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# Mathematical study of free-surface fluids in incompressible dynamics

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par  
**Dena KAZERANI**

**Études mathématiques de fluides à frontières libres  
en dynamique incompressible**

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*The test of a good teacher is not how many questions  
he can ask his pupils that they will answer readily, but  
how many questions he inspires them to ask him which  
he finds it hard to answer.*

Alice Wellington Rollins

*We are what we repeatedly do.  
Excellence then, is not an act, but a habit.*

Aristotle

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# Introduction

This thesis is about the mathematical study of free surface incompressible fluids. An incompressible flow is a fluid with a constant density. This implies a free divergence velocity. Obviously, water like most liquids is incompressible.

In practice, if the density change of a fluid is small, it can be considered incompressible. This happens when the Mach number associated to the flow is low i.e. when the ratio of flow velocity to the speed of sound is small. Particularly, in our framework, air fits this regime and is considered incompressible.

Incompressible fluids satisfy the incompressible Navier–Stokes equations:

$$\begin{cases} \rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) - \mu \Delta \mathbf{u} + \nabla p = \rho \mathbf{f} \\ \operatorname{div} \mathbf{u} = 0 \\ \mathbf{u}(0, x) = \mathbf{u}_0(x). \end{cases} \quad (0.1)$$

Here  $\rho$  is the density whereas  $\mu$  is the viscosity of the fluid. The unknowns of the problem are  $\mathbf{u}$  the velocity flow and  $p$  the pressure. On the other hand,  $\mathbf{f}$  is the density of force exerted on the fluid while  $\mathbf{u}_0$  is the initial velocity flow. This equation is endowed with boundary conditions on the boundary of fluid domain.

An incompressible fluid satisfying the Navier–Stokes equation (0.1) with a viscosity  $\mu > 0$ , is



called a Newtonian fluid. Nevertheless, if the Reynolds number associated to the flow i.e. the ratio of inertial forces to viscous forces is large, we can suppose that the fluid is perfect. In this case, it



satisfies the incompressible Euler equation (Navier–Stokes equation without the Laplace operator):

$$\begin{cases} \rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) + \nabla p = \rho \mathbf{f} \\ \operatorname{div} \mathbf{u} = 0 \\ \mathbf{u}(0, x) = \mathbf{u}_0(x). \end{cases} \quad (0.2)$$

As an example, let us consider water. The viscosity of water is  $0.001 \text{ Pa}\cdot\text{s}$  while its density is  $1000$



$\text{kg}/\text{m}^3$ . Therefore, it is modeled by incompressible Euler equations in most applications specially in oceanography. For this reason, system (0.2) is the equation used in this latter domain, for the derivation of approximative models. These models are called shallow water and concern the regime where the fluid height is small compared to the characteristic wavelength. This regime coincides with the frame of some geophysical phenomena like tsunamis and earthquakes.

Let us mention that in this work, our concern is the fluid's free surface. Indeed, free surface flows satisfy the fluid equation on the fluid domain which moves following the velocity flow. For this reason, the treatment of the free surface raises some challenges which have led to the derivation of shallow water models and to creation of some specific numerical tools. The first part of the thesis is on the theoretical study of one of these approximative models called the Green–Naghdi equation whereas the second part is on the numerical study of System (0.1).

## 1 Some results on a shallow water model called the Green-Naghdi equations

Shallow water models are essentially considered in Oceanography where the water height is small compared to the characteristic wavelength and the rotational effects are negligible. The ratio between the squared characteristic water height and the squared characteristic wavelength is called the shallowness parameter. As explained in [75], the derivation of these models is based on a non dimensionalized Euler equations considered for a free curl velocity flow. As a matter of fact, the fluid considered here is incompressible and perfect (inviscid) and is subjected to the earth gravity. It slips on a flat solid bottom which is defined on the whole  $x$ -axis (resp.  $xoy$  plane) if the fluid is 2-dimensional (resp. 3-dimensional). A free surface separates the fluid from the vacuum or from

air represented by an atmospheric pressure. The fluid is supposed to be a perturbation of the still fluid in the sense that it can be represented with a function defined on the domain of definition of the bottom. Moreover, the water depth is always assumed to be bounded from below by a strictly

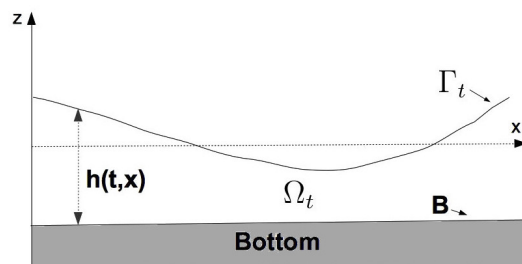


Figure 1.1: An incompressible inviscid (perfect) fluid is considered on a solid flat bottom  $B$ . The fluid domain (resp. the fluid free surface) at  $t$  is represented by  $\Omega_t$  (resp.  $\Gamma_t$ ).

positive constant and to be at rest at infinity. Let us also mention that in the frame of water wave problem, no surface tension is considered.

**Remark 1.1** *During the first part of this thesis, no topography is considered. More precisely, the fluid is always assumed to evolve on a flat bottom. Nevertheless, see for instance [75, 20] for the derivation and analysis of shallow water models with topography.*

This model is described by the following equations:

$$\begin{cases} \rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) + \nabla p = \rho \mathbf{g} & \text{on } \Omega_t \\ \operatorname{div} \mathbf{u} = 0 & \text{on } \Omega_t \\ \operatorname{rot} \mathbf{u} = 0 & \text{on } \Omega_t \\ \mathbf{u}(0, x) = \mathbf{u}_0(x) & \text{on } \Omega_0. \end{cases} \quad (1.1)$$

where  $\Omega_t$  is the domain of fluid at time  $t$  and  $\mathbf{g}$  is the density of the gravity acceleration  $\mathbf{g} = -g\mathbf{e}_z$ . Following the assumptions we consider, this domain evolves with the free surface. Let us also mention that no friction effect is considered here for the derivation of shallow water systems (Saint-Venant system and Green–Naghdi equations). Nevertheless, some classical terms are added, in some references, to these shallow water models to take viscosity and friction into account (see for instance [57, 75]). Therefore, the boundary conditions we consider for the moment are the slip boundary conditions (without friction) on the bottom. Since the bottom is flat and no friction is considered, this condition is equivalent to a homogenous Dirichlet condition for the vertical component of the velocity. In other words, the velocity field is tangent to the bottom. Moreover, since the fluid is inviscid and no surface tension is considered, the free surface condition leads to a Dirichlet condition for the pressure on the free surface. Denoting the atmospheric pressure by  $p^a$  and the vertical axis coordinate by  $z^1$ , these conditions are described by

$$\begin{cases} \mathbf{u}_z = 0 & \text{on } B \\ p = p^a & \text{on } \Gamma_t \end{cases} \quad (1.2)$$

<sup>1</sup>Symbol  $u_z$  represents the norm of the projection of the velocity  $\mathbf{u}$  on the vertical  $z$ -axis.

To complete the fluid model we consider, we just need to add the fact that the free surface is transported by the velocity flow. In other words,

$$\left\{ \begin{array}{l} (1, \mathbf{u}) \text{ is tangential to the free surface } (t, \Gamma_t), t \geq 0. \end{array} \right. \quad (1.3)$$

Equations (1.1)-(1.3) give us the mathematical frame of the physical problem. As we see in the following sections, this is the model we consider to derive shallow water systems.

**Asymptotic regimes.** The complexity of formulation (1.1)-(1.3) leads to consider the asymptotic models in the shallow water and small amplitude (or long wave) regimes. A shallow water system is a long wave model if the ratio between the characteristic wave amplitude and the water height<sup>2</sup> is as small as the shallowness parameter. We consider in this work fluids satisfying the shallow water assumption. However no condition for the waves amplitude is taken into account. Under this assumption, the derived model at the first order is the non linear shallow water model or Saint-Venant system whereas the derived model at the second order is the Green–Naghdi equations. If in addition, the shallow water system is a long wave model, the Green–Naghdi equations will become the Boussinesq equations [29] which leads to the KdV equations in the 2-dimensional case [74, 17]. Let us also mention that in these asymptotic models, the free surface is represented by the water height which is assumed to be unique at all points of the bottom. This water height satisfies a partial differential equation since the kinematic boundary condition becomes a partial differential equation. For this reason, these asymptotic models are simpler than the primitive fluid model. Let us also mention that in this frame, the asymptotic model is defined on the bottom and not on the whole space. This is due to the fact that the velocity components are independent of the vertical variable. In conclusion, shallow water models are interesting essentially for their simplicity as well as the reduction of the dimension.

**Remark 1.2** *During all this thesis, we are interested in smooth solutions of the free surface incompressible fluid problem. For this reason, all the equations are considered in the strong sens. Similarly, all the operations done on the solution are based on its smoothness. Consequently, integrations by part and use of composed function derivative formulas are justified.*

The objective of the first part of this thesis is to prove the asymptotic stability of constant solutions of the Green–Naghdi equations (derived in the next part) with a second order dissipative term. This analysis is done by adapting an approach classically used for hyperbolic systems based on the symmetric structure of the equations. In Chapter 1, we put the Green–Naghdi equation in a generalized symmetric structure. Then, in Chapter 2, we see how this symmetric structure leads to the asymptotic stability of constant solutions of the system with a second order dissipative term.

## 1.1 Zakharov formulation of the water wave problem and some of its properties

The classical formulation of the problem (1.1)-(1.3) which is classically used to derive the shallow water models is a formulation introduced by Zakharov [109]. It is based on the fact that the trace of the velocity as well as the trace of the velocity potential on the free surface is sufficient to obtain the

---

<sup>2</sup>This ratio is also called the nonlinearity parameter since it appears in front of non linear terms of the non-dimensionalized equation.

velocity on the whole fluid domain. Therefore, this formulation consists in considering the equation satisfied by the fluid height  $h(\mathbf{x}, t)$  and the trace of the velocity potential on the free surface. We denote here by  $\mathbf{x}$  in this work  $\mathbf{x} = x$  if the fluid is 2-dimensional and  $\mathbf{x} = (x, y)$  if the fluid is 3-dimensional.

The irrotationality of the fluid leads us to the existence of a velocity potential  $\phi$  such that

$$\mathbf{u} = \nabla\phi. \quad (1.4)$$

Then, thanks to the mass conservation condition (the free divergence condition for the velocity), the velocity potential  $\phi$  is harmonic i.e. it satisfies

$$\Delta\phi = 0. \quad (1.5)$$

Now, we denote by  $\psi$  the trace of the velocity potential  $\phi$  on the free surface  $\Sigma_t$ . This means that we have

$$\psi(\mathbf{x}, t) = \phi(\mathbf{x}, h(\mathbf{x}, t), t) \quad \text{for all } \mathbf{x} \in \mathbb{R}^d \quad \text{with } d = 1, 2.$$

This notation together with the slip condition on the bottom (1.2) leads to the following boundary condition for the velocity potential  $\phi$ .

$$\begin{cases} \phi = \psi & \text{on } \Sigma_t, \\ \nabla\phi \cdot \mathbf{n} = 0 & \text{on } B. \end{cases} \quad (1.6)$$

Obviously, the knowledge of the trace  $\psi$  leads to a non homogenous Dirichlet boundary condition on the free surface and a homogenous Neumann boundary condition on the bottom. Under reasonable regularity assumptions on  $h$  and  $\psi$ , the problem (1.5)-(1.6) admits a unique solution for  $\phi$ . This is why the knowledge of the trace of the velocity potential on the free surface is equivalent to the knowledge of the velocity potential on the whole fluid domain.

Considering this fact, we now derive from (1.1)-(1.3) the equations satisfied by  $h$  and  $\psi$ . To do so, we use the relation between  $\phi$  and  $\psi$  described by the Dirichlet–Neumann operator  $\mathcal{G}(h)$ . This operator relates the velocity potential trace  $\psi$  to the normal derivative of  $\phi$  on the free surface:

$$\mathcal{G}(h)(\psi) = \left(1 + |\partial_{\mathbf{x}}h|^2\right)^{\frac{1}{2}} \nabla\phi \cdot \mathbf{n}. \quad (1.7)$$

Considering the fact that the exterior unit normal satisfies

$$\mathbf{n} = \left(1 + |\partial_{\mathbf{x}}h|^2\right)^{-\frac{1}{2}} \begin{pmatrix} -\partial_{\mathbf{x}}h \\ 1 \end{pmatrix},$$

the expression (1.7) of the operator becomes

$$\mathcal{G}(h)(\psi) = -\partial_{\mathbf{x}}\phi \cdot \partial_{\mathbf{x}}h + \partial_z\phi. \quad (1.8)$$

On the other hand, the momentum equation (the first equation of (1.1), the boundary condition for the pressure on the free surface (the second equation of (1.2)) together with (1.4) gives the following

equation for  $\phi$  on the free surface

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + gh = 0. \quad (1.9)$$

Then, the free surface evolution condition (1.3) satisfied on the free surface becomes the following equation in terms of  $h$  et  $\phi$  :

$$\partial_t h + \partial_{\mathbf{x}} h \cdot \partial_{\mathbf{x}} \phi - \partial_z \phi = 0. \quad (1.10)$$

We can also check by simple arguments and by the chain rule that the following equalities hold true on the free surface

$$\begin{cases} \partial_t \psi = \partial_t \phi + \partial_z \phi \partial_t h, \\ \partial_{\mathbf{x}} \psi = \partial_{\mathbf{x}} \phi + \partial_z \phi \partial_{\mathbf{x}} h, \end{cases} \quad (1.11)$$

Then taking the scalar product of the second equation of (1.11) with  $\partial_{\mathbf{x}} h$  and using the expression (1.8) of the Dirichlet Neumann operator, we find

$$\partial_z \phi = \frac{\mathcal{G}(h)\psi + \partial_{\mathbf{x}} \psi \cdot \partial_{\mathbf{x}} h}{1 + |\partial_{\mathbf{x}} h|^2} \quad (1.12)$$

Then combining (1.8) - (1.12), we find the Zakharov formulation of the water wave problem i.e. the system of equation satisfied by  $h$  and  $\psi$ :

$$\begin{cases} \partial_t h - \mathcal{G}(h)\psi = 0, \\ \partial_t \psi + \frac{1}{2} |\partial_{\mathbf{x}} \psi|^2 - \frac{(\mathcal{G}(h)\psi + \partial_{\mathbf{x}} h \cdot \partial_{\mathbf{x}} \psi)^2}{2(1 + |\partial_{\mathbf{x}} h|^2)} + gh = 0. \end{cases} \quad (1.13)$$

This system admits a Hamiltonian structure which is used in the following section to derive shallow water systems. Indeed, (1.13) can be written under <sup>3</sup>

$$\begin{cases} \partial_t h = \frac{\delta \mathcal{H}}{\delta \psi}, \\ \partial_t \psi = -\frac{\delta \mathcal{H}}{\delta h}, \end{cases} \quad (1.14)$$

where the Hamiltonian  $\mathcal{H}$  is given by

$$\mathcal{H} = \frac{1}{2} \int_{\mathbb{R}} (gh^2 + \psi \mathcal{G}(h)\psi). \quad (1.15)$$

Similarly to [78, 31, 32, 86, 85], we use a Taylor expansion of the Dirichlet–Neumann operator to derive asymptotic models. The shallow water model we obtain, depends on the order of the expansion as well as the assumptions on the wave amplitude, water height and wavelength. Other approaches to derive shallow water models are possible (see [75, 20, 57] for instance).

## 1.2 Derivation of shallow water models

In this part, we derive formally Saint-Venant and Green–Naghdi equations from the water waves problem (1.14). This means that we derive asymptotic models for (1.14) under the shallowness

<sup>3</sup>Symbol  $\frac{\delta \mathcal{H}}{\delta \psi}$  represents the variational derivative of the functional  $\mathcal{H}$  with respect to the variable  $\psi$ .

assumption. This assumption is represented mathematically by

$$\epsilon \ll 1,$$

where the shallowness parameter  $\epsilon$  is defined by

$$\epsilon = \frac{H}{L_x}.$$

Here,  $H$  is the characteristic water height whereas  $L_x$  is the characteristic wavelength in the longitudinal direction. Then, we denote by  $L_y$  the characteristic wavelength in the transverse direction and by  $U$  the characteristic fluid horizontal velocity. Then, we use the following natural scaling for the equation

$$\psi = UL_x\psi', \quad h = Hh', \quad x = L_x x', \quad t = \frac{L_x}{U}t' \text{ and } y = L_y y' \text{ if the fluid is 3-dimensional.}$$

We assume for the sake of simplicity that  $L_x \simeq L_y$ .

Let us also mention that  $g' = gh/U^2$  is the inverse of the Froude number. Then, after omitting prime symbols for the sake of simplicity, we get the following non-dimensionalized equations:

$$\begin{cases} \partial_t h - \frac{1}{\epsilon^2} \mathcal{G}(\epsilon, h)\psi = 0 \\ \partial_t \psi + \frac{1}{2} |\partial_{\mathbf{x}} \psi|^2 - \frac{1}{\epsilon^2} \frac{(\mathcal{G}(\epsilon, h)\psi + \epsilon^2 \partial_{\mathbf{x}} h \cdot \partial_{\mathbf{x}} \psi)^2}{2(1 + \epsilon^2 |\partial_{\mathbf{x}} h|^2)} + gh = 0. \end{cases} \quad (1.16)$$

This system is Hamiltonian with the scaled Hamiltonian

$$\mathcal{H} = \frac{1}{2} \int_{\mathbb{R}^d} gh^2 + \psi \mathcal{G}(\epsilon, h)\psi. \quad (1.17)$$

Symbol  $d$  represents the fluid dimension minus one. In other words,  $d = 2$  if the fluid is 3-dimensional and  $d = 1$  otherwise. Indeed, (1.16) writes

$$\begin{cases} \partial_t h = \frac{\delta \mathcal{H}}{\delta \psi}, \\ \partial_t \psi = -\frac{\delta \mathcal{H}}{\delta h}. \end{cases} \quad (1.18)$$

Then, using the properties of the Dirichlet–Neumann operator, we have the following Taylor expansion for the scaled operator  $\mathcal{G}(\epsilon, h)$  (see [75, 78] for more details).

$$\mathcal{G}(\epsilon, h)(\cdot) = -\partial_{\mathbf{x}}(h\partial_{\mathbf{x}}(\cdot)) - \frac{\epsilon^2}{3} \partial_{\mathbf{x}}^2 (h^3 \partial_{\mathbf{x}}^2(\cdot)) + O(\epsilon^4), \quad (1.19)$$

As mentioned, one of the main advantages of shallow water models is the dimension reduction of the fluid domain as well as the reduction of the components of the velocity. Indeed, in the frame of shallow water models we consider here, the unknown of the problem is the horizontal component of the velocity. The vertical component is related somehow to the horizontal component. For this reason, during all this section, we adopt an abuse of notation: we denote by  $\mathbf{u}$  only the horizontal component of the speed and not the whole velocity vector. Moreover,  $\partial_{\mathbf{x}}$  (resp.  $\partial_{\mathbf{x}}^2$ ) represent the horizontal gradient (resp. the horizontal Laplacian).

### Saint-Venant system

Nonlinear shallow water system, also called Saint-Venant system, is a first order approximation of the scaled water waves problem (1.16). More precisely, it fits the Hamiltonian structure (1.14) with the first order approximation  $\mathcal{H}_{SV}$  of the Hamiltonian  $\mathcal{H}$  of the water waves problem defined by (1.17). In other words, the Hamiltonian  $\mathcal{H}_{SV}$  of Saint-Venant system is obtained by omitting the terms of order  $O(\epsilon^2)$  i.e.

$$\mathcal{H}_{SV} = \frac{1}{2} \int_{\mathbb{R}^d} gh^2 - \psi \partial_{\mathbf{x}} \cdot (h \partial_{\mathbf{x}} \psi).$$

Under convenient regularity assumptions on  $\psi$ , we can rewrite  $\mathcal{H}_{SV}$  as

$$\mathcal{H}_{SV} = \frac{1}{2} \int_{\mathbb{R}^d} gh^2 + h(\partial_{\mathbf{x}} \psi)^2. \quad (1.20)$$

Therefore, the Saint-Venant system writes as

$$\begin{cases} \partial_t h = \frac{\delta \mathcal{H}_{SV}}{\delta \psi} = -\partial_{\mathbf{x}} \cdot (h \partial_{\mathbf{x}} \psi), \\ \partial_t \psi = -\frac{\delta \mathcal{H}_{SV}}{\delta h} = -gh - \frac{(\partial_{\mathbf{x}} \psi)^2}{2}. \end{cases} \quad (1.21)$$

In the frame of Saint-Venant system, we assume that the vertical component of the velocity vanishes. Therefore,  $\partial_z \phi$  vanishes and the second equation of (1.11) becomes

$$\partial_{\mathbf{x}} \psi = \partial_{\mathbf{x}} \phi.$$

We also assume that the horizontal component of the velocity is constant along the vertical lines and are equal to the components of  $\partial_{\mathbf{x}} \psi$  i.e. the value of  $\partial_{\mathbf{x}} \phi$  on the free surface. This leads to the classical writing of the Saint-Venant equation. In fact, we denote by  $\mathbf{u}$  the two first components of the velocity for a 3-dimensional fluid and the first component of the velocity for a 2-dimensional fluid. Then, we replace  $\partial_{\mathbf{x}} \psi$  in the first equation of (1.21) by  $\mathbf{u}$ . This leads to

$$\partial_t h + \partial_{\mathbf{x}} \cdot (h \mathbf{u}) = 0. \quad (1.22)$$

We then take the  $\mathbf{x}$ -derivative of the second equation of (1.21):

$$\partial_t \partial_{\mathbf{x}} \psi = -\partial_{\mathbf{x}}(gh) - (\partial_{\mathbf{x}} \psi \cdot \partial_{\mathbf{x}}) \partial_{\mathbf{x}} \psi.$$

Then, we replace again  $\partial_{\mathbf{x}} \psi$  by  $\mathbf{u}$  to find

$$\partial_t \mathbf{u} = -\partial_{\mathbf{x}}(gh) - (\mathbf{u} \cdot \partial_{\mathbf{x}}) \mathbf{u}. \quad (1.23)$$

Equations (1.22) and (1.23) are the Saint-Venant equation where the unknowns are the water height  $h$  and the horizontal velocity  $\mathbf{u}$  independent of the vertical variable.

Let us also note that the Saint-Venant system under the variable  $(h, \mathbf{u})$  is Hamiltonian of the form

$$\begin{pmatrix} \partial_t h \\ \partial_t \mathbf{u} \end{pmatrix} = \begin{pmatrix} 0 & -\partial_{\mathbf{x}}^t \\ -\partial_{\mathbf{x}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \frac{\delta \mathcal{H}_{SV}}{\delta h} \\ \frac{\delta \mathcal{H}_{SV}}{\delta \mathbf{u}} \end{pmatrix}.$$

This structure can easily get obtained from (1.21) using the chain rule considering the fact that

$$\frac{\delta \mathcal{H}_{SV}}{\delta h} = gh + \mathbf{u}^2/2,$$

and

$$\frac{\delta \mathcal{H}_{SV}}{\delta \mathbf{u}} = h\mathbf{u}.$$

**Remark 1.3** *A rigorous derivation of Saint-Venant system is done in [57].*

### Green–Naghdi system

Green–Naghdi equation is a second order approximation of the water waves problem. More precisely, it fits the Hamiltonian structure (1.14) with the second order approximation  $\mathcal{H}_{GN}$  of the Hamiltonian  $\mathcal{H}$  defined by (1.17). Contrary to the case of Saint-Venant system, we do not assume here that the vertical velocity vanishes. However, we assume that it is deduced by the horizontal velocity and depends linearly to the vertical variable. Thus, we just need to solve the equation on the horizontal velocity, denoted by  $\mathbf{u}$ , in order to get the vertical component. This relation is given through a physical variable  $\mathbf{k}$  called the generalized velocity<sup>4</sup>, introduced for instance in [10, 55], which is given by

$$\mathbf{k} = \frac{\mathbf{m}}{h} \quad \text{with} \quad \mathbf{m} = \mathcal{L}_{\epsilon, h}(\mathbf{u}),$$

where  $\mathcal{L}_{\epsilon, h}$  is the Sturm–Liouville operator defined by

$$\mathcal{L}_{\epsilon, h}(\mathbf{f}) = h\mathbf{f} - \frac{\epsilon^2}{3} \partial_{\mathbf{x}} \cdot (h^3 \partial_{\mathbf{x}} \cdot \mathbf{f}) \quad \text{for all test vector function } \mathbf{f}. \quad (1.24)$$

Indeed, the vertical velocity on the free surface<sup>5</sup>, is obtained by the difference between the generalized velocity and the horizontal velocity as following:

$$u_z \partial_{\mathbf{x}} h = \mathbf{k} - \mathbf{u}. \quad (1.25)$$

On the other hand,  $\phi$  is a potential for  $\mathbf{u}$  and we have  $\partial_z \phi = u_z$ . Therefore, using (1.25), the second equation of (1.11) becomes

$$\partial_{\mathbf{x}} \psi = \mathbf{k} = \frac{\mathbf{m}}{h}. \quad (1.26)$$

Moreover, under convenient regularity assumptions on  $\psi$  and using the second order approximation (1.19) of the Dirichlet-Neumann operator, the Hamiltonian (1.17) of the water waves problem can be written after an integration by part:

$$\mathcal{H} = \frac{1}{2} \int_{\mathbb{R}^d} gh^2 + h (\partial_{\mathbf{x}} \psi)^2 + \frac{\epsilon^2}{3} \partial_{\mathbf{x}} \psi \cdot \partial_{\mathbf{x}} (h^3 \partial_{\mathbf{x}}^2 \psi) + O(\epsilon^4).$$

<sup>4</sup>As mentioned in [82], the generalized velocity  $\mathbf{k}$  is the derivative with respect to the moment of the Lagrangian associated to the Green–Naghdi equation by a total derivative.

<sup>5</sup>The vertical velocity on the bottom is assumed to vanish due to the slip boundary condition (see condition (1.2)).



Then, using (1.26), we replace  $\partial_{\mathbf{x}}\psi$  by  $\frac{\mathbf{m}}{h}$  to find:

$$\mathcal{H} = \frac{1}{2} \int_{\mathbb{R}^d} gh^2 + \frac{\mathbf{m}^2}{h} + \frac{\epsilon^2}{3h} \mathbf{m} \cdot \partial_{\mathbf{x}} \left( h^3 \partial_{\mathbf{x}} \cdot \left( \frac{\mathbf{m}}{h} \right) \right) + O(\epsilon^4). \quad (1.27)$$

Considering Definition of the Sturm–Liouville operator (1.24) of  $\mathcal{L}_{\epsilon,h}$ , we remark that

$$\frac{(\cdot)}{h} + \frac{\epsilon^2}{3h} \partial_{\mathbf{x}} \left( h^3 \partial_{\mathbf{x}} \cdot \left( \frac{(\cdot)}{h} \right) \right) = \mathcal{L}_{\epsilon,h}^{-1}(\cdot) + O(\epsilon^4), \quad (1.28)$$

where  $\mathcal{L}_{\epsilon,h}^{-1}$  is the inverse of the operator  $\mathcal{L}_{\epsilon,h}$ . Using this fact, (1.27) becomes

$$\mathcal{H} = \frac{1}{2} \int_{\mathbb{R}^d} gh^2 + \mathbf{m} \cdot \mathcal{L}_{\epsilon,h}^{-1}(\mathbf{m}) + O(\epsilon^4). \quad (1.29)$$

Therefore, the second order approximation  $\mathcal{H}_{GN}$  of the Hamiltonian  $\mathcal{H}$  is

$$\mathcal{H}_{GN} = \frac{1}{2} \int_{\mathbb{R}^d} gh^2 + \mathbf{m} \cdot \mathcal{L}_{\epsilon,h}^{-1}(\mathbf{m}). \quad (1.30)$$

Then, applying the chain rule, the Hamiltonian structure (1.18) with the Hamiltonian  $\mathcal{H}_{GN}$  becomes

$$\begin{pmatrix} \partial_t h \\ \partial_t \mathbf{m} \end{pmatrix} = - \begin{pmatrix} 0 & \partial_{\mathbf{x}} \cdot (h(\cdot)) \\ h \partial_{\mathbf{x}} & \partial_{\mathbf{x}}(\mathbf{m} \cdot (\cdot)) + \mathbf{m} \partial_{\mathbf{x}} \cdot \end{pmatrix} \begin{pmatrix} \frac{\delta \mathcal{H}_{GN}}{\delta h} \\ \frac{\delta \mathcal{H}_{GN}}{\delta \mathbf{m}} \end{pmatrix} \quad (1.31)$$

System (1.31) is the Green–Naghdi equations written under the variable  $(h, \mathbf{m})$ . The computation of the variational derivative of the Hamiltonian, as an application of the variable  $(h, \mathbf{m})$ , leads to

$$\begin{cases} \partial_t h + \partial_{\mathbf{x}} \cdot (h \mathcal{L}_{\epsilon,h}^{-1}(\mathbf{m})) = 0, \\ \partial_t \mathbf{m} + \partial_{\mathbf{x}} \left( \mathbf{m} \cdot \mathcal{L}_{\epsilon,h}^{-1}(\mathbf{m}) \right) + \mathbf{m} \partial_{\mathbf{x}} \cdot \mathcal{L}_{\epsilon,h}^{-1}(\mathbf{m}) + h \partial_{\mathbf{x}} \left( gh - \frac{(\mathcal{L}_{\epsilon,h}^{-1}(\mathbf{m}))^2}{2} - \frac{\epsilon^2}{2} h^2 \left( \partial_{\mathbf{x}} \cdot \mathcal{L}_{\epsilon,h}^{-1}(\mathbf{m}) \right)^2 \right) = 0. \end{cases} \quad (1.32)$$

In fact,

$$\frac{\delta \mathcal{H}_{GN}}{\delta h} = gh - \frac{(\mathcal{L}_{\epsilon,h}^{-1}(\mathbf{m}))^2}{2} - \frac{\epsilon^2}{2} h^2 \left( \partial_{\mathbf{x}} \cdot \mathcal{L}_{\epsilon,h}^{-1}(\mathbf{m}) \right)^2,$$

and

$$\frac{\delta \mathcal{H}_{GN}}{\delta \mathbf{m}} = \mathcal{L}_{\epsilon,h}^{-1}(\mathbf{m}).$$

Now, under the change of variable  $(h, \mathbf{m}) \mapsto (h, \mathbf{u})$ , the Hamiltonian writes

$$\mathcal{H}_{GN} = \frac{1}{2} \int_{\mathbb{R}^d} gh^2 + h \mathbf{u}^2 + \frac{\epsilon^2}{3} h^3 (\partial_{\mathbf{x}} \cdot \mathbf{u})^2. \quad (1.33)$$

As a consequence, Equation (1.32) becomes

$$\begin{cases} \partial_t h + \partial_{\mathbf{x}} \cdot (h \mathbf{u}) = 0, \\ \partial_t (\mathcal{L}_{\epsilon,h}(\mathbf{u})) + \partial_{\mathbf{x}} (\mathbf{u} \cdot \mathcal{L}_{\epsilon,h}(\mathbf{u})) + (\mathcal{L}_{\epsilon,h}(\mathbf{u})) \partial_{\mathbf{x}} \cdot \mathbf{u} + h \partial_{\mathbf{x}} \left( gh - \frac{\mathbf{u}^2}{2} - \frac{\epsilon^2}{2} h^2 (\partial_{\mathbf{x}} \cdot \mathbf{u})^2 \right) = 0, \end{cases} \quad (1.34)$$

which leads after computations to

$$\begin{cases} \partial_t h + \partial_{\mathbf{x}} \cdot (h\mathbf{u}) = 0, \\ \mathcal{L}_{\epsilon, h} (\partial_t \mathbf{u} + \mathbf{u} \cdot \partial_{\mathbf{x}} \mathbf{u}) + g \partial_{\mathbf{x}} h + \frac{2\epsilon^2}{3h} \partial_{\mathbf{x}} \left( h^3 (\partial_{\mathbf{x}} \cdot \mathbf{u})^2 + h^3 \partial_x \mathbf{u} \cdot \partial_y \mathbf{u}^\perp \right) = 0. \end{cases} \quad (1.35)$$

Let us mention that the approach we use here for the analysis of the system, is suitable for all  $\epsilon$ . Therefore, for the sake of simplicity, we replace the coefficient  $\frac{\epsilon^2}{3}$  by  $\alpha$  which is consequently strictly positive. For a similar reason, we denote  $\mathcal{L}_{\epsilon, h}$  by  $\mathcal{L}_h$ . Hence, we rewrite (1.35) as

$$\begin{cases} \partial_t h + \partial_{\mathbf{x}} \cdot (h\mathbf{u}) = 0, \\ \mathcal{L}_h (\partial_t \mathbf{u} + \mathbf{u} \cdot \partial_{\mathbf{x}} \mathbf{u}) + g \partial_{\mathbf{x}} h + \frac{2\alpha}{h} \partial_{\mathbf{x}} \left( h^3 (\partial_{\mathbf{x}} \cdot \mathbf{u})^2 + h^3 \partial_x \mathbf{u} \cdot \partial_y \mathbf{u}^\perp \right) = 0. \end{cases} \quad (1.36)$$

**Remark 1.4** *Let us note that under the assumption of the regularity of the solution of system (1.36), it can be written under other forms after simple computation. For instance, the system writes also*

$$\begin{cases} \partial_t h + \partial_{\mathbf{x}} \cdot (h\mathbf{u}) = 0, \\ \partial_t (h\mathbf{u}) + (\partial_{\mathbf{x}} \cdot \mathbf{u}) (h\mathbf{u}) + \partial_{\mathbf{x}} (gh^2/2 + \alpha h^2 \ddot{h}) = 0, \end{cases} \quad (1.37)$$

where the superscript  $\dot{(\ )}$  represents the material derivative defined by:

$$\dot{(\ )} = \partial_t(\ ) + \mathbf{u} \cdot \partial_{\mathbf{x}}(\ ).$$

Again, after simple equations, we remark that the Green–Naghdi system in the 2-dimensional case i.e. when  $d = 1$ , can also be written as

$$\begin{cases} \partial_t h + \partial_x (hu) = 0, \\ \partial_t u + u \partial_x u + g \partial_x h = \frac{\alpha}{h} \partial_x \left( h^3 (\partial_{tx} u + u \partial_{xx} u - (\partial_x u)^2) \right). \end{cases} \quad (1.38)$$

**Remark 1.5** *A rigorous justification of the Green–Naghdi equations is done in [80, 78, 75, 30]. In [68], the author derive Green–Naghdi equation using the Hamilton’s principle by approximating the action associated to incompressible Euler equations. A rotational version of Green–Naghdi equations has been derived in [26].*

### 1.3 Symmetric structure of the Green–Naghdi equations

The first part of this thesis concerns the study of the Green–Naghdi equation (1.35) derived in the last subsection. The idea here is to adapt the approach classically used for hyperbolic systems of conservation laws, to the case of Green–Naghdi equation. Indeed, the symmetric structure of hyperbolic systems is used in several references to prove the local well-posedness as well as the global well-posedness with dissipative terms. The first step to adapt the approach to the case of the dispersive Green–Naghdi equations, is to symmetrize the system. Therefore, in the first chapter of this thesis, we consider a general frame the Green–Naghdi equations fit in. We find a general result and then apply it to the Green–Naghdi equation. Let us first recall introduce a brief reminder about classical results on the symmetric structure of hyperbolic systems.

### Well known results on hyperbolic systems

Let us consider a  $N$ -hyperbolic system of conservation laws of the form:

$$\partial_t U + \partial_x F(U) = 0, \quad (1.39)$$

where  $F$  is a smooth function acting on a convex subset of  $\mathbb{R}^N$ .

We say that (1.39) admits an entropy in the sense of Lax if there exists a strictly convex function  $E$  called the entropy and a function  $P$  called the entropy flux such that the gradient of  $E$  and  $P$  are related to each other by the Jacobian of the flux as following [50, 61]:

$$(\nabla F(U))^t \nabla E(U) = \nabla P(U). \quad (1.40)$$

According to Poincaré's theorem, such a pair of function  $(E, P)$  exists if and only if

$$\nabla^2 E(U) \nabla F(U) \text{ is a symmetric matrix,} \quad (1.41)$$

or equivalently, if and only if the smooth solution  $U$  of (1.39) satisfies

$$\partial_t E(U) + \partial_x P(U) = 0. \quad (1.42)$$

We call a system which admits an entropy in the sense of Lax, an entropic system.

The notion of the entropy is related to a structure named after Godunov [61]. Indeed, a hyperbolic system of the form (1.39) is entropic if and only if it is a Godunov system i.e. if and only if there exist a pair of function  $(E^*, \hat{P})$ , and a change of variable  $U \mapsto Q$  such that the system is written

$$\partial_t (\nabla E^*(Q)) + \partial_x (\nabla \hat{P}(Q)) = 0.$$

This relation is highlighted by the Legendre transform. In fact,  $E^*$  is the Legendre transform of the entropy  $E$  and the change of variable to consider is  $U \mapsto Q = \nabla E(U)$ . This can be summarized in the following scheme:

$$\text{System (1.39) admits the entropy } E \iff \text{It admits a Godunov structure through } E^*. \quad (1.43)$$

More precisely, the entropy pair  $(E, P)$  is related to  $(E^*, \hat{P})$  as following:

$$\begin{cases} E^*(Q) = Q \cdot U - E(U), \\ \hat{P}(Q) = Q \cdot F(U) - P(U). \end{cases} \quad (1.44)$$

We also say that (1.39) is symmetrizable if there exists a change of variable  $U \mapsto V$ , a symmetric definite positive matrix  $A_0(V)$  and a symmetric matrix  $A_1(V)$ , such that the system is written under the form

$$A_0(V) \partial_t V + A_1(V) \partial_x V = 0.$$

The hyperbolic result we try to generalize here to the case of Green-Naghdi system is the

following:

System (1.39) is a Godunov system.  $\implies$  System (1.39) is symmetrizable under any variable. (1.45)

Since, we have the equivalence between the Godunov structure and the existence of the entropy for the system, this result writes also

System (1.39) is entropic.  $\implies$  System (1.39) is symmetrizable under any variable. (1.46)

Moreover, the expressions of  $A_0(V)$  and  $A_1(V)$  are given by the Hessian of the entropy  $E(U)$  and the Jacobian of  $F$  and the change of variable  $U \mapsto V$  as following:

$$A_0(V) = (\nabla_V U)^T \nabla^2 E(U) \nabla_V U, \quad A_1(V) = (\nabla_V U)^T \nabla^2 E(U) \nabla F(U) \nabla_V U. \quad (1.47)$$

**Application to the 1-dimensional Saint-Venant equation:** Saint-Venant system (1.22)-(1.23) fits the general frame (1.39). This fact becomes obvious by setting

$$U = \begin{pmatrix} h \\ hu \end{pmatrix}$$

and

$$F(U) = \begin{pmatrix} hu \\ gh^2/2 + hu^2 \end{pmatrix}.$$

It is well known that this system admits the entropy  $E = gh^2/2 + hu^2/2$ . Then, we use the change of variable  $U \mapsto V = (h, u)$  and apply the formulas (1.47) to find the following expressions for  $A_0(V)$  and  $A_1(V)$ .

$$A_0(V) = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix},$$

and

$$A_1(V) = \begin{pmatrix} gu & gh \\ gh & hu \end{pmatrix}.$$

Let us note that  $A_0(V)$  is bloc diagonal since the considered change of variable  $U \mapsto V$  is partial i.e. only some (and not all) components of  $U$  are changed to get  $V$ .

**Remark 1.6** *As detailed in Chapter 1, the presented result can easily get generalized to the case of multi-dimensional systems. Therefore, the 2-dimensional Saint-Venant system is symmetrizable. This symmetric structure under the variable  $V = (h, u, v)$  is*

$$A_0(V) \partial_t V + A_1(V) \partial_x V + A_2(V) \partial_y V = 0,$$

where

$$A_0(V) = \begin{pmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{pmatrix}, \quad A_1(V) = \begin{pmatrix} gu & gh & 0 \\ gh & hu & 0 \\ 0 & 0 & hu \end{pmatrix}, \quad A_2(V) = \begin{pmatrix} gv & 0 & gh \\ 0 & hv & 0 \\ gh & 0 & hv \end{pmatrix}.$$

### Generalization of the notion of symmetry

We now generalize the notion of symmetry presented in the previous part to systems of the form

$$\partial_t U + \partial_x F(U) = 0, \quad (1.48)$$

where  $F$  is not necessarily a function defined on a subset of the finite dimensional space  $\mathbb{R}^N$ . It is an application defined on a convex subset of  $\mathcal{A}$ , with values in  $\mathcal{A}$ , where  $\mathcal{A}$  is a convenient Banach subspace of continuous functions of  $\mathbb{L}^2(\mathbb{R}, \mathbb{R}^N)$ . Moreover, we assume that the solution of (1.48) belongs to  $C([0, T]; \mathcal{A})$  for some  $T > 0$ . In other words,  $F$  can not be seen as a function acting on a finite dimensional space but as an application defined on an infinite dimensional space.

One of the main differences here with the hyperbolic case (1.39) is the fact that the operator version of conditions (1.40) and (1.41) is not equivalent anymore to an equality of the form (1.42). Despite this difference, we generalize these conditions as well as the notion of Godunov structure and symmetric structure such that we are able to find a similar result as (1.46) and (1.45). Nevertheless, we do not use anymore the word "entropy" or "entropic system". Our generalization is basically done by replacing the gradient by the variational derivative and the Hessian by the second variation. Schematically speaking, we can represent this replacement by :

$$\begin{aligned} \text{function } E &\rightarrow \text{functional } \mathcal{H} = \int_{\mathbb{R}} E. \\ \text{gradient } \nabla &\rightarrow \text{variational derivative } \delta. \\ \text{Hessian } \nabla^2 &\rightarrow \text{second variation } \delta^2. \\ E^* &\rightarrow \mathcal{H}^*(Q) = \int_{\mathbb{R}} U \cdot \delta_U \mathcal{H}(U) - E(U). \end{aligned}$$

Let us formally recall that the variational derivative of a functional  $\mathcal{H} = \int_{\mathbb{R}} E$  is such that

$$D\mathcal{H}(U)(\phi) = \int_{\mathbb{R}} \delta\mathcal{H}(U) \cdot \phi \quad \text{for all test function } \phi.$$

Then, the second variation is the differential of the variational derivative. To be more clear, we may sometimes denote the differential (resp. the variational derivative) by  $D_U$  (resp. by  $\delta_U$ ) to insist on the variable we differentiate with respect to.

Let us also mention that we say that an operator is symmetric if it is symmetric for the  $\mathbb{L}^2$  scalar product i.e. we say that  $\mathcal{F}$  is symmetric if it satisfies

$$\int_{\mathbb{R}} \phi \cdot \mathcal{F}(\psi) = \int_{\mathbb{R}} \mathcal{F}(\phi) \cdot \psi \quad \text{for all test functions } \phi \text{ and } \psi.$$

Following the presented approach, condition (1.41) is generalized as : there exists a strictly

convex functional  $\mathcal{H} = \int_{\mathbb{R}} E$  such that

$$\delta_U^2 \mathcal{H}(U) D_U F(U) \text{ is symmetric for the } \mathbb{L}^2 \text{ scalar product.} \quad (1.49)$$

Again, this conditions is equivalent by Poincaré's theorem to : there exists a differentiable application  $\mathcal{N}$  such that

$$D_U \mathcal{H}(U) D_U F(U) \phi = D_U \mathcal{N}(U) \phi \quad \text{for all test function } \phi, \quad (1.50)$$

which can be seen somehow, as the operator version of condition (1.40).

On the other hand, the notion of Godunov structure is generalized as: system (1.48) admits a general Godunov structure if there exist two functionals  $\mathcal{H}^*$  and  $\mathcal{R}$ , as well as a change of unknown  $U \mapsto Q$ , such that the system writes

$$\partial_t(\delta_Q \mathcal{H}^*(Q)) + \partial_x(\delta_Q \mathcal{R}(Q)) = 0.$$

Following this generalization, we prove that the conditions (1.49) and (1.50) are equivalent to the existence of the general Godunov structure under the unknown  $Q = \delta_U \mathcal{H}(U)$  i.e.

$$\text{System (1.48) satisfies (1.49) or equivalently (1.50)} \iff \text{It is a general Godunov structure through } \mathcal{H}^*. \quad (1.51)$$

Again, this equivalence is highlighted using the generalization of the Legendre transform. Indeed,  $\mathcal{H}^*$  and  $\mathcal{R}$  can be expressed by

$$\begin{cases} \mathcal{H}^*(Q) = \int_{\mathbb{R}} Q \cdot \delta_U \mathcal{H}(U) - E(U), \\ \mathcal{R}(Q) = \int_{\mathbb{R}} Q \cdot F(U) - \mathcal{N}(U). \end{cases} \quad (1.52)$$

As mentioned, the notion of symmetry we generalize, is based on the  $\mathbb{L}^2$  scalar product. Indeed, we say that (1.48) is symmetrizable if there exist a change of unknown  $U \mapsto V$ , a symmetric definite positive matrix  $A_0(V)$  and a symmetric matrix  $A_1(V)$ , such that the system is written under the form

$$A_0(V) \partial_t V + A_1(V) \partial_x V = 0,$$

where  $A_0(V)$  is a symmetric definite positive operator and  $A_1(V)$  is a symmetric operator (for the  $\mathbb{L}^2$  scalar product).

Based on these generalized definitions, we prove that the general Godunov structure of the system as well as condition (1.49) imply the symmetric structure of the system under any variable. In other words, we have

$$\text{System (1.48) is a general Godunov system.} \implies \text{System (1.48) is symmetrizable under any unknown.} \quad (1.53)$$

$$\text{System (1.48) satisfies the condition (1.49) or equivalently (1.50)} \implies \text{System (1.48) is symmetrizable under any unknown.}$$

Similarly to the case of the hyperbolic system, the expressions of  $A_0(V)$  and  $A_1(V)$  are given by the Hessian  $\mathcal{H}(U)$  and the differential of  $F$  and the change of variable  $U \mapsto V$  as following:

$$A_0(V) = (D_V U)^T \delta^2 \mathcal{H}(U) D_V U, \quad A_1(V) = (D_V U)^T \delta^2 \mathcal{H}(U) D_U F(U) D_V U.$$

The last question we consider in this chapter is the following: Is it possible to obtain a conservation law similar to (1.42) for (1.48)? Actually, we prove that if system (1.48) satisfies one of the assertions of statement (1.51), then the smooth solution of the system satisfies

$$\int_{\mathbb{R}} \partial_t E(U) + \partial_x N(U) = 0. \quad (1.54)$$

In other words, contrary to the case of hyperbolic systems, the solution does not satisfy a conservation law but we can associate a conserved quantity to the system. We also see that the reciprocal in the general case is false. This means that if all smooth solution of (1.48) satisfies (1.54), we can not conclude that the assertions of statement (1.51) are satisfied.

### Application to the 1-dimensional Green–Naghdi equation :

We then apply our generalized result to the case of 1-dimensional Green–Naghdi equations. We first remark that the 1-dimensional Green–Naghdi equations fits the general frame (1.48) around all constant solution  $(\bar{h}, \bar{u})$  with  $\bar{h} > 0$ . Indeed, using (1.36), it is easy to see that under variable  $U$  defined by

$$U = (\eta, w) \quad \text{with } \eta = h - \bar{h} \text{ and } w = \mathcal{L}_h(u) - \bar{h}\bar{u},$$

the 1-dimensional Green–Naghdi equations write

$$\partial_t U + \partial_x F(U) = 0,$$

where  $F$  is defined by

$$F(U) = \left( \begin{array}{c} (\eta + \bar{h}) \mathcal{L}_h^{-1}(w + \bar{h}\bar{u}) - \bar{h}\bar{u} \\ (w + \bar{h}\bar{u}) \mathcal{L}_h^{-1}(w + \bar{h}\bar{u}) - 2\alpha(\eta + \bar{h})^3 (\partial_x \mathcal{L}_h^{-1}(w + \bar{h}\bar{u}))^2 + \frac{g}{2}(\eta + \bar{h})^2 - \frac{g}{2}\bar{h}^2 - \bar{h}\bar{u}^2 \end{array} \right).$$

Therefore,  $F : \mathcal{A} \rightarrow \mathcal{A}$  is a differentiable application with  $\mathcal{A}$  defined by

$$\mathcal{A} = \mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R}) \quad \text{with } s \geq 2.$$

The next step is to remark that the system owns a Godunov structure. Indeed, this structure is highlighted using the Legendre transform of the energy integral of the system:

$$\mathcal{H}_{\bar{h}, \bar{u}}(U) = \int_{\mathbb{R}} E_{\bar{h}, \bar{u}}(U) \quad \text{with} \quad E_{\bar{h}, \bar{u}} = \frac{gh(h - \bar{h})}{2} + \frac{h(u - \bar{u})^2}{2} + \frac{\alpha h^3 (u_x)^2}{2}.$$

Indeed, if we consider the change of variable

$$U \mapsto Q = \delta_U \mathcal{H}_{\bar{h}, \bar{u}}(U),$$

We can check that

$$F(U) = \delta_Q \mathcal{R}(Q),$$

where

$$\begin{aligned} \mathcal{R}(Q) = & \int_{\mathbb{R}} gu \left( \frac{h^2 - \bar{h}^2}{2} \right) - \alpha h^3 u (u_x)^2 - \bar{h} \bar{u} \left( gh - g\bar{h}/2 - u^2/2 + \bar{u}^2/2 - \frac{3}{2} \alpha h^2 (u_x)^2 \right) \\ & - \bar{h} \bar{u}^2 (u - \bar{u}) + g \bar{h}^2 \bar{u}. \end{aligned}$$

Therefore, the Green–Naghdi equations is a general Godunov system i.e. it writes

$$\partial_t (\delta_Q \mathcal{H}_{h, \bar{u}}^*(Q)) + \partial_x (\delta_Q \mathcal{R}(Q)) = 0.$$

Hence, the system is symmetrizable under any change of variable. The symmetric structure we are using in the next part, is the structure obtained based on the change of variable  $U \mapsto V$  where  $V = (h, u)$ . More precisely, the 1-dimensional Green–Naghdi equation writes

$$A_0(V) \partial_t V + A_1(V) \partial_x V = 0, \quad (1.55)$$

where

$$A_0(V) = \begin{pmatrix} g - 3\alpha h (u_x)^2 & 0 \\ 0 & \mathcal{L}_h \end{pmatrix}, \quad (1.56)$$

and

$$A_1(V) = \begin{pmatrix} gu - 3\alpha hu (u_x)^2 & gh - 3\alpha h^2 (u_x)^2 \\ gh - 3\alpha h^2 (u_x)^2 & hu + 2\alpha \partial_x (h^3 u_x) - \alpha h^3 u_x \partial_x - \alpha u \partial_x (h^3 \partial_x()) \end{pmatrix}. \quad (1.57)$$

We see again that  $A_0(V)$  is bloc diagonal since the considered change of unknown  $U \mapsto V$  is partial i.e. only some (and not all) components of  $U$  are changed to get  $V$ .

**Remark 1.7** *As we will see later, a multi-dimensional generalization of the general result is possible. This means that the 2-dimensional Green–Naghdi equation is symmetrizable under the variable  $V = (h, u, v)$  of the form*

$$A_0(V) \partial_t V + A_1(V) \partial_x V + A_2(V) \partial_y V = 0,$$

where

$$A_0(V) = \begin{pmatrix} g - 3\alpha h (\operatorname{div}(u, v))^2 & 0 & 0 \\ 0 & h - \alpha \partial_x (h^3 \partial_x) & -\alpha \partial_x (h^3 \partial_y) \\ 0 & -\alpha \partial_y (h^3 \partial_x) & h - \alpha \partial_y (h^3 \partial_y) \end{pmatrix}.$$

## 1.4 Global existence for small data of a viscous Green–Naghdi equations

As mentioned previously, the symmetric structure of hyperbolic systems is interesting since it can be used to prove the local well-posedness as well as the global well-posedness of systems with dissipative terms. In fact, we generalized the notion of symmetry for Green–Naghdi equations in order to obtain a global well-posedness result for the system with viscosity. We first briefly present the hyperbolic results found by Kawashima–Shizuta [71, 70, 95], Hanouzet–Natalini [64] and Yong [108]. Then, we will see how we adapt the hyperbolic approach to the case of Green–Naghdi equations. Therefore, Chapter 2 of this thesis is devoted to the rigorous demonstration of the global



existence for initial data close to constant solutions of the Green–Naghdi equations with a second order viscosity. The asymptotic stability of these solutions is also proved as a consequence by a similar approach as for hyperbolic system. Let us also mention that the definition of the asymptotic stability we consider here is the following:

**Asymptotic stability:** A particular global solution  $U_e$  of an evolution system is called asymptotically stable if there exists a neighborhood of  $U_e$  such that for all initial data in this neighborhood, the solution of the system exists for all time and converges (for the norm of local well-posedness) to  $U_e$  while  $t \rightarrow \infty$ .

Let us also mention that to prove the global well-posedness, we assume that the system is locally well-posed. Though, a rigorous proof of the local well-posedness of the Green–Naghdi equations, with the viscosity we consider, is illustrated in an appendix of Chapter 2. The proof is based on the result presented in [69] for Green–Naghdi equations without viscosity. Then, we use similar computations as for hyperbolic case in [15] to deduce the local well-posedness for the system with a second order viscosity.

### Global well-posedness results for hyperbolic systems

The hyperbolic approach we try to generalize is the approach used in [71, 70, 95] for hyperbolic parabolic systems which can be put under the form

$$A_0(U)\partial_t U + A_1(U)\partial_x U = B\partial_x^2 U, \quad (1.58)$$

satisfying

- **Symmetrizability:**  $A_0(U)$  is a symmetric definite positive matrix,  $A_1(U)$  is a symmetric matrix.
- **Entropy dissipativity:**  $B$  is a symmetric constant definite positive matrix such that its kernel is invariant under  $A_0(U)$ .
- **Kawashima–Shizuta condition:** There exists a real matrix  $K$  such that  $KA_0(U_e)$  is skew-symmetric and  $\frac{1}{2}(KA_1(U_e) + A_1(U_e)K^T) + B(U_e)$  is definite positive for a constant solution  $U_e$ .

Let us mention that the symmetric structure of the system together with entropy dissipativity lead to estimates which gives the local well-posedness of the system in the space  $C([0, \infty); \mathbb{H}^s(\mathbb{R}))$  with  $s \geq 2$ . Then, the Kawashima–Shizuta condition gives some other estimates which completes the local well-posedness estimates in a way to obtain the global existence result. Indeed, the results due to Kawashima–Shizuta can be schemed as following:

Let us assume that System (1.58) satisfies the symmetrizability, entropy dissipativity as well as the Kawashima–Shizuta condition on an equilibrium  $U_e$ . Then,  $U_e$  is asymptotically stable. In other words, for all initial data close enough to  $U_e$ , the local solution of (1.58) exists for all time and tends to  $U_e$  when  $t \rightarrow \infty$ .

In [64, 108], the authors find a similar result using similar approach and assumptions as Kawashima–Shizuta. The main difference in these references, is that the dissipative terms are added only to some equations of the symmetric hyperbolic system. Moreover, the added dissipative terms are of 0<sup>th</sup> derivative order. Indeed, their result can be resumed as

Let us consider a  $n$ -symmetric hyperbolic system of the form

$$A_0(U)\partial_t U + A_1(U)\partial_x U = (0, Q(U)), \quad (1.59)$$

where  $U = (U_1, U_2)$  is a  $n$ -component vector. Let us also consider a constant vector  $U_e = (U_e^1, U_e^2)$  such that  $Q(U_e) = 0$ . We also assume that

- **Symmetrizability:**  $A_0(U)$  is a symmetric definite positive matrix,  $A_1(U)$  is a symmetric matrix.
- **Entropy dissipativity:** There exists a definite positive matrix  $B(U)$  such that  $Q(U) = -B(U)(U - U_e)$ .
- **Kawashima–Shizuta condition:** There exists a real matrix  $K$  such that  $KA_0(U_e)$  is skew-symmetric and

$$\frac{1}{2} (KA_1(U_e) + A_1(U_e)K^T) + \begin{pmatrix} 0 & 0 \\ 0 & B(U_e) \end{pmatrix}$$

is definite positive.

Then, the equilibrium  $U_e$  is asymptotically stable for the norm  $C([0, \infty); \mathbb{H}^s(\mathbb{R}))$  (the norm of the local well-posedness space).

In both cases, the proof is based on two classes of estimates found for all integer  $l \in [1, s]$ :

- The first category is obtained by taking the scalar product of the  $l^{\text{th}}$  derivative of the system with the  $l^{\text{th}}$  derivative of the solution using the symmetric structure and the entropy dissipativity.
- The second category is obtained by acting the operator  $K\partial_x^{l-1}$  on the system before taking the scalar product with the  $l^{\text{th}}$  derivative of the solution.

Combining these two estimates, we can find  $\delta > 0$  such that for all initial data in the  $\delta$ -neighborhood of  $U_e$ , the solution belongs to the neighborhood for all time. The asymptotic stability is then obtained by simple computations and using the Gagliardo–Nirenberg inequality.

### Global existence result for the Green–Naghdi equation with a second order dissipative term

During the first part of this thesis, we try to find a global well-posedness result for the Green–Naghdi equation with a dissipative term, using a similar approach as the approach presented in the previous section. We add here a dissipative term of second order of the form  $\mu\partial_x(h\partial_x u)$  (with  $\mu > 0$ ) to the second equation of the system i.e. we consider the following equations<sup>6</sup>

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(hu^2) + \partial_x(gh^2/2 + \alpha h^2 \ddot{h}) = \mu\partial_x(h\partial_x u). \end{cases} \quad (1.60)$$

As mentioned in an appendix of Chapter 2, under the assumption of the strict positivity of  $h$ , the system is locally well-posed under the variable  $(h-h_e, u)$  (for all  $h_e > 0$ ) in space  $C([0, T]; \mathbb{X}^s(\mathbb{R}))$  where  $\mathbb{X}^s(\mathbb{R}) = \mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R})$ .

Then, we use the symmetric structure (1.55) of system and write the equation under the variable  $V = (h, u)$  as following:

$$A_0(V)\partial_t V + A_1(V)\partial_x V = \begin{pmatrix} 0 \\ \mu\partial_x(h\partial_x u) \end{pmatrix},$$

where

$$A_0(V) = \begin{pmatrix} g - 3\alpha h(u_x)^2 & 0 \\ 0 & \mathcal{L}_h \end{pmatrix} \quad \text{and} \\ A_1(V) = \begin{pmatrix} gu - 3\alpha hu(u_x)^2 & gh - 3\alpha h^2(u_x)^2 \\ gh - 3\alpha h^2(u_x)^2 & hu + 2\alpha\partial_x(h^3 u_x) - \alpha h^3 u_x \partial_x - \alpha u \partial_x(h^3 \partial_x()) \end{pmatrix}.$$

Then, we see that it is possible to obtain the first category of estimates based on this structure by a similar way as in the hyperbolic case. In other words, the system satisfies the operator version of the symmetry and dissipativity condition. However, it does not satisfy any operator version of the Kawashima–Shizuta condition. Nevertheless, there exists a constant matrix  $K$  such that  $K\mathcal{A}_1(V)$  is a definite positive matrix by a constant even though  $K\mathcal{A}_0(V)$  is not skew-symmetric. We see that we are able to extract some convenient terms in a total time derivative form and cancel the high order terms together. Therefore, we get similar estimates as the second category of estimates for hyperbolic systems. Then, we get the asymptotic stability by basic computations together with the Gagliardo–Nirenberg inequality. Indeed, here is the main result of Chapter 2:

<sup>6</sup>The right hand side  $\mu\partial_x(h\partial_x u)$  of the second equation of (1.60) is the classical viscosity term for Saint-Venant system.

All constant solution of the form  $(h_e, u_e)$  of System (1.60), with  $h_e > 0$ , is asymptotically stable for the norm of the space of the local well-posedness.

Let us mention that we tried here to adapt the general result on hyperbolic-parabolic systems and hyperbolic systems to the case of dispersive Green–Naghdi equations. We could adapt the computations to this particular case. Nevertheless, the treatment of an abstract general case is an open problem. For instance, the case of 2-d Green-Naghdi equation is not treated. Let us also mention that this approach can not be adapted if we consider a 0<sup>th</sup> order or a 4<sup>th</sup> order derivative dissipative term instead of a second order dissipative term. Indeed, in these cases, the order of the estimates of the first category and second category are incoherent in a way that we are not able to obtain the stability estimate. However, 0<sup>th</sup> order dissipative terms can be considered only if they are considered together with a second order dissipative term.

**Remark 1.8** *Let us remark that all along the first part of this thesis, the problem is considered on the whole horizontal space. This is mainly due to two reasons:*

- *The hyperbolic results we try to generalize concern hyperbolic or hyperbolic-parabolic systems defined on the whole space.*
- *No local well-posedness result is available for the Green–Naghdi equations on a bounded domain. Therefore, the investigation of the global well-posedness of the system on a bounded domain, does not seem to be a priority at this stage.*

## 2 Numerical simulation of the free surface incompressible Navier–Stokes equations

The second part of the thesis concerns the numerical treatment of the free surface Navier–Stokes system. Naturally, contrary to the first part of this thesis, we can not consider the problem on the whole space but only on a bounded domain. Nevertheless, the problem we treat is quite general since no constraint on the form of the fluid domain or on the topography is considered. Moreover, friction and surface tension are taken into account and no assumption on the irrotationality of the fluid is considered. From the physical point of view, the problem can be described by the evolution of an incompressible Newtonian fluid in a closed container or on a bottom defined on a bounded space. The container (or the bottom) is fixed and does not change over time. From the computational point of view, we consider a fixed bounded computational domain containing the free-surface fluid. It satisfies the Navier slip or Dirichlet boundary conditions on the parts of boundary which corresponds to the bottom or to the boundary of the container, the free surface boundary condition on the free surface and homogenous Neumann condition elsewhere on the boundary. The fluid satisfies the incompressible Navier–Stokes equations (0.1) inside the fluid domain. As a matter of fact, the mathematical description of the problem is the following: We denote by  $\Omega_t$  the fluid domain at time  $t$ , by  $\Gamma_t$  the free surface at time  $t$  and by  $D$  the computational domain. Then, the fluid velocity

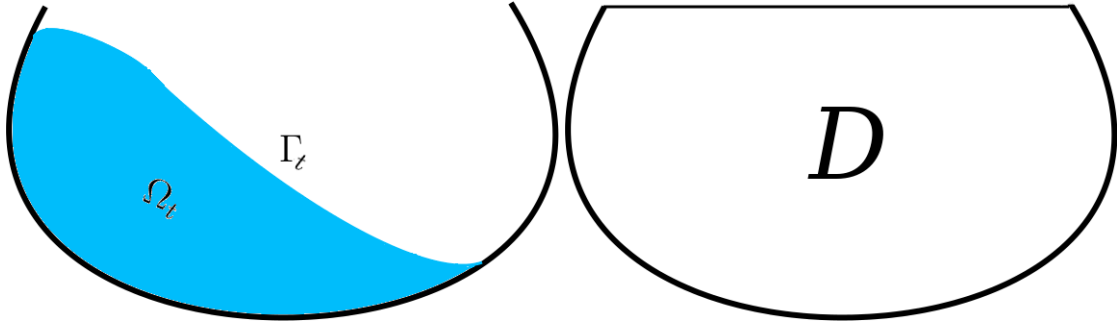


Figure 2.2: An incompressible Newtonian fluid on a curvy bottom (left) and the computational domain  $D$  considered for the simulation of the fluid (right).

$\mathbf{u}$  and the fluid pressure  $p$  satisfy

$$\begin{cases} \rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) - \mu \Delta \mathbf{u} + \nabla p = \rho \mathbf{f} & \text{in } \Omega_t, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega_t, \\ \mathbf{u}(0, x) = \mathbf{u}_0(x) & \text{in } \Omega_0. \end{cases} \quad (2.1)$$

together with the boundary conditions determined by the friction coefficient  $\alpha$ , the surface tension coefficient  $\gamma$  and the atmospheric pressure  $p^a$  as following:

$$\begin{cases} \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \partial \Omega_t \setminus \Gamma_t, \\ [\alpha \mathbf{u} + \mu (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) \mathbf{n}]_{tan} = 0 & \text{on } \partial \Omega_t \setminus \Gamma_t, \\ (\mu (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) - p) \mathbf{n} = -(\gamma \kappa + p^a) \mathbf{n} & \text{in } \Gamma_t, \end{cases} \quad (2.2)$$

and the free surface evolution condition<sup>7</sup>

$$\left\{ (1, \mathbf{u}) \text{ is tangential to the free surface } (t, \Gamma_t), t \geq 0. \right. \quad (2.3)$$

The objective of the second part of the thesis is the numerical resolution of the time dependent problem (2.1)-(2.3) on the time interval  $[0, T]$ . Therefore, the natural approach consists in dividing the time interval  $[0, T]$  into small subintervals and then solving the time discretized problem on each subinterval. Indeed, each iteration of our global algorithm, detailed in Chapter 3, corresponds to the resolution of the time-discretized problem on a subinterval. Besides, the time-discretized fluid equation is resolved only on the fluid domain thanks to an anisotropic mesh adaptation. The algorithm is made by several steps which can be classified in two categories: The first category enables us to advect the fluid domain during the concerned time subinterval, the second category supplies us with the tools which let us to solve the time discretized fluid equation on the advected domain. Let us mention that the basis of the tools of the first category comes from previous works done in [22, 36, 38, 4] whereas the basis of the tools of the second category comes from the work done in [100].

**Level set formulation :** The free surface evolution condition (2.3) here is treated in the frame

<sup>7</sup>Again, the free surface is transported by the fluid velocity.

of level set formulation. We choose this approach since it is good to deal with topology changes and is easy to be implemented on unstructured meshes. Therefore, it is adapted to our problem regarding the use of an adapted unstructured mesh at each iteration. In level set formulation, the fluid free surface is represented by a continuous function  $\phi$  defined on the computational domain  $D$  such that it vanishes on the free surface and only on the free surface, it takes negative values on the domain of fluid and only on the domain of fluid. Therefore, it takes positive values out of the fluid (see Figure (2.3)). In other words, the level set function  $\phi$  satisfies

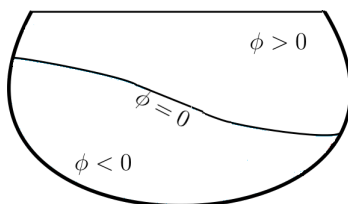


Figure 2.3: The free surface  $\Gamma_t$  corresponds to the 0-level set of  $\phi$  defined on  $D$ .

$$\begin{cases} \phi^{-1}((-\infty, 0)) = \Omega_t, \\ \phi^{-1}(\{0\}) = \Gamma_t, \\ \phi^{-1}((0, +\infty)) = D \setminus \Omega_t. \end{cases} \quad (2.4)$$

Under this representation, the surface evolution condition (2.3) can be replaced by the advection of the level set function  $\phi$  with a flux which takes the same values as the fluid velocity on the free surface. This condition can be described in mathematical terms by the fact that  $\phi$  satisfies

$$\begin{cases} \partial_t \phi + \tilde{\mathbf{u}} \cdot \nabla \phi = 0 & \forall (x, t) \in D \times [0, T], \\ \phi(0, x) = \phi_0(x) & \text{on } D, \end{cases} \quad (2.5)$$

where  $\tilde{\mathbf{u}}$  is a smooth flux defined on  $D$  such that it coincides with the flow velocity on the free surface and  $\phi_0$  is a level set function for the initial fluid domain. Numerically, the advection of the fluid domain at each iteration consists in numerical resolution of (2.5) on the corresponding time subinterval for a convenient flux  $\tilde{\mathbf{u}}$  and a convenient time initial condition. The numerical resolution of (2.5) is done by solving the corresponding characteristic equation using a 4<sup>th</sup> order Runge-Kutta scheme. The construction of the flux  $\tilde{\mathbf{u}}$  and the initial condition are other steps of the algorithm which are explained in following paragraphs.

**Velocity extension:** In order to solve the equation (2.5), we need obviously to construct the flux  $\tilde{\mathbf{u}}$ . This flux must be smooth and defined on the whole domain  $D$  and must coincide with the fluid velocity on the free surface. On the other hand, the fluid velocity  $\mathbf{u}$  is defined only on the fluid domain. Therefore, we need to extend the velocity smoothly outside of the fluid. This extension is done using a Helmholtz operator with non homogenous Dirichlet mixed with homogenous Neumann Dirichlet boundary condition. Indeed, the flux of the advection equation is obtained as the

solution of

$$\begin{cases} -a\Delta\tilde{\mathbf{u}} + \tilde{\mathbf{u}} = 0 & \text{in } D \setminus \Omega_t \\ \nabla\tilde{\mathbf{u}} \cdot \mathbf{n} = 0 & \text{on } \partial D \\ \tilde{\mathbf{u}} = \mathbf{u} & \text{on } \bar{\Omega}_t \end{cases} \quad (2.6)$$

where  $a = 0.5$  is chosen such that  $a$  is neither too small nor too large. In fact, if it is too small, the flux will not be smooth enough for the numerical resolution of (2.5). On the other hand, if  $a$  is too large, the values of the velocity on the interface may interfere with its values inside or outside of the fluid. Therefore, the 0-level set may exceed for instance strictly positive level sets and the level set function may take some positive values on the advected fluid region and may not fit any more in the frame of level set formulation. For the same reason, System (2.6) will lead to a convenient flux for the equation (2.5) only if the Reynolds number associated to the fluid is not too high i.e. if the characteristic fluid velocity is small enough. Therefore, if the fluid Reynolds number is high, we need to solve the Helmholtz problem on the whole computational domain  $D$  with a non-homogenous Dirichlet boundary condition (described by  $\mathbf{u}$ ) on the free surface and homogenous Neumann boundary conditions on the rest of the boundary of the computational domain.

**Distancing:** As mentioned previously, during each iteration of our general algorithm the fluid domain is advected on the corresponding subinterval. In other words, at iteration  $n$  of the algorithm, we solve the advection equation on the subinterval  $[t^n, t^{n+1}]$  of  $[0, T]$  i.e. we solve numerically

$$\begin{cases} \partial_t\phi + \tilde{\mathbf{u}} \cdot \nabla\phi = 0 & \forall (x, t) \in D \times [t^n, t^{n+1}], \\ \phi(t^n, x) = \phi_{t^n}(x) & \text{on } D, \end{cases} \quad (2.7)$$

where  $\phi_{t^n}$  is a level set function for the fluid domain  $\Omega_{t^n}$  at time  $t^n$ . It is well known that in the frame of level set method the initial level set function  $\phi_{t^n}$  must satisfy

$$|\nabla\phi_{t^n}| = 1 \quad (2.8)$$

in the vicinity of the free surface. This condition is necessary because of numerical reasons since it guarantees that the level sets of the initial level set function are neither too close nor too far one from each other. Indeed, if the initial level sets are too close, then they may cut each other during the advection. On the other hand, if they are initially too far from each other, we will maybe find numerically more than one 0-level set line. Therefore, the advected free surface is not any more a null measured subspace of the computational domain. Regarding the fact that the 0-level set line is supposed to represent physically the free surface, consideration of the condition (2.8) is necessary for the initial condition.

For this reason, at each iteration, before advecting the fluid domain along the subinterval  $[t^n, t^{n+1}]$ , we construct an initial level set function  $\phi_{t^n}$  corresponding to the fluid domain  $\Omega_{t^n}$  at time  $t^n$  satisfying (2.8). This step is called distancing and uses the algorithm suggested in [38]. Indeed, the initial level set function the authors suggest, is equal to the signed distance function in the neighborhood of the free surface. Therefore, it satisfies (2.8) in this neighborhood. This function is obtained as a long term limit of an unsteady Eikonal equation.

**Mesh adaptation:** Once the fluid domain is advected during the subinterval  $[t^n, t^{n+1}]$ , we

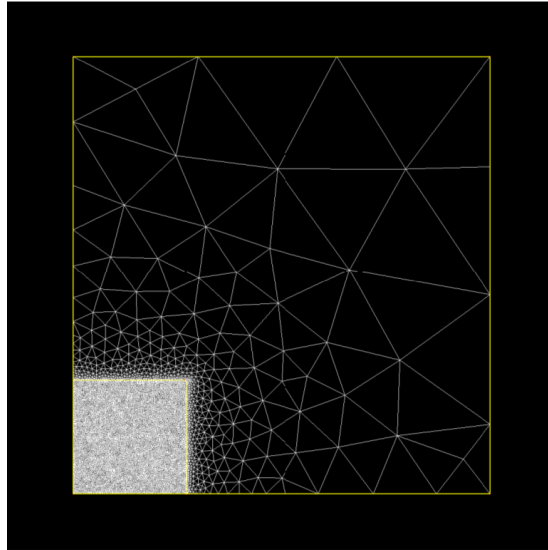


Figure 2.4: Mesh adaptation for the initial fluid domain in the case of dam break problem.

intend to solve the discretized fluid equation on the advected domain. The key tool which enables us to solve the fluid equation only on the fluid domain and not on the whole computational domain is the mesh adaptation. Let us mention that the resolution of the fluid equation is expensive. Therefore, its resolution only on the fluid domain saves computational time. Indeed, this point is the main difference of this work with the work done in [100]. The work in [100] concerns bi-fluid incompressible systems. Therefore the authors solve the fluid equation in the whole computational domain. On the contrary, in our case, the fluid is in contact with vacuum or with a fluid such air we are not interested in the behavior. Nevertheless, the mesh adaptation we consider here, is done on the same principle as in [100]. In fact, after the advection of the domain along  $[t^n, t^{n+1}]$ , we adapt the mesh to the advected domain. This adaptation is done with the aim of minimizing the geometric error, the interpolation error and approximation errors. As shown in Figure 2.4, this adaptation leads to very small elements near the free surface for a precise capture whereas elements out of the fluid domain, far from the free surface and the curvy boundaries of the computational domain, are large since no information in this zone is of interest. Then, inside the domain of fluid, the mesh elements are small enough to get a good approximation of the velocity field and the pressure. Let us also mention that in order to get a good approximation error, it should be a convenience between the smallness of the mesh elements in the fluid and the smallness of the time discretization step. Indeed, a too small time discretization step compared to the mesh elements' size, may lead to a large approximation error.

**Resolution of the fluid equation:** The mentioned steps considered together, are sufficient to enable us to solve the time-discretized Navier–Stokes equation on the fluid domain. This time discretization is done by the method of characteristics. Therefore, at each iteration, the last step consists in solving a generalized steady stokes problem. This problem is resolved by considering the associated variational formulation. Besides, slip boundary conditions are then taken into account by adding a penalization term. Then, we use a convenient finite elements method to discretize the problem and solve the corresponding linear system by Uzawa algorithm.



Chapter 3 contains some numerical results and comparisons with results of bi-fluid systems .

## **Part I**

# **Some results on a shallow water model called the Green-Naghdi equations**



# Chapter 1

## The symmetric structure of Green–Naghdi type equations

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### 1.1 Introduction

Incompressible Euler equations and water waves problem model free surface incompressible fluids under the influence of the gravity. The complexity of these systems leads to consider averaged geophysical models to describe coastal oceanic flows. We focus on a particular type of these reduced models called the Green–Naghdi type model [62], which writes

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(hu^2) + \partial_x(gh^2/2 + \alpha h^2 \ddot{h}) = 0. \end{cases} \quad (1.1.1)$$

The unknown  $h$  represents the fluid height and is assumed to be positive, while  $u$  is the averaged horizontal velocity. Moreover, the material derivative  $\dot{(\cdot)}$  is defined by  $\dot{(\cdot)} = \partial_t(\cdot) + u\partial_x(\cdot)$ ,  $\alpha$  is a positive real number and  $g$  is the gravity constant.

If  $\alpha = 0$ , system (1.1.1) is hyperbolic and equivalent to the Saint-Venant equations (and to the barotropic Euler equations). System (1.1.1) with  $\alpha \neq 0$  is different from the Saint-Venant system by the dispersive term  $\partial_x (\alpha h^2 \ddot{h})$ . It has been rigorously derived for  $\alpha = \frac{1}{3}$  from the water wave problem for irrotational flows by Li [78] and by Alvarez and Lannes [5]. In [68], Ionescu derived the same system by a variational method considering the Lagrangian formulation of the irrotational incompressible Euler equations. In [20], the authors obtain (1.1.1) for  $\alpha = \frac{1}{4}$  by a different but a formal method without any hypothesis on the irrotationality of the fluid.

It is worth remarking that (1.1.1) admits the following conservation law (see for instance [54, 55]),

$$\partial_t E + \partial_x (u(E + p)) = 0, \quad (1.1.2)$$

where the energy  $E$  is defined by

$$E = gh^2/2 + hu^2/2 + \alpha h^3(\partial_x u)^2/2, \quad (1.1.3)$$

and  $p$  by

$$p = gh^2/2 + \alpha h^2 \ddot{h}. \quad (1.1.4)$$

Contrary to the case of hyperbolic systems, the energy  $E$  and the pressure  $p$  are not functions of the unknown but smooth operators acting on the space of functions the unknown belongs to.

The aim of this chapter is to extend the notion of symmetry classically defined for hyperbolic systems, to more general type of equations, including the Green–Naghdi model (1.1.1). We first recall the definition of symmetrizability for hyperbolic systems and its relation with the existence of a convex entropy.

### 1.1.1 Symmetric structure of hyperbolic systems of conservation laws

Let us provide a brief review on the symmetrization of hyperbolic systems of conservation laws. We consider the system

$$\partial_t U + \partial_x F(U) = 0 \quad (1.1.5)$$

where the flux  $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$ ,  $N \geq 1$ , is a smooth function. We only consider in the sequel smooth solutions  $U : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^N$ . The hyperbolic system (1.1.5) is called *symmetrizable* if there exists a change of variable  $U \mapsto Q$  such that (1.1.5) is equivalent to

$$A_0(Q)\partial_t Q + A_1(Q)\partial_x Q = 0, \quad (1.1.6)$$

where  $A_0(Q)$  is a symmetric positive definite matrix and  $A_1(Q)$  is a symmetric one.

Moreover, a pair of smooth functions  $(E, P)$  from  $\mathbb{R}^N$  to  $\mathbb{R}$  such that  $\nabla_U^2 E(U)$  positive definite is an *entropy pair* for system (1.1.5) if any solution  $U$  to (1.1.5) satisfies

$$\partial_t E(U) + \partial_x P(U) = 0, \quad (1.1.7)$$

or equivalently if

$$(\nabla_U F(U))^T \nabla_U E(U) = \nabla_U P(U). \quad (1.1.8)$$

Using Poincaré's theorem [25], the latter condition, which is nothing but an integrability condition,

is equivalent to the symmetry condition

$$\nabla^2 E(U) \nabla F(U) = (\nabla^2 E(U) \nabla F(U))^T.$$

The following classical proposition illustrates how the notions of entropy and symmetry are related. We also provide the associated proof in order to compare it to the generalized case of the next section.

**Proposition 1.1.1** [61, 50, 83, 79, 15] *Let us assume that the hyperbolic system (1.1.5) admits an entropy pair  $(E, P)$ . Then, it is symmetrizable under any change of variable  $U \mapsto V$  under the form*

$$A_0(V) \partial_t V + A_1(V) \partial_x V = 0,$$

where

$$A_0(V) = (\nabla_V U)^T \nabla_U^2 E(U) \nabla_V U \text{ and } A_1(V) = (\nabla_V U)^T \nabla_U^2 E(U) \nabla_U F(U) \nabla_V U. \quad (1.1.9)$$

**Proof** *Considering a change of variable  $U \mapsto V$ , System (1.1.5) becomes*

$$\nabla_V U \partial_t V + \nabla_U F(U) \nabla_V U \partial_x V = 0. \quad (1.1.10)$$

*We now apply  $(\nabla_V U)^T \nabla_U^2 E(U)$  to the left-hand side and obtain*

$$(\nabla_V U)^T \nabla_U^2 E(U) \nabla_V U \partial_t V + (\nabla_V U)^T \nabla_U^2 E(U) \nabla_U F(U) \nabla_V U \partial_x V = 0. \quad (1.1.11)$$

*The symmetric matrix  $A_0(V) = (\nabla_V U)^T \nabla_U^2 E(U) \nabla_V U$  is positive definite due to the strict convexity of the entropy. Therefore, we just need to prove the symmetry of  $\nabla_U^2 E(U) \nabla_U F(U)$ . To do so, we consider the change of variable  $U \mapsto Q$  where  $Q$  is the entropy variable, i.e.*

$$Q = \nabla_U E(U). \quad (1.1.12)$$

*This change of variable is valid since  $E$  is strictly convex. As a consequence, the Legendre transform  $E^*$  of  $E$  defined by*

$$E^*(Q) = Q \cdot (\nabla_U E)^{-1}(Q) - E((\nabla_U E)^{-1}(Q)), \quad (1.1.13)$$

*satisfies*

$$U = \nabla_Q E^*(Q). \quad (1.1.14)$$

*Let us now define the scalar function  $\hat{P}$  by*

$$\hat{P}(Q) = Q \cdot F(U(Q)) - P(U(Q)). \quad (1.1.15)$$

*Then, we use relation (1.1.8) to get*

$$\nabla_Q \hat{P}(Q) = F(U). \quad (1.1.16)$$

*Hence,*

$$\nabla_U^2 E(U) \nabla_U F(U) = \nabla_U^2 E(U) \nabla_Q^2 \hat{P}(Q) \nabla_U Q = \nabla_U^2 E(U) \nabla_Q^2 \hat{P}(Q) \nabla_U^2 E(U)$$

is symmetric. ■

Gathering (1.1.14) and (1.1.16), we remark that (1.1.5) is equivalent to

$$\partial_t (\nabla_Q E^*(Q)) + \partial_x (\nabla_Q \hat{P}(Q)) = 0.$$

In other words, system (1.1.5) admits a so-called *Godunov structure* [61]. Note that such a structure can be used to deduce the existence of an entropy pair since it implies the symmetry of  $\nabla_U^2 E(U) \nabla_U F(U)$ , and thus the integrability of  $(\nabla_U F(U))^T \nabla_U E(U)$ . Let us consider a system of the form (1.1.5) which admits an entropy pair  $(E, P)$ . Assume that there exists a decomposition of the unknown  $U = (U_1, U_2)$  such that the application  $\phi \mapsto \nabla_{U_2} E(U_1, \phi)$  is invertible. Then, the change of variable

$$U \mapsto V = (U_1, \nabla_{U_2} E(U_1, U_2))$$

is particularly interesting since  $A_0(V)$  is block diagonal (this is a direct consequence of the expression (1.1.9) of  $A_0(V)$ ). Indeed, this can be useful to deduce equivalent normal forms of system (1.1.5) when studying for instance parabolic regularizations [71].

In the case of the Saint-Venant equations, with  $U = (h, hu)$  and  $E = gh^2/2 + hu^2/2$ , let us compare two symmetric forms. If we consider the entropy variable  $Q = \nabla_U E(U) = (gh - u^2/2, u)$ , one has

$$A_0(Q) \partial_t Q + A_1(Q) \partial_x Q = 0,$$

where

$$A_0(Q) = \frac{1}{g} \begin{pmatrix} 1 & u \\ u & gh + u^2 \end{pmatrix} \quad \text{and} \quad A_1(Q) = \frac{1}{g} \begin{pmatrix} u & gh + u^2 \\ gh + u^2 & 3ghu + u^3 \end{pmatrix}.$$

On the other hand, using the change of variable  $U \mapsto V = (h, \nabla_{hu} E(U)) = (h, u)$ , the Saint-Venant equations become

$$A_0(V) \partial_t V + A_1(V) \partial_x V = 0,$$

with

$$A_0(V) = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \quad \text{and} \quad A_1(V) = \begin{pmatrix} gu & gh \\ gh & hu \end{pmatrix}.$$

The notion of symmetrizability is crucial to be useful to prove the local well-posedness of hyperbolic systems (see [15] for instance) as well as the stability of constant solutions of hyperbolic systems with dissipative terms [64, 108, 71, 95]. Let us now recall some properties of the Green–Naghdi equations.

### 1.1.2 Hamiltonian structure of the Green–Naghdi equations

Following Li [78], let us consider the unknown  $U = (h, m)$  defined by

$$m = \mathcal{L}_h(u) = hu - \alpha (h^3 u_x)_x. \quad (1.1.17)$$

The change of variable  $(h, u) \mapsto (h, m)$  is valid since the Sturm–Liouville operator  $\mathcal{L}_h$  is an isomorphism from  $\mathbb{H}^s(\mathbb{R})$  to  $\mathbb{H}^{s-2}(\mathbb{R})$ , for  $s \geq 2$ , due to the fact that  $h$  is positively bounded by

below<sup>1</sup>. Let us also mention that the variable  $m$  has been used in [55] to define the generalized velocity  $k = \frac{m}{h}$ .

We illustrate in the following proposition the Hamiltonian structure of the Green–Naghdi equations inherited from the structure of the water waves problem. To state this result, we adopt classical notations of variational derivatives and second variations (see for instance [56, 87]).

**Proposition 1.1.2** [78] *Let  $\bar{h} > 0$  be a real constant. System (1.1.1) is equivalent to*

$$\partial_t U = \mathcal{J}(U) \delta \mathcal{H}_{\bar{h}}(U), \quad (1.1.18)$$

where

$$\begin{aligned} U &= (h, m) = (h, \mathcal{L}_h(u)), \\ \mathcal{H}_{\bar{h}}(h, u) &= \int_{\mathbb{R}} gh(h - \bar{h})/2 + hu^2/2 + \alpha h^3(u_x)^2/2, \end{aligned} \quad (1.1.19)$$

and

$$\mathcal{J}(U) = - \begin{pmatrix} 0 & \partial_x(h(\cdot)) \\ h\partial_x & \partial_x(m(\cdot)) + m\partial_x \end{pmatrix}. \quad (1.1.20)$$

More precisely, we have for all test functions  $(\phi, \psi)$

$$\mathcal{J}(U) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = - \begin{pmatrix} \partial_x(h\psi) \\ h\partial_x\phi + \partial_x(m\psi) + m\partial_x\psi \end{pmatrix}. \quad (1.1.21)$$

By classical calculations, we have

$$\delta \mathcal{H}_{\bar{h}}(U) = (\sigma, u),$$

with

$$\sigma = gh - g\bar{h}/2 - u^2/2 - \frac{3}{2}\alpha h^2(u_x)^2. \quad (1.1.22)$$

The variable  $\sigma$  has been used in [54] for the canonical representation of the Green–Naghdi equations.

The function  $\mathcal{H}_{\bar{h}}$  is the integral of the relative energy

$$E_{\bar{h}} = gh(h - \bar{h})/2 + hu^2/2 + \alpha h^3(u_x)^2/2, \quad (1.1.23)$$

which, following the same calculations as those which lead to (2.1.2), satisfies the conservation law

$$\partial_t E_{\bar{h}} + \partial_x(u(E_{\bar{h}} + p)) = 0, \quad (1.1.24)$$

where  $p$  is given by (1.1.4). The first consequence is the conservation of the Hamiltonian  $\mathcal{H}_{\bar{h}}$  over time by integration in space<sup>2</sup>. This important property can also be obtained using the Hamiltonian structure (1.1.18) of the system and the fact that  $\mathcal{J}(U)$  is a skew-symmetric operator acting on the

<sup>1</sup>Operator  $\mathcal{L}_h$  is a diffeomorphism from  $\mathbb{H}^{s+2}(\mathbb{R})$  to  $\mathbb{H}^s(\mathbb{R})$  if  $h$  is close enough to a constant state  $\bar{h}$  for the norm  $\mathbb{H}^n$  with  $n \geq 2$ . This assumption is considered in Section 1.3.1 while symmetrizing the Green–Naghdi equations.

<sup>2</sup>It has been shown in [78, 69] that the Green–Naghdi equations endowed with the unknown  $(h - \bar{h}, u)$  are well-posed in  $C([0, T]; \mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R}))$  for some  $T > 0$  and  $s \geq 2$ . Hence,  $u$  is a continuous function vanishing at infinity using the Sobolev embedding theorem.



space of vector-valued functions whose second component converges to 0 at infinity. Hence,

$$\frac{d}{dt}\mathcal{H}(U(t)) = \int_{\mathbb{R}} \delta\mathcal{H}(U) \cdot \partial_t U = \int_{\mathbb{R}} \delta\mathcal{H}(U) \cdot \mathcal{J}(U)\delta\mathcal{H}(U) = 0.$$

### 1.1.3 General idea

The generalization of the notion of symmetrizability to dispersive perturbations of hyperbolic systems has been studied by several authors. For instance, Gavriluk and Gouin in [53] (see also [14]) investigate the symmetric structure of Euler–Korteweg models and some  $p$ -systems. Similar ideas can be partially adapted to some generalized  $p$ -systems like bubbly fluid equations and to modified Lagrangian Green–Naghdi [52].

These generalizations are investigated with the hope of extending the results on hyperbolic systems to their dispersive perturbations. In Chapter 2 (published as [72]), we use the generalized symmetric structure presented in this work (more precisely in Section 1.3) to prove the asymptotic stability of constant solutions of the Green–Naghdi equations with viscosity. Let us note that the symmetric structure presented here for Green–Naghdi equations holds only in a small enough neighborhood of constant solutions, this is to say that we consider the symmetrizability as a local notion. As we will see in the following chapter, this is not an obstacle to prove the stability of equilibria since the solution of the viscous Green–Naghdi equations remains close to equilibria for initial data close enough to these solutions.

In this chapter, we consider general systems written under the following conservative form

$$\partial_t U + \partial_x F(U) = 0. \tag{1.1.25}$$

The unknown  $U$  is supposed to belong to  $C([0, T]; \mathcal{A})$  for some  $T > 0$  where  $\mathcal{A}$  is a Banach subspace of continuous functions of  $\mathbb{L}^2(\mathbb{R}, \mathbb{R}^N)$  converging to 0 at infinity. Let us note that  $F$  is not anymore a function of  $\mathbb{R}^N$  but a smooth application defined from  $\mathcal{A}$  to  $\mathcal{A}$ . This is actually the case for the Green–Naghdi equations. As we will see in Subsection 1.3.1, the Green–Naghdi equations under the Hamiltonian variable  $(h, m)$  fits the abstract form (1.1.25) with no loss of derivatives through  $F$ .

For sake of simplicity, we mainly consider the one-dimensional problem (1.1.25). We provide some generalizations of the previous notions used in the hyperbolic case, symmetrizability, Godunov structure, and relate it, in the case of Green–Naghdi equations, to the existence of a Hamiltonian structure. The extension of the results of the next section to the multi-dimensional case will be addressed at the end of the next section. Section 1.3 is devoted to the particular case of the Green–Naghdi equations.

## 1.2 Weak symmetric structure

The aim of this part is to provide a sufficient condition for the symmetrizability of System (1.1.25) under any variable. First, we provide an adapted notion of symmetrizability and define the Legendre transform of a variational function. Then, we will see how a convenient strictly convex function can lead to the symmetrizability.

The notion of symmetry we consider here is based on the  $\mathbb{L}^2$  scalar product, and not on the scalar product of  $\mathbb{R}^N$ . More precisely, an operator  $\mathcal{F} : \mathcal{A} \subset \mathbb{L}^2(\mathbb{R}, \mathbb{R}^N) \rightarrow \mathbb{L}^2(\mathbb{R}, \mathbb{R}^N)$  is said to be *symmetric* if

$$\int_{\mathbb{R}} \phi \cdot \mathcal{F}(\psi) = \int_{\mathbb{R}} \mathcal{F}(\phi) \cdot \psi \quad \forall \phi, \psi \in \mathcal{A},$$

and *positive definite* if, for all  $\phi \in \mathcal{A} \setminus \{0\}$ ,  $\int_{\mathbb{R}} \phi \cdot \mathcal{F}\phi > 0$ .

**Definition 1.2.1 (Weak symmetrizability)** System (1.1.25) is called *weakly symmetrizable* if there exists a change of variable  $U \mapsto V$  such that (1.1.25) is equivalent to

$$A_0(V)\partial_t V + A_1(V)\partial_x(V) = 0, \quad (1.2.1)$$

where  $A_0(V)$  is a symmetric positive definite operator and  $A_1(V)$  is a symmetric one.

**Definition 1.2.2 (Legendre transform)** Let  $\Omega$  be an open convex subset of a Banach space  $\mathcal{A} \subset \mathbb{L}^2(\mathbb{R}, \mathbb{R}^N)$  and consider a smooth application  $E : \Omega \rightarrow \mathbb{L}^1(\mathbb{R})$  together with the variational function  $\mathcal{H} : \Omega \rightarrow \mathbb{R}$  defined by

$$\mathcal{H}(U) = \int_{\mathbb{R}} E(U).$$

Assume that there exists an open set  $\Omega^*$  of a Banach space  $\mathcal{B} \subset \mathbb{L}^2(\mathbb{R}, \mathbb{R}^N)$  such that the application

$$\delta_U \mathcal{H} : \begin{cases} \Omega \rightarrow \Omega^* \\ U \mapsto \delta_U \mathcal{H}(U) \end{cases}$$

is a diffeomorphism. The Legendre transform  $\mathcal{H}^*$  of  $\mathcal{H}$  is defined on  $\Omega^*$  by

$$\mathcal{H}^*(Q) = \int_{\mathbb{R}} Q \cdot (\delta_U \mathcal{H})^{-1}(Q) - E((\delta_U \mathcal{H})^{-1}(Q)). \quad (1.2.2)$$

Let us note that the Legendre transform  $\mathcal{H}^*$  of a function  $\mathcal{H}$  satisfying the assumptions of Definition 1.2.1, also satisfies the assumptions of the definition. Moreover, basic computations show that the Legendre transform of  $\mathcal{H}^*$  is nothing but  $\mathcal{H}$ . In other words,

$$\mathcal{H}^{**} = \mathcal{H}.$$

We now state one of the fundamental properties of the Legendre transform of a strictly convex variational function (i.e. a function with a definite positive second variation). Let us remark here that contrary to the finite dimensional case, the variational derivative of a smooth strictly convex function is not necessarily a diffeomorphism. Therefore, we still need to assume in the sequel that its variational derivative defines a diffeomorphism as in Definition 1.2.2.

**Proposition 1.2.3** *The Legendre transform  $\mathcal{H}^*$  of a strictly convex function  $\mathcal{H}$  which satisfies the assumptions of Definition 1.2.2 is strictly convex.*

**Proof** *Considering the expression (1.2.2) of the Legendre transform, we remark that*

$$\delta_Q \mathcal{H}^*(Q) = (\delta_U \mathcal{H})^{-1}(Q).$$

In other words,

$$Q = \delta_U \mathcal{H}(U) \iff U = \delta_Q \mathcal{H}^*(Q).$$

Hence, the definite positivity of the second variation of  $\mathcal{H}$  implies the definite positivity of the second variation of  $\mathcal{H}^*$ . More precisely, we have

$$\delta_U^2 \mathcal{H}(U) = D_U Q(U),$$

and

$$\delta_Q^2 \mathcal{H}^*(Q) = D_Q U(Q).$$

Therefore,

$$\delta_U^2 \mathcal{H}(U) = (\delta_Q^2 \mathcal{H}^*(Q))^{-1}.$$

■

The following theorem provides the connection between the convexity of  $\mathcal{H}$  and the existence of a general Godunov structure (this notion has been introduced in [54] and is recalled in the following statement).

**Theorem 1.2.4** *We use the same notations and assumptions as in Definition 1.2.2. Assume that  $\mathcal{H}$  is strictly convex on  $\Omega$ . If*

$$\delta_U^2 \mathcal{H}(U) D_U F(U) \text{ is symmetric,} \quad (1.2.3)$$

*then system (1.1.25) admits a general Godunov structure: there exists a change of variable  $U \mapsto Q$  defined on  $\Omega$  and a function  $R$ , together with  $\mathcal{R}(Q) = \int_{\mathbb{R}} R(Q)$ , such that system (1.1.25) is equivalent to*

$$\partial_t (\delta_Q \mathcal{H}^*(Q)) + \partial_x (\delta_Q \mathcal{R}(Q)) = 0, \quad (1.2.4)$$

*as long as the solution  $U$  remains in  $\Omega$ .*

**Proof** *Let us first consider the change of variable  $U \mapsto Q$  defined by*

$$Q = \delta_U \mathcal{H}(U), \quad (1.2.5)$$

*or equivalently by*

$$U = \delta_Q \mathcal{H}^*(Q). \quad (1.2.6)$$

*Considering the fact that  $\delta_U^2 \mathcal{H}(U) D_U F(U)$  is symmetric on the open convex set  $\Omega$ , there exists, by Poincaré's theorem [25], a differentiable application  $\mathcal{N} : \Omega \rightarrow \mathbb{R}$  such that*

$$D_U \mathcal{N}(U) \phi = \int_{\mathbb{R}} \delta_U \mathcal{H}(U) \cdot D_U F(U) \phi \quad \forall \phi \in \mathcal{A}. \quad (1.2.7)$$

*We now define the function  $\mathcal{R}$  by*

$$\mathcal{R}(Q) = \int_{\mathbb{R}} Q \cdot F(U(Q)) - \mathcal{N}(U(Q)). \quad (1.2.8)$$

We differentiate (1.2.8) and take the action on a test function  $\psi$ . This leads to

$$D_Q \mathcal{R}(Q)\psi = \int_{\mathbb{R}} F(U(Q)) \cdot \psi + Q \cdot D_U F(U) D_Q U(\psi) - D_U \mathcal{N}(U) D_Q U(\psi).$$

Then, we have by (1.2.5),

$$D_Q \mathcal{R}(Q)\psi = \int_{\mathbb{R}} F(U(Q)) \cdot \psi + \delta_U \mathcal{H}(U) \cdot D_U F(U) D_Q U(\psi) - D_U \mathcal{N}(U) D_Q U(\psi).$$

Finally, using (1.2.7), we find

$$D_Q \mathcal{R}(Q)\psi = \int_{\mathbb{R}} F(U(Q)) \cdot \psi,$$

or equivalently

$$\delta_Q \mathcal{R}(Q) = F(U(Q)). \quad (1.2.9)$$

Considering system (1.1.25) together with (1.2.6) and (1.2.9), we obtain (1.2.4). ■

The general Godunov structure (1.2.4) directly implies the weak symmetrizability of system (1.1.25) with respect to the unknown  $Q$ , since it lets us write the system under

$$\delta_Q^2 \mathcal{H}^*(Q) \partial_t Q + \delta_Q^2 \mathcal{R}(Q) \partial_x Q = 0.$$

Let us now state in the following theorem, other consequences of a general Godunov structure for system (1.1.25).

**Theorem 1.2.5** *We use the same notations and assumptions as in Definition 1.2.2. Assume that  $\mathcal{H}^*$  is strictly convex on  $\Omega^*$ . Then, the general Godunov system (1.2.4) is weakly symmetrizable for any change of variable  $Q \mapsto V$ . More precisely, it is written under the form*

$$A_0(V) \partial_t V + A_1(V) \partial_x V = 0,$$

where the symmetric operators are given by

$$A_0(V) = (D_V U)^T \delta_U^2 \mathcal{H}(U) D_V U, \quad (1.2.10)$$

$$A_1(V) = (D_V U)^T \delta_U^2 \mathcal{H}(U) D_U F(U) D_V U, \quad (1.2.11)$$

with  $U = \delta_Q \mathcal{H}^*(Q)$ ,  $F(U) = \delta_Q \mathcal{R}(Q)$ , and  $\mathcal{H}$  the Legendre transform of  $\mathcal{H}^*$ .

**Proof** Setting  $U = \delta_Q \mathcal{H}^*(Q)$  and  $F(U) = \delta_Q \mathcal{R}(Q(U))$ , system (1.2.4) writes

$$\partial_t U + \partial_x F(U) = 0. \quad (1.2.12)$$

We now consider the change of variable  $U \mapsto V$  and write (1.2.12) under

$$D_V U \partial_t V + D_U F(U) D_V U \partial_x V = 0. \quad (1.2.13)$$

Then, we denote by  $\mathcal{H}$  the Legendre transform of  $\mathcal{H}^*$  and take the left side action of  $(D_V U)^T \delta_U^2 \mathcal{H}(U)$

on (1.2.13). This leads to

$$(D_V U)^T \delta_U^2 \mathcal{H}(U) D_V U \partial_t V + (D_V U)^T \delta_U^2 \mathcal{H}(U) D_U F(U) D_V U \partial_x V = 0. \quad (1.2.14)$$

Hence, the theorem is proved if we show that  $\delta_U^2 \mathcal{H}(U) D_U F(U)$  is symmetric. To do so, let us differentiate the following application

$$\mathcal{N}(U) := \int_{\mathbb{R}} Q(U) \cdot F(U) - \mathcal{R}(Q(U)),$$

and find

$$D_U \mathcal{N}(U) \phi = \int_{\mathbb{R}} (F(U) - \delta_Q \mathcal{R}(Q)) \cdot D_U Q \phi + Q \cdot D_U F(U) \phi \quad \forall \phi \in \mathcal{A}.$$

On the other hand,  $\delta_Q \mathcal{R}(Q) = F(U)$  and  $Q = \delta_U \mathcal{H}(U)$ . Therefore,

$$D_U \mathcal{N}(U) \phi = \int_{\mathbb{R}} \delta_U \mathcal{H}(U) \cdot D_U F(U) \phi \quad \forall \phi \in \mathcal{A}.$$

The symmetry of the operator  $\delta_U^2 \mathcal{H}(U) D_U F(U)$  is just a consequence of the integrability of  $\phi \mapsto \int_{\mathbb{R}} \delta_U \mathcal{H}(U) \cdot D_U F(U) \phi$ . ■

Let us gather the two previous results in the following corollary.

**Corollary 1.2.6** *We use the same notations and assumptions as in Definition 1.2.2. Assume that  $\mathcal{H}$  is strictly convex on  $\Omega$ . The three following statements are equivalent:*

1. System (1.1.25) owns a general Godunov structure using the Legendre transform  $\mathcal{H}^*$  of  $\mathcal{H}$ .
2. The operator  $\delta_U^2 \mathcal{H}(U) D_U F(U)$  is symmetric.
3. System (1.1.25) is weakly symmetrizable under any change of variable  $U \mapsto V$  with the expressions (1.2.10) and (1.2.11) for symmetric operators.

One can see that these relations are very similar to the case of hyperbolic systems. It remains to check whether or not one can add to these statements the existence of a conservation law.

**Proposition 1.2.7** *Assume any of the three statements of Corollary 1.2.6. Assume also that there exists a pair of functions  $(E, R)$  which defines  $\mathcal{H}(U) = \int_{\mathbb{R}} E(U)$  and  $\mathcal{R}(Q) = \int_{\mathbb{R}} R(Q)$  describing the general Godunov form (1.2.4) of system (1.1.25). Then, the solution  $U$  to system (1.1.25) satisfies*

$$\int_{\mathbb{R}} (\partial_t E(U) + \partial_x N(U)) = 0, \quad (1.2.15)$$

where

$$N(U) = Q(U) \cdot F(U) - R(Q(U)).$$

**Proof** *We take the left action of  $D_U E(U)$  on (1.1.25) and find*

$$D_U E(U) \partial_t U + D_U E(U) D_U F(U) \partial_x U = 0. \quad (1.2.16)$$

We then take the integral on  $\mathbb{R}$  and use the definition of the variational derivative to get

$$\int_{\mathbb{R}} D_U E(U) \partial_t U + \delta_U \mathcal{H}(U) \cdot D_U F(U) \partial_x U = 0. \quad (1.2.17)$$

On the other hand, as done in the proof of Theorem 1.2.5, we have

$$\int_{\mathbb{R}} D_U N(U) \phi = \int_{\mathbb{R}} \delta_U \mathcal{H}(U) \cdot D_U F(U) \phi \quad \forall \phi \in \mathcal{A}. \quad (1.2.18)$$

Therefore,

$$\int_{\mathbb{R}} D_U N(U) \partial_x U = \int_{\mathbb{R}} \delta_U \mathcal{H}(U) \cdot D_U F(U) \partial_x U. \quad (1.2.19)$$

Hence, we can write (1.2.17) as

$$\int_{\mathbb{R}} D_U E(U) \partial_t U + D_U N(U) \partial_x U = 0, \quad (1.2.20)$$

which provides (1.2.15). ■

Let us remark that contrary to the case of hyperbolic systems, the reciprocal of Proposition 1.2.7 is false since (1.2.18) and (1.2.19) are not any more equivalent. Indeed,  $\delta_U \mathcal{H}(U)$  as well as the components of  $D_U N(U)$  depend not only on  $U$  but also on its derivatives.

Let us also remark that the notion of symmetry introduced for (1.1.25) corresponds to the symmetry for the  $\mathbb{L}^2$  scalar product and is a weak notion while the symmetry of hyperbolic system is a strong one. This is due to the fact that the assertion

$$\int_{\mathbb{R}} \phi \cdot \mathcal{F} \psi = \int_{\mathbb{R}} \mathcal{F} \psi \cdot \phi \quad \forall \phi, \psi \text{ test functions}, \quad (1.2.21)$$

does not imply

$$\phi \cdot \mathcal{F} \psi = \mathcal{F} \psi \cdot \phi \quad \forall \phi, \psi \text{ test functions}. \quad (1.2.22)$$

Therefore, the weak symmetry of the system does not lead to a conservation law but to an equality of the form (1.2.15). However, as we can see in [72], this definition is strong enough to allow us to generalize the hyperbolic techniques to the Green–Naghdi equations. In fact, if we considered a stronger definition like the one deduced in (1.2.22) for the symmetric operator and a stronger condition such as the symmetry of  $D_U^2 E(U) D_U F(U)$  for Theorem 1.2.4, we would obtain a conservation law in addition to similar theorems. However, less equations would be covered (i.e. the result would be less general). Moreover, the strong symmetry of  $D_U^2 E(U) D_U F(U)$  is more tedious to be checked than the weak symmetry of  $\delta_U^2 \mathcal{H}(U) D_U F(U)$ . We end this section by two remarks. The first one is about an interesting change of variable (similarly to Remark 1.1.1) while the second deals with the multi-dimensional case. Let us consider system (1.1.25) with a variational function  $\mathcal{H}$  such that  $\delta_U^2 \mathcal{H}(U) D_U F(U)$  is a symmetric operator. Assume that there exists a decomposition of the unknown  $U = (U_1, U_2)$  such that the application  $\phi \mapsto \delta_{U_2} \mathcal{H}(U_1, \phi)$  is invertible. Then, the change of variable

$$U \mapsto (V_1, V_2) = (U_1, \delta_{U_2} \mathcal{H}(U_1, U_2)) \quad (1.2.23)$$

is very interesting since it leads to a block diagonal structure of the matrix operator  $A_0(V)$  defined

by (1.2.10). Using this expression, we have

$$A_0(V) = \begin{pmatrix} A_0^{11} & A_0^{12} \\ A_0^{21} & A_0^{22} \end{pmatrix},$$

where

$$\begin{aligned} A_0^{11} &= \delta_{U_1}^2 \mathcal{H}(U) + \delta_{U_2 U_1}^2 \mathcal{H}(U) D_{V_1} U_2 + (D_{V_1} U_2)^T \delta_{U_1 U_2}^2 \mathcal{H}(U) + (D_{V_1} U_2)^T \delta_{U_2}^2 \mathcal{H}(U) D_{V_1} U_2 \\ A_0^{12} &= \delta_{U_2 U_1}^2 \mathcal{H}(U) D_{V_2} U_2 + (D_{V_1} U_2)^T \delta_{U_2}^2 \mathcal{H}(U) D_{V_2} U_2, \\ A_0^{21} &= (A_0^{12})^T = (D_{V_2} U_2)^T \delta_{U_1 U_2}^2 \mathcal{H}(U) + (D_{V_2} U_2)^T \delta_{U_2}^2 \mathcal{H}(U) D_{V_1} U_2, \\ A_0^{22} &= (D_{V_2} U_2)^T \delta_{U_2}^2 \mathcal{H}(U) D_{V_2} U_2. \end{aligned}$$

Therefore,  $A_0(V)$  is block diagonal since

$$A_0^{21} = (A_0^{12})^T = (D_{V_2} U_2)^T \delta_{U_1 U_2}^2 \mathcal{H}(U) + (D_{V_2} U_2)^T \delta_{U_2}^2 \mathcal{H}(U) D_{V_1} U_2 = 0.$$

This is due to the fact that (1.2.23) implies that

$$\begin{aligned} (D_{V_2} U_2)^T \delta_{U_1 U_2}^2 \mathcal{H}(U) + (D_{V_2} U_2)^T \delta_{U_2}^2 \mathcal{H}(U) D_{V_1} U_2 \\ &= (D_{V_2} U_2)^T D_{U_1} V_2 + (D_{V_2} U_2)^T D_{U_2} V_2 D_{V_1} U_2 \\ &= (D_{V_2} U_2)^T D_{U_1} V_2 D_{V_1} U_1 + (D_{V_2} U_2)^T D_{U_2} V_2 D_{V_1} U_2 \\ &= (D_{V_2} U_2)^T (D_{U_1} V_2 D_{V_1} U_1 + D_{U_2} V_2 D_{V_1} U_2) \\ &= (D_{V_2} U_2)^T D_{V_1} V_2 = 0. \end{aligned}$$

Let us consider the multi-dimensional version of system (1.1.25)

$$\partial_t U + \sum_{i=1}^n \partial_{x_i} F_i(U) = 0. \quad (1.2.24)$$

One can easily extend the previous results. Consider a variational function  $\mathcal{H}(U) = \int_{\mathbb{R}} E(U)$  which admits a Legendre transform, as in Definition 1.2.2. Then the following three statements are equivalent:

1. The operators  $\delta_U^2 \mathcal{H}(U) D_U F_i(U)$  are symmetric for all  $i \in \{1, \dots, n\}$ .
2. System (1.2.24) admits a general Godunov structure. i.e. there exist functions  $R_i$  and the associated  $\mathcal{R}_i(Q) = \int_{\mathbb{R}} R_i(Q)$  such that system (1.2.24) is equivalent to

$$\partial_t (\delta_Q \mathcal{H}^*(Q)) + \sum_{i=1}^n \partial_{x_i} (\delta_Q \mathcal{R}_i(Q)) = 0.$$

3. System (1.2.24) is symmetrizable under any change of variable  $U \mapsto V$  i.e. it is equivalent to

$$A_0(V) \partial_t V + \sum_{i=1}^n A_i(V) \partial_{x_i} V = 0,$$

where the symmetric positive definite operator  $A_0(V)$  is given by

$$A_0(V) = (D_V U)^T \delta_U^2 \mathcal{H}(U) D_V U, \quad (1.2.25)$$

and the symmetric operators  $A_i(V)$  by

$$A_i(V) = (D_V U)^T \delta_U^2 \mathcal{H}(U) D_U F_i(U) D_V U. \quad (1.2.26)$$

Moreover, if one of these statements is satisfied, the solution to system (1.1.25) satisfies

$$\int_{\mathbb{R}} \partial_t E(U) + \sum_{i=1}^n \partial_{x_i} (Q \cdot F_i(U) - R_i(Q)) = 0.$$

## 1.3 Application to Green–Naghdi type equations

### 1.3.1 Symmetrization of the Green–Naghdi system

In this part, we are going to apply the result of the previous section to the Green–Naghdi type system (1.1.1) around constant solutions  $(\bar{h}, \bar{h}\bar{u})$ , with  $\bar{h} > 0$  and  $\bar{u} \in \mathbb{R}$ . First, we show that system (1.1.1) is of the form (1.1.25) under convenient variables.

**Proposition 1.3.1** *Let  $s \geq 2$  be an integer and set  $\mathcal{A} = \mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R})$ . Then, using the variable*

$$U = (\eta, w)$$

*with  $\eta = h - \bar{h}$  and  $w = \mathcal{L}_h(u) - \bar{h}\bar{u}$ , system (1.1.1) is of the form (1.1.25) where  $F : \mathcal{A} \rightarrow \mathcal{A}$  is differentiable.*

**Proof** *We denote  $\mathcal{L}_h(u)$  by  $m$  and  $\bar{h}\bar{u}$  by  $\bar{m}$ . Let us first prove that system (1.1.1) can be written as*

$$\begin{cases} \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \\ \frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(mu) + \frac{\partial}{\partial x}(-2\alpha h^3(\partial_x u)^2 + \frac{g}{2}h^2) = 0 \end{cases} \quad (1.3.1)$$

*This is a consequence of the Hamiltonian structure (1.1.18) of the system. Indeed, developing the first line of (1.1.18) we find easily the first equation of (1.3.1). Then, we develop the second equation of (1.1.18) to get*

$$\partial_t m + h\partial_x \sigma + \partial_x(mu) + m\partial_x u = 0,$$

*where  $\sigma$  is given by (1.1.22). Now, using the expression (1.1.22) of  $\sigma$  together with the fact that  $m = \mathcal{L}_h(u)$ , we find the second equation of (1.3.1). One can deduce*

$$F(U) = \begin{pmatrix} (\eta + \bar{h})\mathcal{L}_h^{-1}(w + \bar{m}) - \bar{h}\bar{u} \\ (w + \bar{m})\mathcal{L}_h^{-1}(w + \bar{m}) - 2\alpha(\eta + \bar{h})^3(\partial_x \mathcal{L}_h^{-1}(w + \bar{m}))^2 + \frac{g}{2}(\eta + \bar{h})^2 - \frac{g}{2}\bar{h}^2 - \bar{m}\bar{u} \end{pmatrix}$$



Let us now check the properties of  $F$ . Assuming that  $h \in \mathbb{H}^s(\mathbb{R}) + \bar{h}$  is positively bounded by below,  $\mathcal{L}_h$  is a diffeomorphism from  $\mathbb{H}^{s+1}(\mathbb{R}) + \bar{u}$  to  $\mathbb{H}^{s-1}(\mathbb{R}) + \bar{h}\bar{u}$ . This together with the fact that  $\mathbb{H}^{s-1}(\mathbb{R})$  is an algebra for  $s \geq 2$  ensures that  $F$  is an application from  $\mathcal{A}$  to  $\mathcal{A}$ . For instance, let us consider the first component of  $F(U)$ . Since  $w \in \mathbb{H}^{s-1}(\mathbb{R})$ , we obtain that  $\mathcal{L}_h^{-1}(w + \bar{m})$  belongs to  $\mathbb{H}^{s+1}(\mathbb{R}) + \bar{u}$  and thus, to  $\mathbb{H}^s(\mathbb{R}) + \bar{u}$ . On the other hand,  $\eta + \bar{h} \in \mathbb{H}^s(\mathbb{R}) + \bar{h}$ . Hence, the product is in  $\mathbb{H}^s(\mathbb{R}) + \bar{h}\bar{u}$ . A similar logic can be applied to get a similar result on the second component of  $F$ .

The differentiability of  $F$  is due to the fact that it is a composition of differentiable applications. ■

We are now going to see that system (1.1.1) satisfies the assumptions of theorems 1.2.4 and 1.2.5 and Corollary 1.2.6 presented in Section 1.2.

**Proposition 1.3.2** *Let us consider a constant solution  $\bar{V} = (\bar{h}, \bar{u})$  with  $\bar{h} > 0$ . Then, there exists a neighborhood in  $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R})$  of  $\bar{V}$ , such that as long as the solution  $V = (h, u)$  remains in this neighborhood, system (1.1.1) is symmetrizable under any change of variable defined on this neighborhood. In other words, (1.1.1) is locally weakly symmetrizable around constant solutions.*

**Proof** *Let us prove that system (1.1.1) admits a general Godunov structure of the form (1.2.4) using the function*

$$\mathcal{H}_{\bar{h}, \bar{u}}(U) = \int_{\mathbb{R}} \frac{gh(h - \bar{h})}{2} + \frac{h(u - \bar{u})^2}{2} + \frac{\alpha h^3(u_x)^2}{2}.$$

Let us first remark that  $\mathcal{H}_{\bar{h}, \bar{u}}(U)$  is strictly convex in a small neighborhood<sup>3</sup> of  $\bar{U} = U(\bar{V})$ . The explicit representation formula of the second variation of  $\mathcal{H}_{\bar{h}, \bar{u}}$  is provided in Appendix 1.A. For all test functions  $\phi_1, \phi_2$ , one has<sup>4</sup>

$$\begin{aligned} \int_{\mathbb{R}} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \cdot \delta^2 \mathcal{H}_{\bar{h}, \bar{u}}(U) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \\ \int_{\mathbb{R}} (g - 3\alpha h(u_x)^2)(\phi_1)^2 + \left( \mathcal{L}_h^{-\frac{1}{2}}(-u\phi_1 + 3\alpha \partial_x(h^2 u_x \phi_1)) + \mathcal{L}_h^{-\frac{1}{2}}(\phi_2) \right)^2. \end{aligned}$$

Now, considering the fact that  $g - 3\alpha h(u_x)^2$  is bounded positively by below for  $(h, u)$  close enough to  $\bar{V}$ <sup>5</sup> (therefore, for  $U$  close enough to  $\bar{U}$ ), the strict convexity of  $\mathcal{H}_{\bar{h}, \bar{u}}$  on the small neighborhood of  $\bar{U}$  is concluded. We can formulate this conclusion as following:

There exists a neighborhood in  $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R})$  of  $\bar{U} = (0, 0)$  such that as long as the solution  $U = (\eta, w)$  is in this neighborhood,  $\delta_{\bar{U}}^2 \mathcal{H}_{\bar{h}, \bar{u}}$  is positive definite. In particular, we have on  $\bar{U}$ ,

$$\delta^2 \mathcal{H}_{\bar{h}, \bar{u}}(\bar{U}) = \begin{pmatrix} g & 0 \\ 0 & \mathcal{L}_h^{-1} \end{pmatrix}. \quad (1.3.2)$$

Let us also remark that  $\delta^2 \mathcal{H}_{\bar{h}, \bar{u}}(U)$  is an isomorphism from  $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R})$  to  $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R})$  if  $U$  is close enough to  $\bar{U}$ . Hence, the variational derivative  $\delta \mathcal{H}_{\bar{h}, \bar{u}}$  defines a diffeomorphism on a

<sup>3</sup>for the classical norm of  $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R})$ .

<sup>4</sup> $\mathcal{L}_h^{-\frac{1}{2}}$  is the symmetric operator such that  $\mathcal{L}_h^{-\frac{1}{2}} \circ \mathcal{L}_h^{-\frac{1}{2}} = \mathcal{L}_h^{-1}$ . The existence of this operator is guaranteed by the symmetry definite positivity of  $\mathcal{L}_h^{-1}$ .

<sup>5</sup>for the classical norm of  $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R})$ .

small enough neighborhood of the equilibrium  $\bar{U}$ . This is a consequence of the inverse function theorem considering the injectivity of  $\delta\mathcal{H}_{\bar{h},\bar{u}}(U)$  for  $U$  close to  $\bar{U}$ .

We now consider the Legendre transform  $\mathcal{H}_{\bar{h},\bar{u}}^*$  which is defined by

$$\mathcal{H}_{\bar{h},\bar{u}}^*(Q) = \int_{\mathbb{R}} Q \cdot U - E_{\bar{h},\bar{u}}, \quad (1.3.3)$$

where

$$E_{\bar{h},\bar{u}} = \frac{gh(h - \bar{h})}{2} + \frac{h(u - \bar{u})^2}{2} + \alpha h^3 (u_x)^2.$$

and

$$Q = \delta_U \mathcal{H}_{\bar{h},\bar{u}}(U). \quad (1.3.4)$$

One can check that  $Q = (\sigma, u - \bar{u})$ , with  $\sigma = gh - g\bar{h}/2 - u^2/2 + \bar{u}^2/2 - \frac{3}{2}\alpha h^2 (u_x)^2$ . This leads to the following expression for  $\mathcal{H}_{\bar{h},\bar{u}}^*$

$$\mathcal{H}_{\bar{h},\bar{u}}^*(Q) = \int_{\mathbb{R}} \frac{g(h - \bar{h})^2}{2} + \frac{\bar{h}(u - \bar{u})^2}{2} - \alpha h^3 (u_x)^2 + \frac{3}{2}\alpha h^2 \bar{h} (u_x)^2.$$

We just now need to remark that there exists a function  $\mathcal{R}$  of  $Q$  such that

$$F(U) = \delta_Q \mathcal{R}(Q).$$

We can get to this equality setting

$$\mathcal{R}(Q) = \int_{\mathbb{R}} gu \left( \frac{h^2 - \bar{h}^2}{2} \right) - \alpha h^3 u (u_x)^2 - \bar{h}\bar{u}\sigma - \bar{h}\bar{u}^2 (u - \bar{u}) + g\bar{h}^2 \bar{u}/2. \quad (1.3.5)$$

Hence, the system is equivalent on a small enough neighborhood of  $\bar{V}$  to

$$\partial_t \left( \delta_Q \mathcal{H}_{\bar{h},\bar{u}}^*(Q) \right) + \partial_x \left( \delta_Q \mathcal{R}(Q) \right) = 0. \quad (1.3.6)$$

Now, using Theorem 1.2.5, we can conclude the weak symmetrizability of the system under any change of variable around constant solutions. ■

Let us remark that the quantity  $E_{\bar{h},\bar{u}}$  introduced in the proof of Proposition 1.3.2, is actually an energy for the system. Indeed, we can check that the solution of (1.1.1) satisfies

$$\partial_t E_{\bar{h},\bar{u}} + \partial_x \left( u E_{\bar{h},\bar{u}} + (u - \bar{u})p \right) = 0. \quad (1.3.7)$$

where  $p$  is defined by (1.1.4).

Proposition 1.3.2 together with Theorem 1.2.5 implies the symmetrizability of the system under any variable around constant solutions. We now provide in the two following propositions some explicit symmetric forms of system (1.1.1).

**Proposition 1.3.3** *The Green–Naghdi type system (1.1.1) can be written under the symmetric form*

$$A_0(Q) \partial_t Q + A_1(Q) \partial_x(Q) = 0, \quad (1.3.8)$$

where  $Q = (\sigma, u - \bar{u})$  is defined by (2.1.6) and

$$A_0(Q) = \left( \begin{array}{c} \frac{1}{g-3\alpha h(u_x)^2} \\ \frac{u}{g-3\alpha h(u_x)^2} - 3\alpha \partial_x \left( \frac{h^2 u_x}{g-3\alpha h(u_x)^2} \right) \end{array} \right) \mathcal{L}_h + \frac{u+3\alpha h^2(u_x)\partial_x}{g-3\alpha h(u_x)^2} - 3\alpha \partial_x \left( h^2 u_x \frac{u()+3\alpha h^2 u_x \partial_x ()}{g-3\alpha h(u_x)^2} \right) \quad (1.3.9)$$

and

$$A_1(Q) = \left( \begin{array}{c} \frac{u}{g-3\alpha h(u_x)^2} \\ h + \frac{u^2}{g-3\alpha h(u_x)^2} - 3\alpha \partial_x \left( \frac{h^2 u(u_x)}{g-3\alpha h(u_x)^2} \right) \end{array} \right) \begin{array}{c} h + \frac{u^2+3\alpha h^2 u u_x \partial_x}{g-3\alpha h(u_x)^2} \\ 3hu + \frac{u^3+3\alpha h^2 u^2 u_x \partial_x}{g-3\alpha h(u_x)^2} - \alpha \partial_x (h^3 u_x()) - \\ \alpha u \partial_x (h^3 \partial_x ()) - 3\alpha \partial_x \left( \frac{h^2 u^2 u_x + 3\alpha h^4 u(u_x)^2 \partial_x ()}{g-3\alpha h(u_x)^2} \right) \end{array}. \quad (1.3.10)$$

**Proof** This is a consequence of the general Godunov structure (1.3.6) of the system. We just need to set  $A_0(Q) = \delta_Q^2 \mathcal{H}_{\bar{h}, \bar{u}}^*(Q)$  and  $A_1(Q) = \delta_Q^2 \mathcal{R}(Q)$  to get the result. ■

Let us remark that the operators  $A_0(Q)$  and  $A_1(Q)$  defined by (1.3.9) and (1.3.10) are second order differential operators.

**Proposition 1.3.4** *The Green–Naghdi type system (1.1.1) is symmetric under the unknown  $V = (h, u)$  of the form*

$$A_0(V) \partial_t V + A_1(V) \partial_x(V) = 0, \quad (1.3.11)$$

with

$$A_0(V) = \begin{pmatrix} g - 3\alpha h(u_x)^2 & 0 \\ 0 & \mathcal{L}_h \end{pmatrix}, \quad (1.3.12)$$

and

$$A_1(V) = \begin{pmatrix} gu - 3\alpha hu(u_x)^2 & gh - 3\alpha h^2(u_x)^2 \\ gh - 3\alpha h^2(u_x)^2 & hu + 2\alpha \partial_x(h^3 u_x) - \alpha h^3 u_x \partial_x - \alpha u \partial_x(h^3 \partial_x()) \end{pmatrix}. \quad (1.3.13)$$

**Proof** This proposition is just a consequence of Theorem 1.2.5 and Proposition 1.3.2. In fact, we check that the change of variable  $U \mapsto \tilde{V}$  such that

$$\begin{cases} U = (\eta, w), \\ \tilde{V} = (\eta, \delta_w \mathcal{H}_{\bar{h}, \bar{u}}(U)), \end{cases}$$

leads to  $\tilde{V} = (h - \bar{h}, u - \bar{u})$  which is nothing but  $V$  within a constant. This fact is true since  $\delta_w \mathcal{H}_{\bar{h}, \bar{u}}(U) = \mathcal{L}_h^{-1}(m) - \bar{u}$ . This change of variable is valid by the properties of the Sturm–Liouville operator  $\mathcal{L}_h$  while  $h$  is positively bounded by below. Hence, the system is symmetric with

$$A_0(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{\bar{h}, \bar{u}}(U) D_V U,$$

and

$$A_1(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{\bar{h}, \bar{u}}(U) \nabla_U F(U) D_V U.$$

Basic computations (similar to those presented in Appendix 1.A) show that their analytic expressions are given by (1.3.12) and (1.3.13). ■

Let us remark that similarly to Proposition 1.3.3, the operators  $A_0(V)$  and  $A_1(V)$  are second order differential operators. However, the analytic expressions of these operators are much simpler than the expressions of  $A_0(Q)$  and  $A_1(Q)$  in Proposition 1.3.3. In fact, as explained in Remark 1.2, the symmetric positive definite operator of Proposition 1.3.4 is diagonal. A similar structure to (1.3.11) (but non symmetric) is used in [69] to study the linearized Green–Naghdi system in order to prove the local well-posedness. Let us now apply Proposition 1.2.7 to the Green–Naghdi type equations to get a conserved quantity. According to this proposition, as long as the solution  $U$  remains close  $\bar{U}$ , it satisfies

$$\int_{\mathbb{R}} \partial_t E_{\bar{h}, \bar{u}}(U) + \partial_x N(U) = 0, \quad (1.3.14)$$

where

$$N(U) = Q \cdot F(U) - R(U),$$

with

$$R(U) = gu \left( \frac{h^2 - \bar{h}^2}{2} \right) - \alpha h^3 u (u_x)^2 - \bar{h} \bar{u} \sigma - \bar{h} \bar{u}^2 (u - \bar{u}) + g \bar{h}^2 \bar{u} / 2$$

given by (1.3.5). Now, we use the expressions of  $Q$ ,  $F(U)$ , and  $R(U)$  and we find

$$N(U) = \frac{ghu(h - \bar{h})}{2} + \left( \frac{gh^2 + hu^2}{2} + 3\alpha h^3 (u_x)^2 \right) (u - \bar{u}) + \frac{\alpha}{2} h^3 u (u_x)^2.$$

Since  $(h - \bar{h}, u - \bar{u}) \in \mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R})$  and  $s$  large, we remark that

$$\lim_{x \rightarrow \pm\infty} N(U) = 0,$$

which gives

$$\frac{d}{dt} \int_{\mathbb{R}} E_{\bar{h}, \bar{u}}(U) = 0.$$

Hence, we conclude the conservation of the energy integral  $\mathcal{H}_{\bar{h}, \bar{u}}(U)$  from the general Godunov structure of the system. Let us note that we could get the conservation of the energy integral simply by integrating the energy conservation law (1.3.7).

### 1.3.2 Two-dimensional extension

Let us fix  $\bar{V} = (\bar{h}, \bar{u}, \bar{v}) \in \mathbb{R}^3$  with  $\bar{h} > 0$  and consider the 2D Green–Naghdi model

$$\partial_t h + \partial_x hu + \partial_y hv = 0, \quad (1.3.15a)$$

$$\partial_t hu + \partial_x hu^2 + \partial_y huv + \partial_x (gh^2/2 + \alpha h^2 \dot{h}) = 0, \quad (1.3.15b)$$

$$\partial_t hv + \partial_x huv + \partial_y hv^2 + \partial_y (gh^2/2 + \alpha h^2 \dot{h}) = 0, \quad (1.3.15c)$$

where  $\dot{h} = \partial_t h + u\partial_x h + v\partial_y h$ .

This system is equivalent to

$$\partial_t U + \partial_x F_1(U) + \partial_y F_2(U) = 0, \quad (1.3.16)$$

where  $U = (h - \bar{h}, m - \bar{h}\bar{u}, n - \bar{h}\bar{v})$ , with  $(m, n) = \mathcal{L}_h(u, v)$  and  $\mathcal{L}_h(u, v) = h(u, v) - \alpha \nabla (h^3 \operatorname{div}(u, v))$ . The transformation  $(m, n) \mapsto (u, v)$  is well-defined if  $h$  is strictly positively bounded by below. Indeed in this case,  $\mathcal{L}_h$  is an isomorphism acting on the space

$$\mathbb{H}^{s+1}(\operatorname{div}) = \{(u, v) \in (\mathbb{H}^s(\mathbb{R}^2) + \bar{u}) \times (\mathbb{H}^s(\mathbb{R}^2) + \bar{v}) \text{ such that } \operatorname{div}(u, v) \in \mathbb{H}^s(\mathbb{R}^2)\}.$$

The fluxes are defined by

$$F_1(U) = \begin{pmatrix} hu \\ gh^2/2 + hu^2 - 2\alpha h^3(\operatorname{div}(u, v))^2 - \alpha u \partial_x (h^3 \operatorname{div}(u, v)) + \alpha h^3 \operatorname{div}(u, v) v_y \\ huv - \alpha \partial_y (h^3 u \operatorname{div}(u, v)) \end{pmatrix},$$

and

$$F_2(U) = \begin{pmatrix} hv \\ huv - \alpha \partial_x (h^3 v \operatorname{div}(u, v)) \\ gh^2/2 + hv^2 - 2\alpha h^3(\operatorname{div}(u, v))^2 - \alpha v \partial_y (h^3 \operatorname{div}(u, v)) + \alpha h^3 \operatorname{div}(u, v) u_x \end{pmatrix}.$$

**Proposition 1.3.5** *The solution of system (1.3.15) satisfies the following conservation law*

$$\partial_t E_{\bar{V}} + \partial_x (u E_{\bar{V}} + (u - \bar{u})p) + \partial_y (v E_{\bar{V}} + (v - \bar{v})p) = 0. \quad (1.3.18)$$

where

$$E_{\bar{V}} = gh(h - \bar{h})/2 + h(u - \bar{u})^2/2 + h(v - \bar{v})^2/2 + \alpha h^3(u_x + v_y)^2/2, \quad (1.3.19)$$

with  $p$  given by (1.1.4).

Let us consider the space integral  $\mathcal{H}_{\bar{V}}$  of the energy  $E_{\bar{V}}$ ,

$$\mathcal{H}_{\bar{V}}(U) = \int_{\mathbb{R}^2} E_{\bar{V}}(U). \quad (1.3.20)$$

Similarly to the one dimensional case, this function is strictly convex as an application of  $U$  while  $V = (h, u, v)$  is close enough to the equilibrium  $\bar{V} = (\bar{h}, \bar{u}, \bar{v})$ , i.e.  $\delta_{\bar{V}}^2 \mathcal{H}_{\bar{V}}(U)$  is positive definite for  $U$  close to  $\bar{U} = U(\bar{V}) = (0, 0, 0)$ . Let us now consider the change of variable

$$U \mapsto Q = \delta_U \mathcal{H}_{\bar{V}}(U),$$

defined around  $\bar{U}$ . Similarly to the 2-dimensional case, this is a diffeomorphism since  $\delta_U \mathcal{H}_{\bar{V}}$  is injective on a small enough neighborhood of  $\bar{U}$ . Moreover,  $\delta_{\bar{V}}^2 \mathcal{H}_{\bar{V}}(U)$  is an isomorphism for all  $U$  close to  $\bar{U}$ . The invertibility of  $\delta_U \mathcal{H}_{\bar{V}}$  is then just a consequence of the inverse function theorem.

One can check that

$$Q = \begin{pmatrix} gh - g\bar{h}/2 - (u^2 - \bar{u}^2)/2 - (v^2 - \bar{v}^2)/2 - 3\alpha h^2(\operatorname{div}(u, v))^2/2 \\ u - \bar{u} \\ v - \bar{v} \end{pmatrix}. \quad (1.3.21)$$

We are going to see in the following proposition that the 2-dimensional Green–Naghdi equation (1.3.15) admits a general Godunov structure using the variable  $Q$ .

**Proposition 1.3.6** *Let  $s > 4$ . There exists a neighborhood for the norm  $\mathbb{H}^s \times \mathbb{H}^{s+1}(\operatorname{div})$  of  $\bar{V}$  such that as long as the solution  $V$  of (1.3.15) remains in this neighborhood, the system is equivalent to*

$$\partial_t(\delta_Q \mathcal{H}_{\bar{V}}^*(Q)) + \partial_x(\delta_Q \mathcal{R}_1(Q)) + \partial_y(\delta_Q \mathcal{R}_2(Q)) = 0, \quad (1.3.22)$$

where  $Q$  is defined by (1.3.21) and  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are two functions defined on a neighborhood of  $\bar{Q} = Q(\bar{V}) = (g\bar{h}/2, 0, 0)$  (see (1.3.24) for some explicit representation formulas).

**Proof** *Let us first remark that the Legendre transform  $\mathcal{H}_{\bar{V}}^*$  of the energy integral  $\mathcal{H}_{\bar{V}}$  is defined by*

$$\begin{aligned} \mathcal{H}_{\bar{V}}^*(Q) &= \int_{\mathbb{R}^2} Q \cdot U - E_{\bar{V}}(U) \\ &= \int_{\mathbb{R}^2} g(h - \bar{h})^2/2 + \bar{h}(u - \bar{u})^2/2 + \bar{h}(v - \bar{v})^2/2 - \alpha h^3(\operatorname{div}(u, v))^2 + \frac{3}{2}\alpha h^2\bar{h}(\operatorname{div}(u, v))^2. \end{aligned}$$

We know by Definition 1.2.2 of the Legendre transform that we have

$$U = \delta_Q \mathcal{H}_{\bar{V}}^*(Q). \quad (1.3.23)$$

Let us now consider the variational functions  $\mathcal{R}_1$  and  $\mathcal{R}_2$  defined by

$$\mathcal{R}_1(Q) = \int_{\mathbb{R}^2} g \left( \frac{uh^2 - \bar{u}\bar{h}^2}{2} \right) + \bar{h}\bar{u}(u^2 - \bar{u}^2) - \alpha h^3 u(\operatorname{div}(u, v))^2, \quad (1.3.24a)$$

and

$$\mathcal{R}_2(Q) = \int_{\mathbb{R}^2} g \left( \frac{vh^2 - \bar{v}\bar{h}^2}{2} \right) + \bar{h}\bar{v}(v^2 - \bar{v}^2) - \alpha h^3 v(\operatorname{div}(u, v))^2. \quad (1.3.24b)$$

We can easily check that

$$F_1(U) = \delta_Q \mathcal{R}_1(Q), \quad (1.3.25)$$

$$F_2(U) = \delta_Q \mathcal{R}_2(Q). \quad (1.3.26)$$

Considering (1.3.23) together with (1.3.25) and (1.3.26) we get the result. ■

Now, according to Remark 1.2, the 2-dimensional Green–Naghdi system (1.3.15) is symmetrizable under any change of variable around any constant solution  $\bar{V}$ . Especially, the general Godunov structure of the system leads directly to the following symmetric structure under the unknown  $Q$ :

$$A_0(Q)\partial_t Q + A_1(Q)\partial_x Q + A_2(Q)\partial_y Q = 0, \quad (1.3.27)$$

where

$$A_0(Q) = \delta_Q^2 \mathcal{H}_{\tilde{V}}^*(Q), \quad (1.3.28a)$$

$$A_1(Q) = \delta_Q^2 \mathcal{R}_1(Q), \quad (1.3.28b)$$

and

$$A_2(Q) = \delta_Q^2 \mathcal{R}_2(Q). \quad (1.3.28c)$$

Considering the fact that we can recover the physical variable  $V = (h, u, v)$  using the partial variational derivative of the energy integral, we have the following corollary.

**Corollary 1.3.7** *The two-dimensional Green–Naghdi equation (1.3.15) is symmetric under the physical variable  $V = (h, u, v)$  of the form*

$$A_0(V)\partial_t V + A_1(V)\partial_x V + A_2(V)\partial_y V = 0. \quad (1.3.29)$$

where

$$A_0(V) = \begin{pmatrix} g - 3\alpha h(\operatorname{div}(u, v))^2 & 0 & 0 \\ 0 & h - \alpha\partial_x(h^3\partial_x) & -\alpha\partial_x(h^3\partial_y) \\ 0 & -\alpha\partial_y(h^3\partial_x) & h - \alpha\partial_y(h^3\partial_y) \end{pmatrix}$$

is block diagonal.

**Proof** We first consider the change of variable  $U \mapsto \tilde{V}$  where

$$U = (h, m, n),$$

and

$$\tilde{V} = (h, \delta_{(m,n)} \mathcal{H}_{\tilde{V}}(U)) = (h, u - \bar{u}, v - \bar{v})$$

is nothing but  $V$  within a constant. This change of variable is valid by the invertibility of  $\mathcal{L}_h$  on  $\mathbb{H}^{s+1}(\operatorname{div})$  since  $h$  is positively bounded by below and the physical speed  $(u, v)$  belongs to  $\mathbb{H}^{s+1}(\operatorname{div})$ . We then use Remark 1.2 to find the following expression for the operators

$$A_0(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{\tilde{V}}(U) D_V U,$$

$$A_1(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{\tilde{V}}(U) D_U F_1(U) D_V U,$$

and

$$A_2(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{\tilde{V}}(U) D_U F_2(U) D_V U.$$

Using Remark 1.2, we could predict the block diagonal structure of  $A_0(V)$ . ■

Let us mention that similarly to the first dimensional case, the conservation over time of the energy integral  $\mathcal{H}_{\tilde{V}}$  can be concluded.

## 1.4 Conclusion

A generalization of the notion of symmetry classically defined for hyperbolic systems has been presented. This generalization is mainly based on the generalization of Godunov systems introduced in

[54]. We prove that all general Godunov systems are symmetrizable under any change of variable. We also see that this structure leads to a conserved quantity. Then, we check that the one and two dimensional Green–Naghdi equations are general Godunov systems as long as the solution remains close enough to equilibriums. Therefore, there are symmetrizable under any change of variable defined on a small neighborhood of constant solutions. Moreover, the conserved quantity deduced by the general Godunov structure of the system is nothing but the energy integral which represents the total physical energy of the system.

Let us also mention that we write the Green–Naghdi equation on a quite simple structure under the physical variable. This is due to the fact that the physical variable can be obtained from the Hamiltonian variable by a partial change of variables. In fact, this leads to a bloc diagonal operator for the symmetric structure. The symmetric structure of the Green–Naghdi equations under the variable  $(h, u)$  is also used in [72] to prove the non linear stability of constant solutions of the system with viscosity.

## 1.A Computation of the second variation

In this part, we compute the second variation with respect to  $U = (h, m)$  of

$$\mathcal{H}_{\bar{h}, \bar{u}}(U) = \int_{\mathbb{R}} \frac{gh(h - \bar{h})}{2} + \frac{h(u - \bar{u})^2}{2} + \frac{\alpha h^3 (u_x)^2}{2}.$$

Let us first compute the variational derivative with respect to  $h$  of  $\mathcal{H}_{\bar{h}, \bar{u}}(U)$ . In fact, fixing the function  $m$ , we have for all test functions  $\phi$ ,

$$\begin{aligned} \mathcal{H}_{\bar{h}, \bar{u}}(h + \phi, m) &= \int_{\mathbb{R}} E_{\bar{h}, \bar{u}}(h + \phi, u) \\ &= \int_{\mathbb{R}} E_{\bar{h}, \bar{u}}(h, u) + D_h E_{\bar{h}, \bar{u}}(h, u)(\phi) + o(\|\phi\|), \\ &= \mathcal{H}_{\bar{h}, \bar{u}}(h, u) + \int_{\mathbb{R}} D_h E_{\bar{h}, \bar{u}}(h, u)(\phi) + o(\|\phi\|) \end{aligned}$$

where  $\lim_{\phi \rightarrow 0} \frac{o(\|\phi\|)}{\|\phi\|} = 0$ . Using the definition of  $E_{\bar{h}, \bar{u}}$ , we have

$$\begin{aligned} \mathcal{H}_{\bar{h}, \bar{u}}(h + \phi, m) &= \mathcal{H}_{\bar{h}, \bar{u}}(h, u) + \int_{\mathbb{R}} gh\phi - \frac{g\bar{h}}{2}\phi + \frac{(u - \bar{u})^2}{2}\phi \\ &+ \int_{\mathbb{R}} h(u - \bar{u})D_h u(\phi) + \frac{3}{2}\alpha h^2 (u_x)^2 \phi + \alpha h^3 u_x \partial_x D_h u(\phi) + o(\|\phi\|). \end{aligned} \quad (1.A.1)$$

In order to compute  $D_h u(\phi)$ , we consider (1.1.17) where  $m$  is defined. We differentiate this relation with respect to  $h$  and take the action on  $\phi$ . We find

$$0 = u\phi + hD_h u(\phi) - \alpha \partial_x (3h^2 \phi u_x) - \alpha \partial_x (h^3 \partial_x (D_h u(\phi))).$$

This leads to

$$D_h u(\phi) = \mathcal{L}_h^{-1} (3\alpha \partial_x (h^2 u_x \phi) - u\phi).$$



Injecting this into (1.A.1), we get

$$\begin{aligned} \mathcal{H}_{\bar{h},\bar{u}}(h + \phi, m) &= \mathcal{H}_{\bar{h},\bar{u}}(h, u) + \int_{\mathbb{R}} gh\phi - \frac{g\bar{h}}{2}\phi + \frac{(u - \bar{u})^2}{2}\phi \\ &+ \int_{\mathbb{R}} \frac{3}{2}\alpha h^2(u_x)^2\phi + (h(u - \bar{u}) + \alpha h^3 u_x \partial_x) \mathcal{L}_h^{-1} (3\alpha \partial_x(h^2 u_x \phi) - u\phi) + o(\|\phi\|). \end{aligned}$$

Hence, we have after an integration by part

$$\begin{aligned} \mathcal{H}_{\bar{h},\bar{u}}(h + \phi, m) &= \mathcal{H}_{\bar{h},\bar{u}}(h, u) + \int_{\mathbb{R}} gh\phi - \frac{g\bar{h}}{2}\phi + \frac{(u - \bar{u})^2}{2}\phi + \frac{3}{2}\alpha h^2(u_x)^2\phi \\ &+ \int_{\mathbb{R}} (h(u - \bar{u}) - \partial_x(\alpha h^3 u_x)) \mathcal{L}_h^{-1} (3\alpha \partial_x(h^2 u_x \phi) - u\phi) + o(\|\phi\|), \end{aligned}$$

or equivalently

$$\begin{aligned} \mathcal{H}_{\bar{h},\bar{u}}(h + \phi, m) &= \mathcal{H}_{\bar{h},\bar{u}}(h, u) + \int_{\mathbb{R}} gh\phi - \frac{g\bar{h}}{2}\phi + \frac{(u - \bar{u})^2}{2}\phi + \frac{3}{2}\alpha h^2(u_x)^2\phi \\ &+ \int_{\mathbb{R}} \mathcal{L}_h(u - \bar{u}) \cdot \mathcal{L}_h^{-1} (3\alpha \partial_x(h^2 u_x \phi) - u\phi) + o(\|\phi\|). \end{aligned}$$

Now considering the fact that  $\mathcal{L}_h$  is symmetric and using another integration by part, we get

$$\mathcal{H}_{\bar{h},\bar{u}}(h + \phi, m) = \mathcal{H}_{\bar{h},\bar{u}}(h, u) + \int_{\mathbb{R}} \left( gh - g\bar{h}/2 - \frac{u^2}{2} + \frac{\bar{u}^2}{2} - \frac{3}{2}\alpha h^2(u_x)^2 \right) \phi + o(\|\phi\|).$$

Then, we have

$$\delta_h \mathcal{H}_{\bar{h},\bar{u}}(U) = gh - g\bar{h}/2 - \frac{u^2}{2} + \frac{\bar{u}^2}{2} - \frac{3}{2}\alpha h^2(u_x)^2,$$

which is nothing but the quantity called  $\sigma$  in Section 1.1.2.

Using exactly the same type of computations, we find

$$\delta_m \mathcal{H}_{\bar{h},\bar{u}}(U) = u - \bar{u}.$$

On the other hand, we know by the definition of the second variation that

$$\delta_U^2 \mathcal{H}_{\bar{h},\bar{u}}(U) = D_U \delta_U \mathcal{H}_{\bar{h},\bar{u}}(U) = \begin{pmatrix} D_h \sigma(U) & D_m \sigma(U) \\ D_h u(U) & D_m u(U) \end{pmatrix}.$$

Then, similar computations lead to

$$\begin{aligned} \delta^2 \mathcal{H}_{\bar{h},\bar{u}}(U) &= \\ &\begin{pmatrix} g - 3\alpha h(u_x)^2 - (u + 3\alpha h^2 u_x \partial_x) \mathcal{L}_h^{-1} (-u) + 3\alpha \partial_x(h^2 u_x) & - (u + 3\alpha h^2 u_x \partial_x) \mathcal{L}_h^{-1} \\ \mathcal{L}_h^{-1} (-u) + 3\alpha \partial_x(h^2 u_x) & \mathcal{L}_h^{-1} \end{pmatrix}. \end{aligned}$$

## Chapter 2

# Global existence for small data of the viscous Green–Naghdi type equations

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## 2.1 Introduction

The Green–Naghdi system is a shallow water approximation of the water waves problem which models incompressible flows. The vertical and horizontal speeds are averaged vertically. Moreover, vertical acceleration is supposed too small to be considered [62]. In other words, Green–Naghdi equations is one order higher in approximation compared to the Saint–Venant (called also isentropic Euler) system [9]. To obtain the latter system, not only the vertical acceleration but also the vertical speed are neglected. This leads to a hyperbolic system of equations whereas the Green–Naghdi equation is dispersive due to the term  $\alpha h^2 \dot{h}$  defined below. In this work, we focus on the Green–

Naghdi type equation with a second order viscosity:

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(hu^2) + \partial_x(gh^2/2 + \alpha h^2\ddot{h}) = \mu\partial_x(h\partial_x u) \end{cases} \quad (2.1.1)$$

We assume that  $h(x, t) > 0$ ,  $\alpha$  and  $\mu$  are strictly positive and  $g$  is the gravity constant. The unknown  $h$  represents the fluid height and  $u$  its average horizontal speed. Moreover, the material derivative  $\dot{(\cdot)}$  is defined by  $\dot{(\cdot)} = \partial_t(\cdot) + u\partial_x(\cdot)$ .

**Remark 2.1.1** *Let us note that the  $\alpha = 0$  case gives us the Saint-Venant system. We can also learn more about the derivation of the system in [78, 5, 68] for  $(\mu, \alpha) = (0, \frac{1}{3})$ , and in [20] for  $(\mu, \alpha) = (0, \frac{1}{4})$ .*

It is worth remarking that (2.1.1) admits the following energy equality [55, 20],

$$\partial_t E + \partial_x(u(E + p)) = \mu u \partial_x(h\partial_x u), \quad (2.1.2)$$

where

$$E(h, u) = gh^2/2 + hu^2/2 + \alpha h^3(\partial_x u)^2/2,$$

and

$$p(h, u) = gh^2/2 + \alpha h^2\ddot{h}.$$

Then, we can check that (2.1.1) admits a family of relative energy conservation equalities given by

$$\partial_t E_{h_e, u_e} + \partial_x P_{h_e, u_e} = \mu(u - u_e)\partial_x(h\partial_x u), \quad (2.1.3)$$

where

$$E_{h_e, u_e}(h, u) = g(h - h_e)^2/2 + h(u - u_e)^2/2 + \alpha h^3(\partial_x u)^2/2, \quad (2.1.4a)$$

and

$$P_{h_e, u_e}(h, u) = uE_{h_e, u_e}(h, u) + (u - u_e)p(h, u) - \frac{gh_e^2}{2}u. \quad (2.1.4b)$$

This family is parametrized by  $(h_e, u_e) \in \mathbb{R}^2$  with  $h_e > 0$ .

**Remark 2.1.2** *Let us assume that  $\alpha = 0$ . Then,  $E(h, u)$  and  $E_{h_e, u_e}(h, u)$  are convex entropies for Saint-Venant system.*

The dissipative term  $\mu\partial_x(h\partial_x u)$  considered here in the right hand side of (2.1.1), is presented in [57] and some other references, as the viscosity for Saint-Venant system. Indeed, Saint-Venant system with this viscosity is derived in [57] from the Navier–Stokes equations under the shallow water assumption. On the one hand, this term is stabilizing for the hyperbolic Saint-Venant system. On the other hand, Green–Naghdi equation is a higher order approximation of the water waves problem and contains Saint-Venant system in addition to some dispersive terms. Therefore, we are interested to learn more about the role this viscosity plays on Green–Naghdi equations. Following the result of this work, we see that the dispersion does not cancel the stabilizing effect of the viscosity.

The aim of this chapter is to study the stability of equilibriums based on the symmetric structure of the system. The intuition comes from the Kawashima–Shizuta works on hyperbolic–parabolic

systems [95, 71] and Hanouzet–Natalini and Yong [64, 108] on entropy dissipative symmetric hyperbolic equations. All these results have been proved using the symmetric structure of hyperbolic systems. In particular, Saint-Venant system with a linear friction can be treated by the general result obtained in [64, 108] whereas Saint-Venant system with viscosity fits the general frame considered in [95, 71]. The notion of symmetric structure and of Godunov systems has been extended to some dispersive systems in the previous chapter as well as in [73]. In particular, the Green–Naghdi equations enter in this framework and then can be written under a symmetric structure which is recalled in Subsection 2.1.1. We show in this work how this structure enables us to extend the techniques used in [71, 64, 108] for symmetric hyperbolic equations to the dispersive Green–Naghdi equations.

**Remark 2.1.3** *The order of the dissipative term  $\mu\partial_x(h\partial_x u)$  plays a very crucial role in this work. Indeed, we can prove the global existence for small initial data only if the dissipative term, considered in the right hand side of (2.1.1), is a second order derivative term with respect to  $u$ . For instance, we are not able to generalize the results presented in Section 2.2, if we replace the dissipative term  $\mu\partial_x(h\partial_x u)$  with a friction type term such as  $-\kappa u$  for some  $\kappa > 0$ . Likewise, if we consider a fourth order dissipation such as  $-\mu\partial_x^2(h\partial_x(h\partial_x u)) / 4$  (suggested in [20]) instead of the second order  $\mu\partial_x(h\partial_x u)$ , the estimates we find are not sufficient to conclude the global existence.*

In all this work, partial derivatives with respect to  $x$  of any differentiable function  $f$  are presented by  $\partial_x f$ . The differential of the application  $F$  at  $U$  is symbolized by  $D_U F(U)$ . The adjoint of the operator  $\mathcal{A}$  is denoted by  $\mathcal{A}^*$ .

### 2.1.1 Symmetric structure

Following Li’s notations in [78], we use the unknown  $U = (h, m)$  defined by a Sturm–Liouville operator called  $\mathcal{L}$ :

$$m = hu - \alpha\partial_x(h^3\partial_x u) = \mathcal{L}_h(u).$$

Let us note that  $\mathcal{L}_h : \mathbb{H}^s(\mathbb{R}) \rightarrow \mathbb{H}^{s-2}(\mathbb{R})$  is an isomorphism if  $h$  is strictly positively bounded by below and  $s \geq 2$ . Therefore, System (2.1.1) can be written under

$$\partial_t U + \partial_x F(U) = Q(U),$$

where

$$F(U) = \left( \begin{array}{c} h\mathcal{L}_h^{-1}(m) \\ m\mathcal{L}_h^{-1}(m) - 2\alpha h^3(\partial_x \mathcal{L}_h^{-1}(m))^2 + \frac{g}{2}h^2 - \frac{g}{2}h_e^2 \end{array} \right), \quad (2.1.5)$$

and

$$Q(U) = \left( \begin{array}{c} 0 \\ \mu\partial_x(h\partial_x u) \end{array} \right) \quad (2.1.6)$$

Based on the structure presented in [78], it is easy to check that the unknown  $U$  enables us to write (2.1.1) under a Hamiltonian structure where the Hamiltonian  $\mathcal{H}_{h_e, u_e}$  is defined by the integral of the relative energy i.e. by

$$\mathcal{H}_{h_e, u_e} := \int_{\mathbb{R}} E_{h_e, u_e}.$$

This unknown presents also another advantage. In fact, we can recover the physical variable  $V = (h, u)$  from  $U$  using the interesting change of variable  $V = (h, \delta_m \mathcal{H}_{h_e, u_e}(U))$ , where  $\delta_m$  denotes the variational derivative with respect to  $m$ <sup>1</sup>. This consideration, as suggested in the following theorem found in the previous chapter, enables us to symmetrize the system in the physical variable with a diagonal locally definite positive operator (See Appendix 2.A for more details).

**Theorem 2.1.4** [73]. *Let  $V_e = (h_e, u_e)$  be a constant solution of (2.1.1) with  $h_e > 0$ . Let also  $s \geq 2$  be an integer. Then, as long as the solution  $V = (h, u)$  remains close to  $V_e$  for the usual norm of  $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R})$ , the system is equivalent to the following symmetric form:*

$$\mathcal{A}_0(V)\partial_t V + \mathcal{A}_1(V)\partial_x V = \begin{pmatrix} 0 \\ \mu\partial_x(h\partial_x u) \end{pmatrix} \quad (2.1.7)$$

where

$$\begin{aligned} \mathcal{A}_0(V) &= D_V U^*(V) (\delta_U^2 \mathcal{H}) D_V U(V) \\ &= \begin{pmatrix} g - 3\alpha h(\partial_x u)^2 & 0 \\ 0 & \mathcal{L}_h \end{pmatrix} \end{aligned} \quad (2.1.8)$$

is a positive definite operator and

$$\begin{aligned} \mathcal{A}_1(V) &= D_V U^*(V) (\delta_U^2 \mathcal{H}) (D_U F(U)) D_V U(V) \\ &= \begin{pmatrix} gu - 3\alpha hu(\partial_x u)^2 & gh - 3\alpha h^2(\partial_x u)^2 \\ gh - 3\alpha h^2(\partial_x u)^2 & hu + 2\alpha\partial_x(h^3\partial_x u) - \alpha\partial_x(h^3u)\partial_x - \alpha h^3u\partial_x^2 \end{pmatrix} \end{aligned} \quad (2.1.9)$$

is a symmetric one.

**Proof** *Let us consider the conservative form*

$$\partial_t U + \partial_x F(U) = Q(U).$$

Obviously, we have

$$D_V U(V)\partial_t V + D_U F(U)D_V U(V)\partial_x V = Q(U).$$

Then, acting  $D_V U^*(V) (\delta_U^2 \mathcal{H}_{h_e, u_e})$  on the system and considering the fact that  $Q(U)$  is an invariant vector of  $D_V U^*(V) (\delta_U^2 \mathcal{H}_{h_e, u_e})$ , we get the result (See Appendix 2.A for more details). ■

Let us note that  $\mathcal{A}_0(V)$  and  $\mathcal{A}_1(V)$  are linear second order differential operators. Therefore, they can be decomposed as

$$\mathcal{A}_0(V) = \mathcal{A}_0^0(V) + \mathcal{A}_0^1(V)\partial_x + \mathcal{A}_0^2(V)\partial_x^2 \quad (2.1.10)$$

<sup>1</sup>We have

$$\delta_h \mathcal{H}_{h_e, u_e}(U) = g(h - h_e) - \frac{u^2 - u_e^2}{2} - \frac{3}{2}\alpha h^2(\partial_x u)^2,$$

and

$$\delta_m \mathcal{H}_{h_e, u_e}(U) = u - u_e.$$

$$\mathcal{A}_1(V) = \mathcal{A}_1^0(V) + \mathcal{A}_1^1(V)\partial_x + \mathcal{A}_1^2(V)\partial_x^2 \quad (2.1.11)$$

where the expressions of symmetric matrix  $\mathcal{A}_i^j(V)$  for  $i, j \in \{0, 1, 2\}$  are given by

$$\begin{aligned} \mathcal{A}_0^0(V) &= \begin{pmatrix} g - 3\alpha h(\partial_x u)^2 & 0 \\ 0 & h \end{pmatrix}, & \mathcal{A}_0^1(V) &= \begin{pmatrix} 0 & 0 \\ 0 & -3\alpha h^2 \partial_x h \end{pmatrix}, & \mathcal{A}_0^2(V) &= \begin{pmatrix} 0 & 0 \\ 0 & -\alpha h^3 \end{pmatrix}, \\ \mathcal{A}_1^0(V) &= \begin{pmatrix} gu - 3\alpha hu(\partial_x u)^2 & gh - 3\alpha h^2(\partial_x u)^2 \\ gh - 3\alpha h^2(\partial_x u)^2 & hu + 2\alpha \partial_x(h^3 \partial_x u) \end{pmatrix}, & \mathcal{A}_1^1(V) &= \begin{pmatrix} 0 & 0 \\ 0 & -\alpha \partial_x(h^3 u) \end{pmatrix}, \\ \mathcal{A}_1^2(V) &= \begin{pmatrix} 0 & 0 \\ 0 & -\alpha h^3 u \end{pmatrix}. \end{aligned}$$

**Remark 2.1.5** *The definite positivity of a real matrix is equivalent to its coercivity. However, this fact does not necessary hold true for definite positive operators i.e. some definite positive operators are not coercive. It is important to point out that, as illustrated in Section 2.3, one of the keys which lets us generalize the hyperbolic methods to our symmetric system is actually the coercivity of  $\mathcal{A}_0(V)$  for the convenient norm. This means that we would not be able to generalize the method if  $\mathcal{A}_0(V)$  was definite positive but not coercive.*

We can also remark that the symmetric structure suggested in this section is similar to the structure used in [69] to study the local well-posedness of the Green–Naghdi equations without viscosity.

## 2.1.2 Outline

We are going to study the global existence of solutions of the viscous Green–Naghdi type equations for smooth initial data close to equilibriums. A local well-posedness result is proved in Appendix 2.B. Let us also mention that some similar writings as (2.1.7) have been used to study the local well-posedness of some dispersive systems (see [93] and [42] for instance). Then, we use the dissipative character of the viscosity as well as the symmetric structure of the system to prove the global existence of the local solution. In fact, the first step of the proof contains some initial estimates obtained by taking the scalar product of the  $s^{\text{th}}$  derivative of the equation with the  $s^{\text{th}}$  derivative of the solution. As it is exposed in Subsections 2.3.1 and 2.3.2, these estimates are obtained by almost the same approach as in the hyperbolic case ([64] and [108]). Then, the second step is to estimate the time integral of the norm of the solution. In the case of hyperbolic systems, this estimate is found by using the Kawashima–Shizuta condition. This condition has been introduced in [95] as a stability condition for constant solutions. It is based on the existence of a constant real matrix such that its product with the definite positive matrix (the one equivalent to  $\mathcal{A}_0$ ) is skew-symmetric at equilibrium while the symmetric part of its product with the symmetric matrix (the one equivalent to  $\mathcal{A}_1$ ), added to the right hand side term matrix, gives a definite positive matrix. However, in the case of Green–Naghdi system, we have not been able to find any operator generalization of the Kawashima–Shizuta condition. Hence, we must use a slightly different approach to find a convenient estimate. Indeed, we can find a null diagonal real matrix  $K$  such that  $K\mathcal{A}_1(V_e)$  is a symmetric definite positive matrix for all equilibriums  $V_e$  with  $u_e = 0$ . However,  $K\mathcal{A}_0(V_e)$  is not a skew-symmetric operator. Nevertheless, we are able to put some non straightforwardly controllable

term under a time integral of a time derivative <sup>2</sup> and estimate the remaining terms in a convenient manner (see Subsection 2.3.3). Then, using a symmetry group of the system, we can generalize the result to the case of equilibria  $V_e$  with  $u_e \neq 0$ .

This chapter is organized in 4 sections. The global existence theorem and its corollaries are presented in Section 2.2. Section 2.3 contains the steps of the proof. Some perspectives are suggested in Section 2.4. The advantages of the symmetric structure used in this study are explained in Appendix 2.A. So we can see why this symmetric structure is more appropriate than others. Appendix 2.B contains the proof of the local well-posedness theorem 2.2.1. Appendix 2.C highlights one of the other utilities of the symmetric structure. In fact, linear stability of equilibrium of non viscous Green–Naghdi can be proved using this structure.

## 2.2 Main results

The local well-posedness of (2.1.1) has been studied in [69] and [78] for the case  $\mu = 0$ . We see here that we can prove the local well-posedness of (2.1.1), around constant solutions, based on the idea used for symmetric hyperbolic systems. To do so, we first note that the set of constant solutions of (2.1.1) is

$$\{V_e = (h_e, u_e); h_e > 0, u_e \in \mathbb{R}\}.$$

We may also call these solutions the equilibria of the system.

We denote the norm associated with the affine space  $\mathbb{X}^s(\mathbb{R}) = (\mathbb{H}^s(\mathbb{R}) + h_e) \times (\mathbb{H}^{s+1}(\mathbb{R}) + u_e)$  by

$$\| (f, g) \|_{\mathbb{X}^s}^2 = \| f \|_{\mathbb{H}^s}^2 + \| g \|_{\mathbb{H}^{s+1}}^2 .$$

Moreover, the  $s$ -neighborhood of radius  $\delta$  and center  $V_e$  is presented by  $B_s(V_e, \delta) = \{V \in \mathbb{X}^s(\mathbb{R}), \| V - V_e \|_{\mathbb{X}^s} \leq \delta\}$  for all integer  $s \in \mathbb{R}$ .

Let us also denote by  $C$  the universal constant of the following Gagliardo–Nirenberg inequality:

$$\| f \|_{\mathbb{L}^\infty} \leq C \| \partial_x f \|_{\mathbb{L}^2}^{\frac{1}{2}} \| f \|_{\mathbb{L}^2}^{\frac{1}{2}} \quad \forall f \in \mathbb{H}^1(\mathbb{R}). \quad (2.2.1)$$

We are now able to announce the local well-posedness theorem,

**Theorem 2.2.1** *Let  $s \geq 2$  be an integer and consider a constant solution  $V_e$  of System (2.1.1). Then, there exists  $0 < \delta < h_e$  such that for all initial data  $V_0 \in B_s(V_e, \delta)$ , there exists  $T > 0$  such that the system admits a unique solution which belongs to  $C([0, T], \mathbb{X}^s(\mathbb{R}))$ .*

The proof of the theorem is given in Appendix 2.B. The steps of the proof are the same as for hyperbolic systems (see [35, 15] for instance). However, the necessary estimate to reach the final result

<sup>2</sup>The skew-symmetry of  $K\mathcal{A}_0$  for hyperbolic systems lets us put the non straightforwardly controllable terms under a time derivative. Therefore, we can deal with them by taking the time integral. Although, we are not able here to obtain a skew-symmetry  $K\mathcal{A}_0$ , we try to deal with non straightforwardly controllable terms by a similar idea.

of each step, is obtained by the same technique used in Section 2.3.2. In fact, we can see again in this part, how the generalized symmetric structure (2.1.7) of the system enables us to generalize the techniques used for symmetric hyperbolic systems.

An immediate corollary for Theorem 2.2.1 is the following. It states the positivity of the water height for small data and for short times.

**Corollary 2.2.2** *Let  $s \geq 2$  be an integer and consider a constant solution  $V_e$  of System (2.1.1). Let us also consider  $\delta \in (0, \frac{h_e}{C})$  and  $0 < T$  both conveniently small, and  $V_0 \in B_s(V_e, \delta)$  such that (2.1.1) admits a unique solution  $(h, u) \in C([0, T], \mathbb{X}^s(\mathbb{R}))$ . Then, for all  $\eta_0 \in (0, \inf_{x \in \mathbb{R}} h_0(x))$ , there exists a time  $\tilde{T} \in (0, T)$  such that*

$$\inf_{x \in \mathbb{R}} h(t, x) \geq \eta_0 \quad \forall t \in [0, \tilde{T}]. \quad (2.2.2)$$

**Proof** *Let us first note that  $\inf_{x \in \mathbb{R}} h_0(x) > 0$ . This is a consequence of the Gagliardo–Nirenberg inequality. Indeed,*

$$\|h_0 - h_e\|_{\mathbb{L}^\infty} \leq C \|\partial_x h_0\|_{\mathbb{L}^2}^{\frac{1}{2}} \|h_0 - h_e\|_{\mathbb{L}^2}^{\frac{1}{2}}.$$

*Considering the fact that  $V_0 \in B_s(V_e, \delta)$  with  $s \geq 2$  and  $\delta < \frac{h_e}{C}$ , the inequality becomes*

$$\|h_0 - h_e\|_{\mathbb{L}^\infty} \leq C\delta < h_e.$$

*Therefore,*

$$0 < h_e - C\delta \leq h_0(x) \leq h_e + C\delta < 2h_e \quad \forall x \in \mathbb{R}.$$

*Then, we conclude that*

$$\inf_{x \in \mathbb{R}} h_0(x) \geq h_e - C\delta > 0.$$

*Let us now fix  $\eta_0 \in (0, \inf_{x \in \mathbb{R}} h_0(x))$ . The unique solution of (2.1.1) belongs to  $C([0, T], \mathbb{X}^s(\mathbb{R}))$ . Hence, there exists  $\tilde{T} \in (0, T)$  such that*

$$\|h(t) - h_0\|_{\mathbb{X}^s} \leq \frac{\inf_{x \in \mathbb{R}} h_0(x) - \eta_0}{C} \quad \forall t \in [0, \tilde{T}].$$

*Again, the Gagliardo–Nirenberg inequality leads us to*

$$\|h(t) - h_0\|_{\mathbb{L}^\infty} \leq \inf_{x \in \mathbb{R}} h_0(x) - \eta_0 \quad \forall t \in [0, \tilde{T}].$$

*Then, we have*

$$\eta_0 - \inf_{x \in \mathbb{R}} h_0(x) \leq h(t, x) - h_0(x) \quad \forall (x, t) \in \mathbb{R} \times [0, \tilde{T}]$$

*and finally*

$$\eta_0 \leq \eta_0 + h_0(x) - \inf_{x \in \mathbb{R}} h_0(x) \leq h(t, x) \quad \forall (x, t) \in \mathbb{R} \times [0, \tilde{T}].$$

■

The main result of this study is the following theorem on the asymptotic stability of equilibriums.



**Theorem 2.2.3** *Let us consider an equilibrium  $V_e = (h_e, u_e)$  of (2.1.1) and  $s \geq 2$  an integer. Then, there exists  $\delta > 0$  such that for all initial data  $V_0 = (h_0, u_0) \in B_s(V_e, \delta)$ , the solution  $V$  exists for all time and converges asymptotically to  $V_e$ .*

*In other words, every constant solution  $V_e = (h_e, u_e)$  of (2.1.1) is asymptotically stable.*

Let us remark that we can prove Theorem 2.2.3 by considering  $u_e = 0$ . This is due to the fact that  $v = t\partial_x + \partial_u$  is a infinitesimal generator of a symmetry group of (2.1.1). This means that

$$V_\beta = (h(x - \beta t, t), u(x - \beta t, t) + \beta)$$

is also a solution of (2.1.1) for all solution  $V = (h, u)$  and all  $\beta \in \mathbb{R}$ . This fact has been mentioned in [77, 8] for the case  $\mu = 0$ . It is easy to check that the second order viscosity right hand side does not change this symmetry group. Hence, from now on, all the equilibriums considered in this work are of the form

$$V_e = (h_e, 0).$$

The key of this study is the following proposition which is a consequence of the primitive estimates in  $\mathbb{X}^s$  and the estimation of the time integral of the  $\mathbb{H}^{s-1}$  norm of  $h_x$  obtained in Section 2.3. In order to understand this study, let us mention that symbol  $C_S(\delta)$  stands for a function of  $\delta$ , defined by the elements of the set  $S$ , which converges to a strictly positive limit while  $\delta$  goes to 0. On the other hand,  $\Theta_S(\delta)$  stands for a function, defined by the elements of the set  $S$ , which converges to zero while  $\delta$  goes to 0. Let us also mention that the estimate suggested in Proposition 2.2.4 has a similar structure to the estimate given in Theorem 3.1 of [108].

**Proposition 2.2.4** *Let us consider an equilibrium  $V_e = (h_e, 0)$  of System (2.1.1), an integer  $s \geq 2$  and  $\delta > 0$  such that the system is locally well-posed for all initial data  $V_0 \in B_s(V_e, \delta)$ . Assume also that there exists  $T > 0$  such that the unique local solution  $V$  satisfies  $V(t) \in B_s(V_e, \delta)$  for all  $0 \leq t < T$ . Then, the following estimate holds true for all  $t \in [0, T)$ ,*

$$(1 - \Theta_{\{h_e, \alpha\}}(\delta)) \| V(t) - V_e \|_{\mathbb{X}^s}^2 + C_{\{h_e, \mu\}}(\delta) \int_0^t \| \partial_x u \|_{\mathbb{H}^s}^2 \leq C_{\{h_e, \alpha\}}(\delta) \| V(0) - V_e \|_{\mathbb{X}^s}^2 + \Theta_{\{h_e, \mu, \alpha\}}(\delta) \int_0^t \| \partial_x u \|_{\mathbb{H}^s}^2$$

*Besides, if  $\delta$  is conveniently small, this inequality leads to*

$$\| V(t) - V_e \|_{\mathbb{X}^s}^2 + C_{\{h_e, \mu\}}(\delta) \int_0^t \| \partial_x u \|_{\mathbb{H}^s}^2 \leq C_{\{h_e, \alpha\}}(\delta) \| V(0) - V_e \|_{\mathbb{X}^s}^2 .$$

Now, we get the global existence theorem as a result. In fact, we have

**Theorem 2.2.5** *Let us consider an equilibrium  $V_e = (h_e, 0)$  of (2.1.1) and an integer  $s \geq 2$ . Then, there exists  $\nu > 0$  such that for all initial data  $V_0 = (h_0, u_0) \in B_s(V_e, \nu)$ , the solution  $V$  exists for all time.*

*In other words, the equilibrium solutions  $V_e = (h_e, 0)$  of (2.1.1) are stable.*

**Proof** Let us first remark that if  $\delta > 0$  is small enough, we have

$$1 - \Theta_{\{h_e, \alpha\}}(\delta) > \frac{1}{2} \quad \text{and} \quad \frac{C_{\{h_e, \mu\}}(\delta) - \Theta_{\{h_e, \mu, \alpha\}}(\delta)}{1 - \Theta_{\{h_e, \alpha\}}(\delta)} > 0.$$

Let us also assume that  $\delta$  satisfies the assumptions of Proposition 2.2.4. Then, as long as  $V \in B_s(V_e, \delta)$ , it satisfies

$$\|V(t) - V_e\|_{\mathbb{X}^s}^2 + \frac{C_{\{h_e, \mu\}}(\delta) - \Theta_{\{h_e, \mu, \alpha\}}(\delta)}{1 - \Theta_{\{h_e, \alpha\}}(\delta)} \int_0^t \|\partial_x u\|_{\mathbb{H}^s}^2 \leq C_{\{g, h_e, \alpha\}}(\delta) \|V_0 - V_e\|_{\mathbb{X}^s}^2$$

Therefore, while  $V \in B_s(V_e, \delta)$ ,

$$\|V(t) - V_e\|_{\mathbb{X}^s}^2 \leq L(\delta) \|V_0 - V_e\|_{\mathbb{X}^s}^2$$

where  $L$  is a function of  $\delta$  such that  $\lim_{\delta \rightarrow 0} L(\delta) = l > 0$ . Setting  $\nu \leq \delta$  such that  $L(\delta)\nu \leq \delta/2$ , we have

$$\|V(t) - V_e\|_{\mathbb{X}^s}^2 \leq \delta/2, \quad \text{while} \quad V(t) \in B_s(V_e, \delta).$$

Then, considering the uniqueness of the local solution as well as its continuity for the norm  $\mathbb{X}^s$  we have the following conclusion: For  $V(0) \in B_s(V_e, \nu)$ , the local solution can not go out from  $B_s(V_e, \delta/2)$  for any time. Therefore, the norm of the local solution does not blow up. Hence, the unique local solution exists for all time. ■

**Corollary 2.2.6 (Asymptotic stability of equilibriums)** Let  $s \geq 2$  be an integer and consider the equilibrium  $V_e = (h_e, 0)$  of (2.1.1). Then, there exists  $\delta > 0$  such that for all initial data  $V_0 = (h_0, u_0)$  in  $B_s(V_e, \delta)$ , the global solution  $V(x, t)$  in  $\mathbb{X}^s(\mathbb{R})$  of (2.1.1) converges asymptotically to  $V_e$ . In other words,  $\lim_{t \rightarrow \infty} V(x, t) = V_e$  for all  $x \in \mathbb{R}$ .

**Proof** We use a similar logic to the one used in [108] for symmetric entropy dissipative hyperbolic systems satisfying the stability condition. We first take the  $x$  derivative of the first equation of (2.1.1), the time integral on  $[t_1, t_2]$  and consider the  $\mathbb{L}^2$  norm. This leads us to

$$\|\partial_x h(t_2) - \partial_x h(t_1)\|_{\mathbb{L}^2} = \left\| \int_{t_1}^{t_2} \partial_{xx}(hu) \right\|_{\mathbb{L}^2}. \quad (2.2.3)$$

Therefore,

$$\|\partial_x h(t_2) - \partial_x h(t_1)\|_{\mathbb{L}^2} \leq |t_2 - t_1| \left( \sup_{t_1 \leq t \leq t_2} \|\partial_{xx}(hu)\|_{\mathbb{L}^2} \right).$$

Considering the fact that  $\|\partial_{xx}(hu)\|_{\mathbb{L}^2}$  is bounded by Proposition 2.2.4, there exists  $\tilde{C} > 0$  such that we have for all  $t_1, t_2$  positive,

$$|\|\partial_x h(t_1)\|_{\mathbb{H}^1 \times \mathbb{L}^2} - \|\partial_x h(t_2)\|_{\mathbb{L}^2}| \leq \|\partial_x h(t_2) - \partial_x h(t_1)\|_{\mathbb{L}^2} \leq \tilde{C}|t_2 - t_1|.$$

This means that  $t \mapsto \|\partial_x h(t)\|_{\mathbb{L}^2}$  is Lipschitz continuous. On the other hand, it is  $\mathbb{L}^2([0, \infty))$  according to the estimate of the same proposition together with Proposition 2.3.12 of Subsection

2.3.3. Therefore,  $\| \partial_x h(t) \|_{\mathbb{L}^2}$  converges to 0 at the limit  $t \rightarrow \infty$ .

Let us now consider the second equation of (2.1.1) which writes ([42]) also

$$u_t = -u\partial_x u - \mathcal{L}_h^{-1}\partial_x (gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h\partial_x u).$$

Again, we derivate with respect to  $x$ , take the  $[t_1, t_2]$  time integral and consider its  $\mathbb{L}^2$  norm :

$$\| \partial_x u(t_2) - \partial_x u(t_1) \|_{\mathbb{L}^2} = \left\| \int_{t_1}^{t_2} \partial_x (-u\partial_x u - \mathcal{L}_h^{-1}\partial_x (gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h\partial_x u)) \right\|_{\mathbb{L}^2}.$$

Therefore,

$$\| \partial_x u(t_2) - \partial_x u(t_1) \|_{\mathbb{L}^2} \leq |t_2 - t_1| \left( \sup_{t_1 \leq t \leq t_2} \| \partial_x (-u\partial_x u - \mathcal{L}_h^{-1}\partial_x (gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h\partial_x u)) \|_{\mathbb{L}^2} \right).$$

Considering the fact that  $\| \partial_x (m\mathcal{L}_h^{-1}(m) - 2\alpha h^3(\partial_x \mathcal{L}_h^{-1}(m))^2 + \frac{g}{2}h^2 - \mu h\partial_x u) \|_{\mathbb{L}^2}$  is bounded, the Lipschitz continuity of  $t \mapsto \| \partial_x u(t) \|_{\mathbb{L}^2}$  is concluded. This together with the fact that  $t \mapsto \| u_x(t) \|_{\mathbb{L}^2}$  is square integrable (according to the estimate of Proposition 2.2.4), leads to

$$\lim_{t \rightarrow \infty} \| \partial_x u(t) \|_{\mathbb{L}^2} = 0.$$

We just now need to consider Gagliardo–Nirenberg inequality

$$\| V(t) - V_e \|_{\mathbb{L}^\infty \times \mathbb{L}^\infty} \leq C \| \partial_x V(t) \|_{\mathbb{L}^2 \times \mathbb{L}^2}^{\frac{1}{2}} \| V(t) - V_e \|_{\mathbb{L}^2 \times \mathbb{L}^2}^{\frac{1}{2}}.$$

Then, considering the facts that  $\| V(t) - V_e \|_{\mathbb{L}^2 \times \mathbb{L}^2}^{\frac{1}{2}}$  is bounded by  $\sqrt{\delta}$  and  $\| \partial_x V(t) \|_{\mathbb{L}^2 \times \mathbb{L}^2}^{\frac{1}{2}}$  converges to 0, the uniform convergence of  $V(x, t)$  to  $V_e$  is concluded. ■

**Remark 2.2.7** In addition to the asymptotic stability of constant solutions, the question of decay rates naturally arises. This point has been studied in [102] for linear symmetric systems of hyperbolic–parabolic type, by means of Fourier techniques in the frame of an energy method. Then, the result is used in [71] for the linearized symmetric hyperbolic–parabolic system to obtain a polynomial decay rate for the non-linear equation. The study of decay rate of linearized Green–Naghdi equations with viscosity, seems to be necessary to obtain a decay rate for the non-linear system and beyonds the scope of this work.

## 2.3 A priori estimates

The goal of this part is to obtain some a priori estimates of (2.1.1) similar to the estimate obtained in [64, 108, 95, 71] for hyperbolic systems. To do so, we use the Hamiltonian dissipation to find a 0<sup>th</sup> order estimate. We then take the  $\ell^{\text{th}}$  order derivative of the symmetric equation and consider the

scalar product with the  $\ell^{\text{th}}$  order spatial derivative of the solution for all  $1 \leq \ell \leq s$ . Then, using the properties of the operators  $\mathcal{A}_0(V)$  and  $\mathcal{A}_1(V)$ , especially the coercivity of  $\mathcal{A}_0(V)$  and their symmetry, we get a  $\ell^{\text{th}}$  order estimate for the solution  $V \in B_s(V_e, \delta)$ . Then, in Subsection 2.3.3, we get an estimation of  $\int_0^T \|\partial_x^\ell h\|_{\mathbb{L}^2}^2$  for all  $1 \leq \ell \leq s$  which together with the first estimates leads us to Proposition 2.2.4. These estimates are obtained by acting a hollow real matrix on the system. The equilibrium  $V_e$  we consider in all this section is of the form  $V_e = (h_e, 0)$  and  $s$  is an integer equal or greater than 2.

### 2.3.1 Estimate in $\mathbb{X}^0$

System (2.1.1) admits a  $\mathbb{X}^0$  estimation which is obtained by using the dissipation of the integral  $\mathcal{H}_{h_e,0}$  of the relative energy  $E_{h_e,0}$  defined in Section 2.1. In fact, the following proposition holds true.

**Proposition 2.3.1** *Let  $\delta, t > 0$  be small and  $V_0 \in B_s(V_e, \delta)$  such that System (2.1.1) admits a unique solution  $(h, u) \in C([0, t], \mathbb{X}^s(\mathbb{R}))$ , with  $h$  uniformly in time, strictly positively bounded by below<sup>3</sup>. Then,*

$$\|u(t)\|_{\mathbb{H}^1}^2 \leq \frac{\mathcal{H}_{h_e,0}(h_0, u_0)}{\min\{\inf_{x \in \mathbb{R}} h(t)/2, \alpha \inf_{x \in \mathbb{R}} h^3(t)/2\}}, \quad (2.3.1)$$

and

$$\|h(t) - h_e\|_{\mathbb{L}^2}^2 \leq \frac{2}{g} \mathcal{H}_{h_e,0}(h_0, u_0) \quad (2.3.2)$$

**Proof** *We take the spatial integral of the both sides of the relative energy equality (2.1.3) with  $u_e = 0$ . On the other hand,  $(h, u) \in (\mathbb{H}^s(\mathbb{R}) + h_e) \times \mathbb{H}^{s+1}(\mathbb{R})$  and  $s \geq 2$ . Therefore, an integration by part leads us to the dissipation of the Hamiltonian  $\mathcal{H}_{h_e,0}$ :*

$$\frac{d}{dt} \mathcal{H}_{h_e,0}(h, u) = -\mu \int_{\mathbb{R}} h(\partial_x u)^2 \leq 0.$$

*In other words,*

$$\mathcal{H}_{h_e,0}(h(t), u(t)) - \mathcal{H}_{h_e,0}(h(0), u(0)) = -\mu \int_0^t \int_{\mathbb{R}} h(\partial_x u)^2 \leq 0. \quad (2.3.3)$$

*Thus,*

$$\mathcal{H}_{h_e,0}(h(t), u(t)) \leq \mathcal{H}_{h_e,0}(h(0), u(0)). \quad (2.3.4)$$

*On the other hand,  $\mathcal{H}_{h_e,0}$  is defined by*

$$\mathcal{H}_{h_e,0}(h, u) = \int_{\mathbb{R}} g(h - h_e)^2/2 + hu^2/2 + \alpha h^3(\partial_x u)^2/2,$$

*and  $h$  is strictly positively bounded by below. Therefore,*

$$\frac{g}{2} \|h(t) - h_e\|_{\mathbb{L}^2}^2 + \left(\inf_{x \in \mathbb{R}} h(t)\right) \|u\|_{\mathbb{L}^2}^2 + \alpha \left(\inf_{x \in \mathbb{R}} h(t)\right)^3 \|\partial_x u\|_{\mathbb{L}^2}^2 \leq \mathcal{H}_{h_e,0}(h(t), u(t)).$$

<sup>3</sup>The existence of such  $\delta$  and  $t$  is guaranteed by Theorem 2.2.1 and Corollary 2.2.2

This together with (2.3.4) gives us the inequalities of the proposition. ■

Let us also remark that the Hamiltonian  $\mathcal{H}_{h_e,0}$  is locally  $\mathbb{X}^0$ -quadratic on  $V_e$ , in the sense that the following relation is satisfied for  $s \geq 2$  and  $\delta > 0$  small:

$$C_{\{h_e,\alpha\}}(\delta) \|V - V_e\|_{\mathbb{X}^0}^2 \leq \mathcal{H}_{h_e,0}(h, u) \leq C_{\{h_e,\alpha\}}(\delta) \|V - V_e\|_{\mathbb{X}^0}^2 \quad \forall V \in B_s(V_e, \delta).$$

This together with the dissipation equality (2.3.3) of  $\mathcal{H}_{h_e,0}$  gives us the following 0<sup>th</sup> order estimate around equilibriums.

**Proposition 2.3.2** *Let  $s \geq 2$  be an integer and  $V_e$  be an equilibrium of (2.1.1). Let us also assume that there exist  $\delta, T > 0$  such that the solution  $V$  of the system satisfies*

$$V(t) \in B_s(V_e, \delta) \quad \forall t \in [0, T].$$

Then, the following estimate holds true for such time:

$$\|V(t) - V_e\|_{\mathbb{X}^0}^2 + C_{\{h_e,\mu,\alpha\}}(\delta) \int_0^t \|\partial_x u\|_{\mathbb{L}^2}^2 \leq C_{\{h_e,\alpha\}}(\delta) \|V(0) - V_e\|_{\mathbb{X}^0}^2. \quad (2.3.5)$$

### 2.3.2 Estimate in $\mathbb{X}^s$

The main objective of this part is to obtain a convenient a priori estimate of  $\ell^{\text{th}}$  order, for all integer  $\ell \in [1, s]$ . This is done by a similar strategy as for hyperbolic systems. This analogy works here due to the structure of differential operators  $\mathcal{A}_0$  and  $\mathcal{A}_1$ . More precisely, the operator  $\mathcal{A}_0$  writes

$$\mathcal{A}_0(V) = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} + \begin{pmatrix} -3\alpha h(\partial_x u)^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\alpha \partial_x(h^3 \partial_x \bullet) \end{pmatrix}. \quad (2.3.6)$$

Likewise,  $\mathcal{A}_1(V)$  writes

$$\mathcal{A}_1(V) = \begin{pmatrix} gu & gh \\ gh & hu \end{pmatrix} + \begin{pmatrix} -3\alpha hu(\partial_x u)^2 & -3\alpha h^2(\partial_x u)^2 \\ -3\alpha h^2(\partial_x u)^2 & 2\alpha \partial_x(h^3 \partial_x u) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\alpha \partial_x(h^3 u \partial_x \bullet) \end{pmatrix}. \quad (2.3.7)$$

Indeed, the first term of the right-hand sides of (2.3.6) and (2.3.7) gives the hyperbolic part of the system i.e. the part which corresponds to Saint-Venant system. Therefore, it can be treated exactly as in [71]. The other terms need a specific treatment but they are not an obstacle to the result. On the one hand, this is due to the fact that the space of local well-posedness for  $u$ , is one order higher in regularity compared to the case of the hyperbolic Saint-Venant system. On the other hand, the conservative structure of the last term of (2.3.6) and (2.3.7) plays an important role in the treatment of the third order terms of (2.1.1), responsible for dispersion. For this reason, all along this subsection, different terms of operators  $\mathcal{A}_0^1 \partial_x$  and  $\mathcal{A}_0^2 \partial_x^2$  (resp.  $\mathcal{A}_1^1 \partial_x$  and  $\mathcal{A}_1^2 \partial_x^2$ ), introduced by (2.1.10) (resp. by (2.1.11)), are matched together to form the conservative term presented in the last part of the right hand side of (2.3.6) (resp. (2.3.7)).

We start the computations by taking the  $\ell^{\text{th}}$  derivative of (2.1.7) with respect to the spatial

variable, taking the scalar product with  $\partial_x^\ell V$  and integrating on  $[0, T) \times \mathbb{R}$ :

$$\int_0^T \int_{\mathbb{R}} \partial_x^\ell (\mathcal{A}_0(V) \partial_t V) \cdot \partial_x^\ell V + \int_0^T \int_{\mathbb{R}} \partial_x^\ell (\mathcal{A}_1(V) \partial_x V) \cdot \partial_x^\ell V = \mu \int_0^T \int_{\mathbb{R}} \partial_x^{\ell+1} (h \partial_x u) \partial_x^\ell u \quad (2.3.8)$$

Then, using basic computations and the Leibniz formula, we remark that<sup>4</sup>

$$\begin{aligned} \int_{\mathbb{R}} \partial_x^\ell (\mathcal{A}_0(V) \partial_t V) \cdot \partial_x^\ell V &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \mathcal{A}_0(V) \partial_x^\ell V \cdot \partial_x^\ell V - \frac{1}{2} \int_{\mathbb{R}} (\mathcal{A}_{0t}^0 + \mathcal{A}_{0t}^1 \partial_x + \mathcal{A}_{0t}^2 \partial_x^2) \partial_x^\ell V \cdot \partial_x^\ell V \\ &\quad + \sum_{i=1}^{\ell} \binom{\ell}{i} \int_{\mathbb{R}} (\mathcal{A}_{0i}^0 + \mathcal{A}_{0i}^1 \partial_x + \mathcal{A}_{0i}^2 \partial_x^2) \partial_t \partial_x^{\ell-i} V \cdot \partial_x^\ell V, \end{aligned}$$

where,  $\mathcal{A}_{0i}^j$  is another notation for  $\partial_x^i (\mathcal{A}_0^j(V))$ , the  $i^{\text{th}}$  spatial derivative of  $\mathcal{A}_0^j(V)$ , for all  $j \in \{0, 1, 2\}$  and for any  $i \in \mathbb{N}$ .

On the other hand, the integration by part and the symmetry of  $\mathcal{A}_1$  imply that

$$\begin{aligned} \int_{\mathbb{R}} \partial_x^\ell (\mathcal{A}_1(V) \partial_x V) \cdot \partial_x^\ell V &= \left( \ell - \frac{1}{2} \right) \int_{\mathbb{R}} (\mathcal{A}_{1x}^0 + \mathcal{A}_{1x}^1 \partial_x + \mathcal{A}_{1x}^2 \partial_x^2) \partial_x^\ell V \cdot \partial_x^\ell V \\ &\quad + \sum_{i=2}^{\ell} \binom{\ell}{i} \int_{\mathbb{R}} (\mathcal{A}_{1i}^0 + \mathcal{A}_{1i}^1 \partial_x + \mathcal{A}_{1i}^2 \partial_x^2) \partial_x^{\ell-i+1} V \cdot \partial_x^\ell V. \end{aligned}$$

We have also

$$\int_{\mathbb{R}} \partial_x^{\ell+1} (h \partial_x u) \partial_x^\ell u = - \int_{\mathbb{R}} h (\partial_x^{\ell+1} u)^2 - \sum_{i=1}^{\ell} \binom{\ell}{i} \int_{\mathbb{R}} (\partial_x^i h) (\partial_x^{\ell-i+1} u) (\partial_x^{\ell+1} u).$$

Hence, (2.3.8) becomes

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{A}_0(V) \partial_x^\ell V(T) \cdot \partial_x^\ell V(T) + 2\mu \int_0^T \int_{\mathbb{R}} h (\partial_x^{\ell+1} u)^2 &= \int_{\mathbb{R}} \mathcal{A}_0(V) \partial_x^\ell V(0) \cdot \partial_x^\ell V(0) \\ &\quad - 2 \sum_{i=1}^{\ell} \binom{\ell}{i} \int_0^T \int_{\mathbb{R}} (\mathcal{A}_{0i}^0 + \mathcal{A}_{0i}^1 \partial_x + \mathcal{A}_{0i}^2 \partial_x^2) \partial_t \partial_x^{\ell-i} V \cdot \partial_x^\ell V \\ &\quad + \int_0^T \int_{\mathbb{R}} (\mathcal{A}_{0t}^0 + \mathcal{A}_{0t}^1 \partial_x + \mathcal{A}_{0t}^2 \partial_x^2) \partial_x^\ell V \cdot \partial_x^\ell V \\ &\quad + (1 - 2\ell) \int_0^T \int_{\mathbb{R}} (\mathcal{A}_{1x}^0 + \mathcal{A}_{1x}^1 \partial_x + \mathcal{A}_{1x}^2 \partial_x^2) \partial_x^\ell V \cdot \partial_x^\ell V \\ &\quad - 2 \sum_{i=2}^{\ell} \binom{\ell}{i} \int_0^T \int_{\mathbb{R}} (\mathcal{A}_{1i}^0 + \mathcal{A}_{1i}^1 \partial_x + \mathcal{A}_{1i}^2 \partial_x^2) \partial_x^{\ell-i+1} V \cdot \partial_x^\ell V \\ &\quad - 2\mu \sum_{i=1}^{\ell} \binom{\ell}{i} \int_{\mathbb{R}} (\partial_x^i h) (\partial_x^{\ell-i+1} u) (\partial_x^{\ell+1} u). \end{aligned} \quad (2.3.9)$$

<sup>4</sup>For sake of simplicity, we use sometimes  $\mathcal{A}_1$  or  $\mathcal{A}_0$  instead of  $\mathcal{A}_1(V)$  or  $\mathcal{A}_0(V)$ .

The two following lemmas present two results which are used several times in the rest of this Section. The first one is on the  $\mathbb{X}^0$ -quadraticity of  $\mathcal{A}_0(V)$ :

**Lemma 2.3.3** *There exists  $\delta > 0$  such that  $\mathcal{A}_0(V)$  is quadratic on  $B_s(V_e, \delta)$ . In other words, we have for all  $V = (h, u) \in B_s(V_e, \delta)$  and all  $f = (f_1, f_2) \in \mathbb{X}^0(\mathbb{R})$ ,*

$$C_{\{h_e\}}(\delta) \|f\|_{\mathbb{X}^0}^2 \leq \int_{\mathbb{R}} \mathcal{A}_0(V) f \cdot f \leq C_{\{h_e\}}(\delta) \|f\|_{\mathbb{X}^0}^2.$$

**Proof** *The expression (2.1.8) of  $\mathcal{A}_0(V)$  leads to*

$$\int_{\mathbb{R}} \mathcal{A}_0(V) f \cdot f = \int_{\mathbb{R}} (g - 3\alpha h (\partial_x u)^2) f_1^2 + f_2 \mathcal{L}_h f_2.$$

*On the other hand, Gagliardo-Nirenberg inequality (2.2.1) leads to*

$$\|h - h_e\|_{\mathbb{L}^\infty} \leq C\delta,$$

*or equivalently to*

$$h_e - C\delta \leq h(x) \leq h_e + C\delta \quad \forall x \in \mathbb{R}.$$

*We also apply this inequality to  $\partial_x u$  to get*

$$\|\partial_x u\|_{\mathbb{L}^\infty} \leq C\delta,$$

*or equivalently*

$$-C\delta \leq \partial_x u(x) \leq C\delta \quad \forall x \in \mathbb{R}.$$

*Thus,*

$$-3\alpha h (\partial_x u)^2 \geq -3\alpha (h_e + C\delta) (C^2 \delta^2),$$

*and, if  $\delta$  is conveniently small,*

$$-3\alpha h (\partial_x u)^2 \geq -\frac{g}{2}.$$

*Consequently*

$$\begin{aligned} \frac{g}{2} \|f_1\|_{\mathbb{L}^2}^2 + \min\{h_e - \delta, \alpha(h_e - \delta)^3\} \|f_2\|_{\mathbb{H}^1}^2 &\leq \int_{\mathbb{R}} \mathcal{A}_0(V) f \cdot f \\ &\leq g \|f_1\|_{\mathbb{L}^2}^2 + \max\{h_e + \delta, \alpha(h_e + \delta)^3\} \|f_2\|_{\mathbb{H}^1}^2. \end{aligned}$$

■

The second lemma is on the smallness of the  $\mathbb{L}^\infty$  norm (in time and space) of  $\partial_t h$  and  $\partial_t u$  and some of their spatial derivatives as long as  $V \in B_s(V_e, \delta)$ . Actually, the following lemma holds true.

**Lemma 2.3.4** *Let us assume that the solution  $V(t)$  of (2.1.1) belongs to  $B_s(V_e, \delta)$  for all  $t \in [0, T)$ .*

Then, we have for all  $0 \leq j \leq s - 2$  and all  $0 \leq l \leq s - 1$ ,

$$\lim_{\substack{\delta \rightarrow 0 \\ V \in B(V_e, \delta)}} \|\partial_x^j \partial_t h\|_{\mathbb{L}^\infty} = 0, \quad \lim_{\substack{\delta \rightarrow 0 \\ V \in B(V_e, \delta)}} \|\partial_x^l \partial_t u\|_{\mathbb{L}^\infty} = 0. \quad (2.3.10)$$

Moreover, we have for all  $2 \leq k \leq s$ ,

$$\|\partial_x^k \partial_t u\|_{\mathbb{L}^2} \leq C_{\{h_e, \mu, \alpha\}}(\delta) (\|\partial_x u\|_{\mathbb{H}^k} + \|\partial_x h\|_{\mathbb{H}^{k-2}}). \quad (2.3.11)$$

**Proof** The first equation of System (2.1.1) gives us  $\partial_t h = -h\partial_x u - u\partial_x h$ . Therefore,

$$\|\partial_x^j \partial_t h\|_{\mathbb{L}^\infty} \leq \left\| \sum_{i=0}^j \partial_x^i h \partial_x^{j-i+1} u \right\|_{\mathbb{L}^\infty} + \left\| \sum_{i=0}^j \partial_x^i u \partial_x^{j-i+1} h \right\|_{\mathbb{L}^\infty} \leq \Theta_{\{h_e\}}(\delta)$$

Likewise, the second equation of the system can be written under the following form,

$$\partial_t u = -u\partial_x u - \mathcal{L}_h^{-1} \partial_x (gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h \partial_x u). \quad (2.3.12)$$

This form can be obtained by applying  $\mathcal{A}_0(V)^{-1}$  to (2.1.7) and coincides with the form suggested in [69, 75].

On the other hand,  $\mathcal{L}_h : \mathbb{H}^m(\mathbb{R}) \rightarrow \mathbb{H}^{m-2}(\mathbb{R})$  is bounded for all  $2 \leq m \leq s$ . This is due to the facts that  $\|h - h_e\|_{\mathbb{H}^s} \leq \delta$  and  $\delta$  is small. Indeed,

$$\|\mathcal{L}_h(u)\|_{\mathbb{H}^{m-2}} = \|hu - 3\alpha h^2 \partial_x h \partial_x u - \alpha h^3 \partial_x^2 u\|_{\mathbb{H}^{m-2}} \leq C_{\{h_e, \alpha\}}(\delta) \|u\|_{\mathbb{H}^m}.$$

Therefore,  $\mathcal{L}_h$  is a linear bijective bounded application from the Banach space  $\mathbb{H}^m(\mathbb{R})$  to the Banach space  $\mathbb{H}^{m-2}(\mathbb{R})$ . We now use the Banach theorem (see [18] for instance) to conclude that  $\mathcal{L}_h^{-1} : \mathbb{H}^{m-2}(\mathbb{R}) \rightarrow \mathbb{H}^m(\mathbb{R})$  is bounded. Thus, there exists  $C > 0$  such that

$$\begin{aligned} \|\partial_x^\ell \partial_t u\|_{\mathbb{L}^\infty} &\leq \|\partial_x^\ell (u\partial_x u)\|_{\mathbb{L}^\infty} + \|\partial_x^\ell \mathcal{L}_h^{-1} \partial_x (gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h \partial_x u)\|_{\mathbb{L}^\infty} \\ &\leq \Theta(\delta) + C \|\partial_x^\ell \mathcal{L}_h^{-1} \partial_x (gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h \partial_x u)\|_{\mathbb{H}^1} \\ &\leq \Theta(\delta) + C \|\mathcal{L}_h^{-1} \partial_x (gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h \partial_x u)\|_{\mathbb{H}^{\ell+1}} \\ &\leq \Theta(\delta) + C_{\{h_e, \alpha\}}(\delta) \|\partial_x (gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h \partial_x u)\|_{\mathbb{H}^{\ell-1}} \\ &\leq \Theta(\delta) + C_{\{h_e, \alpha\}}(\delta) \|\partial_x (gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h \partial_x u)\|_{\mathbb{H}^{s-2}} \\ &\leq \Theta(\delta) + C_{\{h_e, \alpha\}}(\delta) \Theta_{\{h_e, \alpha, \mu\}}(\delta) \leq \Theta_{\{h_e, \alpha, \mu\}}(\delta). \end{aligned}$$

To prove (2.3.11), we use similar computations. Indeed,

$$\begin{aligned} \|\partial_x^k \partial_t u\|_{\mathbb{L}^2} &\leq \|\partial_x^k (u\partial_x u)\|_{\mathbb{L}^2} + \|\partial_x^k \mathcal{L}_h^{-1} \partial_x (gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h \partial_x u)\|_{\mathbb{L}^2} \\ &\leq \sum_{i=0}^k \|\partial_x^{k-i} u \partial_x^{i+1} u\|_{\mathbb{L}^2} + \|\mathcal{L}_h^{-1} \partial_x (gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h \partial_x u)\|_{\mathbb{H}^k} \\ &\leq \Theta(\delta) \|\partial_x u\|_{\mathbb{H}^k} + \|\partial_x (gh^2/2 + 2\alpha h^3(\partial_x u)^2 - \mu h \partial_x u)\|_{\mathbb{H}^{k-2}} \\ &\leq \Theta(\delta) \|\partial_x u\|_{\mathbb{H}^k} + C_{\{h_e\}}(\delta) \|\partial_x h\|_{\mathbb{H}^{k-2}} + (\Theta_{\{\alpha, h_e\}}(\delta) + C_{\{h_e, \mu\}}(\delta)) \|\partial_x u\|_{\mathbb{H}^{k-2}} \\ &\leq C_{\{h_e, \mu, \alpha\}}(\delta) (\|\partial_x u\|_{\mathbb{H}^k} + \|\partial_x h\|_{\mathbb{H}^{k-2}}). \end{aligned}$$



■

We are now able to prove the following lemma which is the key step to achieve the appropriate  $\ell^{\text{th}}$  order estimate.

**Lemma 2.3.5** *Let us consider the solution  $V$  of (2.1.7) and assume that it belongs to  $B_s(V_e, \delta)$  for some  $\delta > 0$ . Then, the following estimates hold true for all integer  $1 \leq \ell \leq s$ ,*

$$\left| \sum_{i=1}^{\ell} \binom{\ell}{i} \int_{\mathbb{R}} \mathcal{A}_{0i}^0 \partial_x^{\ell-i} \partial_t V \cdot \partial_x^{\ell} V \right| \leq \Theta_{\{h_e, \alpha, \mu\}}(\delta) \left( \sum_{j=1}^{\ell+1} \|\partial_x^j u\|_{\mathbb{L}^2}^2 + \sum_{j=1}^{\ell} \|\partial_x^j h\|_{\mathbb{L}^2}^2 \right). \quad (2.3.13)$$

$$\left| \int_{\mathbb{R}} \mathcal{A}_{0t}^0 \partial_x^{\ell} V \cdot \partial_x^{\ell} V \right| \leq \Theta_{\{h_e, \alpha, \mu\}}(\delta) (\|\partial_x^{\ell} u\|_{\mathbb{L}^2}^2 + \|\partial_x^{\ell} h\|_{\mathbb{L}^2}^2). \quad (2.3.14)$$

$$\left| \sum_{i=1}^{\ell} \binom{\ell}{i} \int_{\mathbb{R}} (\mathcal{A}_{0i}^1 \partial_x + \mathcal{A}_{0i}^2 \partial_x^2) \partial_x^{\ell-i} \partial_t V \cdot \partial_x^{\ell} V \right| \leq \Theta_{\{h_e, \alpha, \mu\}}(\delta) \left( \sum_{i=1}^{\ell} \|\partial_x^i h\|_{\mathbb{L}^2}^2 + \|\partial_x u\|_{\mathbb{H}^2}^2 \right). \quad (2.3.15)$$

$$\left| \int_{\mathbb{R}} (\mathcal{A}_{0t}^1 \partial_x + \mathcal{A}_{0t}^2 \partial_x^2) \partial_x^{\ell} V \cdot \partial_x^{\ell} V \right| \leq \Theta_{\{h_e, \alpha, \mu\}}(\delta) (\|\partial_x^{\ell+1} u\|_{\mathbb{L}^2}^2). \quad (2.3.16)$$

$$\left| \int_{\mathbb{R}} (\mathcal{A}_{1x}^0 + \mathcal{A}_{1x}^1 \partial_x + \mathcal{A}_{1x}^2 \partial_x^2) \partial_x^{\ell} V \cdot \partial_x^{\ell} V \right| \leq \Theta_{\{h_e, \alpha\}}(\delta) (\|\partial_x^{\ell} h\|_{\mathbb{L}^2}^2 + \|\partial_x^{\ell} u\|_{\mathbb{H}^1}^2). \quad (2.3.17)$$

$$\left| \sum_{i=2}^{\ell} \binom{\ell}{i} \int_{\mathbb{R}} (\mathcal{A}_{1i}^0 + \mathcal{A}_{1i}^1 \partial_x + \mathcal{A}_{1i}^2 \partial_x^2) \partial_x^{\ell-i+1} V \cdot \partial_x^{\ell} V \right| \leq \Theta_{\{h_e, \alpha\}}(\delta) (\|\partial_x h\|_{\mathbb{H}^{\ell-1}}^2 + \|\partial_x u\|_{\mathbb{H}^{\ell}}^2). \quad (2.3.18)$$

**Proof** *Let us first prove (2.3.13). The expression of  $\mathcal{A}_0^0$  gives us the following equality for all  $1 \leq i \leq \ell$ ,*

$$\int_{\mathbb{R}} \mathcal{A}_{0i}^0 \partial_x^{\ell-i} \partial_t V \cdot \partial_x^{\ell} V = -3\alpha \partial_x^i (h(\partial_x u)^2) \partial_x^{\ell-i} \partial_t h \partial_x^{\ell} h + \partial_x^i h \partial_x^{\ell-i} \partial_t u \partial_x^{\ell} u.$$

Therefore,

$$\begin{aligned} & \left| \int_{\mathbb{R}} \mathcal{A}_{0i}^0 \partial_x^{\ell-i} \partial_t V \cdot \partial_x^{\ell} V \right| \leq \frac{\|\partial_x^{\ell-i} \partial_t u\|_{\mathbb{L}^{\infty}}}{2} \left( \|\partial_x^i h\|_{\mathbb{L}^2}^2 + \|\partial_x^{\ell} u\|_{\mathbb{L}^2}^2 \right) \\ & + \|\partial_x^{\ell-i} \partial_t h\|_{\mathbb{L}^{\infty}} \left( \|\partial_x^{\ell} h\|_{\mathbb{L}^2}^2 + \|3\alpha \partial_x^i h (\partial_x u)^2\|_{\mathbb{L}^2}^2 + \|6\alpha \sum_{j=0}^{i-1} \partial_x^j h \partial_x u \partial_x^{i-j+1} u\|_{\mathbb{L}^2}^2 \right) \\ & \leq \max \left\{ \frac{\|\partial_x^{\ell-i} \partial_t u\|_{\mathbb{L}^{\infty}}}{2}, C_{h_e, \alpha}(\delta) \|\partial_x^{\ell-i} \partial_t h\|_{\mathbb{L}^{\infty}} \right\} (\|\partial_x h\|_{\mathbb{H}^{\ell-1}}^2 + \|\partial_x u\|_{\mathbb{H}^{\ell}}^2). \end{aligned}$$

Then, considering (2.3.10), the proof of (2.3.13) is complete.

We are now going to prove (2.3.14). To do so, we should first remark that

$$\int_{\mathbb{R}} \mathcal{A}_{0t}^0 \partial_x^{\ell} V \cdot \partial_x^{\ell} V = -3\alpha \partial_t (h(\partial_x u)^2) (\partial_x^{\ell} h)^2 + h_t (\partial_x^{\ell} u)^2.$$

Then,

$$\left| \int_{\mathbb{R}} \mathcal{A}_{0t}^0 \partial_x^\ell V \cdot \partial_x^\ell V \right| \leq \| 3\alpha \partial_t (h(\partial_x u)^2) \|_{\mathbb{L}^\infty} \| \partial_x^\ell h \|_{\mathbb{L}^2}^2 + \| h_t \|_{\mathbb{L}^\infty} \| \partial_x^\ell u \|_{\mathbb{L}^2}^2.$$

Now, we use (2.3.10) to get the result.

The first step to prove (2.3.15) is to notice that we have for all  $1 \leq i \leq \ell$

$$\int_{\mathbb{R}} (\mathcal{A}_{0i}^1 \partial_x + \mathcal{A}_{0i}^2 \partial_x^2) \partial_x^{\ell-i} \partial_t V \cdot \partial_x^\ell V = \alpha \int_{\mathbb{R}} (\partial_x^i h^3) \partial_x^{\ell+1} u (\partial_x^{\ell-i+1} \partial_t u).$$

Hence we have for all  $2 \leq i \leq \ell$ ,

$$\begin{aligned} \left| \int_{\mathbb{R}} (\mathcal{A}_{0i}^1 \partial_x + \mathcal{A}_{0i}^2 \partial_x^2) \partial_x^{\ell-i} \partial_t V \cdot \partial_x^\ell V \right| &\leq \frac{\| \alpha \partial_x^{\ell-i+1} \partial_t u \|_{\mathbb{L}^\infty}}{2} \left( \| \partial_x^i h^3 \|_{\mathbb{L}^2}^2 + \| \partial_x^{\ell+1} u \|_{\mathbb{L}^2}^2 \right) \\ &\leq \frac{\| \alpha \partial_x^{\ell-i+1} \partial_t u \|_{\mathbb{L}^\infty}}{2} \left( C_{\{h_e\}}(\delta) \sum_{j=1}^i \| \partial_x^j h \|_{\mathbb{L}^2}^2 + \| \partial_x^{\ell+1} u \|_{\mathbb{L}^2}^2 \right). \end{aligned}$$

Considering (2.3.10), we obtain the estimate on the terms where  $2 \leq i \leq \ell$ . It remains to consider the case  $i = 1$ . This leads to

$$\int_{\mathbb{R}} (\mathcal{A}_{01}^1 \partial_x + \mathcal{A}_{01}^2 \partial_x^2) \partial_x^{\ell-1} \partial_t V \cdot \partial_x^\ell V = \alpha \int_{\mathbb{R}} (\partial_x h^3) \partial_x^{\ell+1} u (\partial_x^\ell \partial_t u).$$

Therefore,

$$\begin{aligned} \left| \int_{\mathbb{R}} (\mathcal{A}_{01}^1 \partial_x + \mathcal{A}_{01}^2 \partial_x^2) \partial_x^{\ell-1} \partial_t V \cdot \partial_x^\ell V \right| &\leq \frac{\| \alpha \partial_x (h^3) \|_{\mathbb{L}^\infty}}{2} \left( \| \partial_x^\ell \partial_t u \|_{\mathbb{L}^2}^2 + \| \partial_x^{\ell+1} u \|_{\mathbb{L}^2}^2 \right) \\ &\leq \Theta_{\{\alpha, h_e\}}(\delta) \left( \| \partial_x^\ell \partial_t u \|_{\mathbb{L}^2}^2 + \| \partial_x^{\ell+1} u \|_{\mathbb{L}^2}^2 \right). \end{aligned}$$

We now use (2.3.11) and find

$$\left| \int_{\mathbb{R}} (\mathcal{A}_{01}^1 \partial_x + \mathcal{A}_{01}^2 \partial_x^2) \partial_x^{\ell-1} \partial_t V \cdot \partial_x^\ell V \right| \leq \Theta_{\{h_e, \alpha, \mu\}}(\delta) \left( \| \partial_x h \|_{\mathbb{H}^{\ell-2}}^2 + \| \partial_x u \|_{\mathbb{H}^\ell}^2 \right).$$

In order to prove (2.3.16), we first remark that

$$\int_{\mathbb{R}} (\mathcal{A}_{0t}^1 \partial_x + \mathcal{A}_{0t}^2 \partial_x^2) \partial_x^\ell V \cdot \partial_x^\ell V = \int_{\mathbb{R}} 3\alpha h^2 \partial_t h (\partial_x^{\ell+1} u)^2.$$

Again, using (2.3.10), we find

$$\left| \int_{\mathbb{R}} (\mathcal{A}_{0t}^1 \partial_x + \mathcal{A}_{0t}^2 \partial_x^2) \partial_x^\ell V \cdot \partial_x^\ell V \right| \leq \Theta_{\{h_e, \alpha\}}(\delta) \| \partial_x^{\ell+1} u \|_{\mathbb{L}^2}^2.$$

To prove (2.3.17), we use an integration by part:

$$\begin{aligned} \int_{\mathbb{R}} (\mathcal{A}_{1x}^1 \partial_x + \mathcal{A}_{1x}^2 \partial_x^2) \partial_x^\ell V \cdot \partial_x^\ell V &= \int_{\mathbb{R}} -\alpha \partial_x^2 (h^3 u) \partial_x^{\ell+1} u \partial_x^\ell u - \alpha \partial_x (h^3 u) \partial_x^{\ell+2} u \partial_x^\ell u \\ &= \int_{\mathbb{R}} -\alpha \partial_x (\partial_x (h^3 u) \partial_x^{\ell+1} u) \partial_x^{\ell+1} u = \int_{\mathbb{R}} \alpha \partial_x (h^3 u) (\partial_x^{\ell+1} u)^2. \end{aligned}$$

Hence,

$$\left| \int_{\mathbb{R}} (\mathcal{A}_{1x}^1 \partial_x + \mathcal{A}_{1x}^2 \partial_x^2) \partial_x^\ell V \cdot \partial_x^\ell V \right| \leq \Theta_{\{h_e, \alpha\}}(\delta) \|\partial_x^{\ell+1} u\|_{\mathbb{L}^2}^2.$$

We have also

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{A}_{1x}^0 \partial_x^\ell V \cdot \partial_x^\ell V &= \int_{\mathbb{R}} \partial_x (gu - 3\alpha hu (\partial_x u)^2) (\partial_x^\ell h)^2 + 2 \int_{\mathbb{R}} \partial_x (gh - 3\alpha h^2 (\partial_x u)^2) \partial_x^\ell h \partial_x^\ell u \\ &\quad + \int_{\mathbb{R}} \partial_x (hu + 2\alpha \partial_x (h^3 \partial_x u)) (\partial_x^\ell u)^2, \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{A}_{1x}^0 \partial_x^\ell V \cdot \partial_x^\ell V &= \int_{\mathbb{R}} \partial_x (gu - 3\alpha hu (\partial_x u)^2) (\partial_x^\ell h)^2 + 2 \int_{\mathbb{R}} \partial_x (gh - 3\alpha h^2 (\partial_x u)^2) \partial_x^\ell h \partial_x^\ell u \\ &\quad - 2 \int_{\mathbb{R}} (hu + 2\alpha \partial_x (h^3 \partial_x u)) \partial_x^\ell u \partial_x^{\ell+1} u. \end{aligned}$$

Then,

$$\left| \int_{\mathbb{R}} \mathcal{A}_{1x}^0 \partial_x^\ell V \cdot \partial_x^\ell V \right| \leq \Theta_{\{h_e, \alpha\}}(\delta) \left( \|\partial_x^\ell h\|_{\mathbb{L}^2}^2 + \|\partial_x^\ell h\|_{\mathbb{H}^1}^2 \right).$$

The last estimate (2.3.18) is just a consequence of the following fact which holds true for all  $2 \leq i \leq \ell$ . It is due to the structure of  $\mathcal{A}_1^1(V)$  and  $\mathcal{A}_1^2(V)$  together with an integration by part:

$$\int_{\mathbb{R}} (\mathcal{A}_{1i}^1 \partial_x + \mathcal{A}_{1i}^2 \partial_x^2) \partial_x^{\ell-i+1} V \cdot \partial_x^\ell V = \alpha \int_{\mathbb{R}} (\partial_x^i (h^3 u)) \partial_x^{\ell-i+2} u (\partial_x^{\ell+1} u).$$

Hence, as long as  $V \in B_s(V_e, \delta)$  and  $2 \leq i \leq \ell - 1$ ,

$$\begin{aligned} \left| \int_{\mathbb{R}} (\mathcal{A}_{1i}^1 \partial_x + \mathcal{A}_{1i}^2 \partial_x^2) \partial_x^{\ell-i+1} V \cdot \partial_x^\ell V \right| &\leq \frac{\|\alpha \partial_x^i (h^3 u)\|_{\mathbb{L}^\infty}}{2} \left( \|\partial_x^{\ell-i+2} u\|_{\mathbb{L}^2}^2 + \|\partial_x^{\ell+1} u\|_{\mathbb{L}^2}^2 \right) \\ &\leq \Theta_{\{h_e, \alpha\}}(\delta) \left( \|\partial_x^{\ell-i+2} u\|_{\mathbb{L}^2}^2 + \|\partial_x^{\ell+1} u\|_{\mathbb{L}^2}^2 \right). \end{aligned}$$

On the other hand,

$$\int_{\mathbb{R}} (\mathcal{A}_{1\ell}^1 \partial_x + \mathcal{A}_{1\ell}^2 \partial_x^2) \partial_x V \cdot \partial_x^\ell V = \alpha \int_{\mathbb{R}} (\partial_x^\ell (h^3 u)) \partial_x^2 u (\partial_x^{\ell+1} u).$$

Therefore,

$$\begin{aligned} \left| \int_{\mathbb{R}} (\mathcal{A}_{1\ell}^1 \partial_x + \mathcal{A}_{1\ell}^2 \partial_x^2) \partial_x V \cdot \partial_x^\ell V \right| &\leq \Theta_{\{\alpha\}}(\delta) \left( \|\partial_x^\ell(h^3 u)\|_{\mathbb{L}^2}^2 + \|\partial_x^{\ell+1} u\|_{\mathbb{L}^2}^2 \right) \\ &\leq \Theta_{\{h_e, \alpha\}}(\delta) \left( \|\partial_x h\|_{\mathbb{H}^{\ell-1}}^2 + \|\partial_x u\|_{\mathbb{H}^\ell}^2 \right). \end{aligned}$$

Let us now treat the remaining terms of the left hand side of the estimate. In fact, we have for all  $2 \leq i \leq \ell - 2$ ,

$$\begin{aligned} \left| \int_{\mathbb{R}} \mathcal{A}_{1i}^0 \partial_x^{\ell-i+1} V \cdot \partial_x^\ell V \right| &\leq 2 \|\mathcal{A}_{1i}^0\|_{\mathbb{L}^\infty} \left( \|\partial_x^{\ell-i+1} u\|_{\mathbb{L}^2}^2 + \|\partial_x^\ell u\|_{\mathbb{L}^2}^2 + \|\partial_x^{\ell-i+1} h\|_{\mathbb{L}^2}^2 + \|\partial_x^\ell h\|_{\mathbb{L}^2}^2 \right) \\ &\leq \Theta_{\{h_e, \alpha\}}(\delta) \left( \|\partial_x^{\ell-i+1} u\|_{\mathbb{L}^2}^2 + \|\partial_x^\ell u\|_{\mathbb{L}^2}^2 + \|\partial_x^{\ell-i+1} h\|_{\mathbb{L}^2}^2 + \|\partial_x^\ell h\|_{\mathbb{L}^2}^2 \right), \end{aligned}$$

since the structure of  $\mathcal{A}_1^0$  gives us for all integer  $i \in [2, \ell - 2]$ ,

$$\lim_{\substack{\delta \rightarrow 0 \\ V \in B_s(V_e, \delta)}} \|\mathcal{A}_{1i}^0\|_{\mathbb{L}^\infty} = 0.$$

On the other hand,

$$\begin{aligned} &\left| \int_{\mathbb{R}} \mathcal{A}_{1(\ell-1)}^0 \partial_x^2 V \cdot \partial_x^\ell V + \mathcal{A}_{1\ell}^0 \partial_x V \cdot \partial_x^\ell V \right| \\ &\leq \max\{\|\partial_x^\ell(gu - 3\alpha hu(\partial_x u)^2)\|_{\mathbb{L}^\infty}, \|\partial_x^{\ell-1}(gu - 3\alpha hu(\partial_x u)^2)\|_{\mathbb{L}^\infty}\} \left( \|\partial_x h\|_{\mathbb{H}^1}^2 + \|\partial_x^\ell h\|_{\mathbb{L}^2}^2 \right) \\ &+ \max\{\|\partial_x^\ell(gh - 3\alpha h^2(\partial_x u)^2)\|_{\mathbb{L}^\infty}, \|\partial_x^{\ell-1}(gh - 3\alpha h^2(\partial_x u)^2)\|_{\mathbb{L}^\infty}\} \left( \|\partial_x h\|_{\mathbb{H}^{\ell-1}}^2 + \|\partial_x u\|_{\mathbb{H}^{\ell-1}}^2 \right) \\ &+ \left| \int_{\mathbb{R}} \partial_x \left( \partial_x^{\ell-1}(hu + 2\alpha \partial_x(h^3 \partial_x u)) \partial_x u \right) \partial_x^\ell u \right|. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \int_{\mathbb{R}} \mathcal{A}_{1(\ell-1)}^0 \partial_x^2 V \cdot \partial_x^\ell V + \mathcal{A}_{1\ell}^0 \partial_x V \cdot \partial_x^\ell V \right| &\leq \Theta_{\{h_e, \alpha\}}(\delta) \left( \|\partial_x h\|_{\mathbb{H}^{\ell-1}}^2 + \|\partial_x u\|_{\mathbb{H}^{\ell-1}}^2 \right) \\ &+ \left| \int_{\mathbb{R}} \partial_x^{\ell-1}(hu + 2\alpha \partial_x(h^3 \partial_x u)) \partial_x u \partial_x^{\ell+1} u \right| \\ &\leq \Theta_{\{h_e, \alpha\}}(\delta) \left( \|\partial_x h\|_{\mathbb{H}^{\ell-1}}^2 + \|\partial_x u\|_{\mathbb{H}^\ell}^2 \right). \end{aligned}$$

Hence, estimate (2.3.18) is totally proved. ■

This lemma together with the coercivity of  $\mathcal{A}_0$  and relation (2.3.9) leads us to the following propositions.

**Proposition 2.3.6** *Let us assume that there exists  $\delta > 0$ ,  $T > 0$  such that the solution  $V$  of (2.1.1) satisfies  $V(t) \in B_s(V_e, \delta)$  for all  $t \in [0, T)$ . Then, we have for all  $1 \leq \ell \leq s$ ,*

$$\|\partial_x^\ell(V(t) - V_e)\|_{\mathbb{X}^0}^2 + C_{\{h_e, \mu\}}(\delta) \int_0^t \|\partial_x^{\ell+1} u\|_{\mathbb{L}^2}^2 \leq C_{\{h_e, \alpha\}}(\delta) \|\partial_x^\ell(V(0) - V_e)\|_{\mathbb{X}^0}^2$$

$$+ \Theta_{\{h_e, \alpha, \mu\}}(\delta) \int_0^t \|\partial_x V\|_{\mathbb{X}^{\ell-1}}^2.$$

Then, considering this proposition together with the 0<sup>th</sup> order estimate of Subsection 2.3.1, we reach the final primary estimate which is given in the following proposition. This estimate together with the result of the next part enables us to prove the main theorem.

**Proposition 2.3.7** *Let us assume that there exists  $\delta > 0$ ,  $T > 0$  such that the solution  $V$  of (2.1.1) satisfies  $V(t) \in B_s(V_e, \delta)$  for all  $t \in [0, T)$ . Then, we have for such  $T$ ,*

$$\begin{aligned} \|V(t) - V_e\|_{\mathbb{X}^s}^2 + C_{\{h_e, \mu\}}(\delta) \int_0^t \|\partial_x u\|_{\mathbb{H}^s}^2 &\leq C_{\{h_e, \alpha\}}(\delta) \|V(0) - V_e\|_{\mathbb{X}^s}^2 \\ &+ \Theta_{\{h_e, \mu\}}(\delta) \int_0^t \|\partial_x V\|_{\mathbb{X}^{s-1}}^2. \end{aligned} \quad (2.3.19)$$

### 2.3.3 Estimate on $\int_0^t \|\partial_x^s h\|_{\mathbb{L}^2}^2$

This part is the final step to prove Proposition 2.2.4. In fact, we need to find a convenient estimate on  $\int_0^t \|\partial_x V\|_{\mathbb{X}^{s-1}}^2$  to be able to control the right hand side of (2.3.19). This idea has been used in [108], [64] and [95]. Actually, Estimate (2.3.19) has a similar appearance as the estimate found in these references for symmetric hyperbolic systems with dissipative terms. Then, they use the Kawashima stability condition to control the norm of spatial derivatives of first components of the solution. Let us note that, as in the case of hyperbolic system, we do not need to control the norm of second components. This is due to the presence of the second term of the left hand side of inequality (2.3.19). Therefore, what we need to control in the case of Green–Naghdi equation, is the time integral of the norm of the spatial derivative of  $h$ . Nevertheless, the main difficulty is the generalization of the Kawashima–Shizuta condition. Actually, we have not been able to find any operator version of the Kawashima–Shizuta condition for Green–Naghdi equation. However, we are going to see that it is possible to find an appropriate upper bound for  $\int_0^t \|\partial_x^s h\|_{\mathbb{L}^2}^2$  by using a slightly different technique from the hyperbolic case. To do so, we consider the  $2 \times 2$  hollow real matrix  $K(V_e)$  defined by

$$K(V_e) = \begin{pmatrix} 0 & 1 \\ -\frac{h_e}{g} & 0 \end{pmatrix}. \quad (2.3.20)$$

As we will see further, the reason why we consider this matrix, is the fact that  $K(V_e)\mathcal{A}_1(V_e)$  is a diagonal real matrix with a strictly positive first component. In other words, there exists a matrix of the form  $B = \begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix}$  with  $L \geq 0$  such that  $K(V_e)\mathcal{A}_1(V_e) + B$  is definite positive. This enables us, as in [108], [64], to get an upper bound for  $\int_0^t \|\partial_x^s h\|_{\mathbb{L}^2}^2$ . This upper bound is convenient even though, unlike the case of hyperbolic systems,  $K(V_e)\mathcal{A}_0(V_e)$  is not a skew-symmetric operator. This is due to the fact that we can extract from  $K(V_e)\mathcal{A}_0(V)$ , a part which plays a quite similar role to a skew-symmetric operator such that the norm of the remaining part is controllable in a suitable manner. So, let us write (2.1.7) under the form

$$\mathcal{A}_0(V)\partial_t V + \mathcal{A}_1(V_e)\partial_x V = H(V), \quad (2.3.21)$$

where  $H(V)$  is defined by

$$H(V) = [\mathcal{A}_1(V_e) - \mathcal{A}_1(V)] \partial_x V + \begin{pmatrix} 0 \\ \mu \partial_x (h \partial_x u) \end{pmatrix}. \quad (2.3.22)$$

We then take the action of the operator  $K(V_e) \partial_x^{\ell-1}$  on (2.3.21) and take the scalar product with  $\partial_x^\ell V$ . This leads us to

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} K(V_e) \partial_x^{\ell-1} (\mathcal{A}_0(V) \partial_t V) \cdot \partial_x^\ell V + \int_0^T \int_{\mathbb{R}} K(V_e) \mathcal{A}_1(V_e) \partial_x^\ell V \cdot \partial_x^\ell V \\ = \int_0^T \int_{\mathbb{R}} K(V_e) \partial_x^{\ell-1} H(V) \cdot \partial_x^\ell V, \end{aligned}$$

or equivalently to

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} K(V_e) \mathcal{A}_1(V_e) \partial_x^\ell V \cdot \partial_x^\ell V = \int_0^T \int_{\mathbb{R}} K(V_e) \partial_x^{\ell-1} H(V) \cdot \partial_x^\ell V \\ - \int_0^T \int_{\mathbb{R}} K(V_e) \partial_x^{\ell-1} (\mathcal{A}_0(V) \partial_t V) \cdot \partial_x^\ell V. \end{aligned} \quad (2.3.23)$$

Let us note that

$$K(V_e) \mathcal{A}_1(V_e) = \begin{pmatrix} gh_e & 0 \\ 0 & -h_e^2 \end{pmatrix}. \quad (2.3.24)$$

Hence,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} K(V_e) \mathcal{A}_1(V_e) \partial_x^\ell V \cdot \partial_x^\ell V = \int_0^T \int_{\mathbb{R}} \left( gh_e (\partial_x^\ell h)^2 - h_e^2 (\partial_x^\ell u)^2 \right) \\ = gh_e \int_0^T \|\partial_x^\ell h\|_{\mathbb{L}^2}^2 - h_e^2 \int_0^T \|\partial_x^\ell u\|_{\mathbb{L}^2}^2. \end{aligned} \quad (2.3.25)$$

Gathering (2.3.23) and (2.3.25), we get

$$g \int_0^T \|\partial_x^\ell h\|_{\mathbb{L}^2}^2 = h_e \int_0^T \|\partial_x^\ell u\|_{\mathbb{L}^2}^2 + \frac{1}{h_e} \int_0^T \int_{\mathbb{R}} \partial_x^{\ell-1} (K(V_e) H(V) - K(V_e) \mathcal{A}_0(V) \partial_t V) \cdot \partial_x^\ell V. \quad (2.3.26)$$

It is now sufficient to give a convenient estimate on the last term of (2.3.26). This estimation is given in the following lemma.

**Lemma 2.3.8** *Let  $V_e = (h_e, 0)$  be an equilibrium (with  $h_e > 0$ ) and  $\delta > 0$  be small such that System (2.1.1) admits a local solution  $V \in C^0([0, T]; \mathbb{X}^s(\mathbb{R}))$  for initial data in  $B_s(V_e, \delta)$ . Then, as long as  $V$  remains in  $B_s(V_e, \delta)$ , we have for all  $1 \leq \ell \leq s$ ,*

$$\begin{aligned} \int_{\mathbb{R}} K(V_e) \partial_x^{\ell-1} (H(V) - \mathcal{A}_0(V) \partial_t V) \cdot \partial_x^\ell V = \int_{\mathbb{R}} \partial_t \left( \partial_x^{\ell-1} \mathcal{L}_h u \cdot \partial_x^\ell h \right) + \mu \partial_x^\ell (h \partial_x u) \partial_x^\ell h \\ + R[h, u], \end{aligned} \quad (2.3.27)$$

where

$$\left| \int_{\mathbb{R}} R[h, u] \right| \leq \Theta_{\{h_e, \alpha\}}(\delta) \|\partial_x h\|_{\mathbb{H}^{\ell-1}}^2 + C_{\{h_e, \alpha\}}(\delta) \|\partial_x u\|_{\mathbb{H}^{\ell}}^2. \quad (2.3.28)$$

**Proof** First of all, we look at the first term of the left hand side of (2.3.27). To do so, we first remark that

$$\mathcal{A}_1(V_e) - \mathcal{A}_1(V) = \begin{pmatrix} -gu + 3\alpha hu(\partial_x u)^2 & g(h_e - h) + 3\alpha h^2(\partial_x u)^2 \\ g(h_e - h) + 3\alpha h^2(\partial_x u)^2 & -hu - 2\alpha \partial_x(h^3 \partial_x u) + \alpha \partial_x(h^3 u \partial_x()) \end{pmatrix}. \quad (2.3.29)$$

Thus, the definition (2.3.22) of  $H(V)$  leads to

$$\begin{aligned} K(V_e) \partial_x^{\ell-1} H(V) \cdot \partial_x^{\ell} V &= \boxed{\mu \partial_x^{\ell}(h \partial_x u) \partial_x^{\ell} h} + \partial_x^{\ell-1} (g(h_e - h)(\partial_x h) + 3\alpha h^2(\partial_x u)^2 \partial_x h) \partial_x^{\ell} h \\ &+ \partial_x^{\ell-1} \left( h_e u \partial_x h - \frac{3\alpha h_e}{g} h u \partial_x h (\partial_x u)^2 \right) \partial_x^{\ell} u - \partial_x^{\ell-1} \left( h_e (h_e - h)(\partial_x u) + \frac{3\alpha h_e}{g} h^2 (\partial_x u)^3 \right) \partial_x^{\ell} u \\ &- \partial_x^{\ell-1} (h u \partial_x u + 2\alpha \partial_x(h^3 \partial_x u) \partial_x u) \partial_x^{\ell} h + \boxed{\alpha \partial_x^{\ell}(h^3 u \partial_x^2 u) \partial_x^{\ell} h}. \end{aligned} \quad (2.3.30)$$

Let us remark here that all of the non boxed terms of (2.3.30) are straightforwardly controllable as in (2.3.28).

We now consider the second term of the left hand side of (2.3.27) and observe that

$$K(V_e) \mathcal{A}_0(V) = \begin{pmatrix} 0 & \mathcal{L}_h \\ -h_e + \frac{3\alpha}{g} h_e h (\partial_x u)^2 & 0 \end{pmatrix}. \quad (2.3.31)$$

Therefore,

$$K(V_e) \partial_x^{\ell-1} (\mathcal{A}_0(V) \partial_t V) \cdot \partial_x^{\ell} V = \partial_x^{\ell-1} \mathcal{L}_h (\partial_t u) \cdot \partial_x^{\ell} h + \partial_x^{\ell-1} \left( \frac{3\alpha h_e}{g} h (\partial_x u)^2 \partial_t h - h_e \partial_t h \right) \partial_x^{\ell} u. \quad (2.3.32)$$

Now, we need the following lemma to deal with non straightforwardly controllable term of the right hand side of (2.3.32).

**Lemma 2.3.9** Assume that  $(h, f) \in C^0([0, T], \mathbb{X}^s(\mathbb{R}))$  for some  $T > 0$ . Then, we have

$$\mathcal{L}_h \partial_t f = \partial_t \mathcal{L}_h f - f \partial_t h + 3\alpha \partial_x (h^2 \partial_t h \partial_x f). \quad (2.3.33)$$

We now use the lemma to rewrite (2.3.32):

$$\begin{aligned} K(V_e) \partial_x^{\ell-1} (\mathcal{A}_0(V) \partial_t V) \cdot \partial_x^{\ell} V &= \partial_t \left( \partial_x^{\ell-1} \mathcal{L}_h u \cdot \partial_x^{\ell} h \right) - \partial_x^{\ell-1} \mathcal{L}_h u \cdot \partial_t \partial_x^{\ell} h \\ &- \partial_x^{\ell-1} (u \partial_t h) \cdot \partial_x^{\ell} h + 3\alpha \partial_x^{\ell} (h^2 \partial_t h \partial_x u) \partial_x^{\ell} h + \partial_x^{\ell-1} \left( \frac{3\alpha h_e}{g} h (\partial_x u)^2 h_t - h_e h_t \right) \partial_x^{\ell} u. \end{aligned} \quad (2.3.34)$$

We then use the mass conservation equation,  $h_t = -\partial_x(hu)$ , to find

$$K(V_e) \partial_x^{\ell-1} (\mathcal{A}_0(V) \partial_t V) \cdot \partial_x^{\ell} V = \boxed{\partial_t \left( \partial_x^{\ell-1} \mathcal{L}_h u \cdot \partial_x^{\ell} h \right)} + \boxed{\partial_x^{\ell-1} \mathcal{L}_h u \cdot \partial_x^{\ell+1}(hu)} +$$

$$\begin{aligned} & \partial_x^{\ell-1}(u\partial_x(hu)) \cdot \partial_x^\ell h - \partial_x^{\ell-1} \left( \frac{3\alpha h_e}{g} h(\partial_x u)^2 \partial_x(hu) - h_e \partial_x(hu) \right) \partial_x^\ell u \\ & - 3\alpha \partial_x^\ell (h^2 \partial_x(hu) \partial_x u) \partial_x^\ell h. \end{aligned} \quad (2.3.35)$$

Considering the fact that all of the non-boxed terms of (2.3.35) are straightforwardly controllable as in (2.3.28), we notice that the form (2.3.35) of  $K(V_e) \partial_x^{\ell-1} (\mathcal{A}_0(V) \partial_t V) \cdot \partial_x^\ell V$  is very interesting. This is due on the one hand to the fact the non desirable term  $g \partial_x^\ell (h^2/2) \partial_x^\ell h$  is hidden in the boxed time derivative term  $\partial_t (\partial_x^{\ell-1} \mathcal{L}_h u \cdot \partial_x^\ell h)$ . Therefore, we can easily deal with this term by a time integration. On the other hand, as detailed in the following lemma, this formulation gathers the other non straightforwardly controllable term under the boxed term  $\partial_x^{\ell-1} \mathcal{L}_h u \cdot \partial_x^{\ell+1}(hu)$  which is cancellable with the boxed term  $\alpha \partial_x^\ell (h^3 u \partial_x^2 u) \partial_x^\ell h$  of (2.3.30).

**Lemma 2.3.10** Assume that  $V \in B_s(V_e, \delta)$ . Then, we have for all  $1 \leq \ell \leq s$ ,

$$\left| \int_{\mathbb{R}} \alpha \partial_x^\ell (h^3 u \partial_x^2 u) \partial_x^\ell h - \partial_x^{\ell-1} \mathcal{L}_h u \cdot \partial_x^{\ell+1}(hu) \right| \leq \Theta_{\{h_e, \alpha\}}(\delta) \|\partial_x h\|_{\mathbb{H}^{\ell-1}}^2 + C_{\{h_e, \alpha\}}(\delta) \|\partial_x u\|_{\mathbb{H}^\ell}^2 \quad (2.3.36)$$

We just now need to consider (2.3.30), (2.3.35) together with lemma 2.3.10 to complete the proof.  $\blacksquare$

**Proof of Lemma 2.3.10** We first use an integration by part and the definition of  $\mathcal{L}_h$  to write

$$\begin{aligned} & \int_{\mathbb{R}} \alpha \partial_x^\ell (h^3 u \partial_x^2 u) \partial_x^\ell h - \partial_x^{\ell-1} \mathcal{L}_h u \cdot \partial_x^{\ell+1}(hu) = \int_{\mathbb{R}} \alpha \partial_x^\ell (h^3 u \partial_x^2 u) \partial_x^\ell h + \partial_x^\ell \mathcal{L}_h u \cdot \partial_x^\ell(hu) \\ & = \int_{\mathbb{R}} \alpha \partial_x^\ell (h^3 u \partial_x^2 u) \partial_x^\ell h + \partial_x^\ell(hu) \cdot \partial_x^\ell(hu) - \int_{\mathbb{R}} \alpha \partial_x^{\ell+1}(h^3 \partial_x u) \cdot \partial_x^\ell(hu). \end{aligned}$$

Then, we use a simple development to get

$$\begin{aligned} & \int_{\mathbb{R}} \alpha \partial_x^\ell (h^3 u \partial_x^2 u) \partial_x^\ell h - \partial_x^{\ell-1} \mathcal{L}_h u \cdot \partial_x^{\ell+1}(hu) = \int_{\mathbb{R}} \alpha \partial_x^\ell h \left( \sum_{j=1}^{\ell} \partial_x^j (h^3 u) \partial_x^{\ell-j+2} u \right) \\ & + \left( \partial_x^\ell(hu) \right)^2 - \alpha \partial_x^\ell(hu) \left( \sum_{j=1}^{\ell} \partial_x^j (h^3) \partial_x^{\ell-j+2} u \right) - \alpha \boxed{\int_{\mathbb{R}} \partial_x^\ell(hu) \partial_x^{\ell+1}(h^3) \partial_x u}. \end{aligned} \quad (2.3.37)$$

We now see that the term  $\int_{\mathbb{R}} \partial_x^\ell(hu) \partial_x^{\ell+1}(h^3) \partial_x u$  may be the only obstacle to the estimate (2.3.36). However, we can treat this term as following to get the desired estimate. Indeed, we use the fact that<sup>5</sup>

$$\partial_x^{\ell+1}(h^3) = 3h^2 \partial_x^{\ell+1} h + [\partial_x^\ell, 3h^2] \partial_x h,$$

<sup>5</sup> As in [108], Symbol  $[\partial_x^\ell, A]U$  represents the commutator of  $A \in \mathbb{H}^s(\mathbb{R})$  and  $U \in \mathbb{H}^{s-1}(\mathbb{R})$ . In other words, we have

$$[\partial_x^\ell, A]U = \partial_x^\ell(AU) - A\partial_x^\ell U.$$



to write

$$\int_{\mathbb{R}} \partial_x^\ell(hu) \partial_x^{\ell+1}(h^3) \partial_x u = 3 \int_{\mathbb{R}} \partial_x^\ell(hu) h^2 \partial_x^{\ell+1}(h) \partial_x u + \partial_x^\ell(hu) \left( [\partial_x^\ell, h^2] \partial_x h \right) \partial_x u.$$

Likewise, we have

$$\begin{aligned} \int_{\mathbb{R}} \partial_x^\ell(hu) \partial_x^{\ell+1}(h^3) \partial_x u &= 3 \int_{\mathbb{R}} uh^2 \partial_x u \partial_x^\ell h \partial_x^{\ell+1} h + h^2 \partial_x u \left( [\partial_x^\ell, u] h \right) \partial_x^{\ell+1} h \\ &\quad + \partial_x^\ell(hu) \left( [\partial_x^\ell, h^2] \partial_x h \right) \partial_x u. \end{aligned}$$

We just now use an integration by part to get

$$\begin{aligned} \int_{\mathbb{R}} \partial_x^\ell(hu) \partial_x^{\ell+1}(h^3) \partial_x u &= -3 \int_{\mathbb{R}} \partial_x (uh^2 \partial_x u) (\partial_x^\ell h)^2 + \partial_x \left( h^2 \partial_x u \left( [\partial_x^\ell, u] h \right) \right) \partial_x^\ell h \\ &\quad + 3 \int_{\mathbb{R}} \partial_x^\ell(hu) \left( [\partial_x^\ell, h^2] \partial_x h \right) \partial_x u. \end{aligned}$$

Therefore,

$$\left| \alpha \int_{\mathbb{R}} \partial_x^\ell(hu) \partial_x^{\ell+1}(h^3) \partial_x u \right| \leq \Theta_{\{h_e, \alpha\}}(\delta) \|\partial_x h\|_{\mathbb{H}^{\ell-1}}^2 + C_{\{h_e, \alpha\}}(\delta) \|\partial_x u\|_{\mathbb{H}^\ell}^2.$$

This together with (2.3.37) leads to

$$\left| \int_{\mathbb{R}} \alpha \partial_x^\ell (h^3 u \partial_x^2 u) \partial_x^\ell h - \partial_x^{\ell-1} \mathcal{L}_h u \cdot \partial_x^{\ell+1}(hu) \right| \leq \Theta_{\{h_e, \alpha\}}(\delta) \|\partial_x h\|_{\mathbb{H}^{\ell-1}}^2 + C_{\{h_e, \alpha\}}(\delta) \|\partial_x u\|_{\mathbb{H}^\ell}^2$$

■

The last step to get the estimate of Proposition 2.2.4 is to give an estimate on the first two terms of the right hand side of (2.3.27). This is done in the following lemma.

**Lemma 2.3.11** *Let  $V = (h, u)$  be in  $C^0([0, T]; \mathbb{X}^s(\mathbb{R}))$  and assume that it belongs to  $B_s(V_e, \delta)$  for all  $t \in [0, T]$ . Then, we have for all  $1 \leq \ell \leq s$ ,*

$$\|\mu \partial_x^\ell (h \partial_x u) \partial_x^\ell h\|_{\mathbb{L}^1} \leq \Theta_{\{\mu\}}(\delta) \|\partial_x V\|_{\mathbb{X}^{\ell-1}}^2 + C_{\{\mu, h_e\}}(\delta) \|\partial_x^{\ell+1} u\|_{\mathbb{L}^2}^2 + \frac{g}{2} \|\partial_x^\ell h\|_{\mathbb{L}^2}^2, \quad (2.3.38)$$

and

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \partial_t \left( \partial_x^{\ell-1} \mathcal{L}_h u \cdot \partial_x^\ell h \right) &\leq C_{\{h_e, \alpha\}}(\delta) \left( \|u(t)\|_{\mathbb{H}^{\ell+1}}^2 + \|\partial_x^\ell h(t)\|_{\mathbb{L}^2}^2 \right) \\ &\quad + C_{\{h_e, \alpha\}}(\delta) \left( \|u(0)\|_{\mathbb{H}^{\ell+1}}^2 + \|\partial_x^\ell h(0)\|_{\mathbb{L}^2}^2 \right). \end{aligned} \quad (2.3.39)$$

**Proof** *The first estimate (2.3.38) is a consequence of Leibniz formula and the fact that*

$$\left| \partial_x^{\ell+1} u \partial_x^\ell h \right| \leq \frac{2\mu(h_e + \delta)}{g} \left( \partial_x^{\ell+1} u \right)^2 + \frac{g}{2\mu(h_e + \delta)} \left( \partial_x^\ell h \right)^2.$$

To prove (2.3.39), we use the definition of  $\mathcal{L}_h$  to write

$$\left| \partial_x^{\ell-1} \mathcal{L}_h u \cdot \partial_x^\ell h \right| = \left| \partial_x^{\ell-1} (hu) \cdot \partial_x^\ell h - \alpha \partial_x^\ell (h^3 \partial_x u) \cdot \partial_x^\ell h \right|.$$

Then, the estimate is obtained by very basic computations. Indeed,

$$\left| \partial_x^{\ell-1} (hu) \cdot \partial_x^\ell h - \alpha \partial_x^\ell (h^3 \partial_x u) \cdot \partial_x^\ell h \right| \leq \left| \partial_x^{\ell-1} (hu) \cdot \partial_x^\ell h \right| + \left| \alpha \partial_x^\ell (h^3 \partial_x u) \cdot \partial_x^\ell h \right|$$

On the other hand, we have

$$\left| \partial_x^{\ell-1} (hu) \cdot \partial_x^\ell h \right| \leq C_{\{h_e\}}(\delta) \left( \|u\|_{\mathbb{H}^{\ell-1}}^2 + \|\partial_x^\ell h\|_{\mathbb{L}^2}^2 \right),$$

and

$$\left| \alpha \partial_x^\ell (h^3 \partial_x u) \cdot \partial_x^\ell h \right| \leq C_{\{h_e\}}(\delta) \left( \|\partial_x u\|_{\mathbb{H}^{\ell-1}}^2 + \|\partial_x^\ell h\|_{\mathbb{L}^2}^2 \right).$$

Hence, the lemma is proved. ■

We now sum (2.3.26) for  $1 \leq \ell \leq s$ . This together with (2.3.27) and Lemma 2.3.11 enables us to give an estimation on  $\int_0^T \|h_x\|_{\mathbb{H}^{s-1}}^2$ :

**Proposition 2.3.12** *Let us assume that there exists  $T > 0$  such that the local solution of (2.1.1) satisfies  $V(t) \in B_s(V_e, \delta)$  for all  $t \in [0, T)$ . Then, we have,*

$$\begin{aligned} \int_0^t \|\partial_x h\|_{\mathbb{H}^{s-1}}^2 &\leq C_{\{h_e, \mu\}}(\delta) \int_0^t \|\partial_x u\|_{\mathbb{H}^s}^2 + C_{\{h_e, \alpha\}}(\delta) \left( \|u(t)\|_{\mathbb{H}^{s+1}}^2 + \|\partial_x h(t)\|_{\mathbb{H}^{s-1}}^2 \right) \\ &\quad + C_{\{h_e, \alpha\}}(\delta) \left( \|u(0)\|_{\mathbb{H}^{s+1}}^2 + \|\partial_x h(0)\|_{\mathbb{H}^{s-1}}^2 \right). \end{aligned} \quad (2.3.40)$$

This proposition together with Proposition 2.3.7 gives the a priori estimate of Proposition 2.2.4.

**Remark 2.3.13** *In this work,  $\alpha$  and  $\mu$  are supposed to be strictly positive. However, we can use the same approach and computations for the viscous Saint-Venant system i.e. for  $\alpha = 0$ . In this case, the system fits the general framework considered in [71] and our approach, as well as our result, is exactly the same. Indeed, the main difference between the case  $\alpha = 0$  (Saint-Venant system) and the case  $\alpha > 0$  (Green–Naghdi system) is the space on which the Hamiltonian  $\mathcal{H}_{h_e, 0}$  and the operator  $\mathcal{A}_0(V)$  are quadratic: this space is  $(\mathbb{H}^s(\mathbb{R}) + h_e) \times \mathbb{H}^s(\mathbb{R})$  when  $\alpha = 0$  whereas it is  $\mathbb{X}^s(\mathbb{R})$  when  $\alpha > 0$ . As a matter of fact, in both cases, the space of quadraticity of  $\mathcal{H}_{h_e, 0}$  and  $\mathcal{A}_0(V)$  are the same as the space on which the system is locally well-posed. For this reason, instead of the estimate of Proposition 2.2.4, we find the following estimate*

$$\begin{aligned} (1 - \Theta_{\{h_e\}}(\delta)) \|V(T) - V_e\|_{\mathbb{H}^s \times \mathbb{H}^s}^2 &+ C_{\{h_e, \mu\}}(\delta) \int_0^T \|\partial_x u\|_{\mathbb{H}^s}^2 \\ &\leq C_{\{h_e\}}(\delta) \|V(0) - V_e\|_{\mathbb{H}^s \times \mathbb{H}^s}^2 + \Theta_{\{h_e, \mu\}}(\delta) \int_0^T \|u_x\|_{\mathbb{H}^s}^2, \end{aligned}$$

which writes for small  $\delta > 0$ ,

$$\| V(T) - V_e \|_{\mathbb{H}^s \times \mathbb{H}^s}^2 + C_{\{h_e, \mu\}}(\delta) \int_0^T \| \partial_x u \|_{\mathbb{H}^s}^2 \leq C_{\{h_e\}}(\delta) \| V(0) - V_e \|_{\mathbb{H}^s \times \mathbb{H}^s}^2 .$$

**Remark 2.3.14** *The dissipative right hand side term,  $\mu \partial_x(h \partial_x u)$ , plays a very important role to obtain the stability result in both hyperbolic and dispersive cases. Indeed, it is well-known that equilibriums of Saint-Venant system without any dissipative term, are unstable<sup>6</sup> (see [35] for instance). Such an instability result does not exist for the Green–Naghdi equations. However, we are not able to prove the global existence result if the dissipative term is absent, i.e. if  $\mu = 0$ . More precisely, the presence of the  $\int_0^T \| \partial_x u \|_{\mathbb{H}^s}^2$  term in the left hand side of the estimate of Proposition 2.2.4 is due to the strict positivity of  $\mu$ . Therefore, this term disappears if  $\mu = 0$ . This means that the estimate of Proposition 2.2.4 becomes*

$$(1 - \Theta_{\{h_e, \alpha\}}(\delta)) \| V(T) - V_e \|_{\mathbb{X}^s}^2 \leq C_{\{h_e, \alpha\}}(\delta) \| V(0) - V_e \|_{\mathbb{X}^s}^2 + \Theta_{\{h_e, \alpha\}}(\delta) \int_0^T \| \partial_x u \|_{\mathbb{H}^s}^2 .$$

Hence,  $\| V(T) - V_e \|_{\mathbb{X}^s}^2$  is not any longer controlled by the norm of the initial data and the global existence for small data can not be concluded.

## 2.4 Conclusion and Perspectives

During this study, we proved the global existence for small data and the asymptotic stability of constant solutions of the Green–Naghdi system with a second order viscosity. This result is obtained by generalizing the technique used for symmetric entropy dissipative hyperbolic equations thanks to the generalized symmetric structure of the system. The study of the rate of convergence to equilibrium is one of the perspectives of this work [71].

Let us however recall that the result found in this study can not be generalized by this method to the Green–Naghdi system with friction  $-\kappa u$  (with  $\kappa > 0$ ), without the viscosity  $\mu u \partial_x(h \partial_x u)$ . In fact, in absence of this term, the first estimations are not coherent with the estimation of  $\int_0^t \| \partial_x h \|_{\mathbb{H}^{s-1}}^2$ , in the sense that there are of one order less than the estimation of  $\int_0^t \| \partial_x h \|_{\mathbb{H}^{s-1}}^2$ . Furthermore, if we add higher order viscous terms (order 4 or more) such as  $-\mu \partial_x^2(h \partial_x(h \partial_x u))$ , we are not able either to generalize the technique used in this work. In fact, in this latter case, the order of the first estimations are always less than the order of the estimates of  $\int_0^t \| \partial_x h \|_{\mathbb{H}^{s-1}}^2$ , with or without  $-\kappa u + \mu u \partial_x(h \partial_x u)$ . This means that the order 2 seems to be the only order of viscosity, our approach can be used for.

One of the other perspectives of this work is to study, in a general frame, the stability of equilibriums of locally-wellposed symmetrizable systems with a convenient friction or viscous term. In fact, the main difficulty of this generalization is to find the condition which leads to convenient estimates on the time integral of the spatial derivative of the solution. Let us note that in the case

<sup>6</sup>in the sens that in all neighborhood of constant solutions, there exists an initial data for which a shock is created in a finite time.

of hyperbolic systems, there are other equivalent formulations of the Kawashima–Shizuta condition [95, 71] which may be more convenient for the generalization. One of these formulations for hyperbolic systems is the emptiness of the intersection of the eigenspaces of the symmetric positive definite matrix (the one equivalent to  $\mathcal{A}_0$ ) and the symmetric matrix (the one equivalent to  $\mathcal{A}_1$ ) with the kernel of the viscosity matrix at equilibriums. It is also interesting to mention that the Kawashima–Shizuta condition is not sharp for hyperbolic systems (see [81] or [11] for instance). A generalization of less sharp conditions may be another way to follow. The answer to this question may let us for instance, investigate the stability of equilibriums of 2D Green–Naghdi system. Let us recall that  $\mathcal{A}_0(V)$  in 2-dimensional case is given in Chapter 1 (as well as in [73]), by

$$\mathcal{A}_0(V) = \begin{pmatrix} g - 3\alpha h(\operatorname{div}(u, v))^2 & 0 & 0 \\ 0 & h - \alpha\partial_x(h^3\partial_x) & -\alpha\partial_x(h^3\partial_y) \\ 0 & -\alpha\partial_y(h^3\partial_x) & h - \alpha\partial_y(h^3\partial_y) \end{pmatrix}$$

where  $u$  (respectively  $v$ ) represents the vertically averaged  $x$ -component (resp.  $y$ -component) of the speed. In this case,  $\mathcal{A}_0(V)$  is quadratic near equilibriums, for the norm  $\|\cdot\|_{\mathbb{X}^0}$  defined by

$$\|f\|_{\mathbb{X}^0}^2 = \|f\|_{\mathbb{L}^2}^2 + \|\operatorname{div}(f)\|_{\mathbb{L}^2}^2.$$

This is also the 0<sup>th</sup> order norm of the local well-posedness space of the 2-dimensional system [5]. Indeed, the symmetric structure is coherent with the well-posedness space.

## 2.A Special Symmetric Structure

In this section, we consider a system of the form

$$\partial_t W + \partial_x F(W) = 0, \tag{2.A.1}$$

The unknown  $W$  is supposed to belong to  $C([0, T]; \mathcal{X})$  for some  $T > 0$  where  $\mathcal{X}$  is a Banach subspace of continuous functions of  $\mathbb{L}^2(\mathbb{R}, \mathbb{R}^N)$  converging to 0 at infinity. We also assume that the derivative of all elements of  $\mathcal{X}$  belongs to  $\mathcal{X}$ . Additionally,  $F$  is not anymore a function of  $\mathbb{R}^N$  but a smooth application defined from  $\mathcal{X}$  to  $\mathcal{X}$ . We also assume that (2.A.1) is a general Godunov system (cf Chapter 1 and [54, 73]). Therefore there exists a strictly convex functional  $\mathcal{H}$  defined on a convex subset  $\Omega$  of  $\mathcal{X}$  such that  $\delta^2 \mathcal{H}(W) D F(W)$  is symmetric. Under these assumptions, System (2.A.1) is symmetrizable under any change of unknown (see [73] or Chapter 1 for more details).

**Proposition 2.A.1** *Let us consider the decomposition  $W = (U, V)$  of the unknown. Assume also that the application*

$$(U, V) \mapsto (U, \delta_V \mathcal{H}(U, \cdot))$$

*is a diffeomorphism. Then, (2.A.1) is written under the unknown  $w = (U, \delta_V \mathcal{H}(W))$ , as following*

$$\mathcal{A}_0(w) \partial_t w + \mathcal{A}_1(w) \partial_x w = 0. \tag{2.A.2}$$

*Moreover,  $\mathcal{A}_0(w) = D_w W^*(w) \delta_W^2 \mathcal{H}(W) D_w W(w)$  is a symmetric definite positive bloc diagonal operator and  $\mathcal{A}_1(w) = D_w W^*(w) \delta_W^2 \mathcal{H}(W) D_W F(W) D_w W(w)$  is a symmetric one.*

**Proof** Let us set  $u = U$  and  $v = \delta_V \mathcal{H}(W)$ . Therefore  $w = (u, v)$ . It is easy to check that we obtain (2.A.2) by acting  $D_w W^*(w) \delta^2 \mathcal{H}(w)$  on System (2.A.1). Let us now remark that

$$D_w W = \begin{pmatrix} 1 & 0 \\ D_u V & D_v V \end{pmatrix},$$

and

$$\delta_W^2 \mathcal{H}(W) = \begin{pmatrix} \delta_U^2 \mathcal{H}(W) & \delta_{VU}^2 \mathcal{H}(W) \\ \delta_{UV}^2 \mathcal{H}(W) & \delta_V^2 \mathcal{H}(W) \end{pmatrix}.$$

Hence,

$$\mathcal{A}_0(w) = \begin{pmatrix} \delta_U^2 \mathcal{H}(W) + \delta_{VU}^2 \mathcal{H}(W) D_u V + (D_u V)^T \delta_{UV}^2 \mathcal{H}(W) + (D_u V)^T \delta_V^2 \mathcal{H}(W) D_u V & \delta_{VU}^2 \mathcal{H}(W) D_v V + (D_u V)^T \delta_V^2 \mathcal{H}(W) D_v V \\ (D_v V)^T \delta_{UV}^2 \mathcal{H}(W) + (D_v V)^T \delta_V^2 \mathcal{H}(W) D_u V & (D_v V)^T \delta_V^2 \mathcal{H}(W) D_v V \end{pmatrix}.$$

Then,  $\mathcal{A}_0(w)$  is bloc diagonal considering the fact that

$$(D_v V)^T \delta_{UV}^2 \mathcal{H}(W) + (D_v V)^T \delta_V^2 \mathcal{H}(W) D_u V = 0.$$

Indeed,  $v = \delta_V \mathcal{H}(W)$  and  $u = U$  give us

$$\begin{aligned} (D_v V)^T \delta_{UV}^2 \mathcal{H}(W) + (D_v V)^T \delta_V^2 \mathcal{H}(W) D_u V &= (D_v V)^T D_U v + (D_v V)^T D_V v D_u V \\ &= (D_v V)^T D_U v D_u U + (D_v V)^T D_V v D_u V = (D_v V)^T (D_U v D_u U + D_V v D_u V) \\ &= (D_v V)^T D_u v = 0. \end{aligned}$$

■

Let us now add a right hand side term of the following form to (2.A.1)

$$\begin{cases} \partial_t U + \partial_x F_1(U) = 0, \\ \partial_t V + \partial_x F_2(V) = q(W), \end{cases} \quad (2.A.3)$$

where  $q$  is a smooth application of  $W$  and  $(U, V)$  is a decomposition of  $W$  satisfying the assumptions of Proposition (2.A.1). Again, we act  $D_w W^*(w) \delta^2 \mathcal{H}(w)$  on System (2.A.1) to find

$$\mathcal{A}_0(w) \partial_t w + \mathcal{A}_1(w) \partial_x w = G(w),$$

with

$$G(w) = (D_w W)^T \delta_W^2 \mathcal{H}(W) Q(W).$$

We are now going to see that  $Q(W) = (0, q(W))$  is an eigenvector for the eigenvalue 1 of  $(D_w W)^T \delta_W^2 \mathcal{H}(W)$ . In fact, the following proposition holds true.

**Proposition 2.A.2** *The right hand side term  $G(w)$  is equal to  $Q(W)$ .*

**Proof** We have by assumptions

$$G(W) = (D_w W)^T \delta_W^2 \mathcal{H}(W) Q(W) = \begin{pmatrix} \delta_{VU}^2 \mathcal{H} q(W) + (D_u V)^T \delta_V^2 \mathcal{H} q(W) \\ (D_v V)^T \delta_V^2 \mathcal{H} q(W) \end{pmatrix}.$$

Considering the fact that the first components (associated to  $U$ ) of  $G(W)$  are the same as the upper non diagonal bloc of the operator  $\mathcal{A}_0(w)$  considered in the proof of Proposition 2.A.1 acting on  $q(W)$ , these components vanish. On the other hand,

$$\begin{aligned} (D_v V)^T \delta_V^2 \mathcal{H} q(W) &= (D_v V)^T (\delta_V^2 \mathcal{H})^T q(W) = (D_v V)^T (D_V v)^T q(W) \\ &= (D_V v D_v V)^T q(W) = (D_V v (D_V v)^{-1})^T q(W) = q(W). \end{aligned}$$

■

## 2.B Local well-posedness

Let us first note that there exists  $0 < \delta < h_e$  such that  $\mathcal{A}_0(V)$  is invertible for all  $V \in B_s(V_e, \delta)$ . Then, consider the associated linear problem

$$\begin{cases} \partial_t V + \mathcal{A}_0^{-1}(V) \mathcal{A}_1(V) \partial_x V = \begin{pmatrix} 0 \\ \mu \mathcal{L}_h^{-1}(\partial_x(h \partial_x u)) \end{pmatrix} \\ V(0, x) = g_0(x) \end{cases} \quad (2.B.1)$$

where  $V \in C([0, T]; \mathbb{X}^s(\mathbb{R}))$  and  $\partial_t V \in \mathbb{X}^{s-1}(\mathbb{R})$  for some  $s \geq 2$  and  $g_0 \in \mathbb{X}^s(\mathbb{R})$ . It is proved in [69] that the problem admits a unique solution  $V$  in  $C([0, T]; \mathbb{X}^s(\mathbb{R}))$ . We now consider the following iteration scheme

$$\begin{cases} \mathcal{A}_0(V^k) \partial_t V^{k+1} + \mathcal{A}_1(V^k) \partial_x V^{k+1} = \begin{pmatrix} 0 \\ \mu \partial_x(h^k \partial_x u^k) \end{pmatrix} \\ V^{k+1}(0, x) = g^{k+1}(x) \end{cases} \quad (2.B.2)$$

where  $g^{k+1} = \epsilon^k V_0 \star \rho(\frac{\cdot}{\epsilon^k})$  for some mollifier  $\rho^7$  with the positive real set  $\epsilon^k = \frac{\beta}{2^k}$ , with  $\beta > 0$ . We initialize the iteration by  $g^0 = V_0$ . We know that (2.B.2) admits a unique solution for all positive integer  $k$ . Let us now assume that  $V^l(t) \in B_s(V_e, \delta)$  for all  $l \leq k$  and all  $t \in [0, T]$ . This implies by triangle inequality that

$$\|V^l - g^0\|_{C([0, T]; \mathbb{X}^s)} \leq 2\delta \quad (2.B.3)$$

for all  $l \leq k$ . We can show that there exists a suitable  $T > 0$  such that the estimate (2.B.3) holds also true for  $l = k + 1$ . In fact, we consider the  $\bar{s}$ th derivative of (2.B.2), take the scalar product with  $\partial_x^{\bar{s}+1}(V^{k+1} - g^0)$  and we sum over  $\bar{s} \in \{0, \dots, s\}$ . Then, using very similar logics as in 2.3.2, we find for all  $0 \leq t \leq T$ ,

$$\begin{aligned} \|V^{k+1}(t) - g^0\|_{\mathbb{X}^s}^2 &\leq C_{\{\|g^0\|_{L^\infty}\}}(\delta) \|g^{k+1} - g^0\|_{\mathbb{X}^s}^2 + C_{\{\|g^0\|_{L^\infty}, \mu\}}(\delta) \int_0^t \|V^{k+1}(t') - g^0\|_{\mathbb{X}^s}^2 dt' \\ &\quad + C_{\{\|g^0\|_{L^\infty}, \mu\}}(\delta) t. \end{aligned}$$

<sup>7</sup> $\rho : \mathbb{R} \rightarrow \mathbb{R}^+$  is infinity derivable compactly supported in the unit ball with  $\int_{\mathbb{R}} \rho = 1$ .

Then, Gronwall lemma leads us, for  $\delta$  small enough, to

$$\|V^{k+1} - g^0\|_{C([0,T];\mathbb{X}^s)}^2 \leq Ce^{\lambda T} \left( \|g^{k+1} - g^0\|_{\mathbb{X}^s}^2 + T \right).$$

where  $C$  and  $\lambda$  are strictly positive reals independent of  $k$ . On the other hand, there exists by assumption,  $\epsilon_0 > 0$  such that

$$\|g^{k+1} - g^0\|_{\mathbb{X}^s} \leq \epsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

Then, choosing  $\beta$  small enough (therefore  $\epsilon_0$  small enough), there exists  $T > 0$  such that the condition (2.B.3) is satisfied for all  $l \in \mathbb{N}$ . We assume from now that  $T$  and  $\beta$  are small enough to give us (2.B.3) for all positive integer. Then, we consider the  $\bar{s}$ th derivative of (2.B.2) for iterations  $k$  and  $k - 1$ , take the scalar product with  $\partial_x^{\bar{s}+1}(V^{k+1} - V^k)$ , subtract the two equations and sum over  $\bar{s} \in \{0, \dots, s\}$ . Likewise, we get

$$\begin{aligned} \|V^{k+1}(t) - V^k(t)\|_{\mathbb{X}^s}^2 &\leq \gamma \|g^{k+1} - g^k\|_{\mathbb{X}^s}^2 + \theta \int_0^t \|V^k(t') - V^{k-1}(t')\|_{\mathbb{X}^s}^2 dt' \\ &\quad + \theta \int_0^t \|V^{k+1}(t') - V^k(t')\|_{\mathbb{X}^s}^2 dt' \end{aligned}$$

for some convenient positive  $\gamma, \theta$ .

Applying the Gronwall lemma, we have for all  $k \in \mathbb{N}$

$$\|V^{k+1} - V^k\|_{C([0,T];\mathbb{X}^s)}^2 \leq e^{\lambda T} \left( \|g^{k+1} - g^k\|_{\mathbb{X}^s}^2 + \theta \int_0^T \|V^k(t') - V^{k-1}(t')\|_{\mathbb{X}^s}^2 dt' \right). \quad (2.B.4)$$

Now, we sum (2.B.4) on  $k \in \mathbb{N}$ . This leads us to

$$(1 - \theta T e^{\lambda T}) \sum_{k \in \mathbb{N}} \|V^{k+1} - V^k\|_{C([0,T];\mathbb{X}^s)}^2 \leq e^{\lambda T} \sum_{k \in \mathbb{N}} \|g^{k+1} - g^k\|_{\mathbb{X}^s}^2.$$

Then, considering the fact the  $T$  is small and the fact that the sum  $\sum_{k \in \mathbb{N}} \|g^{k+1} - g^k\|_{\mathbb{X}^s}^2$  is convergent, we conclude that the set  $V^k$  is convergent in  $C([0, T]; \mathbb{X}^s(\mathbb{R}))$ . The uniqueness can be proved by the same way. In fact, we obtain a very similar approximation to (2.B.4) for  $\|V^1 - V^2\|_{\mathbb{X}^s}$  considering two solutions  $V^1(x, t)$  and  $V^2(x, t)$  for the initial conditions  $V_1(x)$  and  $V_2(x)$ . Hence, the local well-posedness is proved.

## 2.C Linear stability of equilibriums of the Green-Naghdi equation

In this part we are going to see another use of the symmetric structure of the Green-Naghdi equation. In fact, this structure enables us to prove the linear stability of an equilibrium  $V_e = (h_e, u_e)$  with  $h_e > 0$ , for the system without any dissipative right hand side term. To see this, let us consider the solution  $V \in C([0, T]; \mathbb{X}^s(\mathbb{R}))$  of the linearized system

$$\mathcal{A}_0(V_e)\partial_t V + \mathcal{A}_1(V_e)\partial_x V = 0, \quad (2.C.1)$$

act  $\partial_x^\ell$  on (2.C.1) for  $0 \leq \ell \leq s$ , and take the scalar product by  $\partial_x^\ell(V - V_e)$  :

$$\int_0^t \int_{\mathbb{R}} \mathcal{A}_0(V_e) \partial_t \partial_x^\ell V \cdot \partial_x^\ell(V - V_e) + \int_0^t \int_{\mathbb{R}} \mathcal{A}_1(V_e) \partial_x^{\ell+1} V \cdot \partial_x^\ell(V - V_e) = 0. \quad (2.C.2)$$

Now, considering the facts that

$$\int_{\mathbb{R}} \mathcal{A}_0(V_e) \partial_t \partial_x^\ell V \cdot \partial_x^\ell(V - V_e) = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \mathcal{A}_0(V_e) \partial_x^\ell(V - V_e) \cdot \partial_x^\ell(V - V_e),$$

and

$$\int_0^t \int_{\mathbb{R}} \mathcal{A}_1(V_e) \partial_x^{\ell+1} V \cdot \partial_x^\ell(V - V_e) = 0,$$

together with the  $\mathbb{X}^0$ -quadraticity of  $\mathcal{A}_0(V_e)$ , we get the following estimate,

$$\| \partial_x^\ell(V(t) - V_e) \|_{\mathbb{X}^0}^2 \leq C \| \partial_x^\ell(V(0) - V_e) \|_{\mathbb{X}^0}^2. \quad (2.C.3)$$

where  $C$  is a strictly positive constant depending only on  $h_e, \alpha$  and  $g$ . Hence, we have the following proposition,

**Proposition 2.C.1** *Let  $s \geq 2$  be an integer and consider the initial data  $V_0 \in \mathbb{X}^s(\mathbb{R})$ . Then, there exists  $C > 0$  such that the solution  $V$  of (2.C.1) satisfies for all time,*

$$\| V(t) - V_e \|_{\mathbb{X}^s}^2 \leq C \| V_0 - V_e \|_{\mathbb{X}^s}^2. \quad (2.C.4)$$

This gives us the linear stability of the equilibrium of (2.1.1).

**Theorem 2.C.2** *Let  $s \geq 2$  be an integer and consider the Green-Naghdi system,*

$$\begin{cases} \partial_t h + \partial_x h u = 0, \\ \partial_t h u + \partial_x(h u^2) + \partial_x(g h^2/2 + \alpha h^2 \ddot{h}) = 0. \end{cases} \quad (2.C.5)$$

*Then, the equilibrium solutions  $V_e = (h_e, u_e)$ , with  $h_e > 0$ , are linearly stable for the  $\mathbb{X}^s$  norm.*

Let us note that this theorem can be generalized to all locally well-posed symmetrizable system of the form (2.C.1) such that  $\mathcal{A}_0(V_e)$  is quadratic.





## **Part II**

# **Numerical simulation of the free surface incompressible Navier–Stokes equations**



# Chapter 3

## Numerical tools and global algorithm

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### 3.1 Introduction

Similarly to incompressible two-phase flows, modeling and simulation of free surface incompressible fluids are important problems for the scientific community. On the one hand, this is due to the various applications in engineering. On the other hand, the treatment of the free-surface is still a difficult challenge and few results are available on the mathematical analysis of these problems. See [2, 3, 98, 97, 107] and references therein for some well-posedness results for free surface incompressible fluids modeled by Stokes/Navier–Stokes equations. Free surface incompressible problems have been addressed dealt with two different approaches in the literature. The first one,

widely used in oceanography, consists on substituting free surface Euler or Navier–Stokes equation by their shallow water approximation (see for instance [57, 62, 9]). The advantage of this approach is that the obtained shallow water model is usually easier to study theoretically and numerically. A famous example of these models is the Saint-Venant equations which is a particular case of isentropic Euler equations. Nevertheless, these models can be considered only in the shallow water regime. Moreover, the free surface is fully represented by the height of water, which is an unknown of the problem, as a function of time and horizontal space variables. For this reason, the numerical treatment of the free surface is not an extra challenge and the free surface location is found by solving an evolution partial differential equation with boundary conditions. Breaking waves however can not be described using this approach. The second approach is to consider the full free surface Navier-Stokes problem. In this case, the domain on which the Navier–Stokes equations are posed is also an unknown. The first numerical method introduced to treat this problem is the marker and cell method (MAC) [65]. In this method, the authors resolve the problem in a fixed grid using finite-difference method and represent the free surface by markers which move with the fluid (see also [103, 45, 46, 66]). Since then, other techniques have been developed to deal with the simulation of the free surface. Arbitrary Lagrangian-Eulerian method (ALE) [12, 7, 51, 59], Volume of Fluid methods (VOF) [67, 94, 58, 99] and Level Set method (LS) [88, 40] are among the most famous. The ALE method is based on the assumption that each fluid particle is transported by the velocity field. Thus, its trajectory is described by a characteristic equation whose flow is the fluid velocity. In this method, the underlying mesh is moved with the flow velocity in order to deal partially with topology changes and severe displacements i.e. distortions. In the VOF method, the interface is represented by the characteristic function of the fluid domain. This discontinuous function is advected by the flow velocity. Thus, the method is based on the weak formulation of the advection equation. It is able to handle topology changes but is quite tedious to implement on unstructured meshes. In the LS method, the free surface is represented by the 0-level set of a continuous function. Again, this function is advected by the velocity flow but thanks to its continuity, no weak formulation of the advection equation is considered. Therefore, this method can be easily implemented on unstructured meshes. All these three approaches can be also used without difficulty for bi-fluid incompressible flows.

The main purpose of this work is to adapt the work done in [100] on bi-fluid incompressible models, to free surface incompressible fields. Therefore, the free surface is modeled by the level set method with a moving mesh. In other words, the free surface is explicitly discretized by a mesh which is moved with the flow velocity. The main improvement of this work is that Navier–Stokes equations are resolved in the fluid domain only. This approach is reasonable since only one fluid behavior is of interest. The other part of the domain is not used in the resolution of the Navier–Stokes equations, it is only represented by an external pressure which is equal to the atmospheric pressure if the fluid is in contact with air and vanishes for a fluid in the vacuum. This helps to reduce the time of computation consequently since the Navier-Stokes equations is solved only on a small part of the domain of computation. No constraints are considered here on the geometrical structure of the computational domain. Friction and surface tension may easily be taken into account in the formulation.

The first section is devoted to the mathematical formulation of the physical problem. Section 3.3 details the numerical tools used in this work. The adaptation of the global scheme of [100] to our problem is presented in Section 3.4. Finally, Section 3.5 contains some numerical results.

## 3.2 Mathematical model

We consider in this work an incompressible Newtonian fluid with viscosity  $\mu > 0$  and density  $\rho > 0$ . This fluid is contained in a fixed  $d$ -dimensional<sup>1</sup> computational domain  $D$  with boundary  $\partial D$ . The fluid domain is time dependent and is assumed to be included in  $D$ . This domain is denoted by  $\Omega_t$  at time  $t$  and satisfies  $\Omega_t \subset D$ . The boundary  $\partial\Omega_t$  of  $\Omega_t$  is represented by the partition  $\partial\Omega_t = \Gamma_t \cup (\partial\Omega_t \setminus \Gamma_t)$  where the boundary part  $\Gamma_t$  represents the free surface. A surface tension with a coefficient  $\gamma > 0$  is considered on the free surface. We assume that the fluid slips on  $\partial\Omega_t \setminus \Gamma_t$  with a friction coefficient denoted by  $\alpha > 0$ . An atmospheric pressure  $p^a$  is exerted on the fluid. The external normal on the free surface is denoted by  $\mathbf{n}$ .

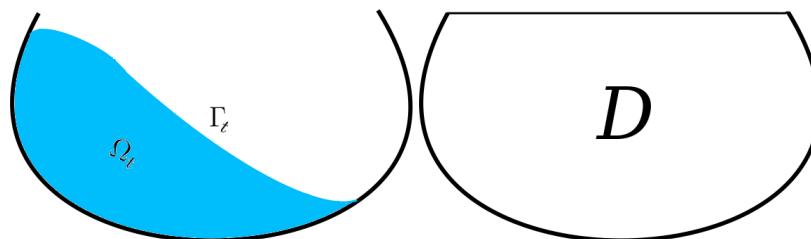


Figure 3.2.1: Illustration of a computational domain  $D$  containing the fluid domain  $\Omega_t$  at time  $t$ .

### 3.2.1 Incompressible Navier-Stokes equations

Incompressible Newtonian fluids satisfy the incompressible Navier–Stokes equations. Moreover, the free surface is advected by the flow velocity. This means that the system is described by the following partial differential equations:

$$\begin{cases} \rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) - \mu \Delta \mathbf{u} + \nabla p = \rho \mathbf{f} & \text{in } \Omega_t, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega_t, \\ \mathbf{u}(0, x) = \mathbf{u}_0(x) & \text{in } \Omega_0, \end{cases} \quad (3.2.1)$$

where  $\mathbf{u}_0$  and  $\Omega_0$  are respectively the initial velocity flow and the initial fluid domain while  $\mathbf{f}$  is the density of external force. The problem unknowns are  $\mathbf{u}$  and  $p$  which represent respectively the velocity flow and the pressure of the fluid. This system is endowed with Navier slip conditions on the computational domain boundary and with a Neumann condition on the free surface:

$$\begin{cases} \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega_t \setminus \Gamma_t, \\ [\alpha \mathbf{u} + \mu (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) \mathbf{n}]_{\tan} = 0 & \text{on } \partial\Omega_t \setminus \Gamma_t, \\ (\mu (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) - p) \mathbf{n} = -(\gamma \kappa + p^a) \mathbf{n} & \text{in } \Gamma_t, \end{cases} \quad (3.2.2)$$

Let us note that  $\kappa(\mathbf{x})$  is the algebraic mean curvature, for  $\mathbf{x}$  on  $\Gamma_t$ , which is positive if the free surface bends towards  $\Omega_t$  and negative otherwise. Moreover, the evolution of the free surface is

<sup>1</sup>with  $d = 2, 3$ .

described by

$$(1, \mathbf{u}) \text{ is tangential to the free surface } (t, \Gamma_t), t \geq 0. \quad (3.2.3)$$

Equations (3.2.1), (3.2.2) and (3.2.3) together, give us a mathematical model representing the fluid dynamic.

### Remark

In this model, Dirichlet boundary conditions are not considered. However, these conditions can be added with no extra difficulty. In fact, the free surface model is presented only with Navier slip boundary conditions mainly for the sake of simplicity. Nevertheless, all of the numerical tools presented in Section 3.3 as well as the global algorithm can get adapted without any technical or mathematical difficulty to mixed Dirichlet-slip boundary conditions. Let us also note that Dirichlet boundary condition can also be obtained by tending the friction coefficient  $\alpha$  to  $\infty$ .

### 3.2.2 Interface capturing with level set methods

As mentioned previously, the fluid domain can be characterized by a continuous level set function  $\phi(\mathbf{x}, t)$  defined on the whole computational domain, such that

$$\begin{cases} \phi(\mathbf{x}, t) < 0 \iff \mathbf{x} \in \Omega_t \\ \phi(\mathbf{x}, t) = 0 \iff \mathbf{x} \in \Gamma_t. \end{cases} \quad (3.2.4)$$

Therefore, the knowledge of the level set function is the same as the knowledge of the fluid domain. This is the approach considered in this work. In this frame, the free surface evolution relation (3.2.3) is substituted by the advection equation

$$\begin{cases} \frac{\partial \phi}{\partial t}(\mathbf{x}, t) + \tilde{\mathbf{u}}(\mathbf{x}, t) \cdot \nabla \phi(\mathbf{x}, t) = 0 \quad \forall (\mathbf{x}, t) \in D \times (0, \infty), \\ \phi(\mathbf{x}, 0) = \phi_0(\mathbf{x}) \quad \forall \mathbf{x} \in D. \end{cases} \quad (3.2.5)$$

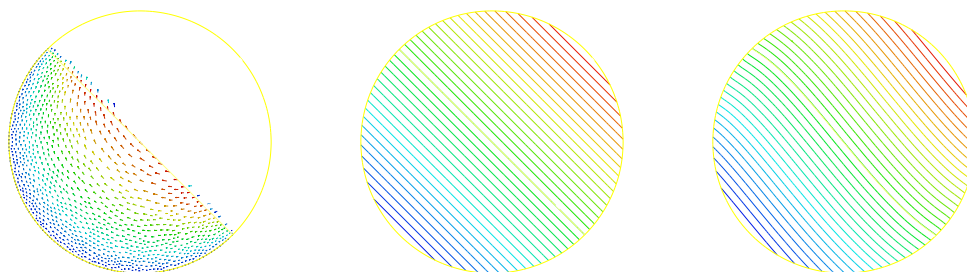
where  $\tilde{\mathbf{u}}$  is a smooth vector field defined on  $D$  which coincides with the fluid velocity of the free surface i.e. it satisfies

$$\tilde{\mathbf{u}}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) \quad \text{on } \Gamma_t \quad \forall t \geq 0.$$

Moreover,  $\phi_0(\mathbf{x})$  is a level set function for the initial fluid domain  $\Omega_0$ . In this work, the initial level set function considered is the signed distance function to  $\Gamma_0$  i.e.

$$\phi_0(\mathbf{x}) = d_{\Gamma_0}(\mathbf{x}) = \begin{cases} -d(\mathbf{x}, \Gamma_0) & \text{if } \mathbf{x} \in \Omega_0 \\ 0 & \text{if } \mathbf{x} \in \Gamma_0 \\ d(\mathbf{x}, \Gamma_0) & \text{if } \mathbf{x} \in D \setminus \Omega_0 \end{cases} \quad (3.2.6)$$

where  $d(\cdot, \Gamma_0)$  denotes the usual Euclidean distance function to  $\Gamma_0$ . As mentioned in [22], the smoothness of the vector field  $\tilde{\mathbf{u}}$  is necessary in this work since we intend to solve the advection method by the method of characteristics. The regularity of  $\tilde{\mathbf{u}}$  guarantees the mathematical well-posedness of the problem as well as its numerical accuracy. This is also why the initial level set function considered is the signed distance function. In fact, an initial signed distance function with



**Figure 3.2.2:** The initial fluid velocity field (left), the initial level set function associated with the initial fluid domain (middle), the advected level set function by the fluid velocity at time  $t = 0.006$ .

too steep or too loose variations in its level sets, may jeopardize the accuracy of computations.

### 3.3 Numerical tools

In this section, we present numerical techniques necessary to understand the global scheme of Section 3.4.

#### 3.3.1 Method of characteristics for the advection equation

The method of characteristics [33, 16, 76, 13] is known to be very efficient for solving advection-diffusion problems, including the Navier-Stokes equations, see [89, 90, 91] for the mathematical and numerical method and its application to fluid mechanics problems.

Here, we consider the Cauchy problem for the advection equation (3.2.5): *given a velocity field  $\tilde{\mathbf{u}}(\mathbf{x}, t)$  defined on  $D$ , find a scalar solution  $\phi(\mathbf{x}, t)$  defined on  $D \times [0, T]$  solving the equation:*

$$\begin{cases} \frac{\partial \phi}{\partial t}(\mathbf{x}, t) + \tilde{\mathbf{u}}(\mathbf{x}, t) \cdot \nabla \phi(\mathbf{x}, t) = 0 & \text{on } D \times [0, T] \\ \phi(\mathbf{x}, 0) = \phi_0(\mathbf{x}) & \text{for } \mathbf{x} \in D. \end{cases} \quad (3.3.1)$$

Under proper hypotheses on the regularity and the growth of  $\tilde{\mathbf{u}}$  and  $\phi_0$  [90], the unique  $C^1$  solution  $\phi$  of (3.3.1) is

$$\phi(\mathbf{x}, t) = \phi_0(\mathbf{X}(\mathbf{x}, t; 0)) \quad (3.3.2)$$

with  $s \mapsto \mathbf{X}(\mathbf{x}, t; s)$  the *characteristic curve* of  $\tilde{\mathbf{u}}$  passing through  $\mathbf{x}$  at time  $t$  and defined as the



solution of the ODE system:

$$\begin{cases} \frac{d\mathbf{X}(\mathbf{x}, t; s)}{ds} = \tilde{\mathbf{u}}(\mathbf{X}(\mathbf{x}, t; s), t) & \text{for } s \in \mathbb{R} \\ \mathbf{X}(\mathbf{x}, t; t) = \mathbf{x}. \end{cases} \quad (3.3.3)$$

This curve describes the trajectory of a particle at position  $\mathbf{x}$  at time  $t$  and transported by the velocity field  $\tilde{\mathbf{u}}$ . The first equation of (3.3.1) implies that  $\phi(\mathbf{x}, t)$  is constant along the characteristics  $\mathbf{X}(\mathbf{x}, t; s)$ . Hence, the solution of the Cauchy problem (3.3.1) reads:

$$\phi(\mathbf{x}, t) = \phi_0(\mathbf{X}(\mathbf{x}, t; 0), 0), \quad \text{on } D \times [0, T] \quad (3.3.4)$$

In the numerical setting of this work, we divide the time interval  $[0, T]$  into a finite number of subintervals of the form  $[t^{n-1}, t^n]$  with  $t^n = t^{n-1} + \Delta t$ , for a time step  $\Delta t$ . Then, we denote  $\phi^0(\mathbf{x}) := \phi_0(\mathbf{x})$  and define  $\phi^n$ , for all integer  $n \in [1, \frac{T}{\Delta t}]$ , as the solution of

$$\begin{cases} \frac{\partial \phi^n}{\partial t}(\mathbf{x}, t) + \tilde{\mathbf{u}}(\mathbf{x}, t) \cdot \nabla \phi^n(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in D \times (t^{n-1}, t^n) \\ \phi(\mathbf{x}, t^n) = \phi^{n-1}(\mathbf{x}), & \forall \mathbf{x} \in D. \end{cases} \quad (3.3.5)$$

Again, we use the characteristic method to solve (3.3.5). In other words, the solution  $\phi^n$  of this equation is approximated by

$$\phi^n(\mathbf{x}) = \phi^{n-1}(\mathbf{X}(\mathbf{x}, t^n; t^{n-1})), \quad \text{for all } \mathbf{x} \in D \quad (3.3.6)$$

where  $\mathbf{X}(\mathbf{x}, t^n; t^{n-1})$  is the position at the time  $t^{n-1}$  of the characteristic line emerging from  $\mathbf{x}$  at the time  $t^n$ . This function is computed by

$$\begin{cases} \frac{d\mathbf{X}(\mathbf{x}, t^n; t)}{dt} = \tilde{\mathbf{u}}(\mathbf{X}(\mathbf{x}, t^n; t), t) & \text{for all } (\mathbf{x}, t) \in D \times (t^{n-1}, t^n) \\ \mathbf{X}(\mathbf{x}, t^n; t^n) = \mathbf{x} & \text{for all } x \in D. \end{cases} \quad (3.3.7)$$

Let us also note that the first equation of (3.3.7) is equivalent to

$$\mathbf{X}(\mathbf{x}, t^n; t) = \mathbf{x} - \int_t^{t^n} \tilde{\mathbf{u}}(\mathbf{X}(\mathbf{x}, t^n; t), t) dt. \quad (3.3.8)$$

This integral can be approximated by any method used for solving ordinary differential equations. For example, introducing a small sub-integration time step  $\delta t \ll \Delta t$  and subdividing the interval  $]t^{n-1}, t^n[ (= \cup_{l=0}^L ]t^l, t^{l+1}[$  with  $t^l = t^{n-1} + l\delta t$ , a fourth order Runge-Kutta schemes yields for all  $\mathbf{x} \in D$  to the following approximation  $\tilde{\mathbf{X}}(\mathbf{x}, t^n, \cdot)$  of  $\mathbf{X}(\mathbf{x}, t^n, \cdot)$ , defined on  $\{t^l\}_{l \in \{0, \dots, L\}}$  computed

by:

$$\left\{ \begin{array}{l} \tilde{\mathbf{X}}(\mathbf{x}, t^n; t^n) = \mathbf{x} \\ \tilde{\mathbf{X}}(\mathbf{x}, t^n; t^l) = \tilde{\mathbf{X}}(\mathbf{x}, t^n; t^{l+1}) - \frac{\delta t}{6}(v_1 + 2v_2 + 2v_3 + v_4), \\ \text{with } v_1 = \tilde{\mathbf{u}}(\tilde{\mathbf{X}}(\mathbf{x}, t^n; t^l)) \\ v_2 = \tilde{\mathbf{u}}(\tilde{\mathbf{X}}(\mathbf{x}, t^n; t^{l+1}) - \frac{\delta t}{2}v_1) \\ v_3 = \tilde{\mathbf{u}}(\tilde{\mathbf{X}}(\mathbf{x}, t^n; t^{l+1}) - \frac{\delta t}{2}v_2) \\ v_4 = \tilde{\mathbf{u}}(\tilde{\mathbf{X}}(\mathbf{x}, t^n; t^{l+1}) - \delta t v_3). \end{array} \right.$$

We are now able to use (3.3.6) to approximate  $\phi^n$  for all integer  $n \in [1, \frac{T}{\Delta t}]$ . This problem can be solved by a Galerkin numerical scheme which involves resolution of a linear system with using quadrature formulas for approximating integrals in the right hand side. Another alternative is the Lagrange interpolation. In fact, both methods have been used in previous works. The first approach in [23] is more expensive than the second one in [27]. Moreover, the second approach leads to an estimate which enables us to control the geometric error on the interface by the interpolation error. This is very interesting for the mesh adaptation step. Therefore, this second approach is considered in the present work. Indeed, we endow here the computational domain  $D$  with a mesh  $D_h$  and we consider a suitable Lagrange finite element space  $V$  (e.g.  $\mathbb{P}^1$  or  $\mathbb{P}^2$ ). Then, we denote by  $\phi_h^0$  the  $V$  projection of the initial level set function  $\phi_0$ . Moreover, at each degree of freedom represented by  $\mathbf{x}$ , we compute the approximation  $\phi_h^n$  of  $\phi^n$  as following:

$$\phi_h^n(\mathbf{x}) = \phi_h^{n-1}(\tilde{\mathbf{X}}(\mathbf{x}, t^{n+1}; t^n)) \quad (3.3.9)$$

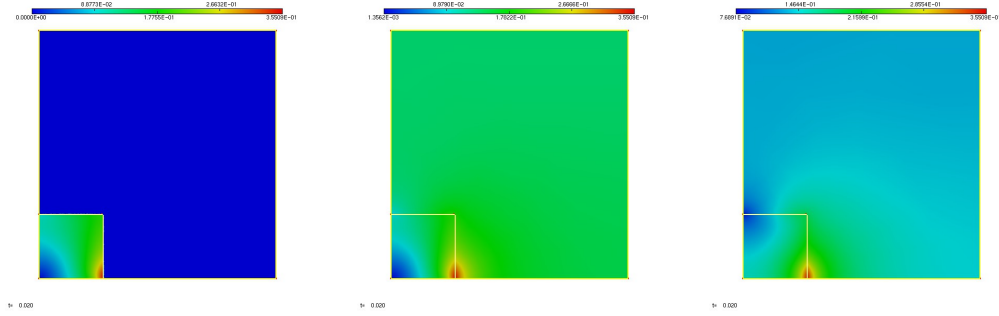
As mentioned in [27], this method requires the resolution of one ODE at each degree of freedom of  $D_h$  and is efficient since no matrix inversion nor quadrature formulas for approximating integrals are considered. This first-order space approximation is sufficient here since in our frame work, only values of the level set function in the vicinity of the free surface are of interest. Moreover, thanks to a mesh adaptation step, mesh elements are very small in the vicinity of the interface. We also refer the reader to the previous works in [21], for more details on these schemes .

**Remark 3.3.1** *It is worth to forecast some difficulties: in the numerical framework, the characteristic curves may go out of the computational domain specially when the computational mesh approximates a computational domain with curvy boundaries. In such cases, we project the final point of the characteristic curve, corresponding to  $\tilde{\mathbf{X}}(\mathbf{x}, t^n; t^{n-1})$  for some  $n$ , on the mesh boundary. This is done by identifying the edge (or the face) of the boundary such that the barycentric coordinate of the final point with respect to this edge (or face) is strictly negative in the element to whom this edge (or face) belongs.*

### 3.3.2 Extension and regularization of the velocity field

In this work, we intend to solve the Navier–Stokes equations only on the fluid domain. Thus, the velocity field is defined only on the subdomain  $\Omega_t$  of  $D$  representing the fluid domain whereas the transport equation needs to be solved on the whole computational domain. As mentioned in Subsection 3.3.1, the flow  $\tilde{\mathbf{u}}$  considered for the advection equation, must be defined on the totality

of the computational domain  $D$  and must coincide with the flow velocity in the vicinity of the free surface. Therefore, an extension of velocity is crucial for the resolution of free-surface flows while using the level set formulation.



**Figure 3.3.3:** Norm of the fluid velocity on the fluid domain (left), Norm of the velocity after the extension (middle), Norm of the velocity after the regularization (right). The Reynolds number associated to the fluid is not large here. For this reason, the resolution of the Helmholtz equation out of the fluid domain is sufficient to obtain a velocity defined on the computational domain.

The approach we use here to construct a convenient flow  $\tilde{\mathbf{u}}$  for advection equation comes from [96, 28, 24, 39]. In fact, it is required that  $\tilde{\mathbf{u}}$  is defined away from the free surface and smoothly approaches the prescribed velocity as the zero level set is approached.

On the other hand, large velocity discrepancies between neighboring points may cause uncontrolled oscillations on the level sets and jeopardize the stability of the numerical algorithm. For this reason, a regularization of the velocity field must be considered as a step of the algorithm if the flow presents such discrepancies. The methods we use to extend and regularize the velocity flow are described in the following paragraphs.

*-Velocity extension:* This step consists in solving a non homogeneous Dirichlet–Neumann problem for a Helmholtz equation out of the fluid domain. The system to solve writes:

$$\begin{cases} -a\Delta\tilde{\mathbf{u}} + \tilde{\mathbf{u}} = 0 & \text{in } D \setminus \Omega_t \\ \nabla\tilde{\mathbf{u}} \cdot \mathbf{n} = 0 & \text{on } \partial D \setminus \partial\Omega_t \\ \tilde{\mathbf{u}} = \mathbf{u} & \text{on } \Gamma_t \end{cases} \quad (3.3.10)$$

where  $a > 0$  is an extension regularization parameter. As mentioned in [39], it must be small enough such that large values of  $\tilde{\mathbf{u}}$  outside of the free surface do not interfere with other values on the other side of the free surface. Indeed, if  $a$  is too large, some strictly positive level sets may cross the 0-level set during the advection. Therefore, the level set function does not represent any more the fluid domain during the advection. For this reason, it is important to consider a Helmholtz operator and not only a Laplace operator. The identity part of the Helmholtz operator together with a conveniently small  $a$ , guarantees a reasonable decrease for the extended velocity flow as the boundary of the domain far from the free surface are approached. On the other hand,  $a$  must obviously

not be too small, since the solution  $\tilde{\mathbf{u}}$  must be smooth enough such that the advection equation can be resolved by the characteristic method. Figure 3.3.3 illustrates the application of this extension (middle) to the velocity flow presented on the left.

-*Velocity regularization*: Regularization of the velocity consists in solving the system on the whole computational domain  $D$  in order to get a convenient field  $\tilde{\mathbf{u}}$  for the advection equation:

$$\begin{cases} -a\Delta\tilde{\mathbf{u}} + \tilde{\mathbf{u}} = 0 & \text{in } D \\ \nabla\tilde{\mathbf{u}} \cdot \mathbf{n} = 0 & \text{on } \partial D \\ \tilde{\mathbf{u}} = \mathbf{u} & \text{on } \Gamma_t \end{cases} \quad (3.3.11)$$

Let us also note that for the same reason as in the previous case,  $a$  must be not too small, not too large. Let us also mention if we solve the advection equation with a flow which is the same as the fluid velocity inside of the fluid domain, it is possible that some values of large velocities inside of the fluid domain interfere with the velocity of the fluid on the free surface i.e. some strictly negative level sets may cross the 0-level set. For this reason, we need to regularize the velocity of the fluid. As an illustration, see Figure 3.3.3 where this regularization is applied to the velocity flow presented on the left. The result of this extension is given on the right of the figure.

**Remark 3.3.2** *The critical threshold beyond which we need to consider (3.3.11) instead of system (3.3.10) depends on the geometry of the computational and the fluid domain as well as the size of time step and mesh elements. In practice, we use the first kind of extension (extension for small Reynolds number) by default. The regularization of the velocity on the whole computational domain is used only if level sets cross each other during the first kind of extension.*

In both cases, the variational problem associated to the Helmholtz problem is solved using a finite element method.

### 3.3.3 Resolution of the Navier–Stokes equations on the fluid domain

We are interested in this part in the numerical resolution of the free surface incompressible Navier–Stokes equation on the fluid domain at some time  $t$ . In other words, we endow the time discretized problem with boundary conditions (3.2.2). This time discretization is done following the pioneering works [13, 89, 60]. More precisely, we consider the term

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right)$$

which actually models the *transport* of the *momentum*  $\rho \mathbf{u}$  by the velocity field  $\mathbf{u}$  with the idea of taking advantage of the *transport nature* of the non linear term. Hence, we use here the *backward method of characteristics* for the time discretization of the Navier–Stokes equations. Then, we will use a convenient finite elements method for the spatial resolution of the time discretized equation on the fluid domain.

### Time discretization by the method of characteristics

In this section, we describe the time discretization of the Navier–Stokes equations based on the method of characteristics. This is not the only possible choice. Nevertheless, we use this scheme because it is *unconditionally stable* (see [89]). Moreover, this method is already implemented in our framework to solve the advection equation (3.3.1). Therefore, we can use the same scheme for the resolution of the advection equation and the time discretization of the Navier–Stokes equation. Moreover, we use obviously the same time step  $\Delta t$  for both advection and Navier–Stokes equation. Indeed, we have the same subdivision  $[t^{n-1}, t^n]$ , with  $t^n = t^{n-1} + \Delta t$ , of the time interval  $[0, T]$ . The main idea is that the nonlinear convection part of Navier–Stokes equations can be hidden in the Cauchy problem (3.3.3) i.e. the operator  $\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$  may be turned into a total derivative (also called Lagrange derivative)  $\frac{d}{ds}$ . Therefore, the first equation of (3.2.1) is recast into the following form:

$$\rho \frac{d\mathbf{u}(\mathbf{X}(\mathbf{x}, t; s), s)}{ds} \Big|_{s=t} - \mu \Delta \mathbf{u} + \nabla p = \rho \mathbf{f}. \quad (3.3.12)$$

Hence, we denote  $\mathbf{u}(\mathbf{x}, t^n)$  by  $\mathbf{u}^n(\mathbf{x})$  for all  $n$  and use the following explicit approximation:

$$\frac{d\mathbf{u}(\mathbf{X}(\mathbf{x}, t^n; s), s)}{ds} \Big|_{s=t^n} \approx \frac{\mathbf{u}(\mathbf{X}(\mathbf{x}, t^n; t^n), t^n) - \mathbf{u}(\mathbf{X}(\mathbf{x}, t^n; t^{n-1}), t^{n-1})}{\Delta t},$$

since

$$\frac{d\mathbf{u}(\mathbf{X}(\mathbf{x}, t^n; s), s)}{ds} \Big|_{s=t^n} = \frac{\mathbf{u}(\mathbf{X}(\mathbf{x}, t^n; t^n), t^n) - \mathbf{u}(\mathbf{X}(\mathbf{x}, t^n; t^{n-1}), t^{n-1})}{\Delta t} + O(\Delta t).$$

This approximation is equivalent, using the latter notation together with the definition (3.3.3) of the characteristic curve, to

$$\frac{d\mathbf{u}(\mathbf{X}(\mathbf{x}, t^n; s), s)}{ds} \Big|_{s=t^n} \approx \frac{\mathbf{u}^n(\mathbf{x}) - \mathbf{u}^{n-1}(\mathbf{X}(\mathbf{x}, t^n; t^{n-1}))}{\Delta t}.$$

Therefore, denoting  $\mathbf{X}(\mathbf{x}, t^n; t^{n-1})$  by  $\mathbf{X}^{n-1}(\mathbf{x})$ , the time discretized Navier–Stokes equation writes

$$\begin{cases} \rho \frac{\mathbf{u}^n(\mathbf{x}) - \mathbf{u}^{n-1} \circ \mathbf{X}^{n-1}(\mathbf{x})}{\Delta t} - \mu \Delta \mathbf{u}^n(\mathbf{x}) + \nabla p^n(\mathbf{x}) = \rho \mathbf{f}^n & \text{in } \Omega_{t^n} \\ \operatorname{div} \mathbf{u}^n(\mathbf{x}) = 0 & \text{in } \Omega_{t^n} \end{cases} \quad (3.3.13)$$

or equivalently

$$\begin{cases} \rho \frac{\mathbf{u}^n(\mathbf{x})}{\Delta t} - \mu \Delta \mathbf{u}^n(\mathbf{x}) + \nabla p^n(\mathbf{x}) = \rho \mathbf{f}^n + \rho \frac{\mathbf{u}^{n-1} \circ \mathbf{X}^{n-1}(\mathbf{x})}{\Delta t} & \text{in } \Omega_{t^n}, \\ \operatorname{div} \mathbf{u}^n(\mathbf{x}) = 0 & \text{in } \Omega_{t^n} \end{cases} \quad (3.3.14)$$

Indeed,  $\mathbf{u}^{n-1} \circ \mathbf{X}^{n-1}(\mathbf{x})$  approximates the velocity at the point  $\mathbf{X}^{n-1}(\mathbf{x})$  at the time  $t^{n-1}$ . We endow problem (3.3.14) with slip boundary and free surface conditions. Indeed, the boundary

conditions (3.2.2) writes here <sup>2</sup>

$$\begin{cases} \mathbf{u}^n \cdot \mathbf{n} = 0 & \text{on } \partial\Omega_{t^n} \setminus \Gamma_{t^n}, \\ [\alpha \mathbf{u}^n + \mu (\nabla \mathbf{u}^n + {}^t \nabla \mathbf{u}^n) \mathbf{n}]_{tan} = 0 & \text{on } \partial\Omega_{t^n} \setminus \Gamma_{t^n}, \\ (\mu (\nabla \mathbf{u}^n + {}^t \nabla \mathbf{u}^n) - p^n) \mathbf{n} = -(\gamma \kappa + p^a) \mathbf{n} & \text{in } \Gamma_{t^n}, \end{cases} \quad (3.3.15)$$

Obviously, at each time step, the Navier–Stokes problem becomes a Stokes problem plus a transport of the previous solution on the characteristic. This problem is resolved in two steps:

1. Approximate the characteristic curves  $\mathbf{X}^{n-1}(\mathbf{x})$ .
2. Solve the resulting generalized Stokes system with the corresponding boundary conditions.

**Remark 3.3.3** - *The approximation of characteristic curves  $\mathbf{X}^{n-1}(\mathbf{x})$  in each time interval  $[t^{n-1}, t^n]$  for the Navier–Stokes problem is implemented by the same way as in Subsection 3.3.1.*

- *Similarly to Subsection 3.3.1, the characteristic curves may cross some elements or go out the computational domain. In this cases we identify the last element the characteristic curve crossed and project the final point of this curve to the corresponding edge (or face). Let us recall that we intend to solve the Navier–Stokes equation on the fluid domain only. However, the flow we consider to solve the characteristic equation is not the flow velocity  $\mathbf{u}$  but the extended flow velocity  $\tilde{\mathbf{u}}$ . This lets us handle situations where the characteristic curves cross the free surface of the fluid but stay in the computational domain  $D$ .*

### Variational formulation

As mentioned previously, we intend to solve the time discretized Navier–Stokes equation (3.3.14) together with boundary conditions (3.3.15) by finite element method for all integer  $n \in [0, \frac{T}{\Delta t}]$ . Equation (3.3.14) is nothing but a generalized Stokes equation. The associated variational formulation is then obtained by the same method as for generalized stokes problems. Therefore, Hilbert spaces we consider here for the velocity flow and the pressure respectively  $V_n$  and  $Q_n$  defined by (see for instance [104, 105, 106, 41]):

$$V_n = \{\mathbf{v} \in (\mathbb{H}^1(\Omega_{t^n}))^d, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_{t^n} \setminus \Gamma_{t^n}\}, \quad (3.3.16)$$

$$Q_n = \mathbb{L}^2(\Omega_{t^n}). \quad (3.3.17)$$

Let us note that we do not need to consider zero-integral functions for  $Q_n$  since we assume that  $\Gamma_{t^n}$  is non empty for all  $n$ . We are now able to find the variational formulation of the problem. We first take the scalar product of a test function  $\mathbf{v} \in V_n$  with the first equation of (3.3.14) and integrate on

<sup>2</sup>Let us mention the the curvature  $\kappa$  at the free surface as well as the normal vector  $\mathbf{n}$  depends on time  $t^n$  since they correspond to the fluid domain  $\Omega_{t^n}$  which travels by time. Nevertheless, for sake of simplicity, we do not use any  $n$  superscript for these symbols in (3.3.14) either in the following section.

$\Omega_{t^n}$  to get:

$$\int_{\Omega_{t^n}} \frac{\rho}{\Delta t} \mathbf{u}^n \cdot \mathbf{v} - \int_{\Omega_{t^n}} \mu \Delta \mathbf{u}^n \cdot \mathbf{v} + \int_{\Omega_{t^n}} \nabla p^n \cdot \mathbf{v} = \int_{\Omega_{t^n}} \rho \left( \mathbf{f} + \frac{\mathbf{u}_*^{n-1}}{\Delta t} \right) \cdot \mathbf{v}, \quad (3.3.18)$$

where  $\mathbf{u}_*^{n-1}$  is set equal to  $\mathbf{u}^{n-1} \circ \mathbf{X}^{n-1}(\mathbf{x})$ . Then, we multiply the second equation of (3.3.14) with a test function  $q \in Q_n$  and integrate on  $\Omega_{t^n}$  to get

$$\int_{\Omega_{t^n}} \operatorname{div} \mathbf{u}^n q = 0. \quad (3.3.19)$$

The following lemma is the next step to the variational formulation.

**Lemma 3.3.4** *Let us consider the solution  $(\mathbf{u}^n, p^n) \in V_n \times Q_n$  of system (3.3.14) with boundary conditions (3.2.2). Then, we have for all  $\mathbf{v} \in V_n$*

$$\begin{aligned} \int_{\Omega_{t^n}} -\mu \Delta \mathbf{u}^n \cdot \mathbf{v} + \nabla p^n \cdot \mathbf{v} &= \int_{\Omega_{t^n}} \mu (\nabla \mathbf{u}^n + {}^t \nabla \mathbf{u}^n) : \nabla \mathbf{v} - \int_{\Omega_{t^n}} p^n \operatorname{div} \mathbf{v} + \int_{\Gamma_{t^n}} (\gamma \kappa + p^a) \mathbf{n} \cdot \mathbf{v} \\ &+ \alpha \int_{\partial \Omega_{t^n} \setminus \Gamma_{t^n}} [\mathbf{u}^n]_{tan} \cdot [\mathbf{v}]_{tan}. \end{aligned}$$

**Proof** *The first step to prove the lemma is the following equality which holds true under the incompressibility condition  $\operatorname{div} \mathbf{u}^n = 0$ ,*

$$\Delta \mathbf{u}^n = \operatorname{div} (\nabla \mathbf{u}^n + {}^t \nabla \mathbf{u}^n). \quad (3.3.20)$$

Next, we use the Green's formula [63] to remark that

$$-\mu \int_{\Omega_{t^n}} \Delta \mathbf{u}^n \cdot \mathbf{v} = \mu \int_{\Omega_{t^n}} (\nabla \mathbf{u}^n + {}^t \nabla \mathbf{u}^n) : \nabla \mathbf{v} - \mu \int_{\partial \Omega_{t^n}} (\nabla \mathbf{u}^n + {}^t \nabla \mathbf{u}^n) \mathbf{n} \cdot \mathbf{v}, \quad (3.3.21)$$

where Symbol  $:$  denotes the usual square matrix product i.e.  $A : B = \sum_{i,j=1}^d A_{ij} B_{ij}$  for all  $d$ -square matrices  $A$  and  $B$ . On the other hand, we have obviously

$$\mu \int_{\partial \Omega_{t^n}} (\nabla \mathbf{u}^n + {}^t \nabla \mathbf{u}^n) \mathbf{n} \cdot \mathbf{v} = \mu \int_{\Gamma_{t^n}} (\nabla \mathbf{u}^n + {}^t \nabla \mathbf{u}^n) \mathbf{n} \cdot \mathbf{v} + \mu \int_{\partial \Omega_{t^n} \setminus \Gamma_{t^n}} (\nabla \mathbf{u}^n + {}^t \nabla \mathbf{u}^n) \mathbf{n} \cdot \mathbf{v} \quad (3.3.22)$$

Then we have by the free surface condition on  $\Gamma_{t^n}$  (the last equation of (3.2.2)):

$$\mu \int_{\Gamma_{t^n}} (\nabla \mathbf{u}^n + {}^t \nabla \mathbf{u}^n) \mathbf{n} \cdot \mathbf{v} = \int_{\Gamma_{t^n}} (p^n - p^a - \gamma \kappa) \mathbf{n} \cdot \mathbf{v}. \quad (3.3.23)$$

On the other hand, we have

$$\begin{aligned} \mu \int_{\partial \Omega_{t^n} \setminus \Gamma_{t^n}} (\nabla \mathbf{u}^n + {}^t \nabla \mathbf{u}^n) \mathbf{n} \cdot \mathbf{v} &= \mu \int_{\partial \Omega_{t^n} \setminus \Gamma_{t^n}} ((\nabla \mathbf{u}^n + {}^t \nabla \mathbf{u}^n) \mathbf{n} \cdot \mathbf{n}) (\mathbf{v} \cdot \mathbf{n}) \\ &+ \mu \int_{\partial \Omega_{t^n} \setminus \Gamma_{t^n}} [(\nabla \mathbf{u}^n + {}^t \nabla \mathbf{u}^n) \mathbf{n}]_{tan} [\mathbf{v}]_{tan}. \end{aligned} \quad (3.3.24)$$

The first integral of the right hand side of (3.3.24) vanishes since  $v \in V_n$ . Then, the slip boundary

conditions on  $\partial\Omega_{t^n} \setminus \Gamma_{t^n}$  (the two first equations of (3.2.2)) leads us to

$$\mu \int_{\partial\Omega_{t^n} \setminus \Gamma_{t^n}} [(\nabla \mathbf{u}^n + {}^t\nabla \mathbf{u}^n) \mathbf{n}]_{tan} [\mathbf{v}]_{tan} = -\alpha \int_{\partial\Omega_{t^n} \setminus \Gamma_{t^n}} [\mathbf{u}]_{tan} [\mathbf{v}]_{tan} \quad (3.3.25)$$

Hence, (3.3.24) becomes

$$\mu \int_{\partial\Omega_{t^n} \setminus \Gamma_{t^n}} (\nabla \mathbf{u}^n + {}^t\nabla \mathbf{u}^n) \mathbf{n} \cdot \mathbf{v} = -\alpha \int_{\partial\Omega_{t^n} \setminus \Gamma_{t^n}} [\mathbf{u}]_{tan} [\mathbf{v}]_{tan} \quad (3.3.26)$$

Gathering (3.3.21), (3.3.25) and (3.3.26), we find

$$-\mu \int_{\Omega_{t^n}} \Delta \mathbf{u}^n \cdot \mathbf{v} = \mu \int_{\Omega_{t^n}} (\nabla \mathbf{u}^n + {}^t\nabla \mathbf{u}^n) : \nabla \mathbf{v} + \alpha \int_{\partial\Omega_{t^n} \setminus \Gamma_{t^n}} [\mathbf{u}]_{tan} [\mathbf{v}]_{tan} - \int_{\Gamma_{t^n}} (p^n - p^a - \gamma\kappa) \mathbf{n} \cdot \mathbf{v}. \quad (3.3.27)$$

We now apply the Green's formula to  $\int_{\Omega_{t^n}} \nabla p^n \cdot \mathbf{v}$  and get

$$\int_{\Omega_{t^n}} \nabla p^n \cdot \mathbf{v} = - \int_{\Omega_{t^n}} p^n \operatorname{div} \mathbf{v} + \int_{\partial\Omega_{t^n}} p^n \mathbf{n} \cdot \mathbf{v}. \quad (3.3.28)$$

Since the test function  $\mathbf{v}$  belongs to  $V_n$ , we get

$$\int_{\Omega_{t^n}} \nabla p^n \cdot \mathbf{v} = - \int_{\Omega_{t^n}} p^n \operatorname{div} \mathbf{v} + \int_{\Gamma_{t^n}} p^n \mathbf{n} \cdot \mathbf{v}. \quad (3.3.29)$$

Adding the both sides of equalities (3.3.27) and (3.3.29) yields the result. ■

Now, we use the lemma to rewrite (3.3.18)-(3.3.19) as following:

$$\begin{aligned} & \int_{\Omega_{t^n}} \frac{\rho}{\Delta t} \mathbf{u}^n \cdot \mathbf{v} + \int_{\Omega_{t^n}} \mu (\nabla \mathbf{u}^n + {}^t\nabla \mathbf{u}^n) : \nabla \mathbf{v} + \alpha \int_{\partial\Omega_{t^n} \setminus \Gamma_{t^n}} [\mathbf{u}^n]_{tan} \cdot [\mathbf{v}]_{tan} - \int_{\Omega_{t^n}} p^n \operatorname{div} \mathbf{v} = \\ & \int_{\Omega_{t^n}} \rho \left( \mathbf{f} + \frac{\mathbf{u}^{n-1}}{\Delta t} \right) \cdot \mathbf{v} - \int_{\Gamma_{t^n}} (\gamma\kappa + p^a) \mathbf{n} \cdot \mathbf{v}, \end{aligned} \quad (3.3.30)$$

$$\int_{\Omega_{t^n}} \operatorname{div} \mathbf{u}^n q = 0. \quad (3.3.31)$$

Let us also remark that

$$(\nabla \mathbf{u}^n + {}^t\nabla \mathbf{u}^n) : \nabla \mathbf{v} = (\nabla \mathbf{u}^n + {}^t\nabla \mathbf{u}^n) : {}^t\nabla \mathbf{v}.$$

Therefore,

$$(\nabla \mathbf{u}^n + {}^t\nabla \mathbf{u}^n) : \nabla \mathbf{v} = 2\mathbf{D}(\mathbf{u}^n) : \mathbf{D}(\mathbf{v}),$$

where  $\mathbf{D}$  denotes the symmetric gradient also called rate of deformation tensor i. e.

$$\mathbf{D}(\cdot) = \frac{\nabla(\cdot) + {}^t\nabla(\cdot)}{2}.$$

The variational formulation is then given in the following proposition.

**Proposition 3.3.5** *The variational formulation corresponding to (3.3.14), (3.2.2) is:*



Find  $(\mathbf{u}^n, p^n)$  in  $V_n \times Q_n$  such that we have for all  $(\mathbf{v}, q) \in V_n \times Q_n$

$$\begin{cases} a(\mathbf{u}^n, \mathbf{v}) + b(\mathbf{v}, p^n) = l(\mathbf{v}), \\ b(\mathbf{u}^n, q) = 0. \end{cases} \quad (3.3.32)$$

In this formulation,  $a(\cdot, \cdot)$  is a continuous bilinear symmetric coercive operator defined on  $V_n$  by

$$a(\mathbf{u}, \mathbf{v}) = \frac{\rho}{\Delta t} \int_{\Omega_{t^n}} \mathbf{u} \cdot \mathbf{v} + 2\mu \int_{\Omega_{t^n}} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) + \alpha \int_{\partial\Omega_{t^n} \setminus \Gamma_{t^n}} [\mathbf{u}]_{tan} \cdot [\mathbf{v}]_{tan}. \quad (3.3.33)$$

Besides, the bilinear operator  $b(\cdot, \cdot)$  is defined on  $V_n \times Q_n$  by

$$b(\mathbf{u}, q) = - \int_{\Omega_{t^n}} \operatorname{div} \mathbf{u} q. \quad (3.3.34)$$

The linear operator  $l(\cdot)$  is continuous on  $V_n$  and given by

$$l(\mathbf{v}) = \int_{\Omega_{t^n}} \rho \left( \mathbf{f} + \frac{\mathbf{u}_*^{n-1}}{\Delta t} \right) \cdot \mathbf{v} - \int_{\Gamma_{t^n}} (\gamma\kappa + p^a) \mathbf{n} \cdot \mathbf{v}.$$

**Remark 3.3.6** Let us mention that the operator  $b$  satisfies the Babuska-Brezzi inf-sup condition and the variational formulation (3.3.32) is well-posed (for instance, see [19]).

### Penalization technique for slip boundary condition

The variational formulation (3.3.32) is defined on the Hilbert space  $V_n \times Q_n$ . However, the construction of a basis of  $V_n$  as well as its finite element approximation is a challenge since  $V_n$  contains only functions whose velocity is tangent to  $\partial\Omega_{t^n} \setminus \Gamma_{t^n}$  and is a strict subspace of  $(\mathbb{H}^1(\Omega_{t^n}))^d$ . We use the penalization technique to overcome with this difficulty (see [41]). In this method, we do not consider any more a strict subspace of  $\mathbb{H}^1(\Omega_{t^n})^d$  but the whole space. The non-penetration condition  $(\mathbf{u} \cdot \mathbf{n})|_{\partial\Omega_{t^n} \setminus \Gamma_{t^n}} = 0$  is considered by adding a penalization term to the first equation of the variational formulation. In fact, we consider the following variational formulation:

Find  $(\mathbf{u}_\epsilon^n, p_\epsilon^n)$  in  $K_n \times Q_n$  such that we have for all  $(\mathbf{v}, q) \in K_n \times Q_n$

$$\begin{cases} a_\epsilon(\mathbf{u}_\epsilon^n, \mathbf{v}) + b(\mathbf{v}, p_\epsilon^n) = l(\mathbf{v}), \\ b(\mathbf{u}_\epsilon^n, q) = 0. \end{cases} \quad (3.3.35)$$

where

$$K_n = (\mathbb{H}^1(\Omega_{t^n}))^d,$$

and  $a_\epsilon$  is defined by

$$a_\epsilon(\mathbf{u}, \mathbf{v}) = \frac{\rho}{\Delta t} \int_{\Omega_{t^n}} \mathbf{u} \cdot \mathbf{v} + 2\mu \int_{\Omega_{t^n}} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) + \alpha \int_{\partial\Omega_{t^n} \setminus \Gamma_{t^n}} [\mathbf{u}]_{tan} \cdot [\mathbf{v}]_{tan} + \frac{1}{\epsilon} \int_{\partial\Omega_{t^n} \setminus \Gamma_{t^n}} (\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n}).$$

The penalization technique is interesting since the solution  $(\mathbf{u}_\epsilon^n, p_\epsilon^n)$  of (3.3.35) converges (for the

usual norm  $K_n \times Q_n$ ) to the solution  $(\mathbf{u}^n, p^n)$  of (3.3.32) when  $\epsilon$  tends to 0 (see [41]).

**Remark 3.3.7** *Simple computations lead to*

$$\alpha \int_{\partial\Omega_{t^n} \setminus \Gamma_{t^n}} [\mathbf{u}]_{tan} \cdot [\mathbf{v}]_{tan} = \alpha \int_{\partial\Omega_{t^n} \setminus \Gamma_{t^n}} \mathbf{u} \cdot \mathbf{v} - (\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n}).$$

Therefore, the bilinear operator  $a_\epsilon$  can also writes

$$a_\epsilon(\mathbf{u}, \mathbf{v}) = \frac{\rho}{\Delta t} \int_{\Omega_{t^n}} \mathbf{u} \cdot \mathbf{v} + 2\mu \int_{\Omega_{t^n}} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) + \alpha \int_{\partial\Omega_{t^n} \setminus \Gamma_{t^n}} \mathbf{u} \cdot \mathbf{v} + \left(\frac{1}{\epsilon} - \alpha\right) \int_{\partial\Omega_{t^n} \setminus \Gamma_{t^n}} (\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n}).$$

### Resolution by finite element method

We use the Galerkin finite element approximation to find the following discrete problem: Find  $(\mathbf{u}_h, p_h) \in K_n^h \times Q_n^h$  such that :

$$\begin{cases} \forall \mathbf{v}_h \in K_n^h & a_\epsilon(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = l(\mathbf{v}_h) \\ \forall q_h \in Q_n^h & b(\mathbf{u}_h, q_h) = 0 \end{cases} \quad (3.3.36)$$

where  $K_n^h \subset K_n$  and  $Q_n^h \subset Q_n$  represent two families of finite dimensional subspaces constructed from a triangulation  $T_h$  covering the fluid domain  $\Omega_{t^n}$ . Symbol  $h$  denotes the space discretization parameter which is nothing but the characteristic element size. The discrete border of the triangulation  $T_h$  is denoted by  $\partial T_h$ . It can also be seen as a discrete approximation of the boundary of  $\Omega_{t^n}$ . Symbol  $\Gamma_h$  represents the part of  $\partial T_h$  which corresponds to a discrete approximation of the free surface. Besides,  $a_\epsilon(\mathbf{u}_h, \mathbf{v}_h)$ ,  $b(\mathbf{v}_h, p_h)$ ,  $l(\mathbf{v}_h)$  are bilinear and linear forms defined on  $K_n^h \times K_n^h$ ,  $K_n^h \times Q_n^h$  and  $K_n^h$  respectively as follows:

$$\begin{aligned} a_\epsilon(\mathbf{u}_h, \mathbf{v}_h) &= \sum_{K \in T_h} \frac{\rho}{\Delta t} \int_K \mathbf{u}_h \cdot \mathbf{v}_h dx + \sum_{K \in T_h} 2\mu \int_K \mathbf{D}(\mathbf{u}_h) : \mathbf{D}(\mathbf{v}_h) dx + \alpha \sum_{E \in \partial T_h} \int_E \mathbf{u}_h \cdot \mathbf{v}_h dx \\ &+ \left(\frac{1}{\epsilon} - \alpha\right) \sum_{E \subset (\partial T_h \setminus \Gamma_h)} \int_E (\mathbf{u}_h \cdot \mathbf{n}(E)) (\mathbf{v}_h \cdot \mathbf{n}(E)) dx; \end{aligned}$$

$$b(\mathbf{v}_h, p_h) = \sum_{K \in T_h} \int_K -p_h \operatorname{div} \mathbf{v}_h;$$

$$l(\mathbf{v}_h) = \sum_{K \in T_h} \int_K \rho \mathbf{f}_h \cdot \mathbf{v}_h dx + \frac{\rho}{\Delta t} \sum_{K \in T_h} \int_K \rho (\mathbf{u}_*^{n-1})_h \cdot \mathbf{v}_h dx + l_{Surface}(\mathbf{v}_h);$$

where  $\mathbf{n}(E)$  is the exterior normal (with respect to  $T_h$ ) of the edge  $E$  for all edge  $E$  of  $\Gamma_h$ . Moreover, the term  $l_{Surface}(\mathbf{v}_h)$  is a discretization of the surface tension and atmospheric pressure term  $-\int_{\Gamma_{t^n}} (\gamma\kappa + p^a) \mathbf{n} \cdot \mathbf{v}$  and will be concerned in the next section.

The existence and uniqueness of the weak formulation for generalized Stokes problem has been proven, see [44], [92] and [41]. This proof dues to: i. the ellipticity of the form  $a_\epsilon(\cdot, \cdot)$ ; ii. the compatibility of the spaces of velocity and pressure results satisfying the Babuska–Brezzi condition,

also called *inf-sup condition* on the form  $b(\cdot, \cdot)$ , i.e. there exists a positive constant  $C$  such that:

$$\inf_{q \in Q_n} \sup_{\mathbf{v} \in K_n} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_1 \|q\|_0} \geq C > 0 \quad (3.3.38)$$

where  $\|\mathbf{v}\|_1 = \left( \sum_{i=1}^d \|v_i\|_1^2 \right)^{1/2}$  and  $\|\cdot\|_1, \|\cdot\|_0$  are standard notations of norms in the Sobolev spaces  $H^1(\Omega), L^2(\Omega)$  respectively.

The approximative problem also requires a compatibility condition, meaning that the discrete spaces of velocity needs to be "rich" enough to compare with the one of pressure. For this reason we choose mini elements ( $\mathbb{P}^1$ -bubble/ $\mathbb{P}^1$ ) as discrete spaces<sup>3</sup>. Hence, the problem (3.3.36) leads to solve the square linear system:

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \quad (3.3.39)$$

where the matrices  $A, B$  correspond to the bilinear forms  $a_\epsilon$  and  $b$ , respectively and  $F$  corresponds to the right hand side  $l$  of the first equation. The system (3.3.39) is sparse, symmetric but not positive and its size is  $\dim K_n^h + \dim Q_n^h$ . In practice, this linear system is solved by classical *Uzawa* method [6] as recalled in the following paragraph.

**Uzawa algorithm :** In order to understand the Uzawa method, let us first remark that the solution of the linear problem (3.3.39) is the unique solution to the following saddle point problem:

$$\begin{cases} \text{Find } (U, P) \in K_n^h \times Q_n^h \text{ such that} \\ \mathcal{L}(U, q) \leq \mathcal{L}(U, P) \leq \mathcal{L}(v, P) \quad \forall (v, q) \in K_n^h \times Q_n^h, \end{cases} \quad (3.3.40)$$

where

$$\mathcal{L}(v, q) = \frac{1}{2} a_\epsilon(v, v) + b(v, q) - l(v),$$

or equivalently

$$\mathcal{L}(v, q) = \frac{1}{2} Av \cdot v + Bq \cdot v - F \cdot v.$$

Then, the Uzawa method consists in solving the saddle point problem (3.3.40) using a gradient method applied to the minimization of the dual function. Indeed, we initialize the algorithm by choosing  $P_0 \in K_n^h$ . Then, for each iteration  $k$ ,

- We compute velocity  $U_k$  satisfying

$$\mathcal{L}(U_k, P_k) \leq \mathcal{L}(v, P_k) \quad \forall v \in K_n^h.$$

- The initial condition  $P_{k+1}$  for the following iteration is computed moving in the gradient direction by

$$P_{k+1} = P_k + \beta \nabla_q \mathcal{L}(U_k, P_k),$$

where  $\beta > 0$  is the gradient method step.

---

<sup>3</sup>Other choices satisfying the inf-sup condition are possible. For instance, we can consider Taylor-Hood elements ( $\mathbb{P}^2/\mathbb{P}^1$ )

Equivalently, this algorithm writes in the matrix version at an iteration  $k$  as

- Solve

$$AU_k = F - B^t P_k.$$

- Set

$$P_{k+1} = P_k + \beta BU_k.$$

It is well-known that Uzawa algorithm converges for  $\beta > 0$  sufficiently small.

### Approximation of the surface tension term

Surface tension plays a crucial role in the regularization of the free surface for incompressible fluids with small viscosity. The accurate computation of this term is one of the critical stages in any interface tracking or capturing technique. Classically, the level set function can be used to calculate the unit normal vector and the mean curvature at the interface by following formulas:

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} \Big|_{\phi=0}; \quad \kappa = \operatorname{div} \left( \frac{\nabla\phi}{|\nabla\phi|} \right) \Big|_{\phi=0}.$$

It can be seen that these formulas require an approximation of the gradient, however, in our approach the interface is explicitly discretized in the triangulation  $T_h$ . This feature gives us an alternative technique to approximate the interface via a set of connected segments (faces in three dimensions). In 2-dimensional case, we denote by  $(\mathbf{x}_i)_{1 \leq i \leq n_s}$  the set of ordered vertices along the discrete curve  $\Gamma_h$  such that  $\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}$  represent its three consecutive points and  $\mathbf{x}_0 \equiv \mathbf{x}_{n_s}, \mathbf{x}_1 \equiv \mathbf{x}_{n_s+1}$  if  $\Gamma_h$  is a closed curve. Using quadrature formula along each edge  $E$  of  $\Gamma_h$ , it has been shown in [23] that the surface tension term can be rewritten as follows, for all  $\mathbf{v}_h \in K_n^h$ :

$$\begin{aligned} \int_{\Gamma_h} \gamma \kappa \mathbf{n} \cdot \mathbf{v}_h ds &= \sum_{E \subset \Gamma_h} \frac{|E|}{2} \sum_{\mathbf{x}_i \in E} \gamma \kappa(\mathbf{x}_i) \mathbf{n}_h(\mathbf{x}_i) \cdot \mathbf{v}_h(\mathbf{x}_i) \\ &= \sum_{\mathbf{x}_i \in \Gamma_h} \gamma \kappa(\mathbf{x}_i) \mathbf{n}_h(\mathbf{x}_i) \cdot \mathbf{v}_h(\mathbf{x}_i) \sum_{E \ni \mathbf{x}_i} \frac{|E|}{2} \end{aligned} \quad (3.3.41)$$

where the unit external normal vector  $\mathbf{n}_h$  is computed from the approximation of unit tangent vector  $\tau = (\tau_1, \tau_2)^t$  at each vertex  $\mathbf{x}_i \in \Gamma_h$ :  $\tau(\mathbf{x})_i = \overrightarrow{\mathbf{x}_{i+1}\mathbf{x}_{i-1}} / \|\overrightarrow{\mathbf{x}_{i+1}\mathbf{x}_{i-1}}\|$ , hence  $\mathbf{n}_h(\mathbf{x}_i) = (\tau_2(\mathbf{x}_i), -\tau_1(\mathbf{x}_i))^t$ . The mean curvature  $\kappa(\mathbf{x}_i)$  is obtained as the inverse of the radius  $r(\mathbf{x}_i)$  which can be computed via the following approximation [48]:

$$r(\mathbf{x}_i) = \frac{1}{4} \left( \frac{\langle \overrightarrow{\mathbf{x}_i\mathbf{x}_{i-1}}, \overrightarrow{\mathbf{x}_i\mathbf{x}_{i-1}} \rangle}{\langle -\mathbf{n}_h(\mathbf{x}_i), \overrightarrow{\mathbf{x}_i\mathbf{x}_{i-1}} \rangle} + \frac{\langle \overrightarrow{\mathbf{x}_i\mathbf{x}_{i-1}}, \overrightarrow{\mathbf{x}_i\mathbf{x}_{i+1}} \rangle}{\langle -\mathbf{n}_h(\mathbf{x}_i), \overrightarrow{\mathbf{x}_i\mathbf{x}_{i+1}} \rangle} \right)$$

Other formulas can be used to approximate  $r(\mathbf{x})$  or  $\kappa(\mathbf{x})$ , see for instance [101].

The atmospheric pressure term can be then added with no difficulty:

$$\int_{\Gamma_h} (\gamma \kappa + p^a) \mathbf{n} \cdot \mathbf{v}_h ds = \sum_{\mathbf{x}_i \in \Gamma_h} (\gamma \kappa + p^a)(\mathbf{x}_i) \mathbf{n}_h(\mathbf{x}_i) \cdot \mathbf{v}_h(\mathbf{x}_i) \sum_{E \ni \mathbf{x}_i} \frac{|E|}{2}$$

This technique can be extended straightforwardly to three dimensions, where the unit normal is then taken as the weighted average value of the unit normals of all triangles sharing vertex  $\mathbf{x}_i$ . Moreover, the mean curvature is computed by approximating the discrete contour of the mesh at each vertex by a quadratic surface. This means that for each vertex on the free surface, we intend to find the quadratic surface which minimize the distance to discrete contour of the mesh at this vertex. Therefore, this is done by a solving a polynomial minimization problem (see [47] for more details).

### 3.3.4 Redistancing

It is well-known, in the context of level set method, that the level-set function must usually satisfy:

$$|\nabla\phi| = 1. \quad (3.3.42)$$

Unfortunately, when  $\phi$  is transported by a physical velocity field using equation (3.3.1), all the isolines do not travel with the same speed. As a consequence, the level set does not preserve the property (3.3.4). A natural choice to reinitialize the level-set function is the signed distance function to the interface for all time iteration  $n \in \mathbf{N}$  in the discrete approximation  $D_h$  of the computational domain  $D$ :

$$\phi(x) = \begin{cases} d(x, \Gamma_h) & \text{if } x \in T_h \\ 0 & \text{if } x \in \Gamma_h \\ -d(x, \Gamma_h) & \text{if } x \in D_h \setminus T_h. \end{cases}$$

In our scheme, this signed distance function is approximated by *redistancing procedure* studied in [37], consists in implementation the following two steps:

- Step 1: Initialization  $\phi^0$  of  $\phi$ : denoting  $\mathcal{T}_\Gamma$  the set of mesh elements intersected by the interface, i.e.  $\mathcal{T}_\Gamma = \{K \in D_h : K \cup \Gamma_h \neq \emptyset\}$ ,  $\phi^0(x)$  is defined as:

$$\phi^0(x) = \begin{cases} \phi(x) & \text{if } x \in \mathcal{T}_\Gamma \\ +\infty & \text{if } x \in T_h \setminus \mathcal{T}_\Gamma \\ -\infty & \text{if } x \in (D_h \setminus T_h) \setminus \mathcal{T}_\Gamma. \end{cases}$$

- Step 2: Numerical computation of  $\phi$  as *steady solution* of so-called *Eikonal equation*:

$$\begin{cases} \frac{\partial\phi}{\partial t}(\mathbf{x}, t) + \text{sgn}(\phi_0)(|\nabla\phi| - 1) = 0 & \forall (\mathbf{x}, t) \in D \times (0, T) \\ \phi(\mathbf{x}, 0) = \phi^0(\mathbf{x}) & \forall \mathbf{x} \in D. \end{cases} \quad (3.3.43)$$

The long term solution of this equation keeps a similar behavior in "vicinity" of the zero isoline i.e. the position of interface  $\Gamma_{t^n}$  is not modified and to ensure the constraint (3.3.4). The numerical computation of the solution of this equation is based on the explicit expression of its unique uniformly continuous viscosity solution, see [37] for more details. Figure 3.3.4 illustrates two level-set functions for a fluid domain. The function on the right of the figure, is obtained using the algorithm described in this part. Therefore, it is equal to the signed distance function in the vicinity of the free surface and satisfies (3.3.4).

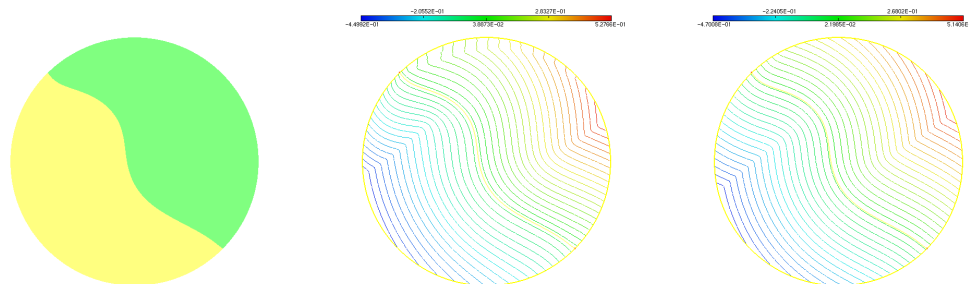


Figure 3.3.4: Illustration of the fluid domain (the yellow part) in the computational domain (left). A level set function for the fluid which does not satisfy the condition in the vicinity of the free surface (middle), A level set function for the fluid satisfying this condition (right). The latter function is obtained using the algorithm presented in this part.

### 3.3.5 Mesh adaptation

The key tool which lets us solve the time-discretized fluid problem on the fluid domain is the anisotropic mesh adaptation of the computational domain  $D$  based on the location of the free surface. Indeed, mesh generation at each step is performed as presented in [100]. The idea of this method is based on a *metric tensor* to prescribe the characteristics (*size*, *shape* and *orientation*) of the mesh elements. The definition of this metric tensor relies on the information related to numerical error estimates as: geometric error, interpolation error, approximation error (see [48] and for details). The estimates we consider here, are from [89] for time-discretized Navier–Stokes problem and from [27] for the level set advection equation as well as for the discrete approximation of the smooth boundary of the computational domain. More precisely, we intend to get a small approximation error based on the error estimate given in [89] for the Navier–Stokes equation with Dirichlet boundary conditions. Indeed, let  $\mathbf{u}$  be the smooth solution of the Navier–Stokes equations (3.2.1) with Dirichlet boundary condition. While  $\mathbf{u}_h$  is the solution of the associated problem discretized temporarily by the characteristic method and spatially by the finite element method as described in the previous section. Then, we have the following error estimate<sup>4</sup>

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2} \leq c(h + \Delta t + h^2/\Delta t), \quad (3.3.44)$$

where  $c$  is a strictly positive constant and  $h$  is the characteristic mesh element size. According to this error estimate, there should be a coherence between the time step and space step in order to give an interesting upper bound for the velocity  $L^2$  error. More precisely, if  $\Delta t$  is too small compared to the mesh elements' size  $h$ , this upper bound is not interesting.

Then, we use the estimates of [27] for the advection equation on the whole space conjecturing

<sup>4</sup>Let us note that for more precision, we must find an error estimate for the free-surface Navier–Stokes problem with slip boundary conditions which is the actual problem treated in this work. Nevertheless, we consider the result in [89] on time dependent Navier–Stokes equation with Dirichlet boundary conditions since in our knowledge, no error estimate result is available on free surface Navier–Stokes equations.

that similar estimates holds true on bounded domains. In fact, we see that the approximation error associated with the level set function is bounded by above by the interpolation error of the level set function:

$$\|\phi - \phi_h\|_{L^\infty(D)} \leq \|\phi - \Pi_h \phi\|_{L^\infty(D)} + \|\phi_0 - \Pi_h \phi_0\|_{L^\infty(D)} + c_1 \|\tilde{\mathbf{u}} - \Pi \tilde{\mathbf{u}}\|_{L^\infty(D)} + c_2 e^{\delta t} \delta t, \quad (3.3.45)$$

where  $c_1, c_2$  are constants depending on initial data  $\phi_0$  and velocity  $\mathbf{u}$  while the operator  $\Pi_h$  is the  $\mathbb{P}^1$  interpolate over the mesh  $D_n^h$  covering the computational domain  $D$  on the interval  $[t^n, t^{n+1}]$ . On the other hand, we have the following geometric estimate. In other words, the Hausdorff distance between the discrete 0 isoline contour  $\Gamma_h$  and the continuous one  $\Gamma$  satisfies

$$d^H(\Gamma_t, \Gamma_n^h) \leq \sup \left( \frac{\sup_{x \in D} |\nabla \phi(x)|}{\inf_{x \in D} |\nabla \phi(x)|^2}, \frac{\sup_{K \in D_n^h} |\nabla \phi_h(x)|_K}{\inf_{K \in D_n^h} |\nabla \phi_h(x)|_K^2} \right) \|\phi - \phi_h\|_{L^\infty(D)} \quad (3.3.46)$$

Therefore, we see that the geometric error is bounded by the level set function approximation error which is itself bounded by the interpolation error of the solution of the problem. Using the result of [49] on the  $\mathbb{L}^\infty$  error estimate for the Lagrange finite element  $\mathbb{P}^1$ -interpolation, we have for all function defined on  $D$  and on all elements  $K$  of the mesh  $D_n^h$ :

$$\begin{aligned} \|\varphi - \Pi_h \varphi\|_{L^\infty(K)} &\leq c_d \max_{x \in K} \max_{\vec{v} \subset K} \langle \vec{v}, |\nabla^2 \varphi(x)| \vec{v} \rangle \\ &\leq c_d \max_{x \in K} \max_{\vec{e} \subset E_K} \langle \vec{e}, |\nabla^2 \varphi| \vec{e} \rangle \end{aligned} \quad (3.3.47)$$

where  $\nabla^2 \varphi$  is the Hessian of  $\varphi$ ,  $E_K$  is the set of edges of the element  $K$  and  $c_d$  is a constant depending on the dimension  $d$ . This estimate implies that we can control the interpolation error on each element by controlling the size of edges of the element. This estimate lets us define an anisotropic metric for each function on each element which leads to a small interpolation error on the element (see [1] for details). According to (3.3.45) and (3.3.46), the geometric error and the approximation error associated with the level set function are conveniently bounded if the mesh is generated under the intersection of the metrics (see [48]) associated with  $\phi, \phi_0$  and  $\mathbf{u}$ . Then, at each step, the generation of the mesh from this metric is then obtained by using a Delaunay-based local mesh modification procedure (see [43]).

As mentioned previously, contrary to the framework of [100], we are interested in only the fluid domain information, we do not use a very sharp metric tensor outside of the fluid far from the free surface. Nevertheless, since we are specially interested in the behavior of the free surface, the mesh elements in the vicinity of the free surface are taken small enough to give a precise approximation for the free surface.

### 3.4 Global numerical scheme

In this part we will describe the general scheme used in this work on the time period  $[0, T]$ . This scheme is based on the numerical tools presented in the previous section. We suppose that  $[0, T]$  is divided by  $N$  interval  $[t^{n-1}, t^n]$ . The main improvement of our work compared with [100] is the fact that the Navier–Stokes equations is solved only in the fluid domain. In discrete terms, the com-

putation domain  $D$  is covered by a time depending mesh  $D_h^n$  at iteration  $n$ . This mesh is adapted at each iteration based on the position of the fluid domain, particularly based on the location of the free surface. Therefore, at each iteration the Navier–Stokes equation is solved only on a strict sub-domain  $T_h^n$  of  $D_h^n$  which corresponds to the fluid domain. Let us also note that contrary to [100], the velocity extension step on the computational domain must be placed before the resolution of the Navier–Stokes equation. The reason is the convection term and its numerical treatment with the characteristic method. In fact, if the velocity extension step is placed after the fluid equation resolution, the velocity considered for the flow when the characteristic curve crosses the free surface and goes to the other side of the computational domain, will be 0. This is not coherent with the physical system nor with the continuous mathematical model which represents it. Indeed, one of the features of this work is to compare the free surface simulation with the bifluid simulation where air represents one of the two phases. In other words, in the case of free-surface flows, the action of the second fluid is describe by an atmospheric pressure. On the other hand, according to the interface condition, the flow velocity is continuous through the bifluid interface. This is why the consideration of a 0 value for velocity when the characteristic curve crosses the free surface is not convenient. Of course, in these latter cases, we could also take the value of the last point of the domain intersected with the advection curve. However, since an extension of the velocity is necessary for the resolution of the advection equation on  $D$ , it is practical to use the same extended velocity to solve the Navier-Stokes equation in the fluid domain. This means that even though the Navier-Stokes equation is solved only on the fluid domain, the velocity taken the Navier-Stokes solver as entry, is the extended velocity defined on the whole computational mesh  $D_h^n$ . However, only values of the same extended velocity on the vicinity of free surface and in the fluid domain which are used to construct the right hand side of the linear system (3.3.39). As a confirmation, we can check by numerical simulations that the behavior of the free surface is different if we give the non-extended velocity to the Navier-Stokes solver. Consequently, if we consider a Stokes flow, the placement of the velocity extension step before the resolution of the fluid equation is not necessary. This is due to the absence of the convection term, therefore to non-use of the characteristic function on the right hand side of the linear system in the case of Stokes flows.

In this part, we omit the subscript  $h$  for all iterations. However, let us recall that the characteristic element size  $h$  is very important in our approach. As explained previously, it is prescribed by an anisotropic metric tensor and is adapted at each iteration. In other words, the scheme generates mobile unstructured meshes.

The algorithm starts with an initial discretization of the computational domain  $D^0$ , an initial velocity flow  $\mathbf{u}^0$  defined on a strict subdomain  $T^0$  of  $D^0$ , which corresponds to the initial fluid domain  $\Omega_{t^0}$ . Then, at each iteration  $n = 0, \dots, N - 1$ , we first extend the discrete velocity  $\mathbf{u}^n$  defined on  $T^n$ , on the whole computational domain  $D^n$  to get the discrete extended velocity  $\tilde{\mathbf{u}}^n$ . We then generate the signed distance function  $\phi^n$  of the discrete domain  $T^n$ . Let us recall that this is a level set function for  $T^n$  and is defined on the whole  $D^n$ . This function is then transported by the resolution of the level set advection equation defined thanks to the extended velocity  $\tilde{\mathbf{u}}^n$ . The new level set function is called  $\tilde{\phi}^{n+1}$ . It is supposed to characterize the discrete fluid domain  $T^{n+1}$  at time  $t^{n+1}$ . The mesh is then adapted according to this level set function. We now have a new mesh  $D^{n+1}$  with a fluid sub domain  $T^{n+1}$ . The extended function  $\tilde{\mathbf{u}}^n$  is now interpolated on the new mesh  $D^{n+1}$ . Finally the time-discretized Navier–Stokes equation (3.3.14) (with boundary



conditions (3.2.2)) is solved on  $T^{n+1}$ . This algorithm is recapitulated in the following table.

1. At time  $t^0 = 0$ , we know the initial mesh  $D^0$  covering the computational domain, as well as the fluid velocity  $\mathbf{u}^0$  defined on the subdomain  $T^0 \subset D^0$  representing the initial fluid domain  $\Omega_{t^0}$ .
2. For  $n = 0, \dots, N - 1$ ,

<b>Smooth velocities</b>	mesh	input	output
Velocity extension	$(D^n \setminus T^n) \cup \Gamma^n$	$\mathbf{u}^n$	$\tilde{\mathbf{u}}^n$
Generation of the signed distance function	$D^n$	$D^n$	$\phi^n$
Resolution of the advection equation	$D^n$	$(\tilde{\mathbf{u}}^n, \phi^n)$	$\tilde{\phi}^{n+1}$
Mesh adaptation	$D^n$	$(\tilde{\phi}^{n+1}, D^n)$	$D^{n+1}$
Interpolation	$D^{n+1}$	$\tilde{\mathbf{u}}^n$	$\tilde{\mathbf{u}}^n$
Resolution of the time-discretized Navier–Stokes system	$T^{n+1}$	$\tilde{\mathbf{u}}^n$	$(\mathbf{u}^{n+1}, p^{n+1})$

<b>Non-smooth velocities</b>	mesh	input	output
Velocity regularization	$D^n$	$\mathbf{u}^n$	$\tilde{\mathbf{u}}^n _{adv}$
Velocity extension	$(D^n \setminus T^n) \cup \Gamma^n$	$\mathbf{u}^n$	$\tilde{\mathbf{u}}^n _{NS}$
Generation of the signed distance function	$D^n$	$D^n$	$\phi^n$
Resolution of the advection equation	$D^n$	$(\tilde{\mathbf{u}}^n _{adv}, \phi^n)$	$\tilde{\phi}^{n+1}$
Mesh adaptation	$D^n$	$(\tilde{\phi}^{n+1}, D^n)$	$D^{n+1}$
Interpolation	$D^{n+1}$	$\tilde{\mathbf{u}}^n _{NS}$	$\tilde{\mathbf{u}}^n _{NS}$
Resolution of the time-discretized Navier–Stokes system	$T^{n+1}$	$\tilde{\mathbf{u}}^n _{NS}$	$(\mathbf{u}^{n+1}, p^{n+1})$

3. We return  $\mathbf{u}^N, p^N, \phi^N, D^N$ .

Let us recall that a velocity regularization step is necessary if the velocity field presents large discrepancies between neighboring points. As mentioned in Subsection 3.3.2, in this case, we first solve the Helmholtz equation (3.3.11) on the whole computational mesh  $D^n$  to get a regularized velocity for the advection equation. This velocity coincides with the physical velocity in the vicinity of the free surface. However, the knowledge of the fluid velocity on the fluid domain  $T^n$  is necessary to solve the time-discretized Navier-Stokes equations. For this reason, we must also solve the Helmholtz equation (3.3.10) out of the fluid domain to find a convenient extended velocity. Let us also mention that the resolution of the problem (3.3.10) is not expensive since mesh elements situated inside of  $D^n \setminus T^n$  are quite large. This is due to the fact that physical quantities as well as the extended velocity far from the fluid part are of few if no interest.

### 3.5 Numerical results

In this section, we present some numerical results that we used to validate our numerical code. The first result concerns a viscous fluid in a circular computational domain. The objective here is to ensure that our work can be used properly for curvy boundary computational domain. In fact, comparing with square or rectangle computational domains, it happens more often to go out from the computational domain along the flow characteristic curves. The second test case is a dam break with shampoo. Our result is compared with the physical experiments and numerical results presented in [34]. Both of these cases concern fluids with quite high viscosity and small Reynolds number. The third example concerns the collapse of a water column. The Reynolds number here is

large and the flow is considered to be turbulent. As mentioned in [84], when the Reynolds number is large, there are usually strong gradients for the velocity and the vorticity in the vicinity of solid walls. For this reason, we may consider some adapted models for turbulent flows. We modify here the viscosity as suggested in [34] to take into account the energy dissipated by turbulence. Our results are then compared with the results presented therein.

### 3.5.1 Viscous fluid in a circular domain

We consider here a fluid with density  $\rho = 1$  and dynamic viscosity  $\mu = 1$ . The surface tension constant is  $\gamma = 7.2$ . and the gravity action is modeled by  $f = -100$ . No atmospheric pressure is considered i.e. the fluid is in the vacuum. The friction with the boundary of the domain is modeled by  $\alpha = 10$ . The initial solution is presented in Figure 3.5.1. Let us note that the initial solution is obtained by solving the steady Stokes equation with a upward gravity. The computational domain

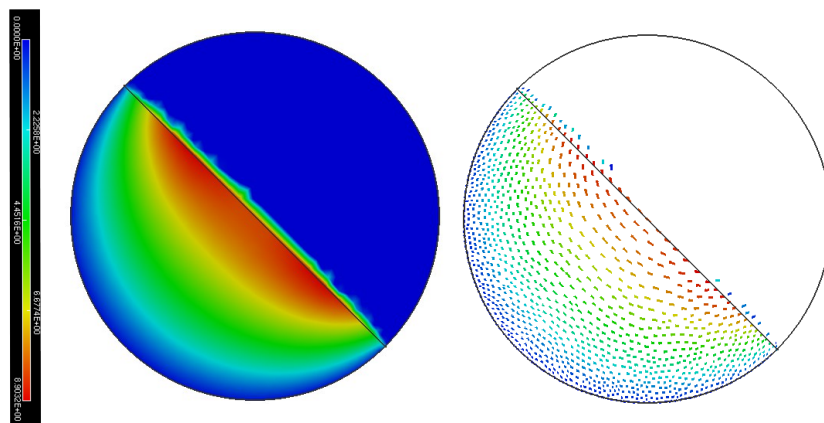


Figure 3.5.5: Norm of the initial velocity (on the left) and the associated vector field (on the right).

considered here is the unit circle i.e. the circle of radius 1 and the center  $(0, 0)$ . Let us note that, as illustrated in Figure 3.5.1, in this test the mesh elements all over the computational boundary, even out of fluid domain, must be small enough to preserve the curvy shape of the computational domain. This is also due to the fact the fluid domain is quite large (half of the computational domain). The characteristic element size of the mesh is about 0.03 and the time step is in the order of  $10^{-3}$ . The test is done over the time interval  $[0, 4.435]$  (over 3300 iterations). Figure 3.5.1 represents the solution at some time steps.

### 3.5.2 Collapse of a shampoo column

We consider here a fluid with density  $\rho = 1024$  and dynamic viscosity  $\mu = 8$ . The surface tension constant is  $\gamma = 0.07$  and the gravity action is modeled by the force density  $f = -9.8$ . The fluid is subjected to no atmospheric pressure i.e.  $p^a = 0$  and the friction constant is equal to  $\alpha = 10^{-2}$ .

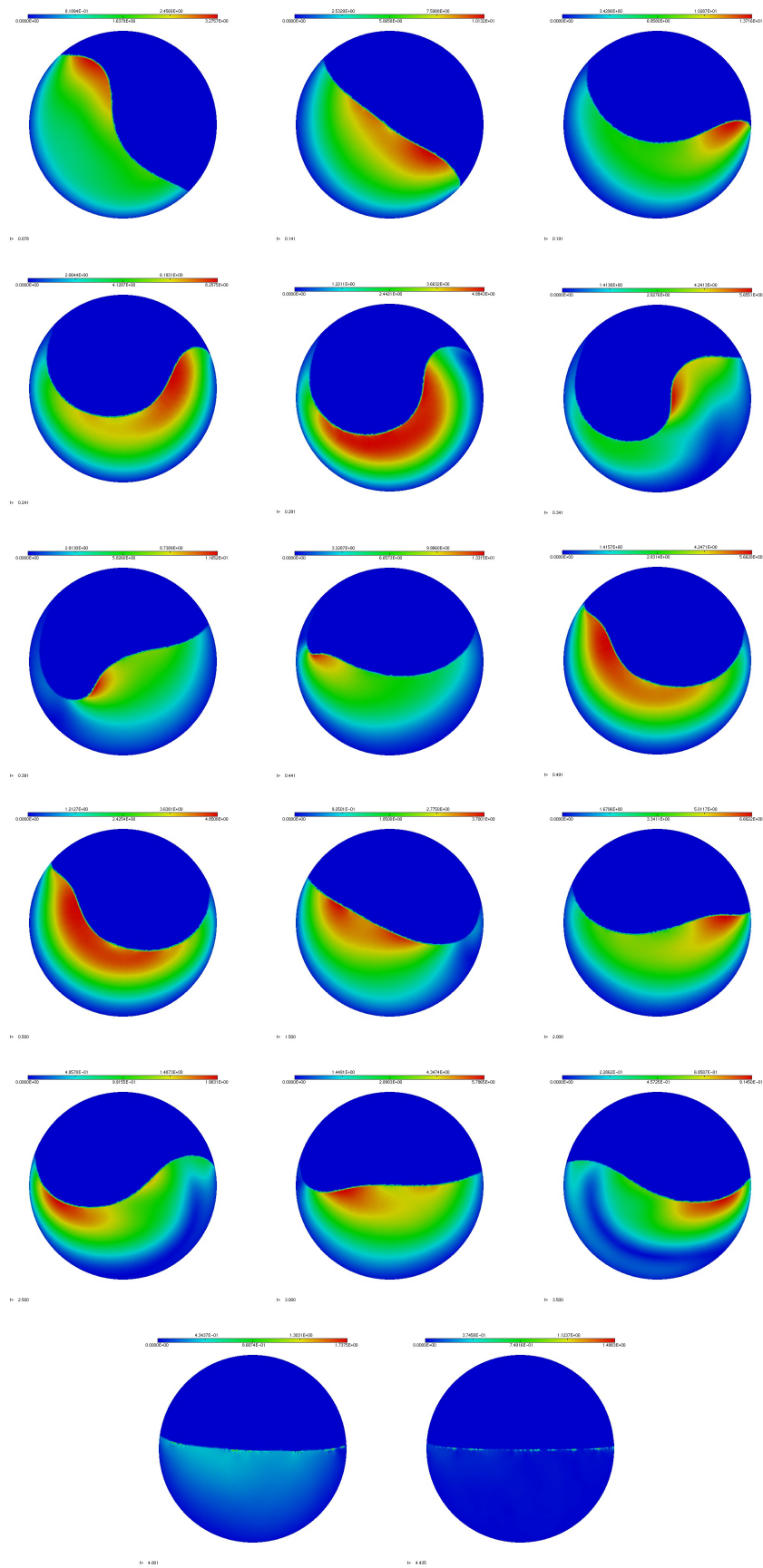


Figure 3.5.6: Evolution of a 2-dimensional viscous fluid in the unit circle. The equilibrium is reached at about time  $T = 4$ .

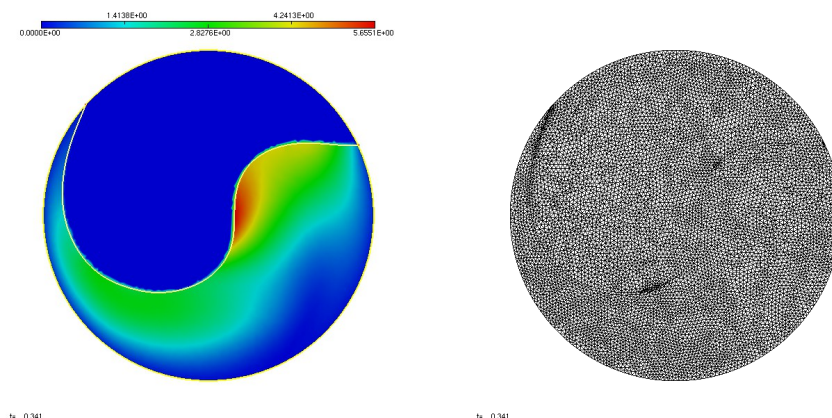


Figure 3.5.7: Norm of the velocity at time 0.341 (left), the associated adapted mesh (right).

This friction is computed following the law

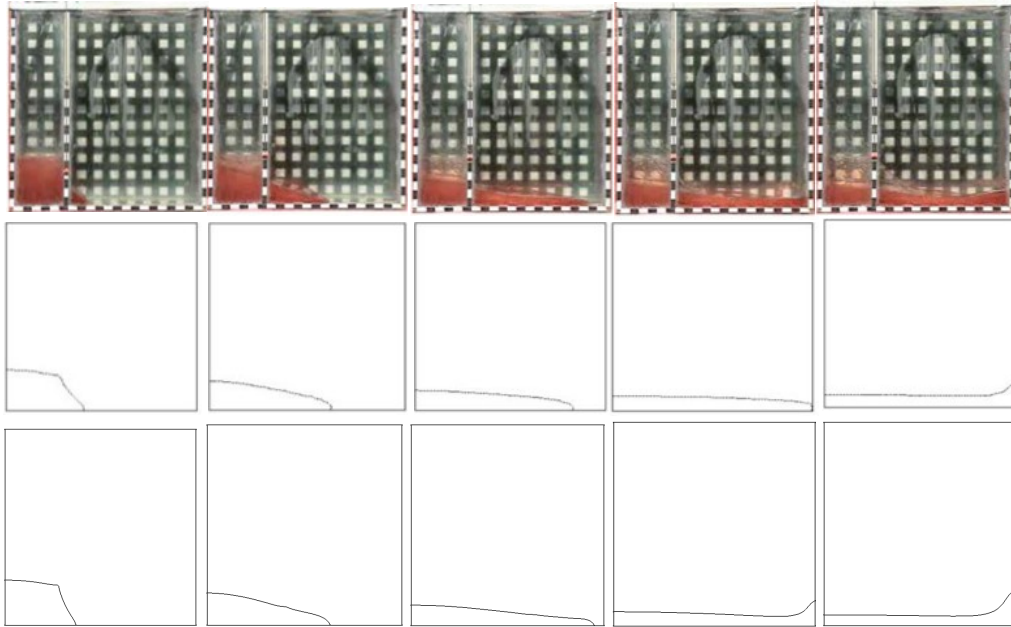
$$\alpha = \frac{\rho |\mathbf{U}|}{C_f^2},$$

where  $C_f = 190$  as in [34] and  $|\mathbf{U}|$  is the order of the norm of the flow velocity. The value which is taken here  $f|\mathbf{U}|$  is 0.4. However, the result does not change significantly if we take for instance  $|\mathbf{U}| = 0.04$  or  $|\mathbf{U}| = 4$ .

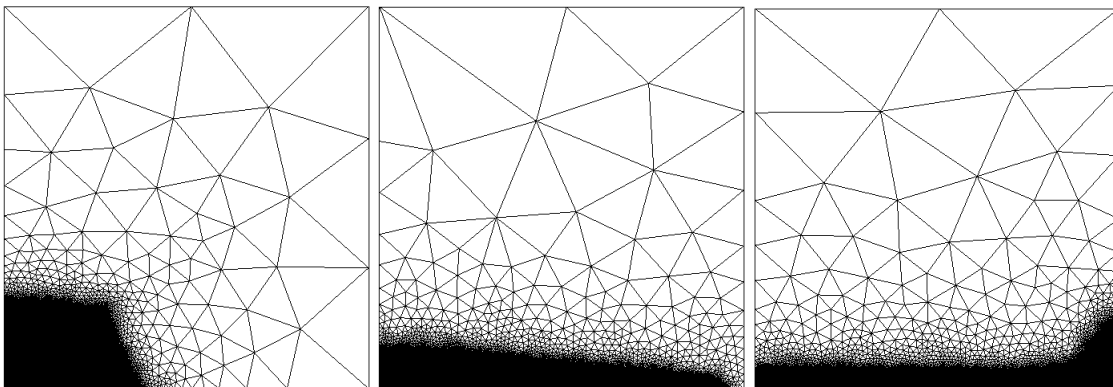
The initial fluid is a 0-velocity fluid column. The computational domain dimension as well as the fluid column width and height are the same as in [34]: The computational domain is  $0.42m \times 0.44m$  whereas the fluid domain is  $0.114m \times 0.114m$ . Figure 3.5.8 compares our result with the physical experiments and the numerical results presented in [34]. The problem is considered in [34] as a bi-fluid air-shampoo problem. Therefore, the fluid equation is solved on the whole computational domain. Similarly to our work, the interface between two flows is captured by solving an advection equation (edge-track interface locator technique). The time step we consider here is  $\Delta t = 0.01$  for the first ten iterations (until time  $t = 0.1$ ) and is  $\Delta t = 0.02$  for other iterations. The values we consider for  $hmin$  and  $hmax$  are 0.0009 and 1.0. More over  $hgrad = 2.5$ . The mesh adaptation at each step leads to large elements out of the fluid and fine elements inside (see Figure 3.5.9). Let us note that the time step in [34] is  $\Delta t = 0.001$ . Nevertheless, the results are similar.

### 3.5.3 Collapse of a water column

This test case is a dam break test case with water whose characteristics are  $\rho = 1000$  and dynamic viscosity  $\mu = 0.001$ . The high value of the Reynolds number of this problem leads to some turbulent



**Figure 3.5.8:** Comparison between physical experiments of [34] (top), the numerical result of [34] with wall friction (middle) and our results (bottom) at times  $t = 0.1, 0.2, 0.3, 0.4, 0.5$  (from left to right).



**Figure 3.5.9:** Adapted mesh for the dam break with shampoo at times  $t = 0.1$  (left),  $t = 0.3$  (middle) and  $t = 0.5$  (right). We see that the mesh elements' size is small inside the fluid, very small in the vicinity of the free surface and large out of the fluid far from the surface.

effects. The model we consider here is the model recalled in [34]. In this simple model, the viscosity of the fluid is modified to take into account the energy dissipated by the turbulent effects. The law according to which the flow is modified is given by

$$\mu_{mod} = \min \left( \mu + l_{mix}^2 \rho \sqrt{\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{u})/2}; \mu_{max} \right),$$

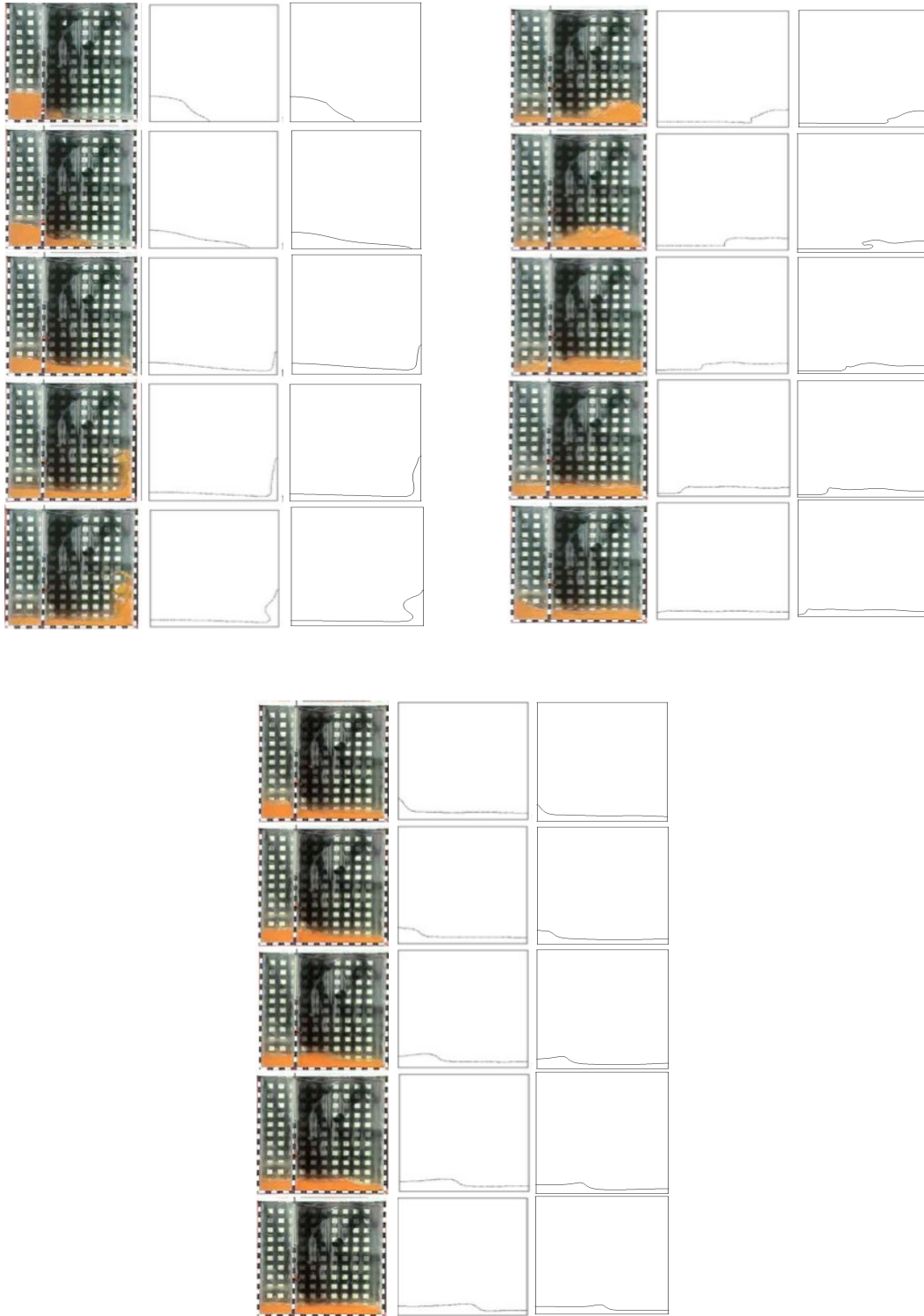
where  $l_{mix} = C_t h_{UGN}$  such that  $C_t$  is a modeling parameter,  $h_{UGN}$  is a characteristic element size and  $\mu_{max}$  is a cut-off value. The values suggested in this test case for this parameters are

$$C_t = 3.57, \quad h_{UGN} = h_{min} = 9 \times 10^{-4}, \quad \mu_{max} = 1.5.$$

In practice, we take here  $\mu = 0.001$  and  $\Delta t = 0.001$  for the first iteration of the algorithm. Then, we take the cut-off value  $\mu_{max}$  for viscosity and set  $\Delta t = 0.005$  for others. Boundary conditions considered here are slip boundary conditions without friction. Besides, no atmospheric pressure is taken into account and the surface tension coefficient is  $\gamma = 0.07$ . Other parameters are set as  $h_{max} = 1$  and  $h_{grad} = 2.5$ .

### Remark

In all dam break test cases, simulations are proved to be very sensitive to the time step. Indeed, even small variations of the time step play a very important role in the behavior of the fluid. Moreover, for a fixed size of the mesh elements, a too small time step may lead to an incoherent simulation. This may be eventually due to the presence of the time step in a denominator of the right hand side of the error estimate of the problem.



**Figure 3.5.10:** Comparison between physical experiments of [34] (left column), the numerical result of [34] with a discontinuity capturing interface dissipation (middle column) and our results (right column) at times  $t = 0.1, 0.2, 0.3, 0.4, 0.5$  (top left), at times  $0.6, 0.7, 0.8, 0.9, 1.0$  (top right) and at times  $1.1, 1.2, 1.3, 1.4, 1.5$  (bottom). (from top to bottom).

### 3.6 Conclusion and perspectives

In this work, we presented a numerical scheme using level set methods on mobile unstructured meshes to simulate free-surface incompressible problems. In the model we consider here, the influence of other fluids is represented by an external pressure on the free surface, denoted by  $p^a$ . As we see in Subsections 3.5.1 and 3.5.1, our dam break numerical experiments seem to be coherent with bifluid simulations in [34] even though the atmospheric pressure  $p^a$  in our case is set to 0. Nevertheless, the selection of the best atmospheric pressure to model the external effect is a perspective for this work. Similarly, the sensitivity of our numerical algorithm to the variations of the atmospheric pressure would be the subject of further investigations. Answers to similar questions on the friction coefficient  $\alpha$  will also lead to more precise simulations.

Besides, the regularization of the velocity in our algorithm is optional unless it represents important fluctuations on small regions of the fluid domain. Nevertheless, the seek of quite precise criteria to evaluate the importance of velocity fluctuations will be help-full to improve the algorithm. Obviously, this point is correlated with the sensitivity of the algorithm to the regularization of the velocity in different practices. Finally, a numerical analysis and the knowledge of some error estimates will be illuminating for the choice of time step and mesh elements size.





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## Mathematical study of free-surface fluids in incompressible dynamics

### Abstract :

This thesis is about theoretical study and numerical treatment of some problems raised in incompressible free-surface fluid dynamics. The first part concerns a model called the Green–Naghdi (GN) equations. Similarly to the non linear shallow water system (called also Saint-Venant system), the Green–Naghdi equations is a shallow water approximation of water waves problem. Indeed, GN equation is one order higher in approximation compared to Saint-Venant system. For this reason, it contains all the terms of Saint-Venant system in addition to some non linear third order dispersive terms. In other words, the GN equations is a dispersive perturbation of the Saint-Venant system. The latter system is hyperbolic and fits the general framework developed in the literature for hyperbolic systems. Particularly, it is entropic (in the sense of Lax) and symmetrizable. Therefore, we can apply the well-posedness results known for symmetric hyperbolic system. During the first part of this work, we generalize the notion of symmetry to a more general type of equations including the GN system. This lets us to symmetrize the GN equation. Then, we use the suggested symmetric structure to obtain a global existence result for the system with a second order dissipative term by adapting the approach classically used for hyperbolic systems.

The second part of this thesis concerns the numerical treatment of the free surface incompressible Navier–Stokes equation with surface tension. We use the level set formulation to represent the fluid free-surface. Thanks to this formulation, the kinematic boundary condition is treated by solving an advection equation satisfied by the level set function. This equation is solved on a computational domain containing the fluid domain over small time subintervals. Each iteration of the algorithm corresponds to the advection of the fluid domain on a small time subinterval and to solve the time-discretized Navier–Stokes equations only on the fluid domain. The time discretization of the Navier–Stokes equation is done by the characteristic method. Then, the key tool which lets us solve this equation on the fluid domain is the anisotropic mesh adaptation. Indeed, at each iteration the mesh is adapted to the fluid domain such that we get convenient approximation and geometric errors in the vicinity of the fluid domain. This resolution is done using the Uzawa algorithm for a convenient finite element method. The slip boundary conditions are considered by adding a penalization term to the variational formulation associated to the problem.

**Keywords :** Incompressible fluid, shallow water models, Green–Naghdi equations, symmetric structure, Navier–Stokes equation, level set method, anisotropic mesh adaptation, characteristic method

## Etudes mathématiques de fluides à frontières libres en dynamique incompressible

### Résumé :

Cette thèse est consacrée à l'étude théorique ainsi qu'au traitement numérique de fluides incompressibles à surface libre. La première partie concerne un système d'équations appelé le système de Green–Naghdi. Comme le système de Saint-Venant, il s'agit d'une approximation d'eaux peu-profondes du problème de Zakharov. La différence est que le système de Green–Naghdi est d'un degré plus élevé en ordre d'approximation. C'est pourquoi il contient tous les termes du système de Saint-Venant plus de termes d'ordre trois non-linéairement dispersives. Autrement dit, le système de Green–Naghdi peut être vu comme une perturbation dispersive du système de Saint-Venant. Ce dernier système étant hyperbolique, il entre dans le cadre classique développé pour des systèmes hyperboliques. En particulier, il est entropique (au sens de Lax) et symétrique. On peut donc lui appliquer les résultats d'existence et d'unicité bien connus pour des systèmes hyperboliques. Dans la première partie de ce travail, on généralise la notion de symétrie à une classe plus générale de systèmes contenant le système de Green–Naghdi. Ceci nous permet de symétriser les équations de Green–Naghdi et d'utiliser la symétrie obtenue pour déduire un résultat d'existence globale après avoir ajouté un terme dissipatif d'ordre 2 au système. Ceci est fait en adaptant l'approche utilisée dans la littérature pour des systèmes hyperboliques.

La deuxième partie de ce travail concerne le traitement numérique des équations de Navier–Stokes à surface libre avec un terme de tension de surface. Ici, la surface libre est modélisée en utilisant la formulation des lignes de niveaux. C'est pourquoi la condition cinématique (condition de l'évolution de surface libre) s'écrit sous la forme d'une équation d'advection satisfaite par la fonction de ligne de niveaux. Cette équation est résolue sur un domaine de calcul contenant strictement le domaine de fluide, sur de petits sous-intervalles du temps. Chaque itération de l'algorithme global correspond donc à l'advection du domaine du fluide sur le sous-intervalle du temps associé et ensuite de résoudre le système de Navier–Stokes discrétisé en temps sur le domaine du fluide. Cette discrétisation en temps est faite par la méthode des caractéristiques. L'outil clé qui nous permet de résoudre ce système uniquement sur le domaine du fluide est l'adaptation de maillage anisotrope. Plus précisément, à chaque itération le maillage est adapté au domaine du fluide tel que l'erreur d'approximation et l'erreur géométrique soient raisonnablement petites au voisinage du domaine du fluide. La résolution du problème discrétisé en temps sur le domaine du fluide est faite par l'algorithme d'Uzawa utilisé dans la cadre de la méthode des éléments finis. Par ailleurs, la condition de glissement de Navier est traité ici en ajoutant un terme de pénalisation à la formulation variationnelle associée.

**Keywords :** Fluide incompressible, Modèle d'eaux peu-profondes, équations de Green–Naghdi, structure symétrique, équation de Navier–Stokes, méthode des lignes de niveaux, adaptation de maillage anisotrope, méthode des caractéristiques