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# The topology of loop braid groups: applications and remarkable quotients

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Oresme



O time, thou must untangle this, not I.  
It is too hard a knot for me to untie!

---

*Twelfth Night*  
SHAKESPEARE



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# Introduction

Les groupes de tresses ont été introduits par Hurwitz [42] en 1891 comme groupes fondamentaux des espaces des configurations de  $n$  points dans le plan complexe. Cependant, ils doivent leur nom à Artin [3] : il les considérait en termes d'automorphismes de  $F_n$ , les groupes libres de rang  $n$ , mais aussi en termes géométriques. L'interprétation géométrique est sans doute la plus intuitive et mieux connue, en particulier grâce à son utilisation dans la théorie des nœuds. Magnus [60] considéra les groupes de tresses d'un point de vue de classes d'un groupe modulaire, tandis que Markov [62] introduisait ces groupes d'un point de vue purement relatif à la théorie des groupes. Il est connu depuis longtemps que tous ces points de vue sont équivalents [84]. Chacune de ces définitions porte à une différente généralisation : par exemple, on peut voir les groupes de tresses comme des cas particuliers de groupes d'Artin-Tits, de groupes de Garside, de groupes modulaire, de groupes de tresses sur surfaces. Peu de ces généralisations partagent avec les groupes de tresses leur propriété principale : une grande famille de définitions équivalentes.

Les *groupes de tresses de cercles* sont une exception remarquable à ce fait. Leur étude a été largement développée au cours des vingt dernières années. Le premier fait curieux est qu'elles apparaissent en littérature avec un grand nombre de noms différents. Ici on choisit d'adopter la terminologie introduite par Baez, Wise, et Crans [8], parce que leur définition de groupes de tresses de cercles en termes de groupes modulaires convient aux propos de cette thèse. On donnera bientôt une idée de cette définition : tout d'abord on remarque qu'il ne s'agit ni de la première appellation, ni de la première interprétation des tresses de cercles qui serait apparue au fil du temps.

Le concept qui se trouve derrière la genèse des tresses de cercles est celui de *mouvements* [angl. : *motions*] d'un sous-espace compact  $N$  dans une variété  $M$  : pour avoir une idée de cette notion on peut penser aux différentes manières de bouger  $N$  dans  $M$  de sorte qu'à la fin du mouvement  $N$  soit revenu en position d'origine. Le premier pas vers les mouvements est dû à Fox et Neuwirth [34]. En 1962 ils donnent une nouvelle preuve de la présentation d'Artin pour les groupes de tresses, en termes de groupes fondamentaux d'espaces des configurations de  $n$  points non-ordonnés sur un plan. La même année Dahm, dans sa thèse de doctorat, généralise ce concept aux variétés quelconques. Il applique cette construction à une collection de  $n$  cercles non noués et non entrelacés dans l'espace à 3 dimensions : ces mouvements représentent la première apparition du groupe de tresses de cercles et ses résultats sont publiés et étendus dans [37]. Avec des termes plus modernes, au lieu de parler de groupes de mouvements, on parlera plutôt

de groupes modulaires [angl. : mapping class groups]. En effet, on peut donner une définition informelle de nos protagonistes comme suit.

**Définition Informelle.** Fixons  $n \geq 1$ , le *groupe de tresses de cercles* avec  $n$  composantes est le groupe modulaire de  $n$  cercles non noués et non entrelacés dans l'espace de dimension 3.

Une définition formelle sera donnée dans le Chapitre 1 (Définition 1.1.5). En 1986 McCool [64] considère les tresses de cercles comme *automorphismes de conjugaison de base* [angl. : *basis-conjugation automorphisms*] et en 1996 Savushkina [73] les considère comme *automorphismes de permutation et conjugaison* [angl. : *permutation-conjugacy automorphisms*] de  $F_n$ . Les tresses de cercles sont aussi connues dans la littérature comme *tresses soudées* [angl. *welded braids*], suivant la nomenclature avec laquelle elles ont été introduites par Fenn, Rimányi et Rourke [30]. Celle-ci est peut être la plus connue des notations. Elle est associée à des diagrammes de chemins monotones dans l'espace de dimension 2, très similaires aux diagrammes qu'on utilise pour représenter les tresses usuelles. Plus récemment, Brendle et Hatcher [21] ont proposé une interprétation de ces groupes comme groupes fondamentaux des espaces des configurations de  $n$  cercles euclidiens. De plus, on peut donner une interprétation topologique des tresses de cercles : grosso modo, on peut les voir comme des immersions dans l'espace de dimension 4 de  $n$  copies disjointes de la couronne orientée  $S^1 \times I$  avec les composantes de bord fixées. Les tresses de cercles peuvent aussi être codées par des diagrammes de Gauss pour tresses : cette dernière est une interprétation combinatoire, utilisée implicitement dans [10]. Cette interprétation peut aussi être vue comme un cas particulier des objets considérés dans [5]. Elle représente également un point de vue qui permet de considérer les groupes de tresses de cercles comme des quotients des *groupes de tresses virtuelles*. Le Chapitre 1 sera dédié à la présentation de tous ces formalismes et à la démonstration de leur équivalence.

Dans le Chapitre 1 on considère aussi les *groupes de tresses de cercles étendus*, une généralisation des groupes de tresses de cercles. Dans la littérature, ces groupes ont été moins étudiés, mais comme il apparaîtra au cours de ce travail, ils sont encore plus intéressants que les groupes de tresses de cercles, et émergent d'une façon plus naturelle quand on considère les objets analogues aux tresses usuelles dans l'espace de dimension 4.

En plus d'apparaître dans tellement de contextes, les groupes de tresses de cercles sont des objets intéressants en eux-mêmes. Tout d'abord, plusieurs calculs et conjectures à propos des algèbres de cohomologie des groupes de tresses de cercles *purs*, des sous-groupes des groupes de tresses de cercles, ont émergé dans la littérature. L'étude des propriétés de ces algèbres, par exemple en relation avec les variétés de résonance et les rangs des suites centrales descendantes, nous permet de comparer les groupes de tresses de cercles avec les groupes de tresses, et avec d'autres généralisations des groupes de tresses. Dans la Section 1.7 on donne des références à propos de ce thème.

Dans cette thèse on se concentrera en particulier sur les aspects topologiques et les applications des groupes de tresses de cercles. Du point de vue de la topologie, les groupes de tresses de cercles peuvent être vus comme la contrepartie tressée d'un type particulier d'objet noué : les *tores-entrelacs ruban* [angl. : *ribbon torus-links*]. On va les définir dans le Chapitre 2 (Définition 2.1.3). Pour le moment on en donne juste une idée :

**Définition Informelle.** Pour  $n \geq 1$ , un *tore-entrelacs ruban* avec  $n$  composantes est un plongement de  $n$  tores orientés  $S^1 \times S^1$  dans l'espace de dimension 4.

Dans la théorie classique de tresses et entrelacs, le théorème d'Alexander nous permet de représenter tout entrelacs comme la clôture d'une tresse. De plus, le théorème de Markov établit que deux tresses, qui peuvent avoir un nombre différent de brins, ont des clôtures isotopes dans l'espace de dimension 3 si et seulement si une peut être obtenue de l'autre avec un nombre fini de mouvements de Markov, appelés conjugaison et stabilisation. Ce théorème permet de décrire toutes les tresses avec clôtures isotopes comme entrelacs dans l'espace de dimension 3. Ces deux théorèmes nous permettent de retrouver certains invariants d'entrelacs comme traces de Markov, des traces sur les algèbres de groupe des groupes de tresses. Pour ce qui concerne les tresses de cercles, on a un théorème de type Alexander et un théorème de type Markov [47] pour les tresses soudées, l'interprétation diagrammatique des tresses de cercles. Toutefois, on n'en a pas une version topologique. Inspirés par ce fait, dans le Chapitre 2, on propose un premier pas vers un théorème de Markov topologique pour les tresses de cercles et les tores-entrelacs ruban.

Une autre application des groupes de tresses de cercles peut être trouvée dans la théorie de Lie, par exemple dans le travail de Bar Natan [9–11]. Cette application concerne un troisième type d'objets noués qui dérivent des tresses de cercles : les *enchevêtrements ruban* [angl. *ribbon tangles*], que Bar Natan formalise en tant qu'*enchevêtrements soudés* [angl. *welded tangles*]. Les enchevêtrements ruban sont la notion analogue des enchevêtrements usuels, mais en termes de couronnes et tores plongés dans l'espace de dimension 4. La définition formelle de ces objets est donnée dans le Chapitre 3 (Définition 3.1.1) ; pour le moment on propose une version informelle.

**Définition Informelle.** Un *enchevêtrement ruban* est un plongement d'une union disjointe de tores orientés  $S^1 \times S^1$  et de couronnes orientées  $S^1 \times I$  dans l'espace de dimension 4.

On traitera d'entrelacs ruban dans le Chapitre 3 pour définir une généralisation du polyôme d'Alexander.

De plus, il ne faut pas oublier de mentionner la théorie des représentations des groupes de tresses de cercles. Les représentations de Burau s'étendent trivialement aux groupes de tresses de cercles en utilisant l'expansion de Magnus et les dérivées de Fox [14], mais la linéarité ou non des groupes de tresses de cercles n'a pas encore été déterminée. Certains nouveaux résultats sur les représentations locales des groupes de tresses de cercles, qui apparaissent comme extensions des représentations des groupes de tresses, peuvent être trouvés en [45] et [22]. Cependant l'étude des quotients de dimension finie des algèbres des groupes de tresses de cercles n'est pas encore présente dans la littérature. Dans [45] les auteurs s'intéressent également à certains quotients remarquables des groupes de tresses de cercles : les *groupes de tresses de cercles symétriques* [angl. : *symmetric loop braid groups*], aussi connus comme *tresses soudées symétriques* [angl. : *unrestricted virtual braid groups*] [53]. On traitera la structure de ces groupes dans le Chapitre 4, et ensuite on décrira quelques applications aux *entrelacs «fused»*.

Dans la suite on propose un aperçu de chaque chapitre :

## Groupes de tresses de cercles

Plusieurs formulations parallèles et plusieurs notations ont été utilisées pour les groupes de tresses de cercles, de la part de chercheurs qui travaillent dans des domaines différents. Bien que l'équivalence de certaines de ces formulations ait déjà été démontrée, ou au moins énoncée, dans la littérature, par exemple dans [10, Section 2, Théorème 2.6], toutes n'ont pas été explicitement reliées. Le propos de ce chapitre est de donner une collection de ces formalismes. De plus, on propose une preuve complète de l'équivalence de ces formulations. Cela nous met dans un cadre théorique clair et bien établi pour les groupes de tresses de cercles, comme celui qu'on a dans le cas des groupes de tresses classiques.

Dans la première partie du Chapitre 1 on donne la définition des groupes de tresses de cercles, notés  $LB_n$ , comme étant les groupes de classes d'isotopie des homéomorphismes de la 3-boule  $B^3$  dans elle-même qui :

- fixent ponctuellement le bord de  $B^3$  ;
- fixent globalement une collection de cercles disjoints, non noués, non entrelacés et orientés dans l'intérieur de  $B^3$  ;
- préservent l'orientation de  $B^3$  et de la collection des cercles.

On définit aussi les groupes de tresses de cercles étendus, notés  $LB_n^{ext}$ , comme les groupes d'isotopie des classes de ce type d'homéomorphismes sans la condition de préserver l'orientation de la collection de cercles.

Ensuite on introduit deux familles d'espaces de configurations de cercles dans la 3-boule  $B^3$ . Les premiers sont les espaces de configurations de  $n$  cercles euclidiens, non ordonnés, disjoints et non entrelacés dans la 3-boule  $B^3$  : ces espaces sont notés  $\mathcal{R}_n$ . Les seconds sont les espaces de configurations d'une collection de cercles avec les susdites caractéristiques, avec en plus la propriété que les cercles soient situés sur des plans parallèles. Ces espaces de configurations sont notés  $\mathcal{UR}_n$ . Leurs groupes fondamentaux sont respectivement appelés *groupes d'anneaux* [angl. *ring groups*] et *groupe d'anneaux parallèles* [angl. *untwisted ring groups*] et sont respectivement notés avec les symboles  $R_n$  et  $UR_n$ . Le premier résultat important de ce chapitre établit que les groupes d'anneaux  $R_n$  sont isomorphes aux groupes de tresses de cercles étendus  $LB_n^{ext}$ . On en énonce ici une version simplifiée :

**Théorème 1.2.11.** Pour  $n \geq 1$ , il existe un isomorphisme naturel entre le groupe d'anneaux  $R_n$  et le groupe de tresses de cercles étendu  $LB_n^{ext}$ .

Ce théorème nous permet de démontrer un résultat analogue pour les versions non-ordonnées de ces groupes. De plus, il représente une première motivation pour considérer les groupes de tresses de cercles étendus.

**Proposition 1.2.12.** Pour  $n \geq 1$ , il existe un isomorphisme naturel entre le groupe d'anneaux parallèles  $UR_n$  et le groupe de tresses de cercles  $LB_n$ .

De plus, Brendle et Hatcher donnent des présentations pour les groupes d'anneaux  $R_n$  et pour les groupes d'anneaux parallèles  $UR_n$ . La correspondance des groupes d'anneaux (parallèles) avec les groupes de tresses de cercles (étendus) nous permet d'énoncer les résultats qui suivent.

**Corollaire 1.2.15, (voir aussi [21]).** Pour  $n \geq 1$ , le groupe  $LB_n$  admet la présentation donnée par les générateurs  $\{\sigma_i, \rho_i \mid i = 1, \dots, n-1\}$ , soumis aux relations :

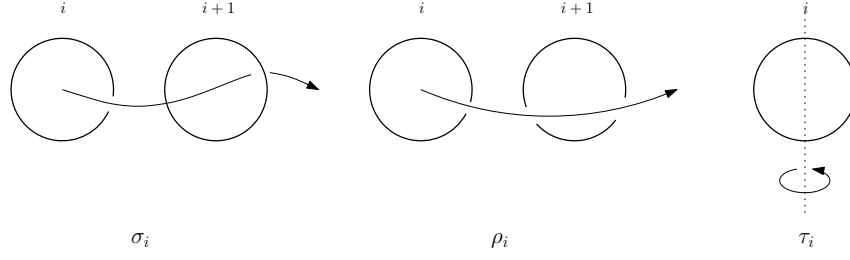
$$\left\{ \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{pour } |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{pour } i = 1, \dots, n-2 \\ \rho_i \rho_j = \rho_j \rho_i & \text{pour } |i-j| > 1 \\ \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} & \text{pour } i = 1, \dots, n-2 \\ \rho_i^2 = 1 & \text{pour } i = 1, \dots, n-1 \\ \rho_i \sigma_j = \sigma_j \rho_i & \text{pour } |i-j| > 1 \\ \rho_{i+1} \rho_i \sigma_{i+1} = \sigma_i \rho_{i+1} \rho_i & \text{pour } i = 1, \dots, n-2 \\ \sigma_{i+1} \sigma_i \rho_{i+1} = \rho_i \sigma_{i+1} \sigma_i & \text{pour } i = 1, \dots, n-2. \end{array} \right.$$

**Corollaire 1.2.17, (voir aussi [21]).** Pour  $n \geq 1$ , le groupe  $LB_n^{ext}$  admet la présentation donnée par les générateurs  $\{\sigma_i, \rho_i \mid i = 1, \dots, n-1\}$ , et  $\{\tau_i \mid i = 1, \dots, n\}$ , soumis aux relations :

$$\left\{ \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{pour } |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{pour } i = 1, \dots, n-2 \\ \rho_i \rho_j = \rho_j \rho_i & \text{pour } |i-j| > 1 \\ \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} & \text{pour } i = 1, \dots, n-2 \\ \rho_i^2 = 1 & \text{pour } i = 1, \dots, n-1 \\ \rho_i \sigma_j = \sigma_j \rho_i & \text{pour } |i-j| > 1 \\ \rho_{i+1} \rho_i \sigma_{i+1} = \sigma_i \rho_{i+1} \rho_i & \text{pour } i = 1, \dots, n-2 \\ \sigma_{i+1} \sigma_i \rho_{i+1} = \rho_i \sigma_{i+1} \sigma_i & \text{pour } i = 1, \dots, n-2 \\ \tau_i \tau_j = \tau_j \tau_i & \text{pour } i \neq j \\ \tau_i^2 = 1 & \text{pour } i = 1, \dots, n \\ \sigma_i \tau_j = \tau_j \sigma_i & \text{pour } |i-j| > 1 \\ \rho_i \tau_j = \tau_j \rho_i & \text{pour } |i-j| > 1 \\ \tau_i \rho_i = \rho_i \tau_{i+1} & \text{pour } i = 1, \dots, n-1 \\ \tau_i \sigma_i = \sigma_i \tau_{i+1} & \text{pour } i = 1, \dots, n-1 \\ \tau_{i+1} \sigma_i = \rho_i \sigma_i^{-1} \rho_i \tau_i & \text{pour } i = 1, \dots, n-1. \end{array} \right.$$

Dans la Figure 1 on montre l'interprétation des générateurs qui apparaissent dans ces présentations comme des éléments des groupes d'anneaux parallèles  $UR_n$  et des groupes d'anneaux  $R_n$ .

On passe à une interprétation des tresses de cercles comme automorphismes de  $F_n$ , le groupe libre de rang  $n$ . On introduit les sous-groupes des *automorphismes de permutation-conjugaison*  $PC_n^*$  de  $F_n$ , et on rappelle un résultat de Dahm, publié par Goldsmith, qui

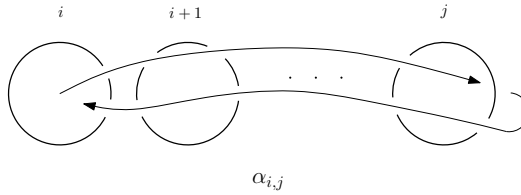
Figure 1: Éléments  $\sigma_i$ ,  $\rho_i$  et  $\tau_i$ .

énonce que les groupes de tresses de cercles sont isomorphes aux groupes  $PC_n^*$  [37]. Ce résultat offre aussi une caractérisation des tresses de cercles comme automorphismes d'une façon similaire à la caractérisation d'Artin des tresses. À présent, on va donner une présentation pour les sous-groupes *purs* des groupes de tresses de cercles étendus  $LB_n^{ext}$ , qu'on note  $PLB_n^{ext}$ .

**Proposition 1.3.8.** Pour  $n \geq 1$ , le groupe  $PLB_n^{ext}$  admet la présentation donnée par les générateurs  $\{\alpha_{ij} \mid 1 \leq i \neq j \leq n\}$  et  $\{\tau_i \mid i = 1, \dots, n\}$  soumis aux relations :

$$\begin{cases} \alpha_{ij}\alpha_{kl} = \alpha_{kl}\alpha_{ij} \\ \alpha_{ij}\alpha_{kj} = \alpha_{kj}\alpha_{ij} \\ (\alpha_{ij}\alpha_{kj})\alpha_{ik} = \alpha_{ik}(\alpha_{ij}\alpha_{kj}) \\ \tau_i^2 = 1 \\ \tau_i\alpha_{ij} = \alpha_{ij}\tau_i \\ \tau_i\alpha_{jk} = \alpha_{jk}\tau_i \\ \tau_i\alpha_{ji}\tau_i = \alpha_{ji}^{-1} \end{cases}$$

où des lettres différentes indiquent des indices différents.

Figure 2: Éléments  $\alpha_{ij}$ .

Dans la Figure 2 on montre l'interprétation des générateurs de  $PLB_n^{ext}$  comme éléments du groupe d'anneaux  $R_n$ .

Dans la suite on passe à un point de vue plus topologique sur les tresses de cercles, vues comme des classes particulières de couronnes nouées dans  $B^4$ , la boule de dimension 4. On appelle ces objets *tresses ruban (étendues)* [angl.: *(extended) ribbon braids*]. Pour  $n \geq 1$ , on note le groupe de tresses ruban avec le symbole  $rB_n$ , et le groupe de tresses ruban



étendu avec  $rB_n^{ext}$ . Les résultats principaux de cette partie établissent des isomorphismes entre groupes de tresses ruban (étendus) et groupes de tresses de cercles (étendus) :

**Théorème 1.4.11 et Théorème 1.4.18.** Pour  $n \geq 1$ , il y a des isomorphismes entre le groupe de tresses ruban  $rB_n$  et le groupe de tresses de cercles  $LB_n$ , et entre leurs versions étendues  $rB_n^{ext}$  et  $LB_n^{ext}$ .

Dans le contexte des tresses ruban, avec les isomorphismes précédents, on peut aussi démontrer que la notion d'isotopie qu'on a choisie pour les tresses ruban coïncide avec la notion d'isotopie de la 4-boule dans laquelle les tresses ruban sont plongées. En particulier, on a le résultat suivant.

**Théorème 1.4.5.** Toute isotopie d'une tresse ruban géométrique dans  $B^3 \times I$  s'étend à une isotopie de  $B^3 \times I$  dans lui-même qui est l'identité sur le bord.

On décrit aussi deux types de représentations graphiques en dimension 3 et 2 des tresses de cercles. On introduit les *surfaces cassées* [angl. : *broken surfaces*], et les *diagrammes soudés* [angl. : *welded diagrams*]. Les surfaces cassées sont des projections en position générale des tresses de cercles dans l'espace de dimension 3. Dans la Figure 3 on peut voir des surfaces cassées équivalentes entre elles. Les diagrammes soudés sont des diagrammes en dimension 2, introduits dans [30].

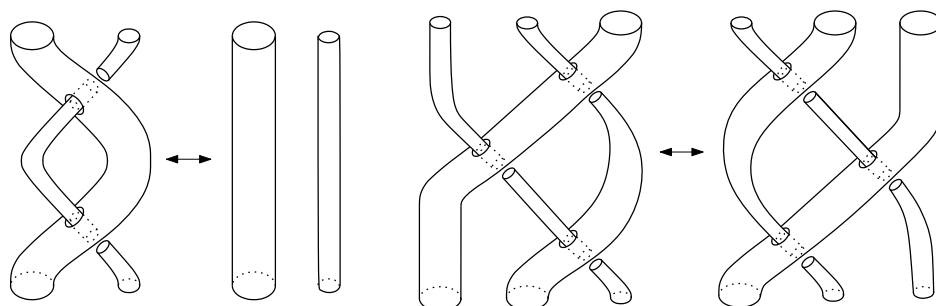


Figure 3: Surfaces cassées.

Enfin on traite d'une description combinatoire des tresses de cercles à travers les *diagrammes de Gauss*. Ces diagrammes, dans le sens de [69], codent l'information d'un objet noué. On utilise les résultats de Cisneros à propos des diagrammes de Gauss pour tresses virtuelles [25] afin de démontrer l'existence d'un isomorphisme entre des quotients de l'ensemble des diagrammes de Gauss et les diagrammes des tresses soudées.

Pour terminer cet aperçu sur les groupes de tresses de cercles, on donne une brève exposition de l'histoire de ces objets, et d'autres références à propos des applications topologiques et de la théorie de représentations de ces groupes.

## Une application topologique : vers une version topologique du théorème de Markov pour les tresses de cercles

Dans le Chapitre 2, on effectue les premiers pas pour adapter des résultats classiques dans la théorie des nœuds aux tores-entrelacs ruban, la contrepartie nouée des tresses de cercles. Pour cette raison, on considère les tresses de cercles comme tresses ruban (Théorème 1.4.11). Dans le cas classique, le théorème d'Alexander énonce que tout entrelacs peut être représenté comme la clôture d'une tresse. Ce résultat a déjà été démontré pour les diagrammes de tresses soudées et leur contrepartie nouée par Kamada [47]. L'isomorphisme entre diagrammes de tresses soudées et tresses ruban, décrit dans le Chapitre 1, garantit que le théorème d'Alexander reste valide quand on passe aux tresses ruban et aux tores-entrelacs ruban.

Le théorème de Markov présente plus de difficultés, à cause du manque d'une correspondance biunivoque entre tores-entrelacs ruban et diagrammes d'entrelacs soudés. Dans la théorie classique de tresses et nœuds ce théorème énonce que deux tresses, pouvant avoir un nombre différent de brins, ont des clôtures isotopes dans  $\mathbb{R}^3$  si et seulement si l'une peut être obtenue de l'autre avec un nombre fini de mouvements de Markov, appelés conjugaison et stabilisation. Ce théorème permet de décrire toutes les tresses avec clôtures isotopes comme entrelacs dans  $\mathbb{R}^3$ . Concernant les tresses de cercles vues comme diagrammes de tresses soudées, une démonstration combinatoire de ce théorème peut être trouvée dans [47]. Dans cette thèse on démontre une version du théorème de Markov pour les tresses de cercles avec clôture dans l'analogue d'un tore solide dans  $\mathbb{R}^4$ , qu'on énonce tout de suite. On rappelle qu'on note  $rB_n$  les groupes de tresses ruban, et  $rB_n^{ext}$  les groupes de tresses ruban étendus.

**Théorème 2.3.1.** Soit  $n \geq 1$  et  $\beta, \beta' \in rB_n$  une paire de tresses ruban. Les tresses ruban clôturées  $\widehat{\beta}, \widehat{\beta}'$  sont isotopes dans  $B^3 \times S^1$  si et seulement si  $\beta$  et  $\beta'$  sont conjuguées dans  $rB_n^{ext}$ .

On remarque que pour avoir le même type d'énoncé qu'on a pour le cas classique, il faut passer aux groupes étendus, dans le cas desquels on a :

**Théorème 2.3.3.** Soit  $n \geq 1$  et  $\beta, \beta' \in rB_n^{ext}$  une paire de tresses ruban étendues. Les tresses ruban étendues clôturées  $\widehat{\beta}, \widehat{\beta}'$  sont isotopes dans  $B^3 \times S^1$  si et seulement si  $\beta$  et  $\beta'$  sont conjuguées dans  $rB_n^{ext}$ .

Pour étendre le résultat dans tout l'espace de dimension 4, et pas seulement dans le tore solide, il faut démontrer l'invariance de la classe d'isotopie d'une tresse ruban (étendue) clôturée par rapport à l'opération de stabilisation sur les tresses ruban (étendues). L'approche qu'on a suivi pour la conjugaison ne nous permet pas de le faire sans passer à une interprétation diagrammatique. Pour cette raison l'exploration d'autres approches fait partie des projets futurs.

## Une généralisation du polynôme d'Alexander : l'invariant d'Alexander pour les enchevêtrements ruban

Le Chapitre 3 concerne un travail en commun avec Vincent Florens [29]. Dans cette partie on considère un troisième type d'objets noués : les enchevêtrements ruban. Dans ce travail on introduit un invariant de type Alexander pour les enchevêtrements ruban, où, quand on dit "de type Alexander" on entend qu'il partage la technique de construction avec le polynôme d'Alexander de la théorie de nœuds classique. On considère un enchevêtrement ruban  $T$  dans la 4-boule  $B^4$ . Les composantes de bord des couronnes qui composent  $T$  sont les composantes d'un entrelacs trivial  $L$  dans  $S^3$ , le bord de  $B^4$ . On fixe un groupe libre abélien  $G$ , et on dit qu'on colore avec  $G$  l'enchevêtrement ruban  $T$  quand on choisit un homomorphisme de groupes  $\varphi$  défini sur le premier groupe d'homologie du complément de  $T$  dans  $B^4$  à valeurs dans  $G$ .

On construit un *invariant d'Alexander*  $A(T)$  de l'enchevêtrement colorié  $T$  comme un élément de l'algèbre extérieure du  $\mathbb{Z}[G]$ -module d'homologie  $H_1^\varphi(S^3 \setminus L)$ , tordue par le morphisme induit par  $\varphi$ . On note ce module  $H_\partial$ .

Ensuite on change de formalisme et on introduit une généralisation fonctorielle du polynôme d'Alexander, inspirée par [19] et [33], qui va de la catégorie  $\mathcal{Rib}_G$  des *cobordismes ruban* à la catégorie des modules gradués. On l'appelle le *foncteur de Burau*, et on le note par  $\rho$ . Dans ce contexte on voit les enchevêtrements ruban plongés dans  $B^3 \times I$ , le cylindre de dimension 4, avec les composantes de bord dans les intérieurs des 3-boules  $B^3 \times \{0\}$  et  $B^3 \times \{1\}$ . Les objets de la catégorie des cobordismes ruban sont des suites de signes associées aux composantes de bord des enchevêtrements ruban, alors que les morphismes sont les enchevêtrements ruban eux-mêmes, vus comme des morphismes qui vont des composantes de bord de la 3-boule supérieure, aux composantes de bord dans la 3-boule inférieure. Le foncteur de Burau associe aux séquences de signes de l'algèbre extérieure des modules gradués  $M_0$  et  $M_1$ . Aux enchevêtrements ruban il associe une somme d'applications linéaires entre les algèbres extérieures de  $M_0$  et  $M_1$ . Informellement, étant donné un enchevêtrement ruban dans la 4-boule  $B^4$  comme dans le premier formalisme, on peut séparer ses composantes de bord, en déformant la 4-boule  $B^4$  dans un cylindre en dimension 4, et en envoyant certaines composantes de bord dans la boule supérieure et les autres dans la boule inférieure. On démontre que le foncteur de Burau n'est qu'un changement de formalismes pour l'invariant d'Alexander  $A$ . Plus précisément, on montre :

**Théorème 3.1.19.** Soit  $(T, \varphi)$  un enchevêtrement ruban  $G$ -colorié, et soit  $(\tilde{T}, \varphi)$  une séparation de  $(T, \varphi)$  dans  $\mathcal{Rib}_G$ . Il existe un isomorphisme, défini à une unité de  $\mathbb{Z}[G]$  près,

$$\wedge^n H_\partial \longrightarrow \text{Hom}(\wedge M_0, \wedge M_1)$$

qui envoie  $A(T, \varphi)$  dans  $\oplus_k (-1)^{k(n_0-k)} \rho_k(\tilde{T}, \varphi)$ , où  $n$  est le nombre de couronnes plongées,  $k$  est la déficience de la présentation du module d'Alexander de  $(T, \varphi)$ ,  $n_0$  est le nombre de composantes de bord de  $(\tilde{T}, \varphi)$  qui se trouvent à l'intérieur de la boule  $B^3 \times \{0\}$ , et  $\rho_k$  est la  $k$ ème composante de  $\rho$ .

Quand on considère les tresses ruban comme un cas particulier des enchevêtrements ruban, on a une autre analogie entre l'invariant d'Alexander et le polynôme d'Alexander, via l'équivalence avec le foncteur de Burau. On l'énonce ici dans une forme simplifiée.

**Proposition 3.1.17.** Soit  $(S, \varphi)$  une tresse ruban  $G$ -coloriée. La somme des applications linéaires de modules gradués  $\rho(S, \varphi)$  coïncide avec les puissances extérieures de la représentation de Burau.

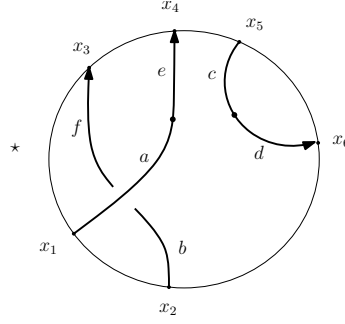


Figure 4: Un diagramme d'enchevêtrement soudé  $\tau$ .

On peut encore avancer d'un pas en donnant une méthode combinatoire pour calculer l'invariant  $A$ . Cela peut se faire grâce à une relation entre l'invariant  $A$  et un invariant défini par Archibald pour les diagrammes d'enchevêtrement virtuels, qui sont des objets proches des diagrammes d'enchevêtrements soudés [2]. En effet, on étend cet invariant diagrammatique aux diagrammes d'enchevêtrements soudés. En particulier, on démontre que pour calculer l'invariant d'Alexander  $A(T)$  d'un enchevêtrement ruban  $T$  associé à un diagramme d'enchevêtrement soudé  $\tau$ , il suffit de calculer l'invariant diagrammatique de  $\tau$ . Ce dernier se calcule à l'aide de règles simples pour construire une matrice d'Alexander avec des étiquettes qu'on place sur le diagramme, puis calculer certains mineurs de cette matrice.

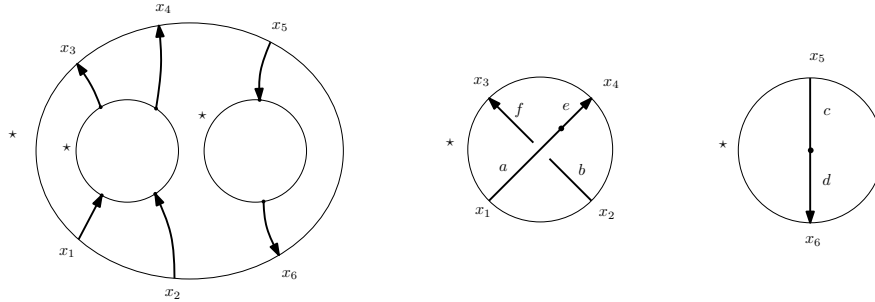


Figure 5: Un diagramme de circuit  $Q$  et deux diagrammes d'enchevêtrement soudés  $\sigma$  et  $\beta$ .

On peut simplifier encore plus la façon combinatoire de calculer l'invariant d'Alexander. On le fait en montrant qu'on peut définir des structures d'*algèbres de circuit* sur l'ensemble

des enchevêtrements ruban et sur l'ensemble des diagrammes d'enchevêtrements soudés. Ces structures, inspirées par les algèbres planaires de Jones [44], sont respectées par l'invariant d'Alexander  $A$  pour les enchevêtrements ruban, et par l'invariant diagrammatique pour les diagrammes d'enchevêtrements soudés. Cela a l'avantage de nous permettre de faire des calculs locaux sur les diagrammes d'enchevêtrements soudés pour calculer l'invariant d'Alexander  $A$  du enchevêtrement ruban associé. Cela signifie que, pour calculer l'invariant d'Alexander  $A(T)$ , où  $T$  est l'enchevêtrement ruban associé au diagramme d'enchevêtrement soudé  $\tau$  dans la Figure 4, il suffit de calculer les invariants diagrammatiques des diagrammes d'enchevêtrements soudés  $\sigma$  et  $\beta$  à droite dans la Figure 5, puis de les composer avec les règles imposées par le diagramme de circuit  $Q$  à gauche dans la Figure 5.

## Un quotient remarquable : les tresses soudées symétriques, et leur relation avec les entrelacs «fused»

On a mentionné précédemment qu'il y a une théorie des représentations des groupes de tresses de cercles. Kadar, Martin, Rowell, et Wang ont exploré les représentations des groupes de tresses qui peuvent s'étendre aux groupes de tresses de cercles  $LB_n$ , mais qui ne s'étendent pas à une famille de quotients des groupes de tresses de cercles appelés *groupes de tresses de cercles symétriques* [angl. : *symmetric loop braid groups*], notés avec le symbole  $SLB_n$  [45]. Si on passe des groupes de tresses de cercles à leur interprétation diagrammatique, les groupes de tresses soudés (à travers l'isomorphisme décrit dans le Chapitre 1), on a que ces quotients sont isomorphes aux *groupes de tresses soudées symétriques* [angl. : *unrestricted virtual braid groups*], notés  $UVB_n$ . Les groupes de tresses soudées symétriques ont été introduits par Kauffman et Lambropoulou, en tant que contreparties tressées des *entrelacs «fused»* [52]. Les auteurs ont aussi étendu le travail de Kamada [47] en présentant une version des théorèmes d'Alexander et Markov pour ces objets [53].

Dans le Chapitre 4 on présente une partie d'un travail en commun avec Valeriy G. Bardakov et Paolo Bellingeri [16] qui concerne ces objets. Les éléments du groupe de tresses soudées symétriques sont des classes d'équivalence de diagrammes de tresses soudées modulo la relation donnée par l'addition d'un mouvement de Reidemeister de plus, le *mouvement interdit* (F2) (voir Figure 6). Il a été démontré que tous les nœuds

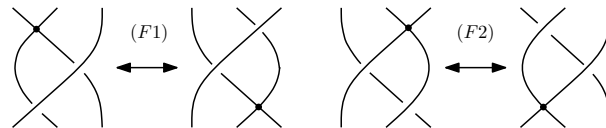


Figure 6: Mouvements interdits de type (F1) (à la gauche) et mouvements interdits de type (F2) (à la droite).

«fused» sont équivalents au nœud trivial [48, 67]. De plus, la preuve de Nelson dans [67] du fait que tout nœud virtuel se dénoue quand on permet les mouvements interdits, peut

être adapté verbatim aux entrelacs avec plusieurs composantes. On remarque que pour obtenir son résultat Nelson utilise les diagrammes de Gauss pour les entrelacs «fused». Alors, tout entrelacs «fused» est «*fused*» isotope à un diagramme d'entrelacs qui n'a que des croisements (classiques ou virtuels) entre composantes différentes. D'autre part, il existe des entrelacs «fused» non-triviaux, et leur classification n'est pas complètement triviale [32]. En particulier, dans [31], Fish et Keyman démontrent que les entrelacs «fused» qui n'ont que des croisements classiques sont caractérisés par les nombres d'enlacements classiques des composantes. Cependant, ce résultat ne se généralise pas aux entrelacs qui ont aussi des croisements virtuels. En effet, il est facile de trouver des entrelacs «fused» non-équivalents qui ont les mêmes nombres d'enlacements classiques (voir Remarque 4.2.6). Ce fait répond à une question de [31, Remark 1], où Fish et Keyman demandent si les nombres d'enlacements classiques des composantes d'un entrelacs «fused» composent un invariant complet pour les entrelacs «fused». On décrit les tresses soudées symétriques et on compare des invariants plus ou moins connus pour les entrelacs «fused».

Le résultat principal de cette partie donne la description de la structure des groupes des tresses soudées symétriques  $UVB_n$ , en répondant à une remarque de Kauffman et Lambropoulou [53] à propos de la non trivialité de ces groupes qui méritent donc d'être étudiés. On reporte ici le résultat.

**Théorème 4.1.4.** Pour  $n \geq 1$ , soit  $X_n$  un groupe d'Artin à angle droit engendré par  $x_{i,j}$ , pour  $1 \leq i \neq j \leq n$ , où tous les générateurs commutent à part pour les paires  $x_{i,j}$  et  $x_{j,i}$  pour  $1 \leq i \neq j \leq n$ . Le groupe  $UVB_n$  est isomorphe à  $X_n \rtimes S_n$  où  $S_n$  agit par permutation sur les indices des générateurs de  $X_n$ .

On propose aussi une application du Théorème 4.1.4 qui montre que tout entrelacs «fused» admet comme représentant la clôture d'une tresse soudée symétrique pure :

**Théorème 4.2.3.** Tout entrelacs «fused» est «fused» isotope à la clôture d'une tresse soudée symétrique pure.

Ce résultat améliore la version du théorème d'Alexander donnée par Kauffman et Lambropoulou, parce qu'elle implique que : étant donné un entrelacs «fused», on peut trouver une tresse soudée symétrique avec clôture isotope à l'entrelacs «fused», et qui a le même nombre de brins que le nombre de composantes de l'entrelacs «fused». Comme corollaire on en déduit une preuve simple du théorème principale de [31], qui donne une classification des entrelacs «fused» dotés uniquement de croisements classiques.

Pour  $n \neq 1$ , on définit aussi une représentation nilpotente pour  $UVB_n$ , qui est une représentation dans  $\text{Aut}(N_n)$ , le groupe des automorphismes du groupe libre nilpotent de pas 2 et rang  $n$  (Proposition 4.3.4). On remarque que cette représentation, une fois restreinte au sous-groupe pur de  $UVB_n$ , coïncide avec l'application d'abélianisation.

À l'aide de cette représentation on définit une notion de *groupe d'entrelacs «fused»*, et on vérifie que c'est un invariant pour les entrelacs «fused». On passe ensuite à comparer cet invariant avec les *nombres d'enlacement virtuels*. On termine ce chapitre en considérant une famille de quotients des groupes de tresses soudées par rapport à une autre relation, différente de (F2). On obtient les groupes des *tresses soudées plates* notés

avec  $FWB_n$ . On utilise les techniques déjà utilisées dans la preuve du Théorème 4.1.4 pour en décrire la structure.

**Proposition 4.4.2.** Soit  $n \geq 1$ , et  $\mathbb{Z}^{n(n-1)/2}$  le groupe abélien libre de rang  $n(n-1)/2$ . On note par  $x_{i,j}$ , pour  $1 \leq i \neq j \leq n$  un système de générateurs pour  $\mathbb{Z}^{n(n-1)/2}$ . Le groupe  $FWB_n$  est isomorphe à  $\mathbb{Z}^{n(n-1)/2} \rtimes S_n$ , où  $S_n$  agit par permutation sur les indices des générateurs de  $\mathbb{Z}^{n(n-1)/2}$  (en posant  $x_{j,i} := x_{i,j}^{-1}$ , pour  $1 \leq i < j \leq n$ ).





# Introduction

Braid groups were introduced by Hurwitz [42] in 1891 as fundamental groups of configuration spaces of  $n$  points in the complex plane. However, they owe their name to Artin [3]: he considered them in terms of braid automorphisms of  $F_n$ , the free group of rank  $n$ , but also in geometric terms. The geometric interpretation certainly is the most intuitive and best known, in particular because of its use in knot theory. Then, Magnus [60] considered braid groups from the point of view of mapping classes, while Markov [62] introduced these groups from a purely group-theoretic point of view. All these points of view have long been known to be equivalent [84]. We can then say that braid groups are ubiquitous objects. Any different definition carries a possible generalization; for instance, we can see braid groups as particular case of Artin-Tits groups, Garside groups, mapping class groups and surface braid groups. Few of these generalizations share with braid groups their principal property: a large family of different equivalent definitions.

*Loop braid groups* are a remarkable exception to this fact. Their study has been widely developed during the last twenty years. The first curious fact about these groups is that they appear in the literature with a large number of different names. We choose adopt the terminology introduced by Baez, Wise, and Crans [8], because their definition of loop braid groups in terms of mapping classes fits particularly well the purpose of this thesis. We will soon give an idea of it; however, let us first remark that this is not the first name and interpretation of loop braids that have appeared in the course of time.

The concept that stands behind the genesis of loop braids is the one of *motions* of a compact subspace  $N$  in a manifold  $M$ : these can be understood as the different ways of continuously moving  $N$  in  $M$  so that at the end of the motion  $N$  has returned in its starting position. The first step towards motions is due to Fox and Neuwirth [34]. In 1962 they give a new proof of the standard presentation of the Artin braid groups, in terms of fundamental groups of the configuration spaces of  $n$  unordered points in the plane, considering these groups as groups of continuous movements of  $n$  points on a plane. The same year Dahm, in his Ph.D thesis, generalizes this concept to general manifolds. He applies this construction to a collection of  $n$  unknotted, unlinked circles in the 3-dimensional space: these motions represent the first appearance of the loop braid groups, and his results are published and extended by Goldsmith [37]. In more modern terms, we would speak of mapping class groups, rather than groups of motions. Indeed, we can give an informal definition of our protagonist in the following way.

**Informal definition.** Let  $n$  be greater or equal to 1. The *loop braid group* on  $n$  compo-

nents is the mapping class group of  $n$  unknotted, unlinked circles in the 3-dimensional space.

A formal definition will be given in Chapter 1 (Definition 1.1.5). In 1986 McCool [64] considers loop braids as *basis-conjugating automorphisms* and in 1996 Savushkina [73] considers them as *permutation conjugacy automorphisms* of  $F_n$ . Loop braids are also known in the literature as *welded braids*, as defined by Fenn, Rimányi and Rourke [30]. This may be the most widely known of the notations, and comes with the interpretation in terms of diagrams made of monotone paths in the 2-dimensional space, similar to usual braid diagrams. More recently, Brendle and Hatcher [21] propose an interpretation of these groups as the fundamental groups of the configuration spaces of  $n$  unlinked Euclidean circles. Moreover, one could give a topological interpretation of loop braids: roughly speaking, they can be seen as immersions in the 4-dimensional space of  $n$  disjoint copies of the oriented annulus  $S^1 \times I$  with fixed boundary components. Loop braids can also be encoded as Gauss braid diagrams: this is a combinatorial interpretation, implicitly used in [10]. This interpretation can also be seen as a particular case of the objects considered in [5], and represents a point of view that allows us to consider loop braid groups as quotients of *virtual braid groups*. Chapter 1 will be devoted to collecting these formalisms and proving their equivalence.

In Chapter 1 we also consider *extended loop braid groups*, a generalization of loop braid groups. In the literature these groups have been less studied, but, as it will appear throughout this work, they are even more interesting, and arising in a more natural way as analogues of classical braids, when considering corresponding objects in the 4-dimensional space.

In addition to appearing in so many contexts, loop braid groups are interesting object of study on their own. For instance, several computations and conjectures about the cohomology algebras of the *pure* loop braid groups, certain subgroups of the loop braid groups, have appeared in the literature. Investigating the properties of these algebras, for example in relation with resonance varieties and lower central series ranks, allows us to compare loop braid groups to braid groups and to other generalizations of braid groups. In Section 1.7 we give references for this topic.

In this thesis we will particularly focus on topological aspects and applications of loop braid groups. From the topological point of view, loop braid groups can be seen as the braid counterpart of a particular kind of knotted objects: *ribbon torus-links*. We define them in Chapter 2 (Definition 2.1.3). For the moment let us just give an idea.

**Informal definition.** For  $n \geq 1$ , a *ribbon torus-link* with  $n$  components is the embedding of a disjoint union of  $n$  oriented tori  $S^1 \times S^1$  in the 4-dimensional space.

We recall that in the theory of classical braids and links, Alexander's theorem allows us to represent every link as the closure of a braid. In addition Markov's theorem states that two braids (possibly with different numbers of strings) have isotopic closures in the 3-dimensional space if and only if one can be obtained from the other by a finite number of Markov moves, called conjugation and stabilization. This theorem allows us to describe all braids with isotopic closures as links in the 3-dimensional space. These

two theorems allow us to recover certain link invariants as Markov traces, *i.e.*, traces on the group algebras of braid groups. For the diagrammatical interpretation of loop braids given in [30] we dispose of theorems of Alexander's and Markov's type [47], but a topological version is still missing. Inspired by this, in Chapter 2, we do a first step towards a topological Markov's theorem for loop braids and ribbon torus-links.

Another application can be found in Lie theory, for example in Bar Natan's work [9–11]. It involves a third kind of knotted objects that we derive from loop braids: *ribbon tangles*, which Bar Natan considers in the interpretation as welded tangles. This is the analogous notion of tangles in terms of annuli and tori embedded in the 4-dimensional space. The formal definition of these objects is given in Chapter 3 (Definition 3.1.1); for the moment we propose an informal version.

**Informal definition.** A *ribbon tangle* is an embedding of a disjoint union of oriented tori  $S^1 \times S^1$ , and oriented annuli  $S^1 \times I$  in the 4-dimensional space.

We will discuss ribbon tangles in Chapter 3 in order to define a generalization of the Alexander polynomial.

In addition, we shall not forget the theory of representations of loop braid groups. Burau representations extend trivially to loop braid groups using Magnus expansion and Fox derivatives [14], but it is still unknown if loop braid groups are linear. Some new results on local representations of loop braid groups, rising as extensions of representations of braid groups, can be found in [45] and [22]. However the study of finite dimensional quotients of algebras of loop braid groups is yet to be found in the literature. In [45] the authors show interest also in certain remarkable quotients of loop braid groups: the *symmetric loop braid groups* (also known as *unrestricted virtual braid groups* [53]). We will discuss the structure of these groups in Chapter 4 and give some applications to *fused links*.

In the following we propose an overview of each chapter.

## Loop Braid Groups

Several parallel formulations and notations are being used for loop braid groups, by researchers working in different fields. Although the equivalence of some of the formulations has already been proved or at least stated in the literature, for example in [10, Section 2, Theorem 2.6], we did not find the explicit relationships for all of them. The purpose of this chapter is to give a collection of these formalisms. Moreover we propose a complete proof of the equivalence of these formulations. This provides a clear and well-established theoretical setting for loop braid groups, as the one that we have in the case of braid groups.

In the first part of Chapter 1 we give the definition of the loop braid groups, denoted by  $LB_n$ , as the groups of isotopy classes of self-homeomorphisms of the 3-ball  $B^3$  that:

- fix pointwise the boundary of  $B^3$ ;
- fix globally a collection of disjoint, unknotted, oriented, unlinked circles in the interior of  $B^3$ ;

- preserve the orientations both on  $B^3$  and on the collection of circles.

We also define the extended loop braid groups, denoted by  $LB_n^{ext}$ , as the groups of isotopy classes of such homeomorphisms without the condition of preserving the orientation on the collection of circles.

Then we introduce some configuration spaces of circles in the 3-ball  $B^3$ . The first ones are the spaces of configurations of  $n$  Euclidean, unordered, disjoint, unlinked circles in a 3-ball  $B^3$ , which are denoted by  $\mathcal{R}_n$ . The second ones are the spaces of configurations of a collection of circles with the above characteristics, with the added property of the circles lying on parallel planes. These configuration spaces are denoted by  $\mathcal{UR}_n$ . Their fundamental groups are called respectively the *ring groups* and the *untwisted ring groups* and are respectively denoted by  $R_n$  and  $UR_n$ . The first main result of this chapter establishes that the ring groups  $R_n$  are isomorphic to the extended loop braid groups  $LB_n^{ext}$ . We state it here in a simplified version.

**Theorem 1.2.11.** For  $n \geq 1$ , there is a natural isomorphism between the ring group  $R_n$  and the extended loop braid group  $LB_n^{ext}$ .

This theorem allows us to prove the analogous result for the unordered version of these groups, and is a first motivation to consider extended loop braid groups.

**Proposition 1.2.12.** For  $n \geq 1$ , there is a natural isomorphism between the untwisted ring group  $UR_n$  and the loop braid group  $LB_n$ .

These results provide a second interpretation of loop braids. In addition, Brendle and Hatcher give presentations for the ring groups  $R_n$  and for the untwisted ring groups  $UR_n$ . The correspondence of (untwisted) ring groups and (extended) loop braid groups allows us to state the following:

**Corollary 1.2.15, (see also [21]).** The group  $LB_n$  admits the presentation given by generators  $\{\sigma_i, \rho_i \mid i = 1, \dots, n-1\}$ , subject to relations:

$$\left\{ \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } i = 1, \dots, n-2 \\ \rho_i \rho_j = \rho_j \rho_i & \text{for } |i-j| > 1 \\ \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} & \text{for } i = 1, \dots, n-2 \\ \rho_i^2 = 1 & \text{for } i = 1, \dots, n-1 \\ \rho_i \sigma_j = \sigma_j \rho_i & \text{for } |i-j| > 1 \\ \rho_{i+1} \rho_i \sigma_{i+1} = \sigma_i \rho_{i+1} \rho_i & \text{for } i = 1, \dots, n-2 \\ \sigma_{i+1} \sigma_i \rho_{i+1} = \rho_i \sigma_{i+1} \sigma_i & \text{for } i = 1, \dots, n-2. \end{array} \right.$$

**Corollary 1.2.17, (see also [21]).** The group  $LB_n^{ext}$  admits the presentation given by

generators  $\{\sigma_i, \rho_i \mid i = 1, \dots, n-1\}$ , and  $\{\tau_i \mid i = 1, \dots, n\}$ , subject to relations:

$$\left\{ \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } i = 1, \dots, n-2 \\ \rho_i \rho_j = \rho_j \rho_i & \text{for } |i-j| > 1 \\ \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} & \text{for } i = 1, \dots, n-2 \\ \rho_i^2 = 1 & \text{for } i = 1, \dots, n-1 \\ \rho_i \sigma_j = \sigma_j \rho_i & \text{for } |i-j| > 1 \\ \rho_{i+1} \rho_i \sigma_{i+1} = \sigma_i \rho_{i+1} \rho_i & \text{for } i = 1, \dots, n-2 \\ \sigma_{i+1} \sigma_i \rho_{i+1} = \rho_i \sigma_{i+1} \sigma_i & \text{for } i = 1, \dots, n-2 \\ \tau_i \tau_j = \tau_j \tau_i & \text{for } i \neq j \\ \tau_i^2 = 1 & \text{for } i = 1, \dots, n \\ \sigma_i \tau_j = \tau_j \sigma_i & \text{for } |i-j| > 1 \\ \rho_i \tau_j = \tau_j \rho_i & \text{for } |i-j| > 1 \\ \tau_i \rho_i = \rho_i \tau_{i+1} & \text{for } i = 1, \dots, n-1 \\ \tau_i \sigma_i = \sigma_i \tau_{i+1} & \text{for } i = 1, \dots, n-1 \\ \tau_{i+1} \sigma_i = \rho_i \sigma_i^{-1} \rho_i \tau_i & \text{for } i = 1, \dots, n-1. \end{array} \right.$$

In Figure 7 we show the interpretation of the generators of these presentations as elements of the untwisted ring groups  $UR_n$  and the ring groups  $R_n$ .

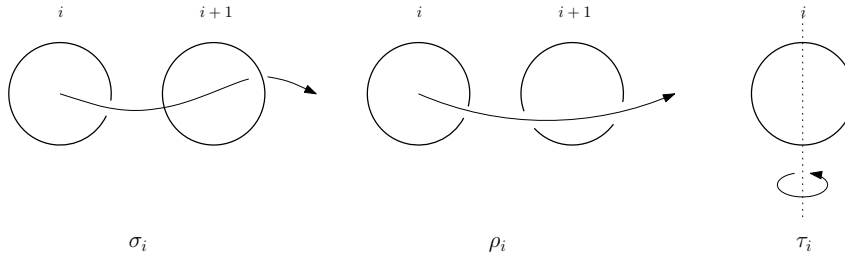


Figure 7: Elements  $\sigma_i$ ,  $\rho_i$  and  $\tau_i$ .

We move on to an interpretation of loop braids as automorphisms of  $F_n$ , the free group of rank  $n$ . We introduce subgroups of *permutation-conjugacy automorphisms*  $PC_n^*$  of  $F_n$ , and recall a result of Dahm, published by Goldsmith, stating that the extended loop braid groups are isomorphic to the groups  $PC_n^*$  [37]. This result also offers a characterization of loop braids as automorphisms similar to Artin's characterization of braids. At this point we also give a presentation for *pure* subgroups of extended loop braid groups  $LB_n^{ext}$ , that we denote by  $PLB_n^{ext}$ .

**Proposition 1.3.8.** For  $n \geq 1$ , the group  $PLB_n^{ext}$  admits the presentation given by

generators  $\{\alpha_{ij} \mid 1 \leq i \neq j \leq n\}$  and  $\{\tau_i \mid i = 1, \dots, n\}$  subject to relations:

$$\begin{cases} \alpha_{ij}\alpha_{kl} = \alpha_{kl}\alpha_{ij} \\ \alpha_{ij}\alpha_{kj} = \alpha_{kj}\alpha_{ij} \\ (\alpha_{ij}\alpha_{kj})\alpha_{ik} = \alpha_{ik}(\alpha_{ij}\alpha_{kj}) \\ \tau_i^2 = 1 \\ \tau_i\alpha_{ij} = \alpha_{ij}\tau_i \\ \tau_i\alpha_{jk} = \alpha_{jk}\tau_i \\ \tau_i\alpha_{ji}\tau_i = \alpha_{ji}^{-1} \end{cases}$$

where distinct letters stand for distinct indices.

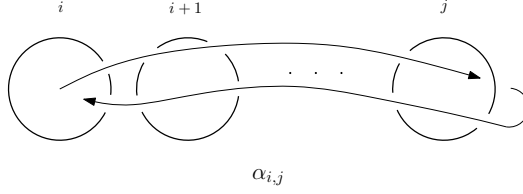


Figure 8: Elements  $\alpha_{ij}$ .

In Figure 8 we show the interpretation of the generators of  $PLB_n^{ext}$  as elements of the ring group  $R_n$ .

Next we pass to a more topological viewpoint on loop braids, as particular classes of braided annuli in a 4-dimensional ball  $B^4$ , that we call *(extended) ribbon braids*. For  $n \geq 1$ , we denote the group of ribbon braids by  $rB_n$ , and the group of extended ribbon braids by  $rB_n^{ext}$ . The main results of this part establish isomorphisms between (extended) ribbon braid groups and (extended) loop braid groups:

**Theorem 1.4.11 and Theorem 1.4.18.** For  $n \geq 1$ , there are isomorphisms between the group of ribbon braids  $rB_n$  and the loop braid group  $LB_n$ , and between their extended versions  $rB_n^{ext}$  and  $LB_n^{ext}$ .

In the context of ribbon braids, using the preceding isomorphisms, we are also able to prove that the isotopy notion we chose for ribbon braids coincides with the notion of ambient isotopy of the 4-ball in which ribbon braids are immersed. In particular, we have the following result.

**Theorem 1.4.5.** Every isotopy of a geometric ribbon braid in  $B^3 \times I$  extends to an isotopy of  $B^3 \times I$  in itself constant on the boundary.

We also describe two kind of representations in dimension 3 and 2 of loop braids. We introduce *broken surfaces* as well as *welded diagrams*. Broken surfaces are projections in general position of loop braids in the 3-dimensional space. See Figure 9 for a representation

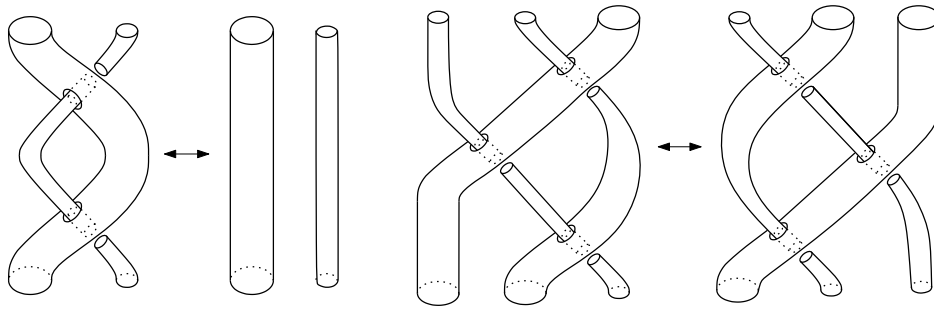


Figure 9: Broken surfaces.

of equivalent broken surfaces. Welded diagrams are 2-dimensional diagrams, introduced in [30].

Finally we discuss a combinatorial description of loop braids through *Gauss diagrams*. These are diagrams, in the spirit of [69], encoding information about a knotted object. We use Cisneros' results on Gauss diagrams for virtual braids from [25] to prove an isomorphism between a quotient of Gauss diagrams and welded braid diagrams.

To close this overview on loop braid groups, we give a brief exposition of the history of these objects, and some other references to find about topological applications and what is known about a representation theory for these groups.

## A topological application: towards a topological Markov's theorem for loop braids

In Chapter 2 we take the first steps in adapting some classical results of knot theory to ribbon torus-links, the knotted counterpart of loop braids. For this reason we consider loop braids as ribbon braids (Theorem 1.4.11). In the classical case, Alexander's theorem states that every link can be represented as the closure of a braid. This result has already been proved for welded braid diagrams and their knotted counterpart by Kamada [47]. The isomorphism between welded braid diagrams and ribbon braids described in Chapter 1 guarantees that Alexander's theorem still holds when passing to ribbon braids and ribbon torus-links.

Markov's theorem presents more difficulties, because of the lack of a one-to-one correspondence between ribbon torus-links and welded link diagrams. In classical braid and knot theory this theorem states that two braids (possibly with different numbers of strings) have isotopic closures in  $\mathbb{R}^3$  if and only if one can be obtained from the other by a finite number of Markov moves, called conjugation and stabilization. This theorem allows us to describe all braids with isotopic closures as links in  $\mathbb{R}^3$ . Concerning loop braids seen as welded braid diagrams, a combinatorial proof also for this theorem can be found in [47]. We have been able to prove a version of Markov's theorem for loop braids with closure in a solid torus in  $\mathbb{R}^4$ , which we state here. We recall that we denote by  $rB_n$  the groups of ribbon braids, and by  $rB_n^{ext}$  the groups of extended ribbon braids.

**Theorem 2.3.1.** Let  $n \geq 1$  and  $\beta, \beta' \in rB_n$  a pair of ribbon braids. The closed ribbon braids  $\widehat{\beta}, \widehat{\beta}'$  are isotopic in  $B^3 \times S^1$  if and only if  $\beta$  and  $\beta'$  are conjugate in  $rB_n^{ext}$ .

We remark that to have the same kind of statement that we have in the classical case, one shall pass to the extended groups, in which case we have:

**Theorem 2.3.3.** Let  $n \geq 1$  and  $\beta, \beta' \in rB_n^{ext}$  a pair of ribbon braids. The closed extended ribbon braids  $\widehat{\beta}, \widehat{\beta}'$  are isotopic in  $B^3 \times S^1$  if and only if  $\beta$  and  $\beta'$  are conjugate as elements in  $rB_n^{ext}$ .

To extend the result in the 4-dimensional space, and not only in the solid torus, we should prove the invariance of the isotopy class of a closed (extended) ribbon braid under the operation known as stabilisation on (extended) ribbon braids. The approach we followed for conjugation does not allow us to do this without passing to a diagrammatical interpretation. This is why in future we plan to follow other approaches.

## Generalizing the Alexander polynomial: Alexander invariants for ribbon tangles

In Chapter 3 we discuss an expanded version of a joint work with Vincent Florens [29]. Here we consider the third kind of knotted objects that we discuss in this thesis: ribbon tangles. This work concerns the introduction of an invariant of Alexander type for ribbon tangles, where, when we say “of Alexander type”, we mean that it shares the construction technique of the Alexander polynomial in classical knot theory. We take a ribbon tangle  $T$  in the 4-ball  $B^4$ . The boundary components of the annuli of  $T$  are the components of a trivial link  $L$  in  $S^3$ , the boundary of  $B^4$ . We fix a free abelian group  $G$ , and we say that we  $G$ -color the ribbon tangle  $T$  when we chose a group homomorphism  $\varphi$  from the first homology group of the complement of  $T$  in  $B^4$  to  $G$ . We construct an *Alexander invariant*  $A(T)$  of the colored ribbon tangle  $T$  as an element of the exterior algebra of the homology  $\mathbb{Z}[G]$ -module  $H_1^\varphi(S^3 \setminus L)$ , twisted by the morphism induced by  $\varphi$ . We denote this module by  $H_\partial$ .

Then we change formalism and we introduce a functorial generalization of the Alexander polynomial, inspired by [19] and [33], which goes from the category  $\mathcal{Rib}_G$  of *ribbon cobordisms* to the category of graded modules. We call it the *Burau functor*, and denote it by  $\rho$ . In this context we see ribbon tangles embedded in a 4-dimensional cylinder  $B^3 \times I$ , with boundary components only in the interiors of the 3-balls  $B^3 \times \{0\}$  and  $B^3 \times \{1\}$ . The objects of the category of ribbon cobordisms are sequences of signs associated to the boundary components of ribbon tangles, while the morphisms are the ribbon tangles themselves, seen as morphisms from the boundary components in the upper ball to the boundary components in the lower ball. The Burau functor associates to the sequences of signs the exterior algebras of some graded modules  $M_0$  and  $M_1$ . To ribbon tangles it associates a sum of linear mappings between the exterior algebras of  $M_0$  and  $M_1$ . Informally, given a ribbon tangle in the 4-ball  $B^4$  as in the first formalism, we can *split* it by squeezing the 4-ball  $B^4$  into the 4-dimensional cylinder, sending some boundary



components to the upper ball, and others to the lower ball. We prove that the Burau functor is indeed just a change of formalism for the Alexander invariant  $A$ . More precisely, we show:

**Theorem 3.1.19.** Let  $(T, \varphi)$  be a  $G$ -colored ribbon tangle, and let  $(\tilde{T}, \varphi)$  be a splitting of  $(T, \varphi)$  in  $\mathcal{Rib}_G$ . There is an isomorphism, defined up to a unit of  $\mathbb{Z}[G]$ ,

$$\wedge^n H_{\partial} \longrightarrow \text{Hom}(\wedge M_0, \wedge M_1)$$

sending  $A(T, \varphi)$  to  $\oplus_k (-1)^{k(n_0-k)} \rho_k(\tilde{T}, \varphi)$ , where  $n$  is the number of embedded annuli,  $k$  is the deficiency of the presentation of the Alexander module of  $(T, \varphi)$ ,  $n_0$  is the number of boundary components of  $(\tilde{T}, \varphi)$  that lie in the interior of the ball  $B^3 \times \{0\}$ , and  $\rho_k$  is the  $k$ -component of  $\rho$ .

When considering ribbon braids as a particular case of ribbon tangles, we have one more analogy between the Alexander invariant and the Alexander polynomial, passing through the equivalence with the Burau functor. We state it here in a simplified form.

**Proposition 3.1.17.** For any  $G$ -colored ribbon braid  $(S, \varphi)$ , the sum of linear applications of graded modules  $\rho(S, \varphi)$  coincides with the exterior powers of the Burau representation.

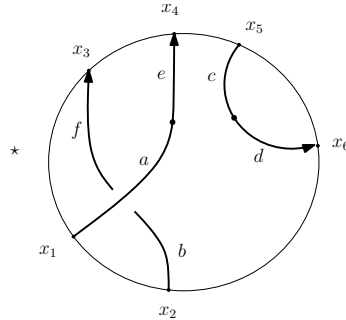


Figure 10: A welded tangle diagram  $\tau$ .

We make a step further giving a combinatorial way of computing the invariant  $A$ , by making a relation with a diagrammatical invariant defined by Archibald on virtual tangle diagrams, which are close relatives to welded tangle diagrams [2]. In fact, we extend this diagrammatical invariant to welded tangle diagrams. In particular, we prove that in order to compute the Alexander invariant  $A(T)$  of a ribbon tangle  $T$  associated to a welded tangle diagram  $\tau$ , it is enough to compute the diagrammatical invariant of  $\tau$ . The latter invariant is computed by following simple rules for building an Alexander matrix using the labels of the diagram, and then computing certain minors of the matrix.

Finally we simplify this combinatorial way of computing by proving that we can define *circuit algebras* structures both on ribbon tangles and on welded tangle diagrams. These structures, inspired by Jones' planar algebras [44], are respected by the Alexander

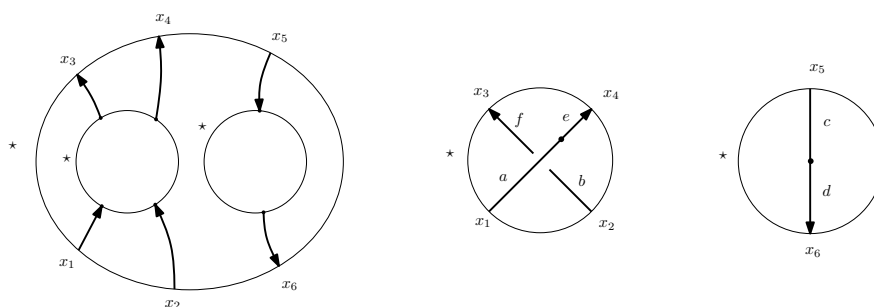


Figure 11: A circuit diagram  $Q$  and two welded tangle diagrams  $\sigma$  and  $\beta$ .

invariant  $A$  on ribbon tangles and by the diagrammatical invariant on welded tangle diagrams. This brings the advantage of allowing local computations on a welded tangle diagram to compute the Alexander invariant  $A$  of the associated ribbon tangle. This means that, to compute the Alexander invariant  $A(T)$ , where  $T$  is the ribbon tangle associated to the welded tangle diagram  $\tau$  in Figure 10, it is enough to compute the diagrammatical invariants of the welded tangle diagrams  $\sigma$  and  $\beta$  on the right hand side of Figure 11, and then compose them with the rules given by the circuit diagram  $Q$  on the left hand side of Figure 11.

## A remarkable quotient: unrestricted virtual braids - and their relation to fused links

We mentioned earlier that there is a theory of representations of loop braid groups: Kadar, Martin, Rowell, and Wang have explored representations of braid groups that can be extended to loop braid groups  $LB_n$ , but do not extend to a family of quotients of the loop braid groups called *symmetric loop braid groups*, and denoted by  $SLB_n$  [45]. If we pass from loop braid groups to their diagrammatic interpretation, welded braid groups (through the isomorphism described in Chapter 1), we have that these quotients are isomorphic to *unrestricted virtual braids groups*, that we denote by  $UVB_n$ . Unrestricted virtual braid groups have been introduced by Kauffman and Lambropoulou, as braided counterparts of *fused links* [52]. The authors also extended Kamada's work [47] by presenting a version of Alexander's and Markov's theorems for these objects [53].

In Chapter 4 we present parts of a work with Valeriy G. Bardakov and Paolo Bellingeri [16] concerning these objects. The elements of the group of unrestricted virtual braids are equivalence classes of welded braid diagrams under the relation given by the addition of one more kind of Reidemeister moves, the *forbidden moves* ( $F2$ ) (see Figure 12).

It has been shown that all fused knots are equivalent to the unknot [48, 67]. Moreover, Nelson's proof in [67] of the fact that every virtual knot unknots when allowing forbidden moves, can be adapted verbatim to links with several components. We remark that to achieve this result Nelson passes to Gauss diagrams for fused knots. So, every fused link

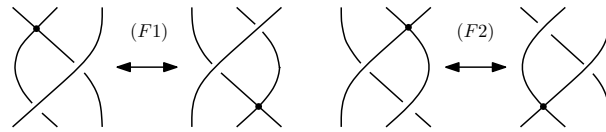


Figure 12: Forbidden moves of type (F1) (on the left) and forbidden moves of type (F2) (on the right).

diagram is fused isotopic to a link diagram where the only crossings (classical or virtual) are the ones involving different components.

On the other hand, there are non-trivial fused links, and their classification is not (completely) trivial [32]. In particular, in [31], Fish and Keyman prove that fused links that have only classical crossings are characterized by the (classical) linking numbers of their components. However, this result does not generalize to links with virtual crossings. In fact it is easy to find non-equivalent fused links with the same (classical) linking numbers (see Remark 4.2.6). This answers a question from [31, Remark 1], where Fish and Keyman ask whether the classical linking numbers of the components of a fused link are a complete invariant for fused links. We describe unrestricted virtual braids and compare more or less known invariants for fused links.

The main result that allows us to do so gives a description of the structure of the groups of unrestricted virtual braids  $UVB_n$ , addressing a remark of Kauffman and Lambropoulou [53] about these groups not being trivial and deserving to be studied. We report here our main result.

**Theorem 4.1.4.** For  $n \geq 1$ , let  $X_n$  be the right-angled Artin group generated by  $x_{i,j}$ , for  $1 \leq i \neq j \leq n$ , where all generators commute except the pairs  $x_{i,j}$  and  $x_{j,i}$  for  $1 \leq i \neq j \leq n$ . The group  $UVB_n$  is isomorphic to  $X_n \rtimes S_n$  where  $S_n$  acts by permutation on the indices of generators of  $X_n$ .

We provide an application of Theorem 4.1.7 showing that any fused link admits as a representative the closure of a *pure* unrestricted virtual braid:

**Theorem 4.2.3.** Any fused link is fused isotopic to the closure of an unrestricted virtual pure braid.

This result improves Kauffman and Lambropoulou's version of Alexander's theorem, because it implies that: given a fused link, we can find an unrestricted virtual braid whose closure is isotopic to the fused link, and which has the same number of strands as the number of components of the fused link. As a corollary we deduce an easy proof of the main theorem of [31], which gives a classification of fused links with only classical crossings.

We also construct a nilpotent representation for  $UVB_n$ , which is a representation in  $\text{Aut}(N_n)$ , the group of automorphisms of the free 2-step nilpotent group of rank  $n$  (Proposition 4.3.4). We remark that this representation, when restricted to the pure subgroups of  $UVB_n$ , coincides with the abelianization map.

Using this representation we define a notion of *group of fused links*, and prove that this is an invariant for fused links. We proceed then to compare this invariant to the *virtual linking numbers*. We close this chapter by considering a family of quotients of the welded braid groups by another relation, different from (F2). We obtain the groups of *flat welded braids*, denoted by  $FWB_n$ . Using the techniques employed in the proof of Theorem 4.1.4, we are able to describe the structure of these groups:

**Proposition 4.4.2.** For  $n \geq$ , let  $\mathbb{Z}^{n(n-1)/2}$  be the free abelian group of rank  $n(n-1)/2$ . Let us denote by  $x_{i,j}$ , for  $1 \leq i \neq j \leq n$  a set of generators of  $\mathbb{Z}^{n(n-1)/2}$ . The group  $FWB_n$  is isomorphic to  $\mathbb{Z}^{n(n-1)/2} \rtimes S_n$ , where  $S_n$  acts by permutation on the indices of generators of  $\mathbb{Z}^{n(n-1)/2}$  (setting  $x_{j,i} := x_{i,j}^{-1}$ , for  $1 \leq i < j \leq n$ ).

# Chapter 1

## Loop Braid Groups

In this chapter we introduce several interpretations of *loop braid groups*, and provide a complete proof of the equivalence of these formulations. This provides a clear and well-established theoretical setting, as the one we have in the case of braid groups  $B_n$ . We will use the equivalence of these formulations in Chapter 2 to give some topological applications. Although the equivalence of some of the formulations has already been proved or at least stated in the literature, for example in [10, Section 2, Theorem 2.6], we did not find the explicit isomorphisms between all of them.

A second reason for working on the equivalence of these definitions is unifying the several parallel notations that are being used for loop braid groups, by researchers working in different fields.

The third purpose of this section is to introduce *extended loop braid groups*  $LB_n^{ext}$ , a generalization of loop braid groups. In the literature these groups have been less studied, but, as it will appear throughout Chapters 1 and 2, they are even more interesting and rising in a more natural way than the groups  $LB_n$ . For this reason we develop in a parallel fashion:

- a) the theory of loop braids, for the sake of unifying and completing existent literature;
- b) the theory of extended loop braids, because they appear to be the most natural analogue of classical braids, when considering corresponding objects in the 4-dimensional space.

This Chapter is organized as follows. In Section 1.1 we give a first definition of loop braid groups in terms of mapping classes.

In Section 1.2 we introduce some configuration spaces of circles in the 3-ball  $B^3$ , and show that their fundamental groups are isomorphic to  $LB_n^{ext}$  and to  $LB_n$  (Theorem 1.2.11 and Proposition 1.2.12). This provides a second interpretation of loop braids. We also recall the presentations given in [21] for  $LB_n$  and  $LB_n^{ext}$ , and for the *pure subgroups* of the first family of groups.

In Section 1.3 we introduce subgroups of *permutation-conjugacy automorphisms*  $PC_n^*$  of the group of automorphisms of  $F_n$ , the free group of rank  $n$ , and recall a result of

Dahm [27], published by Goldsmith [37], stating that extended loop braid groups are isomorphic to  $PC_n^*$  (Theorem 1.3.1). This gives us a third interpretation of  $LB_n^{ext}$  and  $LB_n$ , in terms of automorphisms of  $F_n$ , and also a characterization of loop braids as automorphisms similar to Artin's characterization of braids. At this point we also give a presentation for the pure subgroups of the groups  $LB_n^{ext}$  (Proposition 1.3.8).

Section 1.4 brings a more topological viewpoint on loop braids, as particular classes of braided surfaces in a 4-dimensional ball  $B^4$  that we call *(extended) ribbon braids*. We establish an isomorphism between the groups of (extended) ribbon braids and the (extended) loop braid groups (Theorems 1.4.11 and 1.4.18). This is the fourth interpretation of loop braids that we consider. Sections 1.5 is devoted to representations in dimension 3 and 2 of loop braids. Here we introduce *broken surfaces* as well as *welded diagrams*. Broken surfaces are projections in general position of loop braids in the 3-dimensional space. Welded diagrams are 2-dimensional diagrams, introduced in [30]: passing through broken surfaces and using results of [21] we introduce an isomorphism between the groups of these diagrams and the loop braid groups, seen as groups of ribbon braids (Theorem 1.5.9).

In Section 1.6 we discuss a combinatorial description of loop braids through *Gauss diagrams*. These are diagrams, in the spirit of [69], encoding information about a knotted object. This point of view allows us to see loop braid groups as quotients of the groups of virtual braids. This is done using the isomorphisms between loop braid groups and welded braid groups. Though Gauss diagrams has already been used as an equivalent formulation of welded objects (see for example [7, 10, 20]), to formally prove the isomorphism between the groups of welded Gauss diagrams and welded braid diagrams we need to use results from [25] on virtual braids.

Finally Section 1.7 contains a brief history of these objects, and some other references to find about topological applications and what is known about a representation theory for loop braid groups.

## 1.1 Mapping class groups of a trivial link of unknotted circles in $B^3$

In this section we introduce the mapping class group of a 3-manifold with respect to a submanifold. Then we present a first definition for loop braid groups in terms of mapping classes of a 3-ball with  $n$  circles that are left setwise invariant in its interior. We also introduce three relatives of these groups. To introduce the main tools we follow [37] and [51].

Let  $M$  be a compact, connected, orientable 3-manifold, possibly with boundary, and  $N$  an orientable submanifold contained in the interior of  $M$ , not necessarily connected or non-empty. A *self-homeomorphism* of the pair of manifolds  $(M, N)$  is an homeomorphism  $f: M \rightarrow M$  that fixes  $\partial M$  pointwise, preserves orientation on  $M$ , and globally fixes  $N$ . Every self-homeomorphism of  $(M, N)$  induces a permutation on the connected components of  $N$  in the natural way.

We denote by  $\text{Homeo}(M; N)$  the group of self-homeomorphisms of  $(M, N)$  that

preserve orientation on both  $M$  and  $N$ . The multiplication in  $\text{Homeo}(M; N)$  is given by the usual composition. We denote by  $\text{Homeo}(M)$  the group  $\text{Homeo}(M; \emptyset)$ . Moreover we denote by  $\text{PHomeo}(M; N)$  the subgroup of self-homeomorphisms of  $(M, N)$  that send each connected component of  $N$  to itself.

We remark that  $\text{Homeo}(M; N)$  is a topological group when equipped with the compact-open topology. The embedding of  $\text{Homeo}(M; N)$  into  $\text{Homeo}(M)$  makes  $\text{Homeo}(M; N)$  a closed subgroup of the topological group  $\text{Homeo}(M)$ .

Let  $I$  be the unit interval. Two self-homeomorphisms  $f_0, f_1$  of  $(M, N)$  are said to be *isotopic* if they can be extended to a family  $\{f_t\}_{t \in I}$  of self-homeomorphisms of  $(M, N)$  such that the map  $M \times I \rightarrow M$ , sending  $(x, t) \rightarrow f_t(x)$ , is continuous. The isotopy relation is an equivalence relation and isotopic self-isomorphisms induce the same permutation on the connected components of  $N$ .

**Definition 1.1.1.** The *mapping class group of a 3-manifold  $M$  with respect to a submanifold  $N$* , denoted by  $\text{MCG}(M, N)$ , is the group of isotopy classes of self-homeomorphisms of  $\text{Homeo}(M; N)$ , with multiplication determined by composition. We denote by  $\text{MCG}(M)$  the mapping class group  $\text{MCG}(M, \emptyset)$ .

The *pure mapping class group of a 3-manifold  $M$  with respect to a submanifold  $N$* , denoted by  $\text{PMCG}(M, N)$ , is the subgroup of elements of  $\text{MCG}(M, N)$  that send each connected component of  $N$  to itself.

*Remark 1.1.2.* It is known ([55]) that a map  $f$  from a topological space  $X$  to  $\text{Homeo}(M; N)$  is continuous if and only if the map  $X \times M \rightarrow M$  sending  $(x, y) \mapsto f(x)(y)$  is continuous. Taking  $X$  equal to the unit interval  $I$ , we have that two self-homeomorphisms are isotopic if and only if they are connected by a path in  $\text{Homeo}(M; N)$ . Therefore  $\text{MCG}(M, N) = \pi_0(\text{Homeo}(M; N))$ . In a similar way we have that  $\text{PMCG}(M, N) = \pi_0(\text{PHomeo}(M; N))$ .

We consider now a bigger class of self-homomorphisms of the pair  $(M, N)$ , removing the condition of preserving orientation on  $N$ . We add an “\*” to the notation of the submanifold to indicate this. We denote by  $\text{Homeo}(M; N^*)$  the group of self-homeomorphisms of  $(M, N)$ . The multiplication in  $\text{Homeo}(M; N^*)$  is given by the usual composition. Also this group is a closed subgroup of the topological group  $\text{Homeo}(M)$ . The isotopy relation considered remains the same. We denote by  $\text{PHomeo}(M; N^*)$  the subgroup of  $\text{Homeo}(M; N^*)$  of self-homeomorphisms that send each connected component of  $N$  to itself.

**Definition 1.1.3.** The *extended mapping class group of a 3-manifold  $M$  with respect to a submanifold  $N$* , denoted by  $\text{MCG}(M, N^*)$ , is the group of isotopy classes of self-homeomorphisms of  $\text{Homeo}(M; N^*)$ , with multiplication determined by composition.

The *pure extended mapping class group of a 3-manifold  $M$  with respect to a submanifold  $N$* , denoted by  $\text{PMCG}(M, N^*)$ , is the subgroup of  $\text{MCG}(M, N^*)$  of elements that send each connected component of  $N$  to itself.

As in Remark 1.1.2, we have an equivalent definition of the groups  $\text{MCG}(M, N^*)$  and  $\text{PMCG}(M, N^*)$  in terms of connected components of subgroups of  $\text{Homeo}(M)$ . We have that  $\text{MCG}(M, N)$  can be defined as  $\pi_0(\text{Homeo}(M; N))$ , and  $\text{PMCG}(M, N^*)$  as  $\pi_0(\text{PHomeo}(M; N^*))$ .

**Example 1.1.4.** Fix  $n \geq 1$ . Let us take  $M$  to be the disk  $D^2$ , and  $N$  to be a set of  $n$  distinct points  $P = \{p_1, \dots, p_n\}$  in the interior of  $D^2$ . In this case we have that  $\text{PMCG}(D^2, P)$  and  $\text{PMCG}(D^2, P^*)$  are isomorphic, since there is no choice on the orientation of a point. For the same reason  $\text{MCG}(D^2, P)$  is isomorphic to  $\text{MCG}(D^2, P^*)$ . Then the group  $\text{PMCG}(D^2, P)$  is isomorphic to  $P_n$ , the *pure braid group on  $n$  strands*, and  $\text{MCG}(D^2, P)$  is isomorphic to  $B_n$ , the *braid group on  $n$  strands*, as defined, for example, in [38] or [51, Chapter 1.6].

We conclude this section with the definition of the main objects we are interested in: the loop braid group, the extended loop braid group, and their respective pure subgroups.

**Definition 1.1.5.** Let us fix  $n \geq 1$ , and let  $C = C_1 \sqcup \dots \sqcup C_n$  be a collection of  $n$  disjoint, unknotted, oriented circles, that form a trivial link of  $n$  components in  $\mathbb{R}^3$ . The exact position of  $C$  is irrelevant because of [37, Corollary 3.8]. So in the following we assume that  $C$  is contained in the  $xy$ -plane in the 3-ball  $B^3$ . The *loop braid group on  $n$  components*, denoted by  $LB_n$ , is the mapping class group  $\text{MCG}(B^3, C)$ . The *pure loop braid group on  $n$  components*, denoted by  $PLB_n$ , is the pure mapping class group  $\text{PMCG}(B^3, C)$ . In a similar way the *extended loop braid group*, denoted by  $LB_n^{\text{ext}}$ , is the extended mapping class group  $\text{MCG}(B^3, C^*)$ . The *pure extended loop braid group*, denoted by  $PLB_n^{\text{ext}}$ , is the pure extended mapping class group  $\text{PMCG}(B^3, C^*)$ .

This definition appears in [37], in terms of *motion groups*, which we have reformulated in terms of mapping class groups. We note in particular that the groups  $LB_n^{\text{ext}}$  coincide with the motion groups of a trivial unlink defined in [37]. However the nomenclature “loop braid groups” is due to Baez, Crans, and Wise [8].

## 1.2 The configuration spaces of a trivial link of unknotted circles

The second interpretation of loop braid groups  $LB_n$  that we give is in terms of configuration spaces, and has been introduced in [21]. We recall some notions and results about configuration space. Then we exhibit the isomorphism from the fundamental groups of certain configuration spaces to the loop braid groups  $LB_n$ . At the end of this section we will give presentations for the groups  $LB_n$  and  $LB_n^{\text{ext}}$ .

**Definition 1.2.1.** Let  $n \geq 1$ , and let  $\mathcal{R}_n$  be the space of configurations of  $n$  Euclidean, unordered, disjoint, unlinked circles in  $B^3$ . The *ring group*  $R_n$  is its fundamental group. Let  $\mathcal{UR}_n$  be the space of configurations of  $n$  Euclidean, unordered, disjoint, unlinked circles in  $B^3$  lying on planes parallel to a fixed one. The *untwisted ring group*  $UR_n$  is its fundamental group.

Similarly, let  $\mathcal{PR}_n$  be the space of configurations of  $n$  Euclidean ordered, disjoint, unlinked circles. The *pure ring group*  $PR_n$  is its fundamental group. Finally, let  $\mathcal{PUR}_n$  be the space of configurations of  $n$  Euclidean, ordered, disjoint, unlinked circles lying on planes parallel to a fixed one. The *pure untwisted ring group*  $PUR_n$  is its fundamental group.



*Remark 1.2.2.* The topology on  $\mathcal{R}_n$  can be described considering its covering space consisting of ordered  $n$ -tuples of disjoint oriented circles in  $B^3$ . This covering space can be identified with an open subset of  $\mathbb{R}^{6n}$ , since each circle can be uniquely determined by its center and a vector orthogonal to the plane of the circle. Circles of configurations of  $\mathcal{R}_n$  are unordered, so one has to factor out the free action of the signed permutation group on this space. More details can be found in [21, Section 2].

Also, in [21, Section 2], the path connectedness of the groups  $\mathcal{R}_n$  is proved, using a "shrinking circles" argument from [36]. The authors define a canonical way of shrinking circles simultaneously. Starting with a configuration which is in general position, in the sense that no circle has its center on the disk bounded by another circle, then this shrinking process produces a configuration of circles lying in disjoint balls.

Let  $S_n$  be the group of permutations of an  $n$ -elements set, and  $p$  the orbit projection  $\mathcal{PR}_n \rightarrow \mathcal{R}_n$  that forgets the order of the circles. We observe that  $p$  is a regular  $n!$ -sheeted cover (see Remark 1.2.2), with  $S_n$  as group of deck transformations. This covering is associated to  $p_*(\pi_1(\mathcal{PR}_n))$ . From this follows that  $PR_n$  is a subgroup of  $R_n$ , and we have the short exact sequence

$$1 \longrightarrow PR_n \longrightarrow R_n \longrightarrow S_n \longrightarrow 1.$$

Similarly, for  $PUR_n$  and  $UR_n$  we can consider the short exact sequence

$$1 \longrightarrow PUR_n \longrightarrow UR_n \longrightarrow S_n \longrightarrow 1.$$

*Remark 1.2.3.* Let  $\mathcal{M}_n$  be the configuration space of  $n$  ordered distinct points in the complex plane  $\mathbb{C}$ , i.e., the set of  $n$ -tuples  $(z_1, \dots, z_n)$  such that  $z_i \neq z_j$  for  $i \neq j$ . It is easy to show that its fundamental group is isomorphic to the pure braid group on  $n$  strands  $P_n$ , see for instance [51, Chapter 1.4]. In the same way, the fundamental group of the configuration space of  $n$  unordered distinct points in  $\mathbb{C}$ , meaning the fundamental group of the quotient of  $\mathcal{M}_n$  by the action of  $S_n$ , is one definition of the braid group on  $n$  strands  $B_n$ .

The following result of Brendle and Hatcher shows the relation between the untwisted ring groups  $UR_n$  and the ring groups  $R_n$ .

**Proposition 1.2.4** ([21, Proposition 2.2]). *For  $n \geq 1$ , the natural map  $UR_n \rightarrow R_n$  induced by the inclusion  $\mathcal{UR}_n \rightarrow \mathcal{R}_n$  is injective.*

In order to prove that the groups  $R_n$  are isomorphic to the loop braid groups  $LB_n$ , seen as the mapping class groups  $\text{MCG}(B^3, C^*)$ , we need some results that we will list below.

**Theorem 1.2.5** ([21, Theorem 1]). *For  $n \geq 1$ , the inclusion of  $\mathcal{R}_n$  into the space of configurations of all smooth trivial links of  $n$  components in  $\mathbb{R}^3$ , denoted by  $\mathcal{L}_n$ , is a homotopy equivalence.*

The result allows us to consider the fundamental group of the configuration space of smooth trivial links as isomorphic to  $R_n$ . Let  $\mathcal{PL}_n$  be the space of configurations of all smooth trivial links of  $n$  ordered components in  $\mathbb{R}^3$ . From Theorem 1.2.5 we can deduce the following corollary about ordered condifuration spaces.

**Corollary 1.2.6.** *For  $n \geq 1$ , the inclusion of  $\mathcal{PR}_n$  into the space of configurations of all ordered smooth trivial links of  $n$  components in  $\mathbb{R}^3$ , denoted by  $\mathcal{PL}_n$ , is a homotopy equivalence.*

*Proof.* We remark that the proof of Theorem 1.2.5 is carried on locally, by considering each component in a sphere, disjoint from the other component's spheres, and rounding each component of the configuration. Then, attaching the order information on the components is left unaffected by the transformation.  $\square$

*Remark 1.2.7.* In Section 1.1 we defined the loop braid group  $LB_n$  (and the pure, extended, and pure extended versions) as the *topological* mapping class group of a 3-ball  $B^3$  with respect to a collection  $C$  of  $n$  disjoint, unknotted, oriented circles, that form a trivial link of  $n$  components. This means that we defined it in terms of self-homeomorphisms.

However, as Wilson recalls in [80], it follows from two results of Wattenberg [79][Lemma 1.4 and Lemma 2.4] that the topological mapping class group of the 3-space with respect to a collection of  $n$  disjoint, unknotted, oriented circles, that form a trivial link, is isomorphic to the  $C^\infty$  mapping class group, defined in terms of diffeomorphisms.

A *self-diffeomorphism* of the pair of manifolds  $(B^3, C)$  is a diffeomorphism  $f: B^3 \rightarrow B^3$  that fixes  $\partial B^3$  pointwise, preserves orientation on  $B^3$ , and fixes setwise  $C$ .

We denote by  $\text{Diffeo}(B^3; C)$  the group of self-diffeomorphisms of  $(B^3, C)$  that preserve orientation on both  $B^3$  and  $C$ . We denote by  $\text{Diffeo}(B^3)$  the group  $\text{Diffeo}(B^3; \emptyset)$ . Moreover we denote by  $\text{PDiffeo}(B^3; C)$  the subgroup of self-diffeomorphisms of  $(B^3, C)$  that send each connected component of  $C$  to itself. In the same spirit of Section 1.1, we can define  $\text{Diffeo}(B^3; C^*)$  and its subgroup  $\text{PDiffeo}(B^3; C^*)$ .

Relying on the results of Wattenberg, we have that

$$\pi_0(\text{Homeo}(B^3; C)) \cong \pi_0(\text{Diffeo}(B^3; C)) \text{ and } \pi_0(\text{Homeo}(B^3; C^*)) \cong \pi_0(\text{Diffeo}(B^3; C^*)).$$

*Remark 1.2.8.* In this part we use a bold font to denote a  $n$ -tuple of disjoint, unlinked, trivial knots, and a normal font to denote a single trivial knot. To be consistent with the notation of Section 1.1, an exception will be made when writing  $\text{PHomeo}(B^3; C^*)$ , which is the subgroup of self-homeomorphisms of  $B^3$  with respect with the subset  $\mathbf{C}$ , that send each connected component of  $\mathbf{C}$  to itself.

We take  $\mathbf{C} = (C_1, \dots, C_n)$  to be an ordered tuple of  $n$  disjoint, unlinked, trivial circles living in  $B^3$ . We consider the space of configurations of ordered smooth trivial links of  $n$  components  $\mathcal{PL}_n$  with the topology induced by the distance defined as follows. Each trivial knot of the configuration can be seen in a 3-ball  $B^3$ , and being smooth, it admits parametrizations. Let us fix a distance  $d$  on  $B^3$ . Then taken two trivial knots  $C_1$  and  $C_2$  in  $B^3$ , we denote by  $d(C_1, C_2)$  the  $\min\{\max\{d(p_1(t), p_2(t))\}\}$ , where the min

is considered on the parametrizations  $p_i: S^1 \rightarrow B^3$  with  $i = 1, 2$  for  $C_1$  and  $C_2$ , and the max is considered on the parameter  $t \in S^1$ .

We define an *evaluation map*

$$\varepsilon: \text{Diffeo}(B^3) \longrightarrow \mathcal{PL}_n \quad (1.1)$$

sending a self-diffeomorphism  $f$  to  $f(\mathbf{C})$ . Remark that  $f(\mathbf{C})$  is an ordered tuple of  $n$  disjoint, unlinked, trivial, smooth knots living in  $B^3$ , since the diffeomorphism could have deformed the circles of  $\mathbf{C}$ . This map is surjective and continuous for construction.

**Lemma 1.2.9.** *For  $n \geq 1$ , the evaluation map  $\varepsilon: \text{Diffeo}(B^3) \rightarrow \mathcal{PL}_n$  is a locally trivial fibration, with fibre  $\text{PDiffeo}(B^3; C^*)$ .*

*Proof.* Let us consider a point  $\mathbf{C}^0 = (C_1^0, \dots, C_n^0)$  in  $\mathcal{PL}_n$ , i.e., an ordered tuple of  $n$  disjoint, smooth, unlinked, trivial knots living in  $B^3$ . Note that:

$$\varepsilon^{-1}(\mathbf{C}^0) = \{f \in \text{Diffeo}(B^3) \mid f(C_i^0) = C_i^0 \text{ for } i = 1, \dots, n\} \cong \text{PDiffeo}(B^3; C^{0*}).$$

Moreover, for any  $\mathbf{C}$  in  $\mathcal{PL}_n$  there is an isomorphism between the groups  $\text{PDiffeo}(B^3; C^{0*})$  and  $\text{PDiffeo}(B^3; C^*)$ . This means that  $\text{PDiffeo}(B^3; C^*)$  is the fibre of the fibration.

We have already remarked that  $\varepsilon$  is surjective. For  $\varepsilon$  to be a locally trivial fibration we need to prove that for every point  $\mathbf{C} \in \mathcal{PL}_n$  there is a neighbourhood  $U_{\mathbf{C}} \subset \mathcal{PL}_n$  together with a homeomorphism from  $U_{\mathbf{C}} \times \text{PDiffeo}(B^3; C^*)$  to  $\varepsilon^{-1}(U_{\mathbf{C}})$  whose composition with  $\varepsilon$  is the projection to the first factor  $U_{\mathbf{C}} \times \text{PDiffeo}(B^3; C^*) \rightarrow U_{\mathbf{C}}$ . This can be done concretely constructing the local product structure.

However an alternative proof of this Lemma consists in the following remark. The topological group  $\text{Diffeo}(B^3)$  acts transitively on the disc in the sense that: if  $(C_1, \dots, C_n)$  is a collection of  $n$  circles with the usual conditions, and  $(\gamma_1, \dots, \gamma_n)$  is another collection with the same conditions, then there is a diffeomorphism  $h$  of  $\text{Diffeo}(B^3)$  such that  $h(C_i) = \gamma_i$  for all  $i \in \{1, \dots, n\}$ . As already remarked, if  $h$  is an element of  $\text{Diffeo}(B^3; C^*)$ , then  $h(C_i) = C_i$  for all  $i \in \{1, \dots, n\}$ , and if  $h, h'$  are such that  $\varepsilon(h) = \varepsilon(h')$ , then they are in the same left coset of  $\text{Diffeo}(B^3; C^*)$  in  $\text{Diffeo}(B^3)$ . This means that  $\text{PDiffeo}(B^3; C^*)$  has a local cross-section in  $\text{Diffeo}(B^3)$  with respect to  $\varepsilon$ . Then, for [74][Section 7.4], we have the result.  $\square$

To prove the next theorem, we also need a result from Hatcher on the group  $\text{Diffeo}(B^3)$ .

**Theorem 1.2.10** ([41][Appendix]). *The group  $\text{Diffeo}(B^3)$  is contractible.*

**Theorem 1.2.11.** *For  $n \geq 1$ , there are natural isomorphisms between the pure ring group  $PR_n$  and the pure extended loop braid group  $PLB_n^{\text{ext}}$ , and between their respective unordered versions  $R_n$  and  $LB_n^{\text{ext}}$ .*

*Proof.* Let  $\varepsilon$  be the evaluation map (1.1). For Lemma 1.2.9, we have the short exact sequence

$$1 \longrightarrow \text{PDiffeo}(B^3; C^*) \xrightarrow{i} \text{Diffeo}(B^3) \xrightarrow{\varepsilon} \mathcal{PL}_n \longrightarrow 1.$$

This induces a long exact sequence of homotopy groups:

$$\begin{aligned} \cdots \longrightarrow \pi_1(\text{Diffeo}(B^3)) &\xrightarrow{\varepsilon_*} \pi_1(\mathcal{PL}_n) \xrightarrow{\partial} \pi_0(\text{PDiffeo}(B^3; C^*)) \xrightarrow{i_*} \\ &\xrightarrow{i_*} \pi_0(\text{Diffeo}(B^3)) \longrightarrow \cdots \end{aligned}$$

Both the groups  $\pi_1(\text{Diffeo}(B^3))$  and  $\pi_0(\text{Diffeo}(B^3))$  are trivial: both of these statements follow from Theorem 1.2.10. We recall that  $\pi_1(\mathcal{PL}_n)$  is isomorphic to  $PR_n$  and  $\pi_0(\text{PDiffeo}(B^3; C^*))$  is isomorphic to  $PLB_n^{ext}$ . Then we have an isomorphism between  $PR_n$  and  $PLB_n^{ext}$ .

Let us now consider  $R_n$ . We can construct the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & PLB_n^{ext} & \longrightarrow & LB_n^{ext} & \longrightarrow & S_n \longrightarrow 1 \\ & & \cong \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & PR_n & \longrightarrow & R_n & \longrightarrow & S_n \longrightarrow 1. \end{array}$$

The bijectivity of the central homomorphism follows from the five lemma.  $\square$

**Proposition 1.2.12.** *For  $n \geq 1$ , there are natural isomorphisms between pure untwisted ring group  $PUR_n$  and the pure loop braid group  $PLB_n$ , and between their respective unordered versions  $UR_n$  and  $LB_n$ .*

*Proof.* The group  $PLB_n$  injects as a normal subgroup in the group  $PLB_n^{ext}$ . In particular, we recall that  $PLB_n$  is the subgroup of elements of the mapping class group  $PLB_n^{ext}$  that preserve orientation on the  $n$  connected components of the submanifold  $C$  of  $B^3$ . It is the kernel of the map  $PLB_n^{ext} \rightarrow \mathbb{Z}_2^n$  sending an homeomorphism reversing the orientation on the  $i$ th component of  $C$ , and preserving the orientation on all the other components, to  $(0, \dots, 1, \dots, 0)$ , where the non-zero entry is in position  $i$ . We have the following short exact sequence.

$$1 \longrightarrow PLB_n \longrightarrow PLB_n^{ext} \longrightarrow \mathbb{Z}_2^n \longrightarrow 1 \quad (1.2)$$

On the other hand, from [21, Proposition 2.2], we have the short exact sequence

$$1 \longrightarrow PUR_n \xrightarrow{i} PR_n \xrightarrow[p]{s} \mathbb{Z}_2^n \longrightarrow 1 \quad (1.3)$$

which has an obvious splitting  $s$ , obtained by rotating the circles within disjoint balls. From the isomorphism between  $PR_n$  and  $PLB_n^{ext}$  (Theorem 1.2.11) we get the diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & PLB_n & \longrightarrow & PLB_n^{ext} & \longrightarrow & \mathbb{Z}_2^n \longrightarrow 1 \\ & & \downarrow & & \cong \downarrow & & \parallel \\ 1 & \longrightarrow & PUR_n & \longrightarrow & PR_n & \longrightarrow & \mathbb{Z}_2^n \longrightarrow 1. \end{array}$$

For the universal property of the kernel, the first vertical arrow is an isomorphism, which proves the first part of the statement. To prove the second isomorphism, we consider the

commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & PLB_n & \longrightarrow & LB_n & \longrightarrow & S_n \longrightarrow 1 \\
 & & \cong \downarrow & & \downarrow & & \parallel \\
 1 & \longrightarrow & PUR_n & \longrightarrow & UR_n & \longrightarrow & S_n \longrightarrow 1.
 \end{array}$$

The bijectivity of the central homomorphism follows from the five lemma.  $\square$

*Remark 1.2.13.* Note that to prove the isomorphism of Proposition 1.2.12 for loop braids we need to prove the result on extended loop braids first. This is because the first one relies on the fibration on  $\mathcal{PL}_n$  of Lemma 1.2.9. Indeed, in the context of configuration spaces, we cannot define a notion of “preserving orientation on the circles” when working with a single configuration of  $n$  circles as a point in the topological space of configurations.

Brendle and Hatcher, in [21, Proposition 3.3], also give a presentation for the untwisted ring groups  $UR_n$ , which are isomorphic to the loop braid groups  $LB_n$ .

**Proposition 1.2.14.** *For  $n \geq 1$ , the group  $UR_n$  admits a presentation given by generators  $\{\sigma_i, \rho_i \mid i = 1, \dots, n-1\}$ , subject to relations:*

$$\left\{ \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } i = 1, \dots, n-2 \\ \rho_i \rho_j = \rho_j \rho_i & \text{for } |i-j| > 1 \\ \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} & \text{for } i = 1, \dots, n-2 \\ \rho_i^2 = 1 & \text{for } i = 1, \dots, n-1 \\ \rho_i \sigma_j = \sigma_j \rho_i & \text{for } |i-j| > 1 \\ \rho_{i+1} \rho_i \sigma_{i+1} = \sigma_i \rho_{i+1} \rho_i & \text{for } i = 1, \dots, n-2 \\ \sigma_{i+1} \sigma_i \rho_{i+1} = \rho_i \sigma_{i+1} \sigma_i & \text{for } i = 1, \dots, n-2. \end{array} \right. \quad (1.4)$$

From this proposition it follows:

**Corollary 1.2.15.** *For  $n \geq 1$ , the group  $LB_n$  admits the presentation given in Proposition 1.2.14.*

In [21, Proposition 3.7] they also obtain a presentation for  $R_n$ , adding to the presentation in Proposition 1.2.14 generators  $\{\tau_i \mid i = 1, \dots, n\}$ , and some relations.

**Proposition 1.2.16.** *For  $n \geq 1$ , the group  $R_n$  admits a presentation given by generators*

$\{\sigma_i, \rho_i \mid i = 1, \dots, n-1\}$  and  $\{\tau_i \mid i = 1, \dots, n\}$ , subject to relations:

$$\begin{cases} \tau_i \tau_j = \tau_j \tau_i & \text{for } i \neq j \\ \tau_i^2 = 1 & \text{for } i = 1, \dots, n \\ \sigma_i \tau_j = \tau_j \sigma_i & \text{for } |i - j| > 1 \\ \rho_i \tau_j = \tau_j \rho_i & \text{for } |i - j| > 1 \\ \tau_i \rho_i = \rho_i \tau_{i+1} & \text{for } i = 1, \dots, n-1 \\ \tau_i \sigma_i = \sigma_i \tau_{i+1} & \text{for } i = 1, \dots, n-1 \\ \tau_{i+1} \sigma_i = \rho_i \sigma_i^{-1} \rho_i \tau_i & \text{for } i = 1, \dots, n-1. \end{cases} \quad (1.5)$$

From this proposition it follows:

**Corollary 1.2.17.** *For  $n \geq 1$ , the group  $LB_n^{ext}$  admits the presentation given in Proposition 1.2.16.*

The elements  $\sigma_i$ ,  $\rho_i$ , and  $\tau_i$  of the presentation represent the following loops in  $R_n$ : if we place the  $n$  rings in a standard position in the  $yz$ -plane with centers along the  $y$ -axis, then the  $\sigma_i$  is the loop that permutes the  $i$ -th and the  $(i+1)$ -st circles by passing the  $i$ -th circle through the  $(i+1)$ -st;  $\rho_i$  permutes them passing the  $i$ -th around the  $(i+1)$ -st, and  $\tau_i$  changes the orientation (“flips”) the  $i$ -th circle, see Figure 1.1<sup>1</sup>.

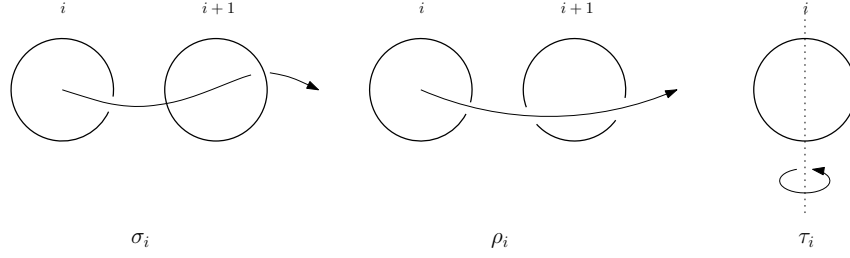


Figure 1.1: Elements  $\sigma_i$ ,  $\rho_i$  and  $\tau_i$ .

Let  $\alpha_{ij}$  be the elements of  $PUR_n$ , representing the movement of the  $i$ -th circle passing through the  $j$ -th circle and going back to its position (see Figure 1.2). Brendle and Hatcher also give a presentation for  $PUR_n$ , the pure subgroups of the groups  $UR_n$ .

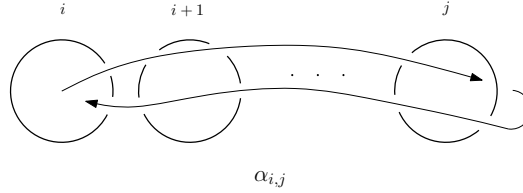


Figure 1.2: Elements  $\alpha_{ij}$ .

<sup>1</sup>We reverse the notations used in [21] for  $\sigma_i$  and  $\rho_i$ , see also, for example, notations used in [15, 30, 37]

**Proposition 1.2.18.** *For  $n \geq 1$ , the group  $PUR_n$  admits a presentation with generators  $\alpha_{ij}$  for  $1 \leq i \neq j \leq n$  and relations:*

$$\begin{cases} \alpha_{ij}\alpha_{kl} = \alpha_{kl}\alpha_{ij} \\ \alpha_{ik}\alpha_{jk} = \alpha_{jk}\alpha_{ik} \\ \alpha_{ij}(\alpha_{ik}\alpha_{jk}) = (\alpha_{ik}\alpha_{jk})\alpha_{ij}. \end{cases} \quad (1.6)$$

**Corollary 1.2.19.** *For  $n \geq 1$ , the group  $PLB_n$  admits the presentation given in Proposition 1.2.18.*

In Corollaries 1.2.15, 1.2.17, and 1.2.19 we showed presentations for the loop braid groups  $LB_n$ , the extended loop braid groups  $LB_n^{ext}$ , and the pure loop braid groups  $PLB_n$ . At the end of next section we will also give a presentation for the pure extended loop braid groups  $PLB_n^{ext}$ .

### 1.3 Conjugating automorphisms of the free group

In this section we give an interpretation of  $LB_n$  and  $LB_n^{ext}$  in terms of automorphisms of  $F_n$ , the free groups of rank  $n$ . Fixing  $n \geq 1$ , we consider automorphisms that send each generator of  $F_n$  to a conjugate of some generator: these are in bijection with elements of  $LB_n$ . When considering automorphisms that send each generator to a conjugate of some generator, or of the inverse of some generator, we have a bijection with elements of  $LB_n^{ext}$ . Finally, when considering automorphisms that send a generator of  $F_n$  to a conjugate of itself, we have a bijection with elements of the pure loop braid group  $PLB_n$ , and if a generator is sent to the conjugate of itself or its inverse, then we have a bijection with elements of  $PLB_n^{ext}$ . We start from a result of Dahm's unpublished thesis [27], that appears in the last section of Goldsmith's paper [37]. We remark that the result is established for  $LB_n^{ext}$ , and that the result for  $LB_n$  is a consequence of it.

**Theorem 1.3.1** ([37, Theorem 5.3]). *For  $n \geq 1$ , there is an injective map from the extended loop braid group  $LB_n^{ext}$  into  $\text{Aut}(F_n)$ , where  $F_n$  is the free group on  $n$  generators  $\{x_1, \dots, x_n\}$ , and its image is the subgroup  $PC_n^*$ , consisting of all automorphisms of the form  $\alpha: x_i \mapsto w_i^{-1} x_{\pi(i)}^{\pm 1} w_i$  where  $\pi$  is a permutation and  $w_i$  is a word in  $F_n$ . Moreover, the group  $PC_n^*$  is generated by the automorphisms  $\{\sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_{n-1}, \tau_1, \dots, \tau_n\}$  defined as:*

$$\sigma_i : \begin{cases} x_i \mapsto x_{i+1}; \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}; \\ x_j \mapsto x_j, \end{cases} \quad \text{for } j \neq i, i+1. \quad (1.7)$$

$$\rho_i : \begin{cases} x_i \mapsto x_{i+1}; \\ x_{i+1} \mapsto x_i; \\ x_j \mapsto x_j, \end{cases} \quad \text{for } j \neq i, i+1. \quad (1.8)$$

$$\tau_i : \begin{cases} x_i \mapsto x_i^{-1}; \\ x_j \mapsto x_j, \end{cases} \quad \text{for } j \neq i. \quad (1.9)$$

*Sketch of the proof.* The main ingredient of the proof is the Dahm homomorphism. This is a map

$$D: \text{MCG}(M, N^*) \longrightarrow \text{Aut}(\pi_1(M \setminus N))$$

where  $M$  and  $N$  are a 3-manifold and a finite subset as in Section 1.1. Consider  $[f] \in \text{MCG}(M, N^*)$ . Dahm homomorphism sends an element  $[f]$  of the mapping class group, represented by a family  $g_t$  for  $t \in [0, 1]$ , to the automorphism induced by  $g_1$  on  $\pi_1(M \setminus N)$ . Applying Dahm homomorphism to  $M = B^3$  and  $N = C$ , we have a map

$$D: LB_n^{\text{ext}} \longrightarrow \text{Aut}(\pi_1(F_n))$$

where  $F_n$  is the free group of rank  $n$ . This homomorphism is injective ([37, Theorem 5.2]). The possible self-homomorphisms of  $B^3$  which move the circles  $C$  are generated by “flips” of the circles, exchanges of two circles, and pulling one circle through another circle. The respective elements of  $LB_n^{\text{ext}}$  are sent by the Dahm homomorphism to the automorphisms in (1.9), (1.8), and (1.7). In [37, Theorem 5.3] it is proved that these automorphisms generate  $PC_n^*$ .  $\square$

*Remark 1.3.2.* The correspondence between the elements of the mapping class group representing the movements  $\sigma_i$ ,  $\rho_i$  and  $\tau_i$  in  $R_n$ , and the automorphisms of  $PC_n^*$  with the same names justifies the abuse of notation.

The following result, which is a consequence of Theorem 1.3.1, establishes an isomorphism between  $LB_n$  and  $PC_n$ , the groups of permutation-conjugacy automorphisms. These groups consist of all automorphisms of the form  $\alpha: x_i \mapsto w_i^{-1} x_{\pi(i)} w_i$  where  $\pi$  is a permutation and  $w_i$  is a word in  $F_n$ .

**Corollary 1.3.3.** *For  $n \geq 1$ , there is an injection from  $LB_n$  to  $\text{Aut}(F_n)$ , where  $F_n$  is the free group on  $n$  generators  $\{x_1, \dots, x_n\}$ , and its image is the subgroup  $PC_n$ . Moreover, the group  $PC_n$  is generated by automorphisms (1.7) and (1.8).*

*Remark 1.3.4.* The elements  $\sigma_i$  in  $PC_n$  generate the *braid subgroup*  $B_n$  of  $\text{Aut}(F_n)$  which is well known to be isomorphic to the classical braid group on  $n$  strings, and the elements  $\rho_i$  generate the *permutation subgroup*  $\bar{S}_n$  of  $\text{Aut}(F_n)$  which is a copy of the symmetric group  $S_n$ . Moreover Artin provided (see for instance [40, Theorem 5.1]) a characterization of usual braids as automorphisms of free groups of which Theorem 1.3.1 is the analogue. Let us recall it: an automorphism  $\beta \in \text{Aut}(F_n)$  lies in  $B_n$  if and only if  $\beta$  satisfies the following conditions:

- i)  $\beta(x_i) = a_i x_{\pi(i)} a_i^{-1}, \quad 1 \leq i \leq n$  ;
- ii)  $\beta(x_1 x_2 \dots x_n) = x_1 x_2 \dots x_n$  ,

where  $\pi \in S_n$  and  $a_i \in F_n$ .



*Remark 1.3.5.* Fenn, Rimányi and Rourke, in [30], consider the subgroups  $BP_n$  of  $\text{Aut}(F_n)$  generated by both sets of elements  $\{\sigma_i \mid i = 1, \dots, n-1\}$  and  $\{\rho_i \mid i = 1, \dots, n-1\}$ . They call these groups the *braid-permutation groups*, and they prove independently from Dahm and Goldsmith that they are isomorphic to the permutation-conjugacy groups  $PC_n$ , and that they admit the presentation given in Proposition 1.2.14.

*Remark 1.3.6.* The presentation given in Proposition 1.2.18 coincides with the presentation given by McCool in [64] for the groups of *basis-conjugating automorphisms* of the free group. Fixed an  $n \geq 1$ , this is the group generated by the automorphisms:

$$\alpha_{ij} : \begin{cases} x_i \mapsto x_j^{-1} x_i x_j; \\ x_k \mapsto x_k, \text{ for } k \neq i, j. \end{cases} \quad (1.10)$$

*Remark 1.3.7.* In [73] Savushkina proves that the centers of the groups  $PLB_n$  are trivial, and gives a presentation for the groups of *permutation-conjugacy automorphisms*, isomorphic to the loop braid groups  $LB_n$ .

To complete the picture we use the conjugating automorphisms point of view to give a presentation for  $PLB_n^{ext}$ , the pure subgroups of the groups of extended loop braids  $LB_n^{ext}$ .

**Proposition 1.3.8.** *For  $n \geq 1$ , the group  $PLB_n^{ext}$  admits the following presentation:*

$$\langle \{\alpha_{ij} \mid 1 \leq i \neq j \leq n\} \cup \{\tau_i \mid i = 1, \dots, n\} \mid R' \rangle \quad (1.11)$$

where  $R'$  is the set of relations:

$$\begin{cases} \alpha_{ij}\alpha_{kl} = \alpha_{kl}\alpha_{ij} \\ \alpha_{ij}\alpha_{kj} = \alpha_{kj}\alpha_{ij} \\ (\alpha_{ij}\alpha_{kj})\alpha_{ik} = \alpha_{ik}(\alpha_{ij}\alpha_{kj}) \\ \tau_i^2 = 1 \\ \tau_i\alpha_{ij} = \alpha_{ij}\tau_i \\ \tau_i\alpha_{jk} = \alpha_{jk}\tau_i \\ \tau_i\alpha_{ji}\tau_i = \alpha_{ji}^{-1} \end{cases} \quad (1.12)$$

where different letters stand for different indices, and  $\alpha_{ij}$  and  $\tau_i$  correspond to the automorphisms (1.10) and (1.9).

*Proof.* By the split short exact sequence (1.3), and the isomorphism between  $PR_n$  and  $PLB_n^{ext}$ , we have that  $PLB_n^{ext}$  is isomorphic to the semidirect product  $PLB_n \rtimes_{\varphi} \mathbb{Z}_2^n$ , where  $\varphi: \mathbb{Z}_2^n \rightarrow \text{Aut}(PLB_n)$  is the map defined by:

$$\varphi(x) = \varphi_x: g \longrightarrow i^{-1}(s(x)i(g)s(x^{-1})).$$

Considering the presentation  $\langle \{\tau_i \mid i = 1, \dots, n\} \mid \tau_i^2 = 1 \rangle$  for  $\mathbb{Z}_2^n$  we have that  $PLB_n^{ext}$  admits the following presentation:

$$\langle (\alpha_{ij})_{1 \leq i \neq j \leq n} \cup (\tau_i)_{1 \leq i \leq n} \mid R, S, \{\tau_i\alpha_{jl}\tau_i^{-1} = \varphi_{\tau_i}(\alpha_{jl})\} \rangle \quad (1.13)$$

where the  $\alpha_{ij}$  are generators of  $PLB_n$ , the  $\tau_i$  are generators of  $\mathbb{Z}_2^n$ , and  $R$  and  $S$  are the respective sets of relations.

The groups  $PLB_n$  and  $\mathbb{Z}_2^n$  can be seen as subgroups of  $\text{Aut}(F_n)$ . Then  $\varphi$  is the action of  $\mathbb{Z}_2^n$  on  $PLB_n$ , so to understand the mixed relation between the  $\alpha_{ij}$  and the  $\tau_k$  it is sufficient to see how  $\tau_k$  act on  $\alpha_{ij}$ , for  $1 \leq i \neq j \leq n$  and  $k = 1, \dots, n$ .

Both families of automorphisms are defined on generators  $x_i$  of  $F_n$ . In particular,  $x_l$  with  $l \neq i, j, k$ , is left unvaried by both automorphisms. Hence it is sufficient to consider the cases where  $l$  is equal to  $i$  or  $j$ , and/or  $k$ . In other words, we are only to consider the mutual positions of three indices. To simplify the calculation we can take  $i, j, k, l \in \{1, 2, 3\}$ . We make a list of all the cases to consider in Table 1.1.

Table 1.1: All the cases we need to consider to write the mixed relations.

$\alpha_{ij}$ 's indices	$\tau_k$ 's index	$i, j, k, l \in \{1, 2, 3\}$
$i < j$	$k = i < j$	$\tau_1 \alpha_{12} \tau_1 = \alpha_{12}$
	$i < j = k$	$\tau_2 \alpha_{12} \tau_2 = \alpha_{12}^{-1}$
	$k < i < j$	$\tau_1 \alpha_{23} \tau_1 = \alpha_{23}$
	$i < k < j$	$\tau_2 \alpha_{13} \tau_2 = \alpha_{13}$
	$i < j < k$	$\tau_3 \alpha_{12} \tau_3 = \alpha_{12}$
$j < i$	$k = j < i$	$\tau_1 \alpha_{21} \tau_1 = \alpha_{21}^{-1}$
	$j < i = k$	$\tau_2 \alpha_{21} \tau_2 = \alpha_{21}$
	$k < j < i$	$\tau_1 \alpha_{32} \tau_1 = \alpha_{32}$
	$j < k < i$	$\tau_2 \alpha_{31} \tau_2 = \alpha_{31}$
	$i < j < k$	$\tau_3 \alpha_{12} \tau_3 = \alpha_{12}$

Let us see the explicit computation for the second case in the table:

$$\begin{aligned} \tau_2 \alpha_{12} \tau_2(x_1) &= \tau_2 \alpha_{12}(x_1) = \tau_2(x_2^{-1} x_1 x_2) = x_2 x_1 x_2^{-1} = \alpha_{12}^{-1}(x_1); \\ \tau_2 \alpha_{12} \tau_2(x_2) &= \tau_2 \alpha_{12}(x_2^{-1}) = \tau_2(x_2^{-1}) = x_2 = \alpha_{12}^{-1}(x_2); \\ \tau_2 \alpha_{12} \tau_2(x_3) &= \tau_2 \alpha_{12}(x_3) = \tau_2(x_3) = x_3 = \alpha_{12}^{-1}(x_3). \end{aligned}$$

So  $\tau_2 \alpha_{12} \tau_2 = \alpha_{12}^{-1}$ . Doing the calculation for all the other cases, we obtain the mixed relations:  $\tau_i \alpha_{ji} \tau_i = \alpha_{ji}^{-1}$ ,  $\tau_i \alpha_{jk} = \alpha_{jk} \tau_i$  and  $\tau_i \alpha_{ij} = \alpha_{ij} \tau_i$ .  $\square$

## 1.4 Ribbon braids and flying rings

We now introduce *ribbon braids* and *extended ribbon braids*. Using the results from Section 1.2, we establish isomorphisms between the groups of (extended) ribbon braids and the (extended) loop braid groups. These isomorphisms give us another interpretation of loop braids as topological knotted objects. We also prove that every isotopy of a ribbon braid in  $B^3 \times I$  extends to an isotopy of  $B^3 \times I$  itself constant on the boundary.

Let  $n \geq 1$ . We recall and adapt notations and definitions from [4] and [5]. Let  $D_1, \dots, D_n$  be a collection of disks in the 2-ball  $B^2$ . Let  $C_i = \partial D_i$  be the oriented boundary of  $D_i$ . Let us consider the 4-ball  $B^4 \cong B^3 \times I$ , where  $I$  is the unit interval. For any submanifold  $X \subset B^m \cong B^{m-1} \times I$ , with  $m = 3, 4$ , we use the following dictionary. To keep the notation readable, here we denote the interior of a topological space by " $\text{int}(\ )$ ", whereas anywhere else it is denoted by " $\circ$ ".

- $\partial_\varepsilon X = X \cap (B^{m-1} \times \{\varepsilon\})$ , with  $\varepsilon \in \{0, 1\}$ ;
- $\partial_* X = \partial X \setminus (\text{int}(\partial_0 X) \sqcup \text{int}(\partial_1 X))$ ;
- $\overset{*}{X} = X \setminus \partial_* X$ .

The image of an immersion  $Y \subset X$  is said to be *locally flat* if and only if it is locally homeomorphic to a linear subspace  $\mathbb{R}^k$  in  $\mathbb{R}^m$  for some  $k \leq m$ , except on  $\partial X$  and/or  $\partial Y$ , where one of the  $\mathbb{R}$  summands should be replaced by  $\mathbb{R}_+$ . Let  $Y_1, Y_2$  be two submanifolds of  $B^m$ . The intersection  $Y_1 \cap Y_2 \subset X$  is called *flatly transverse* if and only if it is locally homeomorphic to the transverse intersection of two linear subspaces  $\mathbb{R}^{k_1}$  and  $\mathbb{R}^{k_2}$  in  $\mathbb{R}^m$  for some positive integers  $k_1, k_2 \leq m$  except on  $\partial X$ ,  $\partial Y_1$  and/or  $\partial Y_2$ , where one of the  $\mathbb{R}$  summands should be replaced by  $\mathbb{R}_+$ . In the next definition we introduce the kind of singularities we consider.

**Definition 1.4.1.** *Ribbon disks* are intersections  $D = Y_1 \cap Y_2 \subset \mathbb{R}^4$  that are isomorphic to the 2-dimensional disk, such that  $D \subset \overset{\circ}{Y}_1$ ,  $\overset{\circ}{D} \subset \overset{\circ}{Y}_2$  and  $\partial D$  is an essential curve in  $\partial Y_2$ .

These singularities are the 4-dimensional analogues of the classical notion of ribbon singularities introduced by Fox in [35].

**Definition 1.4.2.** Let  $A_1, \dots, A_n$  be locally flat embeddings in  $\overset{*}{B}^4$  of  $n$  disjoint copies of the oriented annulus  $S^1 \times I$ . We say that

$$b = \bigsqcup_{i \in \{1, \dots, n\}} A_i$$

is a *geometric ribbon braid* if:

1. the boundary of each annulus  $\partial A_i$  is a disjoint union  $C_i \sqcup C_j$ , for  $C_i \in \partial_0 B^4$  and for some  $C_j \in \partial_1 B^4$ . The orientation induced by  $A_i$  on  $\partial A_i$  coincides with the one of the two boundary circles  $C_i$  and  $C_j$ ;
2. the annuli  $A_i$  are fillable, in the sense that they bound immersed 3-balls  $\subset \mathbb{R}^4$  whose singular points consist in a finite number of ribbon disks;
3. it is transverse to the lamination  $\bigcup_{t \in I} B^3 \times \{t\}$  of  $B^4$ , that is: at each parameter  $t$ , the intersection between  $b$  and  $B^3 \times t$  is a collection of exactly  $n$  circles;
4. the orientations of the circles are concordant, at each parameter  $t$ , to the orientations of the circles that compose the boundary of the annulus.

The group of *ribbon braids*, denoted by  $rB_n$  is the group of equivalence classes of geometric ribbon braids up to continuous deformations through the class of geometric ribbon braids fixing the boundary circles, equipped with the natural product given by stacking and reparametrizing. The unit element for this product is the *trivial ribbon braid*  $U = \bigsqcup_{i \in \{1, \dots, n\}} C_i \times [0, 1]$ .

*Remark 1.4.3.* Let us consider  $I$  in  $B^4 = B^3 \times I$  as a time parameter. If one of the  $n$  circles that we have at each time  $t$  makes a half-turn, we have what is called a *wen* on the corresponding component. One can think of a wen as an embedding in 4-space of a Klein bottle cut along a meridional circle. The last condition of the definition makes sure that there are no wens on the components of a geometric ribbon braid. A detailed treatment of wens can be found in [50].

*Remark 1.4.4.* We recall that also the braid group  $B_n$  can be defined in an analogous way, as equivalence classes of geometric braids, see [51, Chapter 1.2].

The following theorem shows that when two ribbon braids are equivalent in the sense of Definition 1.4.2, there is an ambient isotopy of  $\mathbb{R}^4$  bringing one to the other.

**Theorem 1.4.5.** *Every relative isotopy of a geometric ribbon braid in  $B^3 \times I$  extends to an isotopy of  $B^3 \times I$  in itself constant on the boundary.*

*Proof.* We follow step by step the proof given for the case of usual braids in [51]. Let  $b \subset B^4 \cong B^3 \times I$  be a geometric ribbon braid with  $n$  components, and let us call  $T$  the product  $B^3 \times I$ . Let

$$F: b \times I \longrightarrow T$$

be an isotopy of  $b$ . Thus, for each  $s \in I$ , the map

$$\begin{aligned} F_s: b &\longrightarrow T \\ x &\mapsto F(x, s) \end{aligned}$$

is an embedding whose image is a geometric ribbon braid, and  $F_0 = \text{id}_b$ .

We want to define another continuous map  $G: T \times I \rightarrow T$  such that for each  $s \in I$  the map  $G_s: T \rightarrow T$  sending  $x \in T$  to  $G(x, s)$  is a homeomorphism fixing  $\partial T$  pointwise and extending  $F_s$ , and such that  $G_0 = \text{id}_T$ .

The first step to construct it is to consider a set  $C = \{C_1, \dots, C_n\}$  of  $n$  disjoint, oriented, unlinked, unknotted circles in the interior of  $B^3$ , and another 3-ball  $D^3$  such that:  $C$  is contained in  $\mathring{D}^3$ , and  $\mathring{D}^3$  is contained in  $\mathring{B}^3$ . Moreover, let  $F(b \times I)$  be contained into  $\mathring{D}^3 \times I$ . Remark that  $C \times \{0, 1\}$  is the disjoint union of the circles that compose the boundary components of  $b$  and of  $F_s(b)$  for all  $s \in I$ .

For any  $(s, t) \in I^2$ , denote by  $f(s, t)$  the unique subset of  $n$  circles of  $\mathring{D}^3$  such that

$$F_s(b) \cap (D^3 \times \{t\}) = f(s, t) \times \{t\}.$$

In other words,  $f(s, t)$  is composed by the  $n$  circles of the ribbon braid  $F_s(b)$  at the coordinate  $\{t\}$  of the foliation  $D^3 \times I = \bigcup_{i \in I} D^3 \times \{t\}$ .

So we have a continuous map

$$f: I^2 \longrightarrow \mathcal{UL}_n \subset \mathcal{L}_n$$

where  $\mathcal{UL}_n$  is the configuration space of  $n$  disjoint, oriented, unlinked, unknotted circles that lie on parallel planes.

By definition  $f(s, 0) = f(s, 1) = C$  for all  $s \in I$  and  $b = \bigcup_{t \in I} f(0, t) \times \{t\}$ . Let us consider now the loop

$$\begin{aligned} f_0: I &\longrightarrow D^3 \\ t &\mapsto f(0, t) \end{aligned}$$

that sends the parameter  $t$  to the circles composing  $b$  at the coordinate  $t$ .

Consider the evaluation fibration  $\varepsilon: \text{Diffeo}(D^3) \rightarrow \mathcal{PL}_n$  from Lemma 1.2.9. Composing  $\varepsilon$  with the covering map  $\mathcal{PL}_n \rightarrow \mathcal{L}_n$ , seeing  $\mathcal{L}_n$  as the orbit space with of the action of the symmetric group of  $\mathcal{PL}_n$ , we define a locally trivial fibration

$$\tilde{\varepsilon}: \text{Diffeo}(D^3) \longrightarrow \mathcal{L}_n.$$

By the homotopy lifting property of  $\tilde{\varepsilon}$ , the loop  $f_0$  lifts to a path  $\hat{f}_0$  as in the following diagram.

$$\begin{array}{ccc} & \text{Diffeo}(B^3) & \\ \hat{f}_0 \nearrow & \downarrow \tilde{\varepsilon} & \\ I & \xrightarrow{f_0} & \mathcal{L}_n \end{array}$$

The lifted map is such that  $\hat{f}_0(1) = \text{id}_{D^3}$  and  $\hat{f}_0(0)$  is an element of  $\text{Diffeo}(D^3; C^*)$ . Remark that  $\tilde{\varepsilon} \circ \hat{f}_0(0) = C = f_0(0)$ ,  $\tilde{\varepsilon} \circ \hat{f}_0(1) = C = f_0(1)$ . By the homotopy lifting property of  $\varepsilon$ , considering the topological pair  $(I, \partial I)$ , the latter path extends to a lift

$$\hat{f}: I \times I \longrightarrow \text{Diffeo}(D^3)$$

such that  $\hat{f}(s, 1) = \text{id}_{D^3}$  and  $\hat{f}(s, 0) = \hat{f}(0, 0)$  for all  $s \in I$ . We define a homeomorphism

$$g(s, t): B^3 \longrightarrow B^3$$

defined by

$$g(s, t) = \begin{cases} \text{id}_{B^3} & \text{on } B^3 \setminus D^3, \\ \hat{f}(s, t) \circ (\hat{f}(0, t))^{-1} & \text{on } x \in D^3. \end{cases}$$

This is a continuous function on  $(s, t) \in I^2$  and

$$g(0, t) = g(s, 0) = g(s, 1) = \text{id}_{B^3}.$$

Moreover

$$g(s, t)(f(0, t)) = g(s, t)(\hat{f}(0, t)(C)) = \hat{f}(s, t)(C) = f(s, t).$$

Let us define

$$\begin{aligned} G: T \times I &\longrightarrow T \\ (x, t, s) &\mapsto (g(s, t)(x), t) \end{aligned}$$

for  $x \in B^3$ ,  $(t, s) \in I \times I$ . For each  $s \in I$  we have a homeomorphisms

$$G_s: T \longrightarrow T$$

that fixes  $\partial T$  pointwise, extends  $F_s$ , and  $G_0 = \text{id}_{B^3}$ .  $\square$

*Remark 1.4.6.* This result is true also for *surface links*, which are closed surfaces locally flatly embedded in  $\mathbb{R}^4$ . This is proved with different methods in [46, Theorem 6.7].

**Definition 1.4.7.** A *pure geometric ribbon braid* is a geometric braid as in Definition 1.4.2, for which condition (1) is replaced with

$$(1)' \quad \partial A_i = C_i \times \{0, 1\} \text{ for all } i \in \{1, \dots, n\} \text{ and the orientation induced by } A_i \text{ on } \partial A_i \text{ coincides with that of } C_i.$$

The group of *pure ribbon braids*, denoted by  $PrB_n$ , is the group of equivalence classes of pure geometric ribbon braids up to continuous deformations through the class of geometric ribbon braids fixing the boundary circles, equipped with the natural product given by stacking and reparametrizing.

*Remark 1.4.8.* The group  $PrB_n$  coincides with the kernel of the homomorphism from the group  $rB_n$  to the group of permutation  $S_n$  that associates to a ribbon braid the permutation induced on the boundary circles.

*Remark 1.4.9.* Taken  $b$  a geometric ribbon braid, the transversality forces  $b \cap (B^3 \times \{t\})$  to be the disjoint union of  $n$  circles, for all  $t \in I$ . We can though think to a ribbon braid as a trajectory  $\beta = (C_1(t), \dots, C_n(t))$  of circles in  $B^3 \times I$ . This interpretation corresponds, in the classical case, to the interpretation of braids as particle dance. A visual idea of this can be found in Dalvit's website [28]. This formulation gives rise to the name *groups of flying rings*, denoted by  $FL_n$  and used in [10].

We prove now that, for each  $n \geq 1$ ,  $PrB_n$  and  $rB_n$  are respectively isomorphic to  $PLB_n$  and  $LB_n$ .

**Proposition 1.4.10.** *For  $n \geq 1$ , there is an isomorphism between the pure ribbon braid group  $PrB_n$  and the pure loop braid group  $PLB_n$ .*

*Proof.* Let  $\beta = (C_1(t), \dots, C_n(t))$  be an element of  $PrB_n$  described by a parametrization as in Remark 1.4.9, and set

$$\varphi(\beta): [0, 1] \longrightarrow PLB_n$$

as the morphism defined by  $t \mapsto (C_1(t), \dots, C_n(t))$ . By definition,  $\varphi(\beta)$  is a loop in the configuration space  $PUR_n$  (Definition 1.2.1), and corresponds to an element of  $PLB_n$  through the isomorphism in Proposition 1.2.12. This map induces a bijection

$$\varphi_\star: PrB_n \longrightarrow PLB_n.$$

Indeed, two pure geometric braids  $\beta'$  and  $\beta''$  are equivalent if and only if there is an ambient isotopy of  $\mathbb{R}^3 \times [0, 1]$  from the identity map to a self-homeomorphism that maps  $\beta'$  to  $\beta''$ . That by construction would be an isotopy (so in particular a homotopy) between the two associated loops in  $PLB_n$ . Moreover products are preserved, so  $\varphi_*$  is an isomorphism.  $\square$

**Theorem 1.4.11.** *For  $n \geq 1$ , there is an isomorphism between the ribbon braid group  $rB_n$  and the loop braid group  $LB_n$ .*

*Proof.* We recall that the untwisted ring group  $UR_n$  (Definition 1.2.1) is the fundamental group of  $\mathcal{UR}_n$ , which is the quotient of  $\mathcal{PUR}_n$  by the symmetric group on  $n$  components. As in Proposition 1.4.10 we fix an element  $\beta = (C_1(t), \dots, C_n(t))$  of  $rB_n$  and define a map:

$$\hat{\varphi}(\beta): [0, 1] \longrightarrow LB_n^{ext}$$

by  $t \mapsto [C_1(t), \dots, C_n(t)]$ . The element  $\hat{\varphi}(\beta)$  is a loop in the configuration space  $UR_n$ . This loop corresponds to an element of  $LB_n$  through the isomorphism from Proposition 1.2.12. Then  $\hat{\varphi}$  induces an homomorphism

$$\hat{\varphi}_*: rB_n \longrightarrow LB_n.$$

We consider the following diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & PrB_n & \longrightarrow & rB_n & \longrightarrow & S_n \longrightarrow 1 \\ & & \cong \downarrow \varphi_* & & \downarrow \hat{\varphi}_* & & \parallel \\ 1 & \longrightarrow & PLB_n & \longrightarrow & LB_n & \longrightarrow & S_n \longrightarrow 1. \end{array}$$

It is commutative by construction of  $\varphi$  and  $\hat{\varphi}$ . By applying the five lemma, the statement is proved.  $\square$

Let  $C \subset \mathring{B}^3$  be a collection of  $n$  unordered, disjoint, unknotted, unlinked circles. An isotopy  $\{f_t: B^3 \rightarrow B^3\}_{t \in I}$  in the class of self-diffeomorphisms of  $B^3$  is *normal* with respect to  $C$  if  $f_0(C) = C$  and  $f_1 = \text{id}_{B^3}$ . In other words a normal isotopy is a path in  $\text{Diffeo}(B^3)$  leading from a point of  $\text{Diffeo}(B^3; C)$  to the identity diffeomorphism  $\text{id}_{B^3}$  in  $\text{Diffeo}(B^3)$ . It is clear that, for any normal isotopy  $\{f_t: B^3 \rightarrow B^3\}_{t \in I}$ , the set  $\bigcup_{t \in I} (f_t(C), t) \subset B^3 \times I$  is a geometric ribbon braid on  $n$  components. We say that  $\{f_t\}_{t \in I}$  *parametrizes* the geometric ribbon braid.

In the spirit of the definition of geometric ribbon braids as flying rings (Remark 1.4.9), we prove that given a geometric ribbon braid  $b$  and its set of starting circles  $C \subset \mathring{B}^3$ , we can find a normal isotopy parametrizing it.

**Lemma 1.4.12.** *Let  $n \geq 1$ . For every geometric ribbon braid  $b \in B^4$  on  $n$  components, there is a normal isotopy parametrizing  $b$ .*

*Proof.* We consider the evaluation map

$$\tilde{\varepsilon}: \text{Diffeo}(B^3) \longrightarrow \mathcal{L}_n$$

constructed in the proof of Theorem 1.4.5, sending  $f$  to  $f(C)$ . The ribbon braid  $b$ , through the isomorphism between  $rB_n$  and  $UR_n$  used in Theorem 1.4.11, gives rise to a loop  $f^b: I \rightarrow \mathcal{PL}_n \subset \mathcal{L}_n$  sending  $t \in I$  into the unique  $n$ -circles set  $b_t$  such that

$$b \cap (B^3 \times I) = b_t \times \{t\}.$$

This loop begins and ends at the point  $\tilde{\varepsilon}(\text{id}_{B^3}) \in \mathcal{L}_n$  represented by  $C$ . Being  $\tilde{\varepsilon}$  a fibration, we apply the homotopy lifting property, and lift  $f^b$  to a path  $\hat{f}^b: I \rightarrow \text{Diffeo}(B^3)$  beginning at  $\tilde{\varepsilon}^{-1}(C) = \text{Diffeo}(B^3; C^*)$ , which in particular, being  $b$  a geometric ribbon braid, is in  $\text{Diffeo}(B^3; C)$ , and ending at  $\text{id}_{B^3}$ . The path  $\hat{f}^b$  is a normal isotopy. The commutativity  $\tilde{\varepsilon} \circ \hat{f}^b = f^b$  means that this isotopy parametrizes  $b$ .  $\square$

#### 1.4.1 Extended ribbon braids

We introduce a new topological object, that is the topological realization of extended loop braids.

**Definition 1.4.13.** Let  $n \geq 1$ . An *extended geometric ribbon braid* is a geometric ribbon braid, with condition (4) from Definition 1.4.2 removed, and condition (1) replaced by:

- (1)'' the boundary of each annulus  $\partial A_i$  is a disjoint union  $C_i \sqcup C_j$ , for  $C_i \in \partial_0 B^4$  and for some  $C_j \in \partial_1 B^4$ .

The group of *extended ribbon braids*, denoted by  $rB_n^{\text{ext}}$ , is the group of equivalence classes of geometric ribbon braids up to continuous deformations through the class of extended geometric ribbon braids fixing the boundary circles, equipped with the natural product given by stacking and reparametrizing. The unit element for this product is the *trivial ribbon braid*  $U = \bigsqcup_{i \in \{1, \dots, n\}} C_i \times [0, 1]$ .

**Definition 1.4.14.** The group of *extended pure ribbon braids*, denoted by  $PrB_n^{\text{ext}}$ , is the group of equivalence classes of pure geometric extended ribbon braids, defined as pure geometric ribbon braids (Definition 1.4.7) with condition (4) removed and condition (1)' replaced by

- (1)'''  $\partial A_i = C_i \times \{0, 1\}$  for all  $i \in \{1, \dots, n\}$ .

*Remark 1.4.15.* Theorem 1.4.5 extends to extended ribbon braid groups  $rB_n^{\text{ext}}$ : in fact the proof uses the evaluation filtration  $\tilde{\varepsilon}: \text{Diffeo}(B^3) \rightarrow \mathcal{L}_n$ . We recall that, for  $n \geq 1$  fixed,  $\mathcal{L}_n$  is the space of configurations of  $n$  unordered, disjoint, unlinked trivial links in  $B^3$ , that are not restrained of parallel planes, *i.e.*, can make  $180^\circ$  flips.

*Remark 1.4.16.* The monotony condition allows us to consider the interval  $I$  in  $B^4 = B^3 \times I$  as a time parameter, and to think of an extended ribbon braid as a trajectory  $\beta = (C_1(t), \dots, C_n(t))$  of circles in  $B^3 \times I$ . This trajectory corresponds to a parametrization of the extended ribbon braid. That corresponds to an interpretation in terms of flying rings allowed to “flip”.



The following statements, proved for the case of ribbon braids, hold also in the extended case.

**Proposition 1.4.17.** *For  $n \geq 1$ , there is an isomorphism between the pure extended ribbon braid group  $PrB_n^{ext}$  and the pure extended loop braid group  $PLB_n^{ext}$ .*

*Proof.* To prove this statement it is enough to follow the proof of Proposition 1.4.10, replacing  $PUR_n$  with  $PR_n$ , and considering the isomorphism between  $PR_n$  and  $PLB_n^{ext}$ .  $\square$

**Theorem 1.4.18.** *For  $n \geq 1$ , there is an isomorphism between the extended ribbon braid group  $rB_n^{ext}$  and the extended loop braid group  $LB_n^{ext}$ .*

*Proof.* To prove this statement it is enough to follow the proof of Theorem 1.4.11, replacing  $UR_n$  with  $R_n$ , and considering the isomorphism between  $R_n$  and  $LB_n^{ext}$ .  $\square$

## 1.5 Welded diagrams and broken surfaces

There are two kind of projections of loop braids, seen as ribbon braids: the first one is a 2-dimensional diagrammatical representation, while the second one will be a representation through 3-dimensional surfaces.

### 1.5.1 Welded diagrams

**Definition 1.5.1.** A *strand diagram on  $n$  strings* is a set of oriented arcs in  $\mathbb{R}^2$ , monotone with respect to the second coordinate, from the points  $(0, 1), \dots, (0, n)$  to  $(1, 1), \dots, (1, n)$ . The arcs are allowed to have double points of three kinds, called *classical positive*, *classical negative* and *welded* as in Figure 1.3.

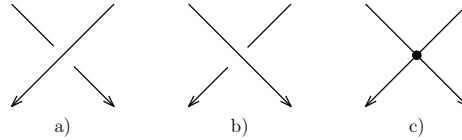
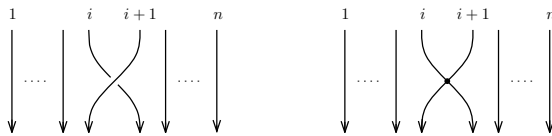


Figure 1.3: a) Classical positive crossing, b) Classical negative crossing, c) Welded crossing.

Let us assume that the double points occur at different  $x$ -coordinates. Then a strand diagram determines a word in the elementary diagrams illustrated in Figure 1.3. We call  $\sigma_i$  the elementary diagram representing the  $(i + 1)$ -th strand passing over the  $i$ -th strand, and  $\rho_i$  the welded crossing of the strands  $i$  and  $(i + 1)$ . The set of strand diagrams on  $n$  strings will be denoted by  $\mathcal{D}_n$ .

**Definition 1.5.2.** A *welded braid* is an equivalence class of strand diagrams under the equivalence relation given by planar isotopy and the following moves:

- Reidemester moves: Figure 1.5;

Figure 1.4: The elementary diagrams  $\sigma_i$  and  $\rho_i$ .

- virtual Reidemeister moves: Figure 1.6;
- mixed Reidemeister moves: Figure 1.7;
- welded Reidemeister moves: Figure 1.8.

This equivalence relation is called *welded Reidemeister equivalence*. We denote classes by representatives. For  $n \geq 1$ , the *group of welded braids* or *welded braid group* on  $n$  strands, denoted by  $WB_n$  is the group of equivalence classes of strand diagrams by welded Reidemeister equivalence. The group structure on these objects is given by: stacking and rescaling as product, mirror image as inverse, and the trivial diagram as identity.



Figure 1.5: Classical Reidemeister moves for braid-like objects.

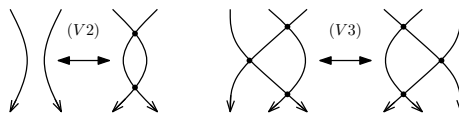


Figure 1.6: Virtual Reidemeister moves for braid-like objects.

*Remark 1.5.3.* We adopt the convention of reading the diagrams from top to bottom, and the corresponding braid words from left to right. Welded braids have been introduced in [30] as a graphic representation of presentation (1.4), given in their paper as a presentation of the braid-permutation groups  $BP_n$ . Then notation used here for elementary diagrams does not carry any ambiguity with the notation used for the presentation (1.4) of  $UR_n$  in Section 1.2, and for the automorphisms of the free group in Section 1.3.

### 1.5.2 Broken surfaces

A diagram of a classical braid is a projection in general position of the braid on the plane, with crossing information specified by deleting a neighbourhood of the underpassing arcs. We introduce similar diagrams for geometric ribbon braids, which are surfaces in the 4-dimensional space: these diagrams are projections in general position of ribbon braids in the 3-dimensional space, and are called *broken surface diagrams*. This representation

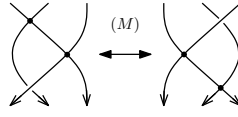


Figure 1.7: Mixed Reidemeister move.

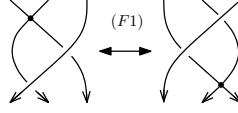


Figure 1.8: Welded Reidemeister move.

has first been introduced for the group of motions of a collection of  $n$  unknotted, unlinked, oriented circle in [8]. They have defined broken surfaces adapting a drawing style from Carter and Saito's work on surfaces in dimension 4 [23]. We use notations introduced at the beginning of Section 1.4.

**Definition 1.5.4.** Let  $A_1, \dots, A_n$  be locally flat embeddings in  $B^3$  of  $n$  disjoint copies of the oriented annulus  $S^1 \times I$ . We say that

$$S = \bigcup_{i \in \{1, \dots, n\}} A_i$$

is a *braid broken surface diagram* if:

1. for each  $i \in \{1, \dots, n\}$ , the oriented boundary  $\partial A_i$  is the disjoint union  $C_i \sqcup C_j$ , for  $C_i$  in  $\partial_0 B^3$  and for some  $C_j$  in  $\partial_1 B^3$ . The orientation induced by  $A_i$  on  $\partial A_i$  coincides with the orientation of one of the two boundary circles  $C_i$  and  $C_j$ ;
2. it is transverse to the lamination  $\bigcup_{t \in I} B^2 \times \{t\}$  of  $B^3$ , that is: at each parameter  $t$ , the intersection between  $S$  and  $B^2 \times t$  is a collection of exactly  $n$  circles, not necessarily disjoint;
3. the set of connected components of singular points in  $S$ , denoted by  $\Sigma(S)$ , consists of flatly transverse disjoint circles in  $(\bigcup_{i=1}^n \mathring{A}_i)$ .

Moreover, for each element of  $\Sigma(S)$ , a local ordering is given on the two circle preimages. By convention this ordering is specified on the diagram by erasing a small neighbourhood of the lower preimage in the interior of the annulus it belongs to. Note that this is the same convention used for classical braid diagrams. Moreover a broken surface diagram is said to be *symmetric* if it is locally homeomorphic to the surfaces in Figure 1.9, which means that, for each pair of preimage circles, the following properties are satisfied:

1. one of the preimage circles is essential in  $\bigcup_{i=1}^n \mathring{A}_i$  and the other is not;
2. there is a pairing of the elements of  $\Sigma(S) = \bigsqcup_r \{c_1^r, c_2^r\}$  such that, for each  $r$ , the essential preimages of  $c_1^r$  and  $c_2^r$

- (a) are respectively lower and higher than their non essential counterparts with respect to the associated order ;
- (b) bound an annulus in  $\bigcup_{i=1}^n \mathring{A}_i$ ;
- (c) this annulus avoids  $\Sigma(S)$ .

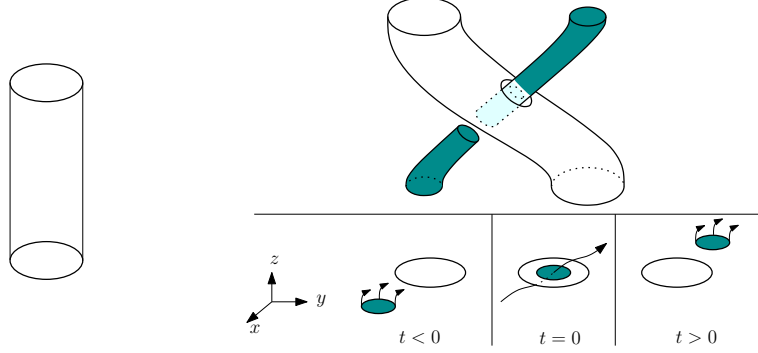


Figure 1.9: Symmetric broken surfaces are locally homeomorphic to a cylinder, or to a crossing with decoration that indicates the order on the preimages of the singularities.

Let  $b$  be a ribbon braid, and consider a projection  $B^4 \rightarrow B^3$  in general position of  $b$ : the following result allows us to consider braid broken surface diagrams as 3-dimensional representations of ribbon braids.

**Lemma 1.5.5.** *Any generic projection of a ribbon braid from  $B^4$  to  $B^3$  is a braid broken surface diagram. Conversely any braid broken surface diagram is the projection of a ribbon braid.*

*Proof.* This statement is proved in [83] for locally flat embedding of 2-spheres in  $\mathbb{R}^4$ . In [5] it is noted that the arguments, which are local, apply to the case of ribbon tubes. Since ribbon tubes and ribbon braids locally behave the same way, the arguments of [83] applies to our case.  $\square$

Moreover, we have the following result:

**Lemma 1.5.6.** *Any ribbon braid can be represented by a symmetric broken surface diagram.*

*Proof.* This statement is proved in [5, Lemma 2.13] for ribbon tubes, adapting results from [50] and [82]. The proof for ribbon tubes adapts without modifications to ribbon braids.  $\square$

*Remark 1.5.7.* If two symmetric braid broken surface diagrams differ by one of the “broken Reidemeister moves” in Figure 1.10, then the associated ribbon braids are isotopic [5, Remark 2.15].

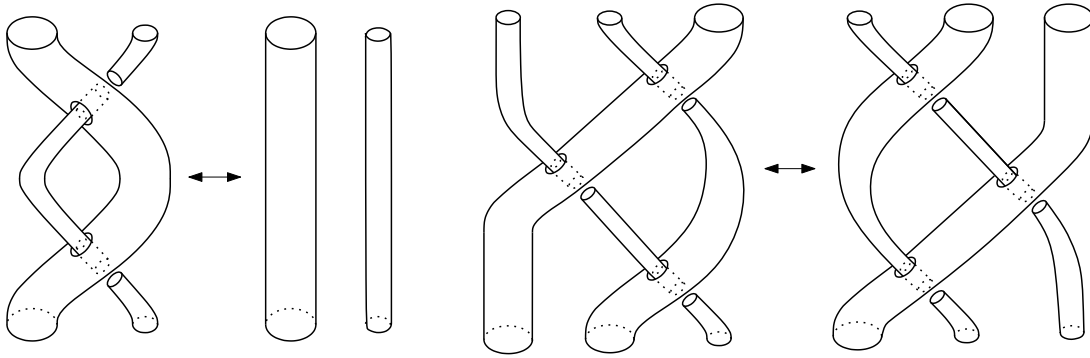


Figure 1.10: Broken Reidemeister moves.

Passing through symmetric braid broken surfaces, 4-dimensional ribbon braids can be described using 2-dimensional welded braids. Let  $b$  be a welded braid. We associate to it a symmetric braid broken surface diagram in the following way (see for details [71] and [81]). Consider  $B^2$  and embed it as  $B^2 \times \{\frac{1}{2}\}$  into  $B^3$ . Taken a tubular neighbourhood  $N(b)$  of  $b$ , we have that  $\partial_\varepsilon N(b) = \sqcup_{i \in \{1, \dots, n\}} D_i \times \varepsilon_i$  where  $\varepsilon_i \in \{0, 1\}$ . Each crossing is sent to a 4-punctured sphere. Then, according to the partial order defined on the double points of welded braid diagrams, we modify the sphere into the broken surfaces shown in Figure 1.11).

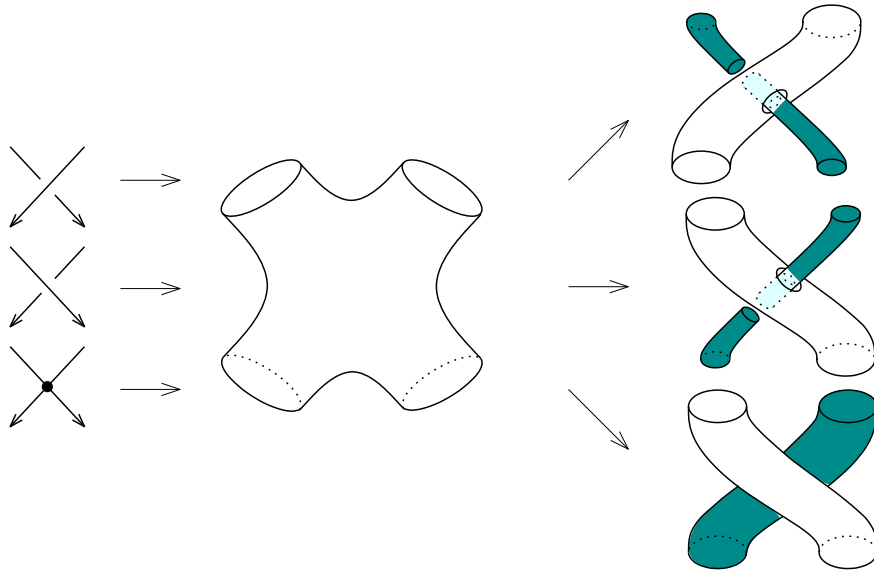


Figure 1.11: Punctured sphere associated to a crossing of the welded braid.

**Definition 1.5.8.** We define a map  $Tube: WB_n \rightarrow rB_n$  that associates to a welded braid  $b$ , the ribbon braid associated to the symmetric broken surface resulting from the preceding construction.

For general ribbon knotted objects, such as ribbon tangles (see Chapter 3), ribbon tubes [5] and knotted spheres [83], the Tube map is well defined and surjective. For these more general objects, it is still unproved if the Tube map is injective [10]. However, Brendle and Hatcher proved that  $UR_n$  is isomorphic to the presented group (1.4), represented by  $WB_n$  [21]. Through the isomorphism between  $UR_n$  and  $LB_n$  (Proposition 1.2.12) and the isomorphism between  $LB_n$  and  $RB_n$  (Theorem 1.4.11), we have the following result, that gives us a graphical interpretation of loop braids.

**Theorem 1.5.9.** *The map  $\text{Tube}: WB_n \rightarrow RB_n$  is an isomorphism.*

Let  $\widetilde{\text{Tube}}$  the composition of  $\text{Tube}$  with the isomorphism between  $RB_n$  and  $LB_n$ . We have:

**Corollary 1.5.10.** *The map  $\widetilde{\text{Tube}}: WB_n \rightarrow LB_n$  is an isomorphism.*

### 1.5.3 Extended welded diagrams and broken surfaces

We introduce new kind of local elementary diagrams, called  $\tau_i$ , for  $i = 1, \dots, n$  as in Figure 1.12,

**Definition 1.5.11.** An *extended welded braid* is an equivalence class of strand diagrams under the equivalence relation given by planar isotopy and the following moves:

- usual Reidemeister moves: Figure 1.5;
- virtual Reidemeister moves: Figure 1.6;
- mixed Reidemeister moves: Figure 1.7;
- welded Reidemeister moves: Figure 1.8;
- extended Reidemeister moves: Figure 1.13.

This equivalence relation is called *extended welded Reidemeister equivalence*. We denote classes by representatives. The *group of extended welded braids* or *extended welded braid group*, denoted by  $WB_n^{\text{ext}}$  is the group of equivalence classes of strand diagrams by extended welded Reidemeister equivalence. The group structure on these objects is given by: stacking and rescaling as product, mirror image as inverse, and the trivial diagram as identity.

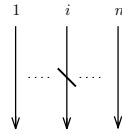


Figure 1.12: The elementary diagram  $\tau_i$ .

The  $\text{Tube}$  map can be extended to  $WB_n^{\text{ext}}$ , associating to elementary diagrams  $\tau_i$  a tube with a wen (Figure 1.14). Brendle and Hatcher proved that  $R_n$  is isomorphic

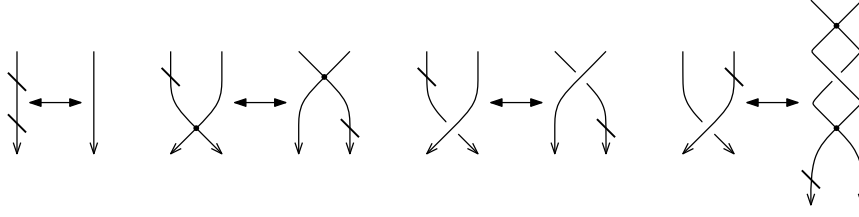


Figure 1.13: Extended Reidemeister moves for braid-like objects.

to the presented group (1.5), this second one represented by  $WB_n^{ext}$  [21]. Through the isomorphism between  $R_n$  and  $LB_n^{ext}$  (Theorem 1.2.11), and the isomorphism between  $LB_n^{ext}$  and  $RB_n^{ext}$  (Theorem 1.4.18), we have the following result, that gives us a graphical interpretation of extended loop braids.

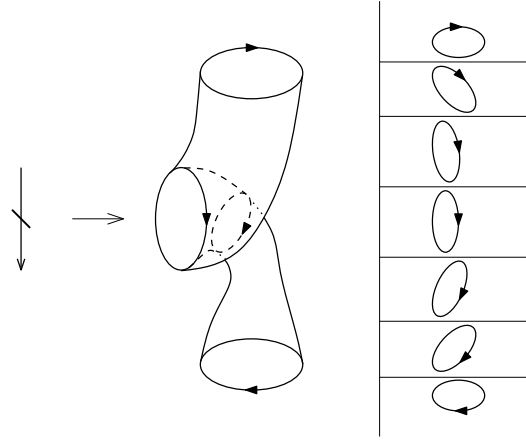


Figure 1.14: The behaviour of the Tube map when applied to wens.

**Theorem 1.5.12.** *The map  $Tube: WB_n^{ext} \rightarrow RB_n^{ext}$  is an isomorphism.*

As before, let  $\widetilde{Tube}$  be the extension of the map from Corollary 1.5.10, i.e., the composition of  $Tube$  with the isomorphism between  $RB_n^{ext}$  and  $LB_n^{ext}$ . We have:

**Corollary 1.5.13.** *The map  $\widetilde{Tube}: WB_n^{ext} \rightarrow LB_n^{ext}$  is an isomorphism.*

## 1.6 Gauss diagrams

In this section we give a combinatorial description of loop braids through Gauss diagrams.

**Definition 1.6.1.** Let  $n \geq 1$ . A *Gauss diagram* is a set of signed and oriented arrows on  $n$  ordered and oriented horizontal intervals, together with a permutation  $\sigma \in S_n$  (see Figure 1.15). The endpoints of the arrows are divided in two sets: the set of *heads* and the set of *tails*, defined by the orientation of the arrow. The right extremity of the  $i$ th horizontal interval is labelled with  $\sigma(i)$ .

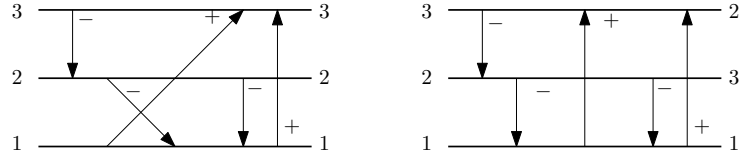


Figure 1.15: Examples of Gauss diagrams.

To make the link between Gauss diagrams and the group of welded braids, we introduce the group of virtual braids. This allows us to see the welded braid group as a quotient of virtual braids. Generally speaking, Gauss diagrams turn out to be a useful tool to investigate properties of various remarkable quotients of the virtual braid group (see for example [6, 7]). Recall Definition 1.5.1 of strand diagrams.

**Definition 1.6.2.** Two strand diagrams on  $n$  strands are *virtual equivalent* if they are related by planar isotopy and a finite number of the following moves:

- virtual Reidemeister moves:  $(V2)$  and  $(V3)$ , Figure 1.6;
- mixed Reidemeister moves:  $(M)$ , Figure 1.7.

Two strand diagrams on  $n$  strands are *welded equivalent* if they are related by planar isotopy and a finite number of the following moves:

- virtual Reidemeister moves:  $(V2)$  and  $(V3)$ , Figure 1.6;
- mixed Reidemeister moves:  $(M)$ , Figure 1.7;
- welded Reidemeister moves:  $(F1)$ , Figure 1.8.

Two strand diagrams on  $n$  strands are *virtual Reidemeister equivalent* if they are related by planar isotopy and a finite number of the following moves:

- Reidemeister moves:  $(R2)$  and  $(R3)$ , Figure 1.5;
- virtual Reidemeister moves:  $(V2)$  and  $(V3)$ , Figure 1.6;
- mixed Reidemeister moves:  $(M)$ , Figure 1.7.

We recall that adding welded Reidemeister moves to virtual Reidemeister equivalence, one obtains welded Reidemeister equivalence (Definition 1.5.2).

**Definition 1.6.3.** For  $n \geq 1$ , the *virtual braid group*  $VB_n$  is the group of equivalence classes of strand diagrams on  $n$  strands with respect to the virtual Reidemeister equivalence. We call *virtual braid diagram* an element of this group.

Other equivalent definitions of virtual braid groups have been introduced for instance in [10, 47, 78].

*Remark 1.6.4.* For  $n \geq 1$ , the welded braid group  $WB_n$  is a quotient of the virtual braid group  $VB_n$  under the relation given by moves of type  $(F1)$ .



To every virtual braid diagram  $\beta$  we can associate a Gauss diagram with the following construction. Let  $\beta$  be a strand diagram on  $n$  strands. The *Gauss diagram* associated to  $\beta$ , denoted by  $G(\beta)$ , is a Gauss diagram on  $n$  intervals satisfying the following properties:

1. for each strand of  $\beta$  there is an associated interval of  $G(\beta)$ ;
2. the endpoints of the arrows of  $G(\beta)$  correspond to the preimages of the classical (positive and negative) crossings of  $\beta$ ;
3. the order of the endpoints of the arrows on an interval of  $G(\beta)$  correspond to the order that the preimages associated to the endpoints on the strand of  $\beta$ ;
4. the arrows are pointing from the overpassing strand to the underpassing strand, when considering the usual convention on strand diagrams of drawing a break on the underpassing strand;
5. the permutation of  $G(\beta)$  corresponds to the permutation defined by  $\beta$ .

*Remark 1.6.5.* We remark that the Gauss diagram associated to a strand diagram has pairwise distinct arrows, and each arrow connects different intervals. By reparametrizing the intervals, the arrows can always be drawn to be vertical and at different  $t$  coordinates, for  $t \in I$ , where  $t$  is the horizontal coordinate.

**Definition 1.6.6.** Gauss diagrams respecting the conditions given in Remark 1.6.5 are called *braid Gauss diagrams*. Their set will be denoted by  $bGD_n$ .

The following result is proved by Cisneros in [25]. We give an example in Figure 1.17.

**Theorem 1.6.7** ([25, Theorem 2.10]). *The following statements hold:*

- (i) *For each braid Gauss diagram  $g$  there exists a braid  $\beta \in \mathcal{D}_n$  such that  $G(\beta) = g$ .*
- (ii) *Given two virtual braids  $\beta_1$  and  $\beta_2$  in  $VB_n$ ,  $G(\beta_1) = G(\beta_2)$  if and only if  $\beta_1$  and  $\beta_2$  are virtual equivalent.*

*Sketch of proof.* Part (i) is proved in [25, Lemma 2.11] with the data of an algorithm that recovers a strand diagram from a braid Gauss diagram. We give the algorithm. Point (ii) is proved verifying that movements (V2), (V3), and (M) do not change the braid Gauss diagram associated to a strand diagram. We recall the algorithm.

Let  $g$  be a braid Gauss diagram, and  $A = \{c_1, \dots, c_k\}$  its set of arrows. Set a parametrization of the intervals so that its arrows are vertical and at different parameters  $t \in I$ . This induces an order in  $A$  given by  $c_i < c_j$  if  $p_i < p_j$ , where  $p_i \in I$  is the endpoint of  $c_i$ . Label the crossings so that  $c_i < c_j$  if  $i < j$ .

For  $j = 1, \dots, k$  let  $d_j$  be the point  $(\frac{j}{k+1}, \frac{1}{2})$  and consider the disk  $D_j$  centered in  $d_j$  and sufficiently small radius for the disks to be disjoint. Draw a crossing inside  $D_j$  according to the sign of  $c_j$ , and label the intersection of the crossing components with the boundary of  $D_j$  as in Figure 1.16.

Let  $\sigma \in S_n$  be the permutation associated to  $g$ . Fix  $i \in \{1, \dots, n\}$ : to draw the  $i$ -th strand of the strand diagram through the disks  $D_j$ , we consider the set  $A_i = \{c_{i_1}, \dots, c_{i_m}\}$

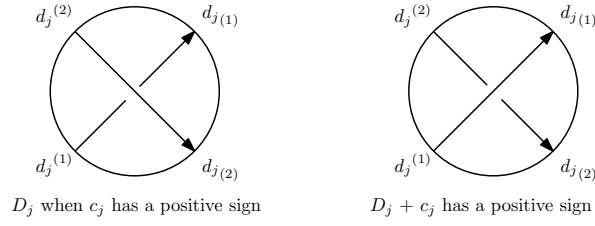


Figure 1.16: How to label crossings in the disks  $D_i$  when recovering a strand diagram from a braid Gauss diagram.

of the arrows starting and ending on the  $i$ -th interval of the braid Gauss diagram. Then for  $s = 0, \dots, m$  we define points  $o_s$  and  $t_s + 1$  as follows:

1. the starting point of the  $i$ -th strand will be  $o_i = a_i$ , and the terminal point will be  $t_{m+1} = b_{\sigma(i)}$ ;
2. for  $l = 1, \dots, m$ , let  $o_l = (d_{i_l})^{(\varepsilon)}$  and  $t_l = (d_{i_l})_{(\varepsilon)}$  where:
  - (i) if  $c_{i_l}$  is a positive arrow starting, or a negative arrow ending at the  $i$ -th interval, then  $\varepsilon = 2$ ;
  - (ii) if  $c_{i_l}$  is a negative arrow starting, or a positive arrow ending at the  $i$ -th interval, then  $\varepsilon = 1$ .

Considering the double points outside the disks as virtual crossings, we have drawn a strand diagram whose associated Gauss diagram is  $g$  as in Figure 1.17. □

Defining equivalence relations on  $bGD_n$ , Cisneros proves the existence of a bijection between a quotient of  $bGD_n$  and  $VB_n$  [25, Proposition 2.24]. This can be extended to  $WB_n$ , giving us the last isomorphism of this paper.

**Definition 1.6.8.** Let  $g$  and  $g'$  be two Gauss diagrams. A *Gauss embedding* is an embedding  $\phi: g' \rightarrow g$  that sends each interval of  $g'$  to a subinterval of  $g$ , and which sends each arrow of  $g'$  to an arrow of  $g$  respecting orientation and sign. Note that there is no condition on the permutations associated to  $g$  and  $g'$ .

**Definition 1.6.9.** Two braid Gauss diagram are *Reidemeister equivalent* if they are related by a finite sequence of moves  $(\Omega_2)$  and  $(\Omega_3)$  as in Figure 1.18. They are *wReidemeister equivalent* if they are related by  $(\Omega_2)$ ,  $(\Omega_3)$ , and Tail Commute moves  $(TC)$  as in Figure 1.19. For  $n \geq 1$ , the group of *welded Gauss diagrams*, denoted by  $wG_n$ , the group of equivalence classes of  $bGD_n$  with respect to wReidemeister equivalence.

*Remark 1.6.10.* Performing one of the moves described in Definition 1.6.9 on a braid Gauss diagram  $g$  means choosing a Gauss embedding in  $g$  of the braid Gauss diagram depicted on one hand of the equivalences in Figures 1.18 and 1.19, and replacing it with the braid Gauss diagram on the other hand. Since there is no condition on the

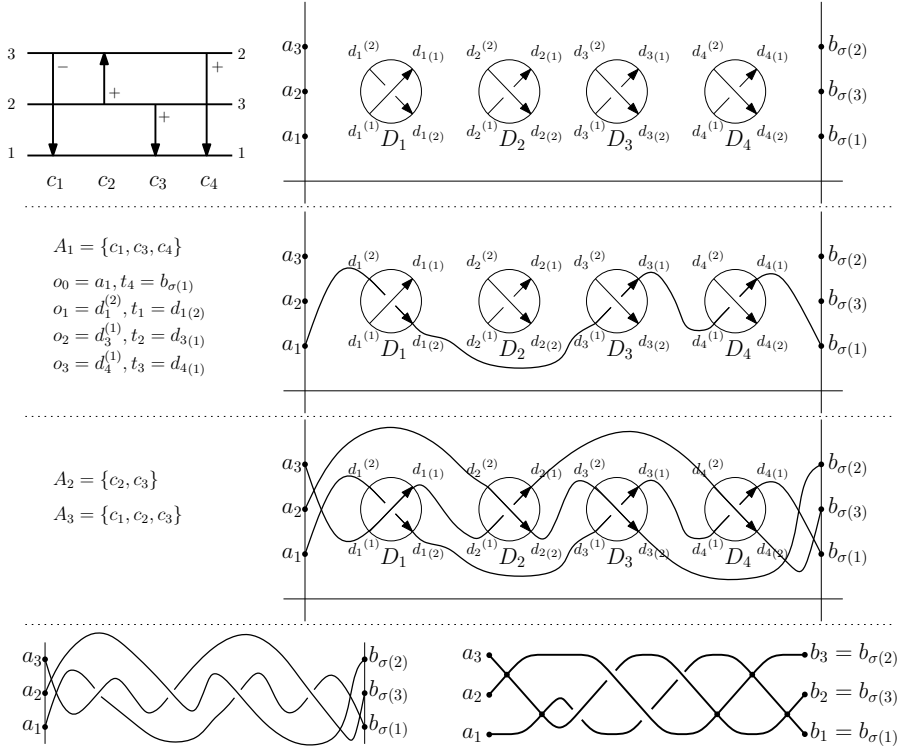
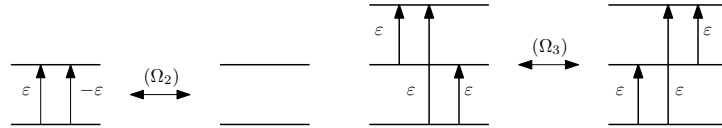


Figure 1.17: From a braid Gauss diagram to a strand diagram.

permutation associated, when embedding a Gauss diagram, this means that in performing the moves the strands can be vertically permuted.

The following result gives us an interpretation of  $LB_n$  in terms of Gauss diagrams, recalling that  $LB_n$  is isomorphic to  $WB_n$ .

Figure 1.18: Equivalence moves on braid Gauss diagrams, where the  $\varepsilon$  is the sign of the arrows.

**Theorem 1.6.11.** *For  $n \geq 1$ , there is a bijective correspondence between the group of welded Gauss diagrams  $wG_n$  and the welded braid group  $WB_n$ .*

*Proof.* By Theorem 1.6.7 we know that there is a bijective correspondence between virtual equivalent braids and braid Gauss diagrams. Therefore we need to prove that if two strand diagrams are related by a classical Reidemeister move  $(R2)$ ,  $(R3)$ , or a welded Reidemeister move  $(F1)$ , then their braid Gauss diagrams are wReidemeister equivalent via moves  $(\Omega_2)$ ,  $(\Omega_3)$  and  $(TC)$ , and viceversa.

Moves  $(\Omega_1)$  and  $(\Omega_2)$  are treated in [25, Proposition 2.24]. Let  $\beta$  and  $\beta'$  be two welded braids that differ by an  $(F1)$  move, and suppose that the strands involved are  $a, b$  and  $c$  in  $\{1, \dots, n\}$ . Then, up to isotopy, we can deform the two braids so that they coincide outside of a portion of the braid diagram which only contains the involved crossings. In this portion the diagrams look like in Figure 1.18, and the Gauss diagrams associated differ by a  $(TC)$  move.

Let  $g$  and  $g'$  be two Gauss diagrams and let  $a, b$  and  $c$  in  $\{1, \dots, n\}$  be pairwise different. Assume they differ by a  $(TC)$  move. Then there exists a sub-portion of the diagram that contains only the arrows involved in the  $(TC)$  move. For Theorem 1.6.7 there exists a strand diagram  $\beta$  that looks like the right or the left handside of move  $(F1)$ , in Figure 4.1. Performing a  $(F1)$  move on  $\beta$ , one obtains a strand diagram  $\beta'$  whose associated braid Gauss diagram is  $g'$ .  $\square$

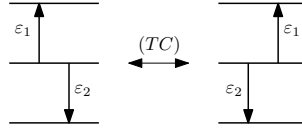


Figure 1.19: Tail Commute move where the  $\varepsilon_i$  are the signs of the arrows.

## 1.7 A historical note and other references

Loop braids have first been considered as *motions* of  $n$  unknotted, unlinked circles in  $\mathbb{R}^3$ , as seen in Section 1.1, a notion that can be translated in terms of mapping classes. More in detail, Fox and Neuwirth give in 1962 a new proof of the standard presentation of the Artin braid groups  $B_n$ , in terms of fundamental groups of the spaces of configurations of  $n$  unordered points in the plane. From this, the interpretation of braids as *motions* on  $n$  points on a plane arises. The same year Dahm, in his Ph.D thesis [27], generalises this concept defining the group of motions of a compact subspace  $N$  in a manifold  $M$ . He applies this construction to a collection of  $n$  unknotted, unlinked circles in  $\mathbb{R}^3$ , and proves the resulting mapping class groups to be isomorphic to particular subgroups of the groups of automorphisms of the free group  $\text{Aut}(F_n)$ . This result appears in [37]. Some insight on the link between motion groups and mapping class groups can be found in Bellingeri-Cattabriga's work [17, Section 5].

In 1986 McCool [64] considers the subgroups of *basis-conjugating automorphisms* of the groups  $\text{Aut}(F_n)$  and gives a presentation for them. We have seen in Section 1.3 that these are isomorphic to the pure loop braid groups  $PLB_n$ . A decade later Savushkina [73] studies the groups of *permutation-conjugacy automorphisms*, isomorphic to the loop braid groups  $LB_n$ . She gives a presentation for these and investigates many properties of these groups.

Loops braids are introduced then as *welded braids*, as defined by Fenn-Rimányi-Rourke [30], in the form of a presented group, with diagrams representing its generators,

and ten years later Baez-Crans-Wise [8] establish an isomorphism between the motion groups of  $n$  non-flipping circles and the groups of welded braids.

More recently, in [21], we find an interpretation of these groups in terms of fundamental groups of the configuration spaces of  $n$  unlinked Euclidean circles. About the topological interpretation of loop braids in terms of braided ribbon tubes, one can find details in [46]: this interpretation is also widely used in [5]. Finally, as explained in [10], they have also an interpretation in terms of Gauss braid diagrams: a point of view that allows us to consider the loop braid groups as quotients of the virtual braid groups.

Several computations and conjectures about the cohomology algebras of the pure loop braid groups appear in the literature. Investigating the properties of these algebra, for example in relation with resonance varieties and lower central series ranks, allows us to compare the loop braid groups to the braid groups, and to other generalizations of the braid groups. About this topic we refer to [26, 43, 75, 77].

A word should be said about the theory of representations of the loop braid groups, which appears to be an upcoming topic. Burau representation extends trivially to loop braid groups using Magnus expansion and Fox derivatives [14], however it is still unknown if the loop braid groups are linear. Some new results on local representations of loop braid groups, rising as extensions of braid groups representations, can be found in [45] and [22]. However the study of finite dimensional quotients of algebras of loop braid groups is yet to be found in the literature. In [45] the authors show interest also in certain remarkable quotients of loop braid groups, the *symmetric loop braid groups* (also known as *unrestricted virtual braid groups* [53]). We will discuss the structure of these groups in Chapter 4, along with giving some applications to *fused links*.



## Chapter 2

# Towards a topological Markov's theorem for loop braids

In this chapter we consider loop braids as ribbon braids (Theorem 1.4.11), and their relationship with ribbon torus-links. This is the beginning of the adaptation of a classical result of braid theory: Markov's theorem. This theorem states that two braids (possibly with different numbers of strings) have isotopic closures in the  $\mathbb{R}^3$  if and only if one can be obtained from the other with a finite number of Markov moves, called conjugation and stabilization. This theorem allows us to describe all braids with isotopic closures as knots in  $\mathbb{R}^3$ . In the classical case many proofs are known: some of them are combinatorial, other topological, and these last ones use all of the isomorphisms among the different definitions of braid groups. Concerning loop braids seen as welded braid diagrams, a combinatorial proof of this theorem can be found in [47]. We prove here a version of Markov's theorem for loop braids with closure in a solid torus in  $\mathbb{R}^4$ .

### 2.1 Ribbon torus-links

We introduce some notions from knot theory of 2-dimensional objects in the 4-dimensional space, for which classical references are [46, 54]. In particular we define ribbon torus-knots and links, introduced under this name in [81].

**Definition 2.1.1.** A *geometric ribbon torus-knot* is an embedded oriented torus  $S^1 \times S^1 \subset \mathbb{R}^4$  which is *fillable*, in the sense that it bounds a *ribbon torus*, *i.e.*, an oriented immersed solid torus  $D^2 \times S^1 \subset \mathbb{R}^4$  whose singular points consist in a finite number of ribbon disks. We denote  $RK$  the set of *ribbon torus-knots*, equivalence classes of geometric ribbon torus-knots defined up to ambient isotopy.

*Remark 2.1.2.* Wens (Remark 1.4.3) can appear on portions of a ribbon knot, but since all ribbon disk ends can be oriented coherently using the co-orientation, there are an even number of them on each component and, they cancel pairwise, as proven in [4].

**Definition 2.1.3.** A *geometric ribbon torus-link* with  $n$  components is the embedding of a disjoint union of  $n$  oriented fillable tori. The set of *ribbon torus-links* is the set of equivalence classes of geometric ribbon torus-knots defined up to ambient isotopy. We denote it by  $RL$ .

### 2.1.1 Welded diagrams and broken surfaces for ribbon torus-links

In Chapter 1 we described ribbon braids with two kind of projections: a 2-dimensional diagrammatical representation through welded diagrams and a 3-dimensional representation through broken surfaces. Here we do the same thing for ribbon torus-links, starting from welded links. We recall that welded braids were introduced in Section 1.5.1.

**Definition 2.1.4.** A *strand link diagram* is the immersion in  $\mathbb{R}^2$  of a collection of disjoint, closed, oriented 1-manifolds such that all multiple points are transverse double points. Double points are decorated with the following information: they can be *classical positive*, *classical negative* or *welded* as in Figure 1.3. We assume that strand link diagrams are the same if they are isotopic in  $\mathbb{R}^2$ .

**Definition 2.1.5.** A *welded link* is an equivalence class of strand link diagrams under the equivalence relation given by planar isotopy, moves from Definition 1.5.2, and by classical and virtual Reidemeister moves of type I (Figure 2.1). This equivalence relation is called *welded isotopy*. We denote classes by representatives. Moreover we denote by  $WL$  the set of welded links.

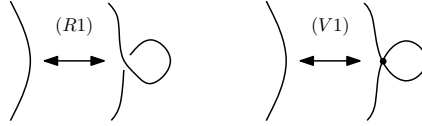


Figure 2.1: Reidemeister moves of type I.

In a parallel fashion with respect to Section 1.5.2, we introduce broken surfaces for ribbon torus-links.

**Definition 2.1.6.** A *broken torus diagram* is a torus immersed in  $\mathbb{R}^3$  whose singular set is a finite number of transverse singular circles, each of which is equipped with an order on its two preimages (Definition 1.5.4). A symmetric broken torus diagram comes with an obvious solid torus filling which is naturally oriented by the ambient space.

As in the ribbon braids case, two broken surface diagrams will be considered equivalent if they differ by one of the braid-like broken Reidemeister moves (Figure 1.10), or by the type I move (Figure 2.2).

The Tube map defined locally in Definition 1.5.8 can be applied also to welded links. On link-like objects there is no result stating that the map is an isomorphism, however, passing through broken tori diagrams, we have the following result:



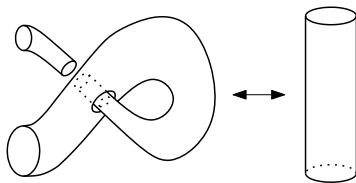


Figure 2.2: Broken Reidemeister move of type I.

**Proposition 2.1.7** ([4, Proposition 2.5]). *The map  $\text{Tube}: W\mathcal{L} \rightarrow R\mathcal{L}$  is a well-defined surjective map.*

A proof and a detailed disquisition on the question of the Tube map injectivity can also be found in [4].

## 2.2 Closed ribbon braids in $V = B^3 \times S^1$

We introduce a particular kind of ribbon torus-links in  $B^3 \times S^1$ .

**Definition 2.2.1.** A torus-link  $L$  in  $V = B^3 \times S^1$  is called a *closed  $n$ -ribbon braid* with  $n \geq 1$  if  $L$  meets each ball  $B^3 \times \{t\}$ ,  $t \in S^1$  transversely in  $n$  circles.

*Remark 2.2.2.* Two closed ribbon braids in  $V$  are isotopic if they are isotopic as oriented torus-links. This implies that the tubes don't necessarily stay transverse to the lamination during the isotopy.

*Remark 2.2.3.* In general a torus-link in  $V$  is not isotopic to a closed ribbon braid in  $V$ . For instance a torus link lying inside a small 4-ball in  $V$  is never isotopic to a closed braid.

**Definition 2.2.4.** Given an  $n$ -ribbon braid  $\beta$ , its *tube closure* is the ribbon torus-knot  $\widehat{\beta}$  obtained by gluing a copy of the trivial ribbon braid  $U$  along  $\beta$ , identifying the pair  $(B^3 \times \{0\}, \partial_0 \beta)$  with  $(B^3 \times \{1\}, \partial_1 U)$  and  $(B^3 \times \{1\}, \partial_1 \beta)$  with  $(B^3 \times \{0\}, \partial_0 U)$ .

A welded link diagram for  $\widehat{\beta}$  in  $S^1 \times I$  is obtained by closing a diagram for  $\beta$  (see Figure 2.3).

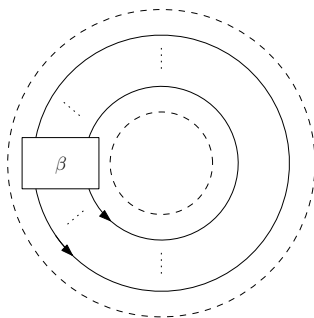


Figure 2.3: Closure of a welded braid diagram.

## 2.3 Conjugation: result in the analogous of the solid torus

In classical braid theory closed braids in the solid torus are classified up to isotopy by the conjugacy classes of braids in  $B_n$ . We give here a classification of this kind for closed ribbon braids: their closures will be classified, up to isotopy in  $B^3 \times S^1$ , by the conjugacy classes of the ribbon braids in the extended ribbon braid group. To do this we use the isomorphism between  $rB_n^{ext}$  and the group of extended loop braids  $LB_n^{ext}$  (Theorem 1.4.18), and the isomorphism between  $LB_n^{ext}$  and the extended permutation-conjugation group  $PC_n^*$  (Theorem 1.3.1). The proof is inspired by the one given for the classical case in [51], which follows in turn [76].

**Theorem 2.3.1.** *Let  $n \geq 1$  and  $\beta, \beta' \in rB_n$  a pair of ribbon braids. The closed ribbon braids  $\widehat{\beta}, \widehat{\beta}'$  are isotopic in  $B^3 \times S^1$  if and only if  $\beta$  and  $\beta'$  are conjugate in  $rB_n^{ext}$ .*

*Proof.* We begin with the "if" part. Suppose first the case that  $\beta$  and  $\beta'$  are conjugate in  $rB_n$ . We recall that  $rB_n$  is isomorphic to the group of welded braid diagrams  $WB_n$ . We call with the same name an element in  $rB_n$  and its image in  $WB_n$ . Conjugate elements of  $WB_n$  give rise to isotopic closed welded braid diagrams, which correspond to isotopic closed braids. This means that, since  $\beta$  and  $\beta'$  are conjugate in  $WB_n$ ,  $\beta' = \alpha\beta\alpha^{-1}$  with  $\alpha \in WB_n$ , and we have that  $\widehat{\alpha\beta\alpha^{-1}} = \widehat{\beta}$ . To see this, it is enough to stack the diagrams of  $\alpha$ ,  $\beta$  and  $\alpha^{-1}$ , close the composed welded braid diagram, and push the upper diagram representing  $\alpha$  along the parallel strands until  $\alpha$  and  $\alpha^{-1}$  are stacked one next to the other at the bottom of the diagram.

Consider now the case that  $\beta$  and  $\beta'$  are conjugate in  $rB_n^{ext}$ . This means that  $\beta' = \alpha\beta\alpha^{-1}$  with  $\alpha \in rB_n^{ext}$ . Then passing through the isomorphism between  $rB_n^{ext}$  and  $WB_n^{ext}$ , we have that  $\alpha$  is in  $WB_n^{ext}$ : doing the same operation of stacking the diagrams, closing them and pushing  $\alpha$  along the strands, one has  $\widehat{\alpha\beta\alpha^{-1}} = \widehat{\beta}$ .

Let us now prove the converse, which is: any pair of ribbon braids with isotopic closures in  $V = B^3 \times S^1$  are conjugate in  $rB_n^{ext}$ . Passing through the isomorphism between  $rB_n$  and  $PC_n$ , and  $rB_n^{ext}$  and  $PC_n^*$ , it will be enough to prove the following: any pair of ribbon braids with isotopic closures in  $V = B^3 \times S^1$  have associated automorphisms of  $PC_n$  that are conjugate in  $PC_n^*$ . Set  $\overline{V} = B^3 \times \mathbb{R}$ . Considering the cartesian product of  $(B^3, \text{id}_{B^3})$  and the universal covering  $(\mathbb{R}, p)$  of  $S^1$  given by

$$\begin{aligned} p: \mathbb{R} &\longrightarrow S^1 \\ t &\longmapsto \exp(2\pi it) \end{aligned}$$

we obtain a universal covering  $(\overline{V}, \text{id}_{B^3} \times p)$  of  $V$ . Denote by  $T$  the covering transformation

$$\begin{aligned} T: \overline{V} &\longrightarrow \overline{V} \\ (x, t) &\longmapsto (x, t + 1) \end{aligned}$$

for all  $x \in B^3$  and  $t \in \mathbb{R}$ . If  $L$  is a closed  $n$ -ribbon braid in  $V$ , then its preimage  $\overline{L} \subset \overline{V}$  is a 2-dimensional manifold meeting each 3-ball  $B^3 \times \{t\}$ , for  $t \in \mathbb{R}$ , transversely in  $n$

disjoint pairwise unlinked circles. This implies that  $\bar{L}$  consists of  $n$  fillable components homeomorphic to  $S^1 \times \mathbb{R}$ .

Being  $L$  a closed ribbon braid, we can present it as a closure of a geometric ribbon braid  $b \subset B^4 = B^3 \times I$  where we identify  $\partial_0 B^4$  with  $\partial_1 B^4$ . Then  $\bar{L} = \bigcup_{m \in \mathbb{Z}} T^m(b)$ , *i.e.*, we can see  $\bar{L}$  as a tiling of an infinite number of copies of  $b$ .

For  $n \geq 1$ , let  $C = (C_1, \dots, C_n)$  be a family of  $n$  disjoint, pairwise unlinked, euclidean circles in  $B^3$ , lying on parallel planes. We consider a parametrization for  $b$ , *i.e.*, a family  $\{\alpha_t: B^3 \rightarrow B^3\}_{t \in I}$  such that  $\alpha_0(C) = C$ ,  $\alpha_1 = \text{id}_{B^3}$ , all  $\alpha_t$  fix  $\partial B^3$  pointwise, and  $b = \bigcup_{t \in I} (\alpha_t(C), t)$  (see Lemma 1.4.12).

We take the self-homeomorphism of  $\bar{V} = B^3 \times \mathbb{R}$  given by

$$(x, t) \mapsto (\alpha_{t - \lfloor t \rfloor} \alpha_0^{-\lfloor t \rfloor}(x), t)$$

where  $x \in B^3$ ,  $t \in \mathbb{R}$ , and  $\lfloor t \rfloor$  is the greatest integer less than or equal to  $t$ . This homeomorphism fixes  $\partial \bar{V} = S^2 \times \mathbb{R}$  pointwise and sends  $C \times \mathbb{R}$  onto  $\bar{L}$ , see Figure 2.4 for an intuitive (although necessarily imprecise) idea.

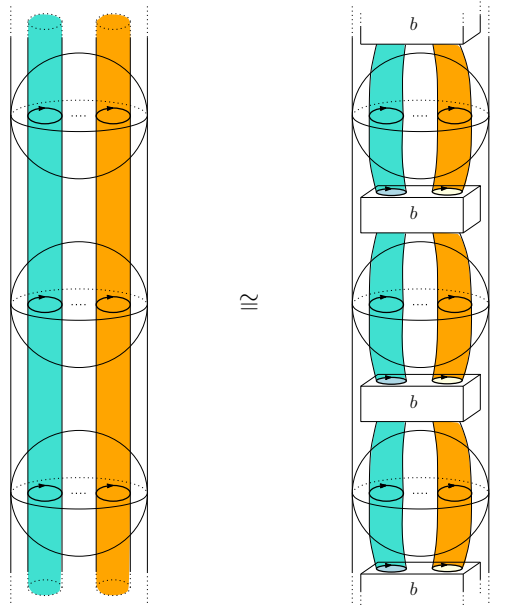


Figure 2.4: A homeomorphism between  $(B^3 \times \mathbb{R}, C \times \mathbb{R})$  and  $(B^3 \times \mathbb{R}, \bar{L})$ .

The induced homeomorphism  $(B^3 \setminus C) \times \mathbb{R} \cong \bar{V} \setminus \bar{L}$  shows that  $B^3 \setminus C = (B^3 \setminus C) \times \{0\} \subset \bar{V} \setminus \bar{L}$  is a deformation retract of  $\bar{V} \setminus \bar{L}$ . Pick a point  $d \in \partial_0 B^4 = B^3$  and set  $\bar{d} = (d, 0) \in \bar{V}$ ; then the inclusion homomorphism

$$i: \pi_1(B^3 \setminus C, d) \longrightarrow \pi_1(\bar{V} \setminus \bar{L}, \bar{d})$$

is an isomorphism.

By definition the image of  $d$  by the covering transformation  $T$  is  $T(d) = (d, 1)$ ; the covering transformation  $T$  restricted to  $\bar{V} \setminus \bar{L}$  induces an isomorphism  $\pi_1(\bar{V} \setminus \bar{L}, \bar{d}) \rightarrow$

$\pi_1(\bar{V} \setminus \bar{L}, T(\bar{d}))$ . Let  $T_*$  be the composition of this isomorphism with the isomorphism  $\pi_1(\bar{V} \setminus \bar{L}, T(\bar{d})) \rightarrow \pi_1(\bar{V} \setminus \bar{L}, \bar{d})$  obtained by conjugating the loops by the path  $d \times [0, 1] \subset \partial B^3 \times \mathbb{R} \subset \bar{V} \setminus \bar{L}$ . Then  $T_*$  is an automorphism of  $\pi_1(\bar{V} \setminus \bar{L}, \bar{d})$ . Therefore the following diagram commutes:

$$\begin{array}{ccc} \pi_1(B^3 \setminus C, d) & \xrightarrow{i} & \pi_1(\bar{V} \setminus \bar{L}, \bar{d}) \\ \tilde{\beta} \downarrow & & \downarrow T_* \\ \pi_1(B^3 \setminus C, d) & \xrightarrow{i} & \pi_1(\bar{V} \setminus \bar{L}, \bar{d}) \end{array}$$

where  $\tilde{\beta}$  is the automorphism induced by the restriction of  $\alpha_0$  to  $B^3 \setminus C$ . The isomorphism between  $rB_n$  and  $\text{MCG}(B^3, C)$  allows us to send the ribbon braid  $\beta$ , represented by  $b$ , to the isotopy class of  $\alpha_0$ .

Identifying  $\pi_1(B^3 \setminus C, d)$  with the free group  $F_n$  with generators  $x_1, x_2, \dots, x_n$ , we conclude that the automorphism  $\tilde{\beta}$  is equal to  $\nu(b)$ , where  $\nu: rB_n \rightarrow BP_n$  is the isomorphism between the group of ribbon braids  $rB_n$  and  $BP_n$ , the braid-permutation subgroup of  $\text{Aut}(F_n)$  (see Remark 1.3.5). Then it is the automorphism of  $F_n$  corresponding to  $\beta$ , the ribbon braid class represented by  $b$ . Thus  $i^{-1}T_*i = \tilde{\beta}$ .

Suppose now that  $\beta, \beta' \in rB_n$  are two ribbon braids with isotopic closures in  $V$ , and that  $b$  and  $b' \subset B^4 = B^3 \times I$  are two geometric ribbon braids that represent them. Let  $L$  and  $L' \subset V = B^3 \times S^1$  be their respective closures.

Then there is a homeomorphism  $g: V \rightarrow V$  such that  $g$  maps  $L$  onto  $L'$ , preserving their canonical orientation along the annuli, but possibly reversing the orientation of the circles at some instant (for example when Reidemeister moves of type I occur (see Remark 1.4.6)). Note that a Reidemeister move of type I is isotopic to the composition of two wens [4, Corollary 3.3], so globally the orientation of the circles at the starting and ending time parameter is preserved). In fact the orientation of the ambient  $V$  is preserved by  $g$ , but when considering a section  $B^3 \times \{t\}$  the orientation of the circles can be concordant or not concordant with the one induced by  $V$ . In addition the restriction of  $g$  to  $\partial V$  is isotopic to the identity  $\text{id}_V$ . This fact, plus the isomorphism of the map induced by the inclusion  $\pi_1(\partial V) = \pi_1(S^2 \times S^1) \rightarrow \pi_1(V) = \pi_1(B^3 \times S^1) \cong \mathbb{Z}$  implies that  $g$  induces an identity automorphism of  $\pi_1(V)$ . Therefore  $g$  lifts to a homeomorphism  $\bar{g}: \bar{V} \rightarrow \bar{V}$  such that  $\bar{g}$  is isotopic to the identity on  $\partial \bar{V}$ ,  $\bar{g}T = T\bar{g}$ , and  $\bar{g}(\bar{L}) = \bar{L}'$ .

Hence  $\bar{g}$  induces an isomorphism

$$\bar{g}_*: \pi_1(\bar{V} \setminus \bar{L}, \bar{d}) \longrightarrow \pi_1(\bar{V} \setminus \bar{L}', \bar{d})$$

commuting with  $T_*$ . The following diagram commutes:

$$\begin{array}{ccc} \pi_1(B^3 \setminus C, d) & \xrightarrow{i} & \pi_1(\bar{V} \setminus \bar{L}, \bar{d}) \\ \varphi \downarrow & & \downarrow \bar{g}_* \\ \pi_1(B^3 \setminus C, d) & \xrightarrow{i'} & \pi_1(\bar{V} \setminus \bar{L}', \bar{d}). \end{array}$$

Consider the automorphism  $\varphi = (i')^{-1}\bar{g}_*i$  of  $F_n = \pi_1(B^3 \setminus C, d)$ , where:

$$i: \pi_1(B^3 \setminus C, d) \longrightarrow \pi_1(\bar{V} \setminus \bar{L}, \bar{d}) \text{ and}$$

$$i': \pi_1(B^3 \setminus C, d) \longrightarrow \pi_1(\bar{V} \setminus \bar{L}', \bar{d})$$

are the inclusion isomorphisms.

Applying the same arguments to  $\beta'$ , we have  $\tilde{\beta}' = (i')^{-1}T_*i'$ , and from the preceding commutative diagram we have:

$$\varphi \tilde{\beta} \varphi^{-1} = ((i')^{-1}\bar{g}_*i) (i^{-1}T_*i) (i^{-1}\bar{g}_*^{-1}i') = (i')^{-1}T_*i' = \tilde{\beta}'$$

We claim that  $\varphi$  is an element of the subgroup of  $\text{Aut}(F_n)$  consisting of all automorphisms of the form  $x_i \mapsto q_i x_{j(i)}^{\pm 1} q_i^{-1}$ , where  $i = 1, \dots, n$ ,  $j(i)$  is some permutation of the numbers  $1, \dots, n$ , and  $q_i$  a word in  $x_1, \dots, x_n$ . Then the isomorphism between this subgroup and  $rB_n^{\text{ext}}$  ([37]) implies that  $\beta$  and  $\beta'$  are conjugate in  $rB_n^{\text{ext}}$ .

We prove this claim. The conjugacy classes of the generators  $x_1, x_2, \dots, x_n$  in  $F_n = \pi_1(B^3 \setminus C, d)$  are represented by loops encircling the circles  $C_i$ . The inclusion  $B^3 \setminus C = (B^3 \setminus C) \times \{0\} \subset \bar{V} \setminus \bar{L}$  maps these loops to some loops in  $\bar{V} \setminus \bar{L}$  encircling at each parameter  $t$  the rings that form the components of  $\bar{L}$ . The homeomorphism  $\bar{g}: \bar{V} \rightarrow \bar{V}$  transforms these loops into loops in  $\bar{V} \setminus \bar{L}'$  encircling the components of  $\bar{L}'$ . The latter represent the conjugacy classes of the images of  $x_1, \dots, x_n$  under the inclusion  $B^3 \setminus C = (B^3 \setminus C) \times \{0\} \subset \bar{V} \setminus \bar{L}'$ .

The automorphism  $\varphi$  transforms the conjugacy classes of  $x_1, \dots, x_n$  into themselves, up to permutation and orientation changes. This verifies the condition. The possible orientation changes are due to the fact that the isotopy of closed braid is not monotone with respect to the time parameter as ribbon braid isotopy is, thus Reidemeister moves of type I can occur. Note that a Reidemeister move of type I is isotopic to the composition of two wens [4, Corollary 3.3].  $\square$

When one ribbon braid is a conjugate in  $rB_n^{\text{ext}}$  of another ribbon braid, we can describe its form.

**Lemma 2.3.2.** *Let  $n \geq 1$  and  $\beta, \beta' \in rB_n$  a pair of ribbon braids. They are conjugates in  $rB_n^{\text{ext}}$  if and only if  $\beta' = \pi_\tau \alpha \beta \alpha^{-1} \pi_\tau^{-1}$ , where  $\pi_\tau$  only composed by wens. Equivalently,  $\pi_\tau$  is represented by a word in the  $\tau_i$  generators of presentation (1.5), and  $\alpha \in rB_n$ .*

*Proof.* Take  $\beta$  and  $\beta'$  in  $rB_n$  conjugate by an element in  $rB_n^{\text{ext}}$ . Then there exists an element  $\gamma$  in  $rB_n^{\text{ext}}$  such that  $\beta = \gamma \beta' \gamma^{-1}$ . Consider  $\gamma$  as an element of the configuration space of  $n$  circles  $R_n$ . We can use relations from presentation (1.5) to push to the right of the word the generators  $\tau_i$ , to obtain an equivalent element  $\gamma' = \pi_\tau \alpha$ , where  $\pi_\tau$  is a word in the  $\tau_i$ s and  $\alpha$  only contains generators  $\sigma_i$  and  $\rho_i$ . This means that  $\alpha$  is in fact an element of  $rB_n$ . Finally, when considering  $\gamma^{-1}$  for the conjugacy, we remark that  $\pi_\tau^{-1}$  is just the mirror image word of  $\pi_\tau$ .  $\square$

The proof of Theorem 2.3.1 adapts to the extended loop braid group with the following statement:

**Theorem 2.3.3.** *Let  $n \geq 1$  and  $\beta, \beta' \in rB_n^{\text{ext}}$  a pair of ribbon braids. The closed extended ribbon braids  $\hat{\beta}, \hat{\beta}'$  are isotopic in  $B^3 \times S^1$  if and only if  $\beta$  and  $\beta'$  are conjugate as elements in  $rB_n^{\text{ext}}$ .*

Here closed extended ribbon braids are realised by tube closure (Definition 2.2.4) demanding that the orientation on the border components of the annuli is coherent.

*Remark 2.3.4.* Although loop braids appear to be more present in literature, it is extended loop braids that allow us to have a result analogous to the one we have in the classical case. This could be expected, since wens are natural phenomena in  $\mathbb{R}^4$ , suggesting that not considering them as allowed movements is not the most natural choice.

*Remark 2.3.5.* We remark that a change of orientation of the ambient space does not affect the lamination. So we have that the result holds if we replace  $V = B^3 \times S^1$  with the solid Klein bottle.

## 2.4 Stabilisation: further developments

To extend the result in  $\mathbb{R}^4$ , outside the solid torus, we shall prove the invariance of the isotopy class of a closed (extended) ribbon braid under the operation known as stabilisation on (extended) ribbon braids.

To follow the approach of [51] would mean to use the diagrammatic description of loop braids given by [54]. However this would not cast a light on the topological structure. Besides, it shall be noted that for welded braid diagrams and welded links there already is a combinatorial version of Markov's theorem, due to Kamada [47]. So in my future plans I will consider Bennequin's proof and use contact structures to prove a complete topological version of Markov's theorem for loop braids and extended loop braids.

## Chapter 3

# Alexander invariants for ribbon tangles

In this chapter we discuss an expanded version of a joint work with Vincent Florens [29]. After considering ribbon braids in Section 1.4 and ribbon torus-links in Section 2.1, we introduce here *ribbon tangles*. These are proper embeddings of disjoint tori and annuli in the ball  $B^4$ , whose singular sets are finite numbers of ribbon disks as in Definition 1.4.1. The intersection of a ribbon tangle  $T$  and  $\partial B^4 = S^3$  is a trivial link  $L$ . For a given free abelian group  $G$ , ribbon tangles are *colored* with a group homomorphism  $\varphi: H_1(B^4 \setminus T) \rightarrow G$ .

We construct an *Alexander invariant*  $A$  of colored ribbon tangles, lying in the exterior algebra of the homology  $\mathbb{Z}[G]$ -module  $H_1^\varphi(S^3 \setminus L)$ , twisted by the morphism induced by  $\varphi$ , see Definition 3.1.5. This invariant corresponds to the Alexander polynomial of long ribbon torus-knots when applied to ribbon tangles with only two boundary components, see Proposition 3.1.9. The construction of  $A$  is based on the *Alexander function* introduced by Lescop [59]. The proof of the invariance and the main properties follow from algebraic and homological arguments developed in a paper by the second author and Massuyeau [33].

Colored ribbon tangles can be *split* into morphisms in a category  $Rib_G$ . We show that the invariant  $A$  induces a functor from  $Rib_G$  to the category of  $\mathbb{Z}[G]$ -graded modules, see Theorem 3.1.19. In the case of ribbon braids, this functor coincides with the exterior powers of the *ad-hoc* colored Burau-Gassner representation, multiplied by a certain relative Alexander polynomial, see Proposition 3.1.17.

The multiplicativity of  $A$  naturally fits in the context of circuit algebras, an extension of the planar algebras introduced by Jones [44]. We construct a circuit algebra  $Cob_G$  over the colored *ball cobordisms* circuit operad, see Theorem 3.2.9. Ball cobordisms are smooth cylinders in  $B^4$  with a finite collection of disjoint internal balls removed, and the compositions are given by identification of some boundary components. We observe that ball cobordisms act on ribbon tangles and that  $A$  commutes with this action, see Theorem 3.2.10.

As we have seen in Sections 1.5 and 2.1, ribbon knotted objects admit representations

through broken surfaces and welded diagrams. This offers two perspectives: ribbon knotted objects provide topological realizations of welded diagrams, and diagrams allow combinatorial computations of invariants of ribbon knotted objects. Moreover, a presentation of the group of ribbon knotted objects is obtained from the diagrams by a Wirtinger type algorithm, see [5].

We give a diagrammatic description of the invariant  $A$  (Theorem 3.3.6) and show that it extends the multivariable Alexander polynomial of virtual tangles developed by Archibald [2]. For further details on these objects see also [12, 68]. The circuit algebra  $Cob_G$  appears as a topological incarnation of a circuit algebra of diagrams  $Weld_\mu$ . The compatibility of  $A$  with these structures allows local calculations on the diagrams. If the diagrams do not have any welded crossing, the construction holds for usual knotted objects, and we obtain a purely local description of the usual Alexander polynomial. This extends the construction of the Alexander representation by Bigelow, Cattabriga, and Florens [19].

Our construction arises in the context of defining generalizations of Alexander polynomials to tangle-like objects. The literature about this topic includes: [19] concerning usual tangles, the works of Archibald [2] and Polyak [68] concerning virtual tangles, the works of Cimasoni and Turaev [24] that use Lagrangian categories, Bigelow [18] and Kennedy [56] who studied diagrammatical invariants of usual tangles, Sartori [70], who defined quantum invariants of framed tangles, and Zibrowius [85]. This last one in particular defines an invariant for usual tangles which consists of a finite set of Laurent polynomials, and states, without explicit calculation, that on usual tangles one can calculate Archibald's invariant from his set of invariants, and vice versa. However, nothing seems to appear in the literature about the 4-dimensional case of ribbon tangles.

In Section 3.1 we recall the definitions of ribbon tangles and construct the invariant  $A$ . Section 3.2 is devoted to the circuit algebra  $Cob_G$  over the ball cobordisms circuit operad, and the properties of  $A$  with respect to the action of ball cobordisms on ribbon tangles. In Section 3.3 we describe the diagrammatic construction of  $A$  and a circuit algebra  $Weld_\mu$  related to  $Cob_G$ . In Section 3.4 we compute some examples.

### 3.1 The Alexander invariant $A$

In this section  $G$  is a free abelian group, and  $R$  is the group ring  $\mathbb{Z}[G]$ .

#### 3.1.1 Ribbon tangles

We use notations from Section 1.4.

**Definition 3.1.1.** Let  $L$  be an oriented trivial link with  $2n$  components in  $S^3 = \partial B^4$ . Consider  $A_1, \dots, A_n$  disjoint copies of the oriented annulus  $S^1 \times I$  and  $E_1, \dots, E_m$  disjoint copies of the torus  $S^1 \times S^1$ . A *ribbon tangle*  $T$  is a locally flat proper immersion in  $B^4$

$$T = \left( \bigsqcup_{i \in \{1, \dots, n\}} A_i \right) \sqcup \left( \bigsqcup_{i \in \{1, \dots, m\}} E_i \right)$$



satisfying the following properties:

1. For  $j = 1, \dots, m$  there exist locally flat immersed solid tori  $F_j$  such that  $\partial F_j = E_j$ .
2. The oriented boundary of each annulus  $\partial A_i$  is the disjoint union of two components of  $L$ . Moreover the orientation induced by  $A_i$  on  $\partial A_i$  coincides with the orientation of the two components of  $L$ .
3. There exist  $n$  locally flat immersed 3-balls  $B_i \simeq B^2 \times I$  such that, for all  $i \in \{1, \dots, n\}$ :

$$\partial B_i = A_i \cup_{\partial} (B^2 \times \{0, 1\}).$$

4. The singular set of the fillings  $B_i$  and  $F_j$  is composed by a finite number of ribbon disks.

A *G-colored ribbon tangle* is a pair  $(T, \varphi)$  where  $T$  is a ribbon tangle with complement  $X_T = B^4 \setminus T$ , equipped with a group homomorphism  $\varphi: H_1(X_T) \rightarrow G$ .

### 3.1.2 Definition of the Alexander invariant A

Let  $(X, Y)$  be a pair of topological spaces. Denote by  $p: \hat{X} \rightarrow X$  the maximal abelian cover. For a ring homomorphism  $\varphi: \mathbb{Z}[H_1(X)] \rightarrow R$ , we define the twisted chain complex

$$C^\varphi(X, Y) = C(\hat{X}, p^{-1}(Y)) \otimes_{\mathbb{Z}[H_1(X)]} R$$

whose homology is denoted by  $H_*^\varphi(X, Y; R)$ , or simply  $H_*^\varphi(X, Y)$ .

Let  $(T, \varphi)$  be a  $G$ -colored ribbon tangle. The homomorphism  $\varphi$  extends to a ring homomorphism  $\varphi: \mathbb{Z}[H_1(X_T)] \rightarrow R$ . For the rest of this section, we set  $H = H_1^\varphi(X_T, *)$ .

**Proposition 3.1.2.** *The  $R$ -module  $H$  admits a presentation with deficiency  $n$ .*

We postpone the proof of Proposition 3.1.2 to the end of Section 3.3.1. Consider a presentation of  $H$  of the form

$$H = \langle \gamma_1, \dots, \gamma_{n+q} \mid r_1, \dots, r_q \rangle.$$

Let  $\Gamma$  be the free  $R$ -module generated by  $\langle \gamma_1, \dots, \gamma_{n+q} \rangle$ . The relators  $\{r_1, \dots, r_q\}$  are words in these generators in  $\Gamma$ . Let us denote by  $r = r_1 \wedge \dots \wedge r_q$  and  $\gamma = \gamma_1 \wedge \dots \wedge \gamma_{n+q}$ .

**Definition 3.1.3.** The *Alexander function*  $\mathcal{A}_T^\varphi: \wedge^n H \rightarrow R$  is the  $R$ -linear application defined by

$$r \wedge \tilde{u} = \mathcal{A}_T^\varphi(u) \cdot \gamma$$

for all  $u = u_1 \wedge \dots \wedge u_n \in \wedge^n H$ , where  $\tilde{u}_1, \dots, \tilde{u}_n$  are arbitrary lifts in  $\Gamma$  of  $u_1, \dots, u_n$ , and  $\tilde{u} = \tilde{u}_1 \wedge \dots \wedge \tilde{u}_n$ . Different  $n$ -deficient presentations will give rise to Alexander functions that differ only by multiplication by a unit in  $R$ . Note that if  $H$  is free of rank  $n$ , then  $\mathcal{A}_T^\varphi$  is a volume form.

To understand how the Alexander function is concretely computed, let us consider the  $q \times (q+n)$ -matrix defined by the presentation of  $H$ . If one adds to this matrix the row vectors giving  $u_1, \dots, u_n$  in the generators  $\{\gamma_1, \dots, \gamma_{q+n}\}$ , then  $\mathcal{A}_T^\varphi(u)$  is the determinant of the resulting  $(q+n) \times (q+n)$ -matrix.

**Example 3.1.4.** Take  $G$  to be the free abelian group of rank 2 generated by  $t_1$  and  $t_2$ . Consider the module  $H$ , associated to some ribbon tangle  $T$ , whose presentation has generators  $\{\gamma_1, \dots, \gamma_4\}$  and two relations given by the matrix

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1-t_1 & t_2 \end{pmatrix}.$$

Then  $H$  admits a presentation of deficiency 2. To determine the Alexander function  $\mathcal{A}_T$ , one needs to compute its values on a chosen basis for  $\wedge^2 H$ . Let us consider the basis  $\{\gamma_1 \wedge \gamma_2, \gamma_1 \wedge \gamma_3, \dots, \gamma_3 \wedge \gamma_4\}$ .

The values of the Alexander function  $\mathcal{A}: \wedge^2 H \rightarrow R$  are computed by adding to the presentation matrix the lines corresponding to the considered generator in terms of  $\{\gamma_1, \dots, \gamma_4\}$ , and then computing the determinant.

Take first  $(\gamma_1 \wedge \gamma_2)$ . Then we have:

$$\mathcal{A}_T(\gamma_1 \wedge \gamma_2) = \det \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1-t_1 & t_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = t_2$$

We remark that adding the lines corresponding to the element  $\gamma_1 \wedge \gamma_2$  in terms of  $\{\gamma_1, \dots, \gamma_4\}$  is the same as considering the minor of the presentation matrix obtained removing the columns corresponding to  $\gamma_1$  and  $\gamma_2$ . Carrying on the computation on the other elements of the basis, one obtains

$$\begin{aligned} \mathcal{A}_T(\gamma_1 \wedge \gamma_3) &= 0, \quad \mathcal{A}_T(\gamma_1 \wedge \gamma_4) = 1, \\ \mathcal{A}_T(\gamma_2 \wedge \gamma_3) &= -t_2, \quad \mathcal{A}_T(\gamma_2 \wedge \gamma_4) = t_1 - 1, \quad \mathcal{A}_T(\gamma_3 \wedge \gamma_4) = 1. \end{aligned}$$

Recall that  $\partial T = L$ . Let  $H_\partial = H_1^\varphi(S^3 \setminus L, *)$ , which is the free  $R$ -module of rank  $2n$ , generated by the meridians of  $L$ . Let  $m_\partial: H_\partial \rightarrow H$  be induced by the inclusion map  $S^3 \setminus L \hookrightarrow X_T$ . For short, for a given  $z \in \wedge^n H_\partial$ , we use the notation  $m_\partial z$  for  $\wedge^n m_\partial(z)$ .

**Definition 3.1.5.** The element  $\mathbf{A}(T, \varphi)$  of  $\wedge^n H_\partial$  is defined by the following property:

$$\forall z \in \wedge^n H_\partial, \quad \mathcal{A}_T^\varphi(m_\partial z) = \omega_\partial(\mathbf{A}(T, \varphi) \wedge z) \quad (3.1)$$

where  $\omega_\partial$  is a volume form on  $H_\partial$ .

By construction of  $\mathbf{A}$  we have the following.

**Proposition 3.1.6.** *Let  $(T, \varphi)$  be a colored ribbon tangle. Then  $\mathbf{A}(T, \varphi)$  is a colored isotopy invariant.*

*Remark 3.1.7.* The Alexander invariant  $\mathbf{A}$  is defined up to a unit of  $R$  because it depends on the choice of the presentation matrix used to compute the Alexander function  $\mathcal{A}_T^\varphi$ .

### 3.1.3 The Alexander polynomial of a $(1 - 1)$ -ribbon tangle

Given a finitely generated  $R$ -module  $H$ , and  $k \geq 0$ , the  $k$ -th Alexander polynomial of  $H$  is the greatest common divisor of all minors of order  $(m - k)$  in a  $(q \times m)$  presentation matrix of  $H$ . This invariant of  $H$ , denoted by  $\Delta_k(H) \in R$ , is defined up to multiplication by a unit of  $R$ .

**Definition 3.1.8.** The Alexander polynomial  $\Delta^\varphi(T) \in R$  of a  $G$ -colored ribbon tangle  $(T, \varphi)$  is  $\Delta_0(H_1^\varphi(X_T))$ .

Similarly to classical knot theory,  $T$  is a  $(1 - 1)$ -ribbon tangle if  $n$ , the number of embedded annuli, is equal to 1. The components of  $T$  in  $B^4$  consist of  $m$  tori and a cylinder whose boundary is a 2-component trivial link  $L$  in  $S^3 = \partial B^4$ . Let  $x_1$  and  $x_2$  be the meridians of the components of  $L$ . Note that in  $X_T$ , both  $x_1$  and  $-x_2$  are homologous to the meridian  $x$  of the cylinder. We use the same notations  $x_1$  and  $x_2$  for the homology classes of their lifts in  $H_1^\varphi(S^3 \setminus L, *)$ .

**Proposition 3.1.9.** Let  $(T, \varphi)$  be a  $G$ -colored  $(1 - 1)$ -ribbon tangle, such that  $\varphi$  is not trivial. Denote  $\varphi(x)$  by  $t$ , and let  $s$  be the rank of  $\varphi(H_1(X_T))$  as  $\mathbb{Z}[H_1(X_T)]$  module. Then the element  $A(T, \varphi)$  of  $H_\partial$  is given by

$$A(T, \varphi) = \begin{cases} (t - 1)\Delta^\varphi(T) \cdot (x_1 - x_2) & \text{if } s \geq 2, \\ \Delta^\varphi(T) \cdot (x_1 - x_2) & \text{if } s = 1. \end{cases}$$

It is worth noticing that, up to a unit in  $R$ , the result is independent of the order chosen on the components of  $L$ . For the reader's convenience, we give a short proof of Proposition 3.1.9. More detailed arguments can be found in [33, Section 3].

*Proof.* Denote  $H_1^\varphi(X_T, *)$  by  $H$ . From the long exact sequence of the pair  $(X_T, *)$ :

$$0 \longrightarrow H_1^\varphi(X_T) \longrightarrow H \longrightarrow H_0^\varphi(*) \longrightarrow H_0^\varphi(X_T) \longrightarrow 0$$

we deduce that  $\text{Tors}H_1^\varphi(X_T) \simeq \text{Tors}H$ . Moreover,  $\text{rk}H = \text{rk}H_1^\varphi(X_T) + 1$ . This implies that

$$\Delta^\varphi(T) = \Delta_1(H).$$

Let  $A$  be the matrix of a presentation of  $H = \langle \gamma_1, \dots, \gamma_{q+1} \mid r_1, \dots, r_q \rangle$ , and  $\mathcal{A}^\varphi$  be the related Alexander function. We have

$$\forall z_1, \dots, z_{q+1} \in R, \mathcal{A}^\varphi(z_1\gamma_1 + \dots + z_{q+1}\gamma_{q+1}) = \sum_{i=1}^{q+1} \det(A_i)z_i,$$

where  $A_i$  is  $A$  with the  $i$ -th column removed. Hence,

$$\Delta_1(H) = \gcd \mathcal{A}^\varphi(H) = \gcd \{ \mathcal{A}^\varphi(h) : h \in H \}.$$

If  $\mathcal{A}^\varphi = 0$ , then  $\Delta^\varphi(T) = 0$ , and the result holds trivially. Consider now the connecting homomorphism  $\partial_*: H \rightarrow H_0^\varphi(*) \simeq R$ . If  $\mathcal{A}^\varphi \neq 0$ , then  $\text{rank}(H) = 1$  and any two linear

maps  $H \otimes_R \mathbb{Q}R \rightarrow \mathbb{Q}R$  are linearly dependent, where  $\mathbb{Q}R$  is the fraction field of  $R$ . Then there exist elements  $P, Q$  in  $R$  such that for each  $h \in H$ ,  $\mathcal{A}^\varphi(h) = (P/Q)\partial_*(h)$ . Hence

$$\mathcal{A}^\varphi(h) = \Delta^\varphi(T) \cdot \frac{\partial_*(h)}{\gcd \partial_*(H)}.$$

For a loop  $\gamma$  based in  $*$  with lift  $\hat{\gamma}$ , one has  $\partial_*\hat{\gamma} = \varphi(\gamma) - 1$ . Hence,  $\gcd \partial_*(H)$  is equal to 1 if  $s \geq 2$  and is equal to  $t - 1$  if  $s = 1$ . We deduce that for all  $h \in H$

$$\mathcal{A}^\varphi(h) = \begin{cases} (t-1)\Delta^\varphi(T) & \text{if } s \geq 2, \\ \Delta^\varphi(T) & \text{if } s = 1. \end{cases}$$

Let  $\omega_\partial$  be the volume form on  $H_\partial$  relative to the choice of the meridians  $x_1, x_2$ . By definition,  $A(T, \varphi)$  verifies  $\mathcal{A}_T^\varphi(m_\partial z) = \omega_\partial(A(T, \varphi) \wedge z)$ . Since  $m_\partial(x_1) = x$  and  $m_\partial(x_2) = -x$ , we obtain the result from

$$\mathcal{A}^\varphi(x) = \omega_\partial(A(T, \varphi) \wedge x_1) = -\omega_\partial(A(T, \varphi) \wedge x_2).$$

□

### 3.1.4 The Burau functor

We begin with defining ribbon cobordisms. We recall we are using notations from Section 1.4.

**Definition 3.1.10.** Let  $L_0$  and  $L_1$  be two trivial links respectively in  $\partial_0 B^4$  and  $\partial_1 B^4$ , of  $n_0$  and  $n_1$  components. Assign to each link a sequence of signs  $\varepsilon_0$  and  $\varepsilon_1$ , of respective lengths  $n_0$  and  $n_1$ . A *ribbon cobordism* from  $\varepsilon_0$  to  $\varepsilon_1$  is a collection  $S$  of ribbon annuli and tori in  $B^4 \simeq B^3 \times I$  whose boundaries are the components of  $L_0$  and  $L_1$ , with according signs. Two cobordisms are equivalent if there is an ambient isotopy fixing the boundary circles  $L_0$  and  $L_1$ . The *degree* of the ribbon cobordism  $S$  is  $\delta = (n_1 - n_0)/2$ .

We can now introduce the category of ribbon cobordisms.

**Definition 3.1.11.** The category  $\mathcal{Rib}$  of ribbon cobordisms is the category whose:

- objects are sequences  $\varepsilon$  of signs  $\pm 1$  of length  $n$ , and correspond to trivial links with  $n$  components in  $B^3$ , such that a sign is affected to each component;
- morphisms  $\varepsilon_0 \rightarrow \varepsilon_1$  are the equivalence classes of ribbon cobordisms between two trivial links  $L_0$  and  $L_1$ .

The composition  $S \circ S'$  of two ribbon cobordisms  $S$  and  $S'$  in  $\mathcal{Rib}$  is defined by identifying  $(B^3 \times \{1\}, S')$  to  $(B^3 \times \{0\}, S)$ , when components and signs are compatible.

The category  $\mathcal{Rib}$  can be refined to the category of colored ribbon cobordisms.

**Definition 3.1.12.** Let us consider a homomorphism  $\varphi: H_1(B^3 \setminus L) \rightarrow G$ . The category  $\mathcal{Rib}_G$  of colored ribbon cobordisms is the category whose:

- objects are pairs  $(\varepsilon, \varphi)$ , where  $\varepsilon$  is like in Definition 3.1.11.
- morphisms  $(\varepsilon_0, \varphi_0) \rightarrow (\varepsilon_1, \varphi_1)$  are pairs  $(S, \varphi)$  such that  $S$  is like in Definition 3.1.11, and  $\varphi$  is such that, for  $i = 0, 1$ :

$$\varphi \circ m_i = \varphi_i$$

where the maps  $m_i$  are induced by the inclusions  $X_{L_i} \hookrightarrow X_S$ ,  $X_{L_i} = (B^3 \times \{i\}) \setminus L_i$  and  $X_S = (B^3 \times I) \setminus S$ .

Composition of colored ribbon cobordisms is defined as in Definition 3.1.11, with the added condition of the morphisms being composable.

Let  $grMod_G$  be the category of  $\mathbb{Z}$ -graded  $R$ -modules, whose morphisms are graded  $R$ -linear maps of arbitrary degree, up to multiplication by an element of  $\pm G$ . We define a map

$$\rho: \mathcal{Rib}_G \longrightarrow grMod_G$$

that we will prove to be a functor, and will be called *Burau functor*. Given a trivial link  $L$  with  $n$  components in the interior of  $B^3$ , and a homomorphism  $\varphi: H_1(B^3 \setminus L) \rightarrow G$ , we denote by  $M$  the free  $R$ -module of  $H_1^\varphi(X_L, *, R)$ , of rank  $n$ . In particular, taken  $L_0$  and  $L_1$  as before on  $\partial_0 B^4$  and  $\partial_1 B^4$ , we denote by  $M_0$  the free  $R$ -module  $H_1^{\varphi_0}(X_{L_0}, *_0; R)$  of rank  $n_0$ , and by  $M_1$  the free  $R$ -module  $H_1^{\varphi_1}(X_{L_1}, *_1; R)$  of rank  $n_1$ , where  $*_0$  and  $*_1$  are base points respectively in  $\partial B^3 \times \{0\}$  and in  $\partial B^3 \times \{1\}$ . Let  $J$  be the interval in  $\partial B^3 \times I$ , which connects the base points of the bottom and top balls  $\partial_0 B^4$  and  $\partial_1 B^4$ . We consider the  $R$ -module  $H_1^\varphi(X_S, J; R)$ . Remark that by Proposition 3.1.2, it admits a presentation with deficiency  $d = (n_0 + n_1)/2$ .

**Definition 3.1.13.** The map  $\rho: \mathcal{Rib}_G \rightarrow grMod_G$  is defined by the following data. Let  $(\varepsilon, \varphi)$  be an object of  $\mathcal{Rib}_G$ , corresponding to a trivial link  $L$ . The image by  $\rho$  of  $(\varepsilon, \varphi)$  is

$$\rho(\varepsilon, \varphi) = \wedge M$$

the exterior algebra of the free  $R$ -module  $M$ . Let us consider a morphism  $(S, \varphi): (\varepsilon_0, \varphi_0) \rightarrow (\varepsilon_1, \varphi_1)$ . We associate to it a  $R$ -linear map

$$\rho(S, \varphi): \wedge M_0 \longrightarrow \wedge M_1,$$

of degree  $\delta$  as follows. Denote by  $\mathcal{A}_S^\varphi$  the Alexander function  $\wedge^d H_1^\varphi(X_S, J; R) \rightarrow R$ . For any integer  $k \geq 0$ , the homomorphism  $\rho_k(S, \varphi)(x): \wedge^k M_0 \rightarrow \wedge^{k+\delta} M_1$  is defined, for any  $x \in \wedge^k M_0$ , by the following property:

$$\forall y \in \wedge^{d-k} M_1, \mathcal{A}_S^\varphi(\wedge^k m_0(x) \wedge \wedge^{d-k} m_1(y)) = \omega_1(\rho(S, \varphi)(x) \wedge y),$$

where  $\omega_1$  is a volume form on  $M_1$ . Then  $\rho(S, \varphi) = \oplus_k \rho_k(S, \varphi)$ , defined up to multiplication by a unit of  $R$ .

**Theorem 3.1.14.** *The map  $\rho$  is a degree preserving functor  $\mathcal{Rib}_G \rightarrow grMod_G$ .*

*Proof.* The proof of Theorem 3.1.14 follows word by word the proof of [33, Theorem I]. Note that the deficiency of the presentation of  $H_1^\varphi(X_S, J; R)$  depends only on  $n_0$  and  $n_1$ .  $\square$

### An equivalent definition for the Burau functor

The Burau functor  $\rho$  can be equivalently defined in a more explicit way. Recall that  $M_1$  is free of rank  $n_1$ . Then, choosing a generator  $w_1$ ,  $\wedge^{n_1} M_1$  is isomorphic to  $Rw_1$ . Fixed such generator  $w_1$ , for  $j \in \{0, \dots, n_1\}$  consider the isomorphism

$$\theta_j: \wedge^j M_1 \longrightarrow \text{Hom}(\wedge^{n_1-j} M_1, R)$$

defined by  $w_1(\theta_j(x)(y)) = w_1(x \wedge y)$  for any elements  $x \in \wedge^j M_1$  and  $y \in \wedge^{n_1-j} M_1$ .

Moreover recall that there exists a presentation of deficiency  $d = \frac{n_1+n_0}{2}$  for the  $R$ -module  $H_1^\varphi(X_S, J; R)$ . For any  $k \in \{0, \dots, d\}$  the choice of generators  $\gamma$  for  $\wedge^{d+k} R$  and  $r$  for  $\wedge^q R$  induce isomorphisms

$$\mathcal{A}_{S_k}^\varphi: \wedge^k H_1^\varphi(X_S, J; R) \longrightarrow \text{Hom}(\wedge^{d-k} H_1^\varphi(X_S, J; R), R)$$

defined by  $(\mathcal{A}_{S_k}^\varphi(x)(y)) = \mathcal{A}_S^\varphi(x \wedge y)$  for any elements  $x \in \wedge^k H_1^\varphi(X_S, J; R)$  and  $y \in \wedge^{d-k} H_1^\varphi(X_S, J; R)$ .

Then, for any integer  $k \geq 0$ , the homomorphism  $\rho_k(S, \varphi)(x): \wedge^k M_0 \rightarrow \wedge^{k+\delta} M_1$  is defined by the composition

$$\begin{aligned} \wedge^k M_0 &\xrightarrow{\wedge^k m_0} \wedge^k H_1^\varphi(X_S, J; R) \xrightarrow{\mathcal{A}_{S_k}^\varphi} \text{Hom}(\wedge^{d-k} H_1^\varphi(X_S, J; R), R) \\ &\xrightarrow{(\wedge^{d-k} m_1)_*} \text{Hom}(\wedge^{d-k} M_1, R) \xrightarrow{\theta_{k+\delta}^{-1}} \wedge^{k+\delta} M_1. \end{aligned}$$

*Remark 3.1.15.* Let us follow an element  $u_0$  in  $\wedge^k M_0$  through the said composition, to see how the various isomorphisms act on it. Consider a presentation matrix  $A$  for the module  $H_1^\varphi(X_S, J; R)$  with deficiency  $d = \frac{n_0+n_1}{2}$ . For some  $k \in \{0, \dots, d\}$ , take an element  $u_0 = u_{0_1} \wedge \dots \wedge u_{0_k} \in \wedge^k M_0$ . To simplify notation, we call  $u_0$  its image by the inclusion in  $\wedge^k H_1^\varphi(X_S, J; R)$ . The application  $\mathcal{A}_{S_k}^\varphi$  will send  $u_0$  to the sum of linear application in  $\text{Hom}(\wedge^{d-k} H_1^\varphi(X_S, J; R), R)$  which take the elements  $y_j$  of the chosen basis of  $\wedge^{d-k} H_1^\varphi(X_S, J; R)$  and send them to the value  $\mathcal{A}_S^\varphi(u_0 \wedge y_j)$  of the Alexander function of the wedge product of  $u_0$  and  $y_j$ . As we remarked in Example 3.1.4, the value  $\mathcal{A}_S^\varphi(u_0 \wedge y_j)$  can be calculated using the presentation matrix  $A$ . First one has to add the lines corresponding to  $u_0 \wedge y_j$  (or a reordering of their components if they do not respect the order on the generating system of  $H_1^\varphi(X_S, J; R)$ , eventually storing the negative sign that can appear, that will be applied to the result). Then, having added  $d$  rows with all 0 entries and one 1 entry, one has to compute the determinant. This is equivalent to calculate  $A$ 's minor  $A^{\widehat{u_0, y_j}}$  obtained by removing the columns relative to the entries  $u_{0_1}, \dots, u_{0_k}, y_{j_1}, \dots, y_{j_{d-k}}$ . So we have that

$$\sum_{j \in \{1, \dots, \frac{(d-k)(d-k-1)}{2}\}} \mathcal{A}_{S_k}^\varphi(u_0)(y_j) = \sum_{j \in \{1, \dots, \frac{(d-k)(d-k-1)}{2}\}} (-1)^{s_j} A^{\widehat{u_0, y_j}}$$

where  $s_j$  is the sign of the permutation of the indices of  $u_{0_1}, \dots, u_{0_k}, y_{j_1}, \dots, y_{j_{d-k}}$ .

Next step is the application induced by the inclusion of  $M_1$  into  $H_1^\varphi(X_S, J; R)$ . Writing  $y_j \in \wedge^{d-k} H_1^\varphi(X_S, J; R)$  as  $m_1(w)$  for some  $w \in \wedge^{d-k} M_1$ , we have that  $u_0$  is sent to

$$\sum_{w \in \wedge^{d-k} M_1} (-1)^{s_w} A^{\widehat{u_0, w}}$$

where  $s_w$  is the sign of the permutation of the indices of  $u_{0_1}, \dots, u_{0_k}, w_1, \dots, w_{d-k}$ . Finally, considering  $\theta_{k+\delta}^{-1}$ , we have that

$$\rho_k(u_0) = \sum_{w \in \wedge^{d-k} M_1} A^{\widehat{u, w}} \star (w)$$

where  $\star$  is the isomorphism  $\wedge^{d-k} M_1 \rightarrow \wedge^k M_1$  such that taken an ordered basis  $x_1, \dots, x_{n_1}$  for  $M_1$ ,  $\star(1) = x_1 \wedge \dots \wedge x_{n_1}$ ,  $\star(x_1 \wedge \dots \wedge x_{n_1}) = 1$ ,  $\star(x_i) = (-1)^{i-1} x_1 \wedge \dots \wedge \widehat{x_i} \wedge \dots \wedge x_{n_1}$  and in general  $\star(x_{l_1} \wedge \dots \wedge x_{l_j}) = (\pm 1) x_{l_{j+1}} \wedge \dots \wedge x_{l_{k_+}}$  where the sign  $(\pm 1)$  depends on the sign of the permutation  $(1, \dots, k_+) \rightarrow (l_1, \dots, l_{k_+})$ . This isomorphism is called, in other contexts, the *Hodge dual* of  $\wedge^{d-k} M_1$ , see [65] for an exhaustive description.

A  $G$ -colored ribbon tube  $(S, \varphi)$  is a morphism such that the applications  $m_0$  and  $m_1$  induced by the inclusions are isomorphisms in homology (with integer coefficients). Ribbon tubes are analogous to string links [5]; the links  $L_0$  and  $L_1$  have the same number of components, and  $S$  has no toric component. For a fixed  $\varphi$ , let us denote by  $\mathcal{T}_\varphi$  the set of  $G$ -colored ribbon tubes. Following [57, Proposition 2.1], one proves that  $m_0$  and  $m_1$  induce isomorphisms

$$(m_i)_*: H^\varphi(X_{L_i}; QR) \longrightarrow H^\varphi(X_S; QR), \text{ for } i = 0, 1$$

where  $QR$  is the quotient field of  $R$ . Set  $H^\varphi$  to be  $H^\varphi(X_{L_0}; QR) = H^\varphi(X_{L_1}; QR)$ . Set  $L = L_0 = L_1$ . Both  $B^3 \times \{0\} \setminus L_0$  and  $B^3 \times \{1\} \setminus L_1$  are balls with  $n$  trivial links removed, and are canonically identified via the homeomorphism  $(x, 0) \rightarrow (x, 1)$ . The composition  $m_1^{-1} \circ m_0$  is an automorphism of  $H^\varphi(X_L; QR)$ .

**Definition 3.1.16.** The *colored Burau representation* is the monoid homomorphism

$$r^\varphi: \mathcal{T}_\varphi \longrightarrow \text{Aut}(H^\varphi).$$

Let us consider  $\Delta^\varphi(X_T, X_L) = \Delta_0(H_1^\varphi(X_T, X_L; R))$ , the Alexander polynomial of the pair  $(X_T, X_L)$ , as defined in Section 3.1.3.

**Proposition 3.1.17.** For any  $G$ -colored ribbon tube  $(S, \varphi) \in \mathcal{T}_\varphi$ , we have

$$\rho(S, \varphi) = \Delta^\varphi(X_T, X_L) \cdot \wedge r^\varphi(S) : \wedge H^\varphi \longrightarrow \wedge H^\varphi.$$

In the case of ribbon braids (which are monotone ribbon tubes),  $\Delta^\varphi(X_T, X_L) = 1$  and  $\rho(S, \varphi)$  coincide with the exterior powers of  $r^\varphi(S)$ . The proof of Proposition 3.1.17 can be obtained similarly to [33, Proposition 7.2], or in the monotone case, to [19, Section 3.1].

We see now how to obtain a morphism in the category of  $\mathcal{Rib}_G$  as the splitting colored ribbon tangle  $(T, \varphi)$ .

**Definition 3.1.18.** Let  $(T, \varphi)$  be a  $G$ -colored ribbon tangle in  $B^4$ . Let  $L = L_0 \cup L_1$  be a splitting of  $L$  into two disjoint (trivial) links, and let  $B_0$  and  $B_1$  be two 3-balls such that

$$S^3 = B_0 \cup_{S^2 \times \{0\}} (S^2 \times [0; 1]) \cup_{S^2 \times \{1\}} B_1$$

and  $L_i \subset B_i$  for  $i = 1, 2$ . Let  $\varphi_i$  be induced by  $\varphi$  on  $H_1(B_i \setminus L_i)$ , and  $\varepsilon_i$  be sequences of signs according to the co-orientations of the components of  $T$ . Then the morphism  $(\tilde{T}, \varphi): (\varepsilon_0, \varphi_0) \rightarrow (\varepsilon_1, \varphi_1)$  is a *splitting* of  $(T, \varphi)$ . Note that  $2n = n_0 + n_1$ .

**Theorem 3.1.19.** Let  $(T, \varphi)$  be a  $G$ -colored ribbon tangle, and  $(\tilde{T}, \varphi): (\varepsilon_0, \varphi_0) \rightarrow (\varepsilon_1, \varphi_1)$  be a splitting of  $(T, \varphi)$  in  $\text{Rib}_G$ . There is an isomorphism, well-defined up to a unit in  $R$ ,

$$\wedge^n H_{\partial} \longrightarrow \text{Hom}_R(\wedge M_0, \wedge M_1)$$

sending  $A(T, \varphi)$  to  $\oplus_k (-1)^{k(n_0-k)} \rho_k(\tilde{T}, \varphi)$ , where  $\rho_k$  is the  $k$ -component of  $\rho$ .

An explicit example is given in Remark 3.4.1.

*Proof.* The decomposition  $H_{\partial} = M_0 \oplus M_1$  induces a natural isomorphism

$$\wedge^n H_{\partial} \simeq \bigoplus_{k=0}^n (\wedge^k M_0 \otimes \wedge^{n-k} M_1).$$

The element  $A(T, \varphi) \in \wedge^n H_{\partial}$  decomposes as  $\sum_k A_k(T, \varphi)$ , where  $A_k(T, \varphi) \in \wedge^k M_0 \otimes \wedge^{n-k} M_1$ . Suppose now that  $k$  is fixed; the element  $A_k(T, \varphi)$  might not be decomposable. There exist a finite sequence of element  $A_0^l \in \wedge^k M_0$  and  $A_1^l \in \wedge^{n-k} M_1$  (depending on  $k$ , and  $(T, \varphi)$ ) such that  $A_k(T, \varphi) = \sum_l A_0^l \otimes A_1^l$ . Let  $\omega_0$  be a volume form  $\wedge^{n_0} M_0 \rightarrow R$ . There is an isomorphism

$$\wedge^k M_0 \otimes \wedge^{n-k} M_1 \simeq \text{Hom}_R(\wedge^{n_0-k} M_0, \wedge^{n-k} M_1)$$

sending  $A_k(T, \varphi)$  to

$$x \mapsto \sum_l \omega_0(x \wedge A_0^l) \cdot A_1^l.$$

We now show that this morphism coincides with  $\rho_{n_0-k}(\tilde{T}, \varphi)$ . Let  $x \in \wedge^{n_0-k} M_0$ . Note that we have  $n_0 - k + \delta = n - k$  and the morphism has degree  $\delta$ . Consider a volume form  $\omega_1: \wedge^{n_1} M_1 \rightarrow R$  and the sum  $\omega_{\partial} = \omega_0 \otimes \omega_1: \wedge^n H_{\partial} \rightarrow R$ . Let  $\mathcal{A}^{\varphi}$  be the Alexander function related to a presentation of  $H_1^{\varphi}(X_T, *; R)$  of deficiency  $n$ . Since  $m_{\partial} = m_0 \oplus m_1$  we have, for all  $y$  in  $\wedge^{k+\delta} M_1$ ,

$$\begin{aligned} \mathcal{A}^{\varphi}(\wedge^{n_0-k} m_0(x) \wedge \wedge^{k+\delta} m_1(y)) &= \omega_{\partial}(A(T, \varphi) \wedge x \wedge y) \\ &= \omega_{\partial}(A_k(T, \varphi) \wedge x \wedge y) \\ &= \sum_l \omega_0(A_0^l \wedge x) \cdot \omega_1(A_1^l \wedge y) \\ &= \omega_1\left(\sum_l \omega_0(A_0^l \wedge x) \cdot A_1^l \wedge y\right). \end{aligned}$$

Hence  $\sum_l \omega_0(x \wedge A_0^l) \cdot A_1^l = (-1)^{k(n_0-k)} \rho_{n_0-k}(\tilde{T}, \varphi)(x)$ . □



## 3.2 The circuit algebra of colored ball cobordisms

In this section we introduce the circuit algebra  $\text{Cob}_G$  of *ball cobordisms* over the diffeomorphism classes of ball cobordisms in a 4-ball. Circuit algebras are algebraic structures inspired by [44, 68], see also [2, 56].

The literature about planar and circuit algebras borrows the categorical language of operads. However, it does so in a non-categorical context, defining structures that can be seen as having partial operad characteristics. For the definition of an operad in a monoidal symmetric category we refer to [63]. We propose here a definition that fits with our framework.

**Definition 3.2.1.** A *circuit operad* is a collection of objects  $\mathcal{C}(j)$ , for  $j \in \mathbb{N}$ , and product morphisms

$$\gamma: \mathcal{C}(j_{i_1}) \otimes \dots \otimes \mathcal{C}(j_{i_k}) \longrightarrow \mathcal{C}\left(\sum_{l=1}^{i_k} j_l\right)$$

for  $i_k \geq 1$  and  $j_i \geq 0$  for all  $i \in \{i_1, \dots, i_k\}$ . The product morphisms are compatible with the composition of the objects, when composition is defined.

**Example 3.2.2.** Let  $\text{Hom}_G$  be the collection of  $R$ -multilinear applications of tensor powers of  $R$ -modules, considered up to an element of  $\pm G$ . Take the composition in  $\text{Hom}_G$  to be the usual composition of maps. Then  $\text{Hom}_G$  is a circuit operad.

**Definition 3.2.3.** A *ball cobordism* consists of the following data:

- A 4-ball  $B = B_0$  and  $B_1, \dots, B_p$  disjoint 4-balls in the interior of  $B$ . For every  $i \in \{0, \dots, p\}$ , let  $L_i$  (with  $L = L_0$ ) be a trivial oriented link with  $2n_i$  ( $n = n_0$ ) components in  $S_i^3 = \partial B_i$  (with  $S^3 = S_0^3$ ).
- A disjoint union  $C$  of oriented locally flat proper embedded annuli in  $B \setminus \{\mathring{B}_1, \dots, \mathring{B}_p\}$ , whose boundary are the links  $L_i$ , with the conditions of Definition 3.1.1 but without singularities.

The complement  $X_C$  of a ball cobordism is defined as  $B^4 \setminus (C \sqcup B_1, \dots, B_p)$ . Under a certain condition on the borders, ball cobordisms can be composed.

**Definition 3.2.4.** Let  $C'$  and  $C''$  be two cobordisms such that  $B'_i$  is a ball of  $C'$  with  $n'_i = n''$ . The *composition*  $C' \circ_i C''$  is the cobordism obtained with the identification of  $B'' = B''_0$  with  $B'_i$ .

As in the previous section,  $G$  is a fixed free abelian group with group ring  $R$ . We can define a colouring on a ball cobordism.

**Definition 3.2.5.** A  $G$ -*colored ball cobordism* is a pair  $(C, \varphi)$  where  $C$  is a ball cobordism with complement  $X_C$ , equipped with a group homomorphism  $\varphi: H_1(X_C) \rightarrow G$ .

**Definition 3.2.6.** The *ball cobordism circuit operad*  $\mathcal{C}_G$  is the collection of orientation-preserving diffeomorphism classes of  $G$ -colored ball cobordisms with composition of compatible ball cobordisms.

We can give her our definition of circuit algebra.

**Definition 3.2.7.** A *circuit algebra* is a morphism of circuit operads.

In our context, we are interested in morphisms which take values in the circuit operad  $\mathcal{H}om_G$  introduced in Example 3.2.2. Giving a circuit algebra structure is equivalent to give the data of a sequence of  $R$ -modules  $\{N_i\}_{i \in \mathbb{N}}$  together with a representation

$$\Phi: \mathcal{C}_G \longrightarrow \mathcal{H}om_G.$$

This means that to a ball cobordism  $C$  one associates a multilinear map

$$\Phi(C): \bigotimes_{i=1}^p N_i \longrightarrow N_n$$

such that, taken two ball cobordisms  $C$  and  $D$ , we have the compatibility

$$\Phi(C \circ D) = \Phi(C) \circ \Phi(D)$$

where the composition on the left hand side is the composition from Definition 3.2.4, and the composition on the right hand side is the usual composition of linear maps.

*Remark 3.2.8.* We recall a construction that we used in the previous sections. Let  $(L, \psi)$  be a  $G$ -colored oriented trivial link with  $k$  components in  $S^3 = \partial B^4$ , with complement  $X_L = S^3 \setminus L$ . The group homomorphism  $\psi: H_1(S^3 \setminus L) \rightarrow G$  induces a ring homomorphism denoted by  $\psi: \mathbb{Z}[H_1(S^3 \setminus L)] \rightarrow R$  too. Let  $*$  be a base point on  $S^3$ . The  $R$ -module  $H_1^\psi(S^3 \setminus L, *; R)$  is free of rank  $k$ , generated by the meridians of  $L$ .

We construct the circuit algebra  $\text{Cob}_G$ . Let  $(C, \varphi)$  be a  $G$ -colored ball cobordism, with complement  $X_C$ . Let  $*$  be again a base point in  $S^3 = \partial B^4$ . For  $i = 1, \dots, p$ , let  $*_i$  be base points in the boundary of  $B_i$  and  $J_i$  be intervals (whose interiors are disjoint, and disjoint from  $C$ ) connecting  $*$  to  $*_i$ . Note that the union of the  $J_i$  is contractible. The homomorphism  $\varphi$  induces a ring homomorphism  $\mathbb{Z}[H_1(X_C)] \rightarrow R$  denoted by  $\varphi$  too.

The inclusion  $m_i: S_i^3 \setminus L_i \hookrightarrow X_C$  induces  $\varphi_i: \mathbb{Z}[H_1(S_i^3 \setminus L_i)] \rightarrow R$ . Set  $H = H_1^\varphi(X_C, J; R)$ ,  $H_\partial = H_1^\varphi(S^3 \setminus L, *; R)$ , and  $H_{\partial_i} = H_1^{\varphi_i}(S_i^3 \setminus L_i, *_i; R)$  for  $i = 1, \dots, p$ . Note that  $H$  is free of rank  $r = n + n_1 + \dots + n_p$  and  $H_{\partial_i}$  are free of rank  $2n_i$ .

Let  $\omega_C^\varphi$  be a volume form  $\omega_C^\varphi: \wedge^r H \rightarrow R$ , and  $\omega_\partial: \wedge^{2n} H_\partial \rightarrow R$ . For  $i = 1, \dots, p$ , we denote again by  $m_i: H_{\partial_i} \rightarrow H$  the maps induced by the inclusion. Let

$$m: \otimes_i (\wedge^{n_i} H_{\partial_i}) \rightarrow \wedge^{n_1 + \dots + n_p} H$$

be defined as  $m = (\wedge^{n_1} m_1) \wedge \dots \wedge (\wedge^{n_p} m_p)$ .

To the ball cobordism  $(C, \varphi)$  we associate

$$\Upsilon_{C, \varphi}: \bigotimes_{i=1}^p \wedge^{n_i} H_{\partial_i} \longrightarrow \wedge^n H_\partial$$

such that, for  $x \in \otimes_i (\wedge^{n_i} H_{\partial_i})$ ,

$$\omega_C^\varphi(m(x) \wedge m_\partial(y)) = \omega_\partial(\Upsilon_{C, \varphi}(x) \wedge y), \quad \forall y \in \wedge^n H_\partial. \quad (3.2)$$

We denote by  $\text{Cob}_G$  the data of:

- a sequence of freely generated  $R$ -modules  $\{H_{\partial_i}\}_{i \in \mathbb{N}}$  of rank  $n_i$ , associated the pairs of colored trivial links  $(L_i, \phi_i)$ , where  $L_i$  has  $n_i$  components;
- $R$ -multilinear maps  $\Upsilon_{C,\varphi}$  associated to each pair  $(C, \varphi)$ .

**Theorem 3.2.9.**  *$\text{Cob}_G$  is a circuit algebra.*

*Proof.* To prove that  $\text{Cob}_G$  is a circuit algebra, we need to prove the compatibility of the applications. We show that  $\Upsilon_{C,\varphi}$  commutes with the composition of ball cobordisms. Let  $(C', \varphi')$  and  $(C'', \varphi'')$  be two ball cobordisms with

$$\Upsilon_{C',\varphi'}: \bigotimes_{k=1}^{p'} \wedge^{k'} H_{\partial'_k} \longrightarrow \wedge^{n'} H_{\partial'}, \quad \text{and} \quad \Upsilon_{C'',\varphi''}: \bigotimes_{l=1}^{p''} \wedge^{n''_l} H_{\partial''_l} \longrightarrow \wedge^{n''} H_{\partial''}.$$

Let  $\varphi$  be the coloring induced by  $\varphi'$  and  $\varphi''$  on  $C' \circ_i C''$ . Then, we have to prove that for all  $u_k \in \wedge_i^k H_{\partial'_k}$  with  $k = 1, \dots, p'$  and  $k \neq i$  and all  $v_l$  for  $l = 1, \dots, p''$ ,

$$\Upsilon_{C' \circ_i C'', \varphi}(u_1 \otimes \dots \otimes (\otimes_1^{p''} v_l) \otimes \dots \otimes u_{p'}) = \Upsilon_{C', \varphi'}(u_1 \otimes \dots \otimes \Upsilon_{C'', \varphi''}(\otimes_1^{p''} v_l) \otimes \dots \otimes u_{p'}).$$

Let  $H' = H_1^{\varphi'}(X_{C'}, J')$  and  $H'' = H_1^{\varphi''}(X_{C''}, J'')$  be the (free) homology modules of the exteriors of the ball cobordisms. Let  $\alpha_1, \dots, \alpha_{2n''}$  be a basis of  $H_{\partial''} \simeq H_{\partial''_i}$ . Consider presentations of  $H'$  and  $H''$ :

$$\begin{aligned} H'' &= \langle m_{\partial''} \alpha_1, \dots, m_{\partial''} \alpha_{2n''}, \beta_1, \dots, \beta_k \mid \rho_1, \dots, \rho_s \rangle, \\ H' &= \langle m'_i \alpha_1, \dots, m'_i \alpha_{2n''}, \zeta_1, \dots, \zeta_l \mid r_1, \dots, r_t \rangle. \end{aligned}$$

Applying Mayer-Vietoris theorem to  $X_C = X_{C'} \cup X_{C''}$ , we obtain that the (free) module  $H$  is generated by

$$m_{\partial''} \alpha_1, \dots, m_{\partial''} \alpha_{2n''}, m'_i \alpha_1, \dots, m'_i \alpha_{2n''}, \beta_1, \dots, \beta_k, \zeta_1, \dots, \zeta_l \quad (3.3)$$

subject to the relations  $\rho_1, \dots, \rho_s, r_1, \dots, r_t, m_{\partial''} \alpha_1 - m'_i \alpha_1, \dots, m_{\partial''} \alpha_{2n''} - m'_i \alpha_{2n''}$ . Let  $\omega'$  and  $\omega''$  be volume forms on  $H'$  and  $H''$ , and  $\omega_{\partial'}$  be the form on  $H_{\partial'}$ . Let  $\omega$  be the form on  $H$  induced by  $\omega'$  and  $\omega''$ . For the computation below, we introduce the notation

$$u \wedge_i v = u_1 \wedge \dots \wedge u_{i-1} \wedge (v_1 \wedge \dots \wedge v_{p''}) \wedge u_{i+1} \wedge \dots \wedge u_{p'}.$$

We want to show that, for all  $y \in \wedge^{n'} H_{\partial'}$ ,

$$\begin{aligned} \omega_{\partial'}(\Upsilon_{C' \circ C''}(u_1 \otimes \dots \otimes (\otimes_1^{p''} v_l) \otimes \dots \otimes u_{p'}) \wedge y) \\ = \omega_{\partial'}(\Upsilon_{C'}(u_1 \otimes \dots \otimes \Upsilon_{C''}(\otimes_1^{p''} v_l) \otimes \dots \otimes u_{p'}) \wedge y). \end{aligned}$$

We have

$$\begin{aligned} \omega_{\partial'}(\Upsilon_{C' \circ C''}(u_1 \otimes \dots \otimes (\otimes_1^{p''} v_l) \otimes \dots \otimes u_{p'}) \wedge y) \cdot m_{\partial''} \alpha \wedge \beta \wedge m'_i \alpha \wedge \zeta \\ = \omega((m' u \wedge_i m'' v) \wedge m_{\partial'} y) \cdot m_{\partial''} \alpha \wedge \beta \wedge m'_i \alpha \wedge \zeta \\ = \rho \wedge r \wedge (m_{\partial''} \alpha - m'_i \alpha) \wedge (m' u \wedge_i m'' v) \wedge m_{\partial'} y \\ = \sum_Q (-1)^{|Q|} \varepsilon_Q \cdot \rho \wedge r \wedge m_{\partial''} \alpha_Q \wedge m'_i \alpha_{\bar{Q}} \wedge (m' u \wedge_i m'' v) \wedge m_{\partial'} y, \end{aligned}$$

where the sum is taken over all subsets  $Q \subset \{1, \dots, 2n''\}$  of cardinal  $n''$ . The number  $t$  of relations  $r_i$  can be chosen arbitrarily to be even and  $m_{\partial''}\alpha_Q \wedge m'_i\alpha_{\bar{Q}} \wedge (m'u \wedge_i m''v)$  coincides with  $m_{\partial''}\alpha_Q \wedge m''v \wedge (m'u \wedge_i m'_i\alpha_{\bar{Q}})$  up to a sign  $\varepsilon_Q$  depending only on  $p''$  and the  $n'_i$ . The sum coincides, up to a sign, with

$$\sum_Q (-1)^{|Q|} \varepsilon_Q \cdot (\rho \wedge m_{\partial''}\alpha_Q \wedge m''v) \wedge (r \wedge (m'u \wedge_i m'_i\alpha_{\bar{Q}}) \wedge m_{\partial'}y)$$

whose summands are equal to

$$(-1)^{|Q|} \varepsilon_Q \cdot \omega''(m_{\partial''}\alpha_Q \wedge m''v) \cdot \omega'((m'u \wedge_i m'_i\alpha_{\bar{Q}}) \wedge m_{\partial'}y) \cdot (m_{\partial''}\alpha \wedge \beta \wedge m'_i\alpha \wedge \zeta).$$

It follows that, up to a sign,

$$\begin{aligned} & \omega_{\partial'}(\Upsilon_{C' \circ C''}(u_1 \otimes \dots \otimes (\otimes_1^{p''} v_l) \otimes \dots \otimes u_{p'}) \wedge y) \\ &= \sum_Q (-1)^{|Q|} \varepsilon_Q \cdot \omega''(m_{\partial''}\alpha_Q \wedge m''v) \cdot \omega'((m'u \wedge_i m'_i\alpha_{\bar{Q}}) \wedge m_{\partial'}y) \\ &= \omega'(\sum_Q (-1)^{|Q|} \varepsilon_Q \cdot \omega''(m_{\partial''}\alpha_Q \wedge m''v) \cdot (m'u \wedge_i m'_i\alpha_{\bar{Q}}) \wedge m_{\partial'}y) \\ &= \omega'(\sum_Q (-1)^{|Q|} \varepsilon_Q \cdot \omega_{\partial''}(\Upsilon_{C''}(v_1 \otimes \dots \otimes v_{p''}) \wedge \alpha_Q) \cdot (m'u \wedge_i m'_i\alpha_{\bar{Q}}) \wedge m_{\partial'}y) \\ &= \omega'((m'u \wedge_i [\sum_Q (-1)^{|Q|} \varepsilon_Q \cdot \omega_{\partial''}(\Upsilon_{C''}(v_1 \otimes \dots \otimes v_{p''}) \wedge \alpha_Q)] m'_i\alpha_{\bar{Q}}) \wedge m_{\partial'}y) \\ &= \omega'((m'u \wedge_i m'_i(\Upsilon_{C''}(v_1 \otimes \dots \otimes v_{p''}))) \wedge m'_{\partial}y) \\ &= \omega_{\partial'}(\Upsilon_{C'}(u_1 \otimes \dots \otimes \Upsilon_{C''}(\otimes_1^{p''} v_l) \otimes \dots \otimes u_{p'}) \wedge y). \end{aligned}$$

□

### 3.2.1 Action of ball cobordisms on ribbon tangles

Given a ball cobordism  $C$  and a collection of ribbon tangles  $T_1, \dots, T_p$ , one may create a new ribbon tangle, if the number of boundary components  $n(T_i)$  of  $T_i$  is equal to  $n_i$ , for all  $i = 1, \dots, p$ , by gluing each  $T_i$  into the internal ball  $B_i$  of  $C$ . The action of  $G$ -colored ball cobordisms on  $G$ -colored ribbon tangles is defined once the colorings coincide on the boundary components. The following theorem states that the invariant  $A$  respects the structure of circuit algebra  $\mathbf{Cob}_G$ .

**Theorem 3.2.10.** *Let  $(T, \psi)$  be the  $G$ -colored ribbon tangle obtained by gluing the  $G$ -colored ribbon tangles  $(T_1, \varphi_1), \dots, (T_p, \varphi_p)$  to a  $G$ -colored ball cobordism  $(C, \varphi)$ . The following equality holds*

$$A(T, \psi) = \Upsilon_{C, \varphi}(A(T_1, \varphi_1) \otimes \dots \otimes A(T_p, \varphi_p)) \in \wedge^n H_{\partial}.$$

*Proof.* For  $i = 1, \dots, p$ , consider a presentation of  $H_{T_i} = H_1^{\varphi_i}(X_{T_i}, *_i)$  of the form

$$H_{T_i} = \langle m_{\partial_i}\gamma_1^i, \dots, m_{\partial_i}\gamma_{2n_i}^i, \beta_1^i, \dots, \beta_{k_i}^i \mid \rho_1^i, \dots, \rho_{s_i}^i \rangle$$

and let  $\mathcal{A}_{T_i}^{\varphi_i}$  be the Alexander function related to this presentation. Let  $H_C = H_1^{\varphi}(X_C, J)$  be the free module, with volume form  $\omega_C^{\varphi}$  associated to the presentation

$$H_C = \langle m_1\gamma_1^1, \dots, m_1\gamma_{2n_1}^1, \dots, m_p\gamma_1^p, \dots, m_p\gamma_{2n_p}^p, \alpha_1, \dots, \alpha_l \mid r_1, \dots, r_t \rangle.$$

By successive Mayer-Vietoris arguments, the module  $H_T = H_1^{\varphi}(X_T, *)$  admits a presentation with generators of the form  $m_{\partial_i}\gamma_j^i$ ,  $m_i\gamma_j^i$ , for  $j = 1, \dots, 2n_i$  and  $i = 1, \dots, p$ , and  $\beta_1^i, \dots, \beta_{k_i}^i, \alpha_1, \dots, \alpha_l$ . They are subject to the relations  $m_{\partial_i}\gamma_j^i - m_i\gamma_j^i$ , for  $j = 1, \dots, 2n_i$  and  $i = 1, \dots, p$ , and  $\rho_1^i, \dots, \rho_{s_i}^i$  for  $i = 1, \dots, p$  and  $r_1, \dots, r_t$ . One has  $A(T, \psi) = \Upsilon_{C, \varphi}(A(T_1, \varphi_1) \otimes \dots \otimes A(T_p, \varphi_p))$  if and only if

$$\forall z \in {}^n H_{\partial}, A(T, \psi) \wedge z = \Upsilon_{C, \varphi}(A(T_1, \varphi_1) \otimes \dots \otimes A(T_p, \varphi_p)) \wedge z$$

By Equation (3.1),  $\omega_{\partial}(A(T, \psi) \wedge z) = \mathcal{A}_T^{\psi}(m_{\partial}z)$ . For short, we write  $A(T_i)$  for  $A(T_i, \varphi_i)$ , for  $i = 1, \dots, p$ . By definition of the Alexander function  $\mathcal{A}_T^{\psi}$ ,

$$\begin{aligned} \mathcal{A}_T^{\psi}(m_{\partial}z) \cdot m_{\partial_*}\gamma \wedge m_*\gamma \wedge \beta \wedge \alpha &= \rho \wedge r \wedge (m_{\partial_i}^i - m_i\gamma^i) \wedge m_{\partial}z \\ &= \sum_{Q_1, \dots, Q_p} (-1)^{|Q|} \varepsilon_Q \cdot \rho \wedge r \wedge m_{\partial_1}\gamma_{Q_1}^1 \wedge \dots \wedge m_{\partial_p}\gamma_{Q_p}^p \wedge m_1\gamma_{\bar{Q}_1}^1 \wedge \dots \wedge m_p\gamma_{\bar{Q}_p}^p \wedge m_{\partial}z, \end{aligned} \quad (3.4)$$

where the sum is taken over the subsets  $Q_i \subset \{1, \dots, 2n_i\}$  of cardinal  $n_i$  (the other terms vanish). We denote by  $\bar{Q}_i$  the complement of  $Q_i$ ,  $|Q| = n_1 + \dots + n_p$  and  $\varepsilon_Q$  the signature of the permutation  $Q_1 \dots Q_p \bar{Q}_1 \dots \bar{Q}_p$ , where the elements of  $Q_1 \dots Q_p$  in increasing order are followed by the elements of  $\bar{Q}_1 \dots \bar{Q}_p$  in increasing order. Moreover, we can decide arbitrarily that the number  $t$  of relations in the presentation of  $H_C$  is even, and get (3.4) to be equal to:

$$\begin{aligned} &\sum_{Q_1, \dots, Q_p} (-1)^{|Q|} \varepsilon_Q \cdot (\rho \wedge m_{\partial_1}\gamma_{Q_1}^1 \wedge \dots \wedge m_{\partial_p}\gamma_{Q_p}^p) \wedge (r \wedge m_1\gamma_{\bar{Q}_1}^1 \wedge \dots \wedge m_p\gamma_{\bar{Q}_p}^p \wedge m_{\partial}z) \\ &= \sum (-1)^{|Q|} \varepsilon_Q \cdot (\rho \wedge m_{\partial_1}\gamma_{Q_1}^1 \wedge \dots \wedge m_{\partial_p}\gamma_{Q_p}^p) \cdot \omega_C^{\varphi}(m_1\gamma_{\bar{Q}_1}^1 \wedge \dots \wedge m_p\gamma_{\bar{Q}_p}^p \wedge m_{\partial}z) \cdot (m_*\gamma \wedge \alpha). \end{aligned}$$

Moreover, since  $s_1, \dots, s_p$  can also be supposed even,

$$\begin{aligned} \rho \wedge m_{\partial_1}\gamma_{Q_1}^1 \wedge \dots \wedge m_{\partial_p}\gamma_{Q_p}^p &= (\rho^1 \wedge m_{\partial_1}\gamma_{Q_1}^1) \wedge \dots \wedge (\rho^p \wedge m_{\partial_p}\gamma_{Q_p}^p) \\ &= A_{T_1}^{\varphi_1}(m_{\partial_1}\gamma_{Q_1}^1) \dots A_{T_p}^{\varphi_p}(m_{\partial_p}\gamma_{Q_p}^p) \cdot m_{\partial_*}\gamma \wedge \beta \\ &= \omega_{\partial_1}(A(T_1) \wedge \gamma_{Q_1}^1) \dots \omega_{\partial_p}(A(T_p) \wedge \gamma_{Q_p}^p) \cdot m_{\partial_*}\gamma \wedge \beta \end{aligned}$$

Then  $\mathcal{A}_T^{\psi}(m_{\partial}z)$  coincides with the sum over  $Q_1 \dots Q_p$  of the summands

$$\begin{aligned} &(-1)^{|Q|} \varepsilon_Q \cdot \omega_{\partial_1}(A(T_1) \wedge \gamma_{Q_1}^1) \dots \omega_{\partial_p}(A(T_p) \wedge \gamma_{Q_p}^p) \cdot \omega_C^{\varphi}(m_{\bar{Q}_1}\gamma_{\bar{Q}_1}^1 \wedge \dots \wedge m_{\bar{Q}_p}\gamma_{\bar{Q}_p}^p \wedge m_{\partial}z) \\ &= \omega_C^{\varphi}(\sum (-1)^{|Q|} \varepsilon_Q \cdot \omega_{\partial_1}(A(T_1) \wedge \gamma_{Q_1}^1) \dots \omega_{\partial_p}(A(T_p) \wedge \gamma_{Q_p}^p) \cdot m_{\bar{Q}_1}\gamma_{\bar{Q}_1}^1 \wedge \dots \wedge m_{\bar{Q}_p}\gamma_{\bar{Q}_p}^p \wedge m_{\partial}z). \end{aligned}$$

Note that for all  $i = 1, \dots, p$  and  $x \in \wedge^{n_i} H_{\partial_i}$ , we have

$$x = \sum_{|Q_i|=n_i} \varepsilon_{\bar{Q}} \omega_{\partial_i}(x \wedge \gamma_{Q_i}^i) \cdot \gamma_{Q_i}^i = \sum_{|Q_i|=n_i} (-1)^{|Q_i|} \varepsilon_{Q_i} \omega_{\partial_i}(x \wedge \gamma_{Q_i}^i) \cdot \gamma_{Q_i}^i.$$

Moreover,  $(-1)^{|Q|} = (-1)^{|Q_1|} \dots (-1)^{|Q_p|}$  and  $\varepsilon_Q \varepsilon_{Q_1} \dots \varepsilon_{Q_p}$  do not depend on  $Q_1, \dots, Q_p$  but only on  $n_1 = |Q_1|, \dots, n_p = |Q_p|$ . Hence, up to a sign

$$\begin{aligned} \mathcal{A}_T^\psi(m_\partial z) &= \omega_C^\varphi(m_1 \mathbf{A}(T_1) \wedge \dots \wedge m_p \mathbf{A}(T_p) \wedge m_\partial z) \\ &= \omega_C^\varphi(m(\mathbf{A}_C(\Upsilon(T_1) \otimes \dots \otimes \mathbf{A}(T_p)) \wedge m_\partial z) \\ &= \omega_\partial(\Upsilon_C(\mathbf{A}(T_1) \otimes \dots \otimes \mathbf{A}(T_p)) \wedge z). \end{aligned}$$

□

### 3.3 A diagrammatic description

We recall from Section 1.5.2 and 2.1.1 that broken surfaces are locally flat immersions in the 3-ball  $B^3$  of disjoint annuli and tori, whose singularities consist of a finite number of circles. We can extend the definitions of broken surfaces for ribbon braids and for ribbon torus-links to ribbon tangles in the natural way. Any ribbon tangle in  $B^4$  can be projected onto a broken surface in a suitable sense (Lemma 1.5.5). These projections can be viewed as a way to represent ribbon tangles similarly to diagrams in usual knot theory. Conversely, every broken surface is the projection of a ribbon tangle. This gives a correspondence between ribbon tangles and broken surfaces.

In the same way welded diagrams have been introduced for braid-like ribbon objects in Section 1.5.1 and for ribbon torus links in Section 2.1.1. We can also consider welded tangle diagrams as a natural extension of the two previous cases. The map *Tube* was defined locally in Definition 1.5.8, and adapts to welded tangle diagrams and ribbon tangles: to each welded tangle one may associate a symmetric broken surface diagram. This defines a map *Tube* sending any welded tangle to the ribbon tangle associated to the symmetric broken surface resulting from the preceding construction.

We recall that the *Tube* map is well defined and surjective, but its injectivity is still an open question [10]. However, we already know from Chapter 1 that there is isomorphisms on ribbon braids and extended ribbon braids.

A *combinatorial* fundamental group can be defined for welded tangle diagrams, with the Wirtinger method (welded crossing are simply ignored). This group coincides with the fundamental group of the complement of an associated ribbon tangle in  $B^4$  [5, 71, 82]. Then, two welded tangle representing the same ribbon tangle have isomorphic fundamental groups (and this isomorphism sends meridian to meridian).

Let  $D$  be the unit disk in  $\mathbb{C}$ , and for any positive integer  $n$ , let  $x_1, \dots, x_{2n}$  be a fixed ordered set of points in  $\partial D$ .

**Definition 3.3.1.** Let  $n$  be a positive integer. A *welded tangle* on  $n$ -strands, or *welded  $n$ -tangle* is a proper immersion  $\tau$  of an oriented 1-manifold in  $D$ . It consists of some

copies of the circle and  $n$  copies of the unit interval whose boundary are  $\{x_1, \dots, x_{2n}\}$ . The singular set of  $\tau$  is a finite number of transversal double points decorated with the following information: they can be *classical positive*, *classical negative* or *welded* as in Figure 1.3.

Let  $\mu$  be a positive integer. A  $\mu$ -colored welded tangle is a pair  $(\tau, \psi)$  where  $\tau$  is a welded tangle and  $\psi$  is a map from the set of strands and circles to the set  $\{t_1, \dots, t_\mu\}$ . Two welded colored tangles are equivalent if they are related by generalized Reidemeister moves (see Definition 2.1.5), respecting the coloring.

### 3.3.1 Computation of the Alexander invariant A

Let  $G$  be the free abelian group generated by  $t_1, \dots, t_\mu$ , and  $R = \mathbb{Z}G$ . Let  $(\tau, \psi)$  be a  $\mu$ -colored welded tangle. It decomposes into a finite union of disjoint oriented arcs. Label the crossings with (formal) letters. Then label each arc with the same letter as the crossing it begins at. If an arc connects points on the boundary of  $\tau$  without meeting any crossing, we use the convention of Figure 3.2. We construct a matrix  $M^\psi(\tau)$  with coefficients in  $R$  where the rows are indexed by crossings (positive, negative and welded) and points interrupting arcs, and the columns by the arcs.

- Fill row corresponding to each positive and negative crossing as shown in Figure 3.1,

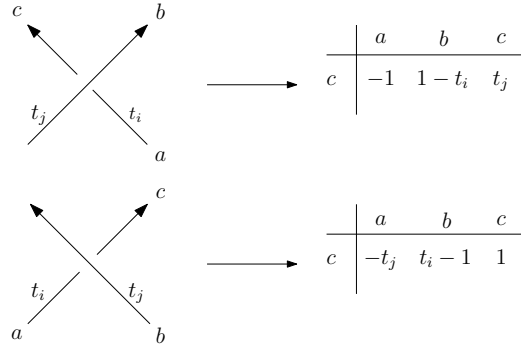


Figure 3.1: The rule to fill the matrix  $M^\psi(T)$ , where  $t_i$  and  $t_j$  are not necessarily different. If  $b = a$  or  $b = c$  we add the contributions.

- At each point on the diagram, fill the row as shown in Figure 3.2.

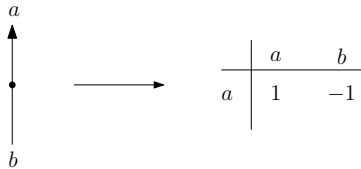


Figure 3.2: Rule for arcs that don't begin at crossings.

The other entries of the rows are zero. Welded crossings can be ignored or considered as divided arcs. Notice that, after some Reidemeister moves of type I, one might suppose that every arc begins at a crossing, and the receipt of Figure 3.2 becomes useless to construct the matrix  $M^\psi(\tau)$ .

*Remark 3.3.2.* Let  $p$  be the number of internal arcs of  $\tau$ . These are the uninterrupted arcs whose extremities are boundary components of the welded tangle. Since  $\tau$  has  $2n$  arcs connected to the boundary, the total number of arcs is  $2n + p$ . One easily observes that the matrix  $M^\psi(\tau)$  has size  $(p + n) \times (p + 2n)$ .

**Definition 3.3.3.** Let  $(\tau, \psi)$  be a  $\mu$ -colored welded  $n$ -tangle and  $H_\partial$  be the module of rank  $2n$  freely generated by the set of marked points  $\{x_1, \dots, x_{2n}\}$ . The invariant  $\alpha$  is defined to be

$$\alpha(\tau, \psi) = \sum_I |M^\psi(\tau)_I| \cdot x_I \in \wedge^n H_\partial,$$

where the sum is taken for all subset  $I \subset \{1, \dots, 2n\}$  of  $n$  elements,  $|M^\psi(\tau)_I|$  is the determinant of the  $(n + p)$ -minor of  $M^\psi(\tau)$  corresponding to the columns indexed by the internal arcs and the columns relative to the arcs indexed by  $I$ , and  $x_I$  is the wedge product of the generators  $x_i$  with  $i \in I$ .

A computation shows that  $\alpha(\tau, \psi)$  is invariant by generalized Reidemeister moves, up to multiplication by a unit. Otherwise the invariance is simply a consequence of Theorem 3.3.6 below.

**Example 3.3.4.** Consider the welded tangle  $\tau$  given by one positive crossing, see Figure 1.3, a). The matrix  $M^\psi(\tau)$  coincides with the matrix of Example 3.1.4. The module  $H_\partial$  is generated by  $x_1, \dots, x_4$  and

$$\alpha(\tau, \psi) = t_2 x_3 \wedge x_4 + x_2 \wedge x_3 - t_2 x_1 \wedge x_4 + (t_1 - 1) x_1 \wedge x_3 + x_1 \wedge x_2 \in \wedge^2 H_\partial.$$

Let  $\tau$  be a welded tangle. We denote by  $\pi(\tau)$  the group defined by the Wirtinger method (ignoring the welded crossings). Then, there is a system of generators of  $\pi(\tau)$  in one-to-one correspondance with the arcs of  $\tau$ . In particular, a  $\mu$ -coloring of a welded tangle  $\tau$  can be viewed as a group homomorphism  $\psi$  from  $\pi(\tau)$  to the abelian group freely generated by  $t_1, \dots, t_\mu$ .

The following proposition follows directly from the results of Satoh and Yajima [71, 82].

**Proposition 3.3.5.** *Let  $\tau$  be a welded tangle. For any ribbon tangle  $T$  such that  $T$  is the image of  $\tau$  by the Tube map, there is an isomorphism*

$$\pi(\tau) \simeq \pi_1(B^4 \setminus T)$$

*sending arcs of  $\tau$  to meridians of  $T$ .*

**Theorem 3.3.6.** *Let  $\mu$  be a positive integer and  $G$  be a free abelian group of rank  $\mu$ . Let  $(\tau, \psi)$  be a  $\mu$ -welded tangle and  $(T, \varphi)$  be a  $G$ -colored ribbon tangle, such that  $T$  is the image of  $\tau$  by the Tube map. Suppose that there are generators  $t_1, \dots, t_\mu$  of  $G$  such that the following diagram commutes:*



$$\begin{array}{ccc}
\pi_1(B^4 \setminus T) & \xrightarrow{\varphi} & G \\
\cong \downarrow & & \downarrow \cong \\
\pi(\tau) & \xrightarrow{\psi} & \langle t_1, \dots, t_\mu \rangle
\end{array}$$

Then,

$$A(T, \varphi) = \alpha(\tau, \psi) \in \wedge^n H_\partial.$$

*Proof.* The points of  $\tau \cap \partial B^2$  are in one-to-one correspondence with the component of the trivial link  $L$  in  $T \cap \partial B^4$ , and  $H_\partial$  is generated by  $x_1, \dots, x_{2n}$ . By Proposition 3.3.5 and Fox calculus, the matrix  $M^\psi(\tau)$  is a presentation matrix of the  $R$ -module  $H_1^\varphi(X_T, *)$ , viewed as a  $\mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$ -module through the choice of generators of  $G$ . Then  $M^\psi(\tau)$  is used to compute  $\mathcal{A}^\varphi$ . By definition, for all  $I \subset \{1, \dots, 2n\}$  with cardinal  $n$ :

$$\omega_\partial(A(T, \varphi) \wedge x_I) = \mathcal{A}^\varphi(m_\partial x_I).$$

To calculate  $\mathcal{A}^\varphi(m_\partial x_I)$ , we consider the matrix  $M^\psi(\tau)$ , add  $n$  row vectors giving the element  $mx_{i_1}, \dots, mx_{i_n}$  and compute the determinant of the resulting square matrix. Hence adding  $mx_{i_j}$  corresponds to add the  $p + n + j^{\text{th}}$  row  $(0, \dots, 0, 1, 0, \dots, 0)$  where 1 is at position  $p + j$ . We obtain  $\mathcal{A}^\varphi(m_\partial x_I) = \varepsilon_{\bar{I}} |M^\psi(\tau)_{\bar{I}}|$ , where  $\bar{I}$  is the complement of  $I$  and  $\varepsilon_{\bar{I}}$  is the signature of the permutation  $I\bar{I}$  (where the elements of  $I$  in increasing order are followed by the elements of  $\bar{I}$  in increasing order). Finally, we get

$$A(T, \varphi) = \sum_I \varepsilon_{\bar{I}} \cdot \omega_\partial(A(T, \varphi) \wedge x_I) \cdot x_{\bar{I}} = \sum_I |M^\psi(\tau)_{\bar{I}}| \cdot x_{\bar{I}} = \alpha(\tau, \psi).$$

□

*Remark 3.3.7.* *Welded string links* are string links that can have welded crossings; for details on these last ones see [57]. Through the Tube map, ribbon tubes can be described by welded string links, see [5, Section 3.3]. By Theorem 3.1.19, Proposition 3.1.17 and Theorem 3.3.6, the invariant  $\alpha$  induces the usual (generalisation of) colored Burau representation - or Gassner, if the coloring is maximal - on (welded) string links [14].

*Remark 3.3.8.* The invariant  $\alpha$  (Definition 3.3.3) coincides up to a unit with the invariant of virtual tangles introduced by Archibald [2]. It is worth mentioning that using a specific canonical choice for the marked points  $x_i$ , her construction is well-defined, not only up to multiplication by a unit (through the multiplication by a correction term that keeps track of the order), up to generalized Reidemeister moves with the exception of moves of type I (see Figure 2.1).

*Proof.* of Proposition 3.1.2. As observed in the proof of Theorem 3.3.6, the matrix  $M^\psi(\tau)$  is a presentation matrix of  $H_1^\varphi(X_T, *)$ , whose size is  $(p + n) \times (p + 2n)$ , see Remark 3.3.2. We get a presentation of deficiency  $n$ . □

### 3.3.2 The circuit algebra $\text{Weld}_\mu$

**Definition 3.3.9.** A *circuit diagram* consists of the following data:

- the unit disk  $D = D_0$  in  $\mathbb{C}$  together with a finite set of disjoint subdisks  $D_1, \dots, D_p$  in the interior of  $D$ . For every  $i$  in  $\{0, \dots, p\}$ , each  $D_i$  have  $2n_i$  distinct marked points with sign on its boundary (with  $n = n_0$ ), and a base point  $*$  on the boundary of each disk.
- a finite set of embedded oriented arcs whose boundary are marked points in the  $D_i$ . They may cross each other along welded crossings *only*. Each marked point is the boundary point of some string -which meets the corresponding disk transversally- and the sign is coherent with the orientation.

We consider circuit diagrams up to equivalence given by the fact that they only encode the matching of marked points, and not the exact position of the arcs, as shown in Figure 3.3.

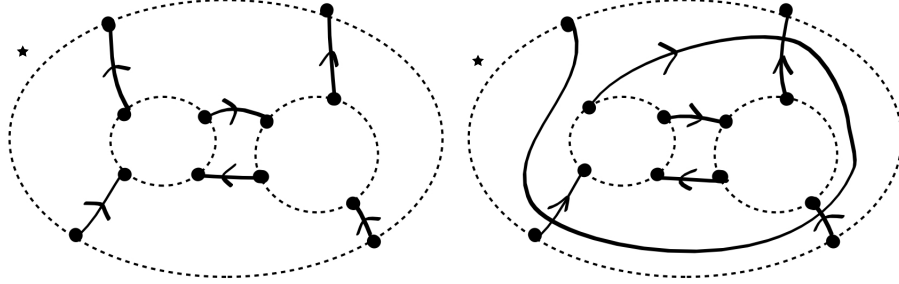


Figure 3.3: Two equivalent circuit diagrams.

**Definition 3.3.10.** Consider a circuit  $p'$ -diagram  $P'$  and a circuit  $p''$ -diagram  $P''$  such that  $D'_i$  is a disk of  $P'$  with  $n'_i = n''$ , for some  $i \in \{1, \dots, p'\}$ . If the signs of the marked points match, we define the diagram  $P = P' \circ_i P''$  by rescaling via isotopy the diagram  $P''$  so that the boundary of  $D''$  is identified with the boundary of  $D'_i$ , and making its marked and base points coincide to those of  $D'_i$ . Then  $D'_i$  is removed to obtain  $P' \circ_i P''$ . This operation is well defined since the starting points eliminate any rotational ambiguity.

A  $\mu$ -colored circuit diagram is a pair  $(P, \psi)$  where  $P$  is a circuit diagram and  $\psi$  is a map from the set of arcs of  $P$  to the set  $\{t_1, \dots, t_\mu\}$ . Two  $\mu$ -colored circuit diagrams can be composed once the coloring match on the boundary components.

**Definition 3.3.11.** The circuit operad of circuit diagrams  $\mathcal{D}_\mu$  is the circuit operad of equivalence classes  $\mu$ -colored circuit diagrams.

Let  $S = \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$  be the Laurent polynomial ring. Let  $\mathcal{H}om_\mu$  be the circuit operad of tensor products of  $S$ -modules and  $S$ -linear maps. The circuit algebra  $\text{Weld}_\mu$  is constructed as a morphism from the circuit operad  $\mathcal{D}_\mu$  to  $\mathcal{H}om_\mu$  as follows.

Consider the unit circle with a base point and a set of marked points  $X = \{x_1, \dots, x_{2k}\}$ , for  $k \geq 0$  (with a sign). To this data, we associate the module  $\wedge^k H_\partial$ , where  $H_\partial$  is the free  $S$ -module of rank  $2k$  generated by  $X$ . Let  $(P, \psi)$  be a  $\mu$ -colored circuit diagram, and  $M = \{c_1, \dots, c_q\}$  be the set of curves of  $P$ . Consider the free module  $H$  generated by  $M$ , and the volume form  $\omega: \wedge^q H \rightarrow S$  related to this basis. For  $i = 1, \dots, p$ , denote by  $H_{\partial_i}$  the module associated to the boundary circle  $\partial D_i$  and  $H_\partial$  the module associated to  $\partial D$ . Let  $m_i: H_{\partial_i} \rightarrow H$  be the morphisms defined by  $m_i(x_j) = \text{sign}(x_j)c_j$  if  $x_j \in \partial c_j$ . The morphism  $m_\partial$  is defined similarly. Let  $\omega_{\partial_i}$  be the volume form on  $H_{\partial_i}$  related to the generating system of points of the circle  $\partial D_i$ . Set  $m = \otimes_i (\wedge^{n_i} m_{\partial_i})$ . To the colored diagram  $(P, \psi)$  we associate

$$\gamma_{P,\psi}: \bigotimes_{i=1}^p \wedge^{n_i} H_{\partial_i} \longrightarrow \wedge^n H_\partial$$

such that, for  $x \in \otimes_i (\wedge^{n_i} H_{\partial_i})$ ,

$$\omega_G^\varphi(m(x) \wedge m_\partial(y)) = \omega_\partial(\gamma_{P,\psi}(x) \wedge y), \quad \forall y \in \wedge^n H_\partial.$$

Then, we can prove Proposition 3.3.12 below, by repeating the arguments of the proof of Theorem 3.2.9.

**Proposition 3.3.12.**  *$\text{Weld}_\mu$  is a circuit algebra.*

Note that  $\text{Weld}_\mu$  is similar to half densities introduced by Archibald, see [2] for the definition of half densities.

The morphism  $\gamma_{P,\psi}$  could be written as the interior product relative to a subset corresponding to interior arcs of  $P$ .

The Tube map and the choice of a set of generators  $\{t_1, \dots, t_\mu\}$  of  $G$  induce a surjective morphism of algebras

$$\text{Weld}_\mu \longrightarrow \text{Cob}_G.$$

Given a colored circuit diagram  $P$  and a collection of colored welded tangle diagrams  $\tau_1, \dots, \tau_p$ , one may create a new tangle, if the data on the boundaries and the colorings match, by gluing  $\tau_i$  into the internal disk  $D_i$  of  $P$ . Similarly to Theorem 3.2.10, the invariant  $\alpha$  commutes with this action of  $\mu$ -colored circuit diagrams on  $\mu$ -colored welded tangles. Indeed, one has the following proposition.

**Proposition 3.3.13.** *Let  $(\tau, \psi)$  be the  $\mu$ -colored welded tangle obtained by gluing the  $\mu$ -colored welded tangles  $(\tau_1, \psi_1), \dots, (\tau_p, \psi_p)$  to a  $\mu$ -colored circuit diagram  $(P, \chi)$ . The following equality holds*

$$\alpha(\tau, \psi) = \gamma_{P,\chi}(\alpha(\tau_1, \psi_1) \otimes \dots \otimes \alpha(\tau_p, \psi_p)) \in \wedge^n H_\partial.$$

### 3.4 Examples

Consider the welded tangle diagram  $\tau$ , given in Figure 3.4. We let  $G = \mathbb{Z} = \langle t \rangle$  and  $\psi$  be the coloring sending all arcs of  $\tau$  to  $t$ . In this section, we compute  $\alpha(\tau, \psi)$  in different ways. By Theorem 3.3.6, this computes the value of  $A(T, \varphi)$  the image of  $(\tau, \psi)$  by the Tube map.

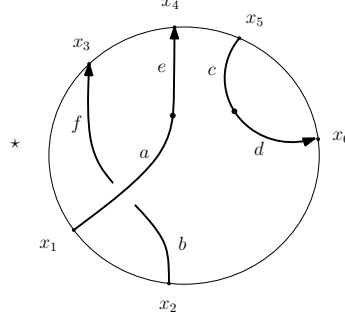


Figure 3.4: A welded tangle  $\tau$ .

First, we compute  $\alpha(\tau, \psi)$  directly. Label the arcs of  $\tau$  with letters  $a$  to  $f$  as in Figure 3.4. We obtain the matrix

$$M^\psi(\tau) = \begin{pmatrix} & a & b & c & d & e & f \\ \begin{pmatrix} 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 1-t & -1 & 0 & 0 & 0 & t \end{pmatrix} \end{pmatrix}$$

The  $\mathbb{Z}[t^{\pm 1}]$ -module  $H_\partial$  is free, generated by  $x_1, \dots, x_6$  and  $\alpha(\tau, \psi) \in \wedge^3 H_\partial$  is given by

$$\begin{aligned} \alpha(\tau, \psi) = & -x_1 \wedge x_2 \wedge x_5 + x_1 \wedge x_2 \wedge x_6 + x_2 \wedge x_5 \wedge x_4 + (t-1)x_1 \wedge x_5 \wedge x_4 - tx_1 \wedge x_5 \wedge x_3 \\ & + (1-t)x_1 \wedge x_6 \wedge x_4 + tx_1 \wedge x_6 \wedge x_3 + tx_6 \wedge x_4 \wedge x_3 - x_2 \wedge x_6 \wedge x_4 - tx_5 \wedge x_4 \wedge x_3. \end{aligned}$$

We now consider  $(\tau, \psi)$  as the composition of the circuit diagram  $(P, \psi)$  with  $(\sigma, \psi)$ , see Figure 3.5. We have to compute  $\gamma_{P, \psi}: \wedge^2 H_{\partial_1} \rightarrow \wedge^3 H_\partial$  (here  $p = 1$ ). Let  $H$

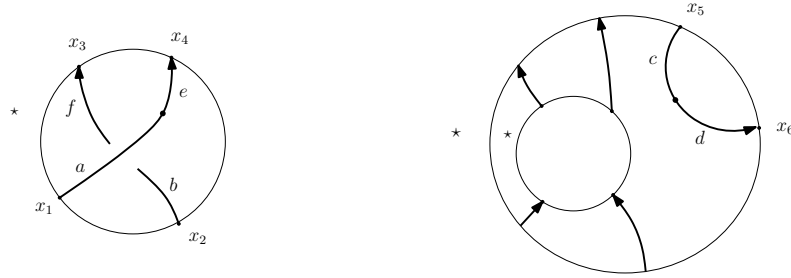


Figure 3.5: A welded tangle  $\sigma$  and a circuit diagram  $P$ .

be the free module generated by the curves of  $\sigma$ , labelled  $a, b, e, f$ . We have that

$H_{\partial_1} = \langle x_1, \dots, x_4 \rangle$  and  $H_{\partial} = \langle x_1, \dots, x_6 \rangle$ . Using the volume form on  $H$  related to the choice of the basis  $a, b, c, e, d, f$ , and the maps induced by the inclusions  $m_1: H_{\partial_1} \rightarrow H$  and  $m_{\partial}: H_{\partial} \rightarrow H$ , we obtain

$$\gamma_{C,\psi}(x_i \wedge x_j) = x_i \wedge x_j \wedge (x_6 - x_5), \quad \forall i, j = 1, \dots, 4,$$

and

$$M^\psi(\sigma) = \begin{pmatrix} a & b & e & f \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1-t & t \end{pmatrix}$$

We get

$$\alpha(\sigma, \psi) = x_1 \wedge x_2 + (t-1)x_1 \wedge x_4 - tx_1 \wedge x_3 + x_2 \wedge x_4 - tx_3 \wedge x_4.$$

The composition  $\alpha(\tau, \psi) = \gamma_{P,\psi}(\alpha(\sigma, \psi))$  gives the result. Finally, we consider  $\tau$  as the composition of the welded tangle  $\sigma \otimes \beta$  with the circuit  $Q$ , see Figure 3.6. Here  $p = 2$ ,

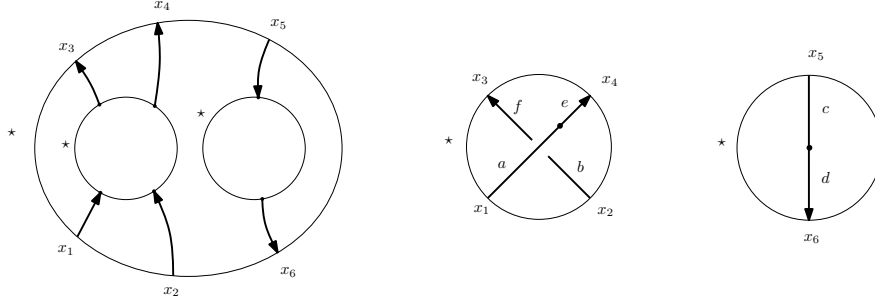


Figure 3.6: A circuit diagram  $Q$  representing a ball cobordism and two welded tangles  $\sigma$  and  $\beta$ .

and

$$\gamma_{Q,\psi}: \wedge^2 H_{\partial_1} \otimes H_{\partial_2} \longrightarrow \wedge^3 H_{\partial}.$$

As previously  $\alpha(\sigma, \psi) = x_1 \wedge x_2 + (t-1)x_1 \wedge x_4 - tx_1 \wedge x_3 + x_2 \wedge x_4 - tx_3 \wedge x_4 \in \wedge^2 H_{\partial_1}$  and  $\gamma(\beta) = x_6 - x_5 \in H_{\partial_2}$ . The composition  $\alpha(\tau, \psi) = \gamma_{Q,\psi}(\alpha(\sigma, \psi) \otimes \alpha(\beta, \psi))$  gives the result again.

*Remark 3.4.1.* In the sense of Subsection 3.1.4, there is a splitting of the welded tangle  $\sigma$  to an (oriented) braid  $\sigma_1$  in  $B_2$ . We have

$$M_0 = \langle x_1, x_2 \rangle \text{ and } M_1 = \langle x_3, x_4 \rangle,$$

and  $\alpha(\sigma, \psi) \in \wedge^2 H_{\partial_1} \simeq \wedge^2(M_0 \oplus M_1) \simeq \bigoplus_{k=0}^2 \wedge^k M_0 \otimes \wedge^{2-k} M_1$ , similarly to the proof of Theorem 3.1.19. Then  $\alpha(\sigma, \psi)$  decomposes as:

$$(-tx_3 \wedge x_4) \oplus ((t-1)x_1 \otimes x_4 - tx_1 \otimes x_3 + x_2 \otimes x_4) \oplus (x_1 \wedge x_2).$$

Let  $\omega_0: \Lambda^2 M_0 \rightarrow R$  be the volume form related to the basis  $\langle x_1, x_2 \rangle$ . For  $k = 1$ , the element  $(t - 1)x_1 \otimes x_4 - tx_1 \otimes x_3 + x_2 \otimes x_4 \in M_0 \otimes M_1$  induces the morphism  $M_0 \rightarrow M_1$ :

$$x \mapsto (t - 1)\omega_0(x \wedge x_1) \cdot x_4 - t\omega_0(x \wedge x_1) \cdot x_3 + \omega_0(x \wedge x_2) \cdot x_4.$$

The image of  $x_1$  is  $x_4$  and the image of  $x_2$  is  $(1 - t)x_4 + tx_3$ . This corresponds to the Burau representation. Similarly, the other values of  $k$  give  $k^{\text{th}}$ -exterior powers of Burau, up to a sign.

### 3.5 Fixing the multiplicative unit indeterminacy

Theorem 3.3.6 allows us to compute the Alexander invariant  $A$  of a ribbon tangle  $T$  by computing the  $\alpha$  invariant of a welded tangle  $\tau$  whose image through the *Tube* map is  $T$ . The Alexander invariant is defined up to a multiplicative unit of  $\mathbb{Z}[G]$  (Section 3.1). This is due to the fact that it depends on the choice of a presentation matrix to compute the Alexander function (Definition 3.1.3).

A first step to eliminate the ambiguity is to compute the invariant  $\alpha$  using a chosen form for the presentation matrix, which is the one constructed in Subsection 3.3.1. However an ambiguity still remains, as the computation of  $\alpha$  depends on the choice of the order given by the labelling, that changes the order of the columns and the rows of the matrix.

As mentioned in Remark 3.3.8, Archibald in [2] defines a stronger form of  $\alpha$ , which we translate in our formalism.

**Definition 3.5.1.** Let  $(\tau, \psi)$  be a  $\mu$ -colored welded  $n$ -tangle and  $H_\partial$  be the module of rank  $2n$  freely generated by the set of ordered marked points  $\{x_1, \dots, x_{2n}\}$ . Let  $\tilde{M}^\psi(\tau)$  the matrix constructed as in Subsection 3.3.1, where the additional rule:

- order of the rows of the matrix in concordance with the order of the arcs whose orientation goes towards the exterior of the diagram ("outgoing arcs").

The invariant  $\tilde{\alpha}$  is defined to be

$$\tilde{\alpha}(\tau, \psi) = \prod_{\mu} t_{\mu}^{\frac{s(\mu)}{2}} \sum_I |M^\psi(\tau)_I| \cdot x_I \in \wedge^n H_\partial,$$

where  $s(\mu)$  is the number of times that the component colored with  $t_{\mu}$  is the over strand in a crossing, and everything else is as in Definition 3.3.3.

Then we have that the invariant  $\tilde{\alpha}$  is well defined for welded tangles under generalized Reidemeister equivalence with the exception of moves of type I (Figure 2.1).

## Chapter 4

# Unrestricted virtual braids and fused links

In this chapter we present a work with Valeriy G. Bardakov and Paolo Bellingeri [16]<sup>1</sup>. We introduce certain remarkable quotients of loop braid groups, and their relation with *fused links*. Fused links were introduced by L. H. Kauffman and S. Lambropoulou in [52]. Afterwards, the same authors introduced their “braided” counterpart: the *unrestricted virtual braids*, and extended S. Kamada’s work [47] (already mentioned in Chapter 2) by presenting a version of Alexander’s and Markov’s theorems for these objects [53]. In the *groups of unrestricted virtual braids*, denoted by  $UVB_n$ , we consider strand diagrams (Definition 1.5.1) where the equivalence relation is given by the welded Reidemeister equivalence from Definition 1.5.2 with the addition of one more *forbidden move* (F2) (see Figure 4.1).

The group  $UVB_n$  appears also in [45], where it is called *symmetric loop braid group*  $SLB_n$ , being isomorphic to a quotient of the loop braid group  $LB_n$  that we presented in Chapter 1.

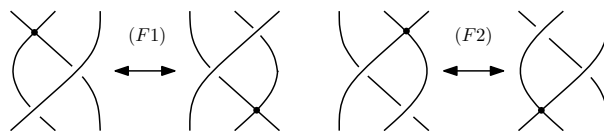


Figure 4.1: Welded Reidemeister moves, in the following called forbidden moves of type (F1) (on the left) and forbidden moves of type (F2) (on the right).

It has been shown that all fused knots are equivalent to the unknot ([48,67]). Moreover, Nelson’s proof in [67] of the fact that every virtual knot unknots when allowing forbidden moves, can be adapted verbatim to links with several components. We remark that to achieve this result Nelson passes to Gauss diagrams for fused knots. So, every fused link diagram is fused isotopic to a link diagram where the only crossings (classical or virtual) are the ones involving different components.

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<sup>1</sup>Sections 5.1 and 5.3 will not be discussed since they are not pertinent with the rest of this thesis.

On the other hand, there are non-trivial fused links and their classification is not (completely) trivial [32]. In particular in [31], A. Fish and E. Keyman proved that fused links that have only classical crossings are characterized by their (classical) linking numbers. However, this result does not generalize to links with virtual crossings. In fact it is easy to find non-equivalent fused links with the same (classical) linking number (see Remark 4.2.6). This answers a question from [31, Remark 1], where Fish and Keyman ask whether the classical linking number is a complete invariant for fused links. We describe unrestricted virtual braids and compare more or less known invariants for fused links.

In Section 4.1 we give a description of the structure of the groups of unrestricted virtual braids  $UVB_n$  (Theorems 4.1.4 and 4.1.7), answering a question of Kauffman and Lambropoulou [53]. In Section 4.2 we provide an application of Theorem 4.1.7 showing that any fused link admits as a representative the closure of a *pure* unrestricted virtual braid (Theorem 4.2.3). As a corollary we deduce an easy proof of the main theorem of [31]. In Section 4.3 we construct a representation for  $UVB_n$  in  $\text{Aut}(N_n)$ , the groups of automorphisms of the free 2-step nilpotent group of rank  $n$  (Proposition 4.3.4). Using this representation we define a notion of *group of fused links* and we compare this invariant to other known invariants (Proposition 4.3.9 and Remark 4.3.10). Finally, in Section 4.4 we describe the structure of the flat welded braid groups (Proposition 4.4.2).

## 4.1 Unrestricted virtual braid groups

In Chapter 1 we introduced the virtual braid groups as the groups of isotopy classes of strand diagrams under virtual Reidemeister equivalence (Definition 1.6.3). In an equivalent way we give a definition for the virtual braid groups as abstract presented groups. Once more generators  $\sigma_i$  and  $\rho_i$  correspond to elementary diagrams from Figure 1.4.

**Definition 4.1.1.** For  $n \geq 1$ , the *virtual braid group*, denoted by  $VB_n$ , is the group defined by the group presentation

$$\langle \{ \sigma_i, \rho_i \mid i = 1, \dots, n-1 \} \mid R \rangle$$

where  $R$  is the set of relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \text{for } i = 1, \dots, n-2; \quad (\text{R1})$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{for } |i-j| \geq 2; \quad (\text{R2})$$

$$\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}, \quad \text{for } i = 1, \dots, n-2; \quad (\text{R3})$$

$$\rho_i \rho_j = \rho_j \rho_i, \quad \text{for } |i-j| \geq 2; \quad (\text{R4})$$

$$\rho_i^2 = 1, \quad \text{for } i = 1, \dots, n-1; \quad (\text{R5})$$

$$\sigma_i \rho_j = \rho_j \sigma_i, \quad \text{for } |i-j| \geq 2; \quad (\text{R6})$$

$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}, \quad \text{for } i = 1, \dots, n-2. \quad (\text{M})$$



Fixing  $n \geq 1$ , the *virtual pure braid group* on  $n$  strands, denoted by  $VP_n$ , is the kernel of the map  $VB_n \rightarrow S_n$  sending generators  $\sigma_i$  and  $\rho_i$  to  $(i, i+1)$ , for every  $i = 1, 2, \dots, n-1$ . A presentation for  $VP_n$  is given in [13]; it will be recalled in the proof of Theorem 4.1.7.

As already noted in another context (see Remark 1.6.4), the welded braid groups  $WB_n$  are quotients of  $VB_n$  by the normal subgroups generated by relations

$$\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, \quad \text{for } i = 1, \dots, n-2. \quad (\text{F1})$$

*Remark 4.1.2.* We will see in Section 4.2 that the symmetrical relations

$$\rho_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \rho_i, \quad \text{for } i = 1, \dots, n-2 \quad (\text{F2})$$

do not hold in  $WB_n$ .

**Definition 4.1.3.** For  $n \geq 1$ , the *group of unrestricted virtual braids*, denoted by  $UVB_n$ , is the group defined by the group presentation

$$\langle \{\sigma_i, \rho_i \mid i = 1, \dots, n-1\} \mid R' \rangle$$

where  $R'$  is the set of relations (R1), (R2), (R3), (R4), (R5), (R6), (M), (F1), (F2).

The main result of this section is to prove that  $UVB_n$  can be described as a semi-direct product of a right-angled Artin group and the symmetric group  $S_n$ . This way we answer a question posed in [53] about the (non-trivial) structure of  $UVB_n$ .

**Theorem 4.1.4.** For  $n \geq 1$ , let  $X_n$  be the right-angled Artin group generated by  $x_{i,j}$ , for  $1 \leq i \neq j \leq n$ , where all generators commute except the pairs  $x_{i,j}$  and  $x_{j,i}$  for  $1 \leq i \neq j \leq n$ . The group  $UVB_n$  is isomorphic to  $X_n \rtimes S_n$  where  $S_n$  acts by permutation on the indices of generators of  $X_n$ .

Before proving Theorem 4.1.4, we introduce some objects and results that we will use in the proof. Let  $\nu: UVB_n \rightarrow S_n$  be the map defined as follows:

$$\nu(\sigma_i) = \nu(\rho_i) = (i, i+1), \quad \text{for } i = 1, 2, \dots, n-1.$$

We will call the kernel of  $\nu$  *unrestricted virtual pure braid group* and we will denote it by  $UVP_n$ . Since  $\nu$  admits a natural section, we have that  $UVB_n = UVP_n \rtimes S_n$ .

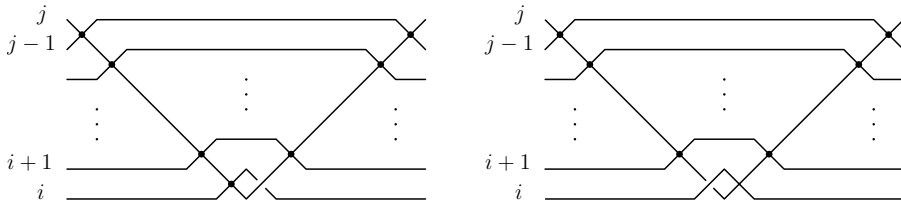


Figure 4.2: Elements  $\lambda_{i,j}$  on the left and  $\lambda_{j,i}$  on the right. Here we adopt the convention of drawing braids from left to right.

For  $i = 1, \dots, n-1$ , we define the following elements of  $UVP_n$  (see Figure 4.2):

$$\begin{aligned}\lambda_{i,i+1} &= \rho_i \sigma_i^{-1}, \\ \lambda_{i+1,i} &= \rho_i \lambda_{i,i+1} \rho_i = \sigma_i^{-1} \rho_i.\end{aligned}\tag{4.1}$$

Then, for  $1 \leq i < j-1 \leq n-1$ :

$$\begin{aligned}\lambda_{i,j} &= \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \lambda_{i,i+1} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1}, \\ \lambda_{j,i} &= \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \lambda_{i+1,i} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1}.\end{aligned}\tag{4.2}$$

The next lemma was proved in [13] for the corresponding elements in  $VB_n$ , and therefore is also true in the quotient  $UVB_n$ .

**Lemma 4.1.5.** *The following conjugating rule is fulfilled in  $UVB_n$ : for all  $1 \leq i \neq j \leq n$  and  $s \in S_n$ ,*

$$\iota(s) \lambda_{i,j} \iota(s)^{-1} = \lambda_{s(i),s(j)}$$

where  $\iota: S_n \rightarrow UVB_n$  is the natural section of the map  $\nu$  defined in Theorem 4.1.4.

**Corollary 4.1.6.** *For  $n \geq 1$ , the symmetric group  $S_n$  acts by conjugation on the set  $\{\lambda_{k,l} \mid 1 \leq k \neq l \leq n\}$ . This action is transitive.*

We prove that the groups generated by  $\{\lambda_{k,l} \mid 1 \leq k \neq l \leq n\}$  coincides with  $UVP_n$ , and then we will find the defining relations. This will show that the groups  $UVP_n$  are a right-angled Artin groups.

**Theorem 4.1.7.** *For  $n \geq 1$ , the group  $UVP_n$  admits a presentation with generators  $\lambda_{k,l}$  for  $1 \leq k \neq l \leq n$ , and defining relations:  $\lambda_{i,j}$  commutes with  $\lambda_{k,l}$  if and only if  $k \neq j$  or  $l \neq i$ . In particular  $UVP_n$  is a right-angled Artin group.*

*Proof.* Since  $UVP_n$  is a finite index subgroup of  $UVB_n$  one can apply the Reidemeister-Schreier method (see, for example, [61, Ch. 2.2]) and check that the given set of relations is complete. Remark that most of the relations were already proven in this way in [13] for the case of the virtual pure braid group  $VP_n$ .

An easier approach is provided by the following commutative diagram:

$$\begin{array}{ccccccc}
 & 1 & & 1 & & & \\
 & \downarrow & & \downarrow & & & \\
 & \ker \pi|_{VP_n} & \longrightarrow & \ker \pi & & & \\
 & \downarrow & & \downarrow & & & \\
 1 & \longrightarrow & VP_n & \longrightarrow & VB_n & \longrightarrow & S_n \longrightarrow 1 \\
 & & \downarrow \pi|_{VP_n} & & \downarrow \pi & & \parallel \\
 1 & \longrightarrow & UVP_n & \longrightarrow & UVB_n & \longrightarrow & S_n \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \\
 & 1 & & 1 & & & 
 \end{array}$$

where  $\pi$  is the canonical projection of  $VB_n$  onto  $UVB_n$  and  $\pi|_{VP_n}$  its restriction to  $VP_n$ . By definition,  $\ker \pi$  is normally generated by the elements  $\sigma_i \sigma_j \rho_i \sigma_j^{-1} \sigma_i^{-1} \rho_j$  for  $|i - j| = 1$  (we will write  $\ker \pi = \langle\langle \sigma_i \sigma_j \rho_i \sigma_j^{-1} \sigma_i^{-1} \rho_j \mid |i - j| = 1 \rangle\rangle$ ). Since  $\sigma_i \sigma_j \rho_i \sigma_j^{-1} \sigma_i^{-1} \rho_j$  belongs to  $VP_n$ , and that  $VP_n$  is normal in  $VB_n$ , we deduce that  $\ker \pi|_{VP_n}$  coincides with  $\ker \pi$ .

As stated in [13],  $VP_n$  is generated by elements (4.1) and (4.2), and has the following set of defining relations:

$$\lambda_{i,j} \lambda_{k,l} = \lambda_{k,l} \lambda_{i,j} \quad (\text{RS1})$$

$$\lambda_{k,i} (\lambda_{k,j} \lambda_{i,j}) = (\lambda_{i,j} \lambda_{k,j}) \lambda_{k,i}. \quad (\text{RS2})$$

Moreover,  $VB_n$  can be seen as a semidirect product  $VP_n \rtimes S_n$ , where the symmetric group  $S_n$  acts by permutations of indices on  $\lambda_{i,j}$ 's (Lemma 4.1.5).

One can easily verify that relators of type (F1), *i.e.*,  $\rho_i \sigma_{i+1} \sigma_i \rho_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1}$ , can be rewritten as:

$$(\rho_i \lambda_{i+1,i+2}^{-1} \rho_i) (\rho_i \rho_{i+1} \lambda_{i,i+1}^{-1} \rho_{i+1} \rho_i) (\rho_{i+1} \lambda_{i,i+1} \rho_{i+1}) \lambda_{i+1,i+2}$$

and using the conjugating rule given above, we get, for  $i = 1, \dots, n-2$ ,

$$\rho_i \sigma_{i+1} \sigma_i \rho_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} = \lambda_{i,i+2}^{-1} \lambda_{i+1,i+2}^{-1} \lambda_{i,i+2} \lambda_{i+1,i+2}.$$

On the other hand one can similarly check that relators of type (F2), which are of the form  $\rho_{i+1} \sigma_i \sigma_{i+1} \rho_i \sigma_{i+1}^{-1} \sigma_i^{-1}$ , can be rewritten as  $\lambda_{i,i+1}^{-1} \lambda_{i,i+2}^{-1} \lambda_{i,i+1} \lambda_{i,i+2}$ .

From these facts and from above description of  $VB_n$  as semidirect product  $VP_n \rtimes S_n$ , it follows that any generator of  $\ker_{VP_n}$  is of the form  $g[\lambda_{i,j}, \lambda_{k,j}]g^{-1}$  or  $g[\lambda_{i,j}, \lambda_{i,k}]g^{-1}$ , for some  $g \in VP_n$  and  $i, j, k$  distinct indices. The group  $UVP_n$  has therefore the following complete set of relations:

$$\lambda_{i,j} \lambda_{k,l} = \lambda_{k,l} \lambda_{i,j} \quad (\text{RS1})$$

$$\lambda_{k,i}(\lambda_{k,j}\lambda_{i,j}) = (\lambda_{i,j}\lambda_{k,j})\lambda_{k,i} \quad (\text{RS2})$$

$$\lambda_{i,j}\lambda_{k,j} = \lambda_{k,j}\lambda_{i,j} \quad (\text{RS3})$$

$$\lambda_{i,j}\lambda_{i,k} = \lambda_{i,k}\lambda_{i,j}. \quad (\text{RS4})$$

Using (RS3) and (RS4) we can rewrite relation (RS2) in the form

$$\lambda_{k,j}(\lambda_{k,i}\lambda_{i,j}) = \lambda_{k,j}(\lambda_{i,j}\lambda_{k,i}). \quad (4.3)$$

After cancelation we have that we can replace relation (RS2) with

$$\lambda_{k,i}\lambda_{i,j} = \lambda_{i,j}\lambda_{k,i} \quad (\text{RS5})$$

This completes the proof.  $\square$

*Proof of Theorem 4.1.4.* The group  $X_n$  is evidently isomorphic to  $UV P_n$  (sending any  $x_{i,j}$  into the corresponding  $\lambda_{i,j}$ ). Recall that  $UV P_n$  is the kernel of the map  $\nu: UV B_n \rightarrow S_n$  defined as  $\nu(\sigma_i) = \nu(\rho_i) = (i, i+1)$  for  $i = 1, \dots, n-1$ . Recall also that  $\nu$  has a natural section  $\iota: S_n \rightarrow UV B_n$ , defined as  $\iota((i, i+1)) = \rho_i$  for  $i = 1, \dots, n-1$ . Therefore  $UV B_n$  is isomorphic to  $UV P_n \rtimes S_n$ , where  $S_n$  acts by permutation on the indices of generators of  $UV P_n$  (Corollary 4.1.6).  $\square$

We recall that, fixed an  $n \geq 1$ , the pure braid group  $P_n$ , first mentioned in Example 1.1.4, is the kernel of the homomorphism from  $B_n$  to the symmetric group  $S_n$  sending every generator  $\sigma_i$  to the permutation  $(i, i+1)$ . It is generated by the set  $\{a_{ij} \mid 1 \leq i < j \leq n\}$ , where

$$a_{i,i+1} = \sigma_i^2, \\ a_{i,j} = \sigma_{j-1}\sigma_{j-2} \cdots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1}\sigma_{j-1}^{-1}, \quad \text{for } i+1 < j \leq n.$$

**Corollary 4.1.8.** *For  $n \geq 1$ , let  $p: P_n \rightarrow UV P_n$  be the canonical map of the pure braid group  $P_n$  in  $UV P_n$ . Then  $p(P_n)$  is isomorphic to the abelianization of  $P_n$ .*

*Proof.* As remarked in [13, page 6], generators  $a_{i,j}$  of  $P_n$  can be written in  $VP_n$  as

$$a_{i,i+1} = \lambda_{i,i+1}^{-1}\lambda_{i+1,i}^{-1}, \quad \text{for } i = 1, \dots, n-1, \\ a_{i,j} = \lambda_{j-1,j}^{-1}\lambda_{j-2,j}^{-1} \cdots \lambda_{i+1,j}^{-1}(\lambda_{i,j}^{-1}\lambda_{j,i}^{-1})\lambda_{i+1,j} \cdots \lambda_{j-2,j}\lambda_{j-1,j}, \quad \text{for } 2 \leq i+1 < j \leq n.$$

Therefore in  $UV P_n$  we have:

$$p(a_{i,i+1}) = \lambda_{i,i+1}^{-1}\lambda_{i+1,i}^{-1}, \quad \text{for } i = 1, \dots, n-1, \\ p(a_{i,j}) = \lambda_{j-1,j}^{-1}\lambda_{j-2,j}^{-1} \cdots \lambda_{i+1,j}^{-1}(\lambda_{i,j}^{-1}\lambda_{j,i}^{-1})\lambda_{i+1,j} \cdots \lambda_{j-2,j}\lambda_{j-1,j}, \quad \text{for } 2 \leq i+1 < j \leq n.$$

According to Theorem 4.1.7,  $UV P_n$  is the cartesian product of the free groups of rank 2  $F_{i,j} = \langle \lambda_{i,j}, \lambda_{j,i} \rangle$  for  $1 \leq i < j \leq n$ .

For every generator  $a_{i,j}$  of  $P_n$ , for  $1 \leq i < j \leq n$ , we have that its image is in  $F_{i,j}$ . Moreover, it is not trivial: in fact,  $p(a_{i,j}) = \lambda_{i,j}^{-1}\lambda_{j,i}^{-1}$ . So  $p(P_n)$  is isomorphic to  $\mathbb{Z}^{n(n-1)/2}$ . Therefore the statement follows readily since the abelianized of  $P_n$  is  $\mathbb{Z}^{n(n-1)/2}$ .  $\square$

## 4.2 Unrestricted virtual braids and fused links

We introduce here fused links, relying on the definition of strand link diagrams in Chapter 2 (see Definition 2.1.4).

**Definition 4.2.1.** A *fused link* is an equivalence class of strand link diagrams under the equivalence relation given by welded isotopy (Definition 2.1.5) with the addition of the forbidden move (F2). This equivalence relation is called *fused isotopy*.

In Chapter 2 we discussed Alexander's and Markov's theorems for loop braids and ribbon torus links. When considering unrestricted virtual braids, we have that the diagrammatical version of the classical Alexander's theorem generalizes to virtual braids and links, and it directly implies that every oriented fused link can be represented by an unrestricted virtual braid, whose Alexander closure is isotopic to the original link. Two braiding algorithms are given in [47] and [52].

Similarly we have a diagrammatical version of Markov's theorem for unrestricted virtual braids and fused links. Before stating it, we recall that the natural map  $UVB_n \rightarrow UVB_{n+1}$ , that adds one strand on the right of an element of  $UVB_n$ , with the convention of considering braids going from the top to the bottom, is an inclusion.

**Theorem 4.2.2** ([53]). *Two oriented fused links are isotopic if and only if any two corresponding unrestricted virtual braids differ by moves defined by braid relations in  $UVB_\infty$  (braid moves) and a finite sequence of the following moves (extended Markov moves):*

- *virtual and classical conjugation:*  $\rho_i \beta \rho_i \sim \beta \sim \sigma_i^{-1} \beta \sigma_i \sim \sigma_i \beta \sigma_i^{-1}$ ;
- *right virtual and classical stabilization:*  $\beta \rho_n \sim \beta \sim \beta \sigma_n^{\pm 1}$ ;

where  $UVB_\infty$  is defined as  $\bigcup_{n=2}^\infty UVB_n$ ,  $\beta$  is a braid in  $UVB_n$ ,  $\sigma_i, \rho_i$  for  $i = 1, \dots, n-1$  are generators of  $UVB_n$ , and  $\sigma_n, \rho_n$  are in  $UVB_{n+1}$ .

Here we give an application to fused links of Theorem 4.1.4.

**Theorem 4.2.3.** *Any fused link is fused isotopic to the closure of an unrestricted virtual pure braid.*

*Proof.* Let us start remarking that the case of knots is trivial because knots are fused isotopic to the unknot (see [48, 67]). Let  $L$  be a fused link with  $n > 1$  components; then there is an unrestricted virtual braid  $\alpha \in UVB_n$  such that  $\hat{\alpha}$  is fused isotopic to  $L$ . Let  $s_{kl} = \rho_{k-1} \rho_{k-2} \dots \rho_l$  for  $l < k$  and  $s_{kl} = 1$  in other cases. We define the set

$$\Lambda_n = \left\{ \prod_{k=2}^n s_{k,j_k} \mid 1 \leq j_k \leq k \right\}.$$

This can be seen as the “virtual part” of  $UVB_n$ , since it coincides with the set of canonical forms of elements in  $\iota(S_n)$ , where  $\iota$  is the map from Lemma 4.1.5. Then using Theorem 4.1.4 we can write  $\alpha$  as

$$\alpha = l_{1,2} l_{1,3} l_{2,3} \dots l_{m-1,m} \pi$$

where  $l_{i,j} \in \langle \lambda_{i,j}, \lambda_{j,i} \rangle$  and  $\pi = s_{2,j_2} \cdots s_{m,j_m} \in \Lambda_n$  (see Figure 4.3).

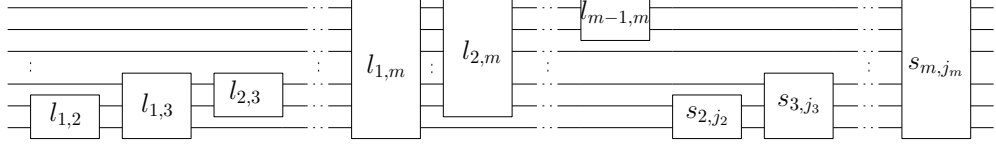


Figure 4.3: The braid  $\alpha$ .

Using Lemma 4.1.5, we can write  $\alpha$  in another way:

$$\alpha = L_2 s_{2,j_2} L_3 s_{3,j_3} \cdots L_m s_{m,j_m},$$

where  $L_i \in \langle \lambda_{1,i}, \lambda_{i,1} \rangle \times \cdots \times \langle \lambda_{i-1,i}, \lambda_{i,i-1} \rangle$ .

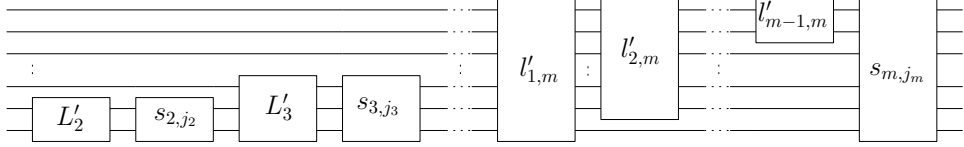


Figure 4.4: The rewriting of the braid  $\alpha$ , with  $L'_i \in \langle \lambda_{1,i}, \lambda_{i,1} \rangle \times \cdots \times \langle \lambda_{i-1,i}, \lambda_{i,i-1} \rangle$ .

Then again we can reorder terms in the  $L_i$ s:

$$\alpha = l'_{1,2} s_{2,j_2} l'_{1,3} l'_{2,3} s_{3,j_3} \cdots l'_{m-1,m} s_{m,j_m}$$

with  $l'_{i,j} \in \langle \lambda_{i,j}, \lambda_{j,i} \rangle$ , see Figure 4.4.

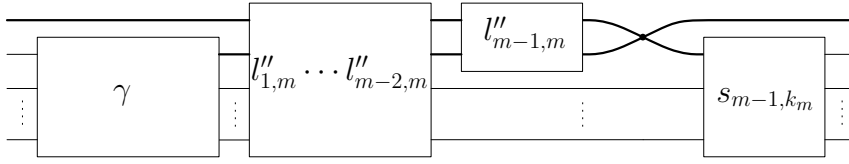
If  $s_{i,j_i} = 1$  for  $i = 2, \dots, m$ , then  $\alpha$  is a pure braid and  $m = n$ .

Suppose then that there is a  $s_{k,j_k} \neq 1$  for some  $k$ , and that  $s_{i,j_i} = 1$  for each  $i > k$ . Conjugating  $\alpha$  for  $s_{m,1}^{m-k}$ , we obtain a braid  $\alpha_1 = s_{m,1}^{k-m} \alpha s_{m,1}^{m-k}$  whose closure is fused isotopic to  $L$  where the  $k$ -th strand of  $\alpha$  is the  $m$ -th strand of  $\alpha_1$ . We write  $\alpha_1$  as:

$$\alpha_1 = \gamma l''_{1,m} l''_{2,m} \cdots l''_{m-1,m} s_{m,k_m}$$

where  $\gamma = l'' s_{2,j_2} \cdots l''_{m-2,m-1} s_{m-1,k_{m-1}}$ , so it does not involve the  $m$ -th strand, and  $l''_{1,m} l''_{2,m} \cdots l''_{m-1,m}$  is pure. For definition  $s_{m,k_m} = \rho_{m-1} s_{m-1,k_m}$ . The  $m$ -th strand and the other strand involved in this occurrence of  $\rho_{m-1}$  that we have just isolated, belong to the same component of  $L_1 = \hat{\alpha}_1$  (see Figure 4.5). Hence also all the crossings in  $l''_{m-1,m}$  belong to that same component.

We virtualize all classical crossings of  $l''_{m-1,m}$  using Kanenobu's technique ([48, Proof of Theorem 1]): it consists in deforming the understrand of one classical crossing at a time, considered in the closure of the link, with a sequence of generalized Reidemeister moves, pushing it along the whole component. At the end of the process, there is a new classical crossing instead of the original one, and  $2j$  new virtual crossings, where  $j$  is the number of crossings the understrand has been pushed through. With generalized Reidemeister moves of braid type, one can change the original classical crossing with a

Figure 4.5: The form of  $\alpha_1$ .

virtual one and remove the new classical crossing with a Reidemeister move of type I. Since our crossings are on the top strand, this Reidemeister move of type I is equivalent to a Markov's classical stabilisation: we obtain a new link  $L'_1$ , fused isotopic to  $L$ , associated to a braid  $\alpha'_1$  who is identical to  $\alpha_1$  except that it has a virtual crossing at the place of the classical crossing considered. This is done for each classical crossing in  $l''_{m-1,m}$ .

Since  $l''_{m-1,m}$  has an even total number of generators  $\sigma_{m-1}$  and  $\rho_{m-1}$ , after virtualizing  $l''_{m-1,m}\rho_{m-1}$  becomes a word composed by an odd number of  $\rho_{m-1}$ . Applying the relation associated with the virtual Reidemeister move of type II we obtain a new link  $L_2$ , fused isotopic to  $L$ , associated to  $\alpha_2 = \gamma l''_{1,m} l''_{2,m} \cdots l''_{m-2,m} \rho_{m-1} s_{m-1,k_m}$ .

Applying once more Lemma 4.1.5,  $\alpha_2$  becomes  $\gamma \rho_{m-1} \overline{l_{1,m}} \overline{l_{2,m}} \cdots \overline{l_{m-2,m}} s_{m-1,k_m}$ , where  $\overline{l_{i,m}}$  is a word in  $\langle \lambda_{m-1,i}, \lambda_{i,m-1} \rangle$ .

In  $\alpha_2$  there is only one (virtual) crossing on the  $m$ -th strand, so, using Markov moves (conjugation and virtual stabilisation) we obtain a new braid  $\alpha_3$ , whose closure is again fused isotopic to  $L$  and has  $(m-1)$  strands. In other words, the braid  $\alpha_3$  is obtained removing from  $\alpha_2$  the only virtual crossing on the  $m$ -th strand, and due to Markov's theorem its closure is fused isotopic to  $L$ .

If we continue this process, eventually we will get to a braid  $\beta$  in  $UVB_n$  whose closure is fused isotopic to  $L$ . At this point, each strand of  $\beta$  corresponds to a different component of  $L$ , so  $\beta$  must be a pure braid.  $\square$

The technique used in the proof of Theorem 4.2.3 was used, associated with braid decomposition in  $B_n$ , by A. Fish and E. Keyman to prove the following result about fused links.

**Theorem 4.2.4** ([31]). *For  $n \geq 1$ , a fused link  $L$  with  $n$  components and with only classical crossings is completely determined by the linking numbers of each pair of components under fused isotopy.*

The proof in [31] is quite technical: it involves several computations on generators of the pure braid group and their images in  $UVP_n$ . The previous result allows us to give an easier proof. The advantage is that no preliminary lemma on the properties of the generators of the pure braid groups is necessary.

*Proof (of Theorem 4.2.4).* We consider a fused link  $L$  with  $n$  components and with only classical crossings: when applying Kanenobu's technique to obtain  $\alpha_2$  (see the proof of Theorem 4.2.3), we get a braid with only one virtual crossing on the  $m$ -strand, and remove it, so that the resulting braid  $\alpha_3$  only has classical crossings. So, continuing the

process, we get that  $L$  is fused isotopic to the closure of an  $n$ -string unrestricted virtual pure braid  $\beta$  which only has classical crossings.

Even though  $B_m$  and  $P_m$  are not subgroups of  $UVB_m$ , since  $\hat{\beta}$  has only classical crossings, we can consider  $B_m$  and  $P_m$ 's images in  $UVB_m$ . We write the pure braid  $\beta$  in terms of  $a_{i,j}$  generators, and defining a group homomorphism  $\delta_{i,j} : P_n \rightarrow \mathbb{Z}$  by

$$a_{s,t} \mapsto \begin{cases} 1 & \text{if } s = i \text{ and } t = j; \\ 0 & \text{otherwise} \end{cases}$$

which is the classical linking number  $lk_{i,j}$  of  $L$ 's  $i$ -th and  $j$ -th components of  $L$ . Any fused link with only classical crossings with  $n$  components can be obtained as a closure of a pure braid  $\beta = x_2 \cdots x_n$  where each  $x_i$  can be written in the form  $x_i = a_{1,i}^{\delta_{1,i}} \cdots a_{i-1,i}^{\delta_{i-1,i}}$  (Corollary 4.1.8). This shows that  $\beta$  only depends on the linking number of the components.  $\square$

*Remark 4.2.5.* In [39, Section 1] a virtual version of the linking number is defined in the following way: to a 2-component link we associate a couple of integers  $(v lk_{1,2}, v lk_{2,1})$ , where  $v lk_{1,2}$  is the sum of signs of classical crossings where the first component passes over the second one, while  $v lk_{2,1}$  is computed by exchanging the components in the definition of  $v lk_{1,2}$ . We call this the *virtual linking number*. Clearly the classical linking number  $lk_{1,2}$  is equal to half the sum of  $v lk_{1,2}$  and  $v lk_{2,1}$ .

Using this definition of virtual linking number, we could be tempted to extend Fish and Keyman's results, claiming that a fused link  $L$  is completely determined by the virtual linking numbers of each pair of components under fused isotopy.

However for the unrestricted case the previous argument cannot be straightforwardly applied: the virtual linking number is able to distinguish  $\lambda_{i,j}$  from  $\lambda_{j,i}$ , but it is still an application from  $UVP_n$  to  $(\mathbb{Z}^2)^{n(n-1)/2} = \mathbb{Z}^{n(n-1)}$  that counts the exponents (*i.e.*, the number of appearances) of the generators. Since  $UVP_n$  is not abelian, this is not sufficient to completely determine the braid.

*Remark 4.2.6.* Fish and Keyman in [31] suggest that their theorem cannot be extended to links with virtual crossings between different components. They consider the unlink on two components  $U_2$  and  $L = \hat{\alpha}$ , where  $\alpha = \sigma_1 \rho_1 \sigma_1^{-1} \rho_1$ . They remark that the classical linking number of both these links is 0, but they conjecture that they are not fused isotopic. In fact, considering the virtual linking number we can see that  $(v lk_{1,2}, v lk_{2,1})(U_2) = (0, 0)$ , while  $(v lk_{1,2}, v lk_{2,1})(L) = (-1, 1)$ .

## 4.3 The fused link group

### 4.3.1 A representation for the unrestricted virtual braid group

We recall that the groups of automorphisms of permutation-conjugacy type, denoted by  $PC_n$  and introduced in Section 1.3, are the group of automorphisms of the form  $\alpha: x_i \mapsto w_i^{-1} x_{\pi(i)} w_i$  where  $\pi$  is a permutation and  $w_i$  is a word in  $F_n$ .



For  $n \geq 1$ , we have that  $WB_n$  is isomorphic to  $PC_n$  passing through the isomorphisms described in Theorem 1.5.9, Theorem 1.4.11, and Corollary 1.3.3. To each generator  $\sigma_i$  of  $WB_n$ , for  $i = 1, 2, \dots, n-1$ , we associate the corresponding automorphism (1.7), and to each generator  $\rho_i$ , for  $i = 1, 2, \dots, n-1$ , we associate automorphism (1.8). We have thus a faithful representation  $\psi: WB_n \rightarrow \text{Aut}(F_n)$ . Equivalently, we can use another convention, exchanging the roles of the generators and their inverses. In this chapter we will use this second convention, to be consistent with the convention used in the article resulted from this work. This convention associates to each generator  $\sigma_i$  and  $\rho_i$  of  $WB_n$ , for  $i = 1, 2, \dots, n-1$  the following automorphisms.

$$\sigma_i : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}; \\ x_{i+1} \mapsto x_i, \\ x_j \mapsto x_j, \text{ for } j \neq i, i+1. \end{cases}$$

$$\rho_i : \begin{cases} x_i \mapsto x_{i+1}; \\ x_{i+1} \mapsto x_i; \\ x_j \mapsto x_j, \text{ for } j \neq i, i+1. \end{cases}$$

*Remark 4.3.1.* Kamada [47] remarks that the classical braid groups  $B_n$  embed in  $VB_n$  through the canonical epimorphism  $VB_n \rightarrow WB_n$ . It can be seen via an argument in [30] that, for a fixed  $n \geq 1$ ,  $B_n$  is isomorphic to the subgroup of  $VB_n$  generated by  $\{\sigma_1, \dots, \sigma_n\}$ .

*Remark 4.3.2.* As a consequence to the isomorphism between  $WB_n$  and  $PC_n$ , we can show that relation (F2) does not hold in  $WB_n$ . In fact applying  $\rho_{i+1}\sigma_i\sigma_{i+1}$  one gets

$$\rho_{i+1}\sigma_i\sigma_{i+1} : \begin{cases} x_i \mapsto x_i \mapsto x_i x_{i+1} x_i^{-1} \mapsto x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1}, \\ x_{i+1} \mapsto x_{i+2} \mapsto x_{i+2} \mapsto x_{i+1}, \\ x_{i+2} \mapsto x_{i+1} \mapsto x_i \mapsto x_i; \end{cases}$$

while applying  $\sigma_i\sigma_{i+1}\rho_i$  one gets

$$\sigma_i\sigma_{i+1}\rho_i : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1} \mapsto x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1} \mapsto x_{i+1} x_i x_{i+2} x_i^{-1} x_{i+1}^{-1}, \\ x_{i+1} \mapsto x_i \mapsto x_i \mapsto x_{i+1}, \\ x_{i+2} \mapsto x_{i+2} \mapsto x_{i+1} \mapsto x_i. \end{cases}$$

Since  $x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1} \neq x_{i+1} x_i x_{i+2} x_i^{-1} x_{i+1}^{-1}$  in  $F_n$ , we deduce that relation (F2) does not hold in  $WB_n$ .

Our aim is to find a representation for unrestricted virtual braids as automorphisms of a group  $G$ . Since the map  $\psi: WB_n \rightarrow \text{Aut}(F_n)$  does not factor through the quotient  $UVB_n$  (Remark 4.3.2) we need to find a representation in the group of automorphisms of a quotient of  $F_n$  in which relation (F2) is preserved.

*Remark 4.3.3.* In [45] the authors look for representations of the braid groups  $B_n$  that can be extended to the loop braid groups  $LB_n$ , but do not factor through  $SLB_n$ , which are isomorphic to  $UVB_n$ . On the contrary, we look for a representation that does factor.

For  $n \geq 1$ , let  $F_n = \gamma_1 F_n \supseteq \gamma_2 F_n \supseteq \cdots$  be the lower central series of  $F_n$ , the free group of rank  $n$ , where  $\gamma_{i+1} F_n = [F_n, \gamma_i F_n]$ . Let us consider its third term,  $\gamma_3 F_n = [F_n, [F_n, F_n]]$ . The free 2-step nilpotent group  $N_n$  of rank  $n$  is defined to be the quotient  $F_n / \gamma_3 F_n$ .

There is an epimorphism from  $F_n$  to  $N_n$  that induces an epimorphism from  $\text{Aut}(F_n)$  to  $\text{Aut}(N_n)$  (see [1]). Then, let  $\phi: UVB_n \rightarrow \text{Aut}(N_n)$  be the composition of  $\varphi: UVB_n \rightarrow \text{Aut}(F_n)$  and  $\text{Aut}(F_n) \rightarrow \text{Aut}(N_n)$ .

**Proposition 4.3.4.** *The map  $\phi: UVB_n \rightarrow \text{Aut}(N_n)$  is a representation for  $UVB_n$ .*

*Proof.* We use the convention  $[x, y] = x^{-1}y^{-1}xy$ . In  $N_n$  we have that  $[[x_i, x_{i+1}], x_{i+2}] = 1$ , for  $i = 1, \dots, n-2$ . This means that  $x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1} = x_{i+1} x_i x_{i+2} x_i^{-1} x_{i+1}^{-1}$ . So, the relation (F2) is preserved.  $\square$

**Proposition 4.3.5.** *The image of the representation  $\phi: UVP_n \rightarrow \text{Aut}(N_n)$  is a free abelian group of rank  $n(n-1)$ .*

*Proof.* From Theorem 4.1.4 we have that the only generators that do not commute in  $UVP_n$  are  $\lambda_{i,j}$  and  $\lambda_{j,i}$ , for  $1 \leq i \neq j \leq n$ . Recalling the expressions of  $\lambda_{i,j}$  and  $\lambda_{j,i}$  in terms of generators  $\sigma_i$  and  $\rho_i$ , we see that the automorphisms associated to  $\lambda_{i,j}$  and  $\lambda_{j,i}$  are

$$\begin{aligned} \phi(\lambda_{i,j}) : & \begin{cases} x_i \mapsto x_j^{-1} x_i x_j = x_i [x_i, x_j]; \\ x_k \mapsto x_k, & \text{for } k \neq i; \end{cases} \\ \phi(\lambda_{j,i}) : & \begin{cases} x_j \mapsto x_i^{-1} x_j x_i = x_j [x_j, x_i] = x_j [x_i, x_j]^{-1}; \\ x_k \mapsto x_k, & \text{for } k \neq i. \end{cases} \end{aligned}$$

It is then easy to check that the automorphisms associated to  $\lambda_{i,j} \lambda_{j,i}$  and to  $\lambda_{j,i} \lambda_{i,j}$  coincide:

$$\phi(\lambda_{i,j} \lambda_{j,i}) = \phi(\lambda_{j,i} \lambda_{i,j}) : \begin{cases} x_i \mapsto x_i [x_i, x_j]; \\ x_j \mapsto x_j [x_i, x_j]^{-1}. \end{cases}$$

To see that in  $\phi(UVP_n)$  there is no torsion, let us consider a generic element  $w$  in  $UVP_n$ . It has the form  $w = l_{1,2} l_{1,3} \cdots l_{n-1,n}$ , where  $l_{i,j}$  is a product of generators  $\lambda_{i,j}$  and  $\lambda_{j,i}$  [58]. Generalizing the calculation done above, we have that

$$\phi(l_{1,2} l_{1,3} \cdots l_{n-1,n}) = \phi(\lambda_{1,2}^{\varepsilon_{1,2}} \lambda_{2,1}^{\varepsilon_{2,1}} \cdots \lambda_{n-1,n}^{\varepsilon_{n-1,n}} \lambda_{n,n-1}^{\varepsilon_{n,n-1}}),$$

where  $\varepsilon_{i,j}$  is the total number of appearances of  $\lambda_{i,j}$  in  $l_{i,j}$ . With another easy calculation (check out also Remark 4.3.10) we have that

$$\phi(\lambda_{1,2}^{\varepsilon_{1,2}} \lambda_{2,1}^{\varepsilon_{2,1}} \cdots \lambda_{n-1,n}^{\varepsilon_{n-1,n}} \lambda_{n,n-1}^{\varepsilon_{n,n-1}}) : \begin{cases} x_1 \mapsto x_1 [x_1, x_2]^{\varepsilon_{12}} [x_1, x_3]^{\varepsilon_{13}} \cdots [x_1, x_n]^{\varepsilon_{1n}}; \\ x_2 \mapsto x_2 [x_2, x_1]^{\varepsilon_{21}} [x_2, x_3]^{\varepsilon_{23}} \cdots [x_2, x_n]^{\varepsilon_{2n}}; \\ \vdots \\ x_n \mapsto x_n [x_n, x_1]^{\varepsilon_{n1}} [x_n, x_2]^{\varepsilon_{n2}} \cdots [x_n, x_{n-1}]^{\varepsilon_{n,n-1}}. \end{cases}$$

So the condition for  $\phi(w)$  to be 1 is equivalent to all exponents being equal to 0, hence  $w = 1$ .  $\square$

*Remark 4.3.6.* As a consequence of the previous calculation, the homomorphism  $\phi$  coincides on  $UVB_n$  with the abelianization map.

As a consequence to Proposition 4.3.5, the representation  $\phi$  is not faithful. However, according to the characterization of  $WB_n$  as subgroups of  $\text{Aut}(F_n)$  (see Section 1.3), it is natural to ask if we can give a characterization of automorphisms of  $\text{Aut}(N_n)$  that belong to  $\phi(UVB_n)$ .

**Proposition 4.3.7.** *For  $n \geq 1$ , let  $\beta$  be an element of  $\text{Aut}(N_n)$ , then  $\beta \in \phi(UVB_n)$  if and only if  $\beta$  satisfies the condition  $\beta(x_i) = a_i^{-1} x_{\pi(i)} a_i$ , for  $1 \leq i \leq n$ , where  $\pi \in S_n$  and  $a_i \in N_n$ .*

*Proof.* Let us denote by  $UVB(N_n)$  the subgroup of  $\text{Aut}(N_n)$  such that any element  $\beta \in UVB(N_n)$  has the form  $\beta(x_i) = g_i^{-1} x_{\pi(i)} g_i$ , denoted by  $x_{\pi(i)}^{g_i}$ , with  $1 \leq i \leq n$ , where  $\pi \in S_n$  and  $g_i \in N_n$ . We want to prove that  $\phi: UVB_n \rightarrow UVB(N_n)$  is an epimorphism. Let  $\beta$  be an element of  $UVB(N_n)$ . Since  $S_n$  is both isomorphic to the subgroup of  $UVB_n$  generated by the  $\rho_i$  generators, and to the subgroup of  $UVB(N_n)$  generated by the permutation automorphisms, we can assume that for  $\beta$  the permutation is trivial, meaning that  $\beta(x_i) = x_i^{g_i}$ . We define  $\varepsilon_{i,j}$  to be  $\phi(\lambda_{i,j})$ , as in Proposition 4.3.5, and we prove that  $\beta$  is a product of such automorphisms. We recall that  $x^{yz} = x^{zy}$  for any  $x, y, z \in N_n$ , therefore

$$\beta(x_i) = x_i^{a_{i,1} \dots a_{i,n}}$$

where  $a_{i,i} = 0$  for all  $i$ . In particular, we can assume that

$$\beta(x_1) = x_1^{a_{1,2} \dots a_{1,n}}.$$

We define a new automorphism  $\beta_1$  multiplying  $\beta$  by  $\varepsilon_{1,2}^{-a_{1,2}} \dots \varepsilon_{1,n}^{-a_{1,n}}$ . We have that  $\beta_1(x_1) = x_1$ , and  $\beta_1(x_j) = \beta(x_j)$  for  $j \neq 1$ . Then again we define a new automorphism  $\beta_2 = \beta_1 \varepsilon_{1,2}^{-a_{2,1}} \varepsilon_{2,3}^{-a_{2,3}} \dots \varepsilon_{2,n}^{-a_{2,n}}$  that fixes  $x_1$  and  $x_2$ . Carrying on in this way for  $n$  steps we get to an automorphism

$$\beta_n = \beta_{n-1} \varepsilon_{n,1}^{-a_{n,1}} \dots \varepsilon_{n,n-1}^{-a_{n,n-1}} = \beta \prod_{j=1}^n \varepsilon_{n,j}^{-a_{1,j}} \prod_{j=1}^n \varepsilon_{n-1,j}^{-a_{2,j}} \dots \prod_{j=1}^n \varepsilon_{1,j}^{-a_{n,j}}$$

where  $\varepsilon_{i,i} = 1$ . The automorphism  $\beta_n$  is the identity automorphism. Then  $\beta$  is a product of  $\varepsilon_{i,j}$  automorphisms. Hence it has a pre-image in  $UVB_n$ .  $\square$

### 4.3.2 The fused link group

Let  $L$  be a fused link. Then there exists an unrestricted virtual braid  $\beta$  such that its closure  $\hat{\beta}$  is equivalent to  $L$ . We use this relation between fused links and unrestricted virtual braids to define a group associated to the fused link. We will see that it is a fused link invariant.

**Definition 4.3.8.** The fused link group  $G(L)$  is the group given by the presentation

$$\left\langle x_1, \dots, x_n \mid \begin{array}{ll} \phi(\beta)(x_i) = x_i & \text{for } i \in \{1, \dots, n\}, \\ [x_i, [x_k, x_l]] = 1 & \text{for } i, k, l \text{ not necessarily distinct} \end{array} \right\rangle,$$

where  $\phi: UVB_n \rightarrow \text{Aut}(N_n)$  is the map from Proposition 4.3.4.

**Proposition 4.3.9.** *The fused link group is invariant under fused isotopy.*

*Proof.* We recall from [53] two unrestricted virtual braids have fused isotopic closures if and only if they are related by *braid moves* and *extended Markov moves*. We should check that, under these moves, the fused link group  $G(L)$  of a fused link  $L$  does not change. This is the case. However a quicker strategy to verify the invariance of this group is to remark that it is a projection of the *welded link group* defined in [15, Section 5]. This last one being an invariant for welded links, we only have to do the verification for the second forbidden braid move, coming from relation (F2). This invariance is guaranteed by the fact that  $\phi$  preserves relation (F2), as seen in Proposition 4.3.4.  $\square$

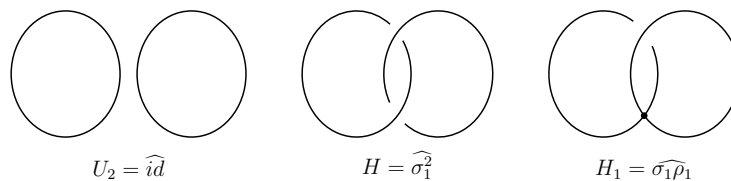


Figure 4.6: The fused link group distinguishes the unlink  $U_2$  from the Hopf link  $H$ . However, it does not distinguish the Hopf link with two classical crossings  $H$  from the one with a classical and a virtual crossing  $H_1$ . In fact:  $G(U_2) = N_2$ , while  $G(H) = G(H_1) = \mathbb{Z}^2$ . We remark however that  $H$  and  $H_1$  are distinguished by the virtual linking number.

*Remark 4.3.10.* Let us recall, from Theorem 4.2.3, that a fused link  $L$  with  $n \geq 1$  components, admits as a representative the closure of an element of  $UVP_n$ , say  $\beta_L$ . Following the proof of Proposition 4.3.7, we deduce that

$$\phi(\beta(x_i)) = x_i^{a_{i,1} x_1^{a_{i,1}} \dots x_n^{a_{i,n}}},$$

where  $a_{i,i} = 0$  and  $a_{i,j} = vl k_{i,j}$ , for  $i \neq j$ . Since the virtual linking number is a fused invariant, we get another easy proof of Proposition 4.3.9. However, it means also that the link group is determined by the virtual linking numbers of the components. Actually, as shown in Figure 4.6, the link group is weaker than the virtual linking number as an

invariant. The relation between the virtual linking number and the link group can be nicely described in the case  $n = 2$  as follows. Let us consider  $\lambda_{1,2}^\alpha \lambda_{2,1}^\beta$  and  $\lambda_{1,2}^\gamma$ , where  $\gamma$  is the greatest common divisor of  $\alpha$  and  $\beta$ , and therefore of  $vlk_{1,2}$  and  $vlk_{2,1}$ . The automorphisms associated to them are

$$\begin{aligned} \phi(\lambda_{1,2}^\alpha \lambda_{2,1}^\beta) : & \begin{cases} x_1 \mapsto x_1 [x_1, x_2 [x_1, x_2]^{-\beta}]^\alpha = x_1 [x_1, [x_1, x_2]^{-\beta}]^\alpha [x_1, x_2]^\alpha = x_1 [x_1, x_2]^\alpha, \\ x_2 \mapsto x_2 [x_1, x_2]^{-\beta}; \end{cases} \\ \phi(\lambda_{1,2}^\gamma) : & \begin{cases} x_1 \mapsto x_1 [x_1, x_2]^\gamma, \\ x_2 \mapsto x_2. \end{cases} \end{aligned}$$

Then

$$\begin{aligned} G(\lambda_{1,2}^\alpha \lambda_{2,1}^\beta) &= G(\lambda_{1,2}^\gamma) = \langle x_1, x_2 \mid [x_1, x_2]^\gamma = 1, [x_i, [x_k, x_l]] = 1 \text{ for } i, k, l \in \{1, 2\} \rangle \\ &= \langle x_1, x_2, t \mid [x_1, x_2] = t, t^\gamma = 1, t \text{ central} \rangle. \end{aligned}$$

This latter group presentation allows us to distinguish the groups for different  $\gamma \in \mathbb{N}$ . This is because  $\gamma$  is the order of the central element  $t$  of these Heisenberg-like groups, setting that  $\gamma = 0$  means that  $t$  has infinite order. In particular, we can set  $G_\gamma := G(\lambda_{1,2}^\gamma)$ .

For instance, the two links considered in [32],  $L = \widehat{\sigma_1 \rho_1 \sigma_1^{-1} \rho_1}$  and  $U_2$ , have corresponding groups  $G_1 = \mathbb{Z}^2$  and  $G_0 = N_2$ . Therefore they are distinguished by  $G_\gamma$ , while, as we saw above, they have the same classical linking number.

## 4.4 Flat welded braids

Several other quotients of virtual braid groups have been studied in the literature. Among these there are the groups of flat welded braids. In particular, these groups are also quotients of the welded braid groups.

**Definition 4.4.1.** For  $n \geq 1$ , the *flat welded braid group*, denoted by  $FWB_n$ , is the quotient of  $WB_n$  obtained by adding relations

$$\sigma_i^2 = 1, \quad \text{for } i = 1, \dots, n-1. \quad (4.4)$$

Let us consider the natural projection map  $g: VB_n \rightarrow FWB_n$ . Set  $g(\rho_i) = \rho_i$  and  $g(\sigma_i) = s_i$ , for  $i = 1, \dots, n-1$ . In  $FWB_n$  we have relations (4.5) and (4.6) coming from the mixed relations, and relations (4.7) coming from the forbidden relations (F1):

$$s_i \rho_j = \rho_j s_i, \quad \text{for } |i - j| \geq 2, \quad (4.5)$$

$$\rho_i \rho_{i+1} s_i = s_{i+1} \rho_i \rho_{i+1}, \quad \text{for } i = 1, \dots, n-2, \quad (4.6)$$

$$s_{i+1} s_i \rho_{i+1} = \rho_i s_{i+1} s_i, \quad \text{for } i = 1, \dots, n-1. \quad (4.7)$$

In  $FWB_n$ , relations (4.4) and (4.7) imply that also relations of type (F2) hold. This is because from  $\rho_i s_{i+1} s_i = s_{i+1} s_i \rho_{i+1}$  one gets  $s_i s_{i+1} \rho_i = \rho_{i+1} s_i s_{i+1}$ . Adapting

Theorem 4.1.7 to this case, one can easily verify that its pure subgroup  $FWP_n$  is isomorphic to  $\mathbb{Z}^{n(n-1)/2}$ . As a straightforward consequence to Theorem 4.1.4, we can describe the structure of  $FWB_n$ .

**Proposition 4.4.2.** *For  $n \geq 1$ , let  $\mathbb{Z}^{n(n-1)/2}$  be the free abelian group of rank  $n(n-1)/2$ . Let us denote by  $x_{i,j}$ , for  $1 \leq i \neq j \leq n$  a set of generators of  $\mathbb{Z}^{n(n-1)/2}$ . The group  $FWB_n$  is isomorphic to  $\mathbb{Z}^{n(n-1)/2} \rtimes S_n$ , where  $S_n$  acts by permutation on the indices of generators of  $\mathbb{Z}^{n(n-1)/2}$  (setting  $x_{j,i} := x_{i,j}^{-1}$ , for  $1 \leq i < j \leq n$ ).*

*Proof.* Let us recall how elements  $\lambda_{i,j}$  in  $UVB_n$  were defined. For  $i = 1, \dots, n-1$ ,

$$\begin{aligned}\lambda_{i,i+1} &= \rho_i \sigma_i^{-1}, \\ \lambda_{i+1,i} &= \rho_i \lambda_{i,i+1} \rho_i = \sigma_i^{-1} \rho_i,\end{aligned}$$

and for  $1 \leq i < j-1 \leq n-1$ :

$$\begin{aligned}\lambda_{i,j} &= \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \lambda_{i,i+1} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1}, \\ \lambda_{j,i} &= \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \lambda_{i+1,i} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1}.\end{aligned}$$

Relations (4.4) are therefore equivalent to relations  $\lambda_{i,j} \lambda_{j,i} = 1$ . Adding these relations and following verbatim the proof of Theorem 4.1.7, we get the proof of the statement.  $\square$

## 4.5 Classifying fused links and other remarkable quotients of the welded braid group: state of the art

Theorem 4.2.4 gives a classification of fused links containing only classical crossings with the linking number of their components. We suggested that the virtual linking number could be used for a classification of all fused links. However Fish and Keyman's argument (and thus the version here used) could not be straightforwardly extended using this invariant (see Remark 4.2.5).

Some months after this work was done, Nasybullov [66], and Audoux, Bellingeri, Meilhan and Wagner [6], independently found the classification of fused links. Nasybullov considers an unrestricted virtual braid whose closure is isotopic to a fixed fused link, and extends the argument here exposed considering case by case what form did the unrestricted virtual braid take. Audoux, Bellingeri, Meilhan and Wagner prove that every fused link is completely determined by the virtual linking numbers of its components. The technique they used was to pass to Gauss diagrams.

In the same paper Audoux, Bellingeri, Meilhan and Wagner consider the quotient of welded links up to forbidden moves, and showed some partial result comparing it with the quotient classical links to  $\Delta$ -moves [72], see Figure 4.7. In the same period Satoh [72] proved that  $\Delta$ -moves are unknotting operations on welded knots. We recall that the fact that  $\Delta$ -moves are unknotting operations on usual knots was already proved in [49]. In this article Kanenobu also considers an operation on 2-knots, called the

$HC$ -move, and proves a relation between the  $\Delta$ -unknotting number for a usual knot and the  $HC$ -unknotting number of its spun 2-knot.

The study of local moves on virtual and welded knotted objects has been useful to many results in relation to finite type invariants, and to the understanding of relations between various quotients of the virtual braid groups, the virtual string links groups and the set of virtual knots. It is thus interesting to consider other remarkable quotients of the welded braid groups, for example the quotient of the welded braid groups up to  $\Delta$ -moves.

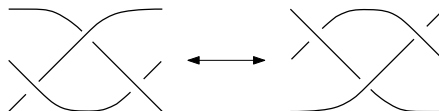


Figure 4.7:  $\Delta$ -move





# Bibliography

- [1] S. Andreadakis. On the automorphisms of free groups and free nilpotent groups. *Proc. London Math. Soc. (3)*, 15:239–268, 1965. 104
- [2] J. Archibald. *The Multivariable Alexander Polynomial on Tangles*. ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)—University of Toronto (Canada). 10, 23, 70, 79, 87, 89, 92
- [3] E. Artin. Theorie der Zöpfe. *Abh. Math. Sem. Univ. Hamburg*, 4(1):47–72, 1925. 1, 15
- [4] B. Audoux. On the welded Tube map. ArXiv: 1408.5479, 2014. 41, 61, 63, 66, 67
- [5] B. Audoux, P. Bellingeri, J.-B. Meilhan, and E. Wagner. Homotopy classification of ribbon tubes and welded string links. ArXiv: 1407.0184, 2014. To appear in *Annali della scuola normale superiore di Pisa*. 2, 16, 41, 50, 52, 59, 70, 77, 84, 87
- [6] B. Audoux, P. Bellingeri, J.-B. Meilhan, and E. Wagner. On forbidden moves and the Delta move. ArXiv: 1510.04237, 2015. 54, 108
- [7] B. Audoux, P. Bellingeri, J.-B. Meilhan, and E. Wagner. On Usual, Virtual and Welded knotted objects up to homotopy. ArXiv: 1507.00202, 2015. To appear in *Journal of the Mathematical Society of Japan*. 28, 54
- [8] J. C. Baez, D. K. Wise, and A. S. Crans. Exotic statistics for strings in 4D  $BF$  theory. *Adv. Theor. Math. Phys.*, 11(5):707–749, 2007. 1, 15, 30, 49, 59
- [9] D. Bar-Natan and Z. Dancso. Finite Type Invariants of w-Knotted Objects II: Tangles, Foams and the Kashiwara-Vergne Problem. ArXiv: 1405.1955, 2014. 3, 17
- [10] D. Bar-Natan and Z. Dancso. Finite-type invariants of w-knotted objects, I: w-knots and the Alexander polynomial. *Algebr. Geom. Topol.*, 16(2):1063–1133, 2016. 2, 3, 4, 16, 17, 27, 28, 44, 52, 54, 59, 84
- [11] D. Bar-Natan, T. T. Q. Le, and D. P. Thurston. Two applications of elementary knot theory to Lie algebras and Vassiliev invariants. *Geom. Topol.*, 7:1–31 (electronic), 2003. 3, 17

- [12] D. Bar-Natan and S. Selmani. Meta-monoids, meta-bicrossed products, and the Alexander polynomial. *J. Knot Theory Ramifications*, 22(10):1350058, 17, 2013. 70
- [13] V. G. Bardakov. The virtual and universal braids. *Fund. Math.*, 184:1–18, 2004. 95, 96, 97, 98
- [14] V. G. Bardakov. Extending representations of braid groups to the automorphism groups of free groups. *J. Knot Theory Ramifications*, 14(8):1087–1098, 2005. 3, 17, 59, 87
- [15] V. G. Bardakov and P. Bellingeri. Groups of virtual and welded links. *J. Knot Theory Ramifications*, 23(3):1450014, 23, 2014. 36, 106
- [16] V. G. Bardakov, P. Bellingeri, and C. Damiani. Unrestricted virtual braids, fused links and other quotients of virtual braid groups. *J. Knot Theory Ramifications*, 24(12):1550063, 23, 2015. 11, 24, 93
- [17] P. Bellingeri and A. Cattabriga. Hilden braid groups. *Journal of Knot Theory and Its Ramifications*, 21(03):1250029, 2012. 58
- [18] S. Bigelow. A diagrammatic Alexander invariant of tangles. *J. Knot Theory Ramifications*, 21(8):1250081, 9, 2012. 70
- [19] S. Bigelow, A. Cattabriga, and V. Florens. Alexander representation of tangles. *Acta Math. Vietnam.*, 40(2):339–352, 2015. 9, 22, 70, 77
- [20] H. U. Boden and A. I. Gaudreau. Bridge numbers for virtual and welded knots. *J. Knot Theory Ramifications*, 24(2):1550008, 15, 2015. 28
- [21] T. E. Brendle and A. Hatcher. Configuration spaces of rings and wickets. *Comment. Math. Helv.*, 88(1):131–162, 2013. 2, 5, 16, 18, 27, 28, 30, 31, 34, 35, 36, 52, 53, 59
- [22] P. Bruillard, L. Chang, S.-M. Hong, J. Y. Plavnik, E. C. Rowell, and M.Y. Sun. Low-dimensional representations of the three component loop braid group. *J. Math. Phys.*, 56(11):111707, 15, 2015. 3, 17, 59
- [23] J. S. Carter and M. Saito. *Knotted surfaces and their diagrams*, volume 55 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998. 49
- [24] D. Cimasoni and V. Turaev. A lagrangian representation of tangles. *Topology*, 44(4):747 – 767, 2005. 70
- [25] B. A. Cisneros de la Cruz. Virtual braids from a topological viewpoint. *J. Knot Theory Ramifications*, 24(6):1550033, 36, 2015. 7, 21, 28, 55, 56, 58
- [26] F. R. Cohen, J. Pakianathan, V. V. Vershinin, and J. Wu. Basis-conjugating automorphisms of a free group and associated Lie algebras. In *Groups, homotopy and configuration spaces*, volume 13 of *Geom. Topol. Monogr.*, pages 147–168. Geom. Topol. Publ., Coventry, 2008. 59

- [27] D. M. Dahm. *A Generalisation of Braid Theory*. PhD thesis, Princeton University, 1962. 28, 37, 58
- [28] E. Dalvit. Visualization of welded objects: <http://science.unitn.it/~dalvit/talks/140211/>, feb 2014. talk at Winterbraid IV. 44
- [29] C. Damiani and V. Florens. Alexander invariants of ribbon tangles and planar algebras. ArXiv: 1602.06191, 2016. 9, 22, 69
- [30] R. Fenn, R. Rimányi, and C. Rourke. The braid-permutation group. *Topology*, 36(1):123–135, 1997. 2, 7, 16, 17, 21, 28, 36, 39, 48, 58, 103
- [31] A. Fish and E. Keyman. Classifying links under fused isotopy. ArXiv: math/0606198, 2006. 12, 25, 94, 101, 102
- [32] A. Fish and E. Keyman. Jones polynomial invariants. *J. Knot Theory Ramifications*, 15(3):339–350, 2006. 12, 25, 94, 107
- [33] V. Florens and G. Massuyeau. A functorial extension of the abelian Reidemeister torsions of three-manifolds. *Enseign. Math.*, 61(1):161–210, 2015. 9, 22, 69, 73, 75, 77
- [34] R. Fox and L. Neuwirth. The braid groups. *Math. Scand.*, 10:119–126, 1962. 1, 15
- [35] R. H. Fox. Characterizations of slices and ribbons. *Osaka J. Math.*, 10:69–76, 1973. 41
- [36] M. H. Freedman and R. Skora. Strange actions of groups on spheres. *J. Differential Geom.*, 25(1):75–98, 1987. 31
- [37] D. L. Goldsmith. The theory of motion groups. *The Michigan Mathematical Journal*, 28(1):3–17, 1981. 1, 6, 15, 19, 28, 30, 36, 37, 38, 58, 67
- [38] J. González-Meneses. Basic results on braid groups. *Ann. Math. Blaise Pascal*, 18(1):15–59, 2011. 30
- [39] M. Goussarov, M. Polyak, and O. Viro. Finite-type invariants of classical and virtual knots. *Topology*, 39(5):1045–1068, 2000. 102
- [40] V. L. Hansen. *Braids and coverings: selected topics*, volume 18 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1989. With appendices by L. Gæde and H. R. Morton. 38
- [41] A. E. Hatcher. A proof of the Smale conjecture,  $\text{Diff}(S^3) \simeq \text{O}(4)$ . *Ann. of Math.* (2), 117(3):553–607, 1983. 33
- [42] A. Hurwitz. Ueber Riemann’sche Flächen mit gegebenen Verzweigungspunkten. *Math. Ann.*, 39(1):1–60, 1891. 1, 15

- [43] C. Jensen, J. McCammond, and J. Meier. The integral cohomology of the group of loops. *Geom. Topol.*, 10:759–784, 2006. 59
- [44] V. F. R. Jones. Planar algebras, I. ArXiv: math/9909027, 1999. 11, 23, 69, 79
- [45] Z. Kadar, P. Martin, E. Rowell, and Z. Wang. Local representations of the loop braid group. ArXiv: 1411.3768, 2014. 3, 11, 17, 24, 59, 93, 103
- [46] S. Kamada. *Braid and knot theory in dimension four*, volume 95 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002. 44, 59, 61
- [47] S. Kamada. Braid presentation of virtual knots and welded knots. *Osaka J. Math.*, 44(2):441–458, 2007. 3, 8, 11, 17, 21, 24, 54, 61, 68, 93, 99, 103
- [48] T. Kanenobu. Forbidden moves unknot a virtual knot. *J. Knot Theory Ramifications*, 10(1):89–96, 2001. 11, 24, 93, 99, 100
- [49] T. Kanenobu. Virtual arc presentations and HC-moves of ribbon 2-knots. *J. Knot Theory Ramifications*, 11(3):387–402, 2002. Knots 2000 Korea, Vol. 1 (Yongpyong). 108
- [50] T. Kanenobu and A. Shima. Two filtrations of ribbon 2-knots. In *Proceedings of the First Joint Japan-Mexico Meeting in Topology (Morelia, 1999)*, volume 121, pages 143–168, 2002. 42, 50
- [51] C. Kassel and V. Turaev. *Braid groups*, volume 247 of *Graduate Texts in Mathematics*. Springer, New York, 2008. With the graphical assistance of Olivier Dodane. 28, 30, 31, 42, 64, 68
- [52] L. H. Kauffman and S. Lambropoulou. Virtual braids. *Fund. Math.*, 184:159–186, 2004. 11, 24, 93, 99
- [53] L. H. Kauffman and S. Lambropoulou. Virtual braids and the  $L$ -move. *J. Knot Theory Ramifications*, 15(6):773–811, 2006. 3, 11, 12, 17, 24, 25, 59, 93, 94, 95, 99, 106
- [54] A. Kawauchi. *A survey of knot theory*. Birkhäuser Verlag, Basel, 1996. Translated and revised from the 1990 Japanese original by the author. 61, 68
- [55] J. L. Kelley. *General topology*. Springer-Verlag, New York-Berlin, 1975. Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics, No. 27. 29
- [56] K. G. Kennedy. *A Diagrammatic Multivariate Alexander Invariant of Tangles*. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)—University of California, Santa Barbara. 70, 79

- [57] P. Kirk, C. Livingston, and Z. Wang. The Gassner representation for string links. *Commun. Contemp. Math.*, 3(1):87–136, 2001. 77, 87
- [58] A. Leibman. A canonical form and the distribution of values of generalized polynomials. *Israel J. Math.*, 188:131–176, 2012. 104
- [59] C. Lescop. A sum formula for the Casson-Walker invariant. *Invent. Math.*, 133(3):613–681, 1998. 69
- [60] W. Magnus. Über Automorphismen von Fundamentalgruppen berandeter Flächen. *Math. Ann.*, 109(1):617–646, 1934. 1, 15
- [61] W. Magnus, A. Karrass, and D. Solitar. *Combinatorial group theory: Presentations of groups in terms of generators and relations*. Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney, 1966. 96
- [62] A. Markov. Foundations of the algebraic theory of tresses. *Trav. Inst. Math. Stekloff*, 16:53, 1945. 1, 15
- [63] J. P. May. Definitions: operads, algebras and modules. In *Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995)*, volume 202 of *Contemp. Math.*, pages 1–7. Amer. Math. Soc., Providence, RI, 1997. 79
- [64] J. McCool. On basis-conjugating automorphisms of free groups. *Canad. J. Math.*, 38(6):1525–1529, 1986. 2, 16, 39, 58
- [65] G. L. Naber. *Topology, geometry, and gauge fields*, volume 25 of *Texts in Applied Mathematics*. Springer, New York, second edition, 2011. Foundations. 77
- [66] T. Nasybullov. The Classification of fused links. ArXiv: 1511.00136, 2015. 108
- [67] S. Nelson. Unknotting virtual knots with Gauss diagram forbidden moves. *J. Knot Theory Ramifications*, 10(6):931–935, 2001. 11, 24, 93, 99
- [68] M. Polyak. Alexander-Conway invariants of tangles. ArXiv: 1011.6200, 2010. 70, 79
- [69] M. Polyak and O. Viro. Gauss diagram formulas for Vassiliev invariants. *Internat. Math. Res. Notices*, (11):445ff., approx. 8 pp. (electronic), 1994. 7, 21, 28
- [70] A. Sartori. The Alexander polynomial as quantum invariant of links. *Ark. Mat.*, 53(1):177–202, 2015. 70
- [71] S. Satoh. Virtual knot presentation of ribbon torus-knots. *J. Knot Theory Ramifications*, 9(4):531–542, 2000. 51, 84, 86
- [72] S. Satoh. Crossing changes, Delta moves, and sharp moves on welded knots, 2015. ArXiv: 1510.02554. 108
- [73] A. G. Savushkina. On the group of conjugating automorphisms of a free group. *Mathematical Notes*, 60(1):68–80, 1996. 2, 16, 39, 58

- 
- [74] N. E. Steenrod. *The Topology of Fibre Bundles*. Princeton Mathematical Series. Princeton University Press, 1951. 33
- [75] A. I. Suciú and H. Wang. The pure braid groups and their relatives. ArXiv: 1602.05291, 2016. 59
- [76] P. Traczyk. A new proof of Markov’s braid theorem. In *Knot theory (Warsaw, 1995)*, volume 42 of *Banach Center Publ.*, pages 409–419. Polish Acad. Sci., Warsaw, 1998. 64
- [77] V. V. Vershinin. Generalizations of braids from the homological point of view. In *Algebra, geometry, analysis and mathematical physics (Russian) (Novosibirsk, 1996)*, pages 40–62, 189. Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 1997. 59
- [78] V. V. Vershinin. On homology of virtual braids and Burau representation. *J. Knot Theory Ramifications*, 10(5):795–812, 2001. Knots in Hellas ’98, Vol. 3 (Delphi). 54
- [79] F. Wattenberg. Differentiable motions of unknotted, unlinked circles in 3-space. *Math. Scand.*, 30:107–135, 1972. 32
- [80] J. C. H. Wilson. Representation stability for the cohomology of the pure string motion groups. *Algebr. Geom. Topol.*, 12(2):909–931, 2012. 32
- [81] T. Yajima. On the fundamental groups of knotted 2-manifolds in the 4-space. *J. Math. Osaka City Univ.*, 13:63–71, 1962. 51, 61
- [82] T. Yajima. On simply knotted spheres in  $R^4$ . *Osaka J. Math.*, 1:133–152, 1964. 50, 84, 86
- [83] T. Yanagawa. On ribbon 2-knots. II. The second homotopy group of the complementary domain. *Osaka J. Math.*, 6:465–473, 1969. 50, 52
- [84] O. Zariski. On the Poincaré Group of Rational Plane Curves. *Amer. J. Math.*, 58(3):607–619, 1936. 1, 15
- [85] C. Zibrowius. On a polynomial Alexander invariant for tangles and its categorification. ArXiv: 1601.04915, 2016. 70

## La topologie des groupes de tresses de cercles : applications et quotients remarquables

**Résumé :** Dans cette thèse nous étudions les groupes de tresses de cercles, nous explorons leurs applications topologiques et certains quotients remarquables. La thèse comporte quatre parties :

- Unification des formalismes pour les groupes de tresses de cercles. Plusieurs formulations ont été utilisées pour les groupes de tresses de cercles en différents domaines ; nous présentons ces interprétations et prouvons leur équivalence.
- Une version topologique du théorème de Markov pour les entrelacs de tores ruban. Avec l'interprétation des tresses de cercles comme objets noués dans l'espace de dimension 4, nous présentons une version du Théorème de Markov pour les groupes de tresses de cercles avec clôture dans l'analogue du tore solide dans l'espace de dimension 4.
- Invariants d'Alexander pour enchevêtrements ruban et algèbres de circuit. Nous définissons un invariant d'Alexander pour enchevêtrements ruban. De cela nous extrayons une généralisation fonctorielle du polynôme d'Alexander. Cet invariant a une signification topologique profonde, mais n'est pas simplement calculable. Nous établissons une correspondance avec le polynôme d'Alexander en plusieurs variables pour enchevêtrements introduit par Archibald pour résoudre ce problème.
- Quotients des groupes de tresses virtuelles. Nous étudions les groupes de tresses de cercles symétriques, et nous en décrivons la structure. Comme conséquence nous montrons que tout entrelacs « fused » admet un représentant comme clôture d'une tresse de cercles symétrique pure.

## The topology of loop braid groups: applications and remarkable quotients

**Abstract:** In this thesis we study loop braid groups, we explore some of their topological applications and some remarkable quotients. The thesis is composed by four parts:

- Unifying the different approaches to loop braid groups. Several formulations are being used by researchers working with loop braid groups in different fields; we present these interpretations and prove their equivalence.
- A topological version of Markov's theorem for ribbon torus-links. Using the understanding of the interpretation of loop braids as knotted objects in the 4-dimensional space, we give a topological proof of a version of Markov theorem for loop braids with closure in a solid torus in the 4-dimensional space.
- Alexander invariants for ribbon tangles. We define an Alexander invariant on ribbon tangles. From this invariant we extract a functorial generalization of the Alexander polynomial. This invariant has a deep topological meaning, but lacks a simple way of computation. To overcome this problem we establish a correspondence with Archibald's multivariable Alexander polynomial for tangles.
- Quotients of the virtual braid group. We study the groups of unrestricted virtual braids, a family of quotients of the loop braid groups, and describe their structure. As a consequence we show that any fused link admits as a representative the closure of a pure unrestricted virtual braid.

**Mots-Clés indexation RAMEAU :** Topologie de basse dimension ; Tresses, Théorie de : Nœuds, Théorie des ; Invariants ; Groupes modulaires.

**Mots-Clés indexation libre :** Tresses soudés ; Tresses de cercles.

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