



HAL
open science

Analyse et contrôle de modèles de dynamique de populations

Yuan He

► **To cite this version:**

Yuan He. Analyse et contrôle de modèles de dynamique de populations. Mathématiques générales [math.GM]. Université Sciences et Technologies - Bordeaux I, 2013. Français. NNT : 2013BOR14918 . tel-01424147

HAL Id: tel-01424147

<https://theses.hal.science/tel-01424147>

Submitted on 2 Jan 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Numéro d'ordre: 4918

THÈSE

PRÉSENTÉE À

L'UNIVERSITÉ BORDEAUX 1

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET
INFORMATIQUE

Par **Yuan, He**

POUR OBTENIR LE GRADE DE

Docteur

Spécialité: Mathématiques Appliquées et Calcul Scientifique

ANALYSE ET CONTRÔLE DE MODÈLES DE DYNAMIQUE DE POPULATIONS

Thèse dirigée par Bedr'Eddine AINSEBA

Soutenue le 22 novembre 2013

Devant la commission d'examen:

M. Mostafa Adimy	Directeur de recherche, INRIA Rhône-Alpes	Rapporteur
M. Bedr'Eddine Aïnseba	Professeur, Université Bordeaux 1	Directeur
M. Mostafa Bendahmane	Maître de Conférence, Université Bordeaux 2	Examineur
MME. Zhilan Feng	Professeur, Purdue University	Examineur
M. Mimmo Iannelli	Professeur, Université de Trento (Italie)	Rapporteur
M. Shigui Ruan	Professeur, University of Miami	Examineur

Acknowledgements

I am glad to express my gratitude to all those who helped me during the writing of this thesis.

First of all, I gratefully acknowledge the help of my supervisor, Professor Bedr'Eddine Ainseba, who has offered me valuable suggestions in the academic studies. I thank him for generously sharing his knowledge and time, for being patient with my questions. The research in this article would not have been possible without his help. In the preparation of the thesis he has spent much time reading through each draft and provided me with meaningful advice.

Secondly, I am also grateful to Professor Mostafa Bendahmane. His vast knowledge, broaden horizons helped me get quick thinking and deep enlightenment.

Besides, I wish to express my gratitude to Professor Pierre Magal, Professor Arnaud Ducrot, Professor Michel Langlais, Professor Jean-Baptiste Burie. I am benefited a lot from their profound knowledge. I should also thank Ingrid Rochel and other colleges in our laboratory. They make me feel warmth in a foreign country.

In addition, I would like to thank my friends: Min Sha, Yanlan Wang, Na Liu, Zhuowei Du, Huihui Ding, Na Zhou, Nianlong Guo, Yuxia Wang, Feiying Yang, Ousmane Seydi, Alejandro Lopez Rincon, Houssein Ayoub, Chahrazed Benosman et al. Thank them for their concern for me both in life and in learning.

I should finally like to thank my beloved parents who have always been helping me out of difficulties and supporting without a word of complaint. My heart swells with gratitude to all the people who helped me.

Résumé

Cette thèse est divisée en deux parties.

La première partie concerne l'analyse mathématique et la contrôlabilité exacte à zéro d'une catégorie de systèmes structurés décrivant la dynamique d'une population d'insectes. L'objectif de ce travail est de développer un modèle mathématique pour l'étude et la compréhension de la dynamique de populations d'insectes ravageurs, l'Eudémis de la vigne (*Lobesia botrana*), dans son écosystème. Le modèle proposé est un système d'équations aux dérivées partielles de type hyperbolique qui décrit les variations au cours du temps de la population en fonction des stades de développement, du sexe des individus et des conditions environnementales. La ressource alimentaire, la température, l'humidité et la prédation sont les principaux facteurs environnementaux du modèle expliquant les fluctuations du nombre d'individus au cours du temps. Les différences de développement qui existent dans une cohorte d'Eudémis sont aussi modélisées.

Dans le chapitre 2 de cette première partie on considère que la population d'adultes diffuse dans le vignoble, la fonction de croissance des individus à chaque stade dépend des variations climatiques et de la variété des raisins.

$$\left\{ \begin{array}{l} \frac{\partial u^e(t,a,x)}{\partial t} + \frac{\partial[v^e(E(t),a)u^e(t,a,x)]}{\partial a} = -\mu^e(E(t),a)u^e(t,a,x) - \beta^e(E(t),a)u^e(t,a,x), \\ \frac{\partial u^l(t,a,x)}{\partial t} + \frac{\partial[v^l(E(t),a)u^l(t,a,x)]}{\partial a} = -\mu^l(P^l(t,x),E(t),a)u^l(t,a,x) - \beta^l(E(t),a)u^l(t,a,x), \\ \frac{\partial u^f(t,a,x)}{\partial t} + \frac{\partial[v^f(E(t),a)u^f(t,a,x)]}{\partial a} = -\mu^f(E(t),a)u^f(t,a,x) + d_f \Delta_x u^f(t,a,x), \\ \frac{\partial u^m(t,a,x)}{\partial t} + \frac{\partial[v^m(E(t),a)u^m(t,a,x)]}{\partial a} = -\mu^m(E(t),a)u^m(t,a,x) + d_m \Delta_x u^m(t,a,x), \end{array} \right. \quad (1)$$

où $(t, a, x) \in (0, T) \times (0, L^k) \times \Omega$ pour k égale à e, l, f et m . On indice les stades de développement par e pour le stade oeuf, l pour le stade larve, f pour les femelles et m pour les mâles. La densité d'individus de ces populations est notée u^k où k est égale à

e, l, f et m . Le calcul de la population totale du stade k est donné par

$$P^k(t, x) = \int_0^{L^k} u^k(t, a, x) da, \quad x \in \Omega, \quad t \geq 0, \quad k = e, l, f, m.$$

Les conditions limites sont données par

$$\begin{cases} v^e(E(t), 0)u^e(t, 0, x) = \int_0^{L^f} \beta^f(P^f(t, x), P^m(t, x), E(t), s)u^f(t, s, x)ds, \\ v^l(E(t), 0)u^l(t, 0, x) = \int_0^{L^e} \beta^e(E(t), s)u^e(t, s, x)ds, \\ v^f(E(t), 0)u^f(t, 0, x) = \int_0^{L^l} \sigma\beta^l(E(t), s)u^l(t, s, x)ds, \\ v^m(E(t), 0)u^m(t, 0, x) = \int_0^{L^l} (1 - \sigma)\beta^l(E(t), s)u^l(t, s, x)ds, \end{cases} \quad (2)$$

où σ est le sexe ratio, $x \in \Omega$, et $t \in (0, T)$. Les conditions initiales et les conditions limites du flux sont données par

$$\begin{cases} u^k(0, a, x) = u_0^k(a, x), & \text{in } (0, L^k) \times \Omega, \quad k = e, l, f, m, \\ \frac{\partial u^f}{\partial \eta} = 0, & \text{on } (0, T) \times (0, L^f) \times \partial\Omega, \\ \frac{\partial u^m}{\partial \eta} = 0, & \text{on } (0, T) \times (0, L^m) \times \partial\Omega. \end{cases} \quad (3)$$

Le vecteur E correspond aux variables climatiques et environnementales et s'écrit sous la forme (T, H, R) où T désigne la température, H l'humidité et R la ressource alimentaire. Les fonctions μ^e, μ^l, μ^f et μ^m sont les taux de mortalité respectivement des stades oeuf, larve, femelle et mâle. Les fonctions β^k denotent les taux de transition d'âge spécifique. Le taux de croissance ne dépend pas de la quantité d'aliments mais de la qualité de celle ci.

On introduit alors les fonctions v^k qui sont les vitesses de croissance des stades d'indice e, l, f, m . Elles ne dépendent pas de la population totale car le nombre d'individus localisé sur un même endroit n'interfère pas avec le processus de croissance de la population [104].

La question de l'existence d'une solution globale est liée au travail de Martin-Pierre pour le système de réaction-diffusion. Cependant le processus démographique ajoute des difficultés supplémentaires. (Voir le travail de Iannelli et Busenberg(1988) sur les cas de paramètres démographiques identiques). Ainsi nous nous sommes intéressés à l'analyse mathématique de ce modèle et son comportement asymptotique. Plus spécialement, dans notre contexte le cadre mathématique tractable pour le problème est l'espace de Banach L^1 . En effet, l'espace L^1 est un choix naturel dans lequel l'interprétation physique de la fonction de densité demande qu'elle soit intégrable, et

le traitement mathématique de cette question demande que la fonction de densité appartienne à un espace vectoriel complet. La norme de densité est une mesure naturelle de la taille de la population. Nous avons obtenu la solution par la méthode des caractéristiques pour ce modèle de *Lobesia botrana*. Selon l'expression de la solution du système, u^e , u^l , u^f et u^m sont liés par leurs conditions aux limites. Remarquons que u^e décrit le nombre de nouveau-nées des femelles papillon u^f . L'existence des solutions du problème est faite grâce à un théorème de point fixe. En utilisant la méthode de point fixe, on obtient l'existence et l'unicité des solutions du modèle. On démontre ensuite l'existence d'un attracteur global pour le système dynamique. Enfin, on utilise la théorie des opérateurs compacts et le théorème de point fixe de Krasnoselskii pour prouver l'existence des états stationnaires.

Dans le chapitre 3, on traite le problème de contrôlabilité exacte du modèle *Lobesia botrana*, lorsque la fonction de croissance est égale à 1. On suppose que les quatre sous-catégories de ce système sont dans une phase statique.

$$\begin{cases} \frac{\partial u^e(t,a)}{\partial t} + \frac{\partial u^e(t,a)}{\partial a} &= -(\mu^e(a) + \beta^e(a))u^e(t,a) + \chi(a)w(t,a), \\ \frac{\partial u^l(t,a)}{\partial t} + \frac{\partial u^l(t,a)}{\partial a} &= -(\mu^l(a) + \beta^l(a))u^l(t,a), \\ \frac{\partial u^f(t,a)}{\partial t} + \frac{\partial u^f(t,a)}{\partial a} &= -\mu^f(a)u^f(t,a), \\ \frac{\partial u^m(t,a)}{\partial t} + \frac{\partial u^m(t,a)}{\partial a} &= -\mu^m(a)u^m(t,a), \end{cases} \quad (4)$$

où $(t, a) \in [0, T] \times [0, A]$, $A = \max\{L^e, L^l, L^f, L^m\}$, L^k est l'âge maximal du stade de développement k pour $k = e, l, f, m$. La fonction $u^k(t, a)$ représente l'âge spécifique de la densité des oeufs, des larves, des papillons femelles et des papillons males respectivement. Pour chaque k , si $A > L^k$, on note $u^k = 0$, $\beta^k = 0$, $\mu^k = 0$. Le terme $\chi(a)w(t, a)$ est un processus de contrôle pour les oeufs: $\chi(a)$ est la fonction caractéristique sur $[0, a^*](0 < a^* < L^e \leq A)$, ce qui signifie que notre intervention peut être limitée à des groupes d'âges petits. Les conditions limites sont données par

$$\begin{cases} u^e(t, 0) &= \int_0^{L^f} \beta^f(s)u^f(t, s)ds, \\ u^l(t, 0) &= \int_0^{L^e} \beta^e(s)u^e(t, s)ds, \\ u^f(t, 0) &= \int_0^{L^l} \sigma\beta^l(s)u^l(t, s)ds, \\ u^m(t, 0) &= \int_0^{L^l} (1 - \sigma)\beta^l(s)u^l(t, s)ds, \end{cases} \quad (5)$$

où σ est le sexe ratio, $t > 0$. Les conditions initiales des équations du système (4) sont

$$u^k(0, a) = u_0^k(a), \quad (6)$$

$k = e, l, f, m$. Les paramètres de démographie μ^k représentent les taux de mortalité à l'âge a , $k = e, l, f, m$. Les fonctions β^k sont les taux de fécondité à l'âge a , $k = e, l, f, m$.

Nous avons étudié la contrôlabilité exacte d'insectes ravageurs en agissant sur les oeufs dans un intervalle d'âge petit. Il faut noter que les méthodes développées dans [19] pour obtenir l'inégalité d'observabilité ne sont pas directement applicables à notre système. Nous avons donc utilisé un argument de point fixe [3, 4] pour obtenir la contrôlabilité exacte. Pour ce faire nous avons d'abord obtenu des estimations a priori des variables du problème adjoint. Ensuite nous avons montrer que la population d'oeufs peut être contrôlée à zéro, grâce à un argument de point fixe.

Lorsque les papillons adultes se dispersent spatialement, on introduit un contrôle sur la population d'oeufs, de larves et de femelles dans une petite région du vignoble. Nous notons la distribution d'oeufs, de larves, de femelles et de males respectivement par $u^k(t, a, x)$ à l'âge $a \geq 0$ au temps $t \geq 0$, et à l'emplacement $x \in \Omega$ avec $k = e, l, f, m$. Le modèle LBM s'écrit

$$\begin{cases} Du^e(t, a, x) &= -(\mu^e(a, x) + \beta^e(a))u^e(t, a, x) + m(a)w^e(t, a, x), \\ Du^l(t, a, x) &= -(\mu^l(a, x) + \beta^l(a))u^l(t, a, x) + m(a)w^l(t, a, x), \\ Du^f(t, a, x) &= -\mu^f(a, x)u^f(t, a, x) + \Delta u^f(t, a, x) + \chi(a, x)w^f(t, a, x), \\ Du^m(t, a, x) &= -\mu^m(a, x)u^m(t, a, x) + \Delta u^m(t, a, x), \end{cases} \quad (7)$$

où $(t, a, x) \in (0, T) \times (0, A) \times \Omega$, et $\Omega \subset \mathbb{R}^3$.

Les conditions limites sont indiquées comme suit

$$\begin{cases} u^e(t, 0, x) &= \int_0^{L^f} \beta^f(s)u^f(t, s, x)ds, \\ u^l(t, 0, x) &= \int_0^{L^e} \beta^e(s)u^e(t, s, x)ds, \\ u^f(t, 0, x) &= \int_0^{L^l} \sigma\beta^l(s)u^l(t, s, x)ds, \\ u^m(t, 0, x) &= \int_0^{L^l} (1 - \sigma)\beta^l(s)u^l(t, s, x)ds, \end{cases} \quad (8)$$

où σ dénote le sexe ratio et $t > 0$. Les conditions initiales sont donnes par

$$u^k(0, a, x) = u_0^k(a, x), \quad k = e, l, f, m. \quad (9)$$

et les conditions au bord par

$$\frac{\partial u^k(t, a, x)}{\partial \eta} = 0, \quad x \in \partial\Omega, \quad k = f, m. \quad (10)$$

On note $A = \max\{L^e, L^l, L^f, L^m\}$. L^k est l'âge maximal du stade de développement k , $k = e, l, f, m$. On note $u^k(t, a, x) = 0, \beta^k(a) = 0, \mu^k(a, x) = 0$ alors que $a \in [L^k, A]$ pour chaque k . Les termes $w^e(t, a, x)$ et $w^l(t, a, x)$ sont des processus de contrôle respectivement pour les oeufs et les larves, et $m(a)$ est la fonction caractéristique sur

$(0, a^*)$ avec $0 < a^* < \min\{L^e, L^l, L^f\} \leq A$, ce qui signifie que notre intervention est limitée à des groupes d'âge très jeunes. Le terme $w^f(t, a, x)$ est le processus de contrôle pour les papillons femelles, et $\chi(a, x)$ est la fonction caractéristique sur $(0, a^*) \times \omega$, avec $\omega \subset \Omega$ étant un sous-ensemble ouvert non vide.

Les paramètres de la démographie μ^k avec $k = e, l, f, m$ correspondent aux taux de mortalités qui dépendent de l'âge et de l'espace. Les fonctions β^k , $k = e, l, f, m$ désignent les fonctions de transition à l'âge a , $k = e, l, f, m$. La dérivé directionnelle de u^k existe pour tout $(t, a, x) \in (0, T) \times (0, A) \times \Omega$ et est donnée par

$$Du^k(t, a, x) = \lim_{h \rightarrow 0} \frac{u^k(t+h, a+h, x) - u^k(t, a, x)}{h},$$

$k = e, l, f, m$. Il est évident que pour u^k suffisamment régulière on a

$$Du^k = \frac{\partial u^k}{\partial t} + \frac{\partial u^k}{\partial a}.$$

A notre connaissance, les méthodes utilisées dans la littérature pour étudier la contrôlabilité à zéro des problèmes paraboliques sont basées sur l'inégalité de Carleman et l'inégalité d'observabilité du système adjoint. En raison des termes non locaux dans (8), nous ne pouvons pas utiliser cette technique pour le système (7)-(10). Ce qui va nous amener à faire appel à des arguments de point fixe. D'abord nous transformons le terme non local $u^e(t, 0, x)$ en un terme local $b^e(t, x)$. Ensuite en combinant une estimation de Carleman avec des estimations à priori du système adjoint, on montre alors la contrôlabilité exacte à zéro pour les femelles par un argument de point fixe [3, 4].

La seconde partie est consacrée à l'étude de la stabilité de la conductivité d'un système de réaction diffusion modélisant l'activité électrique du coeur.

On considère le système linéarisé avec des conditions au bord

$$\begin{cases} c_m \partial_t v^\varepsilon - \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla v^\varepsilon) = -a(t, x) v^\varepsilon + f^\varepsilon \chi_\omega, & \text{in } Q_T, \\ \varepsilon \partial_t u_e^\varepsilon - \operatorname{div}(M(x) \nabla u_e^\varepsilon) = \operatorname{div}(M_i(x) \nabla v^\varepsilon), & \text{in } Q_T, \\ v^\varepsilon(\theta, x) = v_\theta(x), \quad u_e^\varepsilon(\theta, x) = u_{e,\theta}(x), & \text{in } \Omega, \\ v^\varepsilon = 0, \quad u_e^\varepsilon = 0, & \text{on } \Sigma_T, \end{cases}$$

où $a(t, x)$ est une fonction bornée dans Q_T , et sa dérivée par rapport à t existe et est bornée sur Q_T . La variable $u_i = u_i(t, x)$ ($u_e = u_e(t, x)$) représente le potentiel électrique intracellulaire (extracellulaire) à la position spatiale $x \in \Omega$. La différence $v = u_i - u_e$ représente le potentiel transmembranaire. Les conditions semi-initiales $v_\theta(x)$, $u_{e,\theta}(x)$ sont suffisamment régulières pour $\theta \in (0, T)$. Les tenseurs de conductivité inconnus

M et M_e sont supposés être suffisamment lisses. On analyse la stabilité des coefficients de diffusion d'un système parabolique qui modélise l'activité électrique du coeur. On établit une estimation de Carleman pour le système de réaction-diffusion. En combinant cette estimation avec des estimations d'énergie avec poids on obtient le résultat de stabilité. Le résultat de stabilité peut être résumé comme deux activités électriques cardiaques différentes. Il dépend du tenseur de la conductivité extracellulaire M_e dans la première équation, et de la somme du tenseur de la conductivité intracellulaire et extracellulaire. On note M la somme. Précisément, dans un petit domaine ω au cours de la période $(0, T)$ ou dans toute l'espace Ω au temps θ , le potentiel électrique extracellulaire et le potentiel transmembranaire varient assez peu. Alors, le tenseur de conductivité extracellulaire M_e est proche de \tilde{M}_e et M est proche de \tilde{M} sauf dans un petit sous-domaine $\omega_0 \in \omega$.

Mots-clés: Dynamique des populations structurées, contrôlabilité à zéro, méthode des caractéristiques, estimations de Carleman, théorème point de fixe, stabilité, système de réaction-diffusion.

Abstract

This thesis is divided into two parts. One is mainly devoted to make a qualitative analysis and exact null control for a class of structured population dynamical systems, and the other concerns stability of conductivities in an inverse problem of a reaction-diffusion system arising in electrocardiology.

In the first part, we study the dynamics of the European grape moth, which has caused serious damages on the vineyards in Europe, North Africa, and even some Asian countries. To model this grapevine insect, physiologically structured multistage population systems are proposed. These systems have nonlocal boundary conditions arising as nonlocal transition processes in ecosystem. We consider the questions of spatial spread of the population under physiological age and stage structures, and show global dynamical properties for the model. Furthermore, we investigate the control problem for this *Lobesia botrana* model when the growth function is equal to 1. For the case that four subclasses of this system are all in static station, we conclude that the population of eggs can be controlled to zero at a certain moment by acting on eggs. While the adult moths can disperse, we describe a control by a removal of egg and larva population, and also on female moths in a small region of the vineyard. Then the null controllability for female moths in a nonempty open sub-domain at a given time is obtained.

In the second part, a reaction-diffusion system approximating a parabolic-elliptic system was proposed to model electrical activity in the heart. We are interested in the stability analysis of an inverse problem for this model. Then we use the method of Carleman estimates and certain weight energy estimates for the identification of diffusion coefficients for the parabolic system to draw the conclusion.

Keywords: Structured population dynamics, null control, characteristics method, Carleman estimates, fixed point theorem, stability, reaction-diffusion system.

Contents

I	Analysis and control of a stage and age-structured population dynamics system	1
1	Introduction	3
1.1	Background knowledge	3
1.2	Statement of the problem and main results	11
2	Global dynamics of the European grapevine moth model with diffusion	19
2.1	Introduction	19
2.2	Preliminaries	23
2.3	Local existence and uniqueness of the solution	24
2.4	Global existence	32
2.5	Asymptotic behavior	35
2.6	Steady states	40
3	Exact null controllability of a stage and age-structured population dynamics system	51
3.1	Introduction	51
3.2	Preliminaries	54
3.3	Proof of the Main Result	55
3.3.1	Null controllability	55
3.3.2	Nonexistence of the controllability	66

4	Null controllability of the <i>Lobesia botrana</i> model with diffusion	69
4.1	Introduction	69
4.2	Preliminaries	72
4.3	Exact null controllability	74
4.3.1	Weight functions and Carleman inequality	74
4.3.2	An auxiliary optimal control problem	75
4.3.3	Exact null controllability	78
4.3.4	Existence of a fixed point for a multi-valued function	85
4.3.5	Nonexistence of controls	87
 II Stability of conductivities in an inverse problem in the reaction-diffusion system in electrocardiology		89
5	Introduction	91
6	A Carleman type estimate	95
6.1	Weight functions	95
6.2	Main proof of a Carleman type estimate	96
7	Stability of the conductivities	99
 III Conclusions and future works		113
8	Conclusions	115
9	Future works	117
	Bibliography	128

Part I

Analysis and control of a stage and age-structured population dynamics system

Chapter 1

Introduction

1.1 Background knowledge

In 1798, T.R. Malthus proposed a model of population dynamics as follows:

$$\frac{dx}{dt} = rx, \quad (1.1)$$

where x represents the total population size, r is the malthusian parameter of the given population. The solution of (1.1) is given by $x(t) = x_0 e^{rt}$. The exponential function makes apparent the famous exponential growth rate of a malthusian population. But this malthusian model makes no allowance for the effects of crowding or the limitations of resources. In 1838, P. F. Verhulst proposed a more realistic model of population growth, which would allow the malthusian parameter to depend on the size of the total population itself as follows:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right), \quad (1.2)$$

where $r(> 0)$ is the intrinsic growth rate and $K(> 0)$ is the carrying capacity of the population. It is known as the logistic equation. In model (1.2), when x is small the population grows as in the Malthusian model (1.1); when x is large the members of the species compete with each other for the limited resources. Solving (1.2) by separating the variables, we obtain ($x(0) = x_0$)

$$x(t) = \frac{x_0 K}{x_0 - (x_0 - K)e^{-rt}}. \quad (1.3)$$

From the formula (1.3), it is evident that the solutions of the equation (1.2) has the property $\lim_{t \rightarrow \infty} x(t) = K$. Thus the positive equilibrium $x = K$ of the logistic equation (1.2) is globally stable.

The theory of continuous population dynamics has been receiving more attention by mathematical demographers and population biologists. One of the most important theories in this development has been for models which allow for the effects of age structure. For many populations, consideration of the age distribution within the population leads to a more realistic and useful mathematical model.

First, F. R. Sharpe, A. Lotka and A. G. Mckendrick developed the models incorporating age effects with the birth and mortality processes as linear functions of the population densities. Many mathematicians extensively developed the theory of linear age-dependent population dynamics. The classical model is formulated as follows: Let $u(t, a)$ be the density with respect to age of a population at time t . The density function satisfies the aging process of the population

$$Du(t, a) = -\mu(a)u(t, a), \quad (1.4)$$

where μ is a nonnegative function of age-specific mortality modulus, and the differential operator is defined by

$$Du(t, a) = \lim_{h \rightarrow 0^+} \frac{u(t+h, a+h) - u(t, a)}{h}. \quad (1.5)$$

The birth process of the population satisfies the so-called birth law

$$u(t, 0) = \int_0^\infty \beta(a)u(t, a)da, \quad t > 0, \quad (1.6)$$

where $\beta(a)$ is a nonnegative function of age known as the age-specific fertility modulus. Last, the initial age distribution of the population is

$$u(0, a) = \phi(a), \quad a \geq 0, \quad (1.7)$$

where ϕ is a known nonnegative function of age a . The problem (1.4), (1.6) and (1.7) constitute the classical linear age-dependent population dynamics, which is easily solved by the method of characteristics. The main idea is to convert the problem to a Volterra integral equation with the birth rate $u(t, 0)$. This model admits a unique solution which is given by

$$u(t, a) = \begin{cases} \phi(a)\exp\{-\int_{a-t}^a \mu(s)ds\}, & a \geq t, \\ u(t-a, 0)\exp\{-\int_0^a \mu(s)ds\}, & a < t. \end{cases} \quad (1.8)$$

Moreover, the above solution is continue with respect to the age variable a , if this system satisfies the following compatibility condition

$$u(0, 0) = \int_0^\infty \beta(a)\phi(a)da = \phi(0). \quad (1.9)$$

One can refer to W. Feller [49], N. Keyfitz [72] and M. Gurtin [52] to learn about more detailed description. The analysis of linear models of this type is classical (see [63] for an introduction), while nonlinear models have a more recent history and are now undergoing a rapid development. Even though these nonlinear models have inherent mathematical difficulties, there are some broad general classes whose special structure allows a fairly detailed analysis.

The same objection can be raised to the Lotka-Von Forester model as was raised previously to the Malthusian law: the birth and death moduli are independent of the population density. Therefore in 1974, M. Gurtin, R. C. MacCamy and F. Hoppensteadt introduced the first models of nonlinear continuous age-dependent population dynamics. In their study, the effects of crowding were incorporated into the model by allowing the birth and mortality processes to be nonlinear functions of the population densities. Consequently, the equations of these models contained nonlinear terms involving the unknown solutions. Analogously with the model (1.2), these nonlinearity terms provided a mechanism by which the population may stabilize to a nontrivial equilibrium state as time evolved.

In the Gurtin-MacCamy model of age-dependent population dynamics, the fertility function and mortality function are allowed to be density dependent. The nonlinear population dynamics problem is given as follows: Let $P(t)$ be the total population at time t defined by

$$P(t) = \int_0^{\infty} u(t, a) da, \quad (1.10)$$

and $u(t, a)$ is defined as before. The balance law of the Gurtin-MacCamy model is

$$Du(t, a) = -\mu(a, P(t))u(t, a), \quad (1.11)$$

where μ is nonnegative, age- and density-dependent mortality modulus. The differential operator is defined by (1.5). The birth process of the population satisfies the so-called birth law

$$u(t, 0) = \int_0^{\infty} \beta(a, P(t))u(t, a) da, \quad t > 0, \quad (1.12)$$

where β is nonnegative function of age and density known as the fertility modulus. Last, the initial age distribution of the population is

$$u(0, a) = \phi(a), \quad a \geq 0, \quad (1.13)$$

where ϕ is a known initial age distribution. The model (1.11), (1.12) and (1.13) were introduced first by M. Gurtin and R. MacCamy in [54]. The common method for

treating this problem is linearization procedure. Then it is studied analogously to that classical linear case. Similar to the verhulstian age-independent model of population dynamics, the Gurtin-MacCamy model provides a physically more realistic description of the behavior of biological population. Thence, this model and similar models have been investigated by many researchers. One may be interested in them refer to the literatures [30, 39].

It has been advantageous to allow the birth and mortality processes to be nonlinear functions of the age-dependent population. One more interesting generalization has also been investigated by M. E. Gurtin, taking spacial diffusion into account. Considering the diffusion of a biological species in a domain Ω , the evolution of u is governed by the balance law (see M. Langlais [76] and M. E. Gurtin [52])

$$u_t + u_a = -\operatorname{div}q + s, \quad (1.14)$$

where $q = q(t, a, x)$ is the flux of population, div is the divergence operator in \mathbb{R}^N ($\operatorname{div} = \partial_{x_1} + \cdots + \partial_{x_N}$) and $s = s(t, a, x)$ is the supply of individuals. A reasonable assumption for the population-flux vector seems to be that q be a proportional to the gradient of the total population density:

$$q(t, a, x) = k\nabla P, \quad (1.15)$$

where ∇ is the gradient vector with respect to space variables and k is a positive constant. Further, assume that the supply s is due to deaths, and it is proportional to u :

$$s(t, a, x) = -\mu(a)u(t, a, x), \quad (1.16)$$

where μ is the death rate, P is defined as in (1.10). Going back to the balance law, u obeys the partial differential equation

$$u_t + u_a - k\Delta u + \mu u = 0, \quad (1.17)$$

where Δ is the Laplace operator in \mathbb{R}^N . The birth process is given by the equation

$$u(t, 0) = \int_0^\infty \beta(a, P(t))u(t, a)da, \quad t > 0. \quad (1.18)$$

Furthermore, the initial condition is

$$u(0, a) = \phi(a), \quad a \geq 0, \quad (1.19)$$

and the boundary condition is

$$\nabla u(t, a, x) \cdot n = \hat{u}(t, a, x), \quad x \in \partial\Omega, \quad (1.20)$$

while \hat{u} represents the prescribed population supply at the boundary.

However, It has to be emphasized that many population problems involve interactions between population subclasses, so vector system models should also be included, as well as scalar models. The model which allow the interaction to be interpreted as either two species in competition or two species as predator and prey has the form

$$\begin{cases} \frac{\partial u_i(t, a)}{\partial t} + \frac{\partial u_i(t, a)}{\partial a} = -(\mu_{i1}(P_1(t)) + \mu_{i2}(P_2(t)))u_i(t, a), \\ u_i(t, 0) = \int_0^\infty \beta_i(1 - e^{-\alpha_i a})u_i(t, a)da, \\ u_i(0, a) = \phi_i(a), \end{cases}$$

where i equals to 1, 2, respectively.

In the above model, each component of the density function $u = (u_1, u_2)^T$ corresponds to one of two species present in the same environment. Each species has a linear birth process independent of the other species. The form of birth process means that the members of each species reproduce as long as they survive. Each species has a nonlinear mortality process dependent on both species, where the mortality modulus has both an intra-species and inter-species contribution. The form of mortality process corresponds to a harsh environment, since it does not depend on the age variable. It should be noted that age-dependent models of predator-prey interaction have been widely investigated by many researchers about existence, the stability of nontrivial equilibrium solutions. One can see the work of J. M. Cushing, M. Saleem, F. Hoppensteadt, M. E. Gurtin, D. S. Levine and G. F. Webb for more details [38, 39, 53, 63, 109].

Since epidemic models were introduced by Kermack and Mckendrick in 1927, more and more attention has been devoted to the study of their properties [69, 70, 71, 84]. The importance of such a step arises from the fact that for many diseases the rate of infection varies significantly with age. In fact, for exanthematic disease it can be seen the transmission mainly involves early ages, while for sexually transmitted disease the principal mechanism of infection involves mature individuals (see [40, 66]). Moreover, some diseases transmitted from parents to newborns and also immunity is vertically transmitted and lasts up to some age. Because of the epidemics, the population is divided into three classes, such as susceptible class, infective class, and removed class, which are described by their respective age density functions $S(t, a)$, $I(t, a)$, $R(t, a)$.

Denoting by $\gamma(a)$, $\delta(a)$ and $\lambda(a)$ the age specific removal rate, cure rate and infection rate respectively, the following equations describes the transmission dynamics of the disease:

$$\left\{ \begin{array}{l} \frac{\partial s(t,a)}{\partial t} + \frac{\partial s(t,a)}{\partial a} = -\mu(a)s(t,a) - \lambda(t,a)s(t,a) + \delta(a)i(t,a), \\ \frac{\partial i(t,a)}{\partial t} + \frac{\partial i(t,a)}{\partial a} = -\mu(a)i(t,a) + \lambda(t,a)s(t,a) - (\gamma(a) + \delta(a))i(t,a), \\ \frac{\partial r(t,a)}{\partial t} + \frac{\partial r(t,a)}{\partial a} = -\mu(a)r(t,a) + \gamma(a)i(t,a), \\ s(t,0) = \int_0^{a^+} \beta(a)[s(t,a) + (1-q)i(t,a) + (1-w)r(t,a)]da, \\ i(t,0) = q \int_0^{a^+} \beta(a)i(t,a)da, \\ r(t,0) = w \int_0^{a^+} \beta(a)r(t,a)da, \\ s(0,a) = s_0(a), \quad i(0,a) = i_0(a), \quad r(0,a) = r_0(a), \end{array} \right.$$

where $q \in [0, 1]$, $w \in [0, 1]$, are the transmission parameters of infectiveness and immunity, respectively. These parameters indicate the fraction of newborns who are born in the class of their parents. Especially, $q = w = 0$ means newborns are susceptible. In [31], S. Busenberg, K. Cooke, and M. Iannelli investigated an age-structured epidemic model when the fertility, mortality and removal rates depend on age, and obtained endemic threshold criteria and the stability of steady state solutions. A. Ducrot and P. Magal has studied an infection-age structured epidemic model with external supplies, showed the existence and nonexistence of the traveling wave solutions and gave the minimal wave speed [41]. Later, they extended the result to multi-species age infection structured epidemic model [42].

In 1954, Slobodkin [102] experimentally studied a number of laboratory populations of *Daphnia obtuse*. He found that age or size alone was not sufficient to characterize the physiological behavior of an animal. Slobodkin concludes that “age and size taken together can be considered to define a class of physiologically identical animals until proven to the contrary.” Then Sinko and Streifer (1967) proposed an age and size structured model as follows:

$$\frac{\partial \eta(t, a, m)}{\partial t} + \frac{\partial \eta(t, a, m)}{\partial a} + \frac{\partial [g(t, a, m)\eta(t, a, m)]}{\partial m} = -D(t, a, m)\eta(t, a, m), \quad (1.21)$$

where $g(t, a, m)$ is the average rate of growth for an animal of age a and mass m at time t ,

$$g(t, a, m) = \frac{dm}{dt} = \frac{dm}{da},$$

and the function $D(t, a, m)$ is the death rate for animals of age a and mass m at time t . The integral

$$\int_{m_0}^{m_1} \int_{a_0}^{a_1} D(t, a, m)\eta(t, a, m)dadm$$

is the rate at which animals between ages (a_0, a_1) and masses (m_0, m_1) die. It should be noted that the functions g and D generally depend on other factors such as food supply, total biomass and total numbers in the population. To set the mathematical problem well, one must specify boundary conditions for equation (1.21), i.e. the age-mass distribution of the original animals at time zero, $\alpha(a, m)$, and density of newborn, $\beta(t, m)$, i.e.

$$\alpha(a, m) = \eta(0, a, m),$$

$$\beta(t, m) = \eta(t, 0, m).$$

In many situations, $\beta(t, m)$ is given by

$$\beta(t, m) = \int_0^\infty \int_0^\infty f(t, a, m', m) \eta(t, a, m') da dm', \quad (1.22)$$

where $f(t, a, m', m) dm'$ is the rate at which animals of mass m' and age a give birth to neonates with masses between m and $m + dm$. The fact that (1.22) is dependent on density function makes the mathematical problem complicated. It is shown that Von Foerster's equation, the logistic equation and other prior models are special cases of the new model [100].

Since Sinko and Streifer proposed model (1.21), more and more attention has devoted on this kind of a class of size structured model. For example, Farkas obtained stability result for a class of nonlinear models by characteristic method [44]. Later, Farkas et al. studied the linear stability and regularity for a size structure model with density limit, using semigroup theory and the characteristic equation [45]. Farkas et al. investigate a generalized size-structured Daphnia model with inflow, and obtained the stability of steady solution [46].

Since the end of the eighteenth century the European Grapevine moth *Lobesia botrana* has been the most serious grape pest in Europe, North Africa and in many Asian countries, and has caused important economic damages. The life cycle of the EGVM could be divided into four development stages that are egg, larva, pupa and moth. The first three stages correspond to the insect growth and the last adult stage is devoted to reproduction. As during the spring and summer, the period of pupa is very short. Then we can assume that it can be included in larval stage. In order to predict the periods of appearance of the insect in the vineyard, and the mathematical models with age structure maybe very helpful for this objective.

Inspired by (1.21), D. Picart et al. modeled the EGVM population dynamics by a stage-structured population model based on PDEs [12]. To properly describe the reproductive

cycle of the EGVM, they consider a multistage, physiologically structured, population model. Denote u^e, u^l, u^f, u^m the age density distribution at time t of egg, larva, female and male populations respectively. The total population for the k -stage is then defined by

$$P^k(t) = \int_0^{L^k} u^k(t, a) da, \quad t \geq 0,$$

where L^k is the maximum age for the k -stage, and k takes the value e for egg, l for larva, f for female and for m male. The following model describes the dynamics of these populations

$$\begin{cases} \frac{\partial u^e(t, a)}{\partial t} + \frac{\partial [v^e(E(t), a)u^e(t, a)]}{\partial a} = -m^e(E(t), a)u^e(t, a) - \beta^e(E(t), a)u^e(t, a), \\ \frac{\partial u^l(t, a)}{\partial t} + \frac{\partial [v^l(E(t), a)u^l(t, a)]}{\partial a} = -m^l(P^l(t), E(t), a)u^l(t, a) - \beta^l(E(t), a)u^l(t, a), \\ \frac{\partial u^f(t, a)}{\partial t} + \frac{\partial [v^f(E(t), a)u^f(t, a)]}{\partial a} = -m^f(E(t), a)u^f(t, a), \\ \frac{\partial u^m(t, a)}{\partial t} + \frac{\partial [v^m(E(t), a)u^m(t, a)]}{\partial a} = -m^m(E(t), a)u^m(t, a), \end{cases} \quad (1.23)$$

where $(t, a) \in [0, T] \times [0, L^k]$, $k = e, l, f, m$.

The vector $E = (T, H, R)$ corresponds to the climatic and environmental factors where T is the temperature factor, H the humidity factor and R the grape variety factor. The vector E is time-dependent. Stress that R is not the quantity of food eaten by the larva but depending on the species of the vine. The functions $m^k, k = e, l, f, m$, are the k -stage age-specific per capita mortality functions. To model the inter-individual competition between larvae for food, we assume that m^l depends on the total larva population. The functions $\beta^k, k = e, l$, correspond to the k -stage age-specific transition functions. Specially, β^e is the hatching function which models the physiological change between egg and larval stages. The emerging adult function β^l is the transition between the larval and moth stages. The transition function between the moth and egg stages is modeled by the function β^f , i.e. the age-specific per capita birth function. Observations on EGVM indicate that the population does not grow exponentially but reaches a threshold value determined by the carrying capacity of the food. One bunch can hardly bear more than 15 larvae depending upon the bunch size [105]. The growth of the population size is not restricted by the food quantity but by the total number of moths per unit of volume. Therefore, the birth function is dependent on the density of individuals.

The study of laboratory data shows a difference in growth between individuals assembled in cohorts [16, 85, 86]. This phenomenon is classical and has been observed for many other species. Mathematicians often model growth variability by introducing growth function that depends on the physiological age [32]. In LBM, functions

$v^k, k = e, l, f, m$ represent the k -stage age-specific per capita growth functions. These functions are age-dependent and, coupled with the transition functions, allow us to model great variability of growth within a cohort. The set of demographic functions $(m^k, v^k, \beta^e, \beta^l, \beta^f), k = e, l, f, m$ vary with the climatic and environmental factors E . The boundary conditions related to (1.23) are defined by

$$\begin{cases} v^e(E(t), 0)u^e(t, 0) = \int_0^{L^f} \beta^f(P^f(t), P^m(t), E(t), s)u^f(t, s)ds, \\ v^l(E(t), 0)u^l(t, 0) = \int_0^{L^e} \beta^e(E(t), s)u^e(t, s)ds, \\ v^f(E(t), 0)u^f(t, 0) = \int_0^{L^l} \tau\beta^l(E(t), s)u^l(t, s)ds, \\ v^m(E(t), 0)u^m(t, 0) = \int_0^{L^l} (1 - \tau)\beta^l(E(t), s)u^l(t, s)ds, \end{cases} \quad (1.24)$$

where τ describes sex ratio, $t \in (0, T)$; the initial conditions are given by

$$u^k(0, a) = u_0^k(a), a \in (0, L^k), \quad k = e, l, f, m. \quad (1.25)$$

Although from a mathematical view, the equations related to adult stages of (1.23) look like Sinko and Streifers model, the first two equations are not same to their model because of the additional terms modeling the proportion of individuals who change physiological state. Furthermore, the additional terms of the first two equations are also the boundary conditions of the larval and adult stages respectively. Therefore, the fact that all equations in the above system are dependent from each other makes the mathematical analysis more complicated. Note that LBM just exists in the literatures [11, 12, 92, 93]. Also, because of its original and special form, LBM enables us to study new mathematical and biological questions. D. Picart et al. has considered the existence and uniqueness of (1.23) in L^2 , and showed some simulations of experimental field data. Later, they did parameter identification for this system. Finally, they investigated the optimal control problem.

1.2 Statement of the problem and main results

Therefore, considering the economic loss caused by the pest, it is meaningful to study the mathematical analysis for this *Lobesia botrana* model (LBM), and the ways how we can control the size of its population. In this thesis, we shall investigate the questions such as, global dynamics and the exact null controllability of this age and stage structured system with nonlocal boundary conditions arising as transition process.

♠ Global dynamics of the European grapevine moth model with diffusion

We consider a multistage, physiologically structured, population model describing one of the most important grapevine insect pests [12,13]. *Lobesia botrana*, the European grapevine moth (EGVM), is a grape pest causing important economic damages. This kind of moth reduces the amount of berries especially when berries are young in spring, as well as their quality by favoring indirect damages [104]. To predict the population peaks of this insect in vineyards, several ordinary differential equations or discrete equations have been developed to describe the period and the length of the spring and summer population dynamics for egg, larval and adult stages. The temperature is the only environmental factor implicated to predict the population size in time as a growth factor. As a consequence, other relevant aspects of the dynamics, for example the mortality or inter-cohort growth variations, are missing and the predictions are not satisfying [20, 26].

It is known that partial differential equations are also used to describe the dynamics of a single population [89]. These equations enable us to model physiological characteristics such as age or size to differentiate individuals within a cohort. For example, Sinko and Streifer's model is well used to study age-size structured populations [32, 100]. To explore the importance of growth variations within a cohort and properly describe the reproductive cycle of the EGVM, D. Picart et al. proposed a stage-structured population model under nonlinear boundary conditions based on PDEs [11, 12, 92]. Here we consider a more realistic phenomenon that adult moth can diffuse spatially. Then we study a system of four equations each related to the Sinko-Streifer model as follows:

$$\begin{cases} \frac{\partial u^e(t,a,x)}{\partial t} + \frac{\partial[v^e(E(t),a)u^e(t,a,x)]}{\partial a} = -\mu^e(E(t),a)u^e(t,a,x) - \beta^e(E(t),a)u^e(t,a,x), \\ \frac{\partial u^l(t,a,x)}{\partial t} + \frac{\partial[v^l(E(t),a)u^l(t,a,x)]}{\partial a} = -\mu^l(P^l(t,x),E(t),a)u^l(t,a,x) - \beta^l(E(t),a)u^l(t,a,x), \\ \frac{\partial u^f(t,a,x)}{\partial t} + \frac{\partial[v^f(E(t),a)u^f(t,a,x)]}{\partial a} = -\mu^f(E(t),a)u^f(t,a,x) + d_f\Delta_x u^f(t,a,x), \\ \frac{\partial u^m(t,a,x)}{\partial t} + \frac{\partial[v^m(E(t),a)u^m(t,a,x)]}{\partial a} = -\mu^m(E(t),a)u^m(t,a,x) + d_m\Delta_x u^m(t,a,x), \end{cases} \quad (1.26)$$

where $(t, a, x) \in (0, T) \times (0, L^k) \times \Omega$ and $k = e, l, f, m$ represents four stages of development respectively. We denote the age density distribution of individuals in Ω and at time t of egg, larva, female month and male month populations by u^e, u^l, u^f, u^m . The total population for the k stage is then defined by

$$P^k(t, x) = \int_0^{L^k} u^k(t, a, x) da, \quad x \in \Omega, t \geq 0, k = e, l, f, m.$$

The nonlinear boundary conditions are defined by

$$\begin{cases} v^e(E(t), 0)u^e(t, 0, x) = \int_0^{L^f} \beta^f(P^f(t, x), P^m(t, x), E(t), s)u^f(t, s, x)ds, \\ v^l(E(t), 0)u^l(t, 0, x) = \int_0^{L^e} \beta^e(E(t), s)u^e(t, s, x)ds, \\ v^f(E(t), 0)u^f(t, 0, x) = \int_0^{L^l} \sigma\beta^l(E(t), s)u^l(t, s, x)ds, \\ v^m(E(t), 0)u^m(t, 0, x) = \int_0^{L^l} (1 - \sigma)\beta^l(E(t), s)u^l(t, s, x)ds, \end{cases} \quad (1.27)$$

where σ denotes the sex ratio, x is in Ω and $t \in (0, T)$. The system is complete with the initial conditions and no flux boundary conditions as follows

$$\begin{cases} u^k(0, a, x) = u_0^k(a, x), & \text{in } (0, L^k) \times \Omega, \quad k = e, l, f, m, \\ \frac{\partial u^f}{\partial \eta} = 0, & \text{on } (0, T) \times (0, L^f) \times \partial\Omega, \\ \frac{\partial u^m}{\partial \eta} = 0, & \text{on } (0, T) \times (0, L^m) \times \partial\Omega. \end{cases} \quad (1.28)$$

The vector E corresponds to the climatic and environmental factors, and it is time dependent. The functions μ^k are the k-stage age-specific per capita mortality functions. The functions β^k denote the k-stage age-specific transition functions. The growth of the population size is not dependent on the food quantity but on the total number of moths per unit of volume. Therefore, the birth function is dependent on the individual density. The functions v^k represent the k-stage age-specific per capita growth functions which depend on the physiological age, and allow us to model great variability of growth within a cohort [104].

Note that the question of existence of global solutions is connected to the works of Martin-Pierre for reaction diffusion systems, but the demographical processes add new difficulties (see the work of Iannelli and Busenberg (1988) for the case of equal demographical parameters [31]). Thus we become interested in the mathematical analysis of the model (1.26)-(1.28), and inspired to investigate the properties of solutions. Especially, in our context the mathematical tractable setting for the problem is the Banach space L^1 . Actually, L^1 space is the natural choice in which the physical interpretation of the density function requires that it should be integrable, and the mathematical treatment of the problem requires that the density functions belong to a complete norm linear space. The norm of the density is a natural measure of the size of the population. We obtain the solution by the characteristics method for this *Lobesia botrana* model. According to the expression of the solution of the system, u^e , u^l , u^f and u^m are linked with their boundary conditions. Note that u^e is described by the number of newborns laid by the female moths u^f at each egg class. The solutions u^f , u^m are related to the larva density u^l . As the boundary conditions are not null, it is obvious to see that u^e ,

u^f and u^m exist uniquely if and only if u^l exists uniquely. Then a contraction fixed point principle is used to obtain the existence and uniqueness of solution. The result can be extended to be global by the maximum interval of the existence of the solution. We also prove the existence of a global attractor for the trajectories of the dynamical system defined by the solutions of the model. Finally, we use the method based on the theory of compact operators and the Krasnoselskii's fixed point theorem to prove the existence of steady states. This will be based on the manuscript [61].

♠ Exact null controllability of a stage and age-structured population dynamics system

Population control is the process of forcing a population or some of its subclasses to assume a specified-in-advance behavior. The intervention policy which is used to produce the expectable result constitutes the controller. Harvesting of natural or farmed populations, such as plants and fish, controlling of pests and parasites, such as aphids and lice, and containing disease distribution through vaccination and other control measures are just a few of many useful and economically significant applications of the mathematical optimization theory to population control [19].

The problem of control and optimal harvesting of populations has been widely investigated in the literatures with different scope. Early work on the topic considers the problem of deterministic harvesting, ignoring the age structure (see [35] and references cited therein). Discussions for the problem harvesting a discrete age structured population has been investigated by several authors [78, 79]. Discussion of an optimal harvesting strategy for a continuous age structured model was treated by Rorres and Fair in 1980 [96]. The optimal harvesting problem of the population process represented by McKendrick equations has been considered by Murphy and Smith [88]. Then Gurtin and Murphy [55, 57] investigated the optimal control problem for the nonlinear population dynamics introduced in [54]. They reduced the system of nonlinear PDEs to a system of ODEs. The harvesting strategy is restricted to being age-independent. One can refer to the literatures [14, 19, 82, 83] and so on for more results about optimal control problems.

We recall that the internal null controllability of the linear heat equation, when the control acts on a subset of the domain, was established by G. Lebeau and L. Robbiano [77] and was later extended to some semilinear equations by A.V. Fursikov and O.Yu. Imanuvilov [51] in the sublinear case, and by V. Barbu [17] and E. Fernandez-Cara [47] in the superlinear case. The internal null controllability of the age-dependent

population dynamics in the particular case when the control acts in a spatial subdomain ω but for all ages a (this is the particular case corresponding to $a^* = A$) was investigated by B. Ainseba and S. Anița [3]. Let $p(t, a, x)$ be the distribution of individuals of age $a \geq 0$, at time $t \geq 0$, and location $x \in \Omega$, a bounded domain of \mathbb{R}^N , $N \in 1, 2, 3$, with a suitably smooth boundary $\partial\Omega$.

$$\left\{ \begin{array}{l} \frac{\partial p(t,a,x)}{\partial t} + \frac{\partial p(t,a,x)}{\partial a} = -\mu(a,t,x, P(t,x))p(a,t,x) + k\Delta p(t,a,x) \\ \quad + \chi_\omega(x)u(t,a,x), \quad (t,a,x) \in Q_{a^+}, \\ p(t,a,x) = 0, \quad (t,a,x) \in \Sigma_{a^+}, \\ p(t,0,x) = \int_0^{a^+} \beta(t,a,x, P(t,x))p(t,a,x)da, \quad (t,x) \in (0,T) \times \Omega, \\ p(0,a,x) = p_0(a,x), \quad (a,x) \in (0,a^+) \times \Omega, \end{array} \right. \quad (1.29)$$

and

$$P(t,x) = \int_0^{a^+} p(t,a,x)da, \quad (1.30)$$

where u is a control function, χ_ω is the characteristic function of ω , a nonempty, open subset of Ω , and p_0 is the initial distribution of individuals. Here $Q_{a^+} = (0, a^+) \times (0, T) \times \Omega$, and $\Sigma_{a^+} = (0, a^+) \times (0, T) \times \Omega$. B.E. Ainseba et. [7] are concerned with the general nonlinear case of (1.29) in the sense of Gurtin and MacCamy [54], and obtained the exact null controllability result.

It is known that European grapevine moth (EGVM) has caused very serious economical problem not only in Europe, but also in Asian countries and Africa. This kind of moth reduces not only the amount of berries especially when berries are young in spring, as well as their quality by favouring indirect damages as related to different pathogens developing on berries like the grey mold and in several warm vineyards to the black rots on berries [104]. These problems are suspected to increase, and could become more prevalent due to the climatic changes in the future. Therefore, it is meaningful to study the exact null controllability of this *Lobesia botrana* model (LBM). It is worth mentioning that we study the case without adult moths diffusing firstly. The problem is stated as follows:

$$\left\{ \begin{array}{l} \frac{\partial u^e(t,a)}{\partial t} + \frac{\partial u^e(t,a)}{\partial a} = -(\mu^e(a) + \beta^e(a))u^e(t,a) + \chi(a)w(t,a), \\ \frac{\partial u^l(t,a)}{\partial t} + \frac{\partial u^l(t,a)}{\partial a} = -(\mu^l(a) + \beta^l(a))u^l(t,a), \\ \frac{\partial u^f(t,a)}{\partial t} + \frac{\partial u^f(t,a)}{\partial a} = -\mu^f(a)u^f(t,a), \\ \frac{\partial u^m(t,a)}{\partial t} + \frac{\partial u^m(t,a)}{\partial a} = -\mu^m(a)u^m(t,a), \end{array} \right. \quad (1.31)$$

where $(t, a) \in [0, T] \times [0, A]$, $A = \max\{L^e, L^l, L^f, L^m\}$. Here L^k means life expectancy of an individual for $k = e, l, f, m$, and $u^k(t, a)$ represents the age-specific density of the

egg, larva, female moth and male moth respectively. For every k , if $A > L^k$, we denote $u^k = 0, \beta^k = 0, \mu^k = 0$. The term $\chi(a)w(t, a)$ is a control process for egg: $\chi(a)$ is the characteristic function of $[0, a^*](0 < a^* < L^e \leq A)$, which means that our intervention can be restricted to the younger age groups.

The boundary conditions are defined by

$$\begin{cases} u^e(t, 0) &= \int_0^{L^f} \beta^f(s)u^f(t, s)ds, \\ u^l(t, 0) &= \int_0^{L^e} \beta^e(s)u^e(t, s)ds, \\ u^f(t, 0) &= \int_0^{L^l} \sigma\beta^l(s)u^l(t, s)ds, \\ u^m(t, 0) &= \int_0^{L^l} (1 - \sigma)\beta^l(s)u^l(t, s)ds, \end{cases} \quad (1.32)$$

where σ denotes the sex ratio, $t > 0$. The system is complete with the initial conditions as follows

$$u^k(0, a) = u_0^k(a), \quad (1.33)$$

for $k = e, l, f, m$. The demography parameters μ^k are the k -stage per capita mortality functions with respect to age, and the functions β^k denote the k -stage age-specific transition functions, $k = e, l, f, m$.

It is feasible to reduce the number of larva population through taking intervention for the eggs when they are in very young age interval, then to cut back the total number of the moth populations. Here we investigate the controllability for the pest by acting on eggs in a small age interval. Note that the method developed in [19] to the system case to get the key observability inequality cannot be used for our life circle system. In spite of that, considering the fact that the system is a stage and age-dependent life cycle dynamics, we are inspired to apply the fixed point theorem in [3, 4] to study the exact null controllability in finite time of the *Lobesia botrana* model (LBM) with four development stages, by reducing the egg population. The main method is based on the derivation of estimations for the adjoint variables related to an optimal control problem. Finally, a fixed point theorem is used to conclude that in (1.31)-(1.33) $u^e(T, a) = 0$ except the small enough age groups at a certain moment in the future, using an age- and time-dependent control of egg individuals. This will be based on the article [59].

♠ Null controllability of the *Lobesia botrana* model with diffusion

We consider the situation that adult individuals move spatially here. Once grape moths fly around, they will bring about more serious impact for the vineyard. In fact, when these pests appear in a smaller range of the vineyard, the plague of insects will soon spread throughout the whole vineyard, even the surrounding vineyards. It will cause

very serious economic loss. Then one can ask a question of whether or not we can control this population by acting on adults in a small spatial domain. Therefore, our main purpose is to study the null controllability of a stage and age-structured system modeling *Lobesia botrana* growth where adult individuals diffuse.

Let us denote the distribution of egg, larva, female and male individuals, respectively by $u^k(t, a, x)$ of age $a \geq 0$, at time $t \geq 0$, and location $x \in \Omega$ with $k = e, l, f, m$. Then the dynamics of the LBM is given by

$$\begin{cases} Du^e(t, a, x) &= -(\mu^e(a, x) + \beta^e(a))u^e(t, a, x) + m(a)w^e(t, a, x), \\ Du^l(t, a, x) &= -(\mu^l(a, x) + \beta^l(a))u^l(t, a, x) + m(a)w^l(t, a, x), \\ Du^f(t, a, x) &= -\mu^f(a, x)u^f(t, a, x) + \Delta u^f(t, a, x) + \chi(a, x)w^f(t, a, x), \\ Du^m(t, a, x) &= -\mu^m(a, x)u^m(t, a, x) + \Delta u^m(t, a, x), \end{cases} \quad (1.34)$$

where $(t, a, x) \in (0, T) \times (0, A) \times \Omega$, and $\Omega \subset \mathbb{R}^3$.

The boundary conditions are stated as follows

$$\begin{cases} u^e(t, 0, x) &= \int_0^{L^f} \beta^f(s)u^f(t, s, x)ds, \\ u^l(t, 0, x) &= \int_0^{L^e} \beta^e(s)u^e(t, s, x)ds, \\ u^f(t, 0, x) &= \int_0^{L^l} \sigma \beta^l(s)u^l(t, s, x)ds, \\ u^m(t, 0, x) &= \int_0^{L^l} (1 - \sigma)\beta^l(s)u^l(t, s, x)ds, \end{cases} \quad (1.35)$$

where σ denotes the sex ratio, $t > 0$. The system is complete with the initial and boundary conditions as follows

$$u^k(0, a, x) = u_0^k(a, x), \quad k = e, l, f, m, \quad (1.36)$$

$$\frac{\partial u^k(t, a, x)}{\partial \eta} = 0, \quad x \in \partial\Omega, \quad k = f, m. \quad (1.37)$$

Here $A = \max\{L^e, L^l, L^f, L^m\}$, and L^k means life expectancy of an individual with $k = e, l, f, m$. For every k , we denote $u^k(t, a, x) = 0, \beta^k(a) = 0, \mu^k(a, x) = 0$ as $a \in [L^k, A]$. The terms $w^e(t, a, x)$ and $w^l(t, a, x)$ are control processes respectively for eggs and larvae, and $m(a)$ is the characteristic function of $(0, a^*)$ with $0 < a^* < \min\{L^e, L^l, L^f\} \leq A$, which means that our intervention is restricted to the younger age groups. The term $w^f(t, a, x)$ is the control process for female moths, and $\chi(a, x)$ is the characteristic function of $(0, a^*) \times \omega$, with $\omega \subset\subset \Omega$ being a nonempty open subset.

In addition, the functions μ^k are the k -stage per capita mortality functions with respect to age and space. The functions β^k denote the k -stage age-specific transition functions. For each (t, a, x) the directional derivatives of u^k exist, and we can see

$$Du^k(t, a, x) = \lim_{h \rightarrow 0} \frac{u^k(t+h, a+h, x) - u^k(t, a, x)}{h},$$

with $k = e, l, f, m$. It is obvious that for u^k smooth enough

$$Du^k = \frac{\partial u^k}{\partial t} + \frac{\partial u^k}{\partial a}. \quad (1.38)$$

The optimal and exact control problems are widely investigated for age-structured population dynamics by many researchers. Among these literatures, most of the studies are focused on optimal control problems (see [3, 82, 93] and references therein). B. Aïnseba et al. established the exact controllability for age dependent linear and non-linear single-species population model with diffusion (refer to [3, 4, 7, 9]). Viorel Barbu et al. also considered the exact controllability of the linear Lotka-McKendrick model without spatial structure by establishing an observability inequality for the backward adjoint system [19].

However, there are no results dealing with the control problem for a stage and age-dependent life cycle dynamics with diffusion. To control the dynamics of our insect population, it is easier to act on static individuals, eggs and larvas, and manipulate in a certain area for female moths to cut back on the number of butterflies, and then the *Lobesia botrana* population. This controllability problem can be investigated as an exact null controllability in finite time of the diffusive *Lobesia botrana* model.

As far as we know, for the null controllability of the parabolic systems on a subset of the domain, the method is based on the Carleman inequality and an observability inequality for the backward adjoint systems. Due to the non-locality in (1.35), we cannot use this technique for (1.34)-(1.37). Therefore, we have the idea here to apply the fixed point theorem. First we transform the nonlocal term $u^e(t, 0, x)$ to be a local one $b^e(t, x)$. Next we select a family of controls to obtain the null controllability, by combining some estimations and the Carleman inequality for the local backward system related to an optimal control problem. Then choosing a control corresponding to a fixed point of a multi-valued function, we obtain that the solution u^f of (1.34) satisfies

$$u^f(T, a, x) = 0, \quad a.e. \ a \in (\delta, A), \ x \in \Omega, \quad (1.39)$$

where $0 < \delta \leq a_0$ is a small parameter, by getting the existence of the controls w^e , w^l and w^f . This will be based on the article [60].

Chapter 2

Global dynamics of the European grapevine moth model with diffusion

2.1 Introduction

Lobesia botrana, the European grapevine moth (EGVM), is a grape pest causing important economic damages. This kind of moth reduces the amount of berries especially when berries are young in spring, but also their quality by favoring indirect damages [104]. To predict the population peaks of this insect in vineyards, several ordinary differential equations or discrete equations have been developed to describe the period and the length of the spring and summer population dynamics for egg, larval and adult stages. The temperature is the only environmental factor implicated to predict the population size in time as a growth factor. As a consequence, other relevant aspects of the dynamics, for example the mortality or inter-cohort growth variations, are missing and the predictions are not satisfying [20, 26]. Partial differential equations are also used to describe the dynamics of a single population [89]. These equations enable us to model physiological characteristics such as age or size to differentiate individuals within a cohort. For example, Sinko and Streifers model is the well-known model used in the study of age-size structured populations [32]. To study the importance of growth variations within a cohort and properly describe the reproductive cycle of the EGVM, D. Picart et al. propose a stage-structured population model under nonlinear boundary conditions [11, 12, 92].

We denote the age density distribution of individuals in Ω and at time t of egg, larva, male moth and female moth populations by u^e, u^l, u^f, u^m . The total population for the k stage is then defined by

$$P^k(t, x) = \int_0^{L^k} u^k(t, a, x) da, \quad x \in \Omega, t \geq 0.$$

It leads us to write the following system describing the dynamics of the population

$$\begin{cases} \frac{\partial u^e(t, a, x)}{\partial t} + \frac{\partial [v^e(E(t), a)u^e(t, a, x)]}{\partial a} = -\mu^e(E(t), a)u^e(t, a, x) - \beta^e(E(t), a)u^e(t, a, x), \\ \frac{\partial u^l(t, a, x)}{\partial t} + \frac{\partial [v^l(E(t), a)u^l(t, a, x)]}{\partial a} = -\mu^l(P^l(t, x), E(t), a)u^l(t, a, x) - \beta^l(E(t), a)u^l(t, a, x), \\ \frac{\partial u^f(t, a, x)}{\partial t} + \frac{\partial [v^f(E(t), a)u^f(t, a, x)]}{\partial a} = -\mu^f(E(t), a)u^f(t, a, x) + d_f \Delta_x u^f(t, a, x), \\ \frac{\partial u^m(t, a, x)}{\partial t} + \frac{\partial [v^m(E(t), a)u^m(t, a, x)]}{\partial a} = -\mu^m(E(t), a)u^m(t, a, x) + d_m \Delta_x u^m(t, a, x), \end{cases} \quad (2.1)$$

where $(t, a, x) \in (0, T) \times (0, L^k) \times \Omega$ and $k = e, l, f, m$.

The nonlinear boundary conditions are defined by

$$\begin{cases} v^e(E(t), 0)u^e(t, 0, x) = \int_0^{L^f} \beta^f(P^f(t, x), P^m(t, x), E(t), s)u^f(t, s, x)ds, \\ v^l(E(t), 0)u^l(t, 0, x) = \int_0^{L^e} \beta^e(E(t), s)u^e(t, s, x)ds, \\ v^f(E(t), 0)u^f(t, 0, x) = \int_0^{L^l} \sigma \beta^l(E(t), s)u^l(t, s, x)ds, \\ v^m(E(t), 0)u^m(t, 0, x) = \int_0^{L^l} (1 - \sigma) \beta^l(E(t), s)u^l(t, s, x)ds, \end{cases} \quad (2.2)$$

where σ denotes the sex ratio, x is in Ω and $t > 0$. The system is complete with the initial conditions and no flux boundary conditions as follows

$$\begin{cases} u^k(0, a, x) = u_0^k(a, x), & \text{in } (0, L^k) \times \Omega, \\ \frac{\partial u^k}{\partial \eta} = 0, & \text{on } (0, T) \times (0, L^k) \times \partial\Omega \end{cases} \quad (2.3)$$

for $k = e, l, f, m$.

In addition, the vector E corresponds to the climatic and environmental factors, and it is time dependent. The functions μ^k are the k -stage age-specific per capita mortality functions. Because of the inter-individual competition among the larvae for food, we suppose that μ^l depends on the total larva population. The functions β^k denote the k -stage age-specific transition functions. In particular, β^e models the physiological change between the eggs and larvae stage, which is called the hatching function. The function β^l is the flying function describing the transition between the larvae and the moths stage. The function β^f means models the transition between the moths and the eggs stage whose name is the birth function. The growth of the population size is not dependent on the food quantity but on the total number of moths per unit of volume.

Therefore, the birth function is dependent on the individual density. The functions v^k represent the k -stage age-specific per capita growth functions which depend on the physiological age [104].

We first introduce some notations and assumptions for system (2.1)-(2.3).

Suppose that for each (t, a, x) the directional derivatives of u^k exist,

$$Du^k(t, a, x) = \lim_{h \rightarrow 0} \frac{u^k(t+h, X^k(t+h; t, a), x) - u^k(t, a, x)}{h},$$

where $X^k(t+h; t, a)$ with $k = e, l, f, m$ is the solution of the differential equation

$$\begin{cases} X^k(t) = V^k(E(t), X^k(t)), \\ X^k(t_0) = a_0 > 0. \end{cases} \quad (2.4)$$

The definition of a solution of the above system only requires that $u^k(x, t, a)$ is differentiable along the curves defined by $a = X^k(t; t_0, a_0)$, which goes through (t_0, a_0) . Specially, $z^k(t) := X^k(t; 0, 0)$ is the characteristic through the origin. This curve is the trajectory in the (t, a) -plane of the newborn individuals at $t = 0$ and it separates the trajectories of the individuals that were present at the initial time $t = 0$ from the trajectories of those individuals born after the initial time.

Furthermore, diffusion operators A^f and A^m generate uniformly bounded semigroups e^{tA^f} and e^{tA^m} respectively, and satisfy

$$A^k = d_k \Delta_x.$$

Let $L^1 = L^1((0, L^k) \times \Omega; \mathbb{R}^n)$ be the Banach space of equivalence classes of Lebergue integrable functions, from $(0, L^k) \times \Omega$ in \mathbb{R}^n with the norm:

$$\|\varphi\|_{L^1((0, L^k) \times \Omega)} = \int_{\Omega} \int_0^{L^k} |\varphi(a, x)| da dx,$$

where Ω is a bounded open domain with a regular boundary $\partial\Omega$. L^k is the maximum chronological age to each individual.

Let $T > 0$. For all $r \in \mathbb{R}$, we can define the space \mathcal{L}^r by

$$\mathcal{L}^r = \{\varphi \in L^1 \mid \sup_{0 \leq t \leq T} e^{-rt} \|\varphi\|_{L^1}\}.$$

It is obvious that $\mathcal{L}^r \subset L^1$.

Definition 2.1 For all $T > 0$ and all $(a, t, x) \in (0, L^k) \times (0, T) \times \Omega$, (u^e, u^l, u^f, u^m) is called a solution of (2.1) if it belongs to $(\mathcal{C}([0, T], L^1((0, L^k) \times \Omega; \mathbb{R}^n)))^4$ and it satisfies system (2.1), where $k = e, l, f, m$.

In the present work, we impose the following demographic assumptions on system (2.1)-(2.3):

(A1) The function $v^k(E(t), a)$ is bounded, strictly positive and continuously differentiable with respect to a , and it satisfies

$$0 < \underline{v}^k < v^k(E(t), a) < \bar{v}^k, \quad (t, a) \in (0, T) \times (0, L^k),$$

respectively for $k = e, l, f, m$. In addition, there exists a positive constant C_{v^k} such that

$$\left\| \frac{\partial v^k}{\partial a}(E(t), a) \right\|_{\infty} \leq C_{v^k}.$$

(A2) The hatching function β^e and the flying function β^l are bounded and nonnegative functions.

(A3) The birth function $\beta^f(P^f(t, x), P^m(t, x), E(t), a)$ is bounded, nonnegative and Lipschitz continuous with constant β_K^f with respect to the variables P^f and P^m .

(A4) The mortality functions $\mu^e(E(t), a)$, $\mu^f(E(t), a)$ and $\mu^m(E(t), a)$ are nonnegative, locally bounded and satisfy the following conditions:

$$\lim_{a \rightarrow L^k} \int_0^t \mu^k(E(t), X^k(s; a, t)) ds = \infty, \quad a \geq Z^k(t),$$

$$\lim_{a \rightarrow L^k} \int_{\tau^k}^t \mu^k(E(t), X^k(s; a, t)) ds = \infty, \quad a < Z^k(t)$$

with $k = e, f, m$ and t is strictly positive.

(A5) The mortality function of larva stage $\mu^l(P^l(t, x), E(t), a)$ is nonnegative, locally bounded and Lipschitz continuous with constant m_K^l with respect to the first variable and satisfies the following conditions:

$$\lim_{a \rightarrow L^l} \int_0^t \mu^l(P^l(t), E(t), X^l(s; a, t)) ds = \infty, \quad a \geq Z^l(t),$$

$$\lim_{a \rightarrow L^l} \int_{\tau^l}^t \mu^l(P^l(t), E(t), X^l(s; a, t)) ds = \infty, \quad a < Z^l(t),$$

where t is strictly positive.

In a series of papers [2, 10, 48, 67], researchers deal with the analysis of an age-dependent population dynamics model with spatial diffusion. One can also refer to the literature [32]. The authors study the existence and uniqueness of solutions for the scalar size structured population model with a nonlinear growth rate depending on the individual's size and on the total population, and also prove the existence of a (compact) global attractor for the trajectories of the dynamical system defined by the solutions of the model. We are inspired to investigate the mathematical properties of the solution of the EGVM model. To study global existence we consider our problem in L^1 setting which is a natural choice for population dynamics problems.

This chapter is organized as follows: We present some theorems and definitions which are used in Section 2. Local existence and uniqueness of the mild solution of (2.1)-(2.3) are given in Section 3. In Section 4, we give the global result of the mild solution. Section 5 is devoted to study the asymptotic behavior of the solution of the model using asymptotic theory of dissipative systems (see [58]). In Section 6, we give the existence of the steady states of this system, mainly based on the Krasnoselskii's fixed point theorem.

2.2 Preliminaries

We will state some theorems and definitions which are recommended in the previous sections.

Theorem 2.2 (Jack K. Hale [58]) *If $T(t) : X \rightarrow X$ is an asymptotically smooth C^0 -semigroup which is point dissipative and has orbits of bounded sets bounded, then $T(t)$ has a global attractor.*

Theorem 2.3 (Frechet-Kolmogorov) *Let $\Omega \subset R^N$ be an open set, and let \mathcal{F} be a bounded subset of $L^p(\Omega)$ with $1 \leq p \leq \infty$. If the following conditions are satisfied:*

(1) *For any $\varepsilon > 0, \omega \subset\subset \Omega$, there exist $\rho > 0, \rho < \text{dist}(\omega, \partial\Omega)$ such that*

$$\|\tau_h f - f\|_{L^p(\omega)} < \varepsilon,$$

where $\tau_h f$ denotes the translation of f by h , that is, $\tau_h f(x) = f(x+h)$, for $|h| < \rho$ and $h \in R^N, f \in \mathcal{F}$.

(2) *For any $\varepsilon > 0, \omega \subset\subset \Omega$ such that $\|f\|_{L^p(\Omega \setminus \omega)} < \varepsilon$ for $f \in \mathcal{F}$.*

Then \mathcal{F} is relatively compact in $L^p(\Omega)$.

Theorem 2.4 (*M.A.Krasnoselskii [75]*) *Let the positive operator $A(A(0) = 0)$ has a strong Frechet derivative $A'(0)$ and a strong asymptotic derivative $A'(\infty)$ with respect to a cone. Let the spectrum of the operator $A'(\infty)$ lie in the circle $|\lambda| \leq \rho$, where $\rho \leq 1$. Let the operator $A'(0)$ have in K a characteristic vector h_0 , then $A'(0)h_0 = \lambda_0 h_0$, where $\lambda_0 > 1$ does not have in K characteristic vectors to which a characteristic value equal to 1 corresponds. Then it suffices for the existence for the operator A of at one non-zero fixed point in the cone K that one of the following conditions be satisfied:*

- (1) *The operator A is completely continuous.*
- (2) *The operator A is weakly continuous, the space E is weakly complete, the unit sphere in E is weakly compact, the cone K allows plastering.*

Definition 2.5 (*Ikuko Sawashima [99]*) *The cone K is total if the set $\{\psi - \varphi | \varphi \in K\}$ is dense in K . An operator T is a non-supporting operator in K with respect to K if T is positive and for each nonzero $x \in K$ and for each nonzero $f \in K^*$ there exists a natural number $n_0 = n(x, f)$ such that $f(T^n x) > 0$ whenever $n \geq n_0$.*

Theorem 2.6 (*Ikuko Sawashima [99]*) *Let the cone K be total. T is a non-supporting operator with respect to K , and $r(T)$ is a pole of the resolvent of T . The following conditions hold and are equivalent to each other:*

- (1) *Every proper vector corresponding to the proper value $r(T)$ lying in K is a non-supporting point of K and every proper vector corresponding to $r(T)$ lying in K^* is strictly positive.*
- (2) *The proper space corresponding to the proper value $r(T)$ is one-dimensional subspace of E passing through a non-support point K and there exists a strictly positive proper functional corresponding to $r(T)$.*
- (3) *$R(\lambda, T)$ has a pole of order 1 at $\lambda = r(T)$.*

2.3 Local existence and uniqueness of the solution

Integrating along the characteristic curve, we obtain for $k = e, l$

$$u^k(t, a, x) = \begin{cases} u_0^k(X^k(0; t, a), x) e^{-\int_0^t h^k(s, X^k(s; a, t)) ds}, & a \geq Z^k(t), \\ u^k(\tau^k, 0, x) e^{-\int_{\tau^k}^t h^k(s, X^k(s; a, t)) ds}, & a < Z^k(t), \end{cases} \quad (2.5)$$

and for $k = f, m$

$$u^k(t, a, x) = \begin{cases} e^{tA^k} u_0^k(X^k(0; t, a), x) e^{-\int_0^t h^k(s, X^k(s; a, t)) ds}, & a \geq Z^k(t), \\ e^{(t-\tau^k)A^k} u^k(\tau^k, 0, x) e^{-\int_{\tau^k}^t h^k(s, X^k(s; a, t)) ds}, & a < Z^k(t), \end{cases} \quad (2.6)$$

where τ^k is implicitly given by $X^k(t; \tau^k, 0) = a$ or $X^k(\tau^k; t, a) = 0$, that is, τ^k is the initial time of the cohort through (a, t) , and

$$\begin{cases} h^e(t, a) = (\mu^e + \beta^e + \partial_a v^e)(E(t), a), \\ h^l(t, a) = \mu^l(P^l(t, x), E(t), a) + (\beta^l + \partial_a v^l)(E(t), a), \\ h^f(t, a) = (\mu^f + \partial_a v^f)(E(t), a), \\ h^m(t, a) = (\mu^m + \partial_a v^m)(E(t), a). \end{cases} \quad (2.7)$$

Remark 2.7 *Observing the above expression of the solution of the system, u^e, u^l, u^f and u^m are linked with their boundary conditions. Note that u^e is described by the number of newborns laid by the female moths u^f at each egg class. The solutions u^f, u^m are related to the larva density u^l . As the boundary conditions are not null, it is obvious to see that u^e, u^f and u^m exist uniquely if and only if u^l exists uniquely.*

In the following part, we will consider the local existence and uniqueness of the solution of (2.1). Firstly, let a subspace in L^1

$$M := \{\phi : \|\phi\|_{L^\mu} = \sup_{0 \leq t \leq T} e^{-\mu t} \|\phi\|_{L^1} \leq r, \phi(\cdot, 0, \cdot) = u_0^l(\cdot, \cdot) \in L^1, \mu \text{ is a positive constant}\}.$$

It can be easily seen that M is a closed set. Define $\|u\|_{L^1} = \|u^e\|_{L^1} + \|u^l\|_{L^1} + \|u^f\|_{L^1} + \|u^m\|_{L^1}$. Suppose that $\|u_0\|_{L^1} \leq r$ and u_0 is nonnegative.

For all $\phi \in M$ and $t \in (0, T)$, we define $u^l = \Lambda(\phi)$ as follows

$$u^l(t, a, x) = \begin{cases} u_0^l(X^l(0; t, a), x) e^{-\int_0^t \mu^l(P_\phi^l(s, x), E(s), X^l(s; a, t)) + (\beta^l + \partial_a v^l)(E(s), X^l(s; a, t)) ds}, & a \geq Z^l(t), \\ u^l(\tau^l, 0, x) e^{-\int_{\tau^l}^t \mu^l(P_\phi^l(s, x), E(s), X^l(s; a, t)) + (\beta^l + \partial_a v^l)(E(s), X^l(s; a, t)) ds}, & a < Z^l(t), \end{cases} \quad (2.8)$$

and

$$v^f(E(t), 0) u^f(t, 0, x) = \int_0^{L^l} \sigma \beta^l(E(t), s) \phi(t, s, x) ds. \quad (2.9)$$

Before we study the local existence property of the solution, we give the following result firstly.

Lemma 2.8 *Let $T > 0$ and $t \in [0, T]$, then the solution $u = (u^e, u^l, u^f, u^m)$ satisfies*

$$\|u\|_{L^1} = \|u^e\|_{L^1} + \|u^l\|_{L^1} + \|u^f\|_{L^1} + \|u^m\|_{L^1} \leq \mu_1 \|u_0\|_{L^1} e^{\mu_2 t},$$

where $\mu_1 = \max\{1, C_m, C_f\}$ and $\mu_2 = \max\{\max\{\alpha_m, \alpha_f\} \|\beta^l\|_\infty, \|\beta^e\|_\infty, \|\beta^f\|_\infty, 1\}$.

Proof. From now on, we denote $\mu := \min\{\mu^e + \beta^e + \partial_a v^e, \mu^l + \beta^l + \partial_a v^l, \mu^f + \partial_a v^f, \mu^m + \partial_a v^m\}$. Then we obtain

$$\begin{aligned} & \|u^e\|_{L^1} \\ \leq & \int_{Z^e(t)}^{L^e} \int_{\Omega} |u_0^e(X^e(0; a, t), x) \exp\{-\int_0^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds\}| dx da \\ & + \int_0^{Z^e(t)} \int_{\Omega} |u^e(\tau^e, 0, x) \exp\{-\int_{\tau^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds\}| dx da \\ \leq & e^{-\mu t} \|u_0^e\|_{L^1} + \int_0^{Z^e(t)} \int_{\Omega} \int_0^{L^f} |\beta^f(P^f(\tau^e, x), P^m(\tau^e, x), E(\tau^e), s') \\ & u^f(\tau^e, s', x)| ds' \left| \frac{\exp\{-\int_{\tau^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds\}}{v^e(E(\tau^e), 0)} \right| dx da \\ \leq & e^{-\mu t} \|u_0^e\|_{L^1} + \|\beta^f\|_\infty \int_0^t \|u^f\|_{L^1} d\tau \quad (\tau = \tau^e), \end{aligned}$$

$$\begin{aligned} & \|u^l\|_{L^1} \\ \leq & \int_{Z^l(t)}^{L^l} \int_{\Omega} |u_0^l(X^l(0; a, t), x) \exp\{-\int_0^t (\mu^l(P^l(s, x), E(s), a)) \\ & + (\beta^l + \partial_a v^l)(E(s), X^l(s; a, t)) ds\}| dx da + \int_0^{Z^l(t)} \int_{\Omega} |u^l(\tau^l, 0, x) \\ & \exp\{-\int_{\tau^l}^t (\mu^l(P^l(s, x), E(s), a)) + (\beta^l + \partial_a v^l)(E(s), X^l(s; a, t)) ds\}| dx da \\ \leq & e^{-\mu t} \|u_0^l\|_{L^1} + \int_0^{Z^l(t)} \int_{\Omega} \int_0^{L^e} |\beta^e(E(\tau^l), s') u^e(\tau^l, s', x)| ds' \\ & \left| \frac{\exp\{-\int_{\tau^l}^t (\mu^l(P^l(s, x), E(s), a)) + (\beta^l + \partial_a v^l)(E(s), X^l(s; a, t)) ds\}}{v^l(E(\tau^l), 0)} \right| dx da \\ \leq & e^{-\mu t} \|u_0^l\|_{L^1} + \|\beta^e\|_\infty \int_0^t \|u^e\|_{L^1} d\tau \quad (\tau = \tau^l), \end{aligned}$$

$$\begin{aligned}
 & \|u^f\|_{L^1} \\
 \leq & \int_{Z^f(t)}^{L^f} \int_{\Omega} |u_0^f(X^f(0; a, t), x) e^{tA^f} \exp\{-\int_0^t (\mu^f + \partial_a v^f)(E(s), X^f(s; a, t)) ds\}| dx da \\
 & + \int_0^{Z^f(t)} \int_{\Omega} |u^f(\tau^f, 0, x) e^{(t-\tau^f)A^f} \exp\{-\int_{\tau^f}^t (\mu^f + \partial_a v^f)(E(s), X^f(s; a, t)) ds\}| dx da \\
 \leq & C_f e^{-\mu t} \|u_0^f\|_{L^1} + \int_0^{Z^f(t)} \int_{\Omega} \int_0^{L^l} |\sigma \beta^l(E(\tau^f), s') u^l(\tau^f, s', x)| ds' \\
 & \left| \frac{e^{(t-\tau^f)A^f}}{v^f(E(\tau^f), 0)} \exp\{-\int_{\tau^f}^t (\mu^f + \partial_a v^f)(E(s), X^f(s; a, t)) ds\} \right| dx da \\
 \leq & C_f e^{-\mu t} \|u_0^f\|_{L^1} + \|\beta^l\|_{\infty} \sigma \alpha_f \int_0^t \|u^l\|_{L^1} d\tau \quad (\tau = \tau^f, \alpha_f = \max |e^{(-\mu+A^f)(t-\tau^f)}|), \\
 \\
 & \|u^m\|_{L^1} \\
 \leq & \int_{Z^m(t)}^{L^m} \int_{\Omega} |u_0^m(X^m(0; a, t), x) e^{tA^m} \exp\{-\int_0^t (\mu^m + \partial_a v^m)(E(s), X^m(s; a, t)) ds\}| dx da \\
 & + \int_0^{Z^m(t)} \int_{\Omega} |u^l(\tau^l, 0, x) e^{(t-\tau^m)A^m} \exp\{-\int_{\tau^m}^t (\mu^m + \partial_a v^m)(E(s), X^m(s; a, t)) ds\}| dx da \\
 \leq & C_m e^{-\mu t} \|u_0^m\|_{L^1} + \int_0^{Z^m(t)} \int_{\Omega} \int_0^{L^l} |(1-\sigma) \beta^l(E(\tau^m), s') u^l(\tau^m, s', x)| ds' \\
 & \left| \frac{e^{(t-\tau^m)A^m}}{v^m(E(\tau^m), 0)} \exp\{-\int_{\tau^m}^t (\mu^m + \partial_a v^m)(E(s), X^m(s; a, t)) ds\} \right| dx da \\
 \leq & C_m e^{-\mu t} \|u_0^m\|_{L^1} + \|\beta^l\|_{\infty} (1-\sigma) \alpha_m \int_0^t \|u^l\|_{L^1} d\tau \\
 & (\tau = \tau^m, \alpha_m = \max |e^{(-\mu+A^m)(t-\tau^m)}|).
 \end{aligned}$$

Adding the above four inequalities, we deduce that

$$\begin{aligned}
 & \|u^e\|_{L^1} + \|u^l\|_{L^1} + \|u^f\|_{L^1} + \|u^m\|_{L^1} \\
 \leq & \max\{1, C_m, C_f\} (\|u_0^e\|_{L^1} + \|u_0^l\|_{L^1} + \|u_0^f\|_{L^1} + \|u_0^m\|_{L^1}) \\
 & + \max\{\max(\alpha_m, \alpha_f) \|\beta^l\|_{\infty}, \|\beta^e\|_{\infty}, \|\beta^f\|_{\infty}, 1\} \\
 & \cdot \int_0^t (\|u^e\|_{L^1} + \|u^l\|_{L^1} + \|u^f\|_{L^1} + \|u^m\|_{L^1}) d\tau.
 \end{aligned}$$

Applying Gronwall's inequality, it is easy to obtain

$$\|u\|_{L^1} \leq \mu_1 \|u_0\|_{L^1} e^{\mu_2 t}$$

with $\mu_1 = \max\{1, C_m, C_f\}$, $\mu_2 = \max\{\max(\alpha_m, \alpha_f) \|\beta^l\|_{\infty}, \|\beta^e\|_{\infty}, \|\beta^f\|_{\infty}, 1\}$. \square

To get the local existence and uniqueness of solution, we need to prove the following lemma.

Lemma 2.9 *Assume that the above hypothesis (A1)-(A5) hold. Then there exists a value $T > 0$ such that the operator defined by $\Lambda : \phi \rightarrow u^l$ has only one fixed point, where ϕ belongs to M .*

Proof. Since M is a closed set of L^1 , we have only to show that the operator Λ is contractive and maps M into M .

(1) We show Λ is contractive. Let $\phi_i, u_i^e, u_i^l, u_i^f, u_i^m$ satisfy (2.3), (2.5), (2.6), (2.8), (2.9) respectively, and $\phi_i \in M$.

$$\begin{aligned} \|u_1^l - u_2^l\|_{L^1} &= \int_0^{L^l} \int_{\Omega} |u_1^l - u_2^l| dx da \\ &\leq \underbrace{\int_0^{Z^l(t)} \int_{\Omega} |u_1^l - u_2^l| dx da}_{I_1} + \underbrace{\int_{Z^l(t)}^{L^l} \int_{\Omega} |u_1^l - u_2^l| dx da}_{I_2}. \end{aligned}$$

$$\begin{aligned} I_1 &= \int_0^{Z^l(t)} \int_{\Omega} |u_1^l(\tau^l, 0, x)| \cdot \exp\left\{-\int_{\tau^l}^t \mu^l(P_{\phi_1}(s, x), E(s), X^l(s; a, t))\right. \\ &\quad \left. + (\beta^l + \partial_a v^l)(E(s), X^l(s; a, t)) ds\right\} - \exp\left\{-\int_{\tau^l}^t \mu^l(P_{\phi_2}(s, x), E(s), X^l(s; a, t))\right. \\ &\quad \left. + (\beta^l + \partial_a v^l)(E(s), X^l(s; a, t)) ds\right\} dx da + \int_0^{Z^l(t)} \int_{\Omega} |u_1^l(\tau^l, 0, x) - u_2^l(\tau^l, 0, x)| \\ &\quad \exp\left\{-\int_{\tau^l}^t \mu^l(P_{\phi_2}(s, x), E(s), X^l(s; a, t)) + (\beta^l + \partial_a v^l)(E(s), X^l(s; a, t)) ds\right\} dx da \\ &:= I_3 + I_4. \end{aligned}$$

Consider the integral I_3 , we have

$$\begin{aligned} I_3 &\leq \int_0^{Z^l(t)} \int_{\Omega} |u_1^l(\tau^l, 0, x)| e^{-\mu(t-\tau^l)} \int_{\tau^l}^t m_K^l \int_0^{L^l} |\phi_1 - \phi_2| da' ds dx da \\ &\leq \int_0^{Z^l(t)} \int_{\Omega} \frac{\int_0^{L^e} |\beta^e(E(\tau^l), s) u_1^e(\tau^l, s, x)| ds}{v^l(E(\tau^l), 0)} e^{-\mu(t-\tau^l)} \\ &\quad \int_{\tau^l}^t m_K^l \int_0^{L^l} e^{\mu s} e^{-\mu s} |\phi_1 - \phi_2| da' ds dx da \\ &< \int_0^{Z^l(t)} \frac{\|u_1^e\|_{\infty} \cdot \|\beta^e\|_{\infty} \cdot L^e \cdot m_K^l (e^{\mu\tau^l} - e^{2\mu\tau^l - \mu t}) \|\phi_1 - \phi_2\|_{L^{\mu}}}{v^l(E(\tau^l), 0) \cdot \mu} da \\ &\leq \int_0^{Z^l(t)} \frac{\|u_1^e\|_{\infty} \cdot \|\beta^e\|_{\infty} \cdot L^e \cdot m_K^l (e^{\mu\tau^l} - e^{2\mu\tau^l - \mu t}) \|\phi_1 - \phi_2\|_{L^{\mu}}}{\mu} d\tau^l \\ &\leq \frac{\|u_1^e\|_{\infty} \cdot \|\beta^e\|_{\infty} \cdot L^e \cdot m_K^l (e^{\mu t} + e^{-\mu t} - 2) \|\phi_1 - \phi_2\|_{L^{\mu}}}{2\mu^2}. \end{aligned}$$

For I_4 , we can obtain

$$\begin{aligned}
 I_4 &\leq \int_0^{Z^l(t)} \int_{\Omega} \frac{\int_0^{L^e} |\beta^e(E(\tau^l, s))| |u_1^e(\tau^l, 0, x) - u_2^e(\tau^l, 0, x)| ds}{v^l(E(\tau^l), 0)} e^{-\mu(t-\tau^l)} dx da \\
 &\leq \int_0^{Z^l(t)} \frac{\|\beta^e\|_{\infty} \cdot \|u_1^e - u_2^e\|_{L^1} e^{-\mu\tau^l} e^{-\mu t + 2\mu\tau^l}}{v^l(E(\tau^l), 0)} da \\
 &\leq \int_0^t \|\beta^e\|_{\infty} \cdot \|u_1^e - u_2^e\|_{L^{\mu}} e^{-\mu t + 2\mu\tau^l} d\tau^l \\
 &\leq \frac{\|\beta^e\|_{\infty} \cdot \|u_1^e - u_2^e\|_{L^{\mu}} (e^{\mu t} - e^{-\mu t})}{2\mu}.
 \end{aligned}$$

Then we consider the integral I_2 ,

$$\begin{aligned}
 I_2 &\leq \int_{Z^l(t)}^{L^l} |u_0^l(X^l(0; a, t), x)| \exp\left\{-\int_0^t [\mu^l(P_{\phi_1}(s, x), E(s), X^l(s; a, t)) + (\beta^l + \partial_a v^l)(E(s), X^l(s; a, t))] ds\right\} \\
 &\quad - \exp\left\{-\int_0^t [\mu^l(P_{\phi_2}(s, x), E(s), X^l(s; a, t)) + (\beta^l + \partial_a v^l)(E(s), X^l(s; a, t))] ds\right\} | dx da \\
 &\leq \int_{Z^l(t)}^{L^l} \int_{\Omega} |u_0^l(X^l(0; a, t), x)| e^{-\mu t} \int_0^t m_K^l \int_0^{L^l} e^{\mu s} e^{-\mu s} |\phi_1 - \phi_2| da' ds dx da \\
 &\leq \int_{Z^l(t)}^{L^l} \frac{\|u_0^l\|_{\infty} \cdot m_K^l \cdot (1 - e^{-\mu t}) \|\phi_1 - \phi_2\|_{L^{\mu}}}{\mu} da, \\
 &< \frac{\|u_0^l\|_{\infty} \cdot m_K^l \cdot L^l (1 - e^{-\mu t}) \|\phi_1 - \phi_2\|_{L^{\mu}}}{\mu}.
 \end{aligned}$$

According to (2.5), it follows that

$$\begin{aligned}
 \|u_1^e - u_2^e\|_{L^1} &= \int_0^{Z^e(t)} \int_{\Omega} |u_1^e(\tau^e, 0, x) - u_2^e(\tau^e, 0, x)| \cdot \exp\left\{-\int_{\tau^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds\right\} dx da \\
 &\leq \int_0^{Z^e(t)} \int_{\Omega} \int_0^{L^f} |\beta^f(P_1^f(\tau^e, x), P_1^m(\tau^e, x), E(\tau^e), s)u_1^f(\tau^e, s, x) \\
 &\quad - \beta^f(P_2^f(\tau^e, x), P_2^m(\tau^e, x), E(\tau^e), s)u_2^f(\tau^e, s, x)| ds \exp\left\{-\int_{\tau^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s'), X^e(s'; a, t)) ds'\right\} \frac{1}{v^e(E(\tau^e), 0)} dx da \\
 &\leq \int_0^{Z^e(t)} \int_{\Omega} \frac{e^{-\mu(t-\tau^e)}}{v^e(E(\tau^e), 0)} \int_0^{L^f} \|\beta^f\|_{\infty} |u_1^f - u_2^f| + |u_2^f| \\
 &\quad \beta_K^f (|P_1^f - P_2^f| + |P_1^m - P_2^m|) ds dx da \\
 &\leq \int_0^{Z^e(t)} \frac{e^{-\mu(t-\tau^e)}}{v^e(E(\tau^e), 0)} \|\beta^f\|_{\infty} \|u_1^f - u_2^f\|_{L^1} + \|u_2^f\|_{\infty} \\
 &\quad \beta_K^f L (\|u_1^f - u_2^f\|_{L^1} + \|u_1^m - u_2^m\|_{L^1}) da \\
 &\leq \frac{(\|\beta^f\|_{\infty} + \beta_K^f L \|u_2^f\|_{\infty})(e^{\mu t} - e^{-\mu t}) \|u_1^f - u_2^f\|_{L^{\mu}}}{2\mu} \\
 &\quad + \frac{\beta_K^f L \|u_2^f\|_{\infty} (e^{\mu t} - e^{-\mu t}) \|u_1^m - u_2^m\|_{L^{\mu}}}{2\mu},
 \end{aligned}$$

where L is $\max\{L^f, L^m\}$. Analogously we can obtain

$$\begin{aligned}
 \|u_1^f - u_2^f\|_{L^1} &= \int_0^{Z^f(t)} \int_{\Omega} |u_1^f(\tau^f, 0, x) - u_2^f(\tau^f, 0, x)| \exp\left\{A^f(t - \tau^f) - \int_{\tau^f}^t (\mu^f + \beta^f + \partial_a v^f)(E(s), X^f(s; a, t)) ds\right\} dx da \\
 &\leq \int_0^{Z^f(t)} \int_{\Omega} \int_0^{L^l} |\sigma \beta^l(E(\tau^f), s)| |\phi_1(\tau^f, s, x) - \phi_2(\tau^f, s, x)| ds \\
 &\quad \exp\left\{-\int_{\tau^f}^t (\mu^f + \beta^f + \partial_a v^f)(E(s'), X^f(s'; a, t)) ds'\right\} \frac{e^{A^f(t-\tau^f)}}{v^f(E(\tau^f), 0)} dx da \\
 &\leq C_f \int_0^{Z^f(t)} \frac{\|\sigma \beta^l\|_{\infty}}{v^f(E(\tau^f), 0)} e^{-\mu(t-\tau^f)} \|\phi_1 - \phi_2\|_{L^1} da \\
 &= C_f \int_0^t \|\sigma \beta^l\|_{\infty} e^{-\mu(t-2\tau^f)} \|\phi_1 - \phi_2\|_{L^{\mu}} d\tau^f \\
 &\leq C_f \sigma \|\beta^l\|_{\infty} \frac{(e^{\mu t} - e^{-\mu t})}{2\mu} \|\phi_1 - \phi_2\|_{L^{\mu}}.
 \end{aligned}$$

A similar analysis can be made for

$$\begin{aligned}
 \|u_1^m - u_2^m\|_{L^1} &= \int_0^{Z^m(t)} \int_{\Omega} |u_1^m(\tau^m, 0, x) - u_2^m(\tau^m, 0, x)| \exp\{A^m(t - \tau^m) \\
 &\quad - \int_{\tau^m}^t (\mu^m + \beta^m + \partial_a v^m)(E(s), X^m(s; a, t)) ds\} dx da \\
 &\leq \int_0^{Z^m(t)} \int_{\Omega} \int_0^{L^1} |(1 - \sigma)\beta^l(E(\tau^m), s)| |\phi_1(\tau^m, s, x) - \phi_2(\tau^m, s, x)| \\
 &\quad \exp\left\{-\int_{\tau^m}^t (\mu^m + \beta^m + \partial_a v^m)(E(s), X^m(s; a, t)) ds\right\} \frac{e^{A^m(t - \tau^m)}}{v^m(E(\tau^m), 0)} dx da \\
 &\leq C_m \int_0^{Z^m(t)} \frac{\|(1 - \sigma)\beta^l\|_{\infty}}{v^f(E(\tau^m), 0)} e^{-\mu(t - \tau^m)} \|\phi_1 - \phi_2\|_{L^1} da \\
 &= C_m \int_0^t \|(1 - \sigma)\beta^l\|_{\infty} e^{-\mu(t - 2\tau^m)} \|\phi_1 - \phi_2\|_{L^{\mu}} d\tau^m \\
 &\leq C_m (1 - \sigma) \|\beta^l\|_{\infty} \frac{(e^{\mu t} - e^{-\mu t})}{2\mu} \|\phi_1 - \phi_2\|_{L^{\mu}}.
 \end{aligned}$$

Combining the above inequalities, we have

$$\begin{aligned}
 \|u_1^l - u_2^l\|_{L^1} &< \frac{\|u_1^e\|_{\infty} \cdot \|\beta^e\|_{\infty} \cdot L^e \cdot m_K^l (e^{\mu t} + e^{-\mu t} - 2) \|\phi_1 - \phi_2\|_{L^{\mu}}}{2\mu^2} + \frac{\|\beta^e\|_{\infty}}{2\mu} \\
 &\quad \cdot \|u_1^e - u_2^e\|_{L^{\mu}} (e^{\mu t} - e^{-\mu t}) + \frac{\|u_0^l\|_{\infty} \cdot m_K^l \cdot L^l (1 - e^{\mu t}) \|\phi_1 - \phi_2\|_{L^{\mu}}}{\mu} \\
 &\leq \frac{\|u_1^e\|_{\infty} \cdot \|\beta^e\|_{\infty} \cdot L^e \cdot m_K^l (e^{\mu t} + e^{-\mu t} - 2) \|\phi_1 - \phi_2\|_{L^{\mu}}}{2\mu^2} \\
 &\quad + \frac{\|u_0^l\|_{\infty} \cdot m_K^l \cdot L^l (1 - e^{-\mu t}) \|\phi_1 - \phi_2\|_{L^{\mu}}}{\mu} + \frac{\|\beta^e\|_{\infty} (e^{\mu t} - e^{-\mu t})}{2\mu} \\
 &\quad \left\{ \frac{(\|\beta^f\|_{\infty} + \beta_K^f L \|u_2^f\|_{\infty}) (e^{\mu t} - e^{-\mu t})}{2\mu} C_f \sigma \|\beta^l\|_{\infty} \frac{(e^{\mu t} - e^{-\mu t})}{2\mu} \right. \\
 &\quad \left. + \frac{\beta_K^f L \|u_2^f\|_{\infty} (e^{\mu t} - e^{-\mu t})}{2\mu} C_m (1 - \sigma) \|\beta^l\|_{\infty} \frac{(e^{\mu t} - e^{-\mu t})}{2\mu} \right\} \|\phi_1 - \phi_2\|_{L^{\mu}} \\
 &< K \|\phi_1 - \phi_2\|_{L^{\mu}} \\
 &\leq K \|\phi_1 - \phi_2\|_{L^1},
 \end{aligned}$$

where K is defined as follows

$$\begin{aligned}
 K &= \frac{\|u_1^e\|_{\infty} \|\beta^e\|_{\infty} L^e m_K^l (e^{\mu T} - 1)}{2\mu^2} + \frac{\|u_0^l\|_{\infty} m_K^l L^l (1 - e^{-\mu T})}{\mu} \\
 &\quad + \frac{(\|\beta^f\|_{\infty} + \beta_K^f L \|u_2^f\|_{\infty}) C \|\beta^e\|_{\infty} \|\beta^l\|_{\infty} (e^{\mu T} - e^{-\mu T})^3}{8\mu^3}
 \end{aligned}$$

with $C = \max\{C_f, C_m\}$. Note that $K < 1$ provided T is small enough, which implies that Λ is contractive.

(2) It remains to show $\Lambda : M \rightarrow M$. Using (2.5), (2.6), (2.8), (2.9), we obtain

$$\|u^l\|_{L^\mu} \leq \|u^l\|_{L^1} \leq \|u_0^l\|_{L^1} + \|\beta^e\|_\infty \|u^e\|_{L^\mu} \frac{e^{\mu T} - e^{-\mu T}}{2\mu}, \quad (2.10)$$

$$\|u^e\|_{L^\mu} \leq \|u^e\|_{L^1} \leq \|u_0^e\|_{L^1} + \|\beta^f\|_\infty \|u^f\|_{L^\mu} \frac{e^{\mu T} - e^{-\mu T}}{2\mu}, \quad (2.11)$$

$$\|u^f\|_{L^\mu} \leq \|u^f\|_{L^1} \leq \|u_0^f\|_{L^1} + C_f r \sigma \|\beta^l\|_\infty \frac{e^{\mu T} - e^{-\mu T}}{2\mu}. \quad (2.12)$$

Substituting (2.11) and (2.12) into (2.10), we have the following inequality by the assumption of the initial value

$$\begin{aligned} \|u^l\|_{L^\mu} &\leq \|u^l\|_{L^1} \leq \|u_0^l\|_{L^1} + \|\beta^e\|_\infty \frac{e^{\mu T} - e^{-\mu T}}{2\mu} \|u_0^e\|_{L^1} + \|\beta^e\|_\infty \|\beta^f\|_\infty \\ &\quad \frac{(e^{\mu T} - e^{-\mu T})^2}{2\mu} \|u_0^f\|_{L^1} + C_f r \sigma \|\beta^l\|_\infty \|\beta^f\|_\infty \|\beta^e\|_\infty \frac{(e^{\mu T} - e^{-\mu T})^3}{2\mu} \leq r \end{aligned}$$

for small enough values T . That completes the proof of Lemma 2.9. \square

Corollary 2.10 *Under the above assumptions in Lemma 2.9, there exists a unique local solution of system (2.1)–(2.3).*

2.4 Global existence

In this section, we consider the global result of the solution. First we need to prove the continuous property of the solution with respect to t .

Lemma 2.11 *Let $u_0^k \in L^1, T > 0$ and $u^k \in L^T$ be a solution of (2.1), (2.2), (2.3) in the interval $[0, T]$. Let $\hat{u}^k \in L^{\hat{T}}$ with $\hat{T} > 0$ satisfies for $k = e, l$*

$$\hat{u}^k(t, a, x) = \begin{cases} u_T^k(X^k(T; t, a), x) e^{-\int_0^t h^k(s+T, X^k(s+T; a, t)) ds}, & a \geq Z^k(t), \\ \hat{u}^k(\tau^k, 0, x) e^{-\int_{\tau^k}^t h^k(s+T, X^k(s+T; a, t)) ds}, & a < Z^k(t), \end{cases} \quad (2.13)$$

and for $k = f, m$,

$$\hat{u}^k(t, a, x) = \begin{cases} u_T^k(X^k(T; t, a), x) e^{tA^k} e^{-\int_0^t h^k(s+T, X^k(s+T; a, t)) ds}, & a \geq Z^k(t), \\ \hat{u}^k(\tau^k, 0, x) e^{(t-\tau^k)A^k} e^{-\int_{\tau^k}^t h^k(s+T, X^k(s+T; a, t)) ds}, & a < Z^k(t), \end{cases} \quad (2.14)$$

where τ^k is implicitly given by $X^k(t; \tau^k, 0) = a$ or $X^k(\tau^k; t, a) = 0$ and $h^k(t, a)$ is given in (2.7). Define $\hat{u}^k(t, a, x)$ as a continuous extension of the solution $u^k(t, a, x)$ in $[T, T + \hat{T}]$ such that $u^k(t, a, x) = \hat{u}^k(t - T, a, x)$. Then $u^k(t, a, x)$ is a solution of (2.1)–(2.3) on $[0, T + \hat{T}]$, in which $u^k(t, a, x) \in \mathcal{C}([0, T + \hat{T}], L^1((0, L^k) \times \Omega; \mathbb{R}^n))$.

Proof. We have shown that u^k is a solution of system (2.1)–(2.3) on $[0, T]$. It remains to prove that it is a solution for all $t \in [T, T + \hat{T}]$. Now we separate the proof into three steps according to a .

(1) $a \in [0, Z^k(t - T)], k = e, l, f, m$:

$$\begin{aligned} u^e(t, a, x) &= \hat{u}^e(t - T, a, x) \\ &= \hat{u}^e(\tau^e - T, 0, x) \exp\left\{-\int_{\tau^e - T}^{t - T} (\mu^e + \beta^e + \partial_a v^e)(E(s + T), X^e(s + T; a, t)) ds\right\} \\ &= u(\tau^e, 0, x) \exp\left\{-\int_{\tau^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds\right\}. \end{aligned}$$

A similar analysis can be made for u^l, u^f, u^m respectively. We omit the details.

(2) $a \in [Z^k(t - T), Z^k(t)], k = e, l, f, m$:

$$\begin{aligned} u^e(t, a, x) &= \hat{u}^e(t - T, a, x) \\ &= u_T^e(X^e(T; a, t), x) \exp\left\{-\int_0^{t - T} (\mu^e + \beta^e + \partial_a v^e)(E(s + T), X^e(s + T; a, t)) ds\right\} \\ &= u^e(\tau^e, 0, x) \exp\left\{-\int_{\tau^e}^T (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds\right\} \\ &\quad \cdot \exp\left\{-\int_0^{t - T} (\mu^e + \beta^e + \partial_a v^e)(E(s + T), X^e(s + T; a, t)) ds\right\} \\ &= u^e(\tau^e, 0, x) \exp\left\{-\int_{\tau^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds\right\}. \end{aligned}$$

One should note that since $v^e(E(t), a)$ is nonnegative, $X^e(s; a, t)$ is nondecreasing with respect to t . We can see that the result holds for u^l, u^f, u^m respectively by using the same method.

(3) $a \in [Z^k(t), L^k], k = e, l, f, m$:

$$\begin{aligned} u^e(t, a, x) &= \hat{u}^e(t - T, a, x) \\ &= u_T^e(X^e(T; a, t), x) \exp\left\{-\int_0^{t - T} (\mu^e + \beta^e + \partial_a v^e)(E(s + T), X^e(s + T; a, t)) ds\right\} \\ &= u_0^e(X^e(0; a, t), x) \exp\left\{-\int_0^T (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds\right\} \\ &\quad \cdot \exp\left\{-\int_0^{t - T} (\mu^e + \beta^e + \partial_a v^e)(E(s + T), X^e(s + T; a, t)) ds\right\} \\ &= u_0^e(X^e(0; a, t), x) \exp\left\{-\int_0^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds\right\}. \end{aligned}$$

Similarly, for $k = l, f, m$ the equality also holds respectively. Therefore, $u^k(t, a, x)$ is a solution of (2.1)–(2.3) on $[0, T + \hat{T}]$, where $u^k(t, a, x) \in \mathcal{C}([0, T + \hat{T}], L^1((0, L^k) \times \Omega; \mathbb{R}^n))$.

□

Suppose $u_0 = (u_0^e, u_0^l, u_0^f, u_0^m) \in L^1$ through the chapter. Next, we introduce the maximum interval of existence of the solution.

Definition 2.12 *Let $u_0 = (u_0^e, u_0^l, u_0^f, u_0^m) \in L^1$. The maximal interval of existence of the solution, denoted by $[0, T_{\max}]$ is the interval with the property that there exists $u = (u^e, u^l, u^f, u^m) \in L^T$ as the solution of (2.1)–(2.3) for $0 < T < T_{\max}$.*

Lemma 2.13 *Let $u = (u^e, u^l, u^f, u^m)$ be the solution of (2.1)–(2.3) in the interval $[0, T_{\max})$ and $u_0 = (u_0^e, u_0^l, u_0^f, u_0^m) \in L^1$. If $T_{\max} < \infty$, then $\lim_{t \rightarrow T_{\max}} \|u(\cdot, t, \cdot)\|_{L^1} = \infty$.*

Proof. Assume that there exists $r > 0$ such that $\|u(\cdot, t, \cdot)\|_{L^1} \leq r$ for all $t \in [0, T_{\max})$. It suggests that there is a sequence $\{t_n\}_{n=1}^{\infty}$ satisfying

$$\lim_{n \rightarrow \infty} t_n = T_{\max} < \infty$$

and

$$\sup_{n \in \mathbb{N}} \|u(\cdot, t, \cdot)\|_{L^1} \leq r$$

such that $u = (u^e, u^l, u^f, u^m)$ is a solution of (2.1)–(2.3) in $[0, t_n]$. By the Lemma 2.11, we know that for a number $\tau \in (0, \infty)$, $(\hat{u}_n^e, \hat{u}_n^l, \hat{u}_n^f, \hat{u}_n^m)$ is a solution for initial value $(\hat{u}_{t_n}^e, \hat{u}_{t_n}^l, \hat{u}_{t_n}^f, \hat{u}_{t_n}^m)$ as $t \in [t_n, t_n + \tau)$. According to the uniqueness of the solution we get a solution $u = (u^e, u^l, u^f, u^m)$ for the initial value $(u_0^e, u_0^l, u_0^f, u_0^m)$ on the larger interval $[0, T_{\max} + \tau)$. It leads to a contraction with the maximal interval $[0, T_{\max})$. Therefore, $\lim_{t \rightarrow T_{\max}} \|u(\cdot, t, \cdot)\|_{L^1} = \infty$. □

Obviously, we can state the global existence of the solution as follows.

Theorem 2.14 *Let $u_0 = (u_0^e, u_0^l, u_0^f, u_0^m) \in L^1$, and the assumptions (A1)–(A5) hold. Then $u = (u^e, u^l, u^f, u^m)$ is a solution of (2.1)–(2.3) for all $t \in (0, \infty)$.*

Proof. Suppose that there exists a maximal existence interval $[0, T_{\max})$ of the solution $u = (u^e, u^l, u^f, u^m)$. By the above Lemma, we know $\lim_{t \rightarrow T_{\max}} \|u(\cdot, t, \cdot)\|_{L^1} = \infty$. But Lemma 2.8 states $\|u(\cdot, t, \cdot)\|_{L^1} \leq \mu_1 \|u_0\|_{L^1} e^{\mu_2 t}$. It means that $\lim_{t \rightarrow T_{\max}} \|u(\cdot, t, \cdot)\|_{L^1} \leq \lim_{t \rightarrow T_{\max}} \mu_1 \|u_0\|_{L^1} e^{\mu_2 t} < \infty$. It is a contradiction. Then the conclusion that $T_{\max} = \infty$ holds. □

2.5 Asymptotic behavior

The purpose of this section is to study the existence of a global attractor for the trajectories of the dynamical system defined by the solution of (2.1)–(2.3). In other words, we seek for a maximal compact set which attracts every bounded set with bounded initial conditions.

Note that $S(t)u_0 = (S^e(t)u_0^e, S^l(t)u_0^l, S^f(t)u_0^f, S^m(t)u_0^m)$. We need to define a family of maps $\{S^k(t) : X \rightarrow X | S^k(t)u_0^k = u^k(t, a, x), t \in [0, \infty)\}$ with a closed subset X of a Banach space $L^1((0, L^k) \times \Omega)$ for $k = e, l, f, m$. To apply the theory of existence of global attractor, we first recall some definitions from [58].

Definition 2.15 *The semigroup $S(t)$ is asymptotically smooth if, for any nonempty, closed, bounded set $B \subset X$ for which $S(t)B \subset B$, there is a compact set $J \subset B$ such that J attracts B .*

Definition 2.16 *The semigroup $S(t)$ is said to be point dissipative (bounded dissipative) (compact dissipative) if there is a bounded set $B \subset X$ that attracts each point of X (each bounded set of X) (each compact set of X).*

Definition 2.17 *A compact invariant set A is said to be a maximal compact invariant set if every compact invariant set of the semigroup belongs to A . An invariant set A is said to be a global attractor if A is a maximal compact invariant set which attracts each bounded set $B \subset X$.*

The semigroup property of that $S^k(t)u_0 = S^k(t - \tau)S^k(\tau)u_0$ follows from the existence and uniqueness of the solution expression (2.5) and (2.6). Under the assumptions (A1)–(A5), we can define a C_0 -semigroup for the solutions of (2.1). According to Theorem 3.4.6 in [58], we separate the proof of the existence of a global attractor into two parts: $S(t)$ is asymptotically smooth and $S(t)$ is point dissipative.

(i) $S(t)$ is asymptotically smooth.

Note that the semigroup $S^k(t)$ has a decomposition as follows

$$S^k(t) = U^k(t) + W^k(t),$$

where $U^k(t)$ and $W^k(t)$ can be defined by

$$U^k(t)(\phi^k) = \begin{cases} u^k(t, a, x), & a.e. \quad a < Z^k(t), \\ 0, & a.e. \quad a \geq Z^k(t), \end{cases} \quad (2.15)$$

and

$$W^k(t)(\phi^k) = \begin{cases} 0, & a.e. \quad a < Z^k(t), \\ u^k(t, a, x), & a.e. \quad a \geq Z^k(t), \end{cases} \quad (2.16)$$

for all initial $\phi^k \in L^1((0, L^k) \times \Omega)$ with $k = e, l, f, m$. $Z^k(t)$ is the characteristic curve through the origin. We will use the notation $Z_\phi^k(t)$ to indicate the dependence with respect to the initial distribution ϕ^k .

Referring to Lemma 3.2.6 in [58], we need to prove the next lemma.

Lemma 2.18 *Recall the assumptions which β^k and v^k satisfies. If $U^k(t)$ is given by (2.15), , then $U^k(t)$ is compact with $k = e, l, f, m$.*

Before we prove the lemma, we first introduce the following property of compact set in L^1 space (see [43]).

Proposition 2.19 *A closed, bounded set B^k of $L^1((0, L^k) \times \Omega)$ is compact if and only if the following conditions are satisfied for $L^k < \infty$:*

$$\lim_{h \rightarrow 0, \|r\|_{\mathbb{R}^n} \rightarrow 0} \int_{\Omega} \int_0^{L^k} |\phi^k(a+h, x+r) - \phi^k(a, x)| dadx = 0$$

uniformly for any $\phi^k \in B^k$ with $\phi^k(a, x) = 0$ if $a \notin (0, L^k)$ or $x \notin \Omega$.

Proof of Lemma 2.18. It is obvious that $U^k B^k$ is bounded because of the boundedness of B^k and the expression of solution (2.5) and (2.6). $\overline{U^k B^k}$ is closed and bounded in L^1 for any t . The proposition implies that $U^k B^k$ is compact if and only if

$$\lim_{h^k \rightarrow 0, \|r^k\|_{\mathbb{R}^n} \rightarrow 0} \int_{\Omega} \int_0^{L^k} |U^k \phi^k(a+h^k, x+r^k) - U^k \phi^k(a, x)| dadx = 0$$

uniformly for any $\phi^k \in B^k$.

Due to the sign of h^k , we rewrite the above equality as

(1) for $0 < h^k < Z_\phi^k(t)$,

$$\begin{aligned} I(h^k, r^k) &= \int_{\Omega} \int_0^{Z_\phi^k(t)-h^k} |U^k \phi^k(a+h^k, x+r^k) - U^k \phi^k(a, x)| dadx \\ &\quad + \int_{\Omega} \int_{Z_\phi^k(t)-h^k}^{Z_\phi^k(t)} |U^k \phi^k(a, x)| dadx, \end{aligned} \quad (2.17)$$

and

(2) for $-Z_\phi^k(t) < h^k < 0$,

$$\begin{aligned} I(h^k, r^k) &= \int_{\Omega} \int_{-h^k}^{Z_\phi^k(t)} |U^k \phi^k(a + h^k, x + r^k) - U^k \phi^k(a, x)| dadx + \int_{\Omega} \int_0^{-h^k} \\ &\quad |U^k \phi^k(a, x)| dadx + \int_{\Omega} \int_{Z_\phi^k(t)}^{Z_\phi^k(t)-h^k} |U^k \phi^k(a + h^k, x + r^k)| dadx. \end{aligned} \quad (2.18)$$

From Lemma 2.8, we can easily see that the second term in (2.17), the second and the third terms in (2.18) tends to 0 uniformly for any $\phi^k \in B^k$ as $h^k \rightarrow 0$. Hence, we just need to verify that $|U^k \phi^k(a + h^k, x + r^k) - U^k \phi^k(a, x)|$ is uniformly bounded that depends on h . For sake of simplicity we will verify this for $k = e$ here. We remove the superscript e such that $h = h^e < 0$ and $r = r^e$ for convenience.

$$\begin{aligned} & |(U^e(t)\phi^e)(a + h, x + r) - (U^e(t)\phi^e)(a, x)| \\ & \leq |(U^e(t)\phi^e)(a + h, x + r) - (U^e(t)\phi^e)(a, x + r)| \\ & \quad + |(U^e(t)\phi^e)(a, x + r) - (U^e(t)\phi^e)(a, x)| \\ & \leq \left| \frac{\int_0^{L^f} \beta^f(P^f(\bar{\tau}^e, x + r), P^m(\bar{\tau}^e, x + r), s) u^f(\bar{\tau}^e, s, x + r) ds}{v^e(E(\bar{\tau}^e), 0)} \right. \\ & \quad \cdot e^{-\int_{\bar{\tau}^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a + h, t)) ds} \\ & \quad - \frac{\int_0^{L^f} \beta^f(P^f(\tau^e, x + r), P^m(\tau^e, x + r), s) u^f(\tau^e, s, x + r) ds}{v^e(E(\tau^e), 0)} \\ & \quad \left. e^{-\int_{\tau^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds} \right| + \left| e^{-\int_{\tau^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds} \right. \\ & \quad \left. \frac{\int_0^{L^f} \beta^f(P^f(\tau^e, x + r), P^m(\tau^e, x + r), s) u^f(\tau^e, s, x + r) ds}{v^e(E(\tau^e), 0)} \right. \\ & \quad \left. - \frac{\int_0^{L^f} \beta^f(P^f(\tau^e, x), P^m(\tau^e, x), s) u^f(\tau^e, s, x) ds}{v^e(E(\tau^e), 0)} \right. \\ & \quad \left. \cdot e^{-\int_{\tau^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds} \right| := M_1 + M_2, \end{aligned}$$

where τ^e and $\bar{\tau}^e$ are implicitly given by $Z^e(0; a, t) = \tau^e$ and $Z^e(0; a + h, t) = \bar{\tau}^e$.

First we have

$$\begin{aligned} M_2 &\leq e^{-\int_{\tau^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds} \left\{ \int_0^{L^f} |u^f(\tau^e, s, x + r) - u^f(\tau^e, s, x)| \right. \\ &\quad \frac{\beta^f(P^f(\tau^e, x + r), P^m(\tau^e, x + r), s)}{v^e(E(\tau^e), 0)} ds + \int_0^{L^f} \frac{|u^f(\tau^e, s, x)|}{v^e(E(\tau^e), 0)} \\ &\quad \left. |\beta^f(P^f(\tau^e, x + r), P^m(\tau^e, x + r), s) - \beta^f(P^f(\tau^e, x), P^m(\tau^e, x), s)| ds \right\}. \end{aligned}$$

Since β^f is bounded and differentiable, ∇u^f is bounded and u^f uniformly bounded for initial conditions in $B^f \subset L^1((0, L^k) \times \Omega)$, we obtain $M_2 \rightarrow 0$ uniformly as $\|r\|_{\mathbb{R}^n} \rightarrow 0$. Next we continue to study the uniform boundedness of M_1 . Note that hereafter μ is a constant defined as before.

$$\begin{aligned}
 M_1 \leq & \left| \frac{\int_0^{L^f} \beta^f(P^f(\tau^e, x+r), P^m(\tau^e, x+r), s) u^f(\tau^e, s, x+r) ds}{v^e(E(\tau^e), 0)} \right| \\
 & \cdot \left| e^{-\int_{\tau^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds} - e^{-\int_{\bar{\tau}^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds} \right| \\
 & + \left| \frac{\int_0^{L^f} \beta^f(P^f(\tau^e, x+r), P^m(\tau^e, x+r), s) u^f(\tau^e, s, x+r) ds}{v^e(E(\tau^e), 0)} \right| \\
 & \cdot \left| e^{-\int_{\bar{\tau}^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds} - e^{-\int_{\bar{\tau}^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a+h, t)) ds} \right| \\
 & + \left| \frac{\int_0^{L^f} \beta^f(P^f(\bar{\tau}^e, x+r), P^m(\bar{\tau}^e, x+r), s) u^f(\bar{\tau}^e, s, x+r) ds}{v^e(E(\bar{\tau}^e), 0)} \right| \\
 & - \left| \frac{\int_0^{L^f} \beta^f(P^f(\tau^e, x+r), P^m(\tau^e, x+r), s) u^f(\tau^e, s, x+r) ds}{v^e(E(\tau^e), 0)} \right| \\
 & \cdot e^{-\int_{\bar{\tau}^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a+h, t)) ds} := M_3 + M_4 + M_5.
 \end{aligned}$$

For M_3 ,

$$\begin{aligned}
 & \left| e^{-\int_{\tau^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds} - e^{-\int_{\bar{\tau}^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds} \right| \\
 \leq & e^{-\mu(t-\tau^e)} \left| \int_{\tau^e}^{\bar{\tau}^e} (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds \right| \\
 \leq & e^{-\mu(t-\tau^e)} (\|\mu^e\|_\infty + \|\beta^e\|_\infty + \|\partial_a v^e\|_\infty) |\bar{\tau}^e - \tau^e|.
 \end{aligned}$$

For M_4 ,

$$\begin{aligned}
 & \left| e^{-\int_{\bar{\tau}^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds} - e^{-\int_{\bar{\tau}^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a+h, t)) ds} \right| \\
 \leq & e^{-\mu(t-\bar{\tau}^e)} \left| \int_{\bar{\tau}^e}^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a+h, t)) \right. \\
 & \left. - (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds \right| \\
 \leq & C e^{-\mu(t-\bar{\tau}^e)} L_{\bar{m}} \int_{\bar{\tau}^e}^t |X^e(s; a+h, t) - X^e(s; a, t)| ds \\
 \leq & C e^{-\mu(t-\bar{\tau}^e)} L_{\bar{m}} |h| e^{t\|\partial_a v\|_\infty},
 \end{aligned}$$

where C is a constant, $L_{\bar{m}}$ is a Lipschitzian constant of $\beta^e + \mu^e$.

Then

$$\begin{aligned}
 M_5 \leq & L^f \sup_{\tau^e \in [0, t]} \left| \left(\frac{\beta^f(P^f(\tau^e, x+r), P^m(\tau^e, x+r), s) u^f(\tau^e, s, x+r) ds}{v^e(E(\tau^e), 0)} \right)' \right| \\
 & \cdot e^{-\mu(t-\bar{\tau}^e)} |\bar{\tau}^e - \tau^e| := M^* e^{-\mu(t-\bar{\tau}^e)} |\bar{\tau}^e - \tau^e|.
 \end{aligned}$$

Since

$$|\bar{\tau}^e - \tau^e| \leq \frac{|X^e(\bar{\tau}^e, t, a)|}{\underline{v}^e} \leq \frac{|h|e^{t\|\partial_a v\|_\infty}}{\underline{v}^e},$$

it follows that

$$\begin{aligned} & |(U^e(t)\phi^e)(a+h, x+r) - (U^e(t)\phi^e)(a, x+r)| \\ & \leq \frac{L^f \|\beta^f\|_\infty \|u^f\|_\infty}{\underline{v}^e} e^{-\mu(t-\tau^e)} (\|\mu^e\|_\infty + \|\beta^e\|_\infty + \|\partial_a v^e\|_\infty) |\bar{\tau}^e - \tau^e| \\ & \quad + \frac{L^f \|\beta^f\|_\infty \|u^f\|_\infty}{\underline{v}^e} e^{-\mu(t-\bar{\tau}^e)} L_{\bar{m}} |h| e^{t\|\partial_a v\|_\infty} + M^* e^{-\mu(t-\bar{\tau}^e)} |\bar{\tau}^e - \tau^e| \\ & \leq \left\{ \frac{L^f \|\beta^f\|_\infty \|u^f\|_\infty}{\underline{v}^e} (\|\mu^e\|_\infty + \|\beta^e\|_\infty + \|\partial_a v^e\|_\infty) \right\} \\ & \quad \frac{|h| e^{t(\|\partial_a v\|_\infty - \mu) + \mu\tau^e}}{\underline{v}^e} + \left\{ \frac{L^f \|\beta^f\|_\infty \|u^f\|_\infty}{\underline{v}^e} L_{\bar{m}} + M^* \right\} |h| e^{t(\|\partial_a v\|_\infty - \mu) + \mu\bar{\tau}^e}. \end{aligned}$$

By the boundedness of initial conditions B^k , fixed t and the length of the integration interval, we get $I(h, r) \rightarrow 0$ uniformly as $|h|, \|r\|_{\mathbb{R}^n} \rightarrow 0$. According to Proposition 2.19, we conclude that $U^k(t)$ is compact with $k = e, l, f, m$. It concludes the proof of Lemma 2.18. \square

Lemma 2.20 *If $W^k(t)$ is the operator defined as before, then there exists a continuous function $\mu^k(t, r^k)$ satisfying $\mu^k(t, r^k) \rightarrow 0$ when $t \rightarrow \infty$ such that $\|W^k(t)(\phi^k)\| \leq \mu^k(t, r^k)$ if $\|\phi^k\| \leq r^k$ with $k = e, l, f, m$.*

Proof. We just verify the statement for $k = e$. We have

$$\begin{aligned} \|W^e(t)\phi^e\|_{L^1} &= \int_{\Omega} \int_0^{L^e} |u^e(t, a, x)| da dx \\ &= \int_{\Omega} \int_{Z_{\phi}^e(t)}^{L^e} |\phi(X^e(0; t, a), x) e^{-\int_0^t (\mu^e + \beta^e + \partial_a v^e)(E(s), X^e(s; a, t)) ds}| da dx \\ &\leq e^{-\mu t} \|\phi^e\|_{L^1} \leq r^e e^{-\mu t} := \mu^e(t, r^e), \end{aligned}$$

and similar analysis can be made for $k = l, f, m$. Then the statement holds. \square

Let $h = (h^e, h^l, h^f, h^m)$ and $r = (r^e, r^l, r^f, r^m)$. Combining the above two lemmas, clearly the following conclusion holds (see [58]).

Theorem 2.21 *Suppose the conditions imposed on β^k, v^k, μ^k in Sect.1 hold. The semigroup generated by the solutions of (2.1) is asymptotically smooth in L^1 with $k = e, l, f, m$.*

(ii) $S(t)$ is point dissipative.

By the definition of point dissipative, we need to show that

$$\text{dist}(S(t)\phi, B) = \text{dist}((S^e(t)\phi^e, B^e), (S(t)^l\phi^l, B^l), (S^f(t)\phi^f, B^f), (S^m(t)\phi^m, B^m)) = 0,$$

where $\phi = (\phi^e, \phi^l, \phi^f, \phi^m)$. It means that there is an integer $n_0 = n_0(\phi^k, B^k)$ and a bounded set B^k in $L^1((0, L^k) \times \Omega)$ with the property that $S_n^k\phi^k \in B^k$ for $n \geq n_0$, $\phi^k \in B^k$, $k = e, l, f, m$. Under the assumptions in Sect.1, it follows that $S(t)$ is point dissipative even bounded dissipative immediately from the expression of solutions (2.1) and the proof of Lemma 2.9.

In summary, we obtain the main result in this section referring to [58].

Theorem 2.22 *Under the assumptions (A1)-(A5), the C_0 -semigroup $S(t)$ generated by the solutions of (2.1) has a global attractor.*

2.6 Steady states

Inspired by the existence of global attractor, we think about the existence of the steady solution of problem (2.1), (2.2), (2.3) in this section. The system that describes the steady state population is given by:

$$\left\{ \begin{array}{ll} \frac{\partial[v^e(a)u^e(a,x)]}{\partial a} = -\mu^e(a)u^e(a,x) - \beta^e(a)u^e(a,x), & \text{in } (0, L^e) \times \Omega, \\ \frac{\partial[v^l(a)u^l(a,x)]}{\partial a} = -\mu^l(P^l(x), a)u^l(a,x) - \beta^l(a)u^l(a,x), & \text{in } (0, L^l) \times \Omega, \\ \frac{\partial[v^f(a)u^f(a,x)]}{\partial a} = -\mu^f(a)u^f(a,x) + d_f\Delta_x u^f(a,x), & \text{in } (0, L^f) \times \Omega, \\ \frac{\partial[v^m(a)u^m(a,x)]}{\partial a} = -\mu^m(a)u^m(a,x) + d_m\Delta_x u^m(a,x), & \text{in } (0, L^m) \times \Omega, \\ v^e(0)u^e(0,x) = \int_0^{L^f} \beta^f(P^f(x), P^m(x), s)u^f(s,x)ds, & \text{in } \Omega, \\ v^l(0)u^l(0,x) = \int_0^{L^e} \beta^e(s)u^e(s,x)ds, & \text{in } \Omega, \\ v^f(0)u^f(0,x) = \int_0^{L^l} \sigma\beta^l(s)u^l(s,x)ds, & \text{in } \Omega, \\ v^m(0)u^m(0,x) = \int_0^{L^l} (1-\sigma)\beta^l(s)u^l(s,x)ds, & \text{in } \Omega, \\ \frac{\partial u^k}{\partial \eta} = 0, & \text{on } (0, L^k) \times \partial\Omega, \end{array} \right. \quad (2.19)$$

where $k = e, l, f, m$.

Then it is easy to check that

$$\left\{ \begin{array}{l} u^e(a, x) = \frac{\int_0^{L^f} \beta^f(P^f(x), P^m(x), s) u^f(s, x) ds}{v^e(a)} e^{\int_0^a \frac{-(\mu^e + \beta^e)(s)}{v^e(s)} ds}, \\ u^l(a, x) = \frac{\int_0^{L^e} \beta^e(s) u^e(s, x) ds}{v^l(a)} e^{\int_0^a \frac{-(\mu^l(P^l(x), s) + \beta^l(s))}{v^l(s)} ds}, \\ u^f(a, x) = \frac{\int_0^{L^l} \sigma \beta^l(s) u^l(s, x) ds}{v^f(a)} e^{\int_0^a \frac{-\mu^f(s) + A_1}{v^f(s)} ds}, \\ u^m(a, x) = \frac{\int_0^{L^m} (1 - \sigma) \beta^l(s) u^l(s, x) ds}{v^m(a)} e^{\int_0^a \frac{-\mu^m(s) + A_2}{v^m(s)} ds}. \end{array} \right. \quad (2.20)$$

Moreover, since u^k with $k = e, l, f, m$ in (2.20) represents the density, any solution should be nonnegative. In the abstract setting, they belong to the positive cone $L_+^1((0, L^k) \times \Omega)$ respectively. We remark that the steady system has $(u^{*e}, u^{*l}, u^{*f}, u^{*m}) = (0, 0, 0, 0)$ as the extinction. Thus the aim here is to study the existence of nontrivial solutions of (2.20) given some conditions.

In order to deal with the existence of a survival state we define a nonlinear operator J in the Banach cone $L_+^1((0, L^l) \times \Omega)$. Denote $J : L_+^1((0, L^l) \times \Omega) \rightarrow L_+^1((0, L^l) \times \Omega)$ by

$$\left\{ \begin{array}{l} J(u^l)(a, x) = \frac{\int_0^{L^e} \beta^e(s) H^e(H^f(u^l), H^m(u^l))(s, x) ds}{v^l(a)} e^{\int_0^a \frac{-(\mu^l(P^l(x), s) + \beta^l(s))}{v^l(s)} ds}, \\ H^e(u^f, u^m) := u^e(a, x) = \frac{\int_0^{L^f} \beta^f(P^f(x), P^m(x), s) u^f(s, x) ds}{v^e(a)} e^{\int_0^a \frac{-(\mu^e + \beta^e)(s)}{v^e(s)} ds}, \\ H^f(u^l) := u^f(a, x) = \frac{\int_0^{L^l} \sigma \beta^l(s) u^l(s, x) ds}{v^f(a)} e^{\int_0^a \frac{-\mu^f(s) + A_1}{v^f(s)} ds}, \\ H^m(u^l) := u^m(a, x) = \frac{\int_0^{L^m} (1 - \sigma) \beta^l(s) u^l(s, x) ds}{v^m(a)} e^{\int_0^a \frac{-\mu^m(s) + A_2}{v^m(s)} ds}, \end{array} \right. \quad (2.21)$$

where $H^k : L_+^1((0, L^k) \times \Omega) \rightarrow L_+^1((0, L^k) \times \Omega)$, $k = e, f, m$. Therefore, we transfer the problem of looking for positive equilibrium solutions of (2.20) to a problem of studying positive fixed points of J .

To prove the nonlinear operator J has at least one positive fixed point, we would apply the theory of compact operators and the Krasnoselskii's fixed point theorem [75]. Before the main result is stated, it is necessary to verify the following conclusion first:

Lemma 2.23 *Suppose that (A1)-(A5) hold, then J is completely continuous.*

Proof. If J is continuous, and transforms every bounded set into a compact set, then J is completely continuous.

(i) J is continuous.

Let $u_1^l, u_2^l \in L^1_+(\Omega, L^l)$. Assume (A1)-(A5) hold. From (2.21), we have

$$\begin{aligned}
 & \|J(u_1^l) - J(u_2^l)\|_{L^1_+} \\
 = & \left\| \frac{\int_0^{L^e} \beta^e(s) u_1^e(s, x) ds}{v^l(a)} e^{\int_0^a \frac{-(\mu^l(P_1^l(x), s) + \beta^l(s))}{v^l(s)} ds} \right. \\
 & \left. - \frac{\int_0^{L^e} \beta^e(s) u_2^e(s, x) ds}{v^l(a)} e^{\int_0^a \frac{-(\mu^l(P_2^l(x), s) + \beta^l(s))}{v^l(s)} ds} \right\|_{L^1} \\
 \leq & \left\| e^{\int_0^a \frac{-(\mu^l(P_1^l(x), s) + \beta^l(s))}{v^l(s)} ds} \frac{\int_0^{L^e} \beta^e(s) (u_1^e(s, x) - u_2^e(s, x)) ds}{v^l(a)} \right\|_{L^1} \\
 & + \left\| \frac{\int_0^{L^e} \beta^e(s) u_2^e(s, x) ds}{v^l(a)} \left[e^{\int_0^a \frac{-(\mu^l(P_1^l(x), s) + \beta^l(s))}{v^l(s)} ds} - e^{\int_0^a \frac{-(\mu^l(P_2^l(x), s) + \beta^l(s))}{v^l(s)} ds} \right] \right\|_{L^1} \\
 \leq & \int_{\Omega} \int_0^{L^l} e^{\frac{-\mu}{v^l} a} \frac{\|\beta^e\|_{\infty} \int_0^{L^e} |u_1^e(s, x) - u_2^e(s, x)| ds}{\underline{v}^l} da dx + \int_{\Omega} \int_0^{L^l} \frac{\|u_2^e\|_{\infty} \|\beta^e\|_{\infty} L^e}{\underline{v}^l} \\
 & e^{\frac{-\mu}{v^l} a} \int_0^a \frac{|\mu^l(P_1^l(x), s) - \mu^l(P_2^l(x), s)|}{v^l(s)} ds da dx \\
 \leq & \int_0^{L^l} e^{\frac{-\mu}{v^l} a} \frac{\|\beta^e\|_{\infty}}{\underline{v}^l} \|u_1^e - u_2^e\|_{L^1} da + \frac{\|u_2^e\|_{\infty} \|\beta^e\|_{\infty} L^e}{\underline{v}^l} \int_0^{L^l} e^{\frac{-\mu}{v^l} a} m_K^l a \|u_1^l - u_2^l\|_{L^1} da,
 \end{aligned}$$

$$\begin{aligned}
 \|u_1^e - u_2^e\|_{L^1} & = \left\| \frac{\int_0^{L^f} \beta^f(P_1^f, P_1^m, s) u_1^f(s, x) ds}{v^e(a)} e^{\int_0^a \frac{-(\mu^e(s) + \beta^e(s))}{v^e(s)} ds} \right. \\
 & \left. - \frac{\int_0^{L^f} \beta^f(P_2^f, P_2^m, s) u_2^f(s, x) ds}{v^e(a)} e^{\int_0^a \frac{-(\mu^e(s) + \beta^e(s))}{v^e(s)} ds} \right\|_{L^1} \\
 & \leq \int_{\Omega} \int_0^{L^e} \frac{\int_0^{L^f} |\beta^f(P_1^f, P_1^m, s) u_1^f(s, x) - \beta^f(P_2^f, P_2^m, s) u_2^f(s, x)| ds}{v^e(a)} \\
 & e^{\int_0^a \frac{-(\mu^e(s) + \beta^e(s))}{v^e(s)} ds} da dx \\
 & \leq \|\beta^f\|_{\infty} \int_{\Omega} \int_0^{L^e} \frac{\int_0^{L^f} |u_1^f(s, x) - u_2^f(s, x)| ds}{\underline{v}^e} e^{\frac{-\mu}{v^e} a} da dx + \\
 & \|u_2^f\|_{\infty} \int_{\Omega} \int_0^{L^e} \int_0^{L^f} \beta_K^f \left(\frac{\int_0^{L^f} |u_1^f(s, x) - u_2^f(s, x)| ds}{\underline{v}^e} + \right. \\
 & \left. \frac{\int_0^{L^m} |u_1^m(s, x) - u_2^m(s, x)| ds}{\underline{v}^e} \right) ds e^{\frac{-\mu}{v^e} a} da dx \\
 & \leq \|\beta^f\|_{\infty} \|u_1^f - u_2^f\|_{L^1} \int_0^{L^e} \frac{e^{\frac{-\mu}{v^e} a}}{\underline{v}^e} da + \|u_2^f\|_{\infty} \int_0^{L^e} \beta_K^f L^f \frac{e^{\frac{-\mu}{v^e} a}}{\underline{v}^e} da \\
 & (\|u_1^f - u_2^f\|_{L^1} + \|u_1^m - u_2^m\|_{L^1}),
 \end{aligned}$$

and

$$\begin{aligned}
 \|u_1^f - u_2^f\|_{L^1} &= \int_{\Omega} \int_0^{L^f} \left[\frac{\int_0^{L^l} \sigma \beta^l(s) u_1^l(s, x) ds}{v^f(a)} e^{\int_0^a \frac{-\mu^f(s)+A_1}{v^f(s)} ds} - \frac{\int_0^{L^l} \sigma \beta^l(s) u_2^l(s, x) ds}{v^f(a)} \right. \\
 &\quad \left. e^{\int_0^a \frac{-\mu^f(s)+A_1}{v^f(s)} ds} \right] da dx \\
 &\leq \int_{\Omega} \int_0^{L^f} \frac{\int_0^{L^l} \sigma \beta^l(s) |u_1^l(s, x) - u_2^l(s, x)| ds}{v^f(a)} e^{\int_0^a \frac{-\mu^f(s)+A_1}{v^f(s)} ds} da dx \\
 &\leq \sigma \|\beta^l\|_{\infty} \|u_1^l - u_2^l\|_{L^1} \int_0^{L^f} \frac{e^{\frac{-\mu}{\bar{v}^f} a}}{\underline{v}^f} da.
 \end{aligned}$$

Similarly,

$$\|u_1^m - u_2^m\|_{L^1} \leq (1 - \sigma) \|\beta^l\|_{\infty} \|u_1^l - u_2^l\|_{L^1} \int_0^{L^m} \frac{e^{\frac{-\mu}{\bar{v}^m} a}}{\underline{v}^m} da.$$

Combining the above inequalities, we can obtain that

$$\|J(u_1^l) - J(u_2^l)\|_{L^1} \leq M \|u_1^l - u_2^l\|_{L^1},$$

where M is a constant dependent on $\mu, \underline{v}^e, \underline{v}^l, \underline{v}^f, \underline{v}^m, \bar{v}^e, \bar{v}^l, \bar{v}^f, \bar{v}^m, L^e, L^l, L^f, L^m, \sigma, \|u_2^e\|_{\infty}, \|\beta^e\|_{\infty}, \|\beta^l\|_{\infty}, \|\beta^f\|_{\infty}, m_K^l, \beta_K^f$. It leads to the continuity of the operator J .

(ii) J is compact.

Here we refer to the corollary of Frechet-Kolmogorov theorem to prove the compactness of J . Let $D = \mathbb{R} \times \Omega$, $\omega \subset\subset D$ and \mathcal{F} be a bounded subset of $L^1_+(D)$. For each $u^l \in \mathcal{F}$, $(a, x) \in \omega$ and $(h_1, h_2) \in \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned}
 &\|J(u^l)(a + h_1, x + h_2) - J(u^l)(a, x)\|_{L^1(\omega)} \\
 &\leq \|J(u^l)(a + h_1, x + h_2) - J(u^l)(a, x + h_2)\|_{L^1(\omega)} \\
 &\quad + \|J(u^l)(a, x + h_2) - J(u^l)(a, x)\|_{L^1(\omega)} := J_1 + J_2.
 \end{aligned}$$

(1) For J_1 , we have

$$\begin{aligned}
 J_1 &= \|J(u^l)(a + h_1, x + h_2) - J(u^l)(a, x + h_2)\|_{L^1(\omega)} \\
 &= \int_{\omega} \int_0^{L^l} \left| \frac{\int_0^{L^e} \beta^e(s) u^e(s, x + h_2) ds}{v^l(a + h_1)} e^{\int_0^{a+h_1} \frac{-(\mu^l(P^l(x), s) + \beta^l(s))}{v^l(s)} ds} \right. \\
 &\quad \left. - \frac{\int_0^{L^e} \beta^e(s) u^e(s, x + h_2) ds}{v^l(a)} e^{\int_0^a \frac{-(\mu^l(P^l(x), s) + \beta^l(s))}{v^l(s)} ds} \right| da dx \\
 &\leq \int_{\Omega} \int_0^{L^l} \left| \int_0^{L^e} \beta^e(s) u^e(s, x + h_2) ds e^{\int_0^a \frac{-(\mu^l(P^l(x), s) + \beta^l(s))}{v^l(s)} ds} \right. \\
 &\quad \left. \left[\frac{e^{\int_0^{a+h_1} \frac{-(\mu^l(P^l(x), s) + \beta^l(s))}{v^l(s)} ds}}{v^l(a + h_1)} - \frac{1}{v^l(a)} \right] \right| da dx \quad (\beta^l(s) = 0 \text{ for } s \in (-\infty, 0) \cup (L^l, \infty)) \\
 &\leq \int_{\Omega} \int_0^{L^l} \int_0^{L^e} |\beta^e(s) u^e(s, x + h_2)| e^{\int_0^a \frac{-(\mu^l(P^l(x), s) + \beta^l(s))}{v^l(s)} ds} |ds \\
 &\quad \left| \frac{e^{\int_0^{a+h_1} \frac{-(\mu^l(P^l(x), s) + \beta^l(s))}{v^l(s)} ds}}{v^l(a + h_1)} - \frac{1}{v^l(a)} \right| da dx \quad (\beta^l(s) = 0 \text{ for } s \in (-\infty, 0) \cup (L^l, \infty)).
 \end{aligned}$$

As $\frac{e^{\int_a^{a+h_1} \frac{-(\mu^l(P^l(x),s)+\beta^l(s))}{v^l(s)} ds}}{v^l(a+h_1)}$ is continuous with respect to a , and $\int_0^{L^e} |\beta^e(s)u^e(s, x+h_2)| ds$ is bounded, the term on the right hand tends to zero uniformly when $h_1 \rightarrow 0$. It means that for any $\varepsilon > 0$, there exists $\delta > 0$ such that when $|h_1| \leq \delta$ for all $h_1 \in \mathbb{R}$, $J_1 = \|J(u^l)(a+h_1, x+h_2) - J(u^l)(a, x+h_2)\|_{L^1(\omega)} \leq \varepsilon$.

(2) For J_2 , we have

$$\begin{aligned}
 J_2 &= \|J(u^l)(a, x+h_2) - J(u^l)(a, x)\|_{L^1(\omega)} \\
 &= \int \int_{\omega} \left| \frac{\int_0^{L^e} \beta^e(s)u^e(s, x+h_2) ds}{v^l(a)} e^{\int_0^a \frac{-(\mu^l(P^l(x+h_2),s)+\beta^l(s))}{v^l(s)} ds} \right. \\
 &\quad \left. - \frac{\int_0^{L^e} \beta^e(s)u^e(s, x) ds}{v^l(a)} e^{\int_0^a \frac{-(\mu^l(P^l(x),s)+\beta^l(s))}{v^l(s)} ds} \right| dadx \\
 &\leq \int_{\Omega} \int_0^{L^l} \left| \frac{e^{\int_0^a \frac{-\beta^l(s)}{v^l(s)} ds}}{v^l(a)} \left[\int_0^{L^e} \beta^e(s)u^e(s, x+h_2) ds e^{\int_0^a \frac{-\mu^l(P^l(x+h_2),s)}{v^l(s)} ds} \right. \right. \\
 &\quad \left. \left. - \int_0^{L^e} \beta^e(s)u^e(s, x) ds e^{\int_0^a \frac{-\mu^l(P^l(x),s)}{v^l(s)} ds} \right] \right| dadx \\
 &\leq \int_{\Omega} \int_0^{L^l} \left| \frac{e^{\int_0^a \frac{-\beta^l(s)}{v^l(s)} ds}}{v^l(a)} \left[\int_0^{L^e} |\beta^e(s)u^e(s, x+h_2)| e^{\int_0^a \frac{-\mu^l(P^l(x+h_2),s)}{v^l(s)} ds} \right. \right. \\
 &\quad \left. \left. - \beta^e(s)u^e(s, x) e^{\int_0^a \frac{-\mu^l(P^l(x+h_2),s)}{v^l(s)} ds} \right] ds + \int_0^{L^e} |\beta^e(s)u^e(s, x)| ds e^{\int_0^a \frac{-\mu^l(P^l(x+h_2),s)}{v^l(s)} ds} \right. \\
 &\quad \left. - \beta^e(s)u^e(s, x) e^{\int_0^a \frac{-\mu^l(P^l(x),s)}{v^l(s)} ds} \right| ds] dadx \\
 &= \int_{\Omega} \int_0^{L^l} \left| \frac{e^{\int_0^a \frac{-\beta^l(s)}{v^l(s)} ds}}{v^l(a)} \left[\int_0^{L^e} |\beta^e(s)| e^{\int_0^a \frac{-\mu^l(P^l(x+h_2),s)}{v^l(s)} ds} \|u^e(s, x+h_2) - u^e(s, x)\| ds \right. \right. \\
 &\quad \left. \left. + \int_0^{L^e} |\beta^e(s)u^e(s, x)| \left| e^{\int_0^a \frac{-\mu^l(P^l(x+h_2),s)}{v^l(s)} ds} - e^{\int_0^a \frac{-\mu^l(P^l(x),s)}{v^l(s)} ds} \right| ds \right] \right| ds] dadx.
 \end{aligned}$$

Denote

$$\begin{aligned}
 J_3 &= \int_0^{L^e} |\beta^e(s)| e^{\int_0^a \frac{-\mu^l(P^l(x+h_2),s)}{v^l(s)} ds} \|u^e(s, x+h_2) - u^e(s, x)\| ds, \\
 J_4 &= \int_0^{L^e} |\beta^e(s)u^e(s, x)| \left| e^{\int_0^a \frac{-\mu^l(P^l(x+h_2),s)}{v^l(s)} ds} - e^{\int_0^a \frac{-\mu^l(P^l(x),s)}{v^l(s)} ds} \right| ds.
 \end{aligned}$$

On one hand, we deduce that

$$\begin{aligned}
 &|u^e(a, x+h_2) - u^e(a, x)| \\
 &= \left| \frac{e^{\int_0^a \frac{-(\mu^e+\beta^e)(s)}{v^e(s)} ds}}{v^e(a)} \int_0^{L^f} [\beta^f(P^f(x+h_2), P^m(x+h_2), s)u^f(s, x+h_2) \right. \\
 &\quad \left. - \beta^f(P^f(x), P^m(x), s)u^f(s, x)] ds \right| \rightarrow 0, \quad \text{as } h_2 \rightarrow 0,
 \end{aligned}$$

since u^f is continuous with respect to x . It ensures that $J_3 \rightarrow 0$ uniformly as $h_2 \rightarrow 0$. On the other hand the fact that μ^l is Lipschitz continuous with respect to P^l implies $J_4 \rightarrow 0$ uniformly as $h_2 \rightarrow 0$. Hence for any $\varepsilon > 0$, there exists $\delta > 0$ such that when $|h_2| \leq \delta$ for all $h_2 \in \mathbb{R}$, $J_2 = \|J(u^l)(a, x+h_2) - J(u^l)(a, x)\|_{L^1(\omega)} \leq \varepsilon$. We conclude that for any $\varepsilon > 0$, there exists $\delta > 0$ such that when $|h_i| \leq \delta$ for all $h_i \in \mathbb{R}$ with $i = 1, 2$,

$$\|J(u^l)(a+h_1, x+h_2) - J(u^l)(a, x)\|_{L^1(\omega)} \leq 2\varepsilon.$$

Next, let ω be a subset of D such that $\omega = \{(a, x) | a \in (0, L^l), x \in \Omega\}$. Observing that for any $\varepsilon > 0$,

$$\|J(u^l)\|_{L^1(D \setminus \omega)} \leq \varepsilon,$$

since $\beta^l = 0$ implies $u^f = 0$ which entails $u^e = 0$, $J(u^l) = 0$ when $a \in (-\infty, 0) \cup (L^l, \infty)$. According to Frechet-Kolmogorov theorem we deduce that J is compact in $L^1(D)$. In conclusion, J is completely continuous. \square

Theorem 2.24 *The operator J has at least one non-zero fixed point in the positive cone $L^1_+((0, L^l) \times \Omega)$ under the assumptions in (A1)-(A5) for v^k, β^k, μ^k when the following condition holds:*

$$\begin{aligned} R = & \inf_{a \in (0, L^l)} \frac{1}{v^l(a)} e^{\int_0^a \frac{-(\mu^l(P_\varepsilon^l, \zeta) + \beta^l(\zeta))}{v^l(\zeta)} d\zeta} \int_0^{L^e} \frac{\beta^e(s)}{v^e(s)} e^{\int_0^s \frac{-(\mu^e + \beta^e)(t)}{v^e(t)} dt} ds \\ & \int_0^{L^f} \frac{\beta^f(P_\varepsilon^f, P_\varepsilon^m, \eta)}{v^f(\eta)} e^{\int_0^\eta \frac{-\mu^f(\xi) + A_1}{v^f(\xi)} d\xi} d\eta \int_0^{L^l} \sigma \beta^l(\zeta) d\zeta \geq 1 \end{aligned} \quad (2.22)$$

with a small enough number $\varepsilon > 0$.

Proof. As pointed in Krasnoselskii's fixed point theorem, we need to consider the strong asymptotic derivative of J at ∞ . Let

$$\Phi(u^l) = \int_0^{L^f} \beta^f(\tau) \frac{e^{\int_0^\tau \frac{-\mu^f(\eta) + A_1}{v^f(\eta)} d\eta}}{v^f(\tau)} \int_0^{L^l} \sigma \beta^l(\xi) u^l(\xi, x) d\xi d\tau,$$

then $\mu^l(P^l, s) \rightarrow \infty$ as $u^l \rightarrow \infty$ and $u^l \in L^1_+((0, L^l) \times \Omega)$ ensures that

$$\begin{aligned}
 & \langle J'(u^l)|_{u^l=\infty}, w \rangle \\
 = & \int_0^{L^l} \int_{\Omega} \lim_{u^l \rightarrow \infty} \int_0^{L^e} \left(\frac{e^{\int_0^a \frac{-(\mu^l(P^l, s) + \beta^l(s)) ds}{v^l(s)}}}{v^l(a)} \frac{\beta^e(s)}{v^e(s)} e^{\int_0^s \frac{-(\mu^e + \beta^e)(\eta)}{v^e(\eta)} d\eta} \Phi(u^l) \right)' w(a, x) ds dx da \\
 = & \int_0^{L^l} \int_{\Omega} \int_0^{L^e} \lim_{u^l \rightarrow \infty} \left[\left(\frac{e^{\int_0^a \frac{-(\mu^l(P^l, s) + \beta^l(s)) ds}{v^l(s)}}}{v^l(a)} \frac{\beta^e(s)}{v^e(s)} e^{\int_0^s \frac{-(\mu^e + \beta^e)(\eta)}{v^e(\eta)} d\eta} \right)' \Phi(u^l) \right] w(a, x) ds dx da \\
 & + \int_0^{L^l} \int_{\Omega} \int_0^{L^e} \lim_{u^l \rightarrow \infty} \left[\frac{e^{\int_0^a \frac{-(\mu^l(P^l, s) + \beta^l(s)) ds}{v^l(s)}}}{v^l(a)} \frac{\beta^e(s)}{v^e(s)} e^{\int_0^s \frac{-(\mu^e + \beta^e)(\eta)}{v^e(\eta)} d\eta} (\Phi(u^l))' \right] w(a, x) ds dx da \\
 = & \int_0^{L^l} \int_{\Omega} \int_0^{L^e} \lim_{u^l \rightarrow \infty} \left[\frac{e^{\int_0^a \frac{-(\mu^l(P^l, s) + \beta^l(s)) ds}{v^l(s)}}}{v^l(a)} \left(\int_0^a -\frac{\mu^l(P^l, s)}{v^l} ds \right)' \frac{\beta^e(s)}{v^e(s)} e^{\int_0^s \frac{-(\mu^e + \beta^e)(\eta)}{v^e(\eta)} d\eta} \right. \right. \\
 & \left. \left. \Phi(u^l) \right] w(a, x) ds dx da + \int_0^{L^l} \int_{\Omega} \int_0^{L^e} \lim_{u^l \rightarrow \infty} \left[\frac{e^{\int_0^a \frac{-(\mu^l(P^l, s) + \beta^l(s)) ds}{v^l(s)}}}{v^l(a)} \frac{\beta^e(s)}{v^e(s)} e^{\int_0^s \frac{-(\mu^e + \beta^e)(\eta)}{v^e(\eta)} d\eta} \right. \right. \\
 & \left. \left. \int_0^{L^f} \beta^f(\tau) \frac{e^{\int_0^\tau \frac{-\mu^f(\eta) + A_1}{v^f(\eta)} d\eta}}{v^f(\tau)} \int_0^{L^l} \sigma \beta^l(\xi) d\xi d\tau \right] w(a, x) ds dx da \right. \\
 = & 0.
 \end{aligned}$$

Next, we consider the Frechet derivative $J_0 := J'(u^l)|_{u^l=0}$:

$$\begin{aligned}
 \langle J_0, w \rangle &= \left\langle \frac{\partial J(u^l)}{\partial u^l} \Big|_{u^l=0}, w \right\rangle \\
 &= \int_0^{L^l} \int_{\Omega} \left[\frac{e^{\int_0^a \frac{-(\mu^l(P^l, s) + \beta^l(s)) ds}{v^l(s)}}}{v^l(a)} \int_0^{L^e} \beta^e(s) \frac{\partial u^e(s, x)}{\partial u^l} w(a, x) ds dx da \right] \Big|_{u^l=0} \\
 &= \int_0^{L^l} \int_{\Omega} \frac{e^{\int_0^a \frac{-(\mu^l(0, s) + \beta^l(s)) ds}{v^l(s)}}}{v^l(a)} \int_0^{L^e} \beta^e(s) \frac{\partial u^e(s, x)}{\partial u^l} \Big|_{u^l=0} w(a, x) ds dx da.
 \end{aligned}$$

Recall that $H^e(u^f, u^m) := u^e(a, x)$, $H^f(u^l) := u^f(a, x)$, and $H^m(u^l) := u^m(a, x)$.

$$\begin{aligned}
 \frac{\partial u^e}{\partial u^f}(a, x) &= \frac{e^{\int_0^a \frac{-(\mu^e + \beta^e)(s)}{v^e(s)} ds}}{v^e(a)} \left\{ \int_0^{L^f} \beta^f(P^f, P^m, s) ds \right. \\
 & \quad \left. + \int_0^{L^f} (\beta^f(P^f, P^m, s))' u^f(s, x) ds \right. \\
 &= \frac{e^{\int_0^a \frac{-(\mu^e + \beta^e)(s)}{v^e(s)} ds}}{v^e(a)} \int_0^{L^f} \beta^f(P^f, P^m, s) ds \\
 &= \frac{e^{\int_0^a \frac{-(\mu^e + \beta^e)(s)}{v^e(s)} ds}}{v^e(a)} \int_0^{L^f} \beta^f(0, 0, s) ds \quad (u^l = 0).
 \end{aligned}$$

Similarly,

$$\frac{\partial u^e}{\partial u^m}(a, x) = \frac{e^{\int_0^a \frac{-(\mu^e + \beta^e)(s)}{v^e(s)} ds}}{v^e(a)} \int_0^{L^f} (\beta^f(P^f, P^m, s))' u^f(s, x) L^m ds = 0 \quad (u^l = 0).$$

Because

$$\frac{\partial u^f}{\partial u^l} = \frac{1}{v^f(a)} \int_0^{L^l} \sigma \beta^l(s) ds e^{\int_0^a \frac{-\mu^f(s) + A_1}{v^f(s)} ds},$$

we deduce that

$$\begin{aligned} \langle J_0, w \rangle &= \int_0^{L^l} \int_{\Omega} \frac{1}{v^l(a)} e^{\int_0^a \frac{-(\mu^l(0, s) + \beta^l(s))}{v^l(s)} ds} \int_0^{L^e} \frac{\beta^e(s) w(a, x)}{v^e(s)} e^{\int_0^s \frac{-(\mu^e + \beta^e)(t)}{v^e(t)} dt} ds \\ &\quad \int_0^{L^f} \frac{\beta^f(0, 0, \eta)}{v^f(\eta)} e^{\int_0^\eta \frac{-\mu^f(\xi) + A_1}{v^f(\xi)} d\xi} d\eta \int_0^{L^l} \sigma \beta^l(\zeta) d\zeta dx da. \end{aligned}$$

Note that J_0 has no characteristic vector corresponding to the characteristic value equalling to 1. It is easy to check the proof by contradiction. Here we omit the details.

$$\begin{aligned} \langle J_0, w \rangle &= \int_0^{L^l} \int_{\Omega} \frac{1}{v^l(a)} e^{\int_0^a \frac{-(\mu^l(0, s) + \beta^l(s))}{v^l(s)} ds} \int_0^{L^e} \frac{\beta^e(s) w(a, x)}{v^e(s)} e^{\int_0^s \frac{-(\mu^e + \beta^e)(t)}{v^e(t)} dt} ds \\ &\quad \int_0^{L^f} \frac{\beta^f(0, 0, \eta)}{v^f(\eta)} e^{\int_0^\eta \frac{-\mu^f(\xi) + A_1}{v^f(\xi)} d\xi} d\eta \int_0^{L^l} \sigma \beta^l(\zeta) d\zeta dx da > 0 \end{aligned}$$

for any $w(a, x) \in L_+^1((0, L^l) \times \Omega) \setminus \{0\}$. Moreover for any integer n and $w(a, x) \in L_+^1((0, L^l) \times \Omega) \setminus \{0\}$, we have

$$(J_0^{n+1} w)(a, x) = J_0(J_0^n w)(a, x) > 0$$

if $(J_0^n w)(a, x) > 0$ holds. It concludes that $\langle f, J_0^n w \rangle > 0$, for $n \geq 1$ and every pair $f \in (L_+^1(0, L^l) \times \Omega)^* \setminus \{0\}$, $w(a, x) \in L_+^1((0, L^l) \times \Omega) \setminus \{0\}$, that is, T is non-supporting in $L_+^1((0, L^l) \times \Omega)$. Since J_0 is non-supporting and also compact, it has a unique positive eigenvector corresponding to its spectral radius $r(J_0)$. Therefore, we apply Krasnoselskii's fixed point theorem to conclude that J_0 has at more than one non-zero fixed point in the positive cone $L_+^1((0, L^l) \times \Omega)$ if $r(J_0) > 1$. By the definition of spectral radius $r(J_0) = \lim_{n \rightarrow \infty} \sqrt[n]{\|J_0^n\|}$, we just need to verify

$$\begin{aligned} \|J_0^n\| &= \sup_{a \in (0, L^l), \|w\|_{L^1} = 1} \|J_0^n(w)\|_{L^1} \\ &= \sup_{a \in (0, L^l), \|w\|_{L^1} = 1} \left\| \frac{1}{v^l(a)} e^{\int_0^a \frac{-(\mu^l(0, s) + \beta^l(s))}{v^l(s)} ds} \int_0^{L^e} \frac{\beta^e(s) J_0^{n-1}(w)(a, x)}{v^e(s)} \right. \\ &\quad \left. e^{\int_0^s \frac{-(\mu^e + \beta^e)(t)}{v^e(t)} dt} ds \int_0^{L^f} \frac{\beta^f(0, 0, \eta)}{v^f(\eta)} e^{\int_0^\eta \frac{-\mu^f(\xi) + A_1}{v^f(\xi)} d\xi} d\eta \int_0^{L^l} \sigma \beta^l(\zeta) d\zeta \right\|_{L^1} \\ &> 1. \end{aligned}$$

Clearly (2.22) implies that $\|J_0^n\| > 1$ because of $J_0 > R \geq 1$. That completes the proof of Theorem 2.24. \square

If we impose some additional assumptions, then we can prove the uniqueness of the survival steady state. Here we do not investigate the uniqueness of the positive steady state under such restrictive conditions. One can refer to Cha, et al.[33] and S.Fekih, et al.[48] for the uniqueness of positive solution. More important basic observation is that positive solutions of (2.19) bifurcate from the trivial solution zero. In the following, we further discuss the survival steady state of moths from the view of bifurcation.

Now we first give the definition of bifurcation. Let (E, P) be a Banach space ordered by a cone P , and $f : \mathbb{R}_+ \times P \rightarrow P$ be a map such that $f(\cdot, 0) = 0$. Then $\lambda_0 \in \mathbb{R}_+$ is called a bifurcation point for the equation $u = f(\lambda, u)$ (with respect to the trivial solution) if for every neighborhood U of $(\lambda_0, 0)$ in $\mathbb{R}_+ \times P$, there exists a point $(\lambda, u) \in U$ with $u = f(\lambda, u)$ and $u > 0$. In addition, we will denote the Frechet derivative with respect to the second variable by D_2f .

We introduce a bifurcation parameter λ , which determines the transmission rate β^l without changing its structure, by

$$\beta^l(a) := \lambda\beta_0^l(a).$$

Furthermore, we denote

$$J(u^l) := J^*(u^l)$$

as β_0^l is instead of β^l .

Corollary 2.25 *Assume that β_0^l is given such that $R^* = r(\partial(J^*(0))) = 1$. The survival steady states of moths are bifurcated from the extinction steady state at $R^* = 1$.*

Proof. We rewrite the fixed point problem of the first equality in (2.21) as

$$u^l = \lambda J^*(u^l).$$

Now we define a map $f(\lambda, u^l) : \mathbb{R}_+ \times L_+^1((0, L^l) \times \Omega) \rightarrow L_+^1((0, L^l) \times \Omega)$ as follows:

$$f(\lambda, u^l) = \lambda J^*(u^l) - u^l. \tag{2.23}$$

It is easy to check that the derivative of J , i.e. $J'(u^l)$ is continuous. Moreover, we note that $f(\lambda, u^l)$ is C^1 with respect to (λ, u^l) . Observe the structure of solution set

$$f^{-1}(0) := \{(\lambda, u^l) \in \mathbb{R}_+ \times L_+^1((0, L^l) \times \Omega) | f(\lambda, u^l) = 0\}.$$

Inspired by the Implicit function theorem, we investigate the following linear map

$$L(\lambda) := D_2f(\lambda, u^l)|_{u^l=0} = \lambda\partial(J^*(0)) - I,$$

where I is the identity operator.

To obtain that bifurcation occurs at $(\lambda, 0)$, we just need to study the value of λ which can ensure that $L(\lambda)$ does not have a bounded inverse. Since $\partial(J^*(0))$ is compact and non-supporting, from the assumption $R^* = r(\partial(J^*(0))) = 1$, we can deduce that $r(\partial(J^*(0)))$ is the unique positive eigenvalue of $\partial(J^*(0))$, which is the consequence of Theorem [1] in Sawashima [99]. Therefore, we can expect that $\lambda = 1$ is the only possible bifurcation parameter which can lead to the bifurcation occurring at trivial solution.

Let $\chi(\lambda)$ be the simple real principle eigenvalue of $L(\lambda)$ corresponding to the eigenvector $\varphi(\lambda)$. Since $\varphi(\lambda)$ is the Perron-Frobenius eigenvector of non-supporting operator $\partial(J^*(0))$ corresponding to the eigenvalue which equals to 1, it follows immediately that the eigenspace corresponding to $\varphi(\lambda)$ is one-dimensional and spanned by $\varphi(1)$. It is obvious that $L(\lambda)|_{\lambda=1}$ is a Fredholm operator. Hence we can apply the argument of Lyapunov-Schmidt reduction to conclude that there exists a bifurcation point $(1, 0)$ such that (λ, u^l) is a positive solution of (2.23) in every neighbor of $(1, 0)$ (see [34]). It completes the proof of Corollary 2.25. \square

Chapter 3

Exact null controllability of a stage and age-structured population dynamics system

3.1 Introduction

As we know European grapevine moth reduces not only the amount of berries especially when berries are young in spring, but also their quality by favoring indirect damages as related to different pathogens developing on berries like the grey mold and in several warm vineyards to the black rots on berries [104]. These problems are suspected to increase, and could become more prevalent due to the climatic changes in the future. Thus many biological interventions have been developed to control this pest. Currently, the control procedures for this pest rely mainly on chemical insecticides and slightly on mating disruption (no more than 2% of the french vine areas in 2007). But pesticides like growth regulators are used to reduce the population size, so that serious pollution damages environment. Researchers are developing some tools to control these insect populations and also to reduce the application of chemical plant health products. One problem accompanied by these control techniques is that their efficiency depends upon the timing of the treatment and its synchrony with few specific steps of the pest life cycle, e.g. adult flight, oviposition periods. Then their goal is to predict the periods of appearance of the insect in the vineyard, and the mathematical models with age structure maybe very helpful for this objective. Our concerned system is stated as

follows

$$\begin{cases} \frac{\partial u^e(t,a)}{\partial t} + \frac{\partial u^e(t,a)}{\partial a} &= -(\mu^e(a) + \beta^e(a))u^e(t,a) + \chi(a)w(t,a), \\ \frac{\partial u^l(t,a)}{\partial t} + \frac{\partial u^l(t,a)}{\partial a} &= -(\mu^l(a) + \beta^l(a))u^l(t,a), \\ \frac{\partial u^f(t,a)}{\partial t} + \frac{\partial u^f(t,a)}{\partial a} &= -\mu^f(a)u^f(t,a), \\ \frac{\partial u^m(t,a)}{\partial t} + \frac{\partial u^m(t,a)}{\partial a} &= -\mu^m(a)u^m(t,a), \end{cases} \quad (3.1)$$

where $(t, a) \in (0, T) \times (0, A)$, $A = \max\{L^e, L^l, L^f, L^m\}$. Here L^k means life expectancy of an individual for $k = e, l, f, m$, and $u^k(t, a)$ represents the age-specific density of the egg, larva, female moth and male moth respectively. For every k , if $A > L^k$, we denote $u^k = 0, \beta^k = 0, \mu^k = 0$. The term $\chi(a)w(t, a)$ is a control process for egg stage: $\chi(a)$ is the characteristic function of $(0, a^*)$ ($0 < a^* < L^e \leq A$), which means that our intervention can be restricted to the younger age groups.

The boundary conditions are defined by

$$\begin{cases} u^e(t, 0) &= \int_0^{L^f} \beta^f(s)u^f(t, s)ds, \\ u^l(t, 0) &= \int_0^{L^e} \beta^e(s)u^e(t, s)ds, \\ u^f(t, 0) &= \int_0^{L^l} \sigma\beta^l(s)u^l(t, s)ds, \\ u^m(t, 0) &= \int_0^{L^l} (1 - \sigma)\beta^l(s)u^l(t, s)ds, \end{cases} \quad (3.2)$$

where σ denotes the sex ratio, $t > 0$. The system is complete with the initial conditions as follows

$$u^k(0, a) = u_0^k(a), \quad (3.3)$$

for $k = e, l, f, m$.

In addition, we state the following conception for this system.

The functions μ^k are the k-stage age-specific per capita mortality functions. The functions β^k denote the k-stage age-specific transition functions. In particular, β^e models the physiological change between the egg and larva stage, which is called the hatching function. The function β^l is the flying function describing the transition between the larva and the moth stage. The function β^f models the transition between the moth and the egg stage whose name is the birth function. Note that for each (t, a) the directional derivatives of u^k exist, and we can see

$$Du^k(t, a) = \lim_{h \rightarrow 0} \frac{u^k(t+h, a+h) - u^k(t, a)}{h},$$

with $k = e, l, f, m$. It is obvious that for u^k smooth enough

$$Du^k = \frac{\partial u^k}{\partial t} + \frac{\partial u^k}{\partial a}. \quad (3.4)$$

Therefore, considering the economical loss caused by the pest insect, it is meaningful to study the control problem of this *Lobesia botrana* model (LBM). It is well known that optimal and exact control problems are widely investigated for age-structured population dynamics by many researchers. Among these literatures, most of the works are focused on optimal control problems, both almost perfect theory on single species [5, 15] and some results for interacting multi-species. One can see [82, 83, 93] and references therein. It is worth mentioning that B. Ainseba et al. have investigated the local exact controllability for age dependent linear and nonlinear single-species population model with diffusion, where the birth process is nonlocal. The main proof is based on the Carleman's inequality for the adjoint equation [4, 7, 15]. Viorel Barbu et al. also considered the exact controllability of the linear Lotka-McKendrick model without spatial structure by establishing an observability inequality for the backward adjoint system [19].

However, to our knowledge, there are no results dealing with the exact control problem for a stage and age-dependent system. We cannot extend the method developed in [19] to the system case to get the key observability inequality. In spite of that, considering the fact that the system is a stage and age-dependent life cycle dynamics, we are inspired to apply a fixed point theorem in [3, 4] to study the exact null controllability in finite time of the *Lobesia botrana* model (LBM) with four development stages, by reducing the egg population. Roughly speaking, the main result, Theorem 3.1 below, amounts to saying that for $T > A - a^*$ with $A > a^* > 0$, the population u^e can be controlled to zero in a finite time T .

Theorem 3.1 *We denote $\|u_0\|_\infty = \|u_0^e\|_\infty + \|u_0^l\|_\infty + \|u_0^f\|_\infty + \|u_0^m\|_\infty$, and $\|u_0\|_{L^\infty((a^*, A-T))} > 0$. Let $T > A - a^*$ be arbitrary but fixed, and ϱ small enough with $0 < \varrho \leq a_0$. Then there exists $w \in L^2((0, T) \times (0, A))$, such that the solution u^e of (3.1)-(3.3) satisfies*

$$u^e(T, a) = 0, \quad a.e. \quad a \in (\varrho, A). \quad (3.5)$$

If $T < A - a^$, then there is no control w such that u^e satisfies (3.1)-(3.3).*

This chapter is organized as follows: The assumptions are stated in Section 2. Then we give some derivations and the main proof of exact null controllability in Section 3.

3.2 Preliminaries

Let $L^2 = L^2((0, A))$ be the Banach space of equivalence classes of Lebesgue integrable functions, from $(0, A)$ in \mathbb{R} with the norm

$$\|\varphi\|_{L^2((0,A))} = \left(\int_0^A |\varphi(a)|^2 da \right)^{\frac{1}{2}},$$

where $A = \max\{L^e, L^l, L^f, L^m\}$.

Let $T' > 0$. For all $t \in [0, T']$, we define the space $\mathcal{L}^{T'} =: C([0, T'], L^2((0, A)))$ as the Banach space of continuous functions from $[0, T']$, with values in $L^2((0, A))$, which is equipped with the norm

$$\|\varphi\|_{\mathcal{L}^{T'}} = \sup_{0 \leq t \leq T'} \|\varphi(t, \cdot)\|_{L^2}.$$

Definition 3.2 For all $T' > 0$ and all $(t, a) \in (0, T') \times (0, L^k)$, (u^e, u^l, u^f, u^m) is called a solution of (3.1)-(3.3) if and only if it belongs to $(\mathcal{L}^{T'})^4$ and it satisfies system (3.1)-(3.3), where $k = e, l, f, m$.

Integrating along the characteristic lines (see [32]), we obtain the solution of (3.1)-(3.3) for $k = l, f, m$ and $T' > 0$

$$u^k(t, a) = \begin{cases} u_0^k(a-t) \frac{\Pi_{u^k}(a)}{\Pi_{u^k}(a-t)}, & a \geq t, \\ u^k(t-a, 0) \Pi_{u^k}(a), & a < t, \end{cases} \quad (3.6)$$

and

$$u^e(t, a) = \begin{cases} u_0^e(a-t) \frac{\Pi_{u^e}(a)}{\Pi_{u^e}(a-t)} + \int_0^t \frac{\Pi_{u^e}(a)}{\Pi_{u^e}(a-t+s)} \chi(s+a-t) w(s, s+a-t) ds, & a \geq t, \\ u^e(t-a, 0) \Pi_{u^e}(a) + \int_{t-a}^t \frac{\Pi_{u^e}(a)}{\Pi_{u^e}(a-t+s)} \chi(s+a-t) w(s, s+a-t) ds, & a < t, \end{cases} \quad (3.7)$$

where

$$\begin{cases} \Pi_{u^k}(a) = e^{-\int_0^a (\mu^k(\tau) + \beta^k(\tau)) d\tau}, & k = e, l, \\ \Pi_{u^k}(a) = e^{-\int_0^a \mu^k(\tau) d\tau}, & k = f, m. \end{cases} \quad (3.8)$$

Our motivation is to reduce the population of egg, using an age- and time-dependent control of eggs. Especially we are able to find a control w corresponding to a removal (eradication) of egg on $(0, a_0)$ such that $u^e(a, T) = 0$ for a fixed T and $a \in (0, A)$. Throughout this chapter we impose the following assumptions:

(H1) The hatching function β^e , the flying function β^l and the birth function β^f are bounded, non-negative functions. There exist $a_0, a_1 \in (0, L^e)$ such that $\beta^e = 0$, a.e. $a \in (0, a_0) \cup (a_1, L^e)$.

(H2) The mortality functions $\mu^e(a)$, $\mu^l(a)$, $\mu^f(a)$ and $\mu^m(a)$ are non-negative, locally bounded and satisfy the following conditions:

$$\int_0^{L^k} \mu^k(a) ds = \infty$$

with $k = e, f, l, m$.

(H3) The initial distribution $u_0 = (u_0^e, u_0^l, u_0^f, u_0^m)$ is non-negative, a.e. for $a \in (0, A)$.

These assumptions are biologically meaningful [11, 12, 15, 109], so that the existence and uniqueness of a solution of the system (3.1) is guaranteed. Here we omit the proof. One can refer to [11] and [48].

3.3 Proof of the Main Result

We shall divide the proof of Theorem 3.1 into two steps. The first step is to obtain the existence of the control $w \in L^2((0, T) \times (0, A))$ and null controllability of the system (3.1)-(3.3), for $T > A - a^*$. The other is to get the nonexistence of the control w when $T < A - a^*$.

3.3.1 Null controllability

First, we choose a number $T_0 \in (0, \min\{a_0, a^*, A - a^*, T - A + a^*, A - a_1\})$. Define

$$K = L^\infty((0, A - a^* + T_0)).$$

Let $b^l \in K$ arbitrary but fixed and for any $\varepsilon > 0$, we consider the following optimal control problem:

$$\text{Minimize} \left\{ \frac{1}{2} \int_{G^e} |w(t, a)|^2 dt da + \frac{1}{\varepsilon} \int_{\Gamma_0} |u^e(t, a)|^2 dl \right\}, \quad (3.9)$$

where

$$G^e = (0, T_0) \times (0, a^*) \cup (0, A - a^* + T_0) \times (0, T_0),$$

$$G^l = (0, A - a^* + T_0) \times (0, L^l),$$

$$G^f = (0, A - a^* + T_0) \times (0, L^f),$$

$$G^m = (0, A - a^* + T_0) \times (0, L^m),$$

and

$$\Gamma_0 = (T_0, A - a^* + T_0) \times \{T_0\} \cup \{T_0\} \times (T_0, a^*).$$

See Figure 3.1.

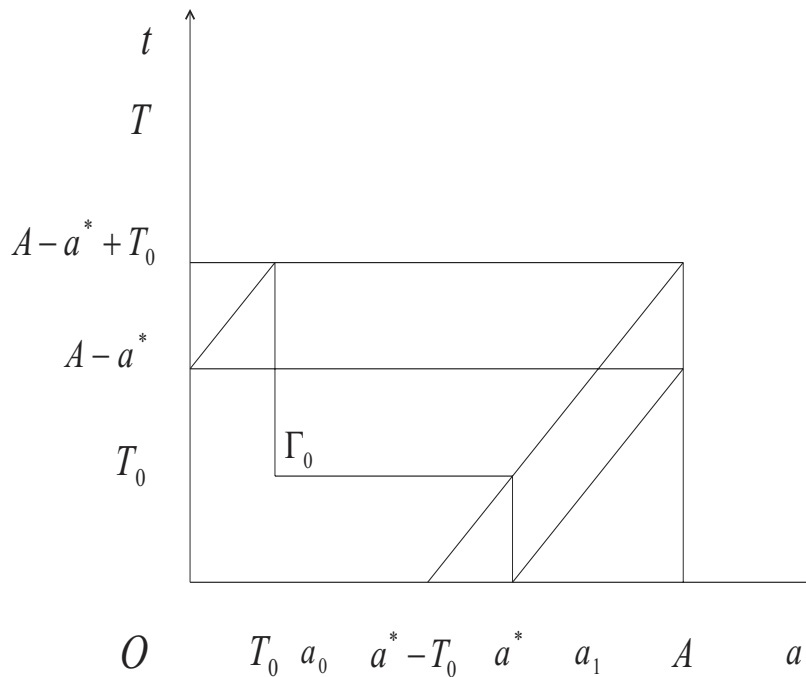


Figure 3.1: An example of domains G and Γ_0 for $T > A - a^*$

In addition, $w \in L^2(G^e)$ and u^e is the solution subject to the following system

$$\begin{cases} \frac{\partial u^e(t,a)}{\partial t} + \frac{\partial u^e(t,a)}{\partial a} = -(\mu^e(a) + \beta^e(a))u^e(t,a) + \chi(a)w(t,a), & (t,a) \in G^e, \\ \frac{\partial u^l(t,a)}{\partial t} + \frac{\partial u^l(t,a)}{\partial a} = -(\mu^l(a) + \beta^l(a))u^l(t,a), & (t,a) \in G^l, \\ \frac{\partial u^f(t,a)}{\partial t} + \frac{\partial u^f(t,a)}{\partial a} = -\mu^f(a)u^f(t,a), & (t,a) \in G^f, \\ \frac{\partial u^m(t,a)}{\partial t} + \frac{\partial u^m(t,a)}{\partial a} = -\mu^m(a)u^m(t,a), & (t,a) \in G^m, \end{cases} \quad (3.10)$$

for $(t, a) \in G^k$ and with the following boundary condition

$$\begin{cases} u^e(t, 0) &= \int_0^{L^f} \beta^f(s) u^f(t, s) ds, \\ u^l(t, 0) &= b^l(t), \\ u^f(t, 0) &= \sigma \int_0^{L^l} \beta^l(s) u^l(t, s) ds, \\ u^m(t, 0) &= (1 - \sigma) \int_0^{L^m} \beta^l(s) u^l(t, s) ds, \\ u^k(0, a) &= u_0^k(a), \end{cases} \quad (3.11)$$

for $t \in (0, A - a^* + T_0)$.

Denote the value of the cost function by $J_\varepsilon(w)$. Since $J_\varepsilon(w) : L^2(G^e) \rightarrow \mathbb{R}^+$ is convex, continuous and

$$\lim_{\|w\|_{L^2(G^e)} \rightarrow \infty} J_\varepsilon(w) = \infty,$$

it means that there is at least one minimum point for $J_\varepsilon(w)$. As a result, an optimal pair $(w_\varepsilon, u_\varepsilon^e)$ exists in (3.9).

We define Lagrange function as follows

$$\mathcal{L}(S) = J_\varepsilon + I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{cases} I_1 = \int_{G^e} q^e(t, a) \left[\frac{\partial u^e(t, a)}{\partial t} + \frac{\partial u^e(t, a)}{\partial a} + (\mu^e(a) + \beta^e(a)) u^e(t, a) - \chi(a) w(t, a) \right] dt da, \\ I_2 = \int_{G^l} q^l(t, a) \left[\frac{\partial u^l(t, a)}{\partial t} + \frac{\partial u^l(t, a)}{\partial a} + (\mu^l(a) + \beta^l(a)) u^l(t, a) \right] dt da, \\ I_3 = \int_{G^f} q^f(t, a) \left[\frac{\partial u^f(t, a)}{\partial t} + \frac{\partial u^f(t, a)}{\partial a} + \mu^f(a) u^f(t, a) \right] dt da, \\ I_4 = \int_{G^m} q^m(t, a) \left[\frac{\partial u^m(t, a)}{\partial t} + \frac{\partial u^m(t, a)}{\partial a} + \mu^m(a) u^m(t, a) \right] dt da. \end{cases} \quad (3.12)$$

Here q^e, q^l, q^f, q^m are the adjoint variables with respect to u^e, u^l, u^f, u^m , representing the fluctuations of the population of *Lobesia botrana* model (LBM). The vector $S^* = (w, q)$ is an optimum of \mathcal{L} if and only if the gradient of the Lagrange function is zero at the optimum, where $q = (q^e, q^l, q^f, q^m)$. The derivative of the Lagrange function with respect to the variables q at the optimum S^* gives the evolution problem (3.10). From integrations by parts for I_i with $i = 1, 2, 3, 4$ and differentiating the Lagrangian at point S^* with respect to the densities u^e, u^l, u^f, u^m (see [92]), we can get the dual (backward) system of (3.10) and (3.11) as follows:

$$\begin{cases} \frac{\partial q^e(t, a)}{\partial t} + \frac{\partial q^e(t, a)}{\partial a} &= (\mu^e(a) + \beta^e(a)) q^e(t, a), \\ \frac{\partial q^l(t, a)}{\partial t} + \frac{\partial q^l(t, a)}{\partial a} &= (\mu^l(a) + \beta^l(a)) q^l(t, a) - \sigma \beta^l(a) q^f(t, 0), \\ \frac{\partial q^f(t, a)}{\partial t} + \frac{\partial q^f(t, a)}{\partial a} &= \mu^f(a) q^f(t, a) - \beta^f(a) q^e(t, 0), \\ \frac{\partial q^m(t, a)}{\partial t} + \frac{\partial q^m(t, a)}{\partial a} &= \mu^m(a) q^m(t, a), \end{cases} \quad (3.13)$$

with $(t, a) \in G^k$, $k = e, l, f, m$, corresponding to the boundary conditions

$$\begin{cases} q^l(t, a) = 0, & t = A - a^* + T_0, \text{ or } a = L^l, \\ q^f(t, a) = 0, & t = A - a^* + T_0, \text{ or } a = L^f, \\ q^m(t, a) = 0, & t = A - a^* + T_0, \text{ or } a = L^m, \\ q^e(t, a) = 0, & (t, a) \in \Gamma \setminus \Gamma_0, \\ q^e(t, a) = -\frac{u^e(t, a)}{\varepsilon}, & (t, a) \in \Gamma_0, \end{cases} \quad (3.14)$$

with $\Gamma = \{A - a^* + T_0\} \times (0, T_0) \cup (0, T_0) \times \{a^*\} \cup \Gamma_0$. Using Ekeland variational principle one can obtain the above optimality system as in the work of Barbu and Iannelli for the scalar population dynamics case [18]. The existence and uniqueness of the dual system are easy to check by the method of characteristics. Here we omit the details, and denote by $u_\varepsilon^k, q_\varepsilon^k$ the solution of (3.10) with (3.11), (3.13) with (3.14) respectively, for $k = e, l, f, m$. One can easily see that q^m is identically zero. At the same time, it is known that q_ε^e satisfies

$$w_\varepsilon(t, a) = \chi(a)q_\varepsilon^e(t, a), \quad \text{a.e. } (t, a) \in (0, A - a^* + T_0) \times (0, A). \quad (3.15)$$

Multiplying the first equation of (3.13) by u_ε^e , and then integrating on G^e :

$$\int_{G^e} u_\varepsilon^e \left[\frac{\partial q_\varepsilon^e(t, a)}{\partial t} + \frac{\partial q_\varepsilon^e(t, a)}{\partial a} - (\mu^e(a) + \beta^e(a))q_\varepsilon^e(t, a) \right] dt da = 0. \quad (3.16)$$

Using integrations by parts, then using (3.10), (3.11) and (3.14), we obtain

$$\begin{aligned} & \int_0^{a^*} \int_0^{A-a^*+T_0} |w_\varepsilon(t, a)|^2 dt da + \frac{1}{\varepsilon} \int_{\Gamma_0} |u_\varepsilon^e(t, a)|^2 da \\ &= - \int_0^{A-a^*+T_0} q_\varepsilon^e(t, 0)u_\varepsilon^e(t, 0) dt - \int_0^{a^*} q_\varepsilon^e(0, a)u_\varepsilon^e(0, a) da. \end{aligned} \quad (3.17)$$

Similarly multiplying the remaining equations of (3.13) by $u_\varepsilon^l, u_\varepsilon^f$, and then integrating on G^k , $k = l, f$ respectively, we have

$$\begin{aligned} & \int_0^{A-a^*+T_0} q_\varepsilon^f(t, 0)u_\varepsilon^f(t, 0) dt \\ &= \int_0^{A-a^*+T_0} q_\varepsilon^l(t, 0)u_\varepsilon^l(t, 0) dt + \int_0^{L^l} q_\varepsilon^l(0, a)u_\varepsilon^l(0, a) da, \end{aligned} \quad (3.18)$$

$$\begin{aligned} & \int_0^{A-a^*+T_0} q_\varepsilon^f(t, 0)u_\varepsilon^f(t, 0) dt + \int_0^{L^f} q_\varepsilon^f(0, a)u_\varepsilon^f(0, a) da \\ &= \int_0^{A-a^*+T_0} q_\varepsilon^e(t, 0)u_\varepsilon^e(t, 0) dt. \end{aligned} \quad (3.19)$$

Combining the above three equations (3.17), (3.18) and (3.19), we obtain

$$\begin{aligned}
 & \int_0^{a^*} \int_0^{A-a^*+T_0} |w_\varepsilon(t, a)|^2 dt da + \frac{1}{\varepsilon} \int_{\Gamma_0} |u_\varepsilon^e(t, a)|^2 dl + \int_0^{L^f} q_\varepsilon^f(0, a) u_\varepsilon^f(0, a) da \\
 = & - \int_0^{A-a^*+T_0} q_\varepsilon^l(t, 0) u_\varepsilon^l(t, 0) dt - \int_0^{L^l} q_\varepsilon^l(0, a) u_\varepsilon^l(0, a) da - \int_0^{a^*} q_\varepsilon^e(0, a) u_\varepsilon^e(0, a) da \\
 = & - \int_0^{A-a^*+T_0} q_\varepsilon^f(t, 0) u_\varepsilon^f(t, 0) dt - \int_0^{a^*} q_\varepsilon^e(0, a) u_\varepsilon^e(0, a) da,
 \end{aligned}$$

which means

$$\begin{aligned}
 & - \int_0^{a^*} \int_0^{A-a^*+T_0} |w_\varepsilon(t, a)|^2 dt da - \frac{1}{\varepsilon} \int_{\Gamma_0} |u_\varepsilon^e(t, a)|^2 dl \\
 = & \int_0^{L^f} q_\varepsilon^f(0, a) u_\varepsilon^f(0, a) da + \int_0^{A-a^*+T_0} q_\varepsilon^f(t, 0) u_\varepsilon^f(t, 0) dt + \int_0^{a^*} q_\varepsilon^e(0, a) u_\varepsilon^e(0, a) da.
 \end{aligned}$$

Let S^e and S^f be arbitrary characteristic lines of the first and third equation respectively in (3.13),

$$S^e = \{(\gamma + t, \theta + t); t \in (0, T_0), (\gamma, \theta) \in \{0\} \times (0, a^* - T_0) \cup (0, A - a^*) \times \{0\}\},$$

$$S^f = \{(t, \pi + t); t \in (0, A - a^* + T_0), \pi \in (0, a^* - T_0)\}.$$

Define

$$\begin{aligned}
 \tilde{w}_\varepsilon(t) &= w_\varepsilon(\gamma + t, \theta + t), & t \in (0, T_0), \\
 \tilde{u}_\varepsilon^e(t) &= u_\varepsilon^e(\gamma + t, \theta + t), & t \in (0, T_0), \\
 \tilde{q}_\varepsilon^e(t) &= q_\varepsilon^e(\gamma + t, \theta + t), & t \in (0, T_0), \\
 \tilde{\mu}^e(t) &= \mu^e(\theta + t), & t \in (0, T_0), \\
 \tilde{\beta}^e(t) &= \beta^e(\theta + t), & t \in (0, T_0), \\
 \tilde{\chi}(t) &= \chi(\theta + t), & t \in (0, T_0), \\
 \tilde{u}_\varepsilon^f(t) &= u_\varepsilon^f(t, \pi + t), & t \in (0, A - a^* + T_0), \\
 \tilde{q}_\varepsilon^f(t) &= q_\varepsilon^f(t, \pi + t), & t \in (0, A - a^* + T_0), \\
 \tilde{\mu}^f(t) &= \mu^f(\pi + t), & t \in (0, A - a^* + T_0), \\
 \tilde{\beta}^f(t) &= \beta^f(\pi + t), & t \in (0, A - a^* + T_0).
 \end{aligned}$$

Note that $(\tilde{u}_\varepsilon^e, \tilde{w}_\varepsilon)$ satisfies

$$\frac{d\tilde{u}_\varepsilon^e(t)}{dt} = -(\tilde{\mu}^e(t) + \tilde{\beta}^e(t))\tilde{u}_\varepsilon^e(t) + \tilde{\chi}(t)\tilde{w}_\varepsilon(t), \quad t \in (0, T_0), \quad (3.20)$$

$$\tilde{u}_\varepsilon^e(0) = \begin{cases} \int_0^{L^f} \beta^f(a) u^f(\gamma, a) da, & \theta = 0, \\ u_0^e(\theta), & \gamma = 0; \end{cases} \quad (3.21)$$

\tilde{q}_ε^e satisfies

$$\begin{cases} \frac{d\tilde{q}_\varepsilon^e(t)}{dt} = (\tilde{\mu}^e(t) + \tilde{\beta}^e(t)) \tilde{q}_\varepsilon^e(t), & t \in (0, T_0), \\ \tilde{q}_\varepsilon^e(T_0) = \frac{-\tilde{u}_\varepsilon^e(T_0)}{\varepsilon}, \end{cases} \quad (3.22)$$

and

$$\tilde{w}_\varepsilon(t) = \tilde{\chi}(t) \tilde{q}_\varepsilon^e(t), \quad a.e. \quad t \in (0, T_0). \quad (3.23)$$

Similarly, \tilde{u}_ε^f and \tilde{q}_ε^f satisfy the following equations respectively

$$\frac{d\tilde{u}_\varepsilon^f(t)}{dt} = -\tilde{\mu}^f(t) \tilde{u}_\varepsilon^f(t), \quad t \in (0, T_0), \quad (3.24)$$

$$\tilde{u}_\varepsilon^f(0) = \begin{cases} \int_0^{L^l} \sigma \beta^l(a) u^l(\gamma, a) da, & \theta = 0, \\ u_0^f(\theta), & \gamma = 0; \end{cases} \quad (3.25)$$

$$\begin{cases} \frac{d\tilde{q}_\varepsilon^f(t)}{dt} = \tilde{\mu}^f(t) \tilde{q}_\varepsilon^f(t) - \tilde{\beta}^f(t) q^e(\gamma, 0), & t \in (0, T_0), \\ \tilde{q}_\varepsilon^f(A - a^* + T_0) = 0. \end{cases} \quad (3.26)$$

Multiplying the first equation in (3.22) by $\tilde{u}_\varepsilon^e(t)$, and the first equation in (3.26) by $\tilde{u}_\varepsilon^f(t)$ respectively, integrating on $(0, A - a^* + T_0)$, then applying (3.20), (3.21), (3.24) and (3.25), we have

$$\int_0^{T_0} |\tilde{w}_\varepsilon(t)|^2 dt + \frac{1}{\varepsilon} |\tilde{u}_\varepsilon^e(T_0)|^2 \leq -\tilde{q}_\varepsilon^f(0) \tilde{u}_\varepsilon^f(0) - q_\varepsilon^e(0, \theta) u_0^e(\theta).$$

By Young's inequality, we obtain

$$\int_0^{T_0} |\tilde{w}_\varepsilon(t)|^2 dt + \frac{1}{\varepsilon} |\tilde{u}_\varepsilon^e(T_0)|^2 \leq \frac{\delta}{2} |\tilde{q}_\varepsilon^f(0)|^2 + \frac{1}{2\delta} |\tilde{u}_\varepsilon^f(0)|^2 + \frac{\delta}{2} |q_\varepsilon^e(0, \theta)|^2 + \frac{1}{2\delta} |u_0^e(\theta)|^2, \quad (3.27)$$

with δ being any positive number.

Using the constant variation formula to (3.22), we have

$$\begin{aligned} & \tilde{q}_\varepsilon^f(A - a^* + T_0) - e^{\int_0^{A-a^*+T_0} \tilde{\mu}^f(s) ds} \tilde{q}_\varepsilon^f(0) \\ &= - \int_0^{A-a^*+T_0} e^{\int_0^t \tilde{\mu}^f(s) ds} \tilde{\beta}^f(A - a^* + T_0 - t) q_\varepsilon^e(\gamma, 0) dt, \end{aligned}$$

which means

$$e^{\int_0^{A-a^*+T_0} \tilde{\mu}^f(s) ds} \tilde{q}_\varepsilon^f(0) = - \int_0^{A-a^*+T_0} e^{\int_0^t \tilde{\mu}^f(s) ds} \tilde{\beta}^f(A - a^* + T_0 - t) q_\varepsilon^e(\gamma, 0) dt.$$

Then we obtain

$$|\tilde{q}_\varepsilon^f(0)|^2 \leq C|q_\varepsilon^e(\gamma, 0)|^2. \quad (3.28)$$

Note that here and after we denote several constants independent of all variables by the same C . Next multiplying the equation in (3.22) by \tilde{q}_ε^e , we obtain

$$\frac{1}{2} \frac{d(\tilde{q}_\varepsilon^e)^2}{dt} = (\tilde{\mu}^e(t) + \tilde{\beta}^e(t))(\tilde{q}_\varepsilon^e(t))^2 \geq 0, \quad a.e. \quad t \in (0, T_0), \quad (3.29)$$

which leads to the result

$$|\tilde{q}_\varepsilon^e(0)|^2 \leq C \int_0^{T_0} |\tilde{q}_\varepsilon^e(\tau)|^2 d\tau = C \int_0^{T_0} |\tilde{w}_\varepsilon(\tau)|^2 d\tau. \quad (3.30)$$

Substituting (3.28) and (3.30) into (3.27),

$$(1 - \frac{\delta C}{2}) \int_0^{T_0} |\tilde{w}_\varepsilon(t)|^2 dt + \frac{1}{\varepsilon} |\tilde{u}_\varepsilon^e(T_0)|^2 \leq \frac{1}{2\delta} (|\tilde{u}_\varepsilon^f(0)|^2 + |u_0^e(\theta)|^2),$$

where C is a constant satisfying $\frac{\delta C}{2} < 1$, independent on ε .

Then we get

$$\int_0^{T_0} |\tilde{w}_\varepsilon(t)|^2 dt \leq \frac{1}{2\delta} (|\tilde{u}_\varepsilon^f(0)|^2 + |u_0^e(\theta)|^2) \leq \frac{1}{2\delta} (|\tilde{u}_\varepsilon^f(0)|^2 + C).$$

Recalling that (3.21) holds, it is obvious that

$$\|\tilde{w}_\varepsilon\|_{L^2((0, T_0))}^2 = \int_0^{T_0} |\tilde{w}_\varepsilon(t)|^2 dt \leq \frac{1}{2\delta} (\|u_0^f\|_{L^\infty((0, A))}^2 + C) \leq M_1,$$

as $\gamma = 0$. If $\pi = 0$, then we substitute u^f given by the formula (3.6) in (3.21), and get

$$\begin{aligned} \|\tilde{w}_\varepsilon\|_{L^2((0, T_0))}^2 &\leq \frac{1}{2\delta} \left(\int_0^{L^l} |u_\varepsilon^l(\gamma, a) \beta^l(a)|^2 da + C \right) \\ &\leq \frac{\|\beta^l\|_{L^\infty((0, A))}^2}{2\delta} \left(\int_0^A |u_\varepsilon^l(\gamma, a)|^2 da + \frac{C}{\|\beta^l\|_{L^\infty((0, A))}^2} \right) \\ &\leq \frac{\|\beta^l\|_{L^\infty((0, A))}^2}{2\delta} \left\{ \int_0^\gamma |u_\varepsilon^l(\gamma - a, 0) \Pi_{u^l}(a)|^2 da \right. \\ &\quad \left. + \int_\gamma^A |u_0^l(a - \gamma) \frac{\Pi_{u^l}(a)}{\Pi_{u^l}(a - \gamma)}|^2 da + \frac{C}{\|\beta^l\|_{L^\infty((0, A))}^2} \right\} \\ &\leq \frac{\|\beta^l\|_{L^\infty((0, A))}^2}{2\delta} \left\{ \|b^l\|_K^2 (A - a^*) + \|u_0^l\|_{L^\infty((0, A))}^2 A + \frac{C}{\|\beta^l\|_{L^\infty((0, A))}^2} \right\} \\ &\leq M_1. \end{aligned}$$

According to the property of relatively weak compactness in L^2 , there exists a subsequence (also denoted by \tilde{w}_ε) such that

$$\tilde{w}_\varepsilon \rightarrow \tilde{w} \quad \text{weakly in } L^2((0, T_0)) \text{ as } \varepsilon \rightarrow 0.$$

In what follows we multiply the equation in (3.24) by $\tilde{u}_\varepsilon^f \geq 0$, thus

$$\frac{1}{2} \frac{d(\tilde{u}_\varepsilon^f)^2}{dt} = -\tilde{\mu}^f(t)(\tilde{u}_\varepsilon^f(t))^2 \leq 0, \quad \text{a.e. } t \in (0, T_0),$$

which implies

$$|\tilde{u}_\varepsilon^f(T_0)|^2 \leq |\tilde{u}_\varepsilon^f(t)|^2 \leq |\tilde{u}_\varepsilon^f(0)|^2.$$

Thus we get

$$\int_0^{T_0} |\tilde{u}_\varepsilon^f(t)|^2 dt \leq C |\tilde{u}_\varepsilon^f(0)|^2.$$

From (3.25), obviously we have

$$\|\tilde{u}_\varepsilon^f\|_{L^2((0, T_0))}^2 = \int_0^{T_0} |\tilde{u}_\varepsilon^f(t)|^2 dt \leq C \|u_0^f\|_{L^\infty((0, A))}^2,$$

as $\gamma = 0$. For $\pi = 0$, applying (3.6) in (3.25), we get

$$\begin{aligned} \|\tilde{u}_\varepsilon^f\|_{L^2((0, T_0))}^2 &\leq C\sigma \int_0^A |u_\varepsilon^l(\gamma, a)\beta^l(a)|^2 da \\ &\leq C\|\beta^l\|_{L^\infty((0, A))}^2 \int_0^A |u_\varepsilon^l(\gamma, a)|^2 da \\ &\leq C\|\beta^l\|_{L^\infty((0, A))}^2 \left\{ \int_0^\gamma |u_\varepsilon^l(\gamma - a, 0)\Pi_{u^l}(a)|^2 da \right. \\ &\quad \left. + \int_\gamma^A |u_0^l(a - \gamma) \frac{\Pi_{u^l}(a)}{\Pi_{u^l}(a - \gamma)}|^2 da \right\} \\ &\leq \frac{C\|\beta^l\|_{L^\infty((0, A))}^2}{2\delta} \{ \|b^l\|_K^2 (A - a^*) + \|u_0^l\|_{L^\infty((0, A))}^2 A \} \\ &\leq M_2, \end{aligned}$$

where C_1, C_2, C_3 are constants independent of u^e, u^l, u^f, u^m .

We also have

$$\left\| \frac{d\tilde{u}_\varepsilon^f}{dt} \right\|_{L^2((0, T_0))}^2 = \int_0^{T_0} |\tilde{\mu}^f(t)\tilde{u}_\varepsilon^f(t)|^2 dt \leq M \|\tilde{u}_\varepsilon^f\|_{L^2((0, T_0))}^2,$$

where M is a constant. Hence there exists a subsequence (also denoted by $\tilde{u}_\varepsilon^f(t)$) such that

$$\tilde{u}_\varepsilon^f(t) \rightarrow \tilde{u}^f(t) \quad \text{weakly in } W^{1,2}((0, T_0)) \text{ as } \varepsilon \rightarrow 0.$$

In addition, we apply (3.6) and get the following result, which is similar to the estimation for $\|\tilde{u}_\varepsilon^f\|_{L^2((0,T_0))}^2$,

$$\begin{aligned}
 \|\tilde{u}_\varepsilon^e\|_{L^2((0,T_0))}^2 &\leq C\left\{\int_\gamma^A |u_0^e(a-\gamma)\frac{\Pi_{u^e}(a)}{\Pi_{u^e}(a-\gamma)} + \int_0^\gamma \frac{\Pi_{u^e}(a)}{\Pi_{u^e}(a-\gamma+s)}\right. \\
 &\quad \left.|\chi(s+a-\gamma)w_\varepsilon(s,s+a-\gamma)ds|^2 da + \int_0^\gamma |u^e(\gamma-a,0)\Pi_{u^e}(a)\right. \\
 &\quad \left. + \int_{\gamma-a}^\gamma \frac{\Pi_{u^e}(a)}{\Pi_{u^e}(a-\gamma+s)}|\chi(s+a-\gamma)w_\varepsilon(s,s+a-\gamma)ds|^2 da\right\} \\
 &\leq C\left\{\int_0^\gamma \int_0^A (1+\zeta)|\beta^f(s)u_\varepsilon^f(\gamma-a,s)|^2 ds da + (1+\zeta)A\|u_0^e\|_{L^\infty((0,A))}^2\right. \\
 &\quad \left. + (1+\frac{1}{\zeta})\int_\gamma^A \int_0^\gamma |\chi(s+a-\gamma)w_\varepsilon(s,s+a-\gamma)|^2 ds da\right. \\
 &\quad \left. + (1+\frac{1}{\zeta})\int_0^\gamma \int_{\gamma-a}^\gamma |\chi(s+a-\gamma)w_\varepsilon(s,s+a-\gamma)|^2 ds da\right\} \\
 &\leq C\{C_1\|\beta^f\|_{L^\infty((0,A))}^2 M_2 + \|u_0^f\|_{L^\infty((0,A))}^2 + C_2\|u_0^e\|_{L^\infty((0,A))}^2 \\
 &\quad + (1+\frac{1}{\zeta})\int_0^A \int_0^\gamma |\chi(s+a-\gamma)w_\varepsilon(s,s+a-\gamma)|^2 ds da\} \\
 &\leq C\{C_1\|\beta^f\|_{L^\infty((0,A))}^2 M_2 + \|u_0^f\|_{L^\infty((0,A))}^2 + C_2\|u_0^e\|_{L^\infty((0,A))}^2 \\
 &\quad + (1+\frac{1}{\zeta})\int_0^A \int_0^{a^*-a+\gamma} |w_\varepsilon(s,s+a-\gamma)|^2 ds da\} \\
 &\leq C\{C_1\|\beta^f\|_{L^\infty((0,A))}^2 M_2 + \|u_0^f\|_{L^\infty((0,A))}^2 + C_2\|u_0^e\|_{L^\infty((0,A))}^2 \\
 &\quad + C_3\|\tilde{w}_\varepsilon\|_{L^2((0,T_0))}^2\} \\
 &\leq C\{C_1\|\beta^f\|_{L^\infty((0,A))}^2 M_2 + C_2\|u_0^e\|_{L^\infty((0,A))}^2 + C_3 M_1\},
 \end{aligned}$$

where C_1, C_2, C_3 are constants independent of u^e, u^l, u^f, u^m , and ζ is a positive number.

We can also show

$$\begin{aligned}
 \left\|\frac{d\tilde{u}_\varepsilon^e}{dt}\right\|^2 &\leq \int_0^{T_0} \left((1+\frac{1}{\delta})|\tilde{\mu}^e(t) + \tilde{\beta}^e(t)\tilde{u}_\varepsilon^e|^2 + (1+\delta)|\tilde{\chi}\tilde{w}_\varepsilon(t)|^2\right) dt \\
 &\leq (1+\frac{1}{\delta})M_2\|\tilde{\mu}^e(t) + \tilde{\beta}^e(t)\|_\infty^2 + (1+\delta)M_1,
 \end{aligned}$$

in which δ is a positive number.

Therefore, there exists a subsequence (also denoted by $\tilde{u}_\varepsilon^e(t)$) such that

$$\tilde{u}_\varepsilon^e(t) \rightarrow \tilde{u}^e(t) \quad \text{weakly in } W^{1,2}((0,T_0)) \text{ as } \varepsilon \rightarrow 0,$$

and (\tilde{u}^e, \tilde{w}) satisfies (3.20), with $\tilde{u}^e(T_0) = 0$, and \tilde{u}^f is a solution of (3.24). One can see [25].

We extend w to \tilde{w} on each characteristic line by 0. Then it is known that $\tilde{w} \in L^2((0, T) \times (0, A))$. Let u^e be the solution of (3.10) and (3.11), which is located on $(0, A - a^* + T_0) \times (0, A)$. We set $S = \{(t, a); T_0 < t < A - a^* + T_0, T_0 < a < t + a^* - T_0\}$. Since $u^e = 0$ on Γ_0 and $w=0$ outside G^e , it can be concluded that

$$\begin{aligned} u^e &= 0, \quad a.e. (t, a) \in S, \\ u^e(A - a^* + T_0, a) &= 0, \quad a.e. a \in (T_0, A). \end{aligned}$$

Due to (3.6), we have

$$\begin{aligned} \|u^e\|_{L^\infty(Q)}^2 &\leq C(\|u^e(0, \cdot)\|_{L^\infty((0, A))}^2 + \|u^e(\cdot, 0)\|_{L^\infty((0, A - a^* + T_0))}^2 + \|w\|_{L^2(Q)}^2) \\ &\leq C\left(\int_0^A \|\beta^f u^f\|_{L^\infty((0, A - a^* + T_0))}^2 ds + \|u_0^e\|_{L^\infty((0, A))}^2 + \|w\|_{L^2(Q)}^2\right) \\ &\leq C(\|u_0^f\|_{L^\infty((0, A))}^2 + \|u_0^e\|_{L^\infty((0, A))}^2 + \|u^f(\cdot, 0)\|_{L^\infty((0, A - a^* + T_0))}^2 \\ &\quad + \|w\|_{L^2(Q)}^2) \\ &\leq C(\|u_0^e\|_{L^\infty((0, A))}^2 + \|u_0^f\|_{L^\infty((0, A))}^2 + \|b^l\|_{L^\infty((0, A - a^* + T_0))}^2 \\ &\quad + \|u_0^l\|_{L^\infty((0, A))}^2 + \|w\|_{L^2(Q)}^2), \end{aligned} \tag{3.31}$$

where $Q = (0, A - a^* + T_0) \times (0, A)$, C represents different constants independent of variables.

In the following part, we prove the exact null controllability result by a fixed point technique. For any $b^l \in K$, we denote by

$$\Phi(b^l) := \left\{ \int_0^{L^e} \beta^e(a) u^e(t, a) da \right\} \subset L^2((0, A - a^* + T_0))$$

such that u^e satisfies (3.31) and

$$u^e = 0, \quad a.e. (t, a) \in S \tag{3.32}$$

with $u^e(A - a^* + T_0, a) = 0, \quad a.e. a \in (T_0, L^e)$.

We consider the following two cases:

(1) $t > T_0$:

It is known that $\beta^e = 0, \quad a.e. a \in (0, a_0) \cup (a_1, L^e)$, which implies that

$$\beta^e(a) = 0 \quad for \quad a \in (0, T_0).$$

Further from the condition (3.32), we have

$$\int_{T_0}^{t+a^*-T_0} \beta^e(a) u^e(t, a) da = 0.$$

Then

$$\int_0^{L^e} \beta^e(a) u^e(t, a) da = \int_{t+a^*-T_0}^{a_1} \beta^e(a) u^e(t, a) da.$$

For $0 < t + a^* - T_0 < a$, we have

$$u^e(t, a) = u_0^e(a - t) \frac{\Pi_{u^e}(a)}{\Pi_{u^e}(a - t)}.$$

Obviously, $\int_0^{L^e} \beta^e(a) u^e(t, a) da$ does not depend on b^l .

(2) $0 < t < T_0$:

Once again we use the fact that $\beta^e = 0$, a.e. $a \in (0, a_0) \cup (a_1, L^e)$. Thus we obtain

$$\begin{aligned} & \int_0^{L^e} \beta^e(a) u^e(t, a) da \\ &= \int_0^{T_0} \beta^e(a) u^e(t, a) da + \int_{T_0}^{L^e-T_0} \beta^e(a) u^e(t, a) da + \int_{L^e-T_0}^{L^e} \beta^e(a) u^e(t, a) da \\ &= \int_{T_0}^{L^e-T_0} \beta^e(a) u^e(t, a) da, \end{aligned}$$

which does not depend on b^l , with $u^e(t, a) = u_0^e(a - t) \frac{\Pi_{u^e}(a)}{\Pi_{u^e}(a - t)}$ holding, for $0 < t < T_0 < a < A - T_0$. Moreover,

$$\left| \int_{T_0}^{L^e-T_0} \beta^e(a) u^e(t, a) da \right| \leq C \|\beta^e\|_{L^\infty((0, A))} \|u_0^e\|_{L^\infty((0, A))}. \quad (3.33)$$

In summary, it is obvious that $\Phi(b^l)$ is a contraction and admits a fixed point.

Next we choose a fixed point for the multivalued function Φ as follows. It is known that

$$\begin{aligned} & \int_0^{L^e} \beta^e(a) u^e(t, a) da \\ &= \int_0^{T_0} \beta^e(a) u^e(t, a) da + \int_{T_0}^{L^e-T_0} \beta^e(a) u^e(t, a) da + \int_{L^e-T_0}^{L^e} \beta^e(a) u^e(t, a) da \\ &= \int_{T_0}^{L^e-T_0} \beta^e(a) u^e(t, a) da, \end{aligned}$$

a.e. $t \in (A - a^*, A - a^* + T_0)$. Furthermore, the condition (3.32) implies

$$\int_{T_0}^{L^e-T_0} \beta^e(a) u^e(t, a) da = 0.$$

Therefore, for any w we can choose

$$b^l(t) = \begin{cases} 0, & t \in (A - a^*, A - a^* + T_0), \\ \int_0^{L^e} \beta^e(a) u^e(t, a) da, & t \in (0, A - a^*), \end{cases}$$

as a fixed point of the multivalued function Φ . We obtain that there exists $w \in L^2((0, A - a^* + T_0) \times (0, A))$ with $w = 0$ in $(A - a^*, A - a^* + T_0) \times (a^*, A)$ such that u^e subject to (3.1) satisfies

$$u^e(A - a^* + T_0, a) = 0, \quad a.e. \quad a \in (T_0, A).$$

Then we denote $\varrho = T_0$ small enough, which is right because of the definition of T_0 . Letting $T = A - a^* + T_0$, it completes the first argument of Theorem 3.1.

3.3.2 Nonexistence of the controllability

Now we consider the second condition if $T < A - a^*$, which implies $a^* < A$. Assume that $\|u_0\|_{L^\infty((a^*, A-T))} > 0$, then there exists $w \in L^2((0, T) \times (0, A))$ such that the solution $u^e(t, a)$ of (3.1) satisfies(3.5).

Since $\chi w = 0$ when $a \in (a^*, A)$, it is concluded that $u^e(t, a)$ independent of $w \in U$, where

$$U =: \{(t, a); 0 < t < a - a^*, a^* < a < A\}.$$

See Figure 3.2.

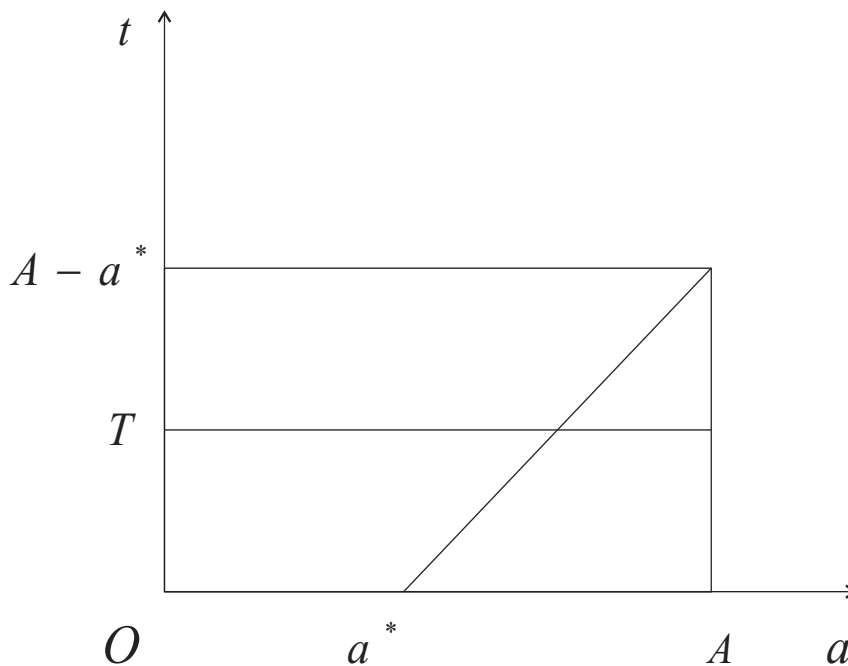


Figure 3.2: An example of domain U when $T < A - a^*$

Moreover, $u^k(t, a)$ with $k = e, l, f, m$ also satisfies the following system

$$\begin{cases} \frac{\partial u^e(t,a)}{\partial t} + \frac{\partial u^e(t,a)}{\partial a} = -(\mu^e(a) + \beta^e(a))u^e(t, a), \\ \frac{\partial u^l(t,a)}{\partial t} + \frac{\partial u^l(t,a)}{\partial a} = -(\mu^l(a) + \beta^l(a))u^l(t, a), \\ \frac{\partial u^f(t,a)}{\partial t} + \frac{\partial u^f(t,a)}{\partial a} = -\mu^f(a)u^f(t, a), \\ \frac{\partial u^m(t,a)}{\partial t} + \frac{\partial u^m(t,a)}{\partial a} = -\mu^m(a)u^m(t, a), \\ u^k(0, a) = u_0^k(a) \quad \text{for } k = e, l, f, m, \end{cases}$$

where $(t, a) \in U$.

Since $\|u_0\|_{L^\infty((a^*, A-T))} > 0$, applying the backward uniqueness result, obviously, it leads to the conclusion that

$$\|u^e(T, \cdot)\|_{L^\infty((0,A))} > 0,$$

which is a contradiction to (3.5). That completes the proof of Theorem 3.1.

Chapter 4

Null controllability of the *Lobesia botrana* model with diffusion

4.1 Introduction

As said in the previous chapters, the European grapevine moth (EGVM) has been the most serious grape pest in Europe, whose life cycle could be divided into four development stages that are egg, larva, pupa and moth. The first three stages correspond to the insect growth and the last adult stage is devoted to the reproduction. This life cycle is repeated two to five times per year according to environmental variations. As a function of temperature and food availability, it lasts about two months during spring and less in summer. In spring or summer the pupa stage lasts one week, and we assume that the pupa stage is included in the larva stage to form a unique stage, the larva stage. This class of moths reduces the quantity of berries especially when the vine are young in spring, as well as their quality by favoring indirect damages. Moreover, adult moths can fly around the vineyard, which is described by the diffusion term. It means that this damage would spread all over the vineyard finally. This diffusion brings about more serious economy loss. In this chapter, we are interested in the dynamics of the *Lobesia botrana* model (LBM) with diffusion terms described by Laplace operators as follows:

$$\begin{cases} Du^e(t, a, x) &= -(\mu^e(a, x) + \beta^e(a))u^e(t, a, x) + m(a)w^e(t, a, x), \\ Du^l(t, a, x) &= -(\mu^l(a, x) + \beta^l(a))u^l(t, a, x) + m(a)w^l(t, a, x), \\ Du^f(t, a, x) &= -\mu^f(a, x)u^f(t, a, x) + \Delta u^f(t, a, x) + \chi(a, x)w^f(t, a, x), \\ Du^m(t, a, x) &= -\mu^m(a, x)u^m(t, a, x) + \Delta u^m(t, a, x), \end{cases} \quad (4.1)$$

where $(t, a, x) \in (0, T) \times (0, A) \times \Omega$, and $\Omega \subset \mathbb{R}^3$. We denote the density of egg, larva, female and male individuals, respectively by $u^k(t, a, x)$ with $k = e, l, f, m$, where t is the time variable, a is the age variable, and x is the spatial variable. The boundary conditions are stated by

$$\begin{cases} u^e(t, 0, x) &= \int_0^{L^f} \beta^f(s)u^f(t, s, x)ds, \\ u^l(t, 0, x) &= \int_0^{L^e} \beta^e(s)u^e(t, s, x)ds, \\ u^f(t, 0, x) &= \int_0^{L^l} \sigma\beta^l(s)u^l(t, s, x)ds, \\ u^m(t, 0, x) &= \int_0^{L^l} (1 - \sigma)\beta^l(s)u^l(t, s, x)ds, \end{cases} \quad (4.2)$$

where σ denotes the sex ratio, $t > 0$. We state the initial condition and the boundary condition for Ω as follows:

$$u^k(0, a, x) = u_0^k(a, x), \quad k = e, l, f, m, \quad (4.3)$$

$$\frac{\partial u^k(t, a, x)}{\partial \eta} = 0, \quad x \in \partial\Omega, \quad k = f, m. \quad (4.4)$$

In the above system, $A = \max\{L^e, L^l, L^f, L^m\}$, and L^k means life expectancy of an individual with $k = e, l, f, m$ standing for development stages: egg, larva, female moth, male moth respectively. For every k , it is easy to see $u^k(t, a, x) = 0, \beta^k(a) = 0, \mu^k(a, x) = 0$ when $a \in [L^k, A]$. The terms $w^e(t, a, x)$ and $w^l(t, a, x)$ are control processes respectively for eggs and larvae, and $m(a)$ is the characteristic function of $(0, a^*)$ with $0 < a^* < \min\{L^e, L^l, L^f\} \leq A$, which means that our intervention is restricted to the younger age groups. The term $w^f(t, a, x)$ is the control process for female moths, and $\chi(a, x)$ is the characteristic function of $(0, a^*) \times \omega$, with ω being a nonempty open subset of Ω .

In addition, the functions μ^k are the k-stage per capita mortality functions with respect to age and space. The functions β^k denote the k-stage age-specific transition functions. In particular, β^e models the physiological change between the egg and larva stage, which is called the hatching function. The flying function β^l describes the transition between the larva and the moth stage. The birth function β^f models the transition

between the moth and the egg stage. For each (t, a, x) the directional derivatives of u^k exist, and we can see

$$Du^k(t, a, x) = \lim_{h \rightarrow 0} \frac{u^k(t+h, a+h, x) - u^k(t, a, x)}{h},$$

with $k = e, l, f, m$. It is obvious that for u^k smooth enough

$$Du^k = \frac{\partial u^k}{\partial t} + \frac{\partial u^k}{\partial a}. \quad (4.5)$$

In order to control the dynamics of the insect population, it is easier to act on static individuals, eggs and larvas, and manipulate in a certain area for female moths to cut back on the number of butterflies, and then the *Lobesia botrana* population. Pesticides can be easily used on eggs and larvas, and avoid serious environment pollution. From economic and environmental views, it is feasible to reduce female moth population by manipulating on egg and larva population, and also on female moths in a small region.

We are interested in the null problem of an age and stage structured system modeling an insect growth where only adult individuals move spatially in finite time. The control corresponds to a removal of individuals by using pesticides. The main purpose in this chapter is to obtain the exact null controllability problem by getting the existence of the controls w^e , w^l and w^f such that the solution u^f satisfies

$$u^f(T, a, x) = 0, \quad a.e. \ a \in (\delta, A), \ x \in \Omega, \quad (4.6)$$

where $0 < \delta \leq a_0$ is a small parameter.

The usual method applied to the null controllability of the parabolic systems on a subset of the domain is based on the Carleman inequality and an observability inequality for the backward adjoint systems. But this technique cannot be extended for the LBM problem, because of the non-locality in (4.2). To overcome the difficulty, the idea here is to apply the fixed point theorem. First we transform the nonlocal term $u^e(t, 0, x)$ to be a local one $b^e(t, x)$. Next we select a family of controls to obtain the null controllability, by combining some estimations and the Carleman inequality for the local backward system related to an optimal control problem. Then choosing a control corresponding to a fixed point of a multi-valued function, we obtain the null controllability of (4.1). The main result is stated in Theorem 4.1 below, saying that for $T > A - a^*$ with $A > a^* > 0$, the population u^f can be steered to zero in a finite time T .

Theorem 4.1 *Let $\|u_0\|_\infty = \|u_0^e\|_\infty + \|u_0^l\|_\infty + \|u_0^f\|_\infty + \|u_0^m\|_\infty$. If $T > A - a^*$ is arbitrary but fixed, and $0 < \delta \leq a_0$ small enough, then there exist controls $w^k \in L^2((0, T) \times (0, A) \times \Omega)$ with $k = e, l, f$ such that the solution u^f of (4.1) satisfies*

$$u^f(T, a, x) = 0, \quad \text{a.e. } (a, x) \in (\delta, A) \times \Omega. \quad (4.7)$$

If $T < A - a^$ and $\|u_0\|_{L^\infty((a^*, A-T) \times \Omega)} > 0$, then there are no controls w^k with $k = e, l, f$ such that u^f satisfies (4.1).*

To illustrate this question, we organize the chapter as follows. The hypothesis is stated in Section 2. Then we study the backward adjoint system associated to (4.1)–(4.4), and prove the exact null controllability stated as (4.6) in Section 3.

4.2 Preliminaries

We first introduce the following notations.

Let $L^2 = L^2((0, A) \times \Omega; \mathbb{R})$ be the Banach space of equivalence classes of Lebesgue integrable functions, from $(0, A) \times \Omega$ to \mathbb{R} with the norm

$$\|\varphi\|_{L^2((0, A) \times \Omega)}^2 = \int_\Omega \int_0^A |\varphi(a, x)|^2 da dx,$$

where $A = \max\{L^e, L^l, L^f, L^m\}$.

Let $T' > 0$. For all $t \in [0, T']$, we can define the space $\mathcal{L}^{T'}$ by $\mathcal{L}^{T'} = C([0, T']; L^2)$ as the Banach space of continuous functions from $[0, T']$, with values in L^2 , which is equipped with the norm

$$\|\varphi\|_{\mathcal{L}^{T'}} = \sup_{0 \leq t \leq T'} \|\varphi(t, \cdot, \cdot)\|_{L^2}.$$

Furthermore, the operator \mathbb{A} defined below generates uniformly bounded semigroup $e^{\mathbb{A}t}$, and satisfies

$$\begin{aligned} \mathbb{A} &= \Delta, \\ \|e^{\mathbb{A}t} u^k\|_{L^2} &\leq C \|u^k\|_{L^2}, \quad k = f, m, \end{aligned}$$

with the Neumann boundary condition

$$\frac{\partial u^k(t, a, x)}{\partial \eta} = 0, \quad x \in \partial\Omega, \quad k = f, m. \quad (4.8)$$

Definition 4.2 For all $T' > 0$ and all $(t, a, x) \in (0, T') \times (0, L^k) \times \Omega$, (u^e, u^l, u^f, u^m) is called a solution of (4.1) if it belongs to $(\mathcal{L}^{T'})^4$ and it satisfies system (4.1), where $k = e, l, f, m$.

Integrating along the characteristic lines (see [32]), we have the solution of (4.1) for $k = e, l$

$$u^k(t, a, x) = \begin{cases} u_0^k(a-t, x) \frac{\Pi_{u^k}(a, x)}{\Pi_{u^k}(a-t, x)} + \int_0^t \frac{\Pi_{u^k}(a, x)}{\Pi_{u^k}(a-t+s, x)} \\ m(s+a-t)w^k(s, s+a-t, x)ds, & a \geq t, \\ u^k(t-a, 0, x)\Pi_{u^k}(a, x) + \int_{t-a}^t \frac{\Pi_{u^k}(a, x)}{\Pi_{u^k}(a-t+s, x)} \\ m(s+a-t)w^k(s, s+a-t, x)ds, & a < t, \end{cases} \quad (4.9)$$

$$u^f(t, a, x) = \begin{cases} e^{t\mathbb{A}}u_0^f(a-t, x) \frac{\Pi_{u^f}(a, x)}{\Pi_{u^f}(a-t, x)} + \int_0^t e^{(t-s)\mathbb{A}} \frac{\Pi_{u^f}(a, x)}{\Pi_{u^f}(a-t+s, x)} \\ \chi(s+a-t, x)w^f(s, s+a-t, x)ds, & a \geq t, \\ e^{a\mathbb{A}}u^f(t-a, 0, x)\Pi_{u^f}(a, x) + \int_{t-a}^t e^{(t-s)\mathbb{A}} \frac{\Pi_{u^f}(a, x)}{\Pi_{u^f}(a-t+s, x)} \\ \chi(s+a-t, x)w^f(s, s+a-t, x)ds, & a < t, \end{cases} \quad (4.10)$$

and

$$u^m(t, a, x) = \begin{cases} e^{t\mathbb{A}}u_0^m(a-t, x) \frac{\Pi_{u^m}(a, x)}{\Pi_{u^m}(a-t, x)}, & a \geq t, \\ e^{a\mathbb{A}}u^m(t-a, 0, x)\Pi_{u^m}(a, x), & a < t, \end{cases} \quad (4.11)$$

with

$$\begin{cases} \Pi_{u^{e,l}}(a, x) & = e^{-\int_0^a (\mu^k(\tau, x) + \beta^k(\tau))d\tau}, & k = e, l, \\ \Pi_{u^{f,m}}(a, x) & = e^{-\int_0^a \mu^k(\tau, x)d\tau}, & k = f, m. \end{cases} \quad (4.12)$$

In the present work, we impose the following reasonable assumptions on the demographic parameters:

(A1) The hatching function β^e , the flying function β^l and the birth function β^f are bounded, nonnegative functions. There exist $a_0, a_1 \in (0, L^k)$ such that $\beta^f = 0$, *a.e.* $a \in (0, a_0) \cup (a_1, L^f)$.

(A2) The mortality functions $\mu^e(a, x)$, $\mu^l(a, x)$, $\mu^f(a, x)$ and $\mu^m(a, x)$ are nonnegative, locally bounded with respect to the first variable, and satisfy the following conditions:

$$\int_0^{L^k} \mu^k(a, x)da = \infty$$

with $k = e, l, f, m$.

(A3) $u_0 = (u_0^e, u_0^l, u_0^f, u_0^m)$ is nonnegative, a.e. $(a, x) \in ((0, A) \times \Omega)$.

Under the above assumptions, the existence and uniqueness of a solution of system (4.1) are guaranteed. One can refer to [11], [48] for the complete proof.

4.3 Exact null controllability

We shall prove Theorem 4.1. To illustrate the proof step by step, we divide this section to five parts as follows.

4.3.1 Weight functions and Carleman inequality

First, we introduce weight functions and the general Carleman inequality for the linear parabolic equations given in [51].

Let $\tilde{\omega} \subset\subset \omega$ be a nonempty bounded subset of Ω , and $\psi \in C^2(\bar{\Omega})$ satisfy

$$\begin{aligned} \psi(x) &> 0, & \text{for any } x \in \Omega, \\ \psi(x) &= 0, & \text{for any } x \in \partial\Omega, \\ |\nabla\psi(x)| &> 0, & \text{for any } x \in \bar{\Omega} \setminus \tilde{\omega}. \end{aligned}$$

Set

$$\alpha(t, x) = \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_{C(\bar{\Omega})}}}{t(T_0 - t)},$$

where λ is an appropriate positive constant and $T_0 \in (0, +\infty)$.

Lemma 4.3 (*Carleman inequality*) Denote $D_{T_0} = (0, T_0) \times \Omega$. There exist positive constants C_1, s_1 such that

$$\begin{aligned} &\frac{1}{s} \int_{D_{T_0}} t(T_0 - t) e^{2s\alpha} (|w_t|^2 + |\Delta w|^2) dx dt \\ &+ s \int_{D_{T_0}} (t(T_0 - t))^{-1} e^{2s\alpha} |\nabla w|^2 dx dt + s^3 \int_{D_{T_0}} (t(T_0 - t))^{-3} e^{2s\alpha} |w|^2 dx dt \\ &\leq C_1 \left[\int_{D_{T_0}} e^{2s\alpha} (|w_t + \Delta w|^2) dx dt + s^3 \int_{(0, T_0) \times \omega} (t(T_0 - t))^{-3} e^{2s\alpha} |w|^2 dx dt \right], \end{aligned} \quad (4.13)$$

for all $w \in C^2(\bar{D}_{T_0})$, $\frac{\partial w}{\partial \eta} = 0$ with $(t, x) \in (0, T_0) \times \partial\Omega$ and $s \geq s_1$.

Refer to [51] for the detailed proof.

4.3.2 An auxiliary optimal control problem

Now we choose a number $T_0 \in (0, \min\{a_0, a^*, A - a^*, T - A + a^*, A - a_1\})$. Define

$$K = L^\infty((0, A - a^* + T_0) \times \Omega).$$

Let $b^e \in K$ be arbitrary but fixed. For any $\varepsilon > 0$, we consider the following optimal control problem:

$$\begin{aligned} \text{Minimize} \quad & \left\{ \frac{1}{2} \int_G \int_\Omega (|w^e(t, a, x)|^2 + |w^l(t, a, x)|^2 + \varphi(t, a, x)|w^f(t, a, x)|^2) dx dt da \right. \\ & \left. + \frac{1}{\varepsilon} \int_{\Gamma_0} \int_\Omega |u^f(t, a, x)|^2 dx dl \right\}, \end{aligned} \quad (4.14)$$

in which

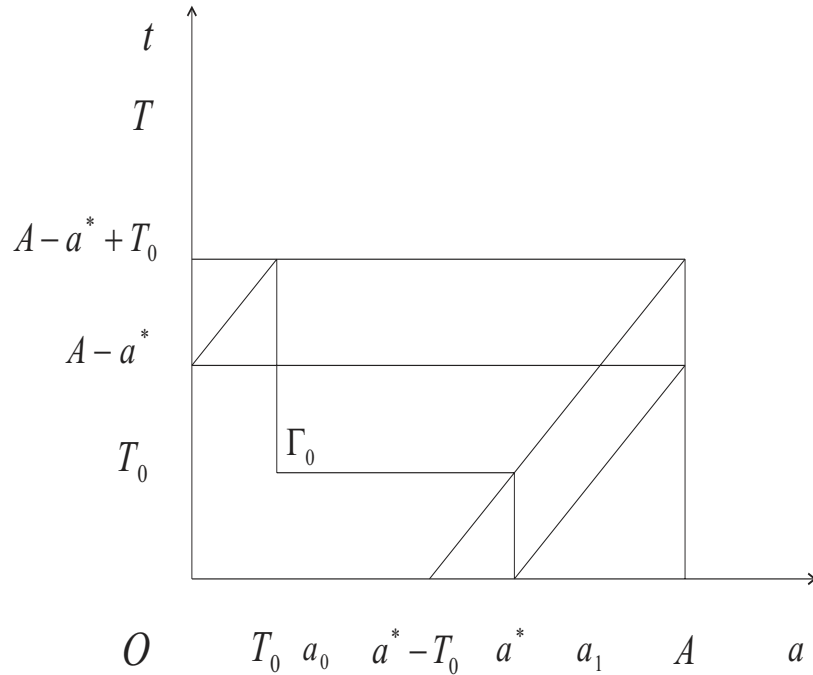
$$G = (0, T_0) \times (0, a^*) \cup (0, A - a^* + T_0) \times (0, T_0),$$

$$\Gamma_0 = (T_0, A - a^* + T_0) \times \{T_0\} \cup \{T_0\} \times (T_0, a^*),$$

and

$$\varphi(t, a, x) = \begin{cases} e^{-2s\alpha(t,x)} t^3 (T_0 - t)^3, & \text{if } a \geq t, (t, a) \in G, \\ e^{-2s\alpha(a,x)} a^3 (T_0 - a)^3, & \text{if } a < t, (t, a) \in G. \end{cases}$$

One can see the figure as follows:



It shows an example of domains G and Γ_0 for $T > A - a^*$.

Moreover, $w^k \in L^2(G)$ with $k = e, l, f$ and u^f is the solution of the system as follows

$$\begin{cases} \frac{\partial u^e(t,a,x)}{\partial t} + \frac{\partial u^e(t,a,x)}{\partial a} &= -(\mu^e(a,x) + \beta^e(a))u^e(t,a,x) + m(a)w^e(t,a,x), \\ \frac{\partial u^l(t,a,x)}{\partial t} + \frac{\partial u^l(t,a,x)}{\partial a} &= -(\mu^l(a,x) + \beta^l(a))u^l(t,a,x) + m(a)w^l(t,a,x), \\ \frac{\partial u^f(t,a,x)}{\partial t} + \frac{\partial u^f(t,a,x)}{\partial a} &= -\mu^f(a,x)u^f(t,a,x) + \Delta u^f(t,a,x) + \chi(a,x)w^l(t,a,x), \\ \frac{\partial u^m(t,a,x)}{\partial t} + \frac{\partial u^m(t,a,x)}{\partial a} &= -\mu^m(a,x)u^m(t,a,x) + \Delta u^m(t,a,x), \end{cases} \quad (4.15)$$

with $(t, a, x) \in G \times \Omega$, which corresponds to the following initial and boundary conditions

$$\begin{cases} u^e(t, 0, x) = b^e(t, x), & x \in \Omega, \\ u^l(t, 0, x) = \int_0^A \beta^e(s)u^e(t, s, x)ds, & x \in \Omega, \\ u^f(t, 0, x) = \int_0^A \sigma\beta^l(s)u^l(t, s, x)ds, & x \in \Omega, \\ u^m(t, 0, x) = \int_0^A (1 - \sigma)\beta^l(s)u^l(t, s, x)ds, & x \in \Omega, \\ u^k(0, a, x) = u_0^k(a, x), & x \in \Omega, \\ \frac{\partial u^f(t,a,x)}{\partial \eta} = 0, & x \in \partial\Omega, \\ \frac{\partial u^m(t,a,x)}{\partial \eta} = 0, & x \in \partial\Omega, \end{cases} \quad (4.16)$$

with $t \in (0, A - a^* + T_0)$, $a \in (0, a^*)$, $k = e, l, f, m$.

Here we study the system (4.15) first in a smaller domain G , with a local boundary condition $u^e(t, 0, x) = b^e(t, x)$, and then extend the result to the system (4.1) in the domain $(0, T) \times (0, A)$ with nonlocal boundary condition (4.2) by a fixed point theorem in the fourth subsection.

Denote the value of the cost function by $J_\varepsilon(w)$ with $w = (w^e, w^l, w^f)$. Since $J_\varepsilon : L^2(G \times \Omega) \times L^2(G \times \Omega) \times L^2(G \times \Omega) \rightarrow \mathbb{R}^+$ is convex, continuous and

$$\lim_{\|w^k\|_{L^2(G \times \Omega)} \rightarrow \infty} J_\varepsilon(w) = \infty, \quad k = e, l, f,$$

there is at least one minimum point for $J_\varepsilon(w)$. As a result, an optimal pair $(w_\varepsilon, u_\varepsilon^f)$ exists for (4.14).

We denote the Lagrange function by

$$\mathcal{L} = J_\varepsilon + I_1 + I_2 + I_3 + I_4,$$

in which

$$\begin{cases} I_1 = \int_G \int_\Omega q^e \left[\frac{\partial u^e}{\partial t} + \frac{\partial u^e}{\partial a} + (\mu^e + \beta^e)u^e - mw^e \right] dx dt da, \\ I_2 = \int_G \int_\Omega q^l \left[\frac{\partial u^l}{\partial t} + \frac{\partial u^l}{\partial a} + (\mu^l + \beta^l)u^l - mw^l \right] dx dt da, \\ I_3 = \int_G \int_\Omega q^f \left(\frac{\partial u^f}{\partial t} + \frac{\partial u^f}{\partial a} + \mu^f u^f - \Delta u^f - \chi w^f \right) dx dt da, \\ I_4 = \int_G \int_\Omega q^m \left(\frac{\partial u^m}{\partial t} + \frac{\partial u^m}{\partial a} + \mu^m u^m - \Delta u^m \right) dx dt da. \end{cases} \quad (4.17)$$

Here q^e, q^l, q^f, q^m are the adjoint variables with respect to u^e, u^l, u^f, u^m . By the condition of Karush-Kuhn-Tucker, the vector $S^* = (w, q)$ is an optimum of \mathcal{L} if and only if

$$\nabla \mathcal{L}(S^*) = 0.$$

Note that $q = (q^e, q^l, q^f, q^m), w = (w^e, w^l, w^f)$. After the calculation of the derivative of the Lagrange function with respect to dual variable q at the optimum S^* , then we get the evolution problem (4.15). Solving this equation by the method of characteristics, an explicit solution is obtained. The adjoint problem is obtained by passing the derivative of the Lagrangian at S^* with respect to variables u^k , where k equals to e, l, f, m . From integrations by parts for $I_j, j = 1, 2, 3, 4$ and standard arguments of the Lagrange function, we deduce the dual (backward) system of (4.15) and (4.16) as follows:

$$\left\{ \begin{array}{l} \frac{\partial q^e(t, a, x)}{\partial t} + \frac{\partial q^e(t, a, x)}{\partial a} = (\mu^e(a, x) + \beta^e(a))q^e(t, a, x) - \beta^e(a)q^l(t, 0, x), \\ \frac{\partial q^l(t, a, x)}{\partial t} + \frac{\partial q^l(t, a, x)}{\partial a} = (\mu^l(a, x) + \beta^l(a))q^l(t, a, x) - \sigma\beta^l(a)q^f(t, 0, x) \\ \quad - (1 - \sigma)\beta^l(a)q^m(t, 0, x), \\ \frac{\partial q^f(t, a, x)}{\partial t} + \frac{\partial q^f(t, a, x)}{\partial a} = \mu^f(a, x)q^f(t, a, x) - \Delta q^f(t, a, x), \\ \frac{\partial q^m(t, a, x)}{\partial t} + \frac{\partial q^m(t, a, x)}{\partial a} = \mu^m(a, x)q^m(t, a, x) - \Delta q^m(t, a, x), \end{array} \right. \quad (4.18)$$

with $(t, a, x) \in G \times \Omega$, as well as the boundary conditions

$$\left\{ \begin{array}{l} q^k(t, a, x) = 0, \quad (t, a, x) \in \Gamma \setminus \Gamma_0 \times \Omega, \quad k = e, l, f, m, \\ q^k(t, a, x) = 0, \quad (t, a, x) \in \Gamma_0 \times \Omega, \quad k = e, l, m, \\ q^f(t, a, x) = -\frac{u^f(t, a, x)}{\varepsilon}, \quad (t, a, x) \in \Gamma_0 \times \Omega, \\ \frac{\partial q^k(t, a, x)}{\partial \eta} = 0, \quad (t, a, x) \in \Gamma_0 \times \partial\Omega, \quad k = f, m, \end{array} \right. \quad (4.19)$$

with $\Gamma = \{A - a^* + T_0\} \times (0, T_0) \cup (0, T_0) \times a^* \cup \Gamma_0$. The existence and uniqueness of the dual system are easy to check by the method of characteristics. We denote by $u_\varepsilon^k, q_\varepsilon^k$ the solutions of (4.15) with (4.16), (4.18) with (4.19) respectively, for $k = e, l, f, m$. One can easily see that q^m is identically zero. Thus the second equation of (4.18) turns out to be

$$\frac{\partial q^l(t, a, x)}{\partial t} + \frac{\partial q^l(t, a, x)}{\partial a} = (\mu^l(a, x) + \beta^l(a))q^l(t, a, x) - \sigma\beta^l(a)q^f(t, 0, x).$$

Simultaneously, it is known that q_ε^k satisfies

$$w_\varepsilon^k(t, a, x) = m(a)q_\varepsilon^k(t, a, x) = q_\varepsilon^k(t, a, x), \quad (4.20)$$

a.e $(t, a, x) \in G \times \Omega$, $k = e, l$, and q_ε^f satisfies

$$w_\varepsilon^f(t, a, x) = \tilde{\chi}(x)q_\varepsilon^f(t, a, x)\varphi^{-1}(t, a, x), \quad (4.21)$$

a.e $(t, a, x) \in G \times \Omega$ and $\tilde{\chi}(x)$ is the characteristic function of ω .

4.3.3 Exact null controllability

Multiply the first equation of (4.18) by u_ε^e , and then integrate on $G \times \Omega$:

$$\int_G \int_\Omega u_\varepsilon^e \left[\frac{\partial q_\varepsilon^e}{\partial t} + \frac{\partial q_\varepsilon^e}{\partial a} - (\mu^e + \beta^e)q_\varepsilon^e + \beta^e q_\varepsilon^l(t, 0, x) \right] dx dt da = 0. \quad (4.22)$$

Using integrations by parts, then substituting (4.15) and (4.16), we can obtain

$$\begin{aligned} \int_G \int_\Omega |w_\varepsilon^e(t, a, x)|^2 dx dt da &= - \int_0^{A-a^*+T_0} \int_\Omega q_\varepsilon^e(t, 0, x) u_\varepsilon^e(t, 0, x) dx dt \\ &\quad - \int_0^{a^*} \int_\Omega q_\varepsilon^e(0, a, x) u_\varepsilon^e(0, a, x) dx da \\ &\quad + \int_0^{A-a^*+T_0} \int_\Omega q_\varepsilon^l(t, 0, x) u_\varepsilon^l(t, 0, x) dx dt. \end{aligned}$$

Similarly multiplying the remaining equations of (4.18) by u^l , u^f respectively and then integrating on G , we have

$$\begin{aligned} \int_G \int_\Omega |w_\varepsilon^l(t, a, x)|^2 dx dt da &= - \int_0^{A-a^*+T_0} \int_\Omega q_\varepsilon^l(t, 0, x) u_\varepsilon^l(t, 0, x) dx dt \\ &\quad - \int_0^{a^*} \int_\Omega q_\varepsilon^l(0, a, x) u_\varepsilon^l(0, a, x) dx da + \int_0^{A-a^*+T_0} \int_\Omega q_\varepsilon^f(t, 0, x) u_\varepsilon^f(t, 0, x) dx dt, \\ &\quad \int_G \int_\omega \varphi(a, t, x) |w_\varepsilon^f(t, a, x)|^2 dx dt da + \frac{1}{\varepsilon} \int_{\Gamma_0} \int_\Omega |u_\varepsilon^f(t, a, x)|^2 dx dl \\ &= - \int_0^{A-a^*+T_0} \int_\Omega q_\varepsilon^f(t, 0, x) u_\varepsilon^f(t, 0, x) dx dt \\ &\quad - \int_0^{a^*} \int_\Omega q_\varepsilon^f(0, a, x) u_\varepsilon^f(0, a, x) dx da. \end{aligned} \quad (4.23)$$

Let S be an arbitrary characteristic line of the system (4.18),

$$S = \{(\gamma + t, \theta + t); t \in (0, T_0), (\gamma, \theta) \in \{0\} \times (0, a^* - T_0) \cup (0, A - a^*) \times \{0\}\}.$$

Denote

$$\begin{aligned}
 \tilde{w}_\varepsilon^k(t, x) &= w_\varepsilon^k(\gamma + t, \theta + t, x), & (t, x) \in (0, T_0) \times \Omega, \quad k = e, l, f, \\
 \tilde{u}_\varepsilon^k(t, x) &= u_\varepsilon^k(\gamma + t, \theta + t, x), & (t, x) \in (0, T_0) \times \Omega, \quad k = e, l, f, \\
 \tilde{q}_\varepsilon^k(t, x) &= q_\varepsilon^k(\gamma + t, \theta + t, x), & (t, x) \in (0, T_0) \times \Omega, \quad k = e, l, f, \\
 \tilde{m}(t) &= m(\theta + t), & t \in (0, T_0), \\
 \tilde{\beta}^k(t) &= \beta^k(\theta + t), & t \in (0, T_0), \quad k = e, l, f, \\
 \tilde{\mu}^k(t, x) &= \mu^k(\theta + t, x), & (t, x) \in (0, T_0) \times \Omega, \quad k = e, l, f, \\
 \tilde{\chi}(t, x) &= \chi(\theta + t, x), & (t, x) \in (0, T_0) \times \omega.
 \end{aligned}$$

Note that $(\tilde{u}_\varepsilon^e, \tilde{w}_\varepsilon^e)$ satisfies

$$(\tilde{u}_\varepsilon^e)_t = -(\tilde{\mu}^e(t, x) + \tilde{\beta}^e(t))\tilde{u}_\varepsilon^e(t, x) + \tilde{m}(t)\tilde{w}_\varepsilon^e(t, x), \quad t \in (0, T_0), \quad x \in \Omega, \quad (4.24)$$

$$\tilde{u}_\varepsilon^e(0, x) = \begin{cases} b^e(\gamma, x), & \theta = 0, \quad x \in \Omega, \\ u_0^e(\theta, x), & \gamma = 0, \quad x \in \Omega; \end{cases} \quad (4.25)$$

\tilde{q}_ε^e satisfies

$$\begin{cases} (\tilde{q}_\varepsilon^e)_t = (\tilde{\mu}^e(t, x) + \tilde{\beta}^e(t))\tilde{q}_\varepsilon^e(t, x) - \tilde{\beta}^e(t)q_\varepsilon^e(\gamma, 0, x), & t \in (0, T_0), \quad x \in \Omega, \\ \tilde{q}_\varepsilon^e(T_0, x) = 0, & x \in \Omega, \end{cases} \quad (4.26)$$

and

$$\tilde{w}_\varepsilon^e(t, x) = \tilde{q}_\varepsilon^e(t, x), \quad a.e. \quad t \in (0, T_0), \quad x \in \Omega. \quad (4.27)$$

Similarly, \tilde{u}_ε^l and \tilde{q}_ε^l satisfy the following equations respectively

$$(\tilde{u}_\varepsilon^l)_t = -(\tilde{\mu}^l(t, x) + \tilde{\beta}^l(t))\tilde{u}_\varepsilon^l(t, x) + \tilde{m}(t)\tilde{w}_\varepsilon^l(t, x), \quad t \in (0, T_0), \quad x \in \Omega, \quad (4.28)$$

$$\tilde{u}_\varepsilon^l(0, x) = \begin{cases} \int_0^A \beta^e(a)u^e(t, a, x)da, & \theta = 0, \quad x \in \Omega, \\ u_0^l(\theta, x), & \gamma = 0, \quad x \in \Omega; \end{cases} \quad (4.29)$$

$$\begin{cases} (\tilde{q}_\varepsilon^l)_t = (\tilde{\mu}^l(t, x) + \tilde{\beta}^l(t))\tilde{q}_\varepsilon^l(t, x) - \sigma\tilde{\beta}^l(t)q_\varepsilon^f(\gamma, 0, x), & t \in (0, T_0), \quad x \in \Omega, \\ \tilde{q}_\varepsilon^l(T_0, x) = 0, & x \in \Omega, \end{cases} \quad (4.30)$$

and

$$\tilde{w}_\varepsilon^l(t, x) = \tilde{q}_\varepsilon^l(t, x), \quad a.e. \quad t \in (0, T_0), \quad x \in \Omega. \quad (4.31)$$

Further, \tilde{u}_ε^f and \tilde{q}_ε^f satisfy the following equations respectively

$$(\tilde{u}_\varepsilon^f)_t = -\tilde{\mu}^f(t, x)\tilde{u}_\varepsilon^f(t, x) + \Delta\tilde{u}_\varepsilon^f(t, x) + \tilde{\chi}(t, x)\tilde{w}_\varepsilon^f, \quad t \in (0, T_0), \quad x \in \Omega, \quad (4.32)$$

$$\tilde{u}_\varepsilon^f(0, x) = \begin{cases} \int_0^A \beta^l(a) u^l(t, a, x) da, & \theta = 0, \quad x \in \Omega, \\ u_0^f(\theta, x), & \gamma = 0, \quad x \in \Omega; \end{cases} \quad (4.33)$$

$$\begin{cases} (\tilde{q}_\varepsilon^f)_t = \tilde{\mu}^f(t, x) \tilde{q}_\varepsilon^f(t, x) - \Delta \tilde{q}_\varepsilon^f(t, x), & t \in (0, T_0), \quad x \in \Omega, \\ \tilde{q}_\varepsilon^f(T_0, x) = \frac{-\tilde{u}_\varepsilon^f(T_0, x)}{\varepsilon}, & x \in \Omega, \end{cases} \quad (4.34)$$

and

$$\tilde{w}_\varepsilon^f(t, x) = \tilde{\chi}(x) \tilde{q}_\varepsilon^f(t, x) \frac{e^{2s\alpha(t, x)}}{t^3(T_0 - t)^3}, \quad a.e. \quad t \in (0, T_0), \quad x \in \Omega. \quad (4.35)$$

Multiplying the first equation in (4.26), (4.30) and (4.34) by $\tilde{u}_\varepsilon^e(t, x)$, $\tilde{u}_\varepsilon^l(t, x)$, $\tilde{u}_\varepsilon^f(t, x)$ respectively, integrating in $(0, T_0) \times \Omega$, then using (4.24), (4.25), (4.28), (4.29), (4.32) and (4.33) we have

$$\begin{aligned} & \int_0^{T_0} \int_\Omega \tilde{q}_\varepsilon^e(t, x) \tilde{w}_\varepsilon^e(t, x) dx dt + \int_0^{T_0} \int_\Omega \tilde{q}_\varepsilon^l(t, x) \tilde{w}_\varepsilon^l(t, x) dx dt + \\ & \int_0^{T_0} \int_\omega \tilde{q}_\varepsilon^f(t, x) \tilde{w}_\varepsilon^f(t, x) dx dt + \frac{1}{\varepsilon} \int_\Omega |\tilde{u}_\varepsilon^f(T_0, x)|^2 dx \\ & \leq - \int_\Omega \tilde{q}_\varepsilon^e(0) \tilde{u}_\varepsilon^e(0, x) dx - \int_\Omega u_0^l(\theta, x) q_\varepsilon^l(0, \theta, x) dx - \int_\Omega u_0^f(\theta, x) q_\varepsilon^f(0, \theta, x) dx. \end{aligned}$$

By Young's inequality, we have

$$\begin{aligned} & \int_0^{T_0} \int_\Omega (|\tilde{w}_\varepsilon^e(t, x)|^2 + |\tilde{w}_\varepsilon^l(t, x)|^2) dx dt + \frac{1}{\varepsilon} \int_\Omega |\tilde{u}_\varepsilon^f(T_0, x)|^2 dx \\ & + \int_0^{T_0} \int_\omega \frac{t^3(T_0 - t)^3}{e^{2s\alpha(t, x)}} |\tilde{w}_\varepsilon^f(t, x)|^2 dx dt \\ & \leq \frac{\delta}{2} \int_\Omega |\tilde{q}_\varepsilon^e(0, x)|^2 dx + \frac{1}{2\delta} \int_\Omega |\tilde{u}_\varepsilon^e(0, x)|^2 dx + \frac{\delta}{2} \int_\Omega |q_\varepsilon^l(0, \theta, x)|^2 dx \\ & + \frac{\delta}{2} \int_\Omega |q_\varepsilon^f(0, \theta, x)|^2 dx + \frac{1}{2\delta} \int_\Omega |\tilde{u}_0^l(\theta, x)|^2 dx + \frac{1}{2\delta} \int_\Omega |\tilde{u}_0^f(\theta, x)|^2 dx, \quad (4.36) \end{aligned}$$

with δ being any positive number.

Using the constant variation formula for (4.26), we have

$$\tilde{q}_\varepsilon^e(t, x) = e^{\int_0^t (\tilde{\mu}^e(s, x) + \tilde{\beta}^e(s)) ds} \tilde{q}_\varepsilon^e(0, x) - \int_0^t e^{\int_0^\tau (\tilde{\mu}^e(s, x) + \tilde{\beta}^e(s)) ds} \tilde{\beta}^e(t - \tau) q_\varepsilon^l(\gamma, 0, x) d\tau. \quad (4.37)$$

Thus

$$\tilde{q}_\varepsilon^e(T_0, x) = e^{\int_0^{T_0} (\tilde{\mu}^e(s, x) + \tilde{\beta}^e(s)) ds} \tilde{q}_\varepsilon^e(0, x) - \int_0^{T_0} e^{\int_0^t (\tilde{\mu}^e(s, x) + \tilde{\beta}^e(s)) ds} \tilde{\beta}^e(T_0 - t) q_\varepsilon^l(\gamma, 0, x) dt.$$

The second equation in (4.26) gives

$$\begin{aligned} q_\varepsilon^l(\gamma, 0, x) & = \frac{e^{\int_0^{T_0} (\tilde{\mu}^e(s, x) + \tilde{\beta}^e(s)) ds}}{\int_0^{T_0} e^{\int_0^t (\tilde{\mu}^e(s, x) + \tilde{\beta}^e(s)) ds} \tilde{\beta}^e(T_0 - t) dt} \tilde{q}_\varepsilon^e(0, x) \\ & := C_1(T_0, x) \tilde{q}_\varepsilon^e(0, x). \end{aligned}$$

Substituting the above equality into (4.37), and denoting $e^{\int_0^t (\tilde{\mu}^e(s,x) + \tilde{\beta}^e(s)) ds}$ by $\tilde{\Pi}_e(t, x)$, we obtain

$$\begin{aligned} \tilde{q}_\varepsilon^e(t, x) &= \tilde{\Pi}_e(t, x) \tilde{q}_\varepsilon^e(0, x) - C_1(T_0, x) \tilde{q}_\varepsilon^e(0, x) \int_0^t \tilde{\Pi}_e(\tau, x) \tilde{\beta}^e(t - \tau) d\tau \\ &= \tilde{q}_\varepsilon^e(0, x) \{ \tilde{\Pi}_e(t, x) - C_1(T_0, x) \int_0^t \tilde{\Pi}_e(\tau, x) \tilde{\beta}^e(t - \tau) d\tau \}. \end{aligned}$$

In view of the positivity of μ^e , it can be concluded that

$$\int_\Omega |\tilde{q}_\varepsilon^e(0, x)|^2 dx \leq M_1(T_0) \int_0^{T_0} \int_\Omega |\tilde{w}_\varepsilon^e(t, x)|^2 dx dt, \quad (4.38)$$

where $M_1(T_0)$ satisfies $M_1(T_0) < \frac{2}{\delta}$.

By the same way, we get

$$\begin{aligned} q_\varepsilon^f(\gamma, 0, x) &= \frac{e^{\int_0^{T_0} (\tilde{\mu}^l(s,x) + \tilde{\beta}^l(s)) ds}}{\sigma \int_0^{T_0} e^{\int_0^t (\tilde{\mu}^l(s,x) + \tilde{\beta}^l(s)) ds} \tilde{\beta}^l(T_0 - t) dt} \tilde{q}_\varepsilon^l(0, x) \\ &:= C_2(T_0, x) \tilde{q}_\varepsilon^l(0, x). \end{aligned}$$

Thus it follows that

$$\begin{aligned} \tilde{q}_\varepsilon^l(t, x) &= \tilde{\Pi}_l(t, x) \tilde{q}_\varepsilon^l(0, x) - C_2(T_0, x) \tilde{q}_\varepsilon^l(0, x) \int_0^t \tilde{\Pi}_l(\tau, x) \tilde{\beta}^l(t - \tau) d\tau \\ &= \tilde{q}_\varepsilon^l(0, x) \{ \tilde{\Pi}_l(t, x) - C_2(T_0, x) \int_0^t \tilde{\Pi}_l(\tau, x) \tilde{\beta}^l(t - \tau) d\tau \} \end{aligned}$$

with denoting $e^{\int_0^t (\tilde{\mu}^l(s,x) + \tilde{\beta}^l(s)) ds}$ by $\tilde{\Pi}_l(t, x)$. It turns out that

$$\int_\Omega |\tilde{q}_\varepsilon^l(0, x)|^2 dx \leq M_2(T_0) \int_0^{T_0} \int_\Omega |\tilde{w}_\varepsilon^l(t, x)|^2 dx dt, \quad (4.39)$$

which means

$$\int_\Omega |q_\varepsilon^l(0, \theta, x)|^2 dx \leq M_2(T_0) \int_0^{T_0} \int_\Omega |\tilde{w}_\varepsilon^l(t, x)|^2 dx dt. \quad (4.40)$$

Here $M_2(T_0)$ satisfies $M_2(T_0) < \frac{2}{\delta}$.

Note that throughout this section we agree to denote several constants independent of all variables by the same C . Multiplying the equation in (4.34) by \tilde{q}_ε^f , and integrating in Ω , we obtain

$$\frac{1}{2} \int_\Omega \frac{d(\tilde{q}_\varepsilon^f)^2}{dt} = \int_\Omega \tilde{\mu}^f(t, x) (\tilde{q}_\varepsilon^f)^2 dx + \int_\Omega |\nabla \tilde{q}_\varepsilon^f|^2 dx \geq 0, \quad a.e. \quad t \in (0, T_0).$$

Integrating the last inequality we get

$$\int_\Omega |\tilde{q}_\varepsilon^f(0, x)|^2 dx \leq C \int_0^{T_0} \int_\Omega \frac{e^{2s\alpha(x,t)}}{t^3(T_0 - t)^3} |\tilde{q}_\varepsilon^f(t, x)|^2 dx dt.$$

Using the Carleman inequality (4.13), we obtain

$$\int_{\Omega} |\tilde{q}_{\varepsilon}^f(0, x)|^2 dx \leq C \int_0^{T_0} \int_{\omega} \frac{e^{2s\alpha(x,t)}}{t^3(T_0-t)^3} |\tilde{q}_{\varepsilon}^f(t, x)|^2 dx dt, \quad (4.41)$$

for $C\delta < 2$ and $s \geq \max(s_1, C\|\mu\|_{\mathcal{C}([0, a^*] \times \Omega)}^{\frac{2}{3}})$. One can see [4] for more details. Substituting (4.38), (4.39), (4.40) and (4.41) into (4.36), we have

$$\begin{aligned} & \int_0^{T_0} \int_{\Omega} (|\tilde{w}_{\varepsilon}^e(t, x)|^2 + |\tilde{w}_{\varepsilon}^l(t, x)|^2) dx dt + \frac{1}{\varepsilon} \int_{\Omega} |\tilde{u}_{\varepsilon}^f(T_0, x)|^2 dx + \\ & \int_0^{T_0} \int_{\omega} \frac{t^3(T_0-t)^3}{e^{2s\alpha(t,x)}} |\tilde{w}_{\varepsilon}^f(t, x)|^2 dx dt \\ & \leq C \left\{ \frac{1}{2\delta} \int_{\Omega} |\tilde{u}_{\varepsilon}^e(0, x)|^2 dx + \frac{1}{2\delta} \int_{\Omega} |\tilde{u}_0^l(\theta, x)|^2 dx + \frac{1}{2\delta} \int_{\Omega} |\tilde{u}_0^f(\theta, x)|^2 dx \right\} \\ & \leq \frac{C}{2\delta} \|\tilde{u}_{\varepsilon}^e(0, \cdot)\|_{L^2(\Omega)}^2 + M. \end{aligned}$$

It is obvious that

$$\begin{aligned} \|\tilde{w}_{\varepsilon}^e\|_{L^2((0, T_0) \times \Omega)}^2 &= \int_0^{T_0} \int_{\Omega} |\tilde{w}_{\varepsilon}^e(t, x)|^2 dx dt \\ &\leq C \|\tilde{u}_{\varepsilon}^e(0, \cdot)\|_{L^2(\Omega)}^2 + M \\ &:= M_1. \end{aligned} \quad (4.42)$$

Analogously, we obtain

$$\|\tilde{w}_{\varepsilon}^l\|_{L^2((0, T_0) \times \Omega)}^2 \leq M_1.$$

According to the property of relatively weak compactness in L^p , there exists a subsequence (also denoted by $\tilde{w}_{\varepsilon}^k, k = e, l$) such that

$$\tilde{w}_{\varepsilon}^k \rightharpoonup \tilde{w}^k, \quad k = e, l, \quad \text{weakly in } L^2((0, T_0) \times \Omega) \text{ as } \varepsilon \rightarrow 0.$$

In addition, we apply (4.9) to get

$$\begin{aligned}
 & \|\tilde{u}_\varepsilon^l\|_{L^2((0,T_0)\times\Omega)}^2 \\
 = & \int_\Omega \left\{ \int_0^{T_0} |u_\varepsilon^l(t, t + \theta, x)|^2 dt + \int_0^{T_0} |u_\varepsilon^l(t + \gamma, t, x)|^2 dt \right\} dx \\
 \leq & C \int_\Omega \left\{ \int_0^{T_0} |u_0^l(\theta, x) \frac{\Pi_{u_\varepsilon^l}(\theta + t, x)}{\Pi_{u_\varepsilon^l}(\theta, x)} + \int_0^t \frac{\Pi_{u_\varepsilon^l}(\theta + t, x)}{\Pi_{u_\varepsilon^l}(s + \theta, x)} m(s + \theta) w_\varepsilon^l(s, s + \theta, x) ds \right|^2 dt \\
 & + \int_0^{T_0} |u_\varepsilon^l(\gamma, 0, x) \Pi_{u_\varepsilon^l}(t, x) + \int_\gamma^{t+\gamma} \frac{\Pi_{u_\varepsilon^l}(t, x)}{\Pi_{u_\varepsilon^l}(s - \gamma, x)} m(s - \gamma) w_\varepsilon^l(s, s - \gamma, x) ds \right|^2 dt \Big\} dx \\
 \leq & C \left\{ \int_0^{T_0} \int_\Omega \int_0^A \frac{\zeta}{2} |\beta^e(s) u_\varepsilon^e(\gamma, s, x)|^2 ds dx dt + \frac{\zeta}{2} \|u_0^l\|_{L^\infty((0,A)\times\Omega)}^2 \right. \\
 & + \frac{1}{2\zeta} \int_0^{T_0} \int_\Omega \int_0^t |m(s + \theta) w_\varepsilon^l(s, s + \theta, x)|^2 ds dx dt \\
 & \left. + \frac{1}{2\zeta} \int_0^{T_0} \int_\Omega \int_0^{t+\gamma} |m(s - \gamma) w_\varepsilon^l(s, s - \gamma, x)|^2 ds dx dt \right\} \\
 \leq & C \{ C_1 \|\beta^e\|_{L^\infty(0,A)}^2 (\|b^e\|_K^2 + \|u_0^e\|_{L^\infty((0,A)\times\Omega)}^2 + M_1) + \|u_0^l\|_{L^\infty(0,A)}^2 + M_1 \} \\
 := & M_2,
 \end{aligned}$$

where C_1 is a constant independent of variables, ζ is a positive number. We can also obtain

$$\begin{aligned}
 \left\| \frac{d\tilde{u}_\varepsilon^l}{dt} \right\|_{L^2((0,T_0)\times\Omega)}^2 & \leq \int_0^{T_0} \int_\Omega \left(\frac{|(\tilde{\mu}^l(t, x) + \tilde{\beta}^l(t)) \tilde{u}_\varepsilon^l(t, x)|^2}{2\delta} + \frac{\delta |\tilde{m} \tilde{u}_\varepsilon^l(t, x)|^2}{2} \right) dx dt \\
 & \leq \frac{M_2 \|\tilde{\mu}^l + \tilde{\beta}^l\|_\infty^2}{2\delta} + \frac{\delta}{2} M_1.
 \end{aligned} \tag{4.43}$$

Therefore, there exists a subsequence (also denoted by $\tilde{u}_\varepsilon^l(t, x)$) such that

$$\tilde{u}_\varepsilon^l(t, x) \rightharpoonup \tilde{u}^l(t, x), \quad \text{weakly in } W^{1,2}((0, T_0) \times \Omega) \text{ as } \varepsilon \rightarrow 0.$$

In a similar way, we also show that there is a subsequence (also denoted by $\tilde{u}_\varepsilon^e(t, x)$)

$$\tilde{u}_\varepsilon^e(t, x) \rightharpoonup \tilde{u}^e(t, x), \quad \text{weakly in } W^{1,2}((0, T_0) \times \Omega) \text{ as } \varepsilon \rightarrow 0.$$

By the Carleman inequality (4.13), for any ε we have

$$\begin{aligned}
 & \int_0^{T_0} \int_\Omega e^{2s\alpha} \left[\frac{t(T_0 - t)}{s} (|\tilde{q}_\varepsilon^f|_t|^2 + |\Delta \tilde{q}_\varepsilon^f|^2) + \frac{s}{t(T_0 - t)} |\nabla \tilde{q}_\varepsilon^f|^2 + \frac{s^3}{t^3(T_0 - t)^3} |\tilde{q}_\varepsilon^f|^2 \right] dx dt \\
 \leq & C \int_0^{T_0} \int_\omega e^{2s\alpha} \frac{s^3}{t^3(T_0 - t)^3} |\tilde{q}_\varepsilon^f|^2 dx dt \\
 \leq & M_1.
 \end{aligned} \tag{4.44}$$

Consequently, for $(t, x) \in (0, T_0) \times \Omega$

$$\|e^{2s\alpha} \frac{s^3}{t^3(T_0 - t)^3} \tilde{q}_\varepsilon^f\|_{W_2^{1,2}((0, T_0) \times \Omega)}^2 \leq M_1.$$

Since

$$W_2^{1,2}((0, T_0) \times \Omega) \subset \begin{cases} L^\infty((0, T_0) \times \Omega), & N = 1, 2, \quad x \in \Omega \subset \mathbb{R}^N, \\ L^{10}((0, T_0) \times \Omega), & N = 3, \quad x \in \Omega \subset \mathbb{R}^N, \end{cases} \quad (4.45)$$

we obtain

$$\|\tilde{w}_\varepsilon^f\|_{L^{10}((0, T_0) \times \Omega)}^2 = \|\tilde{\chi} e^{2s\alpha} \frac{s^3}{t^3(T_0 - t)^3} \tilde{q}_\varepsilon^f\|_{L^{10}((0, T_0) \times \Omega)}^2 \leq M_1.$$

According to the property of relatively weak compactness in L^p , there exists a subsequence (also denoted by \tilde{w}_ε^f) such that

$$\tilde{w}_\varepsilon^f \rightharpoonup \tilde{w}^f, \quad \text{weakly in } L^{10}((0, T_0) \times \Omega) \text{ as } \varepsilon \rightarrow 0.$$

Next, we have

$$\begin{aligned} & \|\tilde{u}_\varepsilon^f\|_{L^2((0, T_0) \times \Omega)}^2 \\ &= \int_\Omega \left\{ \int_0^{T_0} |u_\varepsilon^f(t, t + \theta, x)|^2 dt + \int_0^{T_0} |u_\varepsilon^f(t + \gamma, t, x)|^2 dt \right\} dx \\ &\leq C \int_\Omega \left\{ \int_0^{T_0} \left| \int_0^t \frac{\Pi_{u_\varepsilon^f}(\theta + t, x)}{\Pi_{u_\varepsilon^f}(\theta + s, x)} e^{(t-s)\mathbb{A}} \chi(s + \theta, x) w_\varepsilon^f(s, s + \theta, x) ds \right. \right. \\ &\quad \left. \left. + u_0^f(\theta, x) \frac{\Pi_{u_\varepsilon^f}(\theta + t, x)}{\Pi_{u_\varepsilon^f}(\theta, x)} e^{t\mathbb{A}} \right|^2 dt + \int_0^{T_0} |u_\varepsilon^f(\gamma, 0, x) \Pi_{u_\varepsilon^f}(t, x) e^{t\mathbb{A}} \right. \\ &\quad \left. + \int_\gamma^{t+\gamma} \frac{\Pi_{u_\varepsilon^f}(t, x)}{\Pi_{u_\varepsilon^f}(s - \gamma, x)} e^{(t-s)\mathbb{A}} \chi(s - \gamma, x) w_\varepsilon^f(s, s - \gamma, x) ds \right|^2 dt \Big\} dx \\ &\leq C \left\{ \int_0^{T_0} \int_\Omega \int_0^A \frac{1}{2\zeta} |w_\varepsilon^f(s, s + \theta, x) ds|^2 ds dx dt + \frac{\zeta}{2} \|u_0^f\|_{L^\infty((0, a^* - T_0) \times \Omega)}^2 \right. \\ &\quad \left. + \frac{1}{2\zeta} \int_0^{T_0} \int_\Omega \int_0^{t+\gamma} |w_\varepsilon^f(s, s - \gamma, x)|^2 ds dx dt + \frac{\zeta}{2} \int_0^{T_0} \int_\Omega \int_0^{T_0} |u_\varepsilon^f(\gamma, 0, x)|^2 ds dx dt \right\} \\ &\leq C \left\{ \int_0^{T_0} \int_\Omega \int_0^A \frac{1}{2\zeta} |w_\varepsilon^f(s, s + \theta, x) ds|^2 ds dx dt + \frac{\zeta}{2} \|u_0^f\|_{L^\infty((0, a^* - T_0) \times \Omega)}^2 \right. \\ &\quad \left. + \frac{1}{2\zeta} \int_0^{T_0} \int_\Omega \int_0^A |w_\varepsilon^f(s, s - \gamma, x)|^2 ds dx dt \right. \\ &\quad \left. + \frac{\zeta}{2} \int_0^{T_0} \int_\Omega \int_0^{T_0} \left| \int_0^A \sigma \beta^l(a) u_\varepsilon^l(\gamma, a, x) da \right|^2 ds dx dt \right\} \\ &\leq C \{ C_1 \|\beta^l\|_{L^\infty(0, A)}^2 M_2 + \|u_0^f\|_{L^\infty(0, A)}^2 + M_1 \}, \end{aligned}$$

where $C_1 > 0$ is a constant independent of variables.

By Theorem 9.2.2 in [110], we also have

$$\begin{aligned} & \|(\tilde{u}_\varepsilon^f)_t\|_{L^2((0,T_0)\times\Omega)}^2 + \|\Delta\tilde{u}_\varepsilon^f\|_{L^2((0,T_0)\times\Omega)}^2 + \|\nabla\tilde{u}_\varepsilon^f\|_{L^2((0,T_0)\times\Omega)}^2 \\ & \leq C\|\tilde{\mu}^f\tilde{u}_\varepsilon^f\|_{L^2((0,T_0)\times\Omega)} + \|\tilde{w}_\varepsilon^f\|_{L^2((0,T_0)\times\Omega)}. \end{aligned}$$

Then there exists a subsequence (also denoted by $\tilde{u}_\varepsilon^f(t)$) such that

$$\tilde{u}_\varepsilon^f(t, x) \rightarrow \tilde{u}^f(t, x), \quad \text{weakly in } W_2^{1,2}((0, T_0) \times \Omega) \text{ as } \varepsilon \rightarrow 0.$$

Then $(\tilde{u}^e, \tilde{w}^e)$ and $(\tilde{u}^l, \tilde{w}^l)$ satisfy (4.24) and (4.28), and $(\tilde{u}^f, \tilde{w}^f)$ is a solution of (4.32) with $\tilde{u}^f(T_0, x) = 0$. One can see [25].

We extend w^k to \tilde{w}^k on each characteristic line by 0, $k = e, l, f$. Then it is known that $\tilde{w}^k \in L^2((0, T) \times (0, A) \times \Omega)$, $k = e, l, f$. Let u^k be the solution of (4.15) and (4.16) on $(0, A - a^* + T_0) \times (0, A) \times \Omega$ with $k = e, l, f$. Since $u^f = 0$ on $\Gamma_0 \times \Omega$ and $w^k = 0$ outside $G \times \Omega$ with $k = e, l, f$, it can be concluded that

$$u^f = 0, \quad \text{a.e. } (t, a, x) \in S = \{(t, a, x); T_0 < a < t + a^* - T_0, T_0 < t < A - a^* + T_0, x \in \Omega\},$$

$$u^f(A - a^* + T_0, a) = 0, \quad \text{a.e. } (a, x) \in (T_0, A) \times \Omega.$$

4.3.4 Existence of a fixed point for a multi-valued function

Next, we will prove the exact null controllability using a multi-valued fixed point theorem. For any $b^e \in K$, we denote by

$$\Phi(b^e) := \left\{ \int_0^A \beta^f(a) u^f(t, a, x) da \right\} \subset L^2((0, A - a^* + T_0) \times \Omega)$$

such that u^f satisfies

$$u^f = 0, \quad \text{a.e. } (t, a, x) \in S = \{(t, a, x); T_0 < t < A - a^* + T_0, T_0 < a < t + a^* - T_0, x \in \Omega\}, \quad (4.46)$$

with $u^f(A - a^* + T_0, a, x) = 0$ a.e. for $(a, x) \in (T_0, A) \times \Omega$.

We consider the following two cases:

(1) $t > T_0$

In view of $\beta^f = 0, a.e. a \in (0, a_0) \cup (a_1, L^f)$, and $T_0 \in (0, \min\{a_0, a^*, A - a^*, T - A + a^*, A - a_1\})$, one has

$$\begin{aligned}
 & \int_0^A \beta^f(a) u^f(t, a, x) da \\
 = & \int_0^{T_0} \beta^f(a) u^f(t, a, x) da + \int_{T_0}^{t+a^*-T_0} \beta^f(a) u^f(t, a, x) da \\
 & + \int_{t+a^*-T_0}^A \beta^f(a) u^f(t, a, x) da \\
 = & \int_{t+a^*-T_0}^{a_1} \beta^f(a) u^f(t, a, x) da + \int_{T_0}^{t+a^*-T_0} \beta^f(a) u^f(t, a, x) da.
 \end{aligned}$$

In addition, using condition (4.46), we have

$$\int_0^A \beta^f(a) u^f(t, a, x) da = \int_{t+a^*-T_0}^{a_1} \beta^f(a) u^f(t, a, x) da.$$

We have

$$\begin{aligned}
 u^f(t, a, x) &= \int_0^t e^{(t-s)\mathbb{A}} \frac{\Pi_{u^f}(a, x)}{\Pi_{u^f}(a-t+s, x)} \chi(s+a-t, x) w^f(s, s+a-t, x) ds \\
 &+ u_0^f(a-t, x) \frac{\Pi_{u^f}(a, x)}{\Pi_{u^f}(a-t, x)} e^{t\mathbb{A}} \\
 &= u_0^f(a-t, x) \frac{\Pi_{u^f}(a, x)}{\Pi_{u^f}(a-t, x)} e^{t\mathbb{A}},
 \end{aligned}$$

when $T_0 < t < a$ and $w^f = 0$ outside G . Thus $\int_0^A \beta^f(a) u^f(t, a, x) da$ is independent of b^e .

(2) $0 < t < T_0$

Using $\beta^f = 0, a.e. a \in (0, a_0) \cup (a_1, L^f)$ again, we get

$$\begin{aligned}
 & \int_0^A \beta^f(a) u^f(t, a, x) da \\
 = & \int_0^{T_0} \beta^f(a) u^f(t, a, x) da + \int_{T_0}^{A-T_0} \beta^f(a) u^f(t, a, x) da \\
 & + \int_{A-T_0}^A \beta^f(a) u^f(t, a, x) da \\
 = & \int_{T_0}^{A-T_0} \beta^f(a) u^f(t, a, x) da.
 \end{aligned}$$

Similarly, $\int_0^A \beta^f(a) u^f(t, a, x) da$ does not depend on b^e . Moreover,

$$\left| \int_{T_0}^{A-T_0} \beta^f(a) u^f(t, a, x) da \right| \leq C \|\beta^f\|_{L^\infty(0,A)} \|u^f\|_{L^\infty((0,A) \times \Omega)}, \quad (4.47)$$

a.e. (t, x) in $(0, A - a^* + T_0) \times \Omega$. Finally we show that Φ is a contraction, then there exists a fixed point.

It remains to choose a fixed point for the multi-valued function Φ . Since $u^f(A - a^* + T_0, a, x) = 0$, a.e. for $(a, x) \in (T_0, A) \times \Omega$, it follows that

$$\begin{aligned} & \int_0^A \beta^f(a) u^f(t, a, x) da \\ &= \int_0^{T_0} \beta^f(a) u^f(t, a, x) da + \int_{T_0}^{A-T_0} \beta^f(a) u^f(t, a, x) da \\ &+ \int_{A-T_0}^A \beta^f(a) u^f(t, a, x) da \\ &= 0, \end{aligned}$$

a.e. $(t, x) \in (A - a^*, A - a^* + T_0) \times \Omega$, thanks to $\beta^f = 0$, a.e. $a \in (0, a_0) \cup (a_1, L^f)$ and condition (4.46).

Therefore, for any w^e, w^l, w^f we can take

$$b^e(t, x) = \begin{cases} 0, & (t, x) \in (A - a^*, A - a^* + T_0) \times \Omega, \\ \int_0^A \beta^f(a) u^f(t, a, x) da, & (t, x) \in (0, A - a^*) \times \Omega, \end{cases}$$

as a fixed point of the multi-valued function Φ . It turns out that there exists $w^k \in L^2((0, A - a^* + T_0) \times (0, A) \times \Omega)$ with $w^k = 0$ on $(A - a^*, A - a^* + T_0) \times (a^*, A) \times \Omega$, $k = e, l, f$, such that u^f subject to (4.1) satisfies

$$u^f(A - a^* + T_0, a, x) = 0, \quad a.e. \quad (a, x) \in (T_0, A) \times \Omega.$$

Then we denote $\delta = T_0$ small enough, which is right because of the definition of T_0 . Letting $T = A - a^* + T_0$, it completes the first argument of Theorem 4.1.

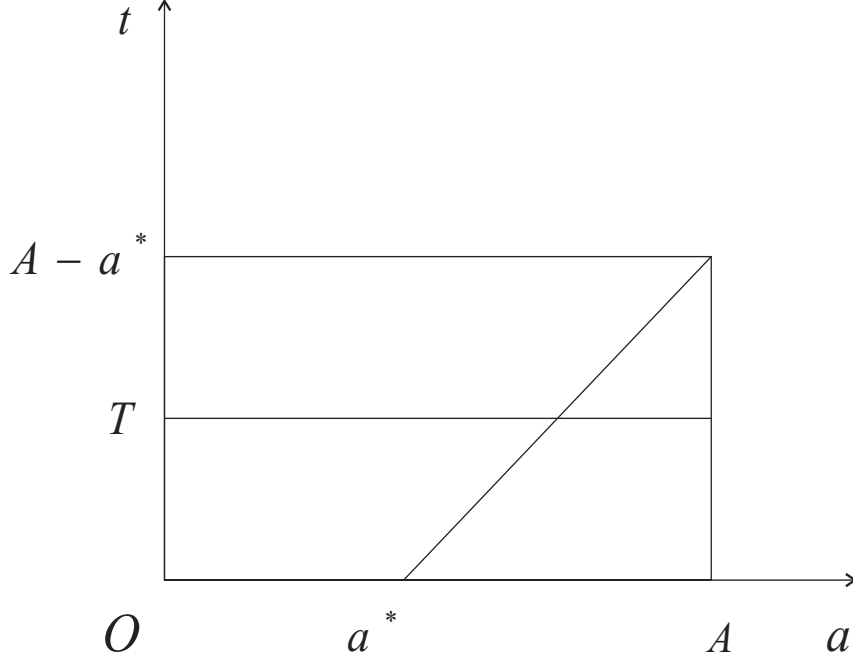
4.3.5 Nonexistence of controls

Now we consider the second condition if $T < A - a^*$, which implies $a^* < A$. Assume that $\|u_0\|_{L^\infty((a^*, A-T) \times \Omega)} > 0$, then there exist $w^k \in L^2((0, T) \times (0, A) \times \Omega)$, $k = e, l, f$, such that the solution $u^f(t, a, x)$ of (4.1) satisfies (4.7).

Since $mw^e = 0$, $mw^l = 0$ and $\chi w^f = 0$ for $(t, a, x) \in (0, T) \times (a^*, A) \times \Omega$, it is concluded that $u^f(t, a, x)$ is independent of $w^k \in U \times \Omega$ with $k = e, l, f$, where

$$U =: \{(t, a); 0 < t < a - a^*, a^* < a < A\}.$$

One can see the following figure, which shows an example of the domain U when $T < A - a^*$.



Furthermore, $u(t, a, x) = (u^e(t, a, x), u^l(t, a, x), u^f(t, a, x), u^m(t, a, x))$ satisfies the following system

$$\begin{cases} \frac{\partial u^e(t, a, x)}{\partial t} + \frac{\partial u^e(t, a, x)}{\partial a} = -(\mu^e(a, x) + \beta^e(a))u^e(t, a, x), \\ \frac{\partial u^l(t, a, x)}{\partial t} + \frac{\partial u^l(t, a, x)}{\partial a} = -(\mu^l(a, x) + \beta^l(a))u^l(t, a, x), \\ \frac{\partial u^f(t, a, x)}{\partial t} + \frac{\partial u^f(t, a, x)}{\partial a} = -\mu^f(a, x)u^f(t, a, x) + \Delta u^f(t, a, x), \\ \frac{\partial u^m(t, a, x)}{\partial t} + \frac{\partial u^m(t, a, x)}{\partial a} = -\mu^m(a, x)u^m(t, a, x) + \Delta u^m(t, a, x), \\ u^k(0, a, x) = u_0^k(a, x) \quad \text{for } k = e, l, f, m, \end{cases}$$

in which $(t, a, x) \in U \times \Omega$, with the boundary condition for the domain Ω

$$\begin{cases} \frac{\partial u^f(t, a, x)}{\partial \eta} = 0, & x \in \partial\Omega, \\ \frac{\partial u^m(t, a, x)}{\partial \eta} = 0, & x \in \partial\Omega. \end{cases}$$

Because of $\|u_0\|_{L^\infty(a^*, A-T)} > 0$, using the backward uniqueness argument, it leads to the conclusion $\|u^f(T, \cdot, \cdot)\|_{L^\infty((0, A) \times \Omega)} > 0$, which is a contradiction to (4.7). That completes the proof of Theorem 4.1.

Part II

Stability of conductivities in an inverse problem in the reaction-diffusion system in electrocardiology

Chapter 5

Introduction

Let $\Omega \in \mathbb{R}^N (N \geq 1)$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough. Let $T > 0$ and ω be a small nonempty subset of Ω . We will denote $(0, T) \times \Omega$ by Q_T and $(0, T) \times \partial\Omega$ by Σ_T .

To state the model of the cardiac electric activity in Ω ($\Omega \in \mathbb{R}^3$ being the natural domain of the heart), we set $u_i = u_i(t, x)$ and $u_e = u_e(t, x)$ to represent the spacial cellular and location $x \in \Omega$ of the intracellular and extracellular electric potentials respectively. Their difference $v = u_i - u_e$ is the transmembrane potential. The anisotropic properties of the two media are modeled by intracellular and extracellular conductivity tensors $M_i(x)$ and $M_e(x)$. The surface capacitance of the membrane is represented by the constant $c_m > 0$. The transmembrane ionic current is represented by a nonlinear function $h(v)$.

The equations governing the cardiac electric activity are given by the coupled reaction-diffusion system:

$$\begin{cases} c_m \partial_t v - \operatorname{div}(M_i(x) \nabla u_i) + h(v) = f \chi_\omega, & \text{in } Q_T, \\ c_m \partial_t v + \operatorname{div}(M_e(x) \nabla u_e) + h(v) = g \chi_\omega, & \text{in } Q_T, \end{cases} \quad (5.1)$$

where f and g are stimulation currents applied to Ω . We complete this model with Dirichlet boundary conditions for the intra- and extracellular electric potentials

$$u_i = 0, \quad u_e = 0, \quad \text{on } \Sigma_T, \quad (5.2)$$

and with initial data for the transmembrane potential

$$v(0, x) = v_0(x), \quad x \in \Omega. \quad (5.3)$$

It is important to point out that realistic models describing electrical activities include a system of ODE's for computing the ionic current as a function of the transmembrane potential and a series of additional "gating variables", which aim to model the ionic transfer across the cell membrane.

If $M_i = \mu M_e$ for some constant $\mu \in \mathbb{R}$, we approximate the above model by the family of parabolic systems

$$\begin{cases} c_m \partial_t v^\varepsilon - \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla v^\varepsilon) = -h(v^\varepsilon) + f^\varepsilon \chi_\omega, & \text{in } Q_T, \\ \varepsilon \partial_t u_e^\varepsilon - \operatorname{div}(M(x) \nabla u_e^\varepsilon) = \operatorname{div}(M_i(x) \nabla v^\varepsilon), & \text{in } Q_T, \\ v^\varepsilon(0, x) = v_0(x), \quad u_e^\varepsilon(0, x) = u_{e,0}(x), & \text{in } \Omega, \\ v^\varepsilon = 0, \quad u_e^\varepsilon = 0, & \text{on } \Sigma_T, \end{cases} \quad (5.4)$$

where $M = M_i + M_e$, ε is a fixed small constant. Note that when $\varepsilon \rightarrow 0$, we obtain the classical monodomain model.

In this work, we consider the following linearized system with semi-initial conditions

$$\begin{cases} c_m \partial_t v^\varepsilon - \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla v^\varepsilon) = -a(t, x) v^\varepsilon + f^\varepsilon \chi_\omega, & \text{in } Q_T, \\ \varepsilon \partial_t u_e^\varepsilon - \operatorname{div}(M(x) \nabla u_e^\varepsilon) = \operatorname{div}(M_i(x) \nabla v^\varepsilon), & \text{in } Q_T, \\ v^\varepsilon(\theta, x) = v_\theta(x), \quad u_e^\varepsilon(\theta, x) = u_{e,\theta}(x), & \text{in } \Omega, \\ v^\varepsilon = 0, \quad u_e^\varepsilon = 0, & \text{on } \Sigma_T, \end{cases} \quad (5.5)$$

where $a(t, x)$ is a bounded function in Q_T , and its derivative with respect to t exists, also bounded in Q_T . For some $\theta \in (0, T)$, the semi-initial conditions $v_\theta(x)$, $u_{e,\theta}(x)$ are sufficiently regular. The unknown conductivity tensors M and M_e are assumed to be sufficiently smooth and shall be kept independent of time t .

It should be mentioned that some works have been devoted to the theoretical and numerical study of the bidomain model (5.1), which is introduced to describe the cardiac electric activity. The existence of weak solutions of (5.1) is proved in [36] by the theory of evolution variational inequalities in Hilbert space. Then Bendahmane and Karlsen [22] proved the existence and uniqueness for a nonlinear version of the bidomain equations (5.1) by a uniformly parabolic regularization of the system and the Faedo-Galerkin method. Moreover, Bendahmane and Chaves-Silva [21] studied exact null controllability to (5.1) for each $\varepsilon > 0$ by establishing estimate for its dual system. To learn more about the cardiac problems, one can refer to the work of Bendahmane et al. [23, 24]. However, it is noted that there is no result of the stability study of conductivities for the bidomain model. Here we are concerned the stability

result of (5.5), which is of two measurements and associated results of the boundary measurements, would be interesting and novel.

To be the best of our knowledge, the inverse question of determining causes for desired or observed result is an interesting problem in many areas. It would stimulate mathematical research, such as on uniqueness questions and on developing stable and efficient numerical methods for solving inverse problem. There have been plenty of literatures which give fundamental study of inverse problems in various directions. The main contributions to this area are due to A.L. Bukhgeimand and M.V. Klibanov [27, 28, 29], who generalized the method of global Carleman estimates in the context of inverse problems. A brief review of inverse problems for partial differential equations about three fundamental issues, uniqueness, stability, and numerical methods by Carleman estimates [73]. Yamamoto et al. have done a lot of work on the inverse problem with respect to the stability result. One can see [64, 65, 74, 94, 106].

In 2004, Isakov describes some general and recent results on Carleman estimates of possible interest for mathematicians working on control theory or inverse problems for partial differential equations [68]. The paper by Cristofol et al. [37] obtains the stability results for reaction-diffusion system of two equations with constant coefficients using a Carleman estimate. Then Sakhivel et al. [98] established the stability results for Lotka-Volterra competition-diffusion system of three equations with variable diffusion coefficients. Inspired by that, we shall discuss the stability of inverse problem for (5.5).

Let $(\tilde{v}^\varepsilon, \tilde{u}_e^\varepsilon)$ be the other solutions of the system (5.5) with new conductivity tensors (\tilde{M}_e, \tilde{M}) given and new semi-initial data $(\tilde{v}_\theta^\varepsilon, \tilde{u}_{e,\theta}^\varepsilon)$. Then setting $A_1 = v^\varepsilon - \tilde{v}^\varepsilon$, $A_2 = u_e^\varepsilon - \tilde{u}_e^\varepsilon$, $g_1 = M_e - \tilde{M}_e$ and $g_2 = M - \tilde{M}$, we obtain

$$\begin{cases} c_m \partial_t A_1 - \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla A_1(t, x)) = -a(t, x) A_1(t, x) + F(g_1, \nabla \tilde{v}^\varepsilon), & \text{in } Q, \\ \varepsilon \partial_t A_2 - \operatorname{div}(M(x) \nabla A_2) = \operatorname{div}(M_i(x) \nabla A_1) + G(g_2, \nabla u_e^\varepsilon), & \text{in } Q, \\ A_1(\theta, x) = A_{1,\theta}(x), \quad A_2(\theta, x) = A_{2,\theta}(x), & \text{in } \Omega, \\ A_1(t, x) = 0, \quad A_2(t, x) = 0, & \text{on } \Sigma, \end{cases} \quad (5.6)$$

where

$$F = \operatorname{div}(g_1(x) \nabla \tilde{v}^\varepsilon)$$

and

$$G = \operatorname{div}(g_2(x) \nabla \tilde{u}_e^\varepsilon).$$

Throughout the paper, we make some assumptions:

Assumption 5.1 Suppose $M_e(x)$, $M_i(x)$ and $M(x)$ are C^∞ , bounded, symmetric, semi-definite, and elliptic matrix (there exists $\beta > 0$ such that $\sum_{i,j}^3 M_{i,j} \xi_i \xi_j \geq \beta |\xi|^2$ for all $\xi \in \mathbb{R}^3$). All their derivatives up to third order are respectively bounded by the positive constants $\gamma_1, \gamma_2, \gamma_3$.

Assumption 5.2 Assume the bounded measurements $\partial_t A_1$ and $\partial_t A_2$ in $(0, T) \times \omega$ are given. Also $A_i(\theta, x)$, $\nabla A_i(\theta, x)$, $\Delta A_i(\theta, x)$ and $\nabla(\Delta A_i(\theta, x))$ for some fixed $\theta \in (0, T)$, where $i = 1, 2$ in Ω are given.

Now the question of interest is whether we can determine the conductivity tensors M_e and M by the two measurements.

In details, let $(v^\varepsilon, u^\varepsilon)$ and $(\tilde{v}^\varepsilon, \tilde{u}^\varepsilon)$ be the solutions of the system (5.5) with two different conductivities. There exists a constant C with $C(\Omega, \omega, T, \gamma_1, \gamma_2, \gamma_3) > 0$, such that the following estimate holds:

$$\begin{aligned} & \int_{\Omega} (|M_e - \tilde{M}_e|^2 + |M - \tilde{M}|^2 + |\nabla(M_e - \tilde{M}_e)|^2 + |\nabla(M - \tilde{M})|^2) dx \\ & \leq C \left(\int_{Q_\omega} (|\partial_t A_1|^2 + |\partial_t A_2|^2) dt dx + \int_{\Omega} |A_1^\theta|^2 + \sum_{j=1}^2 (|\nabla A_j^\theta|^2 + |\Delta A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|^2) dx \right) \\ & \quad + C \int_{\tilde{\omega}} (|M_e - \tilde{M}_e|^2 + |M - \tilde{M}|^2 + |\nabla(M_e - \tilde{M}_e)|^2 + |\nabla(M - \tilde{M})|^2) dx. \end{aligned} \quad (5.7)$$

The stability result can be stated as two different cardiac electric activities, while the extracellular conductivity tensor M_e changes in the first equation, also the sum of intracellular conductivity tensor and extracellular conductivity tensor M changes. To be more precise, when the extracellular electric potential and the transmembrane potential vary small enough in two situations, even on a small domain ω during the period of time $(0, T)$, and in the whole space Ω at time θ , correspondingly, the extracellular conductivity tensor M_e is close to \tilde{M}_e , and also M is close to \tilde{M} , except on a small subdomain ω_0 belong to ω . This will be based on the manuscript [62].

Chapter 6

A Carleman type estimate

In this section, we prove the Carleman estimate based on the standard technique for general parabolic equations. In order to frame a Carleman type estimate, we shall first introduce a particular type of weight functions and in fact, the choice of the function ψ is the key for the derivation of Carleman estimate.

6.1 Weight functions

First, we introduce weight functions for the parabolic equations given in [51].

Let $\tilde{\omega} \subset\subset \omega$ be a nonempty bounded set of Ω , and $\psi \in C^2(\bar{\Omega})$ satisfies

$$\begin{aligned}\psi(x) &> 0, & \text{for any } x \in \Omega, \\ \psi(x) &= 0, & \text{for any } x \in \partial\Omega, \\ |\nabla\psi(x)| &> 0, & \text{for any } x \in \bar{\Omega} \setminus \tilde{\omega}.\end{aligned}$$

Then we introduce another two weight functions:

$$\phi(t, x) = \frac{e^{\lambda\psi(x)}}{\beta(t)}, \quad (6.1)$$

$$\alpha(t, x) = \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_{C(\bar{\Omega})}}}{\beta(t)}, \quad (6.2)$$

where $\lambda > 1$, $t \in (0, T)$ and $\beta(t) = t(T-t)$. Note that the weight function α is positive, and blows up to ∞ as $t = 0$ or $t = T$. As a result, $e^{-2s\alpha}$ and $\phi e^{-2s\alpha}$ are smooth. Even

they vanish when $t = 0$ or $t = T$. It can be seen that $\phi(t, x) \geq C > 0$ for all $(t, x) \in Q$, and $e^{-\epsilon\alpha}\phi^m \leq C < \infty$ for all $\epsilon > 0$ and $m \in \mathbb{R}$.

Before proving the main estimate, we give the following estimates for the two weight functions α and ϕ . Note that throughout the paper we will denote C as a generic positive constant. After some computations, we can obtain such estimates as follows:

$$\begin{cases} |\phi_t| = \frac{|2t-T|}{e^{\lambda\psi}}\phi^2 \leq CT\phi^2, \\ |\alpha_t| = \frac{|2t-T|}{\beta^2}(e^{2\lambda\|\psi\|_{C(\bar{\Omega})}} - e^{\lambda\psi}) \leq CT\phi^2, \\ |\alpha_{tt}| = \frac{2|T^2-3tT+3t^2|}{\beta^3}(e^{2\lambda\|\psi\|_{C(\bar{\Omega})}} - e^{\lambda\psi}) \leq CT\phi^3. \end{cases} \quad (6.3)$$

Furthermore, we also have

$$\begin{cases} \nabla\phi = \lambda\phi\nabla\psi, \\ \nabla\alpha = -\lambda\phi\nabla\psi, \\ \phi^{-1} \leq (\frac{T}{2})^2. \end{cases} \quad (6.4)$$

Refer to [51] for the details.

6.2 Main proof of a Carleman type estimate

Let us set $Q_\omega = (0, T) \times \omega$. For each positive integer m , we denote the Sobolev space of functions in $L^p(\Omega)$ whose weak derivatives of order less than or equal to m are also in $L^p(\Omega)$ with the norm denoted $\|\cdot\|_{L^p(\Omega)}$, by $W^{m,p}(\Omega)$ with $p > 1$ or $p = \infty$. When $p = 2$, we denote $W^{m,p}$ by $H^m(\Omega)$. Moreover, let $L^2(0, T; H^1(\Omega))$ be the space of all equivalent classes of square integrable functions from $(0, T)$ to $H^1(\Omega)$. For the space $L^2(0, T; L^\infty(\Omega))$, we define it in the same way.

Let A_1 be the solution of the first equation of (5.6) with help of using Assumption 5.1. We apply the Carleman estimate (see Theorem 6.1 in [21].) derived for the parabolic equations to the first equation in (5.6). For $\lambda > \lambda_0 \geq 1$, $s \leq s_0(T + T^2 + T^4)$, there exists a constant C depending on Ω, ω, ψ and β so that

$$\mathcal{I}(A_1) \leq C \left(\int_Q e^{-2s\alpha} |F|^2 dt dx + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx \right), \quad (6.5)$$

where $\tilde{\omega} \subset\subset \omega_1 \subset\subset \omega$, and

$$\begin{aligned} \mathcal{I}(A_1) &= \int_Q (s\lambda\phi)^{-1} e^{-2s\alpha} (|\partial_t A_1|^2 + |\Delta A_1|^2) dt dx + \int_Q s\lambda^2 \phi e^{-2s\alpha} |\nabla A_1|^2 dt dx \\ &\quad + s^3 \lambda^4 \int_Q e^{-2s\alpha} \phi^3 |A_1|^2 dt dx. \end{aligned} \quad (6.6)$$

Similarly, we obtain for $\lambda > \lambda_0 \geq 1$, $s \geq s_0(T + T^2 + T^4)$, there exists a constant C depending on Ω, ω, ψ and β satisfying

$$\begin{aligned} \mathcal{I}(A_2) &\leq C \left(\int_Q e^{-2s\alpha} (|G|^2 |\nabla(M_i \nabla A_1)|^2) dt dx + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx \right) \\ &\leq C \left(\int_Q e^{-2s\alpha} |G|^2 dt dx + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx + \int_Q e^{-2s\alpha} |M_i \Delta A_1|^2 dt dx \right) \\ &\quad + C \int_Q e^{-2s\alpha} |\nabla M_i \nabla A_1|^2 dt dx, \end{aligned} \quad (6.7)$$

with

$$\begin{aligned} \mathcal{I}(A_2) &= \int_Q (s\lambda\phi)^{-1} e^{-2s\alpha} (|\partial_t A_2| + |\Delta A_2|^2) dt dx + \int_Q s\lambda^2 \phi e^{-2s\alpha} |\nabla A_2|^2 dt dx \\ &\quad + s^3 \lambda^4 \int_Q e^{-2s\alpha} \phi^3 |A_2|^2 dt dx. \end{aligned} \quad (6.8)$$

Now coupling the above inequalities (6.5) and (6.7), we have

$$\begin{aligned} \mathcal{I}(A_1) + \mathcal{I}(A_2) &\leq C \left(\int_Q e^{-2s\alpha} (|F|^2 + |G|^2) dt dx + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx \right) \\ &\quad + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_2|^2 dt dx + C \int_Q e^{-2s\alpha} |M_i \Delta A_1|^2 dt dx \\ &\quad + C \int_Q e^{-2s\alpha} |\nabla M_i \nabla A_1|^2 dt dx \end{aligned}$$

for sufficiently large $s \geq s_0(T + T^2 + T^4)$ and $\lambda \geq \lambda_0$. From the definition of \mathcal{I}_1 , also M_i and ∇M_i being bounded, we obtain

$$\begin{aligned} \mathcal{I}(A_1) + \mathcal{I}(A_2) &\leq \tilde{C} \left(\int_Q e^{-2s\alpha} (|F|^2 + |G|^2) dt dx + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx \right) \\ &\quad + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_2|^2 dt dx. \end{aligned} \quad (6.9)$$

Then it can be summarized as our desired Carleman estimate as follows.

Theorem 6.1 *Let $\psi(x)$, $\phi(t, x)$ and $\alpha(t, x)$ be defined as in the above subsection, $a(t, x)$ is a bounded function. Moreover, Assumption 5.1 holds. Then there exist λ_0 and s_0 such that for all $\lambda > \lambda_0 \geq 1$ and sufficiently large enough $s > s_0$, the following inequality is true.*

$$\begin{aligned} \mathcal{I}(A_1) + \mathcal{I}(A_2) &\leq \tilde{C} \left(\int_Q e^{-2s\alpha} (|F|^2 + |G|^2) dt dx + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx \right) \\ &\quad + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |A_2|^2 dt dx, \end{aligned}$$

where $\tilde{C} > 0$ is a constant depending on $\Omega, T, \omega, \gamma_2$.

Assumption 7.1 Suppose v_θ^ε and $u_{e,\theta}^\varepsilon$ are C^3 real valued functions. Then all their derivatives up to order three are bounded and also they satisfy $|\nabla\psi \cdot \nabla v_\theta^\varepsilon| \geq \delta > 0$, $|\nabla\psi \cdot \nabla u_{e,\theta}^\varepsilon| \geq \delta > 0$, on $\overline{\Omega \setminus \tilde{\omega}}$, where $\tilde{\omega} \subset\subset \omega \subset\subset \Omega$.

Assumption 7.2 Suppose $|\Delta\tilde{v}^\varepsilon|$, $|\nabla(\Delta\tilde{v}^\varepsilon)|$, $|\nabla(\partial_t\tilde{v}^\varepsilon)|$ and $|\Delta(\partial_t\tilde{v}^\varepsilon)|$ are bounded by a positive constant. For \tilde{u}_e^ε , we also have the assumption of the same form.

Before start proving our main conclusion, we need to give some lemmas first which will be useful in the following part. We define the following operators P_0 and Q_0 :

$$P_0h = \nabla U_\theta \cdot \nabla h, \quad Q_0(e^{-s\alpha^\theta} h) = e^{-s\alpha^\theta} \nabla P_0h.$$

Lemma 7.3 Consider the first order partial differential operator $P_0h = \nabla U_\theta \cdot \nabla h$, where U_θ satisfies Assumption 7.1. Then there exists a constant $C > 0$, such that for sufficiently large enough λ and s , the following result holds:

$$s^2\lambda^2 \int_\Omega \phi^\theta e^{-2s\alpha^\theta} |h|^2 dx \leq C \left(\int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |P_0h|^2 dx + s^2\lambda^2 \int_{\tilde{\omega}} \phi^\theta e^{-2s\alpha^\theta} |h|^2 dx \right),$$

with $\theta \in (0, T)$ and $h \in H_0^1(\Omega)$. Here, we define $\varsigma(\theta, x) = \varsigma^\theta$, where ς represents α and ϕ respectively.

Proof. Let $B_1 = e^{-s\alpha^\theta} h$, we have

$$Q_0B_1 = e^{-s\alpha^\theta} P_0(e^{s\alpha^\theta} B_1) = P_0B_1 + sB_1P_0\alpha^\theta, \tag{7.3}$$

$h \in H_0^1(\Omega)$. Then we take the square of both sides in (7.3), multiply $\frac{1}{\phi^\theta}$ and integrate

by parts with respect to space variable for both sides of (7.3) as follows:

$$\begin{aligned}
\int_{\Omega} \frac{1}{\phi^{\theta}} (Q_0 B_1)^2 dx &= \int_{\Omega} \frac{1}{\phi^{\theta}} (P_0 B_1)^2 dx + \int_{\Omega} s^2 B_1^2 \frac{1}{\phi^{\theta}} (P_0 \alpha^{\theta})^2 dx \\
&\quad + \int_{\Omega} 2s \frac{1}{\phi^{\theta}} B_1 (P_0 B_1) (P_0 \alpha^{\theta}) dx \\
&= \int_{\Omega} \frac{1}{\phi^{\theta}} (P_0 B_1)^2 dx + \int_{\Omega} s^2 B_1^2 \frac{1}{\phi^{\theta}} \lambda^2 (\phi^{\theta})^2 (\nabla U^{\theta} \cdot \nabla \psi^{\theta})^2 dx \\
&\quad - \int_{\Omega} 2\lambda s \frac{1}{\phi^{\theta}} B_1 (\nabla U^{\theta} \cdot \nabla B_1) (\nabla U^{\theta} \cdot \phi^{\theta} \nabla \psi^{\theta}) dx \\
&= \int_{\Omega} \frac{1}{\phi^{\theta}} (P_0 B_1)^2 dx + \int_{\Omega} s^2 B_1^2 \lambda^2 \phi^{\theta} (\nabla U^{\theta} \cdot \nabla \psi^{\theta})^2 dx \\
&\quad - \int_{\Omega} 2\lambda s B_1 (\nabla U^{\theta} \cdot \nabla B_1) (\nabla U^{\theta} \cdot \nabla \psi^{\theta}) dx \\
&= \int_{\Omega} \frac{1}{\phi^{\theta}} (P_0 B_1)^2 dx + \int_{\Omega} s^2 B_1^2 \lambda^2 \phi^{\theta} (\nabla U^{\theta} \cdot \nabla \psi^{\theta})^2 dx \\
&\quad - \int_{\Omega} \lambda s P_0 \psi^{\theta} \nabla U^{\theta} \nabla (B_1^2) dx \\
&= \int_{\Omega} \frac{1}{\phi^{\theta}} (P_0 B_1)^2 dx + \int_{\Omega} s^2 B_1^2 \lambda^2 \phi^{\theta} (\nabla U^{\theta} \cdot \nabla \psi^{\theta})^2 dx \\
&\quad + \int_{\Omega} \lambda s \nabla (P_0 \psi^{\theta} \nabla U^{\theta}) B_1^2 dx \\
&\geq \int_{\Omega} s^2 \lambda^2 B_1^2 \phi^{\theta} |\nabla U^{\theta} \cdot \nabla \psi^{\theta}|^2 dx + \int_{\Omega} s \lambda \nabla (P_0 \psi^{\theta} \nabla U^{\theta}) |B_1|^2 dx \\
&\geq s^2 \lambda^2 \delta^2 \left(\int_{\Omega} |B_1|^2 \phi^{\theta} dx - \int_{\tilde{\omega}} |B_1|^2 \phi^{\theta} dx \right) + \int_{\Omega} s \lambda \nabla (P_0 \psi^{\theta} \nabla U^{\theta}) |B_1|^2 dx,
\end{aligned}$$

where we use Assumption 7.1 in the last step. Thus we obtain

$$s^2 \lambda^2 \delta^2 \left(\int_{\Omega} |B_1|^2 \phi^{\theta} dx - \int_{\tilde{\omega}} |B_1|^2 \phi^{\theta} dx \right) \leq \int_{\Omega} \frac{1}{\phi^{\theta}} |Q_0 B_1|^2 dx + \int_{\Omega} s \lambda |\nabla (P_0 \psi^{\theta} \nabla U^{\theta})| |B_1|^2 dx.$$

From Assumption 7.2 and (6.4), we have

$$\begin{aligned}
s^2 \lambda^2 \delta^2 \int_{\Omega} e^{-2s\alpha^{\theta}} \phi^{\theta} |h|^2 dx &\leq \int_{\tilde{\omega}} s^2 \lambda^2 \delta^2 e^{-2s\alpha^{\theta}} \phi^{\theta} |h|^2 dx + C_1 T^2 \int_{\Omega} s \lambda e^{-2s\alpha^{\theta}} \phi^{\theta} |h|^2 dx \\
&\quad + \int_{\Omega} e^{-2s\alpha^{\theta}} |P_0 h|^2 \frac{1}{\phi^{\theta}} dx. \tag{7.4}
\end{aligned}$$

Taking $\lambda \geq 1$ and $s \geq \frac{2C_1 T^2}{\delta^2}$, we conclude the proof. \square

With the help of the Lemma 7.3, we are proving the following proposition.

Proposition 7.4 *Let (A_1, A_2) be the solution of (5.6), and (B_1, B_2) be the solution of (7.1). Suppose all the conditions of Theorem 6.1 and Assumption 7.1 hold. Then there*

exists a constant $C = C(\gamma_1, \gamma_2, \delta) > 0$ such that for sufficiently large enough s and λ the following estimate is true.

$$\begin{aligned} & s^2 \lambda^2 \int_{\Omega} e^{-2s\alpha^\theta} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx \\ & \leq C \sum_{j=1}^9 E_j + C s^2 \lambda^2 \int_{\tilde{\omega}} e^{-2s\alpha^\theta} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx, \end{aligned} \quad (7.5)$$

for any $g_1, g_2 \in H_0^2(\Omega)$, where the functions E_j , are given as follows:

$$\begin{aligned} E_1 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |B_1^\theta|^2 dx, & E_2 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |B_2^\theta|^2 dx, \\ E_3 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |\nabla B_1^\theta|^2 dx, & E_4 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |\nabla B_2^\theta|^2 dx, \\ E_5 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|A_1^\theta|^2 + |\nabla A_1^\theta|^2 + |\Delta A_1^\theta|^2) dx, \\ E_6 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|\nabla A_2^\theta|^2 + |\Delta A_2^\theta|^2) dx, \\ E_7 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|A_1^\theta|^2 + \sum_{j=1}^2 (|\nabla A_j^\theta|^2 + |\Delta A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|^2) \\ & \quad + |\nabla(g_1 \Delta \tilde{v}_\theta^\varepsilon)|^2 + |\nabla(g_2 \Delta \tilde{u}_{e,\theta}^\varepsilon)|^2) dx, \\ E_8 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |g_1 \Delta \tilde{v}_\theta^\varepsilon|^2 dx, & E_9 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |g_2 \Delta \tilde{u}_{e,\theta}^\varepsilon|^2 dx. \end{aligned}$$

Proof. Due to the value of the solutions satisfying the first equation in (7.1) at $t = \theta$, and $F = \operatorname{div}(g_1(x) \nabla \tilde{v}^\varepsilon)$, from (7.2) we obtain

$$P_0 g_1 = \nabla \tilde{v}_\theta^\varepsilon \cdot \nabla g_1 = c_m B_1^\theta + a(\theta, x) A_1^\theta - \frac{\mu}{\mu + 1} \operatorname{div}(M_e \nabla A_1^\theta) - g_1 \Delta \tilde{v}_\theta^\varepsilon.$$

Note that we replace h by g_1 when choosing U_θ as $\tilde{v}_\theta^\varepsilon$. Therefore, inspired of Lemma 7.3, we get

$$\begin{aligned} s^2 \lambda^2 \int_{\Omega} \phi^\theta e^{-2s\alpha^\theta} |g_1|^2 dx & \leq C \left(\int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |P_0 g_1|^2 dx + s^2 \lambda^2 \int_{\tilde{\omega}} \phi^\theta e^{-2s\alpha^\theta} |g_1|^2 dx \right) \\ & \leq C \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|B_1^\theta|^2 + |A_1^\theta|^2 + \left(\frac{\mu}{\mu + 1}\right)^2 (|\nabla M_e|^2 |\nabla A_1^\theta|^2 \\ & \quad + |M_e|^2 |\Delta A_1^\theta|^2) + |g_1 \Delta \tilde{v}_\theta^\varepsilon|^2) dx + C s^2 \lambda^2 \int_{\tilde{\omega}} \phi^\theta e^{-2s\alpha^\theta} |g_1|^2 dx \\ & \leq C(\gamma_1)(E_1 + E_5 + E_8) + C s^2 \lambda^2 \int_{\tilde{\omega}} \phi^\theta e^{-2s\alpha^\theta} |g_1|^2 dx. \end{aligned} \quad (7.6)$$

Similarly, from the value of the solutions satisfying the second equation in (7.1) at $t = \theta$, and $G = \operatorname{div}(g_2(x)\nabla\tilde{u}_e^\varepsilon)$, we obtain

$$P_0g_2 = \nabla\tilde{u}_{e,\theta}^\varepsilon \cdot \nabla g_2 = \varepsilon B_2^\theta - \operatorname{div}(M\nabla A_2^\theta)\operatorname{div}(M_i\nabla A_1^\theta) - g_2\Delta\tilde{u}_{e,\theta}^\varepsilon.$$

It leads to

$$\begin{aligned} & s^2\lambda^2 \int_{\Omega} \phi^\theta e^{-2s\alpha^\theta} |g_2|^2 dx \\ \leq & C \left(\int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |P_0g_2|^2 dx + s^2\lambda^2 \int_{\tilde{\omega}} \phi^\theta e^{-2s\alpha^\theta} |g_2|^2 dx \right) \\ \leq & C \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|B_2^\theta|^2 + |\nabla M|^2 |\nabla A_2^\theta|^2 + |M|^2 |\Delta A_2^\theta|^2 \\ & + |\nabla M_i|^2 |\nabla A_1^\theta|^2 + |M_i|^2 |\Delta A_1^\theta|^2 + |g_2\Delta\tilde{u}_{e,\theta}^\varepsilon|^2) dx \\ & + Cs^2\lambda^2 \int_{\tilde{\omega}} \phi^\theta e^{-2s\alpha^\theta} |g_2|^2 dx \\ \leq & C(\gamma_2, \gamma_3)(E_2 + E_5 + E_6 + E_9) + Cs^2\lambda^2 \int_{\tilde{\omega}} \phi^\theta e^{-2s\alpha^\theta} |g_2|^2 dx. \end{aligned} \tag{7.7}$$

On the other hand, from the expression of P_0g_1 , we can see that easily,

$$\begin{aligned} P_0\nabla g_1 &= \nabla\tilde{v}_\theta^\varepsilon \cdot \nabla(\nabla g_1) = \nabla(\nabla\tilde{v}_\theta^\varepsilon \cdot \nabla g_1) - \nabla g_1\Delta\tilde{v}_\theta^\varepsilon \\ &= c_m\nabla B_1^\theta + \nabla a^\theta A_1^\theta + a^\theta\nabla A_1^\theta - \frac{\mu}{\mu+1}\Delta(M_e\nabla A_1^\theta) - \nabla(g_1\Delta\tilde{v}_\theta^\varepsilon) - \nabla g_1\Delta\tilde{v}_\theta^\varepsilon. \end{aligned}$$

Similarly, we also have

$$\begin{aligned} P_0\nabla g_2 &= \nabla\tilde{u}_{e,\theta}^\varepsilon \cdot \nabla(\nabla g_2) = \nabla(\nabla\tilde{u}_{e,\theta}^\varepsilon \cdot \nabla g_2) - \nabla g_2\Delta\tilde{u}_{e,\theta}^\varepsilon \\ &= \varepsilon\nabla B_2^\theta - \Delta(M\nabla A_2^\theta) - \Delta(M_i\nabla A_1^\theta) - \nabla(g_2\Delta\tilde{u}_{e,\theta}^\varepsilon) - \nabla g_2\Delta\tilde{u}_{e,\theta}^\varepsilon. \end{aligned}$$

Using the similar method to preceding estimates and Lemma 7.3, it follows that

$$\begin{aligned} & s^2\lambda^2 \int_{\Omega} \phi^\theta e^{-2s\alpha^\theta} |\nabla g_1|^2 dx + s^2\lambda^2 \int_{\Omega} \phi^\theta e^{-2s\alpha^\theta} |\nabla g_2|^2 dx \\ \leq & C(\gamma_1, \gamma_2, \gamma_3)(E_3 + E_4 + E_7) + Cs^2\lambda^2 \int_{\tilde{\omega}} \phi^\theta e^{-2s\alpha^\theta} (|\nabla g_1|^2 + |\nabla g_2|^2) dx. \end{aligned} \tag{7.8}$$

Combing the above three estimates (7.6), (7.7) and (7.8), the proof is complete. \square

In order to prove the main conclusion, we need to get further estimations for E_j , $j = 1, 2, 3, 4$. The Carleman estimate in the previous section plays an important role in obtaining these estimations.

Lemma 7.5 *Assume all the conditions in Theorem 6.1 are satisfied. Then there exists a constant C depending only on \tilde{C} , such that for any $\lambda \geq \lambda_0$ and $s \geq s_1(\Omega, T)$, the following inequality holds:*

$$E_1 + E_2 \leq C\mathcal{E}(g_1, g_2, B_1, B_2), \quad (7.9)$$

where $\mathcal{E}(g_1, g_2, B_1, B_2)$ is defined as follows

$$\mathcal{E}(g_1, g_2, B_1, B_2) = \int_Q e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} (|B_1|^2 + |B_2|^2) dt dx. \quad (7.10)$$

Proof. Note that $\alpha(0, x) = +\infty$. As a result of (6.3) and (6.4), we have

$$\begin{aligned} & \int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|B_1^\theta|^2 + |B_2^\theta|^2) dx \\ &= \int_0^\theta \frac{\partial}{\partial t} \left(\int_\Omega \phi^{-1} e^{-2s\alpha^\theta} (|B_1(t, x)|^2 + |B_2(t, x)|^2) dx \right) dt \\ &= \int_{Q_\theta} \phi^{-1} (-2s) \partial_t e^{-2s\alpha^\theta} (|B_1|^2 + |B_2|^2) dt dx - \int_{Q_\theta} \phi^{-2} \partial_t \phi e^{-2s\alpha^\theta} (|B_1|^2 + |B_2|^2) dt dx \\ & \quad + 2 \int_{Q_\theta} \phi^{-1} e^{-2s\alpha^\theta} (2B_1 \partial_t B_1 + 2B_2 \partial_t B_2) dt dx \\ &\leq C(sT^5 + s\lambda T^8 + T^7) \int_Q \phi^3 e^{-2s\alpha^\theta} (|B_1(t, x)|^2 + |B_2(t, x)|^2) dt dx \\ & \quad + (s\lambda)^{-1} \int_Q \phi^{-1} e^{-2s\alpha^\theta} (|\partial_t B_1|^2 + |\partial_t B_2|^2) dt dx \\ &\leq C(\mathcal{I}(B_1) + \mathcal{I}(B_2)), \end{aligned}$$

where $Q_\theta = (0, \theta) \times \Omega$, $\mathcal{I}(B_j)|_{j=1,2}$ is defined in (6.6) and (6.8), for any $s \geq C(T^{\frac{5}{2}} + T^{\frac{7}{3}} + T^4)$ and $\lambda \geq 1$. Then using the estimate given in 6.1 to the system (7.1), we obtain

$$\begin{aligned} \mathcal{I}(B_1) + \mathcal{I}(B_2) &\leq \tilde{C} \left(\int_Q e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |B_1|^2 dt dx \right. \\ & \quad \left. + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |B_2|^2 dt dx + \int_Q e^{-2s\alpha} |\partial_t a(t, x)| \left(\int_0^t |B_1(s, x)| ds \right. \right. \\ & \quad \left. \left. + |A_1(0, x)| \right)^2 dt dx \right) \\ &\leq \tilde{C} \left(\int_Q e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |B_1|^2 dt dx \right. \\ & \quad \left. + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |B_2|^2 dt dx + C_1 T \int_Q |B_1|^2 dt dx \right). \end{aligned} \quad (7.11)$$

Due to this term $C_1 T \int_Q |B_1|^2 dt dx$ can be absorbed by $\mathcal{I}(B_1)$, we have

$$\mathcal{I}(B_1) + \mathcal{I}(B_2) \leq \tilde{C}_1 \mathcal{E}(g_1, g_2, B_1, B_2),$$

for any $s \geq s_0$ and $\lambda \geq \lambda_0$. Thus for $s \geq s_1 = \max\{s_0, C(T^{\frac{5}{2}} + T^{\frac{7}{3}} + T^4)\}$, The proof is complete. \square

Lemma 7.6 *Let Assumption 7.1 be satisfied. Then there exists $\lambda_1 = \max\{\lambda_0, C(\gamma_1, \gamma_2, \gamma_3)\}$ and $s_2 = \max\{s_1, C(\gamma_1, \gamma_2, \gamma_3)(T + T^2 + T^4)\}$ for all $\lambda \geq \lambda_1$, $s \geq s_2$, the following inequality holds:*

$$E_3 + E_4 \leq C s \lambda^2 \mathcal{E}(g_1, g_2, B_1, B_2),$$

which is defined in (7.10).

Proof. We define

$$\pi(B_1) := e^{-2s\alpha} \phi^{-1} \nabla(M_e \nabla B_1).$$

Multiplying the first equation in (7.1) by it, integrate over Q_θ to get

$$\int_{Q_\theta} c_m \partial_t B_1 \pi(B_1) = \int_{Q_\theta} \pi(B_1) \left(\frac{\mu}{\mu+1} \operatorname{div}(M_e \nabla B_1) - \partial_t a A_1 - a B_1 + F'(g_1, \nabla \tilde{v}^\varepsilon) \right) dt dx. \quad (7.12)$$

We divide the equality into left side and right side integral to estimate separately. Firstly, we integrate the left side integral by parts, and get

$$\begin{aligned} & - \int_{Q_\theta} c_m \partial_t B_1 \pi(B_1) dt dx \\ &= - \int_{Q_\theta} c_m \partial_t B_1 e^{-2s\alpha} \phi^{-1} \nabla(M_e \nabla B_1) dt dx \\ &= \int_{Q_\theta} c_m \partial_t B_1 \nabla(e^{-2s\alpha} \phi^{-1}) M_e \nabla B_1 dt dx + \frac{1}{2} \int_{Q_\theta} c_m \partial_t (|\nabla B_1|^2) e^{-2s\alpha} \phi^{-1} M_e dt dx \\ &= \mathcal{J}_1 + \mathcal{J}_2. \end{aligned} \quad (7.13)$$

Note that $|\nabla(\phi^{-1} e^{-2s\alpha})| \leq s \lambda e^{-2s\alpha}$ for $s \geq CT^2$. Thus we have

$$\begin{aligned} \mathcal{J}_1 &\leq s \lambda (C \|M_e\|_{L^\infty(\Omega)}^2 s \lambda \int_{Q_\theta} \phi e^{-2s\alpha} |\nabla B_1|^2 dt dx + (s \lambda)^{-1} \int_{Q_\theta} \phi^{-1} e^{-2s\alpha} |\partial_t B_1|^2 dt dx) \\ &\leq s \lambda \mathcal{I}(B_1) \end{aligned} \quad (7.14)$$

for any $s \geq C(\gamma_2) T^2$. Integrating by parts with respect to time in \mathcal{J}_2 , we have

$$\begin{aligned} \mathcal{J}_2 &= \frac{1}{2} \int_{Q_\theta} c_m \partial_t (|\nabla B_1|^2) e^{-2s\alpha} \phi^{-1} M_e dt dx \\ &= -\frac{1}{2} \int_{Q_\theta} c_m |\nabla B_1|^2 \partial_t (e^{-2s\alpha} \phi^{-1}) M_e dt dx \\ &\quad + \frac{1}{2} \int_{\Omega} (c_m |\nabla B_1|^2 e^{-2s\alpha} \phi^{-1} M_e) |_{t=\theta} dx. \end{aligned} \quad (7.15)$$

Here,

$$\begin{aligned}
 |\partial_t(e^{-2s\alpha}\phi^{-1})| &= |e^{-2s\alpha}\phi^{-2}\phi_t + e^{-2s\alpha}\phi^{-1}(-2s)\alpha_t| \\
 &= |e^{-2s\alpha}\phi^{-1}(\phi^{-1}\phi_t - 2s\alpha_t)| \\
 &\leq |e^{-2s\alpha}\phi^{-1}| \left(\frac{T^2}{4} + 2s \right) CT\phi^2 \\
 &\leq Cs\lambda^2\phi e^{-2s\alpha},
 \end{aligned}$$

for $\lambda > 1$ and $s \geq CT^3$. Therefore,

$$\mathcal{J}_2 \geq \frac{1}{2} \int_{\Omega} (c_m e^{-2s\alpha} \phi^{-1} M_e |\nabla B_1|^2)|_{t=\theta} dx - Cs\lambda^2 \int_{Q_\theta} c_m \phi e^{-2s\alpha} M_e |\nabla B_1|^2 dt dx. \quad (7.16)$$

Now coming to the right side integrals of (7.12), we have

$$\begin{aligned}
 &\int_{Q_\theta} \pi(B_1) \left(\frac{\mu}{\mu+1} \operatorname{div}(M_e \nabla B_1) - \partial_t a A_1 - a B_1 + F' \right) dt dx \\
 &= \int_{Q_\theta} \pi(B_1) F' dt dx + \int_{Q_\theta} \pi(B_1) \frac{\mu}{\mu+1} \operatorname{div}(M_e \nabla B_1) dt dx \\
 &\quad - \int_{Q_\theta} \pi(B_1) \partial_t a \left(\int_0^t B_1(s, x) ds + A_1(0, x) \right) dt dx - \int_{Q_\theta} \pi(B_1) a B_1 dt dx \\
 &= \sum_{j=1}^4 \mathcal{K}_j. \tag{7.17}
 \end{aligned}$$

Then we estimate the above integrals one by one. Applying the Cauchy inequality, we get the following estimates for $\mathcal{K}_{j=1,2}$.

$$\begin{aligned}
 \mathcal{K}_1 &= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M_e \nabla B_1) F' dt dx \\
 &= \int_{Q_\theta} e^{-s\alpha} \phi^{-\frac{1}{2}} F' (e^{-s\alpha} \phi^{-\frac{1}{2}} \nabla M_e \nabla B_1 + e^{-s\alpha} \phi^{-\frac{1}{2}} M_e \Delta B_1) dt dx \\
 &\leq \int_{Q_\theta} CT^2 e^{-2s\alpha} |F'|^2 dt dx + \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M_e|^2 |\Delta B_1|^2 dt dx \\
 &\quad + \int_{Q_\theta} CT^4 \phi e^{-2s\alpha} |\nabla M_e|^2 |\nabla B_1|^2 dt dx \\
 &\leq s\lambda^2 (\mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha} |F'|^2 dt dx), \tag{7.18}
 \end{aligned}$$

and

$$\begin{aligned}
\mathcal{K}_2 &= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M_e \nabla B_1) \frac{\mu}{\mu+1} \operatorname{div}(M_e \nabla B_1) dt dx \\
&= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \frac{\mu}{\mu+1} |\nabla(M_e \nabla B_1)|^2 dt dx \\
&= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \frac{\mu}{\mu+1} |\nabla M_e \nabla B_1 + M_e \Delta B_1|^2 dt dx \\
&\leq \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \frac{\mu}{\mu+1} (2|\nabla M_e|^2 |\nabla B_1|^2 + 2|M_e|^2 |\Delta B_1|^2) dt dx \\
&\leq C \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M_e|^2 |\Delta B_1|^2 dt dx + CT^4 \int_{Q_\theta} e^{-2s\alpha} \phi |\nabla M_e|^2 |\nabla B_1|^2 dt dx \\
&\leq s\lambda^2 \mathcal{I}(B_1),
\end{aligned} \tag{7.19}$$

where $\lambda \geq 1$ and $s \geq C(\gamma_1)T^4$. Next, we estimate the integral \mathcal{K}_3 , and obtain

$$\begin{aligned}
\mathcal{K}_3 &= - \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M_e \nabla B_1) \partial_t a(t, x) \left(\int_0^t B_1(s, x) ds + A_1(0, x) \right) dt dx \\
&\leq C \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M_e \Delta B_1 + \nabla M_e \nabla B_1|^2 dt dx + CT^8 \int_{Q_\theta} e^{-2s\alpha} \phi^3 \int_0^t |B_1(s, x)|^2 ds dt dx \\
&\leq C \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M_e|^2 |\Delta B_1|^2 dt dx + CT^4 \int_{Q_\theta} e^{-2s\alpha} \phi |\nabla M_e|^2 |\nabla B_1|^2 dt dx \\
&\quad + CT^8 \int_{Q_\theta} e^{-2s\alpha} \phi^3 \int_0^t |B_1(s, x)|^2 ds dt dx \\
&\leq s\lambda^2 \mathcal{I}(B_1).
\end{aligned} \tag{7.20}$$

Similarly, we have

$$\mathcal{K}_4 = - \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} a(t, x) B_1 \nabla(M_e \nabla B_1 + M_e \Delta B_1) dt dx \leq s\lambda^2 \mathcal{I}(B_1). \tag{7.21}$$

Using the assumptions on the conductivity M_e and substituting the inequalities (7.18)-(7.21) into (7.12), we get

$$- \mathcal{J}_1 - \mathcal{J}_2 \leq \sum_{j=1}^4 \mathcal{K}_j \leq s\lambda^2 (\mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha} |F'|^2 dt dx),$$

which means

$$- \mathcal{J}_2 \leq \sum_{j=1}^4 \mathcal{K}_j + \mathcal{J}_1 \leq s\lambda^2 (\mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha} |F'|^2 dt dx) + s\lambda \mathcal{I}(B_1).$$

Substituting (7.16) to the above inequality, we have

$$\begin{aligned} \left| \int_{\Omega} (c_m e^{-2s\alpha} \phi^{-1} M_e |\nabla B_1|^2) |_{t=\theta} dx \right| &\leq |\mathcal{J}_1| + \sum_{j=1}^4 \mathcal{K}_j \\ &\leq s\lambda^2(\mathcal{I}(B_1)) + \int_{Q_\theta} e^{-2s\alpha} |F'|^2 dt dx + s\lambda \mathcal{I}(B_1), \end{aligned}$$

which leads to

$$\left| \int_{\Omega} e^{-2s\alpha\theta} (\phi^\theta)^{-1} |\nabla B_1^\theta|^2 dx \right| \leq s\lambda^2(\mathcal{I}(B_1)) + \int_{Q_\theta} e^{-2s\alpha} |F'|^2 dt dx. \quad (7.22)$$

Next we multiply the second equation of (7.1) by $\xi(B_2) := e^{-2s\alpha} \phi^{-1} \nabla(M \nabla B_2)$, and integrate over Q_θ to get

$$\begin{aligned} &\int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M \nabla B_2) \varepsilon \partial_t B_2 dt dx \\ &= \int_{Q_\theta} \xi(B_2) \nabla(M \nabla B_2) dt dx + \int_{Q_\theta} \xi(B_2) \nabla(M_i \nabla B_1) dt dx + \int_{Q_\theta} \xi(B_2) G' dt dx \\ &= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |\nabla(M \nabla B_2)|^2 dt dx + \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M \nabla B_2) \nabla(M_i \nabla B_1) dt dx \\ &\quad + \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M \nabla B_2) G' dt dx. \end{aligned}$$

We estimate

$$\begin{aligned} &\int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M \nabla B_2) \nabla(M_i \nabla B_1) dt dx \\ &= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} (\nabla M \nabla B_2 + M \Delta B_2) \nabla(M_i \nabla B_1) dt dx \\ &\leq \frac{1}{2} \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |\nabla M \nabla B_2 + M \Delta B_2|^2 dt dx + \frac{1}{2} \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |\nabla M_i \nabla B_1 + M_i \Delta B_1|^2 dt dx \\ &\leq CT^4 \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |\nabla M|^2 |\nabla B_2|^2 dt dx + C \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M|^2 |\Delta B_2|^2 dt dx \\ &\quad + CT^4 \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |\nabla M_i|^2 |\nabla B_1|^2 dt dx + C \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M_i|^2 |\Delta B_1|^2 dt dx \\ &\leq s\lambda^2 \mathcal{I}(B_1) + s\lambda^2 \mathcal{I}(B_2). \end{aligned} \quad (7.23)$$

Continuing the similar computation as the preceding estimates and using Assumption

5.1, we obtain

$$\begin{aligned}
& \left| \int_{\Omega} e^{-2s\alpha^\theta} (\phi^\theta)^{-1} |\nabla B_2^\theta|^2 dx \right| \\
& \leq s\lambda^2 (\mathcal{I}(B_1) + \mathcal{I}(B_2)) + \int_{Q_\theta} e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx \\
& \leq Cs\lambda^2 \left(\int_{Q_\theta} e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} (|B_1|^2 + |B_2|^2) dt dx \right),
\end{aligned} \tag{7.24}$$

for any $s \geq C(\gamma_1, \gamma_2, \gamma_3)(T + T^2 + T^4)$ and $\lambda \geq C(\gamma_1, \gamma_2, \gamma_3)$. Thus combining the estimates (7.22) and (7.24), we obtain the conclusion. \square

Now we shall give the main result of the stability estimate of the conductivities in (5.5) based on the preceding lemmas and proposition.

Theorem 7.7 *Let (A_1, A_2) be the solution of (5.6). Suppose all the assumptions of Theorem 6.1 hold and $g_1, g_2 \in H_0^2(\Omega)$. In addition, suppose Assumption 7.1 and 7.2 are also satisfied. Then there exists a constant C with $C(\Omega, \omega, T, \gamma_1, \gamma_2, \gamma_3) > 0$, such that for sufficiently large $\lambda \geq \lambda_0 \geq 1$ and $s \geq s_4$, the following estimate holds:*

$$\begin{aligned}
& \int_{\Omega} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx \\
& \leq C \left(\int_{Q_\omega} (|\partial_t A_1|^2 + |\partial_t A_2|^2) dt dx + \int_{\Omega} |A_1^\theta|^2 + \sum_{j=1}^2 (|\nabla A_j^\theta|^2 + |\Delta A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|^2) dx \right) \\
& \quad + C \int_{\tilde{\omega}} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx.
\end{aligned} \tag{7.25}$$

Proof. Substituting the results in Lemma 7.5 and Lemma 7.6 into the inequality in

Proposition 7.4, one obtains

$$\begin{aligned}
 & s^2 \lambda^2 \int_{\Omega} \phi^\theta e^{-2s\alpha^\theta} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx \\
 \leq & C s \lambda^2 \mathcal{E}(g_1, g_2, B_1, B_2) + C \sum_{j=5}^9 E_j(\theta) \\
 \leq & C s \lambda^2 \int_Q e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx + C s^4 \lambda^6 \int_{Q_\omega} \phi^3 e^{-2s\alpha} (|B_1|^2 + |B_2|^2) dt dx \\
 & + C \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|\nabla A_1^\theta|^2 + |\Delta A_1^\theta|^2 + |A_1^\theta|^2) dx + C \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|\nabla A_2^\theta|^2 + |\Delta A_2^\theta|^2) dx \\
 & + C \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|g_1 \Delta \tilde{v}_\theta^\varepsilon|^2 + |g_2 \Delta \tilde{u}_{e,\theta}^\varepsilon|^2) dx + C \int_{\tilde{\omega}} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx \\
 & + C \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|A_1^\theta|^2 + \sum_{j=1}^2 (|\Delta A_j^\theta|^2 + |\nabla A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|) + |\nabla(g_1 \Delta \tilde{v}_\theta^\varepsilon)|^2 \\
 & + |\nabla(g_2 \Delta \tilde{u}_{e,\theta}^\varepsilon)|^2) dx \\
 \leq & C s \lambda^2 \int_Q e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx + C s^4 \lambda^6 \int_{Q_\omega} \phi^3 e^{-2s\alpha} (|B_1|^2 + |B_2|^2) dt dx \\
 & + C \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|A_1^\theta|^2 + \sum_{j=1}^2 (|\Delta A_j^\theta|^2 + |\nabla A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|)) dx \\
 & + C \int_{\tilde{\omega}} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx,
 \end{aligned}$$

for large enough $s \geq s_3 = \max\{CT^2, s_2\}$ and $\lambda \geq \lambda_1$. Now for convenience, we set $R_1(t, x) = \nabla \tilde{v}^\varepsilon(t, x)$ and $R_2(t, x) = \nabla \tilde{u}_e^\varepsilon(t, x)$. Then from the regularity of the solutions $(\tilde{v}^\varepsilon(t, x), \tilde{u}_e^\varepsilon(t, x))$, we deduce that there exist $l_j \in L^2(0, T)$, $j = 1, 2, 3, 4$,

$$|\partial_t R_j(t, x)| \leq l_j(t) |R_j^\theta|, \quad j = 1, 2,$$

$$|\partial_t \nabla R_1(t, x)| \leq l_3(t) |\nabla R_1^\theta|,$$

$$|\partial_t \nabla R_2(t, x)| \leq l_3(t) |\nabla R_2^\theta|,$$

for any $(t, x) \in Q$, and the functions $l_j \in L^2(0, T)$, implying $\int_0^T |l_j|^2 dt \leq N < \infty$, $j = 1, 2, 3, 4$. Then we show

$$F' = \partial_t (\nabla(g_1 \nabla \tilde{v}^\varepsilon)) = \nabla g_1 \partial_t R_1 + g_1 \partial_t \nabla R_1,$$

$$G' = \partial_t (\nabla(g_2 \nabla \tilde{u}_e^\varepsilon)) = \nabla g_2 \partial_t R_2 + g_2 \partial_t \nabla R_2.$$

Then in spite of $e^{-2s\alpha(t,x)} \leq e^{-2s\alpha^\theta}$ for all $(t, x) \in Q$, we obtain

$$\begin{aligned}
& s^2 \lambda^2 \int_{\Omega} \phi^\theta e^{-2s\alpha^\theta} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx \\
& \leq C s^4 \lambda^6 \int_{Q_\omega} \phi^3 e^{-2s\alpha} (|B_1|^2 + |B_2|^2) dt dx + C \int_{\tilde{\omega}} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx, \\
& + C \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|A_1^\theta|^2 + \sum_{j=1}^2 (|\Delta A_j^\theta|^2 + |\nabla A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|)) dx,
\end{aligned}$$

for sufficiently large $s \geq s_4 = \max\{CT^2N, s_3\}$. From the properties of α and ϕ , there exist e_0 and e_1 such that

$$\inf_{x \in \Omega} \left(\frac{1}{\phi^\theta} e^{-2s\alpha^\theta} \right) \geq e_0 > 0,$$

$$\sup_{x \in \Omega} \left(\frac{1}{\phi^\theta} e^{-2s\alpha^\theta} \right) \leq e_1 < \infty.$$

Furthermore, $e^{-\epsilon\alpha} \phi^m \leq C < \infty$ for all $\epsilon > 0$ and $m \in \mathbb{R}$ in Q_ω . Thus we obtain

$$\begin{aligned}
& s^2 \lambda^2 \int_{\Omega} \phi^\theta e^{-2s\alpha^\theta} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx \\
& \leq C s^4 \lambda^6 \int_{Q_\omega} (|B_1|^2 + |B_2|^2) dt dx + C \int_{\tilde{\omega}} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx, \\
& + C \int_{\Omega} (|A_1^\theta|^2 + \sum_{j=1}^2 (|\Delta A_j^\theta|^2 + |\nabla A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|)) dx,
\end{aligned}$$

Then we fix the parameters s, λ as $s = s_4, \lambda = \lambda_1$. One complete the proof. □

Part III

Conclusions and future works

Chapter 8

Conclusions

In the first part of this work, we have investigated a multistage, structured population model describing one of the most important grapevine insect pests.

Firstly, we consider a multistage, physiologically age structured dynamics system, with adult moths diffusing around the vineyard. Growth function of the population at each stage is modeled considering the climatic variations and the grape variety, which depends on the physiological age, and allows us to model great variability of growth within a cohort. Based on the contraction fixed point principle, we obtain the existence and uniqueness of solutions for the model equations. Then we also prove the existence of a global attractor for the trajectories of the dynamical system defined by the solutions of the model. Finally, the theory of compact operators and the Krasnoselskii's fixed point theorem are used to prove the existence of steady states.

Next, we continue to think about the control problem of this *Lobesia botrana* model. First of all, we investigate the exact null controllability of an age-dependent life cycle dynamics with nonlocal transition processes arising as boundary conditions. We assume that the four stages of this system: egg, larva, female moth and male moth are all in static station. We obtain the null controllability for the pest by acting on eggs in a small age interval. To get this result, we use the method, which is based on the derivation of estimations for the adjoint variables related to an optimal control problem. Then we apply a fixed point theorem to draw the conclusion that the population of egg except the small enough age groups to zero at a certain moment in the future, using an age- and time-dependent control of eggs.

Inspired by the above result, then we consider the control problem for the *Lobesia botrana* population dynamics system, while the adult moths can be diffusive. Therefore,

we describe a control by a removal of egg and larva population, and also on female moths in a small region. We combine some estimations and the Carleman inequality for the local backward system related to an optimal control problem. Then choosing a control corresponding to a fixed point of a multi-valued function. Finally, we obtain the null controllability for female moths in a nonempty open sub-domain at a given time T except a small enough age interval.

In the second part, we are concerned with the stability result for the conductivities, as diffusion coefficients of a parabolic system modeling electrical activity in the heart. To study the problem, we first establish a Carleman estimate for the reaction-diffusion system. Then the proof is based on the combination of a Carleman estimate and certain weight energy estimates for parabolic system. The stability result can be stated as two different cardiac electric activities, while the extracellular conductivity tensor M_e changes in the first equation, also the sum of intracellular conductivity tensor and extracellular conductivity tensor M changes. To be more precise, when the extracellular electric potential and the transmembrane potential vary small enough in two situations, even on a small domain ω during the period of time $(0, T)$, and in the whole space Ω at time θ , correspondingly, the extracellular conductivity tensor M_e is close to \tilde{M}_e , and also M is close to \tilde{M} , except in a small subdomain ω_0 belong to ω .

Chapter 9

Future works

In our dissertation, we have studied on a nonlocal structured system, which is developed by the most serious European grapevine moth, and made analysis and control on this dynamics system. It is very interested to investigate qualitative properties, such as existence and uniqueness of solutions, asymptotic behavior, traveling waves solutions, bifurcation theory, and so on for the non-local structured systems for the following cases:

- Stage insect growth with velocity depending on the the physiological age;
- Stage cell growth with application to cancer;
- Disease age epidemiological problems.

To understand the spatial spread of the insect population with age structure, it is very important to study the traveling wave solutions of the resulted nonlocal structured reaction-diffusion system. In fact, the theory of traveling wave solutions of reaction-diffusion equations has been successfully applied to biological invasion (Volpert and Petrovskii 2009, Shigesada and Kawasaki 1997). We connect the condition for species survival to that for propagation of traveling wave solutions. Therefore, it will be an meaningful subject to study the existence and nonexistence of nonlocal traveling wave solutions and its threshold conditions.

In the future work, we will also prove the existence and nonexistence of traveling waves, for the non-local structured systems modeling the age-dependent life cycle-dynamics in the three previously cited cases. In addition, taking account into the damages from the epidemic disease spreading, cancer cell growth, and insect multiplying, it will be practical to discuss the control problems of above mentioned topics to reduce the

economic loss. Thence, we will be dedicated to the work about the controllability of the models developed by the above mentioned topics.

We also note that the parameters (such as delay, non-locality, etc) play very important roles in the behavior of the population dynamics system. As a result, we will analyze the effects of the parameters for the existence and stability of positive steady-state of the system on bounded domain for the systems derived from the three previously cited cases. We hope, through these mathematical studies, it will be helpful to find information and ways to control the size and the spatial spread of the pest population.

Bibliography

- [1] B.E. Ainseba, Exact and approximate controllability of the age and space population structured problem, *J. Math. Anal. Appl.* 275 (2002), 562–574.
- [2] B.E. Ainseba, Age-dependent population dynamics diffusive systems, *Discrete Cont. Dyn. Syst. Series B.* 4 (2004), 1233–1247.
- [3] B.E. Ainseba, S. Anița, Local exact controllability of the age-dependent population dynamics with diffusion, *Abstract Appl. Anal.* 6 (2001), 357–368.
- [4] B.E. Ainseba, S. Anița, Internal exact controllability of the linear population dynamics with diffusion, *Electron. J. Differential Equations* 112 (2004), 1–11.
- [5] B.E. Ainseba, S. Anița, M. Langlais, Optimal control for a nonlinear age-structured population dynamics model, *Electron. J. Differential Equations* 28 (2002), 1–9.
- [6] B.E. Ainseba, M. Bendahmane, R. Ruiz-Baier, Analysis of an optimal control problem for the tridomain model in cardiac electrophysiology, *J. Math. Anal. Appl.* 388 (2012), 231–247.
- [7] B.E. Ainseba, M. Iannelli, Exact controllability of a nonlinear population-dynamics problem, *Differential Integral Equations* 16 (2003), 1369–1384.
- [8] B.E. Ainseba, M. Langlais, Sur un problème de contrôle d’une population structurée en âge et en espace, *C. R. Acad. Sci. Paris Sér. I* 323 (1996), 269–274.
- [9] B.E. Ainseba, M. Langlais, On a population dynamics control problem with age dependence and spatial structure, *J. Math. Anal. Appl.* 248 (2000), 455–474.
- [10] B.E. Ainseba, A. Noussair, Existence and uniqueness for a kinetic model of population dynamics with character dependence and spatial structure, *J. Differential Equations* 187 (2003), 293–309.

- [11] B.E. Ainseba, D. Picart, Parameter identification in multistage population dynamics model, *Nonlinear Anal. RWA* 12 (2011), 3315–3328.
- [12] B.E. Ainseba, D. Picart, D. Thiéy, An innovative multistage, physiologically structured, population model to understand the European grapevine moth dynamics, *J. Math. Anal. Appl.* 382 (2011), 34–46.
- [13] H. Amann, On the number of solutions of nonlinear equations in ordered banach spaces, *J. Funct. Anal.* 11 (1972), 346–384.
- [14] S. Anița, Optimal harvesting for a nonlinear age-dependent population dynamics, *J. Math. Anal. Appl.* 226 (1998), 6–22.
- [15] S. Anița, *Analysis and Control of Age-Dependent Population Dynamics*. Kluwer Acad. Publ. 2000.
- [16] O. Auroy, Estimation des traits d’histoire de vie de l’Eudémis de la vigne (*Lobesia botrana* Den. et Schiff.) en vue d’une utilisation dans un modèle mathématique. Mémoire de fin d’études, d’ingénieur de l’ENITA de Bordeaux. 2006.
- [17] V. Barbu, Exact controllability of the superlinear heat equation, *Appl. Math. Optim.* 42 (2000), 73–89.
- [18] V. Barbu, M. Iannelli, Optimal control of population dynamics, *J. Optim. Theory Appl.* 102 (1999), 1–14.
- [19] V. Barbu, M. Iannelli, M. Martcheva, On the controllability of the Lotka-McKendrick model of population dynamics, *J. Math. Anal. Appl.* 253 (2001), 142–165.
- [20] J. Baumgartner, P. Baronio, Modello fenologico di volo di *Lobesia botrana* Den. et Schiff. (Lep. Tortricidae) relativo alla situazione ambiental della Emilia Romagna, *Bollettino dell’Istituto di Entomologia Guido Grandi* 43 (1988), 157–170.
- [21] M. Bendahmane, F.W. Chaves-Silva, Controllability of a degenerating reaction-diffusion system in electrocardiology, *SIAM Journal on Control and Optimization*, to appear. arXiv:1106.1788.
- [22] M. Bendahmane, K.H. Karlsen, Analysis of a class of degenerate reaction-diffusion systems and the bidomain model of cardiac tissue, *Netw. Heterog. Media* 1 (2006), 185–218.

-
- [23] M. Bendahmane, K.H. Karlsen, Convergence of a finite volume scheme for the bidomain model of cardiac tissue, *Appl. Numer. Math.* 59 (2009), 2266–2284.
- [24] M. Bendahmane, R. Bürger, R. Ruiz Baier, A finite volume scheme for cardiac propagation in media with isotropic conductivities, *Math. Comp. Simul.* 80 (2010), 1821–1840.
- [25] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York (2011).
- [26] J.F. Brière, *Analyse mathématique et numérique dun modèle spatio-temporel de la dynamique dun ravageur de la vigne : Lobesia botrana (Eudémis)*, Thèse en Mathématiques et Informatique, Université Bordeaux 1, 1998.
- [27] A.L. Bukhgeim, Carleman estimates for Volterra operators and uniqueness of inverse problems, *Non-classical Problems of Mathematical Physics*, Computing Center of Siberian Branch of Soviet Academy of Sciences, Novosibirsk (1981), 56–69.
- [28] A.L. Bukhgeim, *Introduction to the Theory of Inverse Problems*, VSP, Utrecht (2000).
- [29] A.L. Bukhgeim, M.V. Klibanov, Uniqueness in the large class of multidimensional inverse problems, *Sov. Math. Dokl.* 24 (1981), 244–247.
- [30] S. Busenberg, M. Iannelli, Separable models in age-structured population dynamics, *J. Math. Biology* 22 (1985), 145–173.
- [31] S. Busenberg, K. Cooke, M. Iannelli, Endemic thresholds and stability in a class of age-structured populations, *SIAM J. Applied. Math.* 48 (1988), 1379–1395.
- [32] A. Calsina, J. Saldana, A model of physiologically structured population dynamics with a nonlinear individual growth rate, *J. Math. Biol.* 33 (1995), 335–364.
- [33] Y. Cha, M. Iannelli, F.A. Milner, Existence and uniqueness of endemic states for the age-structured S-I-R epidemic model, *Math. Biosci.* 9 (2000), 361–376.
- [34] K.C. Chang, *Methods in Nonlinear Analysis*, Springer-Verlag Berlin Heidelberg, Netherlands, 2005.

- [35] C. Clark, "Mathematical Bioeconomics: The Optimal Management of Renewable Resources," Wiley, New York, 1976.
- [36] P. Colli Franzone, G. Savaré, Degenerate evolution systems modeling the cardiac electric field at micro- and macroscopic level, in *Evolution equations, semigroups and functional analysis (Milano, 2000)*, *Progr. Nonlinear Differential Equations Appl.* 50, Birkhäuser, Basel, 2002, 49–78.
- [37] M. Cristofol, P. Gaitan, H. Ramoul, Inverse problems for a 2×2 reaction-diffusion system using a Carleman estimate with one observation, *Inverse Problem* 22 (2006), 1561–1573.
- [38] J. M. Cushing, Stability and maturation periods in age structured populations. In *Differential Equations and Applications in Ecology, Epidemics, and Population Problems*, S. Busenberg and K. L. Cooke (Eds.), Academic Press, New York, 1981, pp. 163–182.
- [39] J. M. Cushing, M. Saleem, A predator prey model with age structure. *J. Math. Biol.* 14 (1982), 231–250. Erratum: 16 (1983), 305.
- [40] O. Diekmann, J.A.P. Heesterbeek, *Mathematical Epidemiology of Infectious Diseases* Wiley series in Mathematical and Computational Biology, Wiley (2000).
- [41] A. Ducrot, P. Magal, Travelling wave solutions for an infection-age structured epidemic model with external supplies. *Nonlinearity.* 24 (2011), 2891–2911.
- [42] A. Ducrot, P. Magal, S. Ruan, Travelling Wave Solutions in Multigroup Age-Structured Epidemic Models. *Arch. Rational Mech. Anal.* 195 (2010), 311–331.
- [43] N. Dunford, J.T. Schwartz, *Linear operators, Part I: General theory*, Interscience Pub., New York, 1958.
- [44] J. Z. Farkas, Stability conditions for a non-linear sizestructured model, *Nonlinear Analysis: RWA.* 6 (2005), 962–969.
- [45] J. Z. Farkas, T. Hagen, Stability and regularity results for a size-structured population model, *J. Math. Anal. Appl.* 328 (2007), 119–136.
- [46] J. Z. Farkas, T. Hagen, Linear stability and positivity results for a generalized size-structured Daphnia model with inflow, *Appl. Anal.* 86 (2007), 1087–1103.

-
- [47] E. Fernandez-Cara, Null controllability of the semilinear heat equation, *ESAIM:COCV*, 2 (1997), 87–107.
- [48] S. Fekih, B.E. Ainseba, S.M. Bouguima, Global existence for an age-structured model with vertical transmission. Submitted.
- [49] W. Feller, On the intergral equation of renewal theory, *Ann. Math. Statist.* 12 (1941), 243–267.
- [50] H. Von Foerster, Some Remarks on Changing Population in the Kinetics of Cellular Proliferation, New-York Grune and Station.
- [51] A.V. Fursikov, O.Yu. Imanuvilov, Controllability of Evolution Equations, Lecture Notes Series 34, RIM Seoul National University, Korea, 1996.
- [52] M.E. Gurtin, A system of equations for age dependent population diffusion, *J. Theor. Biol.* 40 (1973), 389–392.
- [53] M.E. Gurtin, D.S. Levine, On predator-prey interactions with predation dependent on age of prey, *Mathematical Biosciences*, 47 (1979), 207–219. Department of Mathematics, Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213, U.S.A.
- [54] M.E. Gurtin, R.C. MacCamy, Nonlinear age dependent population dynamics, *Arch. Ration. Mech. Anal.* 54 (1974), 281–300.
- [55] M. Gurtin, L. Murphy, On the optimal harvesting of age-structured populations: Some simple models, *Math. Biosci.* 55 (1981), 115–136.
- [56] M. Gurtin, R. C. MacCamy, Some simple models for nonlinear age-dependent population dynamics, *Math. Biosci.* 43 (1979), 199–211.
- [57] M. Gurtin, L. Murphy, On the optimal harvesting of persistent age-structured populations, *J. Math. Biol.* 13 (1981), 131–148.
- [58] J.K. Hale, Asymptotic Behavior of Dissipative Systems, *Mathematical Surveys and Monographs* 25. AMS Providence, 1998.
- [59] Y. He, B. E. Ainseba, Exact null controllability of a stage and age-structured population dynamics system, *J. Optim. Theory Appl.* 157 (2013), 918–933.

- [60] Y. He, B. E. Ainseba, Exact null controllability of the *Lobesia botrana* model with diffusion, *J. Math. Anal. Appl.* 409 (2014), 530–543.
- [61] Y. He, B. E. Ainseba, Global dynamics of the European grapevine moth model with diffusion. Submitted.
- [62] Y. He, B. E. Ainseba, M. Bendahmane, Stability of conductivities in an inverse problem in the reaction-diffusion system in electrocardiology. Submitted.
- [63] F. Hoppensteadt, *Mathematical theories of populations: demographics, genetics and epidemics*. SIAM Reg. Conference in Applied Math. Philadelphia (1975).
- [64] O.Yu. Imanuvilov, M. Yamamoto, Lipschitz stability in inverse problems by Carleman estimates, *Inverse Problems* 14 (1998), 1229–1245.
- [65] O.Yu. Imanuvilov, M. Yamamoto, Carleman estimates for the non-stationary Lamé system and the application to an inverse problem, *ESAIM, COCV* 11 (2005), 1–56.
- [66] M. Iannelli, The mathematical modeling of epidemics, Summer school on mathematical models in life science: science and simulation, Dobbiaco (Bolzano-Italy), July 1–5, 2005.
- [67] H. Inaba, Mathematical analysis of an age-structured SIR epidemic model with vertical transmission, *Discrete Contin. Dyn. Syst. Ser. B* 6 (2006), 69–96.
- [68] V. Isakov, Carleman estimates and applications to inverse problems, *Milan J. Math.* 72 (2004), 249–271.
- [69] W. O. Kermack, A. G. McKendrick, A Contribution to the mathematical theory of epidemics. *Proc. Roy. Soc. Lond. A* 115 (1927), 700–721.
- [70] W. O. Kermack, A. G. McKendrick, Contributions to the mathematical theory of epidemics II. The problem of endemicity. *Proc. Roy. Soc. A* 138 (1932), 55–83.
- [71] W. O. Kermack, A. G. McKendrick, Contributions to the mathematical theory of epidemics III. Further studies on the problem of endemicity. *Proc. Roy. Soc. A* 141 (1933), 94–122.
- [72] N. Keyfitz, *Introduction to the Mathematics of Population*, Addison-Wesley, Reading, MA, 1968.

-
- [73] M.V. Klibanov, Carleman estimates and inverse problems in the last two decades, *Surveys on solution methods for inverse problems*, Springer, Berlin (2000), 119–146.
- [74] M.V. Klibanov, M. Yamamoto, Lipschitz stability of an inverse problem for an acoustic equation, *Appl. Anal.* 85 (2006), 515–538.
- [75] M. A. Krasnoselskii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.
- [76] M. Langlais, Large behavior in age-dependent population dynamics problem with spatial diffusion, *J. Math. Biol.* 26 (1988), 319–346.
- [77] G. Lebeau, L. Robbiano, Contrôle exact de l'équation de la chaleur. *Comm. P. D. E.* 20 (1995), 335–356.
- [78] L. Lefkovich, The study of population growth in organisms grouped by stages, *Biometrics* 21 (1965), 1–18.
- [79] L. Lefkovich, A theoretical evaluation of population growth after removing individuals from some age groups, *Bull. Entomolog. Res.* 57 (1966), 437–445.
- [80] J.L. Lions, Remarks on approximate controllability, *J. Anal. Math.* 59 (1992), 103–116.
- [81] Y. Lou, X.Q. Zhao, A reaction-diffusion malaria model with incubation period in the vector population, *J. Math. Biol.* 4 (2011), 543–568.
- [82] Z.X. Luo, Z.R. He, W.T. Li, Optimal birth control for an age-dependent n -dimensional food chain model, *J. Math. Anal. Appl.* 287 (2003), 557–576.
- [83] Z.X. Luo, Z.R. He, W.T. Li, Optimal birth control for predator-prey system of three species with age-structure, *Appl. Math. Comput.* 155 (2004), 665–685.
- [84] A.G. McKendrick, Applications of mathematics to medical problems, *Proc. Edin. Math. Soc.* 44 (1926), 98–130.
- [85] J. Moreau, B. Benrey, D. Thiéry, Grape variety affects larval performance and also female reproductive performance of the European grapevine moth *Lobesia botrana* (Lepidoptera: Tortricidae), *Bull. Entomol. Res.* 96 (2006), 205–212.

- [86] J. Moreau, D. Thiéry, J.P. Troussard, B. Benrey, Grape variety affects female but also male reproductive success in wild European grapevine moth, *Ecol. Entomol.* 332 (2007), 747–753.
- [87] J. Moreau, A. Richard, B. Benrey, D. Thiéry, Host plant cultivar of the grapevine moth *Lobesia botrana* affects the life history traits of an egg parasitoid, *Biol. Control.* 50 (2009), 117–122.
- [88] L. Murphy, S. Smith, Optimal harvesting of an age-structured populations, *J. Math. Biol.* 29, No. 3 (1990), 77–90.
- [89] J.D. Murray, *Mathematical Biology I: An Introduction*, third ed., Springer-Verlag, Berlin, 2002.
- [90] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, (1983).
- [91] O. Nakoulima, A. Omrane, J. Vélin, A nonlinear problem for age-structured population dynamics with spatial diffusion, *Top. Meth. in Nonlinear Analysis* 17 (2001), 307–319.
- [92] D. Picart, *Modélisation et estimation des paramètres liés au succès reproducteur d'un ravageur de la vigne(Lobesia botrana DEN.& SCHIFF.)*, Ph.D Thesis.
- [93] D. Picart, B.E. Ainseba, F. Milner, Optimal control problem on insect pest populations, *Appl. Math. Lett.* 24 (2011), 1160–1164.
- [94] J.-P. Puel, M. Yamamoto, On a global estimate in a linear inverse hyperbolic problem, *Inverse Problem* 12 (1996), 995–1002.
- [95] P.H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* 7 (1971), 487–513.
- [96] C. Rorres, W. Fair, Optimal age specific harvesting policy for continuous-time population model, in "Modeling and Differential Equations in Biology" (T. A. Burton, Ed.), Dekker, New York, 1980.
- [97] S. Ruan, *Delay differential equations in single species dynamics*, *Delay differential equations and applications*, 477–517, NATO Sci. Ser. II Math. Phys. Chem., 205, Springer, Dordrecht, 2006.

-
- [98] K. Sakthivel, N. Baranibalan, J.-H. Kim, K. Balachandran, Stability of diffusion coefficients in an inverse problem for the Lotka-Volterra competition system, *Acta Appl. Math.* 111 (2010), 129–147.
- [99] I. Sawashima, On spectral properties of some positive operators, *Nat. Sci. Report Ochanomizu Univ.* 15 (1964), 53–64.
- [100] J.W. Sinko, W. Streifer, A new model for age-size structure of a population, *Ecology*. 48 (1967), 910–918.
- [101] F.R. Sharp, A.J. Lotka, A problem in age distribution, *Phil. Mag.* 21 (1911), 435–438.
- [102] L. B. Slobodkin, Population dynamics in *Daphnia obtusa* Kurz. *Ecol. Monogr.* 24 (1954), 69–89.
- [103] H. R. Thieme, *Mathematics in Population Biology*, Princeton series in Theoretical and Computational Biology, (2003).
- [104] D. Thiéry, D. Esmenjaud, S. Kreiter, M. Martinez, R. Sforza, M. Van Helden, M. Yvon, Les insectes de la vigne : Les tordeuses nuisibles à la vigne. In: S. Kreiter, ed. *Ravageurs de la vigne*. Bordeaux - France : Féret, 214–246 (2008).
- [105] D. Thiéry, J. Moreau, Relative performance of European grapevine moth (*Lobesia botrana*) on grapes and other hosts, *Oecologia*. 143 (2005), 548–557.
- [106] M. Yamamoto, Uniqueness and stability in multidimensional hyperbolic inverse problems, *J. Math. Pures Appl.* 78 (1999), 65–98.
- [107] C. Walker, Global bifurcation of positive equilibria in nonlinear population models, *J. Differential Equations* 248 (2010), 1756–1776.
- [108] W. Wang, X.Q. Zhao, A nonlocal and time-delayed reaction-diffusion model of dengue transmission, *SIAM J. Appl. Math.* 71 (2011), 147–168.
- [109] G.F. Webb, *Theory of non linear age-dependent population dynamics*, Marcel Dekker, New York, (1985).
- [110] Z.Q. Wu, J.X. Yin, C.P. Wang, *Elliptic and Parabolic Equations*, World Scientific Publishing Co. Pte. Ltd, 2003.

- [111] X.Q. Zhao, *Dynamical Systems in Population Biology*, Springer-Verlag, New York, (2003).
- [112] E. Zuazua, Controllability of partial differential equations and semi-discrete approximations, *Discrete Cont Dyn S.* 8 (2) (2002), 469–513.
- [113] E. Zuazua, Controllability and Observability of Partial Differential Equations: Some results and open problems, in *Handbook of Differential Equations: Evolutionary Differential Equations*, vol 3, C. M. Dafermos and E. Feireisl eds., Elsevier Science, pp. 527–621.