Principal meets agent: a tale in the land of stochastic control and BSDEs

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Principal meets agent: a tale in the land of stochastic control and BSDEs

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[1] Note for self : penser à acheter de quoi nourrir les invités.

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[2]Il doit forcément manquer du monde, mais j'aurai fait de mon mieux !
[3]¡Científicamente hablando, no te preocupes Cel!

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Part I

Introduction

Introduction

In the beginning the Universe was created. This had made many people very angry and has been widely regarded as a bad move.

Douglas Adams

English version

This document is a synthesis of the research that I have been conducting, along with my co–authors, since the defence of my PhD thesis. This encompasses several topics, which although apparently non immediately connected, will turn out to follow, at least from the theoretical point of view, the same breadcrumbs trail, namely stochastic control (and also stochastic differential games). As I will try to make clear, most of my research topics actually share strong connections, be it from the point of view of the applications which drive them, or from the mathematical tools needed to study them. I have nonetheless made the choice to regroup them along four different lines, corresponding to the four parts of the present manuscript

(*i*) 2BSDEs, BSDEs and robust finance.

(*ii*) Regularity and numerics for BSDEs.

(*iii*) Transaction costs and homogenisation of partial differential equations.

(*iv*) Contracting theory.

These four topics contribute, as mentioned above, to the theory of both optimal stochastic control and stochastic differential games, with applications ranging from risk management issues for financial operators, to studies of interactions between economic agents having access to different types of information, and having possibly antagonist aims. Part II is mainly concerned with the problem of model uncertainty in financial markets. This corresponds to the very conservative case where one does not want to make any *a priori* assumption on a model describing the dynamic evolution of economic quantities of interest in a financial market (prices, interest rates, volatilities...). In such a framework, the most prudential approach consists in considering only the so–called super–replicating price of financial instruments. However, this most of the times proves to be too expensive in practice, which generally leads to studying alternative approaches based on utility indifference pricing and utility maximisation problems. The introduction of such issues naturally lead to necessary theoretical developments of the theory of 2BSDEs and BSDEs. To name but a few, we considered this kind of equations in frameworks with reflections, jumps, quadratic growth and unusual time horizons.

Numerical considerations were at the origin of Part III. If my research work described in Part II definitely had a theoretical flavour, for its potential usefulness to be truly realised, it had to go hand in hand with a more numerically orientated endeavour. Hence, one of the first result we obtained with my co–authors was a fully implementable numerical scheme for the computation of solutions to 2BSDEs. This work naturally led to questions regarding the regularity of solutions to these objects, since these properties often turn out to be

crucial in obtaining convergence result or even speed of convergence results for numerical schemes. Hence, the second main question addressed here was concerned with the existence (and estimates) of densities, with respect to the Lebesgue measure, for solutions to BSDEs. The underlying idea is that having access to these densities can be very helpful from a numerical point of view, and since the solutions to these BSDEs, as illustrated in Part II, are often linked to optimal strategies in the associated finance problems, it can also lead to a better understanding of the latter. This work needed, and was then naturally followed by technical considerations related to Malliavin differentiability properties of solutions to BSDEs, as well as new characterisations of necessary and sufficient conditions of the aforementioned Malliavin differentiability.

One of the limitations of the studies in Part II is that it ignores markets frictions, which are endemic nowadays, be it from the introduction of taxes on financial transactions, or from the presence of illiquidity in the market. The latter problem is at the heart of Part IV, where we considered very general pricing problems in the presence of (proportional) transaction costs. These questions lead generally to highly dimensional and complicated partial differential equations (PDEs for short). With the purpose of understanding better the associated effects, we actually addressed some of these problems in the asymptotic setting, assuming that the imperfections are small, and computing the first order correction to the framework without imperfection. This lead to simpler control problems, which sometimes admit explicit solutions, and are in any case often more tractable from a numerical point of view.

Finally, Part V, which represents the majority of my recent research activities, is concerned with another feature that is absent from the problems considered in Part II, that is to say the possibility of interactions between economic agents, and the fact that they do not always share the same information. This corresponds to the so–called contracting theory and Principal–Agent problems, which can be interpreted as a kind of specific Stackelberg equilibrium. We provide a very general analysis of these problems, which points surprisingly[4] out that they are naturally linked to the 2BSDEs and BSDEs studied extensively in Part II. Furthermore, beyond this general setting (which already goes further than what was available in the literature at the time), we were interested as well in several extensions to cases with uncertainty, multiple Agents, mean–field interactions, but also in applications to delegated portfolio management problems and efficient electricity pricing.

I will now put each of these four topics in a bit more perspective, describing also briefly the main results that I have been able to obtain with my co–authors. A more comprehensive description will be given in the subsequent chapters.

2BSDEs, BSDEs and robust finance

This first part, stemming directly from the main subject of my PhD, is related to the theory of backward stochastic differential equations and second order backward stochastic differential equations (the infamous acronyms BSDE and 2BSDE encountered above) as well as their applications in the so–called robust finance, a domain of research that has seen a considerable growth in the academic literature in the last few years.

The financial industry is currently at a turning point. The financial crisis of 2008 has highlighted the risks of model misspecification, especially during troubled periods in the markets. Even more worrisome, the methods used to evaluate prices of financial instruments have been called into question, because they have proved to be often impractical in very illiquid situations. Without clear and reliable reference prices, financial operators have gradually become more and more aware of the notion of model uncertainty. Hence, a stronger demand for more reliable and robust methods from the players in the financial world has naturally emerged. Accordingly, risk management has become more and more demanding. In the insurance industry as well,

[4] At least to us!

model uncertainty is becoming one of the leading risks. Indeed, on the one hand, worldwide insurers and reinsurers are highlighting the growing trend in insured losses arising from natural catastrophes. Unlike traditional financial options, there is no market hedging strategy for insurance products covering natural disasters. To price these contracts, practitioners rely on the physical modelling of natural phenomena such as earthquakes, hurricanes or floods, and on simulations of these events, for which there is no clear consensus. On the other hand, some classes of insurance products are arising, for which there is no, or very few, reliable physical models. This concerns for example terrorism risk, which is legally included in the covers of the French territory. We can also cite nuclear insurance covers, on which insurers and reinsurers focus since the major event in Japan in 2011. These emerging risks are thus inherently subject to model risk.

It would seem more pertinent to try and take into account this uncertainty before the modelling step. Recently, a new approach, usually called robust, has been developed, in response to this requirement. It consists in including the uncertainty from the start in modelling. More precisely, in the absence of certainty about the true dynamics of liquid financial assets, or climatic and emerging insurance risks, the only fundamental law of financial markets is the so-called absence of arbitrage opportunities. One should therefore consider the classical problems of risk management simultaneously in all the models satisfying this law and consistent with the relevant market.

My main contribution to this topic is in the interconnection between robust finance and BSDEs and 2BSDEs. Hence, we have studied wellposedness problems for different type of (2)BSDEs, namely with general reflection, for applications to game options and robust Dynkin games, with jumps, for financial markets with asset prices presenting sudden movements that can only be represented by processes with jumps, with quadratic growth generators, since the latter appear in many practical problems of utility maximisation for instance, and finally with random horizon, to study the investment decisions made by an investor having access to a market that will disappear unexpectedly but with probability one between now and a fixed time horizon. Another important part of my work has been dedicated to the weakening of very restrictive (from the point of view of applications) continuity assumptions that were needed in all the previous literature on 2BSDEs. This required additional technical results concerning Doob–Meyer–Mertens decompositions for non–linear martingales as well as *a priori* estimates for BSDEs in general probability spaces. Let us emphasise right now that being able to lift these continuity requirements has proved crucial in the general theory that we developed for Principal–Agent problems in Part V.

Regularity and numerics for (2)BSDEs

As mentioned above, it is hardly conceivable to study the new mathematical objects introduced in Part II and the associated problems of mathematical finance, without any consideration of the question of how to compute numerically the associated solutions. If numerics for BSDEs have an extremely rich history which dates back to almost 20 years, the same could not be said for the much more recent theory of 2BSDEs. Hence the first contribution of my co–authors and myself was to design the first fully implementable numerical scheme for general 2BSDE, a scheme whose convergence was rigorously proved thanks to weak convergence type arguments.

Another aspect which is intimately linked to the design of efficient numerical schemes, is that of the regularity of the solution that one wishes to compute. One possible aspects of this regularity, which ended up being the one I concerned myself with the most, concerns the differentiability, in the sense of Malliavin, of the solutions to BSDEs and 2BSDEs. Furthermore, thanks to the seminal results of Malliavin himself or Bouleau and Hirsch (see the illuminating book by Nualart [168] for more details on the subject), it is now a well known fact that such a regularity is deeply linked to the question of the existence of densities for the marginal laws

of the processes solving the BSDE or the 2BSDE. To our surprise however, this subject had received very little attention in the literature in the case of BSDEs already, which is why we started our research in that direction, by obtaining the first general sufficient conditions for the existence of such densities. Since these results required a comprehensive understanding of the conditions ensuring the differentiability in the sense of Malliavin of the solution to a BSDE, we also obtained and improved earlier results on the subject, as well as new complete characterisations of the Malliavin–Sobolev spaces, which we think can have a much broader range of applications.

Transaction costs and homogenisation of PDEs

Option pricing and hedging problems in the presence of market imperfections (transaction costs, market impact, constraints on the number of trading interventions...) generally lead to complex impulse or singular control problems which are difficult to analyse and solve numerically. As mentioned above, a quite important and recent trend is to address these problems in the asymptotic setting of small imperfections.

My contribution in this area is mainly devoted to the proportional transaction costs case. In Markovian models, we have used the deep theory of homogenisation of PDEs to treat rigorously the problem of utility maximisation with infinite horizon under small transaction costs, for general (Markovian) financial markets, with arbitrary many assets and arbitrary utility functions. Let us emphasise that this particular problem had remained open with the technology available before for 20 years. In the same vein, we also studied with similar techniques the associated problem in finite horizon with random endowments, allowing to investigate the problem of indifference pricing of European type contingent claims. Nonetheless, we have to point out that this method is fundamentally based on the fact that the underlying stochastic control problems are Markovian, since it uses crucially the Hamilton–Jacobi–Bellman (HJB in the sequel) partial differential equation satisfied by the value function. This is not only a limitation from the purely theoretical point of view, since it generally imposes that the limit frictionless markets are complete, thus ruling out stochastic volatility models for instance. The latter may of course be treated on a case by case basis, but the approach would then lose in generality. As a first step towards understanding better the non–Markovian case, we have also studied the link between general singular stochastic control problems and so–called BSDEs with constraints on the gains process, thus highlighting a possibly fruitful connection with the research summarised in Part II.

Contracting theory

By and large, it has now become common knowledge among the economists that almost everything in economics was to a certain degree a matter of incentives: incentives to work hard, to produce, to study, to invest, to consume reasonably... At the heart of the importance of incentives, lies the fact that, to quote Salanié [198] "asymmetries of information are pervasive in economic relationships, that is to say, customers know more about their tastes than firms, firms know more about their costs than the government, and all agents take actions that are at least partly unobservable". Starting from the 70s, the theory of contracts evolved from this acknowledgment and the fact that such situations could not be reproduced using the general equilibrium theory. In the corresponding typical situation, a Principal (who takes the initiative of the contract) is (potentially) imperfectly informed about the actions of an Agent (who accepts or rejects the contract). The goal is to design a contract that maximises the utility of the Principal while that of the Agent is held at least to a given level. Of course, the form of the optimal contracts typically depends on whether these actions are observable/contractible or not, and on whether there are characteristics of the agent that are unknown to the Principal. There are two main types of such problems: the second best case,

or moral hazard, in which the action of the Agent is hidden or not contractible; the third best case or adverse selection, in which the type of the Agent is hidden. These problems are fundamentally linked to designing optimal incentives, and are therefore present in a very large number of situations.

For a long time, second best problems were only considered in discrete–time or static settings, which are in general quite hard to solve. One had had to wait for the end of the 80s to witness the treatment of specific moral hazard problems in a continuous time framework. This work has then been generalised using a dynamic programming and martingale approach, which is classical in stochastic control theory. Another, and in some cases more general approach, is to use the so–called stochastic maximum principle and associated systems of forward–backward SDEs to characterise the optimal compensation. Nonetheless, none of these standard approaches can solve the problem when the agent also controls the diffusion coefficient of the output, and not just the drift. Building upon this gap in the literature, we have, with my co–authors, developed a general approach to the problem through dynamic programming and BSDEs and 2BSDEs (recall Part II), showing that under mild conditions, the problem of the Principal could always be rewritten in an equivalent way as a standard stochastic control problem involving two state variables, namely the output itself but also the continuation utility (or value function) of the Agent.

We have also used this new approach to treat specific examples related to problems of optimal incentives in delegated portfolio management, and the treatment of the problem for economic agents facing Knightian uncertainty. We have also applied this technology to the optimal design of electricity contracts and prices. We emphasise that this last application has been conducted jointly with research engineers from Électricité de France (EDF) and has lead to the filing of a patent associated to our publication [2]. I also dedicated most of my last research activities to trying to understand how this theory could be extended to context involving not only one but several Agents possibly interacting with each other. We considered two main problems: one with a finite number of Agents as well as the associated mean–field limit corresponding to the infinitely many Agents case.

Version française

Ce document est une synthèse des travaux de recherche que j'ai conduits, avec mes co–auteurs, depuis la soutenance de ma thèse. Cela regroupe un certain nombre de domaines, qui bien qu'au premier abord déconnectés, s'avèrent, tout au moins du point de vue théorique, intimement connectés, ayant pour fil conducteur la théorie du contrôle stochastique (et celle des jeux différentiels stochastiques). Comme je le mettrai en évidence, mes sujets de recherche partagent de nombreux points communs, que ce soit du point de vue des applications qui les sous–tendent ou des outils mathématiques nécessaires à leur étude. J'ai néanmoins fait le choix de les regrouper en quatre catégories correspondants aux quatre parties de ce manuscrit

(*i*) 2EDSR, EDSR et finance robuste.

(*ii*) Régularité et schémas numériques pour les EDSR.

(*iii*) Coûts de transaction et homogénisation d'équations aux dérivées partielles.

(*iv*) Théorie des contrats.

Ces quatre domaines détudes apportent, comme mentionné ci–dessus, un éclairage supplémentaire sur les théories du contrôle optimal stochastique et des jeux différentiels, par le biais d'applications allant de problématiques de gestion des risques pour des opérateurs financiers, à l'étude des interactions entre des agents

économiques ayant accès à des sources d'information disparates, et des intérêts potentiellement antagonistes. La partie II s'attardera essentiellement sur le problème dit d'incertitude de modèle dans les marchés financiers. Cela correspond à la situation extrêmement conservative où l'on ne désire faire aucune hypothèse *a priori* sur un quelconque modèle décrivant les évolutions futures d'un certain nombre de quantité d'intérêt sur le marché (prix, taux d'intérêt, volatilité...). Dans un tel cadre, l'approche la plus prudente pour un investisseur est de s'intéresser à ce que l'on nomme communément le prix de sur–réplication des instruments financiers. Néanmoins, cela heurte généralement au fait que cette attitude conservative mène à des prix trop élevés, et il lui est assez fréquemment préféré une approche basée sur les prix d'indifférence et les problèmes de maximisation d'utilité. L'examen de ces problèmes mène alors naturellement à l'étude et aux développement de la théorie des EDSR et des 2EDSR. Pour ne citer que quelques unes des pistes auant émaillées mes recherches, mentionnons les (2)EDSR réfléchies, à sauts, à croissance quadratique, ou encore avec des horizons d'étude inhabituels.

Ce sont des considérations numériques qui furent à l'origine de la partie III de ce manuscrit. S'il est indéniable que mon travail de recherche décrit dans la première partie a une teneur théorique assez marquée, son utilisation et son application en pratique ne peuvent passer que par une approche plus orientée sur l'aspect numérique. De ce fait, une des premières questions dans cette veine sur laquelle nous nous sommes penchée avec mes co–auteurs concerna l'obtention de schèmas numériques complètement implémentables pour le calcul de solutions de certaines classes de 2EDSR. De manière très naturelle, ce travail nous amena à nous orienter sur des problématiques liées à la régularité de tels objets, puisqu'il est généralement admis que ces propriétés peuvent être d'une aide précieuse dans l'obtention des résultats de convergence, voire de vitesse de convergence, pour des schémas numériques. De ce fait, le reste de cette seconde partie fut consacré à la question de l'existence, et le cas échéant à l'estimation, de densités, par rapport à la mesure de Lebesgue, pour les solutions d'EDSR. L'idée sous–jacente est que la possibilité d'avoir accès à ces densités peut être non seulement extrêmement appréciable du point de vue numérique, mais également, du fait par exemple que, comme illustré dans la partie II, ces EDSR sont intimement liées aux stratégies optimales d'investissement dans les problèmes de finance sus–mentionnés, elle autorise une compréhension plus fine de ces dernières. Du point de vue théorique, nous eurent besoin d'explorer des propriétés de différentiation au sens de Malliavin des solutions d'EDSR, obtenant par la même occasion des nouvelles charactérisations de cette différentiabilité.

Une des importantes limitations des résultats obtenus dans la partie II tient dans le fait que cette dernière ignore totalement la possibilité d'existence de frictions de marché, que ce soit par le biais de taxes sur les transactions financières ou simplement par l'absence de liquidité, alors que ces dernières sont omniprśentes dans les marchés actuels. Ce problème est au cœur même de mes travaux regroupés dans la partie IV, où nous nous sommes intéressés à des questions générales de calculs de prix d'instruments financiers en présence de coûts de transaction proportionnels. Ces dernières mènent la plupart du temps à des systèmes d'équations aux dérivées partielles (EDP) non seulement complexes, mais surtout en grande dimension. Dans l'optique d'une meilleure compréhension des effets associés, nous nous sommes essentiellement attachés à l'étude des ces problèmes dans un contexte asymptotique, présupposant que les imperfections de marché sont d'une amplitude faible, et cherchant à obtenir des corrections au premier ordre par rapport au problème sans imperfection. Cela mène alors à des problèmes de contrôle plus simples, admettant parfois des solutions explicites, et dans tous les cas d'un accès beaucoup plus efficace du point de vue numérique

La dernière partie V de ce manuscrit représente l'essentiel de mes recherches récentes et s'intéresse à une autre caractériqtique brillant par son absence dans les problèmes décrits dans la partie II, en l'occurrence la possibilité d'interactions entre différents agents économiques, ces derniers n'ayant par ailleurs pas nécessairement accès aux mêmes informations. Une telle situation est plus connue dans la littérature sous l'appellation de théorie des contrats ou de problème dit de Principal–Agent, qui n'est en fait rien d'autre qu'un certain

type d'équilibre de Stackelberg. Nous sous sommes attelés à analyser de manière générale ces problèmes, et avons mis en exergue un lien quelque peu surprenant de prime abord[5], avec les 2EDSR et les EDSR ayant fait l'objet d'une étude approfondie dans la partie II. En outre, au-delà même de cette étude général, qui permettait déjà d'accéder à des résultats totalement absents de la littérature pré–existante, nous nous sommes également penchés sur plusieurs pistes d'extension, considérant ainsi l'impact de l'incertitude de modèle sur le problème, la possibilité de présence de plusieurs Agents, ou encore d'interactions de type champ moyen, mais également à des applications plus pratiques de problèmes de délégation de gestion de portefeuille ou de tarification efficace pour l'électricité.

Je vais maintenant décrire de manière un peu plus détaillée le contenu de chacune de ces parties, ainsi que les résultats associés que j'ai pu obtenir avec mes co–auteurs. Une description plus complète et exhaustive fera l'objet des chapitres suivants.

2EDSR, EDSR et finance robuste

Cette première partie, qui est la suite naturelle de ce qui fut le cœur de ma thèse, s'intéresse essentiellement aux théories des équations différentielles stochastiques rétrogrades et des équations différentielles stochastiques rf́rogrades du second ordre (les fameux acronymes EDSR et 2EDSR rencontrés précédemment), ainsi qu'à leurs applications à la finance robuste, un domaine ayant connu ces dernières années un regain d'intérêt considérable dans la litérature académique.

L'industrie financière est aujourd'hui à la croisée des chemins. La crise de 2008 a mis en lumière de manière très claire les risques liés à des choix erronés de modèle, et ce d'autant plus dans des périodes de troubles sur les marchés. Plus inquiétant encore, les méthodes mêmes d'évaluation des instruments financiers ont été remises en cause, car fréquemment inadaptées dans des situations de très faible liquidité. Dès lors, sans prix de référence clairs et fiables, les opérateurs de marché ont graduellement pris conscience da la notion d'incertitude de modèle. Cela a eu pour conséquence l'augmentation drastique de la demande de ces derniers pour des méthodes plus robustes, et permettant une évaluation la plus précise possible de ce risque. L'industrie de l'assurance n'a pas échappé à cette tendance, puisque le risque de modèle s'est imposé, parmi d'autres bien entendu, comme un des risques majeurs. Ainsi, l'importance croissante des risques de catastrophes naturelles, soulignée à la fois par les assureurs et les ré–assureurs, doit être reliée au fait qu'il n'existe aujourd'hui presque pas de stratégie de couverture associée. Ainsi, pour évaluer de tels contrats, les praticiens s'en remettent à des modélisations physiques de phénomènes naturels tels que les séismes, les ouragans ou les inondations, pour lesquels il n'existe pas de consensus clair. Par ailleurs, on constate parallèlement l'émergence de risques pour lesquels de tels modèles n'existent même pas. Tel est le cas du risque d'attentat terroriste, qui est inclus de fait dans les contrats d'assurance sur le territoire français. On peut également citer le risque de catastrophe nucléaire, qui fait l'objet d'une attention toute particulière depuis les évènements survenus au Japon en 2011. Ces risques émergents sont donc, de part leur nature même, sujets au risque de modèle.

Il semble ainsi naturel de chercher à prendre en compte ce risque et cette incertitude avant même la phase de modélisation. Récemment, une approche innovante, qualifiée de robuste, a été dv́eloppée pour répondre exactement à ce besoin, et incorpore l'incertitude dès le départ. Plus précisément, en l'absence d'information sur l'évolution future des valeurs des actifs financiers liquides, ou des risques climatiques et émergents, la seule loi fondamentale du marché est celle de l'absence d'opportunités d'arbitrage. L'idée est alors de considérer les problèmes classiques de gestion du risque simultanément sous tous les modèles satisfaisant cette loi, et consistant avec le marché considéré.

[5]Tout du moins en ce qui nous concerne !

Ma contribution principale à ce domaine tient dans la mise en lumière de l'interconnexion fondamentale existant entre la finance robuste et les EDSR et 2EDSR. Ainsi, nous avons étudié des problèmes d'existence et d'unicité pour différents types de (2)EDSR, parmi lesquels ceux faisant intervenir des réflexions, avec pour application l'étude d'options de jeux et des jeux de Dynkin robustes, des sauts, pour l'étude de marchés financiers où des variations soudaines peuvent être représentées par des sauts, des générateurs à croissance quadratique, puisque ces derniers apparaissent naturellement dans les problèmes de maximisation d'utilité, ou encore des horizons aléatoires, pour étudier les décisions d'investissement faites par un investisseur ayant accès à un marché financier qui est amené de manière certaine à disparaître entre aujourd'hui et une date fixe, mais à une date inconnue à l'avance. Une autre partie importante de mon travail a été consacrée à l'affaiblissement des conditions très restrictives (du point de vue des applications) de continuité qui était nécessaires dans toute la littérature précédente consacrée aux 2EDSR. Cela nécessitat un certain nombre de résultats théoriques supplémentaires, concernant des décompositions de type Doob–Meyer–Mertens pour des martingales non linéaires, ainsi que des estimations, dites *a priori*, pour des EDSR dans des espaces de probabilité généraux. Soulignons dès à présent que le fait d'avoir pu se débarasser de ces hypothèses de continuité s'est avéré absolument crucial pour la théorie générale des problèmes Principal–Agent développée dans la partie V.

Régularité et schémas numériques pour les EDSR

Comme rappelé ci–dessus, il est difficilement concevable de se lancer dans l'étude des nouveaux objets mathématiques et des applications associées introduits dans la partie II, sans qu'elle soit accompagnée de considérations numériques. S'il est indéniable aujourd'hui que le développement de schémas numériques pour les EDSR a une longue et riche histoire, remontant à près de vingt ans, ce n'est clairement pas le cas pour les 2EDSR. De ce fait, la première contribution à ce domaine de mes co–auteurs et moi–même fût d'obtenir le premier schéma numérique complètement implémentable s'appliquant à des solutions de 2EDSR générales, dont la convergence a ensuite été prouvée rigoureusement par des arguments de type convergence faible.

Un autre aspect intimement lié à l'étude et à l'obtention de schémas numériques efficaces concerne la régularité da la solution que l'on cherche à calculer. Une possible approche de cette régularité, qui s'avèra être celle qui nous occupa essentiellement, fût celle de la différentiabilité au sens de Malliavin, des solutions d'EDSR et de 2EDSR. En outre, grâce aux fameux résultats de Malliavin lui–même, mais également de Bouleau et Hirsch (nous référons le lecteur intéressé à l'excellent livre de Nualart [168] sur le sujet), il est aujourd'hui connu que cette notion de régularié est profondément liée à la question de l'existence de densités pour les lois marginales des solutions étudiées. À notre grande surprise, ce sujet n'avait reçu qu'une attention extrêmement limitée dans la littérature des EDSR elles–mêmes, ce qui nous incita à commencer notre étude par chercher à obtenir les premières conditions générales assurant l'existence de telles densités. Par ailleurs, ces résultats nécessitant une compréhension fine des hypothèses impliquant la différentiabilité au sens de Malliavin des solutions d'EDSR, nous avons également obtenu des nouveaux résultats, ainsi qu'amélioré des résultats plus anciens sur le sujet, parmi lesquels une nouvelle caractérisation des espaces de Malliavin–Sobolev, que nous pensons pouvoir appliquer à de nombreux autres domaines.

Coûts de transaction et homogénisation d'équations aux dérivées partielles

Les problèmes d'évaluation et de couverture en présence d'imperfections de marché (coûts de transaction, impact de marché, contraintes sur le nombre de rebalancements...) mènent la plupart du temps à des problèmes de contrôle stochastique impulsionnels ou singuliers ardus, que ce soit du point de vue théorique

ou numérique. Comme expliqué plus haut, une tendance récente a été d'attaquer ces problèmes dans le régime spécifique de petites imperfections.

Ma contribution à ce domaine tient essentiellement à létude de modèle incorporant des coûts de transaction. Dans les cas dits markoviens, nous avons utilisé des méthodes et techniques issues de la théorie de l'homogénisation des équations aux dérivées partielle pour traiter de manière rigoureuse le problème de maximisation d'utilité en horizon infini avec des coûts de transaction faibles, pour des marchés financiers complets généraux, et un nombre d'actifs et des fonctions d'utilité arbitraires. Insistons sur le fait que ce problème spécifique, dans un cadre aussi général était resté ouvert pendant près de vingt ans avec les techniques disponibles. Dans le même ordre d'idées, nous sommes intéressés au même problème en horizon fini, permettant l'étude de la question du prix d'indifférence d'utilité pour des options de type européennes. Nous devons néanmoins mentionner le fait que ces méthodes sont basées de manière cruciale sur le fait que les problèmes de contrôle associés sont markoviens, puisqu'elles utilisent de manière essentielle les équations d'Hamilton–Jacobi–Bellman satisfaites par la fonction valeur. Ce n'est pas qu'une simple limitation du point de vue théorique, mais également pratique, car elle implique que les marchés sans friction considéré doit être complet, ce qui exclu de fait des modèles à volatilité stochastique par exemple. Ces derniers peuvent bien entendu être traités au cas par cas, mais cela ferait perdre à l'approche sa généralité. Nous avons entamé une première étape vers uen compréhension plus fine des problèmes non–markoviens, en nous intéressant aux liens entre les problèmes de contrôle singuliers and les EDSR dites contraintes, mettant ainsi en évidence une connexion potentiellement fructueuse avec les résultats obtenus dans la partie II.

Théorie des contrats

Il est maintenant communément admis parmi les économistes que la très large majorité des problèmes de la litérature économique sont reliées d'une certaine manière à des questions d'incitations : incitations à travailler, à produire, à étudier, à investir, à consommer de manière raisonnable... Au cœur même de cette importance se trouve le fait que, pour citer Salanié [198] "asymmetries of information are pervasive in economic relationships, that is to say, customers know more about their tastes than firms, firms know more about their costs than the government, and all agents take actions that are at least partly unobservable". Depuis les années 1970, la théorie des contrats s'est développée des suites de cette constatation, et du fait que que de telles situations ne pouvaient être reproduites grâce à la théorie de l'équilibre général. Le cas typique traité par la théorie des contrats est celui d'un Principal (prenant l'initiative du contrat), potentiellement imparfaitement informé des actions entreprises par un Agent (qui accepte ou rejette le contrat). Le but est alors de créer un contrat qui maximise l'uilité du Principal, tout en assurant un minimum à l'Agent. Bien entendu, la forme du contrat optimal dépend typiquement de savoir si les actions de l'Agent sont observables ou non, et s'il existe des caractéristiques de l'Agent qui ne sont pas connues par le Principal. On distingue alors généralement deux grandes classes de problèmes : l'aléa moral, pour lequel les actions de l'Agent ne sont pas observables; l'antisélection pour laquelle le type de l'Agent est caché. Ces problèmes sont alors fondamentalement reliés à la conception d'incitations optimales, et sont donc ominprésents dans de nombreuses situations.

Pendant longtemps, les problèmes d'aléa moral ne furent considérés que dans des modèles statiques où à temps discret, et étaient en général difficiles à résoudre. Il fallut attendre la fin des années 1980 pour voire l'émergence de modèle d'aléa moral en temps continu. Ces travaux ont par la suite été généralisé en utilisant une approche basée sur la programmation dynamique et des méthodes de martingale, communément usitée dans la théorie du contrôle stochastique. Une autre approche, s'avérant dans certains cas plus générale, repose sur le principe du maximum stochastique et les systèmes couplés d'équations différentielles stochastiques progressives/rétrogrades caractérisant les compensations optimales. Néanmoins, aucune de ces approches

ne pouvait jusqu'alors résoudre les problèmes pour lesquels l'Agent peut contrôler à la fois la tendance et la volatilité de la quantité d'intérêt du problème. Afin de combler ce manque dans la littérature, nous nous sommes attachés avec mes co–auteurs à développer une approche très générale du problème grâce à la programmation dynamique ainsi que les EDSR et les 2EDSR (souvenez–vous de la partie II), montrant que sous des hypothèses très naturelles, le problème auquel fait face le Principal pouvait se réécrire de manière parfaitement équivalente comme un problème de contrôle stochastique faisant intervenir deux variables d'état, en l'occurrence le processus contrôlé par l'Agent, mais également son utilité de continuation.

Nous avons également cette nouvelle approche pour traiter des exemples spécifiques, notamment de délégation de gestion de portefeuille, ou encore de situations faisant intervenir des agents économiques ayant des croyances ambigües. Nous nous sommes également intéressés à la conception de contrats de fourniture d'électricité efficients. Notons que ce dernier projet a été mené en collaboration avec des ingénieurs d'Électricité de France (EDF) et a été accompagné du dépôt d'un brevet associé à notre publication [2]. Pour conclure, j'ai dédié une grande partie de mes activités de recherche récentes à essayer de comprendre comment cette théorie pouvait être étendue à un contexte faisant intervenir plusieurs Agents en interaction. Nous avons ainsi considéré deux problèmes, l'un avec un nombre fini d'Agents, et l'autre correspondant à la limite dans la régime d'un nombre infini d'Agents en interaction de type champ moyen.

Part II

2BSDEs, BSDEs and robust finance

Chapter 1

2BSDEs, robust finance and Knightian uncertainty

> We demand rigidly defined areas of doubt and uncertainty!
>
> Douglas Adams

As mentioned in the introduction, the years following the financial crisis have given rise to growing concerns from practitioners. In a nutshell, one of the questions raised concerned the possibility of obtaining a correct assessment of the risks they could be taking by choosing a specific model. The theory of robust mathematical finance deals exactly with this, and starts without assuming any specific model on the financial market, and then aims at deriving robust bounds on prices, as well as associated hedging strategies, allowing theoretically to hedge out all of the risk.

From the mathematical point of view, asset prices are usually modelled by jump–diffusion processes, or more generally semi–martingales, and refraining from choosing *a priori* a specific model means that we do not want to describe in more specific terms the three elements of the semi–martingale characteristic triplet of the asset price, namely its drift, its volatility and its jump compensator. We are then lead to considering simultaneously models (or more specifically probability measures) under which this triplet changes. However, a change in the volatility or in the jump compensator of a semi–martingale almost systematically leads to singular probability measures. One of the most serious challenges, from the mathematical point of view at least, of the robust finance theory is therefore to be able to understand and apply stochastic calculus while working simultaneously under families of non–dominated probability measures. This is the so–called quasi–sure stochastic analysis, a term coined in the seminal paper of Denis and Martini [66]. My contribution towards this specific problem concerns Section 1.1 below, which sums up the results of [187] and proves a general version of the robust super–hedging duality for upper–semianalytic (which roughly speaking means that we allow random variables which are Borel measurable) claims.

It has been understood since the illuminating survey paper of El Karoui, Peng and Quenez [78] that BSDEs were intimately linked to the problem of pricing contingent claims in financial markets, especially when the latter present certain sources of incompleteness or frictions. In financial markets with uncertainty however, BSDEs are not the appropriate tool, and one has to use their generalisation in the form of 2BSDEs, which are somehow tailor–made to that specific purpose. Such an approach has been the subject of a rapidly growing strand of literature, and was also a major part of my research activities during my PhD. Hence, Section 1.2 gives an overview of the results that I obtained in [150] considering so-called game options and the associated 2BSDEs, while Section 1.3 covers the papers [132, 131] where 2BSDEs allowing for jumps in the associated processes were introduced for the first time. Section 1.4 is devoted to the paper [151] and presents a new probabilistic interpretation for classical and viscosity solutions to fully non–linear stochastic PDEs using a generalisation of the 2BSDEs to the doubly stochastic case. Finally, the paper [190], presented in Section 1.5, is somehow the (current) theoretical culmination of my work on 2BSDEs, in the sense that it

completely lifts all the regularity assumptions that were made in the previous literature, and gives for the first time a 2BSDE theory that indeed generalises the classical theory of BSDEs. This theoretical result will be at the heart of my contributions in contracting theory presented in Part V.

1.1 Robust super–hedging (see [187])

While the problem of robust super–hedging has been originally introduced by Avellaneda [6] and Lyons [143], its first general formulation is due to Denis and Martini [66] who considered the hedging problem under a non–dominated family of probability measures on the canonical space of continuous trajectories. The next progress was achieved by Soner, Touzi and Zhang [205] who introduced a restriction of the set of non–dominated measures so as to guarantee that the predictable martingale representation property holds true under each measure. However, [205] placed strong regularity conditions on the random variables of interest in order to guarantee the measurability of the value function of some dynamic version of a stochastic control problem, and to derive the corresponding dynamic programming principle. By using the notion of measurable analyticity, Nutz and van Handel [169] and Neufeld and Nutz [166] extended the previous results to general measurable claims, with the only restriction that the non–dominated family of singular measures should satisfy quite general assumptions.

Our first objective in the paper [187] was to point out that the conditions of [166] were not actually satisfied for some sets of probability measures, even though the latter were of importance for certain applications, especially martingale optimal transport (see Galichon, Henry-Labordère and Touzi [92]). The main point is that [166] requires the set of measures to be stable by uncountable concatenation, which implicitly implies some kind of boundedness. Indeed, otherwise the infinite concatenation of measures could actually make you end up out of the original set. Our main result stipulates that one can actually replace the stability by concatenation by a weaker condition involving an appropriate approximating sequence of measures. This condition is satisfied by virtually all the examples considered in the literature so far.

We need to introduce some notations before stating our result. Let $\Omega := \{\omega \in C\left([0,T],\mathbb{R}^d\right),\ \omega_0 = 0\}$ be the canonical space. $\mathcal{F}$ will always be the Borel $\sigma-$field on Ω associated to the uniform topology. We then denote by B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0\leq t\leq T}$ the filtration generated by B. We will denote by $\mathbf{M}(\Omega)$ the set of all probability measures on Ω. We also recall the so–called universal filtration $\mathbb{F}^* := \{\mathcal{F}_t^*\}_{0\leq t\leq T}$ defined by $\mathcal{F}_t^* := \cap_{\mathbb{P}\in\mathbf{M}(\Omega)}\mathcal{F}_t^{\mathbb{P}}$, where $\mathcal{F}_t^{\mathbb{P}}$ is the usual completion under $\mathbb{P}$.

For any subset E of a finite dimensional space and any filtration $\mathbb{X}$ on $(\Omega,\mathcal{F})$, we denote by $\mathbb{H}^0(E,\mathbb{X})$ the set of all $\mathbb{X}-$progressively measurable processes with values in E. Moreover for all $p>0$ and for all $\mathbb{P}\in\mathbf{M}(\Omega)$, we denote by $\mathbb{H}^p(\mathbb{P},E,\mathbb{X})$ the subset of $\mathbb{H}^0(E,\mathbb{X})$ whose elements H satisfy $\mathbb{E}^{\mathbb{P}}\big[\int_0^T |H_t|^p\,dt\big] < +\infty$. The localised versions of these spaces are denoted by $\mathbb{H}^p_{\rm loc}(\mathbb{P},E,\mathbb{X})$.

For all $\alpha\in\mathbb{H}^1_{\rm loc}(\mathbb{P}_0,\mathbb{S}_d^{>0},\mathbb{F})$, where $\mathbb{S}_d^{>0}$ is the set of positive definite matrices of size $d\times d$, we define the probability measure on $(\Omega,\mathcal{F})$

$$\mathbb{P}^\alpha := \mathbb{P}_0\circ(X_\cdot^\alpha)^{-1} \text{ where } X_t^\alpha := \int_0^t \alpha_s^{1/2}dB_s,\ t\in[0,T],\ \mathbb{P}_0-a.s.$$

We denote by $\mathcal{P}_S$ the collection of all such probability measures on $(\Omega,\mathcal{F})$. We are then given a subset $\mathcal{P}\subset\mathcal{P}_S$. A $\mathcal{P}-$polar set is a $\mathbb{P}-$negligible set for all $\mathbb{P}\in\mathcal{P}$, and we say that a property holds $\mathcal{P}-$quasi–surely if it holds outside of a $\mathcal{P}-$polar set. Finally, we introduce the following filtration $\mathbb{G}^{\mathcal{P}} := \{\mathcal{G}_t^{\mathcal{P}}\}_{0\leq t\leq T}$

$$\mathcal{G}_t^{\mathcal{P}} := \mathcal{F}_{t^+}^*\vee\mathcal{N}^{\mathcal{P}},\ t<T \text{ and } \mathcal{G}_T^{\mathcal{P}} := \mathcal{F}_T^*\vee\mathcal{N}^{\mathcal{P}},$$

where the " + " sign indicates the right limit of the filtration, $\mathcal{N}^{\mathcal{P}}$ is the collection of $\mathcal{P}-$polar sets. The set $\mathcal{P}$ and its appropriately defined shifted versions $(\mathcal{P}(s,\omega))_{(s,\omega)\in[0,T]\times\Omega}$ are supposed to satisfy the following assumption[1]

Assumption 1.1.1. *Let $s \in \mathbb{R}_+$, $\tau \geq s$ a stopping time, $\overline{\omega} \in \Omega$, $\mathbb{P} \in \mathcal{P}(s,\overline{\omega})$ and $\theta + s$ be the shifted version of τ.*

(i) The graph $\{(\mathbb{P}',\omega),\ \omega \in \Omega,\ \mathbb{P}' \in \mathcal{P}(t,\omega)\} \subseteq \mathbf{M}(\Omega) \times \Omega$ is analytic.

(ii) For $\mathbb{P}$-a.e. $\omega \in \Omega$, the regular conditional probability distributions associated to $\mathbb{P}$ and $\mathcal{F}_\theta$ belong to the appropriate shifted version of $\mathcal{P}$.

(iii) $\mathcal{P}$ is stable by bifurcation.

(iv) If $\nu : \Omega \longrightarrow \mathbf{M}(\Omega)$ is a $\mathcal{F}_\theta$-measurable kernel belonging to the appropriate shifted version of $\mathcal{P}$, then there exists $\nu^n : \Omega \longrightarrow \mathbf{M}(\Omega)$, which is a $\mathcal{F}_\theta$-measurable kernel such that $\mathbb{P}(\nu^n = \nu) \underset{n\to\infty}{\longrightarrow} 1$ and the measure obtained by concatenating $\mathbb{P}$ and ν^n after θ belongs to $\mathcal{P}(s,\overline{\omega})$

We consider some scalar $\mathcal{G}_T-$measurable random variable ξ, which represents the payoff of the financial claim we which to super–replicate. For any $(s,\omega) \in [0,T] \times \Omega$, we naturally restrict the subset $\mathcal{P}$ to

$$\mathcal{P}^\xi := \left\{\mathbb{P} \in \mathcal{P},\ \mathbb{E}^{\mathbb{P}}[\xi^-] < +\infty\right\}.$$

Notice that such a restriction can be interpreted as suppressing measures which induce arbitrage opportunities in our market. Our interest is on the problem of super-replication under model uncertainty and the corresponding dual formulation. Given some initial capital $X_0 \in \mathbb{R}$, the wealth process is defined by

$$X_t^H := X_0 + \int_0^t H_s dB_s,\ t \in [0,T],$$

where $H \in \mathcal{H}^\xi$, the set of admissible trading strategies, is defined by

$$\mathcal{H}^\xi := \left\{H \in \mathbb{H}^0(\mathbb{R}^d,\mathbb{G}^{\mathcal{P}}) \cap \mathbb{H}^2_{\text{loc}}(\mathbb{P},\mathbb{R}^d,\mathbb{G}^{\mathcal{P}}),\ X^H \text{ is a } \mathbb{P}-\text{super-martingale},\ \forall \mathbb{P} \in \mathcal{P}^\xi\right\}.$$

Our main result is

Theorem 1.1.1. *Let $\mathcal{P}$ be a family of probability measures satisfying Assumption 1.1.1. Let ξ be an upper semi–analytic random variable with $\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi^+] < +\infty$. Then*

$$V(\xi) := \inf\left\{X_0,\ X_T^H \geq \xi,\ \mathcal{P}^\xi - q.s. \text{ for some } H \in \mathcal{H}^\xi\right\} = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi].$$

Moreover, existence holds for the primal problem, i.e. $V(\xi) + \int_0^T H_s dB_s \geq \xi$, $\mathcal{P}^\xi-$q.s. for some $H \in \mathcal{H}^\xi(\mathcal{P})$.

1.2 2BSDEs with general reflection and game options (see [150])

One of the conclusions that can be taken from duality result of Theorem 1.1.1 is that the super–hedging price under model uncertainty can be written as a supremum of an expectation, or of a conditional expectation if one wants to consider dynamic prices. However, conditional expectation are the archetype of BSDEs, since the latter can also be considered as non–linear, time consistent, conditional expectations (see Peng [176]).

[1] Notice that this Assumption is quite technical in nature. We have therefore decided to state it in an informal way and refer the reader to [187] for details.

Therefore, the super–hedging price derived in Section 1.1 is nothing else than the value at time 0 of the supremum of a family of BSDEs. It turns out that this is exactly this intuition which is behind the definition of a second–order BSDE, as introduced by Soner, Touzi and Zhang in [206, 207] (see also the earlier paper by Cheridito, Soner, Touzi and Victoir [47] using a quite different definition). As a supremum of (non–linear) conditional expectations, the solution to a 2BSDE is in fact a (non–linear) super–martingale. Therefore, due to the Doob–Meyer decomposition, a non–decreasing process is naturally added to the definition of a solution, compared to that of classical BSDEs. Intuitively speaking, such a process somehow "pushes" the solution of the 2BSDE upwards so that it remains above all the solutions of the associated BSDEs.

With this intuition in mind, it becomes quite straightforward to transpose classical pricing and hedging problems of finance to the model uncertainty setting, by trying to define a second–order version of the BSDEs associated to the problem at hand. Hence, one of the papers I wrote during my PhD, namely [152], was in part motivated by the extension of the links, proved by El Karoui, Pardoux and Quenez [77] (see also [76]), between prices of American options and reflected BSDEs, to the robust case. Since reflected BSDEs are constrained to stay above a given obstacle, a non–decreasing process is added to the solution and helps satisfying this constraint. This means that for reflected 2BSDEs, the reflection constraint pushes in the same direction as the second order feature, in the sense that they both necessitate to add a non–deceasing process to the solution. However, this would no longer be the case for an upper obstacle, which would require to add a non–increasing process instead, giving us overall simply a bounded variation process. Hence upper and lower reflections for 2BSDEs play fundamentally different roles, and the upper reflection case was left open in [152].

Since the seminal paper of Cvitanić and Karatzas [53], it is common knowledge that so–called doubly reflected BSDEs (that is when the solution is constrained to stay between two obstacles) are fundamentally linked to Dynkin games but also to game or Israeli options, as coined by Kifer [135], and then studied in particular by Hamadène [105] or Hamadène and Zhang [106]. The main goal of my work [150] had therefore been to understand the behaviour of doubly reflected 2BSDEs and their connections with robust games and options.

Before discussing the main findings of [150], let us recall the definition of a doubly reflected 2BSDE. We fix an appropriate subspace $\mathcal{P}$ of the set $\mathcal{P}_S$ introduced in Section 1.1 and introduce a family of standard doubly reflected 2BSDEs. For any $\mathbb{P} \in \mathcal{P}$, $\mathbb{F}-$stopping time τ, and $\mathcal{F}_\tau-$measurable random variable ξ, let $(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P},+}, k^{\mathbb{P},-})$ denote the unique solution to

$$\begin{cases} y_t^{\mathbb{P}} = \xi + \int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} dB_s + k_\tau^{\mathbb{P},-} - k_t^{\mathbb{P},-} - k_\tau^{\mathbb{P},+} + k_t^{\mathbb{P},+}, \ 0 \leq t \leq \tau, \ \mathbb{P}-a.s. \\ L_t \leq y_t^{\mathbb{P}} \leq S_t, \ \mathbb{P}-a.s. \\ \int_0^t \left(y_{s^-}^{\mathbb{P}} - L_{s^-}\right) dk_s^{\mathbb{P},-} = \int_0^t \left(S_{s^-} - y_{s^-}^{\mathbb{P}}\right) dk_s^{\mathbb{P},+} = 0, \ \mathbb{P}-a.s., \ \forall t \in [0,T]. \end{cases}$$

The desired definition is then

Definition 1.2.1. *We say (Y, Z, V) (with appropriate integrability) solves a doubly reflected 2BSDE with upper obstacle S, lower obstacle L, generator $\widehat{F}$ and terminal condition ξ if*

- *$\forall \mathbb{P} \in \mathcal{P}$, the process V has bounded variation, $\mathbb{P}-a.s.$, and*

$$Y_t = \xi - \int_t^T \widehat{F}_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + \int_t^T dV_s, \ 0 \leq t \leq T, \ \mathbb{P}-a.s.$$

- *We have the following minimality condition for $0 \leq t \leq T$*

$$V_t + k_t^{\mathbb{P},+} - k_t^{\mathbb{P},-} = \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}(t^+,\mathbb{P})}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'} \left[V_T + k_T^{\mathbb{P}',+} - k_T^{\mathbb{P}',-} \middle| \mathcal{F}_{t^+} \right], \ \mathbb{P}-a.s., \ \forall \mathbb{P} \in \mathcal{P},$$

where $\mathcal{P}(t^+,\mathbb{P})$ *is the subset of* $\mathcal{P}$ *consisting of measures coinciding with* $\mathbb{P}$ *on* $\mathcal{F}_{t^+}$.

- $L_t \leq Y_t \leq S_t$, $\mathcal{P}-q.s.$

The first main result is a wellposedness one (see [150, Theorems 3.1 and 4.5]). Besides standard regularity and Lipschitz type assumptions, I would like to point out the fact that this results fundamentally needs that the upper barrier is a semi–martingale under all the measures in $\mathcal{P}$. This is linked to the fact that our approach to existence and uniqueness requires to obtain *a priori* estimates for the solution, and even in the standard case, these are in general not available if at least one of the obstacles is not a semi–martingale (see the illuminating works of Crépey and Matoussi [51] or of Pham and Zhang [184] for details). We also obtain the expected result that the super–hedging and sub–hedging prices of Israeli options under uncertainty are obtained as solutions to doubly reflected 2BSDEs (see [150, Theorem 5.4], as well as the fact that, under appropriate Isaac's type conditions, the same holds for robust Dynkin games (see [150, Theorem 5.8]).

1.3 2BSDEs with jumps (see [132, 131])

Another extension of the 2BSDE theory that I have been working on since the defence of my PhD has been to introduce jumps in the dynamics of the processes. The underlying idea, in terms of applications, was that the introduction of jumps allows, for instance, for models of financial markets with uncertainty on the intensity (or the frequency) as well as the size of the jumps of the assets. Such a framework is particularly appealing for applications to insurance problematics, where catastrophic events or accidents, are modelled using jump processes. The robust version of these problems could then give interesting insights on the risks that are actually taken by insurance companies (it is actually the motivation of one of my current work on contracting theory, see Chapter 12).

From a more theoretical point of view, the original motivation for the introduction of 2BSDEs by Cheridito, Soner, Touzi and Victoir [47] was to provide a probabilistic representation for solutions to fully non–linear PDEs. Since it is also known since the works of Buckdahn and Pardoux [33] and Barles, Buckdahn and Pardoux [10], that BSDEs with jumps provide a probabilistic representation for semi–linear partial integro–differential equations (PIDEs for short), it has to be expected that 2BSDEs with jumps will be linked to fully non–linear PIDEs. These two considerations were at the heart of our motivations when working on [132, 131], where we obtained these desired results. However, some important and fundamental differences with the case without jumps deserve to be commented.

(*i*) A crucial issue concerning BSDEs is the aggregation problem, consisting in defining universal version of family of processes indexed by mutually singular measures. In the continuous case, the quadratic variation of the canonical process can always be aggregated, but in general, the aggregation of the jump compensators is not possible. A direct consequence is that the generator of a 2BSDE with jumps must depend explicitly on the considered measure $\mathbb{P}$. This is an important difference which may lead one to think that it might not be possible to define the solution in a universal way, which would be very unfortunate from the point of view of applications, since, the different components of the solution to a 2BSDE are generally associated to prices or optimal trading or replicating strategies.

(*ii*) Another technical aspect is linked to the definition of the set of measures $\mathcal{P}$ that has to be considered, and which represents the allowed domain of volatility and jump uncertainty. Roughly speaking, one has to construct probability measures such that the law of the canonical process (on the Skorokhod space) is the law of a semi–martingale with a specified triplet of characteristics, which means solving semi–martingale problems in the sense of, for instance, Jacod and Shiryaev [120]. The main problem lies in the fact that a very important[2] property that these measures have to satisfy in order to be able to define in a simple

[2]One can actually make do without this, but this requires much more work, and will be explained below in Section 1.5.

way standard BSDEs, is that of the predictable martingale representation property. This means that the set $\mathcal{P}$ cannot be too large. On the other hand, the set $\mathcal{P}$ has to satisfy the generic properties recalled in Assumption 1.1.1, which means that it has to be rich enough. The main difficulty is then to define an adequate set of measures $\mathcal{P}$. This construction is delicate and technical, and we refer the interested reader to [132] for details.

That being said, the definition of a 2BSDE with jumps is as follows

Definition 1.3.1. *We say (Y, Z, U), in appropriate spaces, is a solution of a 2BSDE with jumps and generator $F^{\mathbb{P}}$ if*

- $Y_T = \xi$, $\mathcal{P} - q.s.$
- *For all $\mathbb{P} \in \mathcal{P}$ and $0 \leq t \leq T$, the process $K^{\mathbb{P}}$ defined below is predictable and has non–decreasing paths $\mathbb{P} - a.s.$*

$$K_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t F_s^{\mathbb{P}}(Y_s, Z_s, U_s)ds + \int_0^t Z_s dB_s^{\mathbb{P},c} + \int_0^t \int_E U_s(x)\tilde{\mu}_B^{\mathbb{P}}(dx, ds),$$

where $B^{\mathbb{P},c}$ is the continuous martingale part of B under $\mathbb{P}$ and $\tilde{\mu}_B^{\mathbb{P}}$ is the compensated jump measure of B under $\mathbb{P}$.

- *The family $\left\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\right\}$ satisfies the minimality condition*

$$K_t^{\mathbb{P}} = \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}(t^+,\mathbb{P})}{}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'}\left[K_T^{\mathbb{P}'} \middle| \mathcal{F}_t\right], \ 0 \leq t \leq T, \ \mathbb{P} - a.s., \ \forall \mathbb{P} \in \mathcal{P}.$$

In addition to an existence and uniqueness theory for such equations, we obtained in [132, 131] the following link between these objects and fully non-linear PIDEs in the so–called Markovian case. For simplicity, let us denote $B_s^{t,x}$ the canonical process starting from x at time t, for any $(t, x) \in [0, T] \times \mathbb{R}^d$. For any $\mathbb{P} \in \mathcal{P}^t$, an appropriate shifted version of $\mathcal{P}$, we define the following 2BSDE with jumps

$$u(t, x) := \sup_{\mathbb{P} \in \mathcal{P}^t} \mathcal{Y}_t^{\mathbb{P},t,x}, \ (t, x) \in [0, T] \times \mathbb{R}^d,$$

where $\mathcal{Y}^{\mathbb{P},t,x}$ is the unique solution to the following BSDE with jumps

$$\begin{aligned}\mathcal{Y}_s^{\mathbb{P},t,x} = g(B_T^{t,x}) + \int_s^T f\Big(r, B_r^{t,x}, \mathcal{Y}_r^{\mathbb{P},t,x}, \mathcal{Z}_r^{\mathbb{P},t,x}, \mathcal{U}_r^{\mathbb{P},t,x}, \widehat{a}_r^t, \nu_r^{t,\mathbb{P}}\Big)dr - \int_s^T \mathcal{Z}_r^{\mathbb{P},t,x} dB_r^{t,\mathbb{P},c} \\ - \int_s^T \int_E \mathcal{U}_r^{\mathbb{P},t,x}(e)\widetilde{\mu}_{B^t}^{\mathbb{P}}(de, dr), \ \mathbb{P} - a.s., \ s \in [t, T],\end{aligned}$$

where $\widehat{a}^t$ and $\nu^{t,\mathbb{P}}$ are respectively the density of the quadratic variation and the compensator of the jump measure of $B^{t,x}$, under $\mathbb{P}$.

Consider the following PIDE

$$\begin{cases} -\partial_t u(t, x) = \hat{h}(t, x, u(t, x), Du(t, x), \mathcal{K}u(t, x, \cdot), D^2u(t, x), u(t, x + \cdot)), \ (t, x) \in [0, T) \times \mathbb{R}^d \\ u(T, x) = g(x), \ x \in \mathbb{R}^d, \end{cases} \tag{1.3.1}$$

where the operator $y \longmapsto \mathcal{K}v(t, x, y)$ is defined by $\mathcal{K}v(t, x, y) := v(t, x+y) - v(t^-, x)$ and $\hat{h}$ is a Legendre–type transform of f defined by

$$\hat{h}(t, x, y, z, u, \gamma, v) := \sup_{(a,\nu)} \left\{\frac{1}{2}\mathrm{Tr}(a\gamma) + \int_E (Av)(x, e)\nu(de) - f(t, x, y, z, u, a, \nu)\right\},$$

with

$$(Av)(y,e) := v(e+y) - v(y) - e.(\nabla v)(y).$$

The result is then, see [131, Theorems 5.9 and 5.13].

Theorem 1.3.1. *Under appropriate technical conditions, u is the unique viscosity solution of the PIDE (1.3.1).*

1.4 2BDSDEs and fully nonlinear SPDEs (see [151])

The starting point of this work is the following parabolic fully–nonlinear stochastic partial differential equation (SPDE for short) over the time interval $[0,T]$

$$du_t(x) + H(t,x,u_t(x),Du_t(,x),D^2u_t(x))\,dt + g(t,x,u_t(x),Du_t(x)) \circ d\overleftarrow{W}_t = 0, \tag{1.4.1}$$

with a given final condition $u_T = \Phi$ and where g and H are given nonlinear maps. The differential term involving $d\overleftarrow{W}_t$ refers to the backward stochastic integral with respect to a finite-dimensional Brownian motion

The class of stochastic PDEs similar to (1.4.1) and its extensions arises in a number of applications among which we can mention asymptotic limits of PDEs with rapid oscillations in time, phase transitions and front propagation in random media and with random normal velocities, filtering and stochastic control with partial observations, path–wise stochastic control theory, mathematical finance... The main difficulties with such equations are that, even in the deterministic case, there are no global smooth solutions in general, the fully nonlinear character of the equations seems to make them inaccessible using the classical martingale theory employed for the linear case, and even when smooth solutions may exist, the equations cannot be described in a point–wise sense, because of the everywhere lack of differentiability of the Brownian paths.

Pardoux and Peng [175] have introduced backward doubly stochastic differential equation (BDSDE for short) so as to give a Feynman-Kac's type formula for the following semi–linear SPDEs

$$du_t(x) + [\mathcal{L}u_t(x) + f(t,x,u_t(x),Du_t(x))]\,dt + g(t,x,u_t(x),Du_t(x)) \circ d\overleftarrow{W}_t = 0, \tag{1.4.2}$$

where $\mathcal{L}$ is a second order diffusion operator and f is a nonlinear function, hence allowing for a probabilistic interpretation of classical and Sobolev's solutions of (1.4.2).

Our main aim in this paper was to devise a complete theory of existence and uniqueness of second order BDSDEs (2BDSDEs) and to investigate their links with classical and viscosity (in an appropriate sense) solutions of (1.4.1). The main difficulties in such an approach were obviously due to the presence of the two sources of randomness being mixed together through the nonlinear coefficients of the equation. This called for a meticulous definition and study of the class of probability measures that had to be considered, since the latter had to specifically depend on the forward and the backward noise at the same time. Another particularly difficult point compared to the extensions of the 2BSDE theory described in the previous sections, is that the presence of the extra backward integral term makes it basically impossible to obtain path–wise estimates of the solutions, which were at the heart of our previous works. Furthermore, even though a fully non–linear SPDE and the associated notion of viscosity solutions have received a lot of attention recently in the literature, especially since Hairer as been awarded the Fields medal, to the best of our knowledge, ours was the first probabilistic representation of such solutions.

Nonetheless, similarly to the classical 2BSDEs, the solution of 2BDSDEs can be represented as a supremum of solutions to standard BDSDEs, and we obtain a nice wellposedness result for the following equations (see [151, Theorem 4.2])

Definition 1.4.1. *We say (Y, Z, K) (with appropriate integrability) solves a 2BDSDE with generator $\widehat{F}$ and terminal condition ξ if*

- $\forall \mathbb{P} \in \mathcal{P}$, *the process K is non-decreasing, $\mathbb{P}-a.s.$, and*

$$Y_t = \xi - \int_t^T \widehat{F}_s(Y_s, Z_s)ds - \int_t^T g_s(Y_s, Z_s) \cdot d\overleftarrow{W}_s - \int_t^T Z_s dB_s + \int_t^T dK_s,\ 0 \leq t \leq T,\ \mathbb{P}-a.s.$$

- *We have the following minimality condition for $0 \leq t \leq T$*

$$K_t = \underset{\mathbb{P}' \in \mathcal{P}(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}\left[K_T | \mathcal{F}_{t^+}\right],\ \mathbb{P}-a.s.,\ \forall \mathbb{P} \in \mathcal{P}.$$

Moreover, under regularity conditions on the coefficients, we can indeed show that a probabilistic interpretation of classical solutions of (1.4.1) is obtained via the associated Markovian 2BDSDEs, see [151, Theorem 5.1]. Then, we introduce a notion of stochastic viscosity solution for (1.4.1) in the case when the intensity of the noise g does not depend on the gradient of the solution, similarly to the seminal papers of Buckdahn and Ma [31, 32]. This restriction is due to our approach based on Doss–Sussmann's transformation to convert fully non–linear SPDEs to fully non–linear PDEs with random coefficients. Once this is done, we can use the probabilistic representation of these PDEs thanks to 2BSDEs with quadratic growth that I had studied during my PhD thesis in [191]. The precise result is given in [151, Theorem 5.3].

1.5 Stochastic control for non–linear kernels (see [190])

The last article of this chapter is probably the most technical one, but also contains the most interesting results from the point of view of applications. As mentioned above, all the 2BSDE results that I have been describing so far rely on very stringent regularity assumptions for the generator and the terminal condition, which roughly say that they have to be continuous in ω for the sup (or Skorokhod) topology on the considered path space. This is in stark contrast with the first paper [187] described in this chapter, where the contingent claims, which correspond to the terminal condition for the associated 2BSDE, are merely measurable. These continuity requirements are not only disappointing from the purely theoretical point of view, they are also extremely restrictive when it comes to applications. For instance, there is absolutely no reason why a contingent claim in finance or insurance would have to have this kind of regularity, and such limitations had already been pointed out in one of the articles of my PhD, namely [153].

Roughly speaking, these continuity requirements first appeared in this literature with the so–called $G-$expectations approach to volatility uncertainty of Peng [178]. There, what would later be understood as the equivalent of a 2BSDE with generator equal to 0 was constructed using fully non–linear PDEs, which necessarily implies to assume some regularity. Though the first construction of a solution to a 2BSDE carried out in [206] had definitely probabilistic and stochastic control flavours, regularity assumptions were still made in order to have simple covering type arguments to prove a fundamental step in the construction, that is to say that the value function considered satisfied a dynamic programming principle. These conditions were lifted for the first time by Nutz and van Handel [169] in the case of a null generator, who adapted the classical measurable selection approach to the dynamic programming principle of Bertsekas and Shreve [15] and Dellacherie [63]. However, their result is crucially based on some measurability stability results by integration against linear stochastic kernels, which in plain words means when taking conditional expectations. And this exactly corresponds to what happens when one tries to define 2BSDEs with a generator equal to 0. The general case however imposes to study general non–linear kernels, and this was exactly the first aim of the paper [190].

For every $(t,\omega) \in [0,T] \times \Omega$ and $\mathbb{P} \in \mathcal{P}(t,\omega)$, a family of measures on the Wiener space satisfying an appropriate version of Assumption 1.1.1, we consider the following BSDE

$$\mathcal{Y}_s = \xi - \int_s^T f\left(r, B_{\cdot\wedge r}, \mathcal{Y}_r, \widehat{a}_r^{1/2}\mathcal{Z}_r, \widehat{a}_r\right) dr - \left(\int_s^T \mathcal{Z}_r \cdot dB_r^{c,\mathbb{P}}\right)^{\mathbb{P}} - \int_s^T d\mathcal{M}_r, \ \mathbb{P}-a.s.,$$

where $\mathcal{M}$ is a martingale orthogonal to the canonical process B, which has to be added to the solution since the filtration considered does not in general satisfy the predictable martingale representation property. We then define, for every $(t,\omega) \in [0,T] \times \Omega$,

$$\widehat{\mathcal{Y}}_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\mathcal{Y}_t^{\mathbb{P}}\right].$$

The first main result of [190] is the following dynamic programming principle.

Theorem 1.5.1. *Under some technical assumption, for all $(t,\omega) \in [0,T] \times \Omega$, one has $\widehat{\mathcal{Y}}_t(\omega) = \widehat{\mathcal{Y}}_t(\omega_{t\wedge\cdot})$, and $(t,\omega) \longmapsto \widehat{\mathcal{Y}}_t(\omega)$ is $\mathcal{B}([0,T]) \otimes \mathcal{F}_T-$universally measurable. Moreover, for all $(t,\omega) \in [0,T] \times \Omega$ and $\mathbb{F}-$stopping time τ taking values in $[t,T]$, we have*

$$\widehat{\mathcal{Y}}_t(\omega) = \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\mathcal{Y}_t^{\mathbb{P}}\left(\tau, \widehat{\mathcal{Y}}_\tau\right)\right],$$

where $\mathcal{Y}_t^{\mathbb{P}}\left(\tau, \widehat{\mathcal{Y}}_\tau\right)$ is obtained from the solution to the following BSDE with terminal time τ and terminal condition $\widehat{\mathcal{Y}}_\tau$,

$$\mathcal{Y}_t = \widehat{\mathcal{Y}}_\tau - \int_t^\tau f\left(s, B_{\cdot\wedge s}, \mathcal{Y}_s, \widehat{a}_s^{1/2}\mathcal{Z}_s, \widehat{a}_s\right) ds - \left(\int_t^\tau \mathcal{Z}_s \cdot dB_s^{c,\mathbb{P}}\right)^{\mathbb{P}} - \int_t^\tau d\mathcal{M}_s, \ \mathbb{P}-a.s.$$

This fundamental result allows to prove that a suitably defined right–continuous modification of $\widehat{\mathcal{Y}}$ provides a solution to the 2BSDE defined as in the previous sections, with the notable exception that one has to still add an orthogonal martingale in the solution, see [190, Definition 4.1] for more details. The uniqueness part is basically implied by a stochastic control representation of the solution, or associated *a priori* estimates that are derived from the very general result that I obtained with my co–authors in [24] and which will be detailed in Section 2.4 below.

We also defined a notion of "saturated" solutions to 2BSDEs, when the family of probability measures $\mathcal{P}$ is rich enough for the classical optional decomposition of super–martingales to hold (see El Karoui and Quenez [79, 80] or Kramkov [137]), and used it to obtain the first non–linear version of this decomposition, which we apply to the problem of super–hedging in a non–linear market (for instance with interest rates having non–zero bid–ask spreads) under model uncertainty: the price of any sufficiently integrable contingent claim is given by the value at 0 of the solution to a saturated 2BSDE, see [190, Theorem 5.2].

The last result of [190] links 2BSDEs with continuous data and the recently introduced notion of viscosity solutions of path–dependent PDEs (PPDEs for short), see the series of papers by Ekren, Ren, Touzi and Zhang [74, 73, 193, 194]. In [190, Theorems 6.1 and 6.2], we showed that our notion of 2BSDEs provided the unique viscosity solution of the associated Hamilton–Jacobi–Bellman type PPDEs. More generally, our result implies that any non–Markovian stochastic control control in a diffusion setting which is amenable to dynamic programming, admits a probabilistic representation in the form of a 2BSDE. This result will be at the heart of our treatment of Principal–Agent problems in Part V below. Notice of course that general stochastic control problems have also alternative probabilistic representations, notably in terms of so–called BSDEs with constrained jumps, as studied in illuminating papers by Bandini, Cosso, Fuhrman,

Kharroubi, Pham, Xing and Zeni [9, 50, 90, 91, 134, 179]. They use clever randomization techniques leading to objects that are technically simpler (since they do not involve family of singular measures), but which for instance cannot clearly be linked to the theory of path–dependent PDEs, which is the natural counterpart to classical PDEs in the non–Markovian case, and therefore are of less interest when it comes to Principal–Agent problems.

I would also like to point out that in spite of the fact that the results of [190] were written in a Lipschitz setting, we actually gave a general recipe for proving wellposedness results for *any* type of 2BSDEs, namely

(*i*) Show that the associated (super, minimal or maximal) BSDE solutions satisfy a comparison principle and stability result with respect to their terminal condition.

(*ii*) Show that the solution to the BSDE can be obtained by taking appropriate limits of Lipschitz BSDEs.

(*iii*) Show that a non–linear Doob–Meyer decomposition (first introduce by Peng [177], see also our generalisation in [23], detailed in Section 2.3 below) holds for the associated non–linear martingales.

If all of these properties hold, then we have shown in [190] that all our result are still true. In particular, this implies that in all the results that I obtained during my PhD, namely [152, 153, 191, 185],every strong regularity assumptions made can now be lifted.

Chapter 2

BSDE theory and applications

The grand thing is to be able to reason backwards.

Arthur Conan Doyle

Even though the results presented in the previous chapter were almost exclusively related to the 2BSDE theory, it goes without saying that they had to be based on a deep and precise understanding of the basics, namely the standard BSDE theory itself. Indeed, BSDEs being the building blocks for 2BSDEs, any progress in the second–order case had initially to go through the first–order one. The works described in this chapter follow this line, and were, at least in the beginning, motivated by some of the articles of Chapter 1.

The first three articles summed up in Section 2.1 are devoted to theoretical aspects of ordinary BSDEs in jumps setting. To be more precise, the first two papers [129, 130] propose a comprehensive analysis of BSDEs with jumps whose generator has a quadratic–exponential growth with respect to its control variables. This setting is very important for applications, since a great number of stochastic control problems that can be linked to BSDEs fall in this category (as for instance the utility maximization problem described afterwards in Section 2.2). In addition to wellposedness results, we obtain a certain number of results concerning the non–linear expectations associated to these BSDEs, such as a non–linear Doob–Meyer decomposition, converse comparison theorems or inf–convolution results, the latter being especially useful for applications to risk–sharing problems in insurance. The last paper [172] is in the same spirit as [23] and [24] in the sense that it tries to push as far as possible the generality of the setting considered for the analysis of the BSDEs. Hence, we study there BSDEs in a filtered probability space for which the only assumption made on the filtration is that it satisfies the usual conditions. In particular, the filtration is not necessarily generated by a Brownian motion and/or a Poisson process, and therefore does not by its own nature satisfy the predictable martingale representation property, nor is it quasi–left continuous. Furthermore, the martingale driving the equation is general, and does not have a triplet of characteristics which is assumed to be absolutely continuous with respect to the Lebesgue measure. Finally, the time horizon of the problem is not fixed and is allowed to be an unbounded stopping time.

The time horizon of the problem is exactly the crux of the paper [123] detailed in Section 2.2. There we consider a classical problem of utility maximisation for an investor in a financial market, in line with the seminal work of Merton [155]. But there is a twist in the sense that the investor knows that the whole market is going to default, at a random time which he cannot predict, but which will be for sure less than his investment's time of horizon. Such a problem gives rise to BSDEs with jumps with a singular generator which actually explodes near the default time. In addition to solving the latter investment problem, we give in [123] a comprehensive study of these BSDEs.

The last two sections are devoted to technical results in the same spirit as [172], in terms of their generality. Section 2.3 studies a non–linear version of the Doob–Meyer–Mertens decomposition for làdlàg semi–martingales in general filtrations and some of its potential applications, while Section 2.4 gives a *modus*

operandi to obtain general *a priori* estimates for super–solutions of BSDEs (which include in particular 2BSDEs), again when working in general filtrations. All these theoretical results play a fundamental role in the work [190] detailed in Section 1.5.

2.1 Backward SDEs with jumps (see [129, 130, 172])

The papers [129, 130] study BSDEs with jumps of the form

$$Y_t = \xi + \int_t^T g_s(Y_s, Z_s, U_s)ds - \int_t^T Z_s dB_s - \int_t^T \int_E U_s(x)\widetilde{\mu}(dx, ds),\ t \in [0,T],\ \mathbb{P}-a.s.,$$

where the terminal condition ξ is supposed to be bounded and the generator g is assumed to satisfy

Assumption 2.1.1. (i) For fixed (y,z,u), g is $\mathbb{F}-$progressively measurable.
(ii) For any $p \geq 1$, $\mathcal{T}_0^T$ denoting the set of stopping times valued in $[0,T]$

$$\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_0^T} \mathbb{E}\left[\left(\int_\tau^T |g_t(0,0,0)|\, dt\right)^p \middle| \mathcal{F}_\tau\right] < +\infty,\ \mathbb{P}-a.s. \tag{2.1.1}$$

(iii) There exist $(\beta,\gamma) \in \mathbb{R}_+ \times \mathbb{R}_+^*$ and a positive predictable process α satisfying the same integrability condition (2.1.1) as $g_t(0,0,0)$, such that for all (ω,t,y,z,u)

$$-\alpha_t - \beta|y| - \frac{\gamma}{2}|z|^2 - \frac{j_t(-\gamma u)}{\gamma} \leq g_t(\omega,y,z,u) - g_t(0,0,0) \leq \alpha_t + \beta|y| + \frac{\gamma}{2}|z|^2 + \frac{j_t(\gamma u)}{\gamma}, \tag{2.1.2}$$

where

$$j_t(u) := \int_E \left(e^{u(x)} - 1 - u(x)\right)\nu_t(dx).$$

In such a quadratic–exponential setting, results in the literature were really scarce. Using a clever splitting technic introduced by Tevzadze [215] in the continuous case, we managed to prove, under additional regularity assumptions, existence of a solution in BMO type spaces, see [129, Theorem 5.4]. Uniqueness is then implied by a comparison theorem obtained under concavity type assumptions on the generator, see [129, Theorem 6.3].

The paper [130] pushes this study further, by considering the non–linear expectation operator associated to those BSDEs, in the following sense

Definition 2.1.1. *Let ξ be bounded and let g be such that the BSDE with generator g and terminal condition ξ has a unique solution and such that comparison holds (for instance g could satisfy any of the conditions in* [129, Theorem 6.3]*). Then for every $t \in [0,T]$, we define the conditional $g-$expectation of ξ as $\mathcal{E}_t^g[\xi] := Y_t$, where (Y,Z,U) solves the following BSDE*

$$Y_t = \xi + \int_t^T g_s(Y_s, Z_s, U_s)ds - \int_t^T Z_s dB_s - \int_t^T \int_E U_s(x)\widetilde{\mu}(dx, ds).$$

Then, a process X is called a g-submartingale (resp. g-supermartingale) if $\mathcal{E}_s^g[X_t] \geq$ (resp. $\leq$)X_s, $\mathbb{P}-a.s.$, for any $0 \leq s \leq t \leq T$.

From this definition, we proceeded to show in [130, Theorem 3.1] that any $g-$super–martingale satisfies a non–linear Doob–Meyer decomposition, and obtaines as a corollary a reverse comparison theorem (that is to say that if solutions of BSDEs are ordered whatever the terminal condition, then the generators must

be ordered as well) in [130, Corollary 3.1]. We also obtained generalisations of the classical down–crossing inequalities in [130, Theorem3.2], which imply that $g-$super–martingales always admit càdlàg modifications. When the generator g is also convex, we obtained duality type representations, and computed optimal risk transfers between two economic agents using $g-$expectations as risk measures. If the total risk is modeled by a random variable ξ_T, and at time t, both agents assess their risk using a monotone convex monetary risk measure (resp. $\mathcal{E}_t^{g^1}$ and $\mathcal{E}_t^{g^2}$), Agent 1 will take in charge $\xi_T - F$ and transfer to the second agent the quantity F, and for this he will pay a premium $\pi(F)$. Agent 1 minimizes his risk under the constraint that a transaction takes place, and thus solves :

$$\inf_{F,\pi}\{\mathcal{E}_t^{g^1}(\xi_T - F + \pi)\} \text{ under the constraint } \mathcal{E}_t^{g^2}(F-\pi) \leq \mathcal{E}_t^{g^2}(0) = 0.$$

This is an indifference pricing rule for the first agent, that is to say the price at which he is indifferent (from a risk perspective) between entering and not entering into the transaction. The optimal sharing rule can then be obtained using inf–convolution of $g--$expectations, see [130, Theorem 4.2].

Finally, [172] consider what could arguably be the most general possible form of BSDEs

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s(\cdot))dC_s - \int_t^T Z_s^\top dX_s^c - \int_t^T \int_{\mathbb{R}n} U_s(x)\widetilde{\mu}(ds,dx) - \int_t^T dN_s, \tag{2.1.3}$$

which are driven by a general martingale X, having as continuous part X^c and compensated jump measure $\widetilde{\mu}$, where T is an arbitrary stopping time, and where C is a non–decreasing process absolutely continuous with respect to the angle bracket $\langle X\rangle$ of X. Since the filtration is not assumed to be quasi–left continuous, the process C may have jumps, which complicates significantly the analysis. Our main result basically states that if the jumps of C are bounded and g satisfies a stochastic Lipschitz condition (see [172, Assumption (F3)]), then there is always a unique solution of (2.1.3) in suitably defined spaces. This is a very important improvement on the previous related results in the literature, which assumed that either the jumps of C or the Lipschitz constant of g were small enough. Besides its theoretical interest, this wellposedness result is the first step towards a general robustness theory for BSDEs with jumps, similar to the continuous case considered by Briand, Delyon and Mémin [26], which in turn would be very important for obtaining implementable numerical schemes for the associated BSDEs, as well as 2BSDEs, as will be explained later on in Chapter 5 describing the results of [189].

2.2 Singular BSDEs and utility maximization (see [123])

If the possibly random time horizon in the BSDEs considered in [172] played an important role, it is even more the case in the work [123] considered in this section. There, we revisited a classical problem of mathematical finance, namely the utility maximisation problem faced by an investor having access to a financial market, which was first introduced in the seminal paper by Merton [155]. The main difference with the classical setting is that the investor will no longer have access to the market after an unexpected time τ, that is to say a random time which is not a stopping time for his own filtration, which is also known to have finite support in a given set $[0,T]$. In other words, either the investor of the market will disappear after τ. In that context we think of τ as a death time, either for the agent himself, or for the market (or a specific component of it). Such an assumption can be quite relevant in practice. Indeed, for instance many life–insurance type markets consist of products with very long maturities (up to 95 years for universal life policies and to 120 years for whole life maturity). It is therefore reasonable to consider that during such a period of time an agent in age of investing money in the market will die with probability 1. Another example would be given by markets whose maximal lifetime is finite and known at the beginning of the investment period, like for

instance carbon emission markets in the United States. Mathematically, the investor's optimisation program has to be formulated as

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}[U(X^{\pi}_{T\wedge\tau})], \tag{2.2.1}$$

where for any admissible investment strategy π, X^{π} is the associated wealth process of the investor. With such a random horizon, Kharroubi, Lim and Ngoupeyou [133] have proved that the usual BSDE with jumps that can be associated to this stochastic control problem can actually be solved by considering instead another BSDE without jumps. Loosely stated, their result applied in our context says that the value function of the investor, facing a liability ξ, is given as the initial value of the following BSDE with jumps

$$Y_t = \xi - \int_{t\wedge\tau}^{T^{\wedge}\tau} Z_s \cdot dW_s - \int_{t\wedge\tau}^{T^{\wedge}\tau} U_s dH_s - \int_{t\wedge\tau}^{T^{\wedge}\tau} f(s, Y_s, Z_s, U_s)ds,\ t\in[0,T], \tag{2.2.2}$$

with

$$f(s,\omega,z,u) := -\frac{\alpha}{2}\mathrm{dist}^2\left(z+\frac{1}{\alpha}\theta_s, C_s(\omega)\right) + z\cdot\theta_s + \frac{\|\theta_s\|^2}{2\alpha} - \lambda_s\frac{e^{\alpha u}-1}{\alpha},$$

where H is a counting process associated to the random time τ, λ the associated intensity process, and C a closed set into which the trading strategies of the investor are restrained to live. Then, if one considers the following decompositions, which are direct consequences of the general theory of enlargement of filtrations

$$\xi = \xi^b \mathbf{1}_{T<\tau} + \xi^a_\tau \mathbf{1}_{\tau\leq T},\ f(t,.)\mathbf{1}_{t<\tau} = f^b(t,.)\mathbf{1}_{t<\tau},$$

our main theoretical tool is the following proposition

Proposition 2.2.1. *Let A be a real–valued, $\mathcal{F}_T-$measurable random variable. Assume that the (Brownian) BSDE*

$$Y^b_t = A - \int_t^T f^b(s, Y^b_s, Z^b_s, \xi^a_s - Y^b_s)ds - \int_t^T Z^b_s \cdot dW_s,\ t\in[0,T], \tag{2.2.3}$$

admits a solution (Y^b, Z^b), then (Y,Z,U) defined as

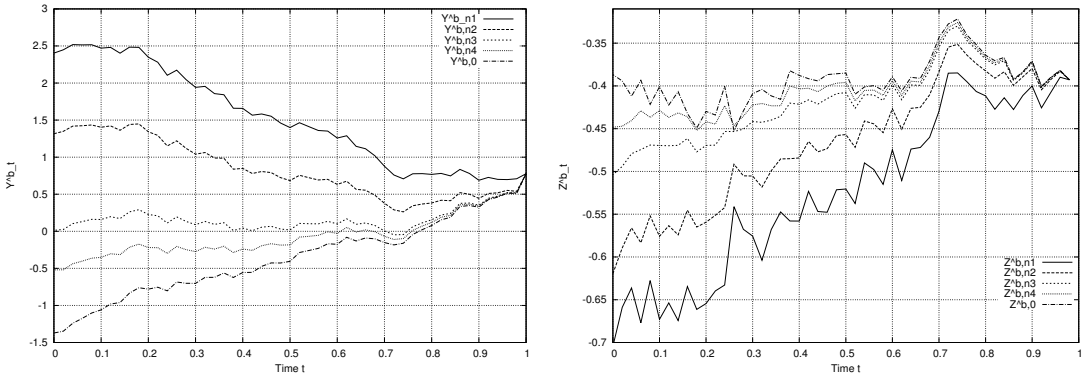
$$Y_t := Y^b_t \mathbf{1}_{t<\tau} + \xi^a_\tau \mathbf{1}_{t\geq\tau},\ Z_t := Z^b_t \mathbf{1}_{t\leq\tau},\ U_t := (\xi^a_t - Y^b_t)\mathbf{1}_{t\leq\tau},$$

is a solution of the BSDE (2.2.2).

The previous result is somehow misleading since the terminal condition A in (2.2.3) plays no role at all. Therefore, if there are solutions for different values of A, this would imply that the problem of the investor would be ill–posed. However, the main consequence of the support of τ being bounded is that the intensity λ is not integrable on $[0,T]$, which makes the generator f^b of the BSDE singular, and thus outside of any of the classes considered previously in the literature. And it is actually this fact that implies indirectly that a necessary condition for BSDE 2.2.3 to have a solution is that $A = \xi^a_T$, hence solving the above apparent paradox.

We provide in [123, Theorem 4.15] a general existence and uniqueness result for such BSDEs and therefore solve the associated utility maximisation problem, using a truncation procedure consisting in replacing λ by $\lambda\wedge n$ and then let n go to infinity and prove that the associated BSDEs converge in the appropriate spaces. This also provides a numerical algorithm to compute the solution of the associated BSDEs, see Figure 2.1. When there is a default time, which corresponds to the case $n\longrightarrow+\infty$, the value function is obviously less than the case without default time (which corresponds to $n=0$). We can interpret this by the fact that the performance of the investor when she knows that her default time appears before the maturity is less than her performance in the case without default time.

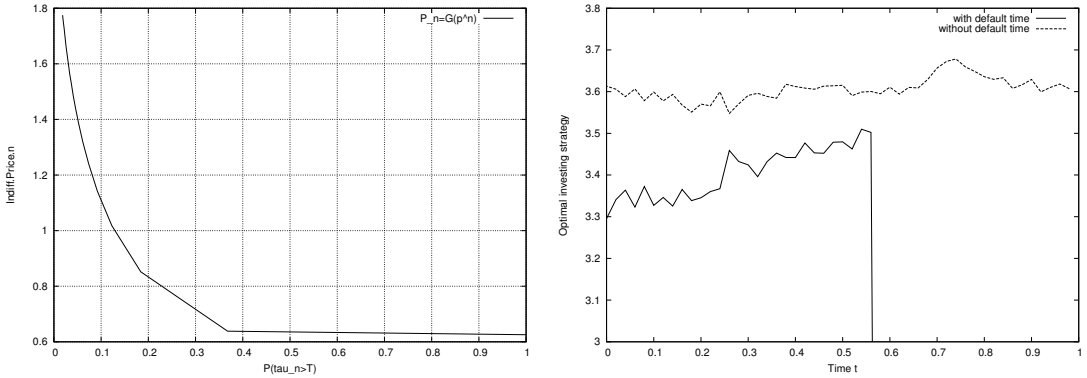
Figure 2.1: Solutions of BSDEs with truncation levels $n_1 = 50$, $n_2 = 10$, $n_3 = 2$, $n_4 = 1$ and $n = 0$.



We can also compute the utility indifference price associated to a given European option in this model. Figure 2.2 below studies the impact of the probability of default p_n on this price P_n, or even the associated optimal trading strategies.

Figure 2.2: Indifference price and optimal strategies with and without default.

(a) P_n as a function of p^n, $n \in \{0, \cdots, 50\}$ (b) Optimal strategies



Notice that when we assume that the default time τ appears almost surely before the maturity, the investor tends to be more cautious by investing less in the risky asset. It is quite reasonable since he knows that he will pay ξ^a_τ which is a non–negative random variable at default. Note that contrary to what happens for small times where the trading strategies are merely mirrors of each other, the strategy in the default problem becomes more and more similar to the one in the non–default case and the former tends to coalesce with the latter.

2.3 Doob–Meyer–Mertens decomposition (see [23])

The Doob–Meyer decomposition is one of the fundamental result of the general theory of processes. As already mentioned in the previous sections, a now classical result by Peng [177] states that it also holds in the semi–linear context of the so–called $g-$expectations. Namely, reminding our reader that an optional process X is said to be a (strong) $g-$super–martingale if for all stopping times $\sigma \leq \tau$, $X_\sigma \geq \mathcal{E}^g_\sigma[X_\tau]$ almost surely, when X is additionally right–continuous, it admits a unique decomposition of the form

$$-dX_t = g_t(X_t, Z^X_t)dt - Z^X_t \cdot dW_t + dA^X_t,$$

in which Z^X is a predictable process, and A^X is non–decreasing predictable. In particular, when $g \equiv 0$, this is the classical Doob–Meyer decomposition in a Brownian filtration framework.

As fundamental as its classical version, this result was used by many authors in various contexts, and is a crucial step in the construction of solutions to any 2BSDE, as I pointed it out before. Nonetheless, virtually all the existing results are limited to right–continuous $g-$super–martingales, while the right–continuity might be very difficult to prove in general, if even correct. The method generally used by the authors is then to work with the right–limit process, which is automatically right–continuous, but they then face important difficulties in trying to prove that it still shares the dynamic programming principle of the original process. This was sometimes overcome at the price of stringent assumptions, in particular in the context of singular optimal control problems.

However, in the classical case, $g \equiv 0$, it is well known that these technical difficulties can actually be avoided by appealing to the version of the Doob-Meyer decomposition for super–martingales with only right and left limits, see El Karoui [75]. This particular result has been established by Mertens [154], while Dellacherie and Meyer [64, Vol. II, Appendice 1] provides an alternative proof. This paper extends Mertens's result to a general family of semi–linear, non–expansive and time consistent expectation operators, which includes $g-$expectations, and is assumed to satisfy stability properties with respect to monotone convergence (see [23, Assumptions (Tc), (S) and (Sld)]).

Let $\mathcal{T}$ be the set of stopping times. Our main result works in general filtrations and writes

Theorem 2.3.1. *Let S be a $g-$super–matingale system s.t. $\{S(\tau),\ \tau \in \mathcal{T}\}$ is uniformly integrable, then there exists a làdlàg $g-$super–matingale X such that $S(\sigma) = X_\sigma$ for all $\sigma \in \mathcal{T}$. If in addition,* esssup$\{S(\tau)\}$ *has moments of order $p > 1$, then there exists Z predictable, A predictable and non–decreasing, and a càdlàg martingale N, orthogonal to W (all in appropriate spaces), such that*

$$S(\sigma) = X_\sigma = X_\tau + \int_\sigma^\tau g_s(X_s, Z_s)ds + A_\tau - A_\sigma - \int_\sigma^\tau Z_s \cdot dW_s - \int_\sigma^\tau dN_s,$$

for all stopping time $\sigma \geq \tau$. This decomposition is unique.

The main use of this result is to simplify greatly many existing arguments in the literature, whenever right continuity *per se* is irrelevant. For instance, we used it to obtain in [23, Theorem 4.2]a general duality for the minimal super–solution of a backward stochastic differential equation with constraints on the gains–process, which we obtain in a framework that could not be considered before in the literature, compare with [21, 54]. We also obtained for the first time a general non–linear optional decomposition result, see [23, Theorem 4.1] which is central in the obtention of the similar result detailed in my work [190]. In both cases, these *a priori* difficult results turn out to be easy consequences of Theorem 2.3.1.

2.4 Estimates for BSDEs in general filtrations (see [24])

The results described in Section 2.3, like the ones of [172] and the ones detailed in the current section are all part of larger project. It should now be getting clear for our reader that the BSDE theory can in some sense be understood as a theory of time–consistent non–linear martingales, and sub– or super–martingales as well. The classical case ranks among the most studied topics of probability theory, and as such is has been fully mature for a long time as part of the so–called general theory of processes. It is my opinion that the same cannot really be said for BSDEs. Indeed, after the seminal papers of Pardoux and Peng [173, 174], most of the papers on BSDEs have concentrated on Brownian filtrations. Granted, there has been numerous works generalising this, by concerning Brownian/Poisson filtrations, or even more general cases, but papers

that I am aware of considering at BSDEs in filtrations satisfying only the usual hypotheses of completeness and right–continuity can be counted on the fingers of one hand (see the backstory and the references given in the introduction of [172] for more details). Not only can this be seen as a bit disappointing from the mathematical point of view, but it can also be quite frustrating in the sense that, at least part of the techniques and ideas used in the great majority of papers on BSDEs share striking similarities. But people most of the time have to redo them all over again for the specific case they are looking at, as it may not have been considered in the literature before.

It was hence my opinion, and that of my co–authors, that this called for a deeper understanding of the basics of the theory, as well as papers trying to study the most general cases possible. This is exactly what had driven the paper [24], where we looked at one extremely important tool for BSDEs, namely *a priori* estimates for the solution. They are ubiquitous in the theory, and extremely useful to obtain existence or uniqueness properties, stability results for associated numerical schemes,... Hence, our main goal was to give a generic and unified way to obtain, in the most general filtration possible, these estimates for any (super–)solution of a BSDE. This framework is general enough to include classical BSDEs, reflected BSDEs, constrained BSDEs, weak BSDEs or 2BSDEs, even though we did not consider the most general situation concerning the martingale driving the BSDE, in the sense that it had no jumps and had to have a quadratic variation absolutely continuous with respect to the Lebesgue measure. However, extensions are among my projects, and some of them have already been achieved in [172].

That being said, if a super–solution of a BSDE is given as

$$Y_t = \xi - \int_t^T g_s(Y_s, Z_s)ds - \int_t^T Z_s \cdot dW_s - \int_t^T dM_s + \int_t^T dK_s,\ t \in [0,T],\ \mathbb{P}-a.s., \tag{2.4.1}$$

where M is a martingale orthogonal to W, then the moral of [24] is that

"It is sufficient to control the norm of Y to control the norm of (Y, Z, M, K)."

Such a result is basically the non–linear equivalent of the estimates of Meyer [156, Theorem 1] that apply to general super–martingales and imply that one can control the non–decreasing process in the Doob–Meyer decomposition of a super–martingale by the super–martingale itself. This result is based deeply on potential theory, and had not been available, *per se*, in the context of BSDEs, except when working in L^2-type spaces, as shown by Schweizer [202, proof of Lemma 6]. This is, at least in my view, an illuminating instance reinforcing what I was saying before, that is to say that despite the now reasonably long history of the BSDE theory[1], we are still lacking, in some sense, a really general framework for them.

Let us introduce the following spaces for some given $p > 1$ and $\alpha > 0$

- L^p is the space of $\mathbb{R}-$valued and $\mathcal{F}_T$-measurable random variables ξ such that $\|\xi\|_{L^p}^p := \mathbb{E}\left[|\xi|^p\right] < +\infty$.
- $\mathbb{S}^p$ (resp. $\mathbb{S}_r^p$) denotes the space of $\mathbb{R}-$valued, $\mathbb{F}-$adapted processes Y, with $\mathbb{P}-a.s.$ làdlàg (resp. càdlàg) paths, such that $\|Y\|_{\mathbb{S}^p}^p := \mathbb{E}\left[\sup_{0\le s\le T}|Y_t|^p\right] < +\infty$.
- $\mathbb{M}^{p,\alpha}$ is the space of $\mathbb{R}-$valued, $\mathbb{F}-$adapted martingales M, with $\mathbb{P}-a.s.$ càdlàg paths, such that M is orthogonal to W and

$$\|M\|_{\mathbb{M}^{p,\alpha}}^p := \mathbb{E}\left[\left(\int_0^T e^{\alpha s} d[M]_s\right)^{\frac{p}{2}}\right] < +\infty.$$

[1] Recall that the first modern exposition of the theory dates back 26 years now!

- $\mathbb{H}^{p,\alpha}$ (resp. $\mathbb{H}_1^{p,\alpha}$) is the space of $\mathbb{R}^d-$valued (resp. $\mathbb{R}-$valued) and $\mathbb{F}-$predictable processes Z such that

$$\|Z\|^p_{\mathbb{H}^{p,\alpha}} := \mathbb{E}\left[\left(\int_0^T e^{\alpha s}\|Z_s\|^2\,ds\right)^{\frac{p}{2}}\right] < +\infty.$$

- $\mathbb{I}^{p,\alpha}$ (resp. $\mathbb{I}_+^{p,\alpha}$, $\mathbb{I}_r^{p,\alpha}$, $\mathbb{I}_{+,r}^{p,\alpha}$) denotes the space of $\mathbb{R}-$valued, $\mathbb{F}-$predictable processes with bounded variations K, with $\mathbb{P}-a.s.$ làdlàg (resp. non–decreasing làdlàg, càdlàg, non–decreasing càdlàg) paths, such that

$$\|K\|^p_{\mathbb{I}^{p,\alpha}} := \mathbb{E}\left[\left(\int_0^T e^{\frac{\alpha}{2}s}d\mathrm{TV}(K)_s\right)^p\right] < +\infty$$

  and $K_0 = 0$. In the above $\mathrm{TV}(K)$ denotes the total variation of K.

- For $\alpha = 0$, we define $\mathbb{M}^p := \mathbb{M}^{p,0}$, $\mathbb{H}^p := \mathbb{H}^{p,0}$, $\mathbb{H}_1^p := \mathbb{H}_1^{p,0}$, $\mathbb{I}^p := \mathbb{I}^{p,0}$, $\mathbb{I}_+^p := \mathbb{I}_+^{p,0}$, $\mathbb{I}_r^p := \mathbb{I}_r^{p,0}$, $\mathbb{I}_{+,r}^p := \mathbb{I}_{+,r}^{p,0}$.

Then our first main result is

Theorem 2.4.1. *Let $(Y,Z,M,K) \in \mathbb{S}^p \times \mathbb{H}^p \times \mathbb{M}^p \times \mathbb{I}_+^p$ be a solution of (2.4.1). Then, under Lipschitz–type assumptions on g, for any $\alpha \geq 0$, there is a constant $C^\alpha_{2.4.1}$ such that*

$$\|Z\|^p_{\mathbb{H}^{p,\alpha}} + \|M\|^p_{\mathbb{M}^{p,\alpha}} + \|K\|^p_{\mathbb{I}^{p,\alpha}} \leq C^\alpha_{2.4.1}\Big(\|\xi\|^p_{L^p} + \|Y\|^p_{\mathbb{S}^p} + \big\|g^0\big\|^p_{\mathbb{H}_1^{p,\alpha}}\Big).$$

Consider next two terminal conditions ξ^1, ξ^2, as well as two generators g^1 and g^2. We then denote by $(Y^i, Z^i, M^i, K^i) \in \mathbb{S}^p \times \mathbb{H}^p \times \mathbb{M}^p \times \mathbb{I}_+^p$ a solution of (2.4.1) with terminal condition ξ^i and generator g^i. For notational simplicity, we also define

$$\delta Y := Y^1 - Y^2,\ \delta Z := Z^1 - Z^2,\ \delta M := M^1 - M^2,\ \delta K := K^1 - K^2.$$

Our second main result gives stability type results, in the sense that it shows that one can control the difference of solutions to the above BSDEs by only the difference the Y parts, and of course the data of the BSDEs.

Theorem 2.4.2. *Under Lipschitz–type assumptions on g, for any $\alpha \geq 0$, there is a constant $C^\alpha_{2.4.2}$ such that*

$$\|\delta Z\|^p_{\mathbb{H}^{p,\alpha}} + \|\delta(M-K)\|^p_{\mathbb{M}^{p,\alpha}} \leq C^\alpha_{2.4.2}\left(\|\delta\xi\|^p_{L^p} + \|\delta Y\|^p_{\mathbb{S}^p} + \|\delta Y\|^{\frac{p}{2}\wedge(p-1)}_{\mathbb{S}^p} + \big\|\delta g(Y^1_\cdot, Z^1_\cdot)\big\|^p_{\mathbb{H}_1^{p,\alpha}}\right).$$

We want to draw the reader's attention to the fact that when it comes to the difference of solutions, one cannot hope to control separately δM and δK, we give a counter–example in [24, Example A.1]. Furthermore, if one has more knowledge on the process K, which is the case for instance for reflected BSDEs, then these estimates can be improved, see [24, Propositions 3.2 and 3.3]. We actually used them to obtain the first general wellposedness result for BSDEs with càdlàg obstacle in general and non–quasi–left continuous filtrations, see [24, Theorem 3.1].

Chapter 3

Perspectives

There are two ways to do great mathematics. The first is to be smarter than everybody else. The second way is to be stupider than everybody else – but persistent.

Raoul Bott

Since so many people have written articles on the BSDE theory, one could naturally be thinking that there is not much left to do on the subject. However, as hinted in Chapter 2, beyond the applications which quite often lead to new types of BSDEs having a specific characteristic which has not been treated yet in the literature, there is still a surprising lack of a general theory encompassing BSDEs, BSDEs with jumps or the so–called semi–martingale BSDEs which are driven by càdlàg martingales. Some of my last contributions were going in this direction, and I intend to pursue it in my future research. First, we are still working with Antonis Papapantoleon (TU Berlin) and Alexandros Saplaouras (TU Berlin) on an application of our wellposedness results in [172] to obtain robustness results (or semi–martingale type stability results) for general BSDEs with jumps, in the spirit of the continuous–time case treated in [26] with in mind that this should be an important first step in devising a numerical scheme for 2BSDEs with jumps, for which there is no results so far in the literature. A second on–going project with Bruno Bouchard (Université Paris–Dauphine), Nicole El Karoui (UPMC Sorbonne Universités) and Xiaolu Tan (Université Paris–Dauphine) aims at generalising the recent approach of Barrieu and El Karoui [11] for quadratic BSDEs to BSDEs with a generator with general type of growth condition. They adopt a purely forward point of view and englobe the quadratic BSDEs in a class of semi–martingales whose triplet of characteristics satisfy special properties, and then use classical inequalities of the general theory of processes to obtain *a priori* estimates and stability results. We would like to understand how this approach could be extended beyond the quadratic setting.

When it comes to 2BSDE however, the theory being after all quite recent (5 years old), there are still lots of exciting opportunities. Beyond applications, especially to contract theory, as will be explained later in Part V, the theory itself can be pushed further. Hence, we have obtained with Halil Mete Soner (ETH Zürich), Nizar Touzi (École Polytechnique) and Jianfeng Zhang (University of Southern California) preliminary results that allow to link the weak formulation (that is to say the so–called case of "control vs control") of general differential games, where both players can control non–Markovian, continuous diffusion processes, to a new class of 2BSDEs, obtained as an infimum of 2BSDEs. The successive inf and sup create an object which is not immediately amenable to measurable selection arguments, as is well know in the differential game literature, but we manage to circumvent this problem by a clever use of the Isaacs's condition together with a link between our formulation of the game and the more classical "strategy versus control" one. We can then have a nice theory, including *a priori* estimates, for these 2BSDEs and prove that they provide probabilistic representations of the associated games.

Part III

Regularity and numerics for (2)BSDEs

Chapter 4

Densities and regularity for BSDEs

Mathematical proofs should only be communicated in private and to consenting adults.

Victor Klee

The first chapter of the second part of this manuscript will be devoted to the contributions of my co–authors and myself to Malliavin calculus and the regularity theory for BSDEs. Our original motivation for looking into these problems was twofold

(*i*) One of the main applications of Malliavin calculus is to study the regularity of the law of random variables, and while this has been applied in numerous instances for SDEs, stochastic PDEs, delay equations or stochastic Volterra equations (see [168, Chapter 2, notes and comments] for more details and references), there are only two papers in the literature addressing this problem for BSDEs, due to Antonelli and Kohatsu–Higa [5] and Aboura and Bourguin [1].

(*ii*) Having access to the marginal laws of the solutions to a BSDE, and more generally knowing estimates on their densities (with respect to the Lebesgue measure) can be a precious tool to design or fine–tune numerical schemes, or to have a deeper understanding of the behaviour of the solutions themselves.

With these motivations in mind, we started working on problem of obtaining a general theory of existence of densities for solutions of BSDEs which would go further than the existing literature. This was achieved in Lipschitz and quadratic settings in [149], whose results will be summed up in Section 4.1. On our way to solving the problems encountered in [149], we obviously had to rely on results ensuring that solutions to BSDEs were Malliavin differentiable, since the classical Bouleau and Hirsch criterion implying existence of a density involves precisely Malliavin derivatives. However, to our surprise, we ended up realising that there seemed to be an important gap in the literature, in terms of required assumptions, between results for Markovian and non–Markovian BSDEs. This lead us to try and understand the reasons for such a difference, and to devise, if not better at least alternative sets of conditions. Incidentally, our approach, which relied on old results on Malliavin–Sobolev spaces, lead us to a completely new characterisation of Malliavin differentiability for random variables. This program was carried out in [148], described in more lengths in Section 4.2. Finally, the criterion obtained in [148] involved a parameter which, roughly speaking, had to be strictly less than some $p > 1$ to ensure the fact that a given random variable belonged to the Malliavin space $\mathbb{D}^{1,p}$. However, the case of whether one could actually take this parameter equal p was left open. We then managed to devise counter–examples giving a negative answer to this question in [118], summed up in Section 4.3. Moreover, we also gained further insights into the fine structure of Malliavin type spaces through other type of counter–examples involving our criterion.

4.1 Density analysis for BSDEs (see [149])

As mentioned before, there is a surprisingly limited number of papers dealing with existence of densities for solutions to BSDEs. The first contribution [5] studies one–dimensional BSDEs and gives sufficient conditions ensuring the existence, as well as smoothness, for the density of Y component of the BSDE. The second paper [1] goes a little bit further by giving Gaussian estimates for this density, as well as an existence result for the density of the Z component, but only for BSDEs with driver which are linear in the z variable, which are, up to a Girsanov transformation, known to be roughly similar to BSDEs with a generator which actually does not depend on z.

The two main tools used to obtain these results are the Bouleau and Hirsch criterion, namely

Theorem 4.1.1. *Let F be Malliavin differentiable, with Malliavin derivative $(D_sF)_{s\in[0,T]}$. Assume that $\int_0^T \|D_sF\|^2\, ds > 0$, $\mathbb{P}-a.s.$ Then F has a probability distribution which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$,*

as well as Nourdin and Viens's formula [167]

Theorem 4.1.2. *F has a density ρ with the respect to the Lebesgue measure if and only if the random variable $g_F(F-\mathbb{E}[F])$ is positive a.s., where if one can write $DF = \Phi_F(W)$ for some map Φ, we have*

$$g_F(x) := \int_0^\infty e^{-u}\mathbb{E}\left[\mathbb{E}^*\left[\int_0^T (\Phi_F(W))_s(\widetilde{\Phi_F^u}(W))_s ds\right]\middle| F-\mathbb{E}(F)=x\right] du,\ x\in\mathbb{R},$$

where $\widetilde{\Phi_F^u}(W) := \Phi_F(e^{-u}W + \sqrt{1-e^{-2u}}W^*)$ with W^* an independent copy of W defined on a probability space $(\Omega^*,\mathcal{F}^*,\mathbb{P}^*)$, and $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$ (Φ_F being extended on $\Omega\times\Omega^*$). In this case, the support of ρ, denoted by $supp(\rho)$, is a closed interval of $\mathbb{R}$ and for all $x\in supp(\rho)$

$$\rho(x) = \frac{\mathbb{E}(|F-\mathbb{E}[F]|)}{2g_F(x-\mathbb{E}[F])}\exp\left(-\int_0^{x-\mathbb{E}[F]}\frac{udu}{g_F(u)}\right).$$

Our take on the problem also uses these two fundamental results, but we managed to pushed them further so as to obtain the first comprehensive theory of existence of densities for both the Y and the Z component of Lipschitz and quadratic BSDEs. The statements are quite technical and necessitate lots of notations, so that I will refrain from presenting them here (the interested reader can look at [149, Theorems 3.7, 3.11, 3.13, 3.16, 4.3, 4.7 and 4.9]), but I will nonetheless illustrate the complexity of the situations that can arise with BSDEs through a simple illustrative example given by

$$Y_t = W_1 + \int_t^T (s-W_s)ds - \int_t^T Z_s dW_s,\quad t\in[0,1],\ (T=1).$$

This equation should be extremely simple in the sense that the driver does not depend on (Y,Z), and indeed it can be solved explicitly to obtain

$$Y_t = W_t\left(-\frac{1}{2}+2t-\frac{t^2}{2}\right),\ t\in[0,1].$$

Hence Y_t is a Gaussian random variable for every time t in $(0,2-\sqrt{3})$, then $Y_{2-\sqrt{3}}=0$, and for t in $(2-\sqrt{3},1]$, Y_t is Gaussian distributed once again. This means that even in very simple cases, Y can already fail to have a density at all times!

Our second contribution in [149] concerns estimates for the densities of Y and Z. In [1], Gaussian tail estimates had been obtained, a result which in our view was the consequence of too stringent assumptions.

Indeed, in general one should not expect Gaussian estimates for solutions to BSDEs. Indeed, consider the following equation

$$Y_t = W_1^3 + \int_t^T 3W_s ds - \int_t^T Z_s dW_s,\ t \in [0,1],$$

which can be solved explicitly

$$Y_t = W_t^3 + 6W_t(1-t),\ Z_t = 3W_t^2 + 6(1-t),\ t \in [0,1],$$

from which we deduce that both Y_t and Z_t admit a density with respect to the Lebesgue's measure for t in $(0,1]$. However, it is clear that neither the law of Y_t nor the one of Z_t admits Gaussian tails. Using Nourdin and Viens's formula and the deep links between solutions to Markovian BSDEs and semi–linear PDEs, we have obtained the following general result, whose range of application can go a lot further than simply BSDEs, since it applies to any random variable of the form $v(t, W_t)$ satisfying appropriate conditions. To state our result, we introduce for any measurable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ the following two kinds of growth rates

$$\overline{\alpha_f} := \inf\left\{\alpha > 0,\ \limsup_{|x|\to+\infty}\left|\frac{f(x)}{x^\alpha}\right| < +\infty\right\},\quad \underline{\alpha_f} := \inf\left\{\alpha > 0,\ \liminf_{|x|\to+\infty}\left|\frac{f(x)}{x^\alpha}\right| < +\infty\right\}.$$

Our result then reads

Theorem 4.1.3. *Fix $t \in (0,T]$. Let $v : [0,T]\times\mathbb{R} \longrightarrow \mathbb{R}$ in $\mathcal{C}^{1,1}$ and let $P_t := v(t, W_t)$. Assume furthermore that $P_t \in L^1(\mathbb{P})$, that v is unbounded in x both from above and from below, that $v' > 0$, $\underline{\alpha_v} \in (0, +\infty)$, $\overline{\alpha_{v'}} < +\infty$ and that there exist $\tilde{\alpha} > 0$ and $K > 0$ such that:*

$$\frac{1}{v'(t,x)} \le K(1 + |x|^{\tilde{\alpha}}),\ \text{for all } x \in \mathbb{R}. \tag{4.1.1}$$

Then, the law of P_t has a density with respect to the Lebesgue measure, denoted by ρ_t, and for all $\varepsilon, \varepsilon' > 0$ and for every $y \in \mathbb{R}$

$$\rho_t(y) \le \frac{\mathbb{E}[|P_t - \mathbb{E}[P_t]|]}{2M(\varepsilon')t}\left(1 + |y|^{2\tilde{\alpha}(\overline{\alpha_{v(-1)}}+\varepsilon')}\right)\exp\left(-\int_0^{y-\mathbb{E}[P_t]} \frac{(M'(\varepsilon,\varepsilon')t)^{-1}x dx}{1 + |x+\mathbb{E}[P_t]|^{2(\overline{\alpha_{v'}}+\varepsilon)(\overline{\alpha_{v(-1)}}+\varepsilon')}}\right),$$

$$\rho_t(y) \ge \frac{(2M'(\varepsilon,\varepsilon')t)^{-1}\mathbb{E}[|P_t - \mathbb{E}[P_t]|]}{1 + |y|^{2(\overline{\alpha_{v'}}+\varepsilon)(\overline{\alpha_{v(-1)}}+\varepsilon')}}\exp\left(-\int_0^{y-\mathbb{E}[P_t]} \frac{x\left(1 + |x+\mathbb{E}[P_t]|^{2\tilde{\alpha}(\overline{\alpha_{v(-1)}}+\varepsilon')}\right)dx}{M(\varepsilon')t}\right),$$

where $M'(\varepsilon,\varepsilon')$ and $M(\varepsilon')$ are explicitly known constants.

The application of this general result is then given in [149, Theorem 5.6], and provides non–Gaussian estimates for the densities of the solutions to BSDEs. Such a result is strikingly different from the usual behaviours observed for diffusive equations like SDEs or SPDEs.

4.2 Malliavin differentiability of BSDEs (see [148])

The question of whether the solution of a BSDE was Malliavin differentiable appeared extremely naturally in the paper presented in the previous section, since Bouleau and Hirsch criterion involves precisely the Malliavin derivative of the random variable for which one is trying to prove that it admits a density. To fix notations, let us consider the following BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s,\ t \in [0,T], \tag{4.2.1}$$

This question was first addressed in a Markovian setting by Pardoux and Peng [174], that is when $\xi := g(X_T)$ and $f(t,\omega,y,z) := h(t,X_t(\omega),y,z)$ where $g : \mathbb{R} \longrightarrow \mathbb{R}$ and $h : [0,T]\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ are regular enough deterministic functions and $X := (X_t)_{t\in[0,T]}$ is the unique solution to some SDE with regular enough coefficients. In that framework, it was proved in [174, Proposition 2.2] that, under (essentially) the following conditions:

(PP1) g is continuously differentiable with bounded derivative,

(PP2) f is continuously differentiable in (x,y,z) with bounded derivatives (uniformly in time),

Y_t is Malliavin differentiable at any time t (with a similar statement for Z) and the Malliavin derivatives of Y and Z provide a solution to an explicit linear BSDE. Assumptions (PP1)–(PP2) look pretty intuitive since they basically require the Malliavin differentiability of the terminal condition ξ and of the generator f once the component (y,z) are frozen. Hence, it is natural to expect that the latter conditions can be easily generalised to the non-Markovian framework. During our review of the literature on the subject while working on [149], we discovered that the first, and surprisingly the most general, result in that direction had been given in the seminal paper of El Karoui, Peng and Quenez [78] and required more stringent conditions than the aforementioned intuitive ones. It involved the following conditions (see [78, Proposition 5.3] for a precise statement)

(EPQ1) ξ is Malliavin differentiable and $\mathbb{E}[|\xi|^4] < +\infty$.

(EPQ2) At any time $t\in[0,T]$, the r.v. $\omega \longmapsto f(t,\omega,Y_t,Z_t)$ is Malliavin differentiable with Malliavin derivative denoted by $D.f(t,Y_t,Z_t)$ such that there exists a predictable process $K^\theta := (K^\theta_t)_{t\in[0,T]}$ with $\int_0^T \mathbb{E}[(\int_0^T |K^\theta_s|^2 ds)^2]d\theta < +\infty$, and such that for any $(y_1,y_2,z_1,z_2)\in\mathbb{R}^4$ it holds for a.e. $\theta\in[0,T]$ that
$$|D_\theta f(t,\omega,y_1,z_1) - D_\theta f(t,\omega,y_2,z_2)| \le K^\theta_t(\omega)(|y_1-y_2|+|z_1-z_2|).$$

Roughly speaking, this means that ξ and $\omega \longmapsto f(t,\omega,y,z)$ have to be Malliavin differentiable on the one hand, which is intuitively the minimal expected requirement, but on the other hand one needs to enforce extra regularity conditions on each of the data, since ξ has to have a finite moment of order 4, and the Malliavin derivative of the driver f has to be Lipschitz continuous in (y,z) with a stochastic Lipschitz constant K which is sufficiently integrable. The most disturbing point there is that one can check that in the Markovian framework, Conditions (EPQ1)–(EPQ2) are strictly stronger than Conditions (PP1)–(PP2).

When we tried to understand the reason for these extra conditions, our conclusion was that they actually stemmed from the method of proof used in [78], which amounted to consider the Picard iterations associated to the BSDE, show that they were Malliavin differentiable and that the derivatives of the Picard iterations themselves solved a BSDE, and then try to pass to the limit in the latter BSDE to obtain the required Malliavin's differentiability of the limit. Even though such an approach seems extremely natural, it has an element of arbitrariness, in the sense that one could choose another approximating sequence than the Picard iteration. This is exactly the road we chose to follow in [148]. The basic idea goes back to old results of Sugita [210]. Before stating it, we need some notations. Let H be he usual Cameron-Martin space of square integrable and absolutely continuous functions, endowed with its usual inner product $\langle\cdot,\cdot\rangle_H$ (corresponding to the classical L^2 inner product but for the Radon–Nykodym densities with respect to the Lebesgue measure). Then Sugita's result states that Malliavin differentiability of a random variable is equivalent to the two following properties

(RAC) For any h in H, there exists a random variable $\tilde{F}_h$ such that $\tilde{F}_h = F$, $\mathbb{P}-a.s.$, and such that for any ω in Ω, $t\in\mathbb{R}\longmapsto \tilde{F}_h(\omega+th)$ is absolutely continuous, where $th := (th^1,\dots,th^d)$.

(SGD) There exists $\mathcal{D}F$ in $L^p(H)$ such that for any h in H,

$$\frac{F \circ \tau_{\varepsilon h} - F}{\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} \langle \mathcal{D}F, h \rangle_H, \text{ in probability.} \tag{4.2.2}$$

This characterisation was not completely suitable for us for two reasons. First the (RAC) condition seems pretty hard to verify in practice. Second, the (SGD) condition involves a limit in probability, which would be less convenient to work with in a BSDE context, since the latter actually function much better with L^p-type limits, v because of the *a priori* estimates that have been mentioned countless times in this manuscript. While trying to overcome these limitations, we actually ended up proving the following new characterisation of Malliavin differentiability, which, at least to us, seems much easier to verify in practice.

Theorem 4.2.1. *Let $p > 1$ and $F \in L^p(\mathbb{R})$. The following properties are equivalent*

(i) F belongs to $\mathbb{D}^{1,p}$.

(ii) There exists $\mathcal{D}F$ in $L^p(H)$ such that for any h in H and any $q \in [1, p)$

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\left| \frac{F \circ \tau_{\varepsilon h} - F}{\varepsilon} - \langle \mathcal{D}F, h \rangle_H \right|^q \right] = 0.$$

(iii) There exists $\mathcal{D}F$ in $L^p(H)$ and there exists $q \in [1, p)$ such that for any h in H

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\left| \frac{F \circ \tau_{\varepsilon h} - F}{\varepsilon} - \langle \mathcal{D}F, h \rangle_H \right|^q \right] = 0.$$

(iv) There exists $\mathcal{D}F$ in $L^p(H)$ such that for any h in H

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\left| \frac{F \circ \tau_{\varepsilon h} - F}{\varepsilon} - \langle \mathcal{D}F, h \rangle_H \right| \right] = 0.$$

In that case, $\mathcal{D}F = \int_0^T D_s F ds$.

Using this result, we then managed to obtain alternative conditions ensuring the Malliavin differentiability of solutions to BSDEs, which apply in the Lipschitz case, but also in the quadratic one, the latter being new in a completely non–Markovian setting in the literature, see [148, Theorems 5.1 and 7.1]. In our view, our set of conditions may be seen as more efficient than the ones in [78], in the sense that they actually are weaker than (PP1)–(PP2) in the Markovian case. In any case, our new characterisation of Malliavin differentiability opens up the way to applications in much broader settings than simply the BSDE one.

4.3 A note on Malliavin–Sobolev spaces (see [118])

The last paper in this chapter concentrates on a point linked to Theorem 4.2.1 that I have not raised until now. The attentive reader may have seen that an immediate corollary of this result is that if a random variable F belongs to the Malliavin space $\mathbb{D}^{1,p}$ for some $p > 1$, then for any $q \in [1, p)$, the following convergence holds

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\left| \frac{F \circ \tau_{\varepsilon h} - F}{\varepsilon} - \langle \mathcal{D}F, h \rangle_H \right|^q \right] = 0.$$

However, our result does not say anything about the extreme case with $q = p$. Let us consider the following definitions

Definition 4.3.1 (Strong Stochastically Gâteaux Differentiability). *Let $p > 1$ and Z be in $L^p(\mathbb{R})$.*

(i) Z is said to be Strongly Stochastically Gâteaux Differentiable of order p if there exist $\mathcal{D}Z \in L^p(H)$ such that for any $q \in (1,p)$ and any h in H

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\left| \frac{Z \circ \tau_{\varepsilon h} - Z}{\varepsilon} - \langle \mathcal{D}Z, h \rangle_H \right|^q \right] = 0.$$

We denote this property by $(SSGD_p)$ and we define $\mathcal{G}_p$ the space of random variables Z which satisfy $(SSGD_p)$ with Gâteaux derivative $\mathcal{D}Z$ in $L^p(H)$.

(ii) We say that Z satisfies $(SSGD_p(p))$ if there exists $\mathcal{D}Z \in L^p(H)$ such that for any h in H

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\left| \frac{Z \circ \tau_{\varepsilon h} - Z}{\varepsilon} - \langle \mathcal{D}Z, h \rangle_H \right|^p \right] = 0.$$

We define $\mathcal{G}_p(p)$ the space of random variables Z which satisfy $(SSGD_p(p))$ with Gâteaux derivative $\mathcal{D}Z$ in $L^p(H)$.

The question raised above translates into whether the spaces $\mathbb{D}^{1,p}$ and $\mathcal{G}_p(p)$ coincide or not. The answer is actually negative, and we provided a counter–example in [118], which we gave for $p = 2$ for simplicity and which takes the following form for some $a > 3/2$

$$Z := f_a(W_1),\ f_a(x) := e^{\frac{x^2}{4}} x^{-a} (2\pi)^{\frac{1}{4}} \mathbf{1}_{x \geq 1} + g(x) \mathbf{1}_{x<1},$$

where g is a bounded continuously differentiable map with bounded derivative, such that f is continuously differentiable on $\mathbb{R}$.

This issue being cleared, we actually pushed our analysis further and studied right–continuity properties of Malliavin–Sobolev spaces and proved

Theorem 4.3.1. *Let $p > 1$. Define $\mathbb{D}^{1,p+} := \bigcup_{\varepsilon > 0} \mathbb{D}^{1,p+\varepsilon}$. We have*

$$\mathbb{D}^{1,p+} \subsetneq \{ Z \in \mathbb{D}^{1,p},\ Z \in \mathcal{G}_p(p) \} \subsetneq \mathbb{D}^{1,p}.$$

Our first result proved that the second inclusion was indeed strict, and we again constructed a counter–example for the first one, which took the form $Z := f(|W_1|)$ where $f : \mathbb{R}_+^* \longrightarrow \mathbb{R}$ is defined by

$$f(x) := F(x) \mathbf{1}_{x \in (0, \frac{1}{2}]} + G(x) \mathbf{1}_{x > 1/2},$$

where F is a smooth map from $\mathbb{R}_+^*$ into $\mathbb{R}$ such that

$$F'(x) = \frac{1}{x^{\frac{1}{2}} \left| \log(x) \right|^{\beta}},\ x \in (0, 1/2],\ \beta > 1,$$

and G is a smooth function on $\mathbb{R}_+^*$ with compact support such that f is continuously differentiable.

Chapter 5

Numerics for 2BSDEs (see [189])

In mathematics you don't understand things. You just get used to them.

John von Neumann

This second chapter of Part III is concerned with the results in [189]. As mentioned in Chapter 4, one of the motivation for the study of regularity of BSDEs stemmed from the design of numerical schemes, since, in my view, one cannot only be interested in theoretical aspects of the objects that I have been describing so far, and leave completely out any numerical considerations. The paper described in this chapter followed the same impetus. Indeed, if there is by now a huge literature on numerical schemes for BSDEs, the same cannot be said for 2BSDEs. Indeed, numerics for BSDEs started with the early works, limited to generators that did not depend on the z variable, of Ma, Protter and Yong [145], Douglas, Ma and Protter [69], Chevance [48], Bally [7], Bally, Pagès and Printems [8], Ma, Protter, San Martín and Torres [144], before the breakthroughs of Zhang [219, 220] and Bouchard and Touzi [25], which allowed to lift the non–dependence on z assumption. Further methods were later developed notably in different papers involving Briand, Chassagneux, Crisan, Delyon, Gobet, Labart, Lemor, Manolarakis, Mémin, Richou, Turkedjiev and Warin [26, 97, 142, 94, 96, 95, 196, 27, 52, 44, 43, 46, 98, 99]. Our approach to the problem of devising a numerical scheme for 2BSDEs could be coined a "weak" approach, in the sense that it followed the original ideas developed by Ma, Protter, San Martín and Torres [144] and Briand, Delyon and Mémin [26], consisting in approximating a BSDE by approximating the driving martingales by discrete–time ones, and then pass to the limit in a weak sense, which is reminiscent of the classical Donsker theorem. Our contribution was to extend this approach to the context of 2BSDEs, which is more challenging due to the necessity of working under a non–dominated family of probability measures $\mathcal{P}$.

Our object of interest is related to the following BSDEs, defined for any $\mathbb{P} \in \mathcal{P}$, and where ":" represents the trace operator for matrices

$$\mathcal{Y}_t^{\mathbb{P}} = \xi(B_\cdot) - \int_t^T g(s, B_\cdot, \mathcal{Y}_s^{\mathbb{P}}, \mathcal{Z}_s^{\mathbb{P}}, \widehat{a}_s) : d\langle B \rangle_s - \int_t^T \mathcal{Z}_s^{\mathbb{P}} \cdot dB_s - \mathcal{N}_T^{\mathbb{P}} + \mathcal{N}_t^{\mathbb{P}}, \tag{5.0.1}$$

The main purpose of the paper was to study the weak approximation of the following optimisation problem, which is nothing else than the initial value of the associated 2BSDE

$$Y_0 := \sup_{\mathbb{P} \in \mathcal{P}} \mathcal{Y}_0^{\mathbb{P}}.$$

Let us fix some notations. Let A be some compact subset of the symmetric definite positive matrices, representing the volatility uncertainty associated to the BSDE. For every $n \geq 1$, we denote by $\Delta_n = (t_k^n)_{0 \leq k \leq n}$ a discretisation of $[0, T]$, such that $0 = t_0^n < t_1^n < \cdots < t_n^n = T$. Wet $|\Delta_n| := \sup_{1 \leq k \leq n}(t_k^n - t_{k-1}^n)$, and suppose that $|\Delta_n| \longrightarrow 0$ as $n \longrightarrow \infty$. For ease of presentation, we shall simplify the notation of the time

step size $\Delta t_k^n := t_k^n - t_{k-1}^n$ into Δt when there is no ambiguity. Similarly, we suppress the dependence in n of t_k^n and write instead t_k.

For every $n \geq 1$, let $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ be a probability space containing n independent random variables $(U_k)_{1\leq k\leq n}$. Moreover, we consider a family of functions $(H_k^n)_{1\leq k\leq n, n\geq 1}$ such that every $H_k^n : A\times[0,1] \longrightarrow \mathbb{R}^d$ is continuous in a and for some $\delta > 0$, we have for any a

$$\mathbb{E}\big[H_k^n(a,U_k)\big] = 0,\ \mathbb{V}\mathrm{ar}[H_k^n(a,U_k)] = a\Delta t,\ \mathbb{E}\big[|H_k^n(a,U_k)|^{2+\delta}\big] \leq C\Delta t^{1+\frac{\delta}{2}},$$

where it is understood that the expectation is taken under $\mathbb{P}^n$.

Define the filtration $\mathbb{F}^n := (\mathcal{F}_{t_k}^n)_{1\leq k\leq n}$, with $\mathcal{F}_{t_k}^n := \sigma(U_1, \cdots, U_k)$ and denote by E_n the collection of all $\mathbb{F}^n-$predictable $A-$valued processes $e = (a_{t_1}^e, \cdots, a_{t_n}^e)$. Then for every $e \in E_n$, the approximating martingale M^e is defined by

$$M_{t_k}^e := \sum_{i\leq k} H_i^n(a_{t_i}^e, U_i).$$

By abuse of notation, we define a continuous time filtration $\mathbb{F}^n = (\mathcal{F}_t^n)_{0\leq t\leq T}$, with $\mathcal{F}_t^n := \mathcal{F}_{t_k}^n,\ \forall t \in [t_k, t_{k+1})$ and a continuous time martingales $M_t^e := M_{t_k}^e$, for all $t \in [t_k, t_{k+1})$ on $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$.

Then for every $e \in E_n$ and $n \geq 1$, we consider the following BSDE approximating (5.0.1), where g_n is an appropriately truncated version of the original generator g

$$\mathcal{Y}_t^e = \xi(\hat{M}_\cdot^e) - \int_t^T g_n(s, \hat{M}_\cdot^e, \mathcal{Y}_{s-}^e, \mathcal{Z}_s^e, a_s^e) : d\langle M^e\rangle_s - \int_t^T \mathcal{Z}_s^e \cdot dM_s^e - \mathcal{N}_T^e + \mathcal{N}_t^e,$$

For every $n \geq 1$, denote now $Y_0^n := \sup_{e\in E_n} \mathcal{Y}_0^e$. Our main result reads

Theorem 5.0.1. *(i) Suppose that g is sufficiently regular. Then*

$$\liminf_{n\to\infty}\ Y_0^n \geq Y_0.$$

(ii) Suppose in addition that (5.0.1) satisfies an appropriate monotonicity assumption and that g does not depend on z. Then

$$\lim_{n\to\infty}\ Y_0^n = Y_0.$$

We were not able to show *(ii)* when the generator depends on z. This was deeply linked to the fact that there are considerable difficulties to obtain any convergence results for the z part of the solution. Since the publication of [189], Ren and Tan [192] have devised numerical schemes for path-dependent PDEs that can be used to obtain a numerical scheme for 2BSDEs with a convergence result that can be apply when the generator does depend on z.

By choosing appropriately the function H, we proposed two fully implementable schemes in [189] as well as associated numerical simulations, one based on finite differences, and one probabilistic scheme, which is the non–Markovian counterpart to the scheme proposed by Fahim, Touzi and Warin [87] for fully non–linear PDEs. Our numerical show that convergence seems indeed to hold, even in cases that go beyond the assumptions made in the paper.

Chapter 6

Perspectives

> And therefore education at the University mostly worked by the age–old method of putting a lot of young people in the vicinity of a lot of books and hoping that something would pass from one to the other, while the actual young people put themselves in the vicinity of inns and taverns for exactly the same reason.
>
> Terry Pratchett

As mentioned in Chapter 3, we are working with Antonis Papantoleon (TU Berlin) and Alexandros Saplaouras (TU Berlin) on generalizing the numerical scheme described in Chapter 5 to 2BSDEs with jumps, using the results that I obtained in [132, 131, 172]. This would naturally lead to probabilistic numerical schemes for fully non–linear PIDEs of Hamilton–Jacobi–Bellman type.

Another avenue of study would be to try and understand how our new criterion on Malliavin differentiability could actually be extended to context for which Malliavin calculus is not available. Indeed, the criterion that we obtained involves Gâteaux derivatives in certain directions, and therefore can always we defined, whereas Malliavin calculus for continuous processes cannot really go further than the class of so-called iso-normal Gaussian processes. If for instance we would like to understand how this could be helpful to study regularity and density existence for 2BSDE or general SPDEs, then one could not in general rely on Malliavin calculus and other techniques would be needed. We are currently exploring this together with Thibaut Mastrolia (Université Paris–Dauphine) and Anthony Réveillac (INSA Toulouse).

Part IV

Transaction costs and homogenisation of partial differential equations

Chapter 7

Transaction costs and homogenisation

For every transaction, there is someone willing to buy and someone willing to sell at an agreed price, both believing that it's good value and that the counter–party is a little crazy.

Coree T. Sol

One of the common points among all the financial applications considered in the first two parts of this manuscript was that they completely ignored the possibility of market frictions. This is an extremely useful simplifying assumption, but it is made at the expense of realism. Indeed, market frictions, be they in the form of illiquidity, transaction costs, taxes, bid–ask spreads or price impacts, are ubiquitous in all the existing financial markets, and as such deserve to be taken into account. The main problem when doing so is that, even without working in the model uncertainty setting described in Part II, this already introduce considerable mathematical difficulties. Since the main problem we will be interested in is the proportional transaction costs problem, let us specify our discussion to this context.

The general problem of investment and consumption under transaction costs, which has received a lot of attention since the seminal papers by Magill and Constantinides [146] and Constantinides [49]. Following these two works which rather concentrated on the numerical aspects of the problem, but contained already the fundamental insight that there exists a no–transaction region with the form of a wedge, Taksar, Klass and Assaf [214] studied an ergodic version of the maximisation, before the classical paper of Davis and Norman [61] put the problem into the modern framework of singular stochastic control theory. Building upon these works, Soner and Shreve [203] proposed a comprehensive analysis of the one-dimensional case (that is to say when there is only one risky asset in the market), using the dynamic programming approach as well as the theory of viscosity solutions. Their approach was then extended to the case of several risky assets by Akian, Menaldi and Sulem [3]. However, these problems are expressed in terms of variational inequalities, and never have an explicit solution. Part of the literature has therefore been concentrated on obtaining asymptotic expansions of quantities of interest, in the limit of small transaction costs. This first chapter of this part is devoted to this problem exactly, and summarises the results obtained in the publications [186, 188], where, along with my co–authors, we explored first the multidimensional consumption–investment problem in infinite horizon, and then the problem of utility indifference pricing of European contingent claims in finite horizon.

7.1 Utility maximisation in infinite horizon (see [188])

As explained above, this paper considers the situation of an investor having the possibility to invest forever in a multi–asset market with transaction costs, and who can also consume. This is similar to the model introduced by Kabanov [125] for currency markets. Furthermore, the utility function of the investor can be

general, as well as the dynamics of the assets, as long as it remains in the class of local volatility models. Let us emphasise that such a level of generality goes far beyond the earlier existing results in the literature. Indeed, the first expansion result had been obtained formally by Whalley and Wllmott [216], and the first rigorous proof of their result, in one dimension[1], in the Black-Scholes model and for power utility, appeared in the appendix of [203]. Since then, several rigorous results [17, 93, 121, 122, 197] (still in the one dimensional case) were also derived. However, the multidimensional problem, which presents intriguing free boundary problems for which regularity results remain scarce, had remained out of reach until the paper by Bichuch and Shreve [19], where they treated the case of two risky assets following arithmetic Brownian motions. Nonetheless, their method of proof required not only to construct sharp sub– and super–solutions to the dynamic programming equation satisfied by the value function, it also needed a lengthy coefficient matching for the formal expansion, which had to be done in 9 regions of the space, which would become 3^n regions for n risky assets. Our approach in the multidimensional setting follows the breakthrough achieved by Soner and Touzi [204], where they connected the asymptotic expansion for small transaction costs to the theory of homogenisation. Hence, the first order term in their expansion is shown to be written in terms of the so–called eigenvalue associated to the dynamic programming equation of an ergodic stochastic control problem. This identification allowed them to construct a rigorous proof similar to the ones in homogenisation theory (see for instance Papanicolaou and Varadhan [171], or Souganidis [208]), even though the setting at hand is not the typical incarnation of such a type of problem, notably because the "oscillatory" (or "fast") variable only appears after a change of variables and is not directly modelled in the original equations.

That being said, the financial market consists of a non–risky asset S^0 and d risky assets with price process $\{S_t = (S_t^1, \dots, S_t^d), t \geq 0\}$ given by the SDEs

$$dS_t^0 = S_t^0 r(S_t) dt, \quad dS_t^i = S_t^i \mu^i(S_t) dt + S_t^i \sum_{j=1}^{d} \sigma^{i,j}(S_t) dW_t^j, \quad 1 \leq i \leq d,$$

The portfolio of an investor is represented by the dollar value X invested in the non–risky asset and the vector process $Y = (Y^1, \dots, Y^d)$ of the value of the positions in each risky asset. These state variables are controlled by the choices of the total amount of transfers $L_t^{i,j}$, $0 \leq i, j \leq d$, from the i–th to the j–th asset cumulated up to time t. Naturally, the control processes $\{L_t^{i,j}, t \geq 0\}$ are defined as càdlàg, non–decreasing, adapted processes with $L_{0^-} = 0$ and $L^{i,i} \equiv 0$. In addition to the trading activity, the investor consumes at a rate determined by a non–negative progressively measurable process $\{c_t, t \geq 0\}$. Here c_t represents the rate of consumption in terms of the non–risky asset S^0. For any initial position $(X_{0^-}, Y_{0^-}) = (x, y) \in \mathbb{R} \times \mathbb{R}^d$, the portfolio positions of the investor are given by

$$dX_t = \big(r(S_t)X_t - c_t\big)dt + \mathbf{R}^0(dL_t), \text{ and } dY_t^i = Y_t^i \frac{dS_t^i}{S_t^i} + \mathbf{R}^i(dL_t), \ i = 1, \dots, d,$$

where

$$\mathbf{R}^i(\ell) := \sum_{j=0}^{d} \big(\ell^{j,i} - (1 + \varepsilon^3 \lambda^{i,j})\ell^{i,j}\big), \ i = 0, \dots, d, \text{ for all } \ell \in \mathcal{M}_{d+1}(\mathbb{R}_+),$$

is the change of the investor's position in the $i-$th asset induced by a transfer policy ℓ, given a structure of proportional transaction costs $\varepsilon^3 \lambda^{i,j}$ for any transfer from asset i to asset j. Here, $\varepsilon > 0$ is a small parameter and the scaling ε^3 is chosen to state the expansion results simpler. For given initial positions $S_0 = s \in \mathbb{R}_+^d$, $X_{0^-} = x \in \mathbb{R}$, $Y_{0^-} = y \in \mathbb{R}^d$, the consumption–investment problem is, U denoting the utility function of the investor, the following

$$v^\varepsilon(s, x, y) := \sup_{(c,L)} \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(c_t) dt\right].$$

[1] That is for one risky asset in the market.

The Merton value function v corresponds to the limiting case $\varepsilon = 0$ where the transfers between assets are not subject to transaction costs. Let us denote by $\mathbf{y}$ and $\mathbf{c}$ the optimal controls in the Merton problem which satisfy

$$\mathbf{c}(s,z) := \left(U'\right)^{-1}\left(v_z(s,z)\right), \ -v_{zz}(s,z)\sigma\sigma^\top(s)\mathbf{y}(s,z) := (\mu - r\mathbf{1}_d)(s)v_z(s,z) + \sigma\sigma^\top(s)\mathbf{D}_s v_z(s,z),$$

where we used the change of variables $z := x + y\cdot\mathbf{1}_d$, and where $\mathbf{D}_s = (\mathbf{D}_s^i)_{1\le i\le d}$, with $\mathbf{D}_s^i := s^i\partial/\partial s^i$. The following change of variable is the one making the fast homogenizing variable appear

$$\xi_i := \xi_i^\varepsilon(s,z) = \varepsilon^{-1}(y_i - \mathbf{y}_i(s,z)).$$

The first order expansion obtained in [188] writes

$$v^\varepsilon(s,x,y) = v(s,z) - \varepsilon^2 u(s,z) - \varepsilon^4 w(s,z,\xi) + \circ(\varepsilon^2),$$

where u is solution of the second corrector equation, with $\mathbf{D}_{ss} := (\mathbf{D}_{ss}^{i,j})_{1\le i,j\le d}$ and $\mathbf{D}_{ss}^{i,j} := s^i s^j\partial^2/(\partial s^i\partial s^j)$

$$\mathcal{A}u := \beta u - \mu\cdot\mathbf{D}_s u - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top\mathbf{D}_{ss}u] - \big(rz + \mathbf{y}\cdot(\mu - r\mathbf{1}_d) - \mathbf{c}\big)u_z - \frac{1}{2}|\sigma^\top\mathbf{y}|^2\,u_{zz} - \sigma\sigma^\top\mathbf{y}\cdot\mathbf{D}_s u_z = a,$$

and where (w,a) is the solution of the first corrector equation

$$\max_{i,j}\max\left\{\frac{\left|\sigma^\top(s)\xi\right|^2}{2}v_{zz}(s,z) - \frac{1}{2}\mathrm{Tr}\left[\alpha\alpha^\top(s,z)w_{\xi\xi}(s,z,\xi)\right] + a(s,z)\ ;\right.$$
$$\left. -\lambda^{i,j}v_z(s,z) + \frac{\partial w}{\partial\xi_i}(s,z,\xi) - \frac{\partial w}{\partial\xi_j}(s,z,\xi)\right\} = 0,$$

where $\xi\in\mathbb{R}^d$ is the dependent variable, while $(s,z)\in(0,\infty)^d\times\mathbb{R}^+$ are fixed, and the diffusion coefficient is given by

$$\alpha(s,z) := \left[\left(\mathrm{I}_d - \mathbf{y}_z(s,z)\mathbf{1}_d^\top\right)\mathrm{diag}[\mathbf{y}(s,z)] - \mathbf{y}_s^\top(s,z)\mathrm{diag}[s]\right]\sigma(s).$$

Notice that this equation is nothing else than the Hamilton–Jacobi–Bellman equation associated to an appropriately defined singular stochastic ergodic control problem.

Unlike in the one-dimensional case, there is no explicit solutions to these corrector equations, so that numerical procedures have to be used. We emphasise nonetheless that the latter are much easier to put into practice that the ones necessary to solve the original problem before taking the limit. Let us now give some of our numerical results, in the case where there are two risky assets. The colour code is as follows, 0 being the bank account, 1 and 2 the risky assets, and NT denoting the no–transaction region.

Transactions	NT	1/2	0/1	0/2	0/1 and 0/2	1/2 and 0/1	1/2 and 0/2	1/2 and 0/1 and 0/2
Color								

The first case in Figure 7.1 corresponds to the only case covered by the previous literature for which transfers between risky assets are forbidden. In line with the known results, the no-transaction region in the uncorrelated case is a rectangle. Then, under a possible correlation between the assets, the shape of the region is modified to a parallelogram, the direction of the deformation depending on the sign of the correlation. The next result in Figure 7.2 presents the same case, but allowing transaction between assets. Our first observation is that transactions between the two assets do occur, and more importantly that as a consequence, the no–transaction region seems to no longer be convex, an observation which, as far as we know, was not made before in the literature. Moreover, the introduction of correlation induces a deformation of the no–transaction region.

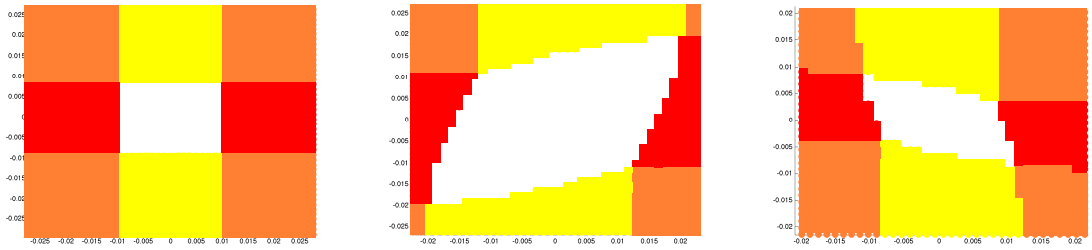


Figure 7.1: Uncorrelated (left), negative correlation (middle), positive correlation (right).

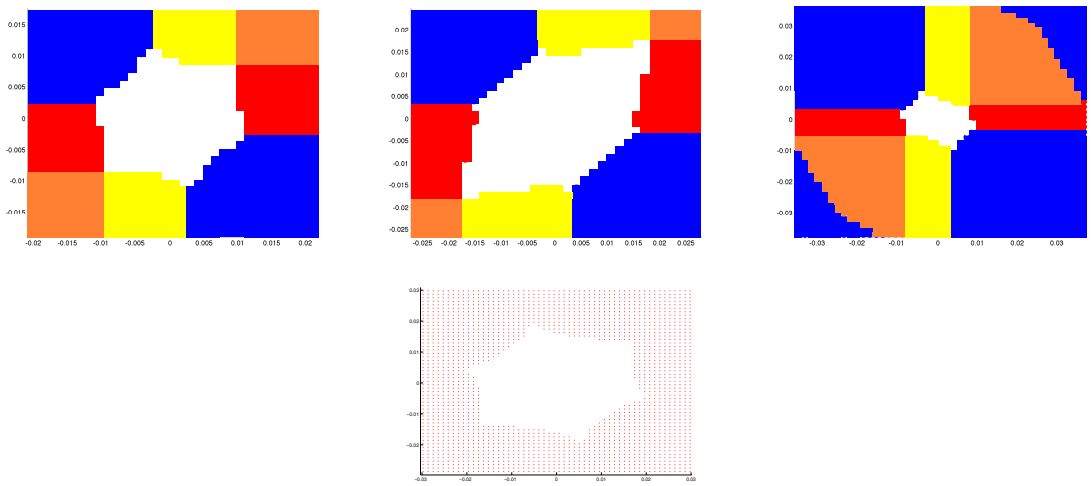


Figure 7.2: Uncorrelated (upper left), negative correlation (upper right), positive correlation (lower left), precise result (lower right).

Our last figure shows the impact of an asymmetric transaction costs structure, where we increase the transaction costs between the cash account and asset 1. As expected, there are almost no transactions between the cash account and asset 1, since they are twice as expensive as the other ones. Surprisingly, we also observe the occurrence of small zones (in black and violet), where transactions are simultaneously performed between the assets and between the assets and the cash account.

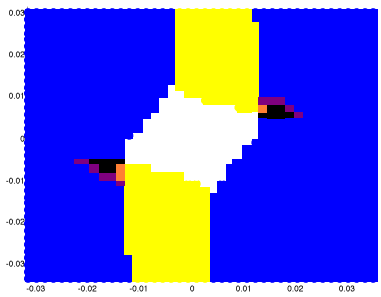


Figure 7.3: Asymmetric transaction costs.

7.2 Utility indifference pricing (see [186])

The problem considered in [186] shares a lot of similarities with the one described in Section 7.1, but concentrates on a finite horizon version of it, for which the problem at hand has finite horizon T, and thus becomes time–dependent

$$v^{\varepsilon,g}(t,s,x,y) := \sup_{\nu} \mathbb{E}_t \left[\int_t^T \kappa e^{-\int_t^\xi k(\iota,S_\iota^{t,s})d\iota} U_1(c_\xi)d\xi + e^{-\int_t^T k(\xi,S_\xi^{t,s})d\xi} \mathcal{U}_2^{\varepsilon,g} \right],$$

where $\kappa \in \{0,1\}$ is here so that we can consider simultaneously the problems with or without consumption and where k is a discount factor, and $\mathcal{U}_2$ is defined by

$$\mathcal{U}_2^{\varepsilon,g} := U_2\left(\ell^\varepsilon\left(X_T^{t,s,x,\nu}, Y_T^{t,s,y,\nu}\right) - g(S_T^{t,s})\right),$$

for some utility function $U_2 : \mathbb{R} \longmapsto \mathbb{R}$ and the liquidation function $\ell^\varepsilon : \mathbb{R}\times\mathbb{R}^d \longmapsto \mathbb{R}$ defined by

$$\ell^\varepsilon(x,y) := \sup\left\{p \in \mathbb{R},\ \left((x,y^T)^\top - p(1,\mathbf{0}_d^\top)^\top\right) \in K_\varepsilon\right\},$$

where K_ε is the solvency cone of admissible positions (see [186] for more details). Such a liquidation function represents the maximum amount, in terms of cash, that the investor can achieve by liquidating his positions in all the risky assets. Furthermore, g is the payoff of a European option that the investor will have to pay at time T.

We are interested in the so–called utility indifference price of the European option g, in both models with or without frictions. They are defined respectively by, v^g denoting the Merton value function

$$\begin{aligned} p^{\varepsilon,g}(t,s,x) &:= \inf\left\{p \in \mathbb{R} : v^{\varepsilon,g}(t,s,x+p,0) \geq v^{\varepsilon,0}(t,s,x,0)\right\} \\ p^{g}(t,s,x) &:= \inf\left\{p \in \mathbb{R} : v^{g}(t,s,x+p) \geq v^{0}(t,s,x)\right\}, \end{aligned}$$

where $v^{\varepsilon,0}$ and v^0 correspond respectively to the value functions without the option, that is to say when $g = 0$. Notice also that we consider here that the initial endowments of the investor are in cash only. This is purely for simplicity and all our results could be easily generalised if we allow the investor to have a non-zero position on the risky assets for the problem with frictions.

This question of utility indifference pricing in markets with transaction costs started with the seminal paper of Davis, Panas and Zariphopoulou [62], where it was shown that the problem of pricing an European option in a market with proportional transaction costs boiled down to solve two stochastic optimal control problems, whose value functions were shown to be the unique viscosity solutions of quasi-linear variational inequalities. Closer to our purpose, let us also point out the work [18] by Bichuch, where the author obtained an asymptotic expansion of the utility indifference price of smooth contingent claims in a market with small transaction costs and on risky asset having geometric Brownian motion dynamics, as well as Bouchard, Moreau and Soner [22], which consider an expected loss framework which includes the utility indifference price. In any case, our framework is multidimensional, unlike these two references, and our method of proof allows to weaken strongly the assumptions, since, for instance, we roughly only need to assume C^1–regularity of the option payoffs we consider, while [18] needed C^4–regularity.

That being said, the approach is similar to the one in [188], albeit made more complicated by having to obtain very precise estimates on the behaviour of all the quantities of interest near the new boundary at time T. In any case, if the expansion for v^g writes

$$v^{\varepsilon,g}(t,s,x,y) = v^g(t,s,x,y) - \varepsilon^2 u^g(t,s,z) - \varepsilon^4 w^g(t,s,z,\xi) + \circ(\varepsilon^2),$$

where u^g and w^g satisfy the same equations as the one described in Section 7.1 then the expansion for he utility indifference price reads

$$p^{\varepsilon,g}(t,s,x) = p^g(t,s,x) + \varepsilon^2 \frac{u^g(t,s,x+p^g(t,s,x)) - u^0(t,s,x)}{v_x^g(t,s,x+p^g(t,s,x))} + \circ(\varepsilon^2).$$

Our main result shows that this indeed holds true under reasonable assumptions that can all be checked in practice. Not only that, but we also give a general way of constructing nearly optimal trading strategies, that achieve the maximum utility up to the first order in the expansion, by means of constructing reflected diffusions in the no–transaction region, see [186, Section 4.1.1].

Chapter 8

Singular stochastic control (see [82])

God does not play dice with the universe; He plays an ineffable game of His own devising, which might be compared, from the perspective of any of the other players (*i.e.* everybody), to being involved in an obscure and complex variant of poker in a pitch-dark room, with blank cards, for infinite stakes, with a Dealer who won't tell you the rules, and who *smiles all the time*.

Terry Pratchett

Let us now move on to the work [82]. The results described in Chapter 7 are proved using almost exclusively PDE type arguments, since they are fundamentally based on Evans [85] viscosity solutions approach to homogenisation. As such, they are not adapted at all to possibly non–Markovian settings, which require novel ideas and approaches. One possibility is the use of duality, since the proportional transaction costs problem still has a convex structure. This was for instance used by Kallsen and Li in [126]. However, for any problem leading to non–convexity, as is the case for instance in the fixed transaction cost problem, see Altarovici, Muhle-Karbe and Soner [4], this approach would also fail. The main idea of our work [82] was to understand how such non–Markovian problems could be tackled, and since the transaction cost framework actually belongs to the more general class of singular stochastic control problems, we actually concentrated our attention on the latter.

More specifically, we studied the singularly controlled multi–dimensional non–Markovian SDE

$$X^{t,\mathbf{x},K} = \mathbf{x}(t) + \int_t^{\cdot} \mu_s^{t,\mathbf{x}}\left(X^{t,\mathbf{x},K}\right) ds + \int_t^{\cdot} f_s dK_s + \int_t^{\cdot} \sigma_s^{t,\mathbf{x}}\left(X^{t,\mathbf{x},K}\right) dB_s^t,$$

where the upper–scripts refer to appropriate shifts, $\mathbf{x}$ is a fixed initial path between 0 and t, and where the control is the vector of non–decreasing processes K. The stochastic control problem we are interested in is then

$$v^{\text{sing}}(t,\mathbf{x}) := \sup_K \mathbb{E}^{\mathbb{P}_0^t}\left[U\left(\mathbf{x} \otimes_t X^{t,\mathbf{x},K}\right)\right].$$

The heart of our approach lies in the fact that in Markovian settings, the HJB variational inequality associated to this problem has a gradient constraint, and corresponds exactly to the variational equality which is naturally associated to a so–called BSDE with constraints on the gains process (that is to say the Z component) as introduced in the seminal paper of Cvitanić, Karatzas and Soner [54]. This fact can be made rigorous by proving that both these functions are viscosity solutions of the variational inequality, and to obtain a comparison principle for the latter, which implies uniqueness.

However, in the non–Markovian setting there is, *a priori*, no longer any PDE associated. But one can then try to rely on the notion of path–dependent PDEs, which has already been introduced in this manuscript. The main issue is that there does not yet exist any theory of path–dependent variational inequalities with gradient constraints, notably because in general the set of laws of the associated controlled SDE solutions fails to have any weak compactness property, a fact which is fundamental for the approach developed by Ekren, Touzi and Zhang [72, 73, 74]. We circumvent this issue by approximating the value function v_{sing} by a sequence of value functions $(v^n)_{n\geq 0}$ when the controls K are restricted to being absolutely continuous with respect to the Lebesgue measure, with a bounded density. Under appropriate continuity assumptions, one can indeed prove convergence (notice that this is not always the case, see the counter–example of Heinricher and Mizel [108]). The associated non–Markovian control problems are now exactly the type amenable to the path–dependent PDE theory. One can then show that v^n and the following BSDE

$$\mathfrak{Y}_\cdot^{t,\mathbf{x},n} = U^{t,\mathbf{x}}\left(B^{t,\mathbf{x}}\right) + \int_\cdot^T n\rho\left[f_s^\top((\sigma_s^{t,\mathbf{x}})^\top)^{-1}\left(B^{t,\mathbf{x}}\right)\mathfrak{Z}_s^{t,\mathbf{x},n}\right]ds - \int_\cdot^T \mathfrak{Z}_s^{t,\mathbf{x},n}dW_s^{t,\mathbf{x}},$$

where the map ρ is defined by $\rho : q \in \mathbb{R}^d \longmapsto q^+ \cdot \mathbf{1}_d$ where for each $q := (q_1, \cdots, q_d)^\top \in \mathbb{R}^d$ we have used the notation: $q^+ := (q_1^+, \cdots, q_d^+)^\top$, are both viscosity solutions of the same semi–linear PPDE, for which a comparison principle holds thanks to the results of Ren, Touzi and Zhang [193]. It then turns out that the above sequence of BSDEs is nothing else than the sequence of penalised BSDEs associated to the constrained BSDE

$$\mathfrak{Y}_\cdot^{t,\mathbf{x}} \geq U^{t,\mathbf{x}}\left(B^{t,\mathbf{x}_t}\right) - \int_\cdot^T \mathfrak{Z}_s^{t,\mathbf{x}} \cdot dW_s^{t,\mathbf{x}},\ ((\sigma_s^{t,\mathbf{x}})^\top)^{-1}\left(B^{t,\mathbf{x}_t}\right)\mathfrak{Z}_s^{t,\mathbf{x}} \in K_s,$$

where the constraint set K is defined by

$$K_t := \left\{q \in \mathbb{R}^d,\ (f_t^\top q)\cdot \mathbf{e}_i \leq 0,\ \forall i \in \{1,\dots,d\}\right\}.$$

Since it is known since the work [54] that the penalised BSDEs do converge to the constrained BSDE, this gives us the desired representation, see [82, Theorem 4.7].

The application to the transaction cost problem actually requires an extension of the result to possibly degenerate volatility matrices σ, which we achieve in [82, Theorem 5.6], using non–Markovian extensions of classical convex ordering results for SDEs. Roughly stated, in this case the value function v_{sing} is obtained as a supremum (corresponding to letting the volatility matrix degenerate) of solutions to constrained BSDEs. Formally speaking, such an object is close in spirit to so–called constrained 2BSDEs, as introduced by Fabre in her PhD thesis [86]. Nonetheless, the special case of transaction costs has a simplifying structure which allows us to claim that the value function in the non–Markovian equivalent of the problem described in Section 7.1 with two risky assets (that is $d = 2$), is equal to the solution of a simply constrained BSDE, see [82, Section 5.2.2]. As far as we know, such a result is new, even in the Markovian setting.

Finally, this representation for singular control problems opens the way to many possible applications, since, as shown for instance by Bouchard, Élie and Moreau [21], it is actually possible to obtain relatively easily (at least more easily than for singular stochastic control problem for which this is extremely hard in general) *a priori* regularity results for the value function, or even dynamic programming type results. We explored such applications in [82, Section 6].

Chapter 9

Perspectives

> The fact that we live at the bottom of a deep gravity well, on the surface of a gas covered planet going around a nuclear fireball 90 million miles away and think this to be normal is obviously some indication of how skewed our perspective tends to be.
>
> Douglas Adams

Since the introduction of homogenisation techniques in the transaction costs literature, there have been a tremendous number of papers using it to obtain similar asymptotic results in related settings, like fixed transaction costs, see Altarovici, Muhle-Karbe and Soner [4], price impact models, see Moreau, Muhle-Karbe and Soner [162], hedging under expected loss constraints, see Bouchard, Moreau and Soner [22], investors with small uncertainty aversion, see Herrman, Muhle-Karbe and Seyfried [110] and so on.

However, since they use PDE techniques, they are confined to Markovian settings. One of the reasons which motivated my paper [82] was to first understand how path–dependent PDEs would be used in a context of non–Markovian singular control problems, and if one could then try to develop a theory suitable for PPDEs with gradient constraints, which should be the natural type of PPDE associated to non–Markovian singular stochastic control problem. As BSDEs provide natural viscosity solution to semi–linear PPDEs and 2BSDEs to fully non–linear PPDEs of Hamilton–Jacobi–Bellman type, given the results of [82], we expect BSDEs and 2BSDEs with constraints on the gains process to play the same role for the aforementioned variational inequalities. This is a quite challenging goal given the current state of the theory, and necessitates in any case a deep understanding of the *a priori* regularity, in both time and path, of the associated value functions. One possible way would be to use the well–known link between optimal stopping problems and singular control problems, see the pioneering works of Karatzas and Shreve on the subject [127, 128], as the associated PPDEs are much more understood, see the recent work of Ekren [71]. In any case, we are exploring these issues with Bruno Bouchard (Université Paris–Dauphine), for a very general class of singular control problems and their associated constrained BSDEs.

Part V

Contracting theory

Chapter 10

Contract theory and dynamic programming

A verbal contract isn't worth the paper it's written on.

Samuel Goldwyn

Contracting theory is the crux of the last part of this manuscript, and has incidentally been my main the topic of my research for the past two years. As recalled in the introduction, the typical instance of this theory is the situation of two economic agents, one (the Principal) wishing to enter into a contract with another (the Agent), who has the opportunity to accept it or not. The contract itself can represent an extremely wide array of possibilities: the Agent executing a task on the Principal's behalf, the Agent consuming a good produced by the Principal, the Agent buying something from the Principal... The main point however is that there is an asymmetry of information between these two parties, in the sense that the Agent has most of the time more knowledge than the Principal, and can take actions that are not observable (or cannot be contracted upon) by the Principal. This leads to what is usually called moral hazard, since the Agent can actually behaves in a way that is not in the best interest of the Principal. The main problem for the Principal now becomes to design contracts (that is wages, prices...) such that

(*i*) The Agent accepts them.

(*ii*) The Agent is given proper incentives and work in a way that allows the Principal to get the most out of the contract.

For a long time, these problems were only considered in discrete-time or static settings. For very early moral hazard models, introducing the so-called first-order approach[1] and then later its rigorous justification, see Zeckhauser [218], Spence and Zeckhauser [209], or Mirrlees [157, 158, 160, 159], as well as the seminal papers by Grossman and Hart [102], Jewitt, [124] or Holmström [111]. These problems were in general quite hard to solve, and one had to wait for the seminal paper by Holmström and Milgrom [113] to witness the treatment of specific moral hazard problems in a continuous time framework. Their work was generalised by Schättler and Sung [200, 201], Sung [211, 212], Müller [163, 164], and Hellwig and Schmidt [109], as well as Sannikov [199] using a dynamic programming and martingales approach, which is classical in stochastic control theory.

Another alternative approach, which in principle can help to tackle more general problems consists in appealing to the Pontryagin stochastic maximum principle to both the problems faced by the Agent and the Principal. This approach was initiated by Williams [217] and Cvitanić, Wan and Zhang [58], and more recently, by Djehiche and Hegelsson [67, 68][2]. However, this literature suffered from the following limitations

(*i*) The maximum principle approach generally leads to very complicated fully coupled systems of forward–backward SDEs, for which a very general theory is still lacking.

[1] The Principal proposes to the Agent a contract and a recommended behaviour, which has to satisfy the so–called incentive compatibility constraint: this behaviour has to be optimal for the Agent. The first–order approach then essentially boils down to replace that constraint with the associated first–order conditions.

[2] The interested reader is urged to read the excellent monograph of Cvitanić and Zhang [60] for a systematic presentation.

(*ii*) The dynamic programming approach fares better, in the sense that the solution in the end is generally obtained through an Hamilton–Jacobi–Bellman equation for the problem of the Principal. However, at least in my view, the way such equations are derived in the economic literature, and especially [199] seems very astute, bordering sometimes to mysterious. For instance, there is no clear mathematical reason why in that paper the only state variable necessary for the Principal is precisely the continuation of the Agent. Such opacity is at the detriment of generality.

(*iii*) Finally, none of these two approaches is able to handle moral hazard problems where the Agent controls both the drift and the volatility of the diffusion process representing whatever the contract is based on. As will be clear in the examples of delegated portfolio management and electricity pricing that will be developed in [2, 55] and described in Sections 10.2 and 10.3 respectively, this is not only a theoretical gap, but a major limitation in terms of applications.

The first section of this chapter is devoted to the paper [56], and gives the general theory for these Principal–Agent problems through dynamic programming, which is then used in the two following examples mentioned above. One of the major advantages of this approach, is that, at least in the view of my co–authors and mine, it lifts the veil of mystery shrouding the previous ones. It gives a clear *modus operandi* to tackle these problems, and shows that in their most general form, as long as the problem of the Agent falls into the spectrum of dynamic programming, so does the problem of the Principal. The latter can therefore always be attacked using the classical tools of stochastic control and HJB equations, which, and this may be the most important point from the practical point of view, gives a clear way to perform numerical computations. Finally, Section 10.4 studies [147] and adds a twist to the problem, linking it even more with the first part of this manuscript, by analyzing the influence of model uncertainty on the behaviours of the Principal and the Agent.

10.1 A general approach (see [56])

Let $T > 0$ be a given terminal time, and $\Omega := C^0([0,T], \mathbb{R}^d)$ the set of all continuous maps from $[0,T]$ to $\mathbb{R}^d$, for a given integer $d > 0$. The canonical process on Ω, representing the output Agent is in charge of, is denoted by X, and the corresponding canonical filtration by $\mathbb{F} := \{\mathcal{F}_t, t \in [0,T]\}$. A control process (Agent's effort/action) $\nu = (\alpha, \beta)$ is an $\mathbb{F}-$adapted process with values in $A \times B$ for some subsets A and B of finite dimensional spaces. The controlled process takes values in $\mathbb{R}^d$, and is defined by means of the controlled coefficients

$$\lambda : \mathbb{R}_+ \times \Omega \times A \longrightarrow \mathbb{R}^n, \text{ bounded, } \lambda(\cdot, a)\ \mathbb{F} - \text{progressive for any } a \in A,$$
$$\sigma : \mathbb{R}_+ \times \Omega \times B \longrightarrow \mathcal{M}_{d,n}(\mathbb{R}), \text{ bounded, } \sigma(\cdot, b)\ \mathbb{F} - \text{progressive for any } b \in B,$$

For all control process ν, and some given initial data $X_0 \in \mathbb{R}^d$, the controlled state equation is defined through weak solutions to an SDE. For such a weak solution $\mathbb{P}$, there is an $n-$dimensional $\mathbb{P}-$Brownian motion $W^{\mathbb{P}}$, as well as $\mathbb{F}-$adapted, and $A \times B-$valued processes $(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$ such that

$$X_t = X_0 + \int_0^t \sigma_r(X, \beta_r^{\mathbb{P}}) \left[\lambda_r(X, \alpha_r^{\mathbb{P}})dr + dW_r^{\mathbb{P}}\right], \ t \in [0,T], \ \mathbb{P} - a.s. \tag{10.1.1}$$

Let $\mathcal{P}_0$ denote the collection of all weak solutions. The Agent is hired at the time origin $t = 0$, and receives the compensation ξ from the Principal at time T. The Principal does not observe the Agent's effort (that's the moral hazard framework), only the output process. Consequently, the compensation ξ, can only be contingent on X, that is ξ is $\mathcal{F}_T-$measurable. Furthermore, the Agent has a reservation utility R, below

which he will not accept the contract. The objective function of the Agent is defined through the following criterion, where k is a discount factor, c represents the cost of effort of the Agent and U_A his utility function

$$J^A(\mathbb{P},\xi) := \mathbb{E}^{\mathbb{P}}\Big[\mathcal{K}_T^{\nu^{\mathbb{P}}} U_A(\xi) - \int_0^T \mathcal{K}_t^{\nu^{\mathbb{P}}} c_t\big(\nu_s^{\mathbb{P}}\big) ds\Big],\ \mathcal{K}_t^{\nu} := \exp\Big(-\int_0^t k_r(\nu_r) dr\Big),\ t\in[0,T],\ \mathbb{P}\in\mathcal{P}_0,\ \xi\in\mathcal{C}_0.$$

The Agent's goal is to choose optimally the effort, given the compensation contract ξ promised by Principal

$$V^A(\xi) := \sup_{\mathbb{P}\in\mathcal{P}_0} J^A(\mathbb{P},\xi).$$

We denote by $\mathcal{P}^\star(\xi)$ the collection of all optimal controls for $V^A(\xi)$. Our approach consists first in restricting the contracts that can be offered by the Principal to those that admit an optimal solution to Agent's problem, *i.e.*, we allow only the contracts ξ for which $\mathcal{P}^\star(\xi)\neq\emptyset$. This means that the Principal will not offer contracts such that he cannot know how the Agent is going to react to it. Thus, the Principal is restricted to choose a contract from the set

$$\Xi := \big\{\xi\in\mathcal{C}_0,\ \mathcal{P}^\star(\xi)\neq\emptyset,\ \text{and } V^A(\xi)\geq R\big\}.$$

As a final ingredient, we need to fix the Agent's optimal strategy in the case in which the set $\mathcal{P}^\star(\xi)$ contains more than one solution. Following the standard convention, we assume that the Agent, when indifferent between such solutions, implements the one that is the best for the Principal. The Principal's problem is then defined by, U being a utility function, $\ell:\Omega\longrightarrow\mathbb{R}$ a liquidation function and $\mathcal{K}_T^P$ a discount factor

$$V^P := \sup_{\xi\in\Xi}\ \sup_{\mathbb{P}^\star\in\mathcal{P}^\star(\xi)} \mathbb{E}^{\mathbb{P}^\star}\Big[\mathcal{K}_T^P U\big(\ell-\xi\big)\Big],$$

The above problem is a typical Stackelberg game between the Principal and the Agent, and as in any of these games, the first step is to understand the best–reaction function of the Agent, that is to say both the value function $V^A(\xi)$ and the set of optimal controls $\mathcal{P}^\star(\xi)$. This is exactly where the 2BSDE theory comes to our help. Indeed, the above problem of the Agent is a general non–Markovian stochastic control problem in weak formulation. And as recalled in Part II, 2BSDEs provide a probabilistic representation for value functions of any of these problems. Let us insist right now that in this context, it is of paramount importance that one should have a 2BSDE theory without regularity assumptions in ω on ξ. Indeed, it would not make sense to restrict the Principal to such contracts, all the more since the classical contracts derived in the literature involve stochastic integrals, which are not continuous in ω in general.

In view of the definition of the Agent's problem it is natural to introduce the Hamiltonian functional, for all $(t,x)\in[0,T)\times\Omega$ and $(y,z,\gamma)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}_d(\mathbb{R})$:

$$H_t(x,y,z,\gamma) := \sup_{u\in A\times B} h_t(x,y,z,\gamma,u),$$
$$h_t(x,y,z,\gamma,u) := -c_t(x,u) - k_t(x,u)y + \sigma_t(x,b)\lambda_t(x,a)\cdot z + \frac{1}{2}(\sigma_t\sigma_t^\top)(x,b):\gamma,$$

for $u:=(a,b)\in A\times B$. We next introduce a subset of contracts which will play a crucial role in our analysis. The idea is to restrict our attention to contracts $U_A(Y_T^{Z,\Gamma})$, for sufficiently integrable processes (Z,Γ) and with

$$Y_t^{Z,\Gamma} := Y_0 + \int_0^t Z_r\cdot dX_r + \frac{1}{2}\int_0^t \Gamma_r : d\langle X\rangle_r - \int_0^t H_r\left(Y_r^{Z,\Gamma}, Z_r, \Gamma_r\right) dr. \tag{10.1.2}$$

The interesting point of this class is that under mild assumptions, $V^A(U_A(Y_T^{Z,\Gamma})) = Y_0$, and one can obtain the corresponding optimal controls by maximising the Hamiltonian $H_r(Y_r^{Z,\Gamma}, Z_r, \Gamma_r)$. This is purely a verification type result, see [56, Proposition 3.3]. With this class of contracts, the contract ξ now appears as

the terminal value of the diffusion process $Y^{Z,\Gamma}$, itself controlled by Z and Γ. The problem of the Principal therefore boils down to a standard stochastic control problem in weak formulation with two state variables, namely X, the output value, and $Y^{Z,\Gamma}$, which is nothing else than the continuation utility of the Agent (that is the dynamic version of his value function).

With this class of contracts, one can therefore write directly an HJB equation for the problem of the Principal, see [56, Theorem 3.6], which in the end makes this problem not more complicated than a standard stochastic control problem, and more importantly opens the way to all the already existing numerical methods for HJB equations to actually compute solutions and optimal contracts.

Of course, one has to ask the question of whether we actually solved the correct problem by restricting our attention to the set of contracts defined above. The answer is that under mild assumptions, this is actually without any loss of generality. Indeed, introducing the following Legendre transform of H

$$F_t(x,y,z,\Sigma) := \sup_{(\alpha,\beta)\in A\times B_t(x,\Sigma)} \big\{ -c_t(x,\alpha,\beta) - k_t(x,\alpha,\beta)y + \sigma_t(x,\beta)\lambda_t(x,\alpha)\cdot z \big\},$$

where we denoted for all $(t,x)\in[0,T]\times\Omega$

$$\Sigma_t(x,b) := (\sigma_t\sigma_t^\top)(x,b),\ b\in B,\ \text{and } B_t(x,\Sigma) := \big\{b\in B,\ \sigma_t\sigma_t^\top(x,b)=\Sigma\big\}.$$

Then the general 2BSDE theory developed in Section 1.5 implies that the value function of the Agent solves the following 2BSDE

$$Y_t = U_A(\xi) + \int_t^T F_s(Y_s,Z_s,\widehat{a}_s^2)ds - \int_t^T Z_s\cdot dX_s + \int_t^T dK_s.$$

Comparing with the restricted class of contracts, we can actually write

$$Y_t^{Z,\Gamma} = U_A(Y_t^{Z,\Gamma}) + \int_t^T F_s(Y_s,Z_s,\widehat{a}_s^2)ds - \int_t^T Z_s\cdot dX_s + \int_t^T k_s^{Z,\Gamma}ds,$$

where $k_t^{Z,\Gamma} := H_t(Y_t,Z_t,\Gamma_t) - F_t\big(Y_t,Z_t,\widehat{a}_t^2\big) - \frac{1}{2}\widehat{a}_t^2 : \Gamma_t$. The main difference between the two classes lies thus in the question of the absolute continuity of K. If one can show that it always holds in the context of drift control only (we recall that this was basically the only case considered in the literature so far), it is not the case in general for volatility control. However, as shown in [56, Theorem 4.9], the supremum over the two classes of contracts are actually the same, so that in the end our approach indeed solves the correct problem.

10.2 Application to delegated portfolio management (see [55])

The first application of the previously described theory that we looked at concerned the problem of delegated portfolio management, which is faced in general by an investor wishing to give his money to be managed by an hedge fund for instance. We consider the following general model

$$X_t^{v,a} = \int_0^t \sigma_s(v_s)\cdot(b_s(a_s)ds + dB_s),$$

where (v,a) represents the control pair of the agent where v and a are adapted processes taking values in some subset of $\mathbb{R}^m\times\mathbb{R}^n$. The example of delegated portfolio management corresponds to the case in which $m=n=d$, and $\sigma_t(v) := v^T\sigma$, $b_t(a) := b$, for some fixed $b\in\mathbb{R}^d$ and some invertible $d\times d$ matrix σ. In that case, the interpretation of the process $X^{v,a}$ is that of the portfolio value process dependent on the Agent's

choice of the vector v of portfolio dollar–holdings in d risky assets with volatility matrix σ, and the vector b of risk premia.

In this model, the Principal and the Agent both have exponential type utilities (with different risk aversions). Thanks to this, the general HJB equation for the Principal's problem can actually be solved explicitly and lead to a contract of the form $Y_T^{Z,\Gamma}$ with constant Z and Γ. In other words, the optimal contract is linear in the terminal value of the output X_T (that is to say here the terminal value of the portfolio) and in its quadratic variation $\langle X \rangle_T$, and is therefore path–dependent, which is a new feature compared to the classical case of Holmström and Milgrom [113]. From the practical point of view, such a remuneration could be achieved by using future contracts (or stocks) for the part involving X_T, as well as variance swaps or stock options for the part involving the quadratic variation, which is very close to what is actually done in practice.

We now present a numerical example showing that first best is not attained (see Figure 10.1), second, that ignoring the quadratic variation in the contract can lead to substantial loss in expected utility (see Figure 10.2), and third, that there are parameter values for which the Principal rewards the Agent for taking high risk, unlike the typical use of portfolio Sharpe ratios in practice (see Figure 10.3). The set–up considers two risky assets for simplicity, and that the Agent only controls the volatility. Furthermore, his cost function is given by

$$c(v_1, v_2) = \frac{1}{2}\beta_1(v_1 - \alpha_1)^2 + \frac{1}{2}\beta_2(v_2 - \alpha_2)^2.$$

Thus, it is costly to move the volatility v_i away from α_i, and the cost intensity is β_i. An interpretation is that α_i are the initial risk exposures of the firm at the time the manager starts his contract.

In Figure 10.1 we plot the percentage loss in the Principal's utility certainty equivalent relative the case without moral hazard, when varying the parameter α_2, and keeping everything else fixed. The loss can be significant for extreme values of initial exposure α_2. That is, when the initial risk exposure is far from desirable, the moral hazard cost of providing incentives to the agent to modify the exposure is high. In Figure 10.2, we compare the Principal's certainty equivalent to the one he would obtain if offering the contract that is optimal among those that are linear in the output, but do not depend on its quadratic variation. Again we see that the corresponding relative percentage loss can be large.

Figure 10.1: Loss of utility compared to the case without moral hazard

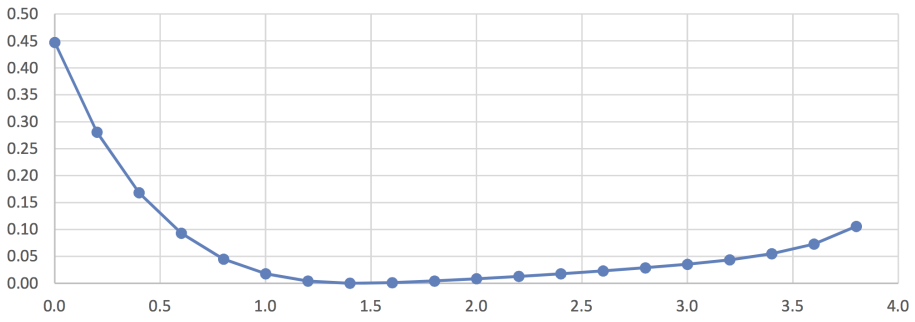


Figure 10.3 plots the values of the coefficient (the sensitivity) multiplying the quadratic variation in the optimal contract. We see that the Principal uses quadratic variation as an incentive tool: for low values of the initial risk exposure α_2 he wants to increase the risk exposure by rewarding higher variation, and for its high values he wants to decrease it by penalising high variation. This is because when the initial risk exposure α_2 is not at the desired value, incentives are needed to make the agent apply costly effort.

Figure 10.2: Loss of utility without quadratic variation

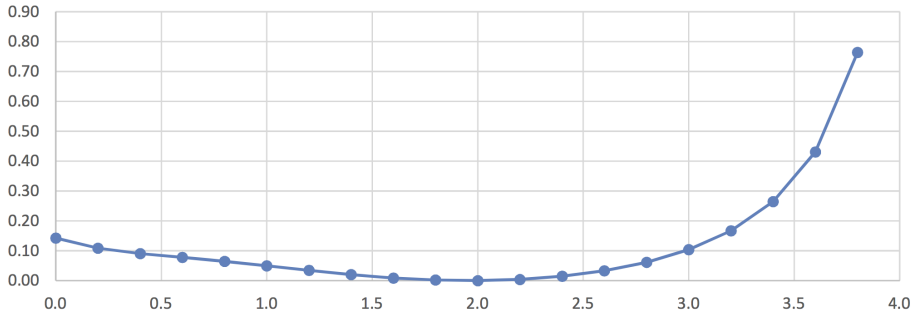
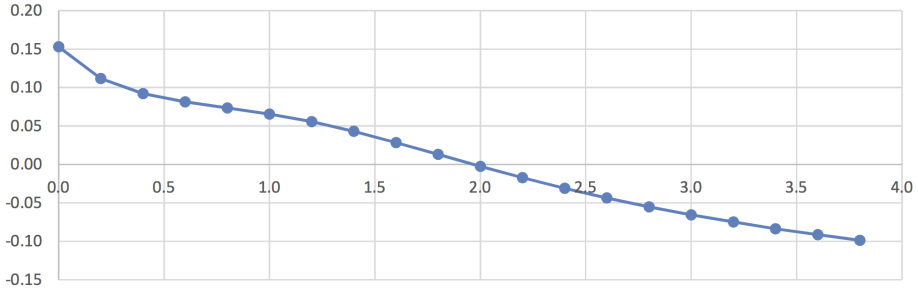


Figure 10.3: Optimal contract's sensitivity to quadratic variation



10.3 Application to electricity pricing (see [2])

The starting point of this work was a series of interrogations that were brought to us by EDF (Électricité de France) R&D. Their basic point was that the massive development of renewable energy sources for electricity generation in the electric systems had renewed the interest for demand response programs. Indeed, one key aspect of renewable energy sources is their irregularity, *i.e.* their volatility. This property has compelled systems operator and electric utilities to increase their storage capacity in order to cope with the important variation over small times of the amount of electricity generated by wind farms or solar panels. Of course one way out was to develop storage solutions, but an alternative solution could be to try and develop demand response programs. In those program, consumers would receive signals (either physical or financial) to reduce or increase their consumption. The problem of the definition and valuation of such demand response to cope with peak load demand has of course long been identified by the key players in the electricity sector, and solutions based on special pricing formulas have been proposed, such as the EDF program "Effacement Jour de Pointe". But, the question of designing a tariff that would incentivise clients to have a more regular load curve was still an open question. Our goal here is exactly this: proposing a simple model of electricity pricing that allows a producer to give proper incentives to a consumer to smooth his consumption over time. Of course, such a problem requires the provider to have an almost complete and real–time access to the consumption of the consumers. Fortunately, and this was actually one the technical points that lead to this projects, thanks to the development of smart meters (Linky in France for instance), electricity providers can now have access not only to the whole consumption of electricity of their clients over a time period, as it has long been the case during the previous monopoly period, but also to the whole trajectory of this consumption.

From the mathematical point of view, such a problem falls precisely into the framework that we described in Section 10.1, and is yet another illustration of the practical interest to understand moral hazard problems for which the Agent is allowed to control both the drift and the volatility, in this case of his electricity consumption. That being said, we consider a simple model where the (log) of the electricity consumption of the Agent (the client) X is controlled and such that, with the same notations as in Section 10.1

$$X_t = x_0 + \int_0^t \lambda_s(X, \alpha_s^{\mathbb{P}})ds + \int_0^t \sigma_s(X, \beta_s^{\mathbb{P}}) \cdot dW_s^{\mathbb{P}},$$

where λ and σ are respectively $\mathbb{R}$ and $\mathbb{R}^N-$valued, for some N representing the different usages of electricity that the Agent can have and may wish to impact differently. Our numerical illustrations will be given in the case where $N = 1$ for simplicity and

$$\lambda_s(x, \alpha) := a - \alpha, \ \sigma_s(x, \beta) = \sqrt{2}\sigma \exp\left(-\frac{\beta}{2}\right),$$

with positive α and β, meaning that the Agent can improve his average consumption a by putting some effort, as well as his average volatility of consumption σ. The Agent pays the electricity price ξ to the Principal at the end of the period T and his problem is therefore

$$U^A(\xi) := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\int_0^T \left(u(X_s) - c(\alpha_s^{\mathbb{P}}, \beta_s^{\mathbb{P}})\right) ds - \xi\right],$$

for some utility function u and some cost function c. Numerical experiments will be performed for the choices $u(x) = -\exp(-R_A x)$, $c(\alpha, \beta) := \alpha^2/2 + \exp(b\beta)$.

As for the Principal, he is the electricity provider and thus faces two kinds of generation costs, a cost linked to the level of generation and a cost linked to the variation of production over time. We sum up this into the following criterion for the Principal

$$U^P := \sup_{\xi} \sup_{\mathbb{P}\in\mathcal{P}^\star(\xi)} \mathbb{E}^{\mathbb{P}}\left[\mathbb{U}\left(\xi - \int_0^T g(X_s)ds - h\langle X\rangle_T\right)\right],$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}_+$ is the generation cost function, supposed to be strictly convex, and h is a positive constant resuming the costs induced by the quadratic variation of the consumption. The higher the volatility of the consumption, the harder it is for the producer to follow the load curve. Numerical experiments will correspond to the choice $\mathbb{U}(x) := -\exp(-R_P x)$ and $g(x) := \exp(\gamma x)$.

From the theoretical point of view, since the controls of the Agent are not supposed to be bounded, this leads to a setting where the BSDEs and 2BSDEs considered have quadratic growth in the z variable, while the results of [56] where given for simplicity in a purely Lipschitz setting. Nonetheless, and this is one of the reasons why we decided to give in [190] a very general *modus operandi* to construct solutions to 2BSDEs, the ideas and proof go, *mutatis mutandis*, following the aforementioned recipe. In any case, the Hamiltonians of both the Agent and the Principal can be computed explicitly in our examples and lead to simple, albeit without explicit solutions, HJB equations for the Agent. Using the exponential utility structure, one state variable can actually be taken out of the problem and lead to a one–dimensional fully non–linear parabolic PDE that can be solved easily using finite difference methods. The Figure 10.4 presents the value function of the Principal for different values of the current time. Regardless of the current consumption of the Agent, his value seems to be increasing with time, and as expected increases as well for large or small values of the consumption.

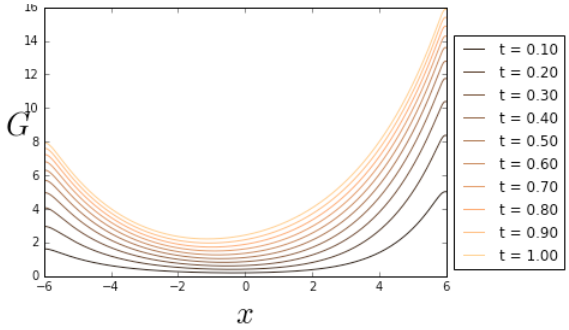


Figure 10.4: Value function of the Principal.

Figure 10.5 explores the difference between the average consumption, as well as the (opposite) of the average price paid by an Agent who actually makes effort and one who does not at all, denominated lazy Agent. As expected, the more time passes, the more the lazy Agent consumes compared to the one making effort, and as such he is penalised by paying more for his electricity. We can also obtain the optimal controls of the Agent and represent in Figure 10.6 the corresponding optimal drift and volatilities of the consumption. Naturally the drift tends towards the nominal drift $a = 2$ as time goes to T, since there are no more incentives for the Agent to reduce his consumption trend when the end of the contract is close. Another interesting observation is that the optimal volatility, although quite erratic for the early times, becomes relatively constant after a while, showing that our goal of reducing the volatility of consumption is indeed achieved in this example.

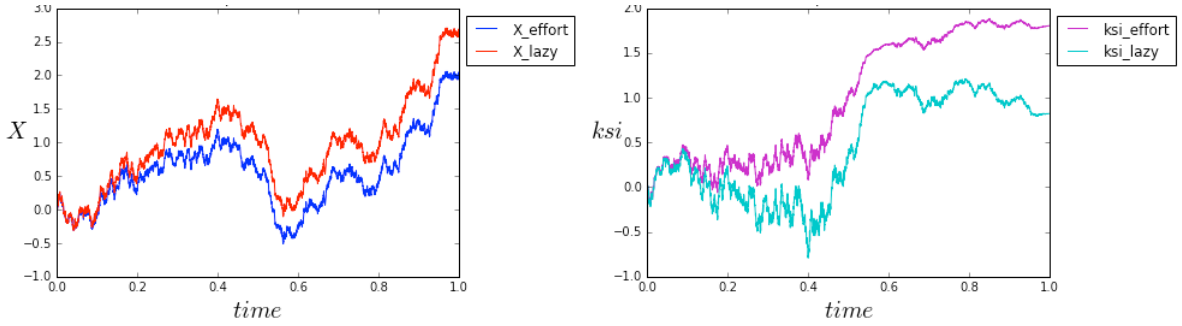


Figure 10.5: Average consumption and (opposite of) price between sophisticated and lazy Agents.

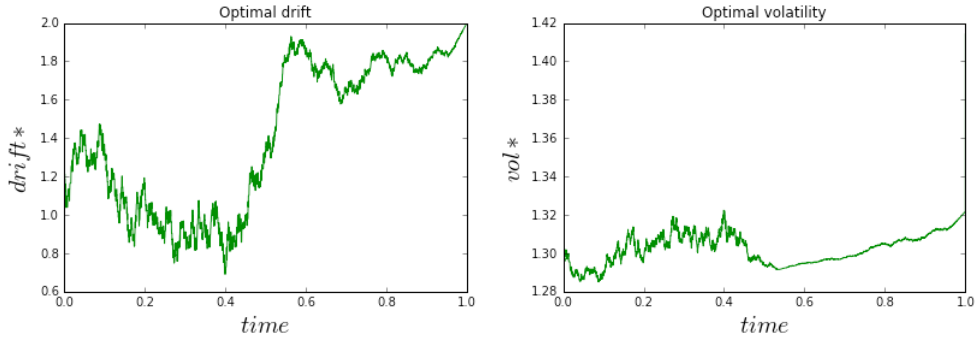


Figure 10.6: Optimal drifts and volatility of the Agent's consumption.

10.4 Moral hazard and ambiguity (see [147])

In many situations, both the Principal and the Agent can be imperfectly informed about some key elements associated to the project to which the contract is attached. This can be understood as economic uncertainties or ambiguities. Our goal in the paper [147] was to try to examine the possible effects of ambiguity on optimal contracting under moral hazard, especially from the point of view of optimal managerial compensation. The idea is that in a simple model similar to the one of Holmström and Milgrom [113], where the Agent only controls the drift of the output X, which represents the value of the firm he manages, this control representing the amount of work that he puts into its management. However, the twist is now that both the Principal and the Agent do not know what is precisely the volatility associated to the value of the firm, but are simply convinced that it must lie in some interval. One interpretation could be that very sophisticated economic agents (or presumptuous ones) must have narrow intervals of volatility uncertainty, and uninformed agents (or prudent ones) have large intervals instead. Of course, there is no reason in particular why the intervals of the Principal and the Agent should *a priori* be the same.

Now, we assume that they both adopt a worst case approach to the problem, and have robust criteria in the sense that they compute their utilities behaving as if there was a third player in the game (the Nature) who is choosing the worst possible volatility for them. In these regards, this problem is a transposition into the Principal–Agent framework of the robust financial problems that I presented in Part II. This is of course a case of extreme ambiguity aversion on the part of the Principal and the Agent that can be argued upon, all the more since it in particular implies that they both gives the same weight to particularly plausible realisations of volatility and to extremely unlikely ones. Nonetheless, as far as we know, our contribution was the first of the kind in this literature, and thus can be seen as a first step to try and understand the impact of uncertainty on the contracts, before studying more realistic models of uncertainty.

Both the Principal and the Agent have exponential type utilities, and the problem faced by the Agent having an ambiguity set $[\underline{\alpha}^A, \overline{\alpha}^A]$ for the volatility can then be written as, with the same notations as before

$$V_0^A(\xi) := \sup_{a\in\mathcal{A}} \inf_{\mathbb{P}\in\mathcal{P}^a_{[\underline{\alpha}^A,\overline{\alpha}^A]}} \mathbb{E}^{\mathbb{P}}\left[-\exp\left(-R_A\left(\xi - \int_0^T c(a_s)ds\right)\right)\right],$$

where $(a_s)_{s\in[0,T]}$ is the chosen effort of the Agent, and $\mathcal{P}^a_{[\underline{\alpha}^A,\overline{\alpha}^A]}$ the set of probability measures associated to this effort and the Agent's ambiguity set, while that of the Principal, if the Agent chooses a given effort a is

$$V_0^P := \sup_{\xi} \inf_{\mathbb{P}\in\mathcal{P}^a_{[\underline{\alpha}^P,\overline{\alpha}^P]}} \mathbb{E}^{\mathbb{P}}\left[-\exp\left(-R_P(X_T-\xi)\right)\right].$$

Since this problem was new in the literature, we first addressed the question of finding the optimal contract in the full information case where the Principal can choose both the contract and the effort of the Agent (subject to the participation constraint of course). This problem is actually an unusual calculus of variations problem, that we solved by finding first a correct *ansatz* for the optimal contract. The idea is that, though different from the problem described in Section 10.1, these two are in fact quite similar in the sense that the volatility of the output is controlled, even if it is not by the Agent. Thus, we actually compute the optimal contract among the class of contracts linear in X_T and $\langle X\rangle_T$ (corresponding to the contracts of the form (10.1.2) with constant Z and Γ) and show that the optimal contract thus derived satisfies the general optimality conditions. We refer to [147] for precise statements (they are too lengthy to state for the present manuscript), but it turns out that the case where the ambiguity sets of the Principal and the Agent are disjoint is particularly surprising, since the problem degenerates and the Principal can actually achieve asymptotically a utility of 0 (recall that the Principal has an exponential utility which is therefore negative). In the other cases, except in very specific situations, the optimal contract is path–dependent and uses the quadratic variation for incentives.

The moral hazard case follows the same ideas as the general ones given in Section 10.1, and start by showing that the value function of the Agent, despite being that of a differential game, can be actually represented as a 2BSDE. This is due to the fact that the Agent can only control the drift, so that for a fixed volatility his value function is a BSDE, and taking the infimum afterwards leads naturally to a 2BSDE. This would no longer work if the Agent controlled the volatility as well, as an infimum of a 2BSDE may no longer be a 2BSDE (this problem would be similar to the one considered in my current project with Soner, Touzi and Zhang already mentioned in Chapter 3). That being said, the idea is then to only look at contracts for which the K process in the 2BSDE is absolutely continuous, and then write the associated HJB equation for the Principal. In this setting it turns out that it can be solved explicitly, and leads once more to contracts linear in X_T and $\langle X \rangle_T$. The major difference with Section 10.1 is that in order to check that the restriction to these contracts was without loss of generality, the same proof cannot be used, because the Principal problem is also a differential game. In this simple setting however, we managed to derive sharp upper bounds on the value function of the Principal, that are attained by the contracts derived through our general method.

A point that deserves to be mentioned again is that the degeneracy mentioned above also appears in the moral hazard case, which means that when the ambiguity sets are disjoint, then the asymmetry of information occurs no loss for the Principal. The reason is that the Principal can use the quadratic variation component in the contract in order to make appear in his exponential utility a term which can be arbitrarily large, but which is not seen at all by the agent in his utility, as it is constructed so that it disappears under the worst–case probability measure of the Agent. This is therefore the combination of this difference between the worst–case measures of the Principal and the Agent, as well as the fact that their uncertainty sets are disjoints which makes the problem degenerate. In terms of sophistication of the Agent, this seems to indicate that the Principal would most of the time be better off with an Agent who has a better knowledge of the volatility, or the economic uncertainty, than himself, since there is a higher chance then that the ambiguity sets will be disjoint, which is consistent with the idea that one should hire bright employees!

Chapter 11

Contract theory with interaction

> The trouble with having an open mind, of course, is that people will insist on coming along and trying to put things in it.
>
> Terry Pratchett

One important limitation of the theory exposed above is that it ignores the fact that in general there is not only one employee, or one client for the Principal, but several, or even many of them, and that in general they can interact with each other. This is the so–called multi–Agent problem. Early works in that direction, first in one–period frameworks, include Holmström [112], Mookherjee [161], Green and Stokey [101], Harris, Kriebel and Raviv [107], Nalebuff and Stiglitz [165] or Demski and Sappington [65]. The first extension to continuous time is due to Koo, Shim and Sung [136]. Of course, as soon as one starts to consider contracting situation involving several agents, in addition to their possible interaction, the question of these agents comparing themselves to each other becomes quite relevant. This led to a large strand of literature which tried to understand the impact of inequity aversion. The gist of it is that several empirical studies have shown that agents working in a firm have a strong tendency to dislike inequity. One possible way to model that is by assuming that an agent suffers a utility loss if another agent conducting a similar task receives a higher wage. The first papers which examined moral hazard problems with inequity aversion are due to Englmaier and Wambach [84], or Fehr et al. [88]. Later studies include Itoh [119], Rey-Biel [195], Bartling and von Siemens [12], or Grund and Sliwka [103], which analyse the consequences of inequity aversion for team incentives. Another related paper is Dur and Glazer [70], where the authors study optimal contracts when workers envy their boss. A continuous–time extension of the problem has also been studied by Goukasian and Wan [100], still in a context where Agents exhibit envy and jealousy towards their co–workers. Almost all these works show that envious behaviour is destructive for organisations.

My works described in this last chapter concentrates on the study of two aspects related to contracting problem involving several agents. The first one [83], detailed in Section 11.1 is a first attempt at understanding how the general approach outlined in Section 10.1 could be extended to a setting with N agents. As will be made clear, the key point is the definition of admissible contracts. We will see that in general the problem of the Principal can once more be reinterpreted as a standard stochastic control problem, but with in general $2N$ state variables, making the corresponding HJB PDE highly dimensional, except in simple cases where explicit solutions can be obtained. This lead us to consider the setting of the so–called mean–field games (MFG for short), corresponding to the asymptotic limit as N goes to infinity, as introduced by Lasry and Lions [138, 139, 140] and independently by Huang, Caines and Malhamé [114, 115, 116, 117]. In this situation, following similar ideas as in [83], we have developed in [81] (see Section 11.2) a general theory showing that a moral hazard problem involving a Principal and Agents with mean–field interactions boiled down in the end to solve a stochastic control problem of McKean–Vlasov SDEs, otherwise known in the literature as a mean–field type control problem. Even though the link between these two notions is extremely strong, we believe that this problem is the first one in the literature involving both MFG and mean–field type

control at the same time. In any case, this is in general a much harder problem than classical stochastic control. Nonetheless, we show that in many situations, we can actually obtain explicit solutions and optimal contracts using our approach.

11.1 Moral hazard with competing agents (see [83])

The situation explored in [83] is that of a Principal wishing to hire N Agents to manage N projects on his behalf. These Agents can impact their own project, or the projects of the others, be it negatively or positively. Mathematically, this translates into saying that the $N-$dimensional output vector X, representing the value of the N projects, is controlled by a matrix–valued process a such that $a^{i,j}$ denotes the impact of the i–th Agent on the j–th project

$$X_t = \int_0^t b(s, a_s, X_s)ds + \int_0^t \Sigma_s dW_s^a,\ 0 \leq t \leq T,\ a.s. \tag{11.1.1}$$

Then, we assume that the utilities of the Agents are exponential and that, for any $1 \leq i \leq N$, given N contracts (that is to say wages) $\xi := (\xi^1, \dots, \xi^N)^\top$ and a choice of actions a made by all the Agents, the utility of Agent i is

$$U_0^i(a^{:,i}, a^{:,-i}, \xi^i) := \mathbb{E}^{\mathbb{P}^a}\left[-\exp\left(-R_A^i\left(\xi^i + \Gamma_i(X_T) - \int_0^T c^i(s, a_s^{:,i}, X_s)ds\right)\right)\right],$$

where $R_i^A > 0$ represents the risk–aversion of Agent i, and the map Γ_i corresponds to Agent i comparing his performance with the performances of the other Agents. This map Γ^i can be quite general and a typical example would be

$$\Gamma_i(x) := \gamma_i\left(x^i - \bar{x}^{-i}\right),\ x \in \mathbb{R}^N, \tag{11.1.2}$$

where γ_i is a given non–negative constant, representing the propensity to compete of Agent i. This setting corresponds to the case where each Agent compares his performance to the average of the other Agents performances. The higher γ^i is, the more competitive Agent i will be.

Then, the problem of the Principal is to first find an equilibrium between the Agents, given some contracts, and then to use this equilibrium to optimise his own criterion over the contracts. The notion of equilibrium that we choose for the Agents is that of a Nash equilibrium. However, unlike in the case of one Agent only, where the problem was simply standard non–Markovian stochastic control, and therefore was easily linked to a BSDE (since we only consider drift control) for which a solution existed as soon as the contract was sufficiently integrable, the existence of Nash equilibria is a much more subtle and complicated problem. What can actually be shown is that the existence of Nash equilibria with sufficient integrability is equivalent to the existence of a solution to a $N-$dimensional system of BSDEs

Theorem 11.1.1. *Under appropriate assumptions, there is a one–to–one correspondence between*

(*i*) *a Nash equilibrium $a^\star(\xi)$ satisfying integrability properties.*

(*ii*) *a solution (Y, Z) in appropriate spaces to the multidimensional BSDE*

$$Y_t^\xi = \xi + \Gamma(X_T) + \int_t^T f(s, Z_s^\xi, X_s)ds - \int_t^T (Z_s^\xi)^\top \Sigma_s dW_s,\ a.s.,$$

where the map $f : [0,T] \times \mathcal{M}_N(\mathbb{R}) \longrightarrow \mathbb{R}^N$ is explicit (see [83, Theorem 4.1]),

the correspondence being given by, for any $i = 1, \dots, N$, $ds \times d\mathbb{P} - a.e.$,

$$(a_s^\star(\xi))^{\cdot,i} \in \underset{a}{\operatorname{argmax}} \left\{ \sum_{j=1}^{N} b^j(s, (a \otimes_i (a_s^\star)^{\cdot,-i}(s, Z_s, X_s))^{\cdot,j}, X_s) Z_s^{i,j} - c^i(s, a, X_s) \right\}.$$

However, the generator f has quadratic growth in z and it is know that even in extremely simple situations, multidimensional quadratic BSDEs can fail to have solutions, see the counter–example of Frei and dos Reis [89]. It is therefore not clear at all that for any contract ξ, an equilibrium between the Agents will indeed exist. Our rationale is therefore that there is no economic meaning for the Principal to offer a contract like that, and therefore that admissibility of a contract should include that it leads to the existence of at least one Nash equilibrium. Of course, since we only assume existence of an equilibrium, we have to specify what happens when there is no uniqueness. Several possibilities can be considered

(*i*) The Principal could be risk–averse and behave as if the Agents were choosing the worst equilibrium from his point of view.

(*ii*) The Agents could also have a collective criterion allowing them to differentiate between possible equilibria, and, if this is not enough to have uniqueness, they would behave as in the standard contract theory literature and let the Principal choose among the equilibria maximising their criterion.

In [83], we have followed the second option, but the first one could also be implemented following a similar approach. With this definition, one can therefore follow the same ideas as in the one Agent case and prove that the problem of the Principal once more becomes a standard stochastic control problem, whose state variables are X and the continuation utilities of all the Agents, that is to say $2N$ variables in total. It also turns out that one can obtain an almost explicit solution to this problem (up to the minimisation of a real valued function of N^2 variables) when Γ is chosen as in (11.1.2).

Besides the above general considerations which are actually valid in any Principal–Agent problem with several interacting agents, our setting also allows to study the impact of exogenous competitiveness between the Agents. Our findings are in a nutshell that, as in the one agent case, the optimal contract is linear in X_T. Moreover each Agent obtains his reservation utility and is paid a given proportion of the value of each project. Each Agent has incentives to help the project of a very competitive colleague, whereas he may have to work against the project of a poorly competitive one. In some sense, a competitive Agent rewards himself via his appetite for competition, and therefore it is in the economic interest of the Principal to provide him a project with higher probability of success, *e.g.* with less volatility or the help of the other Agents.

11.2 Moral hazard and mean–field games (see [81])

As mentioned before, except in very specific cases, basically corresponding to having the drift b in (11.1.1) as well as the cost function c being linear in x, the solution of the Principal's problem in Section 11.1 above requires to solve a possibly very highly dimensional PDE and is therefore impractical as soon as the number of Agents becomes too large. It seems then highly relevant to try and understand how one could work with the mean–field limit of the system instead.

As the reader must have by now realised, Principal–Agent problems are always written in the so–called weak formulation of stochastic control, and a MFG extension should not differ in this respect. However, if the approach to MFG in the literature has both a PDE flavour, in the form of either a coupled system of a forward Kolmogorov PDE and a backward HJB PDE or of the so–called Master equation, see notably the contributions of Bensoussan, Cardaliaguet, Carmona, Chau, Delarue, Frehse, Guéant, Lasry, Lions and Yam

[14, 35, 38, 104, 138, 139, 140] and the references therein, and a more probabilistic one involving systems of FBSDEs, initiated by Carmona and Delarue [36, 37] and more recently by Chassagneux, Crisan and Delarue [45], they all are formulated in a strong sense. To the best of our knowledge, the only exceptions are the contributions of Carmona and Lacker [41, 42] which consider weak or relaxed formulations. Following the latter approach, our model involves a representative Agent who can control the output process X given a choice of action of the other mean–field identical players, summed up in two probability measures μ and q, intended to represent the laws of X and the control respectively

$$X_t = \psi + \int_0^t b(s, X, \mu, q_s, \alpha_s)ds + \int_0^t \sigma_s(X)dW_s^{\mu,q,\alpha},\ t \in [0,T],\ \mathbb{P}-a.s.,$$

where ψ is a fixed initial random variable with a given law, representing the initial state of the system, and $W^{\mu,q,\alpha}$ is a Brownian motion under a measure $\mathbb{P}^{\mu,q,\alpha}$ obtained through a Girsanov transformation. Letting the drift b depend on the laws of the output and the control allows for instance to incorporate the following effects

(i) The average value of the firm can have a positive or negative impact on its future evolution: when things are going well they have a tendency to keep doing so, and conversely. Large disparities in the distribution of X, thus involving the variance, can also negatively impact the future evolutions.

(ii) Similarly, if all the other Agents in the firm are working on average very hard, this could have a ripple effect on the whole firm.

In this framework, we allow the Principal to make both a terminal payment ξ as well inter–temporal ones summed up in the process χ. Given such a contract, the problem of the Agent is then

$$V_0^A(\chi, \xi, \mu, q) := \sup_{\alpha \in \mathcal{A}} v_0^A(\chi, \xi, \mu, q, \alpha),$$

where

$$\begin{aligned} v_0^A(\chi, \xi, \mu, q, \alpha) := \mathbb{E}^{\mathbb{P}^{\mu,q,\alpha}}\left[K_{0,T}^{X,\mu,q}U_A(\xi) + \int_0^T K_{0,s}^{X,\mu,q,\alpha}u_A\big(s, X, \mu, q_s, \chi_s\big)ds\right] \\ - \mathbb{E}^{\mathbb{P}^{\mu,q,\alpha}}\left[\int_0^T K_{0,s}^{X,\mu,q}c(s, X, \mu, q_s, \alpha_s)ds\right], \end{aligned}$$

with the discount factor K defined by $K_{t,s}^{x,\mu,q} := \exp\left(-\int_t^s k(u, x, \mu, q_s)du\right)$. The solution to the mean field problem is then defined as the following fixed point problem

Definition 11.2.1. *A triplet (μ, q, α) is a solution to the mean field game if $V_0^A(\chi, \xi, \mu, q) = v_0^A(\chi, \xi, \mu, q, \alpha)$, $\mathbb{P}^{\mu,q,\alpha} \circ (X)^{-1} = \mu$ and $\mathbb{P}^{\mu,q,\alpha} \circ (\alpha_t)^{-1} = q_t$ for Lebesgue almost every $t \in [0,T]$.*

The problem of existence and uniqueness of the above problem is addressed in [41] and one can, unlike the equivalent Nash equilibrium in the $N-$player game of Section 11.1, give conditions ensuring the wellposedness. However, these conditions are in general only sufficient and can be very restrictive from the point of view of applications. Furthermore, the case of controlled volatility is actually not treatable using their approach. Judging by these facts, and following the intuition of our treatment of the $N-$player game, we therefore enforce in the definition of the admissibility of a contract that there should exist at least one solution to the associated mean–field game (and treat the non–uniqueness problem as in Section 11.1). With this definition in hand, one of our main result is an equivalence result between the MFG system and a special type of BSDE, which we coin of mean–field type, reminiscent of related works by Buckdahn, Djehiche, Li, Peng and Rainer [28, 29, 30], or Carmona and Delarue [36], see [81, Theorem 4.1] for more precise statements.

It then follows as an almost immediate corollary that the problem of the Principal is nothing else than the value function of a stochastic control problem of a bi–dimensional SDE, since it involves the output X and the continuation utility of the Agent as usual, of McKean-Vlasov type, since the law of X appears in the dynamics. This problem is one of the most well–known time inconsistent ones, and therefore is not amenable directly to classical dynamic programming approaches. This is why it has been approached recently through the Pontryagin maximum principle and the associated FBSDEs, see the works of Carmona, Delarue and Lachapelle [40, 39]. However, if one does not mind adding the law of the processes as an (infinite dimensional) state variable, then dynamic programming can come to the rescue, as shown in special cases by Laurière and Pironneau [141], and in more general settings by Bayraktar, Pham, Cosso and Wei [13, 180, 181, 182, 183]. Using this approach and an adapted version of the classical verification theorem of stochastic control we managed to solve completely the case where there is no inter–temporal payments and the drift b and the cost c take the form

$$b(s,x,\mu,q,a) := a + \alpha x + \beta_1 \int_{\mathbb{R}} x d\mu_s(x) + \beta_2 \int_{\mathbb{R}} x dq_s(x) - \gamma V_\mu(s),$$

with $V_\mu(s) := \int_{\mathbb{R}} |x|^2 d\mu_s(x) + \left|\int_{\mathbb{R}} x d\mu_s(x)\right|^2$, and for some $n > 1$

$$c(s,x,\mu,q,a) := c\frac{|a|^n}{n},\ c > 0.$$

We obtain the following result, showing that the contract is in general path–dependent, but still having some nice linearity properties in the value of the output

Theorem 11.2.1. *The optimal contract for the problem of the Principal is*

$$\xi^\star := \delta - \alpha(1+\beta_2)\int_0^T e^{(\alpha+\beta_1)(T-t)} X_t dt + \sigma(1+\beta_2)\int_0^T e^{(\alpha+\beta_1)(T-t)} dX_t,$$

for some explicit constant δ, and the associated optimal effort of the Agent is

$$a_u^\star := (1+\beta_2)^{\frac{1}{n-1}} \left(\frac{e^{(\alpha+\beta_1)(T-u)}}{c}\right)^{\frac{1}{n-1}},\ u \in [0,T].$$

We can also extend these results to the case of a Principal having mean–variance type preferences. We emphasise that these explicit results have the particularity of not being obtained in the linear–quadratic setting, which is, to the best of our knowledge, the only one solved explicitly in the MFG or mean–field type control literature.

Chapter 12

Perspectives

I have learned that human beings are all about incentives.

Harlan Coben

In my opinion, the research presented above on contracting theory is just the tip of the iceberg, and there should an extremely wide array of potential applications. Concerning the theory itself presented in Section 10.1, one useful extension would be to allow the output process to have jumps, and to allow the Agent to control both their intensity and their size. This could serve as model of insurance contracts, where the person insured can take measures to reduce the probability of having an accident, as well as its severity. There are some papers in the literature, notably by Biais, Mariotti, Rochet and Villeneuve [16], Capponi and Frei [34], or myself [170], where the Agent can control the intensity, but there is no paper in the literature allowing for control of the size of the jumps. From the theoretical point of view, as explained in Part II, this problem would intimately be linked to 2BSDEs with jumps. We are working on such a model, both from the practical and theoretical sides with Nicolàs Hernàndez Santibàñez (Universidad de Chile and Université Paris–Dauphine), Nabil Kazi-Tani (ISFA and Université Lyon 1) and Chao Zhou (NUS).

One other extremely important avenue of research would be to try and incorporate adverse selection problems in the theory. These problems stipulate that not only the Principal does not observe the actions of the Agent, he is also imperfectly informed about crucial characteristics of the Agent. This in general much harder to tackle, and there is no real comprehensive theory for contracting theory we both moral hazard and adverse selection yet. In order to understand better the problem, we have already started, with Clémence Alasseur (EDF and FIME), Ivar Ekeland (Université Paris–Dauphine), Romuald Élie (Université Paris–Est Marne–la–Vallée) and Nicolàs Hernàndez Santibàñez, working on such an adverse selection model for electricity pricing, trying to determine how an electricity company should charge for electricity without knowing *a priori* the willingness to consume of the clients. We are also working with Romuald Élie, Nicolàs Hernàndez Santibàñez and Chao Zhou on an extension of my PhD article [170], concerning optimal securitisation of bank loans, when there are different types of banks in the market. From what we have understood so far, and using some earlier advances in the literature, notably by Sung [213], Cvitanić and Zhang [59] and Cvitanić, Wan and Yang [57], there seems to be an important difference between the cases with a finite number of type in the population (that should be related to utility maximization problems with stochastic target constraints, as in Bouchard, Élie and Imbert [20], and with a number of state variables depending on the number of types) and the case with a continuum of types which might be easier to treat. We are investigating this general theory with Bruno Bouchard (Université Paris–Dauphine) and Romuald Élie.

Concerning the problems with several Agents, the theory that we have for now, both in the $N-$player case and the mean–field one are limited to the drift control case, and it is not clear at all how to extend it to volatility (and possibly jump) control as well. Furthermore, the $N-$player case seems to be even harder, since one would naturally expect the problem of the Agents to be linked to some kind of multi–dimensional system of 2BSDEs. However, such objects have not been defined yet, and since they should be related

to multidimensional systems of fully non–linear PDEs, the associated theory, if it indeed can be obtained, is bound to be very complicated. This is the reason why we have chosen with René Carmona (Princeton University) and Nizar Touzi (École Polytechnique) to start with the mean–field problem. Granted, the latter would require an extension of [41] to the controlled volatility case, but this seems more easily attainable thanks to recent progress we made on the 2BSDE theory in [190].

From the theoretical perspective, we have shown that the treatment of Principal–Agent problems with mean–field interactions were intimately linked to general mean–field type control problems. Even though some recent progresses have been made on these problems, using notably an adaptation of the dynamic programming approach, they are still requiring regularity and Markovian assumptions, which are not necessary at all for classical stochastic control. We are working on improving and better understanding these issues with Pierre Cardaliaguet (Université Paris–Dauphine) and Xiaolu Tan (Université Paris–Dauphine).

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Appendix A

Comprehensive list of publications

Publications in peer-reviewed international journals[1]

[1] B. Bouchard, D. Possamaï, X. Tan and C. Zhou. A unified approach to *a priori* estimates for supersolutions of BSDEs in general filtrations. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* (B), to appear, 2016.

[2] B. Bouchard, D. Possamaï, D., X. Tan. A general Doob–Meyer–Mertens decomposition for $g-$supermartingale systems. *Electronic Journal of Probability*, 21(36):1–21, 2016.

[3] J. Cvitanić, D. Possamaï, and N. Touzi. Moral hazard in dynamic risk management. *Management Science*, to appear, 2015.

[4] P. Imkeller, T. Mastrolia, D. Possamaï, and A. Réveillac. A note on the Malliavin-Sobolev spaces. *Statistics & Probability Letters*, 109:45–53, 2015.

[5] M. Jeanblanc, T. Mastrolia, D. Possamaï, and A. Réveillac. Utility maximization with random horizon: a BSDE approach. *International Journal of Theoretical and Applied Finance*, 18(7):1550045, 2015.

[6] N. Kazi-Tani, D. Possamaï, D., and C. Zhou. Quadratic BSDEs with jumps: related nonlinear expectations. *Stochastics and Dynamics*, 16(4):1650012, 2015.

[7] N. Kazi-Tani, D. Possamaï, D., and C. Zhou. Quadratic BSDEs with jumps: a fixed–point approach. *Electronic Journal of Probability*, 20(66):1–28, 2015.

[8] N. Kazi-Tani, D. Possamaï, D., and C. Zhou. Second-order BSDEs with jumps: formulation and uniqueness. *The Annals of Applied Probability*, 25(5):2867–2908, 2015.

[9] N. Kazi-Tani, D. Possamaï, D., and C. Zhou. Second order BSDEs with jumps: existence and probabilistic representation for fully-nonlinear PIDEs. *Electronic Journal of Probability*, 20(65):1–31, 2015.

[10] T. Mastrolia, D. Possamaï, and A. Réveillac. Density analysis of BSDEs. *The Annals of Probability*, to appear, 2014.

[11] T. Mastrolia, D. Possamaï, and A. Réveillac. On the Malliavin differentiability of BSDEs. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* (B), to appear, 2014.

[12] A. Matoussi, L. Piozin, and D. Possamaï. Second–order BSDEs with general reflection and game options under uncertainty. *Stochastic Processes and their Applications*, 124(7):2281–2321, 2014.

[13] A. Matoussi, D. Possamaï, and C. Zhou. Second order reflected backward stochastic differential equations. *The Annals of Applied Probability*, 23(6):2420–2457, 2013.

[14] A. Matoussi, D. Possamaï, and C. Zhou. Robust utility maximization in nondominated models with 2BSDE: the uncertain volatility model. *Mathematical Finance*, 25(2):258–287, 2015.

[15] H. Pagès and D. Possamaï. A mathematical treatment of bank monitoring incentives. *Finance and Stochastics*, 18(1):39–73, 2014.

[16] D. Possamaï. Second order backward stochastic differential equations under a monotonicity condition. *Stochastic Processes and their Applications*, 123(5):1521–1545, 2013.

[1] The publications [13,14,15,16,18,21] of the list below were part of my PhD thesis.

[17] D. Possamaï, G. Royer, and N. Touzi. On the robust superhedging of measurable claims. *Electronic Communications in Probability*, 18(95):1–13, 2013.

[18] D. Possamaï, H.M. Soner, and N. Touzi. Large liquidity expansion of super–hedging costs. *Asymptotic Analysis*, 79(1-2):45–64, 2013.

[19] D. Possamaï, H.M. Soner, and N. Touzi. Homogenization and asymptotics for small transaction costs: the multidimensional case. *Communications in Partial Differential Equations*, 40(11):2005–2046, 2015.

[20] D. Possamaï and X. Tan. Weak approximation of second–order BSDEs. *The Annals of Applied Probability*, 25(2):2535–2562, 2015.

[21] D. Possamaï and C. Zhou. Second order backward stochastic differential equations with quadratic growth. *Stochastic Processes and their Applications*, 123(10):3770–3799, 2013.

Preprints

[22] R. Aïd, D. Possamaï, and N. Touzi. A principal–agent model for pricing electricity volatility demand. preprint, 2016.

[23] J. Cvitanić, D. Possamaï, and N. Touzi. Dynamic programming approach to principal–agent problems. *arXiv preprint arXiv:1510.07111*, 2015.

[24] R. Élie, T. Mastrolia, and D. Possamaï. A tale of a principal and many many agents. *arXiv preprint arXiv:1608.05226*, 2016.

[25] R. Élie, L. Moreau, and D. Possamaï. On a class of non–Markovian singular stochastic control problems. preprint, (2016).

[26] R. Élie and D. Possamaï. Contracting theory with competitive interacting agents. *arXiv preprint arXiv:1605.08099*, 2016.

[27] T. Mastrolia and D. Possamaï. Moral hazard under ambiguity. *arXiv preprint arXiv:1511.03616*, 2015.

[28] A. Matoussi, D. Possamaï, and W. Sabbagh. Probabilistic interpretation for solutions of fully nonlinear stochastic PDEs. *arXiv preprint arXiv:1412.5548*, 2014.

[29] A. Papapantoleon, D. Possamaï, and A. Saplaouras. Existence and uniqueness for BSDEs with jumps: the whole nine yards. *arXiv preprint arXiv:1607.04214*, 2016.

[30] D. Possamaï and G. Royer. General indifference pricing with small transaction costs. *arXiv preprint arXiv:1401.3261*, 2014.

[31] D. Possamaï, X. Tan, and C. Zhou. Stochastic control for a class of nonlinear kernels and applications. *arXiv preprint arXiv:1510.08439*, 2015.

Patents

Together with René Aïd (EDF and FIME) and Nizar Touzi (École Polytechnique), we filed a patent titled **"Procédé de gestion des contraintes de flexibilité d'un système production-consommation d'électricité par le contrôle de la variance de la consommation"**, (demand number 1657868) based on the work [22] of the above list.

Technical reports

[32] P. Gauthier and D. Possamaï (2010). Efficient simulation of the double Heston model. *SSRN Working Paper Series*, http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1434853.

[33] P. Gauthier and D. Possamaï (2010). Efficient simulation of the Wishart model. *SSRN Working Paper Series*, http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1474728.

[34] P. Gauthier and D. Possamaï (2010). Prices expansions in the Wishart model. *SSRN Working Paper Series*, http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1475153.

Appendix B

Advising activities and teaching

Supervised students

⋆ **PhD students**

- Between October 2013 and December 2015: co-supervision, with Anthony Réveillac (INSA Toulouse) of the PhD thesis of Thibaut Mastrolia titled *Une étude de la régularité de solutions d'EDS rétrogrades et de leurs utilisations en finance*. The results of the thesis have lead to the publications [4, 5, 10, 11, 27], which have been all presented as either invited or contributed papers in international conferences. The PhD thesis has been defended in December 2015, and Thibaut Mastrolia has been recruited as an Assistant Professor by École Polytechnique, starting September 2016.
- Since October 2014: co-supervision, with Alejandro Jofre (Universidad de Chile) of the PhD thesis of Nicolás Hernández Santibáñez titled *Some contributions to Principal–Agent problems*. The associated results have been presented in international conferences, and two publications are in preparation. The defence is expected for June 2017.

⋆ **Master's students** I have supervised the following master's students

- For the Master's program MASEF in Université Paris Dauphine: Jean-François Rubinstein, Nicolas Bonne, Walid Fakhfakh, Maxime Cherion and David Weinraub.
- For the Master's program in actuarial sciences in Université Paris Dauphine: Solesne Boutoille, Coralie Lefranc, Maxime Hersent, Julien Goix, Olivia Lafay, Audrey Loiseau, Agatha Ngo, Jeanne Fassier, Kevin Sadoun, Mordehai Roos, Élie Dadoun, Adeline Ung, Ouk-Dom Ek, Jonathan Limbourg, Audrey Ouzillou, Quentin Perales, Manon Valdiguie and Arnaud Witz.
- For the engineering program in applied mathematics at École Polytechnique: Bertrand Rondepierre (co-directed with Nizar Touzi, and lead to the publication of the numerical part of [188]), Loïc Richier (co-directed with Nizar Touzi, and lead to the publication of the numerical part of [188]), Raymond Li and Mohamed Aziz Dhaoui.

Doctoral courses

- Invited course on "Second–order BSDEs" at ETH Zürich, Zürich, Switzerland, in February and March 2013 (8h course, invitation from H.M. Soner).
- Invited course on "Second–order BSDEs with jumps" at TU Berlin and Humboldt University, Berlin, Germany, Switzerland, in April 2013 (8h course, invitation from A. Papapantoleon).
- Invited course on "Contracting theory and principal–agent problems" at ENIT, Tunis, Tunisia, for the 6th CREMMA spring school (10h course).
- Invited course on "Recent progresses on non–Markovian stochastic control" at ETH Zürich, Zürich, Switzerland, from February to April 2017 (24h course, invitation from H.M. Soner).

Teaching experience

⋆ **Université Paris Diderot–Paris 7**

- **Probability Theory**, undergraduate students (teaching assistant) (19h30): *2009–2010*.
- **Analysis and Algebra**, undergraduate students (teaching assistant) (19h30): *2010–2011, 2011–2012*.

⋆ **Université Paris Dauphine**

- **Risk Management and Derivatives**, undergraduate students (39h): *2012–2013, 2013–2014, 2014–2015, 2015–2016, 2016–2017*.
- **Stochastic Calculus and Financial Applications**, graduate students (39h): *2012–2013, 2013–2014*.
- **Time Series**, graduate students (teaching assistant) (19h30): *2012–2013, 2013–2014, 2014–2015, 2015–2016, 2016–2017*.
- **Jump Processes**, graduate students (teaching assistant) (10h): *2013–2014, 2014–2015*.
- **Foundations of Analysis, Algebra and Probability Theory for Finance**, professionals of the finance industry (15h): *2013–2014, 2014–2015, 2015–2016, 2016–2017*.
- **Option Pricing and Stochastic Calculus**, graduate students (24h): *2015–2016, 2016–2017*.
- **Financial Econometrics**, graduate students (21h): *2014–2015, 2015–2016, 2016–2017*.
- **Contract theory (new course)**, graduate students (18h): *2015–2016*.

⋆ **École Polytechnique**

- **Stochastic Calculus**, graduate students (teaching assistant) (18h): *2015–2016*.
- **Management and finance for energy markets**, professionals (24h): *2015–2016*.

⋆ **ENSAE**

- **Stochastic Calculus**, graduate students (18h): *2014–2015, 2015–2016, 2016–2017*.

⋆ **Électricité de France (EDF)**

- **Advanced Applications of Stochastic Calculus**, research engineers (24h): *2013–2014, 2016–2017*.
- **Stochastic Calculus for Energy Markets**, research engineers (24h): *2014–2015*.

Appendix C

Curriculum Vitæ

Employment and education

- **December 2016:** Habilitation à diriger des recherches, Université Paris–Dauphine, Paris , France.
- **September 2015–September 2016:** Professeur chargé de cours at École Polytechnique, Palaiseau, France.
- **Since September 2012:** Assistant Professor at Université Paris–Dauphine, Paris, France.
- **January 2012–August 2012:** postdoctoral student at École Polytechnique, Palaiseau, France.
- **October 2009–December 2011:** PhD thesis at École Polytechnique, Palaiseau, France.
- **September 2008–August 2009:** Master's degree "Probabilités et Finance" at UPMC Sorbonne Universités, Paris, France.
- **August 2005-August 2009:** Engineering degree at École Polytechnique, Palaiseau, France.

Visiting positions and extended visits

- Visiting professor ETH Zürich, Zürich, Switzerland, February–March 2013.
- TU Berlin, Germany, April 2013.
- Visiting professor ETH Zürich, Zürich, Switzerland, February–March 2014.
- NUS, Singapore, April 2014.
- TU Berlin, Germany, April 2015.
- NUS, Singapore, July 2015.
- Visiting professor ETH Zürich, Zürich, Switzerland, February–May 2017.

Research activities

⋆ Grants, projects and awards

- Monge international grant for PhD students from 2009 to 2012.
- PhD award of the École Polytechnique for the PhD thesis "A journey through second–order BSDEs and other contemporary problems of mathematical finance" in 2012.
- Nicola Bruti Liberati prize for the PhD thesis "A journey through second–order BSDEs and other contemporary problems of mathematical finance" in 2013.
- Prime d'encadrement doctoral et de recherche (PEDR) since September 2013.

- Member of the funded PROCOPE 2014 project, led by Antonis Papantoleon (TU Berlin) on "Financial markets in transition: mathematical models and challenges".

- Member of the funded Europlace Institute of Finance project led by Stéphane Crépey (Évry–Val–d'Essonne) on "Collateral management and risk concentration in centrally cleared trading".

- Recipient, with Nabil Kazi-Tani (ISFA) and Didier Rulliére (ISFA), of a grant from the Vietnam Institute for Advanced Study in Mathematics (VIASM) for a funded 2 months stay in Hanoï, Vietnam.

- Scientific coordinator of the funded 2016 ANR project PACMAN (Principal–Agent, Contract theory and Mean-field Games for eNergy).

⋆ Organization of seminars and conferences

- Co–organiser, between 2013 and 2015, with Anthony Réveillac (INSA Toulouse) of the *"Stochastic control"* seminar of the Paris–Dauphine university, https://www.ceremade.dauphine.fr/seminaires.php.

- Co–organiser with Anthony Réveillac (INSA Toulouse), Adrien Richou (Université Bordeaux 1) and Thomas Lim (Université Évry–Val–d'Essonne and ENSIIE) of the *"Second young researchers meeting on BSDEs, numerics and finance"*, which took place in July 2014 in Bordeaux, France, https://indico.math.cnrs.fr/event/199/.

- Co–organizer, in 2013/2014, with Bruno Bouchard (Université Paris–Dauphine), Stéphane Crépey (Université Évry–Val–d'Essonne), Romuald Élie (Université Paris–Est Marne–la–Vallée), Emmanuel Gobet (École Polytechnique), Marc Hoffmann (Université Paris–Dauphine), Anis Matoussi (Université du Maine and École Polytechnique), Mathieu Rosenbaum (École Polytechnique), Xiaolu Tan (Université Paris–Dauphine) and Peter Tankov (Université Paris Diderot–Paris 7 and ENSAE) of a thematic cycle on *"Robust management in finance"*, financially supported by the Louis–Bachelier Institute, https://www.ceremade.dauphine.fr/ModelsRisks/ModelsRisks.html.

- Co–organiser with Anthony Réveillac (INSA Toulouse) and Thibaut Mastrolia (École Polytechnique) of the "Workshop on new advances on Malliavin calculus, SPDEs, BSDEs, and application to finance", which took place in March 2015 in Toulouse, France, https://sites.google.com/site/workshopmalliavinbsdesspdes/overview/overview.

- Co–organiser with Arnaud Lionnet (INRIA Paris–Rocquencourt and École Nationale des Ponts et Chaussées), Thibaut Mastrolia (École Polytechnique) and Wissal Sabbagh (Université du Maine) of the *"3rd young researchers meeting on BSDEs, numerics and finance"*, which took place in July 2016 in Le Mans, France, http://who.rocq.inria.fr/Arnaud.Lionnet/Conf3YRMPNF/Home.html.

- Co–organiser, with Peter Bank (TU Berlin), Roxana Dumitrescu (HU Berlin), Romuald Élie (Université Paris–Est–Marne–la–Vallée), Caroline Hillairet (ENSAE), Noemi Kurt (TU Berlin), Antonis Papapantoleon (TU Berlin), Sergio Pulido (Université Évry–Val–d'Essonne and ENSIIE) and Charline Smadi (IRSTEA), of the *"Berlin–Paris young researchers workshop on stochastic analysis with applications in biology and finance"*, which took place in November 2016 in Berlin, Germany, http://www3.math.tu-berlin.de/stoch/BPY2016/.

- Co–organiser, with Ulrich Horst (HU Berlin) and Alejandro Jofre (Universidad de Chile) of a conference on principal–agent problems which will take place in Chile in March 2017.

- Co–organiser, with Bruno Bouchard (Université Paris–Dauphine), Min Dai (NUS), Arnaud Gloter (Université Évry–Val–d'Essonne), Vathana Ly Vath (Université Évry–Val–d'Essonne) and Chao Zhou (NUS), of the *"Second Paris–Asia conference on mathematical finance"*, which will take place in Suzhou, China, in May 2017, http://www.rmi.nus.edu.sg/events/conferences/2017/.

Collective responsibilities

- In charge, from January 2014 to September 2014, and starting September 2015, with Marc Hoffmann (Université Paris–Dauphine), and since may 2016 with Pierre Cardaliaguet (Université Paris–Dauphine), of the undergraduate and master's program "Actuarial sciences" of Université Paris–Dauphine.
- Member of the laboratory council of the CEREMADE in Université Paris–Dauphine, since September 2014.
- Member of the recruitment committee for an assistant professor position in Université Paris–Dauphine in December 2014.
- Associate Editor for the Bulletin Français d'Actuariat starting December 2016.
- Referee for the journals: Advances in Difference Equations; Bernoulli; Control, Optimization and Calculus of Variations; Discrete and Continuous Dynamical Systems; Electronic Journal of Probability; ESAIM: Probability and Statistics; Finance and Stochastics; International Journal of Theoretical and Applied Finance; Journal of Optimisation Theory and Applications; Mathematical Finance; Mathematics of Operations Research; SIAM Journal on Control and Optimization; SIAM Journal on Financial Mathematics; Springer; Statistics & Probability Letters; Stochastic Processes and their Applications; Stochastics; Stochastics and Dynamics; The Annals of Applied Probability.

Invited talks in international conferences

- 6th SMAI congress, Seignosse, France, May 2013, invitation in a mini-symposium.
- International conference on stochastic analysis and applications, Hammamet, Tunisia, October 2013, invitation in a mini–symposium.
- Conference "New developments in stochastic analysis: probability and PDE interactions", Beijing, China, July 2013.
- Conference "Stochastic analysis in finance and insurance", Oberwolfach, Germany, May 2014.
- 7th international symposium on backward stochastic differential equations, SDU, Weihai, China, June 2014.
- Conference "Stochastic analysis for risk modeling", CIRM, Luminy, France, September 2014.
- Workshop "Information in finance and insurance", Université Paris–Dauphine, Paris, France, February 2015.
- "First Paris–southeast Asia conference in mathematical finance", Siem Reap, Cambodia, February 2015.
- Workshop "Probabilistic numerical methods for non linear PDEs", Imperial College, London, United Kingdom, July 2015.
- Conference on "Martingale optimal transport & robust hedging, control of path–dependent systems, and infinite-dimension stochastic analysis", Hammamet, Tunisia, October 2015.
- Workshop "Optimization and equilibrium in financial and energy markets", Universidad de Chile, Santiago, Chile, January 2016.
- Workshop of the sixth CREMMA spring school, ENIT, Tunis, Tunisia, April 2016.
- Conference "Stochastic analysis and mathematical finance – a fruitful partnership", Casa Matemática Oaxaca, Oaxaca, Mexico, May 2016.
- Conference "Stochastic analysis in finance and insurance", University of Michigan, Ann Arbor, United States of America, June 2016.

- Second international congress on actuarial science and quantitative finance, Universidad de Cartagena, Cartagena, Colombia, June 2016, invitation in a mini–symposium.
- 6th IMS–FIPS workshop, University of Alberta, Edmonton, Canada, July 2016, invitation in a mini–symposium.
- Workshop on "Numerical methods for stochastic processes", VIASM, Hanoï, Cambodia, August 2016.
- Workshop on "Risk measures, XVA analysis, capital allocation and central counter–parties", Shanghaï Advanced Institute for Finance, Shanghaï, China, October 2016.
- Conference "Advances in financial mathematics", Paris, France, January 2017.
- 11th Bachelier colloquium, Métabief, France, January 2017.
- Conference "PDE and probability methods for interactions", Inria Sophia Antipolis, Nice, France, March 2017.
- Conference "Advances in stochastic analysis for risk modeling", CIRM, Luminy, France, November 2017.
- Conference on "Economics and contracting theory", Chile, March 2017.

Invited talks in seminars

- Rencontres des chaires de la FBF, Paris, France, April 2010.
- PhD students seminar, École Polytechnique, Palaiseau, France, October 2010.
- Probability and statistics seminar, Université du Maine, Le Mans, France, February 2011.
- MATHFI seminar, École des Ponts et Chaussées, Noisy, France, March 2011.
- Stochastics models in finance seminar, École Polytechnique, Palaiseau, France, May 2011.
- Mathematical finance seminar, UPMC Sorbonne Universités and Université Paris Diderot–Paris 7, Paris, France, November 2011.
- Probabilities and mathematical finance seminar, Université Évry–Val–d'Essonne, Évry, France, December 2011.
- Bachelier seminar (PhD student's day), Institut Henri Poincaré, Paris, France, December 2011.
- Journée de la chaire FDD, Clamart, France, December 2011.
- Mathematical finance seminar, Université Paris–Est Marne–la–Vallée, Noisy, France, January 2012.
- Stochastic control seminar, Université Paris–Dauphine, Paris, France, September 2012.
- Bachelier seminar, Institut Henri Poincaré, Paris, France, January 2013.
- FIME research seminar, Clamart, France, February 2013.
- Demi–journée du prix de l'Académie des Sciences, fondation Natixis, Paris, France, April 2013.
- Stochastic analysis and stochastic finance seminar, TU Berlin, Berlin, Germany, June 2013.
- Mathematical finance seminar, ETH Zürich, Zürich, Switzerland, March 2014.
- Mathematical finance seminar, Université Paris–Est Marne–la–Vallée, Noisy, France, March 2014.
- Nomura seminar, University of Oxford, Oxford, United Kingdom, May 2014.
- Bruti–Liberati prize presentation, Politecnico di Milano, Milano, Italy, June 2014.

- Bachelier seminar, Institut Henri Poincaré, Paris, France, November 2014.
- London math-finance seminar, London School of Economics, London, United Kingdom, November 2014.
- Matinée scientifique du CEREMADE, Université Paris-Dauphine, Paris, January 2015.
- London math-finance seminar, London School of Economics, London, United Kingdom, March 2015.
- Séminaire MAD, Toulouse School of Economics, Toulouse, France, April 2015.
- Stochastic analysis and stochastic finance seminar, TU Berlin, Berlin, Germany, April 2015.
- 60ème journée de séminaires actuariels, ISFA, Lyon, France, November 2015.
- Mathematical finance seminar, Columbia University, New York, United States of America, February 2016.
- Seminar on financial and actuarial mathematics, University of Michigan, Ann Arbor, United States of America, February 2016.
- Statistics and applied probability seminar, University of California Santa Barbara, Santa Barbara, United States of America, February 2016.
- Atelier de la chaire FBF "Marchés en mutation", Paris, France, March 2016.
- Seminar on financial and insurance mathematics, ETH Zürich, Zürich, Switzerland, April 2016.
- Engineering systems and design seminar, SUTD, Singapore, Singapore, July 2016.
- Mathematical finance seminar, UPMC Sorbonne Universités and Université Paris Diderot–Paris 7, Paris, France, October 2016.
- Mathematical finance seminar, Université Évry–Val–d'Essonne, Évry, France, January 2017.
- Probability and computational finance seminar, Carnegie Mellon university, Pittsburgh, United States of America, January 2017.
- Financial mathematics seminar, University of Texas at Austin, Austin, United States of America, January 2017.
- Seminar on financial and actuarial mathematics, University of Michigan, Ann Arbor, United States of America, February 2017.
- Probability and statistics seminar, University of Southern California, Los Angeles, United States of America, February 2017.

Contributed talks in international conferences

- 6th Bachelier world congress, Toronto, Canada, June 2010.
- 34th conference on stochastic processes and their applications, Osaka, Japan, September 2010.
- New advances in backward SDEs for financial engineering applications, Tamerza, Tunisia, October, 2010.
- 6th symposium on BSDEs, University of Southern California, Los Angeles, United States of America, June 2011.
- 6th Bachelier colloquium, Métabief, France, January 2012.
- 7th Bachelier world congress, Sydney, Australia, June 2012.
- Young researchers meeting on BSDEs, numerics and finance, University of Oxford, Oxford, United Kingdom, July 2012.

- 7th Bachelier colloquium, Métabief, France, January 2013.
- 8th Bachelier world congress, Brussels, Belgium, June 2014.
- 9th Bachelier world congress, New York, United States of America, July 2016.