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THÈSE DE DOCTORAT
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2D quantum gravity in the Kähler formalism.

par Lætitia Leduc

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« Chaque chose au monde porte en elle sa réponse,
ce qui prend du temps ce sont les questions. »
José Saramago in *Le Dieu manchot*.

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Résumé / Abstract

Le but de cette thèse est d'étudier la gravité quantique bidimensionnelle. Nous nous intéressons plus particulièrement aux approches dans le continu. Ces dernières reposent principalement sur l'action de Liouville qui décrit le couplage entre théorie conforme et gravité. Si cette action, bien connue, est très bien comprise, la mesure de l'intégrale fonctionnelle sur l'espace des métriques pose plus de problèmes. Toutefois, sous l'hypothèse simplificatrice d'une mesure de champ libre, la dépendance en l'aire de la fonction de partition de la gravité quantique en présence de matière conforme a pu être établie, permettant ainsi d'accéder à un exposant critique : la "susceptibilité de la corde". Malgré l'hypothèse assez forte sur la mesure d'intégration, cette formule (dite KPZ), a été confirmée par des calculs issus de méthodes discrètes, et ce dans plusieurs cas particuliers. Grâce à une nouvelle méthode de régularisation spectrale en espace courbe, cette mesure d'intégration a récemment pu être proprement définie. En considérant les surfaces de Kähler de genre quelconque (qui coïncident avec l'ensemble des surfaces à deux dimensions), un calcul perturbatif de la fonction de partition à aire fixée a ainsi pu être mené à une boucle, en utilisant à la fois les actions de Liouville et de Mabuchi. L'action de Mabuchi intervient en correction au premier ordre si l'on couple de la matière non conforme à la gravité. La susceptibilité de la corde ainsi calculée correspond, dans le pur cas conforme, au développement de la formule KPZ à une boucle. Ce résultat n'est que peu surprenant, puisque les effets induits par le caractère non-trivial de la mesure d'intégration sur les géométries n'apparaissent qu'à partir de deux boucles.

Afin de déterminer les conséquences de cette mesure d'intégration, un calcul perturbatif de la fonction de partition à aire fixée jusqu'à trois boucles a été conduit dans cette thèse. Nous avons premièrement considéré l'action de Liouville et des surfaces de genre quelconque, puis ces résultats ont été généralisés dans la cas du tore à une action couplant les actions de Liouville et de Mabuchi. Nous avons accédé à l'expression complète de la fonction de partition à aire fixée à deux boucles, puis nous avons calculé les divergences dominantes quadratiques et logarithmiques à trois boucles. La régularisation de la mesure d'intégration génère une "action de mesure", ajoutant ainsi des vertex à ceux issus du développement de l'action de Liouville (et de Mabuchi). Si, à deux boucles, les diagrammes à considérer ne sont qu'au nombre de quatre (dont dix sous-diagrammes), à trois boucles, il faut calculer 29 types de diagrammes totalisant plus de 200 contributions de sous-diagrammes. Des divergences étant présentes, il faut renormaliser les actions (en y ajoutant des contre-termes). Afin de les fixer, nous avons calculé la fonction à deux points complète à une boucle et requis que celle-ci soit à la fois finie et indépendante de notre choix de régularisation. Ceci nous a permis de fixer la plupart des contre-termes et d'obtenir une fonction de partition à aire fixée finie et indépendante de la régularisation à deux boucles. La susceptibilité de corde ainsi obtenue est compatible avec le résultat KPZ, mais plus générale car dépendante d'une constante libre. Nous avons donc poursuivi le calcul à trois boucles afin d'obtenir la contribution complète des contre-termes à la fonction de partition à aire fixée ainsi qu'aux fonctions à n -points au même ordre dans

le développement perturbatif. Au bout du compte, il apparaît que parmi les 15 contre-termes possibles, 4 restent indéterminés et contribuent à la susceptibilité de la corde. On peut attirer l'attention sur le fait que les contre-termes sont soit locaux (comme voulu) soit similaires à la mesure, cette dernière contenant en effet des termes non-locaux en $1/A$. En revanche, il est possible de garantir la localité de l'action globale (somme des actions de Liouville, Mabuchi, de la mesure et des contre-termes) grâce aux contre-termes. Ceci nous suggère d'interpréter les contre-termes comme une renormalisation de la mesure d'intégration.

This thesis is devoted to the study of two-dimensional quantum gravity. After giving a general introduction displaying different approaches of quantum gravity, we highlight the continuous ones, mainly based on the so-called Liouville action which universally describes the coupling of any conformal field theory to gravity. While the Liouville action is relatively well understood, the appropriate functional integral measure is complicated, however. Nevertheless, a formula for the area dependence of the quantum gravity partition function in the presence of conformal matter has been obtained, under the simplifying assumption of a free-field measure. It has then been possible to derive relevant critical exponents for these theories, one of them called the “string susceptibility”. Notwithstanding its non-rigorous derivation, this formula, often referred to as the KPZ formula, has since been verified in many instances and has scored many successes. However, the recent development of efficient multi-loop regularization methods on curved space-times opened the way for a precise and well-defined perturbative computation of the fixed-area partition function in the Kähler formalism. The string susceptibility was therefore previously computed in this framework up to one loop for surfaces of arbitrary genus using a somewhat more general quantum gravity action including the Liouville and Mabuchi actions; the latter corresponds to possible couplings to non-conformal matter. For conformal matter only, the one-loop KPZ result was reproduced. This was to be expected since the non-trivial nature of the quantum gravity integration measure only shows up at two and higher loops.

In this thesis, a first-principles computation of the fixed-area partition function up to three loops, considering the Liouville action, is performed. Among other things this allowed us to appreciate the role of the non-trivial quantum gravity integration measure. Our computation allows us to get the full expression of the partition function at two loops and to access both the leading quadratic and logarithmic divergences at three loops, as well as precisely discuss the diverging structure of the string susceptibility. This was done for Riemann surfaces of any genus. The regularization of the complicated integration measure leads to a “measure action” adding vertices to the ones coming from Liouville. While the two-loop computation only requires to take into account 4 types of diagrams subdivided in 10 subdiagrams, the three-loop calculation is equivalent to consider 29 types of diagrams consisting of more than 200 subdiagrams. As expected, the presence of divergences requires renormalization to obtain a finite result in the end. Therefore a counterterm action has been added and the full one-loop two-point Green’s function computed. This allows to fix the relevant counterterms, resulting in a finite and regularization independent fixed-area partition function at two loops. The resulting string susceptibility is more general than the KPZ result, although compatible. It indeed depends on one of the unfixed counterterms. Therefore, the three-loop computation has been performed for the purpose of accessing to the complete contribution of the counterterms to both the partition function and the n -point functions of the same order in the perturbative expansion. Finally, it seems that among the initial 15 counterterms possible to insert, 4 stay undetermined and contribute to the string susceptibility. One can stress that the allowed counterterms are either local (as fitted) or measure-like. The measure action coming from the regularization gives indeed rise to non-local terms. The local-

ity of the global Liouville, measure and counterterm action is however assured through the counterterms. This hints at an understanding of the measure-like counterterms as a renormalization of the measure, which has been overlooked before. In the last chapter these results are generalised to the coupling to non-conformal matter by considering the Mabuchi action in addition to the Liouville action, in the case of the torus.

Introduction

« La fantaisie et la liberté d'imagination ne s'acquièrent pas comme ça,
qu'il y faut du temps, de l'obstination, de la sévérité, de la rigueur,
des mathématiques, de la raison. »

Philippe Sollers.

Quantum gravity remains one of the most thrilling scientific challenges of the 21st century. General relativity has deeply changed our conception of time and space a century ago and has been successfully experimentally verified using different techniques, including the recent discovery of gravitational waves [1], and from them the first observation of merging black holes. However, physics occurring near the center of black holes or in the early universe involves regions where the curvature becomes large and having at our disposal a theory of quantum gravity undoubtedly is unavoidable [2, 3]. Quantum gravity appears to be the missing piece in the jigsaw puzzle of our understanding of basic interactions.

Due to the complexity of the topic (see e.g. [4–10]), two-dimensional theories can be considered as insightful starting points to unravel the quantum aspect of gravity. First, one may think to look at pure quantum gravity. Yet, at 2D, pure quantum gravity does not contain any propagating degrees of freedom. For this reason, and of course keeping in mind that our universe does contain matter fields, a theory of quantum gravity coupled to matter is highly desirable. Since Polyakov's seminal paper [11], conformal matter coupled to quantum gravity on two-dimensional manifolds has been studied intensely [12, 13], both in the discretized approach [14–21] and in the continuum approach, known as Liouville quantum gravity [22–27]. While the matrix models give a non-perturbative definition, the Liouville theories offer a more transparent physical interpretation. In the continuum approach, most of the computations have been done within the conformal gauge. When the conformally coupled matter is integrated out, one ends up with the Liouville action $S_L[g_0, g]$ as an effective gravity action.

One of the simplest, yet interesting objects to study in this quantum gravity is the partition function at fixed area $Z[A]$. One way to define this partition function $Z[A]$ on a Riemann surface of genus h , with metric g of area A , is to choose the conformal gauge with a background metric g_0 and a conformal factor σ such that $g = e^{2\sigma}g_0$. Then,

$$Z_{\text{grav}} = \int \mathcal{D}g e^{-\mu_c^2 A} Z_{\text{mat}}[g] = \int dA Z[A],$$

and $Z[A]$ can be formally written as

$$Z[A] = \int \mathcal{D}\sigma e^{-\frac{\kappa^2}{8\pi} S_L[g_0, g] - \mu_c^2 A} \delta\left(A - \int d^2x \sqrt{g_0} e^{2\sigma}\right), \quad \kappa^2 = \frac{26 - c}{3},$$

where c is the matter central charge¹ and the -26 accounts for the gauge fixing (ghosts). The delta-function restricts the integration to metrics of area A . The measure $\mathcal{D}\sigma$ for

¹For reference on 2D Conformal Field Theory, see e.g. [28].

the conformal factor can be derived from the standard metric on the space of metrics and is a complicated non-flat measure. Many of the difficulties in dealing with this quantum gravity theory originate from this measure being non-trivial. One of the main difficulties when computing the partition function lies in the complicated non-flat measure $\mathcal{D}\sigma$ for the conformal factor.

Khnik, Polyakov and Zamolodchikov studied two-dimensional gravity for genus zero in the light-cone gauge instead [22]. Using the relation with an $SL(2)$ current algebra they derived a remarkable formula relating the scaling dimensions Δ of conformal primary operators coupled to gravity and their undressed conformal dimensions $\Delta^{(0)}$:

$$\Delta - \Delta^{(0)} = \frac{(\sqrt{25-c} - \sqrt{1-c})^2}{24} \Delta(1-\Delta) ,$$

known as algebraic KPZ relation. The scaling of the partition function is obtained from the dressing of the identity operator ($\Delta_{\text{id}}^{(0)} = 0$) and leads to a scaling

$$Z[A] \sim e^{-\mu_c^2 A} A^{\gamma_{\text{str}}-3} ,$$

where $\gamma_{\text{str}} = \Delta_{\text{id}}$ is the ‘‘string susceptibility’’. The previous equation then yields $\gamma_{\text{str}} = 2 - 2 \frac{\sqrt{25-c}}{\sqrt{25-c} - \sqrt{1-c}}$. This formula gives the correct scaling for certain random lattice models corresponding to specific values of c (see [29] and references therein).

On the other hand, working in the conformal gauge, and using several simplifying assumptions, specifically on the measure $\mathcal{D}\sigma$, as well as consistency conditions, ref. [23] and [24] have extended these remarkable formulae to surfaces of arbitrary genus. In particular they found the following formula for the string susceptibility

$$\gamma_{\text{str}} = 2 + 2(h-1) \frac{\sqrt{25-c}}{\sqrt{25-c} - \sqrt{1-c}} ,$$

referred to as the KPZ formula. More recently, ref. [30–36] have given more rigorous, though more abstract alternative derivations of these formulae for $c \leq 1$ (and $h = 0$). An alternative, more physical derivation can be found in [37]. For recent probabilistic constructions of the free field Liouville measure on peculiar geometries, see [38–40].

Probably the most puzzling property of these formulae is that they only seem to work for $c \leq 1$, since for $c > 1$ (actually $1 < c < 25$) they turn complex. This is the so-called $c = 1$ barrier (see e.g. [41, 42] and references therein), where tachyons appear in the Liouville theory [43]. It has been argued that, for $c > 1$, the two-dimensional geometry is dominated by configurations that no longer are smooth and that the surface develops spikes and a fractal character (see e.g. [24, 44–48]), sometimes considered as a branched polymer phase in which the surfaces collapse to tree-like objects (see e.g. [49–52] and references therein).

However, the recent development of efficient multi-loop regularization methods on curved space-times opened the way for a precise definition of the measure $\mathcal{D}\sigma$ as it follows from the usual metric on the space of metrics, and thus for a first-principles quantum field theory computation of $Z[A]$, on a Riemann surface of arbitrary genus h .

This regularization [53] is a generalization of the ζ -function regularization that works at one loop, to a regularization scheme that works for multi-loop computations on curved manifolds. Its basic objects are the heat kernel and generalized heat kernels defined on the manifold for which exist well-known formulae for the asymptotic “small t ” behaviour.

More precisely, to compute the partition function at fixed area one can conveniently reparametrize the metric in the Kähler formalism, in terms of a fixed background metric, the area A and the Laplacian (in the background metric) of the Kähler potential, so that the area appears explicitly as a “coordinate” on the space of metrics. Since all the metrics are Kähler’s in two dimension, this parametrization is complete. Then, the measure on the space of metrics is given in terms of dA , the standard flat measure on the space of Kähler potentials, and various non-trivial determinants. Expanding these determinants and the interactions in the Liouville action in powers of $\frac{1}{\kappa^2}$ generates the loop-expansion. Then, one can regularize the determinants and the propagators with the spectral cutoff regularization [53]. The first-principles computation has been initiated within this framework in [54] with the computation of the string susceptibility up to one loop, considering a somewhat more general quantum gravity action including the usual Liouville action for the coupling to conformal matter and the Mabuchi action for non-conformal matter. Indeed, it has been shown [55] that the Mabuchi action is involved in the first-order mass correction to the Liouville action. For conformal matter only, the one-loop computation was in agreement with the one-loop semi-classical expansion of the KPZ formula. However, it is only beyond one loop, starting at two loops, that the regularized measure plays a role. Thus, it is from this order in the perturbation theory that one can expect to see a difference with the KPZ formula, derived under the simplifying assumption of a free field measure [23, 24]. In order to highlight the consequence of regularizing the integration measure in a proper way, the aim of this thesis is to continue the first-principles computation of the fixed-area partition function to higher order.

In Chapter 1, we present different approaches to two-dimensional quantum gravity. We discuss some suitable effective gravitational actions, such as the Liouville and Mabuchi actions. Then, we present the one loop computation [54]. We also introduce in this chapter the Kähler formalism and the general smooth spectral cutoff regularization developed in [53]. This regularization scheme amounts to first replacing each propagator $G(x, y) = \sum_n \frac{1}{\lambda_n} \psi_n(x) \psi_n^*(y)$ by $\widehat{K}(t_i, x, y) = \sum_n \frac{e^{-t_i \lambda_n}}{\lambda_n} \psi_n(x) \psi_n^*(y)$ where the ψ_n and λ_n are the eigenfunctions and eigenvalues of the relevant differential operator, then substituting $t_i = \frac{\alpha_i}{\Lambda^2}$ and integrating $\int_0^\infty d\alpha_i \varphi(\alpha_i)$. Thus, Λ is the UV cutoff and $\varphi(\alpha)$ is a fairly general regulator function. This yields a regularized propagator $G_{\text{reg}}(x, y) = \sum_n \frac{f(\lambda_n/\Lambda^2)}{\lambda_n} \psi_n(x) \psi_n^*(y)$ with an almost arbitrary f that is a Laplace transform of the almost arbitrary φ . For large Λ , the t_i are small and to evaluate the diverging, as well as the finite parts of any loop diagram, it is enough to know the small t asymptotic of the \widehat{K} . Of course, $\widehat{K}(t, x, y)$ is related to the heat kernel $K(t', x, y)$ on the manifold, which has a well-known small t' asymptotic expansion. We end this chapter by discussing the background independence of our formalism. Indeed, in any theory of quantum gravity, physical results should not depend on the arbitrarily introduced back-

ground metric. However, the regularization scheme is based on inserting a cutoff $e^{-t_i \lambda_n}$ in the sums of the eigenvalues, and both the eigenfunctions and the eigenvalues depend on the choice of the background metric. Thus, the regularization is expected to induce an explicit background dependence in our computations.

Chapter 2 is devoted to the study of the Liouville theory on a Riemann surface of arbitrary genus, up to three loops in the perturbation theory. It is based on [56] and [57]. First, we perform the full two-loop computations of the pure Liouville gravity in the general case of a surface of arbitrary genus. The loop expansion is an expansion around the classical solution of the Liouville equation of motion, thus around a background metric of constant curvature. We explicitly write the non-trivial measure in the Kähler formalism and we expand the measure and the Liouville action up to the order $\frac{1}{\kappa^2}$ which corresponds to the two-loop contributions to $\ln Z[A]$. Of course, the measure vertex itself is already a “one-loop” effect. The expansion of the Liouville action in terms of the Kähler potential yields the propagator and various n -point vertices. For the two-loop computation we find a cubic and two quartic vertices, with derivatives acting in various ways. Thus, we have to consider 3 types of “two-loop” diagrams and one “one-loop” measure diagram. They get contributions from 10 subdiagrams. We explicitly write their regularized expressions and discuss that they can depend on the dimensionful quantities A and Λ only through the dimensionless combination $A\Lambda^2$. Incidentally, this argument also explains why it is possible to obtain an explicit form for γ_{str} . Indeed, γ_{str} is the coefficient of $\ln A\Lambda^2$ and, hence, it is related to short-distance singularities which, in turn, are determined by local quantities like heat kernel coefficients. The regularized fixed-area partition function depends on the cutoff Λ and the area A through divergent terms of the form $A\Lambda^2$, $\ln A\Lambda^2$, $(\ln A\Lambda^2)^2$ and $A\Lambda^2 \ln A\Lambda^2$. While the first term only contributes to the divergent cosmological constant (which can be adjusted by a corresponding local counterterm), and the coefficient of the second term determines γ_{str} , the third and fourth terms are unwanted, non-local divergences. Quite non-trivially, all contributions to the third term added up to zero!

However, the $A\Lambda^2 \ln A\Lambda^2$ divergences remain. Then, one must introduce local counterterms in addition to the cosmological constant. Such local counterterms also contribute to the two-loop partition function, via one-loop diagrams. In particular, they can cancel the $A\Lambda^2 \ln A\Lambda^2$ divergences, while they could not have cancelled the $(\ln A\Lambda^2)^2$ divergences. The precise coefficients of the counterterms are determined up to regulator-independent finite constants by requiring the two-point function of the Kähler fields, or equivalently of $\langle e^{2\sigma} e^{2\sigma} \rangle$, to be both finite and regulator-independent. Then, the partition function automatically becomes also both finite and regulator-independent. Yet, two finite “renormalization” constants – on which the two-loop contribution to γ_{str} depends – remain undetermined. By a locality argument, one of these renormalization constants can be fixed, precisely to the value consistent with the KPZ formula for γ_{str} . However, the other renormalization constant has no particular reason to be fixed to the KPZ value. Remarkably, all values of this parameter are consistent with a background independent partition function at fixed area. This means that we actually have (at least) a one-parameter family of quantum gravity theories that we can consistently define in

this Kähler approach (at least up to the two-loop order of perturbation theory).

The presence of this free parameter is intriguing and a natural question is whether the structure of the counterterm action introduced at two-loops is enough to also cancel the divergences at three (and higher) loops or whether new counterterms, with additional undetermined finite renormalization constants are required. Thus, in the last section of Chapter 2, we present the three-loop study of the Liouville quantum gravity on surfaces of any genus [57]. We expand the measure and the Liouville action up to the “three-loop” order $\frac{1}{\kappa^4}$ and add the previous counterterm action. This leads to new vertices: sextic, quintic, quartic and cubic (from the measure). Thus, in order to compute both the leading quadratic and logarithmic three-loop divergences of $\ln Z[A]$, we have now to consider 29 types of diagrams consisting of more than 200 subdiagrams. Although the summed leading logarithmic divergence $(\ln A\Lambda^2)^3$ cancels quite remarkably, the leading divergence $A\Lambda^2 (\ln A\Lambda^2)^2$ has a non-vanishing coefficient. Therefore, genuine three-loop counterterms are required. We compute their full contribution to the three-loop part of the partition function and then to equivalent order $\frac{1}{\kappa^4}$ to the n -point function. Indeed, even if, in principle, one would need to compute the full one-loop three (and four)-point functions and the two-loop two-point function, and then to require them to be finite and regulator independent in order to really determine the coefficients of the three-loop counterterms, we can still discuss the degrees of freedom of the counterterms by comparing the independent combinations in which they appear. Then, while the full three-loop computation is beyond the scope of this thesis, the full counterterm parts already encodes some interesting information. For consistency, we also include the “two-loop” counterterms that could not be completely fixed by the previous study. It seems that among the initial 15 counterterms possible to insert, 5 stay undetermined and contribute to the string susceptibility, 4 more than at the two-loop order. We end this chapter by highlighting the relation between the regularized measure and the counterterms.

In Chapter 3, we present similar results for the mixed Liouville and Mabuchi theory in the more specific case of the torus. This chapter is based on [58]. We compute once again the full two-loop partition function. Counterterms are also required, as it should have been expected since the leading singularity comes from the Liouville action. However, quite remarkably, the leading logarithmic divergence $(\ln A\Lambda^2)^2$ once more cancels out between the diagrams, thus allowing a renormalization through only local counterterms! Then, we check that the three-loop leading divergence $(\ln A\Lambda^2)^3$ also cancel and we compute the contribution of the new counterterms to both the partition function and the n -point function. Then, we briefly discuss the finite part of the two-loop two-point function, so that we can reduce the number of the remaining free parameters in our theory.

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The continuum approach to quantum gravity

« Ma cohabitation passionnée avec les mathématiques m'a laissé un amour fou pour les bonnes définitions, sans lesquelles il n'y a que des à-peu-près. »
 Stendhal in *Vie de Henry Brulard*.

1.1 Two-dimensional quantum gravity

In this chapter, we first briefly recall the derivation of the KPZ result for the string susceptibility:

$$Z[A] \sim e^{-\mu_c^2 A} A^{\gamma_{\text{str}}-3}, \quad (1.1.1)$$

$$\gamma_{\text{str}} = 2 + 2(h-1) \frac{\sqrt{25-c}}{\sqrt{25-c} - \sqrt{1-c}}, \quad (1.1.2)$$

and the discrete results confirming this formula. Then, we focus on the specificities of our approach. We first discuss the effective gravity action one may consider, based on [59] and [55], and the Kähler formalism. Then, to complete the presentation of our framework, we present the smooth spectral regularization techniques developed in [53]. This regularization is the one used throughout this thesis to obtain the results presented in Chapter 2 and 3. In order to illustrate this formalism, we then recall the one-loop computation of the fixed-area partition function of two-dimensional quantum gravity done in [54]. Finally, we discuss the background independence of our regularization, and give some key formula to check the background (in)dependence of our results.

1.1.1 The DDK argument for the scaling of the fixed-area partition function

Since we will compare our results with the KPZ relation (1.1.2), we find it useful to briefly present the DDK argument [23, 24] that gives this scaling of the partition function in terms of the area. This argument is amazingly simple and yields a result that has been cross-checked by other methods, at least for some specific models at certain values of the central charge (see section 1.1.2). On the other hand, as already pointed out, it relies on various simplifying assumptions that cannot be the full truth.

Consider two metrics g and g_0 of area $A = \int d^2x \sqrt{g}$ and A_0 related in the conformal gauge by the conformal factor σ :

$$g = e^{2\sigma} g_0 . \quad (1.1.3)$$

In this conformal gauge, the Liouville action takes the simple form

$$S_L[g_0, g] \equiv S_L[g_0, \sigma] = \int d^2x \sqrt{g_0} (\sigma \Delta_0 \sigma + R_0 \sigma) , \quad (1.1.4)$$

and the area A can be written as $A = \int d^2x \sqrt{g_0} e^{2\sigma}$. Instead of using the correct non-trivial measure $\mathcal{D}\sigma$ in the fixed-area partition function, DDK use a *flat free-field* measure $\mathcal{D}_0\sigma$. At the same time they argue that, in the quantum theory, the coefficient κ^2 in front of the Liouville action should be renormalized to $\tilde{\kappa}^2$ and that the definition of the area can no longer simply be $\int d^2x \sqrt{g_0} e^{2\sigma}$: the coefficient in the exponential must also be renormalized so that it becomes $\int d^2x \sqrt{g_0} e^{2\alpha\sigma}$ which should be a conformal primary of weight (1, 1) with respect to the Liouville action. Thus

$$Z_{\text{DDK}}[A] = \int \mathcal{D}_0\sigma e^{-\frac{\tilde{\kappa}^2}{8\pi} S_L[g_0, g] - \mu_c^2 A} \delta\left(A - \int d^2x \sqrt{g_0} e^{2\alpha\sigma}\right) . \quad (1.1.5)$$

To determine the coefficient α , one can switch to standard normalizations by setting $\hat{\sigma} = \tilde{\kappa} \sigma$. Then,

$$\frac{\tilde{\kappa}^2}{8\pi} S_L[g_0, g] = \frac{1}{4\pi} \int d^2x \sqrt{g_0} \left(\frac{1}{2} \hat{\sigma} \Delta_0 \hat{\sigma} + \frac{\tilde{\kappa}}{2} R_0 \hat{\sigma} \right) , \quad (1.1.6)$$

which represents a standard free-field action with a background charge $\frac{\tilde{\kappa}}{2}$. The left and right conformal weights of $: e^{2\alpha\sigma} := e^{\frac{2\alpha}{\tilde{\kappa}} \hat{\sigma}} :$ are well-known. Requiring them to equal unity yields

$$-\frac{1}{2} \left(\frac{2\alpha}{\tilde{\kappa}} \right)^2 + \frac{\tilde{\kappa}}{2} \frac{2\alpha}{\tilde{\kappa}} = 1 , \quad (1.1.7)$$

with solution

$$\alpha = \frac{\tilde{\kappa}^2}{4} \left(1 - \sqrt{1 - \frac{8}{\tilde{\kappa}^2}} \right) , \quad (1.1.8)$$

where the sign has been chosen to match with the semi-classical limit $\tilde{\kappa}^2 \rightarrow \infty$. The central charge of this “free” Liouville theory with background charge is

$$c_L = 1 + 3\tilde{\kappa}^2 . \quad (1.1.9)$$

Background independence requires the total central charge of the matter (c), ghosts (−26) and the Liouville theory to vanish:

$$c - 26 + c_L = 0 \quad \Rightarrow \quad \tilde{\kappa}^2 = \frac{25 - c}{3} = \kappa^2 - \frac{1}{3} . \quad (1.1.10)$$

To obtain the area dependence of $Z_{\text{DDK}}[A]$, one simply changes the integration variable in (1.1.5) from σ to $\sigma' = \sigma - b$ with some constant b . The flat measure $\mathcal{D}_0\sigma$ is invariant by translation, while the Liouville action changes as

$$S_{\text{L}}[g_0, \sigma] = S_{\text{L}}[g_0, \sigma'] + 8\pi b(1-h), \quad (1.1.11)$$

and the delta scales as

$$\delta\left(A - \int d^2x \sqrt{g_0} e^{2\alpha\sigma}\right) = e^{-2\alpha b} \delta\left(e^{-2\alpha b} A - \int d^2x \sqrt{g_0} e^{2\alpha\sigma'}\right). \quad (1.1.12)$$

Putting things together, and letting $e^{2\alpha b} = \frac{A}{A_0}$, one gets

$$Z_{\text{DDK}}[A] = \left(\frac{A}{A_0}\right)^{-1 - \frac{\tilde{\kappa}^2}{2\alpha}(1-h)} e^{-\mu_c^2(A-A_0)} Z_{\text{DDK}}[A_0], \quad (1.1.13)$$

from which we read

$$\gamma_{\text{str}} = 2 - \frac{\tilde{\kappa}^2}{2\alpha}(1-h) = 2 + 2(h-1) \frac{\sqrt{25-c}}{\sqrt{25-c} - \sqrt{1-c}}. \quad (1.1.14)$$

1.1.2 Results from the discrete approaches

General references for the discrete models – lattice models known as dynamical triangulations as well as matrix models – can be found for instance in [12]. Dynamical triangulations are a discretization of quantum geometries. Indeed, it was proposed in e.g. [14, 46, 60–63] that the integral over the internal geometry of a two-dimensional surface can be discretized as a sum over randomly triangulated surfaces. This way, the lattice spacing plays the role of a UV regulator, allowing the theory to be finite.

Another approach consists in expressing the two-dimensional quantum gravity partition function coupled to certain matter systems as the free energy of an associated hermitian matrix model. From it, many quantities can be computed exactly, as this matrix realization can be usually solved using large N techniques. It is interesting to note that matrix models originate from QCD, when in the 70s, t'Hooft [64] realized that planar graphs with a large number of colors could be considered as Feynman diagrams for matrix models. He also noticed that the size of the matrices can be used as an expansion parameter.

Both these discrete models reveal themselves to be in agreement with the Liouville theory. Indeed, considering the KPZ formula on the sphere:

$$\gamma_{\text{str}}^{\text{KPZ}} = 2 - 2 \frac{\sqrt{25-c}}{\sqrt{25-c} - \sqrt{1-c}}, \quad (1.1.15)$$

specific values of it have been exactly derived from dynamical triangulations (see e.g. [22, 29] and references therein) and matrix models (see e.g. [12, 65] and references therein). To exemplify, let us stress the value $c = 0$ corresponding to pure gravity and

yielding $\gamma_{\text{str}} = -\frac{1}{2}$. The case describing a coupling with the Ising model, *i.e.* $c = \frac{1}{2}$ has also been derived [66] and provides a string susceptibility $\gamma_{\text{str}} = -\frac{1}{3}$. The cases $c = -2, 0$, give [29, 62] $\gamma_{\text{str}} = -1, 0$, respectively. All these values are in agreement with (1.1.15).

A continuum limit that includes the sum over topologies of two-dimensional surfaces was then defined [15, 16, 18] for certain matter systems coupled to 2D quantum gravity. Before closing this section, let us add that other approaches to quantum gravity have been developed more recently, like for instance Tensor Field Theory [67–69] or Group Field Theory [70–72]. Details of these approaches are beyond the scope of this thesis.

1.2 The gravitational action

Ever since the seminal paper by Polyakov [11] it has been known that the coupling of conformal matter to gravity in two dimensions gives rise to the Liouville action [26, 73–77] as the effective gravitational action. More precisely,

$$-\ln \frac{Z_{\text{mat}}^{(c)}[g]}{Z_{\text{mat}}^{(c)}[g_0]} = -\frac{c}{24\pi} S_{\text{L}}[g_0, g], \quad (1.2.1)$$

where $Z_{\text{mat}}^{(c)}$ is the partition function of conformal matter of central charge c and g and g_0 two metrics of area A and A_0 related in the conformal gauge by the conformal factor σ : $g = e^{2\sigma} g_0$. Being defined as the logarithm of a ratio of partition functions computed with two different metrics, it is clear that the Liouville action satisfies a co-cycle identity

$$S[g_1, g_2] + S[g_2, g_3] = S[g_1, g_3], \quad (1.2.2)$$

which is a fundamental consistency condition any gravitational action must satisfy [55]. For more general “matter” (plus ghost) partition functions, one defines a general gravitational action as

$$-\ln \frac{Z_{\text{mat}}[g]}{Z_{\text{mat}}[g_0]} = S_{\text{grav}}[g_0, g]. \quad (1.2.3)$$

Any gravitational action defined this way automatically satisfies the co-cycle identity (1.2.2). The simplest example of such a gravitational action is the “cosmological constant action”

$$S_{\mu}[g_0, g] = \mu_c^2 \int d^2x (\sqrt{g} - \sqrt{g_0}) = \mu_c^2 (A - A_0). \quad (1.2.4)$$

This action must actually be added to the Liouville action in (1.2.1) as a counterterm to renormalize the divergences.

Other gravitational actions than the Liouville or cosmological constant actions can be constructed and have been studied mainly in the mathematical literature, like the Mabuchi and Aubin-Yau actions (see e.g. [78–83]). These latter functionals crucially involve not only the conformal factor σ but also directly the Kähler potential ϕ and do admit generalizations to higher-dimensional Kähler manifolds. In the mathematical literature they appear in relation with the characterization of constant scalar curvature

metrics (see e.g. [78, 81, 84–89]). Their roles as two-dimensional gravitational actions in the sense of (1.2.3) have been discussed in some detail in [55, 90]. In particular, ref. [55] has studied the metric dependence of the partition function of non-conformal matter like a massive scalar field and shown that a gravitational action defined by (1.2.3) contains these Mabuchi and Aubin-Yau actions as first-order corrections (first order in $m^2 A$ where m is the mass and A the area of the Riemann surface of metric g) to the Liouville action. The purpose of this (quite technical) section is first to derive a few results valid exactly at finite m . Then, doing a small mass expansion allows us to retrieve some of the results of [55] and in particular the involvement of the Mabuchi and Aubin-Yau actions in the first order in $m^2 A$. This section is quite extensively based on [59].

Ideally one would like to study some general matter action where non-conformal terms $\sim a_i \mathcal{O}^i$ have been added to some conformal theory and obtain exact results in these couplings a_i . We are much less ambitious and simply study a single massive scalar field with action

$$S_{\text{mat}}[g, X] = \frac{1}{2} \int d^2x \sqrt{g} \left[g^{ab} \partial_a X \partial_b X + m^2 X^2 \right] = \frac{1}{2} \int d^2x \sqrt{g} X (\Delta_g + m^2) X . \quad (1.2.5)$$

Here Δ_g is the Laplace operator for the metric g , defined with a minus sign, so that its eigenvalues are non-negative:

$$\Delta_g = -\frac{1}{\sqrt{g}} \partial_a (g^{ab} \sqrt{g} \partial_b) . \quad (1.2.6)$$

Maybe not too surprisingly, some of our massive formula look somewhat similar to those that can be found in [55] for the massless case. However, let us insist that our results are exact in m and valid for any finite mass. Nevertheless, we will write them in a way that immediately allows for a small mass expansion, thus recovering the Liouville action in the zero mass limit and the Mabuchi and Aubin-Yau actions as the first-order corrections.

1.2.1 Massive versus massless matter

One should keep in mind that adding the mass term is *not* just a perturbation by some operator that has a non-zero conformal weight. This is due to the zero mode of the scalar field that is absent from the action for zero mass but obviously plays an important role for non-zero mass. In particular, this means that the relevant quantities of the massive theory are not simply given by those of the massless theory plus order m^2 corrections. This is most clearly exemplified by the Green's function $G(x, y)$ of the operator $\Delta_g + m^2$. We let

$$(\Delta_g + m^2) \psi_n(x) = \lambda_n \psi_n(x) \quad , \quad n = 0, 1, 2, \dots . \quad (1.2.7)$$

In general, if B is any quantity defined for $m \neq 0$, we will denote by $B^{(0)}$ the corresponding quantity for $m = 0$. Clearly, the eigenfunctions ψ_n do not depend on m (*i.e.* $\psi_n = \psi_n^{(0)}$), while $\lambda_n = \lambda_n^{(0)} + m^2$. The eigenfunctions which may be chosen to be real, are orthonormalized as

$$\int d^2x \sqrt{g(x)} \psi_n(x) \psi_k(x) = \delta_{nk} . \quad (1.2.8)$$

As it is clear from (1.2.6), Δ_g always has a zero mode and, hence,

$$\lambda_0 = m^2 \quad , \quad \psi_0 = \frac{1}{\sqrt{A}} . \quad (1.2.9)$$

The Green's function for $m \neq 0$ is given by

$$G(x, y) = \sum_{n \geq 0} \frac{\psi_n(x)\psi_n(y)}{\lambda_n} \quad , \quad (\Delta_g + m^2) G(x, y) = \frac{1}{\sqrt{g}} \delta(x - y) . \quad (1.2.10)$$

For $m = 0$ these definitions must be modified. Since $\lambda_0^{(0)} = 0$, obviously, the zero mode must be excluded from the sum and then

$$\tilde{G}^{(0)}(x, y) = \sum_{n > 0} \frac{\psi_n(x)\psi_n(y)}{\lambda_n^{(0)}} \quad , \quad \Delta_g \tilde{G}^{(0)}(x, y) = \frac{1}{\sqrt{g}} \delta(x - y) - \frac{1}{A} . \quad (1.2.11)$$

The subtraction of $\frac{1}{A}$ on the r.h.s. ensures that, when integrated $\int d^2x \sqrt{g} \dots$, one correctly gets zero. We will consistently put a tilde over the various quantities we will encounter if the zero mode is excluded from the sum.¹ In particular, using (1.2.9), we can write for the massive Green's function

$$G(x, y) = \frac{1}{m^2 A} + \tilde{G}(x, y) \quad , \quad (\Delta_g + m^2) \tilde{G}(x, y) = \frac{1}{\sqrt{g}} \delta(x - y) - \frac{1}{A} . \quad (1.2.12)$$

The smallest eigenvalue contributing in \tilde{G} is $\lambda_1 = \lambda_1^{(0)} + m^2$ with $\lambda_1^{(0)} > 0$ being of order $\frac{1}{A}$ [53]. Thus if $m^2 A \ll 1$ one can expand $\frac{1}{\lambda_n} = \sum_{r=0}^{\infty} (-)^r \frac{m^{2r}}{(\lambda_n^{(0)})^{r+1}}$, resulting in

$$G(x, y) = \frac{1}{m^2 A} + \tilde{G}^{(0)}(x, y) + \sum_{r=1}^{\infty} (-m^2)^r \tilde{G}_{r+1}^{(0)}(x, y) , \quad (1.2.13)$$

where

$$G_r(x, y) = \sum_{n \geq 0} \frac{\psi_n(x)\psi_n(y)}{\lambda_n^r} \quad , \quad \tilde{G}_r^{(0)}(x, y) = \sum_{n > 0} \frac{\psi_n(x)\psi_n(y)}{(\lambda_n^{(0)})^r} . \quad (1.2.14)$$

Clearly, the massive Green's function does not equal the massless one plus order- m^2 corrections since there is a crucial $\frac{1}{m^2 A}$ term in (1.2.13).

Writing $X = \sum_{n \geq 0} a_n \psi_n$, the matter partition function is defined as

$$Z_{\text{mat}}[g] = \int \mathcal{D}_g X e^{-S_{\text{mat}}[g, X]} = \int \prod_{n=0}^{\infty} \frac{da_n}{\sqrt{2\pi}} e^{-\frac{1}{2} \sum_{n \geq 0} \lambda_n a_n^2} = (\det(\Delta_g + m^2))^{-1/2} . \quad (1.2.15)$$

¹Except for determinants missing the zero mode, where we will write \det' , following the usual notation.

In the massless case, since $\lambda_0^{(0)} = 0$, the integration over a_0 would be divergent and instead one replaces it by a factor \sqrt{A} (see e.g. [55]). Thus

$$Z_{\text{mat}}^{(0)}[g] = \int \mathcal{D}_g^{(0)} X e^{-S_{\text{mat}}^{(0)}[g, X]} = \sqrt{A} \int \prod_{n=1}^{\infty} \frac{da_n}{\sqrt{2\pi}} e^{-\frac{1}{2} \sum_{n>0} \lambda_n^{(0)} a_n^2} = \left(\frac{\det' \Delta_g}{A} \right)^{-1/2}. \quad (1.2.16)$$

Of course, the determinants \det and \det' are ill-defined and need to be regularized. We will use the very convenient zeta function scheme to regularize-renormalize them (see e.g. [53, 91, 92]). The spectral ζ -functions are defined as

$$\zeta(s) = \sum_{n \geq 0} \lambda_n^{-s}, \quad \tilde{\zeta}(s) = \sum_{n > 0} \lambda_n^{-s}, \quad (1.2.17)$$

and similarly for $\tilde{\zeta}^{(0)}(s)$. By Weil's law (see e.g. [53]), the asymptotic behaviour of the eigenvalues for large n is $\lambda_n \sim \frac{n}{A}$ and, hence the spectral ζ -functions are defined by converging sums for $\text{Re } s > 1$, and by analytic continuations for all other values. In particular, they are well-defined meromorphic functions for all s with a single pole at $s = 1$ with residue $\frac{1}{4\pi}$ (see e.g. [53]). A straightforward formal manipulation shows that $\zeta'(0) \equiv \frac{d}{ds} \zeta(s)|_{s=0}$ provides a formal definition of $-\sum_{n \geq 0} \ln \lambda_n$, *i.e.* of $-\ln \det(\Delta_g + m^2)$:

$$Z_{\text{mat}}[g] = \exp\left(\frac{1}{2} \zeta'(0)\right), \quad Z_{\text{mat}}^{(0)}[g] = A^{1/2} \exp\left(\frac{1}{2} (\tilde{\zeta}^{(0)})'(0)\right). \quad (1.2.18)$$

There is a slight subtlety one should take into account, see e.g. [53]. While the field X is dimensionless, the ψ_n scale as $A^{-1/2} \sim \mu$ where μ is some arbitrary mass scale (even if $m = 0$), and the a_n as μ^{-1} . It follows that one should write $\mathcal{D}_g X = \prod_n \frac{\mu da_n}{\sqrt{2\pi}}$. This results in $Z_{\text{mat}} = \left(\prod_n \frac{\lambda_n}{\mu^2}\right)^{-1/2}$, so that $\zeta'(0)$ should be changed into

$$\zeta'(0) \rightarrow \zeta'(0) + \zeta(0) \ln \mu^2 \quad (1.2.19)$$

in the previous expressions of Z_{mat} and $Z_{\text{mat}}^{(0)}$ [53]. This writing may seem inhomogeneous since μ encountered in the logarithm is a mass scale. However, this seemingly inhomogeneity is just a trick since this combination stands for $-\sum_{n \geq 0} \ln \frac{\lambda_n}{\mu^2}$. In section 1.2.5, we will get back with quantities of seemingly traceable homogeneity.

The regularization-renormalization of determinants in terms of the ζ -functions may appear as rather ad hoc, but it can be rigorously justified by introducing the spectral regularization [53]. The regularized logarithm of the determinant then equals $\zeta'(0) + \zeta(0) \ln \mu^2$ plus a diverging piece $\sim A \Lambda^2 (\ln \frac{\Lambda^2}{\mu^2} + \text{const})$, where Λ is some cutoff. This diverging piece just contributes to the cosmological constant action (1.2.4), and this is why the latter must be present as a counterterm, to cancel this divergence.

Thus,

$$S_{\text{grav}}[g_0, g] = -\frac{1}{2} \left(\zeta'_g(0) - \zeta'_{g_0}(0) + [\zeta_g(0) - \zeta_{g_0}(0)] \ln \mu^2 \right), \quad (1.2.20)$$

$$S_{\text{grav}}^{(0)}[g_0, g] = -\frac{1}{2} \ln \frac{A}{A_0} - \frac{1}{2} \left((\tilde{\zeta}_g^{(0)})'(0) - (\tilde{\zeta}_{g_0}^{(0)})'(0) + [\tilde{\zeta}_g^{(0)}(0) - \tilde{\zeta}_{g_0}^{(0)}(0)] \ln \mu^2 \right),$$

where the first line refers to the massive case and the second line to the massless one.

1.2.2 The Kähler formalism

Consider a compact Riemann surface with fixed complex structure moduli. As already mentioned, up to diffeomorphisms, any two-dimensional metric g on the surface may be written in conformal gauge as $g = e^{2\sigma}g_0$ where g_0 is a reference metric that can be chosen to be the constant curvature metric associated with some area A_0 . Moreover, in two dimensions all the metrics are Kähler's, so that rather than writing the metric g in terms of this g_0 and the conformal factor σ , one can also parametrize it in terms of g_0 , the area $A = \int d^2x \sqrt{g}$ and the Kähler potential ϕ as follows:

$$g = e^{2\sigma}g_0 \quad , \quad e^{2\sigma} = \frac{A}{A_0} \left(1 - \frac{1}{2}A_0\Delta_0\phi \right) \quad , \quad (1.2.21)$$

where Δ_0 denotes the *positive* Laplacian (1.2.6) for the reference metric g_0 . Of course, $\Delta = e^{-2\sigma}\Delta_0$. This Kähler parametrization (1.2.21) has certain advantages and is certainly most convenient if one wants to consider metrics of fixed area, as it will be done throughout this thesis. Given σ , the above relation actually defines A and ϕ uniquely, up to unphysical constant shifts of ϕ . Moreover, positivity of the metric implies the non-perturbative constraint $\frac{1}{2}A_0\Delta_0\phi < 1$. While being crucial in a non-perturbative definition of the integral over the Kähler potentials [54, 90], this constraint is irrelevant in perturbation theory and thus will be ignored in this thesis. The second relation in (1.2.21) is equivalent to the relation

$$\omega = \frac{A}{A_0}\omega_0 + iA\partial\bar{\partial}\phi \quad (1.2.22)$$

between the volume (Kähler) forms ω and ω_0 of the metrics g and g_0 . Often in the following we will use a rescaled (constant curvature) metric g_* of area A , with corresponding Laplace operator Δ_* and Ricci scalar R_* given by

$$g_* = \frac{A}{A_0}g_0 \quad , \quad \Delta_* = \frac{A_0}{A}\Delta_0 \quad , \quad R_* = \frac{A_0}{A}R_0 = \frac{8\pi(1-h)}{A} \quad . \quad (1.2.23)$$

In particular, since $A_0\Delta_0 = A\Delta_*$, eq. (1.2.21) can also be written as

$$e^{2\sigma} = \frac{A}{A_0} \left(1 - \frac{1}{2}A\Delta_*\phi \right) \quad \Leftrightarrow \quad g = g_* \left(1 - \frac{1}{2}A\Delta_*\phi \right) \quad . \quad (1.2.24)$$

Consequently, one has

$$\int d^2x \sqrt{g} = \int d^2x \sqrt{g_*} = A \quad , \quad \int d^2x \sqrt{g_*} R_* = 8\pi(1-h) \quad . \quad (1.2.25)$$

As compared to the DDK approach (cf section 1.1.1), where in the quantum theory the area is computed as $\int d^2x \sqrt{g_0} e^{2\alpha\sigma}$ (cf (1.1.5)), in the Kähler formalism the area A is a ‘‘coordinate’’ on the space of metrics and (1.2.25) always holds exactly even in the quantum theory where ϕ (or $A\Delta_*\phi$) is the quantum field.

1.2.3 Mabuchi and Aubin-Yau actions

Let us briefly recall the basic properties of the known gravitational actions. While the Liouville action (1.1.4) is simply written in terms of g_0 and the conformal factor σ , the Mabuchi and Aubin-Yau actions are naturally formulated using also the Kähler potential ϕ . The Mabuchi action on a Riemann surface of genus h can be written as [55]

$$S_M[g_0, g] = \int d^2x \sqrt{g_0} \left[2\pi(h-1)\phi\Delta_0\phi + \left(\frac{8\pi(1-h)}{A_0} - R_0 \right) \phi + \frac{4}{A} \sigma e^{2\sigma} \right], \quad (1.2.26)$$

while the Aubin-Yau action takes the form

$$S_{AY}[g_0, g] = - \int d^2x \sqrt{g_0} \left[\frac{1}{4} \phi \Delta_0 \phi - \frac{\phi}{A_0} \right]. \quad (1.2.27)$$

As already mentioned, they both satisfy the co-cycle identity (1.2.2) and were shown [55] to appear as accompanying the term of first order in an expansion in $m^2 A$ of S_{grav} . Note that $S_M = 8\pi(1-h)S_{AY} + \int d^2x \sqrt{g_0} \left(\frac{4}{A} \sigma e^{2\sigma} - R_0 \phi \right)$. Eq. (1.2.21) relates the variations $\delta\sigma$ and $\delta\phi$ as

$$\delta\sigma = \frac{\delta A}{2A} - \frac{A}{4} \Delta \delta\phi \quad \text{and} \quad \delta \left(\frac{e^{2\sigma}}{A} \right) = -\frac{1}{2} \Delta_0 \delta\phi. \quad (1.2.28)$$

It is then straightforward to show that the variations of the Liouville, Mabuchi and Aubin-Yau actions are given by

$$\begin{aligned} \delta S_L[g_0, g] &= 4\pi(1-h) \frac{\delta A}{A} - \frac{A}{4} \int d^2x \sqrt{g} \Delta R \delta\phi, \\ \delta S_M[g_0, g] &= 2 \frac{\delta A}{A} - \int d^2x \sqrt{g} \left(R - \frac{8\pi(1-h)}{A} \right) \delta\phi, \\ \delta S_{AY}[g_0, g] &= \frac{1}{A} \int d^2x \sqrt{g} \delta\phi. \end{aligned} \quad (1.2.29)$$

Thus the Liouville and Mabuchi actions obviously admit the constant scalar curvature metrics as saddle-points at fixed area. Although not obvious from the previous equation, the variation of the Aubin-Yau action when restricted to the space of Bergmann metrics is similarly related to metrics of constant scalar curvature [81].

1.2.4 Heat kernel and zeta functions

Before going forward and computing δS_{grav} as defined in (1.2.20), one must introduce useful quantities such as the heat kernel and the local zeta functions (see e.g. [53, 93–95] and the review [96]).

The heat kernel and the integrated heat kernel for the operator $\Delta_g + m^2$ are defined in terms of the eigenvalues and eigenfunctions (1.2.7) as

$$K(t, x, y) = \sum_{n \geq 0} e^{-\lambda_n t} \psi_n(x) \bar{\psi}_n(y) \quad , \quad K(t) = \int d^2x \sqrt{g} K(t, x, x) = \sum_{n \geq 0} e^{-\lambda_n t}. \quad (1.2.30)$$

The corresponding \tilde{K} , $K^{(0)}$ and $\tilde{K}^{(0)}$ are defined similarly. The heat kernel K is the solution of

$$\left(\frac{d}{dt} + \Delta_g + m^2\right) K(t, x, y) = 0 \quad , \quad K(t, x, y) \sim \frac{1}{\sqrt{g}} \delta(x - y) \text{ as } t \rightarrow 0 . \quad (1.2.31)$$

Note that it immediately follows from either (1.2.30) or (1.2.31) that the massless and massive heat kernels are simply related by

$$K(t, x, y) = e^{-m^2 t} K^{(0)}(t, x, y) . \quad (1.2.32)$$

Since the eigenvalues λ_n of $\Delta_g + m^2$ are positive, it is clear from (1.2.30) that $K(t, x, y)$ is given by a converging sum and remains finite even as $x \rightarrow y$ as long as $t > 0$. For $t \rightarrow 0$ however, one recovers various divergences, and in particular

$$\int_0^\infty dt K(t, x, y) = G(x, y) \quad (1.2.33)$$

exhibits the short distance singularity of the Green's function which is well-known to be logarithmic in two dimensions.

The behaviour of K for small t is related to the asymptotics of the eigenvalues λ_n for large n , which in turn is related to the short-distance properties of the Riemann surface. It is thus not surprising that the small t asymptotic is given in terms of local expressions of the curvature and its derivatives. Indeed, one has the well-known small t expansion :

$$K(t, x, y) = \frac{e^{-(\ell^2/4t) - m^2 t}}{4\pi t} \left[a_0^{(0)}(x, y) + a_1^{(0)}(x, y) t + a_2^{(0)}(x, y) t^2 + \mathcal{O}(t^3) \right] \quad (1.2.34)$$

where $\ell^2 \equiv \ell_g^2(x, y)$ is the geodesic distance squared between x and y . For small t , the exponential forces ℓ^2 to be small (of order \sqrt{t}) and one can use normal coordinates around y . This allows one to obtain quite easily explicit expressions for the $a_r(x, y)$ in terms of the curvature tensor and its derivatives. They can be found e.g. in [53]. Here, we will only need them at coinciding points $y = x$, where

$$K(t, y, y) = \frac{1}{4\pi t} \left[1 + \left(\frac{R}{6} - m^2 \right) t + \dots \right] . \quad (1.2.35)$$

Let us note that in the massless case and if the zero mode is excluded one has instead

$$\tilde{K}^{(0)}(t, y, y) = \frac{1}{4\pi t} \left[1 + \left(\frac{R}{6} - \frac{4\pi}{A} \right) t + \dots \right] . \quad (1.2.36)$$

The spectral regularization scheme developed in [53] and used in this thesis also involves the heat kernel so that we will come back to these quantities later in section 1.3.3 and then give the specific values of the $a_i(x, y)$ for the study carried on in Chapter 2 and 3.

Local versions of the ζ -functions are defined as (see e.g. [53])

$$\zeta(s, x, y) = \sum_{n \geq 0} \frac{\psi_n(x) \psi_n(y)}{\lambda_n^s} , \quad (1.2.37)$$

and similarly for $\tilde{\zeta}(s, x, y)$, etc. Note that $\zeta(1, x, y) = G(x, y)$, while for $s = r = 2, 3, \dots$ these local ζ -functions coincide with the $G_r(x, y)$ defined above in (1.2.14). They are related to the heat kernel by

$$\zeta(s, x, y) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} K(t, x, y) . \quad (1.2.38)$$

For $s = 0, -1, -2, \dots$, $\Gamma(s)$ has poles and the value of $\zeta(s, x, y)$ is entirely determined by the singularities of the integral over t that arise from the small t asymptotic of K . As shown above, the latter is given by local quantities on the Riemann surface. Explicitly (see e.g. [53])

$$\zeta(-k, x, x) = (-1)^k k! \frac{a_{1+k}(x, x)}{4\pi} , \quad k = 0, 1, 2, \dots \quad (1.2.39)$$

so that in particular,

$$\zeta(0, x, x) = \frac{R(x)}{24\pi} - \frac{m^2}{4\pi} \quad \text{and} \quad \tilde{\zeta}(0, x, x) = \frac{R(x)}{24\pi} - \frac{m^2}{4\pi} - \frac{1}{A} . \quad (1.2.40)$$

On the other hand, the values for $s = 1, 2, 3, \dots$ or the derivatives at $s = 0$ cannot be determined just from the small t asymptotic. To access them requires the knowledge of the full spectrum of $\Delta_g + m^2$.

Clearly, $\zeta(1, x, y) = G(x, y)$ is singular as $x \rightarrow y$, while for $s \neq 1$, $\zeta(s, x, y)$ provides a regularization of the propagator. More precisely, it follows from (1.2.38) that $\zeta(s, x, x)$ is a meromorphic function with a pole at $s = 1$ and that the residue of this pole is $\frac{a_0(x, x)}{4\pi} = \frac{1}{4\pi}$. Thus [53]

$$G_\zeta(x) = \lim_{s \rightarrow 1} \left[\mu^{2(s-1)} \zeta(s, x, x) - \frac{1}{4\pi(s-1)} \right] \quad (1.2.41)$$

is well-defined. (Here μ is an arbitrary scale.) This is an important quantity, called the ‘‘Green’s function at coinciding points’’. One can give an alternative definition of G_ζ by subtracting the short distance singularity from $G(x, y)$ and taking $x \rightarrow y$. More precisely

$$G_\zeta(y) = \lim_{x \rightarrow y} \left[G(x, y) + \frac{1}{4\pi} \left(\ln \frac{\ell_g^2(x, y) \mu^2}{4} + 2\gamma \right) \right] \quad (1.2.42)$$

where $\ell_g^2(x, y)$ is the geodesic distance between x and y in the metric g . One can show [53] that both definitions of G_ζ are equivalent and define the same quantity. The same relations hold between $\tilde{G}_\zeta(x)$, $\tilde{G}(x, y)$ and $\tilde{\zeta}(s, x, x)$. Note that $G_\zeta(x)$ contains global information about the Riemann surface and cannot be expressed in terms of local quantities only.

1.2.5 Variation of the gravitational action

From (1.2.20), one directly reads the variation of the gravitational action:

$$\begin{aligned} \delta S_{\text{grav}}[g_0, g] &= -\frac{1}{2} (\delta\zeta'(0) + \delta\zeta(0) \ln \mu^2) , \\ \delta S_{\text{grav}}^{(0)}[g_0, g] &= -\frac{\delta A}{2A} - \frac{1}{2} \left(\delta(\tilde{\zeta}^{(0)})'(0) + \delta\tilde{\zeta}^{(0)}(0) \ln \mu^2 \right) . \end{aligned} \quad (1.2.43)$$

Our goal is to compute $\delta\zeta'(0) \equiv \delta\zeta'_g(0)$ and $\delta\zeta(0) \equiv \delta\zeta_g(0)$ and express them as “exact differentials” so that one can integrate them and obtain the finite differences $\zeta'_{g_2}(0) - \zeta'_{g_1}(0)$ and $\zeta_{g_2}(0) - \zeta_{g_1}(0)$, thus accessing $S_{\text{grav}}[g_1, g_2]$.

To do so, we need to study how the eigenvalues λ_n and eigenfunctions ψ_n change under an infinitesimal change of the metric. Since $g = e^{2\sigma}g_0$, the Laplace operator Δ_g and hence also $\Delta_g + m^2$ only depend on the conformal factor σ and on g_0 : $\Delta_g = e^{-2\sigma}\Delta_0$ and thus under a variation $\delta\sigma$ of σ one has

$$\delta\Delta_g = -2\delta\sigma\Delta_g \quad \Rightarrow \quad \langle\psi_k|\delta\Delta_g|\psi_n\rangle = -2\lambda_n^{(0)}\langle\psi_k|\delta\sigma|\psi_n\rangle = -2(\lambda_n - m^2)\langle\psi_k|\delta\sigma|\psi_n\rangle, \quad (1.2.44)$$

where, $\langle\psi_k|\delta\sigma|\psi_n\rangle = \int d^2x\sqrt{g}\psi_k\delta\sigma\psi_n$. One can then apply standard quantum mechanical perturbation theory. The only subtlety comes from the normalisation condition (1.2.8) which also gets modified when varying σ [53, 55]. One finds

$$\delta\lambda_n = -2(\lambda_n - m^2)\langle\psi_n|\delta\sigma|\psi_n\rangle, \quad (1.2.45)$$

$$\delta\psi_n = -\langle\psi_n|\delta\sigma|\psi_n\rangle\psi_n - 2\sum_{k\neq n}\frac{\lambda_n - m^2}{\lambda_n - \lambda_k}\langle\psi_k|\delta\sigma|\psi_n\rangle\psi_k. \quad (1.2.46)$$

Let us insist that this is first-order perturbation theory in $\delta\sigma$, but it is exact in m^2 . From (1.2.45) one immediately gets,

$$\zeta_{g+\delta g}(s) = \sum_{n\geq 0}\frac{1}{(\lambda_n + \delta\lambda_n)^s} = \zeta_g(s) + 2s\sum_{n\geq 0}\frac{\lambda_n - m^2}{\lambda_n^{s+1}}\langle\psi_n|\delta\sigma|\psi_n\rangle. \quad (1.2.47)$$

Note that $\lambda_0 = m^2$ and, hence, there is no zero mode contribution to the second term. Thus,

$$\delta\zeta(s) = \delta\tilde{\zeta}(s) = 2s\int d^2x\sqrt{g}\delta\sigma(x)\left[\tilde{\zeta}(s, x, x) - m^2\tilde{\zeta}(s+1, x, x)\right]. \quad (1.2.48)$$

For $m \neq 0$, the term in brackets could have been equally well written as $\zeta(s, x, x) - m^2\zeta(s+1, x, x)$, but the writing in terms of the $\tilde{\zeta}$ is valid for all non-zero and zero values of m . It follows that

$$\begin{aligned} \delta\zeta'(0) &= 2\int d^2x\sqrt{g}\delta\sigma(x)\tilde{\zeta}(0, x, x) \\ &\quad - 2m^2\int d^2x\sqrt{g}\delta\sigma(x)\lim_{s\rightarrow 0}\left[\tilde{\zeta}(s+1, x, x) + s\tilde{\zeta}'(s+1, x, x)\right], \\ \delta\zeta(0) &= -2m^2\int d^2x\sqrt{g}\delta\sigma(x)\lim_{s\rightarrow 0}\left[s\tilde{\zeta}(s+1, x, x)\right]. \end{aligned} \quad (1.2.49)$$

As recalled above, $\tilde{\zeta}(s, x, x)$ has a pole at $s = 1$ with residue $\frac{1}{4\pi}$. Hence,

$$\tilde{\zeta}(s, x, x) = \tilde{\zeta}_{\text{reg}}(s, x, x) + \frac{1}{4\pi(s-1)}, \quad (1.2.50)$$

which implies

$$\lim_{s \rightarrow 0} [\tilde{\zeta}(s+1, x, x) + s \tilde{\zeta}'(s+1, x, x)] = \tilde{\zeta}_{\text{reg}}(1, x, x) . \quad (1.2.51)$$

From (1.2.41) one sees that $\tilde{\zeta}_{\text{reg}}(1, x, x) = \tilde{G}_\zeta(x) - \frac{1}{4\pi} \ln \mu^2$. Using also (1.2.40) and (1.2.28), as well as $\delta A = \int d^2x \sqrt{g_0} e^{2\sigma} 2\delta\sigma$, we find

$$\begin{aligned} \delta\zeta'(0) &= \frac{1}{12\pi} \int d^2x \sqrt{g} \delta\sigma(x) R(x) - \frac{\delta A}{A} - 2m^2 \int d^2x \sqrt{g} \delta\sigma(x) \left(\tilde{G}_\zeta(x) + \frac{1}{4\pi} - \frac{1}{4\pi} \ln \mu^2 \right) \\ \delta\zeta(0) &= -\frac{m^2}{2\pi} \int d^2x \sqrt{g} \delta\sigma(x) . \end{aligned} \quad (1.2.52)$$

The result for $\delta\zeta(0)$ can also be directly obtained from (1.2.40), using the fact that

$$\int d^2x \sqrt{g} R(x) = 8\pi(1-h) \quad (1.2.53)$$

is a topological invariant. Since $G_\zeta = \tilde{G}_\zeta + \frac{1}{m^2 A}$, we arrive at two equivalent expressions for $\delta\zeta'(0) + \delta\zeta(0) \ln \mu^2$:

$$\begin{aligned} \delta\zeta'(0) + \delta\zeta(0) \ln \mu^2 &= \delta\tilde{\zeta}'(0) + \delta\tilde{\zeta}(0) \ln \mu^2 \\ &= \frac{1}{12\pi} \int d^2x \sqrt{g} \delta\sigma(x) R(x) - \frac{\delta A}{A} - 2m^2 \int d^2x \sqrt{g} \delta\sigma(x) \left(\tilde{G}_\zeta(x) + \frac{1}{4\pi} \right) \\ &= \frac{1}{12\pi} \int d^2x \sqrt{g} \delta\sigma(x) R(x) - 2m^2 \int d^2x \sqrt{g} \delta\sigma(x) \left(G_\zeta(x) + \frac{1}{4\pi} \right) . \end{aligned} \quad (1.2.54)$$

As it stands, this result is exact in m and holds whether $m^2 A$ is small or not. Let us insist that the G_ζ and \tilde{G}_ζ appearing on the right-hand side are the massive ones. The first writing is the appropriate one to study the small m^2 asymptotic, as \tilde{G}_ζ has a smooth limit for $m \rightarrow 0$.

The massless case

It is now straightforward to recover the Liouville action as the gravitational action in the massless case. Inserting (1.2.54) in (1.2.43) and using (1.2.28) and (1.2.53), one gets:

$$\delta S_{\text{grav}}^{(0)}[g_0, g] = -\frac{1}{24\pi} \int d^2x \sqrt{g} \delta\sigma(x) R(x) = -\frac{1}{24\pi} \delta S_{\text{L}}[g_0, \sigma] . \quad (1.2.55)$$

Note that the term in $\delta A/A$ in (1.2.54) precisely cancels with the one in (1.2.43). Of course, eq. (1.2.55) is just the contribution of one conformal scalar field with $c = 1$. However, writing the metric $g = e^{2\sigma} g_0$ amounts to fixing the diffeomorphism invariance. The corresponding Faddeev-Popov determinant that arises gives also a contribution to S_{grav} in terms of the Liouville action but with a coefficient $+\frac{26}{24\pi}$ (see e.g. [11, 97, 98]), so that one gets the overall contribution:

$$S_{\text{grav}}^{(0)}[g_0, g] \Big|_{\text{ghost} + \text{conf matter}} = \frac{26-c}{24\pi} S_{\text{L}}[g_0, \sigma] . \quad (1.2.56)$$

The massive case

Combining (1.2.43) and (1.2.54), and using (1.2.55), one gets:

$$\delta S_{\text{grav}}[g_0, g] = -\frac{1}{24\pi} \delta S_L[g_0, g] + m^2 \int d^2x \sqrt{g} \delta\sigma(x) \left(G_\zeta(x) + \frac{1}{4\pi} \right). \quad (1.2.57)$$

The task ahead is then to rewrite the second term on the r.h.s as the variation of some local functional.

Let us compute $\delta G_\zeta(x)$. In order to do so, we first establish a formula for $\delta G(x, y)$ under a variation $\delta g = 2\delta\sigma g$ of the metric and thus under a corresponding variation $\delta\Delta_g = -2\delta\sigma\Delta_g$ of the Laplace operator. One can then either use the definition (1.2.10) as an infinite sum and the perturbation theory formula (1.2.45) and (1.2.46), or directly the defining differential equation (1.2.10). In any case one finds

$$\delta G(x, y) = -2m^2 \int d^2z \sqrt{g} G(x, z) \delta\sigma(z) G(z, y). \quad (1.2.58)$$

To obtain the variation of G_ζ , according to (1.2.42) one needs to subtract the variation of the short-distance singularity. Now, the geodesic distance $\ell_g(x, y)$ transforms as (see e.g. appendix A1 of [53])

$$\delta \ell_g^2(x, y) = \ell_g^2(x, y) [\delta\sigma(x) + \delta\sigma(y) + \mathcal{O}((x-y)^2)]. \quad (1.2.59)$$

It follows that

$$\lim_{x \rightarrow y} \delta \ln [\mu^2 \ell_g^2(x, y)] = 2\delta\sigma(x). \quad (1.2.60)$$

Plugging (1.2.58) and (1.2.60) into (1.2.42) one gets

$$\delta G_\zeta(x) = -2m^2 \int d^2z \sqrt{g} (G(x, z))^2 \delta\sigma(z) + \frac{\delta\sigma(x)}{2\pi}. \quad (1.2.61)$$

Upon integrating this over x one encounters

$$\begin{aligned} \int d^2x \sqrt{g} (G(x, z))^2 &= \int d^2x \sqrt{g} \sum_{n, k \geq 0} \frac{\psi_n(x) \psi_n(z) \psi_k(x) \psi_k(z)}{\lambda_n \lambda_k} \\ &= \sum_{n \geq 0} \frac{\psi_n(z) \psi_n(z)}{\lambda_n^2} \\ &= \zeta(2, z, z). \end{aligned} \quad (1.2.62)$$

(Note that $\zeta(2, z, z) = G_2(z, z)$ is finite.) It follows that

$$\begin{aligned} \delta \int d^2x \sqrt{g} G_\zeta(x) &= 2 \int d^2x \sqrt{g} G_\zeta(x) \delta\sigma(x) - 2m^2 \int d^2z \sqrt{g} \zeta(2, z, z) \delta\sigma(z) \\ &\quad + \frac{1}{2\pi} \int d^2x \sqrt{g} \delta\sigma(x). \end{aligned} \quad (1.2.63)$$

One can then rewrite (1.2.57) as

$$\delta S_{\text{grav}}[g_0, g] = \delta \left[-\frac{1}{24\pi} S_{\text{L}}[g_0, g] + \frac{m^2}{2} \int d^2x \sqrt{g} G_\zeta(x) \right] + m^4 \int d^2x \sqrt{g} \zeta(2, x, x) \delta\sigma(x). \quad (1.2.64)$$

Note that we can replace G_ζ by \tilde{G}_ζ in the second term. Indeed, their difference is $\frac{m^2}{2} \int d^2x \sqrt{g} \frac{1}{m^2 A} = \frac{1}{2}$, whose variation vanishes.

Next, we use (1.2.38) to rewrite the last term as $m^4 \int_0^\infty dt t \int d^2x \sqrt{g} K(t, x, x) \delta\sigma(x)$, and establish a formula for the variation of the integrated heat kernel $K(t)$. Since $\lambda_0 = m^2$ its variation vanishes and we have

$$\begin{aligned} \delta K(t) &= \delta \tilde{K}(t) = - \sum_{n>0} t e^{-\lambda_n t} \delta \lambda_n = 2 \sum_{n>0} t e^{-\lambda_n t} (\lambda_n - m^2) \int d^2x \sqrt{g} \psi_n^2(x) \delta\sigma(x) \\ &= -2t \left(\frac{d}{dt} + m^2 \right) \int d^2x \sqrt{g} \tilde{K}(t, x, x) \delta\sigma(x) \\ &= -2t e^{-m^2 t} \frac{d}{dt} \int d^2x \sqrt{g} \tilde{K}^{(0)}(t, x, x) \delta\sigma(x), \end{aligned} \quad (1.2.65)$$

where we used (1.2.45) and (1.2.32). It then follows that

$$\begin{aligned} \frac{1}{2} \int_0^\infty \frac{dt}{t} (e^{m^2 t} - m^2 t - 1) \delta \tilde{K}(t) &= \\ &= - \int_0^\infty dt (e^{m^2 t} - m^2 t - 1) \left(\frac{d}{dt} + m^2 \right) \int d^2x \sqrt{g} \tilde{K}(t, x, x) \delta\sigma(x) \\ &= m^4 \int_0^\infty dt t \int d^2x \sqrt{g} \tilde{K}(t, x, x) \delta\sigma(x) = m^4 \int d^2x \sqrt{g} \tilde{\zeta}(2, x, x) \delta\sigma(x) \\ &= m^4 \int d^2x \sqrt{g} \zeta(2, x, x) \delta\sigma(x) - \frac{1}{A} \int d^2x \sqrt{g} \delta\sigma(x), \end{aligned} \quad (1.2.66)$$

where we integrated by parts and used (1.2.38). The boundary terms do not contribute² since \tilde{K} vanishes at $t = \infty$ as $e^{-\lambda_1 t}$ and $\lambda_1 - m^2 > 0$. Upon inserting this into (1.2.64) we finally get

$$\begin{aligned} \delta S_{\text{grav}}[g_0, g] &= \delta \left[-\frac{1}{24\pi} S_{\text{L}}[g_0, g] + \frac{1}{2} \ln \frac{A}{A_0} + \frac{m^2}{2} \int d^2x \sqrt{g} \tilde{G}_\zeta(x) \right. \\ &\quad \left. + \frac{1}{2} \int_0^\infty \frac{dt}{t} (e^{m^2 t} - m^2 t - 1) \tilde{K}(t) \right]. \end{aligned} \quad (1.2.67)$$

Note that in the last term the t -integral is convergent both at $t = 0$ and at $t = \infty$. This

²Had we started with δK rather than $\delta \tilde{K}$ and written this equation for K and $\zeta(2, x, x)$, the $-\frac{1}{A} \int \sqrt{g} \delta\sigma$ would have appeared as the boundary term.

is immediately integrated as

$$\begin{aligned} S_{\text{grav}}[g_0, g] &= -\frac{1}{24\pi} S_{\text{L}}[g_0, g] + \frac{1}{2} \ln \frac{A}{A_0} + \frac{m^2}{2} \int d^2x (\sqrt{g} \tilde{G}_\zeta(x; g) - \sqrt{g_0} \tilde{G}_\zeta(x; g_0)) \\ &\quad + \frac{1}{2} \int_0^\infty \frac{dt}{t} (e^{m^2 t} - m^2 t - 1) (\tilde{K}(t; g) - \tilde{K}(t; g_0)). \end{aligned} \quad (1.2.68)$$

Thus we have expressed the variation of the gravitational action as the variation of a sum of (local) functionals that are all perfectly well-defined without any need of analytical continuation (contrary to the initial $\zeta'(0)$).

It will be useful to rewrite (1.2.61) to obtain the variation of \tilde{G}_ζ in terms of quantities that all have well-defined limits as $m \rightarrow 0$. Recall that $G = \frac{1}{m^2 A} + \tilde{G}$ and $G_\zeta = \frac{1}{m^2 A} + \tilde{G}_\zeta$. Thus (1.2.61) can be rewritten as

$$\delta \tilde{G}_\zeta(x) = -\frac{4}{A} \int d^2z \sqrt{g} \tilde{G}(x, z) \delta\sigma(z) - 2m^2 \int d^2z \sqrt{g} (\tilde{G}(x, z))^2 \delta\sigma(z) + \frac{\delta\sigma(x)}{2\pi}. \quad (1.2.69)$$

Using (1.2.28), we can replace $\delta\sigma$ in the second term by $-A\Delta\delta\phi/4$. Indeed, the $\delta A/2A$ piece doesn't contribute since \tilde{G} has no zero mode. Using the differential equation (1.2.12) satisfied by \tilde{G} , the second term may then be rewritten as

$$\begin{aligned} -\frac{4}{A} \int d^2z \sqrt{g} \tilde{G}(x, z) \delta\sigma(z) &= \int d^2z \sqrt{g} \delta\phi(z) \Delta_z \tilde{G}(x, z) \\ &= \delta\phi(x) - \delta S_{\text{AY}}[g_0, g] - m^2 \int d^2z \sqrt{g} \tilde{G}(x, z) \delta\phi(z), \end{aligned} \quad (1.2.70)$$

with δS_{AY} given in (1.2.29). Thus

$$\delta \tilde{G}_\zeta(x) = \frac{\delta\sigma(x)}{2\pi} + \delta\phi(x) - \delta S_{\text{AY}}[g_0, g] - m^2 \int d^2z \sqrt{g} \left[\tilde{G}(x, z) \delta\phi(z) + 2 (\tilde{G}(x, z))^2 \delta\sigma(z) \right]. \quad (1.2.71)$$

In exactly the same way we also get

$$\begin{aligned} \delta \tilde{G}(x, y) &= \frac{1}{2} (\delta\phi(x) + \delta\phi(y)) - \delta S_{\text{AY}}[g_0, g] \\ &\quad - m^2 \int d^2z \sqrt{g} \left[\frac{1}{2} (\tilde{G}(x, z) + \tilde{G}(y, z)) \delta\phi(z) + 2 \tilde{G}(x, z) \delta\sigma(z) \tilde{G}(z, y) \right]. \end{aligned} \quad (1.2.72)$$

While these are exact relations valid for all m , they are written in a way that makes the small mass expansions obvious.

Small mass expansion

Eq. (1.2.67) is non-perturbative in m . However, it is also written in a way that immediately allows for a perturbative expansion in m , since \tilde{G}_ζ and \tilde{K} have smooth limits as $m \rightarrow 0$. The $m^2 \rightarrow 0$ limit of (1.2.67) also exhibits an extra $\frac{1}{2} \ln \frac{A}{A_0}$ term which only gets removed for $m^2 = 0$ due to the difference in the definitions (cf (1.2.20)).

Let us determine the order $m^2 A$ -correction $S_{\text{grav}}^{(1)}$ to $S_{\text{grav}}^{(0)}$. Since $\tilde{K}(t) = e^{-m^2 t} \tilde{K}^{(0)}(t)$ and $\lambda_n^{(0)} \sim \frac{1}{A}$ it follows that

$$\int_0^\infty \frac{dt}{t} (e^{m^2 t} - m^2 t - 1) \tilde{K}(t) = \mathcal{O}((m^2 A)^2). \quad (1.2.73)$$

Next, it follows from (1.2.12) and (1.2.13), upon subtracting the short-distance singularity and letting $x \rightarrow y$, that

$$\tilde{G}_\zeta(x) = \tilde{G}_\zeta^{(0)}(x) + \sum_{r=1}^{\infty} (-m^2)^r \tilde{G}_{r+1}^{(0)}(x, x), \quad (1.2.74)$$

so that $\int d^2 x \sqrt{g} \tilde{G}_\zeta(x) = \int d^2 x \sqrt{g} \tilde{G}_\zeta^{(0)}(x) + \mathcal{O}(m^2 A)$. Thus, one reads the terms of order $m^2 A$ in the gravitational action from (1.2.68):

$$S_{\text{grav}}^{(1)}[g_0, g] = \frac{m^2}{2} (A \Psi_G[g] - A_0 \Psi_G[g_0]) = \frac{m^2 A}{2} (\Psi_G[g] - \Psi_G[g_0]) + (A - A_0) \frac{m^2}{2} \Psi_G[g_0], \quad (1.2.75)$$

where, following [55], we have introduced

$$\Psi_G[g] = \frac{1}{A} \int d^2 x \sqrt{g} \tilde{G}_\zeta^{(0)}(x; g). \quad (1.2.76)$$

The variation of \tilde{G}_ζ was given in (1.2.71) and that of $\tilde{G}_\zeta^{(0)}$ immediately follow as

$$\delta \tilde{G}_\zeta^{(0)}(x) = \frac{\delta \sigma(x)}{2\pi} + \delta \phi(x) - \delta S_{\text{AY}}[g_0, g], \quad (1.2.77)$$

so that

$$\tilde{G}_\zeta^{(0)}(x; g) = \tilde{G}_\zeta^{(0)}(x; g_0) + \frac{\sigma(x)}{2\pi} + \phi(x) - S_{\text{AY}}[g_0, g]. \quad (1.2.78)$$

This relation has been derived before in [55]. Using (1.2.21), it is then straightforward to obtain (as in [55])

$$\Psi_G[g] - \Psi_G[g_0] = \frac{1}{8\pi} \int d^2 x \sqrt{g_0} \left[\frac{4}{A} \sigma e^{2\sigma} - 2\pi \phi \Delta_0 \phi - 4\pi \phi \Delta_0 \tilde{G}_\zeta^{(0)}(x; g_0) \right]. \quad (1.2.79)$$

For genus $h = 0$ and choosing g_0 to be the round metric on the sphere, $\tilde{G}_\zeta^{(0)}(x; g_0)$ is a constant, and one directly gets the Mabuchi action for $h = 0$. More generally, for arbitrary genus, it has been shown in [55] that

$$\Psi_G[g] - \Psi_G[g_0] = \frac{1}{8\pi} S_{\text{M}}[g_0, g] + h \left(S_{\text{AY}}[g_0, g] - \int d^2 x \sqrt{g_c} \phi \right), \quad (1.2.80)$$

where g_c is the canonical metric on the Riemann surface. Finally, (1.2.75) becomes

$$S_{\text{grav}}^{(1)}[g_0, g] = \frac{m^2 A}{2} \left[\frac{1}{8\pi} S_{\text{M}}[g_0, g] + h \left(S_{\text{AY}}[g_0, g] - \int d^2 x \sqrt{g_c} \phi \right) \right] + (A - A_0) \frac{m^2}{2} \Psi_G[g_0]. \quad (1.2.81)$$

While the last term contributes to the cosmological constant action, the other terms are to be considered as the genuine order $m^2 A$ correction to the gravitational action, and it involves the Mabuchi and Aubin-Yau actions. This motivates the choice made in this thesis to also consider the Mabuchi action in addition to the Liouville action in the quantum gravity partition function.

One can straightforwardly obtain the expansion in powers of m^2 of the terms in (1.2.67) or (1.2.68). Denoting the term of order $(m^2 A)^r$ by $S_{\text{grav}}^{(r)}$, we have

$$S_{\text{grav}}[g_0, g] = \sum_{r=0}^{\infty} S_{\text{grav}}^{(r)}[g_0, g] , \quad (1.2.82)$$

with

$$S_{\text{grav}}^{(r)}[g_0, g] = \frac{(-)^{r+1}}{2r} m^{2r} \left[\int d^2x \sqrt{g} \tilde{G}_r^{(0)}(x, x; g) - \int d^2x \sqrt{g_0} \tilde{G}_r^{(0)}(x, x; g_0) \right] , \quad r \geq 2 . \quad (1.2.83)$$

While these $S_{\text{grav}}^{(r)}[g_0, g]$ are appropriate local gravitational actions, it would be desirable to express them in terms of more geometric quantities like the conformal factor or the Kähler potential, as was the case for $S_{\text{grav}}^{(0)}[g_0, g]$ and $S_{\text{grav}}^{(1)}[g_0, g]$ with the Liouville, Mabuchi and Aubin-Yau actions. As previously stated, in the following of this thesis, both the Liouville action (for conformal matter) and the Mabuchi action are considered, the later being involved in the first-order mass correction to the gravitational action. A writing of these $S_{\text{grav}}^{(r)}[g_0, g]$ in terms of the conformal factor or the Kähler potential could open a way for a continuation of the present study. To our knowledge however, there does not seem to exist any appropriate functional in the mathematical literature. Nevertheless, since the $\tilde{G}_r^{(0)}$ are entirely determined in terms of the properties of the Riemann surface, they are purely geometric quantities.

1.3 Regularization and computation of the partition function at fixed area

The quantum gravity partition function may be written in terms of the so-called partition function at fixed area $Z[A]$

$$Z_{\text{grav}} = \int \mathcal{D}g e^{-\mu_c^2 A} Z_{\text{mat}}[g] = \int dA Z[A] , \quad (1.3.1)$$

with

$$Z[A] = \int \mathcal{D}g e^{-\mu_c^2 A} Z_{\text{mat}}[g] \delta \left(\int_{\Sigma_h} d^2x \sqrt{g} - A \right) = \int \mathcal{D}g e^{-S_{\text{grav}}[g_0, g]} \delta \left(\int_{\Sigma_h} d^2x \sqrt{g} - A \right) , \quad (1.3.2)$$

and Σ_h is a compact Riemann surface of genus h . Some suitable gravitational actions have been discussed in the previous section. In Chapter 2, we will only consider the

Liouville action while in Chapter 3 and in the remaining part of this chapter both the Liouville and the Mabuchi actions will be considered. As previously stated (see section 1.2.5), fixing the conformal gauge amounts to replacing $\mathcal{D}g$ by $\mathcal{D}\sigma Z_{\text{ghost}}[g]$ and results in a term $+\frac{26}{24\pi}S_L[g]$ in the gravitational action. Thus, following [54] we write

$$Z[A] = \int \mathcal{D}\sigma e^{-\frac{\kappa^2}{8\pi\varepsilon}S_L[g_0,g]-\frac{\beta^2}{\varepsilon}S_M[g_0,g]-\mu_c^2 A} \delta\left(\int_{\Sigma_h} d^2x\sqrt{g} - A\right), \quad (1.3.3)$$

where ε is a loop counting parameter (to be set to 1 in the end) and $\kappa^2 = \frac{26-c}{3}$ plays the role of the coupling constant of the Liouville theory of central charge c and includes the ghost contribution.

Note that, as discussed in [54], a further subtlety arises in the case of the sphere, $h = 0$, because the gauge-fixing (1.2.21) then is incomplete. An additional gauge fixing of the residual $\text{SL}(2, \mathbb{C})/\text{SU}(2)$ group of diffeomorphisms acting non-trivially on the conformal factor σ and Kähler potential ϕ must be performed. The result is to project out the spin-one modes of ϕ in its decomposition in terms of the spherical harmonics Y_l^m , which produces an overall factor of $A^{3/2}$ in the partition function coming from the Faddeev-Popov determinant. It is implicitly assumed in the rest of this thesis that $h > 0$, in order not to explicitly deal with this complication.

1.3.1 The measure on the space of metrics

We consider the quantum gravity integration measure $\mathcal{D}g$. This measure can be derived from a choice of metric on the space of metrics. It is generally assumed that this metric should be ultralocal and, hence, of the form $\|\delta g\|^2 = \int d^2x\sqrt{g}\delta g_{ab}\delta g_{cd}(g^{ac}g^{bd} + a g^{ab}g^{cd})$ for some constant $a > -1/2$ (see e.g. [11, 54, 99]). The authors of [54] argue that ultralocality may not be the most important condition for a theory of quantum gravity, and that background independence is a more fundamental requirement. (Then, other metrics satisfying the constraint of background independence could be used in the path integral, such as the Donaldson-Semmes-Mabuchi metric [79, 85, 100].) Nevertheless, we use the above with $g = e^{2\sigma}g_0$ to get

$$\|\delta g\|^2 = 8(1+2a)\|\delta\sigma\|^2, \quad \|\delta\sigma\|^2 = \int d^2x\sqrt{g_0}e^{2\sigma}(\delta\sigma)^2. \quad (1.3.4)$$

The integration measure $\mathcal{D}\sigma$ over σ is determined from this metric [23]. It is *not* the measure of a free field, because of the non-trivial factor $e^{2\sigma}$. Instead of σ , we will use equivalently the variables (A, ϕ) , using the relation (1.2.21). Following [54], we have

$$\|\delta\sigma\|^2 = \frac{(\delta A)^2}{4A} + \|\delta\sigma\|_A^2, \quad (1.3.5)$$

where $\|\delta\sigma\|_A^2$ is the metric on the space of metrics for fixed area A

$$\|\delta\sigma\|_A^2 = \frac{1}{16} \int d^2x\sqrt{g}(A\Delta\delta\phi)^2. \quad (1.3.6)$$

Using eq. (1.2.21) and (1.2.23), this term can be rewritten as

$$\|\delta\sigma\|_A^2 = \frac{1}{16} \int d^2x \sqrt{g_0} e^{2\sigma} \left(e^{-2\sigma} A \Delta_0 \delta\phi \right)^2 = \frac{1}{16} \int d^2x \sqrt{g_*} \left(1 - \frac{1}{2} A \Delta_* \phi \right)^{-1} (A \Delta_* \delta\phi)^2, \quad (1.3.7)$$

in terms of the rescaled (constant curvature) metric g_* of area A . Formally, (1.3.5) and (1.3.7) thus induce the non-trivial measure [54]

$$\mathcal{D}\sigma = \frac{dA}{\sqrt{A}} \mathcal{D}\phi = \frac{dA}{\sqrt{A}} \left[\text{Det}' \left(1 - \frac{1}{2} A \Delta_* \phi \right)^{-1} \right]^{1/2} \text{Det}'(A \Delta_*) \mathcal{D}_* \phi, \quad (1.3.8)$$

where $\mathcal{D}_* \phi$ is the standard free field integration measure in the background metric g_* deduced from the metric $\|\delta\phi\|_*^2 = \int d^2x \sqrt{g_*} \delta\phi^2$. As previously stated, the notation Det' means that we are not taking into account the zero mode when computing the determinant. Indeed, from (1.2.21), the zero mode of ϕ is unphysical and must not be included in the integration measure over the Kähler potentials. The measure $\mathcal{D}_* \phi$ can be expressed in the traditional way by expanding ϕ in eigenmodes of the Laplace operator Δ_* . Following the notation of section 1.2.1, we have

$$\phi = \sum_{n>0} a_n \psi_n, \quad \Delta_* \psi_n = \lambda_n^{(0)} \psi_n, \quad \int d^2x \sqrt{g_*} \psi_n \psi_m = \delta_{nm}, \quad (1.3.9)$$

and thus

$$\mathcal{D}_* \phi = \prod_{n>0} \frac{\mu da_n}{\sqrt{2\pi}}. \quad (1.3.10)$$

One could instead define an expansion of ϕ with respect to eigenfunctions normalized with the metric g_0 of area A_0 . Then, the integration measure $\mathcal{D}_0 \phi$ in the background metric g_0 is related to $\mathcal{D}_* \phi$ by $\mathcal{D}_* \phi = e^{\frac{1}{2} \sum_n \ln \frac{A}{A_0}} \mathcal{D}_0 \phi$ [55]. All these relations are formal and must be regularized.

Our starting point for the computation of the quantum gravity partition function at fixed area then is (cf (1.3.3))

$$\begin{aligned} Z[A] &= \frac{1}{\sqrt{A}} \int \mathcal{D}\phi \exp \left(-\frac{\kappa^2}{8\pi\varepsilon} S_L[g_0, g] - \frac{\beta^2}{\varepsilon} S_M[g_0, g] - \mu_c^2 A \right) \\ &\equiv \frac{1}{\sqrt{A}} \int \mathcal{D}_* \phi e^{-\mu_c^2 A} \exp \left(-S_{\text{measure}}[g_0, g] - \frac{\kappa^2}{8\pi\varepsilon} S_L[g_0, g] - \frac{\beta^2}{\varepsilon} S_M[g_0, g] \right). \end{aligned} \quad (1.3.11)$$

The measure action is thus defined as

$$S_{\text{measure}}[g_0, g] = -\frac{1}{2} \ln \left[\text{Det}' \left(1 - \frac{1}{2} A \Delta_* \phi \right)^{-1} \right] - \ln \text{Det}'(A \Delta_*). \quad (1.3.12)$$

This expression is a formal writing and needs to be regularized.

1.3.2 One-loop expansion of the partition function at fixed area

The semi-classical partition function at fixed area is dominated by the minimum of the gravitational action at fixed area. As previously stated in section 1.2.3 (see (1.2.29)), the classical saddle-points at fixed area of both the Liouville (1.1.4) and the Mabuchi (1.2.26) actions are the constant curvature metrics of arbitrary area A . In terms of the background metric g_0 of constant curvature and of area A_0 , the saddle-point value of σ is $\sigma_{\text{cl}} = \frac{1}{2} \ln \frac{A}{A_0}$. Expanding around this saddle-point up to quadratic order then leads to

$$\begin{aligned} S_L[g_0, g] &= 4\pi(1-h) \ln \frac{A}{A_0} + \frac{1}{16} \int d^2x \sqrt{g_*} \phi (A\Delta_*)^2 \left[\Delta_* + \frac{8\pi(h-1)}{A} \right] \phi + O(\phi^3) \\ S_M[g_0, g] &= 2 \ln \frac{A}{A_0} + \frac{1}{4} \int d^2x \sqrt{g_*} \phi A \Delta_* \left[\Delta_* + \frac{8\pi(h-1)}{A} \right] \phi + O(\phi^3). \end{aligned} \quad (1.3.13)$$

with respect to the Kähler potential ϕ . The terms $O(\phi^3)$ are relevant beyond the one-loop approximation and will be considered in Chapter 2 and 3. Let us insist that, in this expression, the zero mode is excluded. Indeed, as stressed in the previous subsection, the zero mode of ϕ is unphysical for all h .

As highlighted in [54], the full dependence of these actions in the area A comes from the tree-level term in (1.3.13). Indeed, using (1.2.23), all the apparent A -dependence in the second term can be absorbed, and this remains true to all orders in the expansion. The authors of [54] conclude that the non-trivial area dependence of the quantum gravity partition function is actually due to the requirement of a proper regularization procedure for the measure in the functional integral.

The loop counting parameter ε introduced in (1.3.3) allows us to write a loop-expansion of the form

$$\ln Z[A] \equiv W[A] = \sum_{L \geq 0} W^{(L)}[A] \quad (1.3.14)$$

where $W^{(L)}[A]$ is the sum of connected vacuum diagrams of order ε^{L-1} . The tree-level contribution $W^{(0)}$ can be directly read from the tree-level terms in (1.3.13)

$$W^{(0)}[A] = \frac{1}{\varepsilon} \left(\frac{h-1}{2} \kappa^2 - 2\beta^2 \right) \ln \frac{A}{A_0}. \quad (1.3.15)$$

This yields a tree-level string susceptibility

$$\gamma_{\text{str}} = \frac{h-1}{2} \frac{\kappa^2}{\varepsilon} - 2 \frac{\beta^2}{\varepsilon} + \mathcal{O}(1). \quad (1.3.16)$$

Note that when considering only the Liouville action, the loop counting parameter can be taken to be directly κ^2 so that

$$\gamma_{\text{str}}^L = \frac{h-1}{2} \kappa^2 + \mathcal{O}(1) \quad (1.3.17)$$

which is precisely the KPZ value. Indeed, if one does a loop-expansion of the KPZ result (1.1.2) for the string susceptibility up to one loop, one gets

$$\gamma_{\text{str}} = \frac{h-1}{2} \kappa^2 + \frac{19-7h}{6} + O(\kappa^{-2}). \quad (1.3.18)$$

Hence, eq. (1.3.16) is a generalization in agreement with (1.3.18).

The one-loop computation was made in [54], starting from (1.3.11). The complicated determinant factor $[\text{Det}'(1 - \frac{1}{2}A\Delta_*\phi)^{-1}]^{1/2}$ in the measure action (1.3.12) is irrelevant at one loop. Its contributions will be carefully studied in the following chapters of this thesis.

From eq. (1.3.13) and (1.3.12) one has [54],

$$W^{(1)}[A] = -\frac{1}{2} \ln A + \ln \text{Det}'(A\Delta_*) - \frac{1}{2} \ln \text{Det}' \left[A\Delta_* \left(\Delta_* + \frac{8\pi(h-1)}{A} \right) \left(\frac{\kappa^2}{32\pi} A\Delta_* + \beta^2 \right) \right]. \quad (1.3.19)$$

This expression is formal and must be regularized. In section 1.2 the determinants were regularized using the standard ζ -function scheme. However, as discussed in [54], this method is not adapted when dealing with a product of determinants. This is the so-called multiplicative anomaly [101–106]: the ζ -renormalized determinant of a product of operators does not always equal the product of the ζ -renormalized determinants of the operators. As reminded in [54], the multiplicative anomaly is usually irrelevant since the determinants are defined modulo the addition of arbitrary local counterterms [107–111]. However, in the present computation, this anomaly has to be taken into account when performing the change of variables (1.2.21) from the conformal factor σ to the area A and Kähler potential ϕ if one uses the ζ -function regularization scheme. Otherwise one gets a wrong area dependence of the partition function. This is explained and derived in [54].

1.3.3 Spectral cutoff regularization

The general spectral cutoff approach is a powerful regularization scheme developed in [53]. This is the regularization scheme used throughout this thesis. At one loop it generalizes the zeta function regularization scheme without stumbling into the multiplicative anomaly problem [54]. It also allows a physical discussion of the divergences. At higher loops it amounts to regulating the propagators in a specific way. In particular, it will allow us in the following to regularize the two- and three-loop Feynman diagrams in our semi-classical expansion of the partition function.

A basic feature of the spectral cutoff is to replace any sum over the eigenvalues λ_n^D of the relevant differential operator D_g by a regularized sum as

$$\sum_n F(\lambda_n^D) \rightarrow \left[\sum_n F(\lambda_n^D) \right]_{\varphi, \Lambda} = \int_0^\infty d\alpha \varphi(\alpha) \sum_n e^{-\frac{\alpha}{\Lambda^2} \lambda_n^D} F(\lambda_n^D). \quad (1.3.20)$$

The regulator function φ is relatively arbitrary, except for a normalization condition which ensures that for $\Lambda \rightarrow \infty$ one recovers the unregulated sum, and regularity proper-

ties as $\alpha \rightarrow 0$ or $\alpha \rightarrow \infty$ [53]:

$$\begin{aligned} \int_0^\infty d\alpha \varphi(\alpha) &= 1 \\ \varphi(\alpha) &\underset{\alpha \rightarrow 0}{=} \mathcal{O}(\alpha^n) \quad \text{for any } n \geq 0, \\ \int_0^\infty d\alpha \alpha^n \varphi(\alpha) &< \infty \quad \text{for any } n \geq 0. \end{aligned} \quad (1.3.21)$$

Λ is the cutoff scale (which is eventually sent to infinity). An important physical requirement is that, in the end, all φ -dependence should be only in terms that can be changed by the addition of (local) counterterms, while any physical part should be regulator independent³. This has been checked on several examples in [53].

Then, in particular, the regularized propagator⁴ is

$$\tilde{G}(x, y) \rightarrow \left[\tilde{G}(x, y) \right]_{\varphi, \Lambda} = \int_0^\infty d\alpha \varphi(\alpha) \sum_{n>0} e^{-\frac{\alpha}{\Lambda^2} \lambda_n^D} \frac{\psi_n(x) \psi_n(y)}{\lambda_n^D}. \quad (1.3.22)$$

The right-hand-side is related to the “hatted heat kernel” on the manifold. Recalling the expression of the heat kernel (1.2.30), this quantity is defined in [53] as

$$\begin{aligned} \tilde{K}(t, x, y) &= \sum_{n>0} e^{-\lambda_n^D t} \psi_n(x) \psi_n(y), \\ \hat{\tilde{K}}(t, x, y) &= \int_t^\infty dt' \tilde{K}(t', x, y) = \sum_{n>0} e^{-\lambda_n^D t} \frac{\psi_n(x) \psi_n(y)}{\lambda_n^D}. \end{aligned} \quad (1.3.23)$$

The integration is convergent at $+\infty$ since⁵ $\lambda_n^D > 0$ for all $n > 0$.

Of course, $\tilde{K}(t, x, y)$ and $\hat{\tilde{K}}(t, x, y)$ are symmetric under exchange of x and y and one has the following relations

$$-\frac{d}{dt} \hat{\tilde{K}}(t, x, y) = D_x \hat{\tilde{K}}(t, x, y) = D_y \hat{\tilde{K}}(t, x, y) = \tilde{K}(t, x, y), \quad (1.3.24)$$

as well as

$$\begin{aligned} \hat{\tilde{K}}(0, x, y) &= \tilde{G}(x, y), \\ \tilde{K}(0, x, y) &= \frac{\delta(x - y)}{[g_*(x)g_*(y)]^{1/4}} - \frac{1}{A}. \end{aligned} \quad (1.3.25)$$

³For the partition function (1.3.2) or (1.3.11), this means that all the residual φ -dependence can be absorbed in the cosmological constant μ_c^2 , possibly up to an area-independent global normalization constant.

⁴As previously stated, the zero mode of ϕ is unphysical and has to be excluded. Thus we will only consider quantities with the zero mode excluded. This is indicated by a tilde such as in $\tilde{G}(x, y)$.

⁵Indeed, in this thesis, the operator D_g will be either $A\Delta_*(\Delta_* - R_*)\left(\frac{\kappa^2}{32\pi}A\Delta_* + \beta^2\right)$ or of the form $\Delta_* - R_* + m^2$. In any case, if the zero mode of the Laplacian is excluded then $\lambda_n^D > 0$ for $n > 0$.

As it is clear from the definitions (1.3.23), for $t > 0$, $\tilde{K}(t, x, y)$ and $\hat{K}(t, x, y)$ are given by converging sums and are finite, even as $x \rightarrow y$. For $t \rightarrow 0$, one recovers various divergences. In particular, for an operator $D_g = \Delta_* - R_* + m^2$, $\hat{K}(t, x, y)$ then yields the logarithmic short distance singularity of $\tilde{G}(x, y)$. Also, since $\tilde{K}(t, x, y)$ and $\hat{K}(t, x, y)$ do not include the zero-mode, their integrals over x or over y vanish (as is also the case for $\tilde{G}(x, y)$, of course):

$$\int d^2x \sqrt{g_*} \tilde{K}(t, x, y) = \int d^2x \sqrt{g_*} \hat{K}(t, x, y) = 0. \quad (1.3.26)$$

Also, from the definition (1.3.23) of $\tilde{K}(t, x, y)$ and $\hat{K}(t, x, y)$ and the orthonormality of the eigenfunctions ψ_n , it is straightforward to get⁶

$$\begin{aligned} \int d^2x \sqrt{g_*} \tilde{K}(t_1, u, x) \hat{K}(t_2, x, v) &= \hat{K}(t_1 + t_2, u, v), \\ \int d^2x \sqrt{g_*} \tilde{K}(t_1, u, x) \tilde{K}(t_2, x, v) &= \tilde{K}(t_1 + t_2, u, v), \\ \int d^2x \sqrt{g_*} \frac{d}{dt_1} \tilde{K}(t_1, u, x) \hat{K}(t_2, x, v) &= -\tilde{K}(t_1 + t_2, u, v). \end{aligned} \quad (1.3.27)$$

These formula will be useful in the following chapters. The regularized Green's function (1.3.22) is now seen to be given by

$$\left[\tilde{G}(x, y) \right]_{\varphi, \Lambda} = \int_0^\infty d\alpha \varphi(\alpha) \hat{K}\left(\frac{\alpha}{\Lambda^2}, x, y\right). \quad (1.3.28)$$

If a given Feynman diagram (integral) I_n contains n propagators, we can now define its regularized version as

$$I_n^{\text{reg}} \equiv \left[I_n \right]_{\varphi, \Lambda} = \left(\prod_{i=1}^n \int_0^\infty d\alpha_i \varphi(\alpha_i) \right) I_n(t_1 = \frac{\alpha_1}{\Lambda^2}, \dots, t_n = \frac{\alpha_n}{\Lambda^2}), \quad (1.3.29)$$

where $I_n(t_1, \dots, t_n)$ is the Feynman diagram (integral) with all propagators $\tilde{G}(x_i, y_i)$ replaced by $\hat{K}(t_i, x_i, y_i)$. It is obvious from (1.3.29) that the only part of $I_n(t_i)$ that contributes is the part that is completely symmetric in all t_i . To simplify the notation, we will not write the $\int_0^\infty d\alpha_1 \varphi(\alpha_1) \dots \int_0^\infty d\alpha_n \varphi(\alpha_n)$ in our later computations in Chapter 2 and 3.

In our two- and three-loop computations in Chapter 2 and 3, the strategy is then to compute the relevant $I_n(t_1, \dots, t_n)$ and extract the small t_i asymptotic. Since we always let

$$t_i = \frac{\alpha_i}{\Lambda^2} \quad (1.3.30)$$

the small t_i and large Λ asymptotics are, of course, equivalent. To do this, we will once more use the well-known small t asymptotic of the heat kernel $K(t, x, y)$, resp $\tilde{K}(t, x, y)$

⁶The last two lines may also be deduced from the first relation using (1.3.24).

(cf section 1.2.4). Unfortunately, this does not allow us to get the small t asymptotic of $\widehat{K}(t, x, y)$ since by (1.3.23) the latter involves $\widetilde{K}(t', x, y)$ for all $t' \geq t$ which are not all small. However, using (1.3.23) and (1.3.25) we can write

$$\widehat{K}(t, x, y) = \widetilde{G}(x, y) - \int_0^t dt' \widetilde{K}(t', x, y) . \quad (1.3.31)$$

In the second term $t' \leq t$ is small for small t and we get a useful formula if we can also say something about the un-regularized Green's function $\widetilde{G}(x, y)$. This is the case if x is close to y , where a short distance asymptotic involving $\widetilde{G}_\zeta(x)$ (defined in (1.2.37) and (1.2.38) in section 1.2.4) is available.

In view of our later two- and three-loop computations in Chapter 2 and 3 we will now give some more specific formulae in the small t asymptotic for the operators $D_* = \Delta_* - R_*$ and $D_*^M = \Delta_* + \frac{32\pi}{A} \frac{\beta^2}{\kappa^2} \equiv \Delta_* + T$. Indeed, Chapter 2 is dedicated to the study of the Liouville theory coupled to the regularized quantum gravity measure and thus β will be taken to be zero. In Chapter 3, the Mabuchi action will be added but the study will be restricted to the case of the torus, so that in both cases we will be able to work with the field⁷

$$\widehat{\phi} = \frac{1}{2} A \Delta_* \phi . \quad (1.3.32)$$

Since the zero-mode of ϕ was excluded anyway, the relation between ϕ and $\widehat{\phi}$ is one-to-one. Then,

$$\begin{aligned} \widetilde{G}(x, y)|_L &= \langle x | (\Delta_* - R_*)^{-1} | y \rangle' , \\ \widetilde{G}(x, y)|_{L+M}^{h=1} &= \langle x | (\Delta_* + T)^{-1} | y \rangle' . \end{aligned} \quad (1.3.33)$$

Small t and short-distance expansions

We consider the general operator $D = \Delta_* + \xi R_* + m^2$. For the Liouville theory, $\xi = -1$, $m = 0$ and $\lambda_0 = -R_*$, while for the Liouville and Mabuchi theory on the torus $R_* = 0$, $m^2 = T = \frac{32\pi}{A} \frac{\beta^2}{\kappa^2}$ and $\lambda_0 = T$. Recalling the small t expansion of the heat kernel $K(t, x, y)$, we have the following small t expansion of $\widetilde{K}(t, x, y)$:

$$\widetilde{K}(t, x, y) = \frac{1}{4\pi t} e^{-\ell^2/4t} \left[a_0(x, y) + a_1(x, y) t + a_2(x, y) t^2 + \dots \right] - \frac{e^{-\lambda_0 t}}{A} \quad (1.3.34)$$

where $\ell^2 \equiv \ell_A^2(x, y)$ is once again the geodesic distance squared between x and y . As previously stated in section 1.2.4, for small t , the exponential forces ℓ^2 to be small and we can use normal coordinates around y . Then

$$\ell^2 = (x - y)^2 \equiv z^2 \quad (1.3.35)$$

⁷Using ϕ instead of $\widehat{\phi}$ as the basic integration variable generates a Jacobian determinant that cancels a similar determinant at one loop coming from the Liouville action [54]. This cancellation could be incomplete if the multiplicative anomaly is present. The correct cancellation thus depends on the regularization scheme. As was discussed in detail in [54], when using the spectral regularization no multiplicative anomaly occurs and there is no subtlety associated with the change of integration variables from ϕ to $\widehat{\phi}$.

in normal coordinates. From eq. (1.2.34), one reads

$$\begin{aligned} a_0(x, y) &= a_0^{(0)}(x, y) , \\ a_1(x, y) &= a_1^{(0)}(x, y) - m^2 a_0^{(0)}(x, y) , \\ a_2(x, y) &= a_2^{(0)}(x, y) - m^2 a_1^{(0)}(x, y) + \frac{m^4}{2} a_0^{(0)}(x, y) , \end{aligned} \quad (1.3.36)$$

where, following the notation of section 1.2, the $a_i^{(0)}(x, y)$ are associated with the “massless” operator $D^{(0)} = \Delta_* + \xi R_*$, and (see e.g. the appendix of [53])

$$\begin{aligned} a_0^{(0)}(x, y) &= 1 + \frac{R_*}{24} z^2 + \frac{R_*^2}{640} (z^2)^2 + \dots , \\ a_1^{(0)}(x, y) &= \frac{1 - 6\xi}{6} R_* + \frac{R_*^2}{6} \left(\frac{1}{20} - \frac{\xi}{4} \right) R_*^2 z^2 + \dots , \\ a_2^{(0)}(x, y) &= \frac{R_*^2}{60} (1 - 30\xi + 90\xi^2) + \dots . \end{aligned} \quad (1.3.37)$$

and

$$\sqrt{g_*} = \frac{1}{(a_0(x, y))^2} = 1 - \frac{R_*}{12} z^2 + \frac{R_*^2}{480} (z^2)^2 + \dots . \quad (1.3.38)$$

The specific values of the $a_i(x, y)$ for the Liouville theory and for the Liouville and Mabuchi theory on the torus can be immediately deduced from these formulae.

In particular, the heat kernel at coinciding points $\tilde{K}(t, x, x)$ is independent of the point x since R_* is constant at fixed area. Then, for any function f that does not include the zero-mode, this implies that $\int d^2x \sqrt{g_*} \tilde{K}(t, x, x) f(x) = 0$. In particular,

$$\int d^2x \sqrt{g_*} \tilde{K}(t, x, x) \tilde{\phi}(x) = 0 . \quad (1.3.39)$$

To get the small t expansion of $\tilde{K}(t, x, y)$ we use (1.3.31). This yields [53]

$$\tilde{K}(t, x, y) = \tilde{G}(x, y) - \frac{1}{4\pi} \sum_{k \geq 0} a_k(x, y) t^k E_{k+1} \left(\frac{\ell^2}{4t} \right) + \int_0^t dt' \frac{e^{-\lambda_0 t'}}{A} , \quad (1.3.40)$$

where the exponential integral functions E_n are defined as

$$E_n(w) = \int_1^\infty du u^{-n} e^{-uw} , \quad (1.3.41)$$

with asymptotic behaviours for small a : $E_1(a) = -\gamma - \ln a + a + \mathcal{O}(a^2)$ and $E_2(a) = 1 + (\ln a + \gamma - 1)a + \mathcal{O}(a^2)$ (see e.g. [112, 113]).

We will be mostly interested in the case when x is close to y where one can use the small distance expansion of the Green’s function⁸ $\tilde{G}(x, y)$ (see e.g. [53, 56]). Thus, in

⁸To obtain this expansion, the idea is to first fix the leading singularity so that for x close to y , $\Delta_*^x \tilde{G}(x, y) \sim \delta(x - y)$ (in normal coordinates around y , $\sqrt{g_*(y)} = 1$). Then, one has to adjust the subleading terms so that $(\Delta_*^x - R_*) \tilde{G}(x, y) = -\frac{1}{\lambda}$ for $x \neq y$, and finally, one fixes the “integration constants” in terms of $\tilde{G}_\zeta(y)$.

normal coordinate and for small t , one gets the following expression for the “hatted heat kernel” (1.3.40):

$$\begin{aligned} \widehat{K}(t, x, y) = \frac{1}{4\pi} & \left\{ a_0(z) \left(4\pi \widetilde{G}_\zeta(y) - \ln \frac{\mu^2 z^2}{4} - 2\gamma - E_1 \left(\frac{z^2}{4t} \right) \right) \right. \\ & - a_1(z) \left[\frac{z^2}{4} \left(4\pi \widetilde{G}_\zeta(y) - \ln \frac{\mu^2 z^2}{4} - 2\gamma + 2 \right) + t E_2 \left(\frac{z^2}{4t} \right) \right] \\ & \left. + \frac{1}{A} \left(t + \frac{z^2}{4} \right) \right\} + \mathcal{O}(t^2, tz^2, z^3). \end{aligned} \quad (1.3.42)$$

As long as $t > 0$, this has a smooth limit as $x \rightarrow y$ given by

$$\widehat{K}(t, y, y) = \widetilde{G}_\zeta(y) - \frac{1}{4\pi} \left[\ln \mu^2 t + \gamma + t \left(a_1(y, y) - \frac{4\pi}{A} \right) \right] + \mathcal{O}(t^2). \quad (1.3.43)$$

Note that the only term in $\widehat{K}(t, y, y)$ that depends on the position y is $\widetilde{G}_\zeta(y)$. Thus,

$$\begin{aligned} \int d^2x \sqrt{g_*} \widehat{K}(t, x, x) f(x) &= \int d^2x \sqrt{g_*} \widetilde{G}_\zeta(x) f(x), \\ \int d^2x \sqrt{g_*} \widetilde{K}(t, x, x) f(x) &= 0, \end{aligned} \quad (1.3.44)$$

for any function f that does not include the zero-mode.

For later use, let us quote some explicit formulae for the pure Liouville theory on a Riemann surface of genus $h \geq 1$, studied in Chapter 2:

$$\begin{aligned} \widehat{K}(t, x, y) &= \frac{1}{4\pi} \left[\left(1 - \frac{z^2}{4} R_* \right) \left(4\pi \widetilde{G}_\zeta(y) - \ln \frac{\mu^2 z^2}{4} - 2\gamma \right) - \frac{7}{12} R_* z^2 \right. \\ & \left. - \left(1 + \frac{R_*}{24} z^2 \right) E_1 \left(\frac{z^2}{4t} \right) - \frac{7}{6} R_* t E_2 \left(\frac{z^2}{4t} \right) + \frac{1}{A} \left(t + \frac{z^2}{4} \right) + \dots \right], \\ \widetilde{K}(t, x, y) &= \frac{e^{-z^2/4t}}{4\pi t} \left[1 + \frac{1}{6} R_* t \left(\frac{z^2}{4t} + 7 \right) + \dots \right] - \frac{e^{R_* t}}{A}, \end{aligned} \quad (1.3.45)$$

as well as

$$-\frac{d}{dt} \widetilde{K}(t, x, y) = \frac{e^{-z^2/4t}}{4\pi t^2} \left[\left(1 - \frac{z^2}{4t} \right) \left(1 + \frac{R_*}{24} z^2 \right) - \frac{7}{6} R_* \frac{z^2}{4} + \dots \right] + \frac{R_*}{A} e^{R_* t}. \quad (1.3.46)$$

and

$$\begin{aligned} \widetilde{K}(t, x, x) &= \frac{1}{4\pi t} \left[1 + \left(\frac{7}{6} R_* - \frac{4\pi}{A} \right) t + \left(\frac{41}{60} R_* - \frac{4\pi}{A} \right) R_* t^2 \right] + \mathcal{O}(t^2), \\ \widehat{K}(t, x, x) &= \widetilde{G}_\zeta(x) - \frac{1}{4\pi} \left[\ln \mu^2 t + \gamma + \left(\frac{7}{6} R_* - \frac{4\pi}{A} \right) t \right] + \mathcal{O}(t^2). \end{aligned} \quad (1.3.47)$$

1.3.4 One-loop result for the partition function at fixed area

Let us recall the derivation of $W^{(1)}[A]$ (1.3.19) made in [54] in this framework. The determinant (1.3.19) is of the general form

$$\ln \text{Det}(F_1(\Delta) \cdots F_p(\Delta)), \quad (1.3.48)$$

where the operators $F_i(\Delta)$ can be expressed in terms of the Laplacian. In the prescription (1.3.20), the regularized version of (1.3.48) is the sum of the regularized versions of the individual logarithm of determinants:

$$\begin{aligned} \int_0^\infty d\alpha \varphi(\alpha) \sum_r e^{-\frac{\alpha}{\Lambda^2} \lambda_r} \ln \text{Det}(F_1(\lambda_r) \cdots F_p(\lambda_r)) \\ = \int_0^\infty d\alpha \varphi(\alpha) \sum_{i=1}^p \sum_r e^{-\frac{\alpha}{\Lambda^2} \lambda_r} \ln \text{Det} F_i(\lambda_r). \end{aligned} \quad (1.3.49)$$

and one has no multiplicative anomaly in this case. This simple result is true because it is the same cutoff function $\varphi(\alpha) e^{-\frac{\alpha}{\Lambda^2} \lambda_r}$ that is used to regularize all the determinants. Here, λ_r denotes the eigenvalues of the Laplacian.

Now, the only sums required to compute in order to access $W^{(1)}[A]$ are

$$\begin{aligned} \mathcal{S}_0 &= \sum_{r>0} e^{-t \lambda_r}, \\ \mathcal{S}_1\left(\frac{a}{A}\right) &= \sum_{r>0} e^{-t \lambda_r} \ln\left(\lambda_r + \frac{a}{A}\right), \end{aligned} \quad (1.3.50)$$

so that

$$\left[\ln \text{Det}'\left(z\left(\Delta + \frac{a}{A}\right)\right) \right]_{\varphi, \Lambda} = \int_0^\infty d\alpha \varphi(\alpha) \left[\mathcal{S}_0 \ln z + \mathcal{S}_1\left(\frac{a}{A}\right) \right] \quad (1.3.51)$$

Following the techniques used in [53], these sums are evaluated in [54] such that

$$\mathcal{S}_1\left(\frac{a}{A}\right) = -\frac{1}{4\pi} \left(\frac{A}{t} + a \right) (\gamma + \ln t) - \zeta'\left(0, \frac{a}{A}\right) + O(t), \quad (1.3.52)$$

and

$$\mathcal{S}_0 = \frac{A}{4\pi t} - \frac{h+2}{3} + O(t), \quad (1.3.53)$$

where

$$\zeta\left(s, a/A\right) = \sum_{r>0} \frac{1}{\left(\lambda_r + \frac{a}{A}\right)^s}. \quad (1.3.54)$$

To access the area dependence of these expressions when the determinants are evaluated on the saddle point metric g_* of area A , one uses (1.2.23) and the definition (1.3.54) to

get the scaling relations [53, 54]

$$\begin{aligned}\zeta_*(s, \frac{a}{A}) &= \left(\frac{A}{A_0}\right)^s \zeta_0(s, \frac{a}{A_0}), \\ \zeta'_*(0, \frac{a}{A}) &= -\left(\frac{h+2}{3} + \frac{a}{4\pi}\right) \ln \frac{A}{A_0} + \zeta'_0(0, \frac{a}{A_0}),\end{aligned}\quad (1.3.55)$$

so that eq. (1.3.51) can be rewritten as,

$$\begin{aligned}\left[\ln \text{Det}'\left(z\left(\Delta + \frac{a}{A}\right)\right)\right]_{\varphi, \Lambda} &= \int_0^\infty d\alpha \varphi(\alpha) \left[\frac{1}{4\pi} \left(\frac{A\Lambda^2}{\alpha} + a\right) \left(\ln \frac{z\Lambda^2}{\alpha} - \gamma\right) \right. \\ &\quad \left. + \left(\frac{h+2}{3} + \frac{a}{4\pi}\right) \left(\ln \frac{A}{A_0} - \ln z\right) - \zeta'_0(0, \frac{a}{A_0}) + O(\Lambda^{-2}) \right].\end{aligned}\quad (1.3.56)$$

Using (1.3.56), one immediately obtains the expression of the regularized one-loop quantum gravity partition function (1.3.19)

$$\begin{aligned}[W^{(1)}]_{\varphi, \Lambda} &= \int_0^\infty d\alpha \varphi(\alpha) \left[\frac{A\Lambda^2}{8\pi\alpha} \left(\gamma - \ln \Lambda^2 - \ln \frac{\kappa^2}{32\pi\alpha}\right) \right. \\ &\quad \left. - \left(\frac{1}{2} + \frac{7h-4}{6} + 4\frac{\beta^2}{\kappa^2}\right) \ln \frac{A}{A_0} + C[A_0, \kappa^2, \beta^2, \alpha] \right],\end{aligned}\quad (1.3.57)$$

where C is an A -independent irrelevant coefficient. The first term in (1.3.57), which is divergent and cutoff dependent, is proportional to A and can thus be absorbed into the cosmological constant. The one-loop string susceptibility is read from the coefficient of the $\ln A$ term [54]:

$$\gamma_{\text{str}}^{\text{one-loop}} = \frac{19-7h}{6} - 4\frac{\beta^2}{\kappa^2}.\quad (1.3.58)$$

As required, it is finite and cutoff independent. This expression⁹ generalizes the KPZ result (1.3.18). This was to be expected since the non-trivial nature of the quantum gravity integration measure only shows up at two and higher loops.

1.4 Background independence

Background independence is the statement that physical quantities should not depend on the choice of the background metric g_0 or $g_* = \frac{A}{A_0}g_0$. Necessary conditions for this background independence are various co-cycle identities. As an example, consider three different metrics, g_1 , g_2 and g_3 related by conformal factors as

$$g_2 = e^{2\sigma_{21}} g_1, \quad g_3 = e^{2\sigma_{32}} g_2, \quad g_3 = e^{2\sigma_{31}} g_1. \quad (1.4.1)$$

⁹Although the reasoning presented here has been restricted to surfaces of genus $h \geq 1$, it is shown in [54] that this expression is also valid in the case of the sphere.

One may consider g_3 as the quantum metric and g_2 as the background metric. If one then expresses g_2 in terms of g_1 and uses g_1 as the new background metric, all references to g_2 must disappear. This is only possible if the conformal factors are related by the obvious co-cycle identity

$$\sigma_{32} + \sigma_{21} = \sigma_{31} . \quad (1.4.2)$$

In the Kähler formalism, this identity leads to a corresponding identity for the Kähler potentials. Indeed,

$$g_b = e^{2\sigma_{ba}} g_a \equiv \frac{A_b}{A_a} \left(1 - \frac{1}{2} A_a \Delta_a \phi_{ba} \right) g_a \quad (1.4.3)$$

implies the relation

$$\left(1 - \frac{1}{2} A_2 \Delta_2 \phi_{32} \right) \left(1 - \frac{1}{2} A_1 \Delta_1 \phi_{21} \right) = \left(1 - \frac{1}{2} A_1 \Delta_1 \phi_{31} \right) . \quad (1.4.4)$$

Recalling that $\Delta_b = e^{-2\sigma_{ba}} \Delta_a$ then leads to the co-cycle identity for the Kähler potentials

$$\phi_{32} + \phi_{21} = \phi_{31} , \quad (1.4.5)$$

up to an irrelevant constant. From (1.4.4), one has the following identity in terms of $\widehat{\phi}_{ba} = \frac{1}{2} A_a \Delta_a \phi_{ba}$:

$$(1 - \widehat{\phi}_{32})(1 - \widehat{\phi}_{21}) = (1 - \widehat{\phi}_{31}) . \quad (1.4.6)$$

As emphasized in section 1.2, the gravitational action satisfies the similar co-cycle identity (1.2.2). Let us consider for the present discussion the following gravitational action:

$$S_{\text{grav}}[g_0, g] = \frac{\kappa^2}{8\pi} S_{\text{L}}[g_0, g] + S_{\text{measure}}[g_0, g] + S_{\text{ct}}[g_0, g] . \quad (1.4.7)$$

The counterterm action S_{ct} will be shown to be required in Chapter 2 section 2.1.2.

The gravitational action (1.4.7) must satisfy the co-cycle identity (1.2.2). As previously stated in section 1.2, this is true for the Liouville action. Anticipating the rewriting of integration measure $\mathcal{D}\widehat{\phi} \exp(-S_{\text{measure}})$ in terms of $\widehat{\phi}$ in the next chapter (see section 2.1.1), we have

$$\mathcal{D}\widehat{\phi} = \prod_{n>0} \mu db_n . \quad (1.4.8)$$

with

$$\widehat{\phi} = \sum_{n>0} b_n \psi_n , \quad \Delta \psi_n = \lambda_n^{(0)} \psi_n , \quad \int d^2x \sqrt{g} \psi_n \psi_m = \delta_{nm} , \quad (1.4.9)$$

and

$$S_{\text{measure}}[g_1, \widehat{\phi}] = -\frac{1}{2} \ln \text{Det}'(1 - \widehat{\phi})^{-1} = \frac{1}{2} \text{Tr}' \ln(1 - \widehat{\phi}) . \quad (1.4.10)$$

Eq. (1.4.6) immediately implies that S_{measure} also satisfies the required co-cycle identity, at least naively. Of course, to show that this naive statement is really true, we need to understand how $\text{Tr}' \sim \int d^2x \sqrt{g} \sum_r \psi_r \psi_r$ behaves.

To follow the background independence in our computations, we will now study the variations of different quantities under an infinitesimal change of the background metric. To be precise, we begin by expressing our quantum metric g_3 in terms of the background metric $g_* \equiv g_1$. Then we let

$$g_2 = e^{2\omega} g_1 \quad , \quad g_1 \equiv g_* \quad , \quad (1.4.11)$$

with infinitesimal ω and use g_2 as the new background metric. In section 1.2, the infinitesimal change was in $g_3 = e^{2\delta\sigma} g_1$. Here, the quantum metric g_3 is fixed and hence only the background metric g_1 and the conformal factor $\sigma_{31} \equiv \sigma$ vary. We will simply write

$$\delta_B g = 2\omega g_* \quad , \quad \delta_B \sigma = -\omega \quad , \quad (1.4.12)$$

where g is the background metric varying from its original value g_* and δ_B gives the infinitesimal change of any quantity under this change of background metric. In particular, the following quantities yield

$$\delta_B \sqrt{g} = 2\omega \sqrt{g_*} \quad , \quad \delta_B R = -2\omega R_* + 2\Delta_*\omega \quad , \quad \delta_B \Delta = -2\omega\Delta_* \quad , \quad (1.4.13)$$

with R and Δ respectively the background curvature and Laplacian in the background metric. To simplify, we may assume that g_2 and g_1 correspond to the same area so that by (1.2.24) and (1.4.4)

$$\delta_B \ln \left(1 - \frac{1}{2} A \Delta \phi \right) = \delta_B \ln \left(1 - \widehat{\phi} \right) = -2\omega \quad \Rightarrow \quad \delta_B \widehat{\phi} = 2\omega(1 - \widehat{\phi}) \quad . \quad (1.4.14)$$

Thus $\delta_B \int d^2x \sqrt{g} \widehat{\phi} = \int d^2x \sqrt{g_*} 2\omega$. Since $\widehat{\phi}$ has no zero-mode, the left-hand side vanishes and we see that ω has no zero-mode either:

$$\int d^2x \sqrt{g_*} \omega = 0 \quad . \quad (1.4.15)$$

As a consistency check, it is easy to verify that the area and the genus of the Riemann surface are invariant under δ_B :

$$\begin{aligned} \delta_B A &= \delta_B \int d^2x \sqrt{g} = 2 \int d^2x \sqrt{g_*} \omega = 0 \quad , \\ \delta_B [8\pi(1-h)] &= \delta_B \int d^2x \sqrt{g} R = 2 \int d^2x \sqrt{g_*} \Delta_* \omega = 0 \quad . \end{aligned} \quad (1.4.16)$$

Note that the co-cycle identity for the Liouville action becomes

$$\delta_B S_L \equiv S_L[g_2, \sigma_{32}] - S_L[g_1, \sigma_{31}] = -S_L[g_1, \omega] = - \int d^2x \sqrt{g_*} R_* \omega + \mathcal{O}(\omega^2) \quad . \quad (1.4.17)$$

By (1.4.15) and since R_* is constant at fixed area, we see that, to first order in ω , the Liouville action is background independent:

$$\delta_B S_L = 0 \quad . \quad (1.4.18)$$

Even though the Liouville action is background invariant, the sum of Feynman diagrams it generates at two loops (and at higher order) will not be automatically be background independent. There are two reasons for this. First, the integration measure may or may not be invariant as we will see next. Second, the diagrams have to be regularized. Now, the spectral cutoff regularization is based on inserting $e^{-\lambda_n t}$ into all sums over the eigenvalues λ_n of the operator D . Since the eigenvalues (and eigenfunctions) change under a variation of the background metric, all the regularized quantities have thus an induced additional background dependence. Since $\delta_B e^{-\lambda_r t} = -t e^{-\lambda_r t} \delta_B \lambda_r$, this is an order $t \sim \frac{1}{\Lambda^2}$ effect but, as usual, when multiplied by some other divergence $\sim \Lambda^2$ it could well lead to some finite additional background dependence.

Let us first discuss the integration measure $(\prod_{n>0} \frac{\mu db_n}{\sqrt{2\pi}}) \exp(-S_{\text{measure}})$. Once regularized, the measure action (1.4.10) stands as

$$S_{\text{measure}}[g_1, \hat{\phi}] = \frac{1}{2} \int d^2x \sqrt{g_*} \tilde{K}(t, x, x) \ln(1 - \hat{\phi}(x)) , \quad (1.4.19)$$

since, from the previous discussion (cf section 1.3.3):

$$\tilde{K}(t, x, x) = \left[\sum_{n>0} (\psi_n^{(1)}(x))^2 \right]_{\text{reg}} . \quad (1.4.20)$$

Note that the $\psi_n^{(1)}$ are the eigenfunctions of $D_1 = \Delta_1 - R_1 \equiv \Delta_* - R_* \equiv D_*$ normalized with $g_1 = g_*$. The variation of eq. (1.4.10) is

$$\begin{aligned} \delta_B S_{\text{measure}}[g_1, \hat{\phi}] &= \int d^2x \sqrt{g_*} \left[\left(\omega \tilde{K} + \frac{1}{2} \delta \tilde{K} \right) \ln(1 - \hat{\phi}) - \tilde{K} \omega \right] \\ &= \int d^2x \sqrt{g_*} \left(\omega \tilde{K} + \frac{1}{2} \delta \tilde{K} \right) \ln(1 - \hat{\phi}) \end{aligned} \quad (1.4.21)$$

since ω has no zero mode and $\tilde{K}(t, x, x)$ is a constant (see (1.3.44)). To determine the variation $\delta \tilde{K}$, we need first to know how the eigenvalues $\lambda_n^L \equiv \lambda_n^{(0)} - R_*$ and the eigenfunctions ψ_n behave under an infinitesimal change of the background metric. The relevant formulae are once more given by standard quantum mechanical first-order perturbation theory for $\delta D = D_2 - D_1 = -2\omega D_* - 2(\Delta_* \omega)$, see e.g. [53]. There is one slight subtlety though, since the $\psi_n + \delta \psi_n$ are normalized with respect to g_2 while the ψ_n are normalized with respect to g_1 . This implies that $\langle \psi_n | \delta \psi_n \rangle$ is not vanishing but equals $-\langle \psi_n | \omega | \psi_n \rangle$. One finds

$$\delta \lambda_n^L = \langle \psi_n | \delta D | \psi_n \rangle = -2 \langle \psi_n | (\lambda_n^L \omega + (\Delta_* \omega)) | \psi_n \rangle , \quad (1.4.22)$$

and

$$\delta \psi_n(x) = -\omega(x) \psi_n(x) + \sum_{m \neq n} \frac{\psi_m(x)}{\lambda_m^L - \lambda_n^L} \left[(\lambda_n^L + \lambda_m^L) \langle \psi_m | \omega | \psi_n \rangle + 2 \langle \psi_n | (\Delta_* \omega) | \psi_m \rangle \right] . \quad (1.4.23)$$

Note that the eigenfunctions ψ_m are real and, hence, the square bracket in the last line is symmetric under exchange of n and m . These formulae would in principle allow us to

compute $\delta\tilde{K}$. However, we only need the asymptotic for large cutoff. From eq. (1.3.47) and (1.4.13), we find $\delta_B\tilde{K}(t, x, x) \sim \frac{7}{12\pi}(\Delta_*\omega - \omega R_*) + \mathcal{O}(t)$ and thus (1.4.21) becomes

$$\delta_B S_{\text{measure}}[g_1, \hat{\phi}] = \int d^2x \sqrt{g_*} \left(\frac{\omega}{4\pi t} + \frac{7}{24\pi} \Delta_*\omega - \frac{\omega}{A} + \mathcal{O}(t) \right) \ln(1 - \hat{\phi}) . \quad (1.4.24)$$

This action does not appear to be background independent, as it is also the case for the counterterm action introduced in the section 2.1.3.

We also need to study how $\mathcal{D}_*\hat{\phi} = \prod_{n>0} \frac{\mu db_n}{\sqrt{2\pi}}$ transforms. We define the expansion coefficients $b_n^{(2)}$ and $b_n^{(1)}$ of $\hat{\phi}$ as

$$\hat{\phi} = \sum_{n>0} b_n^{(1)} \psi_n^{(\Delta_1)} = \sum_{m>0} b_m^{(2)} \psi_m^{(\Delta_2)} \quad . \quad \psi_n^{(\Delta_1)} = \psi_n \quad , \quad \psi_m^{(\Delta_2)} = \psi_m + \delta\psi_m^\Delta , \quad (1.4.25)$$

where the $\psi_n^{(\Delta_i)}$ are the eigenfunctions of the Laplacian Δ_i , rather than of $\Delta_i - R_i$. For the constant curvature metric g_1 these are, of course, the same eigenfunctions, but for g_2 this is no longer true. The formula for $\delta\psi_n^\Delta$ is given by (1.2.46) with the $m = 0$ and $\delta\sigma$ replaced by ω . In any case, we will only need $\langle \psi_n | \delta\psi_n^\Delta \rangle = -\langle \psi_n | \omega | \psi_n \rangle$. Note that $\psi_0^{(\Delta_1)} = \psi_0^{(\Delta_2)} = \frac{1}{\sqrt{A}}$ is unchanged, and the zero-mode and the non-zero modes transform into each other separately. Thus, for $m \neq 0$, $b_m^{(2)} = b_m^{(1)} + \delta b_m$ with

$$\delta b_m = \sum_{n>0} M_{mn} b_n^{(1)} \quad , \quad M_{mn} = \langle \delta\psi_m | \psi_n \rangle , \quad (1.4.26)$$

and

$$\prod_{n>0} db_n^{(2)} = \prod_{n>0} db_n^{(1)} e^{-\delta_B \mathcal{S}} \quad , \quad \delta_B \mathcal{S} = -\text{Tr}' \log(1 + M) = -\text{Tr}' M + \mathcal{O}(\omega^2) . \quad (1.4.27)$$

Since $\langle \psi_n | \delta\psi_n^\Delta \rangle = -\langle \psi_n | \omega | \psi_n \rangle$, we get upon regularization

$$\begin{aligned} \delta_B \mathcal{S} &= \left[\sum_{n>0} \langle \psi_n | \omega | \psi_n \rangle \right]_{\text{reg}} = \int d^2x \sqrt{g_*} \left[\sum_{n>0} (\psi_n(x))^2 \right]_{\text{reg}} \omega(x) \\ &= \int d^2x \sqrt{g_*} \tilde{K}(t, x, x) \omega(x) = 0 , \end{aligned} \quad (1.4.28)$$

and hence $\prod_{n>0} \frac{\mu db_n}{\sqrt{2\pi}}$ is invariant to first order in ω .

Let us now remark that even if the regularized measure action (1.4.24) does not seem to be background invariant, this does not mean that the overall computation does not satisfy the background independence requirement. Indeed, as previously stated, the spectral cutoff regularization induces *a priori* a background dependence in all the diagrams, because of the insertion of the $e^{-\lambda_n t}$ terms in the propagators. Thus, the background independence of the actions themselves would not have been enough to ensure the background independence of the result of our computation. Of course, it would be extremely cumbersome to follow the background dependence through the individual

regularized Feynman diagrams that will appear at two (and higher) loops (see Chapter 2 and 3). Hence, we will give here some relevant formula that would allow, in principle to do this. However, we will content ourselves by studying directly the background (in)dependence of the *final* result for the two-loop partition function.

Since we are interested in the dependence in the area of the partition function and anticipating the scaling

$$\tilde{G}_\zeta^A(x) = \tilde{G}_\zeta^{A_0}(x) + \frac{1}{4\pi} \ln \frac{A}{A_0} , \quad (1.4.29)$$

we need to compute the variation of the Green's function at coinciding point $\delta_B \tilde{G}_\zeta(x)$ and thus the variation $\delta_B \tilde{G}(x, y)$ of the Green's function. From the equations (1.4.22) and (1.4.23) giving the transformations of the eigenvalues λ_n^L and eigenfunctions ψ_n . One straightforwardly finds

$$\delta_B \tilde{G}(x, y) = -(\omega(x) + \omega(y)) \tilde{G}(x, y) + B + C , \quad (1.4.30)$$

with

$$\begin{aligned} B &= 2 \sum_{n \neq 0} \frac{\psi_n(x) \psi_n(y) \langle \psi_n | (\lambda_n \omega + \Delta_* \omega) | \psi_n \rangle}{(\lambda_n^L)^2} \\ C &= \sum_{n \neq 0} \sum_{m \neq n} \frac{\psi_n(x) \psi_m(y) + \psi_m(x) \psi_n(y)}{\lambda_m^L - \lambda_n^L} \frac{1}{\lambda_n^L} \langle \psi_m | ((\lambda_n^L + \lambda_m^L) \omega + 2\Delta_* \omega) | \psi_n \rangle . \end{aligned} \quad (1.4.31)$$

Upon separating the terms $m = 0$ from the sum C , we can symmetrize $\frac{1}{\lambda_m^L - \lambda_n^L} \frac{1}{\lambda_n^L} \rightarrow \frac{1}{\lambda_m^L - \lambda_n^L} \frac{1}{2} \left(\frac{1}{\lambda_n^L} - \frac{1}{\lambda_m^L} \right) = \frac{1}{2\lambda_n^L \lambda_m^L}$ and rewrite

$$\begin{aligned} C &= C_1 + C_2 \equiv \frac{1}{2} \sum_{\substack{m \neq n, \\ n, m \neq 0}} \frac{\psi_n(x) \psi_m(y) + \psi_m(x) \psi_n(y)}{\lambda_m^L \lambda_n^L} \langle \psi_m | ((\lambda_n^L + \lambda_m^L) \omega + 2\Delta_* \omega) | \psi_n \rangle \\ &+ \sum_{n \neq 0} \frac{\psi_n(x) \psi_0(y) + \psi_0(x) \psi_n(y)}{\lambda_0^L - \lambda_n^L} \frac{1}{\lambda_n^L} \langle \psi_0 | ((\lambda_n^L + \lambda_0^L) \omega + 2\Delta_* \omega) | \psi_n \rangle . \end{aligned} \quad (1.4.32)$$

The “missing terms” $m = n$ in the first sum C_1 are exactly given by the sum B , so that, recalling $\sum_{n \neq 0} \psi_n(x) \psi_n(z) = \frac{1}{\sqrt{g_*(x)}} \delta(x - z) - \frac{1}{A}$, one has

$$\begin{aligned} C_1 + B &= (\omega(x) + \omega(y)) \tilde{G}(x, y) - \frac{1}{A} \int d^2 z \sqrt{g_*} (\tilde{G}(x, z) + \tilde{G}(y, z)) \omega(z) \\ &+ 2 \int d^2 z \sqrt{g_*} \tilde{G}(x, z) \Delta_* \omega(z) \tilde{G}(z, y) . \end{aligned} \quad (1.4.33)$$

To evaluate the sum C_2 we have to remember that $\lambda_n^L - \lambda_0^L = \lambda_n^L + R_*$ are the eigenvalues of Δ_* and that $\psi_0 = \frac{1}{\sqrt{A}}$ so that $\langle \psi_0 | ((\lambda_n^L + \lambda_0^L) \omega + 2\Delta_* \omega) | \psi_n \rangle = \frac{3\lambda_n^L - \lambda_0^L}{\sqrt{A}} \omega_n$, where ω_n is the expansion coefficient in $\omega = \sum_{n \neq 0} \omega_n \psi_n$. Then, one finds

$$C_2 = -\frac{1}{A} \int d^2 z \sqrt{g_*} (\tilde{G}(x, z) + \tilde{G}(y, z) + 2\tilde{G}_\Delta(x, z) + 2\tilde{G}_\Delta(y, z)) \omega(z) , \quad (1.4.34)$$

where $\tilde{G}_\Delta(x, z) = \sum_{n \neq 0} \frac{\psi_n(x)\psi_n(z)}{\lambda_n^L - \lambda_0^L}$ denotes the Green's function of the Laplacian Δ_* . Combining (1.4.30), (1.4.33) and (1.4.34) we finally get

$$\begin{aligned} \delta_B \tilde{G}(x, y) &= 2 \int d^2 z \sqrt{g_*(z)} \left[\tilde{G}(x, z) \Delta_* \omega(z) \tilde{G}(z, y) \right. \\ &\quad \left. - \frac{1}{A} (\tilde{G}(x, z) + \tilde{G}(y, z) + \tilde{G}_\Delta(x, z) + \tilde{G}_\Delta(y, z)) \omega(z) \right]. \end{aligned} \quad (1.4.35)$$

Note that this has no short-distance singularity as $x \rightarrow y$ and we can safely take the coincidence limit to get $\delta_B \tilde{G}_\zeta(x)$.

The change of the geodesic length $\ell(x, y)$ under the Weyl rescaling of the metric was given in (1.2.59). Then, (as always, to first order in ω):

$$\delta_B \ln(\ell_A^2(x, y) \mu^2) = \omega(x) + \omega(y). \quad (1.4.36)$$

Thus, recalling the definition (1.2.42) of $\tilde{G}_\zeta(x)$, we have

$$\begin{aligned} \delta_B \tilde{G}_\zeta(x) &= \lim_{x \rightarrow y} \delta_B \left[\tilde{G}(x, y) + \frac{1}{4\pi} \ln(\ell_A^2(x, y) \mu^2) \right] \\ &= 2 \int d^2 z \sqrt{g_*} \left[(\tilde{G}(x, z))^2 \Delta_* \omega(z) - \frac{2}{A} (\tilde{G}(x, z) + \tilde{G}_\Delta(x, z)) \omega(z) \right] + \frac{\omega(x)}{2\pi}. \end{aligned} \quad (1.4.37)$$

This complicated-looking variation simplifies when integrated over the Riemann surface, since \tilde{G} , \tilde{G}_Δ and ω have no zero-mode:

$$\begin{aligned} \delta_B \int d^2 x \sqrt{g(x)} \tilde{G}_\zeta(x) &= 2 \int d^2 x \sqrt{g_*} \omega(x) \tilde{G}_\zeta(x) \\ &\quad + 2 \int d^2 x \sqrt{g_*(x)} \int d^2 z \sqrt{g_*(z)} (\tilde{G}(x, z))^2 \Delta_* \omega(z), \end{aligned} \quad (1.4.38)$$

and similarly

$$\begin{aligned} \delta_B \int d^2 x \sqrt{g(x)} R(x) \tilde{G}_\zeta(x) &= 2 \int d^2 x \sqrt{g_*} \Delta_* \omega(x) \tilde{G}_\zeta(x) \\ &\quad + 2R_* \int d^2 x \sqrt{g_*(x)} \int d^2 z \sqrt{g_*(z)} (\tilde{G}(x, z))^2 \Delta_* \omega(z). \end{aligned} \quad (1.4.39)$$

We will also encounter expressions involving an integral of $R \tilde{G}_\zeta^2$. We have

$$\begin{aligned} \delta_B \int d^2 x \sqrt{g(x)} R(x) (\tilde{G}_\zeta(x))^2 &= 2 \int d^2 x \sqrt{g_*} \Delta_* \omega(x) (\tilde{G}_\zeta(x))^2 \\ &\quad + 2 \int d^2 x \sqrt{g_*(x)} R_* \tilde{G}_\zeta(x) \left\{ \frac{\omega(x)}{2\pi} + 2 \int d^2 z \sqrt{g_*(z)} \left[(\tilde{G}(x, z))^2 \Delta_* \omega(z) \right. \right. \\ &\quad \left. \left. - \frac{2}{A} (\tilde{G}(x, z) + \tilde{G}_\Delta(x, z)) \omega(z) \right] \right\}. \end{aligned} \quad (1.4.40)$$

We should remember that we will be interested in the area dependence and that area-independent terms only lead to irrelevant normalization factors and will drop out in

the end upon computing $Z[A]/Z[A_0]$. Anticipating the discussion of the scaling of the Green's function in the next Chapter, $\tilde{G}(x, z)$ does not depend of the area. Then, from eq. (1.2.23) and (1.4.29), and the three preceding formulae, we find that

$$\begin{aligned} \delta_B \frac{1}{A} \int d^2x \sqrt{g(x)} \tilde{G}_\zeta(x) &= \frac{1}{2\pi A} \ln \frac{A}{A_0} \int d^2x \sqrt{g_*(x)} \omega(x) + A\text{-independent} \\ &= 0 + A\text{-independent} , \end{aligned} \quad (1.4.41)$$

$$\begin{aligned} \delta_B \int d^2x \sqrt{g(x)} R(x) \tilde{G}_\zeta(x) &= \frac{1}{2\pi} \ln \frac{A}{A_0} \int d^2x \sqrt{g_*(x)} \Delta_* \omega(x) + A\text{-independent} \\ &= 0 + A\text{-independent} , \end{aligned} \quad (1.4.42)$$

and

$$\begin{aligned} \delta_B \int d^2x \sqrt{g(x)} R(x) (\tilde{G}_\zeta(x))^2 &= \frac{1}{\pi} \ln \frac{A}{A_0} \int d^2x \sqrt{g_*(x)} \Delta_* \omega(x) \tilde{G}_\zeta^{A_0}(x) \\ &+ \frac{R_*}{\pi} \ln \frac{A}{A_0} \int d^2x d^2z \sqrt{g_*(x)g_*(z)} (\tilde{G}(x, z))^2 \Delta_* \omega(z) + A\text{-independent}. \end{aligned} \quad (1.4.43)$$

We conclude that the appearance of the terms $\int \tilde{G}_\zeta$ and $\int R \tilde{G}_\zeta$ is compatible with background independence, while the appearance of $\int R \tilde{G}_\zeta^2$ is not¹⁰.

¹⁰Note that from (1.4.37), we expect the structure $\int R \tilde{G}_\zeta^3$ not to be background independent either.

A multi-loop first principles analysis

« Le courage consiste à donner raison aux choses quand nous ne pouvons les changer. »
Marguerite Yourcenar in *Alexis ou le Traité du Vain Combat*.

This chapter presents a first-principles quantum field theory computation of $Z[A]$ on a Riemann surface of arbitrary genus, using the spectral cutoff regularization scheme introduced in the previous chapter in section 1.3.3. We consider the Liouville action and the measure action which arises from the regularization of the quantum gravity measure (see section 1.3.1). This work has been initiated with the one-loop computation presented in section 1.3.2 (see [54]). Now, in section 2.1, we will present the full two-loop computation of $Z[A]$ in a loop-expansion where $\frac{1}{\kappa^2}$ is the loop-counting parameter (see [56]). This will give us the two-loop contribution to the string susceptibility γ_{str} . Then, in section 2.2, the three-loop contribution of the partition function is partially computed. Indeed, we compute the non-vanishing quadratic leading divergence. Thus, counterterms are required and we compute the full contribution of the counterterms to the partition function, so that we can discuss the renormalization of the partition function (see [57]).

2.1 Two-loop landscape

In ref. [54] the partition function $Z[A]$ was computed, with the spectral cutoff regularization scheme, up to and including the one-loop contributions, using a more general quantum gravity action that is a sum of the Liouville as well as Mabuchi actions, as presented in section 1.3.2 in the previous chapter. This gave a definite result for γ_{str} which for the pure Liouville gravity reduced to $\gamma_{\text{str}}^{0,1\text{-loop}} = \frac{1}{2}(h-1)\kappa^2 + \frac{19-7h}{6}$ in agreement with the loop-expansion of the KPZ result for the string susceptibility (1.3.18). Although satisfying, the agreement was, maybe, not too much a surprise. Indeed, the truly non-trivial nature of the determinants coming from the measure over the space of metrics only shows up beyond one loop, starting at two loops. It is thus quite intriguing to try and compute the two-loop contributions to the fixed-area partition function. This is what we will do in the present section, based on [56].

In the subsection 2.1.1, we expand the Liouville action and the measure up to order $\frac{1}{\kappa^2}$ which corresponds to the two-loop contributions to $\ln Z[A]$. The regularized loop-integrals are then evaluated in subsection 2.1.2. It is remarkable that the individual $(\ln A\Lambda^2)^2$ -divergences are always accompanied by certain structures that are not background independent. The fact that the $(\ln A\Lambda^2)^2$ -divergences cancel is equivalent to the cancellation of these background non-invariant structures! However, the leading divergence $A\Lambda^2 \ln A\Lambda^2$ is non-vanishing and we are led to introduce counterterms. This is

done in the subsection 2.1.3. Note that the appearance of a non-local divergence should not be a surprise. In a local quantum field theory, one-loop divergences always are local, but starting at two loops the divergences are not necessarily local, in particular also due to the so-called overlapping divergences. However, we know from the standard BPHZ proofs [114–116] that they can always be cancelled by one-loop diagrams including local counterterm vertices (as well as tree diagrams including further local counterterm vertices). These counterterm vertices occurring in the one-loop diagrams are themselves determined from the cancellation of the divergences of the corresponding one-loop n -point functions. The same does happen here. This will be worked out in subsection 2.1.4.

The subsection 2.1.4 is devoted to computing the one-loop two-point function of the Kähler field and determining the necessary counterterms to make it finite and regulator independent. This computation is done on an arbitrary Riemann surface and is done consistently in the same spectral cutoff regularization as the computation of the partition function. Note that this two-point function is closely related to the two-point function of the conformal factor $\langle e^{2\sigma(x)} e^{2\sigma(y)} \rangle$. Most of the lengthy computational details are relegated into the appendix. However, the absence of divergences does not fix the counterterms uniquely. To fix the finite parts of the counterterms requires to impose finite renormalization conditions. While in a massive Minkowski space quantum field theory it is usually convenient to impose conditions on the mass shell, already in a massless theory it is often more convenient to impose the conditions at an arbitrary scale μ . In the present theory on a curved Riemann surface, there seems just to be no *natural* finite condition one should impose, rather than any other. Thus there appears to be a whole family of counterterms, depending on two finite parameters. Remarkably, for *all* choices of these finite parameters, the unwanted $AA^2 \ln AA^2$ terms cancel in the two-loop contribution to $\ln Z[A]$, and the coefficient of $\ln AA^2$, which is the two-loop contribution to the string susceptibility, becomes independent of the regulator functions $\varphi(\alpha)$. This result is given (and discussed) in subsection 2.1.5.

For later comparison, we quote the loop-expansion of the KPZ result up to two loops:

$$\gamma_{\text{str}} = \frac{1}{2}(h-1)\kappa^2 + \frac{19-7h}{6} + 2(1-h)\frac{1}{\kappa^2} + \mathcal{O}(\kappa^{-4}) . \quad (2.1.1)$$

2.1.1 Two-loop expansion of the Liouville and measure actions

In the previous chapter, we have expanded the Liouville action up to the relevant order in ϕ for the one-loop computation. To go further and perform the two-loop computation, let us consider again the Liouville action written in the background metric g_* of constant curvature R_* and area A .

$$S_L[\sigma] = \int d^2x \sqrt{g_*} (\sigma \Delta_* \sigma + R_* \sigma) . \quad (2.1.2)$$

Recalling (1.2.24) in terms of $\widehat{\phi}$ (1.3.32), we have

$$\sigma - \sigma_{\text{cl}} = \frac{1}{2} \ln(1 - \widehat{\phi}) \quad , \quad \sigma_{\text{cl}} = \frac{1}{2} \ln \frac{A}{A_0} . \quad (2.1.3)$$

Since the first term in S_L is not affected by constant shifts of σ and the second term is linear, one obviously has

$$S_L[\sigma] = S_L[\sigma_{\text{cl}}] + S_L[\sigma - \sigma_{\text{cl}}] = 4\pi(1-h) \ln \frac{A}{A_0} + S_L\left[\frac{1}{2} \ln(1 - \widehat{\phi})\right]. \quad (2.1.4)$$

Expanding the logarithm in terms of $\widehat{\phi}$ one gets

$$S_L[\sigma - \sigma_{\text{cl}}] = \int d^2x \sqrt{g_*} \left[\frac{1}{4} \widehat{\phi} (\Delta_* - R_*) \widehat{\phi} + \frac{1}{4} \widehat{\phi}^2 (\Delta_* - \frac{2}{3} R_*) \widehat{\phi} + \frac{1}{16} \widehat{\phi}^2 \Delta_* \widehat{\phi}^2 + \frac{1}{6} \widehat{\phi}^3 \Delta_* \widehat{\phi} - \frac{1}{8} R_* \widehat{\phi}^4 + \mathcal{O}(\widehat{\phi}^5) \right]. \quad (2.1.5)$$

When looking at the definition of the quantum gravity partition function (1.3.11) with $\beta = 0$, the quantity $\frac{\varepsilon}{\kappa^2}$ can be seen to be a loop-counting parameter. Thus, in the following of this chapter we absorb ε so that the loop-expansion of $Z[A]$ is now done in powers of $\frac{1}{\kappa^2}$. To emphasize it, we rescale $\widehat{\phi}$ as

$$\widetilde{\phi} = \frac{\kappa}{4\sqrt{\pi}} \widehat{\phi}, \quad (2.1.6)$$

so that

$$\begin{aligned} \frac{\kappa^2}{8\pi} S_L[\sigma] &= \frac{\kappa^2}{2} (1-h) \ln \frac{A}{A_0} + \int d^2x \sqrt{g_*} \frac{1}{2} \widetilde{\phi} (\Delta_* - R_*) \widetilde{\phi} \\ &+ \int d^2x \sqrt{g_*} \left[\frac{\sqrt{4\pi}}{\kappa} \widetilde{\phi}^2 (\Delta_* - \frac{2}{3} R_*) \widetilde{\phi} + \frac{2\pi}{\kappa^2} \widetilde{\phi}^2 \Delta_* \widetilde{\phi}^2 + \frac{16\pi}{3\kappa^2} \widetilde{\phi}^3 \Delta_* \widetilde{\phi} - \frac{4\pi}{\kappa^2} R_* \widetilde{\phi}^4 + \mathcal{O}(\kappa^{-3}) \right]. \end{aligned} \quad (2.1.7)$$

The first term in the first line provides the classical contribution to the partition function. The second term of the first line yields the one-loop determinant studied in [54]. It also provides a standard propagator for the present two-loop computation. The terms of the second and third line provide the cubic and quartic vertices relevant for the two-loop vacuum diagrams. Clearly, the $\mathcal{O}(\kappa^{-3})$ terms correspond to quintic and higher vertices that can only contribute to three (and higher)-loop vacuum diagrams.

The non-trivial measure does not contribute at one-loop. However, it gives a two-loop (as well as higher loop) contribution. In terms of $\widehat{\phi} = \frac{1}{2} A \Delta_* \phi$, eq. (1.3.7) becomes

$$\|\delta\sigma\|_A^2 = \frac{1}{4} \int d^2x \sqrt{g_*} (1 - \widehat{\phi})^{-1} \delta\widehat{\phi}^2, \quad (2.1.8)$$

so that the measure action (1.3.12) changes in¹

$$S_{\text{measure}} = -\frac{1}{2} \ln \text{Det}'(1 - \widehat{\phi})^{-1} = \frac{1}{2} \text{Tr}' \ln(1 - \widehat{\phi}) = \text{Tr}' \left(-\frac{\sqrt{4\pi}}{\kappa} \widetilde{\phi} - \frac{4\pi}{\kappa^2} \widetilde{\phi}^2 + \mathcal{O}(\kappa^{-3}) \right). \quad (2.1.9)$$

¹As previously stated (see [54], the change of variable in the field $\widehat{\phi} = \frac{1}{2} A \Delta_* \phi$ does not induce a multiplicative anomaly when using the spectral cutoff regularization scheme.

More explicitly, in terms of the orthonormal set of eigenfunctions ψ_r of the Laplace operator, cf (1.3.9), the trace of any operator O is given by

$$\mathrm{Tr}' O = \sum_{r>0} \langle \psi_r | O | \psi_r \rangle = \int d^2x \sqrt{g_*} \sum_{r>0} \langle \psi_r | x \rangle \langle x | O | \psi_r \rangle = \int d^2x \sqrt{g_*} \sum_{r>0} \psi_r^2(x) O(x), \quad (2.1.10)$$

(the eigenfunctions ψ_r are chosen to be real) and, hence,

$$S_{\text{measure}} = \int d^2x \sqrt{g_*} \sum_{r>0} \psi_r^2(x) \left(-\frac{\sqrt{4\pi}}{\kappa} \tilde{\phi}(x) - \frac{4\pi}{\kappa^2} \tilde{\phi}^2(x) + \mathcal{O}(\kappa^{-3}) \right). \quad (2.1.11)$$

Of course, $\sum_{r>0} \psi_r^2(x) = \frac{\delta(x-x)}{\sqrt{g_*(x)}} - \psi_0^2$ is a formal writing which translates in $\tilde{K}(t, x, x)$ once regularized (see section 1.3.3). Since $\tilde{K}(t, x, x) = \frac{\Lambda^2}{4\pi\alpha} + \frac{7}{24\pi} R_* - \frac{1}{A} + \dots$ does not depend² on x and since $\tilde{\phi}$ has no zero-mode, the first term in the action drops out (cf eq. (1.3.39)). Thus, at two loops, S_{measure} provides a quadratic vertex with a diverging coefficient. This is very similar to a counterterm, as we will see later-on.

One might wonder what is the advantage of rewriting the simple-looking Liouville action (2.1.2) in terms of $\hat{\phi}$ or $\tilde{\phi}$ which has resulted in a complicated action (2.1.5) or (2.1.7) with cubic and quartic (and higher) interactions that all involve derivatives. Would it not be simpler to use $\frac{1}{2} \ln(1 - \hat{\phi}) = \sigma - \sigma_{\text{cl}}$ as the basic field instead? One important point concerns the zero-mode. The absence of zero-mode is easy to implement for $\hat{\phi}$, while for $\sigma - \sigma_{\text{cl}} = -\frac{1}{2} \hat{\phi} - \frac{1}{4} \hat{\phi}^2 - \frac{1}{8} \hat{\phi}^3 - \dots$ it results in a very complicated constraint. Taking this constraint properly into account is highly non-trivial and probably equivalent in difficulty to working with the complicated actions (2.1.5) or (2.1.7).

Vertices

The Feynman rules are straightforwardly read from the expansions (2.1.7) and (2.1.11). Note that we normalize our vertices without including any symmetry factors (*i.e.* a term $\alpha \hat{\phi}^n$ in the total action gives a vertex with n legs and a factor $-\alpha$, *not* $-\alpha n!$) so that when evaluating the diagrams one has to count all possible contractions. There is a cubic vertex

$$\text{Y} = -\frac{\sqrt{4\pi}}{\kappa} (\Delta_* - \frac{2}{3} R_*), \quad (2.1.12)$$

where $(\Delta_* - \frac{2}{3} R_*)$ is meant to act on the propagator connected to the bold line of the vertex. There are also two quartic vertices,

$$\text{X} = -\frac{2\pi}{\kappa^2} (\Delta_* - 2R_*), \quad (2.1.13)$$

with $(\Delta_* - 2R_*)$ acting on the *two* propagators connected to the bold part of the vertex (of course, upon integrating by parts, it does not matter whether one chooses the two

²Recall that the considered metrics are of constant curvature at fixed area.

lines to the left or the two lines to the right), and

$$\times = -\frac{16\pi}{3\kappa^2} \Delta_* . \quad (2.1.14)$$

The measure yields an unique quadratic vertex:

$$\blacktriangleleft = \frac{4\pi}{\kappa^2} \sum_{n>0} \psi_n^2(x) . \quad (2.1.15)$$

Propagator

These vertices are connected by propagators that are

$$\tilde{G}(x, y) = \langle x | (\Delta_* - R_*)^{-1} | y \rangle' . \quad (2.1.16)$$

We recall that the tilde on G and the prime on the r.h.s. indicate that the zero-mode is not to be included. This propagator can be written explicitly in terms of the eigenvalues λ_n^L and eigenfunctions ψ_n of

$$D = \Delta_* - R_* \quad , \quad D \psi_n = \lambda_n^L \psi_n . \quad (2.1.17)$$

Since $R_* = \frac{8\pi(1-h)}{A}$ is constant, D and Δ_* have the same (real) eigenfunctions ψ_n , while the eigenvalues simply are related by $\lambda_n^L = \lambda_n^{(0)} - R_*$. Since $\tilde{\phi}$ has no zero-mode, the propagator is given by the sum over all non-zero modes as

$$\tilde{G}(x, y) = \sum_{n>0} \frac{\psi_n(x)\psi_n(y)}{\lambda_n^L} . \quad (2.1.18)$$

Furthermore, for genus $h > 1$ one has $\lambda_0^L = -R_* > 0$, and then one can explicitly subtract the zero-mode contribution $\frac{\psi_0^2}{\lambda_0^L} = \frac{1}{8\pi(h-1)}$:

$$\text{for } h > 1 : \quad \tilde{G}(x, y) = G(x, y) - \frac{1}{8\pi(h-1)} \quad , \quad G(x, y) = \sum_{n \geq 0} \frac{\psi_n(x)\psi_n(y)}{\lambda_n^L} . \quad (2.1.19)$$

Note also that for all $h \geq 1$, we have $\lambda_n^L > 0$ for $n > 0$ since the eigenvalues $\lambda_n^{(0)}$ of the Laplacian are positive (for $n > 0$) and $-R_* \geq 0$. It is only for $h = 0$ where $\lambda_n^L \equiv \lambda_{l,m}^L = \frac{4\pi}{A}l(l+1) - \frac{8\pi}{A}$ that we get $\lambda_{0,0}^L < 0$ and $\lambda_{1,m}^L = 0$. As already mentioned, for $h = 0$, these three spin-1 modes are excluded by the $\text{SL}(2, \mathbf{C})/\text{SL}(2, \mathbf{R})$ gauge fixing. In the sequel we will always implicitly assume $h \geq 1$.

2.1.2 Two-loop contributions to the partition function

The purpose of the present computation is to pursue the one-loop expansion made in [54] (see section 1.3). We recall:

$$\ln Z[A] \equiv W[A] = \sum_{L \geq 0} W^{(L)}[A] . \quad (2.1.20)$$

For the present pure Liouville quantum gravity,

$$W^{(0)}[A] = \frac{\kappa^2}{2}(h-1) \ln \frac{A}{A_0} \quad (2.1.21)$$

as can be read off directly from (2.1.7), and [54]

$$W^{(1)}[A] = \frac{1-7h}{6} \ln \frac{A}{A_0} + \mu_{\text{c,div}}^2 A + c_1, \quad (2.1.22)$$

where the divergent piece $\mu_{\text{c,div}}^2 A \sim \Lambda^2 A$ is cancelled by the renormalization of the cosmological constant μ_{c}^2 , and c_1 is an A -independent finite constant that could be eliminated by computing $W^{(1)}[A] - W^{(1)}[A_0]$ instead. The coefficients of $\ln \frac{A}{A_0}$ yield the contributions to $\gamma_{\text{str}} - 3$ and thus from (2.1.21) and (2.1.22)

$$\gamma_{\text{str}} = \frac{h-1}{2} \kappa^2 + \frac{19-7h}{6} + \mathcal{O}(\kappa^{-2}), \quad (2.1.23)$$

in agreement with (2.1.1). As already mentioned, one of the present goals is to determine the order $\frac{1}{\kappa^2}$ term in this expansion which comes from the two-loop contribution. The two-loop contribution to $W[A]$ is

$$W^{(2)}[A] = \sum \left[\text{connected vacuum diagrams} \sim \frac{1}{\kappa^2} \right]. \quad (2.1.24)$$

This includes the genuine two-loop diagrams made with the vertices of S_L , as well as a one-loop diagram made with the vertex from S_{measure} and, as we will see, also one-loop diagrams made with further counterterm vertices.

The vacuum diagrams of order $1/\kappa^2$

The two quartic vertices both give a “figure-eight” diagram $\bigcirc\bigcirc$ with the four lines of a single vertex connected by two propagators. The cubic vertex gives two types of diagrams: the “setting sun” diagram \ominus with two cubic vertices connected by three propagators, and the “glasses” diagram $\bigcirc\bigcirc$ with the two vertices joined by a single propagator and the remaining two lines of each vertex connected by a propagator. Finally the measure vertex gives the “measure” (one-loop) diagram \bigcirc with the two lines of the vertex connected by a single propagator.

The figure-eight diagram: This diagram actually gets three contributions : one from the quartic vertex (2.1.14) and two from the different ways to contract the lines of the quartic vertex (2.1.13). Taking into account the different numbers of contractions in each case (3, 2 and 1) yields

$$\begin{aligned} \bigcirc\bigcirc &= -\frac{8\pi}{\kappa^2} \int d^2x \sqrt{g_*} \left[\frac{1}{4} \tilde{G}(x, x) (\Delta_*^x - 2R_*) \tilde{G}(x, x) \right. \\ &\quad \left. + \frac{1}{2} [(\Delta_*^x - 2R_*) (\tilde{G}(x, z))^2] \Big|_{z=x} + 2 \tilde{G}(x, x) [\Delta_*^x \tilde{G}(x, z)] \Big|_{z=x} \right]. \quad (2.1.25) \end{aligned}$$

The setting sun diagram: This diagram gets two contributions, one with the two bold lines of the two cubic vertices connected by the same propagator (two possible contractions) and one with the two bold lines not connected by the same propagator (four contractions). Overall, there is also a factor $\frac{1}{2}$ from expanding $e^{-S_{\text{int}}}$ to second order. Thus

$$\begin{aligned} \text{Setting Sun Diagram} &= \frac{4\pi}{\kappa^2} \int d^2x \sqrt{g_*(x)} d^2y \sqrt{g_*(y)} \tilde{G}(x, y) \left[\tilde{G}(x, y) \left(\Delta_*^x - \frac{2}{3} R_* \right) \left(\Delta_*^y - \frac{2}{3} R_* \right) \tilde{G}(x, y) \right. \\ &\quad \left. + 2 \left[\left(\Delta_*^x - \frac{2}{3} R_* \right) \tilde{G}(x, y) \right] \left(\Delta_*^y - \frac{2}{3} R_* \right) \tilde{G}(x, y) \right]. \end{aligned} \quad (2.1.26)$$

The glasses diagram: This diagram gets four contributions, since for each of the cubic vertices the bold lines can either be contracted with a line from the same vertex (giving a factor of 2) or with a line of the other vertex (factor 1). Again, overall, there is also a factor $\frac{1}{2}$. Thus

$$\begin{aligned} \text{Glasses Diagram} &= \frac{2\pi}{\kappa^2} \int d^2x \sqrt{g_*(x)} d^2y \sqrt{g_*(y)} \left[\left(\Delta_*^x - \frac{2}{3} R_* \right) \tilde{G}(x, x) + 2 \left(\Delta_*^x - \frac{2}{3} R_* \right) \tilde{G}(x, z) \Big|_{z=x} \right] \\ &\quad \times \tilde{G}(x, y) \left[\left(\Delta_*^y - \frac{2}{3} R_* \right) \tilde{G}(y, y) + 2 \left(\Delta_*^y - \frac{2}{3} R_* \right) \tilde{G}(y, z) \Big|_{z=y} \right]. \end{aligned} \quad (2.1.27)$$

The measure diagram: The vertex (2.1.15) simply gives

$$\text{Measure Diagram} = \frac{4\pi}{\kappa^2} \int d^2x \sqrt{g_*} \sum_{n>0} \psi_n^2(x) \tilde{G}(x, x). \quad (2.1.28)$$

In addition to these diagrams there are also one-loop diagrams involving counterterm vertices that contribute at the same order in $\frac{1}{\kappa^2}$. They are discussed in the next subsection (section 2.1.3). So far, to summarize:

$$\begin{aligned} W^{(2)}[A] &= W^{(2)}[A]_{\text{loops}} + W^{(2)}[A]_{\text{ct}} \\ W^{(2)}[A]_{\text{loops}} &= \text{Setting Sun Diagram} + \text{Glasses Diagram} + \text{Measure Diagram} + \text{Other Diagrams} \\ W^{(2)}[A]_{\text{ct}} &= \text{counterterm contributions at order } \frac{1}{\kappa^2}. \end{aligned} \quad (2.1.29)$$

Of course, these expressions for the Feynman diagrams are formal writings and have to be regularized. To do so, we use the smooth spectral cutoff regularization described in the section 1.3.3 of Chapter 1. In particular, the Green's function $\tilde{G}(x, y)$ are replaced by the $\hat{K}(t, x, y)$ and the diverging sum $\sum_{r>0} \psi_r^2(x)$ by the heat kernel at coinciding points $\tilde{K}(t, x, x)$, so that one immediately gets the following expressions for the regularized integrals I^3 :

³We recall that $I(t_1, \dots, t_n)$ has to be integrated over $\int_0^\infty d\alpha_1 \varphi(\alpha_1) \dots \int_0^\infty d\alpha_n \varphi(\alpha_n)$. These integrations are implicitly understood.

$$\begin{aligned}
I_{\infty} = & -\frac{8\pi}{\kappa^2} \int d^2x \sqrt{g_*} \left[\frac{1}{4} \widehat{K}(t_1, x, x) (\Delta_*^x - 2R_*) \widehat{K}(t_2, x, x) \right. \\
& + \frac{1}{2} \left[(\Delta_*^x - 2R_*) (\widehat{K}(t_1, x, z) \widehat{K}(t_2, x, z)) \Big|_{z=x} \right. \\
& \left. \left. + 2 \widehat{K}(t_1, x, x) [\Delta_*^x \widehat{K}(t_2, x, z)] \Big|_{z=x} \right] \right], \quad (2.1.30)
\end{aligned}$$

$$\begin{aligned}
I_{\ominus} = & \frac{4\pi}{\kappa^2} \int d^2x \sqrt{g_*(x)} d^2y \sqrt{g_*(y)} \widehat{K}(t_1, x, y) \\
& \times \left[\widehat{K}(t_2, x, y) (\Delta_*^x - \frac{2}{3}R_*) (\Delta_*^y - \frac{2}{3}R_*) \widehat{K}(t_3, x, y) \right. \\
& \left. + 2 \left[(\Delta_*^x - \frac{2}{3}R_*) \widehat{K}(t_2, x, y) \right] (\Delta_*^y - \frac{2}{3}R_*) \widehat{K}(t_3, x, y) \right], \quad (2.1.31)
\end{aligned}$$

$$\begin{aligned}
I_{\circ\circ} = & \frac{2\pi}{\kappa^2} \int d^2x \sqrt{g_*(x)} d^2y \sqrt{g_*(y)} \\
& \times \left[(\Delta_*^x - \frac{2}{3}R_*) \widehat{K}(t_1, x, x) + 2 (\Delta_*^x - \frac{2}{3}R_*) \widehat{K}(t_1, x, z) \Big|_{z=x} \right] \widehat{K}(t_2, x, y) \\
& \times \left[(\Delta_*^y - \frac{2}{3}R_*) \widehat{K}(t_3, y, y) + 2 (\Delta_*^y - \frac{2}{3}R_*) \widehat{K}(t_3, y, z) \Big|_{z=y} \right], \quad (2.1.32)
\end{aligned}$$

$$I_{\circ} = \frac{4\pi}{\kappa^2} \int d^2x \sqrt{g_*} \widehat{K}(t_1, x, x) \widehat{K}(t_2, x, x). \quad (2.1.33)$$

One should keep in mind that one can always symmetrize in the $t_i = \alpha_i/\Lambda^2$ because these expressions are multiplied by the symmetric $\prod_i \int d\alpha_i \varphi(\alpha_i)$. Thus, we consider two expressions as identical if they only differ by a permutation of the t_i .

Scalings

Let us briefly discuss some general features about the expected dependence on the area A of the Feynman diagram (integrals) $I_r(t_1, \dots, t_n)$.

Recall that the eigenfunctions ψ_n of $D = \Delta_* - R_*$ are normalized as

$$\int d^2x \sqrt{g_*} \psi_n(x) \psi_m(x) = \delta_{nm}. \quad (2.1.34)$$

It follows from (1.2.23) that ψ_n and $\lambda_n^L \equiv \lambda_n$ scale as

$$\lambda_n \equiv \lambda_n^A = \frac{A_0}{A} \lambda_n^{A_0}, \quad \psi_n \equiv \psi_n^A = \sqrt{\frac{A_0}{A}} \psi_n^{A_0}, \quad (2.1.35)$$

where the quantities with a label A_0 are defined as the eigenfunctions and eigenvalues of $\Delta_0 - R_0$ with respect to the metric g_0 of area A_0 . This immediately implies the scaling relations

$$\tilde{K}_A(t, x, y) = \frac{A_0}{A} \tilde{K}_{A_0}\left(\frac{A_0}{A} t, x, y\right) \quad , \quad \widehat{\tilde{K}}_A(t, x, y) = \widehat{\tilde{K}}_{A_0}\left(\frac{A_0}{A} t, x, y\right) . \quad (2.1.36)$$

The last relation implies in particular that $\tilde{G}(x, y) = \widehat{\tilde{K}}(0, x, y)$ does not depend on A :

$$\tilde{G}_A(x, y) = \tilde{G}_{A_0}(x, y) . \quad (2.1.37)$$

It then straightforwardly follows from (2.1.36) together with (1.2.23) that all the integrals $I_r(t_i)$ as given in (2.1.30)-(2.1.33) satisfy

$$I_r[t_i, A] = I_r\left[\frac{A_0}{A} t_i, A_0\right] . \quad (2.1.38)$$

Of course, this is true only because these integrals are finite convergent integrals (for $t_i > 0$). In particular, since $t_i = \alpha_i/\Lambda^2$, we see that the I_r cannot depend on A and Λ^2 separately but only on the combination $A\Lambda^2$. On the other hand, the small t short-distance expansion for $\widehat{\tilde{K}}$ given in (1.3.45) also depends on an arbitrary scale μ introduced when defining \tilde{G}_ζ in (1.2.42), so one might wonder whether an additional area-dependence of the form $\mu^2 A$ could occur. However, this is not the case. Indeed, in (1.2.42) the scale μ appears in the combination $\ln[\ell_A^2(x, y)\mu^2]$ where $\ell_A(x, y)$ is the geodesic distance between x and y computed with the metric of area A . Thus $\ln[\ell_A^2(x, y)\mu^2] = \ln A\mu^2 + \dots$ where $+\dots$ refers to terms that do not depend on A or μ . It follows that $4\pi\tilde{G}_\zeta - \ln A\mu^2$ does not depend on μ . Since also, as anticipated in the previous chapter and as we will explain shortly,

$$\tilde{G}_\zeta^A = \tilde{G}_\zeta^{A_0} + \frac{1}{4\pi} \ln \frac{A}{A_0} , \quad (2.1.39)$$

it follows that

$$4\pi\tilde{G}_\zeta^A - \ln \frac{\ell_A^2(x, y)\mu^2}{4} = 4\pi\tilde{G}_\zeta^{A_0} - \ln A_0\mu^2 + \dots , \quad (2.1.40)$$

depends neither on A nor on μ . Since \tilde{G}_ζ and μ only ever appear in this combination in $\widehat{\tilde{K}}$ it is clear that there is no real μ dependence in the end. Thus

$$I_r\left[\frac{\alpha_i}{\Lambda^2}, A\right] = f_r\left[A\Lambda^2, \frac{\alpha_i}{\alpha_j}\right] . \quad (2.1.41)$$

For example, a term $(\ln \Lambda^2 A)^2$ will appear as

$$(\ln A\Lambda^2)^2 = \left(\ln \frac{\Lambda^2}{\mu^2} + \ln \frac{A}{A_0} + \ln \mu^2 A_0\right)^2 = \left(\ln \frac{A}{A_0}\right)^2 + 2 \ln \frac{\Lambda^2}{\mu^2} \ln \frac{A}{A_0} + \dots \quad (2.1.42)$$

where $+\dots$ refers to terms independent of the area A . This structure will be indeed explicitly observed below. Similarly, a term $A\Lambda^2 \ln \mu^2 A$ must be accompanied by a term $A\Lambda^2 \ln \frac{\Lambda^2}{\mu^2}$. Again, this will be explicitly the case.

The relation (2.1.39) that gives the area dependence of \tilde{G}_ζ^A will play a most important role below, since it is through this relation that the $\ln \frac{A}{A_0}$ terms appear in the logarithm of the partition function. Let us prove it. To begin with, $\tilde{G}_\zeta(x)$ is defined in terms of the spectral ζ -function (without zero-mode) in (1.2.41) as

$$\tilde{G}_\zeta(x) = \lim_{s \rightarrow 1} \left[\mu^{2(s-1)} \tilde{\zeta}(s, x, x) - \frac{1}{4\pi(s-1)} \right], \quad (2.1.43)$$

which is equivalent to saying

$$\mu^{2(s-1)} \tilde{\zeta}(s, x, x) = \frac{1}{4\pi(s-1)} + \tilde{G}_\zeta(x) + \mathcal{O}(s-1). \quad (2.1.44)$$

From the definition of $\tilde{\zeta}(s, x, y)$ (1.2.37) and eq. (2.1.35) it follows that

$$\tilde{\zeta}_A(s, x, x) = \left(\frac{A}{A_0} \right)^{s-1} \tilde{\zeta}_{A_0}(s, x, x). \quad (2.1.45)$$

Inserting this into (2.1.43) or (2.1.44) for \tilde{G}_ζ^A and rewriting the r.h.s. in terms of $\tilde{G}_\zeta^{A_0}$ yields the desired relation (2.1.39). It remains to show that \tilde{G}_ζ defined by (2.1.43) (and (1.2.41)) is exactly the same quantity as the one defined by (1.2.42) and that appears in the expansion (1.3.42) of \hat{K} . This is done as follows. By the Mellin transformation between $\tilde{\zeta}(s, x, y)$ and the heat kernel, the singularity of the former is related to the small t asymptotic of the latter and one sees that

$$\tilde{\zeta}_R(s, x, y) = \tilde{\zeta}(s, x, y) - \frac{1}{\Gamma(s)} \int_0^{1/\mu^2} \frac{dt}{4\pi t^2} t^s a_0(x, y) e^{-\ell^2(x, y)/4t} \quad (2.1.46)$$

is smooth for $s \rightarrow 1$ and for $y \rightarrow x$. Taking first the limit $y \rightarrow x$ (for $s \geq 1$) and then $s \rightarrow 1$ yields $\tilde{G}_\zeta(x)$. On the other hand, taking first $s \rightarrow 1$ yields $\tilde{G}(x, y) - \frac{1}{4\pi} a_0(x, y) E_1(\frac{\mu^2 \ell^2(x, y)}{4})$ and then letting $y \rightarrow x$ yields the relation (1.2.42).

Note for later use that $\hat{K}(t, x, x)$ in (1.3.47) can now be rewritten as

$$\hat{K}\left(\frac{\alpha}{\Lambda^2}, x, x\right) = \tilde{G}_\zeta^{A_0}(x) + \frac{1}{4\pi} \left[\ln A \Lambda^2 - \ln A_0 \mu^2 - \ln \alpha - \gamma - \left(\frac{7}{6} R_* - \frac{4\pi}{A} \right) \frac{\alpha}{\Lambda^2} + \dots \right]. \quad (2.1.47)$$

Evaluating the regularized two-loop integrals

We will now explicitly compute the regularized two-loop integrals. Before going on, we would like to stress that we are free to drop all terms that vanish as $\Lambda \rightarrow \infty$. Also, the area-independent finite terms are without interest since they drop out when computing $Z[A]/Z[A_0]$. We do, however, keep the area-independent diverging terms in order to check that, in the end, they only show up in the combinations allowed by the above

scaling argument, see (2.1.41). In the following, we will adopt the notation $\int dx$ for $\int d^2x \sqrt{g_*(x)}$.

The figure-eight and the measure diagrams:

First, $\partial_i^x \widehat{K}(t, x, x) = 2 \partial_i^x \widehat{K}(t, x, z)|_{z=x}$ so that

$$\begin{aligned} \Delta_*^x \left(\widehat{K}(t_1, x, z) \widehat{K}(t_2, x, z) \right) \Big|_{z=x} &= 2 \left(\Delta_*^x \widehat{K}(t_1, x, z) \right) \Big|_{z=x} \widehat{K}(t_2, x, x) \\ &\quad - \frac{1}{2} g_*^{ij} \partial_i^x \widehat{K}(t_1, x, x) \partial_j^x \widehat{K}(t_2, x, x) \\ &= 2 \left[\widetilde{K}(t_1, x, x) + R_* \widehat{K}(t_1, x, x) \right] \widehat{K}(t_2, x, x) \\ &\quad - \frac{1}{2} g_*^{ij} \partial_i^x \widehat{K}(t_1, x, x) \partial_j^x \widehat{K}(t_2, x, x) . \end{aligned} \quad (2.1.48)$$

Also,

$$\int dx g_*^{ij} \partial_i^x \widehat{K}(t_1, x, x) \partial_j^x \widehat{K}(t_2, x, x) = \int dx \widehat{K}(t_1, x, x) \Delta_* \widehat{K}(t_2, x, x) \quad (2.1.49)$$

upon integrating by parts. Hence,

$$I_{\infty} = \frac{8\pi}{\kappa^2} \int dx \left[-3 \widetilde{K}(t_1, x, x) \widehat{K}(t_2, x, x) - \frac{3}{2} R_* \widehat{K}(t_1, x, x) \widehat{K}(t_2, x, x) \right] . \quad (2.1.50)$$

From the asymptotic expansions (1.3.47) of $\widehat{K}(t, x, x)$ and $\widetilde{K}(t, x, x)$, it is then straightforward to get

$$\begin{aligned} \int dx \widetilde{K}(t_1, x, x) \widehat{K}(t_2, x, x) &= \frac{1}{(4\pi)^2} \left(\frac{\Lambda^2}{\alpha_1} + \frac{7}{6} R_* - \frac{4\pi}{A} \right) \int dx \left[4\pi \widetilde{G}_\zeta(x) + \ln \frac{\Lambda^2}{\mu^2 \alpha_2} - \gamma \right] \\ &\quad + c(\alpha_i) + \mathcal{O}(\ln A \Lambda^2 / \Lambda^2) \end{aligned} \quad (2.1.51)$$

and

$$\begin{aligned} \int dx \widehat{K}(t_1, x, x) \widehat{K}(t_2, x, x) &= \frac{1}{(4\pi)^2} \int dx \left[4\pi \widetilde{G}_\zeta(x) + \ln \frac{\Lambda^2}{\mu^2 \alpha_1} - \gamma \right] \\ &\quad \times \left[4\pi \widetilde{G}_\zeta(x) + \ln \frac{\Lambda^2}{\mu^2 \alpha_2} - \gamma \right] + \mathcal{O}(\ln A \Lambda^2 / \Lambda^2) . \end{aligned} \quad (2.1.52)$$

Here and in the following, the symbol $c(\alpha_i)$ denotes a *generic* finite and A -independent constant (but that may depend of the α_i). We always write $c(\alpha_i)$, but this same symbol stands for different constants. Using (2.1.39) to explicitly display the area dependence, one can rewrite

$$\int dx \left[4\pi \widetilde{G}_\zeta(x) + \ln \frac{\Lambda^2}{\mu^2} - \gamma \right] = A \left[\ln A \Lambda^2 + G_0 \right] , \quad (2.1.53)$$

with

$$G_0 = \frac{4\pi}{A_0} \int d^2x \sqrt{g_0} \tilde{G}_\zeta^{A_0}(x) - \gamma - \ln A_0 \mu^2 . \quad (2.1.54)$$

Thus,

$$I_\circ = \frac{1}{2\pi\kappa^2} \left\{ \frac{A\Lambda^2}{2\alpha_1} \ln A\Lambda^2 + \left(\frac{7}{12} AR_* - 2\pi \right) \ln A\Lambda^2 + \frac{A\Lambda^2}{2\alpha_1} (G_0 - \ln \alpha_2) + c(\alpha_i) + \mathcal{O}(\ln A\Lambda^2/\Lambda^2) \right\} \quad (2.1.55)$$

and

$$I_\infty = \frac{1}{2\pi\kappa^2} \left\{ -\frac{3}{\alpha_1} A\Lambda^2 \ln A\Lambda^2 + \left[\left(-\frac{7}{2} + 3 \ln \alpha_1 \right) AR_* + 12\pi \right] \ln A\Lambda^2 - \frac{3}{\alpha_1} A\Lambda^2 (G_0 - \ln \alpha_2) - \frac{3}{2} R_* \int dx \left[4\pi \tilde{G}_\zeta(x) + \ln \frac{\Lambda^2}{\mu^2} - \gamma \right]^2 + c(\alpha_i) + \mathcal{O}(\ln A\Lambda^2/\Lambda^2) \right\} . \quad (2.1.56)$$

I_∞ involves the structure $\int R\tilde{G}_\zeta^2$ which is not background independent, as was remarked in section 1.4. However, this structure will be seen to cancel against similar contributions from the other diagrams.

The setting sun diagram:

Rewriting (2.1.31) gives

$$I_\ominus = \frac{4\pi}{\kappa^2} \int dx dy \widehat{K}(t_1, x, y) \times \left[-\widehat{K}(t_2, x, y) \frac{d}{dt} \tilde{K}(t, x, y) \Big|_{t=t_3} + 2 \tilde{K}(t_2, x, y) \tilde{K}(t_3, x, y) + 2 R_* \widehat{K}(t_2, x, y) \tilde{K}(t_3, x, y) + \frac{R_*^2}{3} \widehat{K}(t_2, x, y) \widehat{K}(t_3, x, y) \right] . \quad (2.1.57)$$

First consider the last term in $R_*^2 \widehat{K}(t_1, x, y) \widehat{K}(t_2, x, y) \widehat{K}(t_3, x, y)$. Replacing each of the $\widehat{K}(t_i, x, y)$ by the unregulated Green's function $\tilde{G}(x, y)$ one gets a converging finite integral, since the short-distance singularity $\sim (\ln \ell(x, y))^3$ is integrable. Now, $\tilde{G}(x, y)$ does not depend on the area and, thus

$$R_*^2 \int dx dy \widehat{K}(t_1, x, y) \widehat{K}(t_2, x, y) \widehat{K}(t_3, x, y) = R_*^2 \int dx dy (\tilde{G}(x, y))^3 + \mathcal{O}((\ln A\Lambda^2)^2/\Lambda^2) . \quad (2.1.58)$$

Since $\int dx dy \equiv \int d^2x \sqrt{g_*(x)} d^2y \sqrt{g_*(y)}$ scales as A^2 and $R_*^2 \sim \frac{1}{A^2}$ this term is obviously an A -independent constant, up to vanishing terms.

All the other terms in (2.1.57) involve at least one $\tilde{K}(t_i, x, y)$ or $\frac{d}{dt}\tilde{K}(t, x, y)$. We write

$$\tilde{K}(t, x, y) = K(t, x, y) - \frac{e^{R_*t}}{A} \quad (2.1.59)$$

$$\frac{d}{dt}\tilde{K}(t, x, y) = \frac{d}{dt}K(t, x, y) - \frac{R_*}{A}e^{R_*t} \quad (2.1.60)$$

and use the fact that for small t the heat kernel $K(t, x, y)$ and its derivated $\frac{d}{dt}K(t, x, y)$ both exponentially vanish unless $\ell(x, y)^2$ is of order t or less. Thus we can use normal coordinates $z = x - y$ and use the expressions (1.3.38), (1.3.45) and (1.3.46), so that we can do the various integrations over z explicitly. For convenience, we have listed the relevant integrals in the appendix A. Now, eq. (2.1.57) may once more be rewritten (upon symmetrising the t_i) as

$$\begin{aligned} I_\Theta = \frac{4\pi}{\kappa^2} & \left\{ \int dx dy \widehat{\tilde{K}}(t_1, x, y) \right. \\ & \times \left[-\widehat{\tilde{K}}(t_2, x, y) \frac{d}{dt}K(t, x, y) \Big|_{t=t_3} + 2K(t_2, x, y)K(t_3, x, y) \right. \\ & \qquad \qquad \qquad \left. \left. + 2R_*\widehat{\tilde{K}}(t_2, x, y)K(t_3, x, y) \right] \right. \\ & - \frac{e^{R_*t_3}}{A} \int dx dy \left[4\widehat{\tilde{K}}(t_1, x, y)K(t_2, x, y) + R_*\widehat{\tilde{K}}(t_1, x, y)\widehat{\tilde{K}}(t_2, x, y) \right] \\ & \left. + c(\alpha_i) + \mathcal{O}((\ln A\Lambda^2)^2/\Lambda^2) \right\}. \quad (2.1.61) \end{aligned}$$

The last two terms are easy to compute. Similarly as previously, one may indeed safely replace

$$\frac{R_*}{A}e^{R_*t_3} \int dx dy \widehat{\tilde{K}}(t_1, x, y)\widehat{\tilde{K}}(t_2, x, y) = \frac{R_*}{A} \int dx dy (\tilde{G}(x, y))^2 + \mathcal{O}((\ln A\Lambda^2)^2/\Lambda^2), \quad (2.1.62)$$

ending up with an A -independent term. It is also straightforward to get

$$\frac{e^{R_*t_3}}{A} \int dx dy \widehat{\tilde{K}}(t_1, x, y)K(t_2, x, y) = \frac{1}{4\pi} [\ln A\Lambda^2 - \ln(\alpha_1 + \alpha_2) + G_0] + \mathcal{O}(\ln A\Lambda^2/\Lambda^2). \quad (2.1.63)$$

We consider next $R_* \int dx dy \widehat{\tilde{K}}(t_1, x, y)\widehat{\tilde{K}}(t_2, x, y)K(t_3, x, y)$. For this integral, one only needs to keep the leading term in the small t_i expansion of $K(t_3, x, y)$ (1.3.45) and one can drop all terms $\mathcal{O}(t_i)$ or $\mathcal{O}(z^j)$ in the expansion of the $\widehat{\tilde{K}}(t_i, x, y)$ (1.3.45), or of $\sqrt{g_*}$

(1.3.38). The relevant integrals that multiply terms involving either \tilde{G}_ζ and/or $\ln \mu^2 t_3$ are all listed in the appendix A. Other integrals like $\int d^2 \tilde{z} e^{-\tilde{z}^2} E_1(\tilde{z}^2 \frac{t_3}{t_1}) E_1(\tilde{z}^2 \frac{t_3}{t_2})$ only contribute finite constants that only depend on ratios of the α_i . One gets

$$R_* \int dx dy \widehat{\tilde{K}}(t_1, x, y) \widehat{\tilde{K}}(t_2, x, y) K(t_3, x, y) = \frac{R_*}{(4\pi)^2} \int dx \left[4\pi \tilde{G}_\zeta(x) + \ln \frac{\Lambda^2}{\mu^2} - \gamma \right]^2 - 2 \frac{AR_*}{(4\pi)^2} \ln(\alpha_1 + \alpha_2) \ln A\Lambda^2 + c(\alpha_i) + \mathcal{O}((\ln A\Lambda^2)^2/\Lambda^2). \quad (2.1.64)$$

We now consider the term in $\frac{d}{dt} K(t, x, y) \Big|_{t=t_3}$. This term involves one more factor of t_3^{-1} than $K(t_3, x, y)$ in the previous computation (see (1.3.46)), so that one has to keep the terms $\mathcal{O}(t_i)$ or $\mathcal{O}(z^i z^j)$ in the expansions of $\widehat{\tilde{K}}(t_i, x, y)$. The computation is then quite lengthy but straightforward. Note that a term $\int dx \partial_i \tilde{G}_\zeta \partial^i \tilde{G}_\zeta$ appears. Integrating once more by parts and using (2.1.39), this equals $\int dx \tilde{G}_\zeta \Delta_* \tilde{G}_\zeta = \int dx \tilde{G}_\zeta^{A_0} \Delta_* \tilde{G}_\zeta^{A_0}$. Since $\int dx \equiv \int d^2 x \sqrt{g_*}$ scales as A and $\Delta_* = \frac{A_0}{A} \Delta_0$ this is once more an A -independent constant. Upon using the symmetry under exchange of the α_i , the result then is

$$\int dx dy \widehat{\tilde{K}}(t_1, x, y) \widehat{\tilde{K}}(t_2, x, y) \frac{d}{dt} K(t, x, y) \Big|_{t=t_3} = - \frac{R_*}{(4\pi)^2} \int dx \left[4\pi \tilde{G}_\zeta(x) + \ln \frac{\Lambda^2}{\mu^2} - \gamma \right]^2 - \frac{2}{(4\pi)^2} \frac{A\Lambda^2}{\alpha_1 + \alpha_2} \ln A\Lambda^2 - \frac{2}{(4\pi)^2} \left[\left(\frac{1}{6} - \frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2} - \ln(\alpha_1 + \alpha_2) \right) AR_* - 4\pi \right] \ln A\Lambda^2 + A\Lambda^2 C(\alpha_i) + c(\alpha_i) + \mathcal{O}((\ln A\Lambda^2)^2/\Lambda^2). \quad (2.1.65)$$

The constant $C(\alpha_i)$ is a rather complicated function of the α_i we choose not to explicitly display since it will be absorbed in the cosmological constant in the end. It comes indeed with the combination $A\Lambda^2$ (as allowed by the scaling argument (2.1.41)). Note that one might have expected a leading singularity $\sim A\Lambda^2 (\ln A\Lambda^2)^2$, due to the leading $\frac{1}{t_3^2}$ singularity of $\frac{d}{dt} \tilde{K}(t, x, y) \Big|_{t=t_3}$ multiplying the $\ln A\Lambda^2$ singularities from each of the two $\widehat{\tilde{K}}(t_i, x, y)$. However, this leading term is multiplied by $\int d^2 \tilde{z} e^{-\tilde{z}^2} (1 - \tilde{z}^2)$ which vanishes. Thus the leading singularity is $\sim A\Lambda^2 \ln A\Lambda^2$ as expected from naive power counting for the present two-loop diagrams.

To compute the last remaining integral, we use (1.3.38) and (1.3.45) to get

$$\sqrt{g_*(x)} K(t_2, x, y) K(t_3, x, y) = \frac{e^{-z^2/4T}}{(4\pi)^2 (t_2 + t_3) T} \left[1 + \frac{7}{6} R_*(t_2 + t_3) + \dots \right], \quad (2.1.66)$$

where $T = \frac{t_2 t_3}{t_2 + t_3}$. Thus, one has once again to keep the terms $\mathcal{O}(t_i)$ or $\mathcal{O}(z^i z^j)$ in the

short-distance expansion of $\widehat{K}(t_1, x, y)$. Doing the integrals over z results in

$$\begin{aligned} \int dx dy \widehat{K}(t_1, x, y) K(t_2, x, y) K(t_3, x, y) = \\ \frac{1}{(4\pi)^2} \frac{A\Lambda^2}{\alpha_2 + \alpha_3} \left[\ln A\Lambda^2 + G_0 + \ln \frac{\alpha_2 + \alpha_3}{\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3} \right] \\ + \frac{AR_*}{(4\pi)^2} \left(\frac{7}{6} - \frac{\alpha_2\alpha_3}{(\alpha_2 + \alpha_3)^2} \right) \ln A\Lambda^2 + c(\alpha_i) + \mathcal{O}(\ln A\Lambda^2/\Lambda^2), \end{aligned} \quad (2.1.67)$$

so that one ends up with the following contribution from the setting sun diagram:

$$\begin{aligned} I_\Theta = \frac{1}{2\pi\kappa^2} \left\{ \frac{3}{2} R_* \int dx \left[4\pi \widetilde{G}_\zeta(x) + \ln \frac{\Lambda^2}{\mu^2} - \gamma \right]^2 + \frac{2A\Lambda^2}{\alpha_1 + \alpha_2} \ln A\Lambda^2 \right. \\ \left. + \left[\left(\frac{4}{3} - \frac{2\alpha_1\alpha_2}{(\alpha_1 + \alpha_2)^2} - 3\ln(\alpha_1 + \alpha_2) \right) AR_* - 12\pi \right] \ln A\Lambda^2 \right. \\ \left. + A\Lambda^2 C(\alpha_i) + c(\alpha_i) + \mathcal{O}((\ln A\Lambda^2)^2/\Lambda^2) \right\}. \end{aligned} \quad (2.1.68)$$

The glasses diagram:

Expanding (2.1.32), one has

$$\begin{aligned} I_{\text{O}\text{O}} = \frac{2\pi}{\kappa^2} \int dx dy \left\{ \widehat{K}(t_1, x, x) \widehat{K}(t_3, y, y) \right. \\ \left. \times \left[-\frac{d}{dt} \widetilde{K}(t, x, y) \Big|_{t=t_2} + 2R_* \widetilde{K}(t_2, x, y) + R_*^2 \widehat{K}(t_2, x, y) \right] \right. \\ \left. + 2 \left(\widetilde{K}(t_2, x, y) + R_* \widehat{K}(t_2, x, y) \right) \left(\widetilde{K}(t_1, x, x) \widehat{K}(t_3, y, y) + \widehat{K}(t_1, x, x) \widetilde{K}(t_3, y, y) \right) \right. \\ \left. + 4 \widetilde{K}(t_1, x, x) \widehat{K}(t_2, x, y) \widetilde{K}(t_3, y, y) \right\}. \end{aligned} \quad (2.1.69)$$

However, $\widetilde{K}(t, x, x)$ does not depend of x and the only dependence in x in $\widehat{K}(t, x, x)$ is $\widetilde{G}_\zeta(x)$ (cf (1.3.47)). Then, using (1.3.44), and since $\frac{d}{dt} \widetilde{K}(t, x, y) \Big|_{t=t_2}$, $\widetilde{K}(t_2, x, y)$ and $\widehat{K}(t_2, x, y)$ have no zero-mode, the last two lines vanish and one gets

$$I_{\text{O}\text{O}} = \frac{2\pi}{\kappa^2} \int dx dy \widetilde{G}_\zeta(x) \widetilde{G}_\zeta(y) \left[-\frac{d}{dt} \widetilde{K}(t, x, y) \Big|_{t=t_2} + 2R_* \widetilde{K}(t_2, x, y) + R_*^2 \widehat{K}(t_2, x, y) \right]. \quad (2.1.70)$$

Using the by now familiar argument of replacing the $\widehat{K}(t_2, x, y)$ by $\widetilde{G}(x, y)$, one sees that the last term is just an A -independent constant. Indeed,

$$\begin{aligned} R_*^2 \int dx dy \widetilde{G}_\zeta(x) \widehat{K}(t_2, x, y) \widetilde{G}_\zeta(y) &= R_*^2 \int dx dy \widetilde{G}_\zeta(x) \widetilde{G}(x, y) \widetilde{G}_\zeta(y) + \mathcal{O}(1/\Lambda^2) \\ &= R_*^2 \int dx dy \widetilde{G}_\zeta^{A_0}(x) \widetilde{G}(x, y) \widetilde{G}_\zeta^{A_0}(y) + \mathcal{O}(1/\Lambda^2) \end{aligned} \quad (2.1.71)$$

where we once again used $\int dx \widetilde{G}(x, y) = \int dy \widetilde{G}(x, y) = 0$, together with (2.1.39). Since $\widetilde{G}_\zeta^{A_0}$ and \widetilde{G} do not depend on A , this term is just another A -independent constant, up to vanishing terms.

Writing once more $\widetilde{K}(t_2, x, y) = K(t_2, x, y) - \frac{e^{R_* t_2}}{A}$, we compute

$$\begin{aligned} \int dx dy \widetilde{G}_\zeta(x) \widetilde{K}(t_2, x, y) \widetilde{G}_\zeta(y) &= \int dx \left[\widetilde{G}_\zeta^2(x) (1 + R_* t_2) - t_2 \widetilde{G}_\zeta(x) \Delta_* \widetilde{G}_\zeta(x) \right] \\ &\quad - \frac{e^{R_* t_2}}{A} \left(\int dy \widetilde{G}_\zeta(y) \right)^2 + \mathcal{O}(t_2^2). \end{aligned} \quad (2.1.72)$$

Note that we also keep the subleading terms $\sim t_2$ since the remaining term in (2.1.70) is simply obtained by taking $-\frac{d}{dt_2}$ of (2.1.72). These subleading terms in (2.1.72) yield the finite terms of

$$\begin{aligned} \int dx dy \widetilde{G}_\zeta(x) \frac{d}{dt_2} \widetilde{K}(t_2, x, y) \widetilde{G}_\zeta(y) &= \int dx \left[R_* \widetilde{G}_\zeta^2(x) - \widetilde{G}_\zeta(x) \Delta_* \widetilde{G}_\zeta(x) \right] \\ &\quad - \frac{R_*}{A} \left(\int dy \widetilde{G}_\zeta(y) \right)^2 + \mathcal{O}(1/\Lambda^2). \end{aligned} \quad (2.1.73)$$

As previously seen, $\int dx G_\zeta \Delta_* G_\zeta$ is an A -independent term, so that

$$\begin{aligned} I_{\text{O-O}} &= \frac{2\pi}{\kappa^2} \left\{ R_* \int dx \widetilde{G}_\zeta^2(x) - \frac{R_*}{A} \left(\int dy \widetilde{G}_\zeta(y) \right)^2 + c + \mathcal{O}(1/\Lambda^2) \right\} \\ &= \frac{2\pi}{\kappa^2} \left\{ c + \mathcal{O}(1/\Lambda^2) \right\}, \end{aligned} \quad (2.1.74)$$

where we used again (2.1.39) to show that the explicitly written finite terms do not depend on A .

It is satisfying that this diagram does not give any non-trivial contribution. Indeed, we expect that the propagator connecting the two loops should carry zero ‘‘momentum’’ and, since no zero-mode is present, we expect to obtain zero. The absence of a zero-mode, of course, was the reason that the $\widehat{K}(t_i, x, x)$ could be replaced by $\widetilde{G}_\zeta(x)$, and similarly for y . However, on a non-trivial manifold $\widetilde{G}_\zeta(x)$ is not constant, and the overall contribution does not need to vanish. We also note that each individual integral is finite and area-independent, in agreement with our general argument that A can only appear in the combination $A\Lambda^2$: only divergent integrals can give an area dependence.

Summing the two-loop and measure diagrams

Summing the contributions of the four diagrams, we find the two-loop contribution for the logarithm of the partition function:

$$\begin{aligned}
W^{(2)}[A] \Big|_{\text{loops}} &= I_{\infty} + I_{\odot} + I_{\ominus} + I_{\circ\circ} \\
&= \frac{1}{2\pi\kappa^2} \left\{ \left[\frac{2A\Lambda^2}{\alpha_1 + \alpha_2} - \frac{5A\Lambda^2}{2\alpha_1} + AR_* \left(\frac{3}{2} \ln \frac{\alpha_1\alpha_2}{(\alpha_1 + \alpha_2)^2} - 2 \frac{\alpha_1\alpha_2}{(\alpha_1 + \alpha_2)^2} - \frac{19}{12} \right) - 2\pi \right] \ln A\Lambda^2 \right. \\
&\quad \left. + A\Lambda^2 C(\alpha_i) + c(\alpha_i) + \mathcal{O}((\ln A\Lambda^2)^2/\Lambda^2) \right\}, \quad (2.1.75)
\end{aligned}$$

where $C(\alpha_i)$ and $c(\alpha_i)$ are finite area-independent but regulator-dependent ‘‘constants’’. Quite remarkably, the $R_* \int dx \left(4\pi \tilde{G}_\zeta(x) + \ln \frac{\Lambda^2}{\mu^2} - \gamma \right)^2$ terms and thus the $(\ln A\Lambda^2)^2$ divergences appear in exactly the right combination in I_{∞} and I_{\ominus} to cancel in $W^{(2)}$.

The cancellation of these $(\ln A\Lambda^2)^2$ -terms is equivalent to the cancellation of the $\int R(\tilde{G}_\zeta)^2$ terms. As we have observed in section 1.4 in the previous chapter, the presence of these terms in the logarithm of the partition function would have ruined its background independence. Thus, it is this cancellation which ensures the background independence of the fixed-area partition function! Indeed, the remaining structures $R\tilde{G}_\zeta$ and $\frac{1}{A} \int \tilde{G}_\zeta$ were shown to be background independent, up to irrelevant area-independent terms. Finally, $\Lambda^2 \int \tilde{G}_\zeta$ is background independent modulo adjusting the divergent cosmological constant.

To simplify the notation, we define

$$\begin{aligned}
F(\alpha_i) &= 8\pi(1-h) \left(\frac{3}{2} \ln \frac{\alpha_1\alpha_2}{(\alpha_1 + \alpha_2)^2} - 2 \frac{\alpha_1\alpha_2}{(\alpha_1 + \alpha_2)^2} - \frac{19}{12} \right) - 2\pi, \\
H(\alpha_i) &= \frac{2}{\alpha_1 + \alpha_2} - \frac{5}{2\alpha_1}. \quad (2.1.76)
\end{aligned}$$

If we normalize with respect to A_0 , i.e we compute $\ln \frac{Z[A]}{Z[A_0]}$, we get

$$\begin{aligned}
W^{(2)}[A] \Big|_{\text{loops}} - W^{(2)}[A_0] \Big|_{\text{loops}} &= \frac{1}{2\pi\kappa^2} \left(A\Lambda^2 H(\alpha_i) + F(\alpha_i) \right) \ln \frac{A}{A_0} \\
&\quad + \frac{(A - A_0)\Lambda^2}{2\pi\kappa^2} C(\alpha_i) + \mathcal{O}((\ln A\Lambda^2)^2/\Lambda^2). \quad (2.1.77)
\end{aligned}$$

The last line corresponds to a cosmological constant term, and we may always add a corresponding counterterm to the quantum gravity action to cancel this term. The first line displays an $\ln \frac{A}{A_0}$ -term, as expected, although with a rather complicated, regulator dependent coefficient, as well as a diverging term $\sim A\Lambda^2 \ln \frac{A}{A_0}$. The latter term is certainly not expected to occur. It is non-local, and it cannot be cancelled by any local *two-loop* counterterm. At this point it is useful to mention that the KPZ result is, see

(2.1.1),

$$\left[W^{(2)}[A] - W^{(2)}[A_0] \right] \Big|_{\text{KPZ}} = \frac{2}{\kappa^2} (1-h) \ln \frac{A}{A_0} - \mu_c^2 (A - A_0). \quad (2.1.78)$$

Clearly, as in any quantum field theory, the contributions at order κ^{-2} do not only come from two-loop diagrams, but also from tree and one-loop diagrams involving counterterms. Of course, there is no “vacuum tree-diagram” and the only “tree” contribution is the cosmological constant counterterm we already mentioned. However, the various n -point functions at one loop (that are $\sim \kappa^{-n}$) are also divergent quantities and, to make them finite, one has to introduce various “one-loop” counterterm n -point vertices that will be of order κ^{-n} . In particular, there will be a quadratic counterterm vertex needed to make the renormalized propagator finite, and the one-loop diagram made with this counterterm vertex will contribute at order κ^{-2} to the partition function. Thus, our next task is to determine this counterterm. This will involve the one-loop computation of the two-point function.

2.1.3 Counterterm contribution to the partition function

Any divergence in $W[A] \Big|_{\text{loops}}$ that is simply proportional to the area A (and hence also to Λ^2) can be cancelled by a counterterm of the form $\int d^2x \sqrt{g_*} \Lambda^2 \left(c_i \ln \frac{\Lambda^2}{\mu^2} + c_i \right)$, referred to as cosmological constant. Also, any A -independent constants drop out when computing $\ln \frac{Z[A]}{Z[A_0]} = W[A] - W[A_0]$ and are thus irrelevant. However, the divergences $\sim \Lambda^2 \int dy 4\pi \tilde{G}_\zeta = A\Lambda^2 (\ln \frac{A}{A_0} + c)$ are *non-local* divergences. Of course, they cannot be cancelled by any local two-loop counterterm. However, they can be cancelled by one-loop diagrams involving local (one-loop) counterterm vertices. The same thing happens on a four-dimensional curved manifold with non-derivative ϕ^3 and ϕ^4 couplings [53]. At present, the only such one-loop contribution to the partition function comes from a diagram having a single propagator connecting the two legs of the counterterm vertex that itself is $\sim \frac{1}{\kappa^2}$. This is very similar to the diagram coming from the measure and can thus be seen as renormalizing the measure. In particular, the measure resulted in a vertex $\sim \int dx \tilde{K}(t, x, x) \tilde{\phi}^2(x)$ where $\tilde{K}(t, x, x) = \frac{\Lambda^2}{4\pi\alpha} + \frac{7}{24\pi} R_* - \frac{1}{A} + \dots$ is a constant. Thus any counterterm that looks like an (unwanted) mass renormalization can actually be interpreted as a renormalization of the measure. We thus allow a counterterm action of the form

$$S_{\text{ct}} = \frac{8\pi}{\kappa^2} \int d^2x \sqrt{g_*} \left[\frac{c_\phi}{2} \tilde{\phi} (\Delta_* - R_*) \tilde{\phi} + \frac{c_R}{2} R_* \tilde{\phi}^2 + \frac{c_m}{2} \tilde{\phi}^2 \right]. \quad (2.1.79)$$

Note that a linear counterterm automatically vanishes since $\tilde{\phi}$ has no zero-mode, and higher counterterm vertices only contribute within two-loop vacuum diagrams that are $\sim \frac{1}{\kappa^4}$ (see section 2.2.3). The counterterm coefficients can obviously depend on the cutoff Λ and on the regularization functions $\varphi(\alpha)$. They also have an expansion in powers of $\frac{1}{\kappa}$. Due to the explicit factor of $\frac{1}{\kappa^2}$ in front of the counterterm action, we will only be interested here in the lowest order, κ -independent pieces. Note that in terms of the

original $\widehat{\phi}$ they are of order $\kappa^0 \widehat{\phi}^2$, consistent with the fact that these terms originate at one loop. The dependence on Λ is again dictated by dimensional considerations: c_ϕ and c_R must be dimensionless and will correspond to at most logarithmic divergences, while c_m has dimension of Λ^2 and corresponds to an at most quadratic (times a log) divergence. Then, with $\int d\alpha_i \varphi(\alpha_i)$ implicitly understood⁴, one has:

$$\begin{aligned} c_\phi(\Lambda, \alpha_i) &= c_\phi^{(1)}(\alpha_i) \ln A\Lambda^2 + c_\phi^{(2)}(\alpha_i) , \\ c_R(\Lambda, \alpha_i) &= c_R^{(1)}(\alpha_i) \ln A\Lambda^2 + c_R^{(2)}(\alpha_i) , \\ c_m(\Lambda, \alpha_i) &= c_m^{(1)}(\alpha_i) \Lambda^2 \ln A\Lambda^2 + c_m^{(2)}(\alpha_i) \Lambda^2 + c_m^{(3)}(\alpha_i) \frac{1}{A} . \end{aligned} \quad (2.1.80)$$

Note that counterterms involving explicitly the area A like $c_\phi^{(1)}$, $c_R^{(1)}$, $c_m^{(1)}$ and $c_m^{(3)}$ are non-local counterterms that should not occur in any standard QFT. However, we have already observed the similarity between the counterterms and the measure action. The latter actually corresponds to some well-defined values of $c_m^{(2)}$, $c_R^{(2)}$ and $c_m^{(3)}$, so that a non-vanishing counterterm coefficient $c_m^{(3)}$ is certainly allowed.

Of course, the counterterm action (2.1.79) cannot be expressed in terms of geometric invariants written using only the metric g and curvature R , contrary to the cosmological constant $\sim \int d^2x \sqrt{g}$. This is why it is often considered that such counterterms should not be allowed. However, as repeatedly emphasized, the whole quantization procedure is carried out with respect to some fixed background metric g_* and already the original Liouville action cannot be written in terms of g and R alone but requires reference to the background metric. As discussed before, the real question is whether the whole quantization procedure is independent of the choice of the background metric. As argued in the section 1.4 of the previous chapter, discussing the background independence of the counterterm action (2.1.79) (or of the measure plus counterterm action) is pointless. Indeed, it misses the crucial contributions from the regularization of the two-loop integrals. Thus, as already stated, we will instead study the background (in)dependence of the final contributions to the logarithm of the fixed-area partition function $W^{(2)}[A]$.

The one-loop diagram with the counterterm vertex that follows from (2.1.79) gives a contribution $W^{(2)}[A]_{\text{ct}}$ to be added to (2.1.75) that is

$$\begin{aligned} W^{(2)}[A]_{\text{ct}} &= \frac{8\pi}{\kappa^2} \int dx \left[-\frac{c_\phi}{2} \widetilde{K}(t, x, x) - \left(\frac{c_R}{2} R_* + \frac{c_m}{2} \right) \widehat{K}(t, x, x) \right] \\ &= -\frac{1}{\kappa^2} \left[c_\phi \left(\frac{A\Lambda^2}{\alpha} + \frac{7}{6} R_* A - 4\pi \right) + (c_R R_* A + c_m A) \left(G_0 + \ln \frac{A\Lambda^2}{A_0 \mu^2} - \ln \alpha \right) \right] , \end{aligned} \quad (2.1.81)$$

where we used the small t expansions (1.3.47) of $\widetilde{K}(t, x, x)$ and $\widehat{K}(t, x, x)$, as well as the relation (2.1.53) and the definition (2.1.54) for G_0 .

Before actually computing the counterterm coefficients in the next subsection, let us discuss what are the “desired” values of the c_ϕ , c_R and c_m . In particular, the non-local

⁴This means that if any given $c(\alpha_i)$ depends on any given number of α_i , $i = 1, \dots, r$, one should really think of it as $c[\varphi] = \int_0^\infty d\alpha_1 \dots d\alpha_r \varphi(\alpha_1) \dots \varphi(\alpha_r) c(\alpha_1, \dots, \alpha_r)$.

terms $\sim A\Lambda^2(\ln A\Lambda^2)^2$ and $\sim (\ln A\Lambda^2)^2$ are absent from $W^{(2)}[A]|_{\text{loops}}$, as given in (2.1.75) and, hence, should also be absent from (2.1.81). It is easy to see that this implies

$$c_m^{(1)} = c_R^{(1)} = 0 \quad , \quad (\text{desired values}) . \quad (2.1.82)$$

Below, we will indeed find that $c_m^{(1)} = c_R^{(1)} = 0$, as well as $c_\phi^{(1)} = 0$. Anticipating (2.1.82), as well as

$$c_\phi^{(1)} = 0 \quad , \quad (2.1.83)$$

eq. (2.1.81) becomes

$$\begin{aligned} W^{(2)}[A]|_{\text{ct}} &= -\frac{1}{\kappa^2} \left\{ c_m^{(2)} A\Lambda^2 \ln A\Lambda^2 + [c_m^{(3)} + c_R^{(2)} R_* A] \ln A\Lambda^2 \right. \\ &\quad \left. + \left[\frac{c_\phi^{(2)}}{\alpha} + c_m^{(2)} (G_0 - \ln A_0 \mu^2 - \ln \alpha) \right] A\Lambda^2 \right\} + c_{\text{ct}}(\alpha_i) + \mathcal{O}(\ln A\Lambda^2/\Lambda^2) , \end{aligned} \quad (2.1.84)$$

where $c_{\text{ct}}(\alpha_i)$ is some A -independent finite function of the α_i . The terms in the second line are either $\sim A$ and can again be changed by adding an additional cosmological constant counterterm, or irrelevant area-independent constants. If one adds (2.1.84) to (2.1.75), one gets

$$\begin{aligned} W^{(2)}[A]|_{\text{loops}} + W^{(2)}[A]|_{\text{ct}} &= -\frac{1}{\kappa^2} \left\{ \left[-\frac{1}{2\pi} H(\alpha_i) + c_m^{(2)} \right] A\Lambda^2 \ln A\Lambda^2 \right. \\ &\quad \left. + \left[-\frac{1}{2\pi} F(\alpha_i) + c_m^{(3)} + c_R^{(2)} R_* A \right] \ln A\Lambda^2 \right\} \\ &\quad + A\Lambda^2 [\dots] + c(\alpha_i) + c_{\text{ct}}(\alpha_i) + \mathcal{O}((\ln A\Lambda^2)^2/\Lambda^2) . \end{aligned} \quad (2.1.85)$$

We see that $c_\phi^{(2)}$ only enters in the cosmological constant part and, hence, does not play any role. Cancellation of the $A\Lambda^2 \ln A\Lambda^2$ terms requires

$$c_m^{(2)} = \frac{1}{2\pi} H(\alpha_i) . \quad (2.1.86)$$

Finally, the ‘‘physical’’ coefficient of $\ln A\Lambda^2$ should not depend on the choice of the regulator functions $\varphi(\alpha_i)$, which requires

$$c_m^{(3)} + c_R^{(2)} R_* A \equiv c_m^{(3)} + 8\pi(1-h)c_R^{(2)} = \frac{1}{2\pi} F(\alpha_i) + \text{const} , \quad (2.1.87)$$

where const is a true, α_i -independent constant. One could be tempted to equate separately the terms proportional to $R_* A \sim (1-h)$ and those not involving the curvature, resulting in ‘‘universal’’, genus-independent counterterm coefficients. But since we compute on a surface of fixed genus h , this does not really make sense. In any case, it is

satisfying to remark that, with the possible exception of $c_m^{(3)}$, the non-local counterterms $c_\phi^{(1)}$, $c_m^{(1)}$ and $c_R^{(1)}$ are not required!

Let us come back to the issue of background independence. As one sees from (2.1.81), the counterterms can only give rise to contributions in $W^{(2)}$ that involve at most one \tilde{G}_ζ (contained in G_0). This is obvious since the counterterm action is quadratic in $\tilde{\phi}$ and the resulting one-loop diagrams involve at most one \tilde{K} giving rise to one \tilde{G}_ζ . For this same reason, only $\ln A\Lambda^2$ appears in $W_{\text{ct}}^{(2)}$, and not $(\ln A\Lambda^2)^2$ as did appear in the individual two-loop diagrams. While one could obtain a contribution to $W_{\text{ct}}^{(2)}$ involving $(\ln A\Lambda^2)^2$ by allowing a non-local counterterm coefficient $c_R^{(1)}$, there is no way to produce a term $\sim R(\tilde{G}_\zeta)^2$. For the total two-loop contribution to $W^{(2)}$, the absence of $(\ln A\Lambda^2)^2$ was equivalent to the absence of $R(\tilde{G}_\zeta)^2$ terms. The latter would have violated background independence and, as just shown, it could not have been cancelled by counterterms. We see that the absence of non-local counterterm coefficients is a consequence of background independence of the fixed-area two-loop partition function $W^{(2)}$!

After all these considerations, it is now time to actually compute the counterterm coefficients. Remarkably, we will find that they satisfy all desired relations, in particular (2.1.86) and (2.1.87), which are quite non-trivial.

2.1.4 One-loop computation of the two-point function and determination of the counterterm coefficients

To actually determine the counterterms, one must do a one-loop computation of the two-point Green's function and impose some convenient renormalization conditions to not only cancel the diverging parts but to also ensure the result to be regulator independent. Doing so will constrain the finite contributions.

Note that in ordinary flat space quantum field theory one does not need to compute the full two-point Green's function $\mathcal{G}^{(2)}(u, v)$. Instead one rather computes the amputated or 1PI (which is the same at one loop) two-point function in momentum space. The corresponding contribution of the counterterms can then be read directly from the counterterm action. The analogue of such a momentum space computation is not available, in general, on a curved manifold,⁵ and one has to do the computation directly in real space. In this case it seems that the simplest way to correctly take into account all contributions is to compute $\mathcal{G}^{(2)}(u, v)$.

We will compute the regularized⁶ two-point function $\mathcal{G}^{(2)}(u, v)$ which we define so that

⁵The analogue of momentum space is the mode decomposition with respect to the eigenfunctions $\psi_r(x)$ of Δ , replacing the plane waves e^{ipx} . An important property is momentum conservation that follows from $\int dx e^{i(p_1+p_2+p_3)x} \sim \delta(p_1+p_2+p_3)$. On a curved manifold one then needs the $C_{rst} = \int dx \psi_r \psi_s \psi_t$, etc. While on the round sphere, where the ψ_r are the spherical harmonics, the C_{rst} are the well-known Clebch-Gordan coefficients, on the higher genus surfaces much less is known about the C_{rst} and things are much more complicated.

⁶Contrary to what we did at tree-level where we used \tilde{G} and \tilde{K} , resp $\tilde{G}_{\varphi, \Lambda}$ to denote the Green's function and its regularized version, here we just write $\mathcal{G}^{(2)}$ since we will always deal with the regularized two-point function.

at tree-level it is just $\widehat{K}(t, u, v)$. At order κ^{-2} it receives one-loop Liouville contributions and tree-level contributions from the measure and the counterterms:

$$\mathcal{G}^{(2)}(u, v) = \widehat{K}(t, u, v) + \mathcal{G}^{(2)}(u, v) \Big|_{\text{L}}^{1\text{-loop}} + \mathcal{G}^{(2)}(u, v) \Big|_{\text{measure}}^{\text{tree}} + \mathcal{G}^{(2)}(u, v) \Big|_{\text{ct}}^{\text{tree}} + \mathcal{O}(\kappa^{-4}) . \quad (2.1.88)$$

A first renormalization condition one clearly wants to impose on this full 2-point function is finiteness if $u \neq v$:

$$\text{For } u \neq v \quad : \quad \lim_{\Lambda \rightarrow \infty} \mathcal{G}^{(2)}(u, v) \text{ is finite.} \quad (2.1.89)$$

As will be shown, this condition indeed fixes all diverging parts of the counterterm coefficients, *i.e.* $c_\phi^{(1)}$, $c_R^{(1)}$, $c_m^{(1)}$, as well as $c_m^{(2)}$.

It turns out to be surprisingly difficult to find a sensible renormalization condition to fix the finite parts of $\mathcal{G}^{(2)}$ and thus the finite parts of the counterterm coefficients. The trouble is that there is no analogue of a renormalization condition at some particular value of momentum. The only natural choice seems to be zero momentum, corresponding to the zero-mode of the two-point function. But $\int du \mathcal{G}^{(2)}(u, v) = 0$ automatically, and this condition is empty. Instead, one might be tempted to try to impose some condition at $u = v$ like e.g.

$$\text{For } u = v \quad : \quad \lim_{\Lambda \rightarrow \infty} \left[\mathcal{G}^{(2)}(u, u) - \widehat{K}(t, u, u) \right] \stackrel{?}{=} 0 . \quad (2.1.90)$$

However, this doesn't make sense either. Indeed, the divergence of $\mathcal{G}^{(2)}(u, v)$ as $u \rightarrow v$ will turn out to be different from the one of $\widehat{K}(t, u, v)$: there are additional diverging and additional finite terms. As a matter of fact, absence of the diverging terms would require a non-vanishing counterterm coefficient $c_\phi^{(1)}$, which will be excluded by requiring finiteness of the two-point function at $u \neq v$. Independently of this, the difference of the finite terms also makes it impossible to impose (2.1.90).

Actually, this ‘‘problem’’ was to be expected. The usual renormalization of the two-point function at some finite value of momentum, or equivalently at non-coinciding points, does not, of course yield a finite two-point function at coinciding points, nor does it imply that the loop and counterterm contributions to the two-point function vanish at coinciding points. This is the translation of the fact that to define composite operators like $(\widetilde{\phi}(u))^2$ one needs an independent renormalization constant. Of course, in any ordinary flat-space quantum field theory the finite parts of the counterterm coefficients can be changed by changing the renormalization conditions – this freedom being at the origin of the renormalization group. Hence, one should probably accept that one cannot (completely) fix the finite parts of our counterterm coefficients. In particular, this implies that, at the two-loop level, the finite coefficient of $\ln \frac{A}{A_0}$ in $W^{(2)}[A] - W^{(2)}[A_0]$ is a parameter that can be adjusted!

The computation of the regularized two-point function $\mathcal{G}^{(2)}(u, v)$ for $u \neq v$ is displayed hereafter. Then, the result for the case of $u = v$ is briefly discussed. To keep this subsection readable, many computational details are deferred to the appendix B.1.

Counterterm contributions

The regularized counterterm contribution to $\mathcal{G}^{(2)}(u, v)$ is easy to write down, since it only involves two regularized propagators $\widehat{\widetilde{K}}(t_1, u, x)$ and $\widehat{\widetilde{K}}(t_2, x, v)$ connected by the counterterm vertex as given by (2.1.79):

$$\begin{aligned} \mathcal{G}^{(2)}(u, v) \Big|_{\text{ct}} &= \frac{8\pi}{\kappa^2} \int dx \left[-\frac{c_\phi}{2} \left(\widehat{\widetilde{K}}(t_1, u, x) \widehat{\widetilde{K}}(t_2, x, v) + \widehat{\widetilde{K}}(t_1, u, x) \widehat{\widetilde{K}}(t_2, x, v) \right) \right. \\ &\quad \left. - (c_m + c_R R_*) \widehat{\widetilde{K}}(t_1, u, x) \widehat{\widetilde{K}}(t_2, x, v) \right]. \end{aligned} \quad (2.1.91)$$

The counterterm coefficients are now assumed to have the general form (2.1.80), without any further assumptions. Note that, while c_ϕ and c_R involve at most an $\ln A\Lambda^2$ divergence, the coefficient c_m could have a $\Lambda^2 \ln A\Lambda^2$ divergence. This means that, in order to compute the finite contributions, one should keep terms in $\widehat{\widetilde{K}}(t_1, u, x) \widehat{\widetilde{K}}(t_2, x, v)$ that are $\mathcal{O}(1/\Lambda^2)$. (This is one of the reasons why all external propagators must also consistently be replaced by the regularized ones.) Thanks to (1.3.27), one directly reads:

$$\begin{aligned} \mathcal{G}^{(2)}(u, v) \Big|_{\text{ct}} &= -\frac{8\pi}{\kappa^2} c_\phi \widehat{\widetilde{K}}(t_1 + t_2, u, v) \\ &\quad - \frac{8\pi}{\kappa^2} (c_m + c_R R) \int dx \widehat{\widetilde{K}}(t_1, u, x) \widehat{\widetilde{K}}(t_2, x, v). \end{aligned} \quad (2.1.92)$$

Now, one needs to distinguish two cases. For $u \neq v$ the function $\widehat{\widetilde{K}}(t, u, v)$ is non-singular for $t \rightarrow 0$. More precisely, it equals $\widetilde{G}(u, v)$ plus terms that are either exponentially small or of order $1/\Lambda^2$. Since c_ϕ is at most logarithmically divergent, one may replace $c_\phi \widehat{\widetilde{K}}(t_1 + t_2, u, v)$ by $c_\phi \widetilde{G}(u, v)$ up to terms that vanish as $\Lambda \rightarrow \infty$. Thus,

$$\begin{aligned} \mathcal{G}^{(2)}(u, v) \Big|_{\text{ct}} &= -\frac{8\pi}{\kappa^2} c_\phi \widetilde{G}(u, v) - \frac{8\pi}{\kappa^2} (c_m + c_R R) \int dx \widehat{\widetilde{K}}(t_1, u, x) \widehat{\widetilde{K}}(t_2, x, v) \\ &\quad + \mathcal{O}(\ln A\Lambda^2/\Lambda^2), \quad \text{for } u \neq v. \end{aligned} \quad (2.1.93)$$

We see that the finite $\mathcal{O}(\Lambda^0)$ parts of c_m and c_R , *i.e.* $c_R^{(2)}$ and $c_m^{(3)}$ only give finite contributions to $\mathcal{G}^{(2)} \Big|_{\text{ct}}$. Hence, to determine them, one needs to obtain the finite $\mathcal{O}(\Lambda^0)$ terms in $\mathcal{G}^{(2)} \Big|_{\text{L}}^{1\text{-loop}}$ and in $\mathcal{G}^{(2)} \Big|_{\text{measure}}^{\text{tree}}$.

For $u = v$ however, $\widehat{\widetilde{K}}(t_1 + t_2, u, u)$ is divergent when $t_i \rightarrow 0$, so that:

$$\begin{aligned} \mathcal{G}^{(2)}(u, u) \Big|_{\text{ct}} &= -\frac{8\pi}{\kappa^2} c_\phi \widehat{\widetilde{K}}(t_1 + t_2, u, u) \\ &\quad - \frac{8\pi}{\kappa^2} (c_m + c_R R) \int dx \widehat{\widetilde{K}}(t_1, u, x) \widehat{\widetilde{K}}(t_2, x, u). \end{aligned} \quad (2.1.94)$$

Note that the integral is a finite smooth function of u even in the limit $t_i \rightarrow 0$. However, as mentioned above, one needs to keep the t_i finite, since the subleading terms, multiplied with the divergent pieces from c_m , can lead to finite contributions.

Total “one-loop” contribution to $\mathcal{G}^{(2)}(u, v)$ for $u \neq v$

We will now determine the “one-loop”, *i.e.* all order $1/\kappa^2$ contributions to $\mathcal{G}^{(2)}(u, v)$.

There are three one-loop diagrams contributing to $\mathcal{G}^{(2)}(u, v)$. They are $\text{---}\bigcirc\text{---}$, $\text{---}\bigcirc\text{---}$ and $\text{---}\bigcirc\text{---}$ (including the regularized external propagators). These diagrams contribute at the same order $\sim \frac{1}{\kappa^2}$ as the tree contribution from the measure vertex and the one from the counterterms. The counterterm contribution has been determined above in (2.1.93). The computation of the three two-point one-loop diagrams is quite lengthy, mainly due to the non-symmetric nature of the cubic and quartic vertices. This implies that there are many different contributions to each diagram. The details of the computation are deferred to the appendix B.1 where the results for the three one-loop diagrams are given in (B.1.11), (B.1.15) and (B.1.21) and for the measure contribution in (B.1.22).

First however, we want to highlight the fact that the case $u = v$ and $u \neq v$ have to be considered separately. Indeed, the $\text{---}\bigcirc\text{---}$ diagram for example involves terms such as $\widehat{K}(t_1, u, x)\widetilde{K}(t_2, x, y)\widetilde{K}(t_3, x, y)\widehat{K}(t_4, y, v)$. The terms $\widetilde{K}(t_2, x, y)$ and $\widetilde{K}(t_3, x, y)$ force x and y to be close, due to $e^{-\frac{\ell^2(x, y)}{4}(\frac{1}{t_2} + \frac{1}{t_3})}$. For $u \neq v$, we can assume $\ell^2(u, v) \gg \frac{1}{\Lambda^2}$, so that we can safely consider that x is not close to both u and v and, hence, Taylor expand:

$$\widehat{K}(t_1, u, x) = \widehat{K}(t_1, u, y) + (x - y)^i \partial_y^i \widehat{K}(t_1, u, y) + \frac{1}{2} (x - y)^i (x - y)^j \partial_y^i \partial_y^j \widehat{K}(t_1, u, y) + \dots \quad (2.1.95)$$

This expansion is only valid as long as $\ell^2(u, x) \sim \ell^2(u, y) \gg \frac{1}{\Lambda^2}$, so that $\widehat{K}(t_1, u, x)$ is a smooth function of x in the vicinity of y . This is no longer the case if $y \rightarrow u$. In particular, if $\ell^2(u, x) \sim \ell^2(u, y) \sim \frac{1}{\Lambda^2}$, all terms in the expansion (2.1.95) are similarly large. If $u \rightarrow v$, this occurs, as it will it is discussed in more details in the appendix of [56] and in the appendix C for the three-loop computation.

We now add the contributions of the three one-loop diagrams (B.1.11), (B.1.15), (B.1.21) and the measure contribution (B.1.22), as well as the counterterm contribution (2.1.93). As before, since these expressions have to be multiplied with $\int d\alpha_i \varphi(\alpha_i)$ for every α_i , we may symmetrize all expressions in α_i . We get

$$\begin{aligned}
& \left(\mathcal{G}_{u\text{-}O\text{-}v}^{(2)} + \mathcal{G}_{u\text{-}\underline{O}\text{-}v}^{(2)} + \mathcal{G}_{u\text{-}\underline{O}\text{-}v}^{(2)} + \mathcal{G}_{\text{measure}}^{(2)} + \mathcal{G}_{\text{ct}}^{(2)} \right) (u, v) \\
&= \frac{1}{\kappa^2} \left\{ 4 \tilde{G}(u, v) \left[\frac{3}{2} \ln \frac{\alpha_2 \alpha_3}{(\alpha_2 + \alpha_3)^2} - 1 - \frac{2\alpha_2 \alpha_3}{(\alpha_2 + \alpha_3)^2} - 2\pi c_\phi^{(1)} \ln A \Lambda^2 - 2\pi c_\phi^{(2)} \right] \right. \\
&\quad + 8\pi \tilde{G}(u, v) \left[\tilde{G}_\zeta(u) + \tilde{G}_\zeta(v) - \frac{2}{A} \int dx \tilde{G}_\zeta(x) \right] - \frac{8\pi}{A} \int dx [\tilde{G}(u, x) + \tilde{G}(x, v)] \tilde{G}_\zeta(x) \\
&\quad + 4 \int dx \tilde{K}(t_1, u, x) \tilde{K}(t_4, x, v) \left[\frac{2\Lambda^2}{\alpha_2 + \alpha_3} - \frac{5\Lambda^2}{2\alpha_2} - 2\pi c_m^{(1)} \Lambda^2 \ln A \Lambda^2 - 2\pi c_m^{(2)} \Lambda^2 \right. \\
&\quad \left. + R_* \left(\frac{3}{2} \ln \frac{\alpha_2 \alpha_3}{(\alpha_2 + \alpha_3)^2} - \frac{19}{12} - \frac{2\alpha_2 \alpha_3}{(\alpha_2 + \alpha_3)^2} - 2\pi c_R^{(1)} \ln A \Lambda^2 - 2\pi c_R^{(2)} \right) \right. \\
&\quad \left. - \frac{2\pi}{A} (5 + c_m^{(3)}) + 2\pi R_* \left(\tilde{G}_\zeta(x) - \frac{1}{A} \int dy \tilde{G}_\zeta(y) \right) \right] \\
&\quad + 8\pi \left[(\tilde{G}(u, v))^2 - \frac{1}{A} \int dx (\tilde{G}(u, x))^2 - \frac{1}{A} \int dx (\tilde{G}(x, v))^2 + \frac{1}{A^2} \int dx dy (\tilde{G}(x, y))^2 \right] \\
&\quad + 8\pi R_* \int dx \left[(\tilde{G}(u, x))^2 \tilde{G}(x, v) + \tilde{G}(u, x) (\tilde{G}(x, v))^2 \right] \\
&\quad - \frac{16\pi R_*}{A} \int dx dy \tilde{G}(u, x) \tilde{G}(x, y) \tilde{G}(y, v) - \frac{8\pi R_*}{A} \int dx dy (\tilde{G}(x, y))^2 [\tilde{G}(u, x) + \tilde{G}(y, v)] \\
&\quad + 8\pi R_* \tilde{G}(u, v) \int dx (\tilde{G}(u, x) + \tilde{G}(v, x)) \tilde{G}_\zeta(x) \\
&\quad - \frac{8\pi R_*}{A} \int dx (\tilde{G}(u, x) + \tilde{G}(v, x)) \int dy \tilde{G}(x, y) \tilde{G}_\zeta(y) \\
&\quad \left. + 8\pi R_*^2 \int dx dy \left[\tilde{G}(u, x) (\tilde{G}(x, y))^2 \tilde{G}(y, v) + \tilde{G}(u, x) \tilde{G}(x, v) \tilde{G}(x, y) \tilde{G}_\zeta(y) \right] \right\} \\
&\quad + \mathcal{O}(\ln A \Lambda^2 / \Lambda^2) . \tag{2.1.96}
\end{aligned}$$

Finiteness requires

$$c_\phi^{(1)} = c_m^{(1)} = c_R^{(1)} = 0 \quad , \quad c_m^{(2)} = \frac{1}{2\pi} \left(\frac{2}{\alpha_2 + \alpha_3} - \frac{5}{2\alpha_2} \right) . \tag{2.1.97}$$

These are exactly the “desired values” (2.1.82), (2.1.83) and (2.1.86). In particular, the value of $c_m^{(2)}$ is exactly what is needed to cancel the divergent two-loop contributions in $W[A] = \ln Z[A]!$ As explained above, the value (2.1.97) for $c_m^{(2)}$ really means that $c_m^{(2)}[\varphi] = \int_0^\infty d\alpha_2 d\alpha_3 \varphi(\alpha_2) \varphi(\alpha_3) \frac{1}{2\pi} \left(\frac{2}{\alpha_2 + \alpha_3} - \frac{5}{2\alpha_2} \right)$.

Next, since there are no more divergent coefficients, one may now safely replace the $\widehat{K}(t_1, u, y)\widehat{K}(t_4, y, v)$ by the regulator independent $\widetilde{G}(u, y)\widetilde{G}(y, v)$. We moreover require that the Green's function $\mathcal{G}^{(2)}(u, v)$ should not depend at all on the regulator functions $\varphi(\alpha)$, *i.e.* that (2.1.96) should not depend on the α_i . This yields

$$c_\phi^{(2)} = \frac{1}{2\pi} \left[\frac{3}{2} \ln \frac{\alpha_2 \alpha_3}{(\alpha_2 + \alpha_3)^2} - 1 - \frac{2\alpha_2 \alpha_3}{(\alpha_2 + \alpha_3)^2} \right] + \widehat{c}_\phi, \quad (2.1.98)$$

$$c_R^{(2)} = \frac{1}{2\pi} \left[\frac{3}{2} \ln \frac{\alpha_2 \alpha_3}{(\alpha_2 + \alpha_3)^2} - \frac{19}{12} - \frac{2\alpha_2 \alpha_3}{(\alpha_2 + \alpha_3)^2} \right] + \frac{\widehat{c}_R}{2\pi}, \quad (2.1.99)$$

$$c_m^{(3)} = \widehat{c}_m, \quad (2.1.100)$$

where \widehat{c}_ϕ , \widehat{c}_R and \widehat{c}_m are true (α_i -independent, Λ -independent) constants. With these values of the counterterm coefficients, eq. (2.1.96) reduces to

$$\begin{aligned} & \left(\mathcal{G}_{u\text{-O-}v}^{(2)} + \mathcal{G}_{u\text{-O-}v}^{(2)} + \mathcal{G}_{u\text{-O-}v}^{(2)} + \mathcal{G}_{\text{measure}}^{(2)} + \mathcal{G}_{\text{ct}}^{(2)} \right) (u, v) \\ &= \frac{1}{\kappa^2} \left\{ -8\pi \widehat{c}_\phi \widetilde{G}(u, v) \right. \\ &+ 8\pi \widetilde{G}(u, v) \left[\widetilde{G}_\zeta(u) + \widetilde{G}_\zeta(v) - \frac{2}{A} \int dx \widetilde{G}_\zeta(x) \right] - \frac{8\pi}{A} \int dx [\widetilde{G}(u, x) + \widetilde{G}(x, v)] \widetilde{G}_\zeta(x) \\ &+ 8\pi \int dx \widetilde{G}(u, x) \widetilde{G}(x, v) \left[-\frac{\widehat{c}_m + 5}{A} + R_* \left(-\frac{\widehat{c}_R}{2\pi} + \widetilde{G}_\zeta(x) - \frac{1}{A} \int dy \widetilde{G}_\zeta(y) \right) \right] \\ &+ 8\pi \left[(\widetilde{G}(u, v))^2 - \frac{1}{A} \int dx (\widetilde{G}(u, x))^2 - \frac{1}{A} \int dx (\widetilde{G}(x, v))^2 + \frac{1}{A^2} \int dx dy (\widetilde{G}(x, y))^2 \right] \\ &+ 8\pi R_* \int dx \left[(\widetilde{G}(u, x))^2 \widetilde{G}(x, v) + \widetilde{G}(u, x) (\widetilde{G}(x, v))^2 \right] \\ &- \frac{16\pi R_*}{A} \int dx dy \widetilde{G}(u, x) \widetilde{G}(x, y) \widetilde{G}(y, v) - \frac{8\pi R_*}{A} \int dx dy (\widetilde{G}(x, y))^2 [\widetilde{G}(u, x) + \widetilde{G}(y, v)] \\ &+ 8\pi R_* \widetilde{G}(u, v) \int dx \left(\widetilde{G}(u, x) + \widetilde{G}(v, x) \right) \widetilde{G}_\zeta(x) \\ &- \frac{8\pi R_*}{A} \int dx \left(\widetilde{G}(u, x) + \widetilde{G}(v, x) \right) \int dy \widetilde{G}(x, y) \widetilde{G}_\zeta(y) \\ &+ 8\pi R_*^2 \int dx dy \left[\widetilde{G}(u, x) (\widetilde{G}(x, y))^2 \widetilde{G}(y, v) + \widetilde{G}(u, x) \widetilde{G}(x, v) \widetilde{G}(x, y) \widetilde{G}_\zeta(y) \right] \left. \right\} \\ &+ \mathcal{O}(\ln A \Lambda^2 / \Lambda^2), \quad (2.1.101) \end{aligned}$$

which now is finite and completely regulator independent. As a consistency check, we note that this expression correctly vanishes if integrated over du or over dv .

There seems to be no obvious way to fix the remaining constants \widehat{c}_ϕ , \widehat{c}_R and \widehat{c}_m without imposing some definite value for $\mathcal{G}^{(2)}(u, v)$ at some fixed u, v . As already discussed, imposing a condition at $u = v$ does not help since in this case new divergences appear

that have to be renormalized independently. In any case, our expression (2.1.101) is only valid for $u \neq v$.

Total “one-loop” contribution to $\mathcal{G}^{(2)}(u, u)$

To explicitly see which are the new divergences and finite terms that appear for $u = v$, we now quote the result for the total one-loop plus counterterm contributions to the two-point function at coinciding points $u = v$. This case involves some interesting technical difficulties, similar to some one encounters in the three-loop computation and which are discussed in the appendix C. Since we do not use the expression of $\mathcal{G}^{(2)}(u, u)$ any further we do not spell out the computation and only give the result:

$$\begin{aligned}
& \left(\mathcal{G}_{u \circ u}^{(2)} + \mathcal{G}_{u \underline{\circ} u}^{(2)} + \mathcal{G}_{u \circ u}^{(2)} + \mathcal{G}_{\text{measure}}^{(2)} + \mathcal{G}_{\text{ct}}^{(2)} \right) (u, u) \\
&= \frac{1}{\kappa^2} \left\{ 8\pi \widehat{K}(t_i, u, u) \widehat{K}(t_j, u, u) + 4\pi [\hat{c}_1(\alpha_i) - 2c_\phi] \widehat{K}(t_i + t_j, u, u) + \hat{c}_2(\alpha_i) \right. \\
&\quad + 16\pi \widehat{K}(t_1 + t_4, u, u) \left[\widetilde{G}_\zeta(u) - \frac{1}{A} \int dx \widetilde{G}_\zeta(x) + R_* \int dx \widetilde{G}(u, x) \widetilde{G}_\zeta(x) \right] \\
&\quad + 4 \int dx \widehat{K}(t_1, u, x) \widehat{K}(t_4, x, u) \left[\frac{2\Lambda^2}{\alpha_2 + \alpha_3} - \frac{5\Lambda^2}{2\alpha_2} - 2\pi c_m \right. \\
&\quad \quad \quad \left. + R_* \left(\frac{3}{2} \ln \frac{\alpha_2 \alpha_3}{(\alpha_2 + \alpha_3)^2} - \frac{2\alpha_2 \alpha_3}{(\alpha_2 + \alpha_3)^2} - \frac{19}{12} - 2\pi c_R \right) \right] \\
&\quad - \frac{56\pi}{A} \int dx (\widetilde{G}(u, x))^2 - \frac{16\pi}{A} \int dx \widetilde{G}(u, x) \widetilde{G}_\zeta(x) + \frac{8\pi}{A^2} \int dx dy (\widetilde{G}(x, y))^2 \\
&\quad + 16\pi R_* \int dx (\widetilde{G}(u, x))^3 + 8\pi R_* \int dx (\widetilde{G}(u, x))^2 \widetilde{G}_\zeta(x) \\
&\quad - \frac{16\pi R_*}{A} \int dx dy \left[\widetilde{G}(u, x) \widetilde{G}(x, y) \widetilde{G}(y, u) + \widetilde{G}(u, x) (\widetilde{G}(x, y))^2 \right] \\
&\quad - \frac{8\pi R_*}{A} \int dx dy \left[2 \widetilde{G}(u, x) \widetilde{G}(x, y) \widetilde{G}_\zeta(y) + (\widetilde{G}(u, x))^2 \widetilde{G}_\zeta(y) \right. \\
&\quad \left. + 8\pi R_*^2 \int dx dy \left[\widetilde{G}(u, x) (\widetilde{G}(x, y))^2 \widetilde{G}(y, u) + (\widetilde{G}(u, x))^2 \widetilde{G}(x, y) \widetilde{G}_\zeta(y) \right] \right\} \\
&\quad + \mathcal{O}(\ln A \Lambda^2 / \Lambda^2) . \tag{2.1.102}
\end{aligned}$$

Here, $\hat{c}_1(\alpha_i)$ and $\hat{c}_2(\alpha_i)$ are finite coefficients that we did not determine. Using the sums given in the appendix C we could compute them explicitly. However, this will not give us any further information about the counterterms. Indeed, as discussed above, one should not expect finiteness of $\mathcal{G}^{(2)}(u, u)$ or that it equals $\widehat{K}(t, u, u)$ as $\Lambda \rightarrow \infty$. The first term in (2.1.102), *i.e.* $2\widehat{K}(t_i, u, u) \widehat{K}(t_j, u, u)$ is divergent as $\Lambda \rightarrow \infty$. The same is true for the terms in the second line of the right hand side. Cancelling these divergences would require a non-vanishing counterterm coefficient $c_\phi^{(1)}$, which was excluded by demanding finiteness of the two-point function at $u \neq v$ in the previous computation. Independently

of the value of $c_\phi^{(1)}$, one clearly cannot require that $\mathcal{G}^{(2)}(u, u)$ equals $\widehat{K}(t, u, u)$ in the $\Lambda \rightarrow \infty$ limit. Thus, in the three-loop computation, we will only consider the two-point function at non coinciding point.

2.1.5 Two-loop result for the partition function and discussion

The final result for the logarithm of the partition function is obtained from inserting the values of the counterterm coefficients we have determined in (2.1.97)-(2.1.100) into (2.1.85):

$$W^{(2)}[A] = W^{(2)}[A] \Big|_{\text{loops}} + W^{(2)}[A] \Big|_{\text{ct}} = -\frac{1}{\kappa^2} \left[\widehat{c}_m + 1 + 4(1-h)\widehat{c}_R \right] \ln A \Lambda^2 + A \Lambda^2 [\dots] + c(\alpha_i) + \mathcal{O}((\ln A \Lambda^2)^2 / \Lambda^2) . \quad (2.1.103)$$

We see that not only the terms $\sim A \Lambda^2 \ln A \Lambda^2$ have cancelled, moreover the coefficient of $\ln A \Lambda^2$ now is independent of the α_i , *i.e.* independent of the regulator functions $\varphi(\alpha)$! It only depends on the two finite renormalization constants \widehat{c}_m and \widehat{c}_R . As already repeatedly emphasized, we can also adjust the cosmological constant counterterm to cancel any divergence in W that is proportional to the area A . Most importantly, as discussed in section previously, the absence of $(\ln A \Lambda^2)^2$ divergences is equivalent to the absence of the $R(\widetilde{G}_\zeta)^2$ term that would not have been background independent. Clearly, all area-dependent terms present in (2.1.103) are background independent.

Subtracting from (2.1.103) the same expression evaluated at area A_0 , we finally get

$$\begin{aligned} \ln \frac{Z[A]}{Z[A_0]} \Big|_{\kappa^{-2}} &= W^{(2)}[A] - W^{(2)}[A_0] \\ &= -\frac{1}{\kappa^2} \left[\widehat{c}_m + 1 + 4(1-h)\widehat{c}_R \right] \ln \frac{A}{A_0} + (A - A_0) \Lambda^2 [\dots] + \mathcal{O}((\ln A \Lambda^2)^2 / \Lambda^2) . \end{aligned} \quad (2.1.104)$$

Equivalently, this shows that the area dependence of the partition function is

$$Z[A] \Big|_{\text{two-loop+measure+ct}} \sim e^{-\mu_c^2 A} A^{\gamma_{\text{str}}^{(2)} - 3} , \quad (2.1.105)$$

with

$$\gamma_{\text{str}}^{(2)} = \frac{2}{\kappa^2} \left[-2\widehat{c}_R(1-h) - \frac{\widehat{c}_m + 1}{2} \right] . \quad (2.1.106)$$

We see that our careful, first-principles computation of the 2D quantum gravity partition function has established that, up to two loops, the partition function has indeed the expected form (2.1.105). However, we have also found a maybe unexpected dependence on two finite renormalization constants. By which principle should these counterterm coefficients be fixed? We observe that our result is compatible with the KPZ scaling, since we get agreement with the (two-loop prediction of the) KPZ formula if we choose

$$\widehat{c}_m \Big|_{\text{KPZ}} = -1 \quad , \quad \widehat{c}_R \Big|_{\text{KPZ}} = -\frac{1}{2} . \quad (2.1.107)$$

In the absence of any principle to fix these constants, the area-dependence of the partition function $Z[A]$ appears to involve an arbitrary power of A . One more principle we may invoke is locality. Locality of the counterterms implies that all coefficients $c_\phi^{(1)}$, $c_m^{(1)}$, $c_R^{(1)}$ which multiply a non-local $\ln A\Lambda^2$ should vanish. We have indeed found this. But it would also imply that $c_m^{(3)} = \widehat{c}_m$ which multiplies a non-local $\frac{1}{A}$ should equally vanish. On the other hand, such a $\frac{1}{A}$ -term was already present in the regularized measure action due to the absence of a zero-mode. Indeed, from (2.1.11) and (1.3.47) we read that

$$S_{\text{measure}} = -\frac{4\pi}{\kappa^2} \int d^2x \sqrt{g_*} \left[\frac{1}{4\pi t} + \frac{7}{24\pi} R_* - \frac{1}{A} + \dots \right] \widetilde{\phi}^2(x). \quad (2.1.108)$$

Thus, a $\frac{1}{A}$ -counterterm should certainly also be allowed. However, we may require that the non-local $\frac{1}{A}$ -counterterm should actually cancel the non-local $\frac{1}{A}$ -term in S_{measure} , so that all the terms in the global action are local. This is our “strong locality condition”. Comparing (2.1.108) with (2.1.79), (2.1.80) we see that this cancellation requires the following condition

$$\text{absence of non-local } \frac{1}{A}\text{-terms in } \frac{\kappa^2}{8\pi} S_L + S_{\text{measure}} + S_{\text{ct}} \quad \Rightarrow \quad c_m^{(3)} \equiv \widehat{c}_m = -1. \quad (2.1.109)$$

This is precisely the KPZ-value (2.1.107)!

Let us come back on the background independence. As previously emphasized in section 1.4 in the previous chapter, while the Liouville action is background independent, neither is the measure action, nor the counterterm action. This was, of course, to be expected. This must be so, since it is missing the background dependence induced by the regularization. The fact that the regularization explicitly depends on the background metric makes it difficult to trace this dependence through the intermediate steps of the loop computations. Nevertheless, we were able to determine among the structures that could appear in the final result for the fixed-area partition function which ones are background independent and which ones are not. We found that precisely those structures accompanying the divergences $(\ln A\Lambda^2)^2$ are not background independent. However, quite remarkably, the $(\ln A\Lambda^2)^2$ structures and divergences – that could not be cancelled by counterterms – already cancel among themselves in the two-loop contribution. The remaining structures are all background independent !

To summarize, within the present two-loop, order κ^{-2} computation, there does not seem to be an obvious criterion why to choose the KPZ-value $\widehat{c}_R = -\frac{1}{2}$ rather than any other value. The DDK reasoning – which yields the KPZ value – was based on an all-order conformal invariance argument which encodes background invariance. However, our two-loop computation was also background invariant in the end, despite the regularization, and the counterterms did not play any role to ensure that. One may wonder then if a higher loop computation would be able to fix the remaining constant \widehat{c}_R . Indeed, at the three-loop level, *i.e.* at order $\sim \kappa^{-4}$, the counterterm $\widehat{c}_R \phi^2$ contributes via two-loop diagrams and thus enters into more complicated divergences which need to be cancelled. However, one could expect genuine three-loop counterterms (and thus

even further undetermined parameters) to be required in any case, especially to ensure the “strong locality condition”. Indeed, the measure action gives a cubic and a quartic vertex at three loops, including non-local $\frac{1}{A}$ terms. Thus, if the counterterms are to be understood as a renormalization of the measure, a cubic and a quartic structure are to be expected. Regarding the background independence criterion to fix these (new) parameters, it will most likely be irrelevant. Indeed, the background dependent structures are related to divergences which will be required to vanish. At three loops for example, the $(\ln A\Lambda^2)^2$ divergence which is related to the background dependent term $\int R(\tilde{G}_\zeta)^2$, could be cancelled by contributions from two-loop diagrams involving a counterterm. In any case such a divergence will be absorbed in a counterterm and to ensure both the finiteness of the partition function and its background dependence. The only three-loop divergence that could not be cancelled by a counterterm will be the $(\ln A\Lambda^2)^3$ divergence (also related to a background dependent term), whose cancellation among the diagrams will be needed. This is checked in the next section of this chapter, dedicated to the study of the three-loop divergences and of the genuine three-loop counterterms. Also note that if \widehat{c}_R indeed remained undetermined to all orders, we would be led to consider that there could be different choices of consistent quantizations of this two-dimensional gravity. One could imagine that at least some of these new quantization schemes could be consistent quantum gravities for all matter central charges, thus maybe allowing to go beyond the $c = 1$ barrier.

2.2 Three-loop investigation

The purpose of the present section is to go into the previous study in depth, with the idea of improving our understanding of the counterterms. In the subsection 2.2.1, the Liouville and the measure actions are expanded to the order relevant for the computation of the partition function at three loops. In particular, this leads to new vertices. Then the three-loop vacuum diagrams are enumerated. In the subsection 2.2.2, the allowed divergences are investigated in some detail and the leading divergence $\sim A\Lambda^2 (\ln A\Lambda^2)^2$ is fully computed. Since it does not cancel, genuine three-loop counterterms are required. We end this subsection by stating that the $(\ln A\Lambda^2)^3$ divergence cancels, as required for both finiteness and background independence. Finally, subsection 2.2.3 is dedicated to the three-loop counterterms that contribute via one- and two-loop diagrams to the three-loop partition function. We discuss the degrees of freedom of these additional parameters. This section is based on [57].

2.2.1 Three-loop expansions of the actions

To pursue the investigation to three loops, $\frac{1}{\kappa^4}$ terms need to be included. Since the Liouville action (2.1.2) comes with a factor $\frac{\kappa^2}{8\pi}$, it has to be expanded up to the $\frac{1}{\kappa^8}$ order.

One gets:

$$\begin{aligned}
\frac{\kappa^2}{8\pi} S_L[\sigma] &= \frac{\kappa^2}{2} (1-h) \ln \frac{A}{A_0} + \int dx \frac{1}{2} \tilde{\phi} (\Delta_* - R_*) \tilde{\phi} \\
&+ \int dx \left[\frac{\sqrt{4\pi}}{\kappa} \tilde{\phi}^2 (\Delta_* - \frac{2}{3} R_*) \tilde{\phi} + \frac{2\pi}{\kappa^2} \tilde{\phi}^2 \Delta_* \tilde{\phi}^2 + \frac{16\pi}{3\kappa^2} \tilde{\phi}^3 \Delta_* \tilde{\phi} - \frac{4\pi}{\kappa^2} R_* \tilde{\phi}^4 \right] \\
&+ \int dx \left[\frac{16\pi^{3/2}}{\kappa^3} \left[\tilde{\phi}^4 \left(\Delta_* - \frac{4}{5} R_* \right) \tilde{\phi} + \frac{2}{3} \tilde{\phi}^3 \Delta_* \tilde{\phi}^2 \right] \right. \\
&\quad \left. + \frac{(8\pi)^2}{\kappa^4} \left[\frac{4}{5} \tilde{\phi}^5 \left(\Delta_* - \frac{5}{6} R_* \right) \tilde{\phi} + \frac{1}{2} \tilde{\phi}^4 \Delta_* \tilde{\phi}^2 + \frac{2}{9} \tilde{\phi}^3 \Delta_* \tilde{\phi}^3 \right] + \mathcal{O}(\kappa^{-5}) \right]
\end{aligned} \tag{2.2.1}$$

where $\int dx$ stands for $\int d^2x \sqrt{g_*(x)}$. The last two lines yield the quintic and sextic vertices which appear only at three (or higher)-loop computations. They can be grouped as follows. Two quintic vertices

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = -\frac{16\pi^{3/2}}{\kappa^3} \left(\Delta_* - \frac{4}{5} R_* \right), \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = -\frac{16\pi^{3/2}}{\kappa^3} \frac{2}{3} \Delta_* \tag{2.2.2}$$

and three sextic vertices

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} = -\frac{(8\pi)^2}{\kappa^4} \frac{4}{5} \left(\Delta_* - \frac{5}{6} R_* \right), \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = -\frac{(8\pi)^2}{\kappa^4} \frac{1}{2} \Delta_*, \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = -\frac{(8\pi)^2}{\kappa^4} \frac{2}{9} \Delta_*. \tag{2.2.3}$$

Similarly to the two-loop case, the bold parts of the vertices encode the Δ_* acting on one or several propagators. For example, for the two quintic vertices, the $(\Delta_* - \frac{4}{5} R_*)$ in the first vertex acts on the single propagator connected to the bold line, while in the second one Δ_* may act either on the product of the two propagators connected to the bold part of the vertex on the right or on the three other ones.

The three-loop expansion of the non-trivial measure action (2.1.9) is

$$S_{\text{measure}} = \int dx \tilde{K}(t, x, x) \left(-\frac{4\pi}{\kappa^2} \tilde{\phi}^2 - \frac{32\pi^{3/2}}{3\kappa^3} \tilde{\phi}^3 - \frac{32\pi^2}{\kappa^4} \tilde{\phi}^4 + \mathcal{O}(\kappa^{-5}) \right), \tag{2.2.4}$$

where $\tilde{K}(t, x, x)$ is given in (1.3.47). In addition to the quadratic vertex already relevant for the two-loop computation, this action now provides a cubic and a quartic vertex, leading to the following measure vertices up to three loops:

$$\begin{array}{c} \text{---} \\ | \\ \bullet \end{array} = \frac{4\pi}{\kappa^2} \tilde{K}(t, x, x), \quad \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \end{array} = \frac{16\pi^{3/2}}{\kappa^3} \frac{2}{3} \tilde{K}(t, x, x), \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \bullet \end{array} = \frac{(8\pi)^2}{\kappa^4} \frac{1}{2} \tilde{K}(t, x, x). \tag{2.2.5}$$

The requirement for counterterms has been highlighted by the previous study at two loops, so that both the two-point function and the partition function are finite at

respectively one loop and two loops. Thus, the two-loop counterterm action (2.1.79) is also to be considered for the three-loop computation, leading to the overall action:

$$S = \frac{\kappa^2}{8\pi} S_L + S_{\text{measure}} + S_{\text{ct}} . \tag{2.2.6}$$

The vertices relevant up to three loops are summed up in Tab. 2.1.

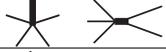
Order	Liouville	Measure	CT
$\frac{1}{\kappa}$			
$\frac{1}{\kappa^2}$			
$\frac{1}{\kappa^3}$			
$\frac{1}{\kappa^4}$			

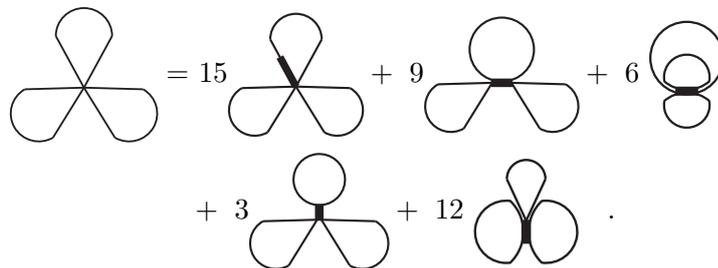
Table 2.1: Vertices relevant for the three-loop computation

Note that all these vertices are normalized without including any symmetry factors so that one has to count all possible contractions when evaluating the diagrams.

Diagrams

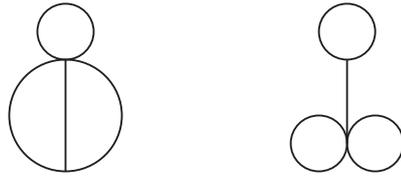
One enumerates now all “three-loop” vacuum diagrams. More precisely, such diagrams are all the diagrams contributing at order $\frac{1}{\kappa^4}$. This involves genuine three-loop diagrams made from the Liouville vertices only, as well as two-loop and one-loop diagrams involving also the vertices from the measure or counterterm action. Combining all these vertices gives twenty-nine types of vacuum diagrams, each of them receiving contributions from subdiagrams. Fifteen of these diagrams come from pure Liouville contributions, nine involve the measure and six the two-loop counterterms . The decomposition of the diagrams is detailed hereafter.

The sextic vertices give one diagram, the “flower diagram”, which may be written as the sum of five subdiagrams:

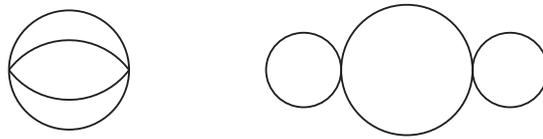


The diagrammatic equation shows a large flower diagram on the left, which is equal to the sum of five smaller diagrams on the right. The coefficients are 15, 9, 6, 3, and 12. The subdiagrams are: a flower diagram with a thick central line; a flower diagram with a thick central line and a dot; a flower diagram with a thick central line and a square; a flower diagram with a thick central line and a dot; and a flower diagram with a thick central line and a square.

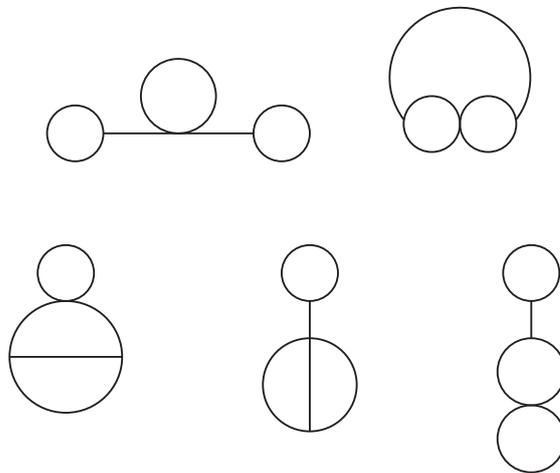
The weight factors in front of the different subdiagrams take into account the multiplicity of the diagram, including the symmetry factors and the contractions. Combining the quintic and cubic vertices yields two types of diagram:



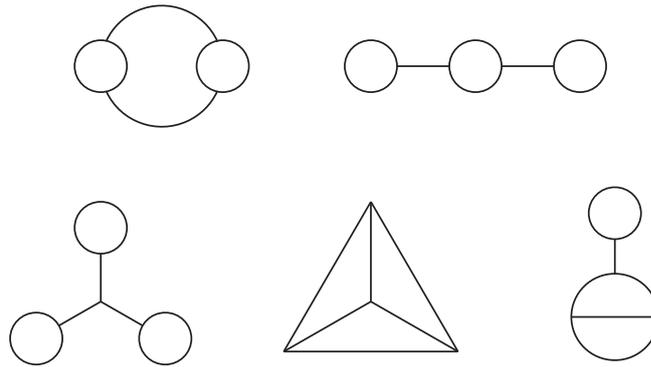
composed of respectively eight and ten subdiagrams. Using two quartic vertices gives two diagrams:



made of respectively five and eleven subdiagrams. Five types of diagrams are built by a quartic vertex and two cubic vertices:

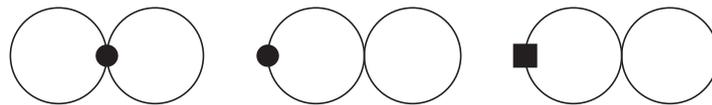


These diagrams consist in thirteen subdiagrams each for the diagrams of the upper line, and of seventeen, ten and eighteen subdiagrams for the bottom line, from left to right. Finally, the last five pure Liouville diagrams come from using four cubic vertices:

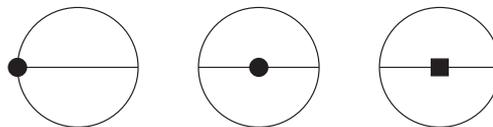


composed of eleven, thirteen, six, six and eighteen subdiagrams, from left to right and from top to bottom.

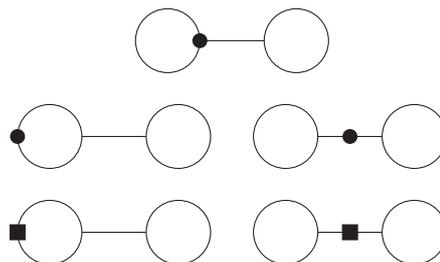
The measure and counterterm vertices contribute to fourteen diagrams. They may be classified according to the corresponding “two-loop” terminology: the “figure-eight”, the “setting sun”, the “glasses” and the “measure” diagrams. At the three-loop order, there are three “figure-eight-like” diagrams,



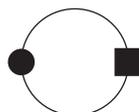
three “setting sun-like” diagrams,



five “glasses-like” diagrams



and finally three “measure-like” diagrams.



From left to right, both the “figure-eight” and “setting sun” diagrams have respectively one, four and five contributions. Concerning the “glasses” diagrams: the upper diagram has two contributions, and, from left to right, the diagrams in the second line have respectively four and three contributions, and the diagrams involving the counterterms six and four respectively. Finally, the “measure” diagram on the right gets two contributions whereas both diagrams involving the measure vertex have no other subdiagram.

2.2.2 Divergences of the partition function

Expected divergence structure

All vacuum diagrams are dimensionless and can depend on A and Λ only through the dimensionless combination $A\Lambda^2$. They contribute to various divergences to the partition function. Standard power counting shows that any loop-diagram has a superficial degree of divergence equal to 2. This means that divergences such as $A\Lambda^2 (\ln A\Lambda^2)^\#$ are allowed. To have a more precise idea of the leading divergence, consider a diagram with I internal lines and V vertices. Each internal line, that is to say each regularized propagator $\widehat{\widetilde{K}}$, gives a logarithmic divergence, according to (2.1.47). Besides, each vertex, carrying a Laplacian, transforms such a propagator into the corresponding heat kernel \widetilde{K} thanks to (1.3.24), leading to a quadratic divergence (1.3.47). Each vertex also implies an integration over the manifold. Due to the term $e^{-l^2/4t}$ in the heat kernel (1.3.45), every integration contributes a factor $t_i \sim \frac{1}{\Lambda^2}$ at most. (The subtraction of the zero-mode terms $\sim \frac{e^{R_*t}}{A}$ does not change the final conclusion.) For the last integration, however, all quantities to be integrated only depend on one point, hence no Gaussian integration can be performed and one just gets a factor of A . Putting everything together, the leading singularity of this L -loop vacuum diagram is

$$(\ln A\Lambda^2)^{I-V} (\Lambda^2)^V A \left(\frac{1}{\Lambda^2} \right)^{V-1} = (\ln A\Lambda^2)^{L-1} A\Lambda^2 \quad (2.2.7)$$

since $I - V = L - 1$ for every diagram. Therefore, the leading divergence at three loops is $A\Lambda^2 (\ln A\Lambda^2)^2$. Note that the vertices not only contain a Laplacian but also terms $\sim R_* \sim \frac{1}{A}$. Picking the contribution coming from $V - V'$ Laplacians and V' terms $\sim R_*$ leads to the divergence

$$(\ln A\Lambda^2)^{I-V+V'} (\Lambda^2)^{V-V'} A^{-V'} A \left(\frac{1}{\Lambda^2} \right)^{V-1} = (\ln A\Lambda^2)^{L-1+V'} (A\Lambda^2)^{1-V'} \quad (2.2.8)$$

For $V' > 1$ this is vanishing. This means that the subleading divergence with the largest power of logarithms is $(\ln A\Lambda^2)^L$. Consequently, the expected divergences in $W^{(3)}[A]$ are

$$\begin{aligned} W^{(3)}[A] \Big|_{\text{loops}} &= d_1 A\Lambda^2 (\ln A\Lambda^2)^2 + d_2 A\Lambda^2 \ln A\Lambda^2 + d_3 A\Lambda^2 + d_4 (\ln A\Lambda^2)^3 + d_5 (\ln A\Lambda^2)^2 \\ &\quad + d_6 \ln A\Lambda^2 + d_7 + \mathcal{O} \left(\frac{\ln A\Lambda^2}{A\Lambda^2} \right). \end{aligned} \quad (2.2.9)$$

Note that the term $\sim \ln A\Lambda^2$, although divergent, has a physical meaning. Indeed, once all other divergences cancelled by appropriate counterterms, one has $W^{(3)}[A] = W^{(3)}[A]|_{\text{loops}} + W^{(3)}[A]|_{\text{ct}} = \tilde{d}_6 \ln A\Lambda^2 + \tilde{d}_7 + \mathcal{O}\left(\frac{\ln A\Lambda^2}{A\Lambda^2}\right)$ so that

$$\lim_{\Lambda \rightarrow \infty} \frac{Z[A]}{Z[A_0]} \Big|_{3\text{-loop+ct}} = \left(\frac{A}{A_0}\right)^{\tilde{d}_6}, \quad (2.2.10)$$

showing that \tilde{d}_6 is the three-loop plus counterterm, order $\frac{1}{\kappa^4}$, contribution to γ_{str} .

Cancellation of the Λ^4 divergence

Moreover, contrary to the preceding, somewhat naive power counting argument, one observes “unexpected” Λ^4 divergences appearing in the diagrams indicated in Tab. 2.2. They appear through the following integrals:

$$\begin{aligned} J_j^i &= \int dx dy \tilde{K}(t_i, x, x) \tilde{K}(t_j, y, y) \hat{\tilde{K}}(t_m, x, y) \hat{\tilde{K}}(t_n, x, y), \\ J_k^{i,j} &= \int dx dy \tilde{K}(t_i + t_j, x, x) \tilde{K}(t_k, y, y) \hat{\tilde{K}}(t_m, x, y) \hat{\tilde{K}}(t_n, x, y), \\ J_{k,l}^{i,j} &= \int dx dy \tilde{K}(t_i + t_j, x, x) \tilde{K}(t_k + t_l, y, y) \hat{\tilde{K}}(t_m, x, y) \hat{\tilde{K}}(t_n, x, y), \end{aligned} \quad (2.2.11)$$

where i, j, k, l, m and n are different. From (1.3.47) one gets the leading divergences

$$J_j^i \sim \frac{\Lambda^4}{\alpha_i \alpha_j} J, \quad J_k^{i,j} \sim \frac{\Lambda^4}{(\alpha_i + \alpha_j) \alpha_k} J \quad \text{and} \quad J_{k,l}^{i,j} \sim \frac{\Lambda^4}{(\alpha_i + \alpha_j)(\alpha_k + \alpha_l)} J, \quad (2.2.12)$$

with $J = \int dx dy \hat{\tilde{K}}(t_m, x, y) \hat{\tilde{K}}(t_n, x, y)$. Thus these Λ^4 divergences come with three different structures in the α_i . All these unwanted divergences are displayed in Tab. 2.2. When summing them up, all three structures (2.2.12) cancel and there is no net Λ^4 divergence !

											Total
J_j^i	9			-3	-15			$\frac{1}{4}$	$\frac{5}{2}$	$\frac{25}{4}$	0
J_k^{i+j}		-12			12	2	10		-2	-10	0
J_{k+l}^{i+j}			4				-8			4	0

Table 2.2: Λ^4 contributions from the diagrams

A simple computation: the flower diagram

The true leading divergence contributing to the partition function at three loops is in $A\Lambda^2 (\ln A\Lambda^2)^2$. As already emphasized, the main goal of this three-loop investigation is to analyse this leading divergence, check that it does not “miraculously” cancel between the diagrams and determine the structure of the required counterterms.

Out of the twenty-nine vacuum diagrams displayed in section 2.2.1, only the fourteen diagrams shown in Fig. 2.1 contribute to the leading divergence in $A\Lambda^2 (\ln A\Lambda^2)^2$. Note

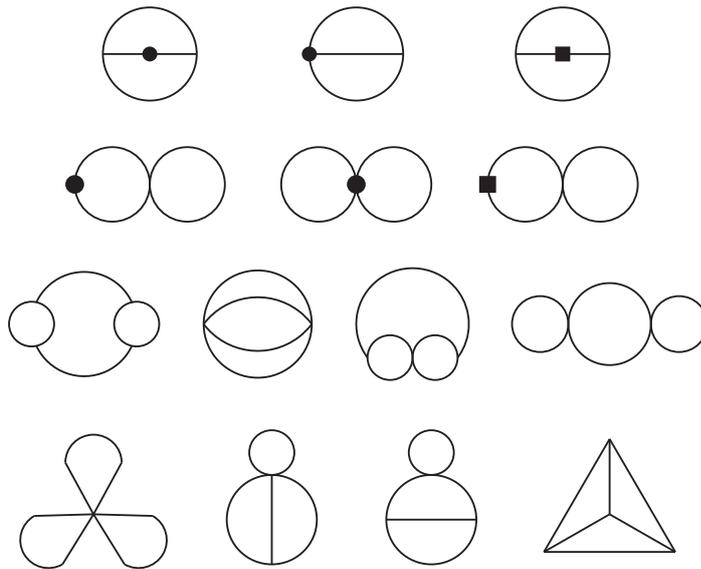


Figure 2.1: Relevant diagrams for the leading divergence in $A\Lambda^2 (\ln A\Lambda^2)^2$

that all the diagrams with a single propagator between two vertices (*i.e.* one-particle reducible) do not contribute, as it was already the case for the two-loop computation. This is because there is no zero-mode and a single propagator connecting two parts of a vacuum diagram should carry only the zero-mode.⁷

Consider again the flower diagram made from the sextic vertices, whose decomposition in subdiagrams was given in the previous subsection. Since only one vertex is involved, no integration has to be done to extract the divergences and it is the second simplest diagram to compute. (The simplest is the figure-eight diagram coming from the quartic

⁷In flat space, by momentum conservation, such a propagator would carry zero momentum. In our curved geometry the argument is more complicated and such one-particle reducible diagrams can still be non-vanishing, but using (1.3.44) one can show that they do not contribute to the present computation.

measure vertex.) The first subdiagram may be written in our regularization as:

$$\begin{aligned} I_{\infty} &= -\frac{4}{5} \frac{(8\pi)^2}{\kappa^4} \int dx \widehat{K}(t_1, x, x) \widehat{K}(t_2, x, x) \left[\left(\Delta_*^x - \frac{5}{6} R_* \right) \widehat{K}(t_3, x, z) \right]_{x=z} \\ &= -\frac{4}{5} \frac{(8\pi)^2}{\kappa^4} \int dx \widehat{K}(t_1, x, x) \widehat{K}(t_2, x, x) \left(\widetilde{K}(t_3, x, x) + \frac{R_*}{6} \widehat{K}(t_3, x, x) \right), \quad (2.2.13) \end{aligned}$$

The second subdiagram is slightly more complicated, because of the Laplacian acting on several propagators:

$$I_{\infty} = -\frac{2}{9} \frac{(8\pi)^2}{\kappa^4} \int dx \widehat{K}(t_1, x, x) \left[\Delta_*^x \left(\widehat{K}(t_2, x, x) \widehat{K}(t_3, x, z) \right) \right]_{z=x}. \quad (2.2.14)$$

The Laplacian term gives

$$\begin{aligned} \left[\Delta_*^x \left(\widehat{K}(t_2, x, x) \widehat{K}(t_3, x, z) \right) \right]_{z=x} &= \widehat{K}(t_2, x, x) \left[\Delta_*^x \widehat{K}(t_3, x, z) \right]_{z=x} \\ &\quad + \Delta_*^x \widehat{K}(t_2, x, x) \widehat{K}(t_3, x, x) \\ &\quad - 2 g_*^{ij} \partial_i^x \widehat{K}(t_2, x, x) \left[\partial_j^x \widehat{K}(t_3, x, z) \right]_{z=x} \\ &= \widehat{K}(t_2, x, x) \widetilde{K}(t_3, x, x) + R_* \widehat{K}(t_2, x, x) \widehat{K}(t_3, x, x) \\ &\quad + \Delta_*^x \widehat{K}(t_2, x, x) \widehat{K}(t_3, x, x) \\ &\quad - g_*^{ij} \partial_i^x \widehat{K}(t_2, x, x) \partial_j^x \widehat{K}(t_3, x, x). \quad (2.2.15) \end{aligned}$$

Inserting (2.2.15) into (2.2.14), and integrating the last term by parts leads to:

$$\begin{aligned} I_{\infty} &= -\frac{2}{9} \frac{(8\pi)^2}{\kappa^4} \int dx \widehat{K}(t_1, x, x) \left[\widehat{K}(t_2, x, x) \widetilde{K}(t_3, x, x) + R_* \widehat{K}(t_3, x, x) \widehat{K}(t_2, x, x) \right. \\ &\quad \left. + \frac{1}{2} \widehat{K}(t_3, x, x) \Delta_*^x \widehat{K}(t_2, x, x) \right]. \quad (2.2.16) \end{aligned}$$

Similarly, the third and fifth subdiagrams give

$$\begin{aligned} I_{\text{Q}} &= -\frac{2}{9} \frac{(8\pi)^2}{\kappa^4} \int dx \left[\Delta_*^x \left(\widehat{K}(t_1, x, z) \widehat{K}(t_2, x, z) \widehat{K}(t_3, x, z) \right) \right]_{z=x} \\ &= -\frac{2}{3} \frac{(8\pi)^2}{\kappa^4} \int dx \widehat{K}(t_1, x, x) \left[\widehat{K}(t_2, x, x) \widetilde{K}(t_3, x, x) + R_* \widehat{K}(t_3, x, x) \widehat{K}(t_2, x, x) \right. \\ &\quad \left. - \frac{1}{4} \widehat{K}(t_3, x, x) \Delta_*^x \widehat{K}(t_2, x, x) \right], \\ I_{\text{O}} &= -\frac{1}{2} \frac{(8\pi)^2}{\kappa^4} \int dx \widehat{K}(t_1, x, x) \left[\Delta_*^x \left(\widehat{K}(t_2, x, z) \widehat{K}(t_3, x, z) \right) \right]_{z=x} \\ &= -\frac{(8\pi)^2}{\kappa^4} \int dx \widehat{K}(t_1, x, x) \left[\widehat{K}(t_2, x, x) \widetilde{K}(t_3, x, x) + R_* \widehat{K}(t_3, x, x) \widehat{K}(t_2, x, x) \right. \\ &\quad \left. - \frac{1}{8} \widehat{K}(t_3, x, x) \Delta_*^x \widehat{K}(t_2, x, x) \right], \quad (2.2.17) \end{aligned}$$

while one reads directly the fourth subdiagram

$$I_{\infty} = -\frac{1}{2} \frac{(8\pi)^2}{\kappa^4} \int dx \widehat{K}(t_1, x, x) \widehat{K}(t_2, x, x) \Delta_*^x \widehat{K}(t_3, x, x). \quad (2.2.18)$$

The overall contribution from the flower diagram is thus

$$\begin{aligned} I_{\infty} &= 15 I_{\infty} + 9 I_{\infty} + 6 I_{\infty} + 3 I_{\infty} + 12 I_{\infty} \\ &= -\frac{(8\pi)^2}{\kappa^4} \int dx \widehat{K}(t_1, x, x) \widehat{K}(t_2, x, x) \left[30 \widehat{K}(t_3, x, x) + 20 R_* \widehat{K}(t_3, x, x) \right]. \end{aligned} \quad (2.2.19)$$

(Note that the $\widehat{K}\widehat{K}\Delta\widehat{K}$ terms have cancelled.) The leading divergence of the second term is in $(\ln A\Lambda^2)^3$. These divergences will be discussed at the end of this subsection where it is shown that all $(\ln A\Lambda^2)^3$ divergences cancel between the different diagrams. The first term on the right-hand-side of (2.2.19) contributes to the leading divergence, giving $-\frac{30}{\pi\kappa^4} A\Lambda^2 (\ln A\Lambda^2)^2 \int_0^\infty d\alpha_3 \frac{\varphi(\alpha_3)}{\alpha_3}$.

Leading divergence of the partition function per diagram

The previous computation already gives the contribution of the flower diagram to the leading divergence of the partition function:

$$I_{\infty} = \frac{A\Lambda^2}{\pi\kappa^4} (\ln A\Lambda^2)^2 \left[-\frac{30}{\alpha_1} \right] + \mathcal{O}(\Lambda^2 \ln A\Lambda^2). \quad (2.2.20)$$

The only other diagram involving only one vertex is one of the measure “figure-eight” diagrams. It contributes

$$I_{\infty} = \frac{A\Lambda^2}{\pi\kappa^4} (\ln A\Lambda^2)^2 \left[\frac{3}{2} \frac{1}{\alpha_1} \right] + \mathcal{O}(\Lambda^2 \ln A\Lambda^2). \quad (2.2.21)$$

There are six diagrams built from two vertices that contribute to the leading singularity: . The integrals to perform are similar to those previously done in order to compute the two-loop vacuum diagrams. It

is rather straightforward to obtain:

$$\begin{aligned}
I_{\ominus} &= \frac{A\Lambda^2}{\pi\kappa^4} (\ln A\Lambda^2)^2 \left[\frac{18}{\alpha_1 + \alpha_2} \right] + \mathcal{O}(\Lambda^2 \ln A\Lambda^2) , \\
I_{\circ\circ} &= \frac{A\Lambda^2}{\pi\kappa^4} (\ln A\Lambda^2)^2 \left[\frac{18}{\alpha_1} + \frac{9}{\alpha_1 + \alpha_2} \right] + \mathcal{O}(\Lambda^2 \ln A\Lambda^2) , \\
I_{\circ\oplus} &= \frac{A\Lambda^2}{\pi\kappa^4} (\ln A\Lambda^2)^2 \left[\frac{24}{\alpha_1} + \frac{48}{\alpha_1 + \alpha_2} \right] + \mathcal{O}(\Lambda^2 \ln A\Lambda^2) , \\
I_{\ominus\ominus} &= \frac{A\Lambda^2}{\pi\kappa^4} (\ln A\Lambda^2)^2 \left[-\frac{2}{\alpha_1} \right] + \mathcal{O}(\Lambda^2 \ln A\Lambda^2) , \\
I_{\bullet\circ\circ} &= \frac{A\Lambda^2}{\pi\kappa^4} (\ln A\Lambda^2)^2 \left[-\frac{3}{\alpha_1} \right] + \mathcal{O}(\Lambda^2 \ln A\Lambda^2) , \\
I_{\circ\circ\circ} &= \frac{A\Lambda^2}{\pi\kappa^4} (\ln A\Lambda^2)^2 \left[-\frac{15}{\alpha_1} + \frac{12}{\alpha_1 + \alpha_2} \right] + \mathcal{O}(\Lambda^2 \ln A\Lambda^2) .
\end{aligned} \tag{2.2.22}$$

As always, according to our regularization scheme (1.3.28), these expressions are to be understood as multiplied with the regulator functions $\prod_i \varphi(\alpha_i)$ and integrated over $\prod_i \int_0^\infty d\alpha_i$. For instance $I_{\circ\oplus}$ contributes as $\frac{A\Lambda^2}{\pi\kappa^4} (\ln A\Lambda^2)^2 c_{\circ\oplus}$, with $c_{\circ\oplus} = 24 \int_0^\infty d\alpha_1 \frac{\varphi(\alpha_1)}{\alpha_1} + 48 \int_0^\infty d\alpha_1 d\alpha_2 \frac{\varphi(\alpha_1)\varphi(\alpha_2)}{\alpha_1 + \alpha_2}$ being a number once the regularization function $\varphi(\alpha)$ is chosen.

Note that the results for the diagrams involving the counterterm vertex, $I_{\circ\circ}$ in (2.2.22) and I_{\oplus} in (2.2.23) below, do not depend on the free (two-loop) renormalization constants \hat{c}_ϕ and \hat{c}_R . In the next subsection, dedicated to the computation of the full contributions of the counterterms to all divergences, we will see indeed that the leading contribution involving \hat{c}_ϕ and \hat{c}_R is in $A\Lambda^2 \ln A\Lambda^2$.

When considering three vertices or more, computations become more technical. While for \ominus and \oplus it is easy to get:

$$\begin{aligned}
I_{\oplus} &= \frac{A\Lambda^2}{\pi\kappa^4} (\ln A\Lambda^2)^2 \left[\frac{7}{2} \frac{1}{\alpha_1} \right] + \mathcal{O}(\Lambda^2 \ln A\Lambda^2) , \\
I_{\ominus} &= \frac{A\Lambda^2}{\pi\kappa^4} (\ln A\Lambda^2)^2 \left[\frac{35}{2} \frac{1}{\alpha_1} - \frac{14}{\alpha_1 + \alpha_2} \right] + \mathcal{O}(\Lambda^2 \ln A\Lambda^2) ,
\end{aligned} \tag{2.2.23}$$

with \circ and $\circ\circ$ already, one stumbles over the same kind of technical difficulties as those faced when computing the one-loop two-point Green's function at coinciding points (see the appendix of [56] and appendix C of this thesis). One of the integrals encountered in $\circ\circ\circ$ is for instance

$$\int dx dy dz \tilde{K}(t_1, x, z) \tilde{K}(t_2, x, z) \tilde{K}(t_3, y, z) \tilde{K}(t_4, y, z) \tilde{K}(t_5, x, y) .$$

Trouble comes from the fact that the three \tilde{K} s in the integral force the three variables x , y and z to be all close to each other. For instance, integrating over y through the term

$\tilde{K}(t_4, y, z)$ requires to Taylor expand

$$\tilde{K}(t_5, x, y) = \tilde{K}(t_5, x, z) + (y - z)_i \partial_z^i \tilde{K}(t_5, x, z) + \frac{1}{2} (y - z)_i (y - z)_j \partial_z^i \partial_z^j \tilde{K}(t_5, x, z) + \dots \quad (2.2.24)$$

When x, y and z are close, such that $l^2(x, y) \sim l^2(x, z) \sim \frac{1}{\Lambda^2}$, all terms in the expansion give contributions of the same order. One gets:

$$\begin{aligned} & \int dx dy dz \hat{\tilde{K}}(t_1, x, z) \tilde{K}(t_2, x, z) \hat{\tilde{K}}(t_3, y, z) \tilde{K}(t_4, y, z) \tilde{K}(t_5, x, y) \\ &= \int dx dz \hat{\tilde{K}}(t_1, x, z) \tilde{K}(t_2, x, z) \hat{\tilde{K}}(t_3 + t_4, z, z) \\ & \quad \times \left[\tilde{K}(t_5, x, z) - t_4 \left(- \frac{d\tilde{K}(t, x, z)}{dt} \Big|_{t=t_5} + R_* \tilde{K}(t_5, x, z) \right) \right. \\ & \quad \left. + \frac{t_4^2}{2} \left(\frac{d^2 \tilde{K}(t, x, z)}{dt^2} \Big|_{t=t_5} - 2R_* \frac{d\tilde{K}(t, x, z)}{dt} \Big|_{t=t_5} + R_*^2 \tilde{K}(t_5, x, z) \right) + \dots \right] \\ & - \frac{1}{A} \int dx dy dz \hat{\tilde{K}}(t_1, x, z) \tilde{K}(t_2, x, z) \hat{\tilde{K}}(t_3, y, z) \tilde{K}(t_5, x, y) . \end{aligned} \quad (2.2.25)$$

As just explained, the terms $+\dots$ contribute at the same order and cannot be dropped. Keeping only the terms that contribute to the leading singularity $A\Lambda^2 (\ln A\Lambda^2)^2$, one has

$$\begin{aligned} & \int dx dy dz \hat{\tilde{K}}(t_1, x, z) \tilde{K}(t_2, x, z) \hat{\tilde{K}}(t_3, y, z) \tilde{K}(t_4, y, z) \tilde{K}(t_5, x, y) \\ &= \int dx dz \hat{\tilde{K}}(t_1, x, z) \tilde{K}(t_2, x, z) \hat{\tilde{K}}(t_3 + t_4, z, z) \sum_{n=0}^{\infty} \frac{t_4^n}{n!} \frac{d^n \tilde{K}(t, x, z)}{dt^n} \Big|_{t=t_5} . \end{aligned} \quad (2.2.26)$$

Furthermore,

$$\int dx \hat{\tilde{K}}(t_1, x, z) \tilde{K}(t_2, x, z) \frac{d^n \tilde{K}(t, x, z)}{dt^n} \Big|_{t=t_5} = \frac{(-1)^n n!}{(4\pi)^2} \left(\frac{\Lambda^2}{\alpha_2 + \alpha_5} \right)^{n+1} \ln A\Lambda^2 + \mathcal{O}(\Lambda^{2n+2}) \quad (2.2.27)$$

so that one can easily resum all the terms. Therefore, the previous integral contributes to the leading divergence by:

$$\frac{A\Lambda^2}{(4\pi)^3} (\ln A\Lambda^2)^2 \sum_{n=0}^{\infty} \frac{(-1)^n \alpha_4^n}{(\alpha_2 + \alpha_5)^{n+1}} = \frac{A\Lambda^2}{(4\pi)^3} (\ln A\Lambda^2)^2 \frac{1}{\alpha_2 + \alpha_5 + \alpha_4} . \quad (2.2.28)$$

Of course, this is valid for $\frac{\alpha_4}{\alpha_2 + \alpha_5} < 1$. However, the initial expression was symmetric under exchange of α_2 and α_4 (upon also exchanging α_1 and α_3). Hence, if $\frac{\alpha_4}{\alpha_2 + \alpha_5} > 1$

one simply exchanges the roles of α_2 and α_4 in the derivation (since now $\frac{\alpha_2}{\alpha_4+\alpha_5} < 1$) and one gets the same result.

Considering carefully each integral, finally one gets⁸ for  and  :

$$\begin{aligned} I_{\bigcirc} &= \frac{A\Lambda^2}{\pi\kappa^4} (\ln A\Lambda^2)^2 \left[-\frac{21}{\alpha_1} - \frac{12}{\alpha_1 + \alpha_2} - \frac{24}{\alpha_1 + \alpha_2 + \alpha_3} \right] + \mathcal{O}(\Lambda^2 \ln A\Lambda^2), \\ I_{\bigcirc\bigcirc} &= \frac{A\Lambda^2}{\pi\kappa^4} (\ln A\Lambda^2)^2 \left[-\frac{42}{\alpha_1 + \alpha_2} - \frac{46}{\alpha_1 + \alpha_2 + \alpha_3} \right] + \mathcal{O}(\Lambda^2 \ln A\Lambda^2). \end{aligned} \quad (2.2.29)$$

One encounters similar problems for the diagrams with four vertices  and  . Taylor expanding leads to series of divergent contributions. In addition to the series (2.2.28), one obtains

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{n+m}{n} (-1)^{n+m} \frac{\alpha_1^n \alpha_2^m}{(\alpha_3 + \alpha_4)^{n+m+1}} = \frac{1}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}. \quad (2.2.30)$$

More details on the integrals generating such series are given in the appendix C. Thus, one gets:

$$\begin{aligned} I_{\bigcirc\bigcirc} &= \frac{A\Lambda^2}{\pi\kappa^4} (\ln A\Lambda^2)^2 \left[\frac{14}{\alpha_1 + \alpha_2} + \frac{16}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} \right] + \mathcal{O}(\Lambda^2 \ln A\Lambda^2), \\ I_{\bigtriangleup} &= \frac{A\Lambda^2}{\pi\kappa^4} (\ln A\Lambda^2)^2 \left[\frac{16}{\alpha_1 + \alpha_2 + \alpha_3} + \frac{8}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} \right] + \mathcal{O}(\Lambda^2 \ln A\Lambda^2). \end{aligned} \quad (2.2.31)$$

Looking at (2.2.21), (2.2.22) and (2.2.23) one observes that the total leading contribution coming from the measure vanishes. Note that this was not the case for the two-loop contribution.

Adding the contributions of all the vacuum diagrams, (2.2.20), (2.2.21), (2.2.22), (2.2.23), (2.2.29) and (2.2.31), one gets the coefficient d_1 of $A\Lambda^2 (\ln A\Lambda^2)^2$ in the logarithm of the partition function, cf (2.2.9):

$$d_1 = \frac{1}{4\pi\kappa^4} \left[-\frac{26}{\alpha_1} + \frac{132}{\alpha_1 + \alpha_2} - \frac{216}{\alpha_1 + \alpha_2 + \alpha_3} + \frac{96}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} \right]. \quad (2.2.32)$$

Clearly, the leading divergence in $A\Lambda^2 (\ln A\Lambda^2)^2$ is not vanishing and new counterterms will be required. They should be determined by ensuring that the one-loop three-point and four-point functions, as well as the two-loop two-point function be all finite. The computation of these one-loop n -point functions is beyond the scope of this thesis, but it is nevertheless already interesting to look at the possible counterterms one could consider and to calculate their contributions to the various divergences of the partition function. This will be done in the next subsection.

⁸Note again that the α_i are to be multiplied with $\varphi(\alpha_i)$ and integrated. This implies that any expression involving several α_i can be symmetrized and that one can also rename the indices. In particular, the $\frac{1}{\alpha_2+\alpha_5+\alpha_4}$ in (2.2.28) has been rewritten as $\frac{1}{\alpha_1+\alpha_2+\alpha_3}$.

Cancellation of the $(\ln A\Lambda^2)^3$ divergence

When the counterterm contributions to the three-loop partition function are computed in the next subsection, it will be shown that local counterterms with local coefficients (*i.e.* not involving explicitly $\ln A\Lambda^2$) cannot give contributions to the $(\ln A\Lambda^2)^3$ divergence. Now, it is easy to see that such $(\ln A\Lambda^2)^3$ divergences are present in individual three-loop diagrams. In particular, this was the case for the flower diagram, see (2.2.19) and the remarks that followed. The only way to ensure finiteness of the partition function then is that these individual divergences cancel between the three-loop vacuum diagrams. Among the twenty-nine diagrams, eight contribute to the $(\ln A\Lambda^2)^3$ divergence. Their contributions are not too difficult to compute. They are displayed in Tab. 2.3. Indeed, when summed, they vanish! This is similar to what happened for the $(\ln A\Lambda^2)^2$ divergence in the two-loop partition function, and one expects the $(\ln A\Lambda^2)^L$ divergence to cancel in the L -loop partition function. Let us insist that this is a requirement not only for the finiteness of the partition function but also for its background independence, as discussed in the end of the previous section.

										Total
$\frac{8(1-h)}{\kappa^4} (\ln A\Lambda^2)^3$	-20	18	18	60	-42	-86	$26 + \frac{4}{3}$	$24 + \frac{2}{3}$	0	

Table 2.3: $(\ln A\Lambda^2)^3$ contributions from the diagrams

2.2.3 Counterterms

There are several types of counterterms one may add in the three-loop computation. Cubic or quartic counterterms lead to diagrams similar to the ones generated by the cubic and quartic measure vertices (cf Tab. 2.1). One may also expand the coefficients of the quadratic counterterms introduced in the two-loop computation (2.1.97)-(2.1.99), (2.1.100), and consider their κ^{-4} contributions. Of course, only local counterterms will be introduced. This means, on the one hand, that the counterterms are polynomial in the Kähler field $\tilde{\phi}$ with only finitely many derivatives acting on them, and, on the other hand, that the coefficients of these counterterms are local expressions. In particular, a counterterm coefficient involving the area e.g. through $\ln A\Lambda^2$ is non-local. However, following the procedure applied in the previous section for the two-loop case, we do allow for counterterm coefficients $\sim \frac{1}{A}$ since they are already present in the measure action due to the absence of the zero-mode. In the previous section (see 2.1.5), we introduced the “strong locality condition”, *i.e.* absence of $\frac{1}{A}$ terms in the total action. Imposing such a condition at two loops led us to fix one of the counterterms (namely \hat{c}_m) precisely to the KPZ value. We also impose the “strong locality condition” at three loops, by fixing the relevant coefficients of the three-loop counterterms. Therefore, the only A -dependent (“non-local”) remaining terms involve a $(\ln A\Lambda^2)^\#$ coefficient.

In this subsection, we will write out the counterterms contributing to the partition function at the same order as the three-loop diagrams, *i.e.* at order $\frac{1}{\kappa^4}$ and give their contributions to $W[A] = \ln Z[A]$. Since the divergences in $A\Lambda^2$ can always be absorbed in the cosmological constant they will be ignored in the following. Similarly, the finite area-independent contributions of the counterterms will not be spelt out.

Cubic counterterms

The new types of counterterms one may introduce are cubic and quartic ones. The allowed cubic counterterm action is

$$S_{\text{ct}}^c = \frac{16\pi^{3/2}}{\kappa^3} \frac{1}{2} \int d^2x \sqrt{g_*} \left[f_\phi \tilde{\phi}^2 (\Delta_* - R_*) \tilde{\phi} + f_R R_* \tilde{\phi}^3 + f_m \tilde{\phi}^3 \right] \quad (2.2.33)$$

where

$$\begin{aligned} f_\phi &= f_\phi^{(1)} , \\ f_R &= f_R^{(1)} , \\ f_m &= f_m^{(1)} \Lambda^2 + \frac{f_m^{(2)}}{A} . \end{aligned} \quad (2.2.34)$$

By dimensional analysis, the coefficients $f_i^{(1)}$ and $f_i^{(2)}$ are dimensionless “numbers”. As already emphasized in the two-loop analysis (see section 2.1.3) they may depend on the regularization through the α_i and are then to be integrated with the given $\varphi(\alpha_i)$, resulting in a number. But they do not depend on the cut-off Λ^2 . The action (2.2.33) contributes via the two two-loop diagrams  and  at the same order in κ^{-4} as the three-loop diagrams studied above.

We first show that the “glasses” diagram  gives no relevant contribution. It may be written as a sum of four subdiagrams. One gets:

$$\begin{aligned} I_{\text{glasses}} &= \frac{1}{4} \frac{(8\pi)^2}{\kappa^4} \int dx dy \widehat{K}(t_1, x, x) \widehat{K}(t_2, y, y) \\ &\quad \times \left\{ f_\phi \left(-\frac{d\widehat{K}(t, x, y)}{dt} \right)_{t=t_3} + \left[f_\phi R_* + 3(f_m + f_R R_*) \right] \widehat{K}(t_3, x, y) \right. \\ &\quad \left. + 3 R_* (f_m + f_R R_*) \widehat{K}(t_3, x, y) \right\} . \end{aligned} \quad (2.2.35)$$

Integrating and taking into account the absence of zero-modes leads to:

$$\begin{aligned}
I_{\text{glass}} = \frac{(4\pi)^2}{\kappa^4} & \left\{ f_\phi \int dx \tilde{G}_\zeta \Delta_* \tilde{G}_\zeta \right. \\
& + 3(f_m + f_R R_*) \int dx \tilde{G}_\zeta(x) \left(\tilde{G}_\zeta(x) - \frac{1}{A} \int dy \tilde{G}_\zeta(y) \right) \\
& \left. + 3(f_m + f_R R_*) R_* \int dx dy \tilde{G}_\zeta(x) \hat{K}(t_3, x, y) \tilde{G}_\zeta(y) \right\}. \quad (2.2.36)
\end{aligned}$$

Using the scaling relation (1.2.23) and (2.1.39), one may rewrite this as

$$\begin{aligned}
I_{\text{glass}} = \frac{(4\pi)^2}{\kappa^4} & \left\{ f_\phi \int d^2x \sqrt{g_0} \tilde{G}_\zeta^{A_0} \Delta_0 \tilde{G}_\zeta^{A_0} \right. \\
& + 3 \left(\frac{A}{A_0} f_m + f_R R_0 \right) \int d^2x \sqrt{g_0} \tilde{G}_\zeta^{A_0}(x) \left(\tilde{G}_\zeta^{A_0}(x) - \frac{1}{A_0} \int d^2y \sqrt{g_0(y)} \tilde{G}_\zeta^{A_0}(y) \right) \\
& \left. + 3 \left(\frac{A}{A_0} f_m + f_R R_0 \right) R_0 \int d^2x d^2y \sqrt{g_0(x)g_0(y)} \tilde{G}_\zeta^{A_0}(x) \hat{K}_0\left(\frac{A_0}{A} t_3, x, y\right) \tilde{G}_\zeta^{A_0}(y) \right\}. \quad (2.2.37)
\end{aligned}$$

The first term is obviously independent of the area A and thus of no interest here. The only A dependence in the second line comes from the $\frac{A}{A_0} f_m$ term through $\frac{f_m^{(1)}}{A_0} A \Lambda^2$. However, the parenthesis being A independent, this term can be included in the cosmological constant and is not significant. The last term is slightly more subtle to handle because of the remaining $\hat{K}_0(\frac{A_0}{A} t_3, x, y)$ term. For the non divergent counterterms $f_R^{(1)}$ and $\frac{f_m^{(2)}}{A}$, the short-distance logarithmic singularity in $\hat{K}_0(\frac{A_0}{A} t_3, x, y)$ being integrable, one may take the limit $t_3 \rightarrow \infty$. Doing so leads to an A independent quantity. Finally, doing a finite expansion in $x - y$ in the integral yields either A -independent or $\frac{1}{\Lambda^2}$ -terms or terms that vanish exponentially as $\Lambda \rightarrow \infty$. Thus, the remaining quadratically divergent counterterm $\Lambda^2 f_m^{(1)}$ only leads to terms finite or to be included in the cosmological constant. None of these terms is of any interest here. This “glasses” diagram thus gives no contribution to the pertinent divergences of the partition function (2.2.9). Note that diagrams with a single propagator joining two or three loops were already discarded from the diagrams contributing to the leading divergence in the previous subsection.

The “setting sun” diagram  gets two contributions according to which line of the cubic counterterm vertex is connected to the bold part of the cubic Liouville vertex.

Thus one obtains

$$\begin{aligned}
I_{\ominus} = \frac{(8\pi)^2}{\kappa^4} \int dx dy \left\{ f_{\phi} \widehat{K}(t_1, x, y) \right. & \quad (2.2.38) \\
& \times \left[\frac{1}{2} \widehat{K}(t_2, x, y) \left(-\frac{d\widehat{K}(t, x, y)}{dt} \right)_{t=t_3} + \widehat{K}(t_2, x, y) \widehat{K}(t_3, x, y) \right] \\
& + \frac{1}{2} [(f_{\phi} + 3f_R) R_* + 3f_m] \widehat{K}(t_1, x, y) \widehat{K}(t_2, x, y) \widehat{K}(t_3, x, y) \\
& \left. + \frac{R_*}{2} (f_R R_* + f_m) \widehat{K}(t_1, x, y) \widehat{K}(t_2, x, y) \widehat{K}(t_3, x, y) \right\}.
\end{aligned}$$

This leads to the following divergences:

$$\begin{aligned}
I_{\ominus} = \frac{1}{\kappa^4} \left\{ 6f_m^{(1)} A\Lambda^2 (\ln A\Lambda^2)^2 + \left[\frac{8}{\alpha_2 + \alpha_3} f_{\phi}^{(1)} + 12 f_m^{(1)} (G_0 - \ln(\alpha_2 + \alpha_3)) \right] A\Lambda^2 \ln A\Lambda^2 \right. \\
+ \left[6 f_m^{(2)} + \left(4 f_{\phi}^{(1)} + 6 f_R^{(1)} - 6 \alpha_3 f_m^{(1)} \right) AR_* \right] (\ln A\Lambda^2)^2 \\
+ \left[16 \left(\frac{f_{\phi}^{(1)}}{3} - \alpha_3 f_m^{(1)} \right) AR_* - \frac{\alpha_2 \alpha_3}{\alpha_2 + \alpha_3} \left(\frac{8}{\alpha_2 + \alpha_3} f_{\phi}^{(1)} + 12 f_m^{(1)} \right) AR_* \right. \\
- 48\pi \left(f_{\phi}^{(1)} - 2 \alpha_3 f_m^{(1)} \right) \\
\left. \left. + 2 \left(6 f_m^{(2)} + \left(4 f_{\phi}^{(1)} + 6 f_R^{(1)} - 6 \alpha_3 f_m^{(1)} \right) AR_* \right) (G_0 - \ln(\alpha_2 + \alpha_3)) \right] \ln A\Lambda^2 \right\} & \quad (2.2.39)
\end{aligned}$$

where G_0 was defined in (2.1.54). The expression (2.2.39) is the full contribution from the cubic counterterms to the diverging part of the partition function.

Quartic counterterms

The quartic counterterm action is

$$S_{\text{ct}}^q = \frac{(8\pi)^2}{\kappa^4} \frac{1}{2} \int d^2x \sqrt{g_*} \left[q_{\phi} \widetilde{\phi}^3 (\Delta_* - R_*) \widetilde{\phi} + \widehat{q}_{\phi} \widetilde{\phi}^2 (\Delta_* - 2R_*) \widetilde{\phi}^2 + q_R R_* \widetilde{\phi}^4 + q_m \widetilde{\phi}^4 \right] \quad (2.2.40)$$

with

$$\begin{aligned}
q_{\phi} &= q_{\phi}^{(1)}, \\
\widehat{q}_{\phi} &= \widehat{q}_{\phi}^{(1)}, \\
q_R &= q_R^{(1)}, \\
q_m &= q_m^{(1)} \Lambda^2 + \frac{q_m^{(2)}}{A}.
\end{aligned} \quad (2.2.41)$$

Again, the coefficients $q_i^{(j)}$ may depend on the α_k but not on the cutoff Λ . This action gives a “figure-eight” diagram  :

$$I_{\text{figure-eight}} = \frac{(8\pi)^2}{\kappa^4} \int dx \left\{ -\frac{3}{2} \left(q_\phi \widehat{K}(t_1, x, x) \widetilde{K}(t_2, x, x) + (q_R R_* + q_m) \widehat{K}(t_1, x, x) \widehat{K}(t_2, x, x) \right) + \widehat{q}_\phi \left(-2\widehat{K}(t_1, x, x) \widetilde{K}(t_2, x, x) + R_* \widehat{K}(t_1, x, x) \widehat{K}(t_2, x, x) \right) \right\}, \quad (2.2.42)$$

which contributes as

$$I_{\text{figure-eight}} = \frac{1}{\kappa^4} \left\{ -6 q_m^{(1)} A \Lambda^2 (\ln A \Lambda^2)^2 - \left[6 q_m^{(2)} + \left(-4 \widehat{q}_\phi^{(1)} + 6 q_R^{(1)} \right) A R_* \right] (\ln A \Lambda^2)^2 - \left[\frac{1}{\alpha_1} \left(6 q_\phi^{(1)} + 8 \widehat{q}_\phi^{(1)} \right) + 12 q_m^{(1)} (G_0 - \ln \alpha_1) \right] A \Lambda^2 \ln A \Lambda^2 - \left[\left(\frac{7}{6} A R_* - 4\pi \right) \left(6 q_\phi^{(1)} + 8 \widehat{q}_\phi^{(1)} - 12 \alpha_1 q_m^{(1)} \right) + 2 \left(6 q_m^{(2)} + \left(-4 \widehat{q}_\phi^{(1)} + 6 q_R^{(1)} \right) A R_* \right) (G_0 - \ln \alpha_1) \right] \ln A \Lambda^2 \right\} \quad (2.2.43)$$

with G_0 given in (2.1.54).

Quadratic two-loop counterterms

The quadratic counterterms (2.1.79) did contribute via one-loop diagrams to the two-loop partition function, but also via two-loop diagrams to the three-loop partition function as shown in the above computation. However, as always, the counterterm coefficients get contributions at different orders in perturbation theory: we can add to c_ϕ , c_R and c_m an additional piece $\frac{1}{\kappa^2} c'_\phi$, $\frac{1}{\kappa^2} c'_R$ and $\frac{1}{\kappa^2} c'_m$, so that $c_\phi^{\text{tot}} = c_\phi + \frac{1}{\kappa^2} c'_\phi + \mathcal{O}(\frac{1}{\kappa^4})$, etc. Overall, the c' are accompanied by a factor $\frac{1}{\kappa^4}$ and they contribute via one-loop diagrams to the three-loop partition function. Thus we also add the following counterterm action

$$S_{\text{ct}}^{\text{quad}} = \frac{8\pi}{\kappa^4} \int d^2x \sqrt{g_*} \left[\frac{c'_\phi}{2} \widetilde{\phi} (\Delta_* - R_*) \widetilde{\phi} + \frac{c'_R}{2} R_* \widetilde{\phi}^2 + \frac{c'_m}{2} \widetilde{\phi}^2 \right], \quad (2.2.44)$$

where, again,

$$\begin{aligned} c'_\phi &= c_\phi^{(1)}, \\ c'_R &= c_R^{(1)}, \\ c'_m &= c_m^{(1)} \Lambda^2 + \frac{c_m^{(2)}}{A}. \end{aligned} \quad (2.2.45)$$

The counterterm action (2.2.44) then provides a new one-loop diagram of order κ^{-4} : 

$$I_{\bullet\bigcirc} = -\frac{1}{2} \frac{8\pi}{\kappa^4} \int dx \left[c'_\phi \widetilde{K}(t, x, x) + (c'_R R_* + c'_m) \widehat{\widetilde{K}}(t, x, x) \right] \quad (2.2.46)$$

leading to the following divergences:

$$I_{\bullet\bigcirc} = \frac{1}{\kappa^4} \left(-c_m^{(1)} A \Lambda^2 \ln A \Lambda^2 - (c_m^{(2)} + c_R^{(1)} A R_*) \ln A \Lambda^2 \right). \quad (2.2.47)$$

Moreover, two parameters of the two-loop counterterms (2.1.80) are still unconstrained: \widehat{c}_ϕ and \widehat{c}_R . Although only \widehat{c}_R appears in the two-loop partition function (2.1.104), both may contribute to the divergent part of the partition function at three loops, through the diagrams , ,  and . Their diverging contributions are displayed below:

$$\begin{aligned} I_{\bullet\bigcirc} &= \frac{1}{\kappa^4} \left\{ -\frac{2}{\alpha_1} \widehat{c}_\phi A \Lambda^2 \ln A \Lambda^2 - 2 \left(\frac{7}{6} A R_* - 4\pi \right) \widehat{c}_\phi \ln A \Lambda^2 \right\}, \\ I_{\bigcirc\bigcirc} &= \frac{1}{\kappa^4} \left\{ \left(-\frac{10}{\alpha_1} + \frac{8}{\alpha_1 + \alpha_2} \right) \widehat{c}_\phi A \Lambda^2 \ln A \Lambda^2 + \left[4A R_* \widehat{c}_\phi \widehat{c}_R - 2 \left(\frac{7}{6} A R_* + 4\pi \right) \widehat{c}_\phi \right. \right. \\ &\quad \left. \left. + 4A R_* \left(\widehat{c}_\phi + \frac{\widehat{c}_R}{2\pi} \right) \left(3 \left(-\ln(\alpha_1 + \alpha_2) + \ln \alpha_1 \right) - 1 - \frac{2\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2} \right) \right] \ln A \Lambda^2 \right\}, \\ I_{\bigcirc\bigcirc} &= \frac{1}{\kappa^4} \left\{ 12 \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_1 + \alpha_2} \right) \widehat{c}_\phi A \Lambda^2 \ln A \Lambda^2 + 12A R_* \left(\widehat{c}_\phi + \frac{\widehat{c}_R}{2\pi} \right) (\ln A \Lambda^2)^2 \right. \\ &\quad \left. + \left[12A R_* \left(\widehat{c}_\phi + \frac{\widehat{c}_R}{2\pi} \right) \left(2 G_0 - \ln \alpha_1 - \ln(\alpha_1 + \alpha_2) \right) + 24 \left(\frac{7}{6} A R_* - 4\pi \right) \widehat{c}_\phi \right. \right. \\ &\quad \left. \left. + 24 R_*^2 \widehat{c}_R \int dx dy \widehat{\widetilde{K}}(t_1, x, y) \widehat{\widetilde{K}}(t_2, x, y) \right] \ln A \Lambda^2 \right\}, \quad (2.2.48) \end{aligned}$$

and

$$\begin{aligned} I_{\bigcirc\oplus} &= \frac{1}{\kappa^4} \left\{ -8 \left(\frac{1}{\alpha_1 + \alpha_2} + \frac{2}{\alpha_1 + \alpha_2 + \alpha_3} \right) \widehat{c}_\phi A \Lambda^2 \ln A \Lambda^2 - \left(18 \widehat{c}_\phi + 7 \frac{\widehat{c}_R}{\pi} \right) A R_* (\ln A \Lambda^2)^2 \right. \\ &\quad \left. + \left[-24 R_*^2 \widehat{c}_R \int dx dy \widehat{\widetilde{K}}(t_1, x, y) \widehat{\widetilde{K}}(t_2, x, y) - 2 G_0 \left(18 \widehat{c}_\phi + 7 \frac{\widehat{c}_R}{\pi} \right) A R_* \right. \right. \\ &\quad \left. \left. + 4A R_* \left[\left(\widehat{c}_\phi + \frac{\widehat{c}_R}{2\pi} \right) \left(1 + \frac{2\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2} + 3 \ln(\alpha_1 + \alpha_2) \right) + 6 \left(\widehat{c}_\phi + \frac{\widehat{c}_R}{3\pi} \right) \ln(\alpha_1 + \alpha_2 + \alpha_3) \right] \right. \right. \\ &\quad \left. \left. + 4 \widehat{c}_\phi \left[36\pi - A R_* \left(\frac{11}{3} + \frac{4\alpha_1^2}{(\alpha_1 + \alpha_2 + \alpha_3)^2} \right) \right] \right] \ln A \Lambda^2 \right\}. \quad (2.2.49) \end{aligned}$$

None of these contains a $A \Lambda^2 (\ln A \Lambda^2)^2$ divergence and this is why these finite counterterm coefficients \widehat{c}_ϕ and \widehat{c}_R did not contribute to the previous computation of the leading three-loop divergence made in section 2.2.2.

Total counterterm contribution to the partition function

Since the “glasses” diagram has no divergence other than in $A\Lambda^2$, the total contribution one could get from the counterterms to the three-loop partition function is given by summing (2.2.39), (2.2.43), (2.2.47), (2.2.48) and (2.2.49). Recalling $AR_* = 8\pi(1-h)$, cf. (1.2.23), one gets:

$$W^{(3)}[A]|_{\text{ct}} = \frac{1}{\kappa^4} \left\{ \Omega_1 A\Lambda^2 (\ln A\Lambda^2)^2 + \Omega_2 A\Lambda^2 \ln A\Lambda^2 + \Omega_3 (\ln A\Lambda^2)^2 + \Omega_4 \ln A\Lambda^2 \right\} \quad (2.2.50)$$

with

$$\begin{aligned} \Omega_1 &= 6 \left(f_m^{(1)} - q_m^{(1)} \right) \\ \Omega_2 &= -\frac{16}{\alpha_1 + \alpha_2 + \alpha_3} \widehat{c}_\phi + \frac{1}{\alpha_1 + \alpha_2} \left(8 f_\phi^{(1)} + 12 \widehat{c}_\phi \right) - \frac{1}{\alpha_1} \left(6 q_\phi^{(1)} + 8 \widehat{q}_\phi^{(1)} \right) - c_m'^{(1)} \\ &\quad + 2 \Omega_1 G_0 - 12 \left(f_m^{(1)} \ln(\alpha_1 + \alpha_2) - q_m^{(1)} \ln \alpha_1 \right) \\ \Omega_3 &= \Omega_3^{(a)} + \Omega_3^{(b)} + \Omega_3^{(c)} \\ \Omega_3^{(a)} &= -6 q_m^{(2)} + \left(4 \widehat{q}_\phi^{(1)} - 6 q_R^{(1)} \right) 8\pi(1-h) \\ \Omega_3^{(b)}(\alpha_1) &= 6 f_m^{(2)} + \left(4 f_\phi^{(1)} + 6 \left(\widehat{c}_\phi + \frac{\widehat{c}_R}{2\pi} + f_R^{(1)} - \alpha_1 f_m^{(1)} \right) \right) 8\pi(1-h) \\ \Omega_3^{(c)} &= -12 \left(\widehat{c}_\phi + \frac{\widehat{c}_R}{3\pi} \right) 8\pi(1-h) \\ \Omega_4 &= 2 \Omega_3 G_0 - 2 \left(\Omega_3^{(a)} \ln \alpha_1 + \Omega_3^{(b)}(\alpha_1) \ln(\alpha_1 + \alpha_2) + \Omega_3^{(c)} \ln(\alpha_1 + \alpha_2 + \alpha_3) \right) \\ &\quad + 4\pi \left(1 - \frac{7}{3}(1-h) \right) \left(6 q_\phi^{(1)} + 8 \widehat{q}_\phi^{(1)} + 12 \alpha_1 f_m^{(1)} - 12 f_\phi^{(1)} + 2 \alpha_1 \Omega_1 \right) \\ &\quad + 8\pi(1-h) \left(\frac{26}{3} \left(\widehat{c}_\phi - f_\phi^{(1)} \right) + \frac{12 \alpha_1^2}{\alpha_1 + \alpha_2} f_m^{(1)} - \frac{8 \alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2} f_\phi^{(1)} - \frac{16 \alpha_1^2}{(\alpha_1 + \alpha_2 + \alpha_3)^2} \widehat{c}_\phi \right) \\ &\quad + 48\pi \widehat{c}_\phi - \left(c_m'^{(2)} + 8\pi(1-h) c_R'^{(1)} \right) + 32\pi(1-h) \widehat{c}_\phi \widehat{c}_R \end{aligned} \quad (2.2.51)$$

where G_0 was defined in (2.1.54). Once again, the divergences in $A\Lambda^2$ (which can always be absorbed in the cosmological constant) as well as the finite area-independent contributions of the counterterms (which drop when considering $W^{(3)}[A] - W^{(3)}[A_0]$) have been ignored here.

This is the total contribution to the three-loop partition function of the counterterms that have not been previously fixed by the order $\frac{1}{\kappa^2}$ (“two-loop”) computation made in section 2.1.4. Requiring the one-loop two-point function to be finite and regulator independent fixed c_m and parts of c_ϕ and c_R . Thus, only their so-far undetermined regularization-independent parts \widehat{c}_ϕ and \widehat{c}_R have been included in (2.2.51).

One way to determine some of these counterterms is to compute the two-loop two-point function (order $\frac{1}{\kappa^4}$) and the one-loop three-point function (order $\frac{1}{\kappa^3}$) and one-loop four-point function (order $\frac{1}{\kappa^4}$) and to require them to be finite and regularization independent. Imposing finiteness will completely determine certain combinations of the counterterm coefficients, while imposing regularization independence of the finite terms will fix certain other combinations up to constants.

The computations of the two-loop two-point function and of the one-loop three-point and four-point functions clearly are beyond the scope of this work. However, there are still interesting remarks that can be made without actually doing these computations. One can rather easily determine the contributions of the counterterms to these n -point functions. This will tell us which combinations of the counterterm coefficient would be fixed by such computations. We will find that the relevant combinations are indeed the same as those appearing in the Ω_i of the three-loop partition function. Although “expected”, this is by no means obvious and constitutes a nice consistency check.

It is straightforward to see that the cubic and quartic counterterms contribute to the diverging parts of the three- and four-point functions as

$$\begin{aligned} \left. \begin{array}{c} \text{CT} \\ \text{div} \end{array} \right| &= -\frac{48\pi^{3/2}}{\kappa^3} f_m^{(1)} \Lambda^2 \int dx \widehat{K}(t_1, a, x) \widehat{K}(t_2, b, x) \widehat{K}(t_3, c, x) , \\ \left. \begin{array}{c} \text{CT} \\ \text{div} \end{array} \right| &= -12 \frac{(8\pi)^2}{\kappa^4} q_m^{(1)} \Lambda^2 \int dx \widehat{K}(t_1, a, x) \widehat{K}(t_2, b, x) \widehat{K}(t_3, c, x) \widehat{K}(t_4, d, x) . \end{aligned} \quad (2.2.52)$$

Thus finiteness of these functions fixes both $f_m^{(1)}$ and $q_m^{(1)}$ and hence, Ω_1 . Finiteness of the two-point function at one loop (order $\frac{1}{\kappa^2}$) was already imposed in section 2.2.3 and resulted in the determination of c_m to this order. Thus, only the two-loop (order $\frac{1}{\kappa^4}$) part of the two-point function⁹ $\mathcal{G}^{(3)}(a, b)$ will be considered here. We find that the contribution of the counterterms to its diverging part is

$$\begin{aligned} \left. \text{CT} \right|_{\frac{1}{\kappa^4}, \text{div}} &= \frac{8\pi}{\kappa^4} \left\{ \left(\rho_1 \Lambda^2 \ln A\Lambda^2 + \rho_2 \Lambda^2 + \rho_3 \frac{\ln A\Lambda^2}{A} \right) \int dx \widehat{K}(t_1, a, x) \widehat{K}(t_2, b, x) \right. \\ &\quad + \rho_4 \Lambda^2 \int d^2x \sqrt{g_*} \widetilde{G}_\zeta^{A_0}(x) \widehat{K}(t_1, a, x) \widehat{K}(t_2, b, x) \\ &\quad \left. + \rho_5 \ln A\Lambda^2 \widetilde{G}(a, b) + 24\pi f_m^{(1)} \Lambda^2 H(a, b) \right\} , \end{aligned} \quad (2.2.53)$$

with

$$\begin{aligned} \rho_1 &= 2 \Omega_1 , \quad \rho_2 = \Omega_2 - 6 \left(3 f_m^{(1)} - 2 q_m^{(1)} \right) (G_0 + \ln A_0 \mu^2 + \gamma) , \quad \rho_3 = 2 \Omega_3 , \\ \rho_4 &= 24\pi \left(3 f_m^{(1)} - 2 q_m^{(1)} \right) , \quad \rho_5 = - \left(6 q_\phi^{(1)} + 8 \widehat{q}_\phi^{(1)} + 12 \alpha_1 f_m^{(1)} + 12 \widehat{c}_\phi - 12 f_\phi^{(1)} \right) \end{aligned} \quad (2.2.54)$$

⁹The two-point function we consider is for $a \neq b$.

and

$$\begin{aligned}
H(a, b) = & R_* \int dx dy \widehat{K}(t_1, a, x) \widehat{K}(t_2, x, y) \widehat{K}(t_3, x, y) \widehat{K}(t_4, y, b) \\
& + \frac{1}{2} \int dx \widehat{K}(t_1, a, x) \widehat{K}(t_2, b, x) \left(\widehat{K}(t_3, a, x) + \widehat{K}(t_3, b, x) \right) \\
& - \frac{1}{A} \int dx dy \left[\frac{1}{2} \widehat{K}(t_1, x, y) \widehat{K}(t_2, x, y) \left(\widehat{K}(t_3, a, x) + \widehat{K}(t_3, y, b) \right) \right. \\
& \quad \left. + 2 \widehat{K}(t_1, a, x) \widehat{K}(t_2, x, y) \widehat{K}(t_3, y, b) \right] \\
& + R_* \int dx dy \widehat{K}(t_1, a, x) \widehat{K}(t_2, x, b) \widehat{K}(t_3, x, y) \widetilde{G}_\zeta(y) \\
& + \int dx \left(\widehat{K}(t_1, a, x) + \widehat{K}(t_2, x, b) \right) \\
& \quad \times \left[\widehat{K}(t_3, a, b) \widetilde{G}_\zeta(x) - \frac{1}{A} \int dy \widehat{K}(t_3, x, y) \widetilde{G}_\zeta(y) \right]. \quad (2.2.55)
\end{aligned}$$

The contributions per diagrams are given in the appendix B.2. Finiteness of the two-point function at order $\frac{1}{\kappa^4}$ then fixes all combinations ρ_1, \dots, ρ_5 and hence Ω_2, Ω_3 and the combination $6 q_\phi^{(1)} + 8 \widehat{q}_\phi^{(1)} + 12 \widehat{c}_\phi - 12 f_\phi^{(1)}$.

Thus, all the coefficients Ω_1, Ω_2 and Ω_3 of the diverging parts of the counterterm contributions to the partition function (2.2.50) are exactly determined by the requirement of the finiteness of the two-loop two-point function and of the one-loop three-point and four-point functions ! Obviously, we expect this determination to be such that (2.2.50) precisely cancels the divergences of the genuine three-loop part of this partition function, as was indeed the case in section 2.1.5 for the two-loop computation.

Let us next discuss Ω_4 which is the counterterm contribution to the order $\frac{1}{\kappa^4}$ part of γ_{str} . With the $f_m^{(1)}, q_m^{(1)}$ and the ρ_i been fixed, also Ω_1, Ω_2 and Ω_3 are fixed and we consider $\Omega_3^{(a)}$ as a function of $\Omega_3^{(b)}$ and $\Omega_3^{(c)}$, *i.e.* of $\Omega_3^{(b)}, \widehat{c}_\phi$ and \widehat{c}_R (cf (2.2.51)). (Note that the second line in the expression of Ω_4 can be expressed through ρ_5, \widehat{c}_ϕ and Ω_1 .) Thus Ω_4 depends on the following six undetermined constants: $\Omega_3^{(b)}, \widehat{c}_\phi, \widehat{c}_R, f_\phi^{(1)}, c_m^{(1)}$ and $c_R^{(1)}$.

Furthermore, one may require the ‘‘strong locality condition’’ that the non-local terms in the measure (2.2.4) and counterterm actions (2.2.44), (2.2.33), (2.2.40) cancel out. This fixes $q_m^{(2)}, f_m^{(2)}$ and $c_m^{(2)}$ as

$$q_m^{(2)} = -1, \quad f_m^{(2)} = -\frac{4}{3}, \quad c_m^{(2)} = 0, \quad (2.2.56)$$

since the corresponding $\frac{1}{A}$ terms in (2.2.45), (2.2.34) and (2.2.41) are

$$\frac{4\pi}{A} \left[\frac{2\sqrt{\pi}}{\kappa^3} \left(f_m^{(2)} + \frac{4}{3} \right) \widetilde{\phi}^3 + \frac{1}{\kappa^4} c_m^{(2)} \widetilde{\phi}^2 + \frac{8\pi^2}{\kappa^4} \left(q_m^{(2)} + 1 \right) \widetilde{\phi}^4 \right] = 0. \quad (2.2.57)$$

Thus, among the six undetermined constants in Ω_4 , only $c_m'^{(2)}$ is fixed, five free finite renormalization constants remain: $\Omega_3^{(b)}$, \widehat{c}_ϕ , \widehat{c}_R , $f_\phi^{(1)}$, and $c_R'^{(1)}$, that is to say four more than the “two-loop” free parameter in γ_{str} : \widehat{c}_R . In the next chapter, we will discuss how the finite part of the two-point function can partly fix these remaining coefficients.

Finally, as anticipated in section 2.2.2, none of the counterterms contributes to the $(\ln A\Lambda^2)^3$ divergence. The only way to generate such divergences would be by introducing non-local counterterm coefficients that already involve a factor of $\ln A\Lambda^2$. However, as repeatedly argued, such counterterms should be forbidden. Then, since there is no possible counterterm for a $(\ln A\Lambda^2)^3$ divergence, such a divergence is required to cancel in the first place between the three-loop vacuum diagrams. As shown above, this is indeed the case. Let us once more remark that the cancellation of this divergence is also required for background independence.

2.2.4 Discussion

The purpose of this section was to check if and which new counterterms are required at three loops. We have therefore computed the leading divergence of the three-loop partition function at fixed area, cf (2.2.32). It does not vanish and thus genuine three-loop counterterms are required. As mentioned before, the two-loop computation already pointed to the insertion of new counterterms at three loops. Indeed, the counterterms inserted at two loops have a strong similarity with the measure terms at two loops, as was discussed in the previous section. Yet, at three loops, the measure action gives rise to cubic and quartic vertices unlike the two-loop counterterm vertices (cf Tab. 2.1). Therefore, one could have expected additional counterterms to be needed. This argument can be generalized to all orders, as the measure action gets additional structures at every order in the loop-expansion. If the counterterms are to be understood as a renormalization of the measure action, the latter itself coming from the regularization of the measure for the metrics, then new counterterms have to be introduced at every order in the perturbation series. On the other hand, what is really surprising and encouraging is that if one requires the counterterms to be local, in particular that no counterterm coefficient with a $\ln A\Lambda^2$ divergence is allowed, then all the divergences may be offset but the $(\ln A\Lambda^2)^3$ divergence. However, as we showed, this divergence cancels out between the three-loop diagrams, meaning that local counterterms are enough to balance all the non-local divergences. Moreover, the required counterterm action has a structure similar to those of the measure action, supporting the understanding of counterterms as a renormalization of the measure.

Nevertheless, with no other way to discriminate the counterterms than to forbid $(\ln A\Lambda^2)$ -like non-local terms, many new free parameters appear. At three loops, doing so gives rise to twelve new parameters. Imposing the divergences to vanish in the one-loop three- and four-point functions and in the two-loop two-point function fixes two parameters and three combinations of the parameters. We found that with these parameters and combinations of parameters fixed, the diverging part of the three-loop partition function is also completely fixed with no additional adjustable parameter remaining. Obviously,

as it was the case at two loops, we expect this to happen precisely in such a way that all divergences in the three-loop partition function cancel, except for the $(\ln A\Lambda^2)$ -piece that yields the three-loop contribution to the string susceptibility. Indeed, this is the only coefficient of the three-loop partition function which contains undetermined finite renormalization constants. To restrain the number of free parameter in this coefficient, we argued that there are two different notions of locality of the counterterm coefficients. While coefficients involving $\ln A\Lambda^2$ were excluded, we did allow coefficients proportional to $\frac{1}{A}$ since such non-local terms already appeared through the measure action. Introducing such $\frac{1}{A}$ counterterms in precisely such a way as to cancel the corresponding $\frac{1}{A}$ terms in the measure action was referred to as “strong locality condition”. Imposing this condition fixes one of the six free parameters in the contribution to the string susceptibility, leaving us with five free renormalization constants on which the three-loop contribution to γ_{str} depends. One of these free renormalization constants was already present in the two-loop string susceptibility, so that at three-loops, four new constants play a role.

Several additional requirements should be considered, such as the condition that neither the n -point functions nor the partition function should depend on the choice of regularization. In particular, the regularization function $\varphi(\alpha_i)$ satisfies $\int_0^\infty d\alpha_i \varphi(\alpha_i) = 1$ and certain regularity conditions at 0 and infinity, but is otherwise arbitrary. Its choice should not impact any final, physical result. This means that all the dependence in the α_i must disappear in the end. Although important, this argument is not enough to fully determine the counterterms, in particular it cannot fix any α -independent pieces. It will be discussed in the next chapter of this thesis.

Background independence does not seem to help fixing the counterterms, since the background dependent quantities appear with the same coefficients as the divergences and hence are automatically removed when the divergences are absorbed. Nevertheless, it could well be that some indirect criteria for background independence fixes some or all of our free renormalization constants. Still, as no obvious criterion has been identified, we will proceed keeping track of the free renormalization constants.

Towards non-conformal matter

« Qu'une chose soit difficile doit nous être une raison de plus pour l'entreprendre. »
Rainer Maria Rilke in *Lettres à un jeune poète*.

In this chapter, following [54], we consider a somewhat more general action including the Mabuchi action:

$$S_{\text{grav}}[g_0, g] = \frac{\kappa^2}{8\pi\varepsilon} S_{\text{L}}[g_0, g] + \frac{\beta^2}{\varepsilon} S_{\text{M}}[g_0, g] + S_{\text{measure}}[g_0, g] , \quad (3.0.1)$$

where ε is a loop counting parameter. As previously discussed in section 1.2, the Mabuchi action

$$S_{\text{M}}[g_0, g] = \int d^2x \sqrt{g_0} \left[2\pi(h-1)\phi\Delta_0\phi + \left(\frac{8\pi(1-h)}{A_0} - R_0 \right) \phi + \frac{4}{A} \sigma e^{2\sigma} \right] \quad (3.0.2)$$

appears (together with the Aubin-Yau action) as the first-order mass correction to the Liouville action. As previously stated in section 1.2.3 (see eq. (1.2.29)), the classical saddle-points at fixed area of the Mabuchi action are also the constant curvature metrics of arbitrary area A^1 . In terms of the rescaled Kähler potential $\phi_\varepsilon = \frac{\kappa}{8\sqrt{\pi\varepsilon}}\phi$, the one-loop expansions (1.3.13) of the Liouville and Mabuchi actions are rewritten as

$$\begin{aligned} \frac{\kappa^2}{8\pi\varepsilon} S_{\text{L}}[g_0, g] &= \frac{\kappa^2}{2\varepsilon} (1-h) \ln \frac{A}{A_0} + \frac{1}{2} \int d^2x \sqrt{g_*} \phi_\varepsilon (A\Delta_*)^2 (\Delta_* - R_*) \phi_\varepsilon + O(\varepsilon) , \\ \frac{\beta^2}{\varepsilon} S_{\text{M}}[g_0, g] &= \frac{2\beta^2}{\varepsilon} \ln \frac{A}{A_0} + 16\pi \frac{\beta^2}{\kappa^2} \int d^2x \sqrt{g_*} \phi_\varepsilon A\Delta_* (\Delta_* - R_*) \phi_\varepsilon + O(\varepsilon) , \end{aligned} \quad (3.0.3)$$

in the background metric g_* of constant curvature R_* and area A . From eq. (3.0.3), one directly reads the propagator for the field ϕ_ε

$$\tilde{G}(x, y) \Big|_{\text{L+M}} = \langle x | \left[A\Delta_* (\Delta_* - R_*) \left(A\Delta_* + 32\pi \frac{\beta^2}{\kappa^2} \right) \right]^{-1} | y \rangle' . \quad (3.0.4)$$

This propagator highly differs from the “pure Liouville theory”. Indeed, if we only consider the Liouville action, the term $\frac{\beta^2}{\kappa^2}$ is vanishing and we can make the change of variable²

$$\tilde{\phi} = A\Delta_* \phi_\varepsilon . \quad (3.0.5)$$

¹We recall that $AR_* = 8\pi(1-h)$.

²In the previous chapter, ε was absorbed in κ or equivalently taken to one.

Thus, we have to consider the much simpler propagator

$$\tilde{G}(x, y)\Big|_{\text{L}} = \langle x | (\Delta_* - R_*)^{-1} | y \rangle' . \quad (3.0.6)$$

The full Liouville and Mabuchi theory is then expected to behave significantly differently from the pure Liouville theory. However, the writings of the vertices in terms of the propagator (3.0.4) is quite challenging and the computation of the two-loop contribution to the string susceptibility for this Liouville and Mabuchi theory on a surface of arbitrary genus is beyond the scope of this thesis. Thus, in this chapter we only present the much simpler case of the torus, where the constant scalar curvature $R_* = \frac{8\pi(1-h)}{A}$ vanishes. Then, the previous change of variable (3.0.5) stands and the propagator associated to the field $\tilde{\phi}$ is simply

$$\begin{aligned} \tilde{G}(x, y)\Big|_{\text{L+M}} &= \langle x | (\Delta_* + \frac{32\pi}{A} \frac{\beta^2}{\kappa^2})^{-1} | y \rangle' , \\ &\equiv \langle x | (\Delta_* + \text{T})^{-1} | y \rangle' , \end{aligned} \quad (3.0.7)$$

where we define

$$\text{T} = \frac{32\pi}{A} \frac{\beta^2}{\kappa^2} . \quad (3.0.8)$$

The coefficients of the small t expansion of the heat kernel associated to the operator $D_*^{\text{M}} = \Delta_* + \text{T}$ are now (see section 1.3.3)

$$a_0^{\text{M}}(x, y) = 1 , \quad a_1^{\text{M}}(x, y) = -\text{T} , \quad a_2^{\text{M}}(x, y) = \frac{\text{T}^2}{2} \dots \quad (3.0.9)$$

and $\sqrt{g_*} = 1$. The regularized propagator $\widehat{\tilde{K}}(t, x, y)$ is to be written in terms of these modified coefficients. In particular, we have

$$\begin{aligned} \tilde{K}(t, x, x) &= \frac{1}{4\pi t} \left[1 - \left(\text{T} + \frac{4\pi}{A} \right) t + \left(\frac{\text{T}}{2} + \frac{4\pi}{A} \right) \text{T} t^2 \right] + \mathcal{O}(t^2) , \\ \widehat{\tilde{K}}(t, x, x) &= \tilde{G}_\zeta(x) - \frac{1}{4\pi} \left[\ln \mu^2 t + \gamma - \left(\text{T} + \frac{4\pi}{A} \right) t \right] + \mathcal{O}(t^2) . \end{aligned} \quad (3.0.10)$$

In the section 3.1, we present the two-loop computation of the partition function in this Liouville and Mabuchi theory, while section 3.2 is dedicated to the three-loop study. Thus, we expand the Mabuchi action around the metric g_* of constant vanishing scalar curvature up to the “three-loop order” ε^2 :

$$\begin{aligned} \frac{\beta^2}{\varepsilon} S_{\text{M}}[\sigma] &= \frac{2\beta^2}{\varepsilon} \ln \frac{A}{A_0} + \frac{32\pi}{A} \frac{\beta^2}{\kappa^2} \int d^2x \sqrt{g_*} \left\{ \frac{1}{2} \tilde{\phi}^2 + \frac{\sqrt{4\pi\varepsilon}}{3\kappa} \tilde{\phi}^3 + \frac{4\pi\varepsilon}{3\kappa^2} \tilde{\phi}^4 \right. \\ &\quad \left. + \frac{16(\pi\varepsilon)^{3/2}}{5\kappa^3} \tilde{\phi}^5 - \frac{(8\pi\varepsilon)^2}{\kappa^4} \frac{2}{15} \tilde{\phi}^6 + \mathcal{O}(\varepsilon^3) \right\} . \end{aligned} \quad (3.0.11)$$

In the following, we adopt the previous notation $\int dx$ for $\int d^2x \sqrt{g_*(x)}$.

3.1 Two-loop partition function

From eq. (2.2.1) and (3.0.11), we get the two-loop vertices (up to the first order in ε) for the Liouville plus Mabuchi theory: a cubic vertex, two quartic vertices and a measure vertex. The measure vertex is unchanged (unless for the changes in $\tilde{K}(t, x, x)$ due to the new a_i 's), whereas the other vertices are modified such that:

$$\begin{aligned} \text{Y} &= -\frac{\sqrt{4\pi\varepsilon}}{\kappa} \left(\Delta_* + \frac{1}{3}\text{T} \right), & \text{X} &= -\frac{4\pi\varepsilon}{\kappa^2} \frac{4}{3} (\Delta_* + \text{T}), & \text{Z} &= -\frac{4\pi\varepsilon}{\kappa^2} \frac{1}{2} (\Delta_* - 2\text{T}). \end{aligned} \quad (3.1.1)$$

3.1.1 Diagrams

We get the same diagrams as in the previous study of the pure Liouville theory: the “figure-eight” diagram $\bigcirc\bigcirc$, the “setting sun” diagram \ominus , the “glasses” diagram $\bigcirc\text{---}\bigcirc$ and the “measure” diagram \bigcirc . In the following, we give the contributions from each diagram (when all its subdiagrams are summed up) and then their contributions to the partition function in terms of the divergences $A\Lambda^2 \ln A\Lambda^2$ and $(\ln A\Lambda^2)^2$ and in $\ln A\Lambda^2$ which leads to the two-loop contribution for γ_{str} . We drop the terms proportional to $A\Lambda^2$ since they will anyway be reabsorbed in the cosmological constant, and the constant terms since they vanish when considering $\ln \frac{Z[A]}{Z[A_0]}$.

The figure-eight diagram

First, we have the figure-eight diagram:

$$\begin{aligned} I_{\infty} &= \frac{4\pi\varepsilon}{\kappa^2} \int dx \left[-6 \widehat{K}(t_1, x, x) \widetilde{K}(t_2, x, x) + 5 \text{T} \widehat{K}(t_1, x, x) \widehat{K}(t_2, x, x) \right], \\ &= \frac{\varepsilon}{\kappa^2} \left\{ \frac{3}{2\pi} \left(-\frac{A\Lambda^2}{\alpha_1} + A\text{T} + 4\pi \right) \ln A\Lambda^2 + \frac{5}{4\pi} A\text{T} \left[(\ln A\Lambda^2)^2 + 2(G_0 - \ln \alpha_1) \ln A\Lambda^2 \right] \right\} \end{aligned} \quad (3.1.2)$$

with G_0 defined in the previous chapter in (2.1.54). Remember that this expression has to be understood with the implicit integration $\int_0^\infty d\alpha_1 \varphi(\alpha_1)$. We insist also on the fact that the coefficients in each \widetilde{K} and \widehat{K} are changed due to the new propagator (3.0.7) and not only the coefficients in front of each integral. Note also that the leading logarithmic divergence $(\ln A\Lambda^2)^2$, which was vanishing when considering the summed contributions in the pure Liouville theory appears through this diagram. As already discussed in the previous chapters, this two-loop divergence has to vanish in order to ensure the background independence of our results, but also the possibility to renormalize through only local (or “measure-like”) counterterms.

The setting sun diagram

Then, we have the setting sun diagram:

$$\begin{aligned}
I_{\ominus} &= \frac{4\pi\varepsilon}{\kappa^2} \int dx dy \widehat{K}(t_1, x, y) \left[- \widehat{K}(t_2, x, y) \frac{d}{dt} \widetilde{K}(t, x, y) \Big|_{t=t_3} + 2 \widetilde{K}(t_2, x, y) \widetilde{K}(t_3, x, y) \right. \\
&\quad \left. - 4 \text{T} \widehat{K}(t_2, x, y) \widetilde{K}(t_3, x, y) \right], \\
&= \frac{\varepsilon}{\kappa^2} \left\{ \frac{1}{\pi} \left(\frac{A\Lambda^2}{\alpha_1 + \alpha_2} + \frac{AT}{2} \left(\frac{2\alpha_1\alpha_2}{(\alpha_1 + \alpha_2)^2} - 1 \right) - 6\pi \right) \ln A\Lambda^2 \right. \\
&\quad \left. - \frac{5}{4\pi} AT \left((\ln A\Lambda^2)^2 + 2(G_0 - \ln(\alpha_1 + \alpha_2)) \ln A\Lambda^2 \right) \right\}, \quad (3.1.3)
\end{aligned}$$

whose contribution to the leading logarithmic divergence $(\ln A\Lambda^2)^2$ is precisely the one required to cancel the contribution from the figure-eight diagram $\bigcirc\bigcirc$.

The glasses diagram

Similarly to the pure Liouville theory case, the glasses diagram:

$$\begin{aligned}
I_{\bigcirc\bigcirc} &= \frac{4\pi\varepsilon}{\kappa^2} \int dx dy \widehat{K}(t_1, x, x) \widehat{K}(t_2, y, y) \times \\
&\quad \left[- \frac{1}{2} \frac{d}{dt} \widetilde{K}(t, x, y) \Big|_{t=t_3} - 2 \text{T} \widetilde{K}(t_3, x, y) + 2 \text{T}^2 \widehat{K}(t_3, x, y) \right] \quad (3.1.4)
\end{aligned}$$

only contributes through constant or $\mathcal{O}(1/\Lambda^2)$ terms.

The measure diagram

Finally, we have the measure diagram, whose contribution to γ_{str} and to the diverging part of the two-loop partition function (up to terms proportional to the cosmological constant) is simply:

$$I_{\bigcirc} = \frac{4\pi\varepsilon}{\kappa^2} \int dx \widetilde{K}(t_1, x, x) \widehat{K}(t_2, x, x) = \frac{\varepsilon}{\kappa^2} \left\{ \frac{1}{4\pi} \left(\frac{A\Lambda^2}{\alpha_1} - AT - 4\pi \right) \ln A\Lambda^2 \right\}. \quad (3.1.5)$$

3.1.2 Total contribution to the two-loop partition function

Summing up the contributions (3.1.2)-(3.1.5) of these diagrams, we get the total contribution to the two-loop (order ε) partition function:

$$\begin{aligned}
\ln Z[A]_{2\text{-loop}}^{\text{L+M}} &= \frac{\varepsilon}{\kappa^2} \left\{ \frac{1}{2\pi} \left(\frac{2}{\alpha_1 + \alpha_2} - \frac{5}{2\alpha_1} \right) A\Lambda^2 \ln A\Lambda^2 \right. \\
&\quad \left. + \left[-1 + 8 \frac{\beta^2}{\kappa^2} \left(3 - 5 \ln \frac{\alpha_1\alpha_2}{(\alpha_1 + \alpha_2)^2} + \frac{4\alpha_1\alpha_2}{(\alpha_1 + \alpha_2)^2} \right) \right] \ln A\Lambda^2 \right\} \quad (3.1.6)
\end{aligned}$$

where we did not include neither the constant terms nor those proportional to $A\Lambda^2$. At this point, we should make a few remarks. First, the leading (non-vanishing) divergence in eq. (3.1.6),

$$\frac{\varepsilon}{2\pi\kappa^2} \left(\frac{2}{\alpha_1 + \alpha_2} - \frac{5}{2\alpha_1} \right) A\Lambda^2 \ln A\Lambda^2 \quad (3.1.7)$$

is the same as in the pure Liouville theory, as it will be the case for the leading three-loop divergence in section 3.2. This is not surprising since the Mabuchi action (or its loop expansion), missing one Laplacian in comparison to the Liouville action, only modifies the subleading terms, through the coefficient $T = \frac{32\pi}{A} \frac{\beta^2}{\kappa^2}$. We are then once more led to introduce counterterms in order to make this partition function finite and regulator independent.

Secondly, the (leading) logarithmic divergence $(\ln A\Lambda^2)^2$ is once again absent in the partition function, although present in the individual diagrams. This was also the case for the pure Liouville theory. Contrary to the leading divergence just discussed, the coefficients of this leading logarithmic divergence $(\ln A\Lambda^2)^2$ involve $AT = 32\pi \frac{\beta^2}{\kappa^2}$ and thus are at present generated from the Mabuchi action. Moreover, in this combined Liouville plus Mabuchi theory, not only the vertices but also the heat kernel coefficients are changed. That the coefficients of the $(\ln A\Lambda^2)^2$ again “miraculously” cancel in the end was all but obvious in the beginning.

3.1.3 Counterterm contributions

We choose to only introduce local or non-local “measure-like” counterterms in $1/A$. Thus, we consider the following action for the counterterms:

$$S_{\text{ct}} = \frac{8\pi\varepsilon}{\kappa^2} \frac{1}{2} \int d^2x \sqrt{g_*} \left[c_\phi \tilde{\phi} (\Delta_* + T) \tilde{\phi} + c_T T \tilde{\phi}^2 + c_m \tilde{\phi}^2 \right] \quad (3.1.8)$$

with

$$c_\phi = c_\phi^{(1)}, \quad c_T = c_T^{(1)}, \quad c_m = c_m^{(1)} \Lambda^2 + \frac{c_m^{(2)}}{A}. \quad (3.1.9)$$

In order to simplify the notation, we do not write the subscript $L + M$ over the counterterms. To determine these counterterms, as previously we compute the full one-loop (order ε) two-point function $\mathcal{G}^{(2)}(u, v)$ (at non coinciding points) and we require it to be both finite and regularization independent.

One-loop two-point function

As previously, we have to compute five diagrams:

$$\begin{aligned} \text{shaded circle} &= \text{unshaded circle} + \text{shaded top circle} + \text{unshaded top circle} + \text{unshaded top circle with vertical line} \\ &+ \text{line with dot} + \text{line with CT square} \end{aligned}$$

We are only interested in the diverging and regulator-dependent part of $\mathcal{G}^{(2)}(u, v)$. Thus, the result for $\mathcal{G}^{(2)}(u, v)$ and the contributions per diagram given hereafter are written without their finite regulator independent parts. The computations themselves are very similar to those done in the pure Liouville theory case and those details are given in appendix B.1. Thus, we only give the results here. As already emphasized, the expressions (1.3.34) and (1.3.42) for $\tilde{K}(t, x, y)$ (and thus for $\frac{d}{dt}\tilde{K}(t, x, y)$) and $\widehat{K}(t, x, y)$ are to be considered with the coefficients (3.0.9).

$$\begin{aligned} \mathcal{G}_{u\text{-}O\text{-}v}^{(2)}(u, v) = \frac{\varepsilon}{\kappa^2} & \left\{ \tilde{G}(u, v) \left[12 \ln A \Lambda^2 - 4 \left(1 + \frac{2\alpha_1\alpha_2}{(\alpha_1 + \alpha_2)^2} + 3 \ln(\alpha_2 + \alpha_3) \right) \right] \right. \\ & + \int dx \widehat{K}(t_1, u, x) \widehat{K}(t_2, x, v) \left(-20 \text{T} (\ln A \Lambda^2 + G_0(x) - \ln(\alpha_3 + \alpha_4)) \right. \\ & \left. \left. + \frac{8\Lambda^2}{\alpha_3 + \alpha_4} + \text{T} \left(-4 + \frac{8\alpha_1\alpha_2}{(\alpha_1 + \alpha_2)^2} \right) - \frac{80\pi}{A} \right) \right\}, \quad (3.1.10) \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{u\text{-}O\text{-}v}^{(2)}(u, v) = \frac{\varepsilon}{\kappa^2} & \left\{ -12 \tilde{G}(u, v) (\ln A \Lambda^2 - \ln \alpha_3) \right. \\ & + \int dx \widehat{K}(t_1, u, x) \widehat{K}(t_2, x, v) \left(20 \text{T} (\ln A \Lambda^2 + G_0(x) - \ln \alpha_3) \right. \\ & \left. \left. - 12 \frac{\Lambda^2}{\alpha_3} + 12 \text{T} + \frac{48\pi}{A} \right) \right\}, \quad (3.1.11) \end{aligned}$$

$$\mathcal{G}_{\text{measure}}^{(2)}(u, v) = \frac{\varepsilon}{\kappa^2} \int dx \widehat{K}(t_1, u, x) \widehat{K}(t_2, x, v) \left(2 \frac{\Lambda^2}{\alpha_3} - 2 \text{T} - \frac{8\pi}{A} \right), \quad (3.1.12)$$

$$\begin{aligned} \mathcal{G}_{\text{ct}}^{(2)}(u, v) = \frac{\varepsilon}{\kappa^2} & \left\{ -8\pi c_\phi^{(1)} \tilde{G}(u, v) \right. \\ & \left. - 8\pi \left(c_{\text{T}}^{(1)} \text{T} + c_m^{(1)} \Lambda^2 + \frac{c_m^{(2)}}{A} \right) \int dx \widehat{K}(t_1, u, x) \widehat{K}(t_2, x, v) \right\}, \quad (3.1.13) \end{aligned}$$

with

$$G_0(x) = 4\pi \tilde{G}_\zeta^{A_0}(x) - \gamma - \ln A_0 \mu^2. \quad (3.1.14)$$

The diagram  has neither diverging nor regulator dependent parts, and is therefore omitted in the following.

Summing up eq. (3.1.10)-(3.1.13), we get the diverging and regulator dependent parts of the full one-loop two-point function:

$$\begin{aligned} \mathcal{G}^{(2)}(u, v) = & \frac{4\varepsilon}{\kappa^2} \left\{ \tilde{G}(u, v) \left[\frac{3}{2} \ln \frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2} - 1 - \frac{2\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2} - 2\pi c_\phi^{(1)} \right] \right. \\ & + \int dx \widehat{\widehat{K}}(t_1, u, x) \widehat{\widehat{K}}(t_2, x, v) \left[\Lambda^2 \left(\frac{2}{\alpha_1 + \alpha_2} - \frac{5}{2\alpha_1} - 2\pi c_m^{(1)} \right) \right. \\ & \left. \left. + \text{T} \left(-\frac{5}{2} \ln \frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2} + \frac{3}{2} + \frac{2\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2} - 2\pi c_T^{(1)} \right) - \frac{10\pi + 2\pi c_m^{(2)}}{A} \right] \right\}. \end{aligned} \quad (3.1.15)$$

Finiteness and regulator dependence partly fix the counterterms. Moreover, imposing the “strong locality condition” also fixes $c_m^{(2)} = -1$ such that we get:

$$\begin{aligned} c_\phi &= \frac{1}{2\pi} \left[\frac{3}{2} \ln \frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2} - 1 - \frac{2\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2} \right] + \widehat{c}_\phi, \\ c_T &= \frac{1}{2\pi} \left[-\frac{5}{2} \ln \frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2} + \frac{3}{2} + \frac{2\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2} \right] + \widehat{c}_T, \\ c_m &= \frac{\Lambda^2}{2\pi} \left(\frac{2}{\alpha_1 + \alpha_2} - \frac{5}{2\alpha_1} \right) - \frac{1}{A}. \end{aligned} \quad (3.1.16)$$

Note that both c_ϕ and c_m take the same value as in the case of the pure Liouville theory. This was expected, since the Mabuchi action comes with a coupling β^2 only present through the T coefficient carried in the counterterm action by c_T .

Two-loop partition function

The counterterms contribute to the partition function through the diagram  :

$$I_{\bigcirc} = -\frac{4\pi\varepsilon}{\kappa^2} \int dx \left[c_\phi \widehat{\widehat{K}}(t, x, x) + (c_T \text{T} + c_m) \widehat{\widehat{K}}(t, x, x) \right], \quad (3.1.17)$$

such that

$$\begin{aligned} W^{(2)}[A] \Big|_{\text{M+L}} &\equiv \ln Z[A]_{2\text{-loop}}^{\text{M+L}} = \frac{\varepsilon}{\kappa^2} \left\{ \frac{1}{2\pi} \left(\frac{2}{\alpha_1 + \alpha_2} - \frac{5}{2\alpha_1} - 2\pi c_m^{(1)} \right) A \Lambda^2 \ln A \Lambda^2 \right. \\ &+ \left[-1 - c_m^{(2)} + 8 \frac{\beta^2}{\kappa^2} \left(3 - 5 \ln \frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2} + \frac{4\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2} - 4\pi c_T^{(1)} \right) \right] \ln A \Lambda^2 \left. \right\} \\ &= -32\pi \frac{\varepsilon \beta^2}{\kappa^4} \widehat{c}_T \end{aligned} \quad (3.1.18)$$

with the values (3.1.16). In the pure Liouville theory, the two-loop contribution to γ_{str} is proportional to $(1-h)$ (see (2.1.106)):

$$\gamma_{\text{str}}^{(2)} = -\frac{4}{\kappa^2} \widehat{c}_R (1-h). \quad (3.1.19)$$

Then, the Liouville contribution to γ_{str} is vanishing on the torus and it is not surprising that our result (3.1.18) is simply proportional to β^2 , the coupling constant to the Mabuchi action. It is remarkable that in this (simplified) Liouville and Mabuchi theory on the torus, the requirement of finiteness and regularization independence of the two-point function is once more sufficient to ensure the finiteness and regularization independence of the partition function (at least at two loops). Note that the counterterms we introduced are either local or non-local in $1/A$ but satisfying the “strong locality condition” that the global action – including the counterterms, the Liouville and Mabuchi actions and the contribution from the measure – does not contain any non-local term.

Once more, at this order in the perturbation theory, we end up with two free finite and regularization independent constants, \widehat{c}_ϕ and \widehat{c}_T , with only one contributing to the string susceptibility.

3.2 Three-loop discussion

We present now the extension to the Liouville and Mabuchi theory on the torus of the results of the three-loop study carried on in the previous chapter for the pure Liouville theory. As already discussed, the leading divergence remains the same. Thus, we will only check in the following that the leading logarithmic divergence in $(\ln A\Lambda^2)^3$ cancels between the diagrams. Indeed, this divergence – which comes together with unwanted background dependent terms – can not be absorbed by the addition of local (or non-local but measure-like) counterterms alone. Thus, this cancellation is a requirement for the consistency of our theory.

Then, we present the contributions of the counterterms to both the three-loop partition function and to the diverging and regulator dependent part of the two-loop two-point function (order ε^2). Indeed, the diverging part of the three- and four-point functions are unchanged and fix uniquely both $f_m^{(1)}$ and $q_m^{(1)}$ as defined in the previous chapter, thus, we only discuss the two-point function in the following.

First, let us write the genuine three-loop (*i.e.* order ε^2) vertices for our Liouville and Mabuchi theory. From eq. (2.2.1) and (3.0.11), one reads the vertices that appear in addition to (3.1.1):

$$\begin{aligned}
 \star &= -\frac{(8\pi\varepsilon)^2}{\kappa^4} \frac{4}{5} \left(\Delta_* + \frac{1}{6} T \right), & \text{---} \times &= -\frac{(8\pi\varepsilon)^2}{\kappa^4} \frac{1}{2} \Delta_*, & \text{---} \times &= -\frac{(8\pi\varepsilon)^2}{\kappa^4} \frac{2}{9} \Delta_*, \\
 \star &= -\frac{16(\pi\varepsilon)^{3/2}}{\kappa^3} \left(\Delta_* + \frac{1}{5} T \right), & \text{---} \times &= -\frac{16(\pi\varepsilon)^{3/2}}{\kappa^3} \frac{2}{3} \Delta_*, & & (3.2.1) \\
 \times &= \frac{(8\pi\varepsilon)^2}{\kappa^4} \frac{1}{2} \widetilde{K}(t, x, x), & \text{---} \text{---} &= \frac{16(\pi\varepsilon)^{3/2}}{\kappa^3} \frac{2}{3} \widetilde{K}(t, x, x).
 \end{aligned}$$

3.2.1 Cancellation of the $(\ln A\Lambda^2)^3$ divergence

Similarly to what happened in the pure Liouville case on surfaces of any genus, the local or measure-like counterterms one can introduce do not contribute to the leading

logarithmic divergence $(\ln A\Lambda^2)^3$. Thus, for consistency, we computed its contribution in each of the 29 types of diagrams at three-loop (see the previous chapter). Indeed, although the coefficients in the vertices and in the regularized propagator are modified in the Liouville plus Mabuchi theory on the torus, their structure remains the same and we can build exactly the same diagram than in the previous study. These diagrams contribute the same leading divergence but their subleading terms differ. However, once more, the “miracle” occurs and the $(\ln A\Lambda^2)^3$ divergence cancels between the diagrams, as one can read in Tab. 3.1.

									Total
$\frac{\varepsilon^2 A_T}{\pi \kappa^4} (\ln A\Lambda^2)^3$	28	-26	-30	-90	80	$136 + \frac{2}{3}$	$-52 - \frac{4}{3}$	$-44 - \frac{4}{3}$	0
$\frac{8(1-h)}{\kappa^4} (\ln A\Lambda^2)^3$	-20	18	18	60	-42	-86	$26 + \frac{4}{3}$	$24 + \frac{2}{3}$	0

Table 3.1: $(\ln A\Lambda^2)^3$ contributions per diagrams. The first line corresponds to the Liouville and Mabuchi theory on the torus whereas the second line corresponds to the Liouville theory on surfaces of any genus.

Let us stress that this cancellation was once more by no means obvious and that it is crucial for the consistency of our theory. Similarly to the “pure” Liouville theory, we expect this to happen for the leading logarithmic divergence at each order in ε .

3.2.2 Counterterm contributions

We introduce the same counterterm actions as in the previous chapter. Namely, inserting the undetermined counterterms from the two-loop computation, we have:

$$\begin{aligned}
S_{\text{ct}}^{\text{quad}'} &= \frac{8\pi\varepsilon}{\kappa^2} \frac{1}{2} \int dx \left[\left(\widehat{c}_\phi + \frac{\varepsilon}{\kappa^2} c'_\phi \right) \widetilde{\phi} (\Delta_* + \text{T}) \widetilde{\phi} + \left(\widehat{c}_\text{T} + \frac{\varepsilon}{\kappa^2} c'_\text{T} \right) \text{T} \widetilde{\phi}^2 + \frac{\varepsilon}{\kappa^2} c'_m \widetilde{\phi}^2 \right], \\
S_{\text{ct}}^{\text{c}} &= \frac{16(\pi\varepsilon)^{3/2}}{\kappa^3} \frac{1}{2} \int dx \left[f_\phi \widetilde{\phi}^2 (\Delta_* + \text{T}) \widetilde{\phi} + f_\text{T} \text{T} \widetilde{\phi}^3 + f_m \widetilde{\phi}^3 \right], \\
S_{\text{ct}}^{\text{q}} &= \frac{(8\pi\varepsilon)^2}{\kappa^4} \frac{1}{2} \int dx \left[q_\phi \widetilde{\phi}^3 (\Delta_* + \text{T}) \widetilde{\phi} + \widehat{q}_\phi \widetilde{\phi}^2 (\Delta_* + 2\text{T}) \widetilde{\phi}^2 + q_\text{T} \text{T} \widetilde{\phi}^4 + q_m \widetilde{\phi}^4 \right].
\end{aligned} \tag{3.2.2}$$

Once again, we include the $\frac{\varepsilon}{\kappa^2}$ terms in the expansion of the quadratic counterterms. These counterterms may be expanded as:

$$\begin{aligned}
c'_\phi &= c_\phi^{(1)}, & f_\phi &= f_\phi^{(1)}, & q_\phi &= q_\phi^{(1)}, \\
c'_\text{T} &= c_\text{T}^{(1)}, & f_\text{T} &= f_\text{T}^{(1)}, & \widehat{q}_\phi &= \widehat{q}_\phi^{(1)}, \\
c'_m &= c_m^{(1)} \Lambda^2 + \frac{c_m^{(2)}}{A}, & f_m &= f_m^{(1)} \Lambda^2 + \frac{f_m^{(2)}}{A}, & q_\text{T} &= q_\text{T}^{(1)}, \\
&&&&& q_m &= q_m^{(1)} \Lambda^2 + \frac{q_m^{(2)}}{A},
\end{aligned} \tag{3.2.3}$$

where we recall that \widehat{c}_ϕ and \widehat{c}_T are regulator independent constants.

Three-loop vacuum diagrams

The diagrams we have to consider are the following: . Indeed, the “glasses-like” diagrams only contribute to the finite part of the partition function or through terms proportional to $A\Lambda^2$ that would be included in the cosmological constant. The contributions per diagrams are given hereafter, up to these irrelevant terms. First, we have the “measure-like” diagrams:

$$\begin{aligned} I_{\bigcirc} &= -\frac{4\pi\varepsilon^2}{\kappa^4} \int dx (c'_T T + c'_m) \widehat{K}(t, x, x) \\ &= -\frac{\varepsilon^2}{\kappa^4} \left\{ c'_m{}^{(1)} A\Lambda^2 \ln A\Lambda^2 + (c'_m{}^{(2)} + c'_T{}^{(1)} AT) \ln A\Lambda^2 \right\}, \end{aligned} \quad (3.2.4)$$

$$\begin{aligned} I_{\bigcirc\bigcirc} &= -\frac{\widehat{c}_\phi (8\pi\varepsilon)^2}{2 \kappa^4} \int dx \widetilde{K}(t_1, x, x) \widehat{K}(t_2, x, y) \widetilde{K}(t_3, x, y) \\ &= \frac{\varepsilon^2}{\kappa^4} \left\{ -\frac{2}{\alpha_1} \widehat{c}_\phi A\Lambda^2 \ln A\Lambda^2 + 2 (AT + 4\pi) \widehat{c}_\phi \ln A\Lambda^2 \right\}, \end{aligned} \quad (3.2.5)$$

$$\begin{aligned} I_{\bigcirc\bigcirc\bigcirc} &= \frac{1}{2} \frac{(8\pi\varepsilon)^2}{\kappa^4} \int dx c_\phi (c_T T + c_m) \widehat{K}(t_1, x, y) \widetilde{K}(t_2, x, y) \\ &= \frac{\varepsilon^2}{\kappa^4} \left\{ \left(-\frac{10}{\alpha_1} + \frac{8}{\alpha_1 + \alpha_2} \right) \widehat{c}_\phi A\Lambda^2 \ln A\Lambda^2 + \ln A\Lambda^2 \left[8\pi AT \widehat{c}_\phi \widehat{c}_T + 2 \left(\frac{1}{4} AT - 4\pi \right) \widehat{c}_\phi \right. \right. \\ &\quad \left. \left. + 4AT \left((5\widehat{c}_\phi - 3\widehat{c}_T) (\ln(\alpha_1 + \alpha_2) - \ln \alpha_1) + (\widehat{c}_\phi - \widehat{c}_T) \left(1 + \frac{2\alpha_1\alpha_2}{(\alpha_1 + \alpha_2)^2} \right) \right) \right] \right\}, \end{aligned} \quad (3.2.6)$$

where we used the values (3.1.16) for the counterterms c_ϕ , c_T and c_m and kept only the parts involving one (or more) counterterm. Then, the “figure-eight” diagrams:

$$\begin{aligned} I_{\bigcirc\bigcirc} &= \frac{(8\pi\varepsilon)^2}{\kappa^4} \int dx dy \left\{ (-5\widehat{c}_\phi + 3\widehat{c}_T) T \widehat{K}(t_1, x, x) \widehat{K}(t_2, x, y) \widetilde{K}(t_3, x, y) \right. \\ &\quad \left. + \frac{3}{2} \widehat{c}_\phi \widehat{K}(t_1, x, x) \left[-\widehat{K}(t_2, x, y) \frac{d}{dt} \widetilde{K}(t, x, y) \Big|_{t=t_3} + \widetilde{K}(t_2, x, y) \widetilde{K}(t_3, x, y) \right] \right. \\ &\quad \left. + 3\widehat{c}_\phi \widetilde{K}(t_1, x, x) \widehat{K}(t_2, x, y) \widetilde{K}(t_3, x, y) - 5\widehat{c}_T T^2 \widehat{K}(t_1, x, x) \widehat{K}(t_2, x, y) \widetilde{K}(t_3, x, y) \right\} \\ &= \frac{\varepsilon^2}{\kappa^4} \left\{ 12 \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_1 + \alpha_2} \right) \widehat{c}_\phi A\Lambda^2 \ln A\Lambda^2 + (12\widehat{c}_T - 20\widehat{c}_\phi) AT (\ln A\Lambda^2)^2 \right. \\ &\quad \left. + \left[(12\widehat{c}_T - 20\widehat{c}_\phi) AT (2G_0 - \ln \alpha_1 - \ln(\alpha_1 + \alpha_2)) - 24 (AT + 4\pi) \widehat{c}_\phi \right. \right. \\ &\quad \left. \left. - 80\pi T^2 \widehat{c}_T \int dx dy \widehat{K}(t_1, x, y) \widehat{K}(t_2, x, y) \right] \ln A\Lambda^2 \right\}, \end{aligned} \quad (3.2.7)$$

and

$$\begin{aligned}
I_{\infty} &= -\frac{(8\pi\varepsilon)^2}{\kappa^4} \int dx \left[\left(\frac{3}{2} q_\phi + 2 \hat{q}_\phi \right) \widehat{K}(t_1, x, x) \widetilde{K}(t_2, x, x) \right. \\
&\quad \left. + \left(\hat{q}_\phi T + \frac{3}{2} (q_T T + q_m) \right) \widehat{K}(t_1, x, x) \widehat{K}(t_2, x, x) \right], \\
&= -\frac{\varepsilon^2}{\kappa^4} \left\{ 6 q_m^{(1)} A \Lambda^2 (\ln A \Lambda^2)^2 + \left[\frac{1}{\alpha_1} (6 q_\phi^{(1)} + 8 \hat{q}_\phi^{(1)}) + 12 q_m^{(1)} (G_0 - \ln \alpha_1) \right] A \Lambda^2 \ln A \Lambda^2 \right. \\
&\quad + \left[6 q_m^{(2)} + A T (4 \hat{q}_\phi^{(1)} + 6 q_T^{(1)}) \right] (\ln A \Lambda^2)^2 - \left[(A T + 4\pi) (6 q_\phi^{(1)} + 8 \hat{q}_\phi^{(1)} - 12 \alpha_1 q_m^{(1)}) \right. \\
&\quad \left. \left. - 2 \left(6 q_m^{(2)} + A T \left(4 \hat{q}_\phi^{(1)} + 6 q_T^{(1)} \right) \right) (G_0 - \ln \alpha_1) \right] \ln A \Lambda^2 \right\}. \quad (3.2.8)
\end{aligned}$$

with $G_0 = \frac{4\pi}{A_0} \int d^2x \sqrt{g_0(x)} \widetilde{G}_\zeta^{A_0}(x) - \gamma - \ln A_0 \mu^2$ given in (2.1.54). Then, finally we have the two “setting sun” diagrams. As previously done for the other diagrams, the first diagram can be written in terms of the regularized propagators and (thus in terms of the heat kernels) as:

$$\begin{aligned}
I_{\oplus} &= \frac{(8\pi\varepsilon)^2}{\kappa^4} \int dx dy dz \left\{ 4 \hat{c}_T T^2 \widetilde{K}(t_1, x, y) \widehat{K}(t_2, x, y) \widehat{K}(t_3, x, z) \widehat{K}(t_4, y, z) \right. \\
&\quad - \left(\hat{c}_\phi \widetilde{K}(t_3, x, z) + \hat{c}_T T \widehat{K}(t_3, x, z) \right) \widehat{K}(t_4, y, z) \times \\
&\quad \left[\widetilde{K}(t_1, x, y) \widetilde{K}(t_2, x, y) - \widehat{K}(t_1, x, y) \frac{d\widetilde{K}}{dt_2}(t_2, x, y) \right] \\
&\quad + T \widehat{K}(t_1, x, y) \widehat{K}(t_2, x, y) \left[\widehat{C} \widetilde{K}(t_3, x, z) \widetilde{K}(t_4, y, z) - \hat{c}_\phi \widehat{K}(t_3, x, z) \frac{d\widetilde{K}}{dt_4}(t_4, y, z) \right] \\
&\quad + \widetilde{K}(t_1, x, y) \widetilde{K}(t_3, x, z) \left[4 \widehat{C} T \widehat{K}(t_2, x, y) \widehat{K}(t_4, y, z) - \hat{c}_\phi \widehat{K}(t_2, x, y) \widetilde{K}(t_4, y, z) \right] \\
&\quad \left. + \hat{c}_\phi \widehat{K}(t_1, x, y) \frac{d\widetilde{K}}{dt_3} \widetilde{K}(t_3, x, z) \left[\widetilde{K}(t_2, x, y) \widehat{K}(t_4, y, z) + \frac{1}{2} \widehat{K}(t_2, x, y) \widetilde{K}(t_4, y, z) \right] \right\}, \quad (3.2.9)
\end{aligned}$$

where we define $\widehat{C} = \hat{c}_\phi - \frac{\hat{c}_T}{2}$ for the sake of brevity.

Then, we get

$$\begin{aligned}
I_{\ominus} = \frac{\varepsilon^2}{\kappa^4} & \left\{ -8 \left(\frac{1}{\alpha_1 + \alpha_2} + \frac{2}{\alpha_1 + \alpha_2 + \alpha_3} \right) \widehat{c}_\phi A\Lambda^2 \ln A\Lambda^2 + (30 \widehat{c}_\phi - 14 \widehat{c}_T) AT (\ln A\Lambda^2)^2 \right. \\
& + \ln A\Lambda^2 \left[80 \pi T^2 \widehat{c}_T \int dx dy \widehat{K}(t_1, x, y) \widehat{K}(t_2, x, y) + 2 G_0 (30 \widehat{c}_\phi - 14 \widehat{c}_T) AT \right. \\
& - 4 AT \left[(\widehat{c}_\phi - \widehat{c}_T) \left(1 + \frac{2\alpha_1\alpha_2}{(\alpha_1 + \alpha_2)^2} \right) - (10 \widehat{c}_\phi - 4 \widehat{c}_T) \ln(\alpha_1 + \alpha_2 + \alpha_3) \right. \\
& \left. \left. \left. - (5 \widehat{c}_\phi - 3 \widehat{c}_T) \ln(\alpha_1 + \alpha_2) \right] + 4 \widehat{c}_\phi \left[36\pi + AT \left(\frac{8}{3} + \frac{4\alpha_1^2}{(\alpha_1 + \alpha_2 + \alpha_3)^2} \right) \right] \right] \left. \right\}, \tag{3.2.10}
\end{aligned}$$

and

$$\begin{aligned}
I_{\ominus} = \frac{(8\pi\varepsilon)^2}{\kappa^4} & \int dx dy \left\{ \left[-f_\phi T + \frac{3}{2} (f_T T + f_m) \right] \widehat{K}(t_1, x, y) \widehat{K}(t_2, x, y) \widetilde{K}(t_3, x, y) \right. \\
& \left. + f_\phi \widehat{K}(t_1, x, y) \left[-\frac{1}{2} \widehat{K}(t_2, x, y) \frac{d}{dt} \widetilde{K}(t, x, y) \Big|_{t=t_3} + \widetilde{K}(t_2, x, y) \widetilde{K}(t_3, x, y) \right] \right\}, \\
= \frac{\varepsilon^2}{\kappa^4} & \left\{ 6 f_m^{(1)} A\Lambda^2 (\ln A\Lambda^2)^2 + \left[\frac{8}{\alpha_2 + \alpha_3} f_\phi^{(1)} + 12 f_m^{(1)} (G_0 - \ln(\alpha_2 + \alpha_3)) \right] A\Lambda^2 \ln A\Lambda^2 \right. \\
& + 6 \left[f_m^{(2)} + AT (-f_\phi^{(1)} + f_T^{(1)} + \alpha_3 f_m^{(1)}) \right] (\ln A\Lambda^2)^2 + \left[-48\pi (f_\phi^{(1)} - 2\alpha_3 f_m^{(1)}) \right. \\
& + (-4 f_\phi^{(1)} + 12\alpha_3 f_m^{(1)}) AT + \frac{\alpha_2\alpha_3}{\alpha_2 + \alpha_3} \left(\frac{8}{\alpha_2 + \alpha_3} f_\phi^{(1)} + 12 f_m^{(1)} \right) AT \\
& \left. \left. \left. + 12 \left(f_m^{(2)} + AT (-f_\phi^{(1)} + f_T^{(1)} + \alpha_3 f_m^{(1)}) \right) (G_0 - \ln(\alpha_2 + \alpha_3)) \right] \ln A\Lambda^2 \right\}. \tag{3.2.11}
\end{aligned}$$

Total contribution of the counterterms to the two-loop partition function

Summing the previous contributions (3.2.4)-(3.2.11), one gets:

$$W^{(3)}[A] \Big|_{\text{ct}} = \frac{\varepsilon^2}{\kappa^4} \left\{ \Omega_1 A\Lambda^2 (\ln A\Lambda^2)^2 + \Omega_2 A\Lambda^2 \ln A\Lambda^2 + \Omega_3 (\ln A\Lambda^2)^2 + \Omega_4 \ln A\Lambda^2 \right\}, \tag{3.2.12}$$

with

$$\Omega_1 = \Omega_1^{\text{Liouville}}, \quad \Omega_2 = \Omega_2^{\text{Liouville}}, \tag{3.2.13}$$

and

$$\begin{aligned}
\Omega_3 &= \Omega_3^{(a)} + \Omega_3^{(b)} + \Omega_3^{(c)} , \\
\Omega_3^{(a)} &= -6 q_m^{(2)} - AT \left(4 \widehat{q}_\phi^{(1)} + 6 q_T^{(1)} \right) , \\
\Omega_3^{(b)}(\alpha_1) &= 6 f_m^{(2)} + AT \left(-6 f_\phi^{(1)} - 10 \widehat{c}_\phi + 6 \widehat{c}_T + 6 f_T^{(1)} + 6 \alpha_1 f_m^{(1)} \right) , \\
\Omega_3^{(c)} &= AT (20 \widehat{c}_\phi - 8 \widehat{c}_T) , \\
\Omega_4 &= 2 \Omega_3 G_0 + \Omega_4^{(a)} + 48\pi \widehat{c}_\phi + 8\pi AT \widehat{c}_\phi \widehat{c}_T , , \\
\Omega_4^{(a)} &= -2 \left(\Omega_3^{(a)} \ln \alpha_1 + \Omega_3^{(b)}(\alpha_1) \ln(\alpha_1 + \alpha_2) + \Omega_3^{(c)} \ln(\alpha_1 + \alpha_2 + \alpha_3) \right) \\
&\quad + (4\pi + AT) \left(6 q_\phi^{(1)} + 8 \widehat{q}_\phi^{(1)} + 12 \alpha_1 f_m^{(1)} - 12 f_\phi^{(1)} + 2 \alpha_1 \Omega_1 \right) \\
&\quad - AT \left(\frac{28}{3} \widehat{c}_\phi - 8 f_\phi^{(1)} + \frac{12 \alpha_1^2}{\alpha_1 + \alpha_2} f_m^{(1)} - \frac{8 \alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2} f_\phi^{(1)} - \frac{16 \alpha_1^2}{(\alpha_1 + \alpha_2 + \alpha_3)^2} \widehat{c}_\phi \right) \\
&\quad - \left(c_m'^{(2)} + AT c_T'^{(1)} \right) . \tag{3.2.14}
\end{aligned}$$

Of course, only the parts involving the coupling to the Mabuchi action β^2 , that is to say $AT = 32\pi \frac{\beta^2}{\kappa^2}$ are modified with respect to the pure Liouville theory. However, as for the pure Liouville theory, these counterterms for the Liouville plus Mabuchi theory do not contribute to the leading logarithmic divergence at three loops $(\ln A\Lambda^2)^3$. Then, its “magic” cancellation through the diagrams (see Tab. 3.1) was indeed crucial for the consistency of our theory.

Finiteness of the two-point function

As previously stated, the diverging parts of the three- and four-point function are not changed with respect to the pure Liouville theory by the inclusion of the Mabuchi action, when studying the torus. Thus, in order to look at the freedom of the three-loop counterterms in (3.2.14), it is sufficient to give their diverging contributions to the two-point function only. We have

$$\begin{aligned}
\text{---} \bigcirc \text{---} \Big|_{\varepsilon^2, \text{div}}^{\text{CT}} &= \frac{8\pi\varepsilon^2}{\kappa^4} \left\{ \left(\rho_1 \Lambda^2 \ln A\Lambda^2 + \rho_2 \Lambda^2 + \rho_3 \frac{\ln A\Lambda^2}{A} \right) \int dx \widehat{K}(t_1, a, x) \widehat{K}(t_2, b, x) \right. \\
&\quad + \rho_4 \Lambda^2 \int dx \widetilde{G}_\zeta^{A_0}(x) \widehat{K}(t_1, a, x) \widehat{K}(t_2, b, x) + \rho_5 \ln A\Lambda^2 \widehat{K}(t_1, a, b) \\
&\quad \left. + 24\pi f_m^{(1)} \Lambda^2 H^M(a, b) \right\} , \tag{3.2.15}
\end{aligned}$$

with

$$\begin{aligned}
\rho_1 &= \rho_1^{\text{Liouville}} , \quad \rho_2 = \rho_2^{\text{Liouville}} , \quad \rho_3 = 2 \Omega_3 , \quad \rho_4 = \rho_4^{\text{Liouville}} , \\
\rho_5 &= - \left(6 q_\phi^{(1)} + 8 \widehat{q}_\phi^{(1)} + 12 \alpha_1 f_m^{(1)} + 12 \widehat{c}_\phi - 12 f_\phi^{(1)} \right) = \rho_5^{\text{Liouville}} \tag{3.2.16}
\end{aligned}$$

and

$$\begin{aligned}
H^M(a, b) = & -2T \int dx dy \widehat{\widehat{K}}(t_1, a, x) \widehat{\widehat{K}}(t_2, x, y) \widehat{\widehat{K}}(t_3, x, y) \widehat{\widehat{K}}(t_4, y, b) \\
& + \frac{1}{2} \int dx \widehat{\widehat{K}}(t_1, a, x) \widehat{\widehat{K}}(t_2, b, x) \left(\widehat{\widehat{K}}(t_3, a, x) + \widehat{\widehat{K}}(t_3, b, x) \right) \\
& - \frac{1}{A} \int dx dy \left[\frac{1}{2} \widehat{\widehat{K}}(t_1, x, y) \widehat{\widehat{K}}(t_2, x, y) \left(\widehat{\widehat{K}}(t_3, a, x) + \widehat{\widehat{K}}(t_3, y, b) \right) \right. \\
& \qquad \qquad \qquad \left. + 2 \widehat{\widehat{K}}(t_1, a, x) \widehat{\widehat{K}}(t_2, x, y) \widehat{\widehat{K}}(t_3, y, b) \right] \\
& - 2T \int dx dy \widehat{\widehat{K}}(t_1, a, x) \widehat{\widehat{K}}(t_2, x, b) \widehat{\widehat{K}}(t_3, x, y) \widetilde{G}_\zeta(y) \\
& + \int dx \left(\widehat{\widehat{K}}(t_1, a, x) + \widehat{\widehat{K}}(t_2, x, b) \right) \\
& \qquad \qquad \qquad \times \left[\widehat{\widehat{K}}(t_3, a, b) \widetilde{G}_\zeta(x) - \frac{1}{A} \int dy \widehat{\widehat{K}}(t_3, x, y) \widetilde{G}_\zeta(y) \right]. \quad (3.2.17)
\end{aligned}$$

Finiteness of the two-point function at order ε^2 then fixes all the combinations ρ_1, \dots, ρ_5 . Namely, ρ_2 fixes Ω_2 and ρ_3 fixes Ω_3 , while Ω_1 is fixed by the finiteness of the three- and four-point functions. Thus, we do not have any freedom in the counterterm coefficients of the divergences of the three-loop partition function. As it was the case in the two-loop study, we expect these combinations to be fixed by the finiteness of the n -point functions in such a way that they absorb the divergences of the partition function.

If we consider now Ω_4 which contributes to the three-loop contribution to γ_{str} , ρ_5 fixes the second line in the coefficient $\Omega_4^{(a)}$. Moreover, the ‘‘strong locality’’ condition impose the values

$$q_m^{(2)} = -1, \quad f_m^{(2)} = -\frac{4}{3}, \quad c_m'^{(2)} = 0, \quad (3.2.18)$$

resulting in five free parameters in Ω_4 (and thus in $\gamma_{\text{str}} : \Omega_3^{(b)}, \widehat{c}_\phi, \widehat{c}_T, f_\phi^{(1)}$ and $c_T'^{(1)}$). However, considering the finite part of the two-point function, and in particular the finite coefficient of $\int dx \widehat{\widehat{K}}(t_1, a, x) \widehat{\widehat{K}}(t_2, b, x)$ and of $\int dx \widetilde{G}_\zeta^{A_0}(x) \widehat{\widehat{K}}(t_1, a, x) \widehat{\widehat{K}}(t_2, b, x)$ can help fix one constant. Indeed, we have

$$\begin{aligned}
\text{---} \bigcirc \text{---} \Big|_{\varepsilon^2, \text{fin}}^{\text{CT}} = & \frac{8\pi}{\kappa^4} \left\{ \frac{1}{A} \int dx \left[\rho_6 \left(4\pi \widetilde{G}_\zeta^{A_0}(x) - \gamma - \ln A_0 \mu^2 \right) + \rho_7 \right] \widetilde{G}(a, x) \widetilde{G}(b, x) \right. \\
& \left. + \rho_8 \frac{4\pi}{A} \int dx \left(\widetilde{G}_\zeta(x) - \frac{1}{A} \int dy \widetilde{G}_\zeta(y) \right) \widetilde{G}(a, x) \widetilde{G}(b, x) + \dots \right\} \quad (3.2.19)
\end{aligned}$$

with

$$\begin{aligned}
\rho_6 &= 2\Omega_3 = \rho_3 , \\
\rho_7 &= \Omega_4^{(a)} - 32\pi f_\phi^{(1)} + 176\pi \widehat{c}_\phi + 16\pi AT \widehat{c}_\phi \widehat{c}_T , \\
\rho_8 &= 6 f_m^{(2)} + AT \left(-f_\phi^{(1)} + 6 f_T^{(1)} + 24 \widehat{c}_\phi - 14 \widehat{c}_T \right) .
\end{aligned} \tag{3.2.20}$$

Imposing this structure to be regulator independent in the full two-loop two-point function result in fixing ρ_7 to a constant \widehat{c} ($\rho_6 = \rho_3$ being already fixed). This means that $\Omega_4^{(a)}$ is fixed in terms of \widehat{c} , $f_\phi^{(1)}$, \widehat{c}_ϕ and \widehat{c}_T . Injecting this into (3.2.14), we end up with four undetermined constants at three loops. This happens also in the pure Liouville theory on surfaces of any genus studied in the previous chapter, where the number of remaining free counterterms is also reduced to four. However, we still need new conditions to impose in order to completely fix these parameters, and hence our theory.

Discussion

The path to a quantized theory of gravity has not been marked out yet. Consequently, several ways are currently explored. One of them relies on the continuous approach of the Liouville theory, which describes the coupling of any conformal field theory to gravity in two dimensions. One object we can characterize in this approach is the partition function at fixed area whose area dependence is described by a critical exponent called the string susceptibility γ_{str} . The formalism of the functional integral appears to be a natural one to compute both the fixed area partition function and the string susceptibility. However, one of the major difficulties lies in the non-flat integration measure over the space of the metrics. Moreover, in a theory of quantum gravity one has to sum over the topologies. Nevertheless, under the simplifying assumption of a free-field measure, a formula for γ_{str} has been obtained in the end of the 1980s through the DDK argument. This formula, referred to as the KPZ formula since it has first been conjectured by Khnizik, Polyakov and Zamolodchikov on the sphere, has since been found in agreement with several results from the discrete methods.

The present work follows these footsteps. Taking advantage of recent advances in efficient multi-loop regularization methods on curved space, it has been possible to precisely regularize the complicated integration measure and thus to treat it without the strong hypothesis of a free-field measure. Then, a first-principles computation of the string susceptibility has been initiated at one loop, in a generalized Liouville theory including the Mabuchi action, which appears as a first order correction to the Liouville theory if the matter is massive and hence non-conformal. The one-loop result for γ_{str} is in agreement with the KPZ formula (in the case of conformal matter only). This thesis goes further with the full two-loop computation in two cases: first in a “pure” Liouville theory on a surface of arbitrary genus, and then, in the special case of the torus, in a Liouville plus Mabuchi theory, thus investigating the coupling to non-conformal matter. The non-trivial nature of the quantum gravity integration measure is revealed beyond one loop, starting at two loops. Then, it generates a “measure” action whose has to be added to the Liouville (and the Mabuchi) actions.

The results obtained from these computations present divergences, and thus one needs to renormalize them. The consistency of this approach is emphasized by the fact that the divergences that would require non-local counterterms, cancel among themselves. It is remarkable that those vanishing terms come together with background dependent structures which are, of course, undesirable in a theory of quantum gravity and which vanish together with the divergences. Consequently, the renormalization procedure is possible through only local counterterms, with the highlighted exception of the so-called “measure-like” counterterms, presenting the same dependence in the area than the measure action. These counterterms are thus understood as a renormalization of the complicated non-flat quantum gravity integration measure. They can be fixed by the requirements of finiteness and regularization independence and by imposing the so-called “strong locality condition” – which imposes the area dependent terms in the “measure

action” to cancel with the counterterms. We obtain an expression for the string susceptibility which depends on one free parameter at two loops and which is compatible with the KPZ formula. Although one of the counterterms involved in γ_{str} is fixed to its KPZ value by the strong locality condition, there seems to be no reason in our formalism to fix the remaining counterterm to its KPZ value.

Then, a three-loop computation has been performed. In particular, the leading divergence is non-vanishing and the renormalization procedure is once more required. Very similarly to the two-loop order, the only divergence which the introduction of local or measure-like counterterms could not cancel is vanishing. This is crucial for the consistency of our theory. Moreover, finiteness of the n -point function automatically fixes the counterterm coefficients of the diverging part of the partition function. If one requires in addition its regularization independence, and if one imposes the strong locality condition, one ends up with four free independent constants in γ_{str} , that is to say three more than at two loops.

As emphasized previously, the fact that only local counterterms are required to guarantee the finiteness of the theory is due to the fact that some divergences vanish. This is far from obvious and may be due to a hidden symmetry of the theory. This symmetry is most probably also the reason why the finiteness of the two-point function seems to automatically imply the finiteness of the partition function. This has been shown at two loops and is hinted at three loops. Unravelling this hidden symmetry is thus the key to ensure renormalizability at all orders.

Some useful integrals

Here we list various integrals of the form

$$I[f] = \frac{1}{\pi} \int d^2\tilde{z} e^{-\tilde{z}^2} f(\tilde{z}^2) = \int_0^\infty d\xi e^{-\xi} f(\xi) , \quad (\text{A.1})$$

where $d^2\tilde{z}$ is the flat measure: any non-trivial expansion of \sqrt{g} is included in f . Then

$$\begin{aligned} I[\xi^n] &= n! , \\ I[\xi^n \ln \xi] &= c_n - (n!) \gamma \end{aligned} \quad (\text{A.2})$$

with

$$c_0 = 0 , \quad c_1 = 1 , \quad c_2 = 3 , \quad c_3 = 11 , \dots \quad (\text{A.3})$$

One has also

$$\begin{aligned} I[(\ln \xi)^2] &= \gamma^2 + \frac{\pi^2}{6} , \\ I[\xi (\ln \xi)^2] &= \gamma^2 - 2\gamma + \frac{\pi^2}{6} , \end{aligned} \quad (\text{A.4})$$

and

$$\begin{aligned} I[E_1(\xi/a)] &= \ln(1+a) , \\ I[\xi E_1(\xi/a)] &= -\frac{a}{1+a} + \ln(1+a) , \\ I[\xi^2 E_1(\xi/a)] &= -\frac{a(2+3a)}{(1+a)^2} + 2\ln(1+a) , \\ I[\xi^3 E_1(\xi/a)] &= -\frac{a(11a^2+15a+6)}{(1+a)^3} + 6\ln(1+a) , \\ I[E_2(\xi/a)] &= 1 - \frac{1}{a} \ln(1+a) , \\ I[\xi E_2(\xi/a)] &= 1 + \frac{1}{1+a} - \frac{2}{a} \ln(1+a) , \\ I[\ln \xi E_1(\xi/a)] &= -\frac{\pi^2}{6} - \gamma \ln(1+a) + Li_2\left(\frac{1}{1+a}\right) , \end{aligned} \quad (\text{A.5})$$

where the E_n are the exponential integral functions defined in (1.3.41) :

$$E_n(w) = \int_1^\infty du u^{-n} e^{-uw} . \quad (\text{A.6})$$

Of course, insertion of any odd number of components of \tilde{z} into any of these integrals gives a vanishing result, while insertions of an even number can be replaced according to the usual rules

$$\tilde{z}^i \tilde{z}^j \rightarrow \frac{1}{2} \tilde{z}^2 \delta^{ij} \quad (\text{A.7})$$

and

$$\tilde{z}^i \tilde{z}^j \tilde{z}^k \tilde{z}^l \rightarrow \frac{1}{8} (\tilde{z}^2)^2 \left(\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right) . \quad (\text{A.8})$$

One- and two-loop contributions to the two-point function

In this appendix are given the details of the computation of the “one-loop” and “two-loop” contributions (order $1/\kappa^2$ and $1/\kappa^4$ respectively) to the two-point function $\mathcal{G}^{(2)}$.

B.1 “One-loop” contributions to the two-point function

In this section we write computational details of the one-loop contributions to the two-point function $\mathcal{G}^{(2)}(u, v)$ for non-coinciding points $u \neq v$.

B.1.1 One-loop contribution from $u - \text{O} - v$

Recall that $\mathcal{G}^{(2)}(u, v)$ always includes the two external propagators that, by consistency, are also regularized, i.e. replaced by \widehat{K} . Then, for the diagram $-\text{O}-$, due to the many ways the derivatives in the cubic vertex can act, one gets many different contributions. They yield

$$\begin{aligned} \mathcal{G}_{u-\text{O}-v}^{(2)}(u, v) = & \frac{8\pi}{\kappa^2} \int dx dy \left[\widetilde{K} \widehat{K} \widehat{K} \widetilde{K} + 2\widetilde{K} \widetilde{K} \widehat{K} \widehat{K} + 2\widehat{K} \widehat{K} \widetilde{K} \widetilde{K} + 2\widehat{K} \widetilde{K} \widetilde{K} \widehat{K} \right. \\ & \left. + 2\widehat{K} \widehat{K} \left(-\frac{d}{dt} \widetilde{K}\right) \widehat{K} + R_* \widetilde{K} \widehat{K} \widehat{K} \widetilde{K} + R_* \widehat{K} \widetilde{K} \widetilde{K} \widehat{K} + 4R_* \widehat{K} \widetilde{K} \widehat{K} \widetilde{K} + R_*^2 \widehat{K} \widehat{K} \widetilde{K} \widetilde{K} \right], \quad (\text{B.1.1}) \end{aligned}$$

where $\widetilde{K} \widehat{K} \widehat{K} \widetilde{K}$ stands for $\widetilde{K}(t_1, u, x) \widehat{K}(t_2, x, y) \widehat{K}(t_3, x, y) \widetilde{K}(t_4, y, v)$ etc.

We now evaluate this for $u \neq v$. More precisely, since we work at finite cutoff, we do not want $\ell^2(u, v)$ to be as small as $\frac{1}{\Lambda^2}$ and require $\ell^2(u, v) \Lambda^2 \gg 1$. Then $K(t = \alpha/\Lambda^2, u, v) \sim \frac{\Lambda^2}{4\pi\alpha} e^{-\ell^2 \Lambda^2 / (4\alpha)}$ is exponentially small and can always be dropped. Also $\widehat{K}(t, u, v) = \widetilde{G}(u, v) + \text{exponentially small} + \mathcal{O}(\frac{1}{\Lambda^2})$, cf (1.3.31) or (1.3.40). Furthermore, in $\int dy \widehat{K}(t_1, u, y) \widehat{K}(t_2, y, v)$ or in $\int dy \widehat{K}(t_1, u, y) \widehat{K}(t_2, y, u)$ we may replace the \widehat{K} by \widetilde{G} since these integrals have finite limits as $\Lambda \rightarrow \infty$. (The logarithmic short-distance singularity $(\ln \mu^2(y - u)^2)^n$ is integrable for any integer n .)

Denoting by $+\dots$ terms that are either $\mathcal{O}(\ln \Lambda \Lambda^2 / \Lambda^2)$ or exponentially small as just

explained, we find

$$\int dx dy \tilde{K} \tilde{K} \tilde{K} \tilde{K} = (\tilde{G}(u, v))^2 - \frac{1}{A} \int dx (\tilde{G}(u, x))^2 - \frac{1}{A} \int dx (\tilde{G}(x, v))^2 + \frac{1}{A^2} \int dx dy (\tilde{G}(x, y))^2 + \dots, \quad (\text{B.1.2})$$

$$\int dx dy \tilde{K} \tilde{K} \hat{\tilde{K}} \hat{\tilde{K}} = \tilde{G}(u, v) \frac{1}{4\pi} \left(-\ln \mu^2(t_2 + t_3) + 4\pi \tilde{G}_\zeta(u) - \gamma \right) - \frac{1}{A} \int dx \tilde{G}(u, x) \tilde{G}(x, v) - \frac{1}{A} \int dx \tilde{G}_\zeta(x) \tilde{G}(x, v) + \dots, \quad (\text{B.1.3})$$

$$\int dx dy \hat{\tilde{K}} \hat{\tilde{K}} \tilde{K} \tilde{K} = \tilde{G}(u, v) \frac{1}{4\pi} \left(-\ln \mu^2(t_2 + t_3) + 4\pi \tilde{G}_\zeta(v) - \gamma \right) - \frac{1}{A} \int dx \tilde{G}(u, x) \tilde{G}(x, v) - \frac{1}{A} \int dx \tilde{G}(u, x) \tilde{G}_\zeta(x) + \dots, \quad (\text{B.1.4})$$

$$\int dx dy \hat{\tilde{K}} \tilde{K} \tilde{K} \hat{\tilde{K}} = -\tilde{G}(u, v) \frac{1}{4\pi} \frac{\alpha_2 \alpha_3}{(\alpha_2 + \alpha_3)^2} + \frac{1}{4\pi} \left(\frac{1}{t_2 + t_3} + \frac{7}{6} R_* - \frac{\alpha_2 \alpha_3}{(\alpha_2 + \alpha_3)^2} R_* - \frac{8\pi}{A} \right) \int dx \hat{\tilde{K}}(t_1, u, x) \hat{\tilde{K}}(t_4, x, v) + \dots, \quad (\text{B.1.5})$$

$$\int dx dy \hat{\tilde{K}} \hat{\tilde{K}} \left(-\frac{d}{dt} \tilde{K} \right) \hat{\tilde{K}} = -\tilde{G}(u, v) \frac{1}{4\pi} \left[\ln \mu^2(t_2 + t_3) + \gamma + 1 + \frac{\alpha_2 \alpha_3}{(\alpha_2 + \alpha_3)^2} \right] + \frac{1}{4\pi} \left[\frac{1}{t_2 + t_3} - R_* \left(-\frac{1}{6} + \gamma + \ln \mu^2(t_2 + t_3) + \frac{\alpha_2 \alpha_3}{(\alpha_2 + \alpha_3)^2} \right) - \frac{4\pi}{A} \right] \times \int dx \hat{\tilde{K}}(t_1, u, x) \hat{\tilde{K}}(t_4, x, v) + \int dx \partial_x^i \tilde{G}(u, x) \tilde{G}_\zeta(x) \partial_x^i \tilde{G}(x, v) + \frac{R_*}{A} \int dx dy \tilde{G}(u, x) \tilde{G}(x, y) \tilde{G}(y, v) + \dots, \quad (\text{B.1.6})$$

$$\int dx dy \tilde{K} \tilde{K} \hat{\tilde{K}} \hat{\tilde{K}} = \int dx (\tilde{G}(u, x))^2 \tilde{G}(x, v) - \frac{1}{A} \int dx dy (\tilde{G}(x, y))^2 \tilde{G}(y, v) + \dots, \quad (\text{B.1.7})$$

$$\int dx dy \hat{\tilde{K}} \hat{\tilde{K}} \hat{\tilde{K}} \tilde{K} = \int dx \tilde{G}(u, x) (\tilde{G}(x, v))^2 - \frac{1}{A} \int dx dy \tilde{G}(u, x) (\tilde{G}(x, y))^2 + \dots, \quad (\text{B.1.8})$$

$$\int dx dy \widehat{\widetilde{K}} \widehat{\widetilde{K}} \widehat{\widetilde{K}} \widehat{\widetilde{K}} = \frac{1}{4\pi} \int dx \left[-\ln \mu^2 (t_2 + t_3) + 4\pi \widetilde{G}_\zeta(x) - \gamma \right] \widehat{\widetilde{K}}(t_1, u, x) \widehat{\widetilde{K}}(t_4, x, v) - \frac{1}{A} \int dx dy \widetilde{G}(u, x) \widetilde{G}(x, y) \widetilde{G}(y, v) + \dots, \quad (\text{B.1.9})$$

$$\int dx dy \widehat{\widetilde{K}} \widehat{\widetilde{K}} \widehat{\widetilde{K}} \widehat{\widetilde{K}} = \int dx dy \widetilde{G}(u, x) (\widetilde{G}(x, y))^2 \widetilde{G}(y, v) + \dots. \quad (\text{B.1.10})$$

Combining everything, we get

$$\begin{aligned} \mathcal{G}_{u-O-v}^{(2)}(u, v) = & \frac{1}{\kappa^2} \left\{ 4 \widetilde{G}(u, v) \left[3 \ln \frac{\Lambda^2}{\mu^2} - 3 \ln(\alpha_2 + \alpha_3) - 3\gamma - 1 - \frac{2\alpha_2\alpha_3}{(\alpha_2 + \alpha_3)^2} \right] \right. \\ & + 8\pi (\widetilde{G}(u, v))^2 - \frac{8\pi}{A} \int dx \left[(\widetilde{G}(u, x))^2 + (\widetilde{G}(x, v))^2 - \frac{1}{A} \int dy (\widetilde{G}(x, y))^2 \right] \\ & + 16\pi \widetilde{G}(u, v) (\widetilde{G}_\zeta(u) + \widetilde{G}_\zeta(v)) - \frac{16\pi}{A} \int dx (\widetilde{G}(u, x) \widetilde{G}_\zeta(x) + \widetilde{G}_\zeta(x) \widetilde{G}(x, v)) \\ & + 4 \int dx \widehat{\widetilde{K}}(t_1, u, x) \widehat{\widetilde{K}}(t_4, x, v) \left[\frac{2\Lambda^2}{\alpha_2 + \alpha_3} + R_* \left(3 \ln \frac{\Lambda^2}{\mu^2} - 3 \ln(\alpha_2 + \alpha_3) - 3\gamma \right. \right. \\ & \quad \left. \left. + \frac{4}{3} - \frac{2\alpha_2\alpha_3}{(\alpha_2 + \alpha_3)^2} \right) + 8\pi R_* \widetilde{G}_\zeta(y) - \frac{20\pi}{A} \right] \\ & + 16\pi \int dx \partial_x^i \widetilde{G}(u, x) \widetilde{G}_\zeta(x) \partial_x^i \widetilde{G}(x, v) \\ & + 8\pi R_* \int dy \left[(\widetilde{G}(u, y))^2 \widetilde{G}(y, v) + \widetilde{G}(u, y) (\widetilde{G}(y, v))^2 \right] \\ & - \frac{16\pi R_*}{A} \int dx dy \left[\widetilde{G}(u, x) \widetilde{G}(x, y) \widetilde{G}(y, v) + \frac{1}{2} (\widetilde{G}(x, y))^2 [\widetilde{G}(u, x) + \widetilde{G}(y, v)] \right] \\ & \left. + 8\pi R_*^2 \int dx dy \widetilde{G}(u, x) (\widetilde{G}(x, y))^2 \widetilde{G}(y, v) + \mathcal{O}(\ln A \Lambda^2 / \Lambda^2) \right\}. \quad (\text{B.1.11}) \end{aligned}$$

B.1.2 One-loop contribution from the tadpole diagram

The diagram $u \text{---} \text{---} \text{---} v$ only gives finite contributions. We first evaluate the tadpole

$$-\frac{2\sqrt{\pi}}{\kappa} B(x) \equiv \text{---} \text{---} \text{---} = -\frac{2\sqrt{\pi}}{\kappa} \int dy \left[\widetilde{K} \widehat{\widetilde{K}} + 2 \widetilde{K} \widetilde{K} + R_* \widehat{\widetilde{K}} \widehat{\widetilde{K}} \right], \quad (\text{B.1.12})$$

where $\widetilde{K} \widehat{\widetilde{K}} \equiv \widetilde{K}(t_3, x, y) \widehat{\widetilde{K}}(t_4, y, y)$ and similarly for the other terms. This simplifies considerably since $\widetilde{K}(t_i, y, y)$ does not depend on y and in $\widehat{\widetilde{K}}(t_j, y, y)$ the only non-

constant term is $\tilde{G}_\zeta(y)$. Thus one finds

$$\begin{aligned} B(x) &= \int dy \widehat{K}(t_3, x, y) \Delta_* \tilde{G}_\zeta(y) \\ &= \tilde{G}_\zeta(x) + R_* \int dy \widehat{K}(t_3, x, y) \tilde{G}_\zeta(y) - \frac{1}{A} \int dy \tilde{G}_\zeta(y). \end{aligned} \quad (\text{B.1.13})$$

The contribution to the Green's function then is

$$\mathcal{G}_{u \ominus v}^{(2)}(u, v) = \frac{8\pi}{\kappa^2} \int dx \left(\tilde{K} B \tilde{K} + \widehat{K} B \tilde{K} + \widehat{K} \Delta_* \tilde{G}_\zeta \widehat{K} + R_* \widehat{K} B \widehat{K} \right), \quad (\text{B.1.14})$$

where $\tilde{K} B \tilde{K} \equiv \tilde{K}(t_1, u, x) B(x) \tilde{K}(t_2, x, v)$ and similarly for the other terms. Inserting the expression for B and evaluating the integrals gives

$$\begin{aligned} \mathcal{G}_{u \ominus v}^{(2)}(u, v) &= \frac{8\pi}{\kappa^2} \left\{ 2 \tilde{G}(u, v) \left[\tilde{G}_\zeta(u) + \tilde{G}_\zeta(v) - \frac{1}{A} \int dx \tilde{G}_\zeta(x) \right] \right. \\ &\quad + R_* \tilde{G}(u, v) \int dx \left(\tilde{G}(u, x) + \tilde{G}(v, x) \right) \tilde{G}_\zeta(x) \\ &\quad - \frac{1}{A} \int dx \left(\tilde{G}(u, x) + \tilde{G}(v, x) \right) \left(2 \tilde{G}_\zeta(x) + R_* \int dy \tilde{G}(x, y) \tilde{G}_\zeta(y) \right) \\ &\quad + 3 R_* \int dx \tilde{G}(u, x) \tilde{G}_\zeta(x) \tilde{G}(x, v) - 2 \int dx \partial_x^i \tilde{G}(u, x) \tilde{G}_\zeta(x) \partial_x^i \tilde{G}(x, v) \\ &\quad + R_*^2 \int dx dy \tilde{G}(u, x) \tilde{G}(x, v) \tilde{G}(x, y) \tilde{G}_\zeta(y) \\ &\quad \left. - \frac{R_*}{A} \int dx \tilde{G}(u, x) \tilde{G}(x, v) \int dy \tilde{G}_\zeta(y) \right\} + \dots \end{aligned} \quad (\text{B.1.15})$$

B.1.3 One-loop contribution from the quartic vertices

The diagram $u \overset{\circlearrowleft}{\text{---}} v$ gets again different contributions:

$$\begin{aligned} \mathcal{G}_{u \circlearrowleft v}^{(2)}(u, v) &= -\frac{8\pi}{\kappa^2} \int dx \left\{ 2 \widehat{K}(t_1, u, x) (\Delta_*^x - 2R_*) \left(\widehat{K}(t_2, z, x) \widehat{K}(t_3, x, v) \right) \Big|_{z=x} \right. \\ &\quad + \widehat{K}(t_1, u, x) \widehat{K}(t_3, x, v) (\Delta_*^x - 2R_*) \widehat{K}(t_2, x, x) \\ &\quad + 2 (\Delta_*^x \widehat{K}(t_1, u, x)) \widehat{K}(t_2, x, x) \widehat{K}(t_3, x, v) \\ &\quad + 2 \widehat{K}(t_1, u, x) \widehat{K}(t_2, x, x) \Delta_*^x \widehat{K}(t_3, x, v) \\ &\quad \left. + 4 \widehat{K}(t_1, u, x) (\Delta_*^x \widehat{K}(t_2, x, z)) \Big|_{z=x} \widehat{K}(t_3, x, v) \right\}. \end{aligned} \quad (\text{B.1.16})$$

One finds, using the fact that $\widehat{K}(t_2, x, x) - \widetilde{G}_\zeta(x)$ does not depend on x ,

$$\begin{aligned} & \int dx \widehat{K}(t_1, u, x) (\Delta_*^x - 2R_*) \left(\widehat{K}(t_2, z, x) \widehat{K}(t_3, x, v) \right) \Big|_{z=x} \\ &= \frac{1}{4\pi} \widetilde{G}(u, v) (-\ln \mu^2 t_2 - \gamma) + \int dx \partial_x^i \widetilde{G}(u, x) \widetilde{G}_\zeta(x) \partial_x^i \widetilde{G}(x, v) \\ &+ \frac{1}{4\pi} \int dx \widehat{K}(t_1, u, x) \widehat{K}(t_3, x, v) \left(\frac{1}{t_2} + \frac{7}{6} R_* - \frac{4\pi}{A} - 4\pi R_* \widetilde{G}_\zeta(x) \right) + \dots \end{aligned} \quad (\text{B.1.17})$$

and

$$\begin{aligned} & \int dx \widehat{K}(t_1, u, x) \widehat{K}(t_3, x, v) (\Delta_*^x - 2R_*) \left(\widehat{K}(t_2, x, x) \right) \\ &= \widetilde{G}(u, v) \left(\widetilde{G}_\zeta(u) + \widetilde{G}_\zeta(v) \right) + \frac{1}{4\pi} \int dx \widehat{K}(t_1, u, x) \widehat{K}(t_3, x, v) 2R_* (\ln \mu^2 t_2 + \gamma) \\ &- 2 \int dx \partial_x^i \widetilde{G}(u, x) \widetilde{G}_\zeta(x) \partial_x^i \widetilde{G}(x, v) - \frac{1}{A} \int dx \left(\widetilde{G}(u, x) + \widetilde{G}(v, x) \right) \widetilde{G}_\zeta(x) \dots \end{aligned} \quad (\text{B.1.18})$$

The three remaining terms in (B.1.16) are straightforward to evaluate:

$$\begin{aligned} & \int dx \left[(\Delta_*^x \widehat{K}(t_1, u, x)) \widehat{K}(t_2, x, x) \widehat{K}(t_3, x, v) + \widehat{K}(t_1, u, x) \widehat{K}(t_2, x, x) \Delta_*^x \widehat{K}(t_3, x, v) \right] \\ &= \widetilde{G}(u, v) \left[\widehat{K}(t_2, u, u) + \widehat{K}(t_2, v, v) \right] - \frac{1}{A} \int dx \widehat{K}(t_2, x, x) \left[\widehat{K}(t_1, u, x) + \widehat{K}(t_3, x, v) \right] \\ &+ \frac{1}{4\pi} \int dx \widehat{K}(t_1, u, x) \widehat{K}(t_3, x, v) 2R_* \left[-\ln \mu^2 t_2 - \gamma + 4\pi \widetilde{G}_\zeta(x) \right], \end{aligned} \quad (\text{B.1.19})$$

and

$$\begin{aligned} & \int dx \widehat{K}(t_1, u, x) (\Delta_*^x \widehat{K}(t_2, x, z)) \Big|_{z=x} \widehat{K}(t_3, x, v) \\ &= \frac{1}{4\pi} \int dx \widehat{K}(t_1, u, x) \widehat{K}(t_3, x, v) \left(\frac{1}{t_2} + \frac{7}{6} R_* - \frac{4\pi}{A} - R_* \ln \mu^2 t_2 - \gamma R_* + 4\pi R_* \widetilde{G}_\zeta(x) \right) \end{aligned} \quad (\text{B.1.20})$$

up to vanishing terms. Combining everything gives

$$\begin{aligned} \mathcal{G}_{u \ominus v}^{(2)}(u, v) &= \frac{1}{\kappa^2} \left\{ \widetilde{G}(u, v) \left[-24\pi \left(\widetilde{G}_\zeta(u) + \widetilde{G}_\zeta(v) \right) + 12 (\ln \mu^2 t_2 + \gamma) \right] \right. \\ &+ 12 \int dx \widehat{K}(t_1, u, x) \widehat{K}(t_3, x, v) \left[-\frac{1}{t_2} - \frac{7}{6} R_* + \frac{4\pi}{A} + R_* \left(\gamma + \ln \mu^2 t_2 - 4\pi \widetilde{G}_\zeta(x) \right) \right] \\ &\left. + \frac{24\pi}{A} \int dx \left[\widetilde{G}(u, x) + \widetilde{G}(v, x) \right] \widetilde{G}_\zeta(x) + \dots \right\}. \end{aligned} \quad (\text{B.1.21})$$

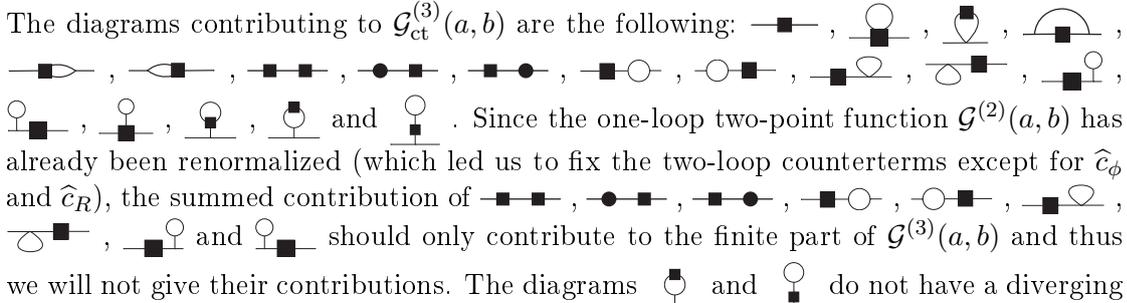
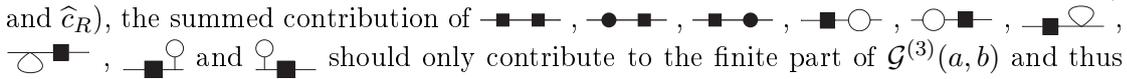
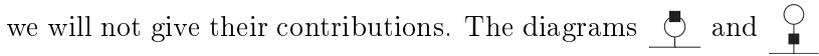
B.1.4 Order $1/\kappa^2$ contribution from the measure

Finally the measure vertex (2.1.15) contributes

$$\begin{aligned} \mathcal{G}_{\text{measure}}^{(2)}(u, v) &= \frac{8\pi}{\kappa^2} \int dx \widehat{K}(t_1, u, x) \widetilde{K}(t_2, x, x) \widehat{K}(t_3, x, v) \\ &= \frac{1}{\kappa^2} \int dx \widehat{K}(t_1, u, x) \widehat{K}(t_3, x, v) \left[\frac{2}{t_2} + \frac{7}{3} R_* - \frac{8\pi}{A} \right] + \dots \quad (\text{B.1.22}) \end{aligned}$$

The total one-loop contributions to $\mathcal{G}^{(2)}(u, v)$ for $u \neq v$ is the sum of the contributions computed here and of the one from the counterterms (2.1.93). It is given and discussed in section 2.1.4.

B.2 “Two-loop” contributions to the two-point function

Here we give the diverging contributions of the counterterm to the two-loop two-point function¹ $\mathcal{G}^{(3)}(a, b)$ per diagrams². We only consider the so far not fixed counterterms, *i.e.* the genuine three-loop counterterms and the “two-loop” counterterms \widehat{c}_ϕ and \widehat{c}_R . The diagrams contributing to $\mathcal{G}_{\text{ct}}^{(3)}(a, b)$ are the following: . Since the one-loop two-point function $\mathcal{G}^{(2)}(a, b)$ has already been renormalized (which led us to fix the two-loop counterterms except for \widehat{c}_ϕ and \widehat{c}_R), the summed contribution of  should only contribute to the finite part of $\mathcal{G}^{(3)}(a, b)$ and thus we will not give their contributions. The diagrams  do not have a diverging part and thus will also not be considered here. Then, we write

$$\mathcal{G}_{\text{ct}}^{(3)}|_{\text{div}} = \mathcal{G}_{\text{ct}}^{(3)} + \mathcal{G}_{\text{ct}}^{(3)}, \quad (\text{B.2.1})$$

¹We consider the two-point function at non coinciding point *i.e.* $a \neq b$.
²The computations are quite lengthy but straightforward and thus left to the motivated reader.

with the understanding that we will only give the diverging contributions. We have first for the cubic counterterms

$$\begin{aligned}
 (\mathcal{G}_{\text{D}}^{(3)} + \mathcal{G}_{\text{R}}^{(3)}) (a, b) &= \frac{8\pi}{\kappa^4} \left\{ 12 \left(f_\phi^{(1)} - \alpha_1 f_m^{(1)} \right) \ln A \Lambda^2 \tilde{G}(a, b) + 24\pi f_m^{(1)} \Lambda^2 D(a, b) \right. \\
 &\quad + \int dx \tilde{K}(t_1, a, x) \tilde{K}(t_2, x, b) \times \left[12 f_m^{(1)} \Lambda^2 \ln A \Lambda^2 \right. \\
 &\quad \left. \left. + \Lambda^2 \left(\frac{8}{\alpha_1 + \alpha_2} f_\phi^{(1)} + 12 f_m^{(1)} (G_0(x) - \ln(\alpha_1 + \alpha_2)) \right) \right] \right. \\
 &\quad \left. + 2 \left[6 f_m^{(2)} + (4 f_\phi^{(1)} + 6 f_R^{(1)} - 6 \alpha_1 f_m^{(1)}) A R_* \right] \frac{\ln A \Lambda^2}{A} \right\} \quad (\text{B.2.2})
 \end{aligned}$$

with

$$\begin{aligned}
 D(a, b) &= R_* \int dx dy \tilde{K}(t_1, a, x) \tilde{K}(t_2, x, y) \tilde{K}(t_3, x, y) \tilde{K}(t_4, y, b) \\
 &\quad + \frac{1}{2} \int dx \tilde{K}(t_1, a, x) \tilde{K}(t_2, b, x) \left(\tilde{K}(t_3, a, x) + \tilde{K}(t_3, b, x) \right) \\
 &\quad - \frac{1}{A} \int dx dy \left[\frac{1}{2} \tilde{K}(t_1, x, y) \tilde{K}(t_2, x, y) \left(\tilde{K}(t_3, a, x) + \tilde{K}(t_3, y, b) \right) \right. \\
 &\quad \left. + 2 \tilde{K}(t_1, a, x) \tilde{K}(t_2, x, y) \tilde{K}(t_3, y, b) \right], \quad (\text{B.2.3})
 \end{aligned}$$

and

$$G_0(x) = 4\pi \tilde{G}_\zeta^{A_0}(x) - \gamma - \ln A_0 \mu^2. \quad (\text{B.2.4})$$

Then, we also have for the cubic counterterms

$$\begin{aligned}
 \mathcal{G}_{\text{D}}^{(3)}(a, b) &= \frac{24\pi}{\kappa^4} f_m^{(1)} \Lambda^2 \left\{ \int dx \tilde{K}(t_1, a, x) \tilde{K}(t_2, x, b) (G_0(x) - G_0) \right. \\
 &\quad \left. + 4\pi R_* \int dx dy \tilde{K}(t_1, a, x) \tilde{K}(t_2, x, b) \tilde{K}(t_3, x, y) \tilde{G}_\zeta(y) \right\} \quad (\text{B.2.5})
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{G}_{\text{R}}^{(3)}(a, b) &= \mathcal{G}_{\text{D}}^{(3)} + \frac{96\pi^2}{\kappa^4} f_m^{(1)} \Lambda^2 \int dx \left(\tilde{K}(t_1, a, x) + \tilde{K}(t_2, x, b) \right) \\
 &\quad \times \left[\tilde{K}(t_3, a, b) \tilde{G}_\zeta(x) - \frac{1}{A} \int dy \tilde{K}(t_3, x, y) \tilde{G}_\zeta(y) \right]. \quad (\text{B.2.6})
 \end{aligned}$$

The unique diagram built from the quartic counterterms gives

$$\begin{aligned}
 \mathcal{G}_{\text{---}}^{(3)}(a, b) &= -\frac{8\pi}{\kappa^4} \left\{ \left(6q_\phi^{(1)} + 8\hat{q}_\phi^{(1)} \right) \ln A\Lambda^2 \tilde{G}(a, b) \right. \\
 &\quad + \int dx \hat{K}(t_1, a, x) \hat{K}(t_2, x, b) \times \left[12q_m^{(1)} \Lambda^2 \ln A\Lambda^2 \right. \\
 &\quad + \Lambda^2 \left(\frac{1}{\alpha_1} \left(6q_\phi^{(1)} + 8\hat{q}_\phi^{(1)} \right) + 12q_m^{(1)} (G_0(x) - \ln \alpha_1) \right) \\
 &\quad \left. \left. + 2 \left[6q_m^{(2)} + (6q_R^{(1)} - 4\hat{q}_m^{(1)}) AR_* \right] \frac{\ln A\Lambda^2}{A} \right] \right\}. \tag{B.2.7}
 \end{aligned}$$

For the c' quadratic counterterms, we have

$$\mathcal{G}_{\text{---}}^{(3)}(a, b) = -\frac{8\pi}{\kappa^4} c_m'^{(1)} \Lambda^2 \int dx \hat{K}(t_1, a, x) \hat{K}(t_2, x, b). \tag{B.2.8}$$

The two remaining diagrams only involves \hat{c}_ϕ and \hat{c}_R . They contribute

$$\begin{aligned}
 \mathcal{G}_{\text{---}}^{(3)}(a, b) &= \frac{8\pi}{\kappa^4} \left\{ 12\hat{c}_\phi \ln A\Lambda^2 \tilde{G}(a, b) \right. \\
 &\quad \left. + \int dx \hat{K}(t_1, a, x) \hat{K}(t_2, x, b) \left[\frac{12\Lambda^2}{\alpha_1 + \alpha_2} \hat{c}_\phi + 12 \left(\hat{c}_\phi + \frac{\hat{c}_R}{2\pi} \right) R_* \ln A\Lambda^2 \right] \right\}, \tag{B.2.9}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{G}_{\text{---}}^{(3)}(a, b) &= -\frac{8\pi}{\kappa^4} \left\{ 24\hat{c}_\phi \ln A\Lambda^2 \tilde{G}(a, b) \right. \\
 &\quad \left. + \int dx \hat{K}(t_1, a, x) \hat{K}(t_2, x, b) \left[\frac{16\Lambda^2}{\alpha_1 + \alpha_2 + \alpha_3} \hat{c}_\phi + 24 \left(\hat{c}_\phi + \frac{\hat{c}_R}{3\pi} \right) R_* \ln A\Lambda^2 \right] \right\}. \tag{B.2.10}
 \end{aligned}$$

Summing (B.2.2) and (B.2.5)-(B.2.10) gives (2.2.53) with the ρ_i in (2.2.54).

Leading divergence of the three-loop partition function

When computing the diagrams $\textcircled{\circ}$, $\textcircled{\infty}$, $\textcircled{\infty}$ and $\textcircled{\triangle}$, one gets infinite series of terms contributing to the coefficients of $A\Lambda^2 (\ln A\Lambda^2)^2$. The details of the computation of these series are described hereafter. For the sake of brevity, notational short-cuts are defined: $dx = d^2x\sqrt{g_*(x)}$, $\widehat{K}_i(x, y) = \widehat{K}(t_i, x, y)$ and $\widehat{K}_{i,j}(x, y) = \widehat{K}(t_i + t_j, x, y)$.

Among the series of diverging contributions appearing in these four diagrams, some may be straightforwardly computed by Taylor expanding the terms, as was done in section 2.2.2. These integrals are hereafter noted by J . Namely, they are:

$$\begin{aligned}
J_{\textcircled{\infty}}^{(1)} &= \int dx dy dz \widehat{K}_1(x, z) \widetilde{K}_2(x, z) \widehat{K}_3(y, z) \widetilde{K}_4(y, z) \widetilde{K}_5(x, y) , \\
J_{\textcircled{\infty}}^{(2)} &= \int dx dy dz \widehat{K}_1(x, z) \widehat{K}_2(x, z) \widehat{K}_3(y, z) \widetilde{K}_4(y, z) \left(-\frac{d}{dt_5} \widetilde{K}_5(x, y) \right) , \\
J_{\textcircled{\infty}}^{(3)} &= \int dx dy dz \widehat{K}_1(x, z) \left(-\frac{d}{dt_2} \widetilde{K}_2(x, z) \right) \widehat{K}_3(y, z) \widetilde{K}_4(y, z) \widehat{K}_5(x, y) , \\
J_{\textcircled{\infty}}^{(4)} &= \int dx dy dz \widehat{K}_1(x, z) \widehat{K}_2(x, z) \widehat{K}_3(y, z) \left(-\frac{d}{dt_4} \widetilde{K}_4(y, z) \right) \widetilde{K}_5(x, y) , \\
J_{\textcircled{\ominus}}^{(1)} &= \int dx dy dz \widehat{K}_1(z, z) \widetilde{K}_2(x, z) \widehat{K}_3(x, y) \widetilde{K}_4(x, y) \widetilde{K}_5(y, z) , \\
J_{\textcircled{\ominus}}^{(2)} &= \int dx dy dz \widehat{K}_1(z, z) \widetilde{K}_2(x, z) \widehat{K}_3(x, y) \left(-\frac{d}{dt_4} \widetilde{K}_4(x, y) \right) \widehat{K}_5(y, z) , \\
J_{\textcircled{\ominus}}^{(3)} &= \int dx dy dz \widehat{K}_1(z, z) \widehat{K}_2(x, z) \widehat{K}_3(x, y) \widetilde{K}_4(x, y) \left(-\frac{d}{dt_5} \widetilde{K}_5(y, z) \right) , \\
J_{\textcircled{\odot}}^{(1)} &= \int dx dy dz dw \widehat{K}_1(x, y) \widetilde{K}_2(x, y) \widehat{K}_3(y, z) \widehat{K}_4(z, w) \widetilde{K}_5(z, w) \widetilde{K}_6(x, w) , \\
J_{\textcircled{\odot}}^{(2)} &= \int dx dy dz dw \widehat{K}_1(x, y) \frac{d}{dt_2} \widetilde{K}_2(x, y) \widehat{K}_3(y, z) \widehat{K}_4(z, w) \frac{d}{dt_5} \widetilde{K}_5(z, w) \widehat{K}_6(x, w) , \\
J_{\textcircled{\odot}}^{(3)} &= \int dx dy dz dw \widehat{K}_1(x, y) \widetilde{K}_2(x, y) \widehat{K}_3(y, z) \widehat{K}_4(z, w) \widetilde{K}_5(z, w) \left(-\frac{d}{dt_6} \widetilde{K}_6(x, w) \right) , \\
J_{\textcircled{\odot}}^{(4)} &= \int dx dy dz dw \widehat{K}_1(x, y) \widetilde{K}_2(x, y) \widehat{K}_3(y, z) \widehat{K}_4(z, w) \left(-\frac{d}{dt_5} \widetilde{K}_5(z, w) \right) \widetilde{K}_6(x, w) .
\end{aligned} \tag{C.1}$$

Note that $J_{\textcircled{\infty}}^{(1)}$ was already computed in section 2.2.2. It is useful to define:

$$\begin{aligned}
B_n(t_a, t_b, t_c; x) &= \int dz \widehat{\widehat{K}}_a(x, z) \widehat{\widehat{K}}_b(x, z) \frac{d^n}{dt_c^n} \widetilde{K}_c(x, z) , \\
C_{n,m}(t_a, t_b, t_c; x) &= \int dz \widehat{\widehat{K}}_a(x, z) \frac{d^n}{dt_b^n} \widetilde{K}_b(x, z) \frac{d^m}{dt_c^m} \widetilde{K}_c(x, z) , \\
D_n(t_a, t_b, t_c; x) &= \int dz \widehat{\widehat{K}}_a(z, z) \widehat{\widehat{K}}_b(x, z) \frac{d^n}{dt_c^n} \widetilde{K}_c(x, z) , \\
E_{n,m}(t_a, t_b, t_c; x) &= \int dz \widehat{\widehat{K}}_a(z, z) \frac{d^n}{dt_b^n} \widetilde{K}_b(x, z) \frac{d^m}{dt_c^m} \widetilde{K}_c(x, z) . \tag{C.2}
\end{aligned}$$

The only divergence investigated here is the one in $A\Lambda^2 (\ln A\Lambda^2)^2$, which cannot appear unless two $\widehat{\widehat{K}}$ s are present, since the logarithmic divergence comes from such terms (2.1.47). The terms without at least two $\widehat{\widehat{K}}$ s after doing the expansions are thus discarded in the following. Remembering that $t = \frac{\alpha}{\Lambda^2}$, the previous integrals may then be rewritten as:

$$\begin{aligned}
J_{\odot}^{(1)} &= \int dx \widehat{\widehat{K}}_{3,4}(x, x) \sum_{n=0}^{\infty} \frac{t_4^n}{n!} C_{0,n}(t_1, t_2, t_5; x) , \\
J_{\odot}^{(2)} &= - \int dx \widehat{\widehat{K}}_{3,4}(x, x) \sum_{n=0}^{\infty} \frac{t_4^n}{n!} B_{n+1}(t_1, t_2, t_5; x) , \\
J_{\odot}^{(3)} &= \int dx \widehat{\widehat{K}}_{3,4}(x, x) \left[-B_1(t_1, t_2, t_5; x) + \sum_{n=0}^{\infty} \frac{t_4^{n+1}}{(n+1)!} C_{1,n}(t_1, t_2, t_5; x) \right] , \\
J_{\odot}^{(4)} &= J_{\odot}^{(2)} + \int dx \frac{1}{4\pi} \frac{\Lambda^2}{\alpha_3 + \alpha_4} B_0(t_1, t_2, t_5; x) , \\
J_{\ominus}^{(1)} &= \int dx \widehat{\widehat{K}}_{3,4}(x, x) \sum_{n=0}^{\infty} \frac{t_4^n}{n!} E_{0,n}(t_1, t_2, t_5; x) , \tag{C.3} \\
J_{\ominus}^{(2)} &= J_{\ominus}^{(1)} + \int dx \frac{1}{4\pi} \frac{\Lambda^2}{\alpha_3 + \alpha_4} D_0(t_1, t_2, t_5; x) , \\
J_{\ominus}^{(3)} &= - \int dx \widehat{\widehat{K}}_{3,4}(x, x) \sum_{n=0}^{\infty} \frac{t_4^n}{n!} D_{n+1}(t_1, t_2, t_5; x) , \\
J_{\odot\odot}^{(1)} &= \int dx \widehat{\widehat{K}}_{1,2}(x, x) \sum_{n,m \geq 0} \frac{t_2^n t_5^m}{n! m!} E_{n,m}(t_4 + t_5, t_3, t_6; x) , \\
J_{\odot\odot}^{(2)} &= J_{\odot\odot}^{(1)} + 2 \int dx \frac{1}{4\pi} \frac{\Lambda^2}{\alpha_1 + \alpha_2} D_0(t_4 + t_5, t_3, t_6; x) + J_{4,5}^{1,2} , \\
J_{\odot\odot}^{(3)} &= \int dx \widehat{\widehat{K}}_{1,2}(x, x) \sum_{n,m \geq 0} \frac{t_2^{n+1}}{(n+1)!} \frac{t_5^m}{m!} E_{n,m+1}(t_4 + t_5, t_3, t_6; x) + J_{\odot}^{(3)} , \\
J_{\odot\odot}^{(4)} &= J_{\odot\odot}^{(3)} + \int dx \frac{1}{4\pi} \frac{\Lambda^2}{\alpha_4 + \alpha_5} D_0(t_1 + t_2, t_3, t_6; x) ,
\end{aligned}$$

where all the terms in $\mathcal{O}(A\Lambda^2 \ln A\Lambda^2)$ are discarded. The fact that the α_i (and thus t_i) are dummy variables that can be renamed and are symmetrized, has been used to simplify the writings of $J_{\odot\odot}^{(2)}$ and $J_{\odot\odot}^{(3)}$. Finally, the term $J_{4,5}^{1,2}$ in $J_{\odot\odot}^{(2)}$ is the term proportional to Λ^4 defined in (2.2.11). All contributions $\sim \Lambda^4$ have been discussed in section 3.2 and are summarized in Tab. 2.2. At present we are only interested in the other types of divergences and thus we will simply drop the term $J_{4,5}^{1,2}$ in the following. We conjecture:

$$B_n(t_a, t_b, t_c; x) = \begin{cases} \frac{(-1)^n \Lambda^{2n}}{(4\pi)^2} (n-1)! \left[\frac{1}{(\alpha_a + \alpha_c)^n} + \frac{1}{(\alpha_b + \alpha_c)^n} \right] \ln A\Lambda^2 + \mathcal{O}(\Lambda^{2n}) & \text{if } n \geq 1, \\ \frac{1}{(4\pi)^2} (\ln A\Lambda^2)^2 + \mathcal{O}(\ln A\Lambda^2) & \text{if } n = 0. \end{cases} \quad (\text{C.4})$$

α_a and α_b being dummy variables, this may be rewritten as

$$B_n(t_a, t_b, t_c; x) = \begin{cases} 2 \frac{(-1)^n (n-1)!}{(4\pi)^2} \left(\frac{\Lambda^2}{\alpha_b + \alpha_c} \right)^n \ln A\Lambda^2 + \mathcal{O}(\Lambda^{2n}) & \text{if } n \geq 1, \\ \frac{1}{(4\pi)^2} (\ln A\Lambda^2)^2 + \mathcal{O}(\ln A\Lambda^2) & \text{if } n = 0. \end{cases} \quad (\text{C.5})$$

Likewise,

$$D_n(t_a, t_b, t_c; x) = \begin{cases} \frac{(-1)^n (n-1)!}{(4\pi)^2} \left(\frac{\Lambda^2}{\alpha_b + \alpha_c} \right)^n \ln A\Lambda^2 + \mathcal{O}(\Lambda^{2n}) & \text{if } n \geq 1, \\ \frac{1}{(4\pi)^2} (\ln A\Lambda^2)^2 + \mathcal{O}(\ln A\Lambda^2) & \text{if } n = 0. \end{cases} \quad (\text{C.6})$$

and

$$C_{n,m}(t_a, t_b, t_c; x) = E_{n,m}(t_a, t_b, t_c; x) = \frac{(-1)^{n+m}}{(4\pi)^2} (n+m)! \left(\frac{\Lambda^2}{\alpha_b + \alpha_c} \right)^{n+m+1} \ln A\Lambda^2 + \mathcal{O}(\Lambda^{2n+2m+2}). \quad (\text{C.7})$$

From (1.3.24), one observes that

$$C_{1,n}(t_a, t_b, t_c; x) = \frac{d}{dt_b} C_{0,n}(t_a, t_b, t_c; x) = -\frac{d^2}{dt_b^2} B_n(t_a, t_b, t_c; x), \quad (\text{C.8})$$

and

$$E_{0,n}(t_a, t_b, t_c; x) = -\frac{d}{dt_b} D_n(t_a, t_b, t_c; x). \quad (\text{C.9})$$

These relations are verified by the above expressions, before considering the symmetries between the α_i . Putting everything together and remembering once more that the α_i are

dummy variables, one gets:

$$\begin{aligned}
J_{\infty}^{(1)} &= \frac{A\Lambda^2}{(4\pi)^3} (\ln A\Lambda^2)^2 \frac{1}{\alpha_1 + \alpha_2 + \alpha_3} , \\
J_{\ominus}^{(1)} &= J_{\ominus}^{(3)} = \frac{1}{2} J_{\infty}^{(2)} = J_{\infty}^{(1)} , \\
J_{\infty}^{(3)} &= J_{\ominus}^{(2)} = J_{\infty}^{(1)} + \frac{A\Lambda^2}{(4\pi)^3} (\ln A\Lambda^2)^2 \frac{1}{\alpha_1 + \alpha_2} , \\
J_{\infty}^{(4)} &= 2J_{\infty}^{(1)} + \frac{A\Lambda^2}{(4\pi)^3} (\ln A\Lambda^2)^2 \frac{1}{\alpha_1 + \alpha_2} , \\
J_{\odot}^{(3)} &= J_{\odot}^{(1)} = \frac{A\Lambda^2}{(4\pi)^3} (\ln A\Lambda^2)^2 \frac{1}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} , \\
J_{\odot}^{(2)} &= J_{\odot}^{(1)} + 2\frac{A\Lambda^2}{(4\pi)^3} (\ln A\Lambda^2)^2 \frac{1}{\alpha_1 + \alpha_2} \\
J_{\odot}^{(4)} &= J_{\odot}^{(1)} + \frac{A\Lambda^2}{(4\pi)^3} (\ln A\Lambda^2)^2 \frac{1}{\alpha_1 + \alpha_2} , \tag{C.10}
\end{aligned}$$

up to subleading divergences. One encounters also integrals such as

$$L_{\ominus} = \int dx dy dz \widehat{\widehat{K}}_1(z, z) \widehat{\widehat{K}}_2(x, y) \widehat{\widehat{K}}_3(x, y) \widetilde{K}_4(x, z) \left(-\frac{d}{dt_5} \widetilde{K}_5(y, z) \right) \tag{C.11}$$

whose explicit computation requires to Taylor expand a product of two $\widehat{\widehat{K}}$ or \widetilde{K} . We denote such integrals by L . Integrating over z around x through the exponential term in $\widetilde{K}_4(x, z)$, see (1.3.45), leads us to Taylor expand $\widehat{\widehat{K}}_1(z, z) \frac{d}{dt_5} \widetilde{K}_5(y, z)$ in $(z - x)$ around x . After integration, one gets terms such as:

$$\int dx dy \widehat{\widehat{K}}_2(x, y) \widehat{\widehat{K}}_3(x, y) \partial_{i_1}^x \dots \partial_{i_r}^x \widehat{\widehat{K}}_1(x, x) \partial_{j_1}^x \dots \partial_{j_s}^x \left(-\frac{d}{dt_5} \widetilde{K}_5(x, y) \right) \tag{C.12}$$

with $r + s$ even. If s is odd, then the function $\widehat{\widehat{K}}_2(x, y) \widehat{\widehat{K}}_3(x, y) \partial_{j_1}^x \dots \partial_{j_s}^x \left(-\frac{d}{dt_5} \widetilde{K}_5(x, y) \right)$ is odd and performing the integral over y kills the contribution: r and s have to be even. Since $\widetilde{K}(t, x, x)$ depends on x only through $\widetilde{G}_\zeta^{A_0}(x)$, see (2.1.47), for $r \geq 2$, $\partial_{i_1}^x \dots \partial_{i_r}^x \widehat{\widehat{K}}_1(x, x)$ does not contribute to the leading divergence by neither a factor $\ln A\Lambda^2$ nor Λ^2 . The diverging contributions may thus only come from the integral over y . However, applying s derivatives on $\frac{d}{dt_5} \widetilde{K}_5(x, y)$ for any even s leads to terms similar to $(-1)^{\frac{s}{2}} \left(\frac{d}{dt_5} \right)^{1+\frac{s}{2}} \widetilde{K}_5(x, y)$. Integrating over y one gets $B_{1+\frac{s}{2}}(t_2, t_3, t_5; x)$ which only produces one of the two $\ln A\Lambda^2$ of the leading divergence. The only terms contributing to $A\Lambda^2 (\ln A\Lambda^2)^2$ are thus the terms with $r = 0$ and s even.

Thus, up to subleading divergences, the previous integral gives:

$$L_{\ominus} = - \int dx \widehat{\widehat{K}}_1(x, x) \sum_{n=0}^{\infty} \frac{t_4^n}{n!} B_{n+1}(t_2, t_3, t_5; x) = J_{\infty}^{(2)} . \tag{C.13}$$

Similarly, one may compute

$$\begin{aligned}
L_{\circlearrowleft}^{(1)} &= \int dx dy dz dw \widehat{K}_1(x, y) \widehat{K}_2(x, y) \widetilde{K}_3(y, z) \widehat{K}_4(z, w) \widetilde{K}_5(z, w) \left(-\frac{d}{dt_6} \widetilde{K}_6(x, w) \right), \\
&= - \int dx dy dz \widehat{K}_1(x, y) \widehat{K}_2(x, y) \widetilde{K}_3(y, z) \widehat{K}_{4,5}(z, z) \sum_{n=0}^{\infty} \frac{t_5^n}{n!} \left(\frac{d}{dt_6} \right)^{n+1} \widetilde{K}_6(x, z), \\
&= - \int dx dy \widehat{K}_1(x, y) \widehat{K}_2(x, y) \widehat{K}_{4,5}(y, y) \sum_{n, m \geq 0} \frac{t_3^m t_5^n}{m! n!} \left(\frac{d}{dt_6} \right)^{n+m+1} \widetilde{K}_6(x, y), \\
&= 2 J_{\circlearrowleft}^{(1)}, \\
L_{\circlearrowleft}^{(2)} &= \int dx dy dz dw \widehat{K}_1(x, y) \widehat{K}_2(x, y) \widetilde{K}_3(y, z) \widehat{K}_4(z, w) \left(-\frac{d}{dt_5} \widetilde{K}_5(z, w) \right) \widetilde{K}_6(x, w), \\
&= L_{\circlearrowleft}^{(1)} + \frac{1}{4\pi} \frac{\Lambda^2}{\alpha_4 + \alpha_5} \sum_{n=0}^{\infty} \frac{t_6^n}{n!} \int dx B_n(t_1, t_2, t_3; x), \\
&= L_{\circlearrowleft}^{(1)} + \frac{A\Lambda^2}{(4\pi)^3} (\ln A\Lambda^2)^2 \frac{1}{\alpha_4 + \alpha_5}, \tag{C.14}
\end{aligned}$$

up to $\mathcal{O}(A\Lambda^2 \ln A\Lambda^2)$ terms.

Since at least two \widehat{K} s are needed to obtain the two $\ln A\Lambda^2$ of the leading divergence, it is easy to compute

$$L_{\triangle}^{(1)} = \int dx dy dz dw \widehat{K}_1(x, w) \widehat{K}_2(y, w) \widetilde{K}_3(z, w) \widetilde{K}_4(x, y) \widetilde{K}_5(x, z) \widetilde{K}_6(y, z). \tag{C.15}$$

Indeed, integrating over w through $\widetilde{K}_3(z, w)$ requires to Taylor expand $\widehat{K}_1(x, w) \widehat{K}_2(y, w)$. The only term in this expansion keeping the structure of the \widehat{K} s and thus contributing to the leading divergence is the first one: $\widehat{K}_1(x, z) \widehat{K}_2(y, z)$. Thus,

$$\begin{aligned}
L_{\triangle}^{(1)} &= \int dx dy dz \widehat{K}_1(x, z) \widehat{K}_2(y, z) \widetilde{K}_4(x, y) \widetilde{K}_5(x, z) \widetilde{K}_6(y, z) + \mathcal{O}(A\Lambda^2 \ln A\Lambda^2) \\
&= J_{\circlearrowleft}^{(1)} + \mathcal{O}(A\Lambda^2 \ln A\Lambda^2). \tag{C.16}
\end{aligned}$$

Similarly, in

$$L_{\triangle}^{(2)} = \int dx dy dz dw \widehat{K}_1(x, w) \widetilde{K}_2(y, w) \widetilde{K}_3(z, w) \widetilde{K}_4(x, y) \widetilde{K}_5(x, z) \widehat{K}_6(y, z), \tag{C.17}$$

if any partial derivative acts on one of the two \widehat{K} s through the Taylor expansion, the $(\ln A\Lambda^2)^2$ are lost. Thus,

$$\begin{aligned}
L_{\triangle}^{(2)} &= \int dx dy dz \widetilde{K}_4(x, y) \widetilde{K}_5(x, z) \widehat{K}_6(y, z) \widehat{K}_1(x, z) \sum_{n=0}^{\infty} \frac{t_3^n}{n!} \frac{d^n}{dt_2^n} \widetilde{K}_2(y, z), \\
&= \int dx \widehat{K}_{1,5}(x, x) \sum_{n, m \geq 0} \frac{t_3^n t_5^m}{n! m!} C_{n, m}(t_6, t_2, t_4; x) = J_{\circlearrowleft}^{(1)}, \tag{C.18}
\end{aligned}$$

up to subleading terms. It is possible to compute

$$\begin{aligned}
L_{\triangle}^{(3)} &= \int dx dy dz dw \widehat{\widetilde{K}}_1(x, y) \widetilde{K}_2(x, z) \left(-\frac{d}{dt_3} \widetilde{K}_3(y, z) \right) \widehat{\widetilde{K}}_4(x, w) \widehat{\widetilde{K}}_5(y, w) \widetilde{K}_6(z, w) , \\
L_{\triangle}^{(4)} &= \int dx dy dz dw \widetilde{K}_1(x, y) \widehat{\widetilde{K}}_2(x, z) \left(-\frac{d}{dt_3} \widetilde{K}_3(y, z) \right) \widehat{\widetilde{K}}_4(x, w) \widehat{\widetilde{K}}_5(y, w) \widetilde{K}_6(z, w) , \\
L_{\triangle}^{(5)} &= \int dx dy dz dw \left(-\frac{d}{dt_3} \widetilde{K}_1(x, y) \right) \widehat{\widetilde{K}}_2(x, z) \widetilde{K}_3(y, z) \widehat{\widetilde{K}}_4(x, w) \widehat{\widetilde{K}}_5(y, w) \widetilde{K}_6(z, w) ,
\end{aligned} \tag{C.19}$$

with the same reasoning. In these three integrals, integrating over w implies to Taylor expand a product of two $\widehat{\widetilde{K}}$ s. Since there are only three $\widehat{\widetilde{K}}$ s in these integrals, this means that, if one wants to extract the leading divergence with two $\ln A\Lambda^2$, one shall only keep in the Taylor expansion the terms with the derivatives acting on one $\widehat{\widetilde{K}}$. Up to subleading terms, one has

$$\begin{aligned}
L_{\triangle}^{(3)} &= J_{\infty}^{(3)} + L_{\triangle}^{(3a)} + L_{\triangle}^{(3b)} , \\
L_{\triangle}^{(3a)} &= \int dx dy dz \widehat{\widetilde{K}}_1(x, y) \widetilde{K}_2(x, z) \frac{d}{dt_3} \widetilde{K}_3(y, z) \widehat{\widetilde{K}}_4(x, z) \sum_{n=0}^{\infty} \frac{t_6^{n+1}}{(n+1)!} \frac{d^n}{dt_5^n} \widetilde{K}_5(y, z) , \\
L_{\triangle}^{(3b)} &= \int dx dy dz \widehat{\widetilde{K}}_1(x, y) \widetilde{K}_2(x, z) \frac{d}{dt_3} \widetilde{K}_3(y, z) \widehat{\widetilde{K}}_5(y, z) \sum_{n=0}^{\infty} \frac{t_6^{n+1}}{(n+1)!} \frac{d^n}{dt_4^n} \widetilde{K}_4(x, z) .
\end{aligned} \tag{C.20}$$

Integrating over x in $L_{\triangle}^{(3a)}$ leads to

$$\int dz \widehat{\widetilde{K}}_{2+4}(z, z) \sum_{n=0}^{\infty} \frac{t_6^{n+1}}{(n+1)!} C_{1,n}(t_1, t_3, t_5; z) = J_{\infty}^{(1)} - \frac{A\Lambda^2}{(4\pi)^3} (\ln A\Lambda^2)^2 \frac{1}{\alpha_1 + \alpha_2} \tag{C.21}$$

while integrating over y in $L_{\triangle}^{(3b)}$ transforms one of the two remaining $\widehat{\widetilde{K}}$ s, resulting in a $\mathcal{O}(A\Lambda^2 \ln A\Lambda^2)$ term. $L_{\triangle}^{(4)}$ and $L_{\triangle}^{(5)}$ are computed in a similar way. Summing up:

$$L_{\triangle}^{(3)} = 2J_{\infty}^{(1)} , \quad L_{\triangle}^{(4)} = J_{\infty}^{(1)} + 2J_{\infty}^{(1)} , \quad L_{\triangle}^{(5)} = J_{\infty}^{(1)} + J_{\infty}^{(1)} . \tag{C.22}$$

The same idea is used to compute

$$\begin{aligned}
L_{\circlearrowleft}^{(3)} &= \int dx dy dz dw \widehat{\widetilde{K}}_1(x, z) \widehat{\widetilde{K}}_2(x, z) \frac{d}{dt_3} \widetilde{K}_3(y, z) \widehat{\widetilde{K}}_4(y, w) \widehat{\widetilde{K}}_5(y, w) \frac{d}{dt_6} \widetilde{K}_6(x, w) , \\
L_{\triangle}^{(6)} &= \int dx dy dz dw \widehat{\widetilde{K}}_1(x, y) \widehat{\widetilde{K}}_2(x, z) \frac{d}{dt_3} \widetilde{K}_3(y, z) \widehat{\widetilde{K}}_4(z, w) \widehat{\widetilde{K}}_5(y, w) \frac{d}{dt_6} \widetilde{K}_6(x, w) ,
\end{aligned} \tag{C.23}$$

the requirement is then to leave two of the four $\widehat{\widetilde{K}}$ s without derivatives. Among such terms, one gets also subleading terms not considered here. In the end, one obtains:

$$L_{\circlearrowleft}^{(3)} = 2L_{\triangle}^{(6)} = 4J_{\circlearrowleft}^{(1)} . \tag{C.24}$$

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Résumé

Le but de cette thèse est d'étudier la gravité quantique bidimensionnelle. Nous nous intéressons plus particulièrement aux approches dans le continu. Ces dernières reposent principalement sur l'action de Liouville qui décrit le couplage entre théorie conforme et gravité. Si cette action, bien connue, est très bien comprise, la mesure de l'intégrale fonctionnelle sur l'espace des métriques pose plus de problèmes. Toutefois, sous l'hypothèse simplificatrice d'une mesure de champ libre, la dépendance en l'aire de la fonction de partition de la gravité quantique en présence de matière conforme a pu être établie. Malgré l'hypothèse assez forte sur la mesure d'intégration, cette formule (dite KPZ), a été confirmée par des calculs issus de méthodes discrètes, et ce dans plusieurs cas particuliers.

Grâce à une nouvelle méthode de régularisation spectrale en espace courbe, cette mesure d'intégration a récemment pu être proprement définie. Dans cette thèse, un calcul perturbatif de la fonction de partition à aire fixée est mené, jusqu'à trois boucles, en considérant l'action de Liouville et des surfaces de Kähler de genre quelconque (qui coïncident avec l'ensemble des surfaces à deux dimensions). Des divergences apparaissant dans les calculs, il est nécessaire de renormaliser les actions. Cette renormalisation peut être interprétée comme une renormalisation de la mesure d'intégration. Nos résultats à deux boucles sont finis, indépendants de la régularisation et compatibles avec le résultat KPZ, mais dépendent d'un paramètre libre. L'étude à trois boucles suggère que la théorie reste renormalisable aux ordres supérieurs mais dépend de nouveaux paramètres à chaque ordre. Ces résultats ont été généralisés dans le cas du tore au couplage à de la matière non-conforme.

Mots Clés

Gravité quantique bidimensionnelle, théorie de Liouville, action de Mabuchi, susceptibilité de la corde, régularisation spectrale, renormalisation.

Abstract

Nowadays, two-dimensional quantum gravity can be studied in two different approaches, one involving discrete theories (triangulation, matrix model...), the other continuous ones, mainly based on the so called Liouville action which universally describes the coupling of any conformal field theory to gravity. While the Liouville action is relatively well understood, the appropriate functional integral measure is however rather complicated. Nevertheless, a formula for the area dependence of the quantum gravity partition function in the presence of conformal matter has been obtained, under the simplifying assumption of a free-field measure. Notwithstanding its non-rigorous derivation, this formula, often referred to as the KPZ formula, has since been verified in many instances and has scored many successes.

Recent developments of efficient multi-loop regularization methods on curved space-times opened the way for a precise and well-defined perturbative computation of the fixed-area partition function in the Kähler formalism. In this work, a first-principles computation of the fixed-area partition function in the Liouville theory is performed, up to three loops. Among other things, the role of the non-trivial quantum gravity integration measure is highlighted. Renormalization is required and may be interpreted as a renormalization of the integration measure. This leads to a finite and regularization-independent result at two loops, that is more general than the KPZ result, although compatible. Finiteness and regularization-independence seem also possible at three loops. These results are generalized to the coupling to non-conformal matter on the torus.

Keywords

2D quantum gravity, Liouville theory, Mabuchi action, string susceptibility, spectral cutoff regularization, renormalization.