}(\mathbb{K})$$

is exactly the quasi-variety:

$$\underbrace{\Big\{\alpha_1+\cdots+\alpha_p+\beta_1+\cdots+\beta_p=\mathbf{0}\Big\}}_{\mathsf{codim}=p}\ \cap\ \Big\{\underbrace{\mathsf{rank}_{\mathbb{K}}\{\alpha_1,\ldots,\alpha_p\}\ =\ p}_{\Leftrightarrow\det(\alpha_1|\cdots|\alpha_2)\neq 0}\Big\},$$

whose codimension is p.

Proof. For every $p\times 2p$ matrix:

$$_p\mathbf{X}_p\ \setminus\ {}_{p-1}\mathbf{X}_p\ \ \ni\ \ \mathsf{X}_p=(\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_p),$$

applying now Lemma 10.3 to condition (157):

$$\mathsf{rank}_{\mathbb{K}}\{\alpha_1,\ldots,\widehat{\alpha_\nu},\ldots,\alpha_p,\alpha_\nu+\underbrace{(\beta_1+\cdots+\beta_p)}_{=:\beta}\}\leqslant p-1\qquad(\nu=1\cdots p),$$

since:

$$\mathsf{rank}_{\mathbb{K}}\{\alpha_1,\ldots,\alpha_p\}\ =\ p,$$

we immediately receive:

$$\alpha_1+\cdots+\alpha_p+\beta_1+\cdots+\beta_p=\mathbf{0}.$$

On the other hand, for all matrices:

$$\mathsf{X}_p=(\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_p)$$

satisfying the above identity, **(ii)** holds immediately. Noting that the $p\times p$ matrix in (158) has a vanishing sum of all its columns, it has rank $\leqslant p-1$, i.e. **(iii)** holds too. □

Now, we give a complete proof of the Codimension Induction Formulas.

Proof of (168). This is a direct consequence of the above corollary. □

Proof of (169). By Observation 10.10, under the map:

$$\mathscr{L}_{\mathsf{G}}\colon\quad D(\alpha_{1,1}+\beta_{1,1})\ \longrightarrow\ \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}),$$

the image of the variety:

$$_{p-1}\mathbf{X}_p\ \cap\ D(\alpha_{1,1}+\beta_{1,1})$$

is contained in the variety:

$$_{p-1}\mathbf{X}_{p-1}\ \subset\ \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}).$$

Now, let us decompose the variety $_{p-1}\mathbf{X}_{p-1}$ into three pieces:

$$_{p-1}\mathbf{X}_{p-1}\ =\ {}_{p-3}\mathbf{X}_{p-1}\ \cup\ ({}_{p-2}\mathbf{X}_{p-1}\ \setminus\ {}_{p-3}\mathbf{X}_{p-1})\ \cup\ ({}_{p-1}\mathbf{X}_{p-1}\ \setminus\ {}_{p-2}\mathbf{X}_{p-1}),\tag{188}$$

where each matrix $(\alpha_1,\ldots,\alpha_{p-1},\beta_1,\ldots,\beta_{p-1})$ in the $\underline{\text{first}}_1$ (resp. $\underline{\text{second}}_2$, $\underline{\text{third}}_3$) piece satisfies:

$$\mathsf{rank}_{\mathbb{K}}(\alpha_1,\ldots,\alpha_{p-1})\underline{\leqslant\ p-3}_1\qquad(\text{resp. }\underline{=\ p-2}_2\,,\ \underline{=\ p-1}_3).\tag{189}$$

Pulling back (188) by the map $\mathscr{L}_{\mathsf{G}}$, we see that:

$$_{p-1}\mathbf{X}_p\ \cap\ D(\alpha_{1,1}+\beta_{1,1})$$

is contained in:

$$\mathscr{L}_{\mathsf{G}}^{-1}({}_{p-3}\mathbf{X}_{p-1})\ \cup\ \mathscr{L}_{\mathsf{G}}^{-1}({}_{p-2}\mathbf{X}_{p-1}\ \setminus\ {}_{p-3}\mathbf{X}_{p-1})\ \cup\ \mathscr{L}_{\mathsf{G}}^{-1}({}_{p-1}\mathbf{X}_{p-1}\ \setminus\ {}_{p-2}\mathbf{X}_{p-1}).\tag{190}$$

Firstly, for every point in the first piece:

$$\mathsf{Y}\ \in\ {}_{p-3}\mathbf{X}_{p-1},$$

thanks to the commutative diagram (187), we receive the fibre dimension:

$$\begin{aligned}\dim \mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) &= \dim \ker \pi_p \,\cap\, D(\alpha_{1,1}+\beta_{1,1}) \\ &= \dim \mathsf{Mat}_{p\times 2p}(\mathbb{K}) - \dim \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}).\end{aligned}$$

Now, applying Corollary 8.3 to the regular map $\mathscr{L}$ restricted on:

$$\mathscr{L}_{\mathsf{G}}^{-1}\left({}_{p-3}\mathbf{X}_{p-1}\right) \;\subset\; \mathsf{Mat}_{p\times 2p}(\mathbb{K})$$

we receive the codimension estimate:

$$\operatorname{codim} \mathscr{L}_{\mathsf{G}}^{-1}\left({}_{p-3}\mathbf{X}_{p-1}\right) \geqslant \underbrace{\operatorname{codim}{}_{p-3}\mathbf{X}_{p-1}}_{{}_{p-3}C_{p-1}}. \tag{191}$$

Secondly, for every point in the second piece:

$$\mathsf{Y} \;\in\; {}_{p-2}\mathbf{X}_{p-1} \setminus {}_{p-3}\mathbf{X}_{p-1},$$

to look at the fibre of $\mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y})$, thanks to the commutative diagram (187), we can use:

$$\mathscr{L}_{\mathsf{G}}^{-1} = \big(\underbrace{\mathsf{R}\circ(L_{\mathsf{G}}\oplus \in)}_{\text{an isomorphism}}\big)^{-1}\circ \pi_p^{-1}, \tag{192}$$

and obtain:

$$\begin{aligned}&\mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) \,\cap\, \left({}_{p-1}\mathbf{X}_p \,\cap\, D(\alpha_{1,1}+\beta_{1,1})\right) \\ \cong\; &\underline{\mathsf{R}\circ(L_{\mathsf{G}}\oplus \in)}\,\mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) \,\cap\, \underline{\mathsf{R}\circ(L_{\mathsf{G}}\oplus \in)}\left({}_{p-1}\mathbf{X}_p \,\cap\, D(\alpha_{1,1}+\beta_{1,1})\right) \\ \cong\; &\underbrace{\pi_p^{-1}(\mathsf{Y}) \,\cap\, \mathsf{R}\circ(L_{\mathsf{G}}\oplus \in)\left({}_{p-1}\mathbf{X}_p \,\cap\, D(\alpha_{1,1}+\beta_{1,1})\right)}_{=:\clubsuit} \qquad \text{[use (192)]}.\end{aligned}$$

Observe now that every matrix:

$$(\alpha_1 \mid \cdots \mid \alpha_p \mid \beta_1 \mid \cdots \mid \beta_p) \;\in\; \clubsuit$$

satisfies the rank estimate:

$$\operatorname{rank}_{\mathbb{K}}(\alpha_1 \mid \cdots \mid \alpha_p) \leqslant p-1.$$

Moreover, noting that the lower-right $(p-1)\times(p-1)$ submatrix J of $(\alpha_1 \mid \cdots \mid \alpha_p)$ — which is the left $(p-1)\times(p-1)$ submatrix of Y — has rank:

$$\operatorname{rank}_{\mathbb{K}} J = p-2 \qquad \text{[see (189)]},$$

by applying Lemma 10.11, we get that:

$$\clubsuit \;\subset\; \pi_p^{-1}(\mathsf{Y})$$

has codimension greater or equal to:

$$\operatorname{codim} {}_{J}S^{\,p-1}_{p,p-2} = p-1-(p-2) = 1.$$

In other words:

$$\mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) \,\cap\, \left({}_{p-1}\mathbf{X}_p \,\cap\, D(\alpha_{1,1}+\beta_{1,1})\right) \;\subset\; \mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y})$$

has codimension $\geqslant 1$. Thus, applying Corollary 8.3 to the map $\mathscr{L}_{\mathsf{G}}$ restricted on:

$$\underbrace{\mathscr{L}_{\mathsf{G}}^{-1}\left({}_{p-2}\mathbf{X}_{p-1} \setminus {}_{p-3}\mathbf{X}_{p-1}\right) \cap \left({}_{p-1}\mathbf{X}_p \,\cap\, D(\alpha_{1,1}+\beta_{1,1})\right)}_{=:\mathbf{II}} \;\subset\; \mathsf{Mat}_{p\times 2p}(\mathbb{K}),$$

we receive the codimension estimate:

$$\begin{aligned}\operatorname{codim}\mathbf{II} &\geqslant \operatorname{codim}\left({}_{p-2}\mathbf{X}_{p-1} \setminus {}_{p-3}\mathbf{X}_{p-1}\right) + 1\\ &\geqslant \underbrace{\operatorname{codim}{}_{p-2}\mathbf{X}_{p-1} + 1}_{{}_{p-2}C_{p-1}+1}.\end{aligned} \tag{193}$$

Thirdly, for every point in the third piece:

$$\mathsf{Y} \in {}_{p-1}\mathbf{X}_{p-1} \setminus {}_{p-2}\mathbf{X}_{p-1},$$

thanks to the diagram (187):

$$\mathscr{L}_{\mathsf{G}}^{-1} = \underbrace{(L_{\mathsf{G}} \oplus \mathbb{\in})^{-1}}_{\cong} \circ (\pi_p \oplus \mathbf{0})^{-1}, \tag{194}$$

we receive:

$$\begin{aligned}&\mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) \cap \left({}_{p-1}\mathbf{X}_p \cap D(\alpha_{1,1}+\beta_{1,1})\right)\\ \cong\ &\underline{(L_{\mathsf{G}} \oplus \mathbb{\in})}\,\mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) \cap \underline{(L_{\mathsf{G}} \oplus \mathbb{\in})}\left({}_{p-1}\mathbf{X}_p \cap D(\alpha_{1,1}+\beta_{1,1})\right)\\ \cong\ &\underbrace{(\pi_p \oplus \mathbf{0})^{-1}(\mathsf{Y}) \cap (L_{\mathsf{G}} \oplus \mathbb{\in})\left({}_{p-1}\mathbf{X}_p \cap D(\alpha_{1,1}+\beta_{1,1})\right)}_{=:\spadesuit} \qquad \text{[use (194)]}.\end{aligned}$$

Recalling Corollary 10.12, the sum of all columns of Y — the bottom $(p-1)$ rows of $(\alpha_2 \mid \cdots \mid \alpha_p \mid \beta_2 \mid \cdots \mid \beta_p)$ — is zero. Thus, every element:

$$(\alpha_1 \mid \cdots \mid \alpha_p \mid \beta_1 \mid \cdots \mid \beta_p) \oplus (s_2, \ldots, s_p)^{\mathsf{T}} \in \spadesuit$$

not only satisfies:

$$\operatorname{rank}_{\mathbb{K}}(\alpha_1 \mid \cdots \mid \alpha_p) \leqslant p-1, \tag{195}$$

but also satisfies:

$$\alpha_2 + \cdots + \alpha_p + \beta_2 + \cdots + \beta_p = (\underbrace{\alpha_{1,2} + \cdots + \alpha_{1,p} + \beta_{1,2} + \cdots + \beta_{1,p}}_{\text{only this first entry could be nonzero}}, \underbrace{0, \ldots, 0}_{(p-1)\text{ copies}})^{\mathsf{T}}.$$

Remembering that:

$$\alpha_1 + \beta_1 = (\alpha_{1,1} + \beta_{1,1}, \underbrace{0, \ldots, 0}_{(p-1)\text{ copies}})^{\mathsf{T}},$$

summing the above two identities immediately yields:

$$\alpha_1 + \cdots + \alpha_p + \beta_1 + \cdots + \beta_p = \alpha_{1,1} + \cdots + \alpha_{1,p} + \beta_{1,1} + \cdots + \beta_{1,p}, \underbrace{0, \ldots, 0}_{(p-1)\text{ copies}})^{\mathsf{T}}. \tag{196}$$

Now, note that (157) ('matrices ranks') in condition **(ii)** are preserved under the map L_{G} ('Gaussian eliminations'), in particular, for $\nu = 1$, the image satisfies:

$$\operatorname{rank}_{\mathbb{K}}\{\alpha_1 + (\beta_1 + \cdots + \beta_p), \alpha_2, \ldots, \alpha_p\} \leqslant p-1,$$

which, by adding the column vectors $2 \cdots p$ to the first one, is equivalent to:

$$\operatorname{rank}_{\mathbb{K}}\{\alpha_1 + \cdots + \alpha_p + \beta_1 + \cdots + \beta_p, \alpha_2 \ldots, \alpha_p\} \leqslant p-1.$$

Remember (196), and recalling Corollary 10.12:

$$\text{the bottom } (p-1)\times(p-1) \text{ submatrix of } (\alpha_2 \mid \cdots \mid \alpha_p) \text{ is of full rank } (p-1), \tag{197}$$

we immediately receive:

$$\underbrace{\alpha_{1,1} + \cdots + \alpha_{1,p} + \beta_{1,1} + \cdots + \beta_{1,p} = 0}_{\mathsf{codim} = 1}.$$

Therefore, by applying Lemma 10.11, the restrictions (195) and (197) contribute one extra codimension:

$$\mathsf{codim}\ {}_{J}S^{p-1}_{p,p-1} = 1.$$

Thus, we see that 'the fibre in fibre':

$$(\pi_p \oplus \mathbf{0})^{-1}(\mathsf{Y}) \cap \spadesuit \ \subset \ (\pi_p \oplus \mathbf{0})^{-1}(\mathsf{Y})$$

has codimension greater or equal to:

$$1 + 1 = 2.$$

Now, applying once again Corollary 8.3 to the map $\mathscr{L}_{\mathsf{G}}$ restricted on:

$$\underbrace{\mathscr{L}_{\mathsf{G}}^{-1}\left({}_{p-1}\mathbf{X}_{p-1} \setminus {}_{p-2}\mathbf{X}_{p-1}\right) \cap \left({}_{p-1}\mathbf{X}_{p} \cap D(\alpha_{1,1} + \beta_{1,1})\right)}_{=:\,\mathbf{III}} \ \subset \ \mathsf{Mat}_{p\times 2p}(\mathbb{K}),$$

we receive the codimension estimate:

$$\begin{aligned} \mathsf{codim}\,\mathbf{III} &\geqslant \mathsf{codim}\left({}_{p-1}\mathbf{X}_{p-1} \setminus {}_{p-2}\mathbf{X}_{p-1}\right) + 2 \\ &\geqslant \underbrace{\mathsf{codim}\ {}_{p-1}\mathbf{X}_{p-1} + 2}_{{}_{p-1}C_{p-1}+2}. \end{aligned} \tag{198}$$

Summarizing (190), (191), (193), (198), we receive the codimension estimate:

$$\mathsf{codim}\ {}_{p-1}\mathbf{X}_{p} \cap D(\alpha_{1,1} + \beta_{1,1}) \geqslant \min\left\{\mathsf{codim}\ {}_{p-3}\mathbf{X}_{p-1},\ \mathsf{codim}\ {}_{p-2}\mathbf{X}_{p-1} + 1,\ \mathsf{codim}\ {}_{p-1}\mathbf{X}_{p-1} + 2\right\}.$$

By permuting the indices, we know that all:

$${}_{p-1}\mathbf{X}_{p} \cap D(\alpha_{i,1} + \beta_{i,1}) \ \subset \ \mathsf{Mat}_{p\times 2p}(\mathbb{K}) \qquad (i = 1 \cdots p)$$

have the same codimension, and so does their union:

$${}_{p-1}\mathbf{X}_{p} \cap D(\alpha_{1} + \beta_{1}) = \cup_{i=1}^{p}\left({}_{p-1}\mathbf{X}_{p} \cap D(\alpha_{i,1} + \beta_{i,1})\right) \ \subset \ \mathsf{Mat}_{p\times 2p}(\mathbb{K}).$$

Finally, taking codimension on both sides of:

$${}_{p-1}\mathbf{X}_{p} = \left({}_{p-1}\mathbf{X}_{p} \cap V(\alpha_{1} + \beta_{1})\right) \cup \left({}_{p-1}\mathbf{X}_{p} \cap D(\alpha_{1} + \beta_{1})\right),$$

Proposition 10.2 and the preceding estimate conclude the proof. □

Proof of (170). If $\ell \geqslant 2$, decompose the variety ${}_{\ell}\mathbf{X}_{p-1}$ into three pieces:

$${}_{\ell}\mathbf{X}_{p-1} = {}_{\ell-2}\mathbf{X}_{p-1} \cup \left({}_{\ell-1}\mathbf{X}_{p-1} \setminus {}_{\ell-2}\mathbf{X}_{p-1}\right) \cup \left({}_{\ell}\mathbf{X}_{p-1} \setminus {}_{\ell-1}\mathbf{X}_{p-1}\right);$$

and if $\ell = 1$, decompose the variety ${}_{\ell}\mathbf{X}_{p-1}$ into two pieces:

$${}_{\ell}\mathbf{X}_{p-1} = {}_{\ell-1}\mathbf{X}_{p-1} \cup \left({}_{\ell}\mathbf{X}_{p-1} \setminus {}_{\ell-1}\mathbf{X}_{p-1}\right).$$

Now, by mimicking the preceding proof, namely by applying Lemma 10.11 and Corollary 8.3, everything goes on smoothly with much less effort, because there is no need to perform delicate codimension estimates such as (198). □

10.6. **Proof of Core Lemma 9.5.** If $N = 1$, there is nothing to prove. Assume now $N \geqslant 2$. Comparing (146) and (155), it is natural to introduce the projection:

$$\begin{aligned} \pi_{2c+r,N}\colon \quad & \mathsf{Mat}_{(2c+r)\times 2(N+1)}(\mathbb{K}) \longrightarrow \mathsf{Mat}_{N\times 2N}(\mathbb{K}) \\ & (\alpha_0,\dots,\alpha_p,\beta_0,\dots,\beta_p) \longmapsto (\widehat{\alpha}_1,\dots,\widehat{\alpha}_p,\widehat{\beta}_1,\dots,\widehat{\beta}_p), \end{aligned}$$

where each widehat vector is obtained by extracting the first N rows (entries) out of the original $2c + r$ rows (entries).

Now, for every point:

$$(\alpha_0,\dots,\alpha_p,\beta_0,\dots,\beta_p) \ \in \ \mathscr{M}^N_{2c+r} \ \subset \ \mathsf{Mat}_{(2c+r)\times 2(N+1)}(\mathbb{K}),$$

in restriction (148), by setting $\nu = 0$, we receive:

$$\mathsf{rank}_{\mathbb{K}}\,\{\alpha_1,\dots,\alpha_N,\alpha_0+(\beta_0+\beta_1+\cdots+\beta_N)\} \leqslant N-1.$$

Dropping the last column and extracting the first N rows, we get:

$$\mathsf{rank}_{\mathbb{K}}\,\{\widehat{\alpha}_1,\dots,\widehat{\alpha}_N\} \leqslant N-1.$$

Similarly, in restriction (149), by dropping the first column and extracting the first N rows, for all $\tau = 0\cdots N-1$ and $\rho = \tau+1\cdots N$, we obtain:

$$\mathsf{rank}_{\mathbb{K}}\,\{\widehat{\alpha}_1+\widehat{\beta}_1,\dots,\widehat{\alpha}_\tau+\widehat{\beta}_\tau,\widehat{\alpha}_{\tau+1},\dots\boxed{\widehat{\alpha}_\rho}\dots,\widehat{\alpha}_N,\widehat{\alpha}_\rho+(\widehat{\beta}_{\tau+1}+\cdots+\widehat{\beta}_N)\} \leqslant N-1,$$

where we omit the column vector $\widehat{\alpha}_\rho$ in the box. Summarizing the above two inequalities, $(\widehat{\alpha}_1,\dots,\widehat{\alpha}_p,\widehat{\beta}_1,\dots,\widehat{\beta}_p)$ satisfies the restriction (156) – (158):

$$\underbrace{\pi_{2c+r,N}\,(\alpha_0,\dots,\alpha_p,\beta_0,\dots,\beta_p)}_{=(\widehat{\alpha}_1,\dots,\widehat{\alpha}_p,\widehat{\beta}_1,\dots,\widehat{\beta}_p)} \in \ {}_{N-1}\mathbf{X}_N \ \subset \ \mathsf{Mat}_{N\times 2N}(\mathbb{K}).$$

Therefore:

$$\pi_{2c+r,N}\,(\mathscr{M}^N_{2c+r}) \ \subset \ {}_{N-1}\mathbf{X}_N.$$

Moreover, for every point $\mathsf{Y} \in {}_{N-1}\mathbf{X}_N$, the 'fibre in fibre':

$$\pi^{-1}_{2c+r,N}(\mathsf{Y}) \cap \mathscr{M}^N_{2c+r} \ \subset \ \pi^{-1}_{2c+r,N}(\mathsf{Y}),$$

thanks to (147), has codimension $\geqslant 2c + r$. Thus a direct application of Corollary 8.3 yields:

$$\begin{aligned} \mathsf{codim}\,\mathscr{M}^N_{2c+r} \ &\geqslant \ \mathsf{codim}\,{}_{N-1}\mathbf{X}_N + 2c + r \\ \text{[use (180)]} \qquad &\geqslant \ N+1+2c+r. \end{aligned}$$

Repeating the same reasoning, we obtain:

$$\begin{aligned} \mathsf{codim}\,\mathscr{M}^{N-\eta}_{2c+r} \ &\geqslant \ \mathsf{codim}\,{}_{N-\eta-1}\mathbf{X}_{N-\eta} + 2c + r \\ \text{[use (180)]} \qquad &\geqslant \ N-\eta+1+2c+r. \end{aligned}$$

Remembering $2c + r \geqslant N$, we conclude the proof. □

10.7. **'Macaulay2', 'Maple' et al.** *vs.* **the Core Lemma.** Believe it or not, concerning the Core Lemma or the Core Codimension Formulas, 'Macaulay2' – a professional software system devoted to supporting research in algebraic geometry and commutative algebra – is not strong enough to compute the precise codimensions of the involved determinantal ideals, even in small dimensions $p \geqslant 4$. And unfortunately, so do other mathematical softwares, like 'Maple'...

This might indicate some weaknesses of current computers. Since the Core Lemma or a variation of it should be a crucial step in the constructions of *ample examples*, the dream of finding explicit examples with rational coefficients, firstly in small dimensional cases, could be kind of a challenge for a moment.

11. A rough estimate of lower degree bound

11.1. **Effective results.** Recalling Subsection 5.3, we first provide an effective

Theorem 11.1. *For all $N \geqslant 3$, for any $\epsilon_1, \ldots, \epsilon_{c+r} \in \{1, 2\}$, Theorem 5.2 holds for $\heartsuit = 1$. and for all $d \geqslant N^{N^2/2} - 1$.*

Proof. Setting $\delta_{c+r+1} = 2$ in (93), and demanding all (94) – (96) to be equalities, we thus receive the desired estimate:

$$[\text{see (97)}] \qquad (N+1)\,\mu_{N,N} \leqslant N^{N^2/2} - 1.$$

For the sake of completeness, we present all computational details in Subsection 11.2 below. □

Hence, the *product coup* in Subsection 5.3 yields

Theorem 11.2. *In Theorem 5.1, for $\heartsuit = 1$, the lower bound $d_0(-1) = N^{N^2}$ works.* □

11.2. **Computational details.** We specify (94) – (97) as follows. Recalling that $\delta_{c+r+1} = 2$ and $\heartsuit = 1$, for every integer $l = c + r + 1 \cdots N$, we choose:

$$\mu_{l,0} \;=\; l\,\delta_l + 4\,l + 1, \tag{199}$$

and inductively we choose:

$$\mu_{l,k} \;=\; \sum_{j=0}^{k-1} l\,\mu_{l,j} + (l-k)\,\delta_l + 4\,l + 1 \qquad (k = 1 \cdots l). \tag{200}$$

Actually, we take the above values in purpose, because they also work in the degree estimates in our coming paper.

For every integer $l = c + r + 1 \cdots N$, for every integer $k = 0 \cdots l$, let:

$$S_{l,k} \;:=\; \sum_{j=0}^{k} \mu_{l,j}. \tag{201}$$

For $k = 1 \cdots l$, we have:

$$[\text{see (200)}] \qquad \underbrace{S_{l,k} - S_{l,k-1}}_{=\,\mu_{l,k}} \;=\; l\,S_{l,k-1} + (l-k)\,\delta_l + 4\,l + 1.$$

Moving the term '$-S_{l,k-1}$' to the right hand side, we receive:

$$S_{l,k} \;=\; (l+1)\,S_{l,k-1} + (l-k)\,\delta_l + 4\,l + 1.$$

Dividing by $(l+1)^k$ on both sides, we receive:

$$\underline{\frac{S_{l,k}}{(l+1)^k}} = \underline{\frac{S_{l,k-1}}{(l+1)^{k-1}}} + \big(l\,\delta_l + 4\,l + 1\big)\frac{1}{(l+1)^k} - \delta_l\,\frac{k}{(l+1)^k}.$$

Noting that the two underlined terms have the same structure, doing induction backwards $k \cdots 1$, we receive:

$$\frac{S_{l,k}}{(l+1)^k} = \frac{S_{l,0}}{(l+1)^0} + \big(l\,\delta_l + 4\,l + 1\big)\sum_{j=1}^{k}\frac{1}{(l+1)^j} - \delta_l\sum_{j=1}^{k}\frac{j}{(l+1)^j}.$$

Now, applying the following two elementary identities:

$$\sum_{j=1}^{k}\frac{1}{(l+1)^j} = \frac{1}{l}\Big(1 - \frac{1}{(l+1)^k}\Big),$$

$$\sum_{j=1}^{k}\frac{j}{(l+1)^j} = \frac{l+1}{l^2}\Big(1 + \frac{k}{(l+1)^{k+1}} - \frac{1+k}{(l+1)^k}\Big),$$

and recalling (199):

$$S_{l,0} = \mu_{l,0} = l\,\delta_l + 4\,l + 1,$$

we obtain:

$$\frac{S_{l,k}}{(l+1)^k} = l\,\delta_l + 4\,l + 1 + \big(l\,\delta_l + 4\,l + 1\big)\frac{1}{l}\Big(1 - \frac{1}{(l+1)^k}\Big) - \delta_l\,\frac{l+1}{l^2}\Big(1 + \frac{k}{(l+1)^{k+1}} - \frac{1+k}{(l+1)^k}\Big).$$

Next, multiplying by $(l+1)^k$ on both sides, we get:

$$\begin{aligned} S_{l,k} &= \big(l\,\delta_l + 4\,l + 1\big)\Big(\underline{(l+1)^k + \frac{(l+1)^k}{l}} - \frac{1}{l}\Big) - \frac{\delta_l}{l^2}\Big((l+1)^{k+1} + k - (1+k)\,(l+1)\Big) \\ &= \big(l\,\delta_l + 4\,l + 1\big)\Big(\underline{\frac{(l+1)^{k+1}}{l}} - \frac{1}{l}\Big) - \frac{\delta_l}{l^2}\Big((l+1)^{k+1} + k - (1+k)\,(l+1)\Big). \end{aligned} \tag{202}$$

Recalling (96), we have:

$$\begin{aligned} \delta_{l+1} &= l\,\mu_{l,l} \\ \text{[use (200) for } k = l\text{]} \quad &= l\Big(\underline{\sum_{j=0}^{l-1} l\,\mu_{l,j} + 4\,l + 1}\Big) \\ \text{[use (201) for } k = l-1\text{]} \quad &= \underline{l^2\,S_{l,l-1}} + l\,(4\,l+1) \\ \text{[use (202) for } k = l-1\text{]} \quad &= \big(l\,\delta_l + 4\,l + 1\big)\big(l\,(l+1)^l - l\big) - \delta_l\Big((l+1)^l + l - 1 - l\,(l+1)\Big) \\ &\qquad + l\,(4\,l+1) \\ &= \delta_l\,\underbrace{\Big(l^2\,(l+1)^l - (l+1)^l + 1\Big)}_{\geqslant 0} + (4\,l+1)\,l\,(l+1)^l. \end{aligned} \tag{203}$$

Throwing away the first positive part, we receive the estimate:

$$\delta_{l+1} > (4\,l+1)\,l\,(l+1)^l. \tag{204}$$

Therefore, for all $l \geqslant c + r + 2$, we have the estimate of (203):

$$\begin{aligned}
\delta_{l+1} &= l^2 (l+1)^l \delta_l - \Big((l+1)^l - 1\Big)\delta_l + (4l+1)\, l\, (l+1)^l \\
\text{[use (204)]} \qquad &< l^2 (l+1)^l \delta_l - \Big((l+1)^l - 1\Big)(4(l-1)+1)(l-1)\, l^{l-1} + (4l+1)\, l\, (l+1)^l \\
&= l^2 (l+1)^l \delta_l - 4\underbrace{\Big[\Big((l+1)^l - 1\Big)(l-1)^2 l^{l-1} - l^2 (l+1)^l\Big]}_{1} \\
&\qquad - \underbrace{\Big[\Big((l+1)^l - 1\Big)(l-1)\, l^{l-1} - l\,(l+1)^l\Big]}_{2}.
\end{aligned}$$

Since $2c + r \geqslant N \geqslant 1$, c, r cannot be both zero, hence $l \geqslant c + r + 2 \geqslant 3$ above, thus we may realize that the first underlined bracket is positive:

$$\begin{aligned}
\Big[\Big((l+1)^l - 1\Big)(l-1)^2 l^{l-1} - l^2 (l+1)^l\Big] &= l^2 (l+1)^l \left[\Big(1 - \frac{1}{(l+1)^l}\Big)(l-1)^2 l^{l-3} - 1\right] \\
&\geqslant l^2 (l+1)^l \left[\Big(1 - \frac{1}{(3+1)^3}\Big)(3-1)^2 3^{3-3} - 1\right] \\
&> 0,
\end{aligned}$$

and that the second underlined bracket is also positive:

$$\begin{aligned}
\Big[\Big((l+1)^l - 1\Big)(l-1)\, l^{l-1} - l\,(l+1)^l\Big] &= l\,(l+1)^l \left[\Big(1 - \frac{1}{(l+1)^l}\Big)(l-1)\, l^{l-2} - 1\right] \\
&\geqslant l\,(l+1)^l \left[\Big(1 - \frac{1}{(3+1)^3}\Big)(3-1)\, 3^{3-2} - 1\right] \\
&> 0.
\end{aligned}$$

Consequently, we have the neat estimate suitable for the induction:

$$\delta_{l+1} \leqslant l^2 (l+1)^l \delta_l \qquad (l = c+r+2 \cdots N-1), \tag{205}$$

which for convenience, we may assume to be satisfied for $l = N$ by just defining $\delta_{N+1} := N\mu_{N,N}$.

In fact, using these estimates iteratively, we may proceed as follows:

$$\begin{aligned}
(N+1)\,\mu_{N,N} &= \frac{N+1}{N} \underbrace{N\,\mu_{N,N}}_{=\delta_{N+1}} \\
&= \frac{N+1}{N}\,\delta_{N+1} \qquad (206) \\
\text{[use (205)]} \qquad &< \frac{N+1}{N}\,\delta_{c+r+2} \prod_{l=c+r+2}^{N} l^2 (l+1)^l.
\end{aligned}$$

Noting that (203) yields:

$$\begin{aligned}
\delta_{c+r+2} &= \Big[\delta_l \Big(l^2 (l+1)^l - (l+1)^l + 1\Big) + (4l+1)\, l\, (l+1)^l\Big]\Big|_{l=c+r+1} \\
\text{[recall } \delta_{c+r+1} = 2\text{]} \qquad &< 6\, l^2 (l+1)^l \Big|_{l=c+r+1},
\end{aligned}$$

thus the above two estimates yield:

$$(N+1)\,\mu_{N,N} \;<\; \frac{N+1}{N}\,6 \prod_{l=c+r+1}^{N} l^2\,(l+1)^l. \tag{207}$$

For the convenience of later integration, we prefer the term $(l+1)^{l+1}$ to $(l+1)^l$, therefore we firstly transform:

$$\begin{aligned}
\prod_{l=c+r+1}^{N} l^2\,(l+1)^l \;&=\; \prod_{l=c+r+1}^{N} \frac{l}{l+1}\,l\,(l+1)^{l+1} \\
&=\; \prod_{l=c+r+1}^{N} \frac{l}{l+1} \prod_{l=c+r+1}^{N} l \prod_{l=c+r+1}^{N} (l+1)^{l+1} \\
&=\; \frac{c+r+1}{N+1} \prod_{l=c+r+1}^{N} l \prod_{l=c+r+1}^{N} (l+1)^{l+1},
\end{aligned}$$

whence (207) becomes:

$$\begin{aligned}
(N+1)\,\mu_{N,N} \;&<\; 6\,\frac{c+r+1}{N} \prod_{l=c+r+1}^{N} l \prod_{l=c+r+1}^{N} (l+1)^{l+1} \\
\text{[recall } c+r \leqslant N-1\text{]} \qquad &\leqslant\; 6 \prod_{l=c+r+1}^{N} l \prod_{l=c+r+1}^{N} (l+1)^{l+1}.
\end{aligned} \tag{208}$$

Now, we estimate the dominant term:

$$\prod_{l=c+r+1}^{N} l \prod_{l=c+r+1}^{N} (l+1)^{l+1}.$$

as follows.

Remembering $2c + r \geqslant N$, we receive:

$$c + r \;\geqslant\; (2c+r)/2 \;\geqslant\; N/2,$$

and hence for $N \geqslant 2$ we obtain:

$$\begin{aligned}
\ln \prod_{l=c+r+1}^{N} l \;&=\; \ln N \;+\; \sum_{l=c+r+1}^{N-1} \ln l \\
&<\; \ln N \;+\; \int_{c+r+1}^{N} \ln x\; dx \\
&\leqslant\; \ln N \;+\; \int_{N/2+1}^{N} \ln x\; dx \\
&=\; \ln N \;+\; (x \ln x - x)\Big|_{N/2+1}^{N}.
\end{aligned}$$

Similarly, when $N \geqslant 4$, noting that $N \geqslant N/2+2$, we get the estimate:

$$\begin{aligned}
\ln \prod_{l=c+r+1}^{N} (l+1)^{l+1} &= (N+1)\ln(N+1) + N\ln N + \sum_{l=c+r+2}^{N-1} l\ln l \\
&\leqslant (N+1)\ln(N+1) + N\ln N + \int_{N/2+2}^{N} x\ln x\, dx \\
&= (N+1)\ln(N+1) + N\ln N + \Big(\frac{1}{2}x^2\ln x - \frac{x^2}{4}\Big)\Big|_{N/2+2}^{N}.
\end{aligned}$$

Summing the above two estimates, for $N \geqslant 4$ we receive:

$$\begin{aligned}
&\ln \prod_{l=c+r+1}^{N} l + \ln \prod_{l=c+r+1}^{N} (l+1)^{l+1} \\
&\leqslant \ln N + (x\ln x - x)\Big|_{N/2+1}^{N} + (N+1)\ln(N+1) + N\ln N + \Big(\frac{1}{2}x^2\ln x - \frac{x^2}{4}\Big)\Big|_{N/2+2}^{N} \\
&= \frac{1}{2}N^2\ln N - \frac{1}{2}(N/2+2)^2\ln(N/2+2) - \Big(\frac{3}{16}N^2 - 2\Big) - (N/2+1)\ln(N/2+1) + \\
&\qquad + (N+1)\ln(N+1) + (2N+1)\ln N \qquad (209) \\
&= \frac{3}{8}N^2\ln N - O(N^2), \qquad \text{as } N\to\infty. \qquad (210)
\end{aligned}$$

In order to have a neat lower bound, we would like to have:

$$(N+1)\,\mu_{N,N} \leqslant N^{N^2/2} - 1. \qquad (211)$$

In fact, using the estimates (208), (209), when $N \geqslant 48$, we can show by hand that (211) holds true. For $N = 14\cdots 47$, we can use a mathematical software 'Maple' to check the above estimate. Finally, for $N = 3\cdots 13$, we ask 'Maple' to compute δ_{N+1} explicitly, and thereby, thanks to (206), we again prove the estimate (211).

12. **Some Improvements of MCM**

12.1. **General core codimension formulas.** In order to lower the degree bound d_0 of MCM, we will modify the hypersurface constructions. Of course, we would like to reduce the number of moving coefficient terms, and this will be based on the General Core Lemma 12.6 below.

For every integers $p \geqslant q \geqslant 2$, for every integer $0 \leqslant \ell \leqslant q$, we first estimate the codimension ${}_{\ell}C_{p,q}$ of the algebraic variety:

$${}_{\ell}\mathbf{X}_{p,q} \subset \mathsf{Mat}_{p\times 2q}(\mathbb{K})$$

which consists of $p \times 2q$ matrices $\mathsf{X}_{p,q} = (\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_q)$ such that:

(i) the first q column vectors have rank:

$$\mathsf{rank}_{\mathbb{K}}\{\alpha_1, \ldots, \alpha_q\} \leqslant \ell;$$

(ii) for every index $\nu = 1\cdots q$, replacing α_ν with $\alpha_\nu + (\beta_1 + \cdots + \beta_q)$ in the collection of column vectors $\{\alpha_1, \ldots, \alpha_q\}$, there holds the rank inequality:

$$\mathsf{rank}_{\mathbb{K}}\{\alpha_1, \ldots, \widehat{\alpha_\nu}, \ldots, \alpha_q, \alpha_\nu + (\beta_1 + \cdots + \beta_q)\} \leqslant p - 1;$$

(iii) for every integer $\tau = 1 \cdots q-1$, for every index $\rho = \tau+1 \cdots q$, replacing α_ρ with $\alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_q)$ in the collection of column vectors $\{\alpha_1 + \beta_1, \ldots, \alpha_\tau + \beta_\tau, \alpha_{\tau+1}, \ldots, \alpha_\rho, \ldots, \alpha_q\}$, there holds the rank inequality:

$$\mathrm{rank}_{\mathbb{K}} \{\alpha_1 + \beta_1, \ldots, \alpha_\tau + \beta_\tau, \alpha_{\tau+1}, \ldots, \widehat{\alpha_\rho}, \ldots, \alpha_q, \alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_q)\} \leqslant q - 1.$$

Repeating the same reasoning as in Section 10, we may proceed as follows. Firstly, here is a very analogue of Proposition 10.1:

Proposition 12.1. *For every integers $p \geqslant q \geqslant 2$, the codimension value ${}_\ell C_{p,q}$ for $\ell = 0$ is:*

$$ {}_0C_{p,q} = p\,q + p - q + 1. \qquad \square$$

Next, we obtain an analogue of Proposition 10.2:

Proposition 12.2. *For every integers $p \geqslant q \geqslant 2$, the codimensions ${}_\ell C^0_{p,q}$ of the algebraic varieties:*

$$\{\alpha_1 + \beta_1 = \mathbf{0}\} \cap {}_\ell\mathbf{X}_{p,q} \subset \mathrm{Mat}_{p\times 2q}(\mathbb{K})$$

read according to the values of ℓ as:

$${}_\ell C^0_{p,q} = \begin{cases} p + \min\{2\,(p-q+1),\ p\} & (\ell = q), \\ p + 2(p-q+1) & (\ell = q-1), \\ p + (p-\ell)\,(q-\ell) & (\ell = 0 \cdots q-2). \end{cases}$$

The last two lines are easy to obtain, while the first line is a consequence of Lemma 10.3. $\square$

Now, we deduce the analogue of Proposition 10.4:

Proposition 12.3 (General Codimension Induction Formulas). (i) *For every positive integers $p \geqslant q \geqslant 2$, for $\ell = q$, the codimension value ${}_q C_{p,q}$ satisfies:*

$${}_q C_{p,q} \ = \ \min\{p,\ {}_{q-1}C_{p,q}\}.$$

(ii) *For every positive integers $p \geqslant q \geqslant 3$, for $\ell = q-1$, the codimension value ${}_\ell C_{p,q}$ satisfies:*

$${}_{q-1}C_{p,q} \ \geqslant \ \min\{{}_{q-1}C^0_{p,q},\ {}_{q-1}C_{p-1,q-1} + p - q + 2,\ {}_{q-2}C_{p-1,q-1} + 1,\ {}_{q-3}C_{p-1,q-1}\}.$$

(iii) *For all positive integers $p \geqslant q \geqslant 3$, for all integers $\ell = 1 \cdots q-2$, the codimension values ${}_\ell C_{p,q}$ satisfy:*

$${}_\ell C_{p,q} \ \geqslant \ \min\{{}_\ell C^0_{p,q},\ {}_\ell C_{p-1,q-1} + (p-\ell) + (q-\ell) - 1,\ {}_{\ell-1}C_{p-1,q-1} + (q-\ell),\ {}_{\ell-2}C_{p-1,q-1}\}. \qquad \square$$

Similar to Proposition 10.5, we have:

Proposition 12.4. *For the initial cases $p \geqslant q = 2$, there hold the codimension values:*

$${}_0C_{p,2} = 3p - 1, \qquad {}_1C_{p,2} = 2p - 1, \qquad {}_2C_{p,2} = p. \qquad \square$$

Finally, by the same induction proof as in Proposition 10.6, we get:

Proposition 12.5 (General Core Codimension Formulas). *For all integers $p \geqslant q \geqslant 2$, there hold the codimension estimates:*

$${}_\ell C_{p,q} \ \geqslant \ (p-\ell)\,(q-\ell) + p - q + l + 1 \qquad (\ell = 0 \cdots q-1)\,,$$

and the core codimension identity:

$${}_q C_{p,q} = p. \qquad \square$$

Actually, we could prove that the above estimates are identities, yet it is not really necessary.

12.2. **General Core Lemma.** Similar to (146), for every integer $k = 1 \cdots N-1$, we introduce the algebraic subvariety:

$$\mathscr{M}_{2c+r}^{N,k} \subset \mathsf{Mat}_{(2c+r)\times(N+1+k+1)}(\mathbb{K})$$

consisting of all $(c+r+c)\times(N+1+k+1)$ matrices $(\alpha_0 \mid \alpha_1 \mid \cdots \mid \alpha_N \mid \beta_0 \mid \beta_1 \mid \cdots \mid \beta_k)$ such that:

(i) the sum of these $(N+1+k+1)$ colums is zero:

$$\alpha_0 + \alpha_1 + \cdots + \alpha_N + \beta_0 + \beta_1 + \cdots + \beta_k = \mathbf{0}; \tag{212}$$

(ii) for every index $\nu = 0 \cdots k$, replacing α_ν with $\alpha_\nu + (\beta_0 + \beta_1 + \cdots + \beta_k)$ in the collection of column vectors $\{\alpha_0, \alpha_1, \ldots, \alpha_N\}$, there holds the rank inequality:

$$\mathsf{rank}_{\mathbb{K}}\,\{\alpha_0, \ldots, \widehat{\alpha_\nu}, \ldots, \alpha_N, \alpha_\nu + (\beta_0 + \beta_1 + \cdots + \beta_k)\} \leqslant N-1;$$

(iii) for every integer $\tau = 0 \cdots k-1$, for every index $\rho = \tau+1 \cdots k$, replacing α_ρ with $\alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_k)$ in the collection of column vectors $\{\alpha_0+\beta_0, \ldots, \alpha_\tau+\beta_\tau, \alpha_{\tau+1}, \ldots, \alpha_\rho, \ldots, \alpha_N\}$, there holds the rank inequality:

$$\mathsf{rank}_{\mathbb{K}}\,\{\alpha_0 + \beta_0, \alpha_1 + \beta_1, \ldots, \alpha_\tau + \beta_\tau, \alpha_{\tau+1}, \ldots, \widehat{\alpha_\rho}, \ldots, \alpha_N, \alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_k)\} \leqslant N-1.$$

Lemma 12.6 (**Sharp Core Lemma of MCM**). *For every positive integers $N \geqslant 3$, for every integers $c, r \geqslant 0$ with $2c + r \geqslant N$, for every integer $k = 1 \cdots N-1$, there holds the codimension estimate:*

$$\begin{aligned} \mathsf{codim}\,\mathscr{M}_{2c+r}^{N,k} &\geqslant {}_{k+1}C_{2c+r-N+k+1,k+1} + (2c+r) \\ &\geqslant 2(2c+r) - N + k + 1. \end{aligned}$$

The term $(2c + r)$ comes from (212). When $k = N-1$, there is nothing to prove. When $k < N-1$, noting that all matrices in **(ii)** and **(iii)** have the same last column α_N, we may do Gaussian eliminations with respect to this column, and then by much the same argument as before, we receive the estimate. □

Actually, these two estimates are identities.

12.3. **Minimum necessary number of moving coefficient terms.** Firstly, letting:

$$\mathsf{codim}\,\mathscr{M}_{2c+r}^{N,k} \geqslant \mathsf{dim}\,\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) = 2N-1,$$

we receive the lower bound:

$$k \geqslant 3N - 2(2c+r) - 2,$$

which indicates that at the step N, the least number of moving coefficient terms, if necessary, should be:

$$3N - 2(2c+r) - 2 + 1.$$

When $3N - 2(2c+r) - 2 \leqslant 0$, no moving coefficient terms are needed, thanks to the following:

Lemma 12.7. (Elementary Core Lemma) *Let:*

$$\mathscr{M}_{2c+r}^{N,-1} \subset \mathsf{Mat}_{(2c+r)\times(N+1)}(\mathbb{K})$$

consist of all $(c+r+c)\times(N+1)$ matrices $(\alpha_0 \mid \alpha_1 \mid \cdots \mid \alpha_N)$ such that:

(i) *the sum of these $(N+1)$ colums is zero:*

$$\alpha_0 + \alpha_1 + \cdots + \alpha_N = \mathbf{0}; \tag{213}$$

(ii) *there holds the rank inequality:*

$$\mathsf{rank}_{\mathbb{K}}\,\{\alpha_0,\ldots,\widehat{\alpha_\nu},\ldots,\alpha_N\}\leqslant N-1\qquad{}_{(\nu=0\cdots N)}.$$

Then one has the codimension identity:

$$\mathsf{codim}\,\mathscr{M}^{N,-1}_{2c+r}\;=\;2(2c+r)-N+1.\qquad\square$$

Next, for $\eta=1\cdots n-1$, in the step $N-\eta$, letting:

$$\mathsf{codim}\,\mathscr{M}^{N-\eta,k}_{2c+r}\;\geqslant\;\mathsf{dim}\,{}_{v_1,\ldots,v_\eta}\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N})\;=\;2N-\eta-1,$$

we receive:

$$2\,(2c+r)-(N-\eta)+k+1\;\geqslant\;2N-\eta-1,$$

that is:

$$k\geqslant\;3N-2\,(2c+r)-2-2\eta,$$

which indicates that, at the step $N-\eta$, the least number of moving coefficient terms, if necessary, should be:

$$3N-2\,(2c+r)-2-2\eta+1.$$

When $3N-2\,(2c+r)-2-2\eta\;\leqslant\;0$, no moving coefficient terms are needed, thanks to the Elementary Core Lemma.

12.4. **Improved Algorithm of MCM.** When $3N-2\,(2c+r)-2>0$, in order to lower the degrees, we improve the hypersurface equations (92) as follows.

Firstly, when $3N-2\,(2c+r)-2=2p$ is even, the following hypersurface equations are suitable for MCM:

$$\begin{aligned}F_i\;=\;\sum_{j=0}^{N}A_i^j\,z_j^d\;+\;\sum_{\eta=0}^{p-1}\;\sum_{0\leqslant j_0<\cdots<j_{N-\eta}\leqslant N}\;\sum_{k=0}^{2p-2\eta}\\ M_i^{j_0,\ldots,j_{N-\eta};j_k}\,z_{j_0}^{\mu_{N-\eta,k}}\cdots\widehat{z_{j_k}^{\mu_{N-\eta,k}}}\cdots z_{j_{2p-2\eta}}^{\mu_{N-\eta,k}}\,z_{j_k}^{d-(2p-2\eta)\mu_{N-\eta,k}-2(N+\eta-2p)}\,z_{j_{2p-2\eta+1}}^2\cdots z_{j_{N-\eta}}^2.\end{aligned}\tag{214}$$

Secondly, when $3N-2\,(2c+r)-2=2p+1$ is odd, the following hypersurface equations are suitable for MCM:

$$\begin{aligned}F_i\;=\;\sum_{j=0}^{N}A_i^j\,z_j^d\;+\;\sum_{\eta=0}^{p}\;\sum_{0\leqslant j_0<\cdots<j_{N-\eta}\leqslant N}\;\sum_{k=0}^{2p+1-2\eta}\\ M_i^{j_0,\ldots,j_{N-\eta};j_k}\,z_{j_0}^{\mu_{N-\eta,k}}\cdots\widehat{z_{j_k}^{\mu_{N-\eta,k}}}\cdots z_{j_{2p+1-2\eta}}^{\mu_{N-\eta,k}}\,z_{j_k}^{d-(2p+1-2\eta)\mu_{N-\eta,k}-2(N+\eta-2p-1)}\,z_{j_{2p-2\eta+2}}^2\cdots z_{j_{N-\eta}}^2.\end{aligned}\tag{215}$$

Of course, all integers $\mu_{\bullet,\bullet}$ and the degree d are to be determined by some *improved Algorithm*, so that all the obtained symmetric forms are negatively twisted. And then we may estimate the lower bound $\mathfrak{d}_0$ accordingly. We leave this standard process to the interested reader.

12.5. **Why is the lower degree bound $\mathfrak{d}_0$ so large in MCM.** Because we could not enter the intrinsic difficulties, firstly of solving some huge linear systems to obtain sufficiently many (negatively twisted, large degree) symmetric differential forms (see [7, Theorem 2.7]), and secondly of proving that the obtained symmetric forms have discrete base locus. What we have done is only focusing on the extrinsic negatively twisted symmetric forms with degrees $\leqslant n$, obtained by some minors of the hypersurface equations/differentials matrix.

Our tool is coarse, based on some robust extrinsic geometric/algebraic structures, yet our goal is delicate, to certify the conjectured intrinsic ampleness. So a large lower degree bound $\mathfrak{d}_0 \gg 1$ is a price we need to pay.

13. Uniform Very-Ampleness of $\mathsf{Sym}^{\kappa}\Omega_X$

13.1. **A reminder.** In [34], Fujita proposed the famous:

Conjecture 13.1. (Fujita) Let M be an n-dimensional complex manifold with canonical line bundle $\mathscr{K}$. If $\mathscr{L}$ is any positive holomorphic line bundle on M, then:

- **(i)** for every integer $m \geqslant n+1$, the line bundle $\mathscr{L}^{\otimes m} \otimes \mathscr{K}$ should be globally generated;
- **(ii)** for every integer $m \geqslant n+2$, the line bundle $\mathscr{L}^{\otimes m} \otimes \mathscr{K}$ should be very ample.

Recall that, given a complex manifold X having ample cotangent bundle Ω_X, the projectivized tangent bundle $\mathbb{P}(\mathrm{T}_X)$ is equipped with the ample Serre line bundle $\mathscr{O}_{\mathbb{P}(\mathrm{T}_X)}(1)$. Denoting $n := \dim X$, one has:

$$\dim \mathbb{P}(\mathrm{T}_X) \;=\; 2n-1.$$

Anticipating, we will show in Corollary 13.3 below that the canonical bundle of $\mathbb{P}(\mathrm{T}_X)$ is:

$$\mathscr{K}_{\mathbb{P}(\mathrm{T}_X)} \;\cong\; \mathscr{O}_{\mathbb{P}(\mathrm{T}_X)}(-n) \otimes \pi^* \mathscr{K}_{\mathscr{X}}{}^{\otimes 2},$$

where $\pi\colon \mathbb{P}(\mathrm{T}_X) \to X$ is the canonical projection. Thus, for the complex manifold $\mathbb{P}(\mathrm{T}_X)$ and the ample Serre line bundle $\mathscr{O}_{\mathbb{P}(\mathrm{T}_X)}(1)$, the Fujita Conjecture implies:

- **(i)** for every integer $m \geqslant 2n$, the line bundle $\mathscr{O}_{\mathbb{P}(\mathrm{T}_X)}(m-n) \otimes \pi^* \mathscr{K}_{\mathscr{X}}{}^{\otimes 2}$ is globally generated;
- **(ii)** for every integer $m \geqslant 2n+1$, the line bundle $\mathscr{O}_{\mathbb{P}(\mathrm{T}_X)}(m-n) \otimes \pi^* \mathscr{K}_{\mathscr{X}}{}^{\otimes 2}$ is very ample.

In other words, we receive the following by-products of the Fujita Conjecture.

A Consequence of the Fujita Conjecture. *For any n-dimensional complex manifold X having ample cotangent bundle Ω_X, there holds:*

- **(i)** *for every integer $m \geqslant n$, the twisted m-symmetric cotangent bundle $\mathsf{Sym}^m\Omega_X \otimes \mathscr{K}_X^{\otimes 2}$ is globally generated;*
- **(ii)** *for every integer $m \geqslant n+1$, the twisted m-symmetric cotangent bundle $\mathsf{Sym}^m\Omega_X \otimes \mathscr{K}_X^{\otimes 2}$ is very ample.*

13.2. **The canonical bundle of a projectivized vector bundle.** In this subsection, we recall some classical results in algebraic geometry.

Let X be an n-dimensional complex manifold, and let E be a holomorphic vector bundle on X having rank e. Let $\mathbb{P}(E)$ be the projectivization of E. Now, we compute its canonical bundle $\mathscr{K}_{\mathbb{P}(E)}$ as follows.

Let π be the canonical projection:

$$\pi\colon\; \mathbb{P}(E) \;\longrightarrow\; X.$$

First, recall the exact sequence which defines the relative tangent bundle T_π:

$$0 \longrightarrow \mathrm{T}_\pi \longrightarrow \mathrm{T}_{\mathbb{P}(E)} \longrightarrow \pi^* \mathrm{T}_X \longrightarrow 0, \tag{216}$$

and recall also the well known Euler exact sequence:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}(E)} \longrightarrow \mathscr{O}_{\mathbb{P}(E)}(1) \otimes \pi^* E \longrightarrow \mathrm{T}_\pi \longrightarrow 0. \tag{217}$$

Next, taking wedge products, the exact sequence (216) yields:

$$\wedge^{n+e-1} \mathrm{T}_{\mathbb{P}(E)} \cong \wedge^{e-1} \mathrm{T}_\pi \otimes \pi^* \wedge^n \mathrm{T}_X, \tag{218}$$

and the Euler exact sequence (217) yields:

$$\mathscr{O}_{\mathbb{P}(E)}(e) \otimes \pi^* \wedge^e E \cong \mathscr{O}_{\mathbb{P}(E)} \otimes \wedge^{e-1} \mathrm{T}_\pi \cong \wedge^{e-1} \mathrm{T}_\pi. \tag{219}$$

Thus, we may compute the canonical line bundle as:

$$\begin{aligned}
\mathscr{K}_{\mathbb{P}(E)} &= \wedge^{n+e-1} \Omega^1_{\mathbb{P}(E)} \\
&\cong \big(\wedge^{n+e-1} \mathrm{T}_{\mathbb{P}(E)} \big)^\vee \\
\text{[use the dual of (218)]} \qquad &\cong \big(\wedge^{e-1} \mathrm{T}_\pi \big)^\vee \otimes \big(\pi^* \wedge^n \mathrm{T}_X \big)^\vee \\
\text{[use the dual of (219)]} \qquad &\cong \big(\mathscr{O}_{\mathbb{P}(E)}(e) \big)^\vee \otimes \big(\pi^* \wedge^e E \big)^\vee \otimes \big(\pi^* \wedge^n \mathrm{T}_X \big)^\vee \\
&\cong \mathscr{O}_{\mathbb{P}(E)}(-e) \otimes \pi^* \wedge^e E^\vee \otimes \pi^* \wedge^n \Omega^1_X \\
&\cong \mathscr{O}_{\mathbb{P}(E)}(-e) \otimes \pi^* \wedge^e E^\vee \otimes \pi^* \mathscr{K}_X,
\end{aligned}$$

where $\mathscr{K}_X$ is the canonical line bundle of X.

Proposition 13.2. *The canonical line bundle $\mathscr{K}_{\mathbb{P}(E)}$ of $\mathbb{P}(E)$ satisfies the formula:*

$$\mathscr{K}_{\mathbb{P}(E)} \cong \mathscr{O}_{\mathbb{P}(E)}(-e) \otimes \pi^* \wedge^e E^\vee \otimes \pi^* \mathscr{K}_X. \qquad \square$$

In applications, first, we are interested in the case where E is the tangent bundle T_X of X.

Corollary 13.3. *One has the formula:*

$$\mathscr{K}_{\mathbb{P}(\mathrm{T}_X)} \cong \mathscr{O}_{\mathbb{P}(\mathrm{T}_X)}(-n) \otimes \pi^* \mathscr{K}_{\mathscr{X}}{}^{\otimes 2}. \qquad \square$$

More generally, we are interested in the case where $X \subset V$ for some complex manifold V of dimension $n + r$, and $E = \mathrm{T}_V\big|_X$.

Corollary 13.4. *One has:*

$$\mathscr{K}_{\mathbb{P}(\mathrm{T}_V|_X)} \cong \mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(-n-r) \otimes \pi^* \mathscr{K}_{\mathscr{V}}\big|_X \otimes \pi^* \mathscr{K}_{\mathscr{X}}. \qquad \square$$

In our applications, X, V are some smooth complete intersections in $\mathbb{P}^N_{\mathbb{C}}$, so their canonical line bundles $\mathscr{K}_X, \mathscr{K}_V$ have neat expressions by the following classical theorem, whose proof is based on the Adjunction Formula.

Theorem 13.5. *For a smooth complete intersection:*

$$Y := D_1 \cap \cdots \cap D_k \subset \mathbb{P}^N_{\mathbb{C}}$$

with divisor degrees:

$$\deg D_i = d_i \qquad (i = 1 \cdots k),$$

the canonical line bundle $\mathscr{K}_X$ of X is:

$$\mathscr{K}_X \cong \mathscr{O}_X\Big(-N - 1 + \sum_{i=1}^k d_i \Big). \qquad \square$$

13.3. **Proof of the Very-Ampleness Theorem 1.4.** Assume for the moment that the ambient field $\mathbb{K} = \mathbb{C}$. Recall that in our Ampleness Theorem 1.3, $V = H_1 \cap \cdots \cap H_c$ and $X = H_1 \cap \cdots \cap H_{c+r}$ with $\dim_{\mathbb{C}} X = n = N - (c + r)$. Then the above Corollary 13.4 and Theorem 13.5 imply:

$$\mathscr{K}_{\mathbb{P}(\mathrm{T}_V|_X)} \;\cong\; \mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(-n-r) \otimes \pi_2^* \, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}\Big(-2\,(N+1) + \sum_{i=1}^{c} d_i + \sum_{i=1}^{c+r} d_i\Big).$$

Also, recalling Theorem 5.1 and Proposition 4.4, for generic choices of $H_1, \ldots, H_{c+r}$, for any positive integers $a > b \geqslant 1$, the negatively twisted line bundle below is ample:

$$\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(a) \otimes \pi_2^* \, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-b).$$

Recall the Fujita Conjecture that, by subsequent works of Demailly, Siu et al. (cf. the survey [21]), it is known that $\mathscr{L}^{\otimes m} \otimes \mathscr{K}^{\otimes 2}$ is very ample for all large $m \geqslant 2 + \binom{3n+1}{n}$. Consequently, the line bundle:

$$\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}\Big(m\,a - 2n - 2r\Big) \otimes \pi_2^* \, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}\Big(-m\,b - 4\,(N+1) + 2\sum_{i=1}^{c} d_i + 2\sum_{i=1}^{c+r} d_i\Big) \tag{220}$$

is very ample.

Also note that, for similar reason as the ampleness of (24), for all large integers $\ell \geqslant \ell_0(N)$:

$$\mathscr{O}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})}(1) \otimes \pi_0^* \, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell)$$

is very ample. Consequently, so is:

$$\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(1) \otimes \pi_2^* \, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell). \tag{221}$$

Now, recall the following two facts:

- **(A)** if $\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(\kappa) \otimes \pi_2^* \, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\star)$ is very ample, then for every $\star' \geqslant \star$, $\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(\kappa) \otimes \pi_2^* \, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\star')$ is also very ample;
- **(B)** the tensor product of any two very ample line bundles remains very ample.

Therefore, thanks to the very-ampleness of (220), (221), we can already obtain the very-ampleness of $\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(\kappa)$ for all large integers $\kappa \geqslant \kappa_0$, for some non-effective κ_0. In other words, the restricted cotangent bundle $\mathsf{Sym}^\kappa \, \Omega_V\big|_X$ is very ample on X for every $\kappa \geqslant \kappa_0$. But to reach an explicit κ_0, one may ask the

Questions. **(i)** Find one explicit $\ell_0(N)$.

(ii) Find one explicit κ_0.

Answer of (i). The value $\ell_0(N) = 3$ works. One can check by hand that the following global sections:

$$z_k \, z_j^{\ell-1} \, d\Big(\frac{z_i}{z_j}\Big) \qquad (i,j,k = 0 \cdots N, i \neq j) \tag{222}$$

guarantee the very-ampleness of $\mathscr{O}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})}(1) \otimes \pi_0^* \, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell)$.

Answer of (ii). The second fact **(B)** above leads us to consider the semigroup $\mathscr{G}$ of the usual Abelian group $\mathbb{Z} \oplus \mathbb{Z}$ generated by elements (ℓ_1, ℓ_2) such that $\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(\ell_1) \otimes \pi_2^* \, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell_2)$ is very

ample. Then, the following elements are contained in $\mathscr{G}$, for all $m \geqslant 2 + \binom{3n+1}{n}$:

$$\text{[see (220), } \forall\, b \geqslant 1, a \geqslant b+1] \qquad \Big(m\,a - 2n - 2r,\ -m\,b - 4\,(N+1) + 2\sum_{i=1}^{c} d_i + 2\sum_{i=1}^{c+r} d_i\Big),$$

$$\text{[see (221), } \ell \geqslant \ell_0(N) = 3] \qquad (1, \ell).$$

Also, the first fact **(A)** above says that if $(\ell_1, \ell_2) \in \mathscr{G}$, then $(\ell_1, \ell_3) \in \mathscr{G}$ for all $\ell_3 \geqslant \ell_2$. Thus, Question **(ii)** becomes to find one explicit κ_0 such that $(\kappa, 0) \in \mathscr{G}$ for all $\kappa \geqslant \kappa_0$.

Paying no attention to optimality, taking:

$$b = 1,\ a = 2,\ m = -4\,(N+1) + 2\sum_{i=1}^{c} d_i + 2\sum_{i=1}^{c+r} d_i + 3,$$

we receive that $(m\,a - 2n - 2r, -3) \in \mathscr{G}$. Adding $(1, 3) \in \mathscr{G}$, we receive $(m\,a - 2n - 2r + 1, 0) \in \mathscr{G}$. Now, also using $(m\,a - 2n - 2r, 0) \in \mathscr{G}$, recalling Observation 5.4, we may take:

$$\kappa_0 \ = \ (m\,a - 2n - 2r - 1)\,(m\,a - 2n - 2r) \ \leqslant \ a^2\,m^2,$$

or the larger neater lower bound:

$$\kappa_0 \ = \ 16\Big(\sum_{i=1}^{c} d_i + \sum_{i=1}^{c+r} d_i\Big)^2.$$

Thus, we have proved the Very-Ampleness Theorem 1.4 for $\mathbb{K} = \mathbb{C}$. Remembering that *very-ampleness (or not) is preserved under any base change obtained by ambient field extension*, and noting the field extensions $\mathbb{Q} \hookrightarrow \mathbb{C}$ and $\mathbb{Q} \hookrightarrow \mathbb{K}$ for any field $\mathbb{K}$ with characteristic zero, by some standard arguments in algebraic geometry, we conclude the proof of the Very-Ampleness Theorem 1.4.

When $\mathbb{K}$ has positive characteristic, by the same arguments, we could also receive the same very-ampleness theorem provided the similar results about the Fujita Conjecture hold over the field $\mathbb{K}$.

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GENERALIZED BROTBEK'S SYMMETRIC DIFFERENTIAL FORMS AND APPLICATIONS

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Abstract. Over an algebraically closed field $\mathbb{K}$ with any characteristic, on an N-dimensional smooth projective $\mathbb{K}$-variety $\mathbf{P}$ equipped with $c \geqslant N/2$ very ample line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_c$, we study the *General Debarre Ampleness Conjecture*, which expects that for all large degrees $d_1, \ldots, d_c \geqslant \mathrm{d} \gg 1$, for generic c hypersurfaces $H_1 \in \left|\mathscr{L}_1^{\otimes d_1}\right|, \ldots, H_c \in \left|\mathscr{L}_c^{\otimes d_c}\right|$, the complete intersection $X := H_1 \cap \cdots \cap H_c$ has ample cotangent bundle Ω_X.

First, we introduce a notion of *formal matrices* and a *dividing device* to produce negatively twisted symmetric differential forms, which extend the previous constructions of Brotbek and the author. Next, we adapt the moving coefficients method (MCM), and we establish that, if $\mathscr{L}_1, \ldots, \mathscr{L}_c$ are *almost proportional* to each other, then the above conjecture holds true. Our method is effective: for instance, in the simple case $\mathscr{L}_1 = \cdots = \mathscr{L}_c$, we provide an explicit lower degree bound $\mathrm{d} = N^{N^2}$.

1. Introduction

Smooth projective varieties having ample cotangent bundle suit well with the phenomenon/philosophy that '*geometry governs arithmetic*', in the sense that, on one hand, over the complex number field $\mathbb{C}$, none of them contain any entire curve, on the other hand, over a number field K, each of them is expected to possess only finitely many K-rational points (Lang's conjecture). For instance in the one-dimensional case, the first property is due to the Uniformization Theorem and the Liouville's Theorem, while the second assertion is the famous Mordell Conjecture/Faltings's Theorem.

For a long time, few such varieties were known, even though they were expected to be reasonably abundant. In this aspect, Debarre conjectured in [4] that the intersection of $c \geqslant N/2$ generic hypersurfaces of large degrees in $\mathbb{P}^N_{\mathbb{C}}$ should have ample cotangent bundle.

By introducing the moving coefficients method (MCM) and the product coup, the Debarre Ampleness Conjecture was first established in [5], with an additional effective lower degree bound.

Theorem 1.1 ([5])**.** *The cotangent bundle Ω_X of the complete intersection $X := H_1 \cap \cdots \cap H_c \subset \mathbb{P}^N_{\mathbb{C}}$ of $c \geqslant N/2$ generic hypersurfaces $H_1, \ldots, H_c$ with degrees $d_1, \ldots, d_c \geqslant N^{N^2}$ is ample.*

The proof there extends the approach of [2], by adding four major ingredients as follows.

(1) Generalizations of Brotbek's symmetric differential forms [2, Lemma 4.5] by means of a geometric approach, and also by a scheme-theoretic approach.

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(2) Make use of 'hidden' symmetric differential forms constructed over any intersection of Fermat-type hypersurfaces with coordinate hyperplanes:

$$\underbrace{H_1 \cap \cdots \cap H_c}_{=X} \cap \{z_{\nu_1} = \cdots = z_{\nu_\eta} = 0\} \qquad (\forall\, \eta = 1 \cdots N-c-1;\ 0 \leqslant \nu_1 < \cdots < \nu_\eta \leqslant N).$$

(3) 'Flexible' hypersurfaces designed by MCM, which produce many more negatively twisted symmetric differential forms than pure Fermat-type ones.

(4) The *product coup*, which produces ample examples of all large degrees $d_1, \ldots, d_c$.

Recently, Brotbek and Darondeau [3] provided another approach to the Debarre Ampleness Conjecture, by means of new constructions and deep theorems in algebraic geometry. As mentioned in [3, p. 2], it is tempting to extend the Debarre Ampleness Conjecture from projective spaces to projective varieties, equipped with several very ample line bundles.

General Debarre Ampleness Conjecture. For any smooth projective $\mathbb{K}$-variety $\mathbf{P}$ of dimension $N \geqslant 1$, for any positive integer $c \geqslant N/2$, for any very ample line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_c$ over $\mathbf{P}$, there exists some lower bound:

$$\mathtt{d} = \mathtt{d}(\mathbf{P}, \mathscr{L}_1, \ldots, \mathscr{L}_c) \gg 1$$

such that, for all large degrees $d_1, \ldots, d_c \geqslant \mathtt{d}$, for c generic hypersurfaces:

$$H_1 \in \left|\mathscr{L}_1^{\otimes d_1}\right|, \ldots, H_c \in \left|\mathscr{L}_c^{\otimes d_c}\right|,$$

the complete intersection $X := H_1 \cap \cdots \cap H_c$ has ample cotangent bundle Ω_X.

Sharing the same flavor as [5, p. 6, Conjecture 1.5], this general conjecture attracts our interest. To this aim, we develop further our previous method in [5], and generalize several results.

We work over an algebraically closed field $\mathbb{K}$ with any characteristic. First of all, by adapting the techniques in [5], we can confirm the General Debarre Ampleness Conjecture in the case $\mathscr{L}_1 = \cdots = \mathscr{L}_c = \mathscr{L}$.

Theorem 1.2. *Let $\mathbf{P}$ be an N-dimensional smooth projective $\mathbb{K}$-variety, equipped with a very ample line bundle $\mathscr{L}$. For any positive integer $c \geqslant N/2$, for all large degrees $d_1, \ldots, d_c \geqslant N^{N^2}$, for c generic hypersurfaces $H_1 \in \left|\mathscr{L}_1^{\otimes d_1}\right|, \ldots, H_c \in \left|\mathscr{L}_c^{\otimes d_c}\right|$, the complete intersection $X := H_1 \cap \cdots \cap H_c$ has ample cotangent bundle Ω_X.*

In fact, we will prove a stronger result, in the case that the c rays:

$$\mathbb{R}_+ \cdot [\mathscr{L}_1],\ \ldots,\ \mathbb{R}_+ \cdot [\mathscr{L}_c] \ \subset\ \text{Ample Cone of } \mathbf{P}$$

have small pairwise angles. More rigorously, we introduce the

Definition 1.3. Let $\mathbf{P}$ be an N-dimensional projective $\mathbb{K}$-variety, and let $\mathscr{L}$, $\mathscr{S}$ be two ample line bundles on $\mathbf{P}$. Then $\mathscr{S}$ is said to be *almost proportional* to $\mathscr{L}$, if there exist two elements $\alpha \in \mathbb{R}_+ \cdot [\mathscr{S}]$ and $\beta \in \mathbb{R}_+ \cdot [\mathscr{L}]$ such that $\beta < \alpha < (1+\epsilon_0)\beta$, i.e. both $\alpha - \beta, (1+\epsilon_0)\beta - \alpha$ lie in the ample cone of $\mathbf{P}$, where $\epsilon_0 := 3/(N^{N^2/2} - 1)$.

The value ϵ_0 is due to the effective degree estimates of MCM, see Proposition 6.1.

Theorem 1.4. *Let $\mathbf{P}$ be an N-dimensional smooth projective $\mathbb{K}$-variety, equipped with a very ample line bundle $\mathscr{L}$. For any integers c, $r \geqslant 0$ with $2c + r \geqslant N$, for any $c + r$ ample line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_{c+r}$ which are almost proportional to $\mathscr{L}$, there exists some integer:*

$$\mathtt{d} = \mathtt{d}(\mathscr{L}_1, \ldots, \mathscr{L}_{c+r}, \mathscr{L}) \gg 1$$

such that, for all large integers $d_1, \dots, d_c, d_{c+1}, \dots, d_{c+r} \geqslant \mathsf{d}$*, for generic* $c + r$ *hypersurfaces:*

$$H_1 \in \left|\mathscr{L}_1^{\otimes d_1}\right|, \dots, H_{c+r} \in \left|\mathscr{L}_{c+r}^{\otimes d_{c+r}}\right|,$$

the cotangent bundle Ω_V *of the intersection of the first* c *hypersurfaces* $V := H_1 \cap \cdots \cap H_c$ *restricted to the intersection of all the* $c + r$ *hypersurfaces* $X := H_1 \cap \cdots \cap H_c \cap H_{c+1} \cap \cdots \cap H_{c+r}$ *is ample.*

We will see that in our proof, the lower degree bound $\mathsf{d} = \mathsf{d}(\mathscr{L}_1, \dots, \mathscr{L}_{c+r}, \mathscr{L})$ is effective. In particular, when $r = 0$ and all $\mathscr{L}_1 = \cdots = \mathscr{L}_c = \mathscr{L}$ coincide, we will obtain the effective degree bound N^{N^2} of Theorem 1.2. See Subsection 6.8 for the details.

This paper is organized as follows. In Section 2, we outline the general strategy for the Debarre Ampleness Conjecture, which serves as a guiding principle of our approach. Next, in Section 3, we introduce a notion of *formal matrices*, and use their determinants to produce symmetric differential forms. Then, for the purpose of making negative twist, we devise a *dividing trick* in Section 4, and thus generalize the aforementioned ingredient **(1)**. Consequently, we are able to generalize **(2)** in Section 5. Thus, by adapting the ingredients **(3)**, **(4)** as well, we establish Theorem 1.4 in Section 6, by means of the moving coefficients method developed in [5]. Lastly, we fulfill some technical details in Section 7.

It is worth to mention that, by means of formal matrices, we can also construct higher order jet differential forms. Therefore, we can also apply MCM to study the ampleness of certain jet subbundle of hypersurfaces in $\mathbb{P}^N_{\mathbb{C}}$, notably when $N = 3$. We will discuss this in our coming paper.

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2. General Strategy

It seems that, up to date, there has been only one strategy to settle the Debarre Ampleness Conjecture. To be precise, for fixed degrees $d_1, \dots, d_c$ of hypersurfaces, the strategy is firstly to choose a certain subfamily of c hypersurfaces, and then secondly to construct sufficiently many negatively twisted symmetric differential forms over the corresponding subfamily of intersections, and lastly to narrow their base locus up to discrete points over a generic intersection. Thus, there exists one desirable ample example in this subfamily, which suffices to conclude the generic ampleness of the whole family thanks to a theorem of Grothendieck.

Following this central idea, the first result [1] was obtained in the case $c = N - 2$ for complex surfaces $X = H_1 \cap \cdots \cap H_c \subset \mathbb{P}^N_{\mathbb{C}}$, by employing a method related to Kobayashi hyperbolicity problems, in which the existence/quantity of negatively twisted symmetric differential forms was guaranteed/measured by the holomorphic Morse inequality. Such an approach would fail in the higher dimensional case, simply because one could not control the base locus of the implicitly given symmetric forms.

To find an alternative approach, the key breakthrough happened when Brotbek constructed explicit negatively twisted symmetric differential forms [2, Lemma 4.5] by a cohomological approach, for the subfamily of pure Fermat-type hypersurfaces of the same degree $d + \epsilon$ defined by:

$$F_i = \sum_{j=0}^{N} A_i^j z_j^d \qquad (i = 1 \cdots c),$$

where $d, \epsilon \geqslant 1$, and where all coefficients A_i^j are some homogeneous polynomials with $\deg A_i^j = \epsilon \geqslant 1$. Then, in the case $4c \geqslant 3N-2$, Brotbek showed that over a generic intersection X, the obtained symmetric forms have discrete base locus, and hence he established the conjectured ampleness.

However, when $4c < 3N-2$, this approach would not work, because the obtained symmetric differential forms keep positive dimensional base locus, for instance in the limiting case $2c = N$, there is only one obtained symmetric form, whereas $\dim \mathbb{P}(\Omega_X) = N-1 \gg 1$.

To overcome this difficulty, the author [5] introduced the moving coefficients method (MCM), the cornerstone of which is a generalization of Brotbek's symmetric differential forms for general Fermat-type hypersurfaces defined by:

$$F_i = \sum_{j=0}^{N} A_i^j z_j^{\lambda_j} \qquad (i = 1 \cdots c), \tag{1}$$

where $\lambda_0, \ldots, \lambda_N \geqslant 1$ and where all polynomial coefficients A_i^j satisfy $\deg A_i^j + \lambda_j = \deg F_i$. Then, by employing the other major ingredients **(2)**, **(3)**, **(4)** mentioned before, the Debarre Ampleness Conjecture finally turned into Theorem 1.1.

Recently, Brotbek and Darondeau [3] discovered a new way to construct negatively twisted symmetric differential forms for a certain subfamily of hypersurfaces, using pullbacks of some Plücker-embedding like morphisms, and they successfully controlled the base loci by means of deep theorems in algebraic geometry. Their approach together with the product coup gives another proof of the Debarre Ampleness Conjecture. Also, it is expected to achieve an effective lower bound on hypersurface degrees, which would ameliorate the preceding bound N^{N^2} of Theorem 1.1.

3. **Formal Matrices Produce Symmetric Differential Forms**

Aiming at the General Debarre Ampleness Conjecture, and following the general strategy above, we would like to first construct negatively twisted symmetric differential forms. Recalling the determinantal structure of Brotbek's symmetric differential forms [2, Lemma 4.5], in fact, we can take any *formal matrices* for construction, regardless of negative twist at the moment.

Take an arbitrary scheme $\mathbf{P}$. For any positive integers $1 \leqslant n \leqslant e$, for any e line bundles $\mathscr{S}_1, \ldots, \mathscr{S}_e$ over $\mathbf{P}$, we construct an $(e+n) \times (e+n)$ formal matrix K such that, for $p = 1 \cdots e$ its p-th row consists of global sections $F_p^1, \ldots, F_p^{e+n} \in \mathsf{H}^0(\mathbf{P}, \mathscr{S}_p)$, and for $q = 1 \cdots n$ its $(e+q)$-th row is the *formal differential* — to be defined — of the q-th row:

$$K := \begin{pmatrix} F_1^1 & \cdots & F_1^{e+n} \\ \vdots & & \vdots \\ F_e^1 & \cdots & F_e^{e+n} \\ dF_1^1 & \cdots & dF_1^{e+n} \\ \vdots & & \vdots \\ dF_n^1 & \cdots & dF_n^{e+n} \end{pmatrix}. \tag{2}$$

We will see later that the determinant of K produces a twisted symmetric differential form on $\mathbf{P}$. First of all, we define the above formal differential entries dF_i^j in a natural way.

Definition 3.1. Let $\mathscr{S}$ be a line bundle over $\mathbf{P}$, with a global section S. For any Zariski open set $U \subset \mathbf{P}$ with a trivialization $\mathscr{S}\big|_U = \mathscr{O}_U \cdot s$ ($s \in \mathsf{H}^0(U, \mathscr{S})$ is invertible), denote S/s for the unique

$\widetilde{s} \in \mathscr{O}_{\mathbf{P}}(U)$ such that $S = \widetilde{s} \cdot s$. Also, define the formal differential $d\,S$ in the local coordinate (U, s) by:

$$dS\,(U,s) \;:=\; \mathrm{d}\,(S/s)\cdot s \;\in\; \mathsf{H}^0(U,\,\Omega^1_{\mathbf{P}}\otimes\mathscr{S}),$$

where 'd' stands for the usual differential.

Let us check that the above definition works well with the usual Leibniz's rule. Indeed, let $\mathscr{S}_1$, $\mathscr{S}_2$ be two line bundles over $\mathbf{P}$, with any two global sections S_1, S_2 respectively. For any Zariski open set $U \subset \mathbf{P}$ with trivializations $\mathscr{S}_1\big|_U = \mathscr{O}_U \cdot s_1$ and $\mathscr{S}_2\big|_U = \mathscr{O}_U \cdot s_2$, we may compute:

$$\begin{aligned} d(S_1\otimes S_2)\,(U, s_1\otimes s_2) \;&=\; \mathrm{d}\,(S_1/s_1\cdot S_2/s_2)\cdot s_1\otimes s_2 \\ &=\; \mathrm{d}\,(S_1/s_1)\cdot S_2/s_2 \;\cdot\; s_1\otimes s_2 \;+\; \mathrm{d}\,(S_2/s_2)\cdot S_1/s_1 \;\cdot\; s_1\otimes s_2 \\ [\,\text{identify } S_1\otimes S_2 \cong S_2\otimes S_1\,] \qquad &=\; \mathrm{d}\,(S_1/s_1)\cdot s_1 \;\otimes\; S_2/s_2\cdot s_2 \;+\; \mathrm{d}\,(S_2/s_2)\cdot s_2 \;\otimes\; S_1/s_1\cdot s_1 \\ &=\; dS_1\,(U,s_1)\;\otimes\; S_2 \;+\; dS_2\,(U,s_2)\;\otimes\; S_1. \end{aligned}$$

Dropping the tensor symbol '$\otimes$' and coordinates (U, s_1, s_2), we abbreviate the above identity as:

$$d(S_1\cdot S_2) \;=\; dS_1\cdot S_2 + dS_2\cdot S_1.$$

Now, let us compute the determinant of (2) in local coordinates. For any Zariski open set $U \subset \mathbf{P}$ with trivializations $\mathscr{S}_1\big|_U = \mathscr{O}_U\cdot s_1, \ldots, \mathscr{S}_e\big|_U = \mathscr{O}_U\cdot s_e$, writing $f_i^j := F_i^j/s_i \in \mathscr{O}_{\mathbf{P}}(U)$, we may factor:

$$K\,(U, s_1, \ldots, s_e) \;:=\; \begin{pmatrix} f_1^1\cdot s_1 & \cdots & f_1^{e+n}\cdot s_1 \\ \vdots & & \vdots \\ f_e^1\cdot s_e & \cdots & f_e^{e+n}\cdot s_e \\ \mathrm{d}\,f_1^1\cdot s_1 & \cdots & \mathrm{d}\,f_1^{e+n}\cdot s_1 \\ \vdots & & \vdots \\ \mathrm{d}\,f_n^1\cdot s_n & \cdots & \mathrm{d}\,f_n^{e+n}\cdot s_n \end{pmatrix} = \underbrace{\begin{pmatrix} s_1 & & & & \\ & \ddots & & & \\ & & s_e & & \\ & & & s_1 & \\ & & & & \ddots \\ & & & & & s_n \end{pmatrix}}_{=:\,\mathsf{T}^U_{s_1,\ldots,s_e}} \cdot \underbrace{\begin{pmatrix} f_1^1 & \cdots & f_1^{e+n} \\ \vdots & & \vdots \\ f_e^1 & \cdots & f_e^{e+n} \\ \mathrm{d}\,f_1^1 & \cdots & \mathrm{d}\,f_1^{e+n} \\ \vdots & & \vdots \\ \mathrm{d}\,f_n^1 & \cdots & \mathrm{d}\,f_n^{e+n} \end{pmatrix}}_{=:\,(K)^U_{s_1,\ldots,s_e}}. \tag{3}$$

Denoting the last two matrices by $\mathsf{T}^U_{s_1,\ldots,s_e}$, $(K)^U_{s_1,\ldots,s_e}$, we obtain:

$$\begin{aligned} \det K\,(U, s_1, \ldots, s_e) \;&=\; \det \mathsf{T}^U_{s_1,\ldots,s_e} \;\cdot\; \det\,(K)^U_{s_1,\ldots,s_e} \\ &=\; s_1\cdots s_e\, s_1\cdots s_n \;\cdot\; \det\,(K)^U_{s_1,\ldots,s_e} \;\in\; \mathsf{H}^0\Big(U,\,\mathsf{Sym}^n\,\Omega^1_{\mathbf{P}}\otimes\mathscr{S}(\heartsuit)\Big), \end{aligned} \tag{4}$$

where for shortness we denote:

$$\mathscr{S}(\heartsuit) \;:=\; \big(\,\otimes_{p=1}^{e}\;\mathscr{S}_p\big)\;\otimes\;\big(\,\otimes_{q=1}^{n}\;\mathscr{S}_q\big). \tag{5}$$

Proposition 3.2. *The local definition:*

$$\det K\big|_U \;:=\; \det K\,(U, s_1, \ldots, s_e) \;\in\; \mathsf{H}^0\Big(U,\,\mathsf{Sym}^n\,\Omega^1_{\mathbf{P}}\otimes\mathscr{S}(\heartsuit)\Big)$$

does not depend on the particular choices of invertible sections $s_1, \ldots, s_e$ over U.

Proof. Assume that $s'_1, \ldots, s'_e$ are any other invertible sections of $\mathscr{S}_1\big|_U, \ldots, \mathscr{S}_e\big|_U$. Abbreviating the p-th row of the formal matrix K by F_p, we may compute:

$$\begin{aligned} F_p/s'_p \;&=\; s_p/s'_p \;\cdot\; F_p/s_p && (p = 1\cdots e), \\ [\,\text{Leibniz's rule}\,] \qquad \mathrm{d}\,(F_q/s'_q) \;&=\; \mathrm{d}\,(s_q/s'_q) \;\cdot\; F_q/s_q \;+\; s_q/s'_q \;\cdot\; \mathrm{d}\,(F_q/s_q) && (q = 1\cdots n). \end{aligned}$$

Thus we receive the transition identity:

$$(K)^U_{s'_1,\dots,s'_e} \;=\; \mathsf{T}^{s_1,\dots,s_e}_{s'_1,\dots,s'_e} \cdot (K)^U_{s_1,\dots,s_e}, \tag{6}$$

where $\mathsf{T}^{s_1,\dots,s_e}_{s'_1,\dots,s'_e}$ is an $(e+n)\times(e+n)$ lower triangular matrix with the diagonal entries $s_1/s'_1,\dots,s_e/s'_e$, $s_1/s'_1,\dots,s_n/s'_n$ in the exact order. Taking determinant on both sides of (6) thus yields:

$$\begin{aligned}\det(K)^U_{s'_1,\dots,s'_e} \;&=\; \det \mathsf{T}^{s_1,\dots,s_e}_{s'_1,\dots,s'_e} \cdot \det(K)^U_{s_1,\dots,s_e}\\ &=\; \underbrace{(\det \mathsf{T}^U_{s'_1,\dots,s'_e})^{-1}}_{=(s'_1\cdots s'_e\, s'_1\cdots s'_n)^{-1}} \cdot \underbrace{(\det \mathsf{T}^U_{s_1,\dots,s_e})}_{=s_1\cdots s_e\, s_1\cdots s_n} \cdot \det(K)^U_{s_1,\dots,s_e}\end{aligned}$$

Multiplying by $\det \mathsf{T}^U_{s'_1,\dots,s'_e}$ on both sides, we conclude the proof. □

Consequently, we receive

Proposition 3.3. *The determinant of the formal matrix* (2) *is globally well defined:*

$$\det K \;\in\; \mathsf{H}^0\Big(\mathbf{P},\, \mathsf{Sym}^n \Omega^1_{\mathbf{P}} \otimes \mathscr{S}(\heartsuit)\Big).$$ □

To grasp the essence of the above arguments, we provide another wholly formal

'Smart Proof'. Suppose that we do not know the meaning of *formal differential* dF, for any global section F of a line bundle $\mathscr{S}$ over $\mathbf{P}$. Nevertheless, we still try to compute the determinant of the formal matrix (2).

First of all, we would like to extract some useful information out of the 'mysterious' dF. *A priori*, we may assume that the formal differential satisfies the Leibniz's rule in a certain sense, and also that when $\mathscr{S} = \mathscr{O}_{\mathbf{P}}$ it coincides with the usual differential d. Thus, starting with any local section z of $\mathscr{S}$, we would have:

$$\begin{aligned} F \;&=\; z\cdot F/z,\\ dF \;&=\; dz\cdot F/z + z\cdot \mathrm{d}(F/z),\end{aligned}$$

that is:

$$\begin{pmatrix} F\\ dF\end{pmatrix} = \underline{\begin{pmatrix} z & 0\\ \star & z\end{pmatrix}} \begin{pmatrix} F/z\\ \mathrm{d}(F/z)\end{pmatrix}, \tag{7}$$

where $\star = dz$ is meaningless/negligible in our coming computations. Indeed, all we need is that the above underlined 2×2 formal matrix is lower triangular, with meaningful diagonal.

Back to our formal proof, we abbreviate every row of K as $F_1,\dots,F_e,dF_1,\dots,dF_n$, and for convenience we write:

$$K \;=\; (F_1,\dots,F_e,dF_1,\dots,dF_n)^{\mathrm{T}}.$$

Over any Zariski open set $U\subset\mathbf{P}$ with invertible sections $z_1,\dots,z_e$ of $\mathscr{S}_1,\dots,\mathscr{S}_e$ respectively, using identity (7), we can dehomogenize K with respect to $z_1,\dots,z_e$ by:

$$K \;=\; \mathsf{T}_{z_1,\dots,z_e} \cdot \underbrace{\Big(F_1/z_1,\dots,F_e/z_e,\mathrm{d}(F_1/z_1),\dots,\mathrm{d}(F_n/z_n)\Big)^{\mathrm{T}}}_{=:(K)_{z_1,\dots,z_e}}, \tag{8}$$

where $\mathsf{T}_{z_1,\dots,z_e}$ is a lower triangular $(e+n)\times(e+n)$ formal matrix with diagonal entries $z_1,\dots,z_e,z_1,\dots,z_n$ in the exact order. Now, it is desirable to notice that, on the right-hand-side of (8), the matrix $(K)_{z_1,\dots,z_e}$ and the diagonal of the formal matrix $\mathsf{T}_{z_1,\dots,z_e}$ are well-defined, thus all the 'mysterious

differentials' of the matrix K appear only in the strict lower-left part of $\mathsf{T}_{z_1,\dots,z_e}$, which would immediately disappear after taking determinant on both sides of (8):

$$\begin{aligned}\det K\Big|_U &= \det \mathsf{T}_{z_1,\dots,z_e} \cdot \det(K)_{z_1,\dots,z_e} \\ &= z_1\cdots z_e\, z_1\cdots z_n \cdot \det(K)_{z_1,\dots,z_e} \in \mathsf{H}^0\Big(U, \mathsf{Sym}^n \Omega^1_{\mathbf{P}} \otimes \mathscr{S}(\heartsuit)\Big).\end{aligned}$$

Bien sûr, it is independent of the choices of $z_1,\dots,z_e$, since the left-hand-side — a formal determinant — is. □

Remark 3.4. The formal differential d is much the same as the usual differential d, in the sense that both of them can be defined locally, and both of them obey the Leibniz's rule. These two facts constitute the essence of Proposition 3.3.

Next, we consider e sections:

$$F_i = \sum_{j=0}^{e+n} F_i^j \in \mathsf{H}^0(\mathbf{P}, \mathscr{S}_i) \qquad (i = 1\cdots e), \tag{9}$$

each F_i being the sum of $e+n+1$ global sections of the same line bundle $\mathscr{S}_i$. Let V be the intersection of the zero loci of the first n sections:

$$V := \{F_1 = 0\} \cap \cdots \cap \{F_n = 0\} \subset \mathbf{P},$$

and let X be the intersection of the zero loci of all the $e \geqslant n$ sections:

$$X := \{F_1 = 0\} \cap \cdots \cap \{F_e = 0\} \subset V \subset \mathbf{P}.$$

Let K be the $(e+n)\times(e+n+1)$ formal matrix whose $e+n$ rows copy the $e+n+1$ terms of $F_1,\dots,F_e,dF_1,\dots,dF_n$ in the exact order:

$$\mathsf{K} := \begin{pmatrix} F_1^0 & \cdots & F_1^{e+n} \\ \vdots & & \vdots \\ F_e^0 & \cdots & F_e^{e+n} \\ dF_1^0 & \cdots & dF_1^{e+n} \\ \vdots & & \vdots \\ dF_n^0 & \cdots & dF_n^{e+n} \end{pmatrix}.$$

Also, for $j = 0\cdots e+n$, let $\widehat{\mathsf{K}}_j$ denote the submatrix of K obtained by omitting the $(j+1)$-th column.

Since the restricted cotangent sheaf $\Omega^1_V\big|_X$ is formally defined by the $e+n$ equations:

$$F_1 = 0,\dots,F_e = 0,\ dF_1 = 0,\dots,dF_n = 0,$$

i.e. the sum of all $e+n+1$ columns of K vanishes, by Observation 3.6 below, we may receive

Proposition 3.5. *For all $j = 0\cdots e+n$, the $e+n+1$ sections:*

$$\psi_j = (-1)^j \det \widehat{\mathsf{K}}_j \in \mathsf{H}^0\Big(\mathbf{P}, \mathsf{Sym}^n \Omega^1_{\mathbf{P}} \otimes \mathscr{S}(\heartsuit)\Big),$$

when restricted to X, give one and the same section:

$$\psi \in \mathsf{H}^0\Big(X, \mathsf{Sym}^n \Omega^1_V \otimes \mathscr{S}(\heartsuit)\Big).$$

Observation 3.6. *In a commutative ring R, for all positive integers $N \geqslant 1$, let $A^0, A^1, \ldots, A^N \in R^N$ be $N+1$ column vectors satisfying:*

$$A^0 + A^1 + \cdots + A^N = \mathbf{0}.$$

Then for all $0 \leqslant j_1, j_2 \leqslant N$, there hold the identities:

$$(-1)^{j_1} \det (A^0, \ldots, \widehat{A^{j_1}}, \ldots, A^N) = (-1)^{j_2} \det (A^0, \ldots, \widehat{A^{j_2}}, \ldots, A^N). \qquad \square$$

Proof of Proposition 3.5. Using the same notation as in (4), for $j = 0 \cdots e+n$, we obtain an $(e+n) \times (e+n)$ matrix $(\widehat{\mathsf{K}}_j)^U_{s_1,\ldots,s_e}$. We also define:

$$\mathsf{K}(U, s_1, \ldots, s_e) := \mathsf{T}^U_{s_1,\ldots,s_e} \cdot (\mathsf{K})^U_{s_1,\ldots,s_e},$$

where the $(e+n) \times (e+n+1)$ matrix $(\mathsf{K})^U_{z_1,\ldots,z_e}$ satisfies that, for $j = 0 \cdots e+n$, the matrix $(\widehat{\mathsf{K}}_j)^U_{s_1,\ldots,s_e}$ is obtained by omitting the $(j+1)$-th column of $(\mathsf{K})^U_{s_1,\ldots,s_e}$.

We may view all entries of $(\mathsf{K})^U_{s_1,\ldots,s_e}$ as sections in $\mathsf{H}^0(U \cap X, \mathsf{Sym}^\bullet \Omega^1_V)$, where:

$$\mathsf{Sym}^\bullet \Omega^1_V := \oplus_{k=0}^\infty \mathsf{Sym}^k \Omega^1_V.$$

Thus the sum of all columns of $(\mathsf{K})^U_{s_1,\ldots,s_e}$ vanishes, and hence Observation 3.6 yields:

$$(-1)^{j_1} \det (\widehat{\mathsf{K}}_{j_1})^U_{s_1,\ldots,s_e} = (-1)^{j_2} \det (\widehat{\mathsf{K}}_{j_2})^U_{s_1,\ldots,s_e} \in \mathsf{H}^0(U \cap X, \mathsf{Sym}^n \Omega^1_V) \qquad (j_1, j_2 = 0 \cdots e+n).$$

By multiplication of $\det \mathsf{T}^U_{s_1,\ldots,s_e}$ on both sides, we conclude the proof. $\square$

Remember that our goal is to construct negatively twisted symmetric differential forms. One idea, foreshadowed by the constructions in [2, 5], is to find some $e+n+1$ line bundles $\mathscr{T}_0, \ldots, \mathscr{T}_{e+n}$ with respective global sections $t_0, \ldots, t_{e+n}$ having empty base locus, such that the line bundle:

$$\mathscr{S}(\heartsuit) \otimes \mathscr{T}_0^{-1} \otimes \cdots \otimes \mathscr{T}_{e+n}^{-1} =: \mathscr{S}(\heartsuit') < 0,$$

is negative, and such that:

$$\widehat{\omega}_j := \frac{\psi_j}{t_0 \cdots t_{e+n}} = \frac{(-1)^j}{t_0 \cdots t_{e+n}} \det \widehat{\mathsf{K}}_j \in \mathsf{H}^0\Big(\mathsf{D}(t_j), \mathsf{Sym}^n \Omega^1_{\mathbf{P}} \otimes \mathscr{S}(\heartsuit')\Big) \qquad (j = 0 \cdots e+n) \tag{10}$$

have no poles. Then, these $e+n+1$ sections, restricted to X, would glue together to make a global negatively twisted symmetric differential form:

$$\omega \in \mathsf{H}^0\Big(X, \mathsf{Sym}^n \Omega^1_V \otimes \mathscr{S}(\heartsuit')\Big).$$

For the purpose of (10), we may require that every $t_0, \ldots, t_{e+n}$ subsequently 'divides' the corresponding column of K in the exact order. With some additional effort, we shall make this idea rigorous in our central applications.

4. A Dividing Trick

Let $\mathscr{L}$ be a line bundle over $\mathbf{P}$ such that it has $N+1$ global sections $\zeta_0, \ldots, \zeta_N$ having empty common base locus. Let $c \geqslant 1$, $r \geqslant 0$ be two integers with $2c + r \geqslant N$ and $c + r < N$. Let $\mathscr{A}_1, \ldots, \mathscr{A}_{c+r}$ be $c+r$ auxiliary line bundles to be determined. Now, we consider $c+r$ *Fermat-type* sections having the same shape as (1):

$$F_i = \sum_{j=0}^N \underline{A_i^j \zeta_j^{\lambda_j}} \in \underline{\mathscr{A}_i \otimes \mathscr{L}^{\epsilon_i^j}} \otimes \mathscr{L}^{\lambda_j} = \underbrace{\mathscr{A}_i \otimes \mathscr{L}^{d_i}}_{=\mathscr{S}_i} \qquad (i = 1 \cdots c+r), \tag{11}$$

where $\epsilon_i^j, \lambda_j, d_i \geqslant 1$ are integers satisfying $\epsilon_i^j + \lambda_j = d_i$, and where every A_i^j is some global section of the line bundle $\mathscr{A}_i \otimes \mathscr{L}^{\epsilon_i^j}$.

For the first c equations of (11), a formal differentiation yields:

$$dF_i \;=\; \sum_{j=0}^{N} d(A_i^j\,\zeta_j^{\lambda_j}) \;=\; \sum_{j=0}^{N} \zeta_j^{\lambda_j-1}\,\underbrace{(\zeta_j\,dA_i^j + \lambda_j\,A_i^j\,d\zeta_j)}_{=:\,\mathsf{B}_i^j} \qquad (i=1\cdots c). \tag{12}$$

Now, we construct the $(c+r+c)\times(N+1)$ matrix M, whose first $c+r$ rows consist of all $(N+1)$ terms in the expressions (11) of $F_1,\ldots,F_{c+r}$ in the exact order, and whose last c rows consist of all $(N+1)$ terms in the expressions (12) of $dF_1,\ldots,dF_c$ in the exact order:

$$\mathsf{M} \;:=\; \begin{pmatrix} A_1^0\,\zeta_0^{\lambda_0} & \cdots & A_1^N\,\zeta_N^{\lambda_N} \\ \vdots & & \vdots \\ A_{c+r}^0\,\zeta_0^{\lambda_0} & \cdots & A_{c+r}^N\,\zeta_N^{\lambda_N} \\ d(A_1^0\,\zeta_0^{\lambda_0}) & \cdots & d(A_1^N\,\zeta_N^{\lambda_N}) \\ \vdots & & \vdots \\ d(A_c^0\,\zeta_0^{\lambda_0}) & \cdots & d(A_c^N\,\zeta_N^{\lambda_N}) \end{pmatrix} = \begin{pmatrix} A_1^0\,\zeta_0^{\lambda_0} & \cdots & A_1^N\,\zeta_N^{\lambda_N} \\ \vdots & & \vdots \\ A_{c+r}^0\,\zeta_0^{\lambda_0} & \cdots & A_{c+r}^N\,\zeta_N^{\lambda_N} \\ \mathsf{B}_1^0\,\zeta_0^{\lambda_0-1} & \cdots & \mathsf{B}_1^N\,\zeta_N^{\lambda_N-1} \\ \vdots & & \vdots \\ \mathsf{B}_c^0\,\zeta_0^{\lambda_0-1} & \cdots & \mathsf{B}_c^N\,\zeta_N^{\lambda_N-1} \end{pmatrix}. \tag{13}$$

Denote $n := N-c-r$, observe that $1 \leqslant n \leqslant c$. For every $1 \leqslant j_1 < \cdots < j_n \leqslant c$, denote by $\mathsf{M}_{j_1,\ldots,j_n}$ the $(c+r+n)\times(N+1)$ submatrix of M consisting of the first upper $c+r$ rows and the selected rows $c+r+j_1,\ldots,c+r+j_n$. Also, for $j=0\cdots N$, denote by $\widehat{\mathsf{M}}_{j_1,\ldots,j_n;\,j}$ the submatrix of $\mathsf{M}_{j_1,\ldots,j_n}$ obtained by omitting the $(j+1)$-th column.

Let $V \subset \mathbf{P}$ be the subvariety defined by the first c sections $F_1,\ldots,F_c$, and let $X \subset \mathbf{P}$ be the subvariety defined by all the $c+r$ sections $F_1,\ldots,F_{c+r}$. Now, applying Proposition 3.5, denoting:

$$\mathscr{A}_{j_1,\ldots,j_n}^{1,\ldots,c+r} \;:=\; \mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_{c+r} \otimes \mathscr{A}_{j_1} \otimes \cdots \otimes \mathscr{A}_{j_n},$$

we receive

Proposition 4.1. *For every $1 \leqslant j_1 < \cdots < j_n \leqslant c$, for all $j = 0\cdots N$, the $N+1$ sections:*

$$\psi_{j_1,\ldots,j_n;\,j} \;=\; (-1)^j \det \widehat{\mathsf{M}}_{j_1,\ldots,j_n;\,j} \;\in\; \mathsf{H}^0\Big(\mathbf{P},\, \mathsf{Sym}^n\,\Omega^1_{\mathbf{P}} \otimes \mathscr{A}_{j_1,\ldots,j_n}^{1,\ldots,c+r} \otimes \mathscr{L}^{\heartsuit_{j_1,\ldots,j_n}}\Big),$$

when restricted to X, give one and the same symmetric differential form:

$$\psi_{j_1,\ldots,j_n} \;\in\; \mathsf{H}^0\Big(X,\, \mathsf{Sym}^n\,\Omega^1_V \otimes \mathscr{A}_{j_1,\ldots,j_n}^{1,\ldots,c+r} \otimes \mathscr{L}^{\heartsuit_{j_1,\ldots,j_n}}\Big),$$

with the twisted degree:

$$\heartsuit_{j_1,\ldots,j_n} \;=\; \sum_{p=1}^{c+r} d_p + \sum_{q=1}^{n} d_{j_q}. \tag{14}$$

Observe in (13) that the $N+1$ columns of M are subsequently divisible by $\zeta_0^{\lambda_0-1},\dots,\zeta_N^{\lambda_N-1}$. Dividing out these factors, we receive the formal matrix:

$$\mathsf{C} := \begin{pmatrix} A_1^0\,\zeta_0 & \cdots & A_1^N\,\zeta_N \\ \vdots & & \vdots \\ A_{c+r}^0\,\zeta_0 & \cdots & A_{c+r}^N\,\zeta_N \\ \mathsf{B}_1^0 & \cdots & \mathsf{B}_1^N \\ \vdots & & \vdots \\ \mathsf{B}_c^0 & \cdots & \mathsf{B}_c^N \end{pmatrix}.$$

By mimicking the notation of the submatrices $\mathsf{M}_{j_1,\dots,j_n}$, $\widehat{\mathsf{M}}_{j_1,\dots,j_n;\,j}$ of M, we analogously define the submatrices $\mathsf{C}_{j_1,\dots,j_n}$, $\widehat{\mathsf{C}}_{j_1,\dots,j_n;\,j}$ of C.

Now, we interpret Proposition 4.1 in terms of the matrix C, starting by the formal computation:

$$\begin{aligned}(-1)^j \det \widehat{\mathsf{M}}_{j_1,\dots,j_n;\,j} &= (-1)^j\,\zeta_0^{\lambda_0-1}\cdots\widehat{\zeta_j^{\lambda_j-1}}\cdots\zeta_N^{\lambda_N-1}\,\det\widehat{\mathsf{C}}_{j_1,\dots,j_n;\,j}\\ &= \frac{(-1)^j}{\zeta_j^{\lambda_j-1}}\,\det\widehat{\mathsf{C}}_{j_1,\dots,j_n;\,j}\cdot\underline{\zeta_0^{\lambda_0-1}\cdots\zeta_N^{\lambda_N-1}}.\end{aligned}$$

Dividing by $\underline{\zeta_0^{\lambda_0-1}\cdots\zeta_N^{\lambda_N-1}}$ on both sides above, we receive the following $N+1$ 'coinciding' forms:

$$\underbrace{\frac{(-1)^j\det\widehat{\mathsf{M}}_{j_1,\dots,j_n;\,j}}{\zeta_0^{\lambda_0-1}\cdots\zeta_N^{\lambda_N-1}}}_{\text{independent of } j} = \underbrace{\frac{(-1)^j}{\zeta_j^{\lambda_j-1}}\,\det\widehat{\mathsf{C}}_{j_1,\dots,j_n;\,j}}_{\text{have no pole over } \mathsf{D}(\zeta_j)} \qquad (j=0\cdots N). \tag{15}$$

This is the aforementioned *dividing trick*.

Proposition 4.2. *For all $j=0\cdots N$, the formal symmetric differential forms:*

$$\widehat{\omega}_{j_1,\dots,j_n;\,j} = \frac{(-1)^j\det\widehat{\mathsf{M}}_{j_1,\dots,j_n;\,j}}{\zeta_0^{\lambda_0-1}\cdots\zeta_N^{\lambda_N-1}}$$

are well-defined sections in:

$$\mathsf{H}^0\Big(\mathsf{D}(\zeta_j),\,\mathsf{Sym}^n\,\Omega^1_{\mathbf{P}}\otimes\mathscr{A}^{1,\dots,c+r}_{j_1,\dots,j_n}\otimes\mathscr{L}^{\heartsuit'_{j_1,\dots,j_n}}\Big),$$

with the twisted degree:

$$\heartsuit'_{j_1,\dots,j_n} := \sum_{p=1}^{c+r} d_p + \sum_{q=1}^{n} d_{j_q} - \sum_{k=0}^{N}(\lambda_k-1).$$

Moreover, when restricted to X, they glue together to make a global section:

$$\omega_{j_1,\dots,j_n} \in \mathsf{H}^0\Big(X,\,\mathsf{Sym}^n\,\Omega^1_V\otimes\mathscr{A}^{1,\dots,c+r}_{j_1,\dots,j_n}\otimes\mathscr{L}^{\heartsuit'_{j_1,\dots,j_n}}\Big).$$

While the formal identity (15) transparently shows the essence of this proposition, it is not yet a proof by itself, since both sides are to be defined. Indeed, to bypass the potential trouble of divisibility, the rigorous proof below is much more involved than one would first expect.

Proof. Without loss of generality, we only prove the case $j_1 = 1, \dots, j_n = n$, and we will often drop the indices $j_1, \dots, j_n$, since no confusion could occur. Here is a sketch of the proof.

Step 1. Over each Zariski open set $U \subset \mathsf{D}(\zeta_j)$ with trivializations $\mathscr{A}_1\big|_U = \mathscr{O}_U \cdot a_1, \dots, \mathscr{A}_{c+r}\big|_U = \mathscr{O}_U \cdot a_{a+r}$, we compute the expression of $\widehat{\omega}_j := \widehat{\omega}_{j_1,\dots,j_n;\,j}$ in coordinates $(U, a_1, \dots, a_{c+r}, \zeta_j)$.

Step 2. We show that the obtained symmetric form $\widehat{\omega}_j\big|_U$ is independent of the choices of trivializations $a_1, \dots, a_{c+r}$, whence we conclude the first claim.

Step 3. For any distinct indices $0 \leqslant \ell_1, \ell_2 \leqslant N$, over any Zariski open set $U \subset \mathsf{D}(\zeta_{\ell_1}) \cap \mathsf{D}(\zeta_{\ell_2})$ with trivializations $\mathscr{A}_1\big|_U = \mathscr{O}_U \cdot a_1, \dots, \mathscr{A}_{c+r}\big|_U = \mathscr{O}_U \cdot a_{a+r}$, we show that:

$$\widehat{\omega}_{\ell_1} \;=\; \widehat{\omega}_{\ell_2} \;\in\; \mathsf{H}^0\Big(U,\, \mathsf{Sym}^n\, \Omega^1_V \otimes \mathscr{A}^{1,\dots,c+r}_{j_1,\dots,j_n} \otimes \mathscr{L}^{\heartsuit'_{j_1,\dots,j_n}}\Big) \tag{16}$$

by computations in coordinates. Thus we conclude the second claim.

Proof of Step 1. Recalling (3), by trivializations:

$$\underbrace{\mathscr{A}_i \otimes \mathscr{L}^{d_i}}_{=:\,\mathscr{S}_i}\Big|_U \;=\; \mathscr{O}_U \cdot \underbrace{a_i\, \zeta_j^{d_i}}_{=:\,s_i} \qquad (i = 1 \cdots c+r),$$

the formal matrix $\mathsf{K} := \mathsf{M}_{j_1,\dots,j_n}$ has coordinates:

$$\mathsf{K} \;=\; \mathsf{T}^{\zeta_j}_{a_1,\dots,a_{c+r}} \cdot \underbrace{\Big(F_1/s_1, \dots, F_{c+r}/s_{c+r}, \mathrm{d}\,(F_1/s_1), \dots, \mathrm{d}\,(F_n/s_n)\Big)^{\mathrm{T}}}_{=:\,(\mathsf{K})^{\zeta_j}_{a_1,\dots,a_{c+r}}}, \tag{17}$$

where $\mathsf{T}^{\zeta_j}_{a_1,\dots,a_{c+r}}$ is an $N \times N$ diagonal matrix with the diagonal $s_1, \dots, s_{c+r}, s_1, \dots, s_n$, and where like (8) we abbreviate the first $c + r$ rows of K by $F_1, \dots, F_{c+r}$. Further computation yields:

$$F_i \,/\, s_i \;:=\; (A^0_i\, \zeta_0^{\lambda_0}, \dots, A^N_i\, \zeta_N^{\lambda_N}) \,/\, s_i = \Big(A^0_i/\alpha^0_i \cdot (\zeta_0/\zeta_j)^{\lambda_0}, \dots, A^N_i/\alpha^N_i \cdot (\zeta_N/\zeta_j)^{\lambda_N}\Big) \qquad (i = 0 \cdots c+r),$$

where $\alpha^j_i := a_i \cdot \zeta_j^{\epsilon^j_i}$. 'Dividing' every column of $(\mathsf{K})^{\zeta_j}_{a_1,\dots,a_{c+r}}$ subsequently by $(\zeta_0/\zeta_j)^{\lambda_0-1}, \dots, (\zeta_N/\zeta_j)^{\lambda_N-1}$, we obtain an $N \times (N+1)$ matrix $(\mathsf{C})^{\zeta_j}_{a_1,\dots,a_{c+r}}$. For every $\ell = 0 \cdots N$, we denote by $(\widehat{\mathsf{C}}_\ell)^{\zeta_j}_{a_1,\dots,a_{c+r}}$ the submatrix of $(\mathsf{C})^{\zeta_j}_{a_1,\dots,a_{c+r}}$ obtained by deleting its $(\ell+1)$-th column. Now, formula (4) yields:

$$\det \widehat{\mathsf{M}}_{j_1,\dots,j_n;\,j} \;=\; s_1 \cdots s_{c+r}\, s_1 \cdots s_n\, (\zeta_0/\zeta_j)^{\lambda_0-1} \cdots (\zeta_N/\zeta_j)^{\lambda_N-1} \,\det (\widehat{\mathsf{C}}_j)^{\zeta_j}_{a_1,\dots,a_{c+r}}. \tag{18}$$

Thus, in coordinates $(U, a_1, \dots, a_{c+r}, \zeta_j)$, we obtain/define:

$$\begin{aligned}
\widehat{\omega}_j \;&=\; \frac{(-1)^j \det \widehat{\mathsf{M}}_{j_1,\dots,j_n;\,j}}{\zeta_0^{\lambda_0-1} \cdots \zeta_N^{\lambda_N-1}} \;:=\; (-1)^j\, \frac{s_1 \cdots s_{c+r}\, s_1 \cdots s_n}{\zeta_j^{(\lambda_0-1)+\cdots+(\lambda_N-1)}} \cdot \det (\widehat{\mathsf{C}}_j)^{\zeta_j}_{a_1,\dots,a_{c+r}} \\
&=\; (-1)^j\, a_1 \cdots a_{c+r} \cdot a_1 \cdots a_n \cdot \zeta_j^{\heartsuit'_{j_1,\dots,j_n}} \cdot \det (\widehat{\mathsf{C}}_j)^{\zeta_j}_{a_1,\dots,a_{c+r}} \\
&\in\; \mathsf{H}^0\Big(U,\, \mathsf{Sym}^n\, \Omega^1_{\mathbf{P}} \otimes \mathscr{A}^{1,\dots,c+r}_{j_1,\dots,j_n} \otimes \mathscr{L}^{\heartsuit'_{j_1,\dots,j_n}}\Big).
\end{aligned} \tag{19}$$

Proof of Step 2. We only need to show that:

$$a_1 \cdots a_{c+r}\, a_1 \cdots a_n \cdot \det (\widehat{\mathsf{C}}_j)^{\zeta_j}_{a_1,\dots,a_{c+r}} \;\in\; \mathsf{H}^0\Big(U,\, \mathsf{Sym}^n\, \Omega^1_{\mathbf{P}} \otimes \mathscr{A}^{1,\dots,c+r}_{j_1,\dots,j_n}\Big)$$

is independent of the choices of $a_1, \dots, a_{c+r}$.

Let $\widetilde{a}_1, \ldots, \widetilde{a}_{c+r}$ be any other choices of invertible sections of $\mathscr{A}_1\big|_U, \ldots, \mathscr{A}_{c+r}\big|_U$. Accordingly, we obtain the matrices $(\mathsf{C})^{\zeta_j}_{\widetilde{a}_1,\ldots,\widetilde{a}_{c+r}}$, $(\widehat{\mathsf{C}}_j)^{\zeta_j}_{\widetilde{a}_1,\ldots,\widetilde{a}_{c+r}}$, and we denote $\widetilde{\alpha}_i^j := \widetilde{a}_i \cdot \zeta_j^{\epsilon_i^j}$. Then, for $i = 1 \cdots c+r$, the i-th row of the matrix $(\mathsf{C})^{\zeta_j}_{\widetilde{a}_1,\ldots,\widetilde{a}_{c+r}}$ is:

$$[\,\alpha_i^k/\widetilde{\alpha}_i^k = a_i/\widetilde{a}_i\,] \qquad \Big(A_i^0/\widetilde{\alpha}_i^0 \cdot (\zeta_0/\zeta_j), \ldots, A_i^N/\widetilde{\alpha}_i^N \cdot (\zeta_N/\zeta_j)\Big) \;=\; a_i/\widetilde{a}_i \cdot \underbrace{\Big(A_i^0/\alpha_i^0 \cdot (\zeta_0/\zeta_j), \ldots, A_i^N/\alpha_i^N \cdot (\zeta_N/\zeta_j)\Big)}_{\text{the } i\text{-th row of the matrix } (\mathsf{C})^{\zeta_j}_{a_1,\ldots,a_{c+r}}}.$$

Also, for $i = 1 \cdots n$, $k = 0 \cdots N$, using:

$$\mathrm{d}\,(A_i^k/\widetilde{\alpha}_i^k) \;=\; \mathrm{d}\,(A_i^k/\alpha_i^k \cdot a_i/\widetilde{a}_i) = a_i/\widetilde{a}_i \cdot \mathrm{d}\,(A_i^k/\alpha_i^k) + A_i^k/\alpha_i^k \cdot \mathrm{d}\,(a_i/\widetilde{a}_i),$$

we see that the $(c+r+i, k+1)$-th entry of $(\mathsf{C})^{\zeta_j}_{\widetilde{a}_1,\ldots,\widetilde{a}_{c+r}}$ satisfies:

$$\begin{aligned}
&(\zeta_k/\zeta_j) \cdot \mathrm{d}\,(A_i^k/\widetilde{\alpha}_i^k) + \lambda_k\,(A_i^k/\widetilde{\alpha}_i^k) \cdot \mathrm{d}\,(\zeta_k/\zeta_j) \\
&= a_i/\widetilde{a}_i \cdot \underbrace{\big[(\zeta_k/\zeta_j) \cdot \mathrm{d}\,(A_i^k/\alpha_i^k) + \lambda_k\,(A_i^k/\alpha_i^k) \cdot \mathrm{d}\,(\zeta_k/\zeta_j)\big]}_{=\text{ the } (c+r+i,\,k+1)\text{-th entry of } (\mathsf{C})^{\zeta_j}_{a_1,\ldots,a_{c+r}}} + \mathrm{d}\,(a_i/\widetilde{a}_i) \cdot \underbrace{A_i^k/\alpha_i^k \cdot (\zeta_k/\zeta_j)}_{=(i,\,k+1)\text{-th entry}}. \qquad (20)
\end{aligned}$$

Hence we receive the transition identity:

$$(\mathsf{C})^{\zeta_j}_{\widetilde{a}_1,\ldots,\widetilde{a}_{c+r}} \;=\; \mathsf{T}^{a_1,\ldots,a_{c+r}}_{\widetilde{a}_1,\ldots,\widetilde{a}_{c+r}} \cdot (\mathsf{C})^{\zeta_j}_{a_1,\ldots,a_{c+r}},$$

where $\mathsf{T}^{a_1,\ldots,a_{c+r}}_{\widetilde{a}_1,\ldots,\widetilde{a}_{c+r}}$ is a lower triangular matrix with the product of the diagonal:

$$\det\,\mathsf{T}^{a_1,\ldots,a_{c+r}}_{\widetilde{a}_1,\ldots,\widetilde{a}_{c+r}} \;=\; \frac{a_1 \cdots a_{c+r} \cdot a_1 \cdots a_n}{\widetilde{a}_1 \cdots \widetilde{a}_{c+r} \cdot \widetilde{a}_1 \cdots \widetilde{a}_n}.$$

In particular, we have:

$$(\widehat{\mathsf{C}}_j)^{\zeta_j}_{\widetilde{a}_1,\ldots,\widetilde{a}_{c+r}} \;=\; \mathsf{T}^{a_1,\ldots,a_{c+r}}_{\widetilde{a}_1,\ldots,\widetilde{a}_{c+r}} \cdot (\widehat{\mathsf{C}}_j)^{\zeta_j}_{a_1,\ldots,a_{c+r}},$$

hence, by taking determinant on both sides above, we obtain:

$$\widetilde{a}_1 \cdots \widetilde{a}_{c+r} \cdot \widetilde{a}_1 \cdots \widetilde{a}_n \cdot \det(\widehat{\mathsf{C}}_j)^{\zeta_j}_{\widetilde{a}_1,\ldots,\widetilde{a}_{c+r}} \;=\; a_1 \cdots a_{c+r}\, a_1 \cdots a_n \cdot \det(\widehat{\mathsf{C}}_j)^{\zeta_j}_{a_1,\ldots,a_{c+r}},$$

which is our desired identity.

Proof of Step 3. First of all, we recall the famous

Cramer's Rule. *In a commutative ring R, for all positive integers $N \geqslant 1$, let $A^0, A^1, \ldots, A^N \in R^N$ be $N+1$ column vectors, and suppose that $z_0, z_1, \ldots, z_N \in R$ satisfy:*

$$A^0\, z_0 + A^1\, z_1 + \cdots + A^N\, z_N \;=\; \mathbf{0}.$$

Then for all indices $0 \leqslant \ell_1, \ell_2 \leqslant N$, there hold the identities:

$$(-1)^{\ell_1} \det\,(A^0, \ldots, \widehat{A^{\ell_1}}, \ldots, A^N)\, z_{\ell_2} \;=\; (-1)^{\ell_2} \det\,(A^0, \ldots, \widehat{A^{\ell_2}}, \ldots, A^N)\, z_{\ell_1}. \qquad \square$$

In the rest of the proof, we shall view all entries of the matrices $(\mathsf{K})^{\zeta_j}_{a_1,\ldots,a_{c+r}}$, $(\mathsf{C})^{\zeta_j}_{a_1,\ldots,a_{c+r}}$ as elements in the ring $\mathsf{H}^0(U \cap X, \mathsf{Sym}^\bullet\,\Omega^1_V)$. Note that the sum of all columns of $(\mathsf{K})^{\zeta_j}_{a_1,\ldots,a_{c+r}}$ vanishes:

$$C_0\Big(\frac{\zeta_0}{\zeta_j}\Big)^{\lambda_0 - 1} + \cdots + C_N\Big(\frac{\zeta_N}{\zeta_j}\Big)^{\lambda_N - 1} \;=\; \mathbf{0},$$

where we denote the $(i+1)$-th row of $(\mathsf{C})^{\zeta_j}_{a_1,\ldots,a_{c+r}}$ by C_i. Applying Cramer's rule, we receive:

$$(-1)^{\ell_1}\,\det(\widehat{\mathsf{C}}_{\ell_1})^{\zeta_j}_{a_1,\ldots,a_{c+r}} \cdot \Big(\frac{\zeta_{\ell_2}}{\zeta_j}\Big)^{\lambda_{\ell_2}-1} \;=\; (-1)^{\ell_2}\,\det(\widehat{\mathsf{C}}_{\ell_2})^{\zeta_j}_{a_1,\ldots,a_{c+r}} \cdot \Big(\frac{\zeta_{\ell_1}}{\zeta_j}\Big)^{\lambda_{\ell_1}-1} \;\in\; \mathsf{H}^0\Big(U \cap X,\, \mathsf{Sym}^n\,\Omega^1_V\Big) \quad (0 \leqslant \ell_1, \ell_2 \leqslant N). \qquad (21)$$

Lastly, we can check the desired identity (16) by the following computation:

$$\begin{aligned}
\widehat{\omega}_{\ell_1} &= (-1)^{\ell_1}\, a_1\cdots a_{c+r}\cdot a_1\cdots a_n\cdot \zeta_{\ell_1}^{\heartsuit'_{j_1,\dots,j_n}}\cdot \det\big(\widehat{\mathsf{C}}_{\ell_1}\big)^{\zeta_{\ell_1}}_{a_1,\dots,a_{c+r}} && \text{[use (19)]}\\
&= (-1)^{\ell_2}\, a_1\cdots a_{c+r}\cdot a_1\cdots a_n\cdot \zeta_{\ell_1}^{\heartsuit'_{j_1,\dots,j_n}}\cdot \det\big(\widehat{\mathsf{C}}_{\ell_2}\big)^{\zeta_{\ell_1}}_{a_1,\dots,a_{c+r}}\cdot \Big(\frac{\zeta_{\ell_1}}{\zeta_{\ell_2}}\Big)^{\lambda_{\ell_2}-1} && \text{[use (21) for } j=\ell_1\text{]}\\
&= (-1)^{\ell_2}\, a_1\cdots a_{c+r}\cdot a_1\cdots a_n\cdot \zeta_{\ell_1}^{\heartsuit'_{j_1,\dots,j_n}}\cdot \Big(\frac{\zeta_{\ell_2}}{\zeta_{\ell_1}}\Big)^{\heartsuit(\ell_2)}\cdot \det\big(\widehat{\mathsf{C}}_{\ell_2}\big)^{\zeta_{\ell_2}}_{a_1,\dots,a_{c+r}}\cdot \Big(\frac{\zeta_{\ell_1}}{\zeta_{\ell_2}}\Big)^{\lambda_{\ell_2}-1} && \text{[use Proposition 4.3 below]}\\
&= (-1)^{\ell_2}\, a_1\cdots a_{c+r}\cdot a_1\cdots a_n\cdot \zeta_{\ell_2}^{\heartsuit'_{j_1,\dots,j_n}}\cdot \det\big(\widehat{\mathsf{C}}_{\ell_2}\big)^{\zeta_{\ell_2}}_{a_1,\dots,a_{c+r}} && [\heartsuit(\ell_2)=\heartsuit'_{j_1,\dots,j_n}+\lambda_{\ell_2}-1]\\
&= \widehat{\omega}_{\ell_2} \qquad [\text{☺}]. && \text{[use (19)]}
\end{aligned}$$

Thus we finish the proof. □

An essential ingredient in the above proof is to compare the same determinant in different trivializations $\zeta_{\ell_1}, \zeta_{\ell_2}$. Now we give general transition formulas.

Proposition 4.3. *For all $0 \leqslant j, \ell_1, \ell_2 \leqslant N$, for any Zariski open set $U \subset \mathsf{D}(\zeta_{\ell_1}) \cap \mathsf{D}(\zeta_{\ell_2})$ with trivializations $\mathscr{A}_1\big|_U = \mathscr{O}_U\cdot a_1, \dots, \mathscr{A}_{c+r}\big|_U = \mathscr{O}_U\cdot a_{a+r}$, there hold the transition formulas:*

$$\det\big(\widehat{\mathsf{C}}_j\big)^{\zeta_{\ell_1}}_{a_1,\dots,a_{c+r}} = \Big(\frac{\zeta_{\ell_2}}{\zeta_{\ell_1}}\Big)^{\heartsuit(j)}\cdot \det\big(\widehat{\mathsf{C}}_j\big)^{\zeta_{\ell_2}}_{a_1,\dots,a_{c+r}} \in \mathsf{H}^0\Big(U, \mathsf{Sym}^n\,\Omega^1_{\mathbf{P}}\Big), \tag{22}$$

with $\heartsuit(j) = \heartsuit'_{j_1,\dots,j_n} + \lambda_j - 1$.

Proof. Our idea is to expand the two determinants and to compare each pair of corresponding terms. Without loss of generality, we may assume $j = 0$.

For $i = 1\cdots N$, $k = 1\cdots N$, we denote the (i,k)-th entry of $\big(\widehat{\mathsf{C}}_0\big)^{\zeta_{\ell_1}}_{a_1,\dots,a_{c+r}}$ (resp. $\big(\widehat{\mathsf{C}}_0\big)^{\zeta_{\ell_2}}_{a_1,\dots,a_{c+r}}$) by $c^1_{i,j}$ (resp. $c^2_{i,j}$). First of all, we recall all the entries:

$$c^\delta_{p,k} := \frac{A^k_p}{a_p\cdot\zeta_{\ell_\delta}^{\epsilon^k_p}}\cdot\frac{\zeta_k}{\zeta_{\ell_\delta}}, \qquad c^\delta_{c+r+q,k} := \mathrm{d}\Big(\frac{A^k_q}{a_q\cdot\zeta_{\ell_\delta}^{\epsilon^k_q}}\Big)\cdot\frac{\zeta_k}{\zeta_{\ell_\delta}} + \lambda_k\frac{A^k_q}{a_q\cdot\zeta_{\ell_\delta}^{\epsilon^k_q}}\cdot\mathrm{d}\Big(\frac{\zeta_k}{\zeta_{\ell_\delta}}\Big) \qquad (\delta=1,2;\ p=1\cdots c+r;\ q=1\cdots n;\ k=1\cdots N).$$

By much the same reasoning as in (20), we can obtain the transition formulas:

$$\begin{aligned}
c^1_{p,k} &= c^2_{p,k}\cdot(\zeta_{\ell_2}/\zeta_{\ell_1})^{\epsilon^k_p+1},\\
c^1_{c+r+q,k} &= c^2_{c+r+q,k}\cdot(\zeta_{\ell_2}/\zeta_{\ell_1})^{\epsilon^k_q+1} + c^2_{q,k}\cdot(\epsilon^k_q+\lambda_k)(\zeta_{\ell_2}/\zeta_{\ell_1})^{\epsilon^k_q}\,\mathrm{d}(\zeta_{\ell_2}/\zeta_{\ell_1}).
\end{aligned}$$

Recalling that $\epsilon^k_p + \lambda_k = d_p$, we thus rewrite the above identities as:

$$\begin{aligned}
c^1_{p,k} &= c^2_{p,k}\cdot(\zeta_{\ell_2}/\zeta_{\ell_1})^{d_p-(\lambda_k-1)},\\
c^1_{c+r+q,k} &= c^2_{c+r+q,k}\cdot(\zeta_{\ell_2}/\zeta_{\ell_1})^{d_q-(\lambda_k-1)} + \underline{c^2_{q,k}\cdot d_q(\zeta_{\ell_2}/\zeta_{\ell_1})^{d_q-\lambda_k}\,\mathrm{d}(\zeta_{\ell_2}/\zeta_{\ell_1})}.
\end{aligned} \tag{23}$$

Now, comparing (23) with the desired formula (22), we may anticipate that, the underlined terms would bring some trouble, since no terms $\mathrm{d}(\zeta_{\ell_2}/\zeta_{\ell_1})$ appear on the right-hand-side of (22). Nevertheless, we can overcome this difficulty firstly by observing:

$$\begin{vmatrix} c^1_{q,k_1} & c^1_{q,k_2}\\ c^1_{c+r+q,k_1} & c^1_{c+r+q,k_2}\end{vmatrix} = \begin{vmatrix} c^2_{q,k_1} & c^2_{q,k_2}\\ c^2_{c+r+q,k_1} & c^2_{c+r+q,k_2}\end{vmatrix}\cdot(\zeta_{\ell_2}/\zeta_{\ell_1})^{d_q-(\lambda_{k_1}-1)}\cdot(\zeta_{\ell_2}/\zeta_{\ell_1})^{d_q-(\lambda_{k_2}-1)} \qquad (q=1\cdots n;\ k_1,k_2=1\cdots N), \tag{24}$$

and secondly by using a tricky Laplace expansion of the determinant:

$$\det\left(\widehat{\mathsf{C}}_0\right)^{\zeta_{\ell_1}}_{a_1,\dots,a_{c+r}} \;=\; \sum \mathsf{Sign}(\pm)\cdot\prod_{q=1}^{n}\begin{vmatrix} c^1_{q,k^1_q} & c^1_{q,k^2_q} \\ c^1_{c+r+q,k^1_q} & c^1_{c+r+q,k^2_q}\end{vmatrix}\cdot\prod_{p=n+1}^{c+r} c^1_{p,k_p}, \tag{25}$$

where the sum runs through all choices of $N = 2n + (c + r - n)$ indices $k^1_1 < k^2_1, \dots, k^1_n < k^2_n, k_{n+1}, \dots, k_{c+r}$ such that their union is exactly $\{1, \dots, N\}$, and where $\mathsf{Sign}(\pm)$ is either 1 or -1 uniquely determined by the choices of indices. Now, using (24), we see that each term in (25) is equal to:

$$\mathsf{Sign}(\pm)\cdot\prod_{q=1}^{n}\begin{vmatrix} c^2_{q,k^1_q} & c^2_{q,k^2_q} \\ c^2_{c+r+q,k^1_q} & c^2_{c+r+q,k^2_q}\end{vmatrix}\cdot\prod_{p=n+1}^{c+r} c^2_{p,k_p}$$

multiplied by $(\zeta_{\ell_2}/\zeta_{\ell_1})^{\heartsuit}$, where:

$$\begin{aligned}\heartsuit \;&:=\; \sum_{q=1}^{n}\left[d_q - (\lambda^1_{k_q} - 1) \;+\; d_q - (\lambda^2_{k_q} - 1)\right] \;+\; \sum_{p=n+1}^{c+r}\left(d_q - (\lambda_{k_p} - 1)\right)\\ &=\; \sum_{p=1}^{c+r} d_p \;+\; \sum_{q=1}^{n} d_q \;-\; \sum_{k=1}^{N}(\lambda_k - 1)\\ &=\; \heartsuit'_{j_1,\dots,j_n} + \lambda_0 - 1.\end{aligned}$$

Thus (25) factors as:

$$\begin{aligned}\det\left(\widehat{\mathsf{C}}_0\right)^{\zeta_{\ell_1}}_{a_1,\dots,a_{c+r}} \;&=\; (\zeta_{\ell_2}/\zeta_{\ell_1})^{\heartsuit}\cdot\sum \mathsf{Sign}(\pm)\cdot\prod_{q=1}^{n}\begin{vmatrix} c^2_{q,k^q_1} & c^2_{q,k^q_2} \\ c^2_{c+r+q,k^q_1} & c^2_{c+r+q,k^q_2}\end{vmatrix}\cdot\prod_{p=n+1}^{c+r} c^2_{p,k_p}\\ \text{[use Laplace expansion again]}\qquad &=\; (\zeta_{\ell_2}/\zeta_{\ell_1})^{\heartsuit}\cdot\det\left(\widehat{\mathsf{C}}_0\right)^{\zeta_{\ell_2}}_{a_1,\dots,a_{c+r}},\end{aligned}$$

whence we conclude the proof. □

5. 'Hidden' Symmetric Differential Forms

Comparing the two approaches in [5, Section 6], the scheme-theoretic one has the advantage in further generalizations, while the geometric one is superior in discovering the 'hidden' symmetric differential forms [5, Proposition 6.12]. Skipping the thinking process, we present the corresponding generalizations of these symmetric forms as follows.

We assume that $\lambda_0, \dots, \lambda_N \geqslant 2$ in this section. For any $\eta = 1 \cdots n - 1$, for any indices $0 \leqslant v_1 < \cdots < v_\eta \leqslant N$ and $1 \leqslant j_1 < \cdots < j_{n-\eta} \leqslant c$, write $\{0, \dots, N\} \setminus \{v_1, \dots, v_\eta\}$ in the ascending order $r_0 < r_1 < \cdots < r_{N-\eta}$, and then denote by ${}_{v_1,\dots,v_\eta}\mathsf{M}_{j_1,\dots,j_{n-\eta}}$ the $(N - \eta) \times (N - \eta + 1)$ submatrix of M determined by the first $c + r$ rows and the selected rows $c + r + j_1, \dots, c + r + j_{n-\eta}$ as well as the $(N - \eta + 1)$ columns $r_0 + 1, \dots, r_{N-\eta} + 1$. Next, for every index $j \in \{0, \dots, N\} \setminus \{v_1, \dots, v_\eta\}$, let ${}_{v_1,\dots,v_\eta}\widehat{\mathsf{M}}_{j_1,\dots,j_{n-\eta};\,j}$ denote the submatrix of ${}_{v_1,\dots,v_\eta}\mathsf{M}_{j_1,\dots,j_{n-\eta}}$ obtained by deleting the column which is originally contained in the $(j + 1)$-th column of M. Lastly, denote by ${}_{v_1,\dots,v_\eta}\mathbf{P} \subset \mathbf{P}$ the subvariety defined by sections $\zeta_{v_1}, \dots, \zeta_{v_\eta}$ ('vanishing coordinates'), and denote ${}_{v_1,\dots,v_\eta}X := X \cap {}_{v_1,\dots,v_\eta}\mathbf{P}$. Setting:

$$\mathscr{A}^{1,\dots,c+r}_{j_1,\dots,j_{n-\eta}} \;:=\; \mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_{c+r} \otimes \mathscr{A}_{j_1} \otimes \cdots \otimes \mathscr{A}_{n-\eta},$$

by much the same reasoning as in Proposition 4.1, we have

Proposition 5.1. *For all $j = 0 \cdots N-\eta$, the following $N+1-\eta$ sections:*

$$\begin{aligned} {}_{v_1,\ldots,v_\eta}\psi_{j_1,\ldots,j_{n-\eta};\,r_j} &:= (-1)^j \det\big({}_{v_1,\ldots,v_\eta}\widehat{\mathsf{M}}_{j_1,\ldots,j_{n-\eta};\,r_j}\big) \\ &\in \mathsf{H}^0\Big({}_{v_1,\ldots,v_\eta}\mathbf{P},\, \mathsf{Sym}^{n-\eta}\,\Omega^1_{\mathbf{P}} \otimes \mathscr{A}^{1,\ldots,c+r}_{j_1,\ldots,j_{n-\eta}} \otimes \mathscr{L}^{\heartsuit_{j_1,\ldots,j_{n-\eta}}}\Big), \end{aligned}$$

when restricted to ${}_{v_1,\ldots,v_\eta}X$, give one and the same symmetric differential form:

$${}_{v_1,\ldots,v_\eta}\psi_{j_1,\ldots,j_{n-\eta}} \in \mathsf{H}^0\Big({}_{v_1,\ldots,v_\eta}X,\, \mathsf{Sym}^{n-\eta}\,\Omega^1_V \otimes \mathscr{A}^{1,\ldots,c+r}_{j_1,\ldots,j_{n-\eta}} \otimes \mathscr{L}^{\heartsuit_{j_1,\ldots,j_{n-\eta}}}\Big),$$

with the twisted degree:

$$\heartsuit_{j_1,\ldots,j_{n-\eta}} = \sum_{p=1}^{c+r} d_p + \sum_{q=1}^{n-\eta} d_{j_q}. \qquad \square$$

Moreover, playing the dividing trick again, we obtain an analogue of Proposition 4.2.

Proposition 5.2. *For all $j = 0 \cdots N-\eta$, the formal symmetric differential forms:*

$${}_{v_1,\ldots,v_\eta}\widehat{\omega}_{j_1,\ldots,j_{n-\eta};\,r_j} = \frac{(-1)^j \det\big({}_{v_1,\ldots,v_\eta}\widehat{\mathsf{M}}_{j_1,\ldots,j_{n-\eta};\,r_j}\big)}{\zeta_{r_0}^{\lambda_{r_0}-1}\cdots\zeta_{r_{N-\eta}}^{\lambda_{r_{N-\eta}}-1}}$$

are well-defined sections in:

$$\mathsf{H}^0\Big(\mathsf{D}(\zeta_{r_j}) \cap {}_{v_1,\ldots,v_\eta}\mathbf{P},\, \mathsf{Sym}^{n-\eta}\,\Omega^1_{\mathbf{P}} \otimes \mathscr{A}^{1,\ldots,c+r}_{j_1,\ldots,j_{n-\eta}} \otimes \mathscr{L}^{{}_{v_1,\ldots,v_\eta}\heartsuit'_{j_1,\ldots,j_{n-\eta}}}\Big),$$

with the twisted degree:

$${}_{v_1,\ldots,v_\eta}\heartsuit'_{j_1,\ldots,j_{n-\eta}} := \sum_{p=1}^{c+r} d_p + \sum_{q=1}^{n-\eta} d_{j_q} - \sum_{k=0}^{N}(\lambda_k - 1) + \sum_{\mu=1}^{\eta}(\lambda_{v_\mu} - 1).$$

Moreover, when restricted to ${}_{v_1,\ldots,v_\eta}X$, they glue together to make a global section:

$${}_{v_1,\ldots,v_\eta}\omega_{j_1,\ldots,j_{n-\eta}} \in \mathsf{H}^0\Big({}_{v_1,\ldots,v_\eta}X,\, \mathsf{Sym}^{n-\eta}\,\Omega^1_V \otimes \mathscr{A}^{1,\ldots,c+r}_{j_1,\ldots,j_{n-\eta}} \otimes \mathscr{L}^{{}_{v_1,\ldots,v_\eta}\heartsuit'_{j_1,\ldots,j_{n-\eta}}}\Big). \qquad \square$$

6. Applications of MCM

6.1. **Motivation.** Recall [5, Section 7] that the moving coefficients method is devised to produce as many negatively twisted symmetric differential forms as possible, by manipulating the determinantal structure of the constructed symmetric differential forms. Since Propositions 4.2, 5.2 exactly share the same determinantal shape, it is possible to adapt MCM for the aim of Theorem 1.2, which coincides with Theorem 1.1 in the case that $\mathbf{P} = \mathbb{P}^N_{\mathbb{K}}$, $\mathscr{L} = \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(1)$. Indeed, by introducing c auxiliary line bundles $\mathscr{A}_1, \ldots, \mathscr{A}_c \approx$ trivial line bundle, we can even treat the case of c ample line bundles $\mathscr{L}+\mathscr{A}_1, \ldots, \mathscr{L}+\mathscr{A}_c \approx \mathscr{L}$, and eventually we will obtain Theorem 1.4.

6.2. **Adaptation.** Let $\mathbf{P}$ be a smooth projective $\mathbb{K}$-variety of dimension N, equipped with a very ample line bundle $\mathscr{L}$. By Bertini's theorem, we may choose $N+1$ simple normal crossing global sections $\zeta_0, \ldots, \zeta_N$ of $\mathscr{L}$, and we shall view them as the 'homogeneous coordinates' of $\mathbf{P}$. Thus, we may 'identify' $(\mathbf{P}, \mathscr{L})$ with $(\mathbb{P}^N_{\mathbb{K}}, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(1))$ in the sense that locally they have the same coordinates $[\xi_0 : \cdots : \zeta_N] \approx [z_0 : \cdots : z_N]$, and therefore we can generalize local computations of the later one to the former one, like what we perform in Section 4. This treatment is also visible in [3].

Let $c \geqslant 1$, $r \geqslant 0$ be two integers with $2c + r \geqslant N$ and $c + r < N$, and let $\mathscr{A}_1, \dots, \mathscr{A}_{c+r}$ be $c + r$ auxiliary line bundles to be determined. Now, we start to adapt the machinery of MCM. First of all, introduce the following $c + r$ 'flexible' sections which copy the major ingredient **(3)**:

$$F_i = \sum_{j=0}^{N} A_i^j \zeta_j^d + \sum_{l=c+r+1}^{N} \sum_{0\leqslant j_0<\cdots<j_l\leqslant N} \sum_{k=0}^{l} M_i^{j_0,\dots,j_l;j_k} \zeta_{j_0}^{\mu_{l,k}} \cdots \widehat{\zeta_{j_k}^{\mu_{l,k}}} \cdots \zeta_{j_l}^{\mu_{l,k}} \zeta_{j_k}^{d-l\mu_{l,k}} \in \mathsf{H}^0\left(\mathbf{P}, \mathscr{A}_i \otimes \mathscr{L}^{\epsilon_i} \otimes \mathscr{L}^d\right) \quad (i = 1 \cdots c+r), \tag{26}$$

where all coefficients $A_i^\bullet, M_i^{\bullet;\bullet}$ are some global sections of $\mathscr{A}_i \otimes \mathscr{L}^{\epsilon_i}$ for some fixed integers $\epsilon_i \geqslant 1$, and where all positive integers $\mu^{l,k}, d$ are to be chosen by a certain *Algorithm*, which is designed to make all the symmetric differential forms obtained later have *negative twist*. For better comprehension, we will make the Algorithm clear in Subsection 6.4 below, and for the time being we roughly summarize it as:

$$1 \leqslant \max\{\epsilon_i\}_{i=1\cdots c+r} \ll \underbrace{\mu_{c+r+1,0} \ll \cdots \ll \mu_{c+r+1,c+r+1}}_{\mu_{c+r+1,\bullet} \text{ grow exponentially}} \ll \cdots\cdots \ll \underbrace{\mu_{N,0} \ll \cdots \ll \mu_{N,N}}_{\mu_{N,\bullet} \text{ grow exponentially}} \ll d. \tag{27}$$

Let $V \subset \mathbf{P}$ be the subvariety defined by the first c sections $F_1, \dots, F_c$, and let $X \subset \mathbf{P}$ be the subvariety defined by all the $c + r$ sections $F_1, \dots, F_{c+r}$. *A priori*, we require all the line bundles $\mathscr{A}_i \otimes \mathscr{L}^{\epsilon_i}$ to be very ample, so that for generic choices of parameters:

$$A_i^\bullet, M_i^{\bullet;\bullet} \in \mathsf{H}^0\left(\mathbf{P}, \mathscr{A}_i \otimes \mathscr{L}^{\epsilon_i}\right) \quad (i = 1 \cdots c+r),$$

both intersections V, X are smooth complete (the proof is much the same as that of Bertini's Theorem, see Subsection 7.2).

6.3. **Manipulations.** Now, we apply MCM to construct a series of negatively twisted symmetric differential forms. For shortness, we will refer to [5, Section 7] for skipped details, in which the canonical setting $\{\mathbb{P}^N_{\mathbb{K}}, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(1), (z_0, \dots, z_N)\}$ there plays the same role as that of $\{\mathbf{P}, \mathscr{L}, (\zeta_0, \dots, \zeta_N)\}$ in our treatment here.

To begin with, we rewrite each section F_i in (26) as (cf. [5, p. 43, (104)]):

$$F_i = \sum_{j=0}^{N} \underbrace{\left(A_i^j \zeta_j^d + \sum_{l=c+r+1}^{N-1} \sum_{\substack{0\leqslant j_0<\cdots<j_l\leqslant N \\ j_k=j \text{ for some } 0\leqslant k\leqslant l}} M_i^{j_0,\dots,j_l;j_k} \zeta_{j_0}^{\mu_{l,k}} \cdots \widehat{\zeta_{j_k}^{\mu_{l,k}}} \cdots \zeta_{j_l}^{\mu_{l,k}} \zeta_{j_k}^{d-l\mu_{l,k}} \right)}_{\text{for each } j=0\cdots N, \text{ we view this whole bracket as one section}} + \sum_{k=0}^{N} \underbrace{M_i^{0,\dots,N;k} \zeta_0^{\mu_{N,k}} \cdots \widehat{\zeta_k^{\mu_{N,k}}} \cdots \zeta_N^{\mu_{N,k}} \zeta_k^{d-N\mu_{N,k}}}_{\text{for each } k=0\cdots N, \text{ we view it as one section}}. \tag{28}$$

Thus, we view each F_i as the sum of $2N + 2 = \sum_{j=0}^{N} 1 + \sum_{k=0}^{N} 1$ sections of the same line bundle, as indicate above.

Next, we construct a $(c + r + c) \times (2N + 2)$ formal matrix M such that, for every $i = 1 \cdots c + r$, $j = 1 \cdots c$, its i-th row copies the $2N + 2$ sections in the sum of F_i in the exact order, and its $(c + r + j)$-th row is the formal differential of the j-th row.

Write the $2N + 2$ columns of M as:

$$\mathsf{M} = \left(\mathsf{A}_0 \mid \cdots \mid \mathsf{A}_N \mid \mathsf{B}_0 \mid \cdots \mid \mathsf{B}_N\right). \tag{29}$$

For every $\nu = 0 \cdots N$, we construct the matrix:

$$\mathsf{K}^\nu := \left(\mathsf{A}_0 \mid \cdots \mid \widehat{\mathsf{A}_\nu} \mid \cdots \mid \mathsf{A}_N \mid \mathsf{A}_\nu + \sum_{j=0}^{N} \mathsf{B}_j\right), \tag{30}$$

where the last column is understood to appear in the 'omitted' column. Also, for every $\tau = 0 \cdots N-1$ and every $\rho = \tau+1 \cdots N$, we construct the matrix:

$$\mathsf{K}^{\tau,\rho} := \Big(\mathsf{A}_0 + \mathsf{B}_0 \mid \cdots \mid \mathsf{A}_\tau + \mathsf{B}_\tau \mid \mathsf{A}_{\tau+1} \mid \cdots \mid \widehat{\mathsf{A}_\rho} \mid \cdots \mid \mathsf{A}_N \mid \mathsf{A}_\rho + \sum_{j=\tau+1}^{N} \mathsf{B}_j\Big). \tag{31}$$

Now, fix a positive integer $\blacktriangledown \geqslant 1$ such that:

$$\mathscr{A}_i \otimes \mathscr{L}^{-\blacktriangledown} < 0 \qquad (i = 1 \cdots c+r). \tag{32}$$

Recalling the rough Algorithm (27), observe in (28), (30) that the $N+1$ columns of K^ν are subsequently divisible by:

$$\zeta_0^{d-\delta_N}, \ldots, \widehat{\zeta_\nu^{d-\delta_N}}, \ldots, \zeta_N^{d-\delta_N}, \zeta_\nu^{\mu_{N,0}},$$

where $\delta_N := (N-1)\,\mu_{N-1,N-1}$. Thus, applying Proposition 4.2, for every $1 \leqslant j_1 < \cdots < j_n \leqslant c$, we obtain a global symmetric differential form:

$$\phi^\nu_{j_1,\ldots,j_n} \in \mathsf{H}^0\big(X, \mathsf{Sym}^n \Omega_V \otimes \underbrace{\mathscr{A}^{1,\ldots,c+r}_{j_1,\ldots,j_n} \otimes \mathscr{L}^{\heartsuit^\nu_{j_1,\ldots,j_n}}}_{<0, \text{ because of } (32),(33)}\big) \qquad (\nu = 0 \cdots N),$$

with negative twist:

$$\begin{aligned} \heartsuit^\nu_{j_1,\ldots,j_n} &= \sum_{p=1}^{c+r}(d+\epsilon_p) + \sum_{q=1}^{n}(d+\epsilon_{j_q}) - \sum_{j=0, j\neq\nu}^{N}(d-\delta_N-1) - (\mu_{N,0}-1) \\ &= -\mu_{N,0} + N\,\delta_N + \sum_{p=1}^{c+r}\epsilon_p + \sum_{q=1}^{n}\epsilon_{j_q} + N + 1 \\ &\leqslant -N\blacktriangledown. \end{aligned} \tag{33}$$

[by Algorithm (27)]

Similarly, observe that the $N+1$ columns of $\mathsf{K}^{\tau,\rho}$ are subsequently divisible by:

$$\zeta_0^{d-N\mu_{N,0}}, \ldots, \zeta_\tau^{d-N\mu_{N,\tau}}, \zeta_{\tau+1}^{d-\delta_N}, \ldots, \widehat{\zeta_\nu^{d-\delta_N}}, \ldots, \zeta_N^{d-\delta_N}, \zeta_\nu^{\mu_{N,\tau+1}},$$

thus by Proposition 4.2 we obtain:

$$\psi^{\tau,\rho}_{j_1,\ldots,j_n} \in \mathsf{H}^0\big(X, \mathsf{Sym}^n \Omega_V \otimes \underbrace{\mathscr{A}^{1,\ldots,c+r}_{j_1,\ldots,j_n} \otimes \mathscr{L}^{\heartsuit^{\tau,\rho}_{j_1,\ldots,j_n}}}_{<0, \text{ because of } (32),(34)}\big) \qquad (\tau = 0 \cdots N-1,\ \rho = \tau+1 \cdots N),$$

with negative twist:

$$\begin{aligned} \heartsuit^{\tau,\rho}_{j_1,\ldots,j_n} &= \sum_{p=1}^{c+r}(d+\epsilon_p) + \sum_{q=1}^{n}(d+\epsilon_{j_q}) - \sum_{k=0}^{\tau}(d-N\mu_{N,k}-1) - \sum_{j=\tau+1, j\neq\rho}^{N}(d-\delta_N-1) - (\mu_{N,\tau+1}-1) \\ &= -\mu_{N,\tau+1} + \sum_{k=0}^{\tau} N\mu_{N,k} + (N-\tau-1)\,\delta_N + \sum_{p=1}^{c+r}\epsilon_p + \sum_{q=1}^{n}\epsilon_{j_q} + N + 1 \\ &\leqslant -N\blacktriangledown. \end{aligned} \tag{34}$$

[by Algorithm (27)]

Recalling the notation in Section 5, for any $\eta = 1 \cdots n-1$, for any 'vanishing' indices $0 \leqslant \nu_1 < \cdots < \nu_\eta \leqslant N$, by applying Proposition 5.2, we can construct a series of negatively twisted symmetric differential forms over the 'coordinates vanishing part':

$$\omega_\ell \in \Gamma\big({}_{\nu_1,\ldots,\nu_\eta}X, \mathsf{Sym}^{n-\eta}\Omega_V \otimes \text{negative twist}\big) \qquad (\ell = 1, 2, \ldots).$$

The procedure is much the same as before. First, we rewrite each section F_i in (26) as:

$$F_i = \sum_{j=0}^{N-\eta} A_i^{r_j}\,\zeta_{r_j}^d + \sum_{l=c+r+1}^{N-\eta}\ \sum_{0\leqslant j_0<\cdots<j_l\leqslant N-\eta}\ \sum_{k=0}^{l} M_i^{r_{j_0},\ldots,r_{j_l};r_{j_k}}\,\zeta_{r_{j_0}}^{\mu_{l,k}}\cdots\widehat{\zeta_{r_{j_k}}^{\mu_{l,k}}}\cdots\zeta_{r_{j_l}}^{\mu_{l,k}}\zeta_{r_{j_k}}^{d-l\mu_{l,k}} + \text{negligible terms},$$

so that F_i has the same structure as (26), in the sense of replacing:

$$N \leftrightarrow N-\eta, \qquad \{0,\ldots,N\} \ \longleftrightarrow\ \{r_0,\ldots,r_{N-\eta}\}.$$

Thus, we can repeat the above manipulations. For shortness, we skip all details (cf. [5, Subsection 7.3]) and only state the results.

For every $1 \leqslant j_1 < \cdots < j_{n-\eta} \leqslant c$, for every $\nu = 0\cdots N-\eta$, we obtain a symmetric differential form:

$$_{v_1,\ldots,v_\eta}\phi^{\nu}_{j_1,\ldots,j_{n-\eta}} \in \Gamma\big(_{v_1,\ldots,v_\eta}X, \mathsf{Sym}^{n-\eta}\,\Omega_V \otimes \underbrace{\mathscr{A}^{1,\ldots,c+r}_{j_1,\ldots,j_{n-\eta}} \otimes \mathscr{L}^{v_1,\ldots,v_\eta\,\heartsuit^{\nu}_{j_1,\ldots,j_{n-\eta}}}}_{<0,\ \text{because of (35)}}\big),$$

with negative twist (set $\delta_{N-\eta} := (N-\eta-1)\,\mu_{N-\eta-1,N-\eta-1}$):

$$_{v_1,\ldots,v_\eta}\heartsuit^{\nu}_{j_1,\ldots,j_{n-\eta}} \ =\ -\mu_{N-\eta,0} + (N-\eta)\,\delta_{N-\eta} + \sum_{i=1}^{c+r}\epsilon_i + \sum_{\ell=1}^{n-\eta}\epsilon_{j_\ell} + (N-\eta) + 1 \ \leqslant\ -(N-\eta)\,\blacktriangledown. \tag{35}$$

Also, for every $\tau = 0\cdots N-\eta-1$ and every $\rho = \tau+1\cdots N-\eta$, we obtain:

$$_{v_1,\ldots,v_\eta}\phi^{\tau,\rho}_{j_1,\ldots,j_{n-\eta}} \in \Gamma\big(_{v_1,\ldots,v_\eta}X, \mathsf{Sym}^{n-\eta}\,\Omega_V \otimes \underbrace{\mathscr{A}^{1,\ldots,c+r}_{j_1,\ldots,j_{n-\eta}} \otimes \mathscr{L}^{v_1,\ldots,v_\eta\,\heartsuit^{\tau,\rho}_{j_1,\ldots,j_{n-\eta}}}}_{<0,\ \text{because of (36)}}\big),$$

with negative twist:

$$_{v_1,\ldots,v_\eta}\heartsuit^{\tau,\rho}_{j_1,\ldots,j_{n-\eta}} \ =\ -\mu_{N-\eta,\tau+1} + \sum_{k=0}^{\tau}(N-\eta)\,\mu_{N-\eta,k} + (N-\eta-\tau-1)\,\delta_{N-\eta} + \sum_{i=1}^{c+r}\epsilon_i + \sum_{\ell=1}^{n-\eta}\epsilon_{j_\ell} + (N-\eta) + 1 \ \leqslant\ -(N-\eta)\,\blacktriangledown. \tag{36}$$

6.4. **A Natural Algorithm.** We will construct $\mu^{l,k}$ in a lexicographic order with respect to indices (l,k), for $l = c+r+1\cdots N, k = 0\cdots l$, together with positive integers δ_l.

For simplicity, we start by setting:

$$\delta_{c+r+1} \ \geqslant\ \max\{\epsilon_1,\ldots,\epsilon_{c+r}\}. \tag{37}$$

For every $l = c+r+1\cdots N$, in this step, we begin with choosing $\mu_{l,0}$ that satisfies:

$$\text{[see (35), (33)]} \qquad \mu_{l,0} \geqslant l\,\delta_l + l\,\delta_{c+r+1} + l + 1 + l\,\blacktriangledown + 1, \tag{38}$$

then inductively we choose $\mu_{l,k}$ satisfying:

$$\text{[see (36), (34)]} \qquad \mu_{l,k} \ \geqslant\ \sum_{j=0}^{k-1} l\,\mu_{l,j} + (l-k)\,\delta_l + l\,\delta_{c+r+1} + l + 1 + l\,\blacktriangledown + 1 \qquad (k = 1\cdots l). \tag{39}$$

If $l < N$, we end this step by setting:

$$\delta_{l+1} := l\,\mu_{l,l} \tag{40}$$

as the starting point for the next step $l+1$. At the end $l = N$, we require that:

$$d \ \geqslant\ (N+1)\,\mu_{N,N}$$

be large enough.

6.5. **Controlling the base loci.** We will provide some technical preparations in Section 7.

By adapting the arguments in [5, Section 9], we can show that, for generic choices of parameters $A_{\bullet}^{\bullet}$, $M_{\bullet}^{\bullet,\bullet}$, firstly, the:

$$\text{Base Locus of } \Big\{\phi^{\nu}_{j_1,\dots,j_n}, \psi^{\tau,\rho}_{j_1,\dots,j_n}\Big\}^{\nu,\tau,\rho}_{1\leqslant j_1<\cdots<j_n\leqslant c} \;=:\; \mathsf{BS} \tag{41}$$

is discrete/empty over the 'coordinates nonvanishing part' $\{\zeta_0\cdots\zeta_N\neq 0\}$, and secondly, for every $1\leqslant\eta\leqslant n-1$, for every $0\leqslant v_1<\cdots<v_\eta\leqslant N$, the:

$$\text{Base Locus of } \Big\{{}_{v_1,\dots,v_\eta}\phi^{\nu}_{j_1,\dots,j_{n-\eta}}, \; {}_{v_1,\dots,v_\eta}\psi^{\tau,\rho}_{j_1,\dots,j_{n-\eta}}\Big\}^{\nu,\tau,\rho}_{1\leqslant j_1<\cdots<j_{n-\eta}\leqslant c} \;=:\; {}_{v_1,\dots,v_\eta}\mathsf{BS} \tag{42}$$

is discrete/empty over the corresponding 'coordinates nonvanishing part' $\{\zeta_{r_0}\cdots\zeta_{r_{N-\eta}}\neq 0\}$.

For the sake of completeness, we sketch the proof in Subsection 7.3 below.

6.6. **Effective degree estimates.** In the Algorithm above, we first set $\blacktriangledown=2$, $\epsilon_1=\cdots=\epsilon_{c+r}=1$, and next we demand all inequalities (37) – (40) to be exactly equalities. Thus we receive the estimate (cf. [5, Section 11]):

$$(N+1)\,\mu_{N,N} \;<\; N^{N^2/2}-1 \;:=\; \mathsf{d}_0 \qquad (\forall\, N\geqslant 3).$$

Now, recall the value $\epsilon_0=3/\mathsf{d}_0$ in Definition 1.3. In fact, the motivation is the following

Proposition 6.1. *Let $\mathscr{L}$, $\mathscr{S}$ be two ample line bundles on $\mathbf{P}$. Then $\mathscr{S}$ is almost proportional to $\mathscr{L}$ if and only if there exist some positive integers $d\geqslant\mathsf{d}_0$, $s,l\geqslant 1$, such that $\mathscr{S}^s=\mathscr{A}\otimes\mathscr{L}^l\otimes\mathscr{L}^{l\,d}$, where the line bundle $\mathscr{A}$ satisfies that $\mathscr{A}\otimes\mathscr{L}^l$ is very ample and that $\mathscr{A}\otimes\mathscr{L}^{-2\,l}<0$ is negative.*

Proof. "$\Longleftarrow$" We can take $\alpha=s\cdot[\mathscr{S}]$ and $\beta=l\,d\cdot[\mathscr{L}]$, so that $\alpha-\beta=[\mathscr{A}\otimes\mathscr{L}^l]>0$, and that $(1+\epsilon_0)\beta-\alpha\geqslant(1+3/d)\beta-\alpha=-\,[\mathscr{A}\otimes\mathscr{L}^{-2\,l}]>0$.

"$\Longrightarrow$" Since $\mathbb{Q}_+$ is dense in $\mathbb{R}_+$, we may assume that $\alpha\in\mathbb{Q}_+\cdot[\mathscr{S}]$ and $\beta\in\mathbb{Q}_+\cdot[\mathscr{L}]$. Next, we can choose a sufficiently divisible integer $m>0$ such that $m\cdot\alpha=s_0\cdot[\mathscr{S}]$ and $m\cdot\beta=l_0\,\mathsf{d}_0\cdot[\mathscr{L}]$ for some positive integers $s_0,l_0>0$. Set the line bundle $\mathscr{A}_0:=\mathscr{S}^{s_0}\otimes(\mathscr{L}^{l_0}\otimes\mathscr{L}^{l_0\,\mathsf{d}_0})^{-1}$, hence $\mathscr{S}^{s_0}=\mathscr{A}_0\otimes\mathscr{L}^{l_0}\otimes\mathscr{L}^{l_0\,\mathsf{d}_0}$. Now, using $\beta<\alpha<(1+\epsilon)\beta$, we receive:

$$0<m\cdot(\alpha-\beta)=m\cdot\alpha-m\cdot\beta=s_0\cdot[\mathscr{S}]-l_0\,\mathsf{d}_0\cdot[\mathscr{L}]=[\mathscr{S}^{s_0}\otimes\mathscr{L}^{-l_0\,\mathsf{d}_0}]=[\mathscr{A}_0\otimes\mathscr{L}^{l_0}],$$

$$0>m\cdot[\alpha-(1+\epsilon_0)\beta]=m\cdot\alpha-(1+3/\mathsf{d}_0)\,m\cdot\beta=s_0\cdot[\mathscr{S}]-(1+3/\mathsf{d}_0)\,l_0\,\mathsf{d}_0\cdot[\mathscr{L}]=[\mathscr{A}_0\otimes\mathscr{L}^{-2\,l_0}].$$

The first line above implies that $(\mathscr{A}_0\otimes\mathscr{L}^{l_0})^{\otimes m'}$ is very ample for some positive integer $m'>0$. Thus we can set $s:=s_0\,m'$, $l:=l_0\,m'$, $\mathscr{A}:=\mathscr{A}_0^{m'}$, then $\mathscr{S}^s=\mathscr{A}\otimes\mathscr{L}^l\otimes\mathscr{L}^{l\,\mathsf{d}_0}$ satisfies that $\mathscr{A}\otimes\mathscr{L}^l$ is very ample and that $\mathscr{A}\otimes\mathscr{L}^{-2\,l}<0$ is negative. □

Remark 6.2. In the above proof, we see that the second assertion holds for $d=\mathsf{d}_0$. In fact, it holds for any positive integer $d'\leqslant d$, since we have:

$$\mathscr{L}^{s\,(1+d')}\;=\;\big(\mathscr{A}\otimes\mathscr{L}^{l\,(1+d)}\big)^{1+d'}\;=\;\mathscr{A}^{1+d'}\otimes\mathscr{L}^{l\,(1+d)}\otimes\big(\mathscr{L}^{l\,(1+d)}\big)^{d'},$$

where:

$$\mathscr{A}^{1+d'}\otimes\mathscr{L}^{l\,(1+d)}\;=\;\big(\underbrace{\mathscr{A}\otimes\mathscr{L}^{l}}_{\text{very ample}}\big)^{1+d'}\otimes\mathscr{L}^{l\,(d-d')}$$

is very ample and where:

$$\mathscr{A}^{1+d'}\otimes\mathscr{L}^{-2\,l\,(1+d)}\;=\;\big(\underbrace{\mathscr{A}\otimes\mathscr{L}^{-2\,l}}_{\text{negative}}\big)^{1+d'}\otimes\mathscr{L}^{-2\,l\,(d-d')}\;<\;0.$$

Thus the second assertion holds not only for (d,s,l) but also for $(d',s\,(1+d'),l\,(1+d))$.

6.7. **Proof of Theorem 1.4.** Summarizing the above Subsections 6.2 – 6.6, we can obtain

Theorem 6.3. *Let* $\mathbf{P}$ *be a smooth projective variety of dimension* N*, and let* $\mathscr{L}$ *be a very ample line bundle over* $\mathbf{P}$*. For any integers* $c, r \geqslant 0$ *with* $2c + r \geqslant N$*, for any integer* $d \geqslant \mathsf{d}_0$*, for any* $c + r$ *line bundles* $\mathscr{A}_i$ $(i = 1 \cdots c + r)$ *such that* $\mathscr{A}_i \otimes \mathscr{L}$ *are very ample and that* $\mathscr{A}_i \otimes \mathscr{L}^{-2} < 0$*, setting:*

$$\mathscr{L}_i \ = \ \mathscr{A}_i \otimes \mathscr{L} \otimes \mathscr{L}^d \qquad (i = 1 \cdots c+r),$$

then, for generic $c + r$ *hypersurfaces:*

$$H_1 \in \left|\mathscr{L}_1\right|, \ldots, H_{c+r} \in \left|\mathscr{L}_{c+r}\right|,$$

the cotangent bundle Ω_V *of the intersection of the first* c *hypersurfaces* $V = H_1 \cap \cdots \cap H_c$ *restricted to the intersection of all the* $c + r$ *hypersurfaces* $X = H_1 \cap \cdots \cap H_c \cap H_{c+1} \cap \cdots \cap H_{c+r}$ *is ample.*

Denote the projectivization of the cotangent bundle $\Omega_{\mathbf{P}}$ of $\mathbf{P}$ by:

$$\mathbb{P}(\Omega_{\mathbf{P}}) \ := \ \mathsf{Proj}\,(\,\oplus_{k\geqslant 0}\,\mathsf{Sym}^k\,\Omega_{\mathbf{P}}),$$

and denote the associated Serre line bundle by $\mathscr{O}_{\mathbb{P}(\Omega_{\mathbf{P}})}(1)$. For any integers $a, b \geqslant 0$, for any $a + b$ global sections $F_1, \ldots, F_a, F_{a+1}, \ldots, F_{a+b}$ of arbitrary $a + b$ line bundles over $\mathbf{P}$, denote by:

$${}_{F_{a+1},\ldots,F_{a+b}}\mathbb{P}_{F_1,\ldots,F_a} \ \subset \ \mathbb{P}(\Omega_{\mathbf{P}}).$$

the unique subscheme defined by equations $F_1, \ldots, F_{a+b}, dF_1, \ldots, dF_a$. Thus, we reformulate the above theorem as:

Theorem 6.3'. *For generic* $c + r$ *sections:*

$$F_1 \in \mathsf{H}^0\,(\mathbf{P}, \mathscr{L}_1), \ldots, F_{c+r} \in \mathsf{H}^0\,(\mathbf{P}, \mathscr{L}_{c+r}),$$

the Serre line bundle $\mathscr{O}_{\mathbb{P}(\Omega_{\mathbf{P}})}(1)$ *is ample over the subvariety* ${}_{F_{c+1},\ldots,F_{c+r}}\mathbb{P}_{F_1,\ldots,F_c}$.

Proof of Theorem 6.3. We may assume that $N \geqslant 3$ and $c + r < N$, otherwise there is nothing to prove. Set $n = N - c - r$, observe that $1 \leqslant n \leqslant c$. Since ampleness is a Zariski open condition in family (Grothendieck), we only need to provide one ample example $H_1, \ldots, H_{c+r}$. In fact, we will construct $c + r$ sections $F_1, \ldots, F_{c+r}$ of the MCM shape (26) to conclude the proof.

Step 1. Since $d \geqslant \mathsf{d}_0$, by the effective degree estimates in preceding subsection, we can construct integers $\{\mu^{l,k}\}$ that satisfy the Algorithm in Subsection 6.4. Now, the structure of (26) is fixed, and we will choose some appropriate coefficients $A_i^{\bullet}, M_i^{\bullet;\bullet}$ for $i = 1 \cdots c + r$.

Step 2. For generic choices of parameters $A_{\bullet}^{\bullet}, M_{\bullet}^{\bullet;\bullet}$, both X, V are smooth complete, and moreover, for all $1 \leqslant \eta \leqslant n = N - c - r$, for all indices $0 \leqslant v_1 < \cdots < v_\eta \leqslant N$, the further intersection varieties ${}_{v_1,\ldots,v_\eta}X$ are all smooth complete. The reasoning is much the same as in Bertini's Theorem. For the sake of completeness, we provide a proof in Subsection 7.2 below.

Step 3. For generic choices of parameters $A_{\bullet}^{\bullet}, M_{\bullet}^{\bullet;\bullet}$, all the constructed negatively twisted symmetric differential forms have discrete based loci outside 'coordinates vanishing part', see Subsection 6.5 for details. This is the core of the moving coefficients method.

Step 4. Choose any generic parameters $A_{\bullet}^{\bullet}, M_{\bullet}^{\bullet;\bullet}$ that satisfy the properties in the above two steps. We claim that the corresponding sections $F_1, \ldots, F_{c+r}$ constitute one ample example.

Proof of the claim. Abbreviate $\mathbb{P} := {}_{F_{c+1},\ldots,F_{c+r}}\mathbb{P}_{F_1,\ldots,F_c}$ and ${}_{v_1,\ldots,v_\eta}\mathbb{P} := {}_{F_{c+1},\ldots,F_{c+r},\zeta_{v_1},\ldots,\zeta_{v_\eta}}\mathbb{P}_{F_1,\ldots,F_c}$. Let $\pi\colon \mathbb{P}(\Omega_{\mathbf{P}}) \longrightarrow \mathbf{P}$ be the canonical projection. Note that all the obtained symmetric differential forms in Step 3 can be viewed as sections (when $\eta = 0$, we agree ${}_{v_1,\ldots,v_\eta}\mathbb{P} = \mathbb{P}$):

$${}_{v_1,\ldots,v_\eta}\omega_{\diamondsuit} \ \in \ \mathsf{H}^0({}_{v_1,\ldots,v_\eta}\mathbb{P}, \mathscr{O}_{\mathbb{P}(\Omega_{\mathbf{P}})}(n-\eta) \otimes \pi^*\,{}_{v_1,\ldots,v_\eta}\mathscr{L}_{\diamondsuit}), \tag{43}$$

where we always use ❄ to denote auxiliary integers, and where all ${}_{v_1,\dots,v_\eta}\mathscr{L}_{\boldsymbol{*}} < 0$ are some negative line bundles. Choose an ample $\mathbb{Q}$-divisor $\mathscr{S} > 0$ over $\mathbf{P}$ such that all ${}_{v_1,\dots,v_\eta}\mathscr{L}_{\boldsymbol{*}}/(n-\eta) + \mathscr{S} < 0$ are still negative. Then we claim that $\mathscr{N} := \mathscr{O}_{\mathbb{P}(\Omega_{\mathbf{P}})}(1) \otimes \pi^* \mathscr{S}^{-1}$ is nef over $\mathbb{P}$.

Indeed, for any irreducible curve $C \subset \mathbb{P}$, if C lies in at least n 'coordinate hyperplanes' defined by $\zeta_{v_1}, \dots, \zeta_{v_n}$, then by Step 2 we see that C must contract to a point by π, thus $\mathscr{N}\big|_C \cong \mathscr{O}_{\mathbb{P}(\Omega_{\mathbf{P}})}(1)\big|_C$ is not only nef but ample. Assume now that C lies in at best $\eta < n$ 'coordinate hyperplanes' defined by $\zeta_{v_1}, \dots, \zeta_{v_\eta}$ (η could be zero). Since the base locus of all sections in (43) is discrete over the 'coordinates nonvanishing part' $\{\zeta_{r_0} \cdots \zeta_{r_{N-\eta}} \neq 0\}$, and $C \cap \{\zeta_{r_0} \cdots \zeta_{r_{N-\eta}} \neq 0\}$ is one-dimensional, we can find some ${}_{v_1,\dots,v_\eta}\omega_{\boldsymbol{*}}$ such that ${}_{v_1,\dots,v_\eta}\omega_{\boldsymbol{*}}\big|_C \neq 0$. Thus the intersection number $C \cdot \mathscr{O}_{\mathbb{P}(\Omega_{\mathbf{P}})}(n-\eta) \otimes \pi^*{}_{v_1,\dots,v_\eta}\mathscr{L}_{\boldsymbol{*}}$ is $\geqslant 0$. Since ${}_{v_1,\dots,v_\eta}\mathscr{L}_{\boldsymbol{*}}/(n-\eta) + \mathscr{S} < 0$, we immediately conclude that $C \cdot \mathscr{N} \geqslant 0$.

Lastly, since $\mathscr{S} > 0$ over $\mathbf{P}$, there exists some large integer $m \gg 1$ such that $\mathscr{P} := \mathscr{O}_{\mathbb{P}(\Omega_{\mathbf{P}})}(1) \otimes \pi^* \mathscr{S}^m > 0$ is positive over $\mathbb{P}(\Omega_{\mathbf{P}})$. In particular, it is also positive over $\mathbb{P}$. Since 'nef+ample=ample', we have $m\,\mathscr{N} + \mathscr{P} > 0$ over $\mathbb{P}$, that is $\mathscr{O}_{\mathbb{P}(\Omega_{\mathbf{P}})}(1)\big|_{\mathbb{P}} > 0$. □

Thus we conclude the proof. □

Finally, using the product coup, we obtain

Proof of Theorem 1.4. For every $i = 1 \cdots c+r$, since $\mathscr{L}_i$ is almost proportional to $\mathscr{L}$, by Proposition 6.1, there exist some positive integers $s_i, l_i \geqslant 1$, $d_i \geqslant \mathsf{d}_0$ such that $\mathscr{L}_i^{s_i} = \mathscr{A}_i \otimes \mathscr{L}^{l_i} \otimes \mathscr{L}^{l_i d_i}$, where the line bundle $\mathscr{A}_i$ satisfies that $\mathscr{A}_i \otimes \mathscr{L}^{l_i}$ is very ample and that $\mathscr{A}_i \otimes \mathscr{L}^{-2\,l_i} < 0$ is negative. In order to apply Theorem 6.3', first of all, we need an

Observation 6.4. *There exist some positive integers $\widetilde{s}_1, \dots, \widetilde{s}_{c+r}, \ell \geqslant 1$ and $d \geqslant \mathsf{d}_0$ such that:*

$$\mathscr{L}_i^{\widetilde{s}_i} \;=\; \widetilde{\mathscr{A}}_i \otimes \widetilde{\mathscr{L}} \otimes \widetilde{\mathscr{L}}^d, \qquad \mathscr{L}_i^{\widetilde{s}_i+1} \;=\; \widetilde{\mathscr{B}}_i \otimes \widetilde{\mathscr{L}} \otimes \widetilde{\mathscr{L}}^d \qquad (i = 1 \cdots c+r),$$

where $\widetilde{\mathscr{L}} := \mathscr{L}^\ell$ is very ample, and where $\widetilde{\mathscr{A}}_i \otimes \widetilde{\mathscr{L}}$, $\widetilde{\mathscr{B}}_i \otimes \widetilde{\mathscr{L}}$ are very ample, and where $\widetilde{\mathscr{A}}_i \otimes \widetilde{\mathscr{L}}^{-2}$, $\widetilde{\mathscr{B}}_i \otimes \widetilde{\mathscr{L}}^{-2}$ are negative.

Proof. First, by Remark 6.2, we may assume that $d_1 = \cdots = d_{c+r} = d \geqslant \mathsf{d}_0$.

Next, we may assume that $l_1 = \cdots = l_{c+r} = l$. Otherwise, we can choose a positive integer l which is divisible by $l_1, \dots, l_{c+r}$, then we receive/rewrite:

$$\mathscr{L}_i^{s_i\,l/l_i} = \big(\mathscr{A}_i \otimes \mathscr{L}^{l_i} \otimes \mathscr{L}^{l_i\,d}\big)^{l/l_i} = \mathscr{A}_i^{l/l_i} \otimes \mathscr{L}^{l} \otimes \mathscr{L}^{l\,d} \qquad (i = 1 \cdots c+r),$$

while $\mathscr{A}_i^{l/l_i} \otimes \mathscr{L}^l = \big(\mathscr{A}_i \otimes \mathscr{L}^{l_i}\big)^{l/l_i}$ remains very ample and also $\mathscr{A}_i^{l/l_i} \otimes \mathscr{L}^{-2\,l} = \big(\mathscr{A}_i \otimes \mathscr{L}^{-2\,l_i}\big)^{l/l_i} < 0$.

Lastly, we can choose one large integer $m \gg 1$ such that, for all $i = 1 \cdots c+r$, not only $\mathscr{L}_i \otimes (\mathscr{A}_i \otimes \mathscr{L}^l)^m$ are very ample, but also $\mathscr{L}_i \otimes (\mathscr{A}_i \otimes \mathscr{L}^{-2\,l})^m < 0$ are negative. Thus, the following data:

$$\ell \;:=\; m\,l, \quad \widetilde{s}_i \;:=\; m\,s_i, \quad \widetilde{\mathscr{A}}_i \;:=\; \mathscr{A}_i^m, \quad \widetilde{\mathscr{B}}_i \;:=\; \mathscr{L}_i \otimes \mathscr{A}_i^m \qquad (i = 1 \cdots c+r)$$

satisfy the claimed observation. □

Now, we can set:

$$\mathsf{d} \;=\; \mathsf{d}(\mathscr{L}_1, \dots, \mathscr{L}_{c+r}, \mathscr{L}) \;=\; \mathsf{max}_{1 \leqslant i \leqslant c+r}\,\{\widetilde{s}_i\,(\widetilde{s}_i - 1)\}. \tag{44}$$

For any integers $d_1, \dots, d_{c+r} \geqslant \mathsf{d}$, all of them can be written as:

$$d_i \;=\; p_i\,\widetilde{s}_i + q_i\,(\widetilde{s}_i + 1) \qquad (i = 1 \cdots c+r)$$

for some integers $p_i, q_i \geqslant 0$. Let every:

$$F_i := f_1^i \cdots f_{p_i}^i\, f_{p_i+1}^i \cdots f_{p_i+q_i}^i \;\in\; \mathsf{H}^0\,(\mathbf{P},\, \mathscr{L}_i^{d_i})$$

be a product of some sections:

$$f_1^i, \dots, f_{p_i}^i \in \mathsf{H}^0(\mathbf{P}, \mathscr{L}_i^{\widetilde{s}_i}), \quad f_{p_i+1}^i, \dots, f_{p_i+q_i}^i \in \mathsf{H}^0(\mathbf{P}, \mathscr{L}_i^{\widetilde{s}_i+1})$$

to be chosen, then the product coup reveals the decomposition:

$$\begin{aligned}{}_{F_{c+1},\dots,F_{c+r}}\mathbb{P}_{F_1,\dots,F_c} = \bigcup_{k=0\cdots c} \bigcup_{1\leqslant i_1<\cdots<i_k\leqslant c} \bigcup_{\substack{1\leqslant v_{i_j}\leqslant p_{v_j}+q_{v_j}\\ j=1\cdots k}} \bigcup_{\substack{\{r_1,\dots,r_{c-k}\}=\{1,\dots,c\}\setminus\{i_1,\dots,i_k\}\\ 1\leqslant w^1_{r_l}<w^2_{r_l}\leqslant p_{r_l}+q_{r_l}\\ l=1\cdots c-k}} \bigcup_{\substack{1\leqslant u_j\leqslant p_j+q_j\\ j=c+1\cdots c+r}}\\ {}_{f^{r_1}_{w^1_{r_1}},f^{r_1}_{w^2_{r_1}},\dots,f^{r_{c-k}}_{w^1_{r_{c-k}}},f^{r_{c-k}}_{w^2_{r_{c-k}}},f^{c+1}_{u_{c+1}},\dots,f^{c+r}_{u_{c+r}}}\mathbb{P}_{f^{i_1}_{v_{i_1}},\dots,f^{i_k}_{v_{i_k}}}.\end{aligned}$$

Now, applying Theorem 6.3', for generic choices of $\{f_\bullet^\bullet\}$, the Serre line bundle $\mathscr{O}_{\mathbb{P}(\Omega_{\mathbf{P}})}(1)$ is ample on every subscheme ${}_{f^{r_1}_{w^1_{r_1}},f^{r_1}_{w^2_{r_1}},\dots,f^{r_{c-k}}_{w^1_{r_{c-k}}},f^{r_{c-k}}_{w^2_{r_{c-k}}},f^{c+1}_{u_{c+1}},\dots,f^{c+r}_{u_{c+r}}}\mathbb{P}_{f^{i_1}_{v_{i_1}},\dots,f^{i_k}_{v_{i_k}}}$, and therefore is also ample on their union ${}_{F_{c+1},\dots,F_{c+r}}\mathbb{P}_{F_1,\dots,F_c}$. Since ampleness is a generic property in family, we conclude the proof. □

6.8. **Effective lower degree bound N^{N^2} of Theorem 1.2.** Now, we provide an effective degree estimate of Theorem 1.4 in the case $\mathscr{L}_1 = \cdots = \mathscr{L}_{c+r} = \mathscr{L}$.

When $N = 1, 2$, Theorem 1.4 holds trivially for $\mathtt{d} = N^{N^2}$. When $N \geqslant 3$, denote the trivial line bundle on $\mathbf{P}$ by $\mathbf{0}_{\mathbf{P}}$. Note that in Observation 6.4 we can take $\widetilde{s}_1 = \cdots = \widetilde{s}_{c+r} = \mathtt{d}_0 + 1$, $\ell = 1$, so that:

$$\mathscr{L}^{\mathtt{d}_0+1} = \mathbf{0}_{\mathbf{P}} \otimes \mathscr{L} \otimes \mathscr{L}^{\mathtt{d}_0}, \qquad \mathscr{L}^{\mathtt{d}_0+2} = \mathscr{L} \otimes \mathscr{L} \otimes \mathscr{L}^{\mathtt{d}_0}$$

satisfy the requirements. Thus by (44) we can set:

$$\mathtt{d} = \mathtt{d}(\mathscr{L}) = \mathtt{max}_{1\leqslant i\leqslant c+r}\{\widetilde{s}_i(\widetilde{s}_i - 1)\} = \mathtt{d}_0(\mathtt{d}_0 + 1) = (N^{N^2/2} - 1)N^{N^2/2} < N^{N^2}.$$

In particular, when $r = 0$, we recover Theorem 1.2.

7. Some Technical Details

7.1. **Surjectivity of evaluation maps.** Recalling the notation in Definition 3.1, at every closed point $z \in \mathbf{P}$, for every tangent vector $\xi \in \mathrm{T}_{\mathbf{P}}\big|_z$, we can choose any local trivialization (U, s) of the line bundle $\mathscr{S}$ near point z, and then evaluate S, dS at (z, ξ) by:

$$\begin{aligned} S(z)(U,s) &:= S/s(z) \in \mathbb{K},\\ dS(z,\xi)(U,s) &:= \mathrm{d}(S/s)(z,\xi) \in \mathbb{K}.\end{aligned}$$

If (U, s') is another local trivialization of $\mathscr{S}$, then we have the transition formula:

$$\begin{pmatrix} S\\ dS\end{pmatrix}(z,\xi)(U,s) = \underbrace{\begin{pmatrix} s'/s & 0\\ \mathrm{d}(s'/s) & s'/s\end{pmatrix}}_{\text{invertible}} \cdot \begin{pmatrix} S\\ dS\end{pmatrix}(z,\xi)(U,s') \tag{45}$$

Thanks to the above identity, in assertions which do not depend on the particular choice of (U, s), we can just write $S(z), dS(z, \xi)$ by dropping (U, s).

Proposition 7.1. *Let $\mathscr{S}$ be a very-ample line bundle over a smooth $\mathbb{K}$-variety $\mathbf{P}$. Then one has:*

(i) *at every closed point $z \in \mathbf{P}$, for any nonzero tangent vector $0 \neq \xi \in \mathrm{T}_{\mathbf{P}}\big|_z$, the evaluation map:*

$$\begin{aligned}\begin{pmatrix} v_z\\ d_z(\xi)\end{pmatrix} : \mathsf{H}^0(\mathbf{P}, \mathscr{S}) &\longrightarrow \mathbb{K}^2\\ S &\longmapsto (S(z), dS(z,\xi))^{\mathrm{T}}\end{aligned}$$

is surjective;

(ii) *at every closed point $z \in \mathbf{P}$, for any $N = \dim \mathbf{P}$ linearly independent tangent vectors $\xi_1, \dots, \xi_N \in \mathrm{T}_{\mathbf{P}}\big|_z$, the evaluation map:*

$$\begin{pmatrix} v_z \\ d_z(\xi_1) \\ \vdots \\ d_z(\xi_N) \end{pmatrix} : \mathsf{H}^0(\mathbf{P}, \mathscr{S}) \longrightarrow \mathbb{K}^{N+1}$$

$$S \longmapsto (S(z), dS(z,\xi_1), \dots, dS(z,\xi_N))^{\mathrm{T}}$$

is surjective.

Proof. We have the following three elementary observations.

(1) By transition formula (45), property **(i)** is independent of the choice of local trivialization (U, s) of $\mathscr{S}$ near z, so it makes sense.

(2) In any fixed local trivialization (U, s) of $\mathscr{S}$ near z, by basic linear algebra, properties **(i)**, **(ii)** are equivalent to each other.

(3) Property **(i)** is the usual property of 'very-ampleness'.

Thus we may conclude the proof by the reasoning 'very-ampleness' $\Longrightarrow$ **(i)** $\Longleftrightarrow$ **(ii)**. □

Proposition 7.2. *Let $\mathscr{S}$ be a very ample line bundle over a smooth $\mathbb{K}$-variety $\mathbf{P}$, and let $\mathscr{A}$ be any line bundle over $\mathbf{P}$ with a nonzero section $A \neq 0$. Then, at every closed point $z \in \mathsf{D}(A) \subset \mathbf{P}$, for any nonzero tangent vector $0 \neq \xi \in \mathrm{T}_{\mathbf{P}}\big|_z$, the evaluation map:*

$$\begin{pmatrix} A \cdot v_z \\ d_z(A\cdot)(\xi) \end{pmatrix} : \mathsf{H}^0(\mathbf{P}, \mathscr{S}) \longrightarrow \mathbb{K}^2$$

$$S \longmapsto \Big((A \cdot S)(z), d(A \cdot S)(z,\xi)\Big)^{\mathrm{T}}$$

is surjective.

Proof. It is a direct consequence of the formula:

$$\begin{pmatrix} A \cdot v_z \\ d_z(A\cdot)(\xi) \end{pmatrix} = \underbrace{\begin{pmatrix} A(z) & 0 \\ dA(z,\xi) & A(z) \end{pmatrix}}_{\text{invertible since } A(z) \neq 0} \cdot \underbrace{\begin{pmatrix} v_z \\ d_z(\xi) \end{pmatrix}}_{\text{surjective}} \qquad \text{[Leibniz's rule]}$$

and of the preceding proposition. □

7.2. **Bertini-type assertions.** Recalling (26) and that for all $i = 1 \cdots c+r$ the line bundles $\mathscr{A}_i \otimes \mathscr{L}^{\epsilon_i}$ are very ample, we now fulfill the *step 2* in the proof of Theorem 6.3. We start with

Observation 1. 'Smooth complete' is a Zariski open condition in family.

Observation 2. We only need to prove that, for generic choices of $A_1^\bullet, M_1^{\bullet;\bullet}$, the hypersurface $H_1 = \{F_1 = 0\} \subset \mathbf{P}$ is smooth complete.

Proof. Indeed, replacing $\mathbf{P}$ by H_1, we can repeat the same argument to choose $A_2^\bullet, M_2^{\bullet;\bullet}$, and so on. Thus we know that there exists at least one choice of parameters $A_\bullet^\bullet, M_\bullet^{\bullet;\bullet}$ such that X, V are both smooth complete. Immediately, by Observation 1 above, it holds for generic choices of parameters.

Next, to show that generically ${}_{v_1,\dots,v_\eta}X$ is smooth complete, we can start with ${}_{v_1,\dots,v_\eta}\mathbf{P}$ instead of $\mathbf{P}$, and use the same reasoning to conclude the proof. □

Observation 3. We can first set all $M_1^{\bullet;\bullet} = 0$, and then thanks to the following proposition, we can find some appropriate $A_1^\bullet$ such that H_1 is smooth complete. Thus we finish the proof of *step 2*.

Proposition 7.3. *Let* $\mathbf{P}$ *be a smooth* $\mathbb{K}$*-variety of dimension* N*, and let* $\mathscr{A}$*,* $\mathscr{B}$ *be two line bundles over* $\mathbf{P}$*. Assume that* $\mathscr{A}$ *is very ample, and that* $\mathscr{B}$ *has* $N+1$ *global sections* $B_0, \ldots, B_N$ *having empty common base locus. Then, for generic choices of parameters* $A_0, \ldots, A_N \in \mathsf{H}^0(\mathbf{P}, \mathscr{A})$*, the section:*

$$F \;=\; \sum_{j=0}^{N} A_j B_j \;\in\; \mathsf{H}^0\,(\mathbf{P},\, \mathscr{A} \otimes \mathscr{B})$$

defines a smooth complete subvariety.

Proof. Denoting $\mathscr{M} := \oplus_{j=0}^{N} \mathsf{H}^0(\mathbf{P}, \mathscr{A})$, then $\mathbb{P}(\mathscr{M})$ stands for the projective parameter space of $t = (A_0, \ldots, A_N)$. Now we introduce the universal subvariety:

$$\mathsf{S} \;:=\; \Big\{([t], z) :\; F_t(z) = 0,\, dF_t\,(z, \xi) = 0,\, \forall\, \xi \in \mathrm{T}_{\mathbf{P}}\Big|_z \Big\} \;\subset\; \mathbb{P}(\mathscr{M}) \times \mathbf{P}$$

consisting of singular points. We claim that $\dim \mathsf{S} < \dim \mathbb{P}(\mathscr{M})$. It suffices to show that, for every closed point $z \in \mathbf{P}$, the fibre $\mathsf{S}_z \subset \mathbb{P}(\mathscr{M}) \times \{z\} \cong \mathbb{P}(\mathscr{M})$ over z satisfies that $\mathrm{codim}\, \mathsf{S}_z > \dim \mathbf{P}$.

Indeed, choose N linearly independent tangent vector $\xi_1, \ldots, \xi_N$ at point z, and then consider the formal $\mathbb{K}$-linear map:

$$\begin{aligned} \mathsf{ev} :\; \mathscr{M} &\longrightarrow \mathbb{K}^{N+1} \\ t &\longmapsto (F_t(z), dF_t\,(z, \xi_1), \ldots, dF_t\,(z, \xi_N))^{\mathrm{T}}. \end{aligned}$$

By Proposition 7.2, ev is surjective. Note that $\mathsf{S}_z \subset \mathbb{P}(\mathscr{M})$ consists of points $[t] \in \mathbb{P}(\mathscr{M})$ such that $\mathsf{ev}(t) = 0$. Thus we see that:

$$\mathrm{codim}\, \mathsf{S}_z \;=\; \underline{\mathrm{codim}\,(\{0\} \subset \mathbb{K}^{N+1}) \;=\; N+1} \;>\; N \;=\; \dim \mathbf{P}. \qquad \square$$

We will see in the proof of Proposition 7.6 that the *Core Lemma* of MCM plays the same role as that of the above underlined codimension equality/estimate.

7.3. **Emptiness of the base loci.** Recalling (41), (42), in order to characterize the base loci

$$\mathsf{BS},\; {}_{\nu_1,\ldots,\nu_\eta}\mathsf{BS} \;\subset\; \mathbb{P}(\Omega_{\mathbf{P}}),$$

we introduce the following subvarieties (cf. [5, p. 62, (148)]):

$$\mathscr{M}_b^a \;\subset\; \mathsf{Mat}_{b\times 2(a+1)}(\mathbb{K}) \qquad (\forall\, 2 \leqslant a \leqslant b)$$

consisting of all $b \times 2(a+1)$ matrices $(\alpha_0 \,|\, \alpha_1 \,|\, \cdots \,|\, \alpha_a \,|\, \beta_0 \,|\, \beta_1 \,|\, \cdots \,|\, \beta_a)$ such that:

(i) the sum of all $2a+2$ colums is zero:

$$\alpha_0 + \alpha_1 + \cdots + \alpha_a + \beta_0 + \beta_1 + \cdots + \beta_a = \mathbf{0}; \tag{46}$$

(ii) for every $\nu = 0 \cdots a$, there holds the rank inequality:

$$\mathrm{rank}_{\mathbb{K}}\,\{\alpha_0, \ldots, \widehat{\alpha_\nu}, \ldots, \alpha_a, \alpha_\nu + (\beta_0 + \beta_1 + \cdots + \beta_a)\} \leqslant a - 1; \tag{47}$$

(iii) for every $\tau = 0 \cdots a-1$, for every $\rho = \tau + 1 \cdots a$, there holds:

$$\mathrm{rank}_{\mathbb{K}}\,\{\alpha_0 + \beta_0, \alpha_1 + \beta_1, \ldots, \alpha_\tau + \beta_\tau, \alpha_{\tau+1}, \ldots, \widehat{\alpha_\rho}, \ldots, \alpha_a, \alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_a)\} \leqslant a - 1. \tag{48}$$

From now on, we only consider the closed points in each scheme. For instance, we shall regard:

$$\mathbb{P}(\Omega_{\mathbf{P}}) \;=\; \Big\{(z,[\xi]) \;:\; \forall\, z\in\mathbf{P},\, \xi\in \mathrm{T}_{\mathbf{P}}\big|_z\Big\}.$$

By the same reasoning as in [5, Proposition 9.3], we get:

Proposition 7.4. *For generic choices of parameters $A_\bullet^\bullet, M_\bullet^{\vdots\bullet}$, a point:*

$$(z,[\xi]) \;\in\; \mathbb{P}(\Omega_{\mathbf{P}}) \setminus \{\zeta_0\cdots\zeta_N\neq 0\}$$

lies in BS *if and only if:*

$$\text{[recall (29)]} \qquad \mathsf{M}(z,\xi)\in \mathscr{M}_{2c+r}^N. \qquad \square$$

Now, we introduce the engine of MCM (slightly different from the original [5, Lemma 9.5]):

Core Lemma. *For all positive integers $2\leqslant a\leqslant b$, there hold the codimension estimates:*

$$\operatorname{codim}\mathscr{M}_b^a \;\geqslant\; a+b-1. \qquad \square$$

'Naive proof'. First of all, the equation (46) eliminates the first variable column:

$$\alpha_0 \;=\; -(\alpha_1+\cdots+\alpha_a+\beta_0+\beta_1+\cdots+\beta_a),$$

so it contributes codimension value b. Next, denoting $S_i:=\sum_{j=i}^a\beta_j$ for $i=0\cdots a$, we may rewrite the restriction (47) as:

$$\operatorname{rank}_{\mathbb{K}}\{\alpha_0,\ldots,\widehat{\alpha_\nu},\ldots,\alpha_a,\alpha_\nu+S_0\}\leqslant a-1 \qquad (\nu=0\cdots a).$$

By (46), the sum of all columns above vanishes, hence we can drop the first column and state it equivalently as:

$$\operatorname{rank}_{\mathbb{K}}\{\alpha_1,\ldots,\widehat{\alpha_\nu},\ldots,\alpha_a,\alpha_\nu+S_0\}\leqslant a-1 \qquad (\nu=0\cdots a). \tag{49}$$

Similarly, we can reformulate (48) equivalently as:

$$\operatorname{rank}_{\mathbb{K}}\{\alpha_1+S_1-S_2,\ldots,\alpha_\tau+S_\tau-S_{\tau+1},\alpha_{\tau+1},\ldots,\widehat{\alpha_\rho},\ldots,\alpha_a,\alpha_\rho+S_{\tau+1}\}\leqslant a-1 \qquad (\tau=0\cdots a-1,\ \rho=\tau+1\cdots a). \tag{50}$$

Observe in (49), (50) that the variable columns $S_0,\ldots,S_a$ have distinct status, and moreover that for $i=1\cdots a$, subsequently, each variable S_i satisfies nontrivial new equations involving only the former variables $\alpha_\bullet, S_1,\ldots,S_{i-1}$. Thus, the restrictions (49), (50) shoud contribute at least $a+1$ codimension value. Summarizing, we should have:

$$\operatorname{codim}\mathscr{M}_b^a \;\geqslant\; b+(a+1) \;\geqslant\; a+b-1. \qquad \square$$

Remark 7.5. However, a rigorous proof (cf. [5, Subsection 10.6]) is much more demanding and delicate, because of the unexpected algebraic complexity behind (cf. [5, Subsection 10.7]).

Thereby, we can exclude positive-dimensional base locus in Proposition 7.4.

Proposition 7.6. *For generic choices of parameters $A_\bullet^\bullet, M_\bullet^{\vdots\bullet}$, the base locus over the 'coordinates nonvanishing part':*

$$\mathsf{BS}\setminus\{\zeta_0\cdots\zeta_N\neq 0\}$$

is discrete or empty.

Proof. The proof goes much the same as that of Proposition 7.3, in which the underlined codimension estimate is replaced by:

$$\begin{aligned} \operatorname{codim} \mathscr{M}^N_{2c+r} &\geqslant N + 2c + r - 1 && \text{[by the Core Lemma]} \\ \text{[use } 2c + r \geqslant N\text{]} \qquad &\geqslant N + N - 1 \\ \text{[exercise]} \qquad &= \dim \mathbb{P}(\Omega_{\mathbf{P}}) \\ \text{[☺]} \qquad &= \dim \left(\mathbb{P}(\Omega_{\mathbf{P}}) \setminus \{\zeta_0 \cdots \zeta_N \neq 0\}\right). \end{aligned}$$

For the remaining details, we refer the reader to [5, Propositions 9.6, 9.7]. □

This is exactly the first emptiness assertion on the base loci in Subsection 6.5. By much the same reasoning, we can also establish the second one there (cf. [5, Proposition 9.11]).

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Appendix

Submitted Concise Version of the First Part

ON THE AMPLENESS OF THE COTANGENT BUNDLES OF COMPLETE INTERSECTIONS

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Abstract. For the intersection family $\mathscr{X}$ of general Fermat-type hypersurfaces in $\mathbb{P}^N_{\mathbb{K}}$ defined over an algebraically closed field $\mathbb{K}$, we extend Brotbek's symmetric differential forms by a geometric approach, and we further exhibit unveiled families of lower degree symmetric differential forms on all possible intersections of $\mathscr{X}$ with coordinate hyperplanes.

Thereafter, we develop what we call the '*moving coefficients method*' to prove a conjecture made by Olivier Debarre: *for a generic choice of $c \geqslant N/2$ hypersurfaces $H_1, \dots, H_c \subset \mathbb{P}^N_{\mathbb{C}}$ of degrees $d_1, \dots, d_c$ sufficiently large, the intersection $X := H_1 \cap \cdots \cap H_c$ has ample cotangent bundle Ω_X*, and concerning effectiveness, the lower bound $d_1, \dots, d_c \geqslant N^{N^2}$ works.

Lastly, thanks to known results about the Fujita Conjecture, we establish the very-ampleness of $\mathsf{Sym}^\kappa \Omega_X$ for all $\kappa \geqslant \kappa_0$, with a uniform lower bound $\kappa_0 = 64 \left(\sum_{i=1}^c d_i \right)^2$.

1. Introduction

The goal of this article is to answer the Debarre Ampleness Conjecture.

Theorem 1.1. *The cotangent bundle Ω_X of the complete intersection $X := H_1 \cap \cdots \cap H_c \subset \mathbb{P}^N_{\mathbb{C}}$ of $c \geqslant N/2$ generic hypersurfaces $H_1, \dots, H_c$ with degrees $\geqslant N^{N^2}$ is ample.*

Precisely, ampleness means that, for all large $k \geqslant \mathtt{k}_0 \gg 1$, for all pairs of distinct points $x, y \in X$, the two evaluation maps:

$$\mathsf{H}^0\left(X,\, \mathsf{Sym}^k \Omega_X\right) \twoheadrightarrow \mathsf{Jet}_1\, \mathsf{Sym}^k \Omega_X\Big|_x, \quad \mathsf{H}^0\left(X, \mathsf{Sym}^k \Omega_X\right) \twoheadrightarrow \mathsf{Sym}^k \Omega_X\Big|_x \oplus \mathsf{Sym}^k \Omega_X\Big|_y$$

are both surjective, where $\mathsf{Jet}_1 E\Big|_x := \mathscr{O}_x(E) \Big/ (\mathfrak{m}_x)^2\, \mathscr{O}_x(E)$ for every vector bundle E over X.

The hypothesis $c \geqslant N/2$ appears *optimal*, for otherwise $\mathsf{H}^0(X, \mathsf{Sym}^k \Omega_X) = 0$ for all $k \geqslant 1$, according to Brückmann-Rackwitz [3] and Schneider [15].

As highlighted in [4], projective algebraic varieties X with ample cotangent bundles have several interesting properties, for instance, the canonical line bundles of all subvarieties of X are ample; there are finitely many nonconstant rational maps from any fixed projective variety to X ([14]); if X is defined over $\mathbb{C}$, then every holomorphic map $\mathbb{C} \to X$ must be constant ([9]); if X is defined over a number field K, the K-rational points of X are conjectured by Lang to be finite ([10], [13]).

In [1], Brotbek reached a proof of the Debarre Ampleness Conjecture for complex surfaces: $\mathsf{dim}_{\mathbb{C}} X = N - c = 2$. Recently, Brotbek [2] constructed explicit global symmetric differential

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forms in coordinates by a cohomological approach, and thereby proved the Debarre Ampleness Conjecture in the case $4c \geqslant 3\,N - 2$ for equal degrees $d_1 = \cdots = d_c \geqslant 2N + 3$.

In this article, we work over an algebraically closed field $\mathbb{K}$ with any characteristic. Our main theorem is the following, which coincides with Theorem 1.1 for $r = 0$ and $\mathbb{K} = \mathbb{C}$.

Theorem 1.2 (**Ampleness**). *For all positive integers $N \geqslant 1$, for any nonnegative integers c, $r \geqslant 0$ with $2c + r \geqslant N$, for all large integers $d_1, \ldots, d_c, d_{c+1}, \ldots, d_{c+r} \geqslant N^{N^2}$, for generic choices of $c + r$ hypersurfaces $H_1, \ldots, H_{c+r} \subset \mathbb{P}^N_{\mathbb{K}}$ with the corresponding degrees $d_1, \ldots, d_{c+r}$, the cotangent bundle Ω_V of the intersection of the first c hypersurfaces $V := H_1 \cap \cdots \cap H_c$ restricted to the intersection of all the $c + r$ hypersurfaces $X := H_1 \cap \cdots \cap H_c \cap H_{c+1} \cap \cdots \cap H_{c+r}$ is ample.*

When $2\,(2c + r) \geqslant 3N - 2$, we obtain a linear bound for equal degrees $d_1 = \cdots = d_{c+r} \geqslant 2N + 3$, hence we recover the previous lower bound of Brotbek in the case $r = 0$, and we also obtain a quadratic bound for all large degrees $d_1, \ldots, d_{c+r} \geqslant (3N + 2)\,(3N + 3)$. More information on the lower degree bound N^{N^2} will be given in Section 12.

Lastly, taking account of known results about the Fujita Conjecture in Complex Geometry (cf. the survey [5]), we will prove in Section 13 the following

Theorem 1.3 (**Effective Very Ampleness**). *Under the same assumption and notation as in the Ampleness Theorem 1.2, if in addition the ambient field $\mathbb{K}$ has characteristic zero, then for generic choices of $H_1, \ldots, H_{c+r}$, the restricted cotangent bundle $\mathsf{Sym}^\kappa\, \Omega_V\big|_X$ is very ample on X, for every $\kappa \geqslant \kappa_0$, with the uniform lower bound $\kappa_0 = 16\Big(\sum_{i=1}^{c} d_i + \sum_{i=1}^{c+r} d_i\Big)^2$.*

Concerning organization, we provide basic preliminaries in Sections 2, 3, 4. In Section 5, we present the proof blueprint of Theorem 1.2, which extends the previous method of [2] by adding new ingredients in four aspects as follows.

As a matter of fact, our initial idea is to reconstruct Brotbek's symmetric differential forms [2, Lemma 4.5] by means of a geometric approach, which turns out to be fruitful in that, not only we obtain symmetric forms on intersections of general Fermat-type hypersurfaces, but also in that we discover unveiled symmetric forms on all possible further intersections with coordinate hyperplanes — see Section 6.

Fundamentally, our central idea is to design more 'flexible' hypersurfaces than the pure Fermat-type ones of [2], so that we can construct many more negatively twisted symmetric differential forms. To this aim, we develop the *moving coefficients method* (MCM) in Section 7. Then, we devote Section 8 to control the base locus of the obtained symmetric forms over the part where all coordinates are nonvanishing. Lastly, the ultimate difficulty lies in the *Core Lemma*, whose proof constitutes the technical heart of this article, and which will be accomplished in Section 11.

Playing an essential role, another key idea is to make use of symmetric differential forms constructed over the part where some coordinates vanish, a new feature which removes the last obstacle towards ampleness, and that therefore determines the shape of our desired hypersurfaces — see Section 9.

Surprisingly, one neat idea, which we term *product coup*, enables us to construct ample examples of intersections of hypersurfaces with any large degrees. This settles the Debarre Ampleness Conjecture and shapes the formulation of our main theorem — see Subsection 5.4.

Finally, we achieve the effective degree estimates of Theorem 1.2 in Section 12.

For shortness, we will often refer to the long version [16] for skipped details.

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2. **Restatements of the Ampleness Theorem 1.2**

Recalling the projective parameter space of $c+r$ hypersurfaces with degrees $d_1, \ldots, d_{c+r} \geqslant 1$ is:

$$\mathbb{P}\Big(\oplus_{i=1}^{c+r}\ \mathsf{H}^0(\mathbb{P}^N_{\mathbb{K}}, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(d_i))\Big) = \mathbb{P}\Big\{\oplus_{i=1}^{c+r}\ \sum_{|\alpha|=d_i} A^i_\alpha\, z^\alpha\ :\ A^i_\alpha \in \mathbb{K}\Big\},$$

for shortness we write it as:

$$\mathbb{P}^\diamond_{\mathbb{K}} = \operatorname{Proj} \mathbb{K}\big[\{A^i_\alpha\}_{\substack{1\leqslant i\leqslant c+r\\ |\alpha|=d_i}}\big]. \tag{1}$$

First, we introduce the two subschemes:

$$\mathscr{X}\ \subset\ \mathscr{V}\ \subset\ \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}},$$

where $\mathscr{X}$ is the family of intersections of all the $c+r$ hypersurfaces:

$$\mathscr{X} := \mathsf{V}\Big(\sum_{|\alpha|=d_1} A^1_\alpha\, z^\alpha, \ldots, \sum_{|\alpha|=d_c} A^c_\alpha\, z^\alpha, \sum_{|\alpha|=d_{c+1}} A^{c+1}_\alpha\, z^\alpha, \ldots, \sum_{|\alpha|=d_{c+r}} A^{c+r}_\alpha\, z^\alpha\Big), \tag{2}$$

and where $\mathscr{V}$ is the family of intersections of the first c hypersurfaces:

$$\mathscr{V} := \mathsf{V}\Big(\sum_{|\alpha|=d_1} A^1_\alpha\, z^\alpha, \ldots, \sum_{|\alpha|=d_c} A^c_\alpha\, z^\alpha\Big). \tag{3}$$

Now, let $\mathrm{pr}_1, \mathrm{pr}_2$ be the two canonical projections of $\mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}}$ onto its two components. By restricting $\mathrm{pr}_1\big|_{\mathscr{V}}\colon \mathscr{V} \to \mathbb{P}^\diamond_{\mathbb{K}}$, we receive the sheaf $\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}}$ of relative differentials of degree 1 of $\mathscr{V}$ over $\mathbb{P}^\diamond_{\mathbb{K}}$, which should be viewed as the family of cotangent bundles of the intersections. Then, we consider the projectivization $\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}})$, equipped with the *Serre line bundle* $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}})}(1)$ on it. Recalling the cotangent bundle $\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}$ of $\mathbb{P}^N_{\mathbb{K}}$ and its projectivization $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$, now looking at:

$$\begin{array}{ccccc}
\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}}) & \xhookrightarrow{\widetilde{i}} & \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) & \xrightarrow{\widetilde{\pi}_2^*} & \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) \\
\downarrow & & \downarrow & & \\
\mathscr{V} & \xhookrightarrow{\ i\ } & \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}}, & &
\end{array} \tag{4}$$

we see that $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}})}(1)$ is the pullback of $\widetilde{\pi}_2^*\, \mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ by $\widetilde{i}$, where $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ is the Serre line bundle of $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$.

Next, let $\widetilde{\pi}\colon\ \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) \to \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}}$ be the canonical projection, and let π_1, π_2 be the compositions of $\widetilde{\pi}$ with $\mathrm{pr}_1, \mathrm{pr}_2$:

$$\begin{array}{ccccc} & & \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) & & \\ & \swarrow^{\pi_1:=\mathrm{pr}_1\circ\widetilde{\pi}} & \downarrow{\scriptstyle \widetilde{\pi}} & \searrow^{\pi_2:=\mathrm{pr}_2\circ\widetilde{\pi}} & \\ & & \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}} & & \\ \mathbb{P}^\diamond_{\mathbb{K}} & \xleftarrow{\mathrm{pr}_1} & & \xrightarrow{\mathrm{pr}_2} & \mathbb{P}^N_{\mathbb{K}}. \end{array} \tag{5}$$

Let:

$$\mathbf{P} \,:=\, \widetilde{\pi}^{-1}(\mathscr{X}) \cap \mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}}) \,\subset\, \mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}}) \,\subset\, \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) \tag{6}$$

be the 'pullback' of $\mathscr{X} \subset \mathscr{V} \subset \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}}$ under the map $\widetilde{\pi}$, and let:

$$\mathscr{O}_{\mathbf{P}}(1) \,:=\, \mathscr{O}_{\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}})}(1)\Big|_{\mathbf{P}} \,=\, \widetilde{\pi}_2^*\, \mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)\Big|_{\mathbf{P}} \tag{7}$$

be the restricted Serre line bundle. Then, Theorem 1.2 can be reformulated as below, with the assumption that the hypersurface degrees are sufficiently large $d_1, \ldots, d_{c+r} \gg 1$.

Theorem 1.2 (Version A). *For a generic point $t \in \mathbb{P}^\diamond_{\mathbb{K}}$, over the fibre $\mathbf{P}_t := \pi_1^{-1}(t) \cap \mathbf{P}$, the restricted Serre line bundle $\mathscr{O}_{\mathbf{P}_t}(1) := \mathscr{O}_{\mathbf{P}}(1)\big|_{\mathbf{P}_t}$ is ample.*

We will abbreviate every closed point $t = [\{A^i_\alpha\}_{1\leqslant i\leqslant c+r}] \in \mathbb{P}^\diamond_{\mathbb{K}}$ as $t = [F_1 \colon \cdots \colon F_{c+r}]$, where $F_i := \sum_{|\alpha|=d_i} A^i_\alpha\, z^\alpha$. Then $\mathbf{P}_t = \{t\} \times_{\mathbb{K}\ F_{c+1},\ldots,F_{c+r}} \mathbf{P}_{F_1,\ldots,F_c}$ for a uniquely defined subscheme:

$$_{F_{c+1},\ldots,F_{c+r}}\mathbf{P}_{F_1,\ldots,F_c} \,\subset\, \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}). \tag{8}$$

Theorem 1.2 (Version B). *For a generic closed point $[F_1 \colon \cdots \colon F_{c+r}] \in \mathbb{P}^\diamond_{\mathbb{K}}$, the Serre line bundle $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ is ample on $_{F_{c+1},\ldots,F_{c+r}}\mathbf{P}_{F_1,\ldots,F_c}$.*

To understand better the above statements, we now show the geometric picture.

3. The Background Geometry

For each scheme mentioned above, we consider its underlying topological space ($\mathbb{K}$-variety) that consists of all the closed points.

3.1. **The geometry of $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$ and $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)$.** Recalling the tangent space of $\mathbb{P}^N_{\mathbb{K}}$ at every point $[z]$ is $\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}\big|_{[z]} = \mathbb{K}^{N+1}/\mathbb{K}\cdot z$, thus the total tangent space of $\mathbb{P}^N_{\mathbb{K}}$ is $\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}} := \mathrm{T}_{\mathrm{hor}}\mathbb{K}^{N+1}/\sim$, where:

$$\mathrm{T}_{\mathrm{hor}}\mathbb{K}^{N+1} := \Big\{(z,[\xi]) \colon\ z \in \mathbb{K}^{N+1}\setminus\{0\} \text{ and } [\xi] \in \mathbb{K}^{N+1}/\mathbb{K}\cdot z\Big\}, \tag{9}$$

and where the quotient relation is $(z,[\xi]) \sim (\lambda z, [\lambda\xi])$, for all $\lambda \in \mathbb{K}^\times$.

Note that the $\mathbb{K}$-variety associated to $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$ is just the projectivized tangent space $\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$. Also, the Serre line bundle $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ on $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$ corresponds to the Serre line bundle $\mathscr{O}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ on $\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$, whose dual is the tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})}(-1)$ defined by:

$$\mathscr{O}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})}(-1)\Big|_{([z],[\xi])} := \mathbb{K}\cdot[\xi] \,\subset\, \mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}\Big|_{[z]} = \mathbb{K}^{N+1}\Big/\mathbb{K}\cdot z \qquad (\forall\,([z],[\xi])\in\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})).$$

3.2. **The geometry of $\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}})$, $\mathbf{P}$ and $\mathbf{P}_t$.** Recalling (3), the $\mathbb{K}$-variety of $\mathscr{V}$ is:

$$\mathbb{V} := \Big\{([F_1,\dots,F_{c+r}],[z]) \in \mathbb{P}^\diamond_{\mathbb{K}} \times \mathbb{P}^N_{\mathbb{K}} : \ F_i(z) = 0,\ \forall\, i = 1\cdots c\Big\}.$$

Moreover, recalling (4), the $\mathbb{K}$-variety of $\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}}) \subset \mathbb{P}^\diamond_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$ is:

$$\mathbb{P}(\mathrm{T}_{\mathbb{V}/\mathbb{P}^\diamond_{\mathbb{K}}}) := \Big\{([F_1,\dots,F_{c+r}],([z],[\xi])) \in \mathbb{P}^\diamond_{\mathbb{K}} \times \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}) : \ F_i(z) = 0,\ dF_i\big|_z(\xi) = 0,\ \forall\, i = 1\cdots c\Big\}.$$

Similarly, the $\mathbb{K}$-variety $\mathbb{P} \subset \mathbb{P}(\mathrm{T}_{\mathbb{V}/\mathbb{P}^\diamond_{\mathbb{K}}})$ associated to $\mathbf{P} \subset \mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^\diamond_{\mathbb{K}}})$ is:

$$\mathbb{P} := \Big\{([F_1,\dots,F_{c+r}],([z],[\xi])) : \ F_i(z) = 0,\ dF_j\big|_z(\xi) = 0,\ \forall\, i = 1\cdots c+r, \forall\, j = 1\cdots c\Big\},$$

and the $\mathbb{K}$-variety ${}_{F_{c+1},\dots,F_{c+r}}\mathbb{P}_{F_1,\dots,F_c} \subset \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$ associated to (8) is:

$${}_{F_{c+1},\dots,F_{c+r}}\mathbb{P}_{F_1,\dots,F_c} := \Big\{([z],[\xi]) : \ F_i(z) = 0,\ dF_j\big|_z(\xi) = 0,\ \forall\, i = 1\cdots c+r,\ \forall\, j = 1\cdots c\Big\}. \tag{10}$$

4. Some Hints on the Ampleness Theorem 1.2

4.1. **Ampleness is Zariski open.** The foundation of our approach is the following theorem due to Grothendieck (cf. [6, III.4.7.1], [11, p. 29, Theorem 1.2.17]).

Theorem 4.1. [Amplitude in families] *Let $f : X \to T$ be a proper morphism of schemes, and let $\mathscr{L}$ be a line bundle on X. For every point $t \in T$, denote by $X_t := f^{-1}(t)$ and $\mathscr{L}_t := \mathscr{L}\big|_{X_t}$. If $\mathscr{L}_0$ is ample on X_0 for some point $0 \in T$, then there exists a Zariski open set $U \subset T$ containing 0 such that $\mathscr{L}_t$ is ample on X_t, for all $t \in U$.*

Note that in (5), $\pi_1 = \mathrm{pr}_1 \circ \pi$ is proper, and so is its restriction to $\mathbf{P}$. Therefore, by virtue of the above theorem, one only needs to find one point $t \in \mathbb{P}^\diamond_{\mathbb{K}}$ such that $\mathscr{O}_{\mathbf{P}_t}(1)$ is ample on $\mathbf{P}_t$.

4.2. **Largely twisted Serre line bundle is (very) ample.** Let $\pi_0 \colon \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) \to \mathbb{P}^N_{\mathbb{K}}$ be the canonical projection. By a classical result [8, p. 161, Proposition 7.10], for all large integers $\ell \gg 1$, the twisted line bundle $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1) \otimes \pi_0^*\,\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell)$ is ample on $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$.

In fact, one can check by hand that, for all $\ell \geqslant 3$, the following global sections:

$$z_k\, z_j^{\ell-1}\, d\Big(\frac{z_i}{z_j}\Big) \qquad (i,j,k = 0\cdots N, i \neq j) \tag{11}$$

even guarantee the very-ampleness.

Consequently, when $\ell \geqslant 3$, for every point $t \in \mathbb{P}^\diamond_{\mathbb{K}}$, the positively twisted Serre line bundle $\mathscr{O}_{\mathbf{P}_t}(1) \otimes \pi_2^*\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell)$ is (very) ample on $\mathbf{P}_t = \pi_1^{-1}(t) \cap \mathbf{P}$.

4.3. **Nefness of negatively twisted cotangent sheaf suffices.** Thanks to the above ampleness, now we may relax the ampleness goal mentioned below Theorem 4.1.

Theorem 4.2. *For every point $t \in \mathbb{P}^\diamond_{\mathbb{K}}$, the following properties are equivalent.*

- **(i)** *$\mathscr{O}_{\mathbf{P}_t}(1)$ is ample on $\mathbf{P}_t$.*
- **(ii)** *There exist two positive integers $a, b \geqslant 1$ such that $\mathscr{O}_{\mathbf{P}_t}(a) \otimes \pi_2^*\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-b)$ is ample on $\mathbf{P}_t$.*
- **(iii)** *There exist two positive integers $a, b \geqslant 1$ such that $\mathscr{O}_{\mathbf{P}_t}(a) \otimes \pi_2^*\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-b)$ is nef on $\mathbf{P}_t$.*

Proof. **(i)** $\Longrightarrow$ **(ii)** $\Longrightarrow$ **(iii)** is clear. To show **(iii)** $\Longrightarrow$ **(i)**, noting that "ample $\otimes$ nef = ample" (cf. [11, p. 53, Corollary 1.4.10]), one can use any ample positively twisted Serre line bundle to compensate the negative twisted degree of the nef line bundle. □

Repeating the same reasoning, we may obtain:

Proposition 4.3. *For every point $t \in \mathbb{P}^\diamond_{\mathbb{K}}$, if $\mathscr{O}_{\mathbf{P}_t}(\ell_1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-\ell_2)$ is nef on $\mathbf{P}_t$ for some positive integers $\ell_1, \ell_2 \geqslant 1$, then for any positive integers $\ell_1', \ell_2' \geqslant 1$ with $\ell_2'/\ell_1' < \ell_2/\ell_1$, the twisted line bundle $\mathscr{O}_{\mathbf{P}_t}(\ell_1') \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-\ell_2')$ is ample on $\mathbf{P}_t$.* □

By definition, the nefness of $\mathscr{S}_t^a(-b) := \mathscr{O}_{\mathbf{P}_t}(a) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-b)$ means that, for every irreducible curve $C \subset \mathbf{P}_t$, the intersection number $C \cdot \mathscr{S}_t^a(-b)$ is $\geqslant 0$. Recalling now a classical result [8, p. 295, Lemma 1.2], one only needs to show $\mathsf{H}^0(C, \mathscr{S}_t^a(-b)) \neq \{0\}$. To this end, one would desire sufficiently many global nonzero sections $s_1, \ldots, s_m \in \mathsf{H}^0(\mathbf{P}_t, \mathscr{S}_t^a(-b))$ such that their base locus is empty or discrete, whence one of $s_1\big|_C, \ldots, s_m\big|_C$ suffices to conclude.

More flexibly, without fixing a and b, we have

Theorem 4.4. *Suppose that there exist some $m \geqslant 1$ nonzero sections:*

$$s_i \in \mathsf{H}^0(\mathbf{P}_t, \mathscr{S}_t^{a_i}(-b_i)) \qquad (i = 1 \cdots m; a_i, b_i \geqslant 1)$$

such that their base locus is discrete or empty, then for all positive integers a, b with:

$$b/a \leqslant \min\{b_1/a_1, \ldots, b_m/a_m\},$$

the negatively twisted Serre line bundle $\mathscr{S}_t^a(-b)$ is nef. □

5. A Proof Blueprint of the Ampleness Theorem 1.2

5.1. **Main Nefness Theorem.** Recalling Theorem 4.2, the Ampleness Theorem 1.2 is a consequence of the theorem below, whose effective bound $\mathsf{d}_0(\heartsuit)$ for $\heartsuit = 1$ will be given in Theorem 12.2.

Theorem 5.1. *Given any positive integer $\heartsuit \geqslant 1$, there exists a lower degree bound $\mathsf{d}_0(\heartsuit) \gg 1$ such that, for all degrees $d_1, \ldots, d_{c+r} \geqslant \mathsf{d}_0(\heartsuit)$, for a very generic[1] $t \in \mathbb{P}^\diamond_{\mathbb{K}}$, the negatively twisted Serre line bundle $\mathscr{O}_{\mathbf{P}_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-\heartsuit)$ is nef on $\mathbf{P}_t$.*

It suffices to find one such $t \in \mathbb{P}^\diamond_{\mathbb{K}}$ to guarantee'very generic' (cf. [11, p. 56, Proposition 1.4.14]).

We will prove Theorem 5.1 in two steps. At first, in Subsection 5.3, we sketch the proof in the central cases when all $c + r$ hypersurfaces are approximately of the same large degrees. Then, in Subsection 5.4, we play a *product coup* to embrace all large degrees.

5.2. **A refined nefness criterion.** In practice, it is delicate to gather enough global sections with empty/discrete base locus to guarantee the desired nefness. To overcome this difficulty, we will also use nonzero sections of the same bundle restricted to proper subvarieties, and such an idea will be a key feature of our approach compared with that of [2].

Definition 5.2. Let X be a variety, and let $Y \subset X$ be a subvariety. A line bundle $\mathscr{L}$ on X is said to be *nef outside Y* if, for every irreducible curve $C \subset X$ with $C \not\subset Y$, the intersection number $C \cdot \mathscr{L}$ is $\geqslant 0$.

A standard reasoning on irreducibility and Noetherian topology gives ([16, p. 16, Theorem 4.6])

Theorem 5.3. *Let X be a Noetherian variety, and let $\mathscr{L}$ be a line bundle on X. Assume that there exists a set $\mathscr{P}$ of closed subvarieties of X satisfying:*

(i) *$\emptyset \in \mathscr{P}$ and $X \in \mathscr{P}$;*

[1] $t \in \mathbb{P}^\diamond_{\mathbb{K}} \setminus \cup_{i=1}^\infty Z_i$ for some countable proper subvarieties $Z_i \subsetneq \mathbb{P}^\diamond_{\mathbb{K}}$.

(ii) *for every element $Y \in \mathscr{P}$ with $Y \neq \emptyset$, there exist finitely many elements $Z_1, \ldots, Z_m \in \mathscr{P}$ with $Z_1, \ldots, Z_m \subsetneq Y$ such that the restricted line bundle $\mathscr{L}\big|_Y$ is nef outside the union $Z_1 \cup \cdots \cup Z_m$.*

Then $\mathscr{L}$ is nef on X. □

5.3. **The central cases of relatively the same large degrees.**

Theorem 5.4. *For any fixed $c + r$ positive integers $\epsilon_1, \ldots, \epsilon_{c+r} \geqslant 1$, for every sufficiently large integer $d \gg 1$, Theorem 5.1 holds with $d_i = d + \epsilon_i$, $i = 1 \cdots c + r$.*

When $c + r \geqslant N$, generically X is discrete or empty, so there is nothing to prove. Assuming $c + r \leqslant N - 1$, we now outline the proof.

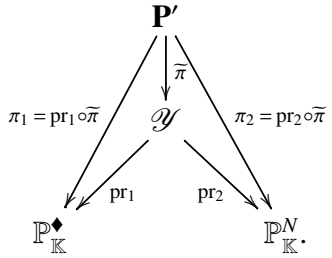
Step 1. In the entire family of $c+r$ hypersurfaces with degrees $d+\epsilon_1, \ldots, d+\epsilon_{c+r}$, whose projective parameter space is $\mathbb{P}^{\diamond}_{\mathbb{K}}$, we will work with some specific subfamily which suits our coming *moving coefficients method*, and whose projective parameter space will be denoted by $\mathbb{P}^{\blacklozenge}_{\mathbb{K}} \subset \mathbb{P}^{\diamond}_{\mathbb{K}}$. The presentation of $\mathbb{P}^{\blacklozenge}_{\mathbb{K}}$ will be made partially in Section 7, and then completely in Subsection 9.2.

Recalling (2) and (5), we then consider the subfamily of intersections $\mathscr{Y} \subset \mathscr{X}$:

$$\mathrm{pr}_1^{-1}\,(\mathbb{P}^{\blacklozenge}_{\mathbb{K}}) \cap \mathscr{X} =: \mathscr{Y} \subset \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}} = \mathrm{pr}_1^{-1}\,(\mathbb{P}^{\blacklozenge}_{\mathbb{K}}).$$

Recalling (5), (6), we introduce the subscheme $\mathbf{P}' := \widetilde{\pi}^{-1}(\mathscr{Y}) \cap \mathbf{P} \subset \mathbf{P}$. By restriction, (5) yields the commutative diagram:

(12)



Introducing the restricted Serre line bundle $\mathscr{O}_{\mathbf{P}'}(1) := \mathscr{O}_{\mathbf{P}}(1)\big|_{\mathbf{P}'}$ over $\mathbf{P}'$, in order to establish Theorem 5.4, it suffices to prove the

Theorem 5.5. *For a generic closed point $t \in \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$, the bundle $\mathscr{O}_{\mathbf{P}'_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-\blacktriangledown)$ is nef on $\mathbf{P}'_t := \mathbf{P}_t$.*

Step 2. The central objects now are the *universal negatively twisted Serre line bundles*:

$$\mathscr{O}_{\mathbf{P}'}(a, b, -c) := \mathscr{O}_{\mathbf{P}'}(a) \otimes \pi_1^*\, \mathscr{O}_{\mathbb{P}^{\blacklozenge}_{\mathbb{K}}}(b) \otimes \pi_2^*\, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-c),$$

where a, c are positive integers such that $c/a \geqslant \blacktriangledown$, and where b are any integers.

Using the moving coefficients method, we will firstly construct a series of global universal negatively twisted symmetric differential n-forms:

$$S_\ell \in \mathsf{H}^0(\mathbf{P}',\, \mathscr{O}_{\mathbf{P}'}(n, N, -\heartsuit_\ell)) \qquad (\ell = 1 \cdots \maltese), \tag{13}$$

where $n := N - (c + r) \geqslant 1$ and all $\heartsuit_\ell / n \geqslant \blacktriangledown$, and where *we always use* the symbol '$\maltese$' to denote auxiliary positive integers, which vary according to the context.

Secondly, for every integer $1 \leqslant \eta \leqslant n - 1$, for every $0 \leqslant v_1 < \cdots < v_\eta \leqslant N$, considering:

$$_{v_1,\ldots,v_\eta}\mathbf{P}' := \mathbf{P}' \cap \pi_2^{-1}\{z_{v_1} = \cdots = z_{v_\eta} = 0\},$$

we will construct a series of universal negatively twisted symmetric differential $(n - \eta)$-forms:

$$_{v_1,\ldots,v_\eta}S_\ell \in \mathsf{H}^0(_{v_1,\ldots,v_\eta}\mathbf{P}',\, \mathscr{O}_{\mathbf{P}'}(n - \eta, N - \eta, -\,_{v_1,\ldots,v_\eta}\heartsuit_\ell)) \qquad (\ell = 1 \cdots \maltese), \tag{14}$$

with all ${}_{v_1,\dots,v_\eta}\heartsuit_\ell/(n-\eta) \geqslant \blacktriangledown$.

This step will be accomplished partially in Section 7, and then completely in Section 9, and thus determines the *Algorithm* of MCM in Subsection 12.1.

Step 3. From now on, we only consider the closed points of every scheme.

Firstly, we will control the base locus of all the global sections obtained in (13):

$$\mathsf{BS} := \text{Base Locus of } \{S_\ell\}_{1\leqslant\ell\leqslant\maltese} \quad \subset \quad \mathbf{P}' \tag{15}$$

by proving that, on the 'coordinates nonvanishing part' $\mathring{\mathbf{P}}' := \mathbf{P}' \cap \pi_2^{-1}\{z_0\cdots z_N \neq 0\}$, there holds:

$$\dim\, \mathsf{BS} \,\cap\, \mathring{\mathbf{P}}' \,\leqslant\, \dim \mathbb{P}^{\blacklozenge}_{\mathbb{K}}. \tag{16}$$

Secondly, for $\eta = 1\cdots n-1$, we will control the base locus of all the sections obtained in (14):

$${}_{v_1,\dots,v_\eta}\mathsf{BS} := \text{Base Locus of } \{{}_{v_1,\dots,v_\eta}S_\ell\}_{1\leqslant\ell\leqslant\maltese} \quad \subset \quad {}_{v_1,\dots,v_\eta}\mathbf{P}'.$$

by proving that, on ${}_{v_1,\dots,v_\eta}\mathring{\mathbf{P}}' := {}_{v_1,\dots,v_\eta}\mathbf{P}' \cap \pi_2^{-1}\{z_{r_0}\cdots z_{r_{N-\eta}} \neq 0\}$, where:

$$\{r_0,\dots,r_{N-\eta}\} \,:=\, \{0,\dots,N\} \,\setminus\, \{v_1,\dots,v_\eta\}\,, \tag{17}$$

there holds:

$$\dim\, {}_{v_1,\dots,v_\eta}\mathsf{BS} \,\cap\, {}_{v_1,\dots,v_\eta}\mathring{\mathbf{P}}' \,\leqslant\, \dim \mathbb{P}^{\blacklozenge}_{\mathbb{K}}. \tag{18}$$

This step will be performed mainly by the *Core Lemma* in Section 8.

Step 4. For the regular map $\pi_1\colon \mathbf{P}' \to \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$, noting the dimension estimates (16), (18), applying now a classical theorem [7, p. 132, Theorem 11.12], we know that there exists a proper subvariety $\Sigma \subsetneqq \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$ such that, for every closed point $t \in \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus \Sigma$:

(i) the base locus of the restricted symmetric differential n-forms:

$$\mathsf{BS}_t := \text{Base Locus of } \{S_\ell(t) \,:=\, S_\ell\big|_{\mathbf{P}'_t}\}_{1\leqslant\ell\leqslant\maltese} \quad \subset \quad \mathbf{P}'_t$$

is discrete or empty over the coordinates nonvanishing part:

$$\dim\, \mathsf{BS}_t \,\cap\, \mathring{\mathbf{P}}'_t \,\leqslant\, 0, \tag{19}$$

where $\mathring{\mathbf{P}}'_t := \mathring{\mathbf{P}}' \cap \pi_1^{-1}(t)$;

(ii) for $\eta = 1\cdots n-1$, the base locus of the restricted symmetric differential $(n-\eta)$-forms:

$${}_{v_1,\dots,v_\eta}\mathsf{BS}_t := \text{Base Locus of } \left\{{}_{v_1,\dots,v_\eta}S_\ell(t) \,:=\, {}_{v_1,\dots,v_\eta}S_\ell\Big|_{{}_{v_1,\dots,v_\eta}\mathbf{P}'_t}\right\}_{1\leqslant\ell\leqslant\maltese} \quad \subset \quad {}_{v_1,\dots,v_\eta}\mathbf{P}'_t$$

is discrete or empty over the coordinates nonvanishing part:

$$\dim\, {}_{v_1,\dots,v_\eta}\mathsf{BS}_t \,\cap\, {}_{v_1,\dots,v_\eta}\mathring{\mathbf{P}}'_t \,\leqslant\, 0, \tag{20}$$

where ${}_{v_1,\dots,v_\eta}\mathring{\mathbf{P}}'_t := {}_{v_1,\dots,v_\eta}\mathring{\mathbf{P}}' \cap \pi_1^{-1}(t)$.

Lastly, for $\eta = n$, there exists a proper subvariety $\Sigma' \subsetneqq \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$ such that, for every closed point $t \in \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus \Sigma'$, the fibre $\mathscr{Y}_t := \mathscr{Y} \cap \mathrm{pr}_1^{-1}(t)$ — corresponding to the intersection of the $c+r$ hypersurfaces defined by t in $\mathbb{P}^N_{\mathbb{K}}$ — is of dimension $n = N-(c+r)$, and moreover all intersections:

$$\mathscr{Y}_t \,\cap\, \mathrm{pr}_2^{-1}\big({}_{v_1,\dots,v_n}\mathbb{P}^N\big) \,=\, \{t\} \,\times\, {}_{v_1,\dots,v_n}\mathscr{Y}_t \qquad (0\leqslant v_1<\cdots<v_n\leqslant N) \tag{21}$$

are just finitely many points.

Now, we shall conclude Theorem 5.5 for every closed point $t \in \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus (\Sigma \cup \Sigma')$.

Proof of Theorem 5.5. For the line bundle $\mathscr{L} = \mathscr{O}_{\mathbf{P}'_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-\heartsuit)$ over the variety $\mathbf{P}'_t$, we claim that the set of subvarieties:

$$\mathscr{V} := \left\{ \emptyset,\ \mathbf{P}'_t,\ {}_{v_1,\dots,v_\eta}\mathbf{P}'_t \right\}_{\substack{1 \leqslant \eta \leqslant n \\ 0 \leqslant v_1 < \cdots < v_\eta \leqslant N}}$$

satisfies the conditions of Theorem 5.3.

Indeed, firstly, recalling (19), the sections $\{S_\ell(t)\}_{\ell=1\cdots\maltese}$ have empty/discrete base locus over the coordinates nonvanishing part, i.e. outside $\cup_{j=0}^N {}_j\mathbf{P}'_t$. Hence, using an adaptation of Theorem 4.4, remembering $\heartsuit/1 \leqslant \min\{\diamondsuit_\ell/n\}_{1\leqslant\ell\leqslant\maltese}$, the line bundle $\mathscr{O}_{\mathbf{P}'_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N}(-\heartsuit)$ is nef outside $\cup_{j=0}^N {}_j\mathbf{P}'_t$.

Secondly, for every integer $\eta = 1 \cdots n-1$, recalling the dimension estimate (20), again by Theorem 4.4, remembering $\heartsuit/1 \leqslant \min\{{}_{v_1,\dots,v_\eta}\diamondsuit_\ell/(n-\eta)\}_{1\leqslant\ell\leqslant\maltese}$, the line bundle $\mathscr{O}_{\mathbf{P}'_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N}(-\heartsuit)$ is nef on ${}_{v_1,\dots,v_\eta}\mathbf{P}'_t$ outside $\cup_{j=0}^{N-\eta} {}_{v_1,\dots,v_\eta,r_j}\mathbf{P}'_t$ (see (17)).

Lastly, for $\eta = n$, noting that under the projection $\pi\colon \mathbf{P}'_t \to \mathscr{Y}_t$, thanks to (21), every subvariety ${}_{v_1,\dots,v_n}\mathbf{P}'_t$ contracts to discrete points ${}_{v_1,\dots,v_n}\mathscr{Y}_t$, we see that on ${}_{v_1,\dots,v_n}\mathbf{P}'_t$, the line bundle $\mathscr{O}_{\mathbf{P}'_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-\heartsuit) \cong \mathscr{O}_{\mathbf{P}'_t}(1)$ is not only nef, but also ample!

Summarizing the above three parts, by Theorem 5.3, we conclude the proof. □

5.4. **Product Coup.** We will use in an essential way Theorem 5.4 with all ϵ_i equal to either 1 or 2. To begin with, we need an elementary

Observation 5.6. *For all positive integers $d \geqslant 1$, every integer $d_0 \geqslant d^2 + d$ is a sum of nonnegative multiples of $d+1$ and $d+2$.* □

Proof of Theorem 5.1. Take one sufficiently large integer d such that Theorem 5.4 holds for any integers $\epsilon_i \in \{1,2\}$, $i = 1 \cdots c+r$. Now, the above observation says that all large degrees $d_1, \dots, d_{c+r} \geqslant d^2 + d$ can be written as $d_i = p_i(d+1) + q_i(d+2)$, with some nonnegative integers $p_i, q_i \geqslant 0$, $i = 1 \cdots c+r$. Let $F_i := f_1^i \cdots f_{p_i}^i f_{p_i+1}^i \cdots f_{p_i+q_i}^i$ be a product of some p_i homogeneous polynomials $f_1^i, \dots, f_{p_i}^i$ each of degree $d+1$ and of some q_i homogeneous polynomials $f_{p_i+1}^i, \dots, f_{p_i+q_i}^i$ each of degree $d+2$, so that F_i has degree d_i.

Recalling (10), a point $([z],[\xi]) \in \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$ lies in ${}_{F_{c+1},\dots,F_{c+r}}\mathbb{P}_{F_1,\dots,F_c}$ if and only if:

$$F_i(z) = 0,\ dF_j\big|_z(\xi) = 0 \qquad (\forall\, i = 1 \cdots c+r, \forall\, j = 1 \cdots c).$$

Note that, for every $j = 1 \cdots c$, the pair of equations:

$$F_j(z) = 0,\ dF_j\big|_z(\xi) = 0 \tag{22}$$

is equivalent to either:

$$\exists\ 1 \leqslant v_j \leqslant p_j + q_j \quad s.t. \quad f_{v_j}^j(z) = 0,\ df_{v_j}^j\big|_z(\xi) = 0, \tag{23}$$

or to:

$$\exists\ 1 \leqslant w_j^1 < w_j^2 \leqslant p_j + q_j \quad s.t. \quad f_{w_j^1}^j(z) = 0,\ f_{w_j^2}^j(z) = 0. \tag{24}$$

Therefore, $([z],[\xi]) \in {}_{F_{c+1},\dots,F_{c+r}}\mathbb{P}_{F_1,\dots,F_c}$ is equivalent to say that there exists a subset $\{i_1,\dots,i_k\} \subset \{1,\dots,c\}$ of cardinality k ($k=0$ for $\emptyset$) such that, firstly, for every index $j \in \{i_1 \cdots i_k\}$, (z,ξ) is a solution of (22) of type (23), secondly, for every index $j \in \{1,\dots,c\} \setminus \{i_1 \cdots i_k\}$, (z,ξ) is a solution of (22) of type (24), and lastly, for every $j = c+1 \cdots c+r$, one of $f_1^j, \dots, f_{p_j+q_j}^j$ vanishes at z. Thus,

we see that the variety ${}_{F_{c+1},\ldots,F_{c+r}}\mathbb{P}_{F_1,\ldots,F_c}$ actually decomposes into a union of subvarieties:

$$ {}_{F_{c+1},\ldots,F_{c+r}}\mathbb{P}_{F_1,\ldots,F_c} \;=\; \cup_{k=0\cdots c}\, \cup_{1\leqslant i_1<\cdots<i_k\leqslant c}\, \cup_{\substack{1\leqslant v_{i_j}\leqslant p_{v_j}+q_{v_j}\\ j=1\cdots k}}\, \cup_{\substack{\{r_1,\ldots,r_{c-k}\}=\{1,\ldots,c\}\setminus\{i_1,\ldots,i_k\}\\ 1\leqslant w^1_{r_l}<w^2_{r_l}\leqslant p_{r_l}+q_{r_l}\\ l=1\cdots c-k}} \cup_{\substack{1\leqslant u_j\leqslant p_j+q_j\\ j=c+1\cdots c+r}} $$
$$ {}_{f^{r_1}_{w^1_{r_1}},f^{r_1}_{w^2_{r_1}},\ldots,f^{r_{c-k}}_{w^1_{r_{c-k}}},f^{r_{c-k}}_{w^2_{r_{c-k}}},f^{c+1}_{u_{c+1}},\ldots,f^{c+r}_{u_{c+r}}}\mathbb{P}_{f^{i_1}_{v_{i_1}},\ldots,f^{i_k}_{v_{i_k}}}\,. $$

Similarly, we can show that the scheme ${}_{F_{c+1},\ldots,F_{c+r}}\mathbf{P}_{F_1,\ldots,F_c}$ also decomposes into a union of subschemes:

$$ {}_{F_{c+1},\ldots,F_{c+r}}\mathbf{P}_{F_1,\ldots,F_c} \;=\; \cup_{k=0\cdots c}\, \cup_{1\leqslant i_1<\cdots<i_k\leqslant c}\, \cup_{\substack{1\leqslant v_{i_j}\leqslant p_{v_j}+q_{v_j}\\ j=1\cdots k}}\, \cup_{\substack{\{r_1,\ldots,r_{c-k}\}=\{1,\ldots,c\}\setminus\{i_1,\ldots,i_k\}\\ 1\leqslant w^1_{r_l}<w^2_{r_l}\leqslant p_{r_l}+q_{r_l}\\ l=1\cdots c-k}} \cup_{\substack{1\leqslant u_j\leqslant p_j+q_j\\ j=c+1\cdots c+r}} $$
$$ {}_{f^{r_1}_{w^1_{r_1}},f^{r_1}_{w^2_{r_1}},\ldots,f^{r_{c-k}}_{w^1_{r_{c-k}}},f^{r_{c-k}}_{w^2_{r_{c-k}}},f^{c+1}_{u_{c+1}},\ldots,f^{c+r}_{u_{c+r}}}\mathbf{P}_{f^{i_1}_{v_{i_1}},\ldots,f^{i_k}_{v_{i_k}}}\,. $$

Note that, for each subscheme on the right hand side, the number of polynomials on the lower-left of '$\mathbf{P}$' is $\#_L = 2(c-k)+r$, and the number of polynomials on the lower-right is $\#_R = k$, whence $2\#_R+\#_L = 2c+r \geqslant N$. Now, applying Theorem 5.4, we can choose one $\{f^\bullet_\bullet\}_{\bullet,\bullet}$ such that the twisted Serre line bundle $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)\otimes\pi_0^*\,\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-\heartsuit)$ is nef on each subscheme ${}_{f^{r_1}_{w^1_{r_1}},f^{r_1}_{w^2_{r_1}},\ldots,f^{r_{c-k}}_{w^1_{r_{c-k}}},f^{r_{c-k}}_{w^2_{r_{c-k}}},f^{c+1}_{u_{c+1}},\ldots,f^{c+r}_{u_{c+r}}}\mathbf{P}_{f^{i_1}_{v_{i_1}},\ldots,f^{i_k}_{v_{i_k}}}$, and therefore is also nef on their union ${}_{F_{c+1},\ldots,F_{c+r}}\mathbf{P}_{F_1,\ldots,F_c}$. Since nefness is a very generic property in family, we conclude the proof. □

6. **Generalizations of Brotbek's Symmetric Differential Forms**

6.1. **Preliminaries on symmetric differential forms in projective space.** For three fixed integers $N \geqslant 2$, $c, r \geqslant 0$ such that $2c + r \geqslant N$ and $c + r \leqslant N - 1$, for $c + r$ positive integers $d_1, \ldots, d_{c+r}$, for all $i = 1 \cdots c + r$, let $H_i \subset \mathbb{P}^N_{\mathbb{K}}$ be $c + r$ hypersurfaces defined by some degree d_i homogeneous polynomials $F_i \in \mathbb{K}[z_0, \ldots, z_N]$. Let V be the intersection of the first c hypersurfaces:

$$ V \;:=\; H_1 \cap \cdots \cap H_c \;=\; \Big\{[z] \in \mathbb{P}^N_{\mathbb{K}} \,:\; F_i(z) = 0,\ \forall\, i = 1 \cdots c\Big\}, \tag{25} $$

and let X be the intersection of all the $c + r$ hypersurfaces:

$$ X \;:=\; H_1 \cap \cdots \cap H_{c+r} \;=\; \Big\{[z] \in \mathbb{P}^N_{\mathbb{K}} \,:\; F_i(z) = 0,\ \forall\, i = 1 \cdots c + r\Big\}. \tag{26} $$

It is well known that, for generic choices of $\{F_i\}_{i=1}^{c+r}$, the intersections $V = \cap_{i=1}^{c} H_i$ and $X = \cap_{i=1}^{c+r} H_i$ are both smooth complete, and we shall assume this henceforth. Set:

$$ n \;:=\; N - (c + r) \;=\; \mathsf{dim}\, X, $$

and observe that $1 \leqslant n \leqslant c$.

Let us denote by $\pi\colon\ \mathbb{K}^{N+1} \setminus \{0\} \to \mathbb{P}^N_{\mathbb{K}}$ the canonical projection. For every $k \in \mathbb{Z}$, over any Zariski open subset $U \subset \mathbb{P}^N_{\mathbb{K}}$, sections in $\mathsf{H}^0\,(U,\, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(k))$ can be defined as regular functions $\widehat{f}$ on $\pi^{-1}(U)$ satisfying $\widehat{f}(\lambda z) = \lambda^k\, \widehat{f}(z)$, for all $z \in \pi^{-1}(U)$ and all $\lambda \in \mathbb{K}^\times$.

For the cone $\widehat{V} := \pi^{-1}(V) = \{z \in \mathbb{K}^{N+1} \setminus \{0\}\colon F_i(z) = 0, \forall\, i = 1 \cdots c\}$ of V, we define the horizontal tangent bundle $\mathrm{T}_{\mathrm{hor}}\widehat{V}$ which has fibre at any point $z \in \widehat{V}$:

$$ \mathrm{T}_{\mathrm{hor}}\widehat{V}\Big|_z = \Big\{[\xi] \in \mathbb{K}^{N+1}\Big/\mathbb{K}\cdot z \,:\; dF_i\Big|_z(\xi) = 0,\ \forall\, i = 1 \cdots c\Big\}. $$

Its total space is:

$$ \mathrm{T}_{\mathrm{hor}}\widehat{V} := \Big\{(z, [\xi]) \,:\; z \in \widehat{V},\, [\xi] \in \mathbb{K}^{N+1}\Big/\mathbb{K}\cdot z,\, dF_i\Big|_z(\xi) = 0,\ \forall\, i = 1 \cdots c\Big\}. \tag{27} $$

Thus the total tangent bundle T_V of V can be viewed as:

$$\mathrm{T}_V = \mathrm{T}_{\mathrm{hor}}\widehat{V}/\sim, \qquad \text{where } (z,[\xi]) \sim (\lambda z, [\lambda\xi]),\ \forall\, \lambda \in \mathbb{K}^\times.$$

Let Ω_V be the dual bundle of T_V, and let $\Omega_{\mathrm{hor}}\widehat{V}$ be the dual bundle of $\mathrm{T}_{\mathrm{hor}}\widehat{V}$. For all $l \geqslant 1$ and for all $\heartsuit \in \mathbb{Z}$, we denote by $\mathsf{Sym}^l \Omega_V(\heartsuit) := \mathsf{Sym}^l \Omega_V \otimes \mathscr{O}_V(\heartsuit)$ the twisted symmetric cotangent bundle.

Proposition 6.1. *Let $Y \subset V$ be any regular subvariety. For every Zariski open set $U \subset Y$ together with its cone $\widehat{U} := \pi^{-1}(U)$, there is a canonical injection:*

$$\mathsf{H}^0(U, \mathsf{Sym}^l \Omega_V(\heartsuit)) \hookrightarrow \mathsf{H}^0(\widehat{U}, \mathsf{Sym}^l \Omega_{\mathrm{hor}}\widehat{V}),$$

whose image is the set of sections Φ satisfying:

$$\Phi(\lambda z, [\lambda\xi]) = \lambda^\heartsuit\, \Phi(z,[\xi]), \tag{28}$$

for all $z \in \widehat{U}$, for all $[\xi] \in \mathrm{T}_{\mathrm{hor}}\widehat{V}\big|_z$ and for all $\lambda \in \mathbb{K}^\times$. □

In future applications, $Y = X$ or $Y = X \cap \{z_{\nu_1} = 0\} \cap \cdots \cap \{z_{\nu_\eta} = 0\}$ for some vanishing coordinate indices $0 \leqslant \nu_1 < \cdots < \nu_\eta \leqslant N$.

6.2. **Symmetric horizontal differential forms.** Now, we introduce the *Fermat-type hypersurfaces* $H_i \subset \mathbb{P}^N_{\mathbb{K}}$ defined by some homogeneous polynomials F_i of the form:

$$F_i = \sum_{j=0}^{N} A_i^j\, z_j^{\lambda_j} \qquad (i = 1\cdots c+r), \tag{29}$$

where $\lambda_0, \ldots, \lambda_N \geqslant 2$ are some positive integers and where $A_i^j \in \mathbb{K}[z_0, z_1, \ldots, z_N]$ are some homogeneous polynomials with $\deg A_i^j + \lambda_j = \deg F_i$ for $j = 0 \cdots N$.

Differentiating F_i for $i = 1 \cdots c$, we receive:

$$\mathrm{d}\, F_i = \sum_{j=0}^{N} \mathsf{B}_i^j\, z_j^{\lambda_j - 1} \qquad (i = 1\cdots c),$$

where $\mathsf{B}_i^j := z_j\, \mathrm{d}\, A_i^j + \lambda_j\, A_i^j\, \mathrm{d}\, z_j$. Now, set $\widetilde{\mathsf{A}}_i^j := A_i^j\, z_j$, so that $F_i = \sum_{j=0}^N \widetilde{\mathsf{A}}_i^j\, z_j^{\lambda_j - 1}$ has the same structure as $\mathrm{d}\, F_i$.

We denote the cone of X by:

$$\widehat{X} := \Big\{ z \in \mathbb{K}^{N+1} \setminus \{0\} :\ F_i(z) = 0,\ \forall\, i = 1 \cdots c + r \Big\}.$$

For all $z \in \widehat{X}$ and for all $[\xi] \in \mathrm{T}_{\mathrm{hor}}\widehat{V}\big|_z$, by the definition (27) of $\mathrm{T}_{\mathrm{hor}}\widehat{V}$, we have:

$$\begin{cases} \sum_{j=0}^N \widetilde{\mathsf{A}}_i^j\, z_j^{\lambda_j - 1}\,(z) = 0 & (i = 1\cdots c+r), \\ \sum_{j=0}^N \mathsf{B}_i^j(z,\xi)\, z_j^{\lambda_j - 1}(z) = 0 & (i = 1\cdots c). \end{cases} \tag{30}$$

Now, we view these $c + r + c$ equations as a linear system with $N + 1$ unknowns $z_0^{\lambda_0 - 1}, \ldots, z_N^{\lambda_N - 1}$:

$$\mathsf{C} \begin{pmatrix} z_0^{\lambda_0-1} \\ \vdots \\ z_N^{\lambda_N-1} \end{pmatrix} = 0, \qquad \text{where} \qquad \mathsf{C} := \begin{pmatrix} \widetilde{\mathsf{A}}_1^0 & \cdots & \widetilde{\mathsf{A}}_1^N \\ \vdots & & \vdots \\ \widetilde{\mathsf{A}}_{c+r}^0 & \cdots & \widetilde{\mathsf{A}}_{c+r}^N \\ \mathsf{B}_1^0 & \cdots & \mathsf{B}_1^N \\ \vdots & & \vdots \\ \mathsf{B}_c^0 & \cdots & \mathsf{B}_c^N \end{pmatrix}. \tag{31}$$

Next, let D be the upper $(c + r + n) \times (N + 1) = N \times (N + 1)$ submatrix of C. For every $j = 0 \cdots N$, let $\widehat{\mathsf{D}}_j$ denote the submatrix of D obtained by omitting the $(j + 1)$-th column, and let D_j denote the $(j + 1)$-th column of D. Introduce the affine $W_j := \{z_j \neq 0\} \subset \mathbb{P}^N_{\mathbb{K}}$ having cone $\widehat{W}_j := \pi^{-1}(W_j) \subset \mathbb{K}^{N+1} \setminus \{0\}$. Denote also $U_j := W_j \cap X$, whose cone is $\widehat{U}_j := \pi^{-1}(U_j) \subset \widehat{X}$. Lastly, let $\Omega_{\mathrm{hor}}\mathbb{K}^{N+1}$ be the dual bundle of $\mathrm{T}_{\mathrm{hor}}\mathbb{K}^{N+1}$, and let $\Omega\,\mathbb{K}^{N+1}$ be the cotangent bundle of $\mathbb{K}^{N+1}$.

Proposition 6.2. *For every $j = 0 \cdots N$, on the affine set $\widehat{W}_j$, the following affine symmetric horizontal differential n-form is well defined:*

$$\widehat{\omega}_j := \frac{(-1)^j}{z_j^{\lambda_j - 1}} \det\left(\widehat{\mathsf{D}}_j\right) \quad \in \quad \mathsf{H}^0\Big(\widehat{W}_j,\, \mathsf{Sym}^n\, \Omega_{\mathrm{hor}}\mathbb{K}^{N+1}\Big). \tag{32}$$

Proof. It suffices to treat the case $j = 0$. First of all, we may view $\widehat{\omega}_0 \in \mathsf{H}^0\Big(\widehat{V}_0,\, \mathsf{Sym}^n\, \Omega\,\mathbb{K}^{N+1}\Big)$, and in order to show $\widehat{\omega}_0 \in \mathsf{H}^0(\widehat{V}_0,\, \mathsf{Sym}^n\, \Omega_{\mathrm{hor}}\mathbb{K}^{N+1})$, we only have to check that:

$$\widehat{\omega}_0(z, \xi + \mu\, z) = \widehat{\omega}_0(z, \xi) \qquad (\forall\, z \in \widehat{V}_0,\, \xi \in \mathbb{K}^{N+1},\, \mu \in \mathbb{K}^\times). \tag{33}$$

Coming back to the definitions of B_i^j and $\widetilde{\mathsf{A}}_i^j$, Euler's identity gives:

$$\mathsf{B}_i^j(z, \xi + \mu\, z) \;=\; \mathsf{B}_i^j(z, \xi) + \mu\,(\lambda_j + \deg A_i^j)\, \widetilde{\mathsf{A}}_i^j(z) \qquad (i = 1 \cdots c,\, j = 0 \cdots N).$$

Therefore, the matrix $\widehat{\mathsf{D}}_0\,(z, \xi + \lambda\, z)$ not only has the same first $c + r$ rows as the matrix $\widehat{\mathsf{D}}_0\,(z, \xi)$, but also for every $\ell = 1 \cdots n$, the $(c+r+\ell)$-th row of the former one is equal to the $(c+r+\ell)$-th row of the latter one plus a multiple of the ℓ-th row. Therefore both matrices have the same determinant, which verifies (33). □

Cramer's Rule. *In a commutative ring R, for all positive integers $N \geqslant 1$, let $A^0, A^1, \ldots, A^N \in R^N$ be $(N + 1)$ column vectors, and suppose that $z_0, z_1, \ldots, z_N \in R$ satisfy:*

$$A^0\, z_0 + A^1\, z_1 + \cdots + A^N\, z_N \;=\; \mathbf{0}.$$

Then for all $0 \leqslant i, j \leqslant N$, there hold the identities:

$$(-1)^j \det\,(A^0, \ldots, \widehat{A^j}, \ldots, A^N)\, z_i = (-1)^i \det\,(A^0, \ldots, \widehat{A^i}, \ldots, A^N)\, z_j. \qquad \square$$

We may thus glue horizontal symmetric differential forms.

Proposition 6.3. *The following $(N + 1)$ affine symmetric horizontal differential n-forms:*

$$\widehat{\omega}_j := \frac{(-1)^j}{z_j^{\lambda_j - 1}} \det\left(\widehat{\mathsf{D}}_j\right) \quad \in \mathsf{H}^0\Big(\widehat{U}_j,\, \mathsf{Sym}^n\, \Omega_{\mathrm{hor}}\widehat{V}\Big) \qquad (j = 0 \cdots N)$$

glue together to make a symmetric horizontal differential n-form on $\widehat{X}$.

Proof. Proposition 6.2 yields that, by restriction, each $\widehat{\omega}_j \in \mathsf{H}^0\big(\widehat{U}_j,\, \mathsf{Sym}^n\, \Omega_{\mathrm{hor}}\widehat{V}\big)$ is well defined. We now show that, for every $0 \leqslant j_1 < j_2 \leqslant N$, the two forms $\widehat{\omega}_{j_1}$ and $\widehat{\omega}_{j_2}$ glue together along $\widehat{U}_{j_1} \cap \widehat{U}_{j_2}$.

Indeed, recalling (31), granted that D consists of the first $(c + r + n)$ rows of C, applying the above Cramer's rule to all the $(N + 1)$ columns of D and the $(N + 1)$ values $z_0^{\lambda_0 - 1}, \dots, z_N^{\lambda_N - 1}$, we receive:

$$(-1)^{j_2} \det\big(\widehat{\mathsf{D}}_{j_2}\big)\, z_{j_1}^{\lambda_{j_1}-1} = (-1)^{j_1} \det\big(\widehat{\mathsf{D}}_{j_1}\big)\, z_{j_2}^{\lambda_{j_2}-1}.$$

When $z_{j_1} \neq 0, z_{j_2} \neq 0$, dividing by $z_{j_1}^{\lambda_{j_1}-1}\, z_{j_2}^{\lambda_{j_2}-1}$ on both sides, we conclude the proof. □

By permuting the indices, the preceding Proposition 6.3 generalizes to any $(c + r + n) \times (N + 1)$ submatrix of C containing the upper $c + r$ rows. Indeed, for every $1 \leqslant j_1 < \cdots < j_n \leqslant c$, denote by $\mathsf{C}_{j_1,\dots,j_n}$ the $(c + r + n) \times (N + 1)$ submatrix of C consisting of the first upper $c + r$ rows and the rows $c + r + j_1, \dots, c + r + j_n$. Also, for $j = 0 \cdots N$, let $\widehat{\mathsf{C}}_{j_1,\dots,j_n;\, j}$ denote the submatrix of $\mathsf{C}_{j_1,\dots,j_n}$ obtained by omitting the $(j + 1)$-th column.

Proposition 6.4. *The following* $(N + 1)$ *affine symmetric horizontal differential n-forms:*

$$\widehat{\omega}_{j_1,\dots,j_n;\, j} := \frac{(-1)^j}{z_j^{\lambda_j - 1}} \det\big(\widehat{\mathsf{C}}_{j_1,\dots,j_n;\, j}\big) \quad \in \mathsf{H}^0\Big(\widehat{U}_j,\, \mathsf{Sym}^n\, \Omega_{\mathrm{hor}}\widehat{V}\Big) \qquad (j = 0 \cdots N)$$

glue together to make a global symmetric horizontal differential n-form $\widehat{\omega}_{j_1,\dots,j_n}$ *on* $\widehat{X}$. □

6.3. **Twisted symmetric differential forms.** Now, applying Proposition 6.1 to the above symmetric forms, we thus generalize [2, Lemma 4.5].

Proposition 6.5. *The symmetric horizontal differential n-form* $\widehat{\omega}_{j_1,\dots,j_n}$ *is the image of a global twisted symmetric differential n-form:*

$$\omega_{j_1,\dots,j_n} \in \mathsf{H}^0(X, \mathsf{Sym}^n\, \Omega_V(\heartsuit))$$

under the canonical injection as a particular case of Proposition 6.1:

$$\mathsf{H}^0(X, \mathsf{Sym}^n\, \Omega_V(\heartsuit)) \hookrightarrow \mathsf{H}^0(\widehat{X}, \mathsf{Sym}^n\, \Omega_{\mathrm{hor}}\widehat{V}),$$

with the twisted degree:

$$\heartsuit := \sum_{p=1}^{c+r} \deg F_p + \sum_{q=1}^{n} \deg F_{j_q} - \sum_{j=0}^{N} \lambda_j + N + 1. \tag{34}$$

For all homogeneous polynomials $P \in \mathsf{H}^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\deg P))$, *by multiplication:*

$$P\, \omega_{j_1,\dots,j_n} \in \mathsf{H}^0(X, \mathsf{Sym}^n\, \Omega_V(\deg P + \heartsuit)).$$ □

Proof. According to the criterion (28), we only need to check, for all $z \in \widehat{X}$, for all $[\xi] \in \mathrm{T}_{\mathrm{hor}}\widehat{V}\big|_z$ and for all $\lambda \in \mathbb{K}^\times$, that:

$$\widehat{\omega}_{j_1,\dots,j_n}(\lambda z, [\lambda\xi]) = \lambda^{\heartsuit}\, \widehat{\omega}_{j_1,\dots,j_n}(z, [\xi]), \tag{35}$$

which follows from direct computations (cf. [16, Proposition 6.10]). □

Now, let K be the $(c+r+c)\times(N+1)$ matrix whose first $c+r$ rows consist of all $(N+1)$ terms in the expressions of $F_1,\dots,F_{c+r}$ in the exact order, i.e. the (i,j)-th entry of K is:

$$\mathsf{K}_{i,j} := A_i^{j-1}\, z_{j-1}^{\lambda_{j-1}} \qquad (i=1\cdots c+r,\, j=1\cdots N+1), \tag{36}$$

and whose last c rows consist of all $(N+1)$ terms in the expressions of $\mathrm{d}F_1,\dots,\mathrm{d}F_c$ in the exact order, i.e. the $(c+r+i,j)$-th entry of K is:

$$\mathsf{K}_{c+r+i,j} := \mathrm{d}\,(A_i^{j-1}\, z_{j-1}^{\lambda_{j-1}}) \qquad (i=1\cdots c,\, j=1\cdots N+1).$$

The j-th column K_j of K and the j-th column C_j of C are proportional:

$$\mathsf{K}_j = \mathsf{C}_j\, z_{j-1}^{\lambda_{j-1}-1} \qquad (j=1\cdots N+1). \tag{37}$$

In later applications, we will be mainly interested in determining the base locus of $\omega_{j_1,\dots,j_n}$ in the coordinates nonvanishing part $\{z_0\cdots z_N\neq 0\}$, thus, using (37), we may compute the corresponding symmetric horizontal differential n-forms $\widehat{\omega}_{j_1,\dots,j_n}$ as:

$$\widehat{\omega}_{j_1,\dots,j_n;\,j} = \frac{(-1)^j}{z_j^{\lambda_j-1}}\det\big(\widehat{\mathsf{C}}_{j_1,\dots,j_n;\,j}\big) = \frac{(-1)^j}{z_0^{\lambda_0-1}\cdots z_N^{\lambda_N-1}}\det\big(\widehat{\mathsf{K}}_{j_1,\dots,j_n;\,j}\big) \qquad (j=0\cdots N), \tag{38}$$

where $\widehat{\mathsf{K}}_{j_1,\dots,j_n;\,j}$ is defined as an analog of $\widehat{\mathsf{C}}_{j_1,\dots,j_n;\,j}$.

6.4. **Twisted symmetric differential forms with some vanishing coordinates.** Investigating further the above construction of symmetric differential forms, for every integer $1\leqslant\eta\leqslant n-1$, for any indices $0\leqslant v_1<\cdots<v_\eta\leqslant N$, by focusing on:

$$_{v_1,\dots,v_\eta}X \;:=\; X\cap\{z_{v_1}=0\}\cap\cdots\cap\{z_{v_\eta}=0\},$$

we can also construct a series of twisted symmetric differential $(n-\eta)$-forms as follows.

For indices $1\leqslant j_1<\cdots<j_{n-\eta}\leqslant c$, denote by ${}_{v_1,\dots,v_\eta}\mathsf{C}_{j_1,\dots,j_{n-\eta}}$ the $(N-\eta)\times(N-\eta+1)$ submatrix of C determined by the first $c+r$ rows and the selected rows $c+r+j_1,\dots,c+r+j_{n-\eta}$ as well as the $(N-\eta+1)$ columns which are complement to the columns $v_1+1,\dots,v_\eta+1$. Next, for every index $j\in\{0,\dots,N\}\setminus\{v_1,\dots,v_\eta\}$, let ${}_{v_1,\dots,v_\eta}\widehat{\mathsf{C}}_{j_1,\dots,j_{n-\eta};\,j}$ denote the submatrix of ${}_{v_1,\dots,v_\eta}\mathsf{C}_{j_1,\dots,j_{n-\eta}}$ obtained by deleting the column which is originally contained in the $(j+1)$-th column of C. Lastly, denote:

$$_{v_1,\dots,v_\eta}U_j \;:=\; {}_{v_1,\dots,v_\eta}X\cap\{z_j\neq 0\},$$

whose cone is ${}_{v_1,\dots,v_\eta}\widehat{U}_j := \pi^{-1}({}_{v_1,\dots,v_\eta}U_j)$.

Now, we have two very analogs of Propositions 6.4 and 6.5. Before, we write the $(N-\eta+1)$ remaining numbers of $\{0,\dots,N\}\setminus\{v_1,\dots,v_\eta\}$ in the ascending order $r_0<r_1<\cdots<r_{N-\eta}$.

Proposition 6.6. *For all $j=0\cdots N-\eta$, the following $(N+1-\eta)$ affine symmetric horizontal differential $(n-\eta)$-forms:*

$${}_{v_1,\dots,v_\eta}\widehat{\omega}_{j_1,\dots,j_{n-\eta};\,r_j} := \frac{(-1)^j}{z_{r_j}^{\lambda_{r_j}-1}}\det\big({}_{v_1,\dots,v_\eta}\widehat{\mathsf{C}}_{j_1,\dots,j_{n-\eta};\,r_j}\big)\;\in\mathsf{H}^0\Big({}_{v_1,\dots,v_\eta}\widehat{U}_{r_j},\,\mathsf{Sym}^{n-\eta}\,\Omega_{\mathrm{hor}}\widehat{V}\Big)$$

glue together to make a symmetric horizontal differential $(n-\eta)$-form:

$${}_{v_1,\dots,v_\eta}\widehat{\omega}_{j_1,\dots,j_{n-\eta}}\;\in\mathsf{H}^0\Big({}_{v_1,\dots,v_\eta}\widehat{X},\,\mathsf{Sym}^{n-\eta}\,\Omega_{\mathrm{hor}}\widehat{V}\Big).$$ □

Proposition 6.7. *The symmetric horizontal differential* $(n-\eta)$*-form* ${}_{v_1,\dots,v_\eta}\widehat{\omega}_{j_1,\dots,j_{n-\eta}}$ *on* ${}_{v_1,\dots,v_\eta}\widehat{X}$ *is the image of a twisted symmetric differential* $(n-\eta)$*-form on* ${}_{v_1,\dots,v_\eta}X$*:*

$$ {}_{v_1,\dots,v_\eta}\omega_{j_1,\dots,j_{n-\eta}} \ \in \ \mathsf{H}^0\left({}_{v_1,\dots,v_\eta}X, \mathsf{Sym}^{n-\eta}\,\Omega_V({}_{v_1,\dots,v_\eta}\heartsuit_{j_1,\dots,j_{n-\eta}})\right) $$

under the canonical injection:

$$ \mathsf{H}^0\left({}_{v_1,\dots,v_\eta}X, \mathsf{Sym}^{n-\eta}\,\Omega_V({}_{v_1,\dots,v_\eta}\heartsuit_{j_1,\dots,j_{n-\eta}})\right) \hookrightarrow \mathsf{H}^0\left({}_{v_1,\dots,v_\eta}\widehat{X}, \mathsf{Sym}^{n-\eta}\,\Omega_{\mathrm{hor}}\widehat{V}\right), $$

with the twisted degree:

$$ {}_{v_1,\dots,v_\eta}\heartsuit_{j_1,\dots,j_{n-\eta}} := \sum_{p=1}^{c+r}\deg F_p + \sum_{q=1}^{n-\eta}\deg F_{j_q} - \Big(\sum_{j=0}^{N}\lambda_j - \sum_{\mu=1}^{\eta}\lambda_{v_\mu}\Big) + (N-\eta) + 1. \tag{39} $$

Furthermore, for all homogeneous polynomials $P \in \mathsf{H}^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\deg P))$*, by multiplication:*

$$ P\,{}_{v_1,\dots,v_\eta}\omega_{j_1,\dots,j_{n-\eta}} \in \mathsf{H}^0\left({}_{v_1,\dots,v_\eta}X, \mathsf{Sym}^{n-\eta}\,\Omega_V\left(\deg P + {}_{v_1,\dots,v_\eta}\heartsuit_{j_1,\dots,j_{n-\eta}}\right)\right). $$ □

In our coming applications, we will be mainly interested in determining the base locus of ${}_{v_1,\dots,v_\eta}\omega_{j_1,\dots,j_{n-\eta}}$ in the coordinates nonvanishing part $\{z_{r_0}\cdots z_{r_{N-\eta}} \neq 0\}$, thus, using (37), we may compute the corresponding symmetric horizontal differential $(n-\eta)$-forms ${}_{v_1,\dots,v_\eta}\widehat{\omega}_{j_1,\dots,j_{n-\eta}}$ as:

$$ {}_{v_1,\dots,v_\eta}\widehat{\omega}_{j_1,\dots,j_{n-\eta};\,r_j} = \frac{(-1)^j}{z_{r_j}^{\lambda_{r_j}-1}}\det\left({}_{v_1,\dots,v_\eta}\widehat{\mathsf{C}}_{j_1,\dots,j_{n-\eta};\,r_j}\right) = \frac{(-1)^j}{z_{r_0}^{\lambda_{r_0}-1}\cdots z_{r_{N-\eta}}^{\lambda_{r_{N-\eta}}-1}}\det\left({}_{v_1,\dots,v_\eta}\widehat{\mathsf{K}}_{j_1,\dots,j_{n-\eta};\,r_j}\right) \qquad (j=0\cdots N-\eta), \tag{40} $$

where ${}_{v_1,\dots,v_\eta}\widehat{\mathsf{K}}_{j_1,\dots,j_{n-\eta};\,r_j}$ is defined as an analog of ${}_{v_1,\dots,v_\eta}\widehat{\mathsf{C}}_{j_1,\dots,j_{n-\eta};\,r_j}$.

The two identities (38), (40) will enable us to efficiently narrow the base loci of the obtained twisted symmetric differential forms, as the matrix K directly copies the original equations/differentials of the hypersurface polynomials $F_1,\dots,F_{c+r}$.

Remark 6.8. We may also give a scheme-theoretic point of view on the obtained twisted symmetric differential forms, over an arbitrary field $\mathbb{K}$, see [16, Subsection 6.5].

7. **Moving Coefficients Method (I)**

7.1. **Motivation and ideas.** It would be natural to try to settle the Debarre Ampleness Conjecture by means of pure Fermat-type hypersurfaces $F_i = \sum_{j=0}^{N} A_i^j z_j^e$ with $\deg A_i^j = \epsilon \geqslant 1$, a strategy which was indeed successful in [2] for the case $4c \geqslant 3N-2$.

However, one could soon realize that this approach would not work, simply because, for instance in the limiting case $N = 2c$ (with $r = 0$), one could only obtain a *single* symmetric form $\omega \in \mathsf{H}^0(X, \mathsf{Sym}^c\,\Omega_X(-\heartsuit))$ for such intersection $X = H_1 \cap \cdots \cap H_c$, since $1 \leqslant j_1 < \cdots < j_n \leqslant c = n$ forces $j_1 = 1,\dots,j_n = n$, whereas $\dim \mathbb{P}(\mathrm{T}_X) = 2\dim X - 1 = N-1 \gg 1$.

To overcome this difficulty, our core idea is to look for more 'flexible' hypersurface equations, such that they can be expressed as general Fermat-type polynomials *in several distinct ways*, while keeping negative twist for all the symmetric differential forms obtained by Proposition 6.5.

To be more precise, assume that $N = 2c$, $r = 0$, and that all hypersurface equations $F_1,\dots,F_c$ have the same degree d. First, every F_i can have a lot of homogeneous monomials, and we arbitrarily gather them in $N+1$ parts as $F_i = \sum_{j=0}^{N} F_i^j$. Next, for every $j = 0\cdots N$, choose the maximum

integer λ_j such that $z_j^{\lambda_j}$ divides $F_1^j, \ldots, F_c^j$, so F_i rewrites under the form $F_i = \sum_{j=0}^N A_i^j z_j^{\lambda_j}$. Hence by applying Proposition 6.5, we receive a symmetric differential form with twisted degree:

$$\heartsuit = N\,\mathtt{d} - \sum_{j=0}^{N} \lambda_j + N + 1 \qquad \text{[use (34)]}.$$

Lastly, we think on how to make $\heartsuit \leqslant -1$. Observe that, although each $\lambda_j \leqslant \mathtt{d}$, yet the total number of $\lambda_\bullet$ is $N+1 > N =$ the multiple number of $\mathtt{d}$, and it is this feature that leaves some room to design the *moving coefficients method* (MCM). Indeed, we can choose all but one $\lambda_j \approx \mathtt{d} \gg 1$, and let the remaining one $\lambda_\nu > \sum_{j\neq\nu}(\mathtt{d}-\lambda_j) + N + 1$ be large enough, so as to ensure:

$$\heartsuit \;=\; -\lambda_\nu + \sum_{j\neq\nu}(\mathtt{d}-\lambda_j) + N + 1 \;\approx\; -\lambda_\nu \;\ll -1. \tag{41}$$

Let us best illustrate this central idea of MCM by the following c hypersurface equations, each being a Fermat-type polynomial plus one *moving coefficient term*:

$$F_i \;=\; \sum_{j=0}^{N} A_i^j\, z_j^d \;+\; M_i\, z_0^{\lambda_0}\cdots z_N^{\lambda_N} \qquad (i=1\cdots c),$$

where A_i^j, M_i are some homogeneous polynomials of degree $\epsilon \geqslant 0$, and where $\lambda_0 + \cdots + \lambda_N = d$ are some positive integers to be determined. For every $\nu = 0\cdots N$, associating the moving coefficient term $M_i\, z_0^{\lambda_0}\cdots z_N^{\lambda_N}$ to the term $A_i^\nu\, z_\nu^d$, we may rewrite:

$$F_i \;=\; \big(A_i^\nu\, z_\nu^d \;+\; M_i\, z_0^{\lambda_0}\cdots z_N^{\lambda_N}\big) \;+\; \sum_{j\neq\nu}^{N} A_i^j\, z_j^d \qquad (i=1\cdots c),$$

and thus by Proposition 6.5, we obtain a symmetric differential form with twisted degree $\heartsuit_\nu = -\lambda_\nu + N\,\epsilon + N + 1$. Therefore we require all $\lambda_\nu \geqslant N\,\epsilon + N + 1 + 1$ to make $\heartsuit_\nu \leqslant -1$.

7.2. **Global moving coefficients method.** Recalling Theorems 5.1, 5.4, we first fix positive integers $\blacktriangledown, \epsilon_1, \ldots, \epsilon_{c+r} \geqslant 1$. Skipping some thinking process which essentially roots in the above example, we construct the following $c+r$ hypersurface equations, each being a Fermat-type polynomial plus $N+1$ *moving coefficient terms*:

$$F_i := \sum_{j=0}^{N} A_i^j\, z_j^d + \sum_{k=0}^{N} M_i^k\, z_0^{\mu_k}\cdots\widehat{z_k^{\mu_k}}\cdots z_N^{\mu_k}\, z_k^{d-N\mu_k} \qquad (i=1\cdots c+r), \tag{42}$$

where all coefficients A_i^j, $M_i^k \in \mathbb{K}[z_0,\ldots,z_N]$ are some degree ϵ_i homogeneous polynomials, and where all positive integers $\mu_\bullet, d$ are to be chosen by a certain *Algorithm*, so as to make all the symmetric differential forms to be obtained later have negative twisted degrees. For the moment, instead of making the Algorithm clear, we prefer to roughly summarize it as:

$$1 \;\leqslant\; \max\{\epsilon_i\}_{i=1\cdots c+r} \;\ll\; \underbrace{\mu_0 \;\ll\; \cdots \;\ll\; \mu_N}_{\mu_\bullet \text{ grow exponentially}} \;\ll\; d. \tag{43}$$

There are two kinds of manipulations in MCM to transform the polynomials $F_1,\ldots,F_{c+r}$ to Fermat-type. The first one is much like the above example, in the sense that we view all the $N+1$

moving coefficient terms as a whole. Precisely, for every $\nu = 0 \cdots N$, for all F_i, we associate all $(N+1)$ moving coefficient terms with the term $A_i^\nu z_\nu^d$ by rewriting:

$$F_i \;=\; \sum_{j\neq\nu} A_i^j z_j^d + \Big(A_i^\nu z_\nu^d + \sum_{k=0}^{N} M_i^k z_0^{\mu_k} \cdots \widehat{z_k^{\mu_k}} \cdots z_N^{\mu_k} z_k^{d-N\mu_k}\Big) \qquad (i = 1 \cdots c+r). \tag{44}$$

By (43), all terms in the bracket are divisible by $z_\nu^{\mu_0}$. Now applying Proposition 6.5, for every $1 \leqslant j_1 < \cdots < j_n \leqslant c$, we receive a twisted symmetric differential n-form:

$$\phi^\nu_{j_1,\ldots,j_n} \;\in\; \mathsf{H}^0(X, \mathsf{Sym}^n \Omega_V(\heartsuit^\nu_{j_1,\ldots,j_n})),$$

whose twisted degree $\heartsuit^\nu_{j_1,\ldots,j_n}$ according to formula (34) is negative:

$$\heartsuit^\nu_{j_1,\ldots,j_n} \;=\; -\mu_0 + \sum_{i=1}^{c+r} \epsilon_i + \sum_{\ell=1}^{n} \epsilon_{j_\ell} + N + 1 \;\leqslant\; -n \,\heartsuit \qquad \text{[compare with (41), use (43)]}. \tag{45}$$

The second kind of manipulations is, for every $\tau = 0 \cdots N-1$, for every $\rho = \tau+1 \cdots N$, for all F_i, to associate each of the first $(\tau+1)$ moving coefficient terms with the corresponding terms $A_i^\bullet z_\bullet^d$, to associate all the remaining $(N-\tau)$ moving coefficient terms with the term $A_i^\rho z_\rho^d$, hence to rewrite:

$$F_i \;=\; \sum_{k=0}^{\tau} \Big(A_i^k z_k^d + M_i^k z_0^{\mu_k} \cdots \widehat{z_k^{\mu_k}} \cdots z_N^{\mu_k} z_k^{d-N\mu_k}\Big) + \sum_{\substack{j=\tau+1 \\ j\neq\rho}}^{N} A_i^j z_j^d + \Big[A_i^\rho z_\rho^d + \sum_{j=\tau+1}^{N} M_i^j z_0^{\mu_j} \cdots \widehat{z_j^{\mu_j}} \cdots z_N^{\mu_j} z_j^{d-N\mu_j}\Big] \qquad (i = 1 \cdots c+r). \tag{46}$$

Each bracket in the first sum is divisible by the corresponding $z_k^{d-N\mu_k}$, and remembering (43), the last square bracket is divisible by $z_\rho^{\mu_{\tau+1}}$. Now applying Proposition 6.5 again, for every $1 \leqslant j_1 < \cdots < j_n \leqslant c$, we receive a twisted symmetric differential n-form:

$$\psi^{\tau,\rho}_{j_1,\ldots,j_n} \;\in\; \mathsf{H}^0(X, \mathsf{Sym}^n \Omega_V(\heartsuit^{\tau,\rho}_{j_1,\ldots,j_n})),$$

whose twisted degree $\heartsuit^{\tau,\rho}_{j_1,\ldots,j_n}$ according to formula (34) is negative:

$$\heartsuit^{\tau,\rho}_{j_1,\ldots,j_n} \;=\; -\mu_{\tau+1} + \sum_{k=0}^{\tau} N\mu_k + \sum_{i=1}^{c+r} \epsilon_i + \sum_{\ell=1}^{n} \epsilon_{j_\ell} + N + 1 \;\leqslant\; -n \,\heartsuit \qquad \text{[compare with (41), use (43)]}. \tag{47}$$

8. **Controlling The Base Locus**

Recalling the claimed step (16), at first we shall determine the:

$$\text{Base Locus of } \Big\{\phi^\nu_{j_1,\ldots,j_n}, \psi^{\tau,\rho}_{j_1,\ldots,j_n}\Big\}^{\nu,\tau,\rho}_{1\leqslant j_1<\cdots<j_n\leqslant c} \;=:\; \mathsf{BS} \;\subset\; \mathbb{P}^\blacklozenge_{\mathbb{K}} \times \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}). \tag{48}$$

To keep fluidity, we leave some technical preparations in Section 10, and start with

8.1. **Characterization of** BS**.** For a fixed $\nu = 0 \cdots N$, we first study:

$$\text{Base Locus of } \Big\{\phi^\nu_{j_1,\ldots,j_n}\Big\}_{1\leqslant j_1<\cdots<j_n\leqslant c} \;=:\; \mathsf{BS}^\nu \;\subset\; \mathbb{P}^\blacklozenge_{\mathbb{K}} \times \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}).$$

For every $1 \leqslant j_1 < \cdots < j_n \leqslant c$, by mimicking the construction of the matrices $\mathsf{K}, \widehat{\mathsf{K}}_{j_1,\ldots,j_n;\,j}$ at the end of Subsection 6.3, in accordance with the manipulation (44), we construct the $(c+r+c) \times (N+1)$ matrix K^ν analogously, i.e. by copying terms, differentials, and then we also define the corresponding $\widehat{\mathsf{K}}^\nu_{j_1,\ldots,j_n;\,j}$.

Now, let us look at points $([A_\bullet^\bullet, M_\bullet^\bullet], [z], [\xi]) \in \mathsf{BS}^\nu$ with $z_0 \cdots z_N \neq 0$. For the sake of readability, we will keep in mind the parameter $(A_\bullet^\bullet, M_\bullet^\bullet)$ without writing it. Recalling Propositions 6.4, 6.5, for every $\widehat{\phi}^\nu_{j_1,\ldots,j_n}$ corresponding to $\phi^\nu_{j_1,\ldots,j_n}$, for every $j = 0 \cdots N$, we have:

$$0 = \widehat{\phi}^\nu_{j_1,\ldots,j_n;\,j}(z,\xi) = \frac{(-1)^j}{z_0^\star \cdots z_N^\star} \det\big(\widehat{\mathsf{K}}^\nu_{j_1,\ldots,j_n;\,j}\big)(z,\xi) \qquad \text{[use (38)]},$$

where all integers $\star$ are of no importance here. Indeed, we can drop the nonzero factor $\frac{(-1)^j}{z_0^\star \cdots z_N^\star}$:

$$\det\big(\underbrace{\widehat{\mathsf{K}}^\nu_{j_1,\ldots,j_n;\,j}}_{N\times N \text{ matrix}}\big)(z,\xi) = 0.$$

In other words:

$$\mathsf{rank}_{\mathbb{K}}\, \widehat{\mathsf{K}}^\nu_{j_1,\ldots,j_n;\,j}(z,\xi) \leqslant N-1.$$

Now, letting the index j run from 0 to N, we receive:

$$\mathsf{rank}_{\mathbb{K}}\, \underbrace{\mathsf{K}^\nu_{j_1,\ldots,j_n}(z,\xi)}_{N\times(N+1) \text{ matrix}} \leqslant N-1, \tag{49}$$

where $\mathsf{K}^\nu_{j_1,\ldots,j_n}$ is defined analogously to $\mathsf{C}^\nu_{j_1,\ldots,j_n}$ before Proposition 6.4.

Now, denote by H^ν the same submatrix of all $\mathsf{K}^\nu_{j_1,\ldots,j_n}$ that consists of the first $c+r$ rows. Then Proposition 10.9 below asserts that:

$$\mathsf{rank}_{\mathbb{K}}\, \mathsf{H}^\nu(z) = c + r, \tag{50}$$

for all parameter $[A_\bullet^\bullet, M_\bullet^\bullet] \in \mathbb{P}^\blacklozenge_{\mathbb{K}} \setminus \Sigma^\nu$, where $\Sigma^\nu \subsetneqq \mathbb{P}^\blacklozenge_{\mathbb{K}}$ is some exceptional proper subvariety.

Now, in (49), letting $1 \leqslant j_1 < \cdots < j_n \leqslant c$ vary, by Lemma 10.2 below, we receive:

$$\mathsf{rank}_{\mathbb{K}}\, \mathsf{K}^\nu(z,\xi) \leqslant N-1. \tag{51}$$

Also, note that $([z],[\xi])$ lies in the variety (10) defined by $[A_\bullet^\bullet, M_\bullet^\bullet]$, in other words:

$$\textit{the sum of all columns of } \mathsf{K}^\nu(z,\xi) \textit{ vanishes}. \tag{52}$$

Conversely, it is direct to check that any $([A_\bullet^\bullet, M_\bullet^\bullet], [z], [\xi]) \in \mathbb{P}^\blacklozenge_{\mathbb{K}} \times \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$ satisfying the above two conditions (51), (52) must lie in the base locus BS^ν, where:

$$\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}) \;:=\; \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}) \cap \{z_0 \cdots z_N \neq 0\}.$$

Thus we conclude the following generic characterization of $\mathsf{BS}^\nu \cap \{z_0 \cdots z_N \neq 0\}$.

Proposition 8.1. *A point* $([A_\bullet^\bullet, M_\bullet^\bullet], [z], [\xi]) \in (\mathbb{P}^\blacklozenge_{\mathbb{K}} \setminus \Sigma^\nu) \times \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$ *lies in the base locus* BS^ν *if and only if:*

$$\mathsf{rank}_{\mathbb{K}}\, \mathsf{K}^\nu(z,\xi) \leqslant N-1, \ \textit{and the sum of all columns vanishes}. \qquad \square$$

Now, for every $\tau = 0 \cdots N-1$ and $\rho = \tau+1 \cdots N$, we may repeat the same reasoning to analyze the base locus $\mathsf{BS}^{\tau,\rho}$ of $\{\psi^{\tau,\rho}_{j_1,\ldots,j_n}\}_{1\leqslant j_1<\cdots<j_n\leqslant c}$. Naturally, we define the matrix $\mathsf{K}^{\tau,\rho}$ in accordance with the manipulation (46), and we receive some analogous $\Sigma^{\tau,\rho} \subsetneqq \mathbb{P}^\blacklozenge_{\mathbb{K}}$.

Proposition 8.2. *A point* $([A_\bullet^\bullet, M_\bullet^\bullet], [z], [\xi]) \in (\mathbb{P}^\blacklozenge_{\mathbb{K}} \setminus \Sigma^{\tau,\rho}) \times \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$ *lies in* $\mathsf{BS}^{\tau,\rho}$ *if and only if:*

$$\mathsf{rank}_{\mathbb{K}}\, \mathsf{K}^{\tau,\rho}(z,\xi) \leqslant N-1, \ \textit{and the sum of all columns vanishes}. \qquad \square$$

To give a neat characterization of BS, which is equal to the intersection of all BS^{ρ}, $\mathsf{BS}^{\tau,\rho}$, it is now time to clarify the (uniform) structures of the matrices K^{ν}, $\mathsf{K}^{\tau,\rho}$.

First of all, we construct the $(c+r+c)\times(2N+2)$ ambient matrix M such that, for every $i=1\cdots c+r$, $j=1\cdots c$, its i-th row copies the $(2N+2)$ terms of F_i in (42) in the exact order, and its $(c+r+j)$-th row is the differential of the j-th row. In order to distinguish the first $(N+1)$ 'dominant' columns from the last $(N+1)$ 'moving coefficients' columns, we write M as:

$$\mathsf{M}=\big(\mathsf{A}_0\,|\,\cdots\,|\,\mathsf{A}_N\,|\,\mathsf{B}_0\,|\,\cdots\,|\,\mathsf{B}_N\big).$$

For every $\nu=0\cdots N$, comparing (44) with (42), the matrix K^{ν} is nothing but:

$$\mathsf{K}^{\nu}=\Big(\mathsf{A}_0\,|\,\cdots\,|\,\widehat{\mathsf{A}_{\nu}}\,|\,\cdots\,|\,\mathsf{A}_N\,|\,\mathsf{A}_{\nu}+\sum_{j=0}^{N}\mathsf{B}_j\Big),\tag{53}$$

where the last column is understood to appear in the 'omitted' column. Similarly, for every $\tau=0\cdots N-1$ and every $\rho=\tau+1\cdots N$, comparing (46) with (42), the matrix $\mathsf{K}^{\tau,\rho}$ is nothing but:

$$\mathsf{K}^{\tau,\rho}=\Big(\mathsf{A}_0+\mathsf{B}_0\,|\,\cdots\,|\,\mathsf{A}_{\tau}+\mathsf{B}_{\tau}\,|\,\mathsf{A}_{\tau+1}\,|\,\cdots\,|\,\widehat{\mathsf{A}_{\rho}}\,|\,\cdots\,|\,\mathsf{A}_N\,|\,\mathsf{A}_{\rho}+\sum_{j=\tau+1}^{N}\mathsf{B}_j\Big).\tag{54}$$

Next, the two preceding propositions lead us to consider the subvariety:

$$\mathscr{M}^{N}_{2c+r}\ \subset\ \mathsf{Mat}_{(2c+r)\times 2(N+1)}(\mathbb{K})\tag{55}$$

consisting of all $(c+r+c)\times 2(N+1)$ matrices $(\alpha_0\,|\,\alpha_1\,|\,\cdots\,|\,\alpha_N\,|\,\beta_0\,|\,\beta_1\,|\,\cdots\,|\,\beta_N)$ such that:

(i) the sum of all $2N+2$ colums is zero:

$$\alpha_0+\alpha_1+\cdots+\alpha_N+\beta_0+\beta_1+\cdots+\beta_N=\mathbf{0};\tag{56}$$

(ii) for every $\nu=0\cdots N$, there holds the rank inequality:

$$\mathsf{rank}_{\mathbb{K}}\,\{\alpha_0,\ldots,\widehat{\alpha_{\nu}},\ldots,\alpha_N,\alpha_{\nu}+(\beta_0+\beta_1+\cdots+\beta_N)\}\leqslant N-1;\tag{57}$$

(iii) for every $\tau=0\cdots N-1$, for every $\rho=\tau+1\cdots N$, there holds:

$$\mathsf{rank}_{\mathbb{K}}\,\{\alpha_0+\beta_0,\alpha_1+\beta_1,\ldots,\alpha_{\tau}+\beta_{\tau},\alpha_{\tau+1},\ldots,\widehat{\alpha_{\rho}},\ldots,\alpha_N,\alpha_{\rho}+(\beta_{\tau+1}+\cdots+\beta_N)\}\leqslant N-1.\tag{58}$$

Finally, letting $\Sigma'\subsetneq\mathbb{P}^{\blacklozenge}_{\mathbb{K}}$ be the union of all exceptional loci Σ^{ν} and $\Sigma^{\tau,\rho}$, we thus receive

Proposition 8.3. *A point* $([A^{\bullet}_{\bullet},M^{\bullet}_{\bullet}],[z],[\xi])\in(\mathbb{P}^{\blacklozenge}_{\mathbb{K}}\setminus\Sigma')\times\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$ *lies in* BS *if and only if:*

$$\mathsf{M}(z,\xi)\in\mathscr{M}^{N}_{2c+r}.$$ □

8.2. **Emptiness of the base locus.** Now, it is natural to introduce the subvariety:

$$M^{N}_{2c+r}\hookrightarrow\mathbb{P}^{\blacklozenge}_{\mathbb{K}}\times\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}),$$

which is defined according to Proposition 8.3 by:

$$M^{N}_{2c+r}:=\Big\{([A^{\bullet}_{\bullet},M^{\bullet}_{\bullet}];\,[z],[\xi])\in\mathbb{P}^{\blacklozenge}_{\mathbb{K}}\times\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N})\colon\ \mathsf{M}(z,\xi)\in\mathscr{M}^{N}_{2c+r}\Big\}.\tag{59}$$

In order to estimate the dimension of M^{N}_{2c+r}, we now claim the following *Core Lemma* in advance, whose proof will constitute the most technical part of this article (see Section 11 below).

Core Lemma. *For all positive integers* $2\leqslant N\leqslant M$*, there hold the codimension estimates:*

$$\mathsf{codim}\,\mathscr{M}^{N}_{M}\ \geqslant\ M+N-1.$$ □

Proposition 8.4. *There holds the dimension estimate:*

$$\dim M^N_{2c+r} \leqslant \dim \mathbb{P}^{\blacklozenge}_{\mathbb{K}}. \tag{60}$$

Proof. Let π_1, π_2 be the two canonical projections of $\mathbb{P}^{\blacklozenge}_{\mathbb{K}} \times \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N})$ onto two components. For every $\zeta = ([z], [\xi]) \in \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N})$, we now estimate the dimension of:

$$\pi_2^{-1}(\zeta) \cap M^N_{2c+r} \subset \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \times \{\zeta\},$$

which is isomorphic to its image ${}_\zeta M^N_{2c+r} \subset \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$ under π_1.

Let $\widehat{\pi}\colon \mathbb{K}^{\blacklozenge+1} \setminus \{0\} \to \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$ be the canonical projection. From (59) we see that:

$$\widehat{\pi}^{-1}({}_\zeta M^N_{2c+r}) \cup \{0\} \subset \mathbb{K}^{\blacklozenge+1}$$

is nothing but the inverse image of $\mathscr{M}^N_{2c+r}$ under the $\mathbb{K}$-linear map:

$$\begin{aligned} \mathsf{M}_{z,\xi}\colon \qquad \mathbb{K}^{\blacklozenge+1} &\longrightarrow \mathsf{Mat}_{(2c+r)\times 2(N+1)}(\mathbb{K}) \\ (A^{\bullet}_{\bullet}, M^{\bullet}_{\bullet}) &\longmapsto \mathsf{M}(z,\xi). \end{aligned}$$

Moreover, Lemma 10.4 below asserts that $\mathsf{M}_{z,\xi}$ is surjective. Thus we have:

$$\operatorname{codim} \mathscr{M}^N_{2c+r} = \operatorname{codim} \widehat{\pi}^{-1}({}_\zeta M^N_{2c+r}) \cup \{0\} = \operatorname{codim} {}_\zeta M^N_{2c+r} \qquad \text{[use Theorem 10.7 below]},$$

and hence:

$$\dim \pi_2^{-1}(\zeta) \cap M^N_{2c+r} = \dim {}_\zeta M^N_{2c+r} = \dim \mathbb{P}^{\blacklozenge}_{\mathbb{K}} - \operatorname{codim} \mathscr{M}^N_{2c+r}.$$

Finally, applying the Core Lemma:

$$\operatorname{codim} \mathscr{M}^N_{2c+r} \geqslant 2c + r + N - 1 \geqslant \dim \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}),$$

we receive the desired estimate:

$$\dim \pi_2^{-1}(\zeta) \cap M^N_{2c+r} \leqslant \dim \mathbb{P}^{\blacklozenge}_{\mathbb{K}} - \dim \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}).$$

By the standard argument on dimensional counting, we finish the proof. □

Now, by the projection $\pi_1\colon M^N_{2c+r} \to \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$ and the dimension estimate (60), thanks to a familiar result [7, p. 138, Theorem 11.12] in algebraic geometry, we obtain

Proposition 8.5. *There exists a proper algebraic subset $\Sigma'' \subsetneqq \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$ such that, for all $t \in \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus \Sigma''$, the fibre $\pi_1^{-1}(t) \cap M^N_{2c+r}$ is discrete or empty.*

Combining Propositions 8.3 and 8.5, we successfully control the base locus $\mathsf{BS}_t := \mathsf{BS} \cap \pi_1^{-1}(t)$.

Proposition 8.6. *For all parameters $t = [A^{\bullet}_{\bullet}, M^{\bullet}_{\bullet}] \in \mathbb{P}^{\blacklozenge}_{\mathbb{K}} \setminus (\Sigma' \cup \Sigma'')$, the base locus BS_t is discrete or empty over the coordinates nonvanishing part $\{z_0 \cdots z_N \neq 0\}$.*

9. Moving Coefficients Method (II)

9.1. **Obstacles of MCM when some coordinates vanish.** It would be desirable to find one parameter t such that $\mathsf{BS}_t \cap \{z_0 \cdots z_N = 0\}$ is discrete or empty, whence the semicontinuity theorem would guarantee the same property for all generic t. Then, together with the above Proposition 8.6, we would prove the Debarre Ampleness Conjecture! However, this is impossible because, for instance when $z_N = 0$, we can check that all the obtained global symmetric differential forms either vanish or become the same one, up to a scalar of the form $z_0^{\bullet} \cdots z_{N-1}^{\bullet}$. Moreover, all hypersurface equations in (42) become exactly Fermat-type $F_i = \sum_{j=0}^{N-1} A_i^j z_j^d$, so even with the further help of

Propositions 6.6, 6.7, we cannot construct enough negatively twisted symmetric forms. Nevertheless, supposing at this moment that we still have abundant moving coefficient terms:

$$F_i\big|_{z_N=0} \;\approx\; \sum_{j=0}^{N-1} A_i^j\, z_j^d + \sum_{k=0}^{N-1} M_i^k\, z_0^{\mu_k} \cdots \widehat{z_k^{\mu_k}} \cdots z_{N-1}^{\mu_k}\, z_k^{d-N\mu_k},$$

we may then continue to move on by adapting MCM … Yet when further coordinates vanish, the same obstacle might appear …

9.2. **Refined construction of hypersurfaces for MCM.** The above reasoning process leads us to construct the following $c+r$ refined homogeneous polynomials $F_1, \ldots, F_{c+r}$, each being the sum of a *dominant* Fermat-type polynomial plus an 'army' of *moving coefficient* terms:

$$F_i \;=\; \sum_{j=0}^{N} A_i^j\, z_j^d \;+\; \sum_{l=c+r+1}^{N} \sum_{0\leqslant j_0<\cdots<j_l\leqslant N} \sum_{k=0}^{l} M_i^{j_0,\ldots,j_l;j_k}\, z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}}\, z_{j_k}^{d-l\mu_{l,k}} \qquad (i=1\cdots c+r), \tag{61}$$

where all coefficients $A_i^\bullet, M_i^{\bullet;\bullet} \in \mathbb{K}[z_0, \ldots, z_N]$ are some degree $\epsilon_i \geqslant 1$ homogeneous polynomials, and where all positive integers $\mu^{l,k}, d$ are to be chosen by a certain *Algorithm*, which is designed to make all the symmetric differential forms to be obtained later have *negative twisted degrees*. For the moment, we just roughly summarize the Algorithm as:

$$1 \leqslant \max\{\epsilon_i\}_{i=1\cdots c+r} \ll \underbrace{\mu_{c+r+1,0} \ll \cdots \ll \mu_{c+r+1,c+r+1}}_{\mu_{c+r+1,\bullet}\text{ grow exponentially}} \ll \cdots\cdots \ll \underbrace{\mu_{N,0} \ll \cdots \ll \mu_{N,N}}_{\mu_{N,\bullet}\text{ grow exponentially}} \ll d, \tag{62}$$

and we will state it explicitly in Subsection 12.1 below when it is really necessary.

The above construction fulfills the claimed Step 1 in Subsection 5.3. Now, we illustrate how to transform (61) into the form (42), so as to apply MCM that we have developed in Section 7.

First, we rewrite the polynomial F_i by extracting the terms for which $l = N$:

$$\begin{aligned} F_i = \sum_{j=0}^{N} A_i^j\, z_j^d + \sum_{l=c+r+1}^{N-1} \sum_{0\leqslant j_0<\cdots<j_l\leqslant N} \sum_{k=0}^{l} M_i^{j_0,\ldots,j_l;j_k}\, z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}}\, z_{j_k}^{d-l\mu_{l,k}} + \\ + \sum_{k=0}^{N} M_i^{0,\ldots,N;k}\, z_0^{\mu_{N,k}} \cdots \widehat{z_k^{\mu_{N,k}}} \cdots z_N^{\mu_{N,k}}\, z_k^{d-N\mu_{N,k}} \qquad (i=1\cdots c+r). \end{aligned}$$

Next, we associate each term $M_i^{j_0,\ldots,j_l;j_k}\, z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}}\, z_{j_k}^{d-l\mu_{l,k}}$ in the second sums with the corresponding term $A_i^{j_k}\, z_{j_k}^d$ in the first sum by rewriting F_i as:

$$F_i \;=\; \sum_{j=0}^{N} C_i^j\, z_j^{d-\delta_N} + \sum_{k=0}^{N} M_i^{0,\ldots,N;k}\, z_0^{\mu_{N,k}} \cdots \widehat{z_k^{\mu_{N,k}}} \cdots z_N^{\mu_{N,k}}\, z_k^{d-N\mu_{N,k}} \qquad (i=1\cdots c+r), \tag{63}$$

where $\delta_N := (N-1)\,\mu_{N-1,N-1}$ and C_i^j are uniquely determined by gathering:

$$C_i^j\, z_j^{d-\delta_N} = A_i^j\, z_j^d + \sum_{l=c+r+1}^{N-1} \sum_{\substack{0\leqslant j_0<\cdots<j_l\leqslant N \\ j_k=j \text{ for some } 0\leqslant k\leqslant l}} M_i^{j_0,\ldots,j_l;j_k}\, z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}}\, z_{j_k}^{d-l\mu_{l,k}},$$

namely, after dividing out the common factor $z_j^{d-\delta_N}$ — guaranteed by (62) — of both sides above:

$$C_i^j := A_i^j\, z_j^{\delta_N} + \sum_{l=c+r+1}^{N-1} \sum_{\substack{0\leqslant j_0<\cdots<j_l\leqslant N\\ j_k=j \text{ for some } 0\leqslant k\leqslant l}} M_i^{j_0,\ldots,j_l;j_k}\, z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}} z_{j_k}^{\delta_N - l\mu_{l,k}}.$$

Note that by the Algorithm (62), we now have the rough estimates:

$$\deg\, C_i^j = \epsilon_i + \delta_N = \epsilon_i + (N-1)\,\mu_{N-1,N-1} \;\ll\; \mu_{N,0} \;\ll\; \cdots \;\ll\; \mu_{N,N} \ll\; d, \tag{64}$$

with the same growth structure as the previous Algorithm (43)!

Thus, we are able to adapt the MCM in Subsection 7.2 to produce a series of negatively twisted symmetric differential forms. Indeed, using the same kind of manipulations and the corresponding notation, for every $1 \leqslant j_1 < \cdots < j_n \leqslant c$, for every $\nu = 0 \cdots N$, we receive:

$$\phi^{\nu}_{j_1,\ldots,j_n} \;\in\; \mathsf{H}^0(X, \mathsf{Sym}^n\, \Omega_V(\heartsuit^{\nu}_{j_1,\ldots,j_n})),$$

with negative twisted degree:

$$\heartsuit^{\nu}_{j_1,\ldots,j_n} \;=\; -\mu_{N,0} + N\,\delta_N + \sum_{i=1}^{c+r} \epsilon_i + \sum_{\ell=1}^{n} \epsilon_{j_\ell} + N + 1 \;\leqslant\; -n\,\blacktriangledown \qquad \text{[compare with (45), use (64)]}. \tag{65}$$

Also, for every $\tau = 0 \cdots N-1$, for every $\rho = \tau+1 \cdots N$, we receive:

$$\psi^{\tau,\rho}_{j_1,\ldots,j_n} \;\in\; \mathsf{H}^0(X, \mathsf{Sym}^n\, \Omega_V(\heartsuit^{\tau,\rho}_{j_1,\ldots,j_n})),$$

with negative twisted degree:

$$\heartsuit^{\tau,\rho}_{j_1,\ldots,j_n} \;=\; -\mu_{N,\tau+1} + \sum_{k=0}^{\tau} N\,\mu_{N,k} + (N-\tau-1)\,\delta_N + \sum_{i=1}^{c+r} \epsilon_i + \sum_{\ell=1}^{n} \epsilon_{j_\ell} + N + 1 \;\leqslant\; -n\,\blacktriangledown \qquad \text{[see (47), use (64)]}. \tag{66}$$

Thus, we complete the claimed construction (13).

To respect the coherence of notation, we will continue to denote by $\mathbb{P}^{\blacklozenge}_{\mathbb{K}}$ the projective parameter space of $F_1, \ldots, F_{c+r}$ in (61), and we will also use the same notation as in Section 8, and then by exactly the same arguments, everything goes on smoothly, and finally we get the analogue Proposition 8.6, which is our claimed estimate (19).

9.3. **MCM with vanishing coordinates.** For every $1 \leqslant \eta \leqslant n-1$, for every $0 \leqslant \nu_1 < \cdots < \nu_\eta \leqslant N$, consider the intersection of X with the η coordinate hyperplanes:

$$_{\nu_1,\ldots,\nu_\eta}X := X \cap \{z_{\nu_1} = 0\} \cap \cdots \cap \{z_{\nu_\eta} = 0\}.$$

Recalling the claimed step (14), we now investigate further the MCM to construct a series of:

$$\omega_\ell \in \mathsf{H}^0\left({}_{\nu_1,\ldots,\nu_\eta}X, \mathsf{Sym}^{n-\eta}\, \Omega_V(\heartsuit_\ell)\right) \qquad (\ell = 1 \cdots \maltese),$$

with all $\heartsuit_\ell \leqslant -(n-\eta)\,\blacktriangledown$, as follows.

First, we write the complement elements of $\{\nu_1, \ldots, \nu_\eta\}$ in $\{0, \ldots, N\}$ as $r_0 < \cdots < r_{N-\eta}$. Observe that in Propositions 6.6, 6.7, the polynomial terms in $F_1, \ldots, F_{c+r}$ involving $z_{\nu_1}, \ldots, z_{\nu_\eta}$ eventually play no role — because they just vanish. Thus, we may decompose F_i into two parts:

$$\begin{aligned} F_i = \sum_{j=0}^{N-\eta} A_i^{r_j}\, z_{r_j}^d + \sum_{l=c+r+1}^{N-\eta} \sum_{0\leqslant j_0<\cdots<j_l\leqslant N-\eta} \sum_{k=0}^{l} M_i^{r_{j_0},\ldots,r_{j_l};r_{j_k}}\, z_{r_{j_0}}^{\mu_{l,k}} \cdots \widehat{z_{r_{j_k}}^{\mu_{l,k}}} \cdots z_{r_{j_l}}^{\mu_{l,k}} z_{r_{j_k}}^{d-l\mu_{l,k}} + \\ + \underbrace{(\text{Residue Terms})_i^{\nu_1,\ldots,\nu_\eta}}_{\text{negligible}} \qquad\qquad (i = 1 \cdots c+r), \end{aligned} \tag{67}$$

where the first part (1st line above) only involves the variables $z_{r_0},\dots,z_{r_{N-\eta}}$, and where the second part (2nd line above) collects all terms involving at least one of $z_{\nu_1},\dots,z_{\nu_\eta}$.

Next, before applying Propositions 6.6, 6.7 to (67), it is appropriate to write:

$$F_i = \sum_{j=0}^{N-\eta} A_i^{r_j}\, z_{r_j}^d + \sum_{l=c+r+1}^{N-\eta} \sum_{0\leqslant j_0<\cdots<j_l\leqslant N-\eta} \sum_{k=0}^{l} M_i^{r_{j_0},\dots,r_{j_l};r_{j_k}}\, z_{r_{j_0}}^{\mu_{l,k}}\cdots\widehat{z_{r_{j_k}}^{\mu_{l,k}}}\cdots z_{r_{j_l}}^{\mu_{l,k}} z_{r_{j_k}}^{d-l\mu_{l,k}} \qquad (i=1\cdots c+r). \quad (68)$$

Now, we see that the above (68) has the same structure as (61), in the sense of replacing:

$$N \leftrightarrow N-\eta, \qquad \{0,\dots,N\} \;\longleftrightarrow\; \{r_0,\dots,r_{N-\eta}\}.$$

Thus, we are able to adapt the MCM in the preceding Subsection 9.2 to produce negatively twisted symmetric differential forms.

Indeed, using the same kind of manipulations and the corresponding notation, for every $1\leqslant j_1<\cdots<j_{n-\eta}\leqslant c$, for every $\nu=0\cdots N-\eta$, we receive:

$$_{\nu_1,\dots,\nu_\eta}\phi^{\nu}_{j_1,\dots,j_{n-\eta}} \in \Gamma(_{\nu_1,\dots,\nu_\eta}X, \mathsf{Sym}^{n-\eta}\,\Omega_V\,(_{\nu_1,\dots,\nu_\eta}\heartsuit^{\nu}_{j_1,\dots,j_{n-\eta}})),$$

with negative twisted degree:

$$_{\nu_1,\dots,\nu_\eta}\heartsuit^{\nu}_{j_1,\dots,j_{n-\eta}} = -\mu_{N-\eta,0}+(N-\eta)\,\delta_{N-\eta}+\sum_{i=1}^{c+r}\epsilon_i+\sum_{\ell=1}^{n-\eta}\epsilon_{j_\ell}+(N-\eta)+1 \;\leqslant\; -(n-\eta)\,\heartsuit \qquad \text{[compare with (65)]}, \quad (69)$$

where $\delta_{N-\eta}$ is defined by replacing:

$$(N-1)\,\mu_{N-1,N-1}=\delta_N \;\longleftrightarrow\; \delta_{N-\eta}:=(N-\eta-1)\,\mu_{N-\eta-1,N-\eta-1}.$$

Also, for every $\tau=0\cdots N-\eta-1$, for every $\rho=\tau+1\cdots N-\eta$, we receive:

$$_{\nu_1,\dots,\nu_\eta}\psi^{\tau,\rho}_{j_1,\dots,j_{n-\eta}} \in \Gamma(_{\nu_1,\dots,\nu_\eta}X, \mathsf{Sym}^{n-\eta}\,\Omega_V(_{\nu_1,\dots,\nu_\eta}\heartsuit^{\tau,\rho}_{j_1,\dots,j_{n-\eta}})),$$

with negative twisted degree:

$$_{\nu_1,\dots,\nu_\eta}\heartsuit^{\tau,\rho}_{j_1,\dots,j_{n-\eta}} = -\mu_{N-\eta,\tau+1}+\sum_{k=0}^{\tau}(N-\eta)\,\mu_{N-\eta,k}+(N-\eta-\tau-1)\,\delta_{N-\eta}+\sum_{i=1}^{c+r}\epsilon_i+\sum_{\ell=1}^{n-\eta}\epsilon_{j_\ell}+(N-\eta)+1 \;\leqslant\; -(n-\eta)\,\heartsuit. \quad (70)$$

For all details, we refer the reader to [16, Section 7].

The above construction fulfills our claimed step (14). So now, we complete the Step 2 in Subsection 5.3.

To prove the claimed estimate (18), we now determine the:

$$\text{Base Locus of } \{_{\nu_1,\dots,\nu_\eta}\phi^{\nu}_{j_1,\dots,j_{n-\eta}},\ _{\nu_1,\dots,\nu_\eta}\psi^{\tau,\rho}_{j_1,\dots,j_{n-\eta}}\}^{\nu,\tau,\rho}_{1\leqslant j_1<\cdots<j_{n-\eta}\leqslant c} =: {}_{\nu_1,\dots,\nu_\eta}\mathsf{BS} \;\subset\; \mathbb{P}^{\blacklozenge}_{\mathbb{K}}\times {}_{\nu_1,\dots,\nu_\eta}\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}), \quad (71)$$

where:

$$_{\nu_1,\dots,\nu_\eta}\mathbb{P}(\mathrm{T}_{\mathbb{P}^N}) := \mathbb{P}(\mathrm{T}_{\mathbb{P}^N})\cap\{z_{\nu_1}=\cdots=z_{\nu_\eta}=0\}.$$

Employing the same arguments, here is an analog of Proposition 8.6.

Proposition 9.1. *There exists some proper subvariety* $_{\nu_1,\dots,\nu_\eta}\Sigma \subsetneq \mathbb{P}^{\blacklozenge}_{\mathbb{K}}$ *such that, for all parameters* $t\in \mathbb{P}^{\blacklozenge}_{\mathbb{K}}\setminus {}_{\nu_1,\dots,\nu_\eta}\Sigma$, *the base locus* $_{\nu_1,\dots,\nu_\eta}\mathsf{BS}_t := {}_{\nu_1,\dots,\nu_\eta}\mathsf{BS}\cap\pi_1^{-1}(t)$ *is discrete or empty over the coordinates nonvanishing part* $\{z_{r_0}\cdots z_{r_{N-\eta}}\neq 0\}$.

Proof. We sketch the proof in two parts. First, by mimicking the reasoning of Proposition 8.3, we characterize $_{\nu_1,\dots,\nu_\eta}\mathsf{BS}$ over the coordinates nonvanishing part:

$$_{\nu_1,\dots,\nu_\eta}\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) \;:=\; {}_{\nu_1,\dots,\nu_\eta}\mathbb{P}(\mathrm{T}_{\mathbb{P}^N})\cap\{z_{r_0}\cdots z_{r_{N-\eta}}\neq 0\}$$

by the variety $\mathscr{M}^{N-\eta}_{2c+r}$ — see (55).

Next, we prove an analog estimate of Proposition 8.4, in which the essential step:

$$\mathsf{codim}\, \mathscr{M}_{2c+r}^{N-\eta} \geqslant \mathsf{dim}\, {}_{v_1,\ldots,v_\eta}\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) = 2N-\eta-1$$

is provided by the Core Lemma.

We refer the reader to [16, p. 66, Proposition 9.9] for details. □

Thus, we finish the claimed Step 3 in Subsection 5.3. Now, the proof of Theorem 5.5 is complete, and so are the proofs of Theorems 5.4, 5.1.

10. Basic Technical Preparations

10.1. **Matrix-rank estimates.** First, we recall a basic fact in linear algebra.

Lemma 10.1. *Let $\mathbb{K}$ be a field and let W be a finite-dimensional $\mathbb{K}$-vector space generated by a set of vectors $\mathscr{B}$. Then every subset $\mathscr{B}_1 \subset \mathscr{B}$ that consists of $\mathbb{K}$-linearly independent vectors can be extended to a bigger subset $\mathscr{B}_2 \subset \mathscr{B}$ which forms a basis of W.* □

Lemma 10.2. *Let $\mathbb{K}$ be a field, and let V be a $\mathbb{K}$-vector space. For all positive integers $e, k, l \geqslant 1$ with $k \geqslant l$, let $v_1, \ldots, v_e, v_{e+1}, \ldots, v_{e+k}$ be $(e+k)$ vectors such that:*

- **(i)** *$v_1, \ldots, v_e$ are $\mathbb{K}$-linearly independent;*
- **(ii)** *for every $e+1 \leqslant i_1 < \cdots < i_l \leqslant e+k$, there holds:*

$$\mathsf{rank}_{\mathbb{K}}\, \{v_1, \ldots, v_e, v_{i_1}, \ldots, v_{i_l}\} \leqslant e+l-1.$$

Then one has the rank estimate:

$$\mathsf{rank}_{\mathbb{K}}\, \{v_1, \ldots, v_e, v_{e+1}, \ldots, v_{e+k}\} \leqslant e+l-1.$$

Proof. Using the preceding lemma and reasoning by contradiction, we conclude the proof. □

10.2. **Surjectivity of evaluation maps.** For every $N \geqslant 1$, denote:

$$\mathscr{A}(\mathbb{K}^{N+1}) := \mathbb{K}[z_0, \ldots, z_N].$$

For every $\lambda \geqslant 1$, the $\mathbb{K}$-linear space spanned by all degree λ homogeneous polynomials is:

$$\mathscr{A}_\lambda(\mathbb{K}^{N+1}) \subset \mathscr{A}(\mathbb{K}^{N+1}).$$

For every $z \in \mathbb{K}^{N+1}$, denote by v_z the $\mathbb{K}$-linear evaluation map:

$$v_z : \mathscr{A}(\mathbb{K}^{N+1}) \longrightarrow \mathbb{K}, \qquad f \longmapsto f(z),$$

and for every $\xi \in \mathrm{T}_z\mathbb{K}^{N+1} \cong \mathbb{K}^{N+1}$, denote by $d_z(\xi)$ the $\mathbb{K}$-linear differential evaluation map:

$$d_z(\xi) : \mathscr{A}(\mathbb{K}^{N+1}) \longrightarrow \mathbb{K}, \qquad f \longmapsto df\big|_z(\xi).$$

For every $g \in \mathscr{A}(\mathbb{K}^{N+1})$, for every $z \in \mathbb{K}^{N+1}$, denote by $(g \cdot v)_z$ the $\mathbb{K}$-linear evaluation map:

$$(g \cdot v)_z : \mathscr{A}(\mathbb{K}^{N+1}) \longrightarrow \mathbb{K}, \qquad f \longmapsto (gf)(z),$$

and for every $\xi \in \mathrm{T}_z\mathbb{K}^{N+1} \cong \mathbb{K}^{N+1}$, denote by $d_z(g\cdot)(\xi)$ the $\mathbb{K}$-linear differential evaluation map:

$$d_z(g\cdot)(\xi) : \mathscr{A}(\mathbb{K}^{N+1}) \longrightarrow \mathbb{K}, \qquad f \longmapsto d(gf)\big|_z(\xi).$$

Lemma 10.3. *For all $\lambda \geqslant 1$, at every $z \in \mathbb{K}^{N+1} \setminus \{0\}$, if $\xi \in \mathbb{K}^{N+1} \setminus \mathbb{K} \cdot z$, then the $\mathbb{K}$-linear map:*

$$\begin{pmatrix} v_z \\ d_z(\xi) \end{pmatrix} : \mathscr{A}_\lambda(\mathbb{K}^{N+1}) \longrightarrow \mathbb{K}^2$$

is surjective.

A similar result was obtained in [2, p. 36, Claim 3].

Proof. We present a proof which works for all fields $\mathbb{K}$.

For $\lambda = 1$, it is clear.

For $\lambda \geqslant 2$, first, choose one $g \in \mathscr{A}_{\lambda-1}(\mathbb{K}^{N+1})$ with $g(z) \neq 0$. Note that Leibniz's rule yields:

$$\begin{pmatrix} (g \cdot v)_z \\ d_z(g\cdot)(\xi) \end{pmatrix} = \underbrace{\begin{pmatrix} g(z) & 0 \\ d\, g\big|_z(\xi) & g(z) \end{pmatrix}}_{\text{invertible, since } g(z) \neq 0} \begin{pmatrix} v_z \\ d_z(\xi) \end{pmatrix}. \tag{72}$$

Now, restricting (72) on the $\mathbb{K}$-linear subspace:

$$\mathscr{A}_1(\mathbb{K}^{N+1}) \subset \mathscr{A}(\mathbb{K}^{N+1}),$$

we see that the map $(v_z, d_z(\xi))^{\mathrm{T}}$ is surjective on the $\mathbb{K}$-linear subspace:

$$g \cdot \mathscr{A}_1(\mathbb{K}^{N+1}) \subset \mathscr{A}_\lambda(\mathbb{K}^{N+1}),$$

which suffices to conclude the proof. □

Lemma 10.4. *For all $\lambda \geqslant 1$, for all $g \in \mathscr{A}(\mathbb{K}^{N+1})$, at any $z \in \mathbb{K}^{N+1} \setminus \{0\}$ with $g(z) \neq 0$, for every $\xi \in \mathbb{K}^{N+1} \setminus \mathbb{K} \cdot z$, the $\mathbb{K}$-linear map:*

$$\begin{pmatrix} (g \cdot v)_z \\ d_z(g\cdot)(\xi) \end{pmatrix} : \mathscr{A}_\lambda(\mathbb{K}^{N+1}) \longrightarrow \mathbb{K}^2$$

is surjective.

Proof. This is a direct consequence of formula (72) and of the preceding lemma. □

10.3. **Fibre dimension estimates.** The following classical theorem will prove fundamental in dimensional estimations of all base loci in this article.

Theorem 10.5. *Let X, Y be two $\mathbb{K}$-varieties, and let $f : X \to Y$ be a morphism. Then one has:*

$$\dim X \leqslant \dim Y + \max_{y \in Y} \dim f^{-1}(y).$$ □

10.4. **Classical codimension formulas for determinantal ideals.** For all $p, q \geqslant 1$, denote by:

$$\mathsf{Mat}_{p\times q}(\mathbb{K}) = \mathbb{K}^{p\times q}$$

the space of all $p \times q$ matrices with entries in $\mathbb{K}$. For every $0 \leqslant \ell \leqslant \max\{p, q\}$, denote by:

$$\Sigma_\ell^{p,q} \subset \mathsf{Mat}_{p\times q}(\mathbb{K})$$

all $p \times q$ matrices with rank $\leqslant \ell$.

Theorem 10.6. *There holds the codimension formula:*

$$\operatorname{codim} \Sigma_\ell^{p,q} = \max\{(p-\ell)(q-\ell),\, 0\}.$$ □

10.5. **Affine cones preserve codimensions.** In our practice, it would be more convenient to count dimension in an Euclidean space rather than in a projective space.

Theorem 10.7. *Let $\pi\colon \mathbb{K}^{N+1} \to \mathbb{P}^N_{\mathbb{K}}$ be the canonical projection, and let $Y \subset \mathbb{P}^N_{\mathbb{K}}$ be a nonempty algebraic set defined by a homogeneous ideal $I \subset \mathbb{K}[z_0, \ldots, z_N]$. Denote by $C(Y)$ the affine cone:*

$$C(Y) := \pi^{-1}(Y) \cup \{\mathbf{0}\} \subset \mathbb{K}^{N+1}.$$

Then $C(Y)$ is an algebraic set in $\mathbb{K}^{N+1}$, and it preserves codimension:

$$\operatorname{codim} C(Y) = \operatorname{codim} Y. \qquad \square$$

10.6. **Full rank of hypersurface equation matrices.** For all $N \geqslant 2$, for every $e = 1 \cdots N$, given integers $\epsilon_1, \ldots, \epsilon_e \geqslant 1$ and $d \geqslant 1$, consider the following e hypersurfaces:

$$H_1, \ldots, H_e \subset \mathbb{P}^N_{\mathbb{K}},$$

each being defined by the degree $(d + \epsilon_i)$ Fermat-type homogeneous polynomial:

$$F_i = \sum_{j=0}^{N} A_i^j \, z_j^d \qquad (i = 1 \cdots e), \tag{73}$$

where all A_i^j are some degree ϵ_i homogeneous polynomials.

Now, denote by H the $e \times (N+1)$ matrix whose every i-th row copies the $(N+1)$ terms of F_i:

$$\mathsf{H} := \begin{pmatrix} A_1^0 \, z_0^d & \cdots & A_1^N \, z_N^d \\ \vdots & \ddots & \vdots \\ A_e^0 \, z_0^d & \cdots & A_e^N \, z_N^d \end{pmatrix}. \tag{74}$$

Also introduce:

$$\mathbb{P}(\mathscr{M}) := \mathbb{P}\Big(\underbrace{\oplus_{\substack{1 \leqslant i \leqslant e \\ 0 \leqslant j \leqslant N}} \mathscr{A}_{\lambda_i}(\mathbb{K}^{N+1})}_{=: \mathscr{M}} \Big)$$

the projectivized parameter space of $F_1, \ldots, F_e$.

Proposition 10.8. *There exists a proper subvariety $\Sigma \subsetneq \mathbb{P}(\mathscr{M})$ such that, for every parameter:*

$$\Big[(A_i^j)_{\substack{1 \leqslant i \leqslant e \\ 0 \leqslant j \leqslant N}} \Big] \in \mathbb{P}(\mathscr{M}) \setminus \Sigma,$$

on the corresponding intersection:

$$X = H_1 \cap \cdots \cap H_e \subset \mathbb{P}^N_{\mathbb{K}},$$

the matrix H has full rank everywhere:

$$\operatorname{rank}_{\mathbb{K}} \mathsf{H}(z) = e \qquad (\forall\, [z] \in X).$$

A similar result was obtained in [2, p. 35, Claim 1].

Proof. We refer the reader to [16, p. 55, Lemma 8.15] for details. $\square$

Recalling the matrix H^ν before (50), now we prove the desired full rank identities.

Proposition 10.9. *For every $\nu = 0 \cdots N$, there exists a subvariety $\Sigma^\nu \subsetneq \mathbb{P}^\blacklozenge_{\mathbb{K}}$ such that, for every:*

$$[A^\bullet_\bullet, M^\bullet_\bullet] \in \mathbb{P}^\blacklozenge_{\mathbb{K}} \setminus \Sigma,$$

on the corresponding intersection:

$$X = H_1 \cap \cdots \cap H_{c+r} \subset \mathbb{P}^N_{\mathbb{K}},$$

the matrix H^ν have full rank:

$$\mathsf{rank}_{\mathbb{K}}\, \mathsf{H}^\nu(z) = c + r \qquad (\forall\, [z] \in X).$$

Proof. 'Full-rank' is a Zariski open condition, so we only need to provide one example.

First, by setting all $M^\bullet_\bullet = 0$, the equations (44) become exactly the equations (73), thus the matrix H^ν becomes the matrix H of Proposition 10.8 with $e = c + r$. Now, a direct application of Proposition 10.8 clearly yields more than one example. □

11. The Engine of MCM

11.1. **Core Codimension Formulas.** To prove the Core Lemma, as an essential step, by induction on positive integers $p \geqslant 2$ and $0 \leqslant \ell \leqslant p$, we first estimate the codimension ${}_\ell C_p$ of:

$${}_\ell \mathbf{X}_p \subset \mathsf{Mat}_{p\times 2p}(\mathbb{K}) \tag{75}$$

which consists of $p \times 2p$ matrices $\mathsf{X}_p = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p)$ such that:

(i) the first p column vectors have rank:

$$\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \ldots, \alpha_p\} \leqslant \ell; \tag{76}$$

(ii) for every $\nu = 1 \cdots p$, there holds:

$$\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \ldots, \widehat{\alpha_\nu}, \ldots, \alpha_p, \alpha_\nu + (\beta_1 + \cdots + \beta_p)\} \leqslant p - 1; \tag{77}$$

(iii) for every $\tau = 1 \cdots p - 1$, for every $\rho = \tau + 1 \cdots p$, there holds:

$$\mathsf{rank}_{\mathbb{K}} \{\alpha_1 + \beta_1, \ldots, \alpha_\tau + \beta_\tau, \alpha_{\tau+1}, \ldots, \widehat{\alpha_\rho}, \ldots, \alpha_p, \alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_p)\} \leqslant p - 1. \tag{78}$$

Let us start with the easy case $\ell = 0$.

Proposition 11.1. *For every integer $p \geqslant 2$, the codimension value ${}_\ell C_p$ for $\ell = 0$ is:*

$${}_0 C_p = p^2 + 1. \tag{79}$$

Proof. It is a direct application of Theorem 10.6. □

For the general case $\ell = 1 \cdots p$, we will use Gaussian eliminations and do inductions on p, ℓ. First, we count the codimension of the exceptional locus of Gaussian eliminations.

Proposition 11.2. *For every integer $p \geqslant 2$, the codimensions ${}_\ell C^0_p$ of the subvarieties:*

$$\{\alpha_1 + \beta_1 = \mathbf{0}\} \cap {}_\ell \mathbf{X}_p \subset \mathsf{Mat}_{p\times 2p}(\mathbb{K})$$

read according to the values of ℓ as:

$${}_\ell C^0_p = \begin{cases} p + 2 & (\ell = p-1, p), \\ p + (p - \ell)^2 & (\ell = 0 \cdots p-2). \end{cases}$$

The following lemma is the key ingredient for the proof.

Lemma 11.3. *Let W be a $\mathbb{K}$-vector space, and let $p \geqslant 1$ be a positive integer. For any $p+1$ vectors $\alpha_1, \ldots, \alpha_p, \beta \in W$, the rank restriction:*

$$\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \ldots, \widehat{\alpha_\nu}, \ldots, \alpha_p, \alpha_\nu + \beta\} \leqslant p - 1 \qquad (\nu = 1 \cdots p), \tag{80}$$

is equivalent to either:

$$\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \ldots, \alpha_p, \beta\} \leqslant p - 1,$$

or to:

$$\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \ldots, \alpha_p\} = p, \qquad (\alpha_1 + \cdots + \alpha_p) + \beta = \mathbf{0}.$$

Proof. We refer the reader to [16, p. 68, Lemma 10.3] for details. □

Proof of Proposition 11.2. We refer to [16, p. 68, Proposition 10.2] for details. □

Next, we claim the following *Codimension Induction Formulas*, the proof of which will appear in Subsection 11.5. To make sense of ${}_{\ell-2}C_{p-1}$ in (83) when $\ell = 1$, we henceforth agree ${}_{-1}C_{p-1} := \infty$.

Proposition 11.4. *For all $p \geqslant 3$, there hold the codimension induction formulas:*

$$ {}_pC_p = \min\{p, \ {}_{p-1}C_p\} \qquad (\ell = p), \tag{81}$$

$$ {}_{p-1}C_p \geqslant \min\{{}_{p-1}C_p^0, \ {}_{p-1}C_{p-1} + 2, \ {}_{p-2}C_{p-1} + 1, \ {}_{p-3}C_{p-1}\} \qquad (\ell = p-1), \tag{82}$$

$$ {}_\ell C_p \geqslant \min\{{}_\ell C_p^0, \ {}_\ell C_{p-1} + 2(p-\ell) - 1, \ {}_{\ell-1}C_{p-1} + (p-\ell), \ {}_{\ell-2}C_{p-1}\} \qquad (\ell = 1 \cdots p-2). \tag{83}$$

Now, we may claim the initial data for the induction process.

Proposition 11.5. *For the initial case $p = 2$, there hold the codimension values:*

$$ {}_0C_2 = 5, \qquad {}_1C_2 = 3, \qquad {}_2C_2 = 2.$$

Proof. We refer the reader to [16, p. 70, Proposition 10.5] for details. □

Admitting temporally Proposition 11.4, we may use induction to obtain

Proposition 11.6. *For all $p \geqslant 2$, there hold:*

$$ {}_pC_p = p, \qquad {}_\ell C_p \geqslant \ell + (p-\ell)^2 + 1 \qquad (\ell = 0 \cdots p-1). \tag{84}$$

Proof. We refer the reader to [16, p. 71, Proposition 10.6] for details. □

11.2. **Gaussian eliminations.** Following the notation in (75), we denote by:

$$\mathsf{X}_p = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p)$$

the coordinate columns of $\mathsf{Mat}_{p\times 2p}(\mathbb{K})$, where each of the first p columns explicitly writes as:

$$\alpha_i = (\alpha_{1,i}, \ldots, \alpha_{p,i})^{\mathrm{T}},$$

and where each of the last p columns explicitly writes as:

$$\beta_i = (\beta_{1,i}, \ldots, \beta_{p,i})^{\mathrm{T}}.$$

First, observing the structures of the matrices in (77), (78):

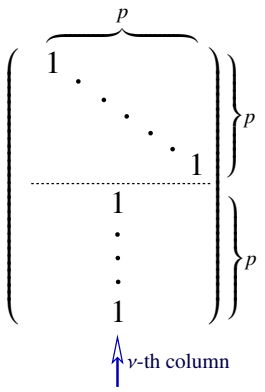
$$\mathsf{X}_p^{0,\nu} := (\alpha_1 \mid \cdots \mid \underline{\widehat{\alpha_\nu}} \mid \cdots \mid \alpha_p \mid \underline{\alpha_\nu + (\beta_1 + \cdots + \beta_p)}),$$

$$\mathsf{X}_p^{\tau,\rho} := (\alpha_1 + \beta_1 \mid \cdots \mid \alpha_\tau + \beta_\tau \mid \alpha_{\tau+1} \mid \cdots \mid \underline{\widehat{\alpha_\rho}} \mid \cdots \mid \alpha_p \mid \underline{\alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_p)}),$$

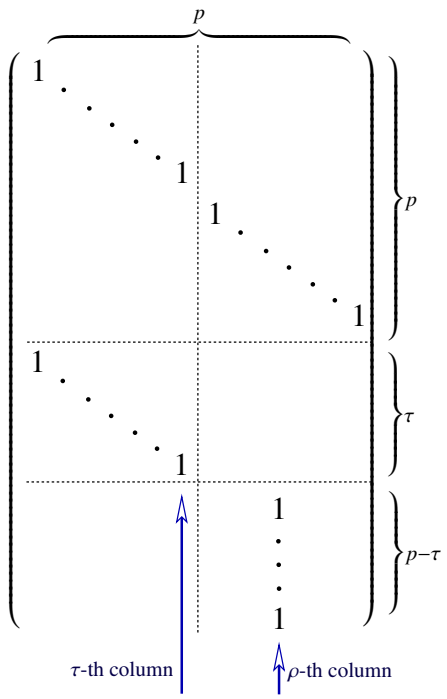
where the second underlined columns are understood to appear in the first underlined places, we realize that they have the uniform shapes:

$$\mathsf{X}_p^{0,\nu} = \mathsf{X}_p\, \mathsf{I}_p^{0,\nu}, \qquad \mathsf{X}_p^{\tau,\rho} = \mathsf{X}_p\, \mathsf{I}_p^{\tau,\rho}, \tag{85}$$

where the $2p \times p$ matrices $\mathsf{I}_p^{0,\nu}$ explicitly read as:



the upper $p \times p$ submatrix being the identity, the lower $p \times p$ submatrix being zero except its ν-th column being a column of 1, and where lastly, the $2p \times p$ matrices $\mathsf{I}_p^{\tau,\rho}$ explicitly read as:



the upper $p \times p$ submatrix being the identity, the lower $p \times p$ submatrix being zero except τ copies of 1 in the beginning diagonal and $p - \tau$ copies of 1 at the end of the ρ-th column.

Next, observe that all matrices $\mathsf{X}^{\tau,\rho}$ have the same first column:

$$\alpha_1 + \beta_1 = (\, \alpha_{1,1} + \beta_{1,1} \mid \cdots \mid \alpha_{p,1} + \beta_{p,1})^{\mathrm{T}}.$$

Therefore, when $\alpha_{1,1}+\beta_{1,1}\neq 0$, operating Gaussian eliminations by means of the matrix:

$$\mathsf{G} := \begin{pmatrix} 1 & & & \\ -\frac{\alpha_{2,1}+\beta_{2,1}}{\alpha_{1,1}+\beta_{1,1}} & 1 & & \\ \vdots & & \ddots & \\ -\frac{\alpha_{p,1}+\beta_{p,1}}{\alpha_{1,1}+\beta_{1,1}} & & & 1 \end{pmatrix}, \tag{86}$$

these matrices $\mathsf{X}^{\tau,\rho}$ become simpler:

$$\mathsf{G}\,\mathsf{X}_p^{\tau,\rho} = \begin{pmatrix} \alpha_{1,1}+\beta_{1,1} & \bullet & \cdots & \bullet \\ 0 & \star & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \star & \cdots & \star \end{pmatrix}, \tag{87}$$

where we will soon see the $(p-1)\times(p-1)$ star submatrices enjoy amazing structural properties.

Observation 11.7. *Let $p\geqslant 1$ be a positive integer, let A be a $p\times 2p$ matrix, let B be a $2p\times p$ matrix such that both its 1-st, $(p+1)$-th rows are $(1,0,\ldots,0)$. Then there holds:*

$$(A\,B)' = A''\,B''',$$

where $(A\,B)'$ means the $(p-1)\times(p-1)$ matrix obtained by deleting the first row and column of $A\,B$, and where A'' means the $(p-1)\times 2(p-1)$ matrix obtained by deleting the first row and the columns $1, p+1$ of A, and where B''' means the $2(p-1)\times(p-1)$ matrix obtained by deleting the first column and the rows $1, p+1$ of B. □

Observation 11.8. *For all $p\geqslant 3$, $\tau = 1\cdots p-1$, $\rho = \tau+1\cdots p$, the matrices $\mathsf{I}_p^{\tau,\rho}$ transform to $\mathsf{I}_{p-1}^{\tau-1,\rho-1}$ after deleting the first column and the rows $1, p+1$, i.e. $\mathsf{I}_{p-1}^{\tau-1,\rho-1} = (\mathsf{I}_p^{\tau,\rho})'''$.* □

Now, thanks to the above two Observations, noting that:

$$\mathsf{G}\,\mathsf{X}_p^{\tau,\rho} = \mathsf{G}\,(\mathsf{X}_p\,\mathsf{I}_p^{\tau,\rho}) = (\mathsf{G}\,\mathsf{X}_p)\,\mathsf{I}_p^{\tau,\rho},$$

denoting $\mathsf{X}_p^{\mathsf{G}} := (\mathsf{G}\,\mathsf{X}_p)''$, the $(p-1)\times(p-1)$ star submatrices in (87) thus have the forms:

$$\begin{pmatrix} \star & \cdots & \star \\ \vdots & \ddots & \vdots \\ \star & \cdots & \star \end{pmatrix} = \mathsf{X}_p^{\mathsf{G}}\,\mathsf{I}_{p-1}^{\tau-1,\rho-1}. \tag{88}$$

Comparing (88) and (85), we immediately see that the star submatrices have the same structures as $\mathsf{X}_p^{0,\nu}, \mathsf{X}_p^{\tau,\rho}$.

11.3. **Study of the morphism of left-multiplication by G.** Let us denote by:

$$D(\alpha_{1,1}+\beta_{1,1}) \;\subset\; \mathsf{Mat}_{p\times 2p}(\mathbb{K})$$

the Zariski open set defined by $\alpha_{1,1}+\beta_{1,1}\neq 0$. Now, consider the morphism of left-multiplication by the regular function matrix G:

$$\begin{aligned} L_{\mathsf{G}}\colon \quad D(\alpha_{1,1}+\beta_{1,1}) &\longrightarrow D(\alpha_{1,1}+\beta_{1,1}) \\ \mathsf{X}_p &\longmapsto \mathsf{G}\,\mathsf{X}_p. \end{aligned}$$

Of course, it is not surjective, as (87) shows that its image lies in the variety:

$$\cap_{i=2}^{p}\{\alpha_{i,1}+\beta_{i,1}=0\}.$$

In order to compensate this loss of surjectivity, combining with:

$$\begin{aligned} \in\colon \quad D(\alpha_{1,1}+\beta_{1,1}) &\longrightarrow \mathsf{Mat}_{(p-1)\times 1}(\mathbb{K}) \\ \mathsf{X}_p &\longmapsto (\alpha_{2,1}+\beta_{2,1} \mid \cdots \mid \alpha_{p,1}+\beta_{p,1})^{\mathsf{T}}, \end{aligned}$$

we construct a morphism:

$$L_{\mathsf{G}} \oplus \in\colon \quad D(\alpha_{1,1}+\beta_{1,1}) \longrightarrow \underbrace{(\cap_{\ell=2}^{p} \{\alpha_{i,1}+\beta_{i,1}=0\} \cap D(\alpha_{1,1}+\beta_{1,1})) \oplus \mathsf{Mat}_{(p-1)\times 1}(\mathbb{K})}_{=:\maltese},$$

which turns out to be an isomorphism. Indeed, it has the inverse morphism:

$$\begin{aligned} \maltese &\longrightarrow D(\alpha_{1,1}+\beta_{1,1}) \\ \mathsf{Y} \oplus (s_2,\ldots,s_p)^{\mathsf{T}} &\longmapsto {}^{-1}\mathsf{G} \cdot \mathsf{Y}, \end{aligned}$$

where the regular function matrix ${}^{-1}\mathsf{G}$ is the 'inverse' of the matrix G in (86):

$$\begin{pmatrix} 1 & & & \\ \frac{s_2}{\alpha_{1,1}+\beta_{1,1}} & 1 & & \\ \vdots & & \ddots & \\ \frac{s_p}{\alpha_{1,1}+\beta_{1,1}} & & & 1 \end{pmatrix}. \tag{89}$$

Now, denote by:

$$\pi_p\colon \quad \mathsf{Mat}_{p\times 2p}(\mathbb{K}) \longrightarrow \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K})$$

the projection obtained by deleting the first row and the columns $1, p+1$. Denote also:

$$\mathscr{L}_{\mathsf{G}} := \pi_p \circ L_{\mathsf{G}}.$$

We can define an isomorphism:

$$\begin{aligned} \mathsf{R}\colon \qquad \maltese &\xrightarrow{\sim} D(\alpha_{1,1}+\beta_{1,1}), \\ \mathsf{Y} \oplus (s_2,\ldots,s_p)^{\mathsf{T}} &\longmapsto \ ? \end{aligned}$$

where ? is Y but replacing $(b_{2,1},\ldots,b_{p,1})^{\mathsf{T}}$ by $(s_2,\ldots,s_p)^{\mathsf{T}}$, and thus we obtain a commutative diagram:

$$\begin{array}{ccccc} D(\alpha_{1,1}+\beta_{1,1}) & \xrightarrow{L_{\mathsf{G}}\oplus\in} & \maltese & \xrightarrow{\mathsf{R}} & D(\alpha_{1,1}+\beta_{1,1}) \\ & {\scriptstyle \mathscr{L}_{\mathsf{G}}} \searrow & \downarrow {\scriptstyle \pi_p\oplus \mathbf{0}} & \swarrow {\scriptstyle \pi_p} & \\ & & \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}), & & \end{array} \tag{90}$$

where the horizontal maps are isomorphisms.

Recalling the end of Subsection 11.2, we in fact receive the following key

Observation 11.9. *For every $p \geqslant 3$, for every $\ell = 1 \cdots p-1$, the image of the variety:*

$$_{\ell}\mathbf{X}_p \cap D(\alpha_{1,1}+\beta_{1,1}) \subset D(\alpha_{1,1}+\beta_{1,1})$$

under the map:

$$\mathscr{L}_{\mathsf{G}}\colon \quad D(\alpha_{1,1}+\beta_{1,1}) \longrightarrow \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K})$$

is contained in the variety:

$$_{\ell}\mathbf{X}_{p-1} \subset \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}). \qquad \square$$

11.4. **A technical lemma.** For all $p \geqslant 3$, for every $\ell = 0 \cdots p-1$, for every $(p-1)\times(p-1)$ matrix J of rank ℓ, denote by ${}_JS_{p,\ell} \cong \mathbb{K}^{2p-1}$ the space which consists of all the $p \times p$ matrices of the form:

$$\left(\begin{array}{c|ccc} z_{1,1} & z_{1,2} & \cdots & z_{1,p} \\ \hline z_{2,1} & & & \\ \vdots & & J & \\ z_{p,1} & & & \end{array}\right).$$

For every $j = \ell, \ell+1$, denote by ${}_JS^j_{p,\ell} \subset {}_JS_{p,\ell}$ all the matrices having rank $\leqslant j$.

Lemma 11.10. *The codimensions of ${}_JS^j_{p,\ell}$ are:*

$$\operatorname{codim}{}_JS^j_{p,\ell} = \begin{cases} 2(p-1-\ell)+1 & (j=\ell), \\ p-1-\ell & (j=\ell+1). \end{cases}$$

Proof. We refer the reader to [16, p. 76, Lemma 10.11] for details. □

11.5. **Proof of Proposition 11.4.** First, applying Lemma 11.3, we receive:

Corollary 11.11. *For every $p \geqslant 1$, the difference of the varieties:*

$$ {}_p\mathbf{X}_p \ \setminus\ {}_{p-1}\mathbf{X}_p \ \subset\ \mathsf{Mat}_{p\times 2p}(\mathbb{K})$$

is exactly the quasi-variety:

$$\Big\{\alpha_1+\cdots+\alpha_p+\beta_1+\cdots+\beta_p=\mathbf{0}\Big\} \ \cap\ \Big\{\mathsf{rank}_{\mathbb{K}}\,\{\alpha_1,\ldots,\alpha_p\}=p\Big\},$$

whose codimension is p. □

Now, we give a complete proof of Proposition 11.4.

Proof of (81). This is a direct consequence of the above corollary. □

Proof of (82). By Observation 11.9, under the map:

$$\mathscr{L}_{\mathsf{G}}\colon \quad D(\alpha_{1,1}+\beta_{1,1}) \longrightarrow \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}),$$

the image of the variety:

$$ {}_{p-1}\mathbf{X}_p \ \cap\ D(\alpha_{1,1}+\beta_{1,1})$$

is contained in the variety:

$$ {}_{p-1}\mathbf{X}_{p-1} \ \subset\ \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}).$$

Now, we decompose the variety ${}_{p-1}\mathbf{X}_{p-1}$ into three pieces:

$$ {}_{p-1}\mathbf{X}_{p-1} \ =\ {}_{p-3}\mathbf{X}_{p-1} \ \cup\ ({}_{p-2}\mathbf{X}_{p-1} \ \setminus\ {}_{p-3}\mathbf{X}_{p-1}) \ \cup\ ({}_{p-1}\mathbf{X}_{p-1} \ \setminus\ {}_{p-2}\mathbf{X}_{p-1}), \tag{91}$$

where each matrix $(\alpha_1,\ldots,\alpha_{p-1},\beta_1,\ldots,\beta_{p-1})$ in the $\underline{\text{first}}_1$ (resp. $\underline{\text{second}}_2$, $\underline{\text{third}}_3$) piece satisfies:

$$\mathsf{rank}_{\mathbb{K}}\,(\alpha_1,\ldots,\alpha_{p-1}) \underline{\leqslant\ p-3}_1 \qquad (\text{resp. } \underline{=\ p-2}_2\,,\ \underline{=\ p-1}_3). \tag{92}$$

Pulling back (91) by the map $\mathscr{L}_{\mathsf{G}}$, we see that:

$$ {}_{p-1}\mathbf{X}_p \ \cap\ D(\alpha_{1,1}+\beta_{1,1})$$

is contained in:

$$\mathscr{L}_{\mathsf{G}}^{-1}\,({}_{p-3}\mathbf{X}_{p-1}) \ \cup\ \mathscr{L}_{\mathsf{G}}^{-1}\,({}_{p-2}\mathbf{X}_{p-1} \ \setminus\ {}_{p-3}\mathbf{X}_{p-1}) \ \cup\ \mathscr{L}_{\mathsf{G}}^{-1}\,({}_{p-1}\mathbf{X}_{p-1} \ \setminus\ {}_{p-2}\mathbf{X}_{p-1}). \tag{93}$$

Firstly, for every point in the first piece:

$$\mathsf{Y} \;\in\; {}_{p-3}\mathbf{X}_{p-1},$$

thanks to the commutative diagram (90), we receive the fibre dimension:

$$\begin{aligned}\dim \mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) &= \dim \ker \pi_p \,\cap\, D(\alpha_{1,1}+\beta_{1,1}) \\ &= \dim \mathsf{Mat}_{p\times 2p}(\mathbb{K}) - \dim \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}).\end{aligned}$$

Now, applying Theorem 10.5 to the regular map $\mathscr{L}$ restricted to:

$$\mathscr{L}_{\mathsf{G}}^{-1}\,({}_{p-3}\mathbf{X}_{p-1}) \;\subset\; \mathsf{Mat}_{p\times 2p}(\mathbb{K})$$

we receive the codimension estimate:

$$\operatorname{codim} \mathscr{L}_{\mathsf{G}}^{-1}\,({}_{p-3}\mathbf{X}_{p-1}) \geqslant \operatorname{codim} {}_{p-3}\mathbf{X}_{p-1}. \tag{94}$$

Secondly, for every point in the second piece:

$$\mathsf{Y} \;\in\; {}_{p-2}\mathbf{X}_{p-1} \setminus {}_{p-3}\mathbf{X}_{p-1},$$

looking at the fibre $\mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y})$, thanks to the commutative diagram (90), we can use:

$$\mathscr{L}_{\mathsf{G}}^{-1} = (\underbrace{\mathsf{R}\circ(L_{\mathsf{G}}\oplus \in)}_{\text{an isomorphism}})^{-1}\circ \pi_p^{-1}, \tag{95}$$

and obtain:

$$\begin{aligned}&\mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) \,\cap\, ({}_{p-1}\mathbf{X}_p \,\cap\, D(\alpha_{1,1}+\beta_{1,1})) \\ \cong\; &\underline{\mathsf{R}\circ(L_{\mathsf{G}}\oplus \in)}\,\mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) \,\cap\, \underline{\mathsf{R}\circ(L_{\mathsf{G}}\oplus \in)}\,({}_{p-1}\mathbf{X}_p \,\cap\, D(\alpha_{1,1}+\beta_{1,1})) \qquad [\underline{\text{isomorphism}}] \\ \cong\; &\underbrace{\pi_p^{-1}(\mathsf{Y}) \,\cap\, \mathsf{R}\circ(L_{\mathsf{G}}\oplus \in)\,({}_{p-1}\mathbf{X}_p \,\cap\, D(\alpha_{1,1}+\beta_{1,1}))}_{=:\clubsuit} \qquad [\text{use (95)}].\end{aligned}$$

Observe now that every matrix:

$$(\alpha_1 \mid \cdots \mid \alpha_p \mid \beta_1 \mid \cdots \mid \beta_p) \;\in\; \clubsuit$$

satisfies the rank estimate:

$$\operatorname{rank}_{\mathbb{K}}(\alpha_1 \mid \cdots \mid \alpha_p) \leqslant p-1.$$

Moreover, noting that the lower-right $(p-1)\times(p-1)$ submatrix J of $(\alpha_1 \mid \cdots \mid \alpha_p)$ is exactly the left $(p-1)\times(p-1)$ submatrix of Y, it follows:

$$\operatorname{rank}_{\mathbb{K}} J = p-2 \qquad [\text{see (92)}].$$

By applying Lemma 11.10, we get that:

$$\clubsuit \;\subset\; \pi_p^{-1}(\mathsf{Y})$$

has codimension greater than or equal to:

$$\operatorname{codim} {}_{J}S^{p-1}_{p,p-2} = p-1-(p-2) = 1.$$

In other words:

$$\mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) \,\cap\, ({}_{p-1}\mathbf{X}_p \,\cap\, D(\alpha_{1,1}+\beta_{1,1})) \;\subset\; \mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y})$$

has codimension $\geqslant 1$. Thus, applying Theorem 10.5 to the map $\mathscr{L}_{\mathsf{G}}$ restricted to:

$$\underbrace{\mathscr{L}_{\mathsf{G}}^{-1}\,({}_{p-2}\mathbf{X}_{p-1} \setminus {}_{p-3}\mathbf{X}_{p-1}) \cap ({}_{p-1}\mathbf{X}_p \,\cap\, D(\alpha_{1,1}+\beta_{1,1}))}_{=:\mathbf{II}} \;\subset\; \mathsf{Mat}_{p\times 2p}(\mathbb{K}),$$

we receive the codimension estimate:

$$\begin{aligned}\operatorname{codim}\mathbf{II} &\geqslant \operatorname{codim}\left({}_{p-2}\mathbf{X}_{p-1} \setminus {}_{p-3}\mathbf{X}_{p-1}\right) + \operatorname{codim}{}_{J}S^{p-1}_{p,p-2}\\ &\geqslant \operatorname{codim}{}_{p-2}\mathbf{X}_{p-1} + 1.\end{aligned} \tag{96}$$

Thirdly, for every point in the third piece:

$$\mathsf{Y} \in {}_{p-1}\mathbf{X}_{p-1} \setminus {}_{p-2}\mathbf{X}_{p-1}, \tag{97}$$

thanks to the diagram (90):

$$\mathscr{L}_{\mathsf{G}}^{-1} = (L_{\mathsf{G}} \oplus \mathbb{C})^{-1} \circ (\pi_p \oplus \mathbf{0})^{-1}, \tag{98}$$

we receive:

$$\begin{aligned}&\mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) \cap \left({}_{p-1}\mathbf{X}_p \cap D(\alpha_{1,1}+\beta_{1,1})\right)\\ \cong\ &\underline{(L_{\mathsf{G}} \oplus \mathbb{C})}\,\mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) \cap \underline{(L_{\mathsf{G}} \oplus \mathbb{C})}\left({}_{p-1}\mathbf{X}_p \cap D(\alpha_{1,1}+\beta_{1,1})\right) \qquad [\underline{\text{isomorphism}}]\\ \cong\ &\underbrace{(\pi_p \oplus \mathbf{0})^{-1}(\mathsf{Y}) \cap (L_{\mathsf{G}} \oplus \mathbb{C})\left({}_{p-1}\mathbf{X}_p \cap D(\alpha_{1,1}+\beta_{1,1})\right)}_{=:\spadesuit} \qquad [\text{use (98)}].\end{aligned}$$

Recalling Corollary 11.11, the sum of all columns of Y — the bottom $(p-1)$ rows of $(\alpha_2 \mid \cdots \mid \alpha_p \mid \beta_2 \mid \cdots \mid \beta_p)$ — is zero. Thus, every element:

$$(\alpha_1 \mid \cdots \mid \alpha_p \mid \beta_1 \mid \cdots \mid \beta_p) \oplus (s_2, \ldots, s_p)^{\mathsf{T}} \in \spadesuit$$

not only satisfies:

$$\operatorname{rank}_{\mathbb{K}}(\alpha_1 \mid \cdots \mid \alpha_p) \leqslant p-1, \tag{99}$$

but also satisfies:

$$\alpha_2 + \cdots + \alpha_p + \beta_2 + \cdots + \beta_p = (\alpha_{1,2} + \cdots + \alpha_{1,p} + \beta_{1,2} + \cdots + \beta_{1,p},\, 0, \ldots, 0)^{\mathsf{T}}.$$

Remembering that:

$$\alpha_1 + \beta_1 = (\alpha_{1,1} + \beta_{1,1},\, 0, \ldots, 0)^{\mathsf{T}},$$

summing the above two identities immediately yields:

$$\alpha_1 + \cdots + \alpha_p + \beta_1 + \cdots + \beta_p = (\alpha_{1,1} + \cdots + \alpha_{1,p} + \beta_{1,1} + \cdots + \beta_{1,p}, 0, \ldots, 0)^{\mathsf{T}}. \tag{100}$$

Now, note that matrices ranks (77) in condition **(ii)** are preserved under the map L_{G}, in particular, for $\nu = 1$, the image satisfies:

$$\operatorname{rank}_{\mathbb{K}}\{\alpha_1 + (\beta_1 + \cdots + \beta_p), \alpha_2, \ldots, \alpha_p\} \leqslant p-1,$$

which, by adding the columns $2 \cdots p$ to the first one, is equivalent to:

$$\operatorname{rank}_{\mathbb{K}}\{\alpha_1 + \cdots + \alpha_p + \beta_1 + \cdots + \beta_p, \alpha_2 \ldots, \alpha_p\} \leqslant p-1. \tag{101}$$

Recalling (97) and Corollary 11.11:

$$\text{the bottom } (p-1)\times(p-1) \text{ submatrix of } (\alpha_2 \mid \cdots \mid \alpha_p) \text{ is of full rank } (p-1), \tag{102}$$

combining (100), (101), we immediately receive:

$$\alpha_{1,1} + \cdots + \alpha_{1,p} + \beta_{1,1} + \cdots + \beta_{1,p} = 0.$$

Therefore, by applying Lemma 11.10, the restrictions (99) and (102) contribute one extra codimension:

$$\operatorname{codim}{}_{J}S^{p-1}_{p,p-1} = 1.$$

Thus, we see that 'the fibre in fibre':

$$\spadesuit \cap (\pi_p \oplus \mathbf{0})^{-1}(\mathsf{Y}) \quad \subset \quad (\pi_p \oplus \mathbf{0})^{-1}(\mathsf{Y})$$

has codimension greater than or equal to:

$$1 + 1 = 2.$$

Now, applying once again Theorem 10.5 to the map $\mathscr{L}_{\mathsf{G}}$ restricted to:

$$\underbrace{\mathscr{L}_{\mathsf{G}}^{-1}\left({}_{p-1}\mathbf{X}_{p-1} \setminus {}_{p-2}\mathbf{X}_{p-1}\right) \cap \left({}_{p-1}\mathbf{X}_{p} \cap D(\alpha_{1,1} + \beta_{1,1})\right)}_{=: \mathbf{III}} \quad \subset \quad \mathsf{Mat}_{p\times 2p}(\mathbb{K}),$$

we receive the codimension estimate:

$$\begin{aligned} \mathsf{codim}\, \mathbf{III} &\geqslant \mathsf{codim}\left({}_{p-1}\mathbf{X}_{p-1} \setminus {}_{p-2}\mathbf{X}_{p-1}\right) + 2 \\ &\geqslant \mathsf{codim}\, {}_{p-1}\mathbf{X}_{p-1} + 2. \end{aligned} \tag{103}$$

Summarizing (93), (94), (96), (103), we receive the codimension estimate:

$$\mathsf{codim}\, {}_{p-1}\mathbf{X}_{p} \cap D(\alpha_{1,1} + \beta_{1,1}) \geqslant \min\left\{\mathsf{codim}\, {}_{p-3}\mathbf{X}_{p-1},\ \mathsf{codim}\, {}_{p-2}\mathbf{X}_{p-1} + 1,\ \mathsf{codim}\, {}_{p-1}\mathbf{X}_{p-1} + 2\right\}.$$

By permuting the indices, we know that all:

$${}_{p-1}\mathbf{X}_{p} \cap D(\alpha_{i,1} + \beta_{i,1}) \quad \subset \quad \mathsf{Mat}_{p\times 2p}(\mathbb{K}) \qquad {\scriptstyle (i = 1 \cdots p)}$$

have the same codimension, and so does their union:

$${}_{p-1}\mathbf{X}_{p} \cap D(\alpha_1 + \beta_1) = \cup_{i=1}^{p}\left({}_{p-1}\mathbf{X}_{p} \cap D(\alpha_{i,1} + \beta_{i,1})\right) \quad \subset \quad \mathsf{Mat}_{p\times 2p}(\mathbb{K}).$$

Finally, taking codimension on both sides of:

$${}_{p-1}\mathbf{X}_{p} = \left({}_{p-1}\mathbf{X}_{p} \cap V(\alpha_1 + \beta_1)\right) \cup \left({}_{p-1}\mathbf{X}_{p} \cap D(\alpha_1 + \beta_1)\right),$$

Proposition 11.2 and the preceding estimate conclude the proof. □

Proof of (83). If $\ell \geqslant 2$, decompose the variety ${}_{\ell}\mathbf{X}_{p-1}$ into three pieces:

$${}_{\ell}\mathbf{X}_{p-1} = {}_{\ell-2}\mathbf{X}_{p-1} \cup \left({}_{\ell-1}\mathbf{X}_{p-1} \setminus {}_{\ell-2}\mathbf{X}_{p-1}\right) \cup \left({}_{\ell}\mathbf{X}_{p-1} \setminus {}_{\ell-1}\mathbf{X}_{p-1}\right);$$

and if $\ell = 1$, decompose the variety ${}_{\ell}\mathbf{X}_{p-1}$ into two pieces:

$${}_{1}\mathbf{X}_{p-1} = {}_{0}\mathbf{X}_{p-1} \cup \left({}_{1}\mathbf{X}_{p-1} \setminus {}_{0}\mathbf{X}_{p-1}\right).$$

Now, by mimicking the preceding proof, namely by applying Lemma 11.10 and Theorem 10.5, everything goes on smoothly with much less effort, because there is no need to perform delicate codimension estimates such as (103). □

11.6. **Proof of Core Lemma.** Comparing (55) and (75), it is natural to introduce the projection:

$$\begin{aligned} \pi_{M,N}\colon \quad \mathsf{Mat}_{M\times 2(N+1)}(\mathbb{K}) &\longrightarrow \mathsf{Mat}_{N\times 2N}(\mathbb{K}) \\ (\alpha_0, \ldots, \alpha_p, \beta_0, \ldots, \beta_p) &\longmapsto (\widehat{\alpha}_1, \ldots, \widehat{\alpha}_p, \widehat{\beta}_1, \ldots, \widehat{\beta}_p), \end{aligned}$$

where each widehat vector is obtained by extracting the first N entries.

It is direct to check that:

$$\pi_{M,N}\left(\mathscr{M}_{M}^{N}\right) \quad \subset \quad {}_{N-1}\mathbf{X}_{N}.$$

Moreover, for every $\mathsf{Y} \in {}_{N-1}\mathbf{X}_{N}$, the 'fibre in fibre':

$$\pi_{M,N}^{-1}(\mathsf{Y}) \cap \mathscr{M}_{M}^{N} \quad \subset \quad \pi_{M,N}^{-1}(\mathsf{Y}),$$

thanks to (56), has codimension $\geqslant M$. Thus a direct application of Theorem 10.5 yields:

$$\operatorname{codim} \mathscr{M}_M^N \ \geqslant\ \operatorname{codim}_{N-1}\mathbf{X}_N + M \ \geqslant\ N+1+M \qquad \text{[use (84)]}. \tag{104}$$

12. Algorithm of MCM and Estimates on Hypersurface Degrees

12.1. **A must design.** Given the integer $\heartsuit \geqslant 1$ in Theorem 5.1, recalling the Step 2 in Subsection 5.3, we naturally carry out the *Algorithm* of MCM as follows.

The procedure is to first construct $\mu^{l,k}$ in a lexicographic order with respect to indices (l,k), for $l = c+r+1\cdots N, k = 0\cdots l$, along with a set of positive integers δ_l.

For simplicity, we start by setting:

$$\delta_{c+r+1} \ \geqslant\ \max\{\epsilon_1,\ldots,\epsilon_{c+r}\}. \tag{105}$$

For every integer $l = c+r+1\cdots N$, in this step, we begin with choosing $\mu_{l,0}$ that satisfies:

$$\text{[see (69), (65)]} \qquad \mu_{l,0} \geqslant l\,\delta_l + l\,(\delta_{c+r+1}+1) + 1 + (l-c-r)\,\heartsuit, \tag{106}$$

then inductively we choose $\mu_{l,k}$ with:

$$\text{[see (70), (66)]} \qquad \mu_{l,k} \ \geqslant\ \sum_{j=0}^{k-1} l\,\mu_{l,j} + (l-k)\,\delta_l + l\,(\delta_{c+r+1}+1) + 1 + (l-c-r)\,\heartsuit \qquad (k=1\cdots l). \tag{107}$$

If $l < N$, we end this step by setting:

$$\delta_{l+1} := l\,\mu_{l,l} \tag{108}$$

as the starting point for the next step $l+1$. At the end $l = N$, we make the integer d large enough:

$$d \ \geqslant\ (N+1)\,\mu_{N,N}. \tag{109}$$

12.2. **Effective lower degree bounds.** Recalling Subsection 5.4, we first provide an effective

Theorem 12.1. *For all $N \geqslant 3$, for any $\epsilon_1,\ldots,\epsilon_{c+r} \in \{1,2\}$, Theorem 5.4 holds for all $d \geqslant N^{N^2/2}-1$.*

Proof. Setting $\delta_{c+r+1} = 2$ in (105), and demanding all (106) – (109) to be equalities, we thus receive the desired estimate without much difficulty. See [16, Section 11] for details. □

Hence, the *product coup* in Subsection 5.4 yields

Theorem 12.2. *In Theorem 5.1, for $\heartsuit = 1$, the lower bound $\mathfrak{d}_0(-1) = N^{N^2}$ works.* □

12.3. **Some improvements of MCM.** In fact, the hypersurface equations (61) contain many more moving coefficient terms than required — as indicated by (104) which strengthens the Core Lemma — which make the lower bound $\mathfrak{d}_0(-1)$ big.

Indeed, for instance when $2\,(2c+r) \geqslant 3N-2$, pure Fermat-type hypersurfaces are adequate for Theorem 5.1. Thus, by setting $\epsilon_1 = \cdots = \epsilon_{c+r} = 1$, in order to make (41) negative, we only need to require $d_1 = \cdots = d_{c+r} \geqslant 2N+3$. Also, the product coup asserts that the lower degree bound $d_1,\ldots,d_{c+r} \geqslant (3N+2)\,(3N+3)$ works.

More improvements and estimates of lower degree bounds can be found in [16, Section 12].

13. Uniform Very-Ampleness of $\mathsf{Sym}^\kappa \Omega_X$

13.1. **The canonical bundle of a projectivized vector bundle.** Let X be an n-dimensional complex manifold, with canonical line bundle $\mathscr{K}_X$. Let E be a holomorphic vector bundle on X having rank e, with dual bundle $E^\vee$, and let $\pi\colon \mathbb{P}(E) \longrightarrow X$ be the projectivization of E.

Theorem 13.1. (See [16, Proposition 13.2]) *The canonical line bundle $\mathscr{K}_{\mathbb{P}(E)}$ of $\mathbb{P}(E)$ satisfies:*

$$\mathscr{K}_{\mathbb{P}(E)} \;\cong\; \mathscr{O}_{\mathbb{P}(E)}(-e) \otimes \pi^* \wedge^e E^\vee \otimes \pi^* \mathscr{K}_X. \qquad \square$$

In applications, we are interested in the case where $X \subset V$ for some complex manifold V of dimension $n + r$, and $E = \mathrm{T}_V\big|_X$.

Corollary 13.2. *One has:*

$$\mathscr{K}_{\mathbb{P}(\mathrm{T}_V|_X)} \;\cong\; \mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(-n-r) \otimes \pi^* \mathscr{K}_V\big|_X \otimes \pi^* \mathscr{K}_{\mathscr{X}}. \qquad \square$$

Moreover, assume that X, V are some smooth complete intersections in $\mathbb{P}^N_{\mathbb{C}}$, so their canonical line bundles $\mathscr{K}_X, \mathscr{K}_V$ have neat expressions by the following classical theorem.

Theorem 13.3. *For a smooth complete intersection:*

$$Y \;:=\; D_1 \cap \cdots \cap D_k \;\subset\; \mathbb{P}^N_{\mathbb{C}}$$

with divisor degrees:

$$\deg D_i \;=\; d_i \qquad (i = 1 \cdots k),$$

the canonical line bundle $\mathscr{K}_Y$ of Y satisfies:

$$\mathscr{K}_Y \;\cong\; \mathscr{O}_Y\Big(-N-1+\sum_{i=1}^{k} d_i\Big). \qquad \square$$

13.2. **Proof of the Very-Ampleness Theorem 1.3.** Now, assume $\mathbb{K} = \mathbb{C}$. Recall our Ampleness Theorem 1.2, $\dim_{\mathbb{C}} X = N - (c + r) = n$. Then the above Corollary 13.2 and Theorem 13.3 imply:

$$\mathscr{K}_{\mathbb{P}(\mathrm{T}_V|_X)} \;\cong\; \mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(-n-r) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}\Big(-2\,(N+1)+\sum_{i=1}^{c} d_i+\sum_{i=1}^{c+r} d_i\Big).$$

Also, recalling Theorem 5.1 and Proposition 4.3, for generic choices of $H_1, \ldots, H_{c+r}$, for any positive integers $a > b \geqslant 1$, the negatively twisted line bundle below is ample:

$$\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(a) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-b).$$

It is now time to recall (cf. the survey [5])

A known result about the Fujita Conjecture. *Let M be an n-dimensional complex manifold with canonical line bundle $\mathscr{K}_M$. If $\mathscr{L}$ is any positive holomorphic line bundle on M, then $\mathscr{L}^{\otimes m} \otimes \mathscr{K}_M^{\otimes 2}$ is very ample for all large $m \geqslant 2 + \binom{3n+1}{n}$.*

Consequently, the line bundle below is very ample:

$$\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}\Big(m\,a-2n-2r\Big) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}\Big(-m\,b-4\,(N+1)+2\sum_{i=1}^{c} d_i+2\sum_{i=1}^{c+r} d_i\Big). \tag{110}$$

Also recalling (11), for all $\ell \geqslant 3$, the line bundle below is very ample:

$$\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell). \tag{111}$$

Note the following two facts:

(A) if $\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(\kappa) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\star)$ is very ample, then so it is for every $\star' \geqslant \star$;

(B) the tensor product of any two very ample line bundles remains very ample.

We now consider the semigroup $\mathscr{G}$ of the usual Abelian group $\mathbb{Z} \oplus \mathbb{Z}$ generated by elements (ℓ_1, ℓ_2) such that $\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(\ell_1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell_2)$ is very ample. Then, for all $m \geqslant 2 + \binom{3n+1}{n}$, $\mathscr{G}$ contains:

$$\text{[see (110)]} \qquad \Big(m\,a - 2n - 2r,\ -m\,b - 4\,(N+1) + 2\sum_{i=1}^{c} d_i + 2\sum_{i=1}^{c+r} d_i\Big),$$

$$\text{[see (111)]} \qquad (1, \ell), \qquad \forall\, \ell \geqslant 3.$$

Paying no attention to optimality, taking:

$$b = 1,\ a = 2,\ m = -4\,(N+1) + 2\sum_{i=1}^{c} d_i + 2\sum_{i=1}^{c+r} d_i + 3,$$

we receive that $(m\,a - 2n - 2r, -3) \in \mathscr{G}$. Adding $(1, 3) \in \mathscr{G}$, we receive $(m\,a - 2n - 2r + 1, 0) \in \mathscr{G}$. Now, also using $(m\,a - 2n - 2r, 0) \in \mathscr{G}$, recalling Observation 5.6, we may take:

$$\kappa_0 \ = \ (m\,a - 2n - 2r - 1)\,(m\,a - 2n - 2r) \ \leqslant \ a^2\,m^2,$$

or the larger neater lower bound:

$$\kappa_0 \ = \ 16\Big(\sum_{i=1}^{c} d_i + \sum_{i=1}^{c+r} d_i\Big)^2.$$

Thus, we have proved the Very-Ampleness Theorem 1.3 for $\mathbb{K} = \mathbb{C}$. Remembering that *very-ampleness (or not) is preserved under any base change obtained by ambient field extension*, and noting the field extensions $\mathbb{Q} \hookrightarrow \mathbb{C}$ and $\mathbb{Q} \hookrightarrow \mathbb{K}$ for any field $\mathbb{K}$ with characteristic zero, by some standard arguments in algebraic geometry, we conclude the proof of the Very-Ampleness Theorem 1.3. □

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Titre: Sur l'amplitude des fibrés cotangents d'intersections complètes

Mots clés: Conjecture d'amplitude de Debarre, Intersection complète, Forme différentielles symétrique, Méthode des coefficients mobiles, Coup du produit, Règle de Cramer, Lieu-base

Résume: Nous établissons la *Conjecture d'amplitude de Debarre: Le fibré cotangent Ω_X d'une intersection $X = H_1 \cap \cdots \cap H_c$ de $c \geqslant N/2$ hypersurfaces génériques $H_i \subset \mathbb{P}^N_{\mathbb{C}}$ de degrés élevés $d_1, \ldots, d_c \gg 1$ est ample.*

Tout d'abord, nous élaborons une interprétation géométrique des différentielles symétriques sur les espaces projectifs. De cette manière, nous reconstruisons les différentielles symétriques de Brotbek sur X, lorsque les équations définissantes des hypersurfaces $H_1, \ldots, H_c$ sont de type Fermat généralisé. De plus, nous dévoilons des familles nouvelles de différentielles symétriques de degré inférieur sur toutes les intersections possibles de X avec des hyperplans de coordonnées.

Ensuite, nous introduisons ce que nous appelons la '*Méthode des Coefficients Mobiles*' ainsi que le '*Coup du Produit*' afin d'accomplir une démonstration de la conjecture d'amplitude de Debarre. De plus, nous obtenons une borne effective inférieure sur les degrés: $d_1, \ldots, d_c \geqslant N^{N^2}$. Enfin, grâce à des résultats connus au sujet de la conjecture de Fujita, nous établissons que $\mathsf{Sym}^\kappa \Omega_X$ est très ample pour tout $\kappa \geqslant 64 \left(\sum_{i=1}^c d_i \right)^2$.

Title: On the ampleness of the cotangent bundles of complete intersections

Keywords: Debarre Ampleness Conjecture, Complete intersection, Symmetric differential form, Moving Coefficients Method, Product coup, Cramer's rule, Base loci

Abstract: We establish the *Debarre Ampleness Conjecture: The cotangent bundle Ω_X of the intersection $X = H_1 \cap \cdots \cap H_c$ of $c \geqslant N/2$ generic hypersurfaces $H_i \subset \mathbb{P}^N_{\mathbb{C}}$ of high degrees $d_1, \ldots, d_c \gg 1$ is ample.*

First of all, we provide a geometric interpretation of symmetric differential forms in projective spaces. Thereby, we reconstruct Brotbek's symmetric differential forms on X, where the defining hypersurfaces $H_1, \ldots, H_c$ are generalized Fermat-type. Moreover, we exhibit unveiled families of lower degree symmetric differential forms on all possible intersections of X with coordinate hyperplanes.

Thereafter, we introduce what we call the '*moving coefficients method*' and the '*product coup*' to settle the Debarre Ampleness Conjecture. In addition, we obtain an effective lower degree bound: $d_1, \ldots, d_c \geqslant N^{N^2}$. Lastly, thanks to known results about the Fujita Conjecture, we establish the very-ampleness of $\mathsf{Sym}^\kappa \Omega_X$ for all $\kappa \geqslant 64 \left(\sum_{i=1}^c d_i \right)^2$.

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