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# Time-varying consensus : application to formation control of vehicles

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UNIVERSITÉ PARIS-SUD

ÉCOLE DOCTORALE 422:  
SCIENCES ET TECHNOLOGIES DE L'INFORMATION DES  
TÉLÉCOMMUNICATIONS ET DES SYSTÈMES

Laboratoire des Signaux et systèmes (L2S)

THÈSE DE DOCTORAT

PHYSIQUE

par

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CONSENSUS VARIANT DANS LE TEMPS: APPLICATION À LA  
FORMATION DE VÉHICLES.

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# *Abstract*

The multiple applications related to networked multi-agent systems such as satellite formation flying, coupled oscillators, air traffic control, unmanned air vehicles, cooperative transport, among others, has been undoubtedly a watershed for the development of this thesis. The study of cooperative control of multi-agent systems is of great interest for his extensive field work and applications. This thesis is devoted to the study of consensus seeking of multi-agents systems and trajectory tracking of nonholonomic mobile robots.

In the context of consensus seeking, first we study a ring topology of dynamic agents with time-dependent communication links which may disconnect for long intervals of time. Simple checkable conditions are obtained by using small-gain theorem to guarantee the achievement of consensus. Then, we deal with a network of dynamic agents with time-dependent communication links interconnected over a time-varying topology. We establish that consensus is reached provided that there always exists a spanning tree for a minimal dwell-time by using stability theory of time-varying and switched systems.

In the context of trajectory tracking, we investigate a simple leader-follower tracking controller for autonomous vehicles following straight lines. We show that global tracking may be achieved by a controller which has a property of persistency of excitation tailored for nonlinear systems. Roughly speaking the stabilisation mechanism relies on exciting the system by an amount that is proportional to the tracking error. Moreover, the method is used to solve the problem of formation tracking of multiple vehicles interconnected on the basis of a spanning-tree topology. We derive stability conditions for the kinematic and dynamic model by using a Lyapunov approach.

**Keywords:** consensus, time-varying systems, mobile robot, persistence of excitation (PE), trajectory tracking, dwell time, stability analysis, Lyapunov theory.



# Résumé

Les multiples applications liées aux systèmes multi-agents en réseau, tels que les satellites en formation, les oscillateurs couplés, les véhicules aériens sans pilote, entre autres, ont été sans aucun doute, une motivation majeure dans le développement de cette thèse qui est consacrée à l'étude du consensus de systèmes dynamiques et à la commande en formation de robots mobiles non holonomes.

Dans le contexte du consensus, nous étudions la topologie en anneau avec de liens de communication variant dans le temps. Notamment, la communication peut être perdue pendant de longs intervalles de temps. Nous donnons de conditions suffisantes pour le consensus qui restent simples à vérifier, par exemple, en utilisant le théorème du petite gain. En suite, nous abordons le problème de consensus en supposant que la topologie de communication est variable. Nous établissons que le consensus est atteint à condition qu'il existe toujours un chemin de communication du type « spanning-tree » pendant un temps de séjour minimal. L'analyse, s'appuie sur la théorie de stabilité des systèmes variant dans le temps et les systèmes variant dans le temps et les systèmes à commutation.

Dans le contexte de la commande en formation de véhicules autonomes nous adressons le problème de commande en suivi de trajectoire sur ligne droite en suivant une approche type maître-esclave. Nous montrons que le suivi global peut être obtenu à partir d'un contrôleur qui possède la propriété d'excitation persistante. En gros, le mécanisme de stabilisation dépend de l'excitation du système par une quantité qui est proportionnelle à l'erreur de suivi. Ensuite, la méthode est utilisée pour résoudre le problème de suivi de formation de plusieurs véhicules interconnectés sur la base d'une topologie « spanning-tree ». Nous donnons des conditions de stabilité concernant les modèles cinématique et dynamique, en utilisant la seconde méthode de Lyapunov.

**Mot clés:** consensus, systèmes variant dans le temps, robot mobile, excitation persistante (PE), suivi de trajectoire, temps de séjour, analyse de la stabilité, théorie de Lyapunov.



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*"I have no special talents. I am only passionately curious."*

Albert Einstein



# TABLE OF CONTENTS

<b>Contents</b>	<b>ix</b>
<b>List of Figures</b>	<b>xi</b>
<b>List of Tables</b>	<b>xiii</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Motivation . . . . .	1
1.2 State of the art . . . . .	3
1.2.1 Consensus seeking . . . . .	3
1.2.2 Trajectory tracking . . . . .	4
1.3 Summary of results and organization . . . . .	5
<b>2 Mathematical Preliminaries</b>	<b>7</b>
2.1 Graph Theory . . . . .	8
2.1.1 Adjacency Matrix . . . . .	8
2.1.2 Laplacian Matrix . . . . .	9
2.2 Stability theory . . . . .	10
2.2.1 Linear time-varying systems . . . . .	11
2.3 Switching systems . . . . .	12
2.3.1 State-dependent switching . . . . .	12
2.3.2 Time dependent switching . . . . .	13
2.3.3 Autonomous switching . . . . .	14
2.3.4 Controlled switching . . . . .	14
2.3.5 Dwell time . . . . .	14
<b>3 Consensus seeking under Persistent Interconnections</b>	<b>15</b>
3.1 Consensus seeking under Persistent Interconnections in a Ring Topology .	16
3.1.1 Stability analysis . . . . .	17
3.1.2 Illustrative example and Simulation Results . . . . .	23
3.2 Consensus for multi-agent systems under persistent interconnections over a time-varying topology. . . . .	26
3.2.1 The network model . . . . .	26
3.2.2 Fixed topology with time-varying interconnections . . . . .	29
3.2.3 Time-varying topology . . . . .	30
3.2.4 Illustrative example and simulation results . . . . .	35
3.3 Discussion . . . . .	38

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<b>4</b>	<b>Leader-follower formation and tracking control of mobile robots along the straight paths</b>	<b>39</b>
4.1	Introduction . . . . .	39
4.2	Leader-follower tracking controller . . . . .	40
4.3	Leader-follower formation control . . . . .	49
4.4	Numerical example and simulation results . . . . .	64
4.4.1	Kinematic formation control . . . . .	64
4.4.2	Dynamic formation control . . . . .	67
4.5	Discussion . . . . .	69
<b>5</b>	<b>Conclusions</b>	<b>71</b>
5.1	Conclusions . . . . .	71
5.2	Perspectives . . . . .	72
<b>A</b>	<b>Appendix</b>	<b>73</b>
	<b>Bibliography</b>	<b>75</b>

# List of Figures

1.1	Biological examples of multi agent systems in nature: (a) Fireflies (b) neuron firing, (c) flocking of migrating birds, and (d) fish schooling. . . .	2
2.1	Three examples of directed graph with rooted spanning trees. Only the graph (3) is strongly connected and has a cyclic path. . . . .	9
2.2	State-dependent switching. . . . .	12
2.3	Switching signal. . . . .	13
3.1	Ring topology with time dependent communication links. . . . .	16
3.2	Persistently exciting signal $a$ with $T = 3$ and $\mu = 1$ . . . . .	18
3.3	Property of a persistently exciting signal. . . . .	18
3.4	Interconnected system . . . . .	19
3.5	Feedback system. . . . .	20
3.6	Ring topology with four interconnected agents . . . . .	23
3.7	Persistently exciting signals $a_{12}(t)$ (top), $a_{23}(t)$ (middle) and $a_{34}(t)$ (bottom). . . . .	24
3.8	Persistently exciting signals $a_{41}(t)$ . . . . .	25
3.9	Trajectories of states $x_1(t)$ , $x_2(t)$ , $x_3(t)$ and $x_4(t)$ . . . . .	25
3.10	Example of 3 agents, where by changing their positions, we obtained six possible topologies. . . . .	27
3.11	A spanning-tree topology with time dependent communication links between $\Psi_{\pi_k}$ and $\Psi_{\pi_{k+1}}$ . . . . .	27
3.12	A spanning-tree topology with time dependent communication links between $\Psi_\lambda$ and $\Psi_{\lambda+1}$ . . . . .	28
3.13	A topology with 3 agents where $\pi_1 = 1$ , $\pi_2 = 2$ and $\pi_3 = 3$ . . . . .	31
3.14	The second topology with 3 agents where $\pi_1 = 1$ , $\pi_2 = 3$ and $\pi_3 = 2$ . . . .	31
3.15	Persistently exciting interconnection gains for the topologies $\{\Psi_1, \Psi_2, \Psi_3\}$ , $\{\Psi_2, \Psi_1, \Psi_3\}$ and $\{\Psi_3, \Psi_1, \Psi_2\}$ . . . . .	36
3.16	Persistently exciting interconnection gains for the topologies $\{\Psi_1, \Psi_3, \Psi_2\}$ , $\{\Psi_2, \Psi_3, \Psi_1\}$ and $\{\Psi_3, \Psi_2, \Psi_1\}$ . . . . .	37
3.17	A switching signal $\sigma(t)$ satisfying the dwell-time condition . . . . .	37
3.18	Trajectories of $x_1(t)$ , $x_2(t)$ and $x_3(t)$ . . . . .	38
4.1	A two-wheel mobile robot. . . . .	39
4.2	Generic representation of a leader-follower configuration. For a swarm of $n$ vehicles, any geometric topology may be easily defined by determining the position of each vehicle relative to its leader. This does not affect the kinematic model. . . . .	41

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4.3	Small gain feedback representation of the closed-loop system with a persistently exciting controller. . . . .	43
4.4	Communication topology: a spanning directed tree with permanent communication between $R_i$ and $R_{i+1}$ for all $i \in [0, n - 1]$ . . . . .	49
4.5	Feedback representation of the closed-loop system (4.31). . . . .	55
4.6	Angular reference velocity. . . . .	65
4.7	Generated paths described by the mobile robots. . . . .	65
4.8	Position errors in $x$ coordinates with kinematic control algorithm. . . . .	66
4.9	Position errors in $y$ coordinates with kinematic control algorithm. . . . .	66
4.10	Heading errors with kinematic control algorithm. . . . .	67
4.11	Position errors in $x$ coordinates. . . . .	68
4.12	Position errors in $y$ coordinates. . . . .	68
4.13	Heading errors. . . . .	69
4.14	Motion and positioning of the robots in aligned formation on the plane. . .	69

# List of Tables

- 3.1 Parameters  $T_i$  and  $\mu_i$  corresponding to the signal  $a_{ii+1}$ ,  $i = 1, 2, 3$ . . . . . 23
- 3.2 Parameters  $\bar{k}_i$  and  $k_i$  corresponding to the signal  $a_{ii+1}$ ,  $i = 1, 2, 3$ . . . . . 24



# Chapter 1

## Introduction

### 1.1 Motivation

During the last twenty years, considerable advances in communications technology and electronics have enabled the development of multiple agents systems to help accomplish tasks that cannot be completed with individual ones acting alone. The tasks previously performed with a big, expensive and complicated to operate equipment, today can be achieved using a certain number of systems, less expensive and much smaller with greater reliability.

The idea of multiple agents working and cooperating was inspired by many examples in biology and life science. Figure 1.1 illustrates a few biological examples of multi-agent systems in nature: Fireflies, neuron firing, flocking of migrating birds and fish schooling. In certain species of the tropical far east the fireflies gather in trees by thousands and the males flash in rhythmic synchrony with the sole purpose of sexual adaptation [5]. Many species of birds fly together in V formation to reduce energy expenditure and enhance the locomotor performance individuals in the assemblage [48].

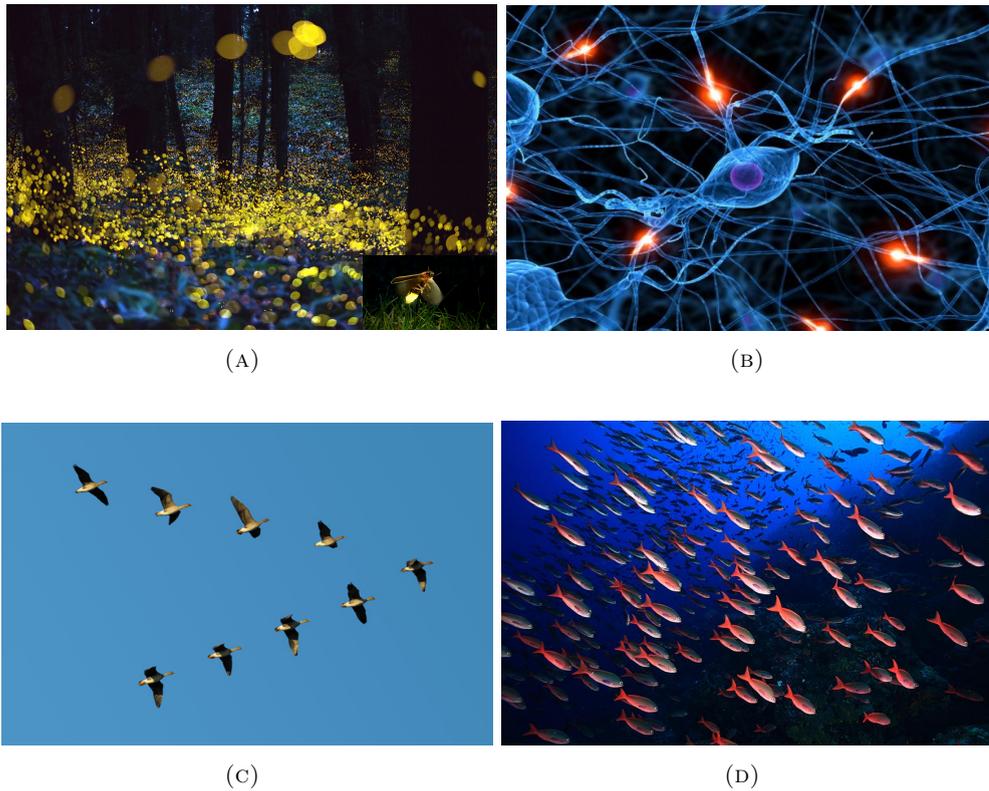


FIGURE 1.1: Biological examples of multi agent systems in nature: (a) Fireflies (b) neuron firing, (c) flocking of migrating birds, and (d) fish schooling.

Around the year 2000, the consensus control of swarm mobile agents increased the attention and is still of great interest for researchers of several disciplines related to networked multi-agent systems, due to the multiple applications that exist. The consensus problem consists in establishing conditions under which the differences between any two motions among a group of dynamic systems, converge to zero asymptotically. Satellite formation flying [7, 55], coupled oscillators [10, 17], formation tracking control for mobile robots [11], coupled air traffic control [58], cooperative control of unmanned air vehicles (UAVS) [20, 49], spacecraft [51], just to mention a few. These applications justify the design of appropriate consensus protocols to drive all dynamic agents to a common value.

Nowadays, studies of multi agent systems and their cooperative controls are a popular research area [50, 57]. We are interested in the study of two important issues related to cooperative control: consensus seeking and trajectory tracking. Consensus seeking is the study of consensus in a network of agents that communicates with each other and need to agree upon a certain objective of interest. In a network of multi-agents systems interconnected the information exchange among the agents plays an important role since of it depends that the objective of control is carried out. However, sometimes, due to

failures or an obstacle between them, the communication links can fail for long intervals of time and is necessary that the group of agents achieve a consensus data.

The control techniques for the tracking control problem of mobile robots have been studied during the last years. One of the drawbacks to these techniques is that the controller relies on the assumption that the angular velocity of the leader robot must be different from zero. Ruling out straight-line paths. This motivates the research of new approaches to design new nonlinear control techniques that can help to the mobile robots to follow straight lines.

## 1.2 State of the art

### 1.2.1 Consensus seeking

Consensus pertains to the case in which a set of individuals have a synchronised motion towards a common goal. One of the main features is that the motion of no individual has priority over that of any other one. Although such type of synchronisation is known for many years in the scientific community, the term consensus stems from the literature on computer science. Scientific studies strictly following the scientific method date centuries back but have boomed at least from the 19th century, among physicists and biologists. Numerous attempts to throw in rigorous mathematics have also been made. Exhaustive simulations triggered by the exponential evolution of computing power have led to the development of so-called consensus theory as an area of study in its own right, within computer science.

The case of constant fixed topologies with permanent all-to-all interconnections is completely solved by various means –see [53], the problem of consensus under intermittent interconnections is still of actual interest. It is clearly motivated, for instance, by scenarios in which communication failures appear. In the pioneer paper [39] the author presents a coordination problem for a network of agents with single integrator dynamics. The network coupling is allowed to be time-dependent and non-bidirectional. It is represented by a linear time-varying system in continuous time with a Metzler matrix. Stability analysis shows that all state components converge to a common value as time grows unbounded using a Lyapunov function.

In [26] the authors provide a theoretical explanation for the observed behavior of Vicsek model [59]. Explicitly takes into account possible changes of each agent's nearest neighbors over time, can be thought of as a consensus problem. Using a classical

convergence result show that all agents converge to a common steady state provided all agents are “linked together” with sufficient frequency as the system evolves.

The approaches in [39] and [26] are based on undirected graphs. When the information exchange is unidirectional, that is, consensus may to be achieved in the presence of limited information exchange. In [52] the authors present both continuous and discrete time consensus problem for switching directed graphs. Considering that the Laplacian matrix is piecewise constant it is shown that consensus is achieved if and only if the topology has a directed spanning tree. In [43] a class of nonlinear consensus protocol for networks of dynamic agents whose dynamics are simple integrators is prescribed. A Lyapunov function is introduced and quantifies the total disagreement among the agents, guaranteeing that consensus problem is globally asymptotically achieved.

### 1.2.2 Trajectory tracking

For many years the study of mechanical control systems called *nonholonomic systems* has been very active research field. An example of a nonholonomic systems is a mobile robot. The stabilization problem at a given position requires a nontrivial controller [54]. The tracking problem for mobile robots has been studied by [29, 40]. Several control approaches have been used to design formation-tracking controllers such as: backstepping [8], sliding mode [1], artificial neural network [18], feedback linearization [14, 29], etc. The seminal paper [29] shows how a feedback controller guarantees that a mobile robot follows a desired reference which is generated by “virtual” robot; the convergence proof is based on local stability results for time-varying systems.

One of the most popular control approaches is the leader-follower technique which consists in specifying one or several leader robots and several followers. For instance, there may be one single leader which specifies the trajectory for the formation and all the rest are set to follow the leader, modulo a position and orientation offset determined by the physical configuration. Then, following the seminal work [29] on tracking control of mobile robots, one can use a variety of nonlinear control techniques to ensure individual tracking control on each follower. Alternatively, one may form a cascade of leader-follower configurations in which each robot follows one leader [12], [56], [9]. Backstepping control is used in [8] and the problem under additive disturbances is solved via sliding mode in [12]. The control approach based on the construction of transverse functions is presented in [45] which guarantees the practical stabilization of the nonholonomic system to any desired trajectory. Another approach is that of virtual structure control, in which the swarm is regarded as a virtual rigid structure advancing as a unit. This approach is tractable for small groups of autonomous robots [16], [13].

In [46] a very simple cascades-based controller was introduced to solve the leader-follower control problem for two robots. The approach was used subsequently, for instance in [32], [21],[6]. The control design is very simple to implement, roughly, it relies on a separation principle by which it is demonstrated that the translational and orientational kinematics may be stabilized independently of each other. The disadvantage of this method is that the controller relies on the assumption that the angular velocity of the leader robot must be different from zero. This rules out straight-line paths. As a matter of fact, there is a structural impediment to stabilize the robot on straight-line paths due to loss of controllability. Only very few works address the problem of formation control along straight-line paths; in [6], [31] where complex nonlinear time varying controls are designed to allow for reference velocity trajectories that converge to zero. It is worth to emphasize that [31] covers the case when also the forward velocity  $v_0$  may converge to zero that is, tracking control towards a fixed point. In [6] the controller is designed so as to make the robot converge to the straight-line trajectory resulting in a path that makes it go back and forth on the path.

In [21], the formation control problem in a leader-follower configuration without the need of knowing the leader's velocity is studied. There are two leaders which govern the group's motion. The stability analysis shows that the triangular formation is stable while the colinear one is not. In [28] the method of feedback linearisation is used to design control laws to solve the problem of leader-follower for multiple robots under different geometries of formation. The motion of the leader robot is computed by minimizing a suitable cost function. In [19] the nonlinear formation control law for the coordination of a group of  $N$  mobile robots force the robots' relative positions with respect to the centre of the virtual structure. Using the backstepping technique and Lyapunov's direct method the control problem is solved for the follower robot. The proposed method guarantees asymptotic stability for the closed-loop error system dynamics. The authors of [49] use consensus-based controllers combined with a cascades-based approach to tracking control, resulting in a group of linearly coupled dynamical systems. Stability analysis relies on cascaded systems and nonlinear synchronization theory.

### 1.3 Summary of results and organization

The aim of this Thesis is to contribute to the study of consensus of multi-agent systems. Taking as a starting point the work carried out by [39], that considers a coordination problem for a network of agents with single integrator dynamics where the network coupling is allowed to be time-dependent and non-bidirectional, the first result of our research, which are presented in this thesis, consist in the study of consensus problem

over a network of dynamic agents with time-dependent communication links which may disconnect for long intervals of time. We assume that the nodes are interconnected in a ring topology. The originality of our results lays, in part, in our method of proof: we leave behind graph theory for linear time-invariant systems and use instead, stability theory. In particular, we employ the small-gain theorem to establish simple checkable conditions on the network interconnections, to guarantee the achievement of consensus [2]. Moreover, we also studied the consensus problem over a network of dynamic agents with time-dependent communication links interconnected over a time-varying topology. We establish that consensus is reached provided that there always exists a spanning tree for a minimal dwell-time. We leave behind graph theory for linear time-invariant systems and use stability theory of time-varying and switched systems [3].

The second part of this thesis is dedicated to the study of trajectory tracking of nonholonomic mobile robots. We assume that only one swarm leader robot has the information of the reference trajectory and each robot receives information from one intermediary leader. This is, the communication graph forms a simple spanning directed tree. The main goal of our control approach is to solve the problem of stabilisation over straight line paths. We do this via nonlinear smooth time-varying controls which rely on a property of persistency of excitation, tailored for nonlinear systems. Our main results ensure uniform global asymptotic stabilisation of the closed-loop system [35, 36]. Inspired by this ideas, we easily extend this result to the case when the mathematical models of the mobile robots are dynamic [34].

The work consist of five chapters. In Chapter 2 we provide an overview of standard mathematical results Control Theory used throughout the thesis. Chapter 3 is devoted to studying the consensus seeking problem in the following sense:

1. dynamic agents with time-dependent communication links over a ring topology, and
2. dynamic agents over a spanning tree topology with time-dependent communication links interconnected over a time-varying topology.

Then, in the Chapter 4, first, we present a result on leader-follower tracking control (two robots only) and describe the control approach. Then, we present a result for a cascade-like configuration of leader-follower mobile robots following straight lines. We derive stability conditions for the kinematic and dynamic model by using a Lyapunov approach. Finally, in Chapter 5 a conclusion sums up the results of this thesis.

## Chapter 2

# Mathematical Preliminaries

In this Chapter, we recall a few notions and results that we use throughout this thesis. First, we consider some fundamental mathematical definitions [42]. Next some results in adaptive systems and some introductory concepts about graph theory are given [15, 25]. Then, stability in the sense of Lyapunov is stated and some useful theorem for showing stability are revisited [30, 60]. Finally, some concepts and lemmas of the switching systems are presented [33].

First, we introduce the definition of a norm.

A norm  $\|x\|$  of an  $n$ -dimensional vector  $x = (x_1, \dots, x_n)^\top$  is a real valued function with the following properties

- $\|x\| \geq 0$  for all  $x \in \mathbb{R}^n$  with  $\|x\| = 0$  if and only if  $x = 0$ ;
- $\|\alpha x\| = |\alpha| \|x\|$  for all  $\alpha \in \mathbb{R}$  and  $x \in \mathbb{R}^n$ ;
- $\|x + y\| \leq \|x\| + \|y\|$  for all  $x, y \in \mathbb{R}^n$ .

The  $p$ -norm and infinity norm are

$$\|x\|_p := (|x_1|^p + \dots + |x_n|^p)^{1/p} \quad 1 \leq p < \infty$$

and

$$\|x\|_\infty := \max_{i=1, \dots, n} |x_i|.$$

## 2.1 Graph Theory

A graph is characterised by a set of *edges*,  $\mathcal{E}$  and a set of *nodes*,  $\mathcal{N} = \{1, \dots, n\}$ . Edges are also called arcs and nodes are better known as vertexes. Thus, edges connect two end-vertexes or endpoints. Edges denoted by  $i, j$  with  $i, j \in \mathcal{N}$  are represented by lines or arrows depending on whether the direction in which the interconnection of two nodes is relevant. If it is so, the arc is directed, otherwise it is undirected. Hence, a graph constituted of directed vertexes and nodes is called a directed graph and undirected if the sense of interconnections is irrelevant. The edge  $(i, j)$  in the edge set of a directed graph denotes that agent  $j$  can obtain information from agent  $i$ , but not necessarily vice versa. Figure 2.1 shown 3 different examples of directed graphs.

An alternating sequence of arcs and nodes constitutes a *walk*. Its *length* equals the number of edges that it contains. It may be directed or undirected. It may be closed or open. It may contain repeated nodes and edges or not. It is open if its first and last vertexes are different and it is closed if they are equal. An open walk is usually called a *path* and its length equals the number of nodes minus 1. If the starting and ending vertexes are the same, the walk is commonly called *circuit* and its length equals the number of nodes it contains. A walk that starts and ends at the same vertex but otherwise has no repeated vertexes or edges, is called a *cycle*. If no vertex is visited more than once during the walk then, it is *simple*.

Graphs may be represented mathematically by matrices with specific properties and they are defined as follows.

### 2.1.1 Adjacency Matrix

The adjacency matrix  $A = [a_{ij}] \in \mathbb{R}^{n \times n}$  collects all information regarding the incidences of the edges in a graph. Its entries  $a_{ij}$  determine the adjacency of the parent node  $j$  to the child node node  $i$ . The index  $i$  relates to the *ith* row and respectively  $j$  refers to the *jth* column of A.

- *Directed Graph.*

$$\begin{cases} a_{ij} > 0 & \text{if } (j, i) \in \mathcal{E} \\ a_{ij} = 0 & \text{if } (j, i) \notin \mathcal{E} \end{cases}$$

- *Indirected Graph.*

$$\begin{aligned} a_{ij} &= a_{ji} & \forall i \neq j \\ (j, i) \in \mathcal{E} &\Rightarrow (i, j) \in \mathcal{E} \end{aligned}$$

that is

$$\begin{cases} a_{ij} > 0 & \text{if } (j, i) \in \mathcal{E} \\ a_{ij} = 0 & \text{if } (j, i) \notin \mathcal{E} \end{cases}$$

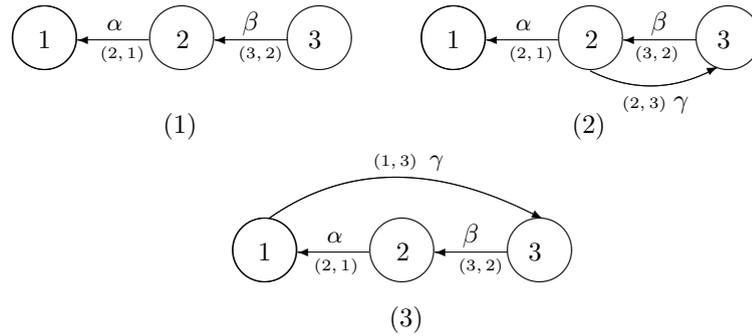


FIGURE 2.1: Three examples of directed graph with rooted spanning trees. Only the graph (3) is strongly connected and has a cyclic path.

For the three graphs shown in figure 2.1 their Adjacency matrices are:

$$A_1 = \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ 0 & \gamma & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ \gamma & 0 & 0 \end{bmatrix} \quad (2.1)$$

### 2.1.2 Laplacian Matrix

The Laplacian matrix  $L = [l_{ij}] \in \mathbb{R}^{n \times n}$  is defined by

$$\begin{cases} l_{ij} = -a_{ij}, & \text{if } i \neq j \\ l_{ii} = \sum_{j=1, i \neq j}^p a_{ij} \end{cases}$$

Note that if  $(j, i) \notin \mathcal{E}$  then  $l_{ij} = -a_{ij} = 0$ . The Laplacian matrix satisfies

$$\begin{aligned} l_{ij} &\leq 0, & i &\neq j \\ \sum_{j=1}^n l_{ij} &= 0 & i &= 1, \dots, p. \end{aligned}$$

For an undirected graph,  $L$  is symmetric and is called the *Laplacian matrix*. However, for a directed graph,  $L$  is not necessarily symmetric and is sometimes called the *nonsymmetric Laplacian matrix* or *directed Laplacian matrix*.

For the three graphs shown in the figure 2.1 their Laplacian matrices are:

$$L_1 = \begin{bmatrix} \alpha & -\alpha & 0 \\ 0 & \beta & -\beta \\ 0 & 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} \alpha & -\alpha & 0 \\ 0 & \beta & -\beta \\ 0 & -\gamma & \gamma \end{bmatrix}, \quad L_3 = \begin{bmatrix} \alpha & -\alpha & 0 \\ 0 & \beta & -\beta \\ -\gamma & 0 & \gamma \end{bmatrix} \quad (2.2)$$

*Remark 1.* Note that  $L$  can be equivalent defined as

$$L := D - A$$

where  $D = [d_{ij}] \in \mathbb{R}^{p \times p}$  is the *in-degree matrix* given as  $d_{ij} = 0, i \neq j$  and  $d_{ii} = \sum_{j=1}^p a_{ij}, i = 1, \dots, p$ .

$$\begin{cases} d_{ij} = 0, & i \neq j \\ d_{ii} = \sum_{j=1}^p a_{ij} & i = 1, \dots, p \end{cases}$$

## 2.2 Stability theory

Consider the non autonomous system

$$\dot{x} = f(t, x) \quad (2.3)$$

where  $f : [0, \infty) \times D \rightarrow \mathbb{R}^n$  is piecewise continuous in  $t$  and locally Lipschitz in  $x$  on  $[0, \infty) \times D$  and  $D \subset \mathbb{R}^n$  is a domain that contains the origin  $x = 0$ . The origin is an equilibrium point for (2.3) at  $t = 0$  if

$$f(t, 0) = 0, \quad \forall t \geq 0.$$

Uniform stability and asymptotic stability of the system (2.3) can be characterized in terms of special scalar functions, known as class  $\mathcal{K}$  and class  $\mathcal{KL}$  functions.

*Definition 1.* A continuous function  $\alpha : [0, a) \rightarrow [0, \infty)$  is said to belong to class  $\mathcal{K}$  if it is strictly increasing and  $\alpha(0) = 0$ . It is said to belong to class  $\mathcal{K}$ , if  $\alpha = \infty$  and  $\alpha(r) \rightarrow \infty$  as  $r \rightarrow \infty$ .

*Definition 2.* A continuous function  $\beta : [0, a) \times [0, \infty) \rightarrow [0, \infty)$  is said to belong to class  $\mathcal{KL}$  if, for each fixed  $s$ , the mapping  $\beta(r, s)$  belongs to class  $\mathcal{K}$  with respect to  $r$  and, for each fixed  $r$ , the mapping  $\beta(r, s)$  is decreasing with respect to  $s$  and  $\beta(r, s) \rightarrow \infty$  as  $s \rightarrow \infty$ .

*Definition 3.* The equilibrium point  $x = 0$  of (2.3) is

**Uniformly stable** if there exist a class  $\mathcal{K}$  function  $\alpha(\cdot)$  and a positive constant  $c$  independent of  $t_0$  such that

$$\|x(t)\| \leq (\|x(t_0)\|), \quad \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c \quad (2.4)$$

**Uniformly asymptotically stable** if there exist a class  $\mathcal{KL}$  function  $\beta(\cdot, \cdot)$  and a positive constant  $c$ , independent of  $t_0$ , such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c \quad (2.5)$$

**Globally uniformly asymptotically stable** if inequality (2.5) is satisfied for any initial state  $x(t_0)$ .

*Definition 4.* The equilibrium point  $x = 0$  of (2.3) is exponentially stable if inequality (2.5) is satisfied with

$$\beta(r, s) = kre^{-\gamma s}, \quad k > 0, \gamma > 0$$

and is globally exponentially stable if this condition is satisfied for any initial state.

### 2.2.1 Linear time-varying systems

The stability behavior of the origin as an equilibrium point for the linear time-varying system

$$\dot{x} = A(t)x \quad (2.6)$$

can be completely characterised in terms of the state transition matrix of the system. From linear system theory, we know that the solution of (2.6) is given by

$$x(t) = \Phi(t, t_0)x(t_0) \quad (2.7)$$

where  $\Phi(t, t_0)$  is the state transition matrix. The following theorem characterizes uniform asymptotic stability in terms of  $\Phi(t, t_0)$ .

*Theorem 1.* The equilibrium point  $x = 0$  of (2.6) is (globally) uniformly asymptotically stable if and only if the state transition matrix satisfies the inequality

$$\|\Phi(t, t_0)\| \leq \bar{\alpha}e^{-\alpha(t-t_0)}, \quad \forall t \geq t_0 \geq 0$$

for some positive constants  $\bar{\alpha}$  and  $\alpha$ .

## 2.3 Switching systems

A switched system is a continuous-time system with (isolated) switching events [33]. The switching systems can be classified into

- State-dependent versus time-dependent;
- Autonomous (uncontrolled) versus controlled.

*Remark 2.* One can have combinations of several types of switching.

### 2.3.1 State-dependent switching

Suppose that the continuous state space  $\mathbb{R}^n$  is partitioned into a finite or infinite number of *operating regions* by means of a family of *switching surfaces*, or *guards*. In each of these regions, a continuous-time dynamical system (described by differential equations, with or without controls) is given. Whenever the system trajectory hits a switching surface, the continuous state jumps instantaneously to a new value, specified by a *reset map*. In the simplest case, this is a map whose domain is the union of the switching surfaces and whose range is the entire state space, possibly excluding the switching surfaces (more general reset maps can also be considered, as explained below). In summary, the system is specified by

- The family of switching surfaces and the resulting operating regions;
- The family of continuous-time subsystems, one for each operating region;
- The reset map.

In figure 2.2, the thick curves denote the switching surfaces, the thin with arrows denote the continuous portions of the trajectory, and the dashed lines symbolise the jumps.

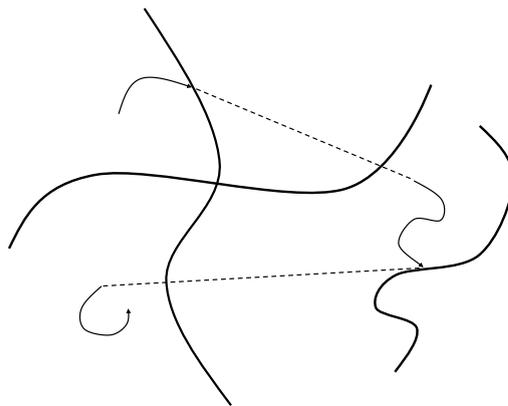


FIGURE 2.2: State-dependent switching.

### 2.3.2 Time dependent switching

Suppose that we have a family  $\{f_i : \mathbb{R}^n \rightarrow \mathbb{R}^n : i \in \mathcal{I}\}$ , where  $\mathcal{I}$  is some index set.<sup>1</sup> This gives rise to a family of systems describe by the state representation.

$$\dot{x} = f_i(x), \quad i \in \mathcal{I} \quad (2.8)$$

The functions  $f_i$  are assumed to be sufficiently regular (at least locally Lipschitz). The easiest case to think about is when all these systems are linear

$$f_i = A_i x, \quad A_i \in \mathbb{R}^{n \times n}, \quad i \in \mathcal{I} \quad (2.9)$$

and the index set  $\mathcal{I}$  is finite, *e.g.*  $\mathcal{I} = \{1, 2, \dots, n\}$ .

To define a switched system generated by the above family, we need the notion of a *switched signal*. This is a piecewise constant  $\sigma : [0, \infty) \rightarrow \mathcal{I}$ . Such a function  $\sigma$  has a finite number of discontinuities –which we call the *switching times*– on every bounded time interval and takes a constant value on every interval between two consecutive switching times. The role of  $\sigma$  is to specify, at each time instant  $t$ , the index  $\sigma(t) \in \mathcal{I}$  of the active *subsystem*, *e.g.*, the system for the family (2.8) that is currently being followed. We assume more concreteness that  $\sigma$  is continuous from the right everywhere:  $\sigma(t) = \lim_{\tau \rightarrow t^+} \sigma(\tau)$  for each  $t \geq 0$ . An example of such a switching signal for the case,  $\mathcal{I} = \{1, 2\}$ , is describe in the figure 2.3.

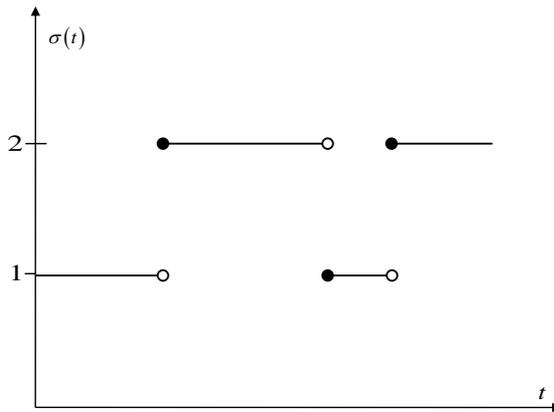


FIGURE 2.3: Switching signal.

Thus, a switched system with-time dependent switching, can be described by equation:

$$\dot{x}(t) = f_{\sigma(t)}(x(t)). \quad (2.10)$$

<sup>1</sup>Typically,  $\mathcal{I}$  is a subset of a finite-dimensional linear vector space.

A particular case is a *switched linear system*

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad (2.11)$$

which arises when all individual subsystems are linear, as in (2.9). For the sake of simplicity we omit the time arguments and write

$$\dot{x} = f_{\sigma}(x) \quad (2.12)$$

and

$$\dot{x} = A_{\sigma}x, \quad (2.13)$$

respectively.

### 2.3.3 Autonomous switching

By autonomous switching, we mean a situation where we have no direct control over the switching mechanism that triggers the discrete events. This category includes systems with state-dependent switching in which locations of the switching surfaces are predetermined, as well as systems with time-dependent switching in which the rule that defines the switching signal is unknown (or was ignored at the modelling stage). For example abrupt changes in system dynamics may be caused by unpredictable environmental factors or component failures.

### 2.3.4 Controlled switching

In many situations the switching is actually imposed by the designer in order to achieve a desired behaviour of the system. In this case, we have direct control over the switching mechanism (which can be state-dependent or time-dependent) and may adjust it as the system evolves.

### 2.3.5 Dwell time

The simplest way to specify slow switching is to introduce a number  $\tau_d > 0$  and restrict the class of admissible switching signals to signals with the property that the switching times  $t_1, t_2, \dots$  satisfy the inequality  $t_{i+1} - t_i \geq \tau_d$  for all  $i$ . This number  $\tau_d$  is usually called the *dwell time* (because  $\sigma$  “dwells” on each of its values for at least  $\tau_d$  units of time).

## Chapter 3

# Consensus seeking under Persistent Interconnections

Consider  $N$  dynamic agents

$$\Psi_i : \dot{x}_i = u_i, \quad i \in \{1, 2, \dots, N\} \quad (3.1)$$

where  $u_i$  represents a protocol of interconnection. The most common continuous consensus protocol under an all-to-all communication assumption, has been studied in [44, 61], and is given by

$$u_i = - \sum_{j=1}^N a_{ij}(t)(x_i - x_j), \quad \forall i \in \{1, \dots, N\} \quad (3.2)$$

where  $a_{ij}(t)$  is the  $(i, j)$  entry of the adjacency matrix (-see Chapter 2) and  $x_i$  is the information state of the  $i$ -th agent.

In this Chapter, we present two new results on consensus seeking by using the consensus protocol (3.2) where the  $a_{ij}(t)$  gains are represented by persistent exciting signals. We summarize our work as follows:

1. In Section, 3.1 We study the consensus problem in a ring topology; we establish simple checkable conditions on the network interconnections guaranteeing the achievement of consensus and,
2. In Section, 3.2 we analyse consensus under time-varying topologies; we establish that consensus is reached provided that there always exists a spanning tree for a minimal dwell-time.

### 3.1 Consensus seeking under Persistent Interconnections in a Ring Topology

We study consensus under the assumption of a ring-communication topology, that is, for each  $i$ , we define

$$u_i = \begin{cases} -a_{ii+1}(t)(x_i - x_{i+1}), & \forall i \in \{1, \dots, N-1\} \\ -a_{i1}(t)(x_i - x_1), & i = N \end{cases} \quad (3.3)$$

where  $a_{ii+1} \geq 0$ , it is strictly positive whenever information flows from the  $(i+1)$ th node to the  $i$ th node. Under (3.30), the system (3.1) has the form

$$\dot{x} = -L(t)x \quad (3.4)$$

where  $L(t)$  is the following *Laplacian* matrix

$$L(t) := \begin{bmatrix} a_{12}(t) & -a_{12}(t) & 0 & 0 & 0 \\ 0 & a_{23}(t) & -a_{23}(t) & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{N-1N}(t) & -a_{N-1N}(t) \\ -a_{N1}(t) & 0 & \cdots & \cdots & a_{N1}(t) \end{bmatrix} \quad (3.5)$$

and  $x := (x_1, \dots, x_N)^\top$  is the vector containing the state of each agent. The Laplacian  $L(t)$  has associated a graphic representation which is shown in Figure 3.1.

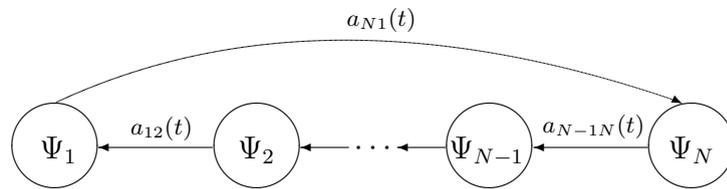


FIGURE 3.1: Ring topology with time dependent communication links.

The system (3.4) reaches consensus if for any initial condition all the states reaches a common value as  $t$  tends to infinity. The consensus problem has been thoroughly studied both for the case of constant and time-varying interconnections, mostly under the assumption of an all-to-all communication topology. Typically, graph theory is used to establish that consensus is reached if there exists a directed spanning tree (any node may be reached from any node). In the case that the interconnections are time-varying, a similar result was established in [39].

### 3.1.1 Stability analysis

We take a different approach to the analysis of consensus: stability theory. Indeed, our analysis builds upon the elementary observation that consensus is equivalent to the asymptotic stability of the origin of

$$\dot{z} = A(t)z \quad (3.6)$$

where we defined the error states

$$z_i = x_i - x_{i+1} \quad \forall i \in \{1, \dots, N-1\}, \quad (3.7)$$

the vector containing all the errors  $z_i$  is defined as  $z = (z_1, \dots, z_{N-1})^\top$  and

$$A(t) := \begin{bmatrix} -a_{12}(t) & a_{23}(t) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & -a_{N-2N-1}(t) & a_{N-1N}(t) \\ -a_{N1}(t) & \cdots & -a_{N1}(t) & -(a_{N1} + a_{N-1N})(t) \end{bmatrix} \quad (3.8)$$

The latter follows from evaluating the time derivative of (3.7) to obtain, substituting the dynamics of the agents  $i$ ,

$$\dot{z}_i = \begin{cases} -a_{ii+1}(t)z_i + a_{i+1i+2}(t)z_{i+1} & i \in \{1, \dots, N-2\} \\ -a_{N-1N}(t)z_{N-1} + a_{N1}(t)z_N & i = N-1. \end{cases} \quad (3.9)$$

We establish sufficient and necessary conditions for the origin of (3.6) to be uniformly globally exponentially stable therefore, for the system (3.4) to reach consensus uniformly and exponentially fast.

We assume that the functions  $a_{ii+1}$  may be equal to zero for intervals of time whose length is uniformly bounded. That is, we assume that each coefficient  $a_{ii+1}$  is *persistently exciting*. The latter property, which is well-known in the literature of adaptive control systems –see [4], is defined as follows.

**Definition 5. Persistence of Excitation.** A locally integrable function  $a : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is said to be persistently exciting (PE) if there exist positive numbers  $T$  and  $\mu$  such that

$$\int_t^{t+T} a(\tau) d\tau \geq \mu, \quad \forall t \geq 0 \quad (3.10)$$

It is an elementary but important fact (see [25]) that if  $t \mapsto a(t)$  is persistently exciting then

$$e^{-\int_{t_1}^t a(\tau) d\tau} \leq \bar{k} e^{-k(t-t_1)}, \quad \forall t \geq t_1 \geq 0 \quad (3.11)$$

where  $\bar{k} = e^\mu$  and  $k = \mu/T$ .

*Example 1.* Consider a PE signal  $a$  with  $T = 3$  and  $\mu = 1$  which is shown in the Figure (3.2). The signal  $a$  satisfies the property (3.11) with  $\bar{k} = 2.78$  and  $k = 0.33$ , see Figure (3.3).

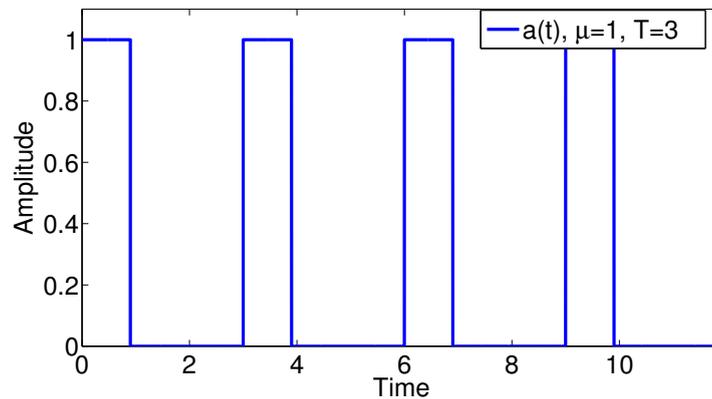


FIGURE 3.2: Persistently exciting signal  $a$  with  $T = 3$  and  $\mu = 1$ .

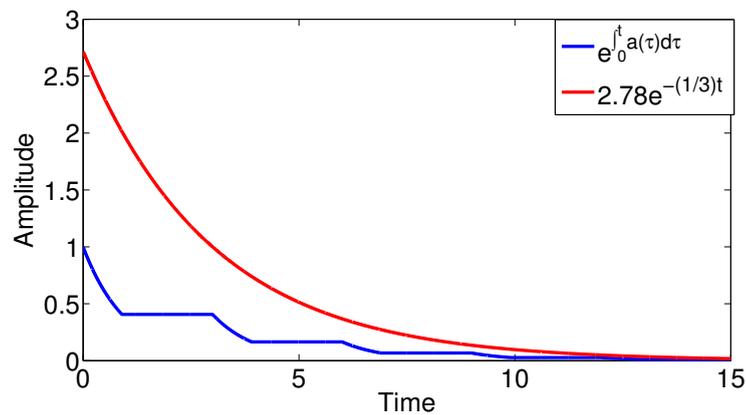


FIGURE 3.3: Property of a persistently exciting signal.

*Theorem 1.* If

$$|a_{N1}|_\infty := \sup_{t \geq 0} |a_{N1}(t)|$$

is sufficiently small, the system (3.4) reaches consensus uniformly in the initial times if and only if  $a_{ii}$  is persistently exciting.

We give a quantification of *sufficiently small* later in this chapter (Lemma 2).

To prove the theorem we start by observing that the matrix  $A(t)$  in 3.6 may be partitioned as

$$A(t) = A_1(t) + A_2(t) \quad (3.12)$$

with

$$A_1(t) := \begin{bmatrix} -a_{12}(t) & a_{23}(t) & 0 & \cdots & 0 \\ 0 & -a_{23}(t) & a_{34}(t) & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -a_{N-2N-1}(t) & a_{N-1N}(t) \\ 0 & 0 & 0 & \cdots & -a_{N-1N}(t) \end{bmatrix} \quad (3.13)$$

and

$$A_2(t) := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -a_{N1}(t) & -a_{N1}(t) & -a_{N1}(t) & \cdots & -a_{N1}(t) \end{bmatrix} \quad (3.14)$$

Therefore, (3.6) is equivalent to the feedback-interconnected system

$$\Sigma : \begin{cases} \Sigma_1 : \begin{cases} \dot{z} = A_1(t)z + y_2 \\ y_1 = z \end{cases} \\ \Sigma_2 : y_2 = A_2(t)y_1 \end{cases} \quad (3.15)$$

—see the illustration in Figure 3.4, below.

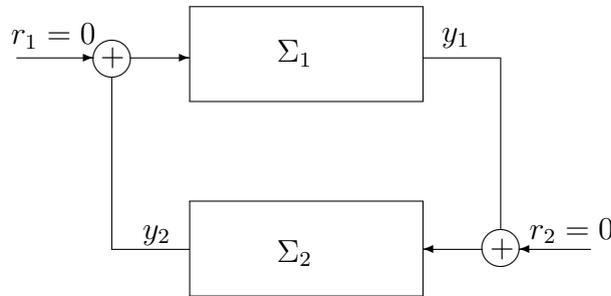


FIGURE 3.4: Interconnected system

The interest of this observation is that the stability conditions for (3.15), hence for (3.6), may be derived by computing the norms of the systems  $\Sigma_1$  and  $\Sigma_2$  and invoking the *small gain theorem* that gives sufficient conditions under which bounded inputs produce bounded outputs in the feedback system of Figure 3.5.

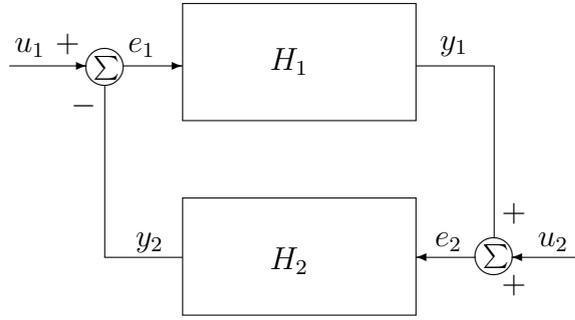


FIGURE 3.5: Feedback system.

*Theorem 2.* [25] Consider the system shown in Figure 3.5. Suppose  $H_1, H_2 : \mathcal{L}_e \mapsto \mathcal{L}_e$ ;  $e_1, e_2 \in \mathcal{L}_e$ . Suppose that for some constants  $\gamma_1, \gamma_2 \geq 0$  and  $\beta_1, \beta_2$ , the operators  $H_1, H_2$  satisfy

$$\begin{aligned} 1) \quad & \| (H_1 e_1)_t \| \leq \gamma_1 \| e_{1t} \| + \beta_1 \\ 2) \quad & \| (H_2 e_2)_t \| \leq \gamma_2 \| e_{2t} \| + \beta_2 \end{aligned} \quad (3.16)$$

$\forall t \in \mathbb{R}^+$ . If

$$\gamma_1 \gamma_2 < 1$$

then

(i)

$$\begin{aligned} \| e_{1t} \| & \leq (1 - \gamma_1 \gamma_2)^{-1} (\| u_{1t} \| + \gamma_2 \| u_{2t} \| + \beta_2 + \gamma_2 \beta_1) \\ \| e_{2t} \| & \leq (1 - \gamma_1 \gamma_2)^{-1} (\| u_{2t} \| + \gamma_1 \| u_{1t} \| + \beta_1 + \gamma_1 \beta_2) \end{aligned} \quad (3.17)$$

for any  $t \geq 0$

(ii) If in addition,  $\| u_1 \|, \| u_2 \| < \infty$ , then  $e_1, e_2, y_1, y_2$  have finite norms, and the norms of  $e_1, e_2$  are bounded by the right-hand sides of (3.17) with all subscripts  $t$  dropped.

Indeed, the system  $\Sigma$  is a particular case of that covered by the small gain theorem with the inputs  $r_1(t) = 0$  and  $r_2(t) = 0$  for every nonnegative  $t$  –see Figure 3.4.

In other words, in view of (4.23), the system (3.6) may be studied as a perturbed system with nominal dynamics

$$\Sigma_1 : \dot{z} = A_1(t)z \quad (3.18)$$

and perturbation (output injection)  $A_2(t)y_1$ . Moreover, note that the perturbation only depends on the interconnection function  $a_{N1}(t)$  hence, we shall establish exponential

stability under the condition that the  $\|\cdot\|_\infty$  norm of  $a_{N1}(t)$  is sufficiently small and the origin of (3.18) is exponentially stable. The stability of system (3.18) is given by the following result.

*Lemma 1.* Let

$$\dot{\Phi}(t, t_o) = A_1(t)\Phi(t, t_o), \quad \Phi(t_o, t_o) = I_{N-1} \quad \forall t \geq t_o > 0 \quad (3.19)$$

where

$$A_1(t) := \begin{bmatrix} -a_{12}(t) & a_{23}(t) & 0 & \cdots & 0 \\ 0 & -a_{23}(t) & a_{34}(t) & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -a_{N-2N-1}(t) & a_{N-1N}(t) \\ 0 & 0 & 0 & \cdots & -a_{N-1N}(t) \end{bmatrix} \quad (3.20)$$

Assume that, for every  $i = 1, \dots, N-1$ ,  $a_{ii+1}(t)$  is a bounded persistently exciting signal. Then, there exist  $\bar{\alpha} > 0$ ,  $\alpha > 0$  such that

$$\|\Phi(t, t_o)\| \leq \bar{\alpha} e^{-\alpha(t-t_o)} \quad \forall t \geq t_o \geq 0. \quad (3.21)$$

*Proof.* Note that the solution of the differential equation (3.19) is given by  $\Phi(t, t_o) = [\phi_{ij}(t, t_o)]$  where

$$\phi_{ij}(t, t_o) = \begin{cases} 0 & i > j \\ e^{-\int_{t_o}^t a_{ii+1}(s) ds} & i = j \\ \int_{t_o}^t \phi_{ii}(t, s) a_{i+1, i+2}(s) \phi_{i+1, j}(s, t_o) ds & i < j \end{cases} \quad (3.22)$$

We show that every element of  $\Phi(t, t_o)$  is bounded by an exponential function. Taking the absolute value of (3.22) and using the property (3.11) for its diagonal entries, we have

$$\begin{aligned} |\phi_{ij}(t, t_o)| &= 0 & i > j \\ |\phi_{ii}(t, t_o)| &\leq \bar{k}_i e^{-k_i(t-t_o)} & i = j \\ |\phi_{ij}(t, t_o)| &\leq \int_{t_o}^t |\phi_{ii}(t; s)| |a_{i+1, i+2}(s)| |\phi_{i+1, j}(s; t_o)| ds & i < j \end{aligned} \quad (3.23)$$

For each  $j = i + 1$  such that  $i < N - 1$  the integral (3.22) depends on  $\phi_{ii}$  and  $\phi_{i+1i+1}$  which are bounded by  $\bar{k}_i e^{-k_i(t-\sigma)}$  and  $\bar{k}_{i+1} e^{-k_{i+1}(t-\sigma)}$ , respectively. Consequently,

$$|\phi_{ij}(t, t_o)| \leq \bar{k}_i \bar{k}_j |a_{i+1, i+2}|_\infty \left[ \frac{1}{|k_i - k_j|} e^{-\min\{k_j, k_i\}(t-t_o)} \right]$$

where by assumption,  $|a_{i+1, i+2}|_\infty$  is bounded. Thus, all elements of  $\Phi(t, t_o)$  are bounded in norm by a decaying exponential.  $\square$

Now, we can compute the input-output gain of the system  $\Sigma_1$  as follows. The norm of the output of  $\Sigma_1$  is given by

$$\|y_1(t)\| \leq \|\Phi(t, t_o)\| \|z(t_o)\| + \int_{t_o}^t \|\Phi(t, \tau)\| \|y_2(\tau)\| d\tau.$$

By Lemma 1 we have  $\|\Phi(t, t_o)\| \leq \bar{\alpha} e^{-\alpha(t-t_o)}$ , then

$$\|y_1(t)\| \leq \bar{\alpha} e^{-\alpha(t-t_o)} \|z(t_o)\| + \int_{t_o}^t \bar{\alpha} e^{-\alpha(t-\tau)} \|y_2(\tau)\| d\tau.$$

Solving the integral and taking the initial condition for  $z$  equals to zero, we obtain

$$\|y_1(t)\| \leq \frac{\bar{\alpha}}{\alpha} (1 - e^{-\alpha(t-t_o)}) \|y_2\|_\infty.$$

Thus,

$$\|\Sigma_1\|_\infty \leq \frac{\bar{\alpha}}{\alpha}. \quad (3.24)$$

On the other hand, the norm of the output of  $\Sigma_2$  is

$$\|y_2\|_\infty \leq \|A_2(t)\|_\infty \|y_1\|_\infty.$$

Hence,

$$\|\Sigma_2\|_\infty \leq \|A_2(t)\|_\infty \leq (N-1) |a_{N1}|_\infty. \quad (3.25)$$

Finally, the proof of the theorem is completed by the following lemma, which establishes the stability of the feedback interconnected system (3.4) based on the computed input/output gains.

*Lemma 2.* Consider the interconnected system

$$\Sigma : \begin{cases} \Sigma_1 & : \begin{cases} \dot{z} & = A_1(t)z + y_2 \\ y_1 & = z \end{cases} \\ \Sigma_2 & : y_2 = A_2(t)y_1 \end{cases} \quad (3.26)$$

Under the assumptions of Lemma 1 the interconnected system (3.26) is stable if  $|a_{N1}|_\infty \leq \frac{\alpha}{(N-1)\bar{\alpha}}$ . In particular, all components of  $x(t)$  converge to a common value as  $t \rightarrow \infty$ .

*Proof.* We apply the small gain theorem to the interconnected system (3.26) using the upper bounds (3.24) and (3.25) of  $\|\Sigma_1\|$  and  $\|\Sigma_2\|$ , respectively, this is

$$\|\Sigma_1\| \|\Sigma_2\| \leq (N-1) |a_{N1}|_\infty \frac{\bar{\alpha}}{\alpha} < 1. \quad (3.27)$$

It follows that

$$|a_{N1}|_\infty < \frac{\alpha}{\bar{\alpha}} \left( \frac{1}{N-1} \right) \quad (3.28)$$

is a sufficient condition for the asymptotically stability of (3.26).  $\square$

### 3.1.2 Illustrative example and Simulation Results

For the sake of illustration, let us study a system with four interconnected agents (see. Figure 3.6).

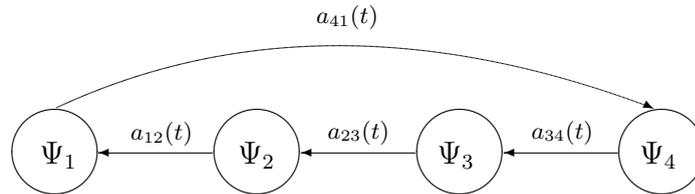


FIGURE 3.6: Ring topology with four interconnected agents

The information exchange among agents is represented by the persistently exciting signals  $a_{12}$ ,  $a_{23}$  and  $a_{34}$  shown in Figure 3.7 with parameters in Table 1.

TABLE 3.1: Parameters  $T_i$  and  $\mu_i$  corresponding to the signal  $a_{ii+1}$ ,  $i = 1, 2, 3$ .

	$T_i$	$\mu_i$
$a_{12}(t)$	2	1
$a_{23}(t)$	3	0.3
$a_{34}(t)$	2.5	1.25

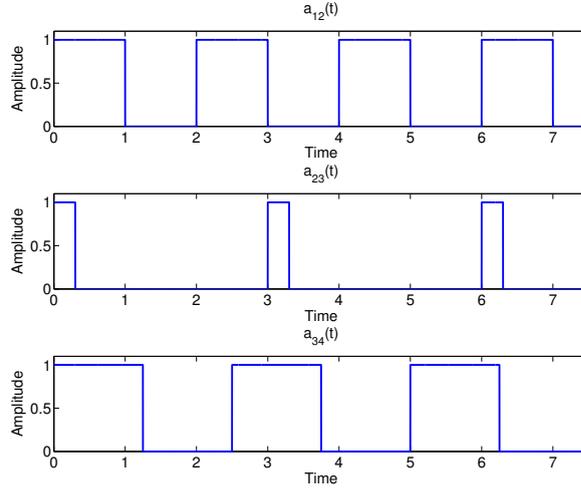


FIGURE 3.7: Persistently exciting signals  $a_{12}(t)$  (top),  $a_{23}(t)$  (middle) and  $a_{34}(t)$  (bottom).

For this example, the system  $\dot{z} = A(t)z(t)$  is partitioned into two matrices  $A_1$  and  $A_2$  defined as

$$A_1(t) = \begin{bmatrix} -a_{12}(t) & a_{23}(t) & 0 \\ 0 & -a_{23}(t) & a_{34}(t) \\ 0 & 0 & -a_{34}(t) \end{bmatrix}$$

and

$$A_2(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -a_{41}(t) & -a_{41}(t) & -a_{41}(t) \end{bmatrix}$$

To apply Theorem 1, is necessary to compute  $\bar{\alpha}$  and  $\alpha$ . By Lemma 1 we have

$$\begin{aligned} \bar{\alpha} &= \bar{k}_1 + |a_{34}|_{\infty} \frac{\bar{k}_2 \bar{k}_3}{|k_2 - k_3|} \\ &\quad + |a_{23}|_{\infty} |a_{34}|_{\infty} \frac{\bar{k}_1 \bar{k}_2 \bar{k}_3}{|k_2 - k_3| |k_1 - \min(k_2, k_3)|}, \\ \alpha &= \min(k_1, k_2, k_3) \end{aligned}$$

where the values of  $\bar{k}_i$  and  $k_i$  are shown in Table 3.2.

TABLE 3.2: Parameters  $\bar{k}_i$  and  $k_i$  corresponding to the signal  $a_{ii+1}$ ,  $i = 1, 2, 3$ .

	$\bar{k}_i$	$k_i$
$a_{12}(t)$	2.71	0.5
$a_{23}(t)$	1.34	0.1
$a_{34}(t)$	3.49	0.5

Therefore, we have

$$\alpha = 13.1285 \quad \text{and} \quad \bar{\alpha} = 47.1147.$$

By applying Theorem 1, we know that the origin of  $\dot{z}(t) = A(t)z(t)$  is exponentially stable if

$$|a_{41}(t)| \leq 0.092. \quad (3.29)$$

To illustrate the feasibility of the results obtained we performed simulations using SIMULINK of MATLAB. The initial conditions of the agents are  $x_1(0) = -1$ ,  $x_2(0) = 3$ ,  $x_3(0) = -2$  and  $x_4(0) = -0.5$ . The persistently exciting signal  $a_{41}$ , satisfying (3.29), is shown in Figure 3.8.

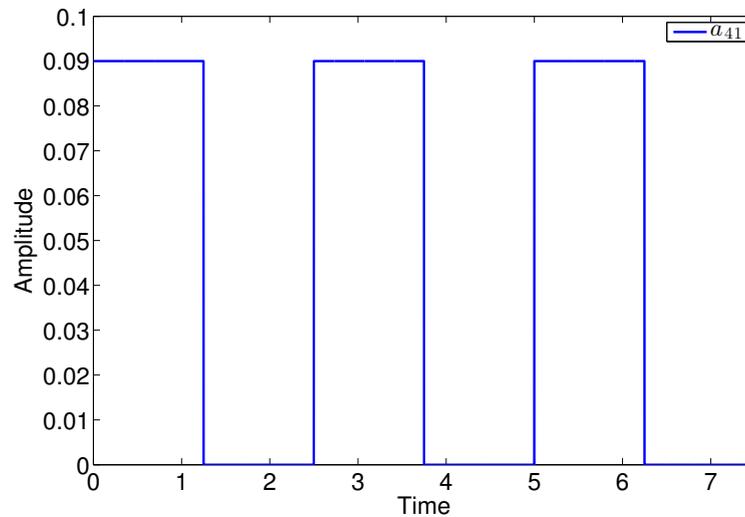


FIGURE 3.8: Persistently exciting signals  $a_{41}(t)$ .

Figure 3.9 depicts that all trajectories converge to a common value.

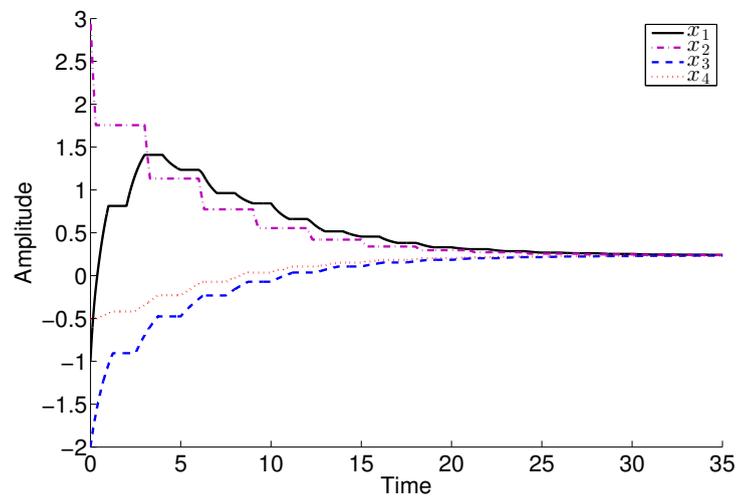


FIGURE 3.9: Trajectories of states  $x_1(t)$ ,  $x_2(t)$ ,  $x_3(t)$  and  $x_4(t)$ .

In [52], it is shown that having a spanning tree topology is a sufficient condition to achieve consensus under the assumption of piecewise constant communication links among the agents. Therefore, by using this result we can also conclude that the foregoing example reaches consensus due to the persistently exciting signals  $a_{12}$ ,  $a_{23}$ ,  $a_{34}$  and  $a_{41}$  satisfy this condition. However, we want to stress that our result is based on the assumptions that the time-varying communication links among the agents are represented by persistently exciting signals and consequently our result is different with respect to the one presented in [10].

## 3.2 Consensus for multi-agent systems under persistent interconnections over a time-varying topology.

We analyze consensus under time-varying topologies; as opposed to the more traditional graph-theory based analysis [53], we adopt a stability theory approach.

### 3.2.1 The network model

With little loss of generality, let us consider the following consensus protocol

$$u_\lambda = \begin{cases} -a_{\lambda\lambda+1}(t)[x_\lambda(t) - x_{\lambda+1}(t)] & \forall \lambda \in [1, N-1] \\ 0 & \lambda = N \end{cases} \quad (3.30)$$

where  $a_{\lambda\lambda+1} \geq 0$  and it is strictly positive whenever information flows from the  $(\lambda+1)$ th node to the  $\lambda$ th node. This protocol leads to a spanning-tree configuration topology; the closed-loop equations are

$$\begin{cases} \dot{x}_1 & = -a_{12}(t)[x_1 - x_2] \\ & \vdots \\ \dot{x}_\lambda & = -a_{\lambda,\kappa}(t)[x_\lambda - x_{\lambda+1}] \\ & \vdots \\ \dot{x}_{N-1} & = -a_{N-1N}(t)[x_{N-1} - x_N] \\ \dot{x}_N & = 0 \end{cases} \quad (3.31)$$

In a leader-follower configuration, the  $N$ th node may be considered as a “swarm master” with its own dynamics. For simplicity, here we consider it to be static.

It is clear that there are many other possible spanning-tree configurations; the one showed above is considered conventionally. Actually, there exist a total number of  $N!$

spanning-tree configurations; for instance, for a group of three agents there exist six possible spanning-tree configuration topologies which determine six different sequences  $\{\Psi_3, \Psi_2, \Psi_1\}$ ,  $\{\Psi_2, \Psi_1, \Psi_3\}$ , *etc.* –see Figure 3.10.

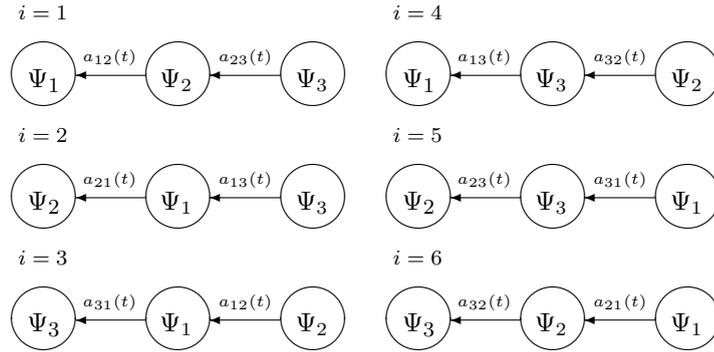


FIGURE 3.10: Example of 3 agents, where by changing their positions, we obtained six possible topologies.

Thus, to determine the  $N!$  possible spanning-tree communication topologies, among  $N$  agents, we introduce the following notation. For each  $k \leq N$  we define a function  $\pi_k$  which takes integer values in  $\{1, \dots, N\}$ . We also introduce the sequence of agents  $\{\Psi_{\pi_k}\}_{k=1}^N$  with the following properties: 1) every agent  $\Psi_\lambda$  is in the sequence; 2) no repetitions of agents in the sequence is allowed 3) the root agent is labeled  $\Psi_{\pi_N}$  and it communicates with the agent  $\Psi_{\pi_{N-1}}$ , the latter is parent of  $\Psi_{\pi_{N-2}}$  and so on down to the leaf agent  $\Psi_{\pi_1}$ . That is, the information flows with interconnection gain  $a_{\pi_k \pi_{k+1}}(t) \geq 0$ , from the agent  $\Psi_{\pi_{k+1}}$  to the agent  $\Psi_{\pi_k}$ . The subindex  $k$  represents the position of the agent  $\Psi_{\pi_k}$  in the sequence. Note that any sequence  $\{\Psi_{\pi_1}, \Psi_{\pi_2}, \dots, \Psi_{\pi_{N-1}}, \Psi_{\pi_N}\}$  of the agents may be represented as a spanning-tree topology which is depicted in Figure 3.11.

Thus, in general, each possible fixed topology labeled  $i \in 1, \dots, N!$  is generated by a protocol of the form (3.30) which we write as

$$u_{\pi_k}^i = \begin{cases} -a_{\pi_k \pi_{k+1}}^i(t) [x_{\pi_k} - x_{\pi_{k+1}}], & k \in \{1, \dots, N-1\} \\ 0, & k = N \end{cases} \quad (3.32)$$

where  $k$  denotes the position of the agent  $\Psi_\lambda$  in the sequence  $\{\Psi_{\pi_k}\}_{k=1}^N$  and  $\pi_k$  represents



FIGURE 3.11: A spanning-tree topology with time dependent communication links between  $\Psi_{\pi_k}$  and  $\Psi_{\pi_{k+1}}$ .

which agent  $\Psi_\lambda$  is in the position  $k$ , this is,  $\pi_k = \lambda$ . Under (3.32), the system (3.1) takes the form

$$\dot{x}_{di} = -L_i(t)x_{di}, \quad i \in \{1, \dots, N!\} \quad (3.33)$$

where to each topology  $i \leq N!$  corresponds a state vector

$$x_{di} = [x_{\pi_1}, x_{\pi_2}, \dots, x_{\pi_N}]^\top$$

which contains the states of all interconnected agents in a distinct order, depending on the topology. For instance, referring to Figure 3.10, for  $i = 1$  we have  $x_{d1} = [x_1, x_2, x_3]^\top$  while  $x_{d4} = [x_1, x_3, x_2]^\top$  while for  $i = 4$ .

Accordingly, to each topology we associate a distinct Laplacian matrix  $L_i(t)$  which is given by

$$L_i(t) := \begin{bmatrix} a_{\pi_1\pi_2}^i(t) & -a_{\pi_1\pi_2}^i(t) & 0 & 0 & 0 \\ 0 & a_{\pi_2\pi_3}^i(t) & -a_{\pi_2\pi_3}^i(t) & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a_{\pi_{k-1}\pi_k}^i(t) & -a_{\pi_{k-1}\pi_k}^i(t) \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix} \quad (3.34)$$

Since any of the  $N!$  configurations is a spanning tree, which is a necessary and sufficient condition for consensus, all configurations may be considered equivalent in some sense, to the first topology, *i.e.*, with  $i = 1$ . As a convention, for the purpose of analysis we denote the state of the latter by  $x = [x_1, x_2, \dots, x_N]^\top$  and refer to it as an *ordered* topology. See Figure 3.12.

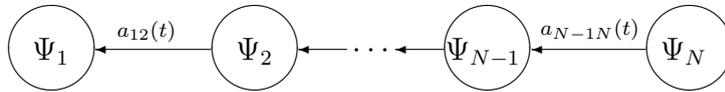


FIGURE 3.12: A spanning-tree topology with time dependent communication links between  $\Psi_\lambda$  and  $\Psi_{\lambda+1}$ .

It is clear (at least intuitively) that consensus of all systems (3.33) is equivalent to that of  $\dot{x} = L_1(t)x$  where

$$L_1(t) := \begin{bmatrix} a_{12}(t) & -a_{12}(t) & 0 & 0 & 0 \\ 0 & a_{23}(t) & -a_{23}(t) & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{N-1N}(t) & -a_{N-1N}(t) \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \quad (3.35)$$

More precisely, the linear transformation from a “disordered” vector  $x_{di}$  to the ordered vector  $x$  is defined via a permutation matrix  $P_i$  that is,

$$x_{di} = P_i x \quad (3.36)$$

where  $P_i \in \mathbb{R}^{n \times n}$  is defined as

$$P_i = \begin{bmatrix} E_{\pi_1} \\ E_{\pi_2} \\ \vdots \\ E_{\pi_N} \end{bmatrix}, \quad i \in \{1, \dots, N!\}$$

and the rows

$$E_{\pi_k} = [0, 0, \dots, \underbrace{1}_{\lambda\text{th position}}, \dots, 0].$$

The permutation matrix  $P_i$  is a nonsingular matrix with  $P_i^{-1} = P_i^\top$  (see [23]). For instance, relative to Figure 3.10 we have  $x_{d2} = [x_2, x_1, x_3]^\top$  and

$$P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In order to study the consensus problem for (3.33) for any  $i$  it is both sufficient and necessary to study that of any configuration topology. Moreover, we may do so by studying the error dynamics corresponding to the differences between any pair of states.

### 3.2.2 Fixed topology with time-varying interconnections

For clarity of exposition we start with the case of a fixed but arbitrary topology. In view of the previous discussion, without loss of generality, we focus on the study of the ordered topology depicted in Figure 3.12. Consensus may be established using an argument on

stability of cascaded systems. To see this, let  $z_1$  denote the vector of ordered errors corresponding to this first topology that is,

$$z_{1\lambda} := x_\lambda - x_{\lambda+1} \quad \forall \lambda \in \{1, \dots, N-1\}$$

Then, the systems in (3.33) with  $i = 1$  reach consensus if and only if the origin of

$$\dot{z}_{11} = -a_{12}(t)z_{11} + a_{23}(t)z_{12} \quad (3.37)$$

$$\dot{z}_{12} = -a_{23}(t)z_{12} + a_{34}(t)z_{13} \quad (3.38)$$

$$\vdots \quad (3.39)$$

$$\dot{z}_{1N} = -a_{N-1N}(t)z_{1N} \quad (3.40)$$

is (globally) uniformly exponentially stable.

In a fixed topology we have  $a_{\lambda,\lambda+1}(t) > 0$  for all  $t \geq 0$  that is, the  $\lambda$ th node in the sequence always receives information from its parent labeled  $\lambda + 1$ , albeit with varying intensity. The origin of the decoupled bottom equation, which corresponds to the dynamics of the root node, is uniformly exponentially stable if  $a_{N-1N}(t) > 0$  for all  $t$ . Each of the subsystems in (4.57) from the bottom to the top is input to state stable. Uniform exponential stability of the origin  $\{z = 0\}$  follows provided that  $a_{\lambda\lambda+1}$  is bounded (-see lemma 3.12).

In compact form, the consensus dynamics becomes

$$\dot{z}_1 = A_1(t)z_1, \quad z_1 = [z_{11} \cdots z_{1N-1}]^\top \quad (3.41)$$

where the matrix  $A_1(t) \in \mathbb{R}^{N-1 \times N-1}$  is defined as

$$A_1(t) := \begin{bmatrix} -a_{12}(t) & a_{23}(t) & 0 & \cdots & 0 \\ 0 & -a_{23}(t) & a_{34}(t) & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -a_{N-2N-1}(t) & a_{N-1N}(t) \\ 0 & 0 & 0 & \cdots & -a_{N-1N}(t) \end{bmatrix} \quad (3.42)$$

### 3.2.3 Time-varying topology

In a changing topology we have  $a_{\lambda,\kappa}(t) \geq 0$  for all  $t$  however, we assume that there always exist an interval of time of non-negligible length (this will be made precise later) during

which a spanning tree communication is established. The topology may be randomly chosen as long as there always is a spanning tree which lasts for at least a dwell-time.

For the purpose of analysis we aim at identifying, with each possible topology, a linear time-varying system of the form (4.1) with a stable origin and to establish stability of the switched system. To that end, let  $i$  determine one among the  $N!$  topologies schematically represented by a graph as showed in Figure 3.11. Let  $x_\lambda$  denote the state of system  $\Psi_\lambda$  then, for the  $i$ th topology, we define the error

$$z_i = [z_{i1} \cdots z_{iN-1}]^\top \quad (3.43)$$

$$z_{ik} = x_{\pi_k} - x_{\pi_{k+1}} \quad k \in \{1, \dots, N-1\} \quad (3.44)$$

where  $k$  denotes the graphical position of the agent  $\Psi_\lambda$  in the sequence  $\{\Psi_{\pi_k}\}_{k=1}^N$  and  $\pi_k$  represents which agent  $\Psi_\lambda$  is in the position  $j$ , this is,  $\pi_k = \lambda$ .

*Example 2.* Consider two possible topologies among those showed in Figure 3.10 represented in more detail in Figure 3.13 (for  $i = 1$ ) and Figure 3.14 (for  $i = 4$ ). Then, we have

$$z_{11} = x_{\pi_1} - x_{\pi_2} = x_1 - x_2$$

$$z_{12} = x_{\pi_2} - x_{\pi_3} = x_2 - x_3$$

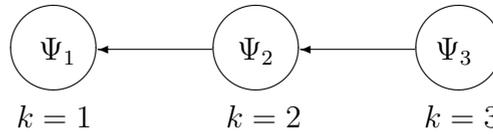


FIGURE 3.13: A topology with 3 agents where  $\pi_1 = 1$ ,  $\pi_2 = 2$  and  $\pi_3 = 3$ .

whereas in the second case, when  $i = 4$ ,

$$z_{21} = x_{\pi_1} - x_{\pi_2} = x_1 - x_3$$

$$z_{22} = x_{\pi_2} - x_{\pi_3} = x_3 - x_2$$

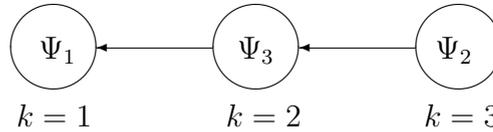


FIGURE 3.14: The second topology with 3 agents where  $\pi_1 = 1$ ,  $\pi_2 = 3$  and  $\pi_3 = 2$ .

That is, for each topology  $i$  the dynamics of the interconnected agents is governed by the equation

$$\dot{z}_i = A_i(t)z_i \quad (3.45)$$

where

$$A_i(t) := \begin{bmatrix} -a_{\pi_1\pi_2}^i(t) & a_{\pi_2\pi_3}^i(t) & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & -a_{\pi_{N-2}\pi_{N-1}}^i(t) & a_{\pi_{N-1}\pi_N}^i(t) \\ 0 & 0 & \cdots & -a_{\pi_{N-1}\pi_N}^i(t) \end{bmatrix} \quad (3.46)$$

According to Lemma 1 the origin  $\{z_i = 0\}$  is uniformly globally exponentially stable provided that  $a_{\pi_k\pi_{k+1}}^i(t)$  is strictly positive for all  $t$ . It is clear that consensus follows if the origin  $\{z_i = 0\}$  for any of the systems (3.45) (with  $i$  fixed for all  $t$ ) is uniformly exponentially stable. Actually, there exist  $\alpha_i$  and  $\bar{\alpha}_i$  such that

$$\|z_i(t)\| \leq \bar{\alpha}_i e^{-\alpha_i t} \quad \forall t \geq 0 \quad (3.47)$$

Observing that all the systems (3.45) are equivalent up to a linear transformation, we establish consensus under the assumption that topology changes, provided that there exists a minimal dwell-time. Indeed, the coordinates  $z_i$  are related to  $z_1$  by the transformation

$$z_i = W_i z_1 \quad (3.48)$$

where  $W_i := TP_iT^{-1}$ ,  $P_i$  is defined in (3.37),  $T \in \mathbb{R}^{N-1 \times N}$  is given by

$$T = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix} \quad (3.49)$$

and  $T^{-1} \in \mathbb{R}^{N \times N-1}$  denotes a right inverse of  $T$ , given by

$$T^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \quad (3.50)$$

Note that the matrix  $W_i \in \mathbb{R}^{N-1 \times N-1}$  is invertible for each  $i \leq N!$  since each of its rows consists in a linear combination of two different rows of  $T^{-1}$ , which contains  $N-1$  linearly-independent rows. Actually, using (3.48) in (3.45) we obtain

$$\dot{z}_1 = \bar{A}_i(t)z_1 \quad (3.51)$$

where

$$\bar{A}_i(t) := W_i^{-1}A_i(t)W_i. \quad (3.52)$$

We conclude that

$$\|z_1(t)\| \leq \tilde{\alpha}_i e^{-\alpha_i t}, \quad \tilde{\alpha}_i := \|W_i^{-1}\| \bar{\alpha}_i, \quad \forall t \geq 0. \quad (3.53)$$

Based on this fact we may now state the following result for the switched error systems which model the network of systems with switching topology.

**Lemma 3.1.** *Consider the switched system*

$$\dot{z}_1 = \bar{A}_{\sigma(t)}(t)z_1 \quad (3.54)$$

where  $\sigma : \mathbb{R}_{\geq 0} \rightarrow \{1, \dots, N!\}$  and for each  $i \in \{1, \dots, N!\}$ ,  $\bar{A}_i$  is defined in (3.52). Let the dwell time

$$\tau_d > \frac{\ln \left( \prod_{i=1}^{N!} \tilde{\alpha}_i \right)}{\sum_{i=1}^{N!} \alpha_i}. \quad (3.55)$$

Then, the equilibrium  $\{z_1 = 0\}$  of (3.51) is uniformly globally exponentially stable for any switching sequence  $\{t_p\}$  such that  $t_{p+1} - t_p > \tau_d$  for every switching time  $t_p$ .

*Proof.* Let  $t_p$  be an arbitrary switching instant. For all  $t \geq t_p$  such that  $\sigma(t) = i$  we have

$$\|z_1(t)\| \leq \tilde{\alpha}_i e^{-\alpha_i(t-t_p)} \|z_1(t_p)\| \quad \forall t_p \leq t < t_{p+1} \quad (3.56)$$

Since by hypothesis  $\tau_d \in [t_p, t_{p+1})$ , from (3.56) we have

$$\|z_1(t_p + \tau_d)\| \leq \tilde{\alpha}_i e^{-\alpha_i \tau_d} \|z_1(t_p)\| \quad (3.57)$$

Using the property of continuity of both the norm function and the state  $z(t)$ , we have

$$\|z_1(t_{p+1})\| \leq \|z_1(t_p + \tau_d)\| \quad (3.58)$$

and therefore

$$\|z_1(t_{p+1})\| \leq \tilde{\alpha}_i e^{-\alpha_i \tau_d} \|z_1(t_p)\| \quad (3.59)$$

Note that to guarantee asymptotic stability of (3.54) it is sufficient that for every pair of switching times  $t_p$  and  $t_q$

$$\|z_1(t_q)\| - \|z_1(t_p)\| < 0 \quad (3.60)$$

whenever  $p < q$  and  $\sigma(t_p) = \sigma(t_q)$ .

Now consider the sequence of switching times  $t_p, t_{p+1}, \dots, t_{p+N!-1}, t_{p+N!}$  satisfying  $\sigma(t_p) \neq \sigma(t_{p+1}) \neq \dots \neq \sigma(t_{p+N!-1})$  and  $\sigma(t_p) = \sigma(t_{p+N!})$  which corresponds to a switching signal in which all the  $N!$  switched are chosen.

From (3.59) it follows that

$$\|z_1(t_{p+N!})\| \leq \left( \prod_{i=1}^{N!} \tilde{\alpha}_i e^{-\alpha_i \tau_d} \right) \|z_1(t_p)\| \quad (3.61)$$

To ensure that

$$\|z_1(t_{p+N!})\| - \|z_1(t_p)\| < 0 \quad (3.62)$$

it is sufficient that

$$\left( \prod_{i=1}^{N!} \tilde{\alpha}_i e^{-\alpha_i \tau_d} - 1 \right) \|z_1(t_p)\| < 0 \quad (3.63)$$

Therefore, since the norm is a non-negative function we obtain

$$\prod_{i=1}^{N!} \tilde{\alpha}_i e^{-\alpha_i \tau_d} < 1 \quad (3.64)$$

and the proof follows.  $\square$

Finally, in view of Lemma 3.1 we can make the following statement.

*Theorem 3.* Let  $\{t_p\}$  denote a sequence of switching instants  $p \in \mathbb{Z}_{\geq 0}$  and let  $\sigma : \mathbb{R}_{\geq 0} \rightarrow \{1, \dots, N!\}$  be a piecewise constant function satisfying  $\sigma(t) \equiv i$  for all  $t \in [t_p, t_{p+1})$  with  $t_p - t_{p+1} \geq \tau_d$  and  $\tau_d$  satisfying (3.55).

Consider the system (3.1) in closed loop with

$$u_{\pi_k}^{\sigma(t)} = \begin{cases} -a_{\pi_k \pi_{k+1}}^{\sigma(t)}(t) [x_{\pi_k} - x_{\pi_{k+1}}], & k \in \{1, \dots, N-1\} \\ 0, & k = N \end{cases} \quad (3.65)$$

Let the interconnection gains  $a_{\lambda\kappa}^i$ , for all  $i \in \{1, \dots, N!\}$  and all  $\lambda, \kappa \in \{1, \dots, N-1\}$ , be persistently exciting. Then, the system reaches consensus with uniform exponential convergence.

### 3.2.4 Illustrative example and simulation results

For illustration, we consider a network of three agents hence, with six possible topologies, as showed in Figure 3.10. The information exchange among agents in each topology is ensured via channels with persistently-exciting communication intensity; the corresponding parameters are shown in Table I.

i=1	$T$	$\mu$	i=2	$T$	$\mu$	i=3	$T$	$\mu$
$a_{12}(t)$	0.25	0.5	$a_{21}(t)$	2.0	1.6	$a_{31}(t)$	0.3	0.1
$a_{23}(t)$	0.2	1	$a_{13}(t)$	0.8	0.2	$a_{12}(t)$	0.7	0.6
i=4	$T$	$\mu$	i=5	$T$	$\mu$	i=6	$T$	$\mu$
$a_{13}(t)$	2	1	$a_{23}(t)$	0.4	0.1	$a_{32}(t)$	0.5	0.3
$a_{32}(t)$	4	0.4	$a_{31}(t)$	0.5	0.4	$a_{21}(t)$	4.2	1.8

Table I. Parameters of the nterconnection gains

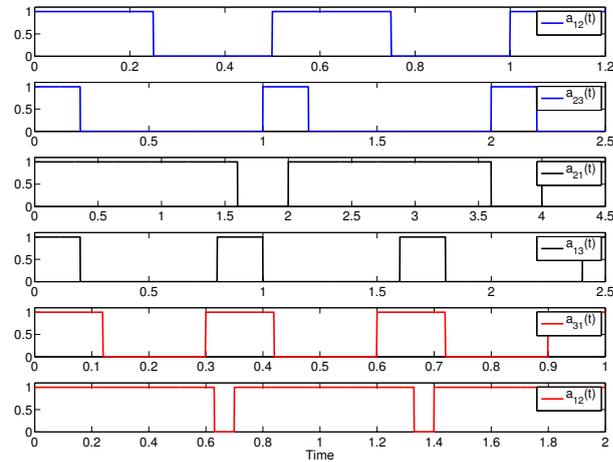


FIGURE 3.15: Persistently exciting interconnection gains for the topologies  $\{\Psi_1, \Psi_2, \Psi_3\}$ ,  $\{\Psi_2, \Psi_1, \Psi_3\}$  and  $\{\Psi_3, \Psi_1, \Psi_2\}$

The graphs corresponding to the interconnection gains are showed in Figures 3.15 and 3.16. By applying Lemma 1, we can compute  $\tilde{\alpha}_i$  and  $\alpha_i$  for each topology  $i$ , see Table II. Substituting the values of  $\tilde{\alpha}_i$  and  $\alpha_i$  into (3.55), we find that the dwell time must satisfy  $\tau_d > 7.92$ .

i	1	2	3	4	5	6
$\tilde{\alpha}_i$	6.51	10.26	6.11	12.85	4.71	3.98
$\alpha$	0.2	0.3	0.4	0.1	0.35	0.1

Table II. Parameters corresponding to the exponential bounds

We performed some numerical simulations using SIMULINK of MATLAB. In a first test, the initial conditions are set to  $x_1(0) = -2$ ,  $x_2(0) = 1.5$  and  $x_3(0) = -0.5$ ; the switching signal  $\sigma(t)$  is illustrated in Figure 3.17.

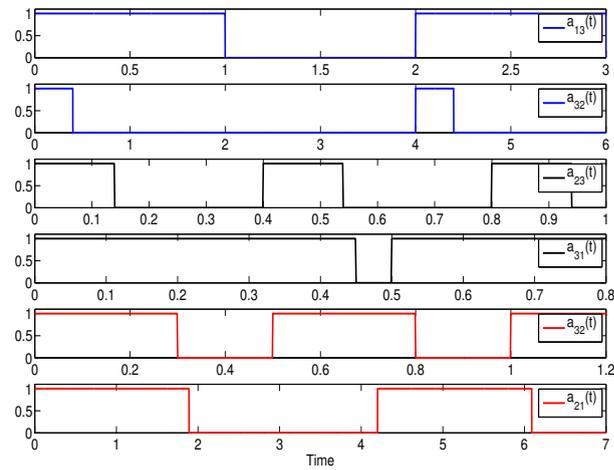


FIGURE 3.16: Persistently exciting interconnection gains for the topologies  $\{\Psi_1, \Psi_3, \Psi_2\}$ ,  $\{\Psi_2, \Psi_3, \Psi_1\}$  and  $\{\Psi_3, \Psi_2, \Psi_1\}$

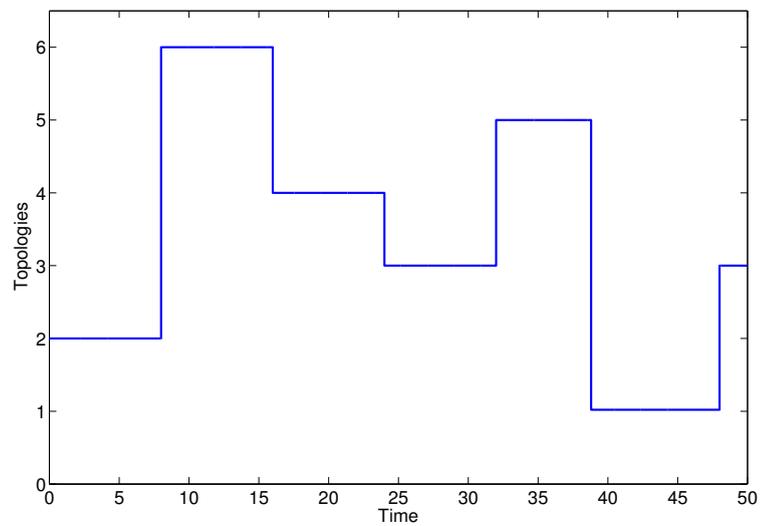
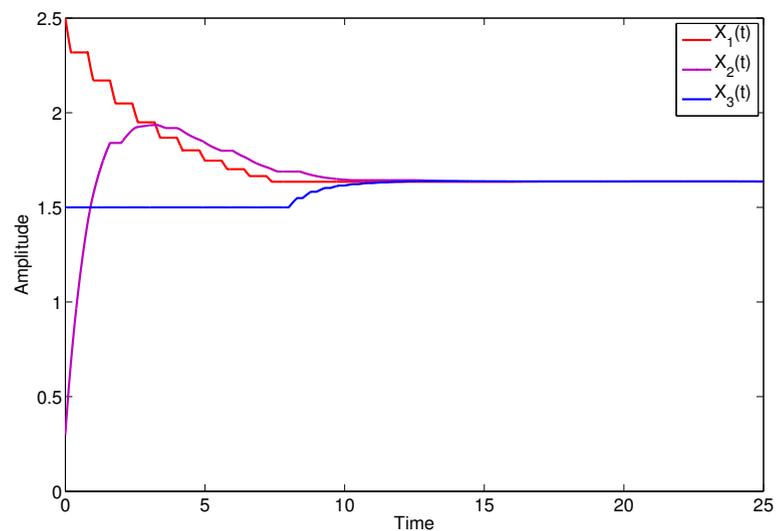


FIGURE 3.17: A switching signal  $\sigma(t)$  satisfying the dwell-time condition

The systems' trajectories, converging to a consensus equilibrium, are showed in Figure 3.18.

FIGURE 3.18: Trajectories of  $x_1(t)$ ,  $x_2(t)$  and  $x_3(t)$ 

### 3.3 Discussion

At first, we discuss the consensus problem for a network of dynamic agents with a ring topology under the assumption that each interconnection between any pair of agents is represented by bounded persistently exciting signals. By using the small-gain theorem we obtain that the system reached consensus if the intensity of *one* of the interconnections is relatively small.

The second result deals with the consensus problem for networks with changing communication topology and with time-dependent communication links. That is, the network changes in two dimensions: *geographical* and *temporal*. We establish that consensus is reached provided that there always exists a spanning tree topology for a minimal dwell-time. The interconnection gains are persistently exciting signals. The originality of our work lies in the method of proof, based on stability theory of time-varying and switched systems.

## Chapter 4

# Leader-follower formation and tracking control of mobile robots along the straight paths

### 4.1 Introduction

In this Chapter, we discuss the problem of leader-follower formation and tracking control of mobile robots along the straight paths. The model of a mobile robot of the unicycle type is shown in Figure 4.1.

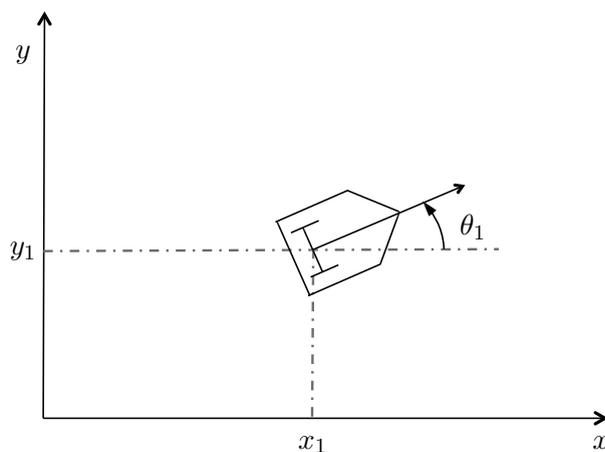


FIGURE 4.1: A two-wheel mobile robot.

We assume that the masses and inertias of the wheels are negligible and that both the forward velocity  $v_1$  and the angular velocity  $w_1$  are controlled independently by motors. Let  $(x, y)$  denote the co-ordinates of the centre of mass, and  $\theta_1$  the angle between the

heading direction and the  $x$ -axis. We assume that the wheels do not slide, which results in the following equations

$$\dot{x}_1 = v_1 \cos(\theta_1) \quad (4.1a)$$

$$\dot{y}_1 = v_1 \sin(\theta_1) \quad (4.1b)$$

$$\dot{\theta}_1 = w_1. \quad (4.1c)$$

where  $v_1$  and  $w_1$  are considered as inputs.

We wish to examine the problem of formation control of multiple mobile robots, using a leader-follower approach. In Section 4.2, for clarity of exposition, we firstly present a result on leader-follower tracking control (two robots only) and describe the control approach. Then, in Section 4.3 we present a result for a cascade-like configuration of leader-follower mobile robots. In the communication graphs, each robot becomes leader to one robot and follower of another. There is a unique swarm leader robot which receives the information of the reference trajectory and there is a unique tail robot which is leader to one. The performance of the derived controllers is illustrated by means of simulations in Section 4.4. The Chapter ends with concluding remarks in Section 4.5.

## 4.2 Leader-follower tracking controller

### A. Kinematic formulation

After the seminal paper [29], the tracking control problem for mobile robots may be reformulated as that of controlling a robot in a leader-follower configuration as shown in Figure 4.2. Hence, the tracking control problem consists, for a mobile robot with kinematic model (4.1), in following a fictitious vehicle

$$\dot{x}_0 = v_0 \cos(\theta_0) \quad (4.2a)$$

$$\dot{y}_0 = v_0 \sin(\theta_0) \quad (4.2b)$$

$$\dot{\theta}_0 = w_0. \quad (4.2c)$$

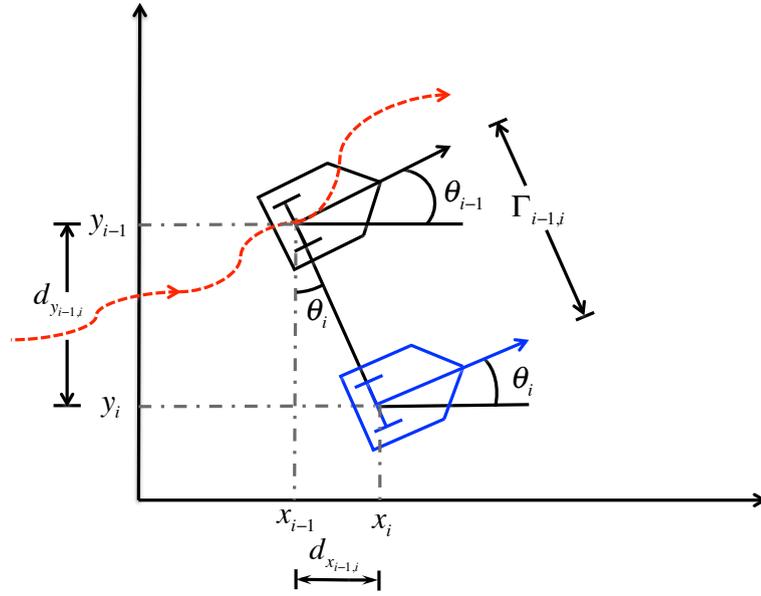


FIGURE 4.2: Generic representation of a leader-follower configuration. For a swarm of  $n$  vehicles, any geometric topology may be easily defined by determining the position of each vehicle relative to its leader. This does not affect the kinematic model.

That is,  $v_0$  and  $w_0$  are, respectively, forward and angular velocity references. From a control viewpoint, the goal is to steer to zero the differences between the Cartesian coordinates of the two robots, as well as orientation angles; in other words, to steer the following quantities to zero:

$$\begin{aligned} p_{1x} &= x_0 - x_1 + d_{x(i-1),i} \\ p_{1y} &= y_0 - y_1 - d_{y(i-1),i} \\ p_{1\theta} &= \theta_0 - \theta_1. \end{aligned}$$

where  $d_x$  and  $d_y$  denoted design parameters imposed by the topology and path planner. Using geometry in Fig. 4.2 we can write the relationship between the turning radius  $\Gamma_{i-1,i}$ , the angle of turn  $\theta_i$  and the translation  $(d_{x(i-1),i}, d_{y(i-1),i})$  as the following equations:

$$\begin{aligned} d_{x(i-1),i} &= \Gamma_{i-1,i} \sin \theta_i \\ d_{y(i-1),i} &= \Gamma_{i-1,i} \cos \theta_i \end{aligned}$$

In this case, we have

$$\begin{aligned} d_{x(0),1} &= \Gamma_{0,1} \sin \theta_1 \\ d_{y(0),1} &= \Gamma_{0,1} \cos \theta_1 \end{aligned}$$

Then, for the purpose of analysis we transform the error coordinates  $[p_{1x}, p_{1y}, p_{1\theta}]$  of the leader robot from the global coordinate frame to local coordinates fixed on the robot that is,

$$\begin{bmatrix} e_{1x} \\ e_{1y} \\ e_{1\theta} \end{bmatrix} = \begin{bmatrix} \cos \theta_1 & \sin \theta_1 & 0 \\ -\sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{1x} \\ p_{1y} \\ p_{1\theta} \end{bmatrix}. \quad (4.3)$$

In the new coordinates, the error dynamics between the virtual reference vehicle and the follower becomes

$$\dot{e}_{1x} = w_1 e_{1y} - v_1 + v_0 \cos e_{1\theta} + \Gamma_{0,1} w_1 \quad (4.4a)$$

$$\dot{e}_{1y} = -w_1 e_{1x} + v_0 \sin e_{1\theta} \quad (4.4b)$$

$$\dot{e}_{1\theta} = w_0 - w_1. \quad (4.4c)$$

The tracking control problem is transformed into that of stabilising the origin for the error dynamics (4.4). It is commonly assumed that the reference angular velocity  $w_0$  is different from zero. Indeed, otherwise the system loses controllability in the  $y$  coordinate –see Eq. (4.4b). For instance, the results in [46], and consequently those of [32] which rely on the former, are based on the assumption that the angular reference velocity satisfies a persistency of excitation condition that is,  $w_0(s) := \psi(s)^2$  where

$$\int_t^{t+T} \psi(s)^2 ds \geq \mu, \quad \forall t \geq 0 \quad (4.5)$$

for some positive constants  $\mu$  and  $T$ . In [6, 31] where complex nonlinear time varying controls are designed to allow for reference velocity trajectories that converge to zero. Furthermore, in [31] the authors cover the case when also the forward velocity  $v_0$  may converge to zero that is, tracking control towards a fixed point. In [6] the controller is designed so as to make the robot *converge* to the straight-line trajectory resulting in a path that makes it go back and forth on the path.

Our control approach is inspired by the cascades-based controllers originally presented in [46], in which persistency of excitation is used to guarantee exponential stabilisation of the origin for the error dynamics. We extend this approach to the case in which the reference angular velocity fails to satisfy the persistency of excitation condition. As a matter of fact, we allow for the case in which  $w_0 \equiv 0$ . Although structurally similar, the control laws are given by

$$v_1 = v_0(t) + c_2 e_{1x} + \Gamma_{0,1} w_1, \quad c_2 > 0 \quad (4.6a)$$

$$w_1 = h(t, e_{1y}) + c_1 e_{1\theta}, \quad c_1 > 0 \quad (4.6b)$$

where  $h$  is bounded, locally of linear order in  $e_{1y}$ , and continuously differentiable. It is the term  $h$  above which replaces the zero angular velocity in the controller introduced in [46] which relies on the assumption that  $w_0$  is persistently exciting. In the present context, we impose as condition that  $h(t, 0) \equiv 0$  and  $\dot{h}$  is persistently exciting for any  $e_{1y} \neq 0$ ; a precise definition is given farther below.

We show that the controller (4.6) stabilizes globally and uniformly the error dynamics. In order to understand the stabilisation mechanism of the controller (4.6) it is convenient to examine the closed-loop equations, which result from using (4.6) in (4.4) to obtain

$$\dot{e}_{1x} = w_1 e_{1y} - c_2 e_{1x} + v_0 [\cos e_{1\theta} - 1] \quad (4.7a)$$

$$\dot{e}_{1y} = -w_1 e_{1x} + v_0 \sin e_{1\theta} \quad (4.7b)$$

$$\dot{e}_{1\theta} = -c_1 e_{1\theta} - h(t, e_{1y}). \quad (4.7c)$$

This system may be rewritten in compact form as

$$\begin{bmatrix} \dot{e}_{1x} \\ \dot{e}_{1y} \end{bmatrix} = \begin{bmatrix} -c_2 & w_1 \\ -w_1 & 0 \end{bmatrix} \begin{bmatrix} e_{1x} \\ e_{1y} \end{bmatrix} + d(t, e_{1\theta}) \quad (4.8a)$$

$$\dot{e}_{1\theta} = -c_1 e_{1\theta} - h(t, e_{1y}) \quad (4.8b)$$

where we purposefully dropped the arguments of  $w_1$  and defined the interconnection term

$$d(t, e_{1\theta}) := \begin{bmatrix} v_0(t)(\cos e_{1\theta} - 1) \\ v_0(t) \sin e_{1\theta} \end{bmatrix}. \quad (4.9)$$

We are interested in establishing uniform global asymptotic stability of the origin of  $(e_{1x}, e_{1y}, e_{1\theta}) = (0, 0, 0)$ . To that end, we observe that the system (4.8) consists in the feedback interconnection of two systems as illustrated in Figure 4.3.

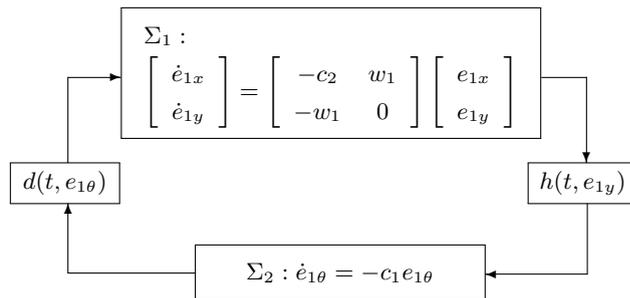


FIGURE 4.3: Small gain feedback representation of the closed-loop system with a persistently exciting controller.

Roughly speaking, after adaptive control systems theory, the system in the centre upper block is uniformly asymptotically stable at the origin, provided that  $c_2 > 0$  and  $w_1$  is persistently exciting, globally Lipschitz and bounded. On the other hand, the origin of the system in the lower-centre block is, clearly, exponentially stable if  $c_1 > 0$ . As a matter of fact, it may also be established that each of these subsystems is input to state stable. Moreover, the interconnection terms  $h$  and  $d$  are both uniformly bounded and satisfy  $d(t, 0) \equiv 0$ ,  $h(t, 0) \equiv 0$ . Thus, the interconnected system (4.8) may be regarded as the feedback interconnection of two input to state stable (ISS) systems. Consequently, stability of the origin of (4.8) may be concluded invoking the small-gain theorem for ISS systems.

Although intuitive, the previous arguments hide certain difficulties in the analysis that we intend to clarify next. Firstly, the function  $w_1$  depends on the states and time hence, persistency of excitation must be appropriately defined. We use a relaxed notion of persistency of excitation, originally introduced in [38]; the following is a refined definition reported in [47].

*Definition 6* (u $\delta$ -Persistency of excitation). Let  $f(\cdot, \cdot)$  be such that the system  $\dot{x} = f(t, x)$ , with state  $x = [x_1^\top \ x_2^\top]^\top$  and solution  $x(t) = x(t, t_o, x_o)$  starting at  $(t_o, x_o) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$  is forward complete.

The pair  $(\phi, f)$  is called uniformly  $\delta$ -persistently exciting (u $\delta$ -PE) with respect to  $x_1$  if, for each  $r$  and  $\delta > 0$ , there exist constants  $T(r, \delta)$  and  $\mu(r, \delta) > 0$  such that, for all  $(t_o, x_o) \in \mathbb{R}_{\geq 0} \times B_r$ , all corresponding solutions satisfy<sup>1</sup>

$$\left\{ \min_{s \in [t, t+T]} \|x_1(s)\| \geq \delta \right\} \Rightarrow \left\{ \int_t^{t+T} \phi(\tau, x(\tau, t_o, x_o)) \phi(\tau, x(\tau, t_o, x_o))^\top d\tau \geq \mu I \right\} \quad (4.10)$$

for all  $t \geq t_o$ .

In words, the pair  $(\phi, f)$  is u $\delta$ -PE if the function  $\phi(\cdot, x(\cdot))$  is PE in the usual sense of adaptive control, uniformly in initial conditions  $(t_o, x_o) \in \mathbb{R}_{\geq 0} \times B_r$ , whenever the trajectory  $x(\cdot)$  is away from a  $\delta$ -neighbourhood of the origin. For simplicity we may also say, with an abuse of terminology, that the *function*  $\phi$  is u $\delta$ -PE in the understanding that the *pair* satisfies Definition 6. For instance, the function  $\phi(t, x) := \psi(t) \|x\|$  is u $\delta$ -PE if  $\psi$  satisfies (4.5). In particular, the function  $\phi(t) = \sin(t)\alpha(x)$  with  $\alpha$  continuous, zero at zero, is u $\delta$ -PE.

There are several properties of u $\delta$ -PE functions which are useful in control design for nonholonomic systems; these are reported in [37]. One of them is that if  $w_1$  is u $\delta$ -PE then there exists a function  $\tilde{w}_1$  which depends only on time and which is persistently

<sup>1</sup>Notice that, in what the definition concerns, unicity of solutions is not required.

exciting in the usual sense that is,

$$\int_t^{t+T'} \|\tilde{w}_1(\tau)\|^2 d\tau \geq \mu' \quad \forall t \geq 0 \quad (4.11)$$

for some  $T'$  and  $\mu' > 0$ . Moreover,  $\tilde{w}_1$  may be purposefully constructed to satisfy

$$\tilde{w}_1(t) := h(t, e_{1y}(t)) + c_1 e_{1\theta}(t) \quad \forall t : \|e_{1y}(t)\| \geq \delta. \quad (4.12)$$

Even though the function  $\tilde{w}_1$  is parameterized by  $\delta$  it is guaranteed that for any  $\delta > 0$  there exists  $\tilde{w}_1$  satisfying all of the above.

This property is useful because, for any  $\delta$  and for all  $t$  such that  $\|e_{1y}(t)\| \geq \delta$ , the trajectories of  $\Sigma_1$  in Figure 4.3 coincide with those of

$$\begin{bmatrix} \dot{e}_{1x} \\ \dot{e}_{1y} \end{bmatrix} = \underbrace{\begin{bmatrix} -c_2 & \tilde{w}_1(t) \\ -\tilde{w}_1(t) & 0 \end{bmatrix}}_{\tilde{A}_1(t)} \underbrace{\begin{bmatrix} e_{1x} \\ e_{1y} \end{bmatrix}}_{z_1} \quad (4.13)$$

which is linear. The clear advantage is that the behaviour of the trajectories of (4.8a) with  $d \equiv 0$  may be analysed as those of a linear system, at least while the trajectories are away from the origin (strictly speaking away of *any*  $\delta$ -neighbourhood). In particular, global exponential stability of the origin of (4.13) is easily concluded invoking classical results on adaptive control systems –see [24]. Consequently, one may use an intuitive contradiction argument to establish uniform global asymptotic stability of (4.8a) with  $d \equiv 0$ : assume that the origin is not attractive then, the trajectories (tend to) remain away of an arbitrary  $\delta$ -neighbourhood of the origin<sup>2</sup>. In that case, since they coincide with those generated by (4.13) which is exponentially stable, it follows that the trajectories of (4.8a) must converge to zero. The argument may be repeated for any arbitrarily small  $\delta$  hence, the “exponential” rate of convergence diminishes but remains uniform in the initial conditions.

Precise general statements for nonlinear time-varying systems are reported in [47]. For the purpose of the system (4.8) we proceed by showing that

- the origin is uniformly stable;
- the solutions are uniformly globally bounded;
- the origin is uniformly globally attractive.

<sup>2</sup>An “oscillating” behaviour which would consist in the trajectories entering and exiting the  $\delta$ -neighbourhood is excluded since the origin is stable.

The first bullet comes from the fact that the system corresponds to the feedback interconnection of two *locally* input to state stable systems. For the first block,  $\Sigma_1$ , the origin is uniformly globally asymptotically stable provided that  $w_1$  is uniformly  $\delta$ -PE with respect to  $e_{1y}$ , bounded and with bounded derivatives –see Theorem 4 in the appendix. On the other hand, local input to state stability (also known as total stability) with respect to the additive input  $d$  is a direct consequence of uniform global asymptotic stability –see [22]. For  $\Sigma_2$  it is evident that the origin is globally exponentially stable and that  $\Sigma_2$  is input-to-state stable with respect to  $h$ . Actually, the interconnected system showed in Figure 4.3 is (locally) uniformly asymptotically stable.

The boundedness property follows from the fact that the trajectories of (4.13) coincide with those of  $\Sigma_1$  in Figure 4.3 (which are globally uniformly bounded) for all  $t$  such that  $\|e_{1y}(t)\| \geq \delta$ ; in particular, if the trajectories tend to grow unboundedly. To see more clearly we remark that in view of (4.11) the origin of (4.13) is globally exponentially stable, this implies that, for any  $\delta$ , there exist positive definite symmetric matrices  $P_\delta$  and  $Q_\delta$  such that  $Q_\delta(t) = \tilde{A}_{1\delta}(t)^\top P_\delta(t) + P_\delta(t)\tilde{A}_{1\delta}(t) + \dot{P}_\delta(t)$  and the total derivative of

$$V_{1\delta}(t, z_1) = z_1^\top P_\delta(t) z_1$$

along the trajectories of (4.8a) satisfies

$$\dot{V}_{1\delta}(t, z_1) \leq -z_1^\top Q_\delta(t) z_1 + z_1^\top P_\delta(t) d(t, e_{1\theta})$$

for all  $t$  such that  $\|e_{1y}(t)\| \geq \delta$ . In turn, we have

$$\begin{aligned} \dot{V}_{1\delta}(t, z_1) &\leq -q_m \|z_1\|^2 + p_M \|z_1\| \|d(t, e_{1\theta})\| \\ &\leq -\frac{q_m}{2} \|z_1\|^2 + \frac{p_M^2}{2q_m} \|d(t, e_{1\theta})\|^2 \end{aligned} \quad (4.14)$$

where we used  $p_M I \geq P_\delta(t)$  and  $Q_\delta(t) \geq q_m I$ . Since  $d(t, e_{1\theta}(t))$  is bounded –see (4.9), it is clear that if  $\|z_1(t)\| \rightarrow \infty$  then  $\dot{V}_{1\delta}(t, z_1(t)) \leq 0$  for sufficiently large  $t$ . We argue in a similar way for the trajectories of (4.8b); the total derivative of  $V_{2\delta}(e_{1\theta}) := 0.5 \|e_{1\theta}\|^2$  yields

$$\begin{aligned} \dot{V}_{2\delta}(e_{1\theta}) &\leq -c_1 \|e_{1\theta}\|^2 + \|e_{1\theta}\| \|h(t, e_{1y})\| \\ &\leq -\frac{\lambda c_1}{2} \|e_{1\theta}\|^2 + \frac{\|h(t, e_{1y})\|^2}{2c_1 \lambda} \end{aligned} \quad (4.15)$$

for any  $\lambda > 0$ . Recall that, by assumption,  $h$  is bounded.

Next, we show that the origin of (4.8) is uniformly globally attractive; that is, we must show that for any  $r$  and  $\sigma > 0$ , there exists  $T$  such that

$$\|e_1(t_0)\| \leq r \implies \|e_1(t)\| \leq \sigma \quad \forall t \geq t_0 + T. \quad (4.16)$$

So let  $r$  and  $\sigma$  be arbitrary given positive constants and define  $\delta := \sigma$ . To establish the convergence property (4.16) we study the behaviour of the solutions of

$$\begin{bmatrix} \dot{e}_{1x} \\ \dot{e}_{1y} \end{bmatrix} = \begin{bmatrix} -c_2 & \tilde{w}_1(t) \\ -\tilde{w}_1(t) & 0 \end{bmatrix} \begin{bmatrix} e_{1x} \\ e_{1y} \end{bmatrix} + d(t, e_{1\theta}) \quad (4.17a)$$

$$\dot{e}_{1\theta} = -c_1 e_{1\theta} - h(t, e_{1y}) \quad (4.17b)$$

whose trajectories, as we have emphasised, coincide with those of (4.8) for all  $t$  such that  $\|e_{1y}(t)\| \geq \delta$ . Therefore, it suffices to establish global exponential stability of the origin of (4.17). To that end, let

$$\lambda := \sqrt{\frac{5v_0^M}{2}} \frac{p_M}{q_m c_1} \quad \varepsilon := \frac{\lambda c_1}{4} \quad \eta := \frac{2q_m}{p_M^2} \varepsilon \quad (4.18)$$

and consider the Lyapunov function  $V_\delta := \eta V_{1\delta} + V_{2\delta}$ . Its total derivative satisfies

$$\dot{V}_\delta(t, z_1, e_{1\theta}) \leq -\left(\frac{q_m^2}{p_M^2} \varepsilon - \frac{v_0^M}{2c_1 \lambda}\right) \|z_1\|^2 - \left(\frac{c_1 \lambda}{2} - \varepsilon\right) \|e_{1\theta}\|^2$$

where we introduced the bound  $v_0^M \geq \|v_0(t)\|$  and we used the fact that  $\|h(t, e_{1y})\| \leq v_0^M \|z_1\|$  and  $\|d(t, e_{1\theta})\| \leq \|e_{1\theta}\|$ . In view of the expressions in (4.18),  $\dot{V}_\delta$  is negative definite, actually,

$$\dot{V}_\delta(t, z_1, e_{1\theta}) \leq -\alpha \|z_1\|^2 - \varepsilon \|e_{1\theta}\|^2, \quad \alpha > 0$$

We conclude that the trajectories of (4.8), which coincide with those of (4.17) for all  $t$  such that  $\|e_{1y}(t)\| \geq \delta$ , tend to zero exponentially fast as long as the latter inequality holds. In view of this there exists a finite time  $T$  such that for any  $\delta' \in (0, \delta]$ , we have  $\|e_1(t_0 + T)\| \leq \delta'$ . From uniform stability, we have  $\|e_1(t)\| \leq \delta$  for all  $t \geq t_0 + T$ . Since  $\delta = \sigma$  by definition, the statement follows.

*Remark 3.* It is worth noticing that this reasoning is reminiscent of ultimate boundedness: namely, the solutions tend to a ball of radius  $\delta$ . However, in this case, opposite to arguments leading to ultimate boundedness, the number  $\delta$  is arbitrarily given and the previous arguments continue to hold for any  $\delta$  and *fixed* values of the control gains.

In the following lemma we obtain the uniformly globally asymptotically stability for the case leader-follower tracking.

**Lemma 4.1.** *The origin of the system (4.8) is uniformly globally asymptotically stable if  $c_1 > 0$ ,  $c_2 > 0$ ,  $v_0$  is bounded and  $w_1$  is  $u\delta$ -PE, bounded and locally Lipschitz in  $e_{1y}$  uniformly in  $t$ . Moreover,  $u\delta$ -PE of  $w_1$  is also a necessary condition.*

The previous lemma establishes a strong, yet intermediary, convergence result in the pursuit of our main objective: tracking control of nonholonomic robots. It is left to state under which conditions  $w_1$  is  $u\delta$ -PE. As a matter of fact, this has been established in the context of set-point stabilization, in [37]. The control input  $w_1$  satisfies the differential equation

$$\dot{w}_1 = -c_1 w_1 + \dot{h}(t, e_{1y})$$

which corresponds to the equation of a low-pass filter. That is, a stable strictly proper linear system with input  $\dot{h}$ . It is well-known from adaptive control textbooks that the output of a low-pass filter driven by an input that is persistently exciting, is also persistently exciting –see [24, 41]. Now, for nonlinear functions we have the analogous Property 1 from [37], which is recalled in the Appendix. Therefore,  $w_1$  which corresponds to a “filtered version” of  $\dot{h}$ , is  $u\delta$ -PE if so is  $\dot{h}$ .

We conclude that the following statement holds.

*Proposition 1.* Consider the system (4.4) in closed-loop with the controller (4.6). Let  $h$  be bounded, once continuously differentiable, such that  $h(t, e_{1y})$  has a unique zero at  $e_{1y} = 0$  for each fixed  $t$ ,

$$\sup_{t, e_{1y}} \left\{ \|h(t, e_{1y})\|, \left\| \frac{\partial h(t, e_{1y})}{\partial e_{1y}} \right\|, \left\| \frac{\partial h(t, e_{1y})}{\partial t} \right\| \right\} \leq c \quad (4.19)$$

for some positive constant  $c$  and assume that for any  $\delta > 0$  there exist positive numbers  $\mu$  and  $T$  such that

$$\|e_{1y}\| \geq \delta \implies \int_t^{t+T} \|\dot{h}(\tau, e_{1y})\| d\tau \geq \mu, \quad \forall t \geq 0. \quad (4.20)$$

Then, the origin of the closed-loop system is uniformly globally asymptotically stable.

*Remark 4.* The function  $h$  may be defined as a monotonic locally linear function of  $e_{1y}$  and smooth, persistently exciting in  $t$ ; for instance,  $h(t, e_{1y}) = \psi(t)\text{sat}(e_{1y})$  where  $\text{sat}(\cdot)$  is a saturation function and  $\psi$  is persistently exciting.

*Proof.* . The closed-loop system is given by Eqs. (4.8) and it may be easily showed, using  $V_1$  and  $V_2$  above, that the system is forward complete. Now, since  $\dot{h}$  is a scalar function (4.20) holds if and only if the following condition, along complete trajectories,

$$\min_{\tau \in [t, t+T]} \|e_{1y}(\tau)\| \geq \delta \implies \int_t^{t+T} \|\dot{h}(\tau, e_{1y}(\tau))\| d\tau \geq \mu, \quad t \geq 0$$

holds. Therefore,  $\dot{h}$  satisfies the properties in Definition 6 and, in view of Property 1, it follows that  $w_1$  is  $u\delta$ -PE. The result follows from Lemma 4.1  $\square$

### 4.3 Leader-follower formation control

We extend the previous result to the case of formation-tracking control. Consider a group of  $n$  mobile robots with kinematic models,

$$\dot{x}_i = v_i \cos(\theta_i) \quad (4.21a)$$

$$\dot{y}_i = v_i \sin(\theta_i) \quad (4.21b)$$

$$\dot{\theta}_i = w_i, \quad i \in [1, n] \quad (4.21c)$$

where, for the  $i$ -th robot,  $x_i$  and  $y_i$  determine the position with respect to a globally-fixed frame,  $\theta_i$  defines the heading angle –see Figure 4.2, and the linear and angular velocities are denoted by  $v_i$  and  $w_i$  respectively.

The control objective is to make the  $n$  robots take specific postures determined by the topology designer, and to make the swarm follow a path determined by a virtual reference vehicle labelled  $R_0$ . Any physically feasible geometrical configuration may be achieved and one can choose any point in the Cartesian plane to follow the virtual reference vehicle.

We solve the problem using a spanning-tree communication topology and a recursive implementation of the tracking leader-follower controller (4.6). That is, the swarm has only one ‘leader’ robot tagged  $R_1$  whose local controller uses knowledge of the reference trajectory generated by the virtual leader  $R_0$ . Therefore, in the communications graph,  $R_1$  is the child of the root-node robot  $R_0$  and the other robots are intermediate nodes labeled  $R_2$  to  $R_{n-1}$  that is,  $R_i$  acts as leader for  $R_{i+1}$  and follows  $R_{i-1}$ . The last robot in the communication topology is denoted  $R_n$  and has no followers that is, it constitutes the tail node of the spanning tree –see Figure 4.4. We remark that the notation  $R_{i-1}$  refers to the graph *communication* topology and not to the *formation* topology.

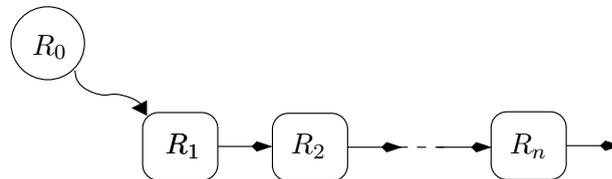


FIGURE 4.4: Communication topology: a spanning directed tree with permanent communication between  $R_i$  and  $R_{i+1}$  for all  $i \in [0, n - 1]$ .

The fictitious vehicle, which serves as reference to the swarm, describes the reference trajectory defined by (4.2); the desired linear and angular velocities  $v_0$  and  $w_0$  are communicated to the leader robot,  $R_1$ , only. According to this communication topology, and following the setting for tracking control, the formation control problem reduces to that of stabilisation of the error dynamics between any pair of leader-follower robots. For each  $i \leq N$ , this is

$$\dot{e}_{ix} = w_i e_{iy} - v_i + v_{i-1} \cos e_{i\theta} + \Gamma_{i-1,i} w_i \quad (4.22a)$$

$$\dot{e}_{iy} = -w_i e_{ix} + v_{i-1} \sin e_{i\theta} \quad (4.22b)$$

$$\dot{e}_{i\theta} = w_{i-1} - w_i \quad (4.22c)$$

and for each  $i \geq 1$  we define the control inputs  $v_i$  and  $w_i$  as

$$v_i = v_{i-1} + c_{2i} e_{ix} + \Gamma_{i-1,i} w_i \quad (4.23a)$$

$$w_i = w_{i-1} + c_{1i} e_{i\theta} + h_i(t, e_{iy}) \quad (4.23b)$$

where  $h_i$  is once continuously differentiable, bounded and with bounded derivative. Then, the closed-loop equations yield

$$\begin{bmatrix} \dot{e}_{ix} \\ \dot{e}_{iy} \end{bmatrix} = \begin{bmatrix} -c_{2i} & w_i \\ -w_i & 0 \end{bmatrix} \begin{bmatrix} e_{ix} \\ e_{iy} \end{bmatrix} + \begin{bmatrix} v_{i-1}[1 - \cos e_{i\theta}] \\ v_{i-1} \sin e_{i\theta} \end{bmatrix} \quad (4.24a)$$

$$\dot{e}_{i\theta} = -c_{1i} e_{i\theta} + h_i(t, e_{iy}) \quad (4.24b)$$

which has the form of (4.8) and inherits similar properties; actually, similarly to Lemma 4.1 we have the following.

**Lemma 4.2.** *The origin of the system (4.24) is uniformly globally asymptotically stable, for any  $i \leq N$ , if  $c_{1i} > 0$ ,  $c_{2i} > 0$ ,  $v_0$  is bounded and  $w_i$  is  $u\delta$ -PE, bounded and locally Lipschitz in  $e_{iy}$  uniformly in  $t$ . Moreover,  $u\delta$ -PE of  $w_i$  is also a necessary condition.*

The proof of this statement follows *mutatis mutandis* along the proof-lines of Lemma 4.1 observing that: 1) the function  $h_i$  is, by assumption, continuous and bounded; 2) for (4.24a) with  $e_{i\theta} = 0$ , the origin is uniformly globally asymptotically stable provided that  $w_i$  is  $u\delta$ -PE and 3) the interconnection term

$$d_i := \begin{bmatrix} v_{i-1}[1 - \cos e_{i\theta}] \\ v_{i-1} \sin e_{i\theta} \end{bmatrix}$$

is also bounded, along trajectories. To see the latter, consider first  $i = 2$  then,

$$d_2 := \begin{bmatrix} v_1[1 - \cos e_{2\theta}] \\ v_1 \sin e_{2\theta} \end{bmatrix}$$

where  $v_1 = v_0(t) + c_{21}e_{1x}$  is a function of  $t$  and  $e_{1x}$ . Hence, the function  $\tilde{d}_2$  defined along trajectories as

$$\tilde{d}_2(t, e_{i\theta}) = \begin{bmatrix} v_1(t, e_{1x}(t))[1 - \cos e_{2\theta}] \\ v_1(t, e_{1x}(t)) \sin e_{2\theta} \end{bmatrix},$$

is also continuous and bounded if so is  $v_1(t, e_{1x}(t))$ . On the other hand,  $e_{1x}(t)$  is part of the solution of (4.8) whose origin, after Lemma 4.1, is uniformly globally asymptotically stable. Therefore,  $e_{1x}(t)$  is uniformly globally bounded and so is  $v_1(t, e_{1x}(t))$ . The statement of Lemma 4.2 for the case  $i = 2$  follows hence,  $v_2(t, \bar{e}_{2x}(t))$  where  $\bar{e}_{2x} := [e_{1x} \ e_{2x}]^\top$ , is uniformly bounded for any  $t$ . Using this and proceeding by induction, we conclude that the result of the lemma holds for any  $i \geq 2$ .

*Proposition 2.* Consider the system (4.22) in closed loop with the controllers (4.8) and (4.23). Assume that, for each  $i \leq N$ ,  $h_i(t, e_{iy})$  has an isolated zero at  $e_{iy} = 0$ ,

$$\sup_{t, e_{iy}} \left\{ \|h_i(t, e_{iy})\|, \left\| \frac{\partial h_i(t, e_{iy})}{\partial e_{iy}} \right\|, \left\| \frac{\partial h_i(t, e_{iy})}{\partial t} \right\| \right\} \leq c, \quad (4.25)$$

$\sum_{j=1}^i \dot{h}_j$  is  $u\delta$ -persistently exciting and the control gains  $c_{1i}$ ,  $c_{2i}$  are positive. Then, the origin of the closed-loop system is uniformly globally asymptotically stable.

*Remark 5.* In most cases, the condition that  $\sum_{j=1}^i \dot{h}_j$  is  $u\delta$ -persistently exciting for any  $i \leq N$  holds if each  $\dot{h}_j$  is  $u\delta$ -persistently exciting. For instance, it suffices to introduce  $N$  different harmonics:

$$h_j(t, e_{ey}) = \psi_j(\varpi_j t) \alpha(e_{iy})$$

where, for simplicity only,  $\psi_j$  is a periodic function of period  $2\pi\varpi_j$ .

*Proof.* We must establish that under the conditions of the proposition, the control input  $w_i$  defined in (4.23a) is  $u\delta$ -PE with respect to  $e_{iy}$ . We proceed by induction. Let  $i = 2$  then

$$w_2 = w_1 + c_{12}e_{2\theta} + h_2(t, e_{2y})$$

which satisfies

$$\begin{aligned}
\dot{w}_2 &= \dot{w}_1 + c_{12}\dot{e}_{2\theta} + \dot{h}_2(t, e_{2y}) \\
&= -c_{11}w_1 + \dot{h}_1(t, e_{1y}) + c_{12}w_1 - c_{12}w_2 + \dot{h}_2(t, e_{2y}) \\
&= -c_{12}w_2 - [c_{11} - c_{12}]w_1 + \dot{h}_1(t, e_{1y}) + \dot{h}_2(t, e_{2y}) \\
&=: -c_{12}w_2 + \Phi_2(t, \bar{e}_{2y})
\end{aligned}$$

where  $\bar{e}_{2y} := [e_{1y} \ e_{2y}]^\top$ . Under the conditions of Proposition 2 and since  $w_1$  is u $\delta$ -PE with respect to  $e_{1y}$ , the function  $\Phi_2$  is u $\delta$ -PE with respect to  $\bar{e}_{2y}$ . Hence, by the filtering property –see the Appendix, so is  $w_2$ . It follows that

$$\begin{aligned}
\Phi_i(t, \bar{e}_{iy}) &= \sum_{j=1}^{i-1} [c_{1j+1} - c_{1j}]w_j + \dot{h}_j(t, e_{jy}) + \dot{h}_i(t, e_{iy}) \\
\bar{e}_{i-1x} &:= [e_{1x} \ \cdots \ e_{i-1x}]^\top,
\end{aligned}$$

with  $i = 3$  is u $\delta$ -PE with respect to  $\bar{e}_{3y}$  and, consequently, by the filtering Property 1, so is  $w_3$ . By induction, it follows that  $\Phi_i(t, \bar{e}_{iy})$  is u $\delta$ -PE with respect to  $\bar{e}_{iy}$  and so is  $w_i$ , which satisfies

$$\dot{w}_i = -c_{1i}w_i + \Phi_i(t, \bar{e}_{iy}),$$

for any  $i \geq 2$ . □

## B. Dynamic formulation

### A. Leader-follower tracking controller

This problem has been studied thoroughly by [6] and [21], however, we focus in the dynamic extension of (4.1) as studied in [32], the error dynamics between the virtual reference vehicle and the follower are the following

$$\begin{aligned}
\dot{e}_{1x} &= w_1 e_{1y} - v_1 + v_0 \cos e_{1\theta}, \\
\dot{e}_{1y} &= -w_1 e_{1x} + v_0 \sin e_{1\theta}, \\
\dot{e}_{1\theta} &= -w_1, \\
\dot{v}_1 &= \frac{u_{11}}{m_1}, \\
\dot{w}_1 &= \frac{u_{21}}{j_1},
\end{aligned} \tag{4.26}$$

where the control inputs  $u_{11}$  and  $u_{21}$  are regarded as force and torque respectively and  $m_1$  denotes the mass of the first robot, while  $j_1$  is the moment of inertia.

Our aim is to find a control law  $u_1 = [u_{11}, u_{21}]^T$  of the form

$$\begin{aligned} u_{11} &= u_{11}(t, e_{1x}, e_{1y}, e_{1\theta}, v, w), \\ u_{21} &= u_{21}(t, e_{1x}, e_{1y}, e_{1\theta}, v, w), \end{aligned} \quad (4.27)$$

such that the closed loop trajectories of (4.26,4.27) are uniformly globally asymptotically stable.

To solve this problem, we first define the velocity error variables for the local control inputs

$$\begin{aligned} e_{1v} &= v_1 - v_0, \\ e_{1w} &= w_1, \end{aligned} \quad (4.28)$$

where the error dynamics between the virtual reference vehicle and the follower becomes

$$\begin{aligned} \dot{e}_{1x} &= w_1 e_{1y} - e_{1v} + v_0(\cos e_{1\theta} - 1), \\ \dot{e}_{1y} &= -w_1 e_{1x} + v_0 \sin e_{1\theta}, \\ \dot{e}_{1\theta} &= -e_{1w}, \\ \dot{e}_{1v} &= \frac{u_{11}}{m_1} - \dot{v}_0, \\ \dot{e}_{1w} &= \frac{u_{21}}{j_1}, \end{aligned} \quad (4.29)$$

We solve the formation and tracking control problems on straight lines with fairly simple time-variant control laws. The control inputs are

$$\begin{aligned} u_{11} &= m_1(\dot{v}_0 + c_{31}e_{1x} - c_{41}e_{1v}), \quad c_{31} > 0, \quad c_{41} > 0 \\ u_{21} &= j_1(c_{51}e_{1\theta} + \dot{h}_1(t, e_{1y}) - c_{61}e_{1w}), \quad c_{51} > 0, \quad c_{61} > 0 \end{aligned} \quad (4.30)$$

where  $\dot{h}$  is bounded, locally of linear order in  $e_{1y}$  and continuously differentiable. We set that  $\dot{h}(t, 0) \equiv 0$  and  $\dot{h}$  is persistently exciting for any  $e_{1y} \neq 0$ .

We show that the controller (4.30) stabilizes globally and uniformly the error dynamics. Replacing the controller (4.30) in (4.29), leads to a set of equations corresponding to the

error dynamics between the leader and the follower robot:

$$\begin{aligned}
\dot{e}_{1x} &= w_1 e_{1y} - e_{1v} + v_0(\cos e_{1\theta} - 1), \\
\dot{e}_{1y} &= -w_1 e_{1x} + v_0 \sin e_{1\theta}, \\
\dot{e}_{1\theta} &= -e_{1w}, \\
\dot{e}_{1v} &= c_{31} e_{1x} - c_{41} e_{1v}, \\
\dot{e}_{1w} &= c_{51} e_{1\theta} + \dot{h}_1(t, e_{1y}) - c_{61} e_{1w}.
\end{aligned} \tag{4.31}$$

We can write the equations (4.31) in compact form. The system  $\Sigma_3$  and  $\Sigma_4$  are defined as

$$\Sigma_3 : \quad \begin{bmatrix} \dot{e}_{1x} \\ \dot{e}_{1v} \\ \dot{e}_{1y} \end{bmatrix} = \begin{bmatrix} 0 & -1 & w_1 \\ c_{31} & -c_{41} & 0 \\ -w_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{1x} \\ e_{1v} \\ e_{1y} \end{bmatrix} + d(t, e_{1\theta}) \tag{4.32a}$$

$$\Sigma_4 : \quad \begin{bmatrix} \dot{e}_{1\theta} \\ \dot{e}_{1w} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ c_{51} & -c_{61} \end{bmatrix} \begin{bmatrix} e_{1\theta} \\ e_{1w} \end{bmatrix} + h(t, e_{1y}) \tag{4.32b}$$

where the interconnection terms are

$$d(t, e_{1\theta}) := \begin{bmatrix} v_0(\cos e_{1\theta} - 1) \\ 0 \\ v_0 \sin e_{1\theta} \end{bmatrix} \tag{4.33}$$

and

$$h(t, e_{1y}) := \begin{bmatrix} 0 \\ \dot{h}_1(t, e_{1y}) \end{bmatrix}. \tag{4.34}$$

We are interested in stabilising uniform global asymptotic stability of the origin of  $(e_{1x}, e_{1y}, e_{1\theta}, e_{1v}, e_{1w}) = (0, 0, 0, 0, 0)$ . The system (4.32) consists of the feedback interconnection of two systems. as is illustrated in the Figure 4.5.

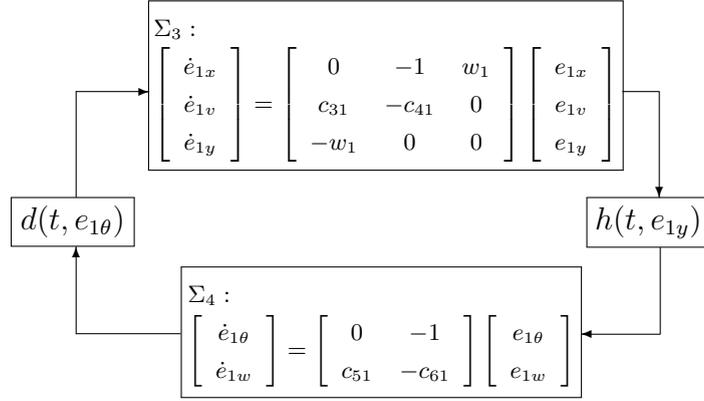


FIGURE 4.5: Feedback representation of the closed-loop system (4.31).

There are several properties of  $u\delta$ -PE functions which are useful in control design for nonholonomic systems; these are reported in [37]. One of the properties of  $u\delta$ -PE functions is that if  $w_1$  is  $u\delta$ -PE then there exists a function  $\tilde{w}_1$  which depends only on time and which is persistently exciting signals in the usual sense that is,

$$\int_t^{t+T'} \|\tilde{w}_1(\tau)\|^2 d\tau \geq \mu' \quad \forall t \geq 0 \quad (4.35)$$

for some  $T'$  and  $\mu' > 0$ . Moreover,  $\tilde{w}_1$  may be purposefully constructed to satisfy

$$\begin{aligned} \tilde{w}_1(t) &:= h_1(t, e_{1y}(t)) + \int (c_{51}e_{1\theta} - c_{61}e_{1w})d\tau \\ &\forall t : \|e_{1y}(t)\| \geq \delta. \end{aligned} \quad (4.36)$$

Even though the function  $\tilde{w}_1$  is parameterized by  $\delta$  it is guaranteed that for any  $\delta > 0$  there exists  $\tilde{w}_1$  satisfying all of the above. This property is useful because, for any  $\delta$  and for all  $t$  such that  $|e_{1y}(t)| \geq \delta$ , the trajectories of  $\Sigma_3$  coincide with those of

$$\underbrace{\begin{bmatrix} \dot{e}_{1x} \\ \dot{e}_{1v} \\ \dot{e}_{1y} \end{bmatrix}}_{\dot{z}_1} = \underbrace{\begin{bmatrix} 0 & -1 & \tilde{w}_1(t) \\ c_{31} & -c_{41} & 0 \\ -\tilde{w}_1(t) & 0 & 0 \end{bmatrix}}_{\tilde{A}_1(t)} \underbrace{\begin{bmatrix} e_{1x} \\ e_{1v} \\ e_{1y} \end{bmatrix}}_{z_1} \quad (4.37)$$

which is linear. The clear advantage is that the behaviour of the trajectories of (4.32a) with  $d \equiv 0$  may be analysed as those of a linear system, at least while the trajectories are away from the origin (strictly speaking away of *any*  $\delta$ -neighbourhood). In particular, global exponential stability of the origin of (4.37) is easily concluded invoking classical results on adaptive control systems –see [24]. Consequently, one may use an intuitive contradiction argument to establish uniform global asymptotic stability

of (4.32a) with  $d \equiv 0$ : assume that the origin is not attractive then, the trajectories (tend to) remain away of an arbitrary  $\delta$ -neighbourhood of the origin<sup>3</sup>. In that case, since they coincide with those generated by (4.37) which is exponentially stable, it follows that the trajectories of (4.32a) must converge to zero. The argument may be repeated for any arbitrarily small  $\delta$  hence, the “exponential” rate of convergence diminishes but remains uniform in the initial conditions.

Precise general statements for nonlinear time-varying systems are reported in [47]. For the purpose of the system (4.32) we proceed by showing that

1. the origin is uniformly stable;
2. the solutions are uniformly globally bounded;
3. the origin is uniformly globally attractive.

The system (4.32) corresponds to the feedback interconnection of two *locally* input to state stable systems. For the first block,  $\Sigma_3$ , the origin is uniformly globally asymptotically stable provided that  $w_1$  is uniformly  $\delta$ -PE with respect to  $e_{1y}$ , bounded and with bounded derivatives –see Theorem 4 in the appendix. In this particular case,  $x_1 = \begin{bmatrix} e_{1x} & e_{1v} \end{bmatrix}^\top$ ,  $x_2 = \begin{bmatrix} e_{1y} \end{bmatrix}$ ,  $H^\top(t, x_2) = \begin{bmatrix} \tilde{w}_1(t) & 0 \end{bmatrix}$  and the matrix  $A(t, x_1)$  is defined by

$$A(t, x_1) := \begin{bmatrix} 0 & -1 \\ c_{31} & -c_{41} \end{bmatrix} \quad (4.38)$$

where  $c_{31} > 0$  and  $c_{41} > 0$ . For the matrix  $P$  to be definite positive  $P^\top > 0$

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_2 \end{bmatrix} = \begin{bmatrix} \frac{c_3^2 + c_4^2 + c_3}{2c_3c_4} & -\frac{c_3}{2} \\ -\frac{c_3}{2} & \frac{c_3 + 1}{2c_3c_4} \end{bmatrix} \quad (4.39)$$

must satisfy

$$p_{11} > 0 \quad p_{11}p_{12} - p_{12}^2 > 0$$

On the other hand, local input to state stability (also known as total stability) with respect to the additive input  $d$  is a direct consequence of uniform global asymptotic stability –see [22]. For  $\Sigma_4$  it is evident that the origin is globally exponentially stable and that  $\Sigma_4$  is input-to-state stable with respect to  $h$ . Actually, the interconnected system showed in Figure 4.5 is (locally) uniformly asymptotically stable.

<sup>3</sup>An “oscillating” behaviour which would consist in the trajectories entering and exiting the  $\delta$ -neighbourhood is excluded since the origin is stable.

The boundedness property follows from the fact that the trajectories of (4.37) coincide with those of  $\Sigma_3$  in Figure 4.5 (which are globally uniformly bounded) for all  $t$  such that  $\|e_{1y}(t)\| \geq \delta$ ; in particular, if the trajectories tend to grow unboundedly. To see more clearly we remark that in view of (4.35) the origin of (4.37) is globally exponentially stable, this implies that, for any  $\delta$ , there exist positive definite symmetric matrices  $P_{1\delta}$  and  $Q_{1\delta}$  such that  $Q_{1\delta}(t) = \tilde{A}_{1\delta}(t)^\top P_{1\delta}(t) + P_{1\delta}(t)\tilde{A}_{1\delta}(t) + \dot{P}_{1\delta}(t)$  and the total derivative of

$$V_{1\delta}(t, z_1) = z_1^\top P_{1\delta}(t) z_1$$

is

$$\dot{V}_{1\delta}(t, z_1) = \dot{z}_1^\top P_{1\delta}(t) z_1 + z_1^\top P_{1\delta}(t) \dot{z}_1 + z_1^\top \dot{P}_{1\delta}(t) z_1$$

along the trajectories of (4.32a) satisfies

$$\dot{V}_{1\delta}(t, z_1) \leq -z_1^\top Q_{1\delta}(t) z_1 + 2z_1^\top P_{1\delta}(t) d(t, e_{1\theta})$$

for all  $t$  such that  $\|e_{1y}(t)\| \geq \delta$ . In turn, we have

$$\dot{V}_{1\delta}(t, z_1) \leq -q_{1m} \|z_1\|^2 + 2p_{1M} z_1^\top d(t, e_{1\theta})$$

where we used  $p_{1M}I \geq P_{1\delta}(t)$  and  $Q_{1\delta}(t) \geq q_{1m}I$ . By using the lambda inequality  $2x^\top y \leq x^\top \lambda x + y^\top \lambda^{-1} y$  with  $x = z_1$ ,  $y = p_{1M} d(t, e_{1\theta})$  and  $\lambda = q_{1m}$  we have

$$\begin{aligned} \dot{V}_{1\delta}(t, z_1) &\leq -q_{1m} \|z_1\|^2 + \frac{q_{1m}}{2} \|z_1\|^2 + \frac{p_{1M}^2}{2q_{1m}} \|d(t, e_{1\theta})\|^2 \\ &\leq -\frac{q_{1m}}{2} \|z_1\|^2 + \frac{p_{1M}^2}{2q_{1m}} \|d(t, e_{1\theta})\|^2 \end{aligned} \quad (4.40)$$

Since  $d(t, e_{1\theta}(t))$  is bounded –see (4.33), it is clear that if  $\|z_1(t)\| \rightarrow \infty$  then  $\dot{V}_{1\delta}(t, z_1(t)) \leq 0$  for sufficiently large  $t$ .

We argue in a similar way for the trajectories of (4.32b); the total derivative of

$$V_{2\delta}(t, z_2) = z_2^\top P_{2\delta}(t) z_2$$

is

$$\dot{V}_{2\delta}(t, z_2) = \dot{z}_2^\top P_{2\delta}(t) z_2 + z_2^\top P_{2\delta}(t) \dot{z}_2 + z_2^\top \dot{P}_{2\delta}(t) z_2$$

along the trajectories of (4.32b) satisfies

$$\begin{aligned} \dot{V}_{2\delta}(t, z_2) &\leq -q_{2m} \|z_2\|^2 + \frac{q_{2m}}{2} \|z_2\|^2 + \frac{p_{2M}^2}{2q_{2m}} \|h(t, e_{1y})\|^2 \\ &\leq -\frac{q_{2m}}{2} \|z_2\|^2 + \frac{p_{2M}^2}{2q_{2m}} \|h(t, e_{1y})\|^2 \end{aligned} \quad (4.41)$$

Recall that, by assumption,  $h$  is bounded.

Then, we show that the origin of (4.32) is uniformly globally attractive; that is, we must show that for any  $r$  and  $\sigma > 0$ , there exists  $T$  such that

$$\|e_1(t_o)\| \leq r \implies \|e_1(t)\| \leq \sigma \quad \forall t \geq t_o + T. \quad (4.42)$$

Let  $r$  and  $\sigma$  be arbitrary given positive constants and define  $\delta = \sigma$ . To establish the convergence property of (4.42), we study the behaviour of the solutions of

$$\begin{aligned} \begin{bmatrix} \dot{e}_{1x} \\ \dot{e}_{1v} \\ \dot{e}_{1y} \end{bmatrix} &= \begin{bmatrix} 0 & -1 & \tilde{w}_1 \\ c_{31} & -c_{41} & 0 \\ -\tilde{w}_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{1x} \\ e_{1v} \\ e_{1y} \end{bmatrix} + d(t, e_{1\theta}) \\ \begin{bmatrix} \dot{e}_{1\theta} \\ \dot{e}_{1w} \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ c_{51} & -c_{61} \end{bmatrix} \begin{bmatrix} e_{1\theta} \\ e_{1w} \end{bmatrix} + h(t, e_{1\theta}) \end{aligned} \quad (4.43)$$

whose trajectories, as we have emphasised, coincide with those of (4.32) for all  $t$  such that  $\|e_{1y}(t)\| \geq \delta$ . Therefore, it suffices to establish global exponential stability of the origin of (4.43). To that end, let

$$\varepsilon := \frac{q_{2m}}{4} \quad \eta := \frac{2q_{1m}}{p_{1M^2}} \varepsilon \quad (4.44)$$

and consider the Lyapunov function  $V_\delta := \eta V_{1\delta} + V_{2\delta}$ . Its total derivative satisfies

$$\dot{V}_\delta(t, z_1, z_2) \leq -\left(\frac{q_{1m}^2}{p_{1M}^2} \varepsilon - \frac{p_{2M} v_0^M}{2q_{2m}}\right) \|z_1\|^2 - \left(\frac{q_{2m}}{2} - \varepsilon\right) \|z_2\|^2$$

where we introduced the bound  $v_0^M \geq \|v_0(t)\|$  and we used the fact that  $\|h(t, e_{1y})\| \leq v_0^M \|z_1\|$  and  $\|d(t, e_{1\theta})\| \leq \|z_2\|$ . In view of the expressions in (4.44),  $\dot{V}_\delta$  is negative definite, actually,

$$\dot{V}_\delta(t, z_1, e_{1\theta}) \leq -\alpha \|z_1\|^2 - \varepsilon \|z_2\|^2, \quad \alpha > 0$$

We conclude that the trajectories of (4.32), which coincide with those of (4.43) for all  $t$  such that  $\|e_{1y}(t)\| \geq \delta$ , tend to zero exponentially fast as long as the latter inequality holds. In view of this there exists a finite time  $T$  such that for any  $\delta' \in (0, \delta]$ , we have  $\|e_1(t_o + T)\| \leq \delta'$ . From uniform stability, we have  $\|e_1(t)\| \leq \delta$  for all  $t \geq t_o + T$ . Since  $\delta = \sigma$  by definition, the statement follows.

*Remark 4.3.* It is worth noticing that this reasoning is reminiscent of ultimate boundedness: namely, the solutions tend to a ball of radius  $\delta$ . However, in this case, opposite to arguments leading to ultimate boundedness, the number  $\delta$  is arbitrarily given

and the previous arguments continue to hold for any  $\delta$  and *fixed* values of the control gains.

**Lemma 4.4.** *The origin of the system (4.32) is uniformly globally asymptotically stable if  $c_{31} > 0$ ,  $c_{41} > 0$ ,  $c_{51} > 0$ ,  $c_{61} > 0$ ,  $v_0$  is bounded and  $w_1$  is  $u\delta$ -PE, bounded and locally Lipschitz in  $e_{1y}$  uniformly in  $t$ . Moreover,  $u\delta$ -PE of  $w_1$  is also a necessary condition.*

The previous lemma establishes a strong, yet intermediary, convergence result in the pursuit of our main objective: tracking control of nonholonomic robots. It is left to state under which conditions  $w_1$  is  $u\delta$ -PE. As a matter of fact, this has been established in the context of set-point stabilization, in [37]. The control input  $w_1$  satisfies the differential equation

$$\dot{w}_1 = -c_{61}w_1 + \dot{h}(t, e_{1y}) + c_{51}e_{1\theta}$$

which corresponds to the equation of a low-pass filter. That is, a stable strictly proper linear system with input  $\dot{h} + c_{51}e_{1\theta}$ . It is well-known from adaptive control textbooks that the output of a low-pass filter driven by an input that is persistently exciting, is also persistently exciting –see [24, 41]. Now, for nonlinear functions we have the analogous Property 1 from [37], which is recalled in the Appendix. Therefore,  $w_1$  which corresponds to a “filtered version” of  $\dot{h}$ , is  $u\delta$ -PE if so is  $\dot{h}$ .

**Proposition 4.5.** *Consider the system (4.26) in closed-loop with the controller (4.30). Let  $h$  be bounded, once continuously differentiable, such that  $h(t, e_{1y})$  has a unique zero at  $e_{1y} = 0$  for each fixed  $t$ ,*

$$\sup_{t, e_{1y}} \left\{ \|h(t, e_{1y})\|, \left\| \frac{\partial h(t, e_{1y})}{\partial e_{1y}} \right\|, \left\| \frac{\partial h(t, e_{1y})}{\partial t} \right\| \right\} \leq c \quad (4.45)$$

for some positive constant  $c$  and assume that for any  $\delta > 0$  there exist positive numbers  $\mu$  and  $T$  such that

$$\|e_{1y}\| \geq \delta \implies \int_t^{t+T} \|\dot{h}(\tau, e_{1y})\| d\tau \geq \mu, \quad \forall t \geq 0. \quad (4.46)$$

Then, the origin of the closed-loop system is uniformly globally asymptotically stable.

*Remark 4.6.* The function  $h$  may be defined as a monotonic locally linear function of  $e_{1y}$  and smooth, persistently exciting in  $t$ ; for instance,  $h(t, e_{1y}) = \psi(t)\text{sat}(e_{1y})$  where  $\text{sat}(\cdot)$  is a saturation function and  $\psi$  is persistently exciting.

*Proof.* . The closed-loop system is given by Eqs. (4.32) and it may be easily showed, using  $V_1$  and  $V_2$  above, that the system is forward complete. Now, since  $\dot{h}$  is a scalar

function (4.20) holds if and only if the following condition, along complete trajectories,

$$\min_{\tau \in [t, t+T]} \|e_{1y}(\tau)\| \geq \delta \implies \int_t^{t+T} \|\dot{h}(\tau, e_{1y}(\tau))\| d\tau \geq \mu, \quad t \geq 0$$

holds. Therefore,  $\dot{h}$  satisfies the properties in Definition 6 and, in view of Property 1, it follows that  $w_1$  is  $u\delta$ -PE. The result follows from Lemma 4.4  $\square$

### B. Leader-follower formation control

We address the dynamic extension of the proposed method based on a simple dynamic model of a mobile robot presented in [27, 32]:

$$\dot{x}_i = v_i \cos(\theta_i) \quad (4.47a)$$

$$\dot{y}_i = v_i \sin(\theta_i) \quad (4.47b)$$

$$\dot{\theta}_i = w_i \quad (4.47c)$$

$$\dot{v}_i = \frac{u_{1i}}{m_i} \quad (4.47d)$$

$$\dot{w}_i = \frac{u_{2i}}{j_i}. \quad (4.47e)$$

where the control inputs  $u_{1i}$  and  $u_{2i}$  are regarded as force and torque respectively and  $m_i$  denotes the mass of the  $i$ th robot, while  $j_i$  is the moment of inertia.

The control objective is to find a control law  $u_{1i} = [u_{1i}, u_{2i}]^T$  of the form

$$\begin{aligned} u_{1i} &= u_{1i}(t, e_{ix}, e_{iy}, e_{i\theta}, v, w) \\ u_{2i} &= u_{2i}(t, e_{ix}, e_{iy}, e_{i\theta}, v, w) \end{aligned} \quad (4.48)$$

such that the closed loop error dynamics is uniformly globally asymptotically stable.

To solve this problem, we first define the velocity error variables for the local control inputs as in the previous section:

$$\begin{aligned} e_{iv} &= v_i - v_{i-1} \\ e_{iw} &= w_i - w_{i-1} \end{aligned} \quad (4.49)$$

which leads to the following error dynamics,

$$\begin{aligned}
\dot{e}_{ix} &= w_i e_{iy} - e_{iv} + v_{i-1} \cos e_{i\theta} - v_{i-1} \\
\dot{e}_{iy} &= -w_i e_{ix} + v_{i-1} \sin e_{i\theta} \\
\dot{e}_{i\theta} &= -e_{iw} \\
\dot{e}_{iv} &= \frac{u_{1i}}{m_i} - \dot{v}_{i-1} \\
\dot{e}_{iw} &= \frac{u_{2i}}{j_i} - \dot{w}_{i-1}.
\end{aligned} \tag{4.50}$$

and for each  $i \geq 1$  we define the control inputs  $u_{1i}$  and  $u_{2i}$  as

$$u_{1i} = m_i(\dot{v}_{i-1} + c_{3i}e_{ix} - c_{4i}e_{iv}) \tag{4.51a}$$

$$u_{2i} = j_i(\dot{w}_{i-1} + \dot{h}_i(t, e_{iy}) + c_{5i}e_{i\theta} - c_{6i}e_{iw}) \tag{4.51b}$$

where  $\dot{h}_i(t, e_{iy})$  is bounded.

The equation (4.51) is replaced in (4.50), leading to a set of equations correspond to the error dynamics between a leader and a follower robot,

$$\dot{e}_{ix} = w_i e_{iy} - e_{iv} + v_{i-1}(\cos e_{i\theta} - 1) \tag{4.52a}$$

$$\dot{e}_{iy} = -w_i e_{ix} + v_{i-1} \sin e_{i\theta} \tag{4.52b}$$

$$\dot{e}_{i\theta} = -e_{iw} \tag{4.52c}$$

$$\dot{e}_{iv} = c_{3i}e_{ix} - c_{4i}e_{iv} \tag{4.52d}$$

$$\dot{e}_{iw} = c_{5i}e_{i\theta} + \dot{h}_i(t, e_{iy}) - c_{6i}e_{iw}. \tag{4.52e}$$

These system of equations can be rewritten in compact form. The system  $\Sigma_3$  is defined as

$$\begin{bmatrix} \dot{e}_{ix} \\ \dot{e}_{iv} \\ \dot{e}_{iy} \end{bmatrix} = \begin{bmatrix} 0 & -1 & w_i \\ c_{3i} & -c_{4i} & 0 \\ -w_i & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{ix} \\ e_{iv} \\ e_{iy} \end{bmatrix} + d(t, e_{i\theta}) \tag{4.53}$$

and

$$\Sigma_4 : \begin{bmatrix} \dot{e}_{i\theta} \\ \dot{e}_{iw} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ c_{5i} & -c_{6i} \end{bmatrix} \begin{bmatrix} e_{i\theta} \\ e_{iw} \end{bmatrix} + h(t, e_{iy}) \tag{4.54}$$

where

$$d_i(t, e_{i\theta}) := \begin{bmatrix} v_{i-1}(\cos e_{i\theta} - 1) \\ 0 \\ v_{i-1} \sin e_{i\theta} \end{bmatrix} \tag{4.55}$$

and

$$h_i(t, e_{iy}) := \begin{bmatrix} 0 \\ \dot{h}_i(t, e_{iy}) \end{bmatrix} \quad (4.56)$$

In the following lemma, we show that the control law (4.51) ensures GUAS of the formation-tracking error dynamics (4.52).

**Lemma 4.7.** *The origin of the system (4.52) is uniformly globally asymptotically stable, for any  $i \leq N$ ,  $c_{3i} > 0$ ,  $c_{4i} > 0$ ,  $c_{5i} > 0$ ,  $c_{6i} > 0$ ,  $v_0$  is bounded and  $w_i$  is  $u\delta$ -PE, bounded and locally Lipschitz in  $e_{iy}$  uniformly in  $t$ . Moreover,  $u\delta$ -PE of  $w_i$  is also a necessary condition.*

The proof of this statement follows *mutatis mutandis* along the proof-lines of Lemma 4.1 observing that: 1) the function  $h_i$  is, by assumption, continuous and bounded; 2) for (4.32a) with  $e_{i\theta} = 0$ , the origin is uniformly globally asymptotically stable provided that  $w_i$  is  $u\delta$ -PE and 3) the interconnection term

$$d_i := \begin{bmatrix} v_{i-1}(\cos e_{i\theta} - 1) \\ 0 \\ v_{i-1} \sin e_{i\theta} \end{bmatrix}$$

is also bounded, along trajectories. To see the latter, consider first  $i = 2$  then,

$$d_2 := \begin{bmatrix} v_1[\cos e_{2\theta} - 1] \\ 0 \\ v_1 \sin e_{2\theta} \end{bmatrix}$$

where  $v_1 = v_0(t) + \int (c_{31}e_{1x} - c_{41}e_{1v})d\tau$  is a function of  $t$ ,  $e_{1x}$  and  $e_{1v}$ . Hence, the function  $\tilde{d}_2$  defined along trajectories as

$$\tilde{d}_2(t, e_{i\theta}) = \begin{bmatrix} v_1(t, e_{1x}(t), e_{1v}(t))[\cos e_{2\theta} - 1] \\ 0 \\ v_1(t, e_{1x}(t), e_{1v}(t)) \sin e_{2\theta} \end{bmatrix},$$

is also continuous and bounded if so is  $v_1(t, e_{1x}(t), e_{1v}(t))$ . On the other hand,  $e_{1x}(t)$  and  $e_{1v}(t)$  are part of the solution for two robots whose origin, is uniformly globally asymptotically stable. Therefore,  $e_{1x}(t)$  and  $e_{1v}(t)$  are uniformly globally bounded and so is  $v_1(t, e_{1x}(t), e_{1v}(t))$ . The statement of Lemma 4.4 for the case  $i = 2$  follows hence,  $v_2(t, \bar{e}_{2x}(t), \bar{e}_{2v}(t))$  where  $\bar{e}_{2x} := [e_{1x} \ e_{2x}]^\top$  and  $\bar{e}_{2v} := [e_{1v} \ e_{2v}]^\top$ , is uniformly bounded for any  $t$ . Using this and proceeding by induction, we conclude that the result of the lemma holds for any  $i \geq 2$ .

*Proposition 3.* Consider the system (4.50) in closed loop with the controller (4.30) and (4.51). Assume that, for each  $i \leq N$ ,  $h_i(t, e_{iy})$  has an isolated zero at  $e_{iy} = 0$ ,

$$\sup_{t, e_{iy}} \left\{ \|h_i(t, e_{iy})\|, \left\| \frac{\partial h_i(t, e_{iy})}{\partial e_{iy}} \right\|, \left\| \frac{\partial h_i(t, e_{iy})}{\partial t} \right\| \right\} \leq c, \quad (4.57)$$

$\sum_{j=1}^i \dot{h}_j$  is  $u\delta$ -persistently exciting and the control gains  $c_{3i}$ ,  $c_{4i}$ ,  $c_{5i}$  and  $c_{6i}$  are positive. Then, the origin of the closed-loop system is uniformly globally asymptotically stable.

*Remark 6.* In most cases, the condition that  $\sum_{j=1}^i \dot{h}_j$  is  $u\delta$ -persistently exciting for any  $i \leq N$  holds if each  $\dot{h}_j$  is  $u\delta$ -persistently exciting. For instance, it suffices to introduce  $N$  different harmonics:

$$\dot{h}_j(t, e_{ey}) = \psi_j(\varpi_j t) \alpha(e_{iy})$$

where, for simplicity only,  $\psi_j$  is a periodic function of period  $2\pi\varpi_j$ .

*Proof.* . We must establish that under the conditions of the proposition, the control input  $w_i$  defined in (4.51b) is  $u\delta$ -PE with respect to  $e_{iy}$ . We proceed by induction. Let  $i = 2$  then

$$w_2 = w_1 + h_2(t, e_{2y}) + \int (c_{52}e_{2\theta} - c_{62}e_{2w})d\tau$$

which satisfies

$$\begin{aligned} \dot{w}_2 &= \dot{w}_1 + \dot{h}_2(t, e_{2y}) + c_{52}e_{2\theta} + c_{62}e_{2w} \\ &= \sum_{k=1}^i \dot{h}_k(t, e_{ky}) + \sum_{k=1}^i c_{5k}e_{k\theta} - c_{61}e_{1w} - c_{62}e_{2w} \\ &= \sum_{k=1}^i \dot{h}_k(t, e_{ky}) + \sum_{k=1}^i c_{5k}e_{k\theta} - c_{61}w_1 - c_{62}(w_2 - w_1) \\ &= -c_{62}w_2 + [c_{62} - c_{61}]w_1 + \sum_{k=1}^i \dot{h}_k(t, e_{ky}) + \sum_{k=1}^i c_{5k}e_{k\theta} \\ &=: -c_{62}w_2 + \Phi_2(t, \bar{e}_{2y}) \end{aligned}$$

where  $\bar{e}_{2y} := [e_{1y} \ e_{2y}]^\top$ . Under the conditions of Proposition 2 and since  $w_1$  is  $u\delta$ -PE with respect to  $e_{1y}$ , the function  $\Phi_2$  is  $u\delta$ -PE with respect to  $\bar{e}_{2y}$ . Hence, by the filtering property –see the Appendix, so is  $w_2$ . It follows that

$$\begin{aligned} \Phi_i(t, \bar{e}_{iy}) &= \sum_{k=1}^{i-1} [c_{6k+1} - c_{6k}]w_k + \sum_{k=1}^i \dot{h}_k(t, e_{ky}) + \sum_{k=1}^i c_{5k}e_{k\theta} \\ \bar{e}_{i-1x} &:= [e_{1x} \ \cdots \ e_{i-1x}]^\top, \end{aligned}$$

with  $i = 3$  is  $u\delta$ -PE with respect to  $\bar{e}_{3y}$  and, consequently, by the filtering Property 1, so is  $w_3$ . By induction, it follows that  $\Phi_i(t, \bar{e}_{iy})$  is  $u\delta$ -PE with respect to  $\bar{e}_{iy}$  and so is

$w_i$ , which satisfies

$$\dot{w}_i = -c_{6i}w_i + \Phi_i(t, \bar{e}_{iy}),$$

for any  $i \geq 2$ . □

## 4.4 Numerical example and simulation results

We illustrate our theoretical findings via some simulation results obtained using SIMULINK<sup>TM</sup> of MATLAB<sup>TM</sup>. We consider a group of five mobile robots. In a first stage of the simulation, the desired formation shape of the mobile robots is linear and they follow a straight line trajectory with initial conditions:  $[x_1(0), y_1(0), \theta_1(0)]^\top = [0, -1, \pi/15]$ ,  $[x_2(0), y_2(0), \theta_2(0)]^\top = [20, -4, \pi/12]$  and  $[x_3(0), y_3(0), \theta_3(0)]^\top = [20, 4, \pi/10]$  and  $[x_4(0), y_4(0), \theta_4(0)]^\top = [30, -5, \pi/8]$  and  $[x_5(0), y_5(0), \theta_5(0)]^\top = [30, 8, \pi/6]$ .

The linear formation shape with a certain desired distance between the robots is obtained by defining  $[d_{x1,2}, d_{y1,2}] = [0, 1]$  and  $[d_{x2,3}, d_{y2,3}] = [0, -2]$  and  $[d_{x3,4}, d_{y3,4}] = [0, 3]$  and  $[d_{x4,5}, d_{y4,5}] = [0, -4]$ . In order to obtain the reference trajectory of the leader robot, we set the reference linear velocity to  $v_0(t) = 10m/s$ , while the angular reference velocity is set to zero.

### 4.4.1 Kinematic formation control

To show the flexibility of the formation and effectiveness of the proposed controller, we allow the formation shape to be linear and the desired path changes from linear to circular and circular to linear every 10s. The reference circular trajectory if the leader robot is obtained by setting the linear and angular velocities to  $[v_0(t), \omega_0(t)] = [10 m/s, 0.3 rad/s]$ ; the latter and the resulting paths are showed in Figure 4.7. The total simulation time is set to 40s.

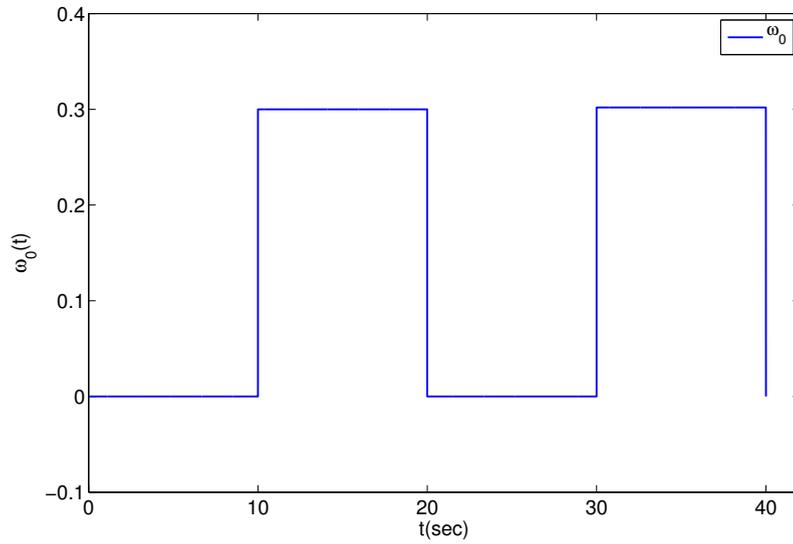


FIGURE 4.6: Angular reference velocity.

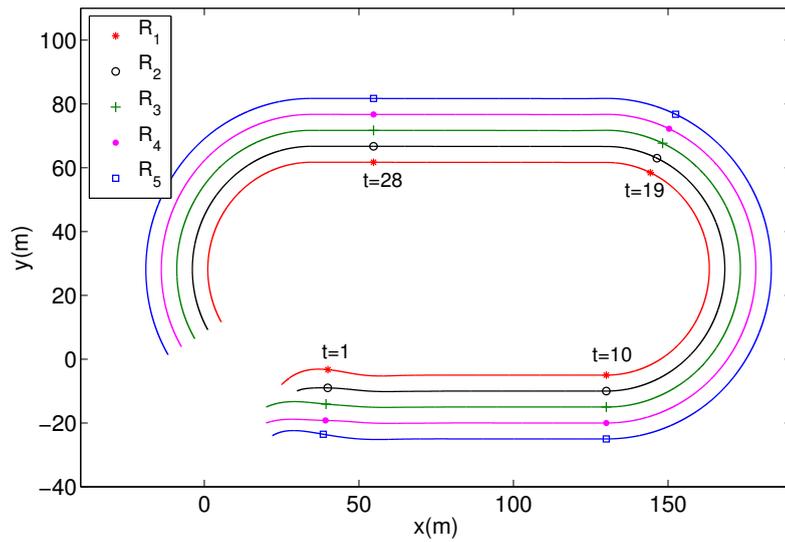


FIGURE 4.7: Generated paths described by the mobile robots.

The control laws are given by

$$\begin{aligned} v_i &= v_{(i-1)} + c_{2i}e_{ix} \\ \omega_i &= \omega_{(i-1)} + c_{1i}e_{i\theta} + \varphi(t) \tanh(e_{iy}) \end{aligned}$$

with control gains  $c_{1i} = 2$  and  $c_{2i} = 5$ . The function  $\varphi$  is generated as a square-pulse train signal of amplitude 0.5, period of four seconds and pulse width of 3.2 s. Note that this function is not smooth but it is persistently exciting hence, the term  $\varphi(t) \tanh(e_{iy})$

induces enough excitation to stabilize the system in the  $y$  direction, as long as there is an error in this coordinate.

The rapid response and excellent performance may be appreciated from the plots of the formation-tracking errors, depicted in Figures 4.8-4.10.

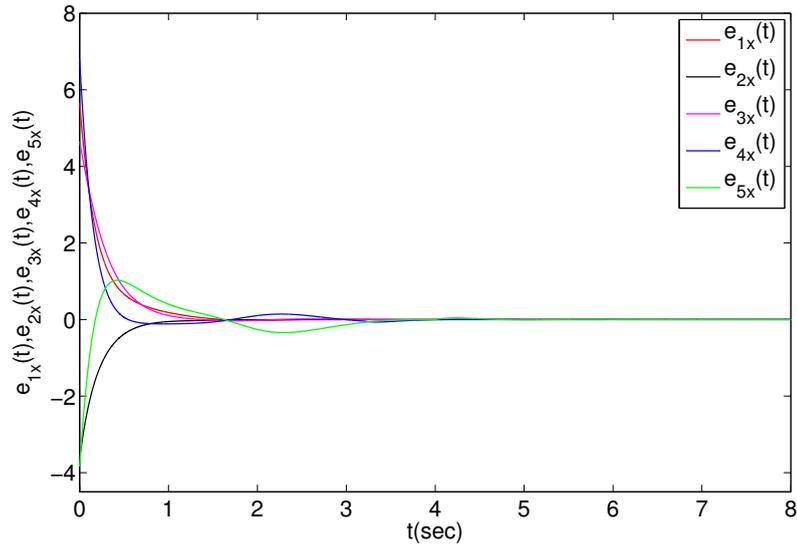


FIGURE 4.8: Position errors in  $x$  coordinates with kinematic control algorithm.

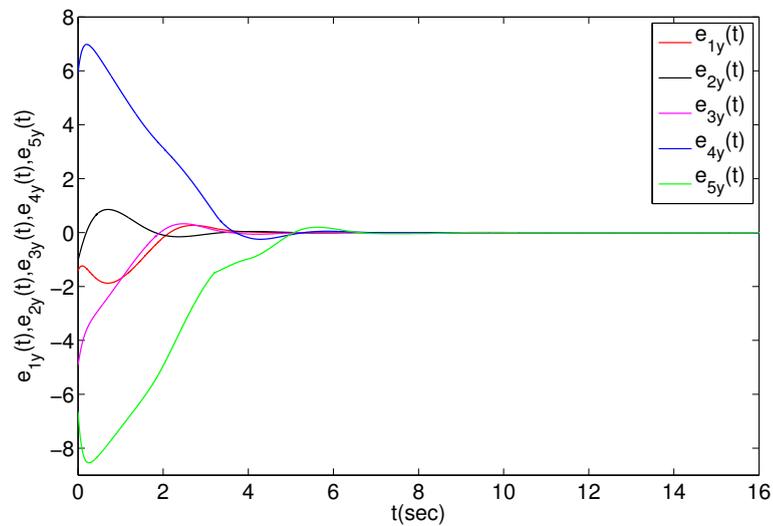


FIGURE 4.9: Position errors in  $y$  coordinates with kinematic control algorithm.

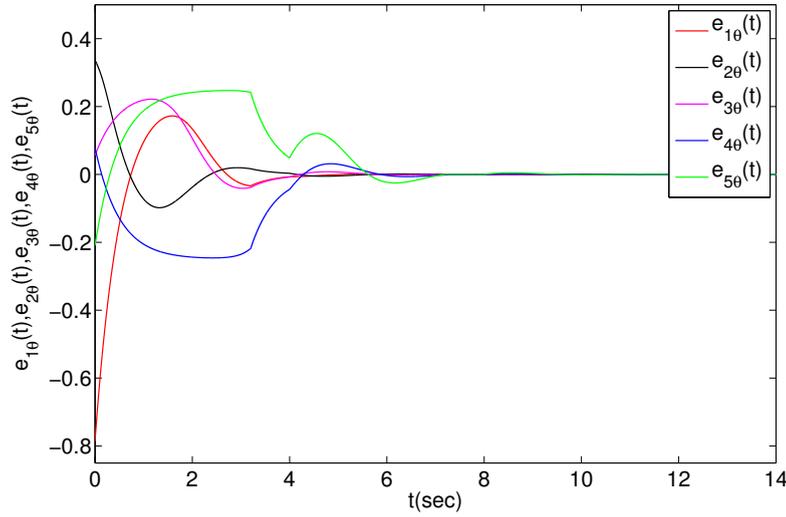


FIGURE 4.10: Heading errors with kinematic control algorithm.

#### 4.4.2 Dynamic formation control

We consider a group of four mobile robots, where one of them is the leader which knows the reference trajectory and the other three as followers. The desired formation shape of the mobile robots is linear and they follow a straight line trajectory with initial conditions:  $[x_1(0), y_1(0), \theta_1(0), v_1(0), w_1(0)]^\top = [25, 1.2, \pi/7, 1.5, 0.2]^\top$ ,  $[x_2(0), y_2(0), \theta_2(0), v_2(0), w_2(0)]^\top = [23, -1.5, \pi/3, 2.4, 4]^\top$ ,  $[x_3(0), y_3(0), \theta_3(0), v_3(0), w_3(0)]^\top = [15, -2.5, \pi/4, 1, 2]^\top$  and  $[x_4(0), y_4(0), \theta_4(0), v_4(0), w_4(0)]^\top = [10, 1, \pi/6, 0.5, 1.3]^\top$ . In order to obtain the reference trajectory of the leader robot, we set the linear and angular velocities as  $[v_0(t), w_0(t)]^\top = [12[m/s], 0[rad/s]]$ . The desired distance between the robots are  $[d_{x1,2}, d_{y1,2}] = [0, 1]$ ,  $[d_{x2,3}, d_{y2,3}] = [0, 1]$  and  $[d_{x3,4}, d_{y3,4}] = [0, -3]$ .

In Figures 4.11-4.13 the trajectory errors of the robots are depicted. It is clear that with the proposed control method the desired formation tracking is successfully ensured.

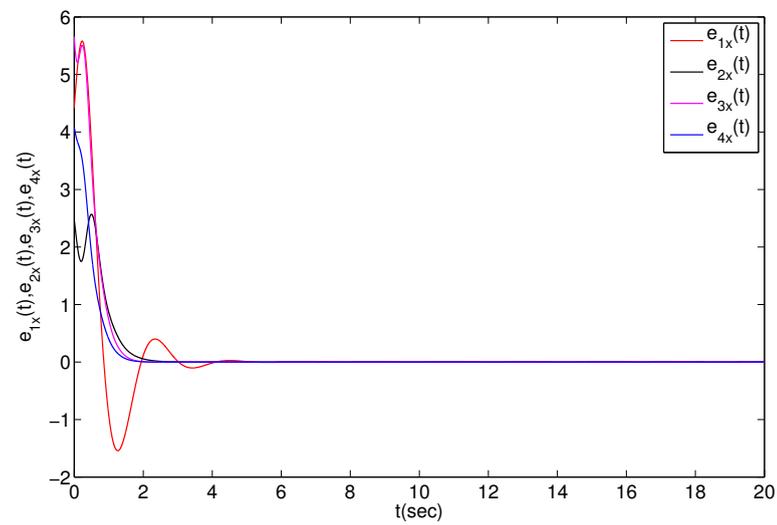


FIGURE 4.11: Position errors in x coordinates.

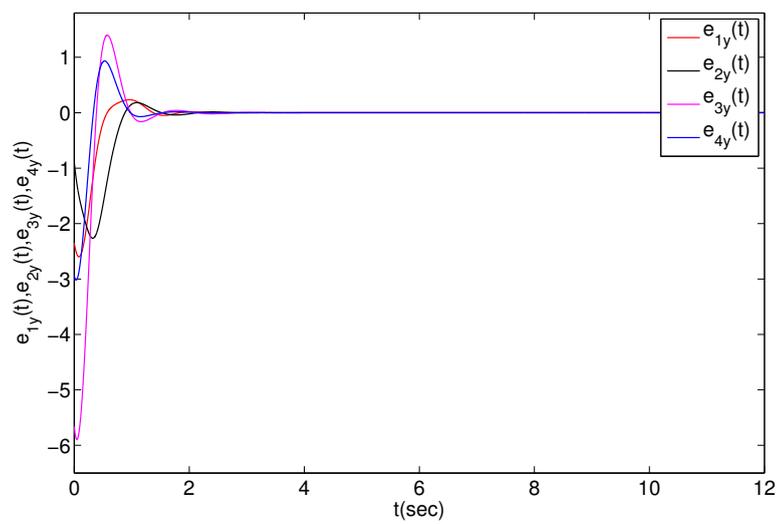


FIGURE 4.12: Position errors in y coordinates.

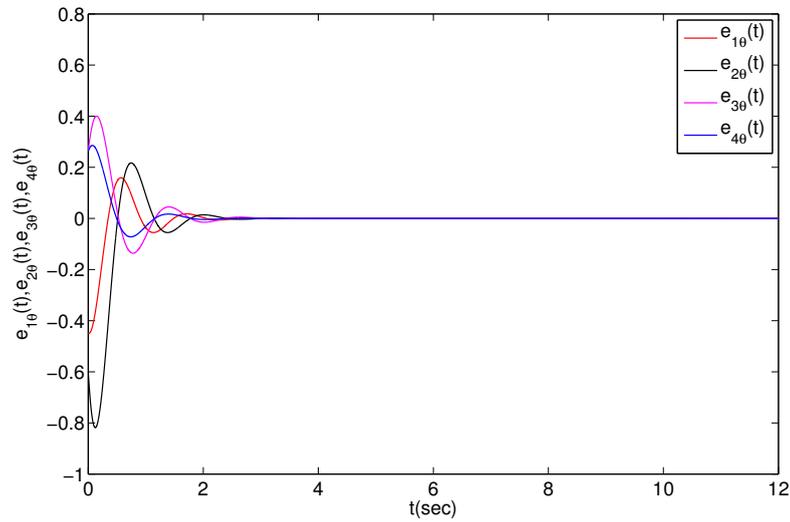


FIGURE 4.13: Heading errors.

In Fig 4.14 we show the motion and relative positioning of the robots. It is easy to see that after a few seconds the formation shape is established and each robot tracks its neighbour with its desired off-set, while the leader  $R_1$  tracking the reference trajectory with a satisfactory performance.

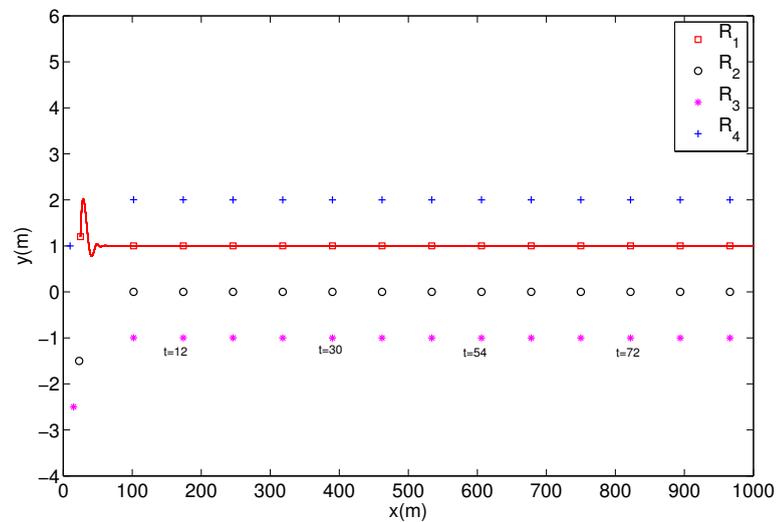


FIGURE 4.14: Motion and positioning of the robots in aligned formation on the plane.

## 4.5 Discussion

We presented a very simple decentralized controller for the problem of formation-tracking control of mobile robots in order to follow straight paths. Our approach relies on a

simple idea which consists in maintaining the reference angular velocity different from zero by an amount proportional to the translation error. In future works, we intend to vary this approach to more complex models and under relaxed assumptions such as time-varying topologies, state dependent interconnection gains, and the case of force-controlled robots.

# Chapter 5

## Conclusions

### 5.1 Conclusions

In this Thesis, we study two important issue related to cooperative control:

- consensus seeking, and
- trajectory tracking.

In Chapter 3 we investigate the consensus problem for a network of dynamic agents with a ring topology. The assumption that each interconnection between any pair of agents is represented by bounded persistently exciting signals models the presence of failures or an obstacle among them. The team of agents reach consensus if the intensity of one of the interconnections is relatively small.

The consensus problem for networks with changing communication topology and with time-dependent communication links is studied in Chapter 3. That is, the network changes in two dimensions: *geographical* and *temporal*. Our result states that having a spanning tree topology for a minimal dwell-time is a sufficient condition to reach consensus.

A new approach to design nonlinear controllers that help to mobile robots to follow straight lines is studied in Chapter 4. We present a decentralized controller for the problem of formation-tracking control of mobile robots. Our approach relies on a simple idea which consists in maintaining the reference angular velocity different from zero by an amount proportional to the translation error.

## 5.2 Perspectives

1. For the problem of consensus seeking, we assumed single integrator dynamics in our previous study. It is possible to extend the results to double integrator dynamics. Our results also suggest that the same framework could be applied to different network topologies.
2. For the problem of tracking control, future work consist in: analyzing more complex models and studying time-varying topologies, state dependent interconnection gains, and the case of force-controlled robots.

# Appendix A

## Appendix

For the sake of self-containedness in this appendix we recall some of the technical tools that we use to establish our main results. The reader is invited to see the cited references for further detail.

The following property roughly states that the output of a stable filter is u $\delta$ -PE if so is its input.

*Property 1. [37] (Filtering)* Let  $\phi_u : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}$ , the pair  $(\phi_u, f)$  is u $\delta$ -PE with respect to  $x$  and consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{\phi}_f \end{bmatrix} = \begin{bmatrix} f(t, x) \\ -a\phi_f + \phi_u(t, x) \end{bmatrix} =: F(t, x_\phi) \quad (\text{A.1})$$

with state  $x_\phi := [x^\top, \phi_f^\top]^\top$ . If  $\phi_u(\cdot, \cdot)$  is locally Lipschitz and there exist continuous non-decreasing functions  $\kappa_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ , ( $i = 1, 2, 3$ ) such that for almost all  $(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ ,

$$\max \left\{ |\phi_u(t, x)|, \left| \frac{\partial \phi_u(t, x)}{\partial t} \right|, \left| \frac{\partial \phi_u(t, x)}{\partial(x)_i} \right| \right\} \leq \kappa_1(\|x\|) \quad (\text{A.2})$$

$$\|f(t, x)\| \leq \kappa_2(\|x_\phi\|) \quad (\text{A.3})$$

and all solutions  $x_\phi(\cdot, t_o, x_{\phi_o})$  are defined on  $[t_o, \infty)$  and satisfy

$$\|x_\phi(t, t_o, x_{\phi_o})\| \leq \kappa_3(\|x_{\phi_o}\|) \quad \forall t \geq t_o \quad (\text{A.4})$$

then, the pair  $(\phi_f, f)$  is u $\delta$ -PE, with respect to  $x$ .

## A stability theorems for time-varying systems

The following theorem is a consequence of Theorem 1 in [47].

*Theorem 4* (Theorem under u $\delta$ -Persistence of excitation). Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A(t, x_1) & H(t, x) \\ -H(t, x)^\top & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (\text{A.5})$$

Let  $A$  be such that there exist a matrix  $P(t, x_1)$  positive definite symmetric and functions  $\alpha_1, \alpha_2 \in \mathcal{K}$  such that

$$\left\| x_1^\top [A(t, x_1)^\top P(t, x_1) + P(t, x_1)A(t, x_1)] x_1 \right\| \geq \alpha_1(\|x_1\|) \quad (\text{A.6})$$

$$\int_{t_0}^{\infty} \alpha_1(\|x_1(t)\|) dt \leq \alpha_2(\|x(t_0)\|) \quad \forall t \geq t_0, t_0 \geq 0. \quad (\text{A.7})$$

Then, the origin is uniformly globally asymptotically stable if  $H$  is u $\delta$ -PE with respect to  $x_2$ ,  $H$  is at least once continuously differentiable and both  $H$  and  $\dot{H}$  are uniformly bounded in  $t$ .

*Remark 7.* The conditions (A.6), (A.7) hold, for instance, if  $A$  is constant and Hurwitz. Indeed, in this case, for any  $Q > 0$  there exists  $P = P^\top > 0$  such that

$$A^\top P + PA = -Q$$

therefore, by choosing  $Q = I$  we obtain, using

$$V = \frac{1}{2} [x_1^\top P x_1 + \|x_2\|^2],$$

$\dot{V} = -\|x_1\|^2 \leq 0$ . That is,

$$\int_{t_0}^{\infty} \|x_1(t)\|^2 dt \leq p_M \|x_1(t_0)\|^2, \quad \forall t \geq t_0 \geq 0$$

where  $p_M I \geq P$ . That is, the conditions of the theorem hold with  $\alpha_1(\|x_1\|) = \|x_1\|^2$  and  $\alpha_2(\|x(t_0)\|) = p_M \|x(t_0)\|^2$ .

# Bibliography

- [1] Aguilar, L., Hamel, T., and Soueres, P. (1997). Robust path following control for wheeled robots via sliding mode techniques. In *Intelligent Robots and Systems, 1997. IROS '97., Proceedings of the 1997 IEEE/RSJ International Conference on*, volume 3, pages 1389–1395.
- [2] Alvarez, N. and Loría, A. (2014). Consensus under persistent interconnections in a ring topology: a small gain approach. *21st International Symposium on Mathematical Theory of Networks and Systems*, pages 525–529.
- [3] Alvarez, N. and Loría, A. (2015). On consensus under time-varying topology. *Conference on Decision and Control (Submitted)*.
- [4] Anderson, B. (1977). Exponential stability of linear equations arising in adaptive identification. *IEEE Trans. Autom. Control*, 22:83–88.
- [5] Buck, J. and Buck, E. (1978). Toward a functional interpretation of synchronous flashing by fireflies. *The American Naturalist*, 112(985):471–492.
- [6] Cao, K.-C. and Tian, Y.-P. (2007). A time-varying cascaded design for trajectory tracking control of non-holonomic systems. *Int. J. of Control*, 80(3):416–429.
- [7] Carpenter, J. (2002). Decentralized control of satellite formations. *Int. J. Robust Nonlinear Control*, 12(2):141–161.
- [8] Chen, Y. and Tian, Y. (2009). A backstepping design for directed formation control of three-coleader agents in the plane. *Int. J. of Robust and Nonlinear Control*, 19:729–745.
- [9] Consolini, L., Morbidib, F., and D. (2008). Leader-follower formation control of nonholonomic mobile robots with input constraints. *Automatica*, 44(5):1343–1349.
- [10] Contevelle, L. and Panteley, E. (2012). Linear reformulation of the kuramoto model: Asymptotic mapping and stability properties. *2nd Conference on Communications, Computing and Control Applications (CCCA)*, pages 1–6.

- [11] Dasdemir, J. and Loria, A. (2012). A simple formation-tracking controller of mobile robots based on a “spanning-tree” communication. *51st IEEE Conf. Decision and Control*, pages 1796–1801.
- [12] Deefort, M., Floquet, T., and Perruquetti, V. (2008). Sliding-mode formation control of three-coleader agents in the plane. *IEEE Trans. Industrial Electronics*, 55:3944–3953.
- [13] den Broek, T. V., van de Wouw, N., and Nijmeijer, N. (2009). Formation control of unicycle mobile robots: a virtual structure approach. *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, pages 8328–8333.
- [14] Desai, J., Ostrowski, J., and Kumar, V. (2001). Modeling and control of formations of nonholonomic mobile robots. *Robotics and Automation, IEEE Transactions on*, 17(6):905–908.
- [15] Diestel, R. (2000). *Graph theory*. Springer verlag.
- [16] Do, K. and Pan, J. (2007). Nonlinear formation control of unicycle-type mobile robots. *Robotics and Autonomous Systems*, 55(3):191–204.
- [17] Fardad, M., Fu, L., and Jovanovic, M. (2014). Design of optimal sparse interconnection graphs for synchronization of oscillator networks. *Automatic Control, IEEE Transactions on*, 59(9):2457–2462.
- [18] Fierro, R. and Lewis, F. (1998). Control of a nonholonomic mobile robot using neural networks. *IEEE Trans. on Neural Networks*, pages 589–600.
- [19] Ghommam, J., Mehrjerdi, H., Saad, M., and Mnif, F. (2010). Formation path following control of unicycle-type mobile robots. *Journal of Robotics and Autonomous Systems*, 58:727–736.
- [20] Giulietti, F., Pollini, L., and Innocenti, M. (2000). Autonomous formation flight. *Control Systems, IEEE*, 20(6):34–44.
- [21] Guo, J., Lin, Z., Cao, M., and Yan, G. (2010). Adaptive leader-follower formation control for autonomous mobile robots. In *American Control Conference (ACC), 2010*, pages 6822–6827.
- [22] Hahn, W. (1967). *Stability of motion*. New York: Springer-Verlag.
- [23] Horn, R. A. and Johnson, C. R. (1985). *Matrix Analysis*. Cambridge University Press.

- 
- [24] Ioannou, P. and Sun, J. (1996). *Robust adaptive control*. Prentice Hall, New Jersey, USA.
- [25] Ioannou, P. A. and Sun, J. (1995). *Robust Adaptive Control*. Prentice Hall.
- [26] Jadbabaie, A., Lin, J., and Morse, A. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. Automat. Contr.*, 48:988–1001.
- [27] Jiang, Z.-P. and Nijmeijer, H. (1997). Tracking control of mobile robots: A case study in backstepping. *Automatica*, 33(7):1393–1399.
- [28] J.P. Desai, J. O. and Kumar, V. (2001). Modeling and control of formations of nonholonomic mobile robots. *Robotics and Automation, IEEE Transactions on*, 17(6):905–908.
- [29] Kanayama, Y., Kimura, Y., Miyazaki, F., and Noguchi, T. (1990). A stable tracking control method for an autonomous mobile robot. In *Robotics and Automation, 1990. Proceedings., 1990 IEEE International Conference on*, pages 384–389 vol.1.
- [30] Khalil, H. (1996). *Nonlinear Systems*. Prentice-Hall, Englewood Cliffs, NJ, 2 edition.
- [31] Lee, T. C., Song, K., Lee, C. H., and Teng, C. (2001). Tracking control of unicycle-modeled mobile robots using a saturation feedback controller. *Control Systems Technology, IEEE Transactions on*, 9:305–318.
- [32] Lefeber, A. A. J. (2000). *Tracking control of nonlinear mechanical systems*. PhD thesis, University of Twente, Enschede, The Netherlands.
- [33] Liberzon, D. (2003). *Switching in systems and control*. Birkhauser.
- [34] Loría, A. and Alvarez, N. (2015). Leader-follower formation and tracking control of autonomous vehicles along straight paths via dynamic model. *Conference on Decision and Control. Osaka, Japan (Submitted)*.
- [35] Loría, A., Dardemir, J., and Alvarez, N. (2014). Decentralize formation tracking control of autonomous vehicles on straight paths. *53rd IEEE Conference on Decision and Control*.
- [36] Loría, A., Dardemir, J., and Alvarez, N. (2015). Leader-follower formation and tracking control of mobile robots along straight paths. *Transactions on Control Systems Technology. (Provisionally accepted)*.

- [37] Loria, A., Panteley, E., and Melhem, K. (2002). UGAS of skew-symmetric time-varying systems: application to stabilization of chained form systems. *European J. of Contr.*, 8(1):33–43.
- [38] Loria, A., Panteley, E., and Teel, A. (1999). A new persistency-of-excitation condition for UGAS of NLTV systems: Application to stabilization of nonholonomic systems. In *Proc. 5th. European Contr. Conf.*
- [39] Moreau, L. (2004). Stability of continuous-time distributed consensus algorithms. *43rd IEEE Conf. Decision and Control*, 4:3998–4003.
- [40] Morin, P. and Samson, C. (2006). Trajectory tracking for nonholonomic vehicles. In Kozcowski, K., editor, *Robot Motion and Control*, volume 335 of *Lecture Notes in Control and Information Sciences*, pages 3–23. Springer London.
- [41] Narendra, K. S. and Annaswamy, A. M. (1989). *Stable adaptive systems*. Prentice-Hall, Inc., New Jersey.
- [42] Natanson, I. P. (1956). *Theory of functions of a real variable*, volume 1. Bull. Amer. Math. Soc.
- [43] Olfati-Saber, R. and Murray, R. M. (2003). Consensus protocols for networks of dynamic agents. in *Proc. 2003 Am. Control Conf. pp. 951–956.*, pages 951–956.
- [44] Olfati-Saber, R. and Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. on Automatic Control*, 49:1520–1533.
- [45] P. Morin, P. and Samson, C. (2009). Control of nonholonomic mobile robots based on the transverse function approach. *Robotics, IEEE Transactions on*, 25(5):1058–1073.
- [46] Panteley, E., Lefeber, E., and Nijmeijer, H. (1998). Exponential tracking of a mobile car using a cascaded approach. in *IFAC Workshop on Motion Control*, pages 221–226.
- [47] Panteley, E., Loria, A., and Teel, A. (2001). Relaxed persistency of excitation for uniform asymptotic stability. *IEEE Trans. Automat. Contr.*, 46(12):1874–1886.
- [48] Portugal, S., Hubel, T., Fritz, J., Heese, S., Trobe, D., Voelkl, B., Hailes, S., Wilson, A., and Usherwood, J. (2014). Upwash exploitation and downwash avoidance by flap phasing in ibis formation flight. *Nature*, 505:399–402.
- [49] Qu, Z., Wang, J., and Hull, R. A. (2008). Cooperative control of dynamical systems with application to autonomous vehicles cooperative control of dynamical systems with application to autonomous vehicles. *IEEE Trans. Autom. Contr.*, 53(4):894–911.

- [50] Rao, S. and Ghose, D. (2014). Sliding mode control-based autopilots for leaderless consensus of unmanned aerial vehicles. *Control Systems Technology, IEEE Transactions on*, 22(5):1964–1972.
- [51] Ren, W. (2006). Distributed attitude consensus among multiple networked spacecraft. In *American Control Conference, 2006*, pages 6 pp.–.
- [52] Ren, W. and Beard, R. W. (2005). Consensus seeking in multi-agent systems under dynamically changing interaction topologies. *IEEE Trans. on Automatic Control*, 50(5):655–661.
- [53] Ren, W. and beard, R. W. (2005). *Distributed consensus in multivehicle cooperative control*. Springer verlag.
- [54] Samson, C. (1991). Velocity and torque feedback control of a nonholonomic cart. In de Wit Carlos, C., editor, *Advanced Robot Control*, volume 162 of *Lecture Notes in Control and Information Sciences*, pages 125–151. Springer Berlin Heidelberg.
- [55] Sarlette, A., Sepulchre, R., and Leonard, N. (2007). Cooperative attitude synchronization in satellite swarms: a consensus approach. in *Proc. 17th IFAC Symp. Autom. Control Aerosp.*, pages 223–228.
- [56] Shao, J., Xie, G., and Wang, L. (2005). Leader-following formation control of multiple mobile robots. *IEEE International Symposium on, Mediterrean Conference on Control and Automation*, pages 808–813.
- [57] Shuanghe, Y. and Xiaojun, L. (2015). Finite-time consensus for second-order multi-agent systems with disturbances by integral sliding mode. *Automatica*, 54(0):158 – 165.
- [58] Tomlin, C., Pappas, G., and Sastry, S. (1998). Conflict resolution for air traffic management: A study in multiagent hybrid systems communication. *IEEE Trans. Autom. Contr.*, 43(4):509–521.
- [59] Vicsek, T., Czirok, A., Jacob, E. B., Cohen, I., and Schochet, O. (1995). Novel type of phase transitions in a system of self-driven particles. *Phys. Rev. Lett.*, 75:1226–1229.
- [60] Vidyasagar, M. (1993). *Nonlinear systems analysis*. Englewood Cliffs, NJ: Prentice-Hall, 2 edition.
- [61] Xiao, F. and Wang, L. (2006). State consensus for multi-agent systems with switching topologies and time-variant delays. *International Journal of Control*, 79(10):1277–1284.