Comportement en temps long des solutions de quelques équations de Hamilton-Jacobi du premier et second ordre, locales et non-locales, dans des cas non-périodiques

Thi-Tuyen Nguyen

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Thi Tuyen Nguyen

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devant le jury composé de :

Annalisa CESARONI
Ricercatore, Univ. de Padoue, Italie/ *Rapportrice*

Nicolas FORCADEL
Professeur, INSA de Rouen/ *Rapporteur*

Ying HU
Professeur, Univ. de Rennes 1/ *Examinateur*

Cyril IMBERT
Directeur de recherche CNRS, ENS Paris/ *Examinateur*

Paola MANNUCCI
Professore associato, Univ. de Padoue, Italie/ *Examinatrice*

Nicoletta TCHOU
MCF HDR, Univ. de Rennes 1/ *Examinatrice*

Emmanuel CHASSEIGNE
MCF HDR, Univ. de Tours/ Co-*directeur de thèse*

Olivier LEY
Professeur, INSA de Rennes / *Directeur de thèse*

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Résumé

La motivation principale de cette thèse est l'étude du comportement en temps grand des solutions non-bornées d'équations de Hamilton-Jacobi visqueuses dans $I\!R^N$ en présence d'un terme de Ornstein-Uhlenbeck

$$\begin{cases} u_t - \mathcal{F}(x,[u]) + \langle b(x), Du\rangle + H(x,Du) = f(x) \text{ in } I\!R^N \times (0,\infty), \\ u(\cdot,0) = u_0(\cdot) \text{ in } I\!R^N, \end{cases} \tag{1}$$

où $\mathcal{F}$ peut être *local* ou *nonlocal*:

$$\begin{cases} \mathcal{F}(x,[u]) = \mathrm{tr}(A(x)D^2u) \ \ \textbf{(opérateur local)}, \\ \mathcal{F}(x,[u]) = \mathcal{I}(x,u,Du) \ \ \textbf{(opérateur non-local)}, \end{cases}$$

A est une matrice de diffusion dans $I\!R^N$, $\mathcal{I}$ est un opérateur non-local intégro-différentiel et le terme $\mathcal{F} - \langle b, D\rangle$ est appelé opérateur de Ornstein-Uhlenbeck.

Nous généralisons les résultats de Fujita, Ishii & Loreti (2006) dans plusieurs directions. La première est de considérer des opérateurs de diffusion plus généraux en remplaçant le Laplacien par une matrice de diffusion quelconque. Nous considérons ensuite des opérateurs non-locaux intégro-différentiels de type Laplacien fractionnaire. Le second type d'extension concerne le Hamiltonien qui peut dépendre de x et est seulement supposé sous-linéaire par rapport au gradient, c'est à dire $|H(x,p)| \leq C(1+|p|)$, pour tous $x, p \in I\!R^N$.

Nous considérons la même question dans le cas d'une équation de Hamilton-Jacobi du premier ordre

$$\begin{cases} \frac{\partial u}{\partial t} + |Du| = l(x) & x \in I\!R^N, t > 0 \\ u(x,0) = u_0(x) & x \in I\!R^N, \end{cases} \tag{2}$$

pour $l \geq 0$ tel que $\mathcal{K} := \mathrm{argmin}_{I\!R^N}\, l$ est une compact non vide de $I\!R^N$.

Dans les deux cas (1) et (2), notre objectif est d'obtenir la convergence suivante qui est appelée comportement en temps grand de la solution

$$u(\cdot,t) - (ct + v) \to 0 \text{ localement uniformément dans } I\!R^N \text{ quand } t \to \infty, \tag{3}$$

où $u \in C(I\!\!R^N \times [0,\infty))$ est une solution de viscosité non-borné de (1) (respectivement (2)) et $(c,v) \in I\!\!R \times C(I\!\!R^N)$ est une solution de viscosité non-borné du problème ergodique associé

$$c - \mathcal{F}(x,[v]) + \langle b(x), Dv\rangle + H(x,Dv) = f(x) \quad \text{dans } I\!\!R^N,$$

respectivement,

$$c + |Dv| = l(x) \quad \text{dans } I\!\!R^N.$$

La thèse constituée de cinq chapitres, les chapitre 3, 4, 5, 6 sont consacrés à l'équation (1) et le dernier chapitre traite de (2).

Chapitre 3: Nous introduisons la définition et les propriétés de base des processus de Ornstein-Uhlenbeck gouvernés par un mouvement brownien (cas local) ou par un processus de Lévy (cas non-local). De plus, nous expliquons le lien entre des problèmes de contrôle optimal stochastique et nos équations de Hamilton-Jacobi à la fois dans le cas local et non-local. En particulier nous montrons que nos hypothèses sur les Hamiltoniens dans cette thèse sont vérifiées dans ces problèmes de contrôle optimal. Enfin, nous décrivons quels sont le rôle et l'effet de l'opérateur de Ornstein-Uhlenbeck dans nos équations. Avec la présence de ce terme, les trajectoires sont confinées dans quelques grosses boules limitant les effets de la non-compacité de $I\!\!R^N$.

Chapitre 4: (Résultats principaux). Nous établissons des bornes a priori de gradient uniformes pour les solutions de l'équation locale et non-locale. Ces résultats sont une étape essentielle pour obtenir les résultats de convergence (3). Les difficultés des preuves proviennent du caractère non-borné des solutions et des hypothèses faibles sur l'Hamiltonien (H est seulement sous-linéaire).

Nous considérons dans un premier temps des équations elliptiques dans le cas local et non-dégénérées dans le cas non-local (ce qui signifie que l'ordre dans la mesure du terme intégro-différentiel est $\beta \in (1,2)$). Les preuves sont basées sur des idées de Ishii & Lions (1990) dans le cas local et de Barles, Chasseigne, Ciomaga & Imbert (2008) dans le cas non-local.

Étant donnée une solution continue u de l'équation qui satisfait une condition de croissance convenable à l'infini, nous considérons la fonction

$$\Psi(x,y) = u(x) - u(y) - \psi(|x-y|)(\phi(x) + \phi(y)),$$

où ψ est une fonction concave positive appropriée qui est construite strictement concave sur un intervalle $[0, r_0]$ et linéaire sur $[r_0, +\infty)$ et ϕ est une fonction croissant plus vite que u et sur-solution de l'équation. L'idée est de montrer que Ψ a un maximum non positif et ceci implique la borne de gradient uniforme pour u. On procède donc par l'absurde, en supposant que le maximum est positif. Afin de obtenir une contradiction, nous tirons profit de l'équation. En premier lien, l'opérateur de Ornstein-Uhlenbeck donne la propriété sur-solution pour ϕ ce qui permet de contrôler des mauvais termes de l'équation. Puis, ensuite lorsque $|x-y|$ est petit, nous profitons de l'ellipticité de l'équation. Lorsque $|x-y|$ est grand au point de maximum, nous utilisons l'effet de l'opérateur de Ornstein-Uhlenbeck pour obtenir à la contraduction. C'est la preuve pour le cas local. Les idées de preuve dans le cas non-local suivent celles du cas local dans le sens où nous pouvons utiliser la méthode Ishii & Lions [51] lorsque $\beta \in (1,2)$ (l'ordre du terme non-local). Mais les estimations des termes non-locaux sont très différentes de celles des termes locaux et inspirées par Barles, Chasseigne, Ciomaga & Imbert (2008). Il existe de nombreuses difficultés techniques. Nous n'avons pas réussi à obtenir directement des bornes Lipschitz pour l'équation comme dans le cas local. La preuve comporte donc deux étapes. La première étape établit la régularité τ-Hölder pour tout $\tau \in (0,1)$. Ensuite, on utilise cette régularité Hölder pour obtenir la régularité et les bornes Lipschitz. À chaque étape on construit des fonctions test concaves différentes, également différentes du cas local afin de parvenir à une contradiction.

Ensuite, nous considérons le cas des équations elliptiques dégénérées, c'est-à-dire quand l'équation locale est elliptique dégénérée et quand l'équation non-locale a un opérateur d'ordre $\beta \in (0,2)$. Dans ce cas, les hypothèses sur l'Hamiltonien doivent être renforcées par des hypothèses Lipschitz en espace et en gradient. De plus, il est également nécessaire d'ajouter une condition supplémentaire sur la taille de l'opérateur d'Ornstein-Uhlenbeck qui doit compenser la dégénérescence de l'opérateur du second ordre et de l'Hamiltonien.

Enfin, tous ces résultats sont également établis pour les problèmes paraboliques dans la même façon.

Chapitre 5: Nous étudions l'existence et l'unicité de solutions dans le cadre local et non-local, stationnaire et d'évolution. La technique habituelle dans la théorie des solutions de viscosité est d'appliquer la méthode de Perron pour construire d'abord des solutions de viscosité discontinues, puis avec des résultats de comparaison forts nous obtenons une solution continue. Mais pour des équations avec des Hamiltoniens seulement sous-linéaires, il est difficile voir impossible d'obtenir un principe de comparaison. Nous utilisons donc une autre méthode qui est basée sur des troncatures de H et de la donnée initiale (pour l'équation d'évolution) comme dans Barles & Souganidis (2001) et qui utilise les estimations a priori du Chapitre 4. En effet, la construction d'une solution du problème est obtenue en approchant l'équation par un argument de troncature et en obtenant ensuite la convergence de la solution du problème approché vers une solution du problème initial. Pour obtenir une telle convergence, les estimations de régularité a priori sont utilisées. L'unicité de la solution est la conséquence directe du résultat de la comparaison.

Chapitre 6: Nous appliquons nos résultats de régularité Lipschitz pour résoudre le problème ergodique associé et étudier le comportement en temps grand des solutions dans le cas local uniformément parabolique et dans le cas non-local non-dégénéré (l'ordre β est dans $(1,2)$). Dans la première partie du Chapitre, nous résolvons d'abord le problème ergodique, qui permet d'identifier la limite en temps du système. En utilisant les résultats a priori de régularité de Lipschitz et les arguments de principe maximum nous obtenons qu'il existe une unique vitesse de propagation asymptotique et une solution unique à translation près par des constantes. La deuxième partie du Chapitre est consacrée à montrer la convergence (3) des solutions du problème parabolique vers ces solutions du problème ergodique et les arguments sont basés sur l'application du principe du maximum fort.

Chapitre 7: Le comportement en temps grand des solutions non-bornées d'équations de Hamilton-Jacobi du premier ordre est considéré. Les équations sont de type eikonal avec un coût instantanné l non-borné, c'est à dire (2). La fonction l et la donnée initiale u_0 sont prises à croissance quadratique. Quand $l \geq 0$ et $\operatorname{argmin} l$ est compact, nous établissons l'existence de solutions (c, v) du problème ergodique, où c est une constante ergodique. De plus, s'il existe une solution v bornée inférieurement, nous obtenons le comportement en temps grand des solutions de l'équations de Hamilton-Jacobi quand on démarre d'une donnée initiale u_0 suffisamment proche de v. Des exemples illustrent ces résultats et leur limitation.

Perspectives: Basés sur l'étude des résultats obtenus dans la thèse, nous présentons des problèmes ouverts à la fin de l'introduction.

Mots clés. Solutions de viscosité, opérateur de Ornstein-Uhlenbeck, régularité des solutions, principe du maximum fort, problème ergodique, comportement asymptotique, équations de Hamilton-Jacobi locales et non-locales.

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Chapter 1

Introduction

1.1 Motivation: comportement en temps long des équations de Hamilton-Jacobi

La motivation principale de cette thèse est l'étude du comportement en temps long des solutions de viscosité d'équations de Hamilton-Jacobi avec un terme d'Ornstein-Uhlenbeck posées dans l'espace $I\!R^N$ tout entier et en présence, soit d'une diffusion classique, soit d'une diffusion non-locale.

Commençons par rappeler la problématique en nous plaçant, pour simplifier, dans le cas d'une diffusion locale classique.

Considérons l'équation aux dérivées partielles (EDP) de Hamilton-Jacobi semilinéaire

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \mathrm{tr}(A(x)D^2u) + H(x, Du) = 0 \\ u(x,0) = u_0(x), \end{array} \right. \tag{1.1}$$

où Du, D^2u sont respectivement le gradient et la matrice hessienne de la fonction inconnue $u : I\!R^N \times [0,\infty) \to I\!R$. Les fonctions $H : I\!R^N \times I\!R^N \to I\!R$ et $A : I\!R^N \to \mathcal{S}^N$ sont l'Hamiltonien et la matrice de diffusion, $\mathcal{S}^N$ étant l'ensemble des matrices carrées symétriques de taille N. La fonction continue u_0 est la donnée initiale. L'hypothèse de base que nous ferons toujours est que A est une matrice positive (définie ou pas suivant les cas) pour avoir une EDP parabolique (éventuellement dégénérée).

Pour expliquer le problème de la manière la plus simple possible, supposons ici que (1.1) admet une unique solution *bornée* dans $I\!R^N \times [0,T]$, pour tout $T > 0$ et toute donnée initiale u_0 *bornée* et localement lipschitzienne.

Nous cherchons à prouver qu'il existe une constante $c \in I\!R$ appelée *constante ergodique* et une fonction $v(x)$ localement lipschitzienne telle que la paire (c, v) est solution du *problème ergodique* suivant,

$$c - \mathrm{tr}(A(x)D^2v) + H(x, Dv) = 0 \tag{1.2}$$

et que le développement asymptotique suivant de la solution u est valide,

$$u(x,t) = ct + v(x) + o(\frac{1}{t}), \tag{1.3}$$

où $o(\frac{1}{t}) \to 0$ quand $t \to +\infty$, localement uniformément par rapport à x. Ce comportement asymptotique en temps est le cas typique de ce qui est attendu pour ce genre de problème.

La comportement en temps long (1.3) est en général démontré à l'aide de deux étapes fondamentales que nous décrivons maintenant.

La première étape consiste à établir des bornes de gradient uniformes pour les solutions de (1.1) et du problème stationnaire approché

$$\lambda v^\lambda - \mathrm{tr}(A(x)D^2v^\lambda) + H(x, Dv^\lambda) = 0, \quad \lambda \in (0,1), \tag{1.4}$$

c'est-à-dire qu'on veut obtenir

$$|Du(x,t)| \leq C, \tag{1.5}$$
$$|Dv^\lambda(x)| \leq C. \tag{1.6}$$

Soulignons que nous recherchons des bornes *uniformes* dans le sens que la constante C dans (1.5) doit être indépendante de t et C dans (1.6) ne doit pas dépendre de λ. Cette uniformité sera cruciale pour la suite.

En effet, (1.6) est la propriété principale pour permettre de passer à la limite $\lambda \to 0$ dans (1.4) pour résoudre (1.2). Plus précisément, si l'on pose $z^\lambda(x) := \lambda v^\lambda(x)$ et $w^\lambda(x) := v^\lambda(x) - v^\lambda(0)$, alors w^λ est une solution de viscosité de

$$\lambda w^\lambda - \mathrm{tr}(A(x)D^2w^\lambda) + H(x, Dw^\lambda) = -z^\lambda(0), \quad \lambda \in (0,1).$$

De plus, par (1.6), $\{w^\lambda\}_{\lambda\in(0,1)}$ est une famille uniformément bornée et équicontinue dans tout compact de $I\!R^N$. Pour la même raison, la famille $\{z^\lambda\}_{\lambda\in(0,1)}$ est également équicontinue. Elle est aussi bornée par application du principe du maximum : pour $M > 0$ suffisamment grand et indépendant de λ, les fonctions $\pm\frac{M}{\lambda}$ sont respectivement sur- et sous-solutions de (1.4). Par comparaison, on obtient $|z^\lambda(x)| \leq M$ ce qui donne le résultat. On est donc en position d'appliquer le théorème d'Ascoli : à sous-suites près, il existe $(c, v) \in I\!R \times C(I\!R^N)$ tel que $z^\lambda(0) \to c$ et $w^\lambda \to v$ quand $\lambda \to 0$, localement uniformément. Par les résultats de stabilité sur les solutions de viscosité (voir par exemple Crandall, Ishii & Lions [30]), nous obtenons que (c, v) est une solution de (1.2). Remarquons au passage que ce premier résultat obtenu correspond à la résolution du "problème dans la cellule" en homogénéisation ce qui suffit souvent à résoudre le problème d'homogénéisation quand le principe de comparaison est valide (voir Lions, Papanicolaou & Varadhan [63] et Lions & Souganidis [64]). Mais cette étape n'est pas suffisante pour obtenir le comportement en temps grand.

D'un autre côté, pour L suffisamment grand, comme u_0 est supposée bornée, $v^\pm(x,t) = v(x) + ct \pm L$ sont respectivement sur- et sous-solutions de (1.1). Ceci entraîne, par comparaison,

$$v(x) + ct - L \leq u(x,t) \leq v(x) + ct + L. \tag{1.7}$$

Une conséquence immédiate de cette estimation est

$$\frac{u(x,t)}{t} \to c, \qquad \text{quand } t \to \infty,$$

ce qui est une autre manière de retrouver la constante ergodique (voir Arisawa & Lions [4]). Remarquons qu'une autre conséquence de (1.7) associée à (1.5) est d'obtenir la compacité de la famille

$$\{u(\cdot,t) - ct, t \geq 0\}. \tag{1.8}$$

Cela entraîne que l'ensemble ω-limite n'est pas vide et on obtient déjà (1.3) à sous-suite près, ce qui n'est toujours pas suffisant pour montrer la convergence car le problème n'a pas une unique solution.

Pour continuer nous avons besoin, dans une seconde étape, de prouver que toute la suite converge dans (1.8) quand $t \to +\infty$. Cette étape est délicate quand A est dégénérée car la structure de l'ensemble des solutions de (1.2) peut-être très compliquée. On peut en général obtenir facilement que la constante ergodique c est unique mais en revanche, il peut y avoir beaucoup de solutions de (1.2). Il est trivial que, étant donné une solution v de (1.2), toutes les translations par des constantes de v sont encore solutions mais il peut en exister d'autres très différentes, voir par exemple Lions, Papanicolaou & Varadhan [63], Roquejoffre [75] et le Chapitre 7 dans le cas d'équations de Hamilton-Jacobi totalement dégénérées (du premier ordre).

Quand l'équation est non-dégénérée, ce qui sera une hypothèse dans toute la thèse (hormis dans le chapitre 7), la structure des solutions de (1.2) est plus simple : la solution de (1.2) est unique à translation près par une constante. Précisons que l'EDP (1.1) est non-dégénérée quand elle est uniformément parabolique, i.e., la diffusion est elliptique,

$$A(x) \geq \rho Id, \quad \rho > 0. \tag{1.9}$$

Nous considérerons aussi des équations non-locales et nous donnerons la définition de la non-dégénérescence dans ce cadre quand nous en aurons besoin, voir le section 1.5.1.

Donnons une idée de la convergence désirée lorsque l'EDP (1.1) est uniformément parabolique et dans le cadre (plus simple) périodique, i.e., dans $\mathbb{T}^N = I\!R^N/\mathbb{Z}^N$ le tore N-dimensionnel $I\!R^N$. Pour les détails voir par exemple Barles & Souganidis [21] et Ley & Nguyen [65].

L'argument principal est basé sur le principe du maximum fort. Pour pouvoir l'appliquer, nous avons besoin de l'hypothèse supplémentaire sur H

$$H(x,\cdot) \in \mathrm{Lip}(I\!R^N), \ \ \text{pour tout } x \in \mathbb{T}^N,$$

pour pouvoir linéariser l'EDP.

Posons

$$k(t) = \max_{\mathbb{T}^N}\{u(x,t) - ct - v(x)\}.$$

À l'aide du principe de comparaison, on montre que $k(t)$ est borné et décroissant (voir [21]), d'où l'existence d'une constante $\bar{k}$ telle que

$$k(t) \to \bar{k} \ \ \text{quand } t \to \infty.$$

De plus, une nouvelle application du principe de comparaison implique que pour tous $n, p \in \mathbb{N}$,

$$|u(x,t+t_n) - ct_n - (u(x,t+t_p) - ct_p)| \leq \max_{\mathbb{T}^N}\{|u(x,t_n) - ct_n - (u(x,t_p) - ct_p)|\}.$$

Ainsi, $\{u(\cdot, \cdot + t_n)\}_{n\in\mathbb{N}}$ est une suite de Cauchy dans $C(\mathbb{T}^N \times [0,\infty))$; appelons $w(\cdot,\cdot)$ sa limite. Par stabilité, w est une solution de viscosité de

$$\frac{\partial u}{\partial t} - \operatorname{tr}(A(x)D^2u) + H(x, Du) = -c \tag{1.10}$$

avec $\bar{u}$ comme condition initiale. Par passage à la limite le long d'une sous-suite $u(x, t+t_n)$, on obtient alors

$$\bar{k} = \max_{\mathbb{T}^N}\{w(x,t) - v(x)\}.$$

Remarquons que v est aussi solution de viscosité de (1.11). Comme H, w et v sont lipschitziens, on en déduit que $w - v$ est une sous-solution de viscosité de

$$\frac{\partial u}{\partial t} - \operatorname{tr}(A(x)D^2u) - C|Du| = 0. \tag{1.11}$$

Mettant à profit l'uniforme ellipticité de l'équation, par le principe du maximum fort pour les solutions de viscosité (voir Da Lio [32], Ley & Nguyen [65]), on en déduit que $w - v$ est constant dans $\mathbb{T}^N \times [0,\infty)$. Donc $\bar{k} = w(x,t) - v(x)$ pour tous $(x,t) \in \mathbb{T}^N \times [0,\infty)$. Tout ceci ne dépend pas de la sous-suite choisie et permet de conclure que $u(x,t) \to v(x) + \bar{k}$ quand $t \to \infty$ uniformément par rapport à x.

1.2 Résultats connus

Pour établir le développement asymptotique (1.3) pour la solution de (1.1), nous venons de voir que nous avons besoin d'hypothèses supplémentaires. Commençons par décrire les hypothèses et les résultats connus avant d'énoncer les principaux résultats de cette thèse.

Une première batterie de résultats concerne le cas des équations totalement dégénérées ou équations de Hamilton-Jacobi du premier ordre dans le cas périodique,

$$\begin{cases} \frac{\partial u}{\partial t} + H(x, Du) = 0 & x \in \mathbb{T}^N, t > 0 \\ u(x,0) = u_0(x) & x \in \mathbb{T}^N, \end{cases} \tag{1.12}$$

et le problème ergodique associé

$$c + H(x, Dv) = 0 \quad x \in \mathbb{T}^N, \tag{1.13}$$

c'est-à dire l'EDP (1.1) avec $A \equiv 0$. Il y a de nombreux travaux sur le comportement en temps long des solutions. Les premiers résultats de convergence généraux ont été établis par Fathi [38] et Namah & Roquejoffre [69] sur une variété compacte sans bord

$\mathcal{M}$ (prenons $\mathcal{M} = \mathbb{T}^N$ pour simplifier) pour des Hamiltoniens strictement convexes. Dans [38], H est de plus supposé lisse et strictement convexe. Dans [69], $H(x,p) = F(x,p) - \ell(x)$ avec $H \in \mathrm{Lip}(\mathbb{T}^N \times I\!R^N)$, $f \in C(\mathbb{T}^N)$ and

$$\ell \geq 0, \qquad \min_{\mathbb{T}^N} \ell = 0, \qquad H(x,p) > H(x,0) = 0 \text{ pour } p \neq 0, \tag{1.14}$$

$$H(x,p) \underset{|p|\to+\infty}{\to} +\infty \text{ uniformément pour } x \in \mathbb{T}^N. \tag{1.15}$$

Le résultat prend la même forme que (1.3), i.e.,

$$u(x,t) - ct \to v(x) \text{ uniformément sur } \mathcal{M} \text{ quand } t \to \infty,$$

où u est la solution de (1.12) et (c, v) est une solution de (1.13) (sous les hypothèses de [69], $c = 0$). La preuve de [69] utilise des arguments EDP, nous y reviendrons dans le Chapitre 7. L'approche de Fathi est basée sur la théorie KAM faible (voir Fathi [37], Fathi [39] et Fathi & Siconolfi [40]) qui utilise la formulation lagrangienne du problème de calcul des variations sous-jacent à (1.12) dans le cas où H est convexe et les ensembles d'Aubry-Mather. Cette approche a aussi été utilisée par Roquejoffre [74] et Davini & Siconolfi [34]. Une approche EDP pour ces deux de problèmes a été développée dans Barles & Souganidis [20] pour des Hamiltoniens qui peuvent aussi être non-convexes ; voir aussi Barles, Ishii & Mitake [16] pour une approche simplifiée et plus générale. Enfin, dans le cadre des systèmes, voir Camilli, Ley, Loreti & Nguyen [24], Mitake & Tran [60] et Nguyen [71]. Il existe aussi des résultats dans le cadre des ouverts bornés avec des conditions aux bord, voir par exemple Mitake [57, 58, 59].

Il est important de remarquer que tous les résultats cités ne prennent pas en compte le cas des équations de Hamilton-Jacobi du second ordre, ni le cas non-périodique et non borné, deux choses qui nous intéresseront particulièrement dans cette thèse.

Pour les équations de Hamilton-Jacobi du second ordre (1.1), le premier résultat important et général est dû à Barles & Souganidis [21] dans le cas périodique. Rappelons que la première étape pour obtenir la convergence (1.3) nécessite des bornes de gradient uniformes pour (1.5)-(1.6). Dans [21], de telles bornes sont obtenues sous deux types d'hypothèses sur H et la diffusion A. Le premier cas concerne (1.1) dans le cas uniformément parabolique (voir (1.9)) avec H *sous-linéaire* (voir (1.23) introduite plus tard). Dans ce cas, la borne de gradient est établie en utilisant l'ellipticité et la méthode de Ishii & Lions [51]. Dans le second cas, les bornes de gradient sont établies pour des équations éventuellement dégénérées lorsque H satisfait une propriété de type *sur-linéaire*, le cas typique étant

$$H(x,p) = a(x)|p|^{1+\theta} + \ell(x), \quad \theta > 0, a, \ell \in C(\mathbb{T}^N) \text{ and } a > 0.$$

La preuve repose sur des arguments de type méthode de Bernstein faible introduite par Barles [9]. Dans les deux cas, la preuve de la convergence repose ensuite sur le principe du maximum fort pour les équations non-dégénérées. Ces résultats ont été récemment généralisés dans Ley & Nguyen [65] pour pouvoir être appliqués aux systèmes. Dans le cas des équations dégénérées, des résultats ont été obtenus par Cagnetti, Gomes, Mitake & Tran [25] et Ley & Nguyen [67] avec des méthodes et sous des hypothèses différentes.

Tous les résultats décrits jusqu'à présent concernent la cas périodique (ou les ensembles bornés). Dans l'espace $I\!R^N$ tout entier, il y a quelques résultats mais la situation est plus délicate à cause du manque de compacité et du caractère possiblement non-borné des solutions. Des hypothèses supplémentaires sont nécessaires ou des cas particuliers sont considérés.

Pour les équations de Hamilton-Jacobi du premier ordre, il existe principalement deux travaux. Celui de Barles & Roquejoffre [19] qui est plutôt basé sur des techniques EDP et celui de Ishii [52] avec une approche dynamique. Ces deux travaux vont nous intéresser tout particulièrement dans le cadre des résultats du chapitre 7 donc nous renvoyons au paragraphe *Résultats de la thèse* pour une description.

Concernant d'autres travaux pour des équations du premier ordre, citons Fujita, Ishii & Loreti [43] and Fujita & Loreti [44] qui peuvent se passer de plusieurs des hypothèses des résultats précédents dans le cadre non-borné pour l'EDP

$$u_t + \alpha\langle x, Du\rangle + H(Du) = f(x) \quad \text{dans } I\!R^N \times (0, +\infty), \tag{1.16}$$

avec $\alpha > 0$ et H convexe $I\!R^N$. Plus particulièrement, $H(p) = \beta|p|^2$, $\beta > 0$ et f est semiconvexe dans [44] et, en plus de la convergence, un taux de convergence est établi. Mais l'hypothèse cruciale de (1.16) qui permet de pallier au manque de compacité est la présence du terme de Ornstein-Uhlenbeck $\alpha\langle x, Du\rangle$ qui est un des ingrédients principaux de cette thèse et que nous décrivons en détail plus loin. Enfin mentionnons également les travaux de Ishii [52] et Ichihara & Ishii [48] pour des Hamiltoniens strictement convexes ou convexes et coercifs avec des hypothèses plus restrictives sur les conditions initiales.

Dans le cas des équations du second ordre, il y a peu de résultats dans le cas non-borné, essentiellement les travaux de Fujita, Ishii & Loreti [42] (qui ont inspiré [43, 44]) et Ghilli [45], tous deux pour des équations de Hamilton-Jacobi visqueuses (du second ordre non-dégénérées) en présence du terme de Ornstein-Uhlbenbeck. Citons également les travaux de Debussche, Hu & Tessitore [35] et Madec [56] sur le sujet en dimension infinie en utilisant une approche probabilité. Comme notre thèse se place essentiellement dans ce cadre, nous décrivons maintenant en détail [42].

1.3 Équations de Hamilton-Jacobi avec opérateur de Ornstein-Uhlenbeck

Le travail de Fujita, Ishii & Loreti [42] concerne l'équation

$$\begin{cases} u_t(x,t) - \Delta u(x,t) + \alpha\langle x, Du(x,t)\rangle + H(Du(x,t)) = f(x) \;\text{ dans } I\!R^N \times (0,\infty), \\ u(.,0) = u_0 \;\text{ dans } I\!R^N, \end{cases} \tag{1.17}$$

où $\alpha > 0$ est une constante donnée, $H \in \mathrm{Lip}(I\!R^N)$, f et u_0 sont des fonctions définies sur $I\!R^N$ satisfaisant

$$|g(x) - g(y)| \le C|x-y|(\Gamma_\mu(x) + \Gamma_\mu(y)), \tag{1.18}$$

pour $g = f$ ou $g = u_0$, avec

$$\Gamma_\mu(x) = e^{\mu|x|^2}, \mu > 0. \tag{1.19}$$

Comme ce travail concerne des solutions non-bornées dans un espace non-borné, la fonction exponentielle (1.19) sera la condition de croissance pour les solutions de (1.17). Plus précisément, ils obtiennent leurs résultats quand les solutions de (1.17) appartiennent à la classe

$$\mathcal{E}_\mu(\overline{Q}) = \left\{ v : \overline{Q} \to I\!R : \lim_{|x|\to+\infty} \sup_{0\le t<T} \frac{v(x,t)}{\Gamma_\mu(x)} = 0, \text{ pour } T > 0 \right\}, \tag{1.20}$$

où $Q = I\!R^N \times (0,\infty)$. L'opérateur $\Delta - \alpha\langle x, D\rangle$ est appelé *opérateur de Ornstein-Uhlenbeck*. En termes de problèmes de contrôle stochastique, le Laplacien correspond à une diffusion classique associée au mouvement brownien qui étale les trajectoires dans tout $I\!R^N$ alors que le terme $-\alpha\langle x, D\rangle$ permet de les confiner dans une grosse boule et de retrouver de la compacité, cf. Chapitre 3 pour le lien avec les problèmes de contrôle stochastique et plus de détails.

La présence de cet opérateur implique que Γ_μ est une sur-solution de l'équation pour $|x|$ grand,

$$-\Delta\Gamma_\mu(x) + \alpha\langle x, D\Gamma_\mu(x)\rangle - C|D\Gamma_\mu(x)| \ge \Gamma_\mu(x) - K, \qquad x \in I\!R^N, \tag{1.21}$$

ici $C > 0$, $K = K(\mu, \alpha, N, C) > 0$. La propriété (1.21) est extrêmement utile et est la base pour obtenir leurs résultats : existence, unicité, régularité des solutions. De plus, elle aide aussi dans l'application du principe du maximum fort pour établir le comportement asymptotique des solutions de (1.17).

Comme expliqué dans le premier paragraphe, pour obtenir un développement asymptotique du type (1.3), les auteurs doivent préalablement établir des bornes de gradient uniformes

$$|Du(x,t)|, |Dv^\lambda(x)| \le C|x-y|(\Gamma_\mu(x) + \Gamma_\mu(y)) \tag{1.22}$$

avec C indépendant de t et λ, où v^λ est solution de viscosité de l'équation approchée correspondant à (1.17).

Soulignons que l'équation (1.17) est écrite dans des termes simples avec de fortes hypothèses sur certaines données. En premier lieu, le terme du second ordre Δu est un cas particulier d'un opérateur de diffusion plus général $\mathrm{tr}(A(x)D^2u)$. Dans le cas du Laplacien pur, il n'y a pas de termes embêtants apparaissant dans les preuves et dûs à A. De plus, pour le laplacien, la théorie classique de Schauder pour la régularité des EDP uniformément paraboliques s'applique (voir Ladyzhenskaya, Solonnikov & Uralseva [61]) et on peut travailler avec des solutions régulières ce qui simplifie considérablement les preuves. Ceci n'est plus vrai lorsque A peut être dégénéré. D'autre part, l'Hamiltonien H est indépendant de la variable x et est lipschitzien,

$$|H(p) - H(q)| \le C|p-q|, \quad p, q \in I\!R^N.$$

Grâce aux propriétés de symétries dans les preuves de viscosité, ce terme disparaît rapidement dans les preuves, évitant le présence de termes provenant de H délicats à traiter en particulier dans ce cadre non-borné.

Le but des premiers chapitres de cette thèse est de généraliser les résultats de [42] dans plusieurs directions. La première est de pouvoir prendre en compte des Hamiltoniens généraux du type Bellman-Isacs qui sont seulement sous-linéaires,

$$|H(x, Du)| \leq C(1 + |Du|) \tag{1.23}$$

sans hypothèse supplémentaire. Ensuite, nous voulons traiter le cas de diffusions générales. Cela amène des difficultés supplémentaires, qui, dans ce cadre non-bornés, ne sont pas seulement techniques. Nous reviendrons sur ces problèmes lorsque nous décrirons plus précisément nos résultats.

Enfin, une autre nouveauté de cette thèse est de remplacer la diffusion $\mathrm{tr}(A(x)D^2u)$, qui est un opérateur local, par un opérateur non-local. Introduisons maintenant les opérateurs non-locaux que nous allons considérer.

1.4 Opérateurs non-locaux intégro-différentiels de type Laplacien fractionnaire

Soient $x \in I\!R^N$ et $\psi : I\!R^N \to I\!R$ une fonction continue bornée de classe C^2 dans un voisinage de x. Nous introduisons l'opérateur intégro-différentiel associé à la partie "saut" du générateur infinitésimal d'un processus de Lévy comme suit:

$$\mathcal{I}(x, \psi, D\psi) = \int_{I\!R^N} \{\psi(x+z) - \psi(x) - D\psi(x) \cdot z\mathbb{I}_B(z)\}\nu(dz), \tag{1.24}$$

où $\mathbb{I}_B$ est la fonction indicatrice de la boule unité, ν est une mesure de Lévy, qui est régulière et positive. Un exemple important de ce type d'opérateur est le cas où

$$\nu(dz) = \frac{C_{N,\beta}}{|z|^{N+\beta}} dz, \tag{1.25}$$

avec $\beta \in (0, 2)$ et $C_{N,\beta}$ une constante de normalisation. Alors $-\mathcal{I} = (-\Delta)^{\beta/2}$ est le Laplacien fractionnaire d'ordre β, voir [36].

Comme nous travaillons avec des solutions non-bornées dans des espaces non-bornés, nous devons considérer des mesures de Lévy pour lesquelles (1.24) est bien défini pour nos solutions :

(M1) Il existe une constante $C_\nu^1 > 0$ telle que

$$\int_B |z|^2 \nu(dz), \int_{B^c} \phi_\mu(z)\nu(dz) \leq C_\nu^1,$$

où $\phi_\mu \in C^2(I\!R^N)$, $\mu > 0$, est la fonction de croissance maximale définie par (2.42).

Un exemple typique de ce genre de mesure est alors : pour $\mu > 0$ et $\beta \in (0, 2)$,

$$\nu(dz) = \frac{e^{-\mu|z|}}{|z|^{N+\beta}} dz, \tag{1.26}$$

qui est connue sous le nom de *loi β-stable tempérée*, cf. Chapitre 3 pour le lien de cet opérateur avec des problèmes de contrôle de processus de Lévy.

1.5 Résultats de la thèse

Dans cette thèse, nous étudions des problèmes de comportement en temps long d'équations de Hamilton-Jacobi. Le premier type de problèmes, qui constitue l'essentiel des travaux présentés, concerne les EDP semilinéaires avec un opérateur de Ornstein-Uhlenbeck,

$$\begin{cases} u_t - \mathcal{F}(x,[u]) + \langle b(x), Du\rangle + H(x,Du) = f(x) \ \text{ dans } I\!\!R^N \times (0,\infty), \\ u(.,0) = u_0(.) \ \text{ dans } I\!\!R^N, \end{cases} \tag{1.27}$$

où $\mathcal{F}$ peut être *local* ou *nonlocal*:

$$\begin{cases} \mathcal{F}(x,[u]) = \mathrm{tr}(A(x)D^2u) \ \ \textbf{(opérateur local)}, \\ \mathcal{F}(x,[u]) = \mathcal{I}(x,u,Du) \ \ \textbf{(opérateur non-local)}. \end{cases} \tag{1.28}$$

Le deuxième type de problèmes abordés dans le dernier chapitre (indépendant) de la thèse s'intéresse au comportement en temps grand des solutions non-bornées d'équations de Hamilton-Jacobi du premier ordre (1.12) posées dans tout l'espace $I\!\!R^N$.

Commençons par introduire les résultats pour l'équation (1.27). Le terme $\mathcal{F} - \langle b, D\rangle$ dans (1.28) est appelé opérateur de Ornstein-Uhlenbeck. Dans le cas local, nous supposerons toujours que la matrice de diffusion A peut être écrite $A(x) = \sigma(x)\sigma^T(x)$ avec σ bornée et lipschitzienne dans $I\!\!R^N$ (remarquer que A est alors automatiquement positive). Dans le cas non-local, nous considérerons des opérateurs définis par (1.24) avec des mesures ν du type (1.26). Concernant le terme d'Ornstein-Uhlenbeck, nous supposerons que

$$\text{Il existe } \alpha > 0 \text{ tel que } \langle b(x) - b(y), x - y\rangle \geq \alpha|x-y|^2, \tag{1.29}$$

et α sera appelé *magnitude* de l'opérateur d'Ornstein-Uhlenbeck.

Les résultats sur ce sujet sont divisés en 3 parties qui constituent chacune un chapitre. La première partie, qui est la partie principale, contient les bornes *a priori* pour le gradient des solutions de (1.27) et de l'équation stationnaire approchée

$$\lambda u^\lambda(x) - \mathcal{F}(x,[u^\lambda]) + \langle b(x), Du^\lambda(x)\rangle + H(x, Du^\lambda(x)) = f(x) \quad \text{dans } I\!\!R^N, \tag{1.30}$$

où $\lambda > 0$ est destiné à tendre vers 0 à la fois dans le cas local et non-local, sous des hypothèses très générales. Cette partie constitue le Chapitre 4. La seconde partie (Chapitre 5) étudie la caractère bien posé des équations (1.27) et (1.30). Dans ce contexte non-borné et sous l'hypothèse assez faible (1.23) sur H, ces résultats ne sont pas classiques dans la théorie des solutions de viscosité et demandent à être prouvés soigneusement. Finalement la dernière partie sur ce sujet revient à la motivation initiale, c'est-à-dire le comportement en temps long des solutions. Cette partie est plus classique car elle repose sur la preuve usuelle utilisant le principe du maximum fort associé aux idées de [42] ; c'est une application assez directe des résultats précédents une fois le principe du maximum fort dans le cas non-local clairement énoncé. Nous renvoyons au Chapitre 6.

Le Chapitre 7 étudie le comportement en temps grand des solutions non-bornées de

$$\begin{cases} \frac{\partial u}{\partial t} + |Du| = \ell(x) & x \in I\!R^N, t > 0 \\ u(x,0) = u_0(x) & x \in I\!R^N, \end{cases} \tag{1.31}$$

pour $\ell \geq 0$ tel que $\mathcal{K} := \operatorname{argmin}_{I\!R^N} \ell$ est un compact non vide de $I\!R^N$.

Insistons encore une fois sur le fait que nous travaillons avec des solutions non-bornées dans un ensemble non-borné. Il n'est donc pas possible d'espérer obtenir des résultats sans imposer de conditions de croissance sur les données et les solutions. Nous devons les restreindre à des ensembles fonctionnels à croissances limitées comme l'ensemble (1.20) avec Γ_μ donnée par (1.19) dans [42].

Dans le cas de (1.27), la fonction (1.19) dictant la croissance ne semble pas satisfaisante car nous voulons pouvoir travailler avec des matrices de diffusion générales ce qui génère des termes délicats à traiter dans les preuves. Pour pouvoir nous en tirer sans avoir à imposer de condition sur la magnitude α du terme de Ornstein-Uhlenbeck, nous avons choisi, après différents essais, de restreindre un peu la croissance des fonctions considérées en choisissant

$$\phi_\mu(x) = e^{\mu|x|} \quad \text{for } x \in I\!R^N. \tag{1.32}$$

Nous travaillerons avec f, u_0 et les solutions de (1.27)-(1.30) dans (1.20) avec ϕ_μ à la place de Γ_μ.

Dans le cas de (1.31), nous travaillons sous des conditions de croissance quadratique. Plus précisément, nous supposerons que u_0, l appartiennent à

$$\begin{aligned} \mathcal{Q} := \{u : I\!R^N \times [0,+\infty) \to I\!R : \text{il existe } C > 0 \text{ tel que pour tout } x \in I\!R^N, \\ |u(x,t)| \leq C(1+|x|^2)\}, \end{aligned} \tag{1.33}$$

et

$$\begin{aligned} \mathcal{L} := \{u : I\!R^N \times [0,+\infty) \to I\!R : \text{il existe } C > 0 \text{ tel que pour tout } x, y \in I\!R^N, \\ |u(x,t) - u(y,t)| \leq C(1+|x|^2+|y|^2)|x-y|\} \end{aligned} \tag{1.34}$$

et nous recherchons des solutions dans les mêmes ensembles.

Avant d'arriver à une description plus précise de nos résultats, rappelons que de manière générale nous travaillons avec des solutions de viscosité des équations considérées. Nous renvoyons le lecteur à la littérature classique sur le sujet : [30, 10, 5, 54] pour les définitions et les résultats principaux de la théorie et [15, 13, 2, 77, 78] pour le cadre non-local.

1.5.1 Estimations a priori des gradients

Le résultat le plus important que nous cherchons à obtenir dans cette thèse concerne des estimations de gradient locales uniformes pour les solutions des équations (1.27) et (1.30). L'un des intérêts de notre travail est de les obtenir pour des classes assez générales d'Hamiltoniens i.e., sous la condition (1.23). Cette hypothèse générale entraîne l'apparition de nombreux termes difficiles à traiter puisque nous le faisons dans un cadre non borné. Pour surmonter cette difficulté, il n'est pas suffisant de ne compter que sur l'effet provenant du terme de Ornstein-Uhlenbeck, nous devons combiner cela avec l'ellipticité de l'équation en suivant des idées de Ishii & Lions [51] dans le cas local, et de Barles, Chasseigne, Ciomaga & Imbert [11] dans le cas non-local. Rappelons et définissons maintenat ce que nous entendons par ellipticité aussi bien dans le cas local que non-local.

Dans le cas local, le définition classique est donnée par (1.9), qui conduit au principe de maximum fort, voir Gilbarg & Trudinger [46].

Dans le cas non-local, il n'existe pas de définition exacte et générale, nous allons donc en donner une adaptée à notre contexte. Dans cette thèse, nous travaillons avec une classe de mesures de Lévy qui satisfont (M1) et

(M2) Il existe $\beta \in (0,2)$ tel que pour tout $a \in I\!R^N$ il existe $0<\eta<1$ et une constante C_ν^2 tels que

$$\forall \gamma > 0 \quad \int_{\mathcal{C}_{\eta,\gamma}(a)} |z|^2 \nu(dz) \geq C_\nu^2 \eta^{\frac{N-1}{2}} \gamma^{2-\beta},$$

où $\mathcal{C}_{\eta,\gamma}(a) := \{z \in B_\gamma; (1-\eta)|z||a| \leq |a.z|\}$. On renvoie à Barles, Chasseigne, Ciomaga & Imbert [11] pour plus de détails au sujet de cette propriété. L'hypothèse (M2) peut être prise comme une définition de l'ellipticité mais pour $\beta \in (0,1)$, l'effet de cette ellipticité n'est pas assez fort pour aider l'opérateur de Ornstein-Uhlenbeck à gérer les mauvais termes provenant de l'Hamiltonien. C'est pourquoi dans la suite, afin d'avoir des énoncés plus simples de nos résultats qui couvrent à la fois le cas local et non-local, nous dirons que *l'équation non-locale est elliptique si (M2) est vérifiée avec* $\beta \in (1,2)$.

Donnons maintenant le résultat principal pour l'équation stationnaire (1.30). On a une version similaire pour le cas non stationaire (1.27), voir Théorème 4.2.1 au Chapitre 4.

Théorème 1.5.1 (Bornes de gradient locales uniformes pour les solutions de (1.30) dans le cas elliptique, Théorème 4.1.1). *Soit u^λ une solution de viscosité continue de* (1.30) *dans la classe* (1.20)*, de croissance exponentielle ϕ_μ définie par* (1.32)*. On suppose que* (1.30) *est uniformément elliptique dans le sens local et non-local, et que les hypothèses* (1.23), (1.29)*, et sur f sont satisfaites. Alors il existe une constante $C > 0$ indépendante de λ telle que*

$$|u^\lambda(x) - u^\lambda(y)| \leq C|x-y|(\phi_\mu(x) + \phi_\mu(y)). \tag{1.35}$$

Donnons un squelette de preuve :

Afin d'utiliser ϕ_μ comme fonction test pour u dans (1.32), nous devons la régulariser. Plus précisément, posons

$$\phi_\mu(x) = e^{\mu\sqrt{|x|^2+1}}, \quad \mu > 0, \forall x \in I\!R^N.$$

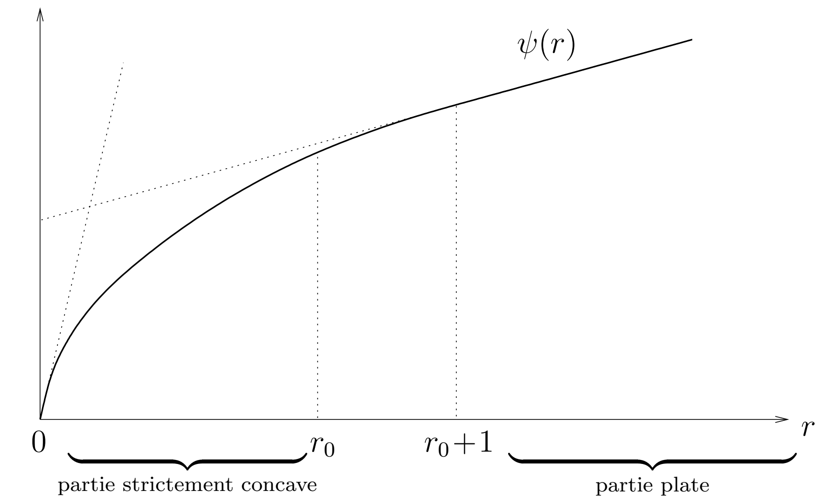

Figure 1.1: La fonction concave ψ

Soit $\delta > 0$ et $\psi : I\!R_+ \to I\!R_+$ une fonction C^2, concave et croissante telle que $\psi(0) = 0$ et bien choisie (voir Figure 1.1). Considérons

$$M = \sup_{x,y \in I\!R^N} \left\{ u^\lambda(x) - u^\lambda(y) - \sqrt{\delta} - (\psi(|x-y|) + \delta)\Phi(x,y) \right\}, \tag{1.36}$$

où $\Phi(x,y) = C_1(\phi_\mu(x) + \phi_\mu(y) + A)$ pour certains $A, C_1 > 0$. Définissons

$$\varphi(x,y) = \sqrt{\delta} + (\psi(|x-y|) + \delta)\Phi(x,y).$$

Si $M \leq 0$ pour un bon choix de A, C_1, ψ indépendants de $\delta > 0$, alors nous obtenons (1.35) en faisant tendre $\delta \to 0$.

On procède donc par l'absurde, en supposant que pour δ assez petit, $M > 0$. Puisque $u^\lambda \in \mathcal{E}_\mu(I\!R^N)$ et $\delta > 0$, le sup est atteint en un point (x,y) tel que $x \neq y$, de par le fait que $\delta > 0$ et la continuité de u. La théorie des solutions de viscosité développée par Crandall, Ishii & Lions [30] pour le cas local, et Barles & Imbert [15] pour le cas non-local nous fournit deux matrices $X, Y \in \mathcal{S}^N$ qui vérifient l'inégalité de viscosité suivante

$$\begin{aligned} & \lambda(u^\lambda(x) - u^\lambda(y)) - (\mathcal{F}(x,[u^\lambda]) - \mathcal{F}(y,[u^\lambda])) \\ & +\langle b(x), D_x\varphi\rangle - \langle b(y), -D_y\varphi\rangle + H(x, D_x\varphi) - H(y, -D_y\varphi) \\ \leq\ & f(x) - f(y), \end{aligned} \tag{1.37}$$

où $\mathcal{F}(x,[u^\lambda]) = \mathrm{tr}(A(x)X)$ et $\mathcal{F}(y,[u^\lambda]) = \mathrm{tr}(A(y)Y)$ dans le cas local ; $\mathcal{F}(x,[u^\lambda]) = \mathcal{I}(x, u^\lambda, D_x\varphi)$ et $\mathcal{F}(y,[u^\lambda]) = \mathcal{I}(y, u^\lambda, -D_y\varphi)$ dans le cas non-local.

Il nous faut maintenant estimer tous les termes dans (1.37) afin d'obtenir une contradiction. La principale difficulté de la preuve réside dans l'estimation de $\mathcal{F}(x,[u^\lambda]) - \mathcal{F}(y,[u^\lambda])$. Afin d'y parvenir, les idées dans le cas local sont inspirées de Ishii & Lions [51] qui permettent d'obtenir un terme strictement positif grace à l'ellipticité de l'équation et un bon choix de fonction ψ concave, croissante (voir le Lemme 4.1.2 pour plus de détails). Dans le cas non-local, les idées viennent de Barles, Chasseigne, Ciomaga & Imbert [11]. Puisque nous sommes dans un cadre non borné, les estimations deviennent compliquées et dépendent du choix de la fonction concave ψ (voir la Proposition 4.1.1 et les lemmes 4.1.4 et 4.1.5 pour plus de détails).

Pour simplifier la présentation, nous allons décrire la preuve seulement dans le cas local.

Puisque $M > 0$, en utilisant (1.9), (1.29), (1.23) et l'hypothèse sur f pour estimer les différents termes, l'inégalité (1.37) entraîne

$$\begin{aligned}
& \lambda\sqrt{\delta} + \Phi(x,y)\left(-4\rho\psi''(|x-y|) + \alpha\psi'(|x-y|)|x-y|\right) \\
\leq\ & \Phi(x,y)C\psi'(|x-y|) + C + C_f(\phi_\mu(x) + \phi_\mu(y))|x-y| \\
& + C_1(\psi(|x-y|)+\delta)\left(\mathrm{tr}(A(x)D^2\phi_\mu(x)) - \langle b(x), D\phi_\mu(x)\rangle + C|D\phi_\mu(x)|\right) \\
& + C_1(\psi(|x-y|)+\delta)\left(\mathrm{tr}(A(y)D^2\phi_\mu(y)) - \langle b(y), D\phi_\mu(y)\rangle + C|D\phi_\mu(y)|\right),
\end{aligned}$$

où $C > 0$ ne dépend que de σ, H, μ. Nous utilisons alors la condition de sur-solution pour ϕ_μ comme dans (1.21) (voir Lemme 4.1.1 pour les détails) et nous obtenons

$$\begin{aligned}
& \lambda\sqrt{\delta} - 4\rho\psi''(|x-y|) + \alpha\psi'(|x-y|)|x-y| \\
\leq\ & C\psi'(|x-y|) + 2K(\psi(|x-y|)+\delta) + C + C_f|x-y|.
\end{aligned} \tag{1.38}$$

Ici nous nous sommes débarrassés des termes en Φ des deux cotés de l'inégalité pour simplifier les explications.
But: obtenir une contradiction dans (1.38).

- *Cas 1:* $r := |x-y| \leq r_0$. La fonction ψ a été choisie pour être strictement concave pour $r \leq r_0$, donc nous profitons de l'ellipticité de l'équation qui donne une contribution strictement positive "$-4\rho\psi$"(r)" dans (1.38), permettant de contrôler tous les autres termes et arriver à une contradiction.

- *Cas 2:* $r := |x-y| \geq r_0$. Dans ce cas, la dérivée seconde $\psi''(r)$ de la fonction ψ est petite ou nulle et l'ellipticité n'est pas assez puissante pour contrôler les mauvais termes. À la place, on utilise la contribution positive du terme $\alpha\psi'(r)r$, issu de l'opérateur de the Ornstein-Uhlenbeck.

Notons qu'en utilisant les "bons termes" comme expliqué ci-dessus, nous pouvons choisir les paramètres indépendamment de λ afin d'obtenir une contradiction dans (1.38).

Les idées de preuve dans le cas non-local suivent celles du cas local dans le sens où nous pouvons utiliser la méthode Ishii & Lions [51] lorsque $\beta \in (1,2)$ (l'ordre du terme non-local). Mais nous n'avons pas réussi à obtenir directement des bornes Lipschitz pour l'équation comme dans le cas local. La preuve se complique à cause de l'opérateur non-local dans un cadre non borné. En fait les estimations des termes non-locaux sont très différentes de celles des termes locaux (voir Proposition 4.1.1 pour le détail, avec une fonction-test générale, concave). La preuve comporte donc deux étapes. La première étape établit la régularité τ-Hölder pour tout $\tau \in (0,1)$. Ensuite, on utilise cette régularité Hölder pour obtenir la régularité et les bornes Lipschitz. À chaque étape on construit des fonctions test concaves différentes, également différentes du cas local.

Utiliser l'ellipticité non-locale pour appliquer les idées de la méthode de Ishii & Lions [51] est très délicat (on peut voir cela dans les lemmes 4.1.4 et 4.1.5).

Dans le cas où nous n'avons pas la propriété d'ellipticité, on parle d'équations dégénérées. Dans de tels cas, il est naturel de se poser plusieurs questions : quels resultats pouvons-nous onbtenir? sous quelles hypothèses sur les données? et plus particulièrement, quelles hypothèses les plus générales sur l'Hamiltonien pouvons-nous gérer? Le théorème suivant et les remarques qui suivent répondent à ces questions. Pour obtenir des bornes de gradient uniformes (1.35) sans utiliser l'ellipticité de l'équation, nous devons renforcer les hypothèses sur H:

$$\begin{cases} \text{Il existe } L_{1H}, L_{2H} > 0 \text{ tels que pour tous } x, y, p, q \in I\!R^N \\ |H(x,p) - H(y,p)| \leq L_{1H}|x-y|(1+|p|), \\ |H(x,p) - H(x,q)| \leq L_{2H}|p-q|(1+|x|). \end{cases} \tag{1.39}$$

Il s'agit des hypothèses classiques satisfaites par un Hamiltonien provenant d'un problème de contrôle optimal, voir Chapitre 3.

Théorème 1.5.2 (Bornes locales de gradient uniformes pour les solutions de (1.30) dans le cas dégénéré, Théorème 4.1.2). *Soit u^λ une solution de viscosité continue de* (1.30) *dans la classe* (1.20)*, de croissance exponentielle ϕ_μ donnée par* (1.32)*. On suppose que* (1.30) *est elliptique dégénéré dans le cas local et non-local, et que* (1.29)*,* (1.39) *sont vérifiées. Alors il existe*

$$C(\mathcal{F}, H) = \begin{cases} C(\sigma, H) \text{ dans le cas local} \\ C(H) \text{ dans le cas non-local} \end{cases}$$

telle que pour tout $\alpha > C(\mathcal{F}, H)$, il existe $C > 0$ indépendente de λ telle que (1.35) *a lieu.*

Dans ce théorème, la constante $C(\sigma, H)$ dépend de la matrice de diffusion σ car celle-ci dépend de la variable x. Dans le cas d'une diffusion homogène, on obtiendrait la même conclusion avec $\alpha > C(H)$ (la constante ne dépend ici que de H), aussi bien dans le cas local que non-local, voir la remarque 4.1.3.

De plus, si la matrice de diffusion est constante dans le cas local et $H(x,p) = H(p) \in \mathrm{Lip}(I\!R^N)$, alors (1.35) a lieu pour tout $\alpha > 0$ dans le cas local et non-local, et on retrouve les résultats contenus dans [42], voir la remarque 4.1.4.

1.5.2 Existence et unicité des solutions

Afin d'être en mesure d'utiliser les résultats évoqués dans la partie précédente, nous devons construire des solutions continues de (1.27) et (1.30), aussi bien dans le cas local que non-local. Les techniques classiques en solutions de viscosité utilisent la méthode de Perron (voir Barles [10], Crandall, Ishii & Lions [30] et Ishii [50]). Plus précisément, nous construisons d'abord des solutions de viscosité discontinues. Puis, la continuité est obtenue à l'aide d'un principe de comparaison fort. Cependant, dans le cas des équations

(1.27) et (1.30) sous l'hypothèse (1.23), il est très difficile d'obtenir le résultat de comparaison. Ce type de résultat peut même être faux, y compris dans le cas périodique (voir Ley & Nguyen [65]). Sous l'hypothèse faible (1.23), nous n'obtenons la comparaison entre sous-solution et sur-solution discontinues que lorsque l'une au moins des deux est Lipschitzienne. Il faut donc une autre approche pour construire les solutions continues de (1.27) et (1.30). Les idées proviennent de Barles & Souganidis [21] et [65], qui sont basées sur une troncature de l'Hamiltonien et du terme non borné f (ainsi que de la donnée initiale pour l'équation d'évolution).

Afin d'obtenir l'unicité des solutions de (1.30) et (1.27), il nous faut encore une hypothèse supplémentaire sur l'Hamiltonien,

$$\begin{cases} \text{Il existe } L_H > 0 \text{ tel que} \\ |H(x,p) - H(x,q)| \leq L_H |p-q|, \quad \text{pour tout } x,p,q \in I\!\!R^N. \end{cases} \tag{1.40}$$

Ceci est vérifié lorsque H provient d'un problème de contrôle optimal avec un terme de drift borné, voir Chapitre 3. L'unicité des solutions est en fait une conséquence directe du principe de comparaison et sera donné plus tard. Nous donnons d'abord le résultat d'existence de solutions de (1.30).

Théorème 1.5.3 (Existence de solutions continues de (1.30), Théorème 5.1.2) *Sous les hypothèses du Théorème 2.5.1, pour tout* $\lambda \in (0,1)$, *il existe une solution de viscosité continue* u^λ *de* (1.30) *dans la classe de fonctions*

$$\begin{aligned} &u^\lambda \in \mathcal{E}_\mu(I\!\!R^N), \\ &|u^\lambda(x) - u^\lambda(y)| \leq C|x-y|(\phi_\mu(x) + \phi_\mu(y)), \quad x, y \in I\!\!R^N, \end{aligned}$$

où $C > 0$ *est une constante indépendante de* λ.
De plus, si (1.40) *a lieu, alors la solution est unique dans la classe* $\mathcal{E}_\mu$.

Décrivons rapidement la procédure de troncature utilisée pour obtenir ce Théorème. Soit $m, n \geq 1$,

$$f_m(x) = \min\{f(x) + \frac{1}{m}\phi(x), m\},$$

$$H_{mn}(x,p) = \begin{cases} H_m(x,p) & \text{si } |p| \leq n \\ H_m(x, n\frac{p}{|p|}) & \text{si } |p| \geq n, \end{cases}$$

avec

$$H_m(x,p) = \begin{cases} H(x,p) & \text{si } |x| \leq m \\ H(m\frac{x}{|x|}, p) & \text{si } |x| \geq m. \end{cases}$$

Ceci génère un Hamiltonien BUC (borné uniformément continu) et la théorie classique des solutions de viscosité (voir [30]) fournit l'existence et l'unicité d'une solution de viscosité continue et bornée de (1.30). Grace aux estimation Lipschitz uniformes et au Théorème d'Ascoli, nous pouvons faire tendre $m, n \to \infty$ et obtenir à la limite une solution de viscosité continue de (1.30).

L'unicité des solutions provient du théorème qui suit. Pour plus de détails on renvoie à la preuve du Théorème 5.1.2 au Chapitre 5.

Théorème 1.5.4 (Principe de comparaison, Théorème 5.1.1) *Soit u^λ et v^λ, respectivement une sous-solution de viscosité semi-continue supérieurement et une sur-solution de viscosité semi-continue inférieurement de* (1.30)*, toutes deux dans la classe $\mathcal{E}_\mu$. On suppose* (1.40) *et que u ou v est localement Lipschitz dans $\mathbb{R}^N$. Alors $u \leq v$ dans $\mathbb{R}^N$.*

Des résultats similaires sont aussi obtenus pour l'équation d'évolution, voir les Théorèmes 5.2.1 et 5.2.2 du Chapitre 5.

Dans cette thèse, nous étudions de plus le caractère bien posé de l'équations (1.27) et (1.30), sous différentes hypothèses sur l'Hamiltonien, par exemple (1.39) ou

$$\begin{cases} \text{Il existe } L_{1H}, L_{2H} > 0 \text{ tels que pour tous } x, y, p, q \in \mathbb{R}^N \\ |H(x,p) - H(y,p)| \leq L_{1H}|x-y|(1+|p|), \\ |H(x,p) - H(x,q)| \leq L_{2H}|p-q|. \end{cases} \tag{1.41}$$

Dans de tels cas, on peut obtenir un principe de comparaison et donc l'existence et l'unicité de solutions de façon classique, voir Chapitre 5.

1.5.3 Problème ergodique et comportement asymptotique

Revenons à la motivation initiale expliquée dans le premier paragraphe et rappelons qu'étudier le problème ergodique associé à (1.27) consiste à trouver une constante $c \in \mathbb{R}$ et une fonction $v \in C(\mathbb{R}^N)$ solutions de

$$c - \mathcal{F}(x, [v]) + \langle b(x), Dv(x) \rangle + H(x, Dv(x)) = f(x) \qquad \text{dans } \mathbb{R}^N. \tag{1.42}$$

La résolution de ce problème est l'une des principales applications du résultat sur les bornes Lipschitz locales (1.35), ainsi que nous l'avons déjà expliqué. Ce type de résultat a déjà été étudié par de nombreux auteurs, citons par exemple [4], [19], [42].

Théorème 1.5.5 (Problème ergodique, Théorème 6.1.1, Théorème 6.1.2). *On suppose que les hypothèses du Théorème 2.5.1 sont vérifiées. Alors il existe une solution $(c, v) \in \mathbb{R} \times C(\mathbb{R}^N)$ de* (1.42)*. Si* (1.40) *a lieu, alors la constante ergodique c est unique et si de plus* (1.41) *a lieu, alors v est unique à une constante additive près.*

L'unicité de c résulte de l'utilisation du résultat de comparaison, c'est pour cette raison que nous devons supposer (1.40). De plus, nous devons linéariser l'équation afin d'appliquer le principe de maximum fort pour obtenir l'unicité de v à constante près, d'où la nécessité de supposer (1.41).

Théorème 1.5.6 (Comportement en temps long des solutions, Théorème 6.2.1) *Soit u*

l'unique solution de (1.27) *et* (c, v) *une solution de* (1.42)*. On suppose* (1.29), (1.41), *les hypothèses sur* f *et que soit*

(i) $\mathcal{F}(x, [u]) = \mathrm{tr}(A(x)D^2u)$ *et* (2.9) *a lieu,*

ou

(ii) $\mathcal{F}(x, [u]) = \mathcal{I}(x, u, Du)$ *et que* $(M1), (M2)$ *ont lieu avec* $\beta \in (1, 2)$.

Alors il existe une constante $a \in I\!R$ *telle que*

$$\lim_{t\to\infty} \max_{B(0,R)} |u(x,t) - (ct + v(x) + a)| = 0 \qquad \textit{pour tout } R > 0.$$

Ce qui est bien le résultat attendu.

1.5.4 Comportement en temps long de solutions non-bornées d'équations de Hamilton-Jacobi du premier ordre dans $I\!R^N$

Ce travail comporte un premier résultat de convergence pour les solutions non-bornées de (1.31) quand $\ell, u_0 \in \mathcal{Q} \cap \mathcal{L}$ (les ensembles $\mathcal{Q}, \mathcal{L}$ sont introduits dans (1.33)-(1.34)) et

$$\ell \geq 0, \qquad \liminf_{|x|\to+\infty} \ell(x) > \inf_{I\!R^N} \ell,$$

la dernière condition impliquant que $\operatorname{argmin} \ell$ est un compact de $I\!R^N$.

Nous commençons par établir l'existence et l'unicité d'une solution de

$$\lambda v^\lambda(x) + |Dv^\lambda(x)| = \ell(x) \qquad \text{dans } I\!R^N$$

dans l'ensemble $\mathcal{Q} \cap \mathcal{L}$ en prouvant que la constante de Lipschitz de v^λ dans $\mathcal{L}$ est indépendante de λ.

Comme expliqué au paragraphe 1.1, cela permet de faire tendre λ vers 0 et d'obtenir une solution du problème ergodique

$$c + |Dv(x)| = \ell(x) \qquad \text{dans } I\!R^N,$$

associé à (1.31). De plus, nous établissons que, si deux solutions ont une différence bornée, alors elles sont associées à une même constante ergodique c, voir Theorem 7.1.1 pour les détails. Une première difficulté du cas non-borné est la possible non-unicité de la constante ergodique, ce qui n'arrive pas en périodique par exemple. Nous produisons un exemple où cette non-unicité est effective dans la Section 7.6. De plus, les solutions v du problème ergodique n'ont plus forcément une croissance quadratique ; elles ne sont plus forcément dans $\mathcal{Q}$ (mais restent dans $\mathcal{L}$).

Nous étudions ensuite le problème d'évolution (1.31). Nous obtenons l'existence et l'unicité d'une solution de viscosité pour toute condition initiale u_0. Si, de plus, u_0 est suffisamment proche de v, où (c,v) est solution du problème ergodique, alors nous avons des propriétés de compacité pour la famille

$$\{u(x,t)-ct, t \geq 0\},$$

plus précisément,

$$\begin{aligned}
&|u(x,t)-ct| \leq C + |v(x)|,\\
&|(u(x,t)-ct)_t| \leq C,\\
&|D(u(x,t)-ct)| \leq C + \ell(x).
\end{aligned}$$

Nous pouvons alors prouver un comportement asymptotique du type (1.3) pour la solution u sous l'hypothèse supplémentaire qu'il existe une solution (c,v) du problème ergodique telle que v est *bornée inférieurement* et $c = \min \ell$. Dans ce cas, nous avons convergence pour toute donnée initiale u_0 assez proche de v (Theorem 7.1.3). Ce résultat signifie que, démarrant suffisamment proche d'une solution du problème ergodique qui est bornée inférieurement, la solution de (1.31) est attirée par cette solution stationnaire. En nous appuyant sur l'interprétation de (1.31) en théorie du contrôle optimal, nous produisons des exemples suggérant les choses suivantes. Il est d'abord assez naturel de pouvoir construire des solutions du problème ergodique qui sont bornées inférieurement sous nos hypothèses. Cependant, lorsque u_0 est proche d'une solution v qui n'est pas bornée inférieurement, la convergence peut ne pas avoir lieu rendant difficile toute généralisation du résultat.

De tels problèmes ont déjà été étudiés dans la littérature (même si cela reste beaucoup plus rare que dans le cas périodique). Barles & Roquejoffre [19] généralisent Namah & Roquejoffre [69] dans le cas non-borné mais ils travaillent avec des Hamiltoniens $H(x,p)$ qui sont bornés uniformément continus par rapport à x interdisant de prendre $\ell \in \mathcal{Q}$. Un autre travail important sur le sujet est celui de Ishii [52] qui se place dans le cadre des Hamiltoniens strictement convexes alors que notre Hamiltonien est le modèle des Hamiltoniens convexes sans être strictement convexes. Les preuves de [52] sont basées sur une approche dynamique et la théorie KAM faible alors que les nôtres sont purement EDP. Citons pour terminer les travaux de Ichihara & Ishii [53, 48] dont nous sommes également proches.

1.6 Perspectives

Décrivons quelques problèmes que nous aimerions étudier prochainement.

1. Dans cette thèse, nous avons obtenu des bornes de gradient uniformes pour les solutions d'EDP elliptiques à la fois pour des opérateurs locaux et non-locaux. Nous aimerions étendre ces résultats en utilisant une autre approche, la méthode de Bernstein faible introduite par Barles [9]. Cela nécessitera un autre type d'hypothèses sur H (coercivité, hypothèses de type convexité) mais cette méthode devrait s'appliquer à des équations dégénérées et on ne devrait plus avoir besoin du terme d'Ornstein-Uhlenbeck. La principale difficulté est de résussir à localiser

convenablement les arguments de la méthode de Bernstein faible dans ce contexte non-borné.

Une autre question est de réussir à établir le comportement en temps long des solutions d'équations dégénérées, pour lesquelles le principe du maximum fort n'est plus valable. Récemment des avancées ont été obtenues dans le cas périodique dans Ley & Nguyen [67] et Cagnetti et al. [25]. Nous souhaiterions voir si elles peuvent s'appliquer dans notre cas.

2. Dans le contexte non-local, nous aimerions obtenir des résultats avec des mesures de Lévy plus générales, par exemple lorsque la mesure dépend de x. Un cas particulier par lequel nous pourrions commencer est le cas des opérateurs de Lévy-Itô.

   Un autre problème qui est très naturel est de considérer des équations avec à la fois une diffusion locale et non-locale, les deux n'étant pas nécessairement non-dégénérées. Il serait intéressant d'étendre nos résultats de régularité à ce cadre. Bien sûr, les principes du maximum fort et la convergence en temps grand seraient aussi étudiés.

3. Les résultats obtenus pour les équations de Hamilton-Jacobi du premier ordre sont encore embryonnaires et demanderaient à être mieux situés dans la littérature par rapport à ce qui existe déjà. Nous souhaitons continuer nos recherches dans plusieurs directions. L'équation que nous considérons est la plus simple possible et des généralisation faciles devraient pouvoir être faites pour des Hamiltoniens plus généraux. Concernant le problème ergodique, existe-t-il des solutions pour toute constante ergodique $c \leq \bar{c} := \min \ell$? Est-il toujours possible de construire une solution du problème ergodique qui est bornée inférieurement ? Enfin, nous aimerions comprendre plus profondément ce qu'il se passe quand les solutions ne sont ni bornées supérieurement ni inférieurement.

Chapter 2

Introduction

2.1 Motivation: the large time behavior of Hamilton-Jacobi equations

The main motivation of this thesis is the study of the large time behavior of Hamilton-Jacobi equations in $I\!R^N$ with a drift of Ornstein-Uhlenbeck type both with local and nonlocal diffusion.

Let us start by recalling the classical statement of the problem in the local case.

Consider the semilinear evolution Hamilton-Jacobi PDE

$$\begin{cases} \frac{\partial u}{\partial t} - \mathrm{tr}(A(x)D^2u) + H(x, Du) = 0 \\ u(x,0) = u_0(x), \end{cases} \tag{2.1}$$

where Du, D^2u are the (spatial) gradient and Hessian matrix of the real-valued unknown function u defined on $I\!R^N \times [0, \infty)$. The functions $H : I\!R^N \times I\!R^N \to I\!R$ and $A : I\!R^N \to \mathcal{S}^N$ are the Hamiltonian and the diffusion matrix, respectively, where $\mathcal{S}^N$ is the set of $N \times N$ real symmetric matrices. The given continuous function u_0 is the initial condition. We will always assume that A is a nonnegative matrix in order that (2.1) is a parabolic (sometimes possibly degenerate) PDE.

To explain the problem in a simple way, assume for the moment that (2.1) has a unique *bounded* solution in $I\!R^N \times [0, T]$, $\forall T > 0$ solution for any given *bounded* locally Lipschitz continuous initial data u_0.

We aim at proving that there exists an *ergodic constant* $c \in I\!R$ and a locally Lipschitz continuous function $v(x)$ such that (c, v) is solution to the *ergodic problem*

$$c - \mathrm{tr}(A(x)D^2v) + H(x, Dv) = 0 \tag{2.2}$$

and the following asymptotic expansion holds for u,

$$u(x,t) = ct + v(x) + o(\frac{1}{t}), \tag{2.3}$$

where $o(\frac{1}{t}) \to 0$ as $t \to +\infty$, locally uniformly with respect to x. This is the typical behavior which is expected for such an evolution problem.

The large time behavior (2.3) is usually established through the following two steps explained below.

The first step is to obtain uniform gradient bounds for (2.1) and the approximate stationary problem

$$\lambda v^\lambda - \mathrm{tr}(A(x)D^2 v^\lambda) + H(x, Dv^\lambda) = 0, \quad \lambda \in (0,1), \tag{2.4}$$

that is

$$|Du(x,t)| \leq C, \tag{2.5}$$
$$|Dv^\lambda(x)| \leq C. \tag{2.6}$$

We look for *uniform* bounds in the sense that C in (2.5) has to be independent of t and C in (2.6) has to be independent of λ. This uniformity is crucial for our purpose.

Indeed, (2.6) is the main step allowing to send $\lambda \to 0$ in (2.4) to solve (2.2). Set $z^\lambda(x) := \lambda v^\lambda(x)$ and $w^\lambda(x) := v^\lambda(x) - v^\lambda(0)$, then w^λ is a viscosity solution of

$$\lambda w^\lambda - \mathrm{tr}(A(x)D^2 w^\lambda) + H(x, Dw^\lambda) = -z^\lambda(0), \quad \lambda \in (0,1).$$

Moreover, by (2.6), $\{w^\lambda\}_{\lambda\in(0,1)}$ is a uniformly bounded and equi-continuous family on any balls of $I\!R^N$. On the other hand, $\{z^\lambda\}_{\lambda\in(0,1)}$ is also an equi-continuous family on any balls of $I\!R^N$. For $M > 0$ big enough and independent of λ, it is easy to verify that $\pm\frac{M}{\lambda}$ are, respectively, a super and subsolution of (2.4). By comparison principle, we get $|z^\lambda(x)| \leq M$. Therefore, thanks to Ascoli Theorem, up to some subsequence, there is $(c, v) \in I\!R \times C(I\!R^N)$ such that $z^\lambda(0) \to c$ and $w^\lambda \to v$ as $\lambda \to 0$ locally uniformly. By the stability of viscosity solutions (see Crandall, Ishii & Lions [30]) we find that (c, v) is a viscosity solution of (2.2). Notice that this kind of result is sufficient to prove homogenisation for such kind of problems if the comparison holds (see Lions, Papanicolaou & Varadhan [63] and Lions & Souganidis [64]). But it is not enough in our case.

On the other hand, for L big enough since u_0 is bounded, $v^\pm(x,t) = v(x) + ct \pm L$ are respectively super and subsolutions of (2.1). By comparison, we obtain

$$v(x) + ct - L \leq u(x,t) \leq v(x) + ct + L. \tag{2.7}$$

This estimate proves that

$$\frac{u(x,t)}{t} \to c, \qquad \text{as } t \to \infty,$$

which is one kind of ergodic control problem (see Arisawa & Lions [4]). Moreover, the estimate (2.7) together with (2.5) give the compactness of the set

$$\{u(\cdot,t) - ct, t \geq 0\}. \tag{2.8}$$

It follows that the ω-limit set is nonempty, allowing to prove (2.3) at least for subsequences. Indeed, thanks to Ascoli Theorem, we have the convergence of $u(\cdot, t_n) - ct_n$

for some subsequence $t_n \to +\infty$ to a function $\bar{u}$. This is a first result in the direction of (2.3).

To continue, we need, in a second step, to prove the convergence of the whole sequence in (2.8) when $t \to +\infty$. This step is delicate when A is degenerate since the structure of the solutions of (2.2) can be very complicated. One can obtain easily in general that the ergodic constant c is unique but there can be a lot of solutions of (2.2). It is obvious that, given a solution v of (2.2), all the translations by constants are still solutions but there can exist even solutions which are not translation of a given one by constants, see Lions, Papanicolaou & Varadhan [63], Roquejoffre [75] and the Examples of Chapter 7 for the case of totally degenerate Hamilton-Jacobi equations.

When the equation is nondegenerate, which will be an assumption when looking for (2.3) in this thesis (except in Chapter 7), the structure of the solutions of (2.2) is simpler: the solution of (2.2) is unique up to translations by constants. Then PDE (2.1) is nondegenerate when it is uniformly parabolic, i.e., the diffusion is elliptic,

$$A(x) \geq \rho Id, \quad \rho > 0. \tag{2.9}$$

We will also consider some nonlocal case and give the definition of nondegeneracy in the nonlocal case in Section 2.5.1.

Let us give a sketch of proof of the long time behavior of solutions of uniformly parabolic equation (2.1) in the periodic setting (i.e., in $\mathbb{T}^N = I\!\!R^N/\mathbb{Z}^N$ the N-dimensional torus in $I\!\!R^N$). For more precise proof, we refer to Barles & Souganidis [21] and Ley & Nguyen [65].

The main key of the proof is to use a strong maximum principle. To do so, we need the following assumption on H to linearize the equation

$$H(x, \cdot) \in \mathrm{Lip}(I\!\!R^N), \quad \text{for all } x \in \mathbb{T}^N.$$

Set

$$k(t) = \max_{\mathbb{T}^N}\{u(x,t) - ct - v(x)\}.$$

By comparison principle we have $k(t)$ is bounded and nonincreasing (see [21]), hence there exists $\bar{k}$ such that

$$k(t) \to \bar{k} \ \text{ as } t \to \infty.$$

Moreover, thanks to comparison again we have, for all $n, p \in \mathbb{N}$,

$$|u(x, t+t_n) - ct_n - (u(x, t+t_p) - ct_p)| \leq \max_{\mathbb{T}^N}\{|u(x, t_n) - ct_n - (u(x, t_p) - ct_p)|\}.$$

Then, $\{u(\cdot, \cdot + t_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(\mathbb{T}^N \times [0,\infty))$ and we call $w(\cdot,\cdot)$ its limit. By stability result [30], w is a viscosity solution of

$$\frac{\partial u}{\partial t} - \mathrm{tr}(A(x)D^2u) + H(x, Du) = -c \tag{2.10}$$

with the initial data $\bar{u}$. After passing to the limit of a subsequence $u(x, t+t_n)$, we now obtain

$$\bar{k} = \max_{\mathbb{T}^N}\{w(x,t) - v(x)\}.$$

Note that v is also a viscosity solution of (2.10). Thanks to to Lipschitz continuity of H on p, we obtain that $w - v$ is a viscosity subsolution of

$$\frac{\partial u}{\partial t} - \operatorname{tr}(A(x)D^2u) - C|Du| = 0.$$

Because of the uniform ellipticity of the equation, by the strong maximum principle for viscosity solutions (see Da Lio [32], Ley & Nguyen [65]) we find that $w-v$ is constant in $\mathbb{T}^N \times [0, \infty)$. Therefore, $\bar{k} = w(x,t) - v(x)$, for all $(x,t) \in \mathbb{T}^N \times [0,\infty)$. This does not depend on the choice of subsequence. Thus we get the conclusion, i.e., $u(x,t) \to v(x) + \bar{k}$ as $t \to \infty$ uniformly in x.

2.2 Existing results

To complete the asymptotic expansion (2.3) of the solutions of (2.1) successfully, we have seen that we need additional assumptions. Let us describe the existing assumptions and results in the literature before giving the main results of this thesis.

A first set of results concerns the totally degenerate Hamilton-Jacobi equation in the periodic setting,

$$\begin{cases} \frac{\partial u}{\partial t} + H(x, Du) = 0 & x \in \mathbb{T}^N, t > 0 \\ u(x,0) = u_0(x) & x \in \mathbb{T}^N, \end{cases} \tag{2.11}$$

and the associated ergodic problem

$$c + H(x, Dv) = 0 \quad x \in \mathbb{T}^N, \tag{2.12}$$

i.e., when $A \equiv 0$ in (2.1). There have been many works concerning to long time behavior of solutions of (2.11). Among them, the first convergence results for (2.11) were established by Fathi [38] and Namah & Roquejoffre [69] on a compact manifold $\mathcal{M}$ (take $\mathcal{M} = \mathbb{T}^N$ for simplicity) with strictly convex Hamiltonians H. In [38], H is further assumed to be smooth and strictly convex. In [69], $H(x,p) = F(x,p) - \ell(x)$ with $H \in \operatorname{Lip}(\mathbb{T}^N \times I\!R^N)$, $\ell \in C(\mathbb{T}^N)$ and

$$\ell \geq 0, \qquad \min_{\mathbb{T}^N} \ell = 0, \qquad H(x,p) > H(x,0) = 0 \text{ for } p \neq 0, \tag{2.13}$$

$$H(x,p) \underset{|p| \to +\infty}{\to} +\infty \text{ uniformly for } x \in \mathbb{T}^N. \tag{2.14}$$

The result takes the same form as in (2.3), i.e.,

$$u(x,t) - ct \to v(x) \text{ uniformly on } \mathcal{M} \text{ as } t \to \infty,$$

where u is the solution of (2.11) and (c, v) is the solution of (2.12) (under the hypothesis of [69], $c = 0$). The proof of [69] use the PDE arguments, we will come back to this problem in Chapter 7. Fathi's approach to this asymptotic problem is based on the weak KAM theorem (see Fathi [37], Fathi [39] and Fathi & Siconolfi [40]) which is concerned with the Hamilton-Jacobi equation as well as with the Lagrangian or Hamiltonian dynamical structures behind it and especially on Aubry-Mather sets. This approach has been developed by Roquejoffre [74] and Davini & Siconolfi [34]. A PDE approach in the context of viscosity solutions to the same asymptotic problem has been developed by Barles & Souganidis [20] for possibly non-convex Hamiltonians. Afterwards, Barles, Ishii & Mitake [16] simplified the ideas in [20] under more general assumptions. Finally, in the framework of systems, see Camilli, Ley, Loreti & Nguyen [24], Mitake & Tran [60] and Nguyen [71]. Some of these results have been also extended for problems with periodic boundary conditions, see for instance Mitake ([57], [58], [59]).

It is important to note that all the results quoted above do not take into the case of the second order Hamilton-Jacobi equations, or the non-periodic and unbounded case, two things that we are particularly interested in this thesis.

For second order Hamilton-Jacobi equations (2.1), the first important result is due to Barles & Souganidis [21] in the periodic setting. Recall that to obtain the convergence (2.3) we first need to establish uniform gradient bounds (2.5)-(2.6). In [21], the uniform gradient bounds are obtained under two types of hypothesis on H and the diffusion matrix A. The first case concerns with (2.1) in the case uniform parabolic (see (2.9)) with H *sublinear* (see Assumption (2.22) later). In this case, the uniform gradient bounds are obtained thanks to the elliptic term and by using Ishii & Lions' method [51]. In the second case, the gradient bounds are established for possibly degenerate equations when the Hamiltonian satisfies a type of *superlinear* property, the typical case is

$$H(x,p) = a(x)|p|^{1+\theta} + l(x), \quad \theta > 0, a, l \in C(\mathbb{T}^N) \text{ and } a > 0.$$

The proof is based on arguments of a weak Bernstein-type method introduced by Barles [9]. The convergence is then established for the uniformly parabolic equation and is based on the strong maximum principle. These results were recently generalized in Ley & Nguyen [65] to be applied to systems. In the case of the degenerate equations, the results were obtained by Cagnetti, Gomes, Mitake & Tran [25] and Ley & Nguyen [67] under different assumptions and with different methods.

All the results described so far relate to the periodic case (or the bounded sets). In the whole space $\mathbb{R}^N$, there are some results but the situation is more delicate due to the lack of compactness and the possible unboundedness of solutions. The extra assumptions are needed or particular cases are considered.

For first order Hamilton-Jacobi equations, there are few works. One is of Barles & Roquejoffre [19] which is based on PDE approach and the other one is of Ishii [52] with a dynamic approach. These two works will interest us particularly in the framework of the results of chapter 7, hence we refer to Section 2.5.4 for a description.

As far as other works for first order equation are concerned, Fujita, Ishii & Loreti [43] and Fujita & Loreti [44] can drop many assumptions of the previous results in the unbounded framework for the EDP

$$u_t + \alpha\langle x, Du\rangle + H(Du) = f(x) \quad \text{dans } \mathbb{R}^N \times (0,+\infty), \tag{2.15}$$

with $\alpha > 0$, H is convex on $\mathbb{R}^N$. More particularly, [44] consider $H(p) = \beta|p|^2$, $\beta > 0$ and f is assumed to be semiconvex. The authors prove the convergence and are able to determine the rate of convergence. But the crucial assumption of (2.15) allowing to overcome the lack of compactness is the presence of the Ornstein-Uhlenbeck term $\alpha\langle x, Du\rangle$ which is one of the main ingredients of this thesis and we will describe in detail below. Besides, by dynamical approach, the result has been investigated by Ishii [52], Ichihara & Ishii [48] for strictly convex or convex and coercive Hamiltonians with more restrictive conditions on the initial data.

In the case of second order equations, there are few results in the unbounded setting, essentially the works of Fujita, Ishii & Loreti [42] (which is inspired by [43, 44]) and Ghilli [45], both are for the viscous Hamilton-Jacobi equations (nondegenerate second order) with Ornstein-Uhlbenbeck term. There are also the works of Debussche, Hu & Tessitore [35] and Madec [56] on the same topic but infinite dimension, using a probabilistic approach. Since our thesis is mainly concerned with this case, we describe now in detail.

2.3 Hamilton-Jacobi equations with Ornstein-Uhlenbeck operator

The work of Fujita, Ishii & Loreti [42] is concerned with

$$\begin{cases} u_t(x,t) - \Delta u(x,t) + \alpha\langle x, Du(x,t)\rangle + H(Du(x,t)) = f(x) \;\text{ in } I\!R^N \times (0,\infty), \\ u(\cdot,0) = u_0(\cdot) \;\text{ on } I\!R^N, \end{cases} \tag{2.16}$$

here $\alpha > 0$ is a given constant, $H \in \mathrm{Lip}(I\!R^N)$, f and u_0 are given functions on $I\!R^N$ satisfying

$$|g(x) - g(y)| \le C|x-y|(\Gamma_\mu(x) + \Gamma_\mu(y)), \tag{2.17}$$

for $g = f$ or $g = u_0$, with

$$\Gamma_\mu(x) = e^{\mu|x|^2}, \mu > 0. \tag{2.18}$$

Since they deal with unbounded solutions in unbounded domains, the exponential function (2.18) is considered as the growth condition for the solutions of (2.16). More precisely, they obtain their results for solutions of (2.16) which belong to the class of functions

$$\mathcal{E}_\mu(\overline{Q}) = \left\{ v : \overline{Q} \to I\!R : \lim_{|x|\to+\infty} \sup_{0\le t<T} \frac{v(x,t)}{\Gamma_\mu(x)} = 0, \text{ for } T > 0 \right\}, \tag{2.19}$$

where Q denotes the space $I\!R^N \times (0,\infty)$. The operator $\Delta - \alpha\langle x, D\rangle$ is called the *Ornstein-Uhlenbeck operator*. In terms of stochastic optimal control, the Laplacian is related to the usual diffusion which spreads the trajectories while the term $-\alpha\langle x, D\rangle$ is the one which confines the trajectories in a big ball and allows to recover some compactness, see next chapter for the link with optimal control.

With this operator, Γ_μ is a supersolution of the equation for large x, i.e.,

$$-\Delta\Gamma_\mu(x) + \alpha\langle x, D\Gamma_\mu(x)\rangle - C|D\Gamma_\mu(x)| \geq \Gamma_\mu(x) - K, \quad x \in I\!\!R^N \tag{2.20}$$

here $C > 0$, $K = K(\mu, \alpha, N, C) > 0$. Property (2.20) is very useful and is the crucial tool to obtain many important results: existence, uniqueness, regularity of solution. Moreover, it is also helpful for applying the strong maximum principle for the equations and then establishing the long time behavior of solutions of (2.16).

As explained in Section 2.1, to obtain the asymptotic like (2.3), the authors of course need to get some local uniform bounds

$$|Du(x,t)|, |Dv^\lambda(x)| \leq C|x-y|(\Gamma_\mu(x) + \Gamma_\mu(y)) \tag{2.21}$$

with C independent of t and λ, where v^λ is viscosity solution of the approximate equation corresponding to (2.16).

Recall that in [42], they consider (2.16) with simple terms and strong assumptions on the datas. At first, the second order term Δu is the special case of the more general diffusion operator $\mathrm{tr}(A(x)D^2u)$ when $A(x) \equiv \mathrm{Id}$. In this simple case, no extra (bad) terms coming from the inhomogeneity of A arise in the proofs. Besides, the equation (2.16) is one of the easiest type of uniformly parabolic equation. By Schauder theory for uniformly parabolic PDE (see Ladyzhenskaya, Solonnikov & Uralseva [61]), there exists a classical solution for (2.16). Then the proofs for (2.21) become less technical which is not anymore the case for degenerate A. Secondly, the Hamiltonian does not depend on the x variable and satisfies a Lipschitz condition, i.e.,

$$|H(p) - H(q)| \leq C|p-q|, \quad p, q \in I\!\!R^N.$$

Thanks to the mutually annulling property of this assumption in the proofs, bad terms coming from the Hamiltonians are also avoided. Dealing with those bad terms is one of the main difficulties, especially in the unbounded setting.

The main aim of this thesis is to generalize the results of [42] in several directions. The first is to be able to deal with more general Hamiltonian of Bellman-Isacs-type which are merely sublinear,

$$|H(x, Du)| \leq C(1 + |Du|) \tag{2.22}$$

without further assumptions. Moreover, we want to be able to use a general diffusion not only the Laplacian. General diffusions bring additional terms in the proof which are delicate to treat. Because of the unbounded setting, it is more than technical difficulties. We will come back to these issues below when describing our results.

Another main novelty is to consider a nonlocal diffusion instead of a local one. This is one of the main results of the thesis and we introduce now the nonlocal operators we consider.

2.4 Nonlocal integro-differential operators of fractional Laplacian type

Let $x \in I\!\!R^N$, $\psi : I\!\!R^N \to I\!\!R$ be a bounded continuous function which is C^2 in a neighborhood of x. We introduce the integral-differential operator, taken on the whole space

$I\!R^N$, associated to the jump part of the generator of a Lévy process

$$\mathcal{I}(x,\psi,D\psi) = \int_{I\!R^N} \{\psi(x+z) - \psi(x) - D\psi(x)\cdot z\mathbb{I}_B(z)\}\nu(dz), \tag{2.23}$$

where $\mathbb{I}_B$ denotes the indicator function of the unit ball, ν is a Lévy type measure, which is regular and nonnegative. An important example of such nonlocal operator is the case

$$\nu(dz) = \frac{C_{N,\beta}}{|z|^{N+\beta}}dz, \tag{2.24}$$

where $\beta \in (0,2)$ and $C_{N,\beta}$ is normalizing constant. In this case, $-\mathcal{I} = (-\Delta)^{\beta/2}$ is the fractional Laplacian of order β, see [36].

Since we deal with unbounded solutions on unbounded setting, we need to make the following assumption on Lévy measure to be sure that (2.23) is well-defined for our solutions:

$$\text{(M1)} \qquad \begin{array}{c} \text{There exists a constant } C_\nu^1 > 0 \text{ such that} \\ \displaystyle\int_B |z|^2\nu(dz), \int_{B^c} \phi_\mu(z)\nu(dz) \le C_\nu^1, \end{array}$$

where $\phi_\mu \in C^2(I\!R^N)$, $\mu > 0$, is the growth function defined by (2.42).

A typical example for this kind of measure is the following: for $\mu > 0$ and $\beta \in (0,2)$, let

$$\nu(dz) = \frac{e^{-\mu|z|}}{|z|^{N+\beta}}dz, \tag{2.25}$$

which is known as *tempered β-stable law*, see next chapter for the link with control problem with Lévy processes.

2.5 Results of the thesis

In this thesis, we study problems related to the long time behavior of solutions of Hamilton-Jacobi equations. The first set of results, which is the core of the thesis, is concerned with the semilinear PDE with an Ornstein-Uhlenbeck operator

$$\begin{cases} u_t - \mathcal{F}(x,[u]) + \langle b(x), Du\rangle + H(x,Du) = f(x) \ \text{ in } I\!R^N \times (0,\infty), \\ u(\cdot,0) = u_0(\cdot) \ \text{ in } I\!R^N, \end{cases} \tag{2.26}$$

where $\mathcal{F}$ can be either *local* or *nonlocal*:

$$\begin{cases} \mathcal{F}(x,[u]) = \mathrm{tr}(A(x)D^2u) \ \ \textbf{(local case)}, \\ \mathcal{F}(x,[u]) = \mathcal{I}(x,u,Du) \ \ \textbf{(nonlocal case)}. \end{cases} \tag{2.27}$$

The last independent chapter is devoted to the large time behvior of unbounded solutions to first order Hamilton-Jacobi equations (2.11) set in the whole space $I\!R^N$.

Let us start by a discussion about (2.26). The term $\mathcal{F} - \langle b, D\rangle$ in (2.27) is called Ornstein-Uhlenbeck operator. In the local case, we assume throughout the work that the diffusion matrix A can be written $A(x) = \sigma(x)\sigma^T(x)$, where σ is bounded Lipschitz continuous in $I\!\!R^N$ (notice that A is nonnegative definite). In the nonlocal case, we will consider nonlocal operators defined by (2.23) with some measure ν like (2.25). As far as the Ornstein-Uhlenbeck drift is concerned, we will assume that

$$\text{There exists } \alpha > 0 \text{ such that } \langle b(x) - b(y), x - y\rangle \geq \alpha|x - y|^2, \tag{2.28}$$

and α will be called the *size* of the Ornstein-Uhlenbeck operator.

The results about this topics are divided into three parts. The first one, which is the main part, consists in establishing *a priori* Lipschitz uniform gradient bounds for (2.26) and the approximate stationary problem

$$\lambda u^\lambda(x) - \mathcal{F}(x, [u^\lambda]) + \langle b(x), Du^\lambda(x)\rangle + H(x, Du^\lambda(x)) = f(x) \quad \text{in } I\!\!R^N, \tag{2.29}$$

where $\lambda > 0$ will be sent to 0, both in the local and nonlocal case, under quite general assumptions. This part is related to Chapter 4 and described here in Section 2.5.1. The second part (Chapter 5) studies the well-posedness of (2.26) and (2.29). Due to the unbounded setting and the weak assumption (2.22) on the Hamiltonian H, these results are beyond the classical theory of viscosity solutions and shall be proved rigorously. Finally, the last part concerns the initial motivation, namely the large time behavior. This part is more classical since it relies on the strong maximum principle together with the ideas of [42] and it becomes a classical application of the previous parts, up to a careful statement of the nonlocal maximum principle in our setting, see Chapter 6.

Chapter 7 deals with the large time behavior of

$$\begin{cases} \frac{\partial u}{\partial t} + |Du| = l(x) & x \in I\!\!R^N, t > 0 \\ u(x, 0) = u_0(x) & x \in I\!\!R^N, \end{cases} \tag{2.30}$$

where $l \geq 0$ is such that $\mathcal{K} := \mathrm{argmin}_{I\!\!R^N}\, l$ is a nonempty compact subset of $I\!\!R^N$.

Let us recall that we work with unbounded solutions in unbounded domains. It is not possible to obtain interesting results without any growth assumptions on the datas and the solutions. We need to restrict them in some space with limited growth like (2.19) with Γ_μ given by (2.18) in [42].

In the case of (2.26), the growth function (2.18) seems not good enough in our framework since we want to deal with general diffusion matrices, which will bring us more bad terms in the study of the equation. To treat them without additional conditions on the size α of the Ornstein-Uhlenbeck operator, we need to use a suitable growth function. After testing many kinds of growth functions, the following seems the best for us: for $\mu > 0$ we consider

$$\phi_\mu(x) = e^{\mu|x|} \quad \text{for } x \in I\!\!R^N. \tag{2.31}$$

We will work with f, u_0 and the solutions of (2.26)-(2.29) in (2.19) with ϕ_μ in place of Γ_μ.

In the case of (2.30), we work with some quadratic polynomial growth conditions. More precisely, we will assume that u_0, l belong to

$$\mathcal{Q} := \{u : I\!R^N \times [0,+\infty) \to I\!R : \text{there exists } C > 0 \text{ such that for all } x \in I\!R^N,$$
$$|u(x,t)| \le C(1+|x|^2)\}, \quad (2.32)$$

and

$$\mathcal{L} := \{u : I\!R^N \times [0,+\infty) \to I\!R : \text{there exists } C > 0 \text{ such that for all } x, y \in I\!R^N,$$
$$|u(x,t) - u(y,t)| \le C(1+|x|^2+|y|^2)|x-y|\} \quad (2.33)$$

and we look for solutions in the same sets.

Before describing more precisely our results, let us recall that we consider mainly viscosity solutions in this thesis. We refer to [30, 10, 5, 54] for definitions and main results of the classical theory, and to [15, 13, 2, 77, 78] for the nonlocal setting. It will be introduced more precisely later for the nonlocal equation.

2.5.1 The *a priori* gradient estimates

The most important result that we want to obtain in this thesis is local uniform bounds for solutions of (2.26) and (2.29). One of the interests in our work is to deal with very general class of Hamiltonians, i.e., when (2.22) holds. This general assumption gives us many terms which are difficult to treat since we are concerned with the unbounded setting. To overcome this difficulty, using only the effect of the Ornstein-Uhlenbeck term is not always enough, we have to combine it with the ellipticity of the equation following some ideas of Ishii & Lions [51] in the local case and Barles, Chasseigne, Ciomaga & Imbert [11] in the nonlocal one. Let us recall and define what the ellipticity is in both local and nonlocal equation.

In the local case, when (2.9) holds, it is the classical definition leading to strong maximum principle, see Gilbarg & Trudinger [46].

In the nonlocal one, there is no exact definition of ellipticity in general. Then we have to state in our context. In fact, in this thesis we will use the classes of Lévy measure which satisfy (M1) and

(M2) There exists $\beta \in (0,2)$ such that for every $a \in I\!R^N$ there exist $0 < \eta < 1$ and a constant C_ν^2 such that the following holds

$$\forall \gamma > 0 \quad \int_{\mathcal{C}_{\eta,\gamma}(a)} |z|^2 \nu(dz) \ge C_\nu^2 \eta^{\frac{N-1}{2}} \gamma^{2-\beta},$$

with $\mathcal{C}_{\eta,\gamma}(a) := \{z \in B_\gamma; (1-\eta)|z||a| \le |a.z|\}$. For more detail about this property we refer to Barles, Chasseigne, Ciomaga & Imbert [11]. Assumption (M2) is a type of ellipticity. But for $\beta \in (0,1)$, the effect of the ellipticity is not strong enough to help Ornstein-Uhlenbeck operator treating the bad terms which come from the Hamiltonian. Therefore, hereafter, to be more convenient in the statement of results for the local and nonlocal equation at the same time, we will say *nonlocal equation is elliptic when (M2) holds with* $\beta \in (1,2)$.

We now state the main result but for the stationary equation (2.29). It is the same for the nonstationary one (2.26), see Theorem 4.2.1 Chapter 4.

Theorem 2.5.1. *(*Local uniform gradient bound for solutions of (2.29) in the ellipticity case, Theorem 4.1.1*). Let u^λ be a continuous viscosity solution of (2.29) which belongs to class (2.19) with the exponential growth ϕ_μ defined by (2.31). Suppose that (2.29) is uniformly elliptic in the both local and nonlocal case and the assumptions (2.22), (2.28), and on f hold. Then there is a constant $C > 0$ independent of λ such that*

$$|u^\lambda(x) - u^\lambda(y)| \leq C|x-y|(\phi_\mu(x) + \phi_\mu(y)). \tag{2.34}$$

Let us give a sketch of proof:

In order to use ϕ_μ in (2.31) as a test function for u, we need to modify it be smooth enough. More precisely, we set

$$\phi_\mu(x) = e^{\mu\sqrt{|x|^2+1}}, \;\; \mu > 0, \text{ for all } x \in I\!R^N.$$

Let $\delta > 0$ and a C^2 concave and increasing function $\psi : I\!R_+ \to I\!R_+$ with $\psi(0) = 0$ which is constructed as follows

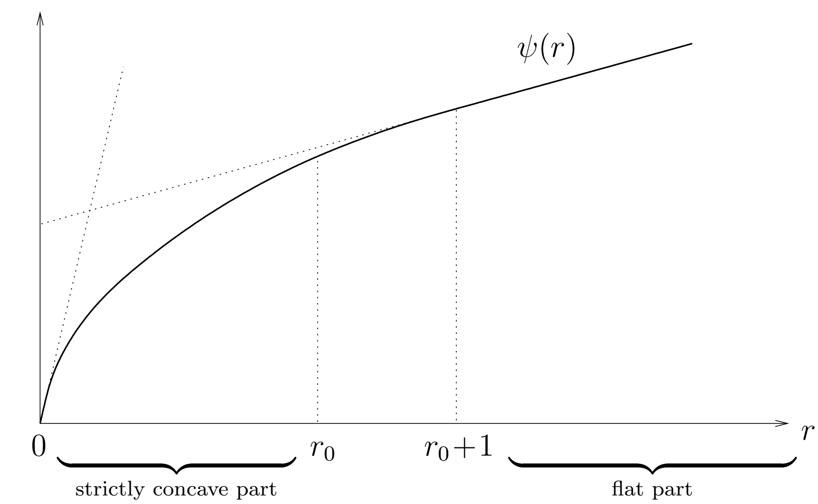

Figure 2.1: The concave function ψ

We consider

$$M = \sup_{x,y\in I\!R^N}\left\{u^\lambda(x) - u^\lambda(y) - \sqrt{\delta} - (\psi(|x-y|) + \delta)\Phi(x,y)\right\},$$

where $\Phi(x,y) = C_1(\phi_\mu(x) + \phi_\mu(y) + A)$ for some $A, C_1 > 0$. Set

$$\varphi(x,y) = \sqrt{\delta} + (\psi(|x-y|) + \delta)\Phi(x,y).$$

If $M \leq 0$ for some good choice of A, C_1, ψ independent of $\delta > 0$, then we get (2.34) by letting $\delta \to 0$.

We argue by contradiction, assuming that for δ small enough $M > 0$. Since $u^\lambda \in \mathcal{E}_\mu(I\!R^N)$ and $\delta > 0$, the supremum is achieved at some point (x,y) with $x \neq y$, thanks to $\delta > 0$ and the continuity of u. In view of viscosity theory in Crandall, Ishii & Lions [30] for the local equation and Barles & Imbert [15] for the nonlocal one, there exist $X, Y \in \mathcal{S}^N$ such that the following viscosity inequality holds

$$\begin{aligned}
& \lambda(u^\lambda(x) - u^\lambda(y)) - (\mathcal{F}(x, [u^\lambda]) - \mathcal{F}(y, [u^\lambda])) \\
& +\langle b(x), D_x\varphi\rangle - \langle b(y), -D_y\varphi\rangle + H(x, D_x\varphi) - H(y, -D_y\varphi) \\
\leq\ & f(x) - f(y),
\end{aligned} \tag{2.35}$$

where $\mathcal{F}(x, [u^\lambda]) = \mathrm{tr}(A(x)X)$ and $\mathcal{F}(y, [u^\lambda]) = \mathrm{tr}(A(y)Y)$ in the local case and $\mathcal{F}(x, [u^\lambda]) = \mathcal{I}(x, u^\lambda, D_x\varphi)$ and $\mathcal{F}(y, [u^\lambda]) = \mathcal{I}(y, u^\lambda, -D_y\varphi)$ in the nonlocal one.

Now we need to estimate all of the terms in (2.35) in order to get a contradiction. The main difficulty of the proof is the estimate of $\mathcal{F}(x, [u^\lambda]) - \mathcal{F}(y, [u^\lambda])$. To get the estimate for this term, the ideas in the local case are inspired by Ishii-Lions [51] to get the strictly positive term thanks to the ellipticity of the equation and the the choice of C^2 concave increasing function ψ (see Lemma 4.1.2 for more details). In the nonlocal one, the ideas come from Barles, Chasseigne, Ciomaga & Imbert [11]. Since we are dealing with the unbounded setting, the estimates become very complicated and depend on the choice of concave function ψ (see Proposition 4.1.1, Lemma 4.1.4 and Lemma 4.1.5 for details).

For simplicity of explanation, let us describe only the proof in the local case.

Since $M > 0$, using the assumptions (2.9), (2.28), (2.22) and assumption on f to estimate for all the different terms, inequality (2.35) becomes

$$\begin{aligned}
& \lambda\sqrt{\delta} + \Phi(x, y)\left(-4\rho\psi''(|x-y|) + \alpha\psi'(|x-y|)|x-y|\right) \\
\leq\ & \Phi(x, y)C\psi'(|x-y|) + C + C_f(\phi_\mu(x) + \phi_\mu(y))|x-y| \\
& +C_1(\psi(|x-y|) + \delta)\left(\mathrm{tr}(A(x)D^2\phi_\mu(x)) - \langle b(x), D\phi_\mu(x)\rangle + C|D\phi_\mu(x)|\right) \\
& +C_1(\psi(|x-y|) + \delta)\left(\mathrm{tr}(A(y)D^2\phi_\mu(y)) - \langle b(y), D\phi_\mu(y)\rangle + C|D\phi_\mu(y)|\right),
\end{aligned}$$

where $C > 0$ depending only on σ, H, μ. Now we use the supersolution property of ϕ_μ as explained in (2.20) (see Lemma 4.1.1 for details) we get

$$\begin{aligned}
& \lambda\sqrt{\delta} - 4\rho\psi''(|x-y|) + \alpha\psi'(|x-y|)|x-y| \\
\leq\ & C\psi'(|x-y|) + 2K(\psi(|x-y|) + \delta) + C + C_f|x-y|.
\end{aligned} \tag{2.36}$$

Here we droped Φ-terms on both sides of the above inequality to simplify the explanation.
Goal: reach a contradiction in (2.36).

- *Case 1:* $r := |x-y| \leq r_0$. The function ψ above was chosen to be strictly concave for small $r \leq r_0$, we take profit of the ellipticity of the equation "$-4\rho\psi''(r)$" which is a positive contribution in (2.36) to control all the others terms and lead to the contradiction.

- *Case 2:* $r := |x-y| \geq r_0$. In this case, the second derivative $\psi''(r)$ of the increasing concave function ψ is small or even 0 and the ellipticity is not powerful enough to control the bad terms. Instead, we use the positive term $\alpha\psi'(r)r$ coming from the Ornstein-Uhlenbeck operator to control everything.

Note that by using the good terms as explained in the two above cases, we can choose all of parameters independently of λ to reach a contradiction in (2.36).

The ideas of the proof in nonlocal case then follow the one of the local one in order that we can apply Ishii-Lions method's ideas [51] when $\beta \in (1,2)$ (order of the nonlocal term). But we could not obtain local Lipschitz bounds for solutions of the equation directly as in the local case. The proof is more complicated because we deal with the nonlocal operator in the unbounded setting. Actually, the estimates for the nonlocal terms are very different from those for local terms (see Proposition 4.1.1 for the detail by using general concave test function). The proof will consist in two main steps. The first one establishes τ-Hölder regularity for any $\tau \in (0,1)$. Then the second step is to use that Hölder regularity to reach Lipschitz regularity. For each step we have to build different concave test functions which are also different from the one in the local case. Using the term called ellipticity here in order to use the Ishii-Lions method's ideas is very delicate (we can see this fact in Lemma 4.1.4 and Lemma 4.1.5).

Recall that when the ellipticity does not necessarily hold, the equations are called degenerate equations. In these cases, it is natural to ask some questions like: What are the results we can obtain? What are the assumptions on the datas to obtain those results? And especially, how general the assumptions on the Hamiltonian can we deal with? The following Theorem and Remarks will answer these questions. To obtain the uniform gradient bound (2.34) without using the ellipticity of the equation, we need to strengthen the hypotheses on H, that is

$$\begin{cases} \text{There exist } L_{1H}, L_{2H} > 0 \text{ such that for all } x, y, p, q \in I\!R^N \\ |H(x,p) - H(y,p)| \leq L_{1H}|x-y|(1+|p|), \\ |H(x,p) - H(x,q)| \leq L_{2H}|p-q|(1+|x|). \end{cases} \tag{2.37}$$

This is the classical assumption satisfied by a Hamiltonian coming from an optimal control problem, see Chapter 3.

Theorem 2.5.2. *(*Local uniform gradient bound for solutions of (2.29) in the degenerate case, Theorem 4.1.2). *Let u^λ be a continuous viscosity solution of* (2.29) *which belongs to class* (2.19) *with the exponential growth ϕ_μ defined by* (2.31). *Suppose that* (2.29) *is degenerate elliptic in the both local and nonlocal case and* (2.28), (2.37) *hold. There is*

$$C(\mathcal{F}, H) = \begin{cases} C(\sigma, H) \ \textit{in the local case} \\ C(H) \ \textit{in the nonlocal case} \end{cases}$$

such that, for any $\alpha > C(\mathcal{F}, H)$, there is a constant $C > 0$ independent of λ such that (2.34) *holds.*

In this Theorem, the constant $C(\sigma, H)$ depends on σ since the diffusion matrix depends on x variable. If the diffusion matrix is constant in the local equation, then we obtain the same conclusion as in Theorem 2.5.2 with $\alpha > C(H)$ (the constant depends only on H) in both local and nonlocal case, see Remark 4.1.3.

Moreover, if the diffusion matrix is constant in the local equation and $H(x,p) = H(p) \in \mathrm{Lip}(I\!R^N)$, then (2.34) holds for any $\alpha > 0$ in both local and nonlocal case and we recover the result obtained in [42]), see Remark 4.1.4.

2.5.2 Well-posedness of the equations

To use the results as explained in the previous part, we have to build continuous solutions for (2.26) and (2.29) in both local and nonlocal case. Classical techniques in viscosity theory usually use Perron's method (see Barles [10], Crandall, Ishii & Lions [30] and Ishii [50]). More precisely, we first build discontinuous viscosity solutions. Then the continuity of the solutions is obtained by using a strong comparison principle. But when we deal with the equations (2.26) and (2.29) using hypothesis (2.22) on the Hamiltonian, it is very difficult to obtain the comparison result and such kind of result may not hold in this case even in the periodic setting (see Ley & Nguyen [65]). Under the weak assumption (2.22), we only have the comparison result between discontinuous sub or supersolutions when at least one of them is Lipschitz continuous. Therefore we need another approach to build continuous solutions for (2.26) and (2.29). The ideas come from Barles & Souganidis [21] and [65]. They are based on a truncation of the Hamiltonian and the unbounded f term (and the initial data u_0 for the nonstationary equation).

In order to get the uniqueness of solutions of (2.29) and (2.26), we need one more assumption on the Hamiltonian,

$$\begin{cases} \text{There exists } L_H > 0 \text{ such that} \\ |H(x,p) - H(x,q)| \leq L_H |p-q|, \quad \text{for all } x,p,q \in \mathbb{R}^N. \end{cases} \tag{2.38}$$

This is true when H comes from an optimal control problem with a bounded drift term, see Chapter 3. The uniqueness of solution in fact is a consequence of comparison principle and it will be given later. We first give the existence result for solutions of (2.29)

Theorem 2.5.3. (Existence of continuous solutions of (2.29), Theorem 5.1.2) *Under the assumptions of Theorem 2.5.1, for all $\lambda \in (0,1)$, there exists a continuous viscosity solution u^λ of (2.29) in the class of functions*

$$\begin{aligned} & u^\lambda \in \mathcal{E}_\mu(\mathbb{R}^N), \\ & |u^\lambda(x) - u^\lambda(y)| \leq C|x-y|(\phi_\mu(x) + \phi_\mu(y)), \quad x,y \in \mathbb{R}^N, \end{aligned}$$

where $C > 0$ is a constant independent of λ.
In addition, if (2.38) holds then the solution is unique in the growth class $\mathcal{E}_\mu$.

Let us describe briefly the truncation on the datas in order to prove Theorem 2.5.3. Let $m, n \geq 1$,

$$f_m(x) = \min\{f(x) + \frac{1}{m}\phi(x), m\},$$

$$H_{mn}(x,p) = \begin{cases} H_m(x,p) & \text{if } |p| \leq n \\ H_m(x, n\frac{p}{|p|}) & \text{if } |p| \geq n, \end{cases}$$

with

$$H_m(x,p) = \begin{cases} H(x,p) & \text{if } |x| \leq m \\ H(m\frac{x}{|x|}, p) & \text{if } |x| \geq m. \end{cases}$$

This gives bounded uniformly continuous Hamiltonian and then classical theory of viscosity solution (see [30]) gives the existence and uniqueness of a bounded continuous viscosity solutions of (2.29). Because of the uniformity of the Lipschitz estimate (Theorem 2.5.1) and thanks to Ascoli Theorem, we can send $m, n \to \infty$ to get a continuous viscosity solution for (2.29).

Uniqueness of solutions follows from the following theorem. For more detail of the proof we can see in the one of Theorem 5.1.2 Chapter 5.

Theorem 2.5.4. (Comparison principle, Theorem 5.1.1) *Let u^λ and v^λ be an upper semicontinuous viscosity subsolution and a lower semicontinuous viscosity supersolution of (4.1), respectively, which belong to growth class $\mathcal{E}_\mu$. Suppose that (2.38) holds. Assume either u or v is locally Lipschitz continuous in $\mathbb{R}^N$. Then $u \le v$ in $\mathbb{R}^N$.*

Similar results are also obtained for the nonstationary equation, see Theorems 5.2.1 and 5.2.2 Chapter 5.

Moreover, in this thesis we study also the well-posedness of solutions for (2.26) and (2.29) when using different assumptions on the Hamiltonian, for instance (2.37) or

$$\begin{cases} \text{There exist } L_{1H}, L_{2H} > 0 \text{ such that for all } x, y, p, q \in \mathbb{R}^N \\ |H(x,p) - H(y,p)| \le L_{1H}|x-y|(1+|p|), \\ |H(x,p) - H(x,q)| \le L_{2H}|p-q|. \end{cases} \tag{2.39}$$

In these cases, we can obtain comparison principles and then the existence and uniqueness of solution are proved as usual way, see Chapter 5.

2.5.3 Ergodic problem and large time behavior

Let us come back to the initial motivation explained in Section 2.1. We recall that studying the ergodic problem associated with (2.26) consists in finding a pair of constant $c \in \mathbb{R}$ and a function $v \in C(\mathbb{R}^N)$ which is a solution of

$$c - \mathcal{F}(x, [v]) + \langle b(x), Dv(x) \rangle + H(x, Dv(x)) = f(x) \qquad \text{in } \mathbb{R}^N. \tag{2.40}$$

Solving this problem is one of the main applications of local Lipschitz bound result (2.34) as explained before. This kind of result has been studied by many authors, see for instance [4, 19, 42].

Theorem 2.5.5. (Ergodic problem, Theorem 6.1.1, Theorem 6.1.2). *Suppose that the assumptions of Theorem 2.5.1 hold. Then there is a solution $(c, v) \in \mathbb{R} \times C(\mathbb{R}^N)$ of (2.40). If (2.38) holds, the ergodic constant c is unique and if, in addition, (2.39) holds, then v is unique up to an additive constant.*

The uniqueness of c will be proved by using comparison result (see Theorem 2.5.4) that is why (2.38) is required. Moreover, to linearize the equation in order to apply the strong maximum principle to prove the uniqueness of v up to some constant, we need (2.39).

Theorem 2.5.6. (Long time behavior of solution, Theorem 6.2.1) *Let u be the unique solution of (2.26) and (c, v) a solution of (2.40). Assume that (2.28), (2.39), assumption of f and either*

(i) $\mathcal{F}(x, [u]) = \mathrm{tr}(A(x)D^2u)$ and (2.9) hold

or

(ii) $\mathcal{F}(x, [u]) = \mathcal{I}(x, u, Du)$ and suppose that $(M1), (M2)$ hold with $\beta \in (1, 2)$.

Then there is a constant $a \in I\!R$ such that

$$\lim_{t\to\infty} \max_{B(0,R)} |u(x,t) - (ct + v(x) + a)| = 0 \qquad \text{for all } R > 0.$$

This is the expected result.

2.5.4 Long time behavior of unbounded solutions of first order Hamilton-Jacobi equations in $I\!R^N$

We obtain a first result of convergence for the unbounded solutions of (2.30) when $l, u_0 \in \mathcal{Q} \cap \mathcal{L}$ (the sets $\mathcal{Q}, \mathcal{L}$ are introduced in (2.32)-(2.33)) and

$$l \geq 0, \qquad \liminf_{|x|\to+\infty} l(x) > \inf_{I\!R^N} l,$$

the last condition ensuring that $\operatorname{argmin} l$ is a compact subset of $I\!R^N$.

We first establish the existence and uniqueness of a solution of

$$\lambda v^\lambda(x) + |Dv^\lambda(x)| = l(x) \qquad \text{in } I\!R^N$$

in $\mathcal{Q} \cap \mathcal{L}$, such that the Lipschitz constant C of v^λ in $\mathcal{L}$ is independent of λ. As explained in Section 2.1, this allows to send $\lambda \to 0$ and solve the ergodic problem

$$c + |Dv(x)| = l(x) \qquad \text{in } I\!R^N,$$

associated with (2.30). Moreover, two solutions whose difference is bounded must have the same ergodic constant, see Theorem 7.1.1. A first difficulty is the possible nonuniqueness of the ergodic constant as illustrated in Section 7.6. Moreover, the solutions v to the ergodic problem are not anymore in $\mathcal{Q}$ (but are still in $\mathcal{L}$).

We then solve the Cauchy problem. There exists a unique viscosity solution of (2.30) for any initial condition u_0 and, if u_0 is sufficiently close to v where (c, v) is solution to the ergodic problem, then we obtain some compactness properties of the family

$$\{u(x,t) - ct, t \geq 0\},$$

more precisely,

$$\begin{aligned} &|u(x,t) - ct| \leq C + |v(x)|, \\ &|(u(x,t) - ct)_t| \leq C, \\ &|D(u(x,t) - ct)| \leq C + l(x). \end{aligned}$$

As a consequence, we can obtain the asymptotic expansion (2.3) for the solution u under the additional assumption that there exists a solution (c, v) to the ergodic problem such that v is *bounded from below* and $c = \min l$. In this case, we have the convergence for all initial condition u_0 close enough to v (Theorem 7.1.3). It means, that, starting sufficiently close to a bounded from below solution to the ergodic problem, the solution u of (2.30) is attracted by this stationary solution. We produce some examples based on the interpretation of (2.30) in terms of optimal control problems. These examples suggest that it is natural to expect the existence of a bounded from below solution to the ergodic problem. But, when u_0 is close to a solution v which is not bounded from below, the convergence may fail so the extension of our result seems delicate.

Such kind of problems were already studied in Barles & Roquejoffre [19] who generalize Namah & Roquejoffre [69] in the unbounded setting but for $H(x, p)$ which is bounded uniformly continuous with respect to x so they cannot deal with $l \in \mathcal{Q}$. Another important work is the one of Ishii [52]. One of the main difference with his result is that we do not need the strict convexity of the Hamiltonian. Moreover, the proofs in [52] are based to the interpretation of (7.1)-(7.8) in terms of a control problem and uses weak KAM theory. Instead, we produce a pure PDE proof. For other related works on the subject, see Ichihara & Ishii [53, 48].

2.6 Perspectives

Let us describe some open problems that we want to study in the future:

1. In this thesis, we have obtained some local uniform Lipschitz bounds for solutions of elliptic equations in both local and nonlocal case. We want to investigate these results by using the weak Bernstein method introduced in Barles [9]. It will require another kind of assumption on H (coercivity, convexity-type assumptions) but it may work for degenerate equations and for equations without Ornstein-Uhlenbeck operator. The main difficulty is to localize successfully the Bernstein method in this unbounded setting.

   Another issue in this context is to be able to prove long time behavior for degenerate equations for which the strong maximum principle does not hold. Recently, Ley & Nguyen [67] and Cagnetti et al. [25] found a new results for large time behavior of solutions of degenerate PDE. Hence we want to study their approach in our cases.

2. In the nonlocal case, we want to get additional results with more general Lévy measures, for instance when the measures depend on x. In particular, we would like to study the nonlocal equation with Lévy-Itô operators.

   Another problem which is quite natural is when the operator is the sum of the local and nonlocal one in which one of them does not necessarily have the ellipticity. We would like to obtain the results as done in this thesis for this kind of equations without any condition on α (the size of Ornstein-Uhlenbeck operator). Of course, strong maximum principles and long time behavior would be studied also.

3. Concerning the first-order Hamilton-Jacobi equation, we have for the moment only a first result of convergence and we want to continue our study in several directions. We consider the simplest equation and some easy generalizations should be true for more general Hamiltonians. As far as the ergodic problem is concerned: do we have existence of solutions for every $c \leq \bar{c} := \min l$? Is it always possible to build some bounded from below solutions? About the convergence result, we would like to understand more deeply what happens when the solutions are neither bounded from below nor above.

2.7 Notations and definitions

Notations : We first introduce some basis notations:

$x \cdot y$ (or $\langle x, y\rangle$)	The scalar product of two vectors in $I\!R^N$
$x \otimes x$	The Tensor product in $I\!R^N$.
Ω	be a subset of $I\!R^N$
$C(\Omega)$	The space of continuous functions on Ω.
$BC(\Omega)$	The space of bounded and continuous functions on Ω.
$BUC(\Omega)$	The space of bounded and uniformly continuous functions on Ω.
$USC(\Omega)$	The space of upper semicontinuous functions on Ω.
$LSC(\Omega)$	The space of lower semicontinuous functions on Ω.
$BUSC(\Omega)$	The space of bounded upper semicontinuous functions on Ω.
$BLSC(\Omega)$	The space of bounded lower semicontinuous functions on Ω.
$\mathrm{Lip}(\Omega)$	The space of Lipschitz continuous functions on Ω.
$\mathcal{M}^N$	The space of $N \times N$ matrix and symmetric matrix in $I\!R^N$.
$\mathcal{S}^N$	The space of $N \times N$ symmetric matrix in $I\!R^N$.
$\|f\|_\infty$	The supremum norm $\sup_{x\in\Omega} \|f(x)\|$ of a function $f : \Omega \to I\!R$.
$\|A\|, A = (a_{ij}) \in \mathcal{M}^N$	The Frobenius norm $\|A\| = \sqrt{\sum_{i=1}^N \sum_{j=1}^N \|a_{ij}\|^2}$.
$B_\delta, B(x, \delta)$	The closed ball centered at 0 and x, respectively, with radius δ.
B_δ^c (or $I\!R^N \setminus B_\delta$)	The complement of B_δ.

For $x, y \in I\!R^N, \langle x\rangle_\delta$ denotes $(|x|^2 + |\delta|^2)^{1/2}$, where $\delta > 0$. We call a function $\omega : [0,\infty) \to [0,\infty)$ a modulus if it is upper semi-continuous and non-decreasing on $[0,\infty)$ and satisfies $\omega(0) = 0$. Let $T \in (0,\infty)$, we write

$$Q = I\!R^N \times (0,\infty), \qquad Q_T = I\!R^N \times (0,T), \qquad \text{and } R_T = I\!R^N \times [0,T)$$

and introduce the spaces $\mathcal{E}_\mu^+(\Omega)$, $\mathcal{E}_\mu^-(\Omega)$, $\mathcal{E}_\mu(\Omega)$ of functions on $\Omega = I\!R^N$, R_T or $\bar{Q}$ as follows:

$$\mathcal{E}_\mu^+(I\!R^N) = \{v : I\!R^N \to I\!R : \limsup_{|x|\to+\infty} \frac{v(x)}{\phi_\mu(x)} \leq 0\}.$$

$$\mathcal{E}_\mu^+(R_T) = \{v : R_T \to I\!R : \limsup_{|x|\to+\infty} \sup_{0\leq t<T} \frac{v(x,t)}{\phi_\mu(x)} \leq 0\}.$$

$$\mathcal{E}_\mu^+(\bar{Q}) = \{v : \bar{Q} \to I\!R : \limsup_{|x|\to+\infty} \sup_{0\leq t<T} \frac{v(x,t)}{\phi_\mu(x)} \leq 0 \ \text{ for all } \ T > 0\}.$$

and for $\Omega = I\!R^N$, R_T or $\bar{Q}$,

$$\mathcal{E}_\mu^-(\Omega) := -\mathcal{E}_\mu^+(\Omega), \qquad \mathcal{E}_\mu(\Omega) := \mathcal{E}_\mu^+(\Omega) \cap \mathcal{E}_\mu^-(\Omega). \tag{2.41}$$

Throughout this thesis we work with solutions which belong to the above classes. Recall that ϕ_μ is the growth function defined in (2.42). In order to use ϕ_μ as the test function when doing the proofs, we need to modify it smooth enough. That is

$$\phi_\mu(x) = e^{\mu\sqrt{|x|^2+1}}, \ \ \mu > 0, \ x \in I\!R^N. \tag{2.42}$$

Hence, hereafter we use (2.42) instead of (2.31) standing for growth function ϕ_μ.

We now give the definition of viscosity solutions for stationary equation, for instance (2.29) (see [11, 15] for more general). It is the same in the non-stationary equation. Before giving it, we introduce two operators $\mathcal{I}[B_\kappa]$ and $\mathcal{I}[B^\kappa]$ as following

$$\begin{aligned}
\mathcal{I}[B_\kappa](x,u,p) &= \int_{|z|\leq\kappa} [u(x+z) - u(x) - p\cdot z\mathbb{I}_B(z)]\nu(dz)\\
\mathcal{I}[B^\kappa](x,u,p) &= \int_{|z|>\kappa} [u(x+z) - u(x) - p\cdot z\mathbb{I}_B(z)]\nu(dz),
\end{aligned}$$

for $0 < \kappa \leq 1$.

Definition 2.7.1. *An upper semi-continuous (in short usc) function u^λ is a subsolution of (2.29) if for any $\psi \in C^2(I\!R^N)$ such that $u^\lambda - \psi$ attains a maximum on $B(x,\kappa)$ at $x \in I\!R^N$*

$$\lambda u^\lambda(x) - \mathcal{I}[B_\kappa](x,\psi,p) - \mathcal{I}[B^\kappa](x,u^\lambda,p) + \langle b(x), p\rangle + H(x,p) \leq f(x),$$

where $p = D\psi(x)$, $0 < \kappa \leq 1$.
An lower semi-continuous (in short lsc) function u^λ is a supersolution of (2.29) if for any $\psi \in C^2(I\!R^N)$ such that $u^\lambda - \psi$ attains a minimum on $B(x,\kappa)$ at $x \in I\!R^N$

$$\lambda u^\lambda(x) - \mathcal{I}[B_\kappa](x,\psi,p) - \mathcal{I}[B^\kappa](x,u^\lambda,p) + \langle b(x), p\rangle + H(x,p) \geq f(x),$$

where $p = D\psi(x)$, $0 < \kappa \leq 1$.
Then, u^λ is a viscosity solution of (2.29) if it is both a viscosity subsolution and a viscosity supersolution of (2.29).

Chapter 3

Stochastic optimal control and Ornstein-Uhlenbeck operator

3.1 Ornstein-Uhlenbeck process driven by Brownian Motion or Lévy process

The Ornstein-Uhlenbeck process was proposed by Uhlenbeck and Ornstein around 1930 in a physical modeling context, see [73]. It is a typical exemple of *mean-reverting* process, which means that the process tends to go back to its mean value (like an oscillating pendulum, for instance). We refer to [70] and [76] for introductory and general information on the Ornstein-Uhlenbeck process.

3.1.1 Ornstein-Uhlenbeck process driven by Brownian motion

The N-dimensional Ornstein-Uhlenbeck process driven by Brownian motion is defined as the solution $(X_t)_{t\geq 0}$ of the following stochastic differential equation

$$\begin{cases} dX_t = -\alpha X_t dt + \sigma dW_t, \quad t > 0, \\ X_0 = x \in I\!R^N, \end{cases} \tag{3.1}$$

where $\alpha > 0$ and $\sigma \geq 0$ are real constants, W_t is a standard Brownian Motion on $I\!R^N$ with $\mathbb{E}[W_t] = 0$ and $\mathrm{Var}[W_t^2] = t$ and x is the starting point of the process X. It is easy to verify that

$$X_t = xe^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dW_s \tag{3.2}$$

is a solution of (3.1) with the conditional expectation

$$\mathbb{E}[X_t | X_0 = x] = \mathbb{E}\left[xe^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dW_s\right] = xe^{-\alpha t}.$$

Using Itô's isometry, we obtain the conditional variance

$$
\begin{aligned}
\mathrm{Var}[X_t|X_0=x] &= \mathbb{E}\left[\left(\sigma\int_0^t e^{-\alpha(t-s)}dW_s\right)^2\right] \\
&= \sigma^2\mathbb{E}\left[\left(\sigma\int_0^t e^{-2\alpha(t-s)}ds\right)^2\right] \\
&= \frac{\sigma^2}{2\alpha}\left(1-e^{-2\alpha t}\right).
\end{aligned}
$$

Thus

$$
X_t \sim \mathrm{Normal}\left(xe^{-\alpha t}, \frac{\sigma^2}{2\alpha}\left(1-e^{-2\alpha t}\right)\right).
$$

As $t \to \infty$, the mean and variance converge to 0 and $\sigma^2/2\alpha$, respectively. This means that the Ornstein-Uhlenbeck process has a *stationary distribution* which is the main and crucial property of Ornstein-Uhlenbeck process. In terms of stochastic optimal control we take the advantage of this process to confine the trajectories in some big balls limiting the effects of the noncompactness of $I\!R^N$, see Section 3.3 for details.

3.1.2 Ornstein-Uhlenbeck process driven by a Lévy process

The Ornstein-Uhlenbeck process driven by standard Brownian Motion is a continuous time Markov process (with continuous sample paths). But recently, there has been a lot of interest in studying *Lévy processes*, which are Markov processes having path discontinuities (or jumps). The interest for such processes comes from their high applicability in particular in financial modelling situations. We refer to [3, 23] for a quite complete overview of Lévy processes. Let us now present the Ornstein-Uhlenbeck process driven by such processes.

Let $(L_t)_{t\geq 0}$ be a Lévy process, we define a random measure on a compact set $A \subset I\!R$ by $N(t,A) := \#\{0 \leq s \leq t, L_s - L_{s_-} \in A\}$ for all $t \in [0,T]$. In other words, $N(t,A)$ counts the number of jumps of size $L_s - L_{s_-} \in A$ which occur before or at time t. For all A, $N(\cdot, A)$ is called the Poisson random measure (or jump measure) of L_t with intensity $\nu(A) := \mathbb{E}[N(1,A)]$ called the Lévy measure. We denote $\tilde{N}(t,A) = N(t,A) - t\nu(A)$ the compensated Poisson random measure of L_t.

A general Lévy measure ν has to satisfy the following integrability condition:

$$
\int_{I\!R^N\setminus\{0\}} \inf\left(|z|^2; 1\right)\nu(dz) < +\infty,
$$

hence the measure can be singular near the origin while it has to be of finite mass away from it. A typical case is when $\nu(dz) = \frac{dz}{|z|^{N+\beta}}$ with $\beta \in (0,2)$: we have a *β-stable Lévy process*. In financial modelling, the *tempered* version is also widely used:

$$
\nu(dz) = \frac{e^{-c|z|}}{|z|^{N+\beta}}, \quad c > 0,
$$

and this is the kind of measure that we consider in this thesis.

Now, apart from the jumps, a general Lévy process may also have a Brownian (continuous) part and a drift (deterministic) part. Actually, it can be shown that any Lévy process can be decomposed as the sum of such three processes (jump, brownian, drift). Since here we restrict ourselves to pure jump Lévy processes, (L_t) can be decomposed as follows

$$L_t = \int_0^t \int_{|z|\le 1} z\tilde{N}(ds,dz) + \int_0^t \int_{|z|>1} zN(ds,dz), \tag{3.3}$$

in which we see that the measure has to be compensated for the small jumps, $|z| < 1$, because of the singularity of the measure.

We shall not enter into details here which is not the subject of this thesis, but let us mention that this decomposition allows to define a notion of integral with respect to (L_t). Then, the Ornstein-Uhlenbeck process $(X_t)_{t\ge 0}$ driven by a Lévy process is defined as a solution of the stochastic differential equation (3.1) where formally, dW_t is replaced by dL_t. The precise meaning of this statment can be found in [3, 23].

3.2 Stochastic optimal control and the associated Hamilton-Jacobi-Bellman equation

In this section we give the link between our equations (i.e., (2.26) and (2.29)) and stochastic optimal control problems.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space, $(W_t)_{t\ge 0}$ be a $\mathcal{F}_t$-adapted standard N-Brownian motion such that $W_0 = 0$ a.s. We consider the stochastic control problem for controlled diffusion processes X_t whose dynamic is governed by a stochastic differential equation of the form

$$\begin{cases} dX_t = (g(X_t,\gamma_t) - b(X_t))dt + 2\sigma(X_t)dW_t \\ X_0 = x \in I\!R^N, \end{cases} \tag{3.4}$$

where the control process $\gamma_t : [0,\infty) \to \Theta$ (Θ is a compact metric space) which is $\mathcal{F}_t$-progressively measurable; $g : I\!R^N \times \Theta \to I\!R^N$ and $b : I\!R^N \to I\!R^N$ are continuous vector fields; σ is a continuous real $N \times N$ matrix satisfying

$$|\sigma(x)| \le C, \quad |\sigma(x) - \sigma(y)| \le C|x-y|, \quad x, y \in I\!R^N. \tag{3.5}$$

We define the following cost functions

$$J(x,\gamma) = \mathbb{E}_x \left(\int_0^\infty e^{-\lambda s} L(X_s,\gamma_s) ds \right) \quad \forall x \in I\!R^N, \tag{3.6}$$

$$J(x,t,\gamma) = \mathbb{E}_x \left(\int_0^t L(X_s,\gamma_s) ds + u_0(X_t) \right) \quad \forall x \in I\!R^N, t \in [0,\infty), \tag{3.7}$$

where $\mathbb{E}_x$ denotes the expectation for a trajectory starting at x, $\lambda > 0$ is the so-called *discount factor*. In our work, the running cost (or Lagrangian function) L takes the form

$$L(X_s, \gamma_s) = l(X_s, \gamma_s) + f(X_s),$$

where $l(\cdot, \gamma)$, f and u_0 are continuous and possibly unbounded functions in $\mathbb{R}^N$. Problems (3.6) and (3.7) are called *finite* and *infinite-horizon* problems, respectively.

We then introduce the value functions (optimal cost functions) by

$$\begin{aligned} u^\lambda(x) &= \inf_\gamma J(x, \gamma), \\ u(x,t) &= \inf_\gamma J(x, t, \gamma). \end{aligned}$$

By the Dynamic Programming Principle introduced by Fleming & Soner [41] and Yong & Zhou [79], one can find that at least formally, $u^\lambda(x)$ and $u(x,t)$ satisfy

$$\lambda u^\lambda(x) - \operatorname{tr}(\sigma(x)\sigma^T(x)D^2u^\lambda(x)) + \langle b(x), Du^\lambda(x)\rangle + H(x, Du^\lambda(x)) = f(x) \tag{3.8}$$

and

$$\begin{cases} u_t(x,t) - \operatorname{tr}(\sigma(x)\sigma^T(x)D^2u(x,t)) + \langle b(x), Du(x,t)\rangle + H(x, Du(x,t)) = f(x) \\ u(x,0) = u_0(x), \end{cases} \tag{3.9}$$

respectively, where

$$H(x,p) = \max_{a\in\Theta}\{-g(x,a)\cdot p - l(x,a)\}. \tag{3.10}$$

Let us now discuss a little bit why we choose assumptions (2.22), (2.37) and (2.38) for the Hamiltonian H in (2.26) and (2.29). Observe that

$$|H(x,p)| \le \max_{a\in\Theta}\{|g(x,a)||p| + |l(x,a)|\}.$$

Hence H satisfies (2.22) if $g(\cdot, a)$ and $l(\cdot, a)$ are bounded functions. With these assumptions, H satisfies (2.38) as well. Besides, if $g(\cdot,a), l(\cdot,a) \in \operatorname{Lip}(\mathbb{R}^N)$, then we have

$$\begin{aligned} |H(x,p) - H(y,p)| &\le \max_{a\in\Theta}\{|g(x,a) - g(y,a)||p| + |l(x,a) - l(y,a)|\} \\ &\le C|x-y|(1+|p|), \quad \forall x, y, p \in \mathbb{R}^N. \end{aligned}$$

If, in addition, $|g(x,a)|, |l(x,a)| \le C(1+|x|)$, for all $x \in \mathbb{R}^N$, we have

$$\begin{aligned} |H(x,p) - H(x,q)| &\le \max_{a\in\Theta}\{|g(x,a)||p-q| + |l(x,a)|\} \\ &\le C|p-q|(1+|x|), \quad \forall x, p, q \in \mathbb{R}^N. \end{aligned}$$

This is why we consider (2.37).

We have explained the relation between stochastic optimal control and the associated Hamilton-Jacobi-Bellman equation in the local case. It is formulated in the same way for the nonlocal one if we replace $2\sigma(X_t)dW_t$ in (3.4) by dL_t, where L_t is the pure jump

Lévy process given by (3.3). We then define the cost functionals and the value functions as above. By the Dynamic Programming Principle, we find that u^λ and $u(x,t)$ are solutions of the Hamilton-Jacobi-Bellman equation (3.8) and (3.9), respectively, where the diffusion is replaced by the nonlocal operator

$$\mathcal{I}(x,u,Du) = \int_{I\!R^N} \{u(x+z) - u(x) - Du(x)\cdot z\mathbb{I}_B(z)\}\nu(dz),$$

where B is the unit ball. Details can be found for instance in [72].

3.3 Compactness properties of the Ornstein-Uhlenbeck operator

In this section we briefly explain the effect of the Ornstein-Uhlenbeck operator in our equations, which allows to recover some compactness in an unbounded setting.

Let us consider a simple PDE (local case)

$$-\Delta u + \alpha\langle x, Du\rangle + H(x,Du) = 0 \qquad \text{in } I\!R^N, \tag{3.11}$$

where $\alpha > 0$ and $H(x,p)$ is uniformly Lipschitz continuous with respect to p. Notice that this implies that there exists $L_H > 0$ such that for any $x \in I\!R^N$, $|D_pH(x,p)| \leq L_H|p|$. We have seen that the operator $\Delta - \alpha\langle x, D\rangle$ is called the Ornstein-Uhlenbeck operator. The stochastic differential equation associated with this equation is

$$\begin{cases} dX_t = \gamma_t dt - \alpha X_t dt + 2dW_t \\ X_0 = x \in I\!R^N, \end{cases}$$

where (X_t) is the stochastic process controlled by γ_t (we can see this case in Ichihara [47]). The optimal control is given by $\gamma_t = -D_pH(X_t, Du(X_t))$, where $D_pH(x,p)$ stands for the gradient of H with respect to p and u is the solution of (3.11) which defines the stochastic process (X_t).

We have seen in (2.20) that $\Gamma_\mu(x) := e^{\mu|x|^2}$, $\mu > 0$, satisfies

$$\mathcal{A}[\Gamma_\mu] := -\Delta\Gamma_\mu + \alpha\langle x, D\Gamma_\mu\rangle - L_H|\Gamma_\mu| \geq \Gamma_\mu - K, \tag{3.12}$$

for some $K > 0$, hence using $\varphi := \Gamma_\mu - K$ we have $\mathcal{A}[\varphi] \geq \varphi$.

Now we apply Itô's formula to $\varphi(X_t)$ and use (3.12). We have

$$\begin{aligned}
& \mathbb{E}[\varphi(X_t)] \\
= \;& \mathbb{E}[\varphi(X_0)] + \mathbb{E}\int_0^t \{\Delta\varphi(X_s) - \alpha\langle X_s, D\varphi(X_s)\rangle - D_pH(X_s, Du(X_s))D\varphi(X_s)\}ds \\
\leq \;& \varphi(x_0) + \mathbb{E}\int_0^t \{-\varphi(X_s) - L_H|D\varphi(X_s)| - D_pH(X_s, Du(X_s))D\varphi(X_s)\}ds \\
\leq \;& \varphi(x_0) - \int_0^t \mathbb{E}[\varphi(X_s)]ds.
\end{aligned}$$

Denoting $\psi(t) := \mathbb{E}[\varphi(X_t)]$ and using Gronwall's inequality, we get $\psi(t) \leq \varphi(x_0)e^{-t}$ which yields

$$\mathbb{E}[e^{\mu|X_t|^2}] \leq K + \left(e^{\mu|x_0|^2} - K\right) e^{-t}.$$

This estimate shows a compactness property of the trajectories of (X_t): for t big enough, the trajectory is essentially localized in the ball of radius $\sqrt{\ln(K)/\mu}$.

From the PDE viewpoint, the compactness property is related to the existence of a kind of supersolution in (3.12), see (2.20) or Lemma 4.1.1, which allows to have some bounds on the solutions we consider. Without the $\alpha\langle x, D\phi\rangle$-term, it would not be possible to control the diffusion (the Laplacian term) for x big, at least in this class of unbounded functions. So, this drift term is essential in the construction of solutions of the ergodic problem, as well as in comparison results and thus, in obtaining the long-time behaviour in this unbounded setting.

Chapter 4

A priori regularity of solutions

4.1 Regularity of solutions for stationary problem

For $\lambda > 0$, we consider the stationary problem

$$\lambda u^\lambda(x) - \mathcal{F}(x, [u^\lambda]) + \langle b(x), Du^\lambda(x)\rangle + H(x, Du^\lambda(x)) = f(x) \quad \text{in } I\!R^N, \tag{4.1}$$

where $\mathcal{F}$ may be either the local or the nonlocal operator defined by (2.27). Throughout this section we always assume that: the diffusion matrix $A(x) = \sigma(x)\sigma(x)^T$ with σ satisfying

$$\begin{cases} \sigma \in C(I\!R^N, \mathcal{M}_N), \text{there exists } C_\sigma,\ L_\sigma > 0 \text{ such that} \\ |\sigma(x)| \le C_\sigma, \quad |\sigma(x) - \sigma(y)| \le L_\sigma |x-y| \quad x \in I\!R^N; \end{cases} \tag{4.2}$$

the nonlocal operator is an integro differential operator defined by (2.23) satisfying (M1) and (M2) with $\beta \in (0, 2)$; the Ornstein-Uhlenbeck drift term b is a continuous vector field in $I\!R^N$ which satisfies

$$\begin{cases} \text{There exists } \alpha > 0 \text{ such that} \\ \langle b(x) - b(y), x - y\rangle \ge \alpha |x-y|^2 \quad x, y \in I\!R^N \end{cases} \tag{4.3}$$

and the $f : I\!R^N \to I\!R$ term is a continuous function such that:

$$\begin{cases} f \in \mathcal{E}_\mu(I\!R^N) \text{ and there exists } C_f, \mu > 0 \text{ such that} \\ |f(x) - f(y)| \le C_f(\phi_\mu(x) + \phi_\mu(y))|x-y|, \quad x, y \in I\!R^N, \end{cases} \tag{4.4}$$

where $\mathcal{E}_\mu(I\!R^N)$ denotes the space introduced in (2.41) with exponential limited growth ϕ_μ defined in (2.42).

4.1.1 Regularity of solutions for uniformly elliptic equations.

In this section, we suppose that (4.1) is uniformly elliptic, i.e., in the local case, $\mathcal{F}(x, [u^\lambda]) = \mathrm{tr}(\sigma(x)\sigma^T(x)D^2u)$ with σ satisfying

$$\begin{cases} \text{There exists } \rho > 0 \text{ such that} \\ \rho I \le \sigma(x)\sigma(x)^T, \quad x \in I\!R^N. \end{cases} \tag{4.5}$$

and in the nonlocal one, $\mathcal{F}(x,[u^\lambda]) = \mathcal{I}(x,u^\lambda,Du^\lambda)$ with Lévy measure satisfying (M1) and (M2) with $\beta \in (1,2)$.

In these cases, we may deal with the class of sublinear Hamiltonians, i.e.,

$$\begin{cases} \text{There exists } C_H > 0 \text{ such that} \\ |H(x,p)| \le C_H(1+|p|), \quad x,p \in \mathbb{R}^N. \end{cases} \tag{4.6}$$

We state now the main result namely local lipschitz estimates which are uniform with respect to λ for solution of (4.1).

Theorem 4.1.1. *Let $u^\lambda \in C(\mathbb{R}^N) \cap \mathcal{E}_\mu(\mathbb{R}^N)$, $\mu > 0$, be a solution of (4.1). Assume that (4.3), (4.4) and (4.6) hold. Assume in addition one of the following assumption:*

(i) $\mathcal{F}(x,[u^\lambda]) = \mathrm{tr}(A(x)D^2u^\lambda(x))$ and (4.2), (4.5) hold.

(ii) $\mathcal{F}(x,[u^\lambda]) = \mathcal{I}(x,u^\lambda,Du^\lambda)$ and suppose that (M1) and (M2) hold with $\beta \in (1,2)$.

Then there exists a constant C independent of λ such that

$$|u^\lambda(x) - u^\lambda(y)| \le C|x-y|(\phi_\mu(x) + \phi_\mu(y)), \quad x,y \in \mathbb{R}^N, \ \lambda \in (0,1). \tag{4.7}$$

Proof. 1. Test-function and maximum point. For simplicity, we skip the λ superscript for u^λ writing u instead and we write ϕ for ϕ_μ. Let $\delta, A, C_1 > 0$, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(0) = 0$ be a C^2 concave and increasing function which will be defined later depending on the two different cases.

Consider

$$M_{\delta,A,C_1} = \sup_{x,y\in\mathbb{R}^N} \left\{ u(x) - u(y) - \sqrt{\delta} - C_1(\psi(|x-y|)+\delta)(\phi(x)+\phi(y)+A) \right\} \tag{4.8}$$

and set

$$\begin{aligned} \Phi(x,y) &= C_1(\phi(x)+\phi(y)+A), \\ \varphi(x,y) &= \sqrt{\delta} + (\psi(|x-y|)+\delta)\Phi(x,y). \end{aligned} \tag{4.9}$$

All the constants and functions will be chosen to be independent of $\lambda > 0$.

We will prove that for any fixed $\lambda \in (0,1)$, there exists a $\delta_0(\lambda) > 0$ such that for any $0 < \delta < \delta_0(\lambda)$, $M_{\delta,A,C_1} \le 0$.

Indeed, if $M_{\delta,A,C_1} \le 0$ for some good choice of A, C_1, ψ independent of $\delta > 0$, then we get (4.7) by letting $\delta \to 0$. So we argue by contradiction, assuming that for δ small enough $M_{\delta,A,C_1} > 0$. Since $u \in \mathcal{E}_\mu(\mathbb{R}^N)$ and $\delta > 0$, the supremum is achieved at some point (x,y) with $x \neq y$, thanks to $\delta > 0$ and the continuity of u.

2. Viscosity inequalities. We first compute derivatives of φ. For the sake of simplicity of notations, we omit (x,y), we write ψ, Φ for $\psi(|x-y|)$, $\Phi(x,y)$ respectively.

Set

$$p = \frac{x-y}{|x-y|}, \qquad \mathcal{C} = \frac{1}{|x-y|}(I - p\otimes p). \tag{4.10}$$

We have

$$
\begin{aligned}
D_x\varphi &= \psi'\Phi p + C_1(\psi+\delta)D\phi(x) && (4.11)\\
D_y\varphi &= -\psi'\Phi p + C_1(\psi+\delta)D\phi(y) && (4.12)\\
D^2_{xx}\varphi &= \psi''\Phi p\otimes p + \psi'\Phi\mathcal{C} + C_1\psi'(p\otimes D\phi(x) + D\phi(x)\otimes p) && (4.13)\\
&\quad +C_1(\psi+\delta)D^2\phi(x)\\
D^2_{yy}\varphi &= \psi''\Phi p\otimes p + \psi'\Phi\mathcal{C} - C_1\psi'(p\otimes D\phi(y) + D\phi(y)\otimes p) && (4.14)\\
&\quad +C_1(\psi+\delta)D^2\phi(y)\\
D^2_{xy}\varphi &= -\psi''\Phi p\otimes p - \psi'(|x-y|)\Phi\mathcal{C} + C_1\psi'(|x-y|)(D\phi(y)\otimes p - p\otimes D\phi(x))\\
D^2_{yx}\varphi &= -\psi''\Phi p\otimes p - \psi'\Phi\mathcal{C} + C_1\psi'(p\otimes D\phi(y) - D\phi(x)\otimes p).
\end{aligned}
$$

Then applying [30, Theorem 3.2] in the local case and [15, Corollary 1] in the nonlocal one we obtain, for any $\zeta > 0$, there exist $X, Y \in \mathcal{S}^N$ such that $(D_x\varphi(x,y), X) \in \overline{J}^{2,+}u(x)$, $(-D_y\varphi(x,y), Y) \in \overline{J}^{2,-}u(y)$ and

$$
\begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq A + \zeta A^2, \tag{4.15}
$$

where

$$
\begin{aligned}
A = D^2\varphi(x,y) &= \psi''\Phi\begin{pmatrix} p\otimes p & -p\otimes p \\ -p\otimes p & p\otimes p \end{pmatrix} + \psi'\Phi\begin{pmatrix} \mathcal{C} & -\mathcal{C} \\ -\mathcal{C} & \mathcal{C} \end{pmatrix}\\
&\quad + C_1\psi'\begin{pmatrix} p\otimes D\phi(x) + D\phi(x)\otimes p & D\phi(y)\otimes p - p\otimes D\phi(x) \\ p\otimes D\phi(y) - D\phi(x)\otimes p & -(p\otimes D\phi(y) + D\phi(y)\otimes p) \end{pmatrix}\\
&\quad + C_1(\psi+\delta)\begin{pmatrix} D^2\phi(x) & 0 \\ 0 & D^2\phi(y) \end{pmatrix}
\end{aligned}
$$

and $\zeta A^2 = O(\zeta)$ (ζ will be sent to 0 first).

Writing the viscosity inequalities at (x,y) in the local and nonlocal case we have

$$
\begin{aligned}
&\lambda u(x) - \mathcal{F}(x,[u]) + \langle b(x), D_x\varphi(x,y)\rangle + H(x, D_x\varphi(x,y)) \leq f(x)\\
&\lambda u(y) - \mathcal{F}(y,[u]) + \langle b(y), -D_y\varphi(x,y)\rangle + H(y, -D_y\varphi(x,y)) \geq f(y),
\end{aligned}
$$

where $\mathcal{F}(x,[u]) = \mathrm{tr}(A(x)X)$ and $\mathcal{F}(y,[u]) = \mathrm{tr}(A(y)Y)$ in the local case and $\mathcal{F}(x,[u]) = \mathcal{I}(x,u,D_x\varphi)$ and $\mathcal{F}(y,[u]) = \mathcal{I}(y,u,-D_y\varphi)$ in the nonlocal one.

Subtracting the viscosity inequalities, we have

$$
\begin{aligned}
&\lambda(u(x)-u(y)) - (\mathcal{F}(x,[u]) - \mathcal{F}(y,[u])) && (4.16)\\
&+\langle b(x), D_x\varphi(x,y)\rangle - \langle b(y), -D_y\varphi(x,y)\rangle\\
&+H(x, D_x\varphi(x,y)) - H(y, -D_y\varphi(x,y))\\
\leq\ & f(x) - f(y).
\end{aligned}
$$

We estimate separately the different terms in order to reach a contradiction.

3. Monotonicity of the equation with respect to u. Using that $M_{\delta,A,C_1} > 0$, we get

$$\lambda(u(x) - u(y)) > \lambda\sqrt{\delta} + \lambda(\psi + \delta)\Phi \geq \lambda\sqrt{\delta}. \tag{4.17}$$

4. b-terms. From (4.3), (4.11) and (4.12), we have

$$\begin{aligned} & \langle b(x), D_x\varphi\rangle - \langle b(y), -D_y\varphi\rangle \\ = \; & \psi'\Phi\langle b(x) - b(y), p\rangle + (\psi + \delta)(\langle b(x), D\phi(x)\rangle + \langle b(y), D\phi(y)\rangle) \\ \geq \; & \alpha\psi'\Phi|x - y| + C_1(\psi + \delta)(\langle b(x), D\phi(x)\rangle + \langle b(y), D\phi(y)\rangle). \end{aligned} \tag{4.18}$$

5. H-terms. From (4.6), (4.11) and (4.12), we have

$$H(x, D_x\varphi) - H(y, -D_y\varphi) \geq -C_H[2 + 2\psi'\Phi + C_1(\psi + \delta)(|D\phi(x)| + |D\phi(y)|)]. \tag{4.19}$$

6. f-terms. From (4.4), we have

$$|f(x) - f(y)| \leq C_f(\phi(x) + \phi(y))|x - y|. \tag{4.20}$$

7. An estimate for the ϕ-terms. To estimate the ϕ-terms we use the following lemma the proof of which is postponed here.

Lemma 4.1.1. *Let $L_0 > 0$, $L(x, y) := L_0(1 + |x| + |y|)$. Define*

$$\mathcal{L}_L[\phi](x, y) := \mathcal{F}(x, [\phi]) + \mathcal{F}(y, [\phi]) + L(x, y)(|D\phi(x)| + |D\phi(y)|). \tag{4.21}$$

There exists a constant $K = K(\alpha, L_0, \mathcal{F}) > 0$ such that for any $\alpha > 2L_0$

$$-\mathcal{L}_L[\phi](x, y) + \langle b(x), D\phi(x)\rangle + \langle b(y), D\phi(y)\rangle \geq \phi(x) + \phi(y) - 2K. \tag{4.22}$$

If $L(x, y) = L_0$, then (4.22) holds for any $\alpha > 0$ and there exists $R = R(\alpha, L_0, \mathcal{F})$ such that

$$-\mathcal{F}(x, [\phi]) + \langle b(x), D\phi(x)\rangle - L_0|D\phi(x)| \geq \begin{cases} -K & \text{for } |x| \leq R, \\ K & \text{for } |x| \geq R. \end{cases} \tag{4.23}$$

We now come back to the proof of Theorem by giving it first in the local case then in the nonlocal one.

8. **Local case:** *Hypothesis i. holds, i.e., $\mathcal{F}(x, [u^\lambda]) = \mathrm{tr}(A(x)D^2u^\lambda(x))$ and (4.2), (4.5) hold.*

8.1. Estimate for second order terms. We use the following lemma the proof of which is given in Section 4.3.2.

Lemma 4.1.2. ***(Estimates on $\mathcal{F}$ in the local case).***

- *Degenerate case: Under assumption (4.2),*

$$\begin{aligned} & -\mathrm{tr}(A(x)X - A(y)Y) \\ \geq \; & -C_1(\psi(|x - y|) + \delta)\{\mathrm{tr}(A(x)D^2\phi(x)) + \mathrm{tr}(A(y)D^2\phi(y)) \\ & + \mathcal{C}_\sigma(|D\phi(x)| + |D\phi(y)|)\} - \mathcal{C}_\sigma|x - y|\psi'(|x - y|)\Phi(x, y) + O(\zeta). \end{aligned}$$

- *Elliptic case: In addition, if (4.5) holds, we have*

$$
\begin{aligned}
& -\mathrm{tr}(A(x)X - A(y)Y) \\
\geq \;& -[4\rho\psi''(|x-y|) + \mathcal{C}_\sigma \psi'(|x-y|)]\Phi(x,y) \\
& -C_1(\psi(|x-y|)+\delta)\{\mathrm{tr}(A(x)D^2\phi(x)) + \mathrm{tr}(A(y)D^2\phi(y)) \\
& +\mathcal{C}_\sigma(|D\phi(x)|+|D\phi(y)|)\} - C_1\mathcal{C}_\sigma\psi'(|x-y|)(|D\phi(x)|+|D\phi(y)|) + O(\zeta),
\end{aligned}
$$

where $\mathcal{C}_\sigma = \mathcal{C}_\sigma(N,\rho,\sigma)$ is given by (4.89).

This Lemma is a crucial tool giving the estimates for the second order terms. The first part is a basic application of Ishii's Lemma (see [30]) in an unbounded context with the test function φ. The second part takes profit of the ellipticity of the equation and allows to apply Ishii-Lions' method ([51]).

8.2. Global estimate. Since (4.2) and (4.5) hold, by Lemma 4.1.2, we have

$$
\begin{aligned}
-(\mathcal{F}(x,[u]) - \mathcal{F}(y,[u])) \;&=\; -\mathrm{tr}(A(x)X - A(y)Y) \\
&\geq\; -[4\rho\psi''(|x-y|) + \mathcal{C}_\sigma\psi'(|x-y|)]\Phi(x,y) \\
&\quad -C_1\mathcal{C}_\sigma\psi'(|x-y|)(|D\phi(x)|+|D\phi(y)|) \\
&\quad -C_1(\psi(|x-y|)+\delta)\{\mathrm{tr}(A(x)D^2\phi(x)) + \mathrm{tr}(A(y)D^2\phi(y)) \\
&\quad +\mathcal{C}_\sigma(|D\phi(x)|+|D\phi(y)|)\} + O(\zeta).
\end{aligned} \tag{4.24}
$$

Plugging (4.17), (4.18), (4.19), (4.20) and (4.24) into (4.16) we obtain

$$
\begin{aligned}
& \lambda\sqrt{\delta} + \Phi(x,y)\left[-4\rho\psi''(|x-y|) + \alpha\psi'(|x-y|)|x-y| - (2C_H + \mathcal{C}_\sigma)\psi'(|x-y|)\right] \\
& +C_1(\psi+\delta)\left(-\mathrm{tr}(A(x)D^2\phi(x)) + \langle b(x), D\phi(x)\rangle - (\mathcal{C}_\sigma + C_H)|D\phi(x)|\right) \\
& +C_1(\psi+\delta)\left(-\mathrm{tr}(A(y)D^2\phi(y)) + \langle b(y), D\phi(y)\rangle - (\mathcal{C}_\sigma + C_H)|D\phi(x)|\right) \\
& -C_1\mathcal{C}_\sigma\psi'(|D\phi(x)|+|D\phi(y)|) - 2C_H \\
\leq \;& C_f(\phi(x)+\phi(y))|x-y| + O(\zeta).
\end{aligned}
$$

Letting $\zeta \to 0$, using $|D\phi| \leq \mu\phi$ and introducing $\mathcal{L}_L$ given by (4.21) with $L(x,y) = \mathcal{C}_\sigma + C_H$ and $\mathcal{F}$ is the local operator in (2.27), we obtain

$$
\begin{aligned}
& \lambda\sqrt{\delta} + C_1(\phi(x)+\phi(y)+A)\left(-4\rho\psi''(|x-y|) + \alpha\psi'(|x-y|)|x-y|\right) \\
\leq \;& C_1(\phi(x)+\phi(y)+A)(2C_H + (1+\mu)\mathcal{C}_\sigma)\psi'(|x-y|) \\
& +C_1(\psi(|x-y|)+\delta)\left(\mathcal{L}_L[\phi](x,y) - \langle b(x), D\phi(x)\rangle - \langle b(y), D\phi(y)\rangle\right) \\
& +2C_H + C_f(\phi(x)+\phi(y))|x-y|.
\end{aligned} \tag{4.25}
$$

The rest of the proof consists in reaching a contradiction in (4.25) by taking profit of the positive terms in the left hand-side of the inequality.

8.3. Construction of the concave test-function ψ. For $r_0, C_2 > 0$ to be fixed later, we define the C^2 concave increasing function $\psi : [0,\infty) \to [0,\infty)$ as follows

$$
\psi(r) = 1 - e^{-C_2 r} \quad \text{for } r \in [0, r_0],
$$

$\psi(r)$ is linear on $[r_0+1,+\infty)$ with derivative $\psi'(r) = C_2 e^{-C_2(r_0+1)}$, and ψ is extended in a smooth way on $[r_0, r_0+1]$ such that, for all $r \geq 0$,

$$\psi'_{\min} := C_2 e^{-C_2(r_0+1)} \leq \psi'(r) \leq \psi'_{\max} := \psi'(0) = C_2.$$

Notice that ψ is chosen such that

$$\psi''(r) + C_2\psi'(r) = 0 \quad \text{for } r \in [0, r_0]. \tag{4.26}$$

8.4. Choice of the parameters to reach a contradiction in (4.25). We now fix in a suitable way all the parameters to conclude that (4.25) cannot hold, which will end the proof. Before rigorous computations, let us explain roughly the main ideas. We set $r := |x-y|$. The function ψ above was chosen to be strictly concave for small $r \leq r_0$. For such r and for a suitable choice of r_0, we will take profit of the ellipticity of the equation, which appears through the positive term $-4\nu\psi''(r)$ in (4.25), to control all the others terms. Since we are in $\mathbb{R}^N$ and we cannot localize anything, r may be large. In this case, the second derivative $\psi''(r)$ of the increasing concave function ψ is small and the ellipticity is not powerful enough to control the bad terms. Instead, we use the positive term $\alpha\psi'(r)r$ coming from the Ornstein-Uhlenbeck operator to control everything for $r \geq r_0$.

At first, we set

$$C_1 = \frac{3C_f}{\alpha\psi'_{\min}} + 1 = \frac{3C_f e^{C_2(r_0+1)}}{\alpha C_2} + 1, \tag{4.27}$$

where C_2 and r_0 will be chosen later. This choice of C_1 is done in order to get rid of the f-terms. Indeed, for every $r \in [0, +\infty)$,

$$\begin{aligned} C_1(\phi(x)+\phi(y)+A)\frac{\alpha}{3}\psi'(r)r &\geq \frac{\alpha C_1 \psi'_{\min}}{3}(\phi(x)+\phi(y))r \\ &\geq C_f(\phi(x)+\phi(y))r. \end{aligned} \tag{4.28}$$

Secondly, we fix r_0 which separates the range of the ellipticity action and the one of the Ornstein-Uhlenbeck term. We fix

$$r_0 = \max\{\frac{3(2C_H + (1+\mu)\mathcal{C}_\sigma)}{\alpha}, 2R\}, \tag{4.29}$$

where R comes from (4.23) with $L(x,y) = \mathcal{C}_\sigma + C_H$.

8.5. Contradiction in (4.25) *for $r \geq r_0$ thanks to the Ornstein-Uhlenbeck term.* We assume that $r \geq r_0$. With the choice of r_0, in (4.29) we have

$$\frac{\alpha}{3}\psi'(r)r \geq \frac{\alpha}{3}\psi'(r)r_0 \geq (2C_H + (1+\mu)\mathcal{C}_\sigma)\psi'(r).$$

Moreover, $2R \leq r_0 \leq r = |x-y| \leq |x| + |y|$ implies that either $|x| \geq R$ or $|y| \geq R$, so by using (4.23) we have

$$-\mathcal{L}_L[\phi](x,y) + \langle b(x), D\phi(x)\rangle + \langle b(y), D\phi(y)\rangle \geq K - K \geq 0.$$

Therefore, taking into account (4.28), inequality (4.25) reduces to

$$0 < \lambda\sqrt{\delta} + C_1(\phi(x) + \phi(y) + A)\frac{\alpha}{3}\psi'(r)r \leq 2C_H.$$

To obtain a contradiction, it is then sufficient to ensure

$$\frac{C_1 A\alpha}{3}\psi'_{\min} r_0 \geq 2C_H,$$

which leads to

$$A \geq \frac{2C_H}{C_f r_0} \tag{4.30}$$

because of the choice of C_1 and the value of $\psi'_{\min}$.

Finally, (4.25) can not hold for $r \geq r_0$ if C_1, r_0, A are chosen as above. Notice that we did not impose yet any condition on $C_2 > 0$.

8.6. Contradiction in (4.25) *for $r \leq r_0$ thanks to ellipticity.* One of the main role of the ellipticity is to control the first term in the right hand-side of (4.25) for small r. More precisely, by setting

$$C_2 \geq \frac{2C_H + (1+\mu)\mathcal{C}_\sigma}{\rho}, \tag{4.31}$$

and using (4.26), we have

$$-\rho\psi''(r) \geq (2C_H + (1+\mu)\mathcal{C}_\sigma)\psi'(r).$$

Since both $|x|$ and $|y|$ may be smaller than R, we cannot estimate $\mathcal{L}_L[\phi](x,y) - \langle b(x), D\phi(x)\rangle - \langle b(y), D\phi(y)\rangle$ from above in a better way than $2K$. Taking into account (4.28), inequality (4.25) reduces to

$$0 < \lambda\sqrt{\delta} + C_1(\phi(x) + \phi(y) + A)(-3\rho)\psi''(r) \leq C_1(\psi(r) + \delta)2K + 2C_H.$$

Using that $\psi(r) \leq 1$, and $C_1 \geq 1$, we then increase A from (4.30) in order that

$$-3\rho A\psi''(r) \geq 2(K + C_H) \geq 2K\psi(r) + \frac{2C_H}{C_1},$$

which leads to the choice

$$A \geq \max\{\frac{2C_H}{C_f r_0}, \frac{2(K+C_H)e^{C_2 r_0}}{3\rho C_2^2}\}. \tag{4.32}$$

Notice that, by the choice of r_0 in (4.29) and C_2 in (4.31), A and C_1 depend only on the datas $\sigma, \rho, b, H, f, \mu$ of the equation.

Finally, inequality (4.25) becomes

$$0 < \lambda\sqrt{\delta} \leq 2C_1 K\delta, \tag{4.33}$$

which is absurd for δ small enough. It ensures the claim of Step 8.6.

8.7. Conclusion. We have proved that $M_{\delta,A,C_1} \leq 0$ in (4.8), for δ small enough. It follows that, for every $x, y \in I\!R^N$,

$$|u(x) - u(y)| \leq \sqrt{\delta} + (\psi(|x-y|) + \delta)C_1(\phi(x) + \phi(y) + A).$$

Since A, ψ do not depend on δ, letting $\delta \to 0$ and using the fact that ψ is a concave increasing function, so we have $\psi(r) \leq \psi'(0)r = C_2 r$. Hence, we get

$$|u(x) - u(y)| \leq C_1C_2|x-y|(\phi(x) + \phi(y) + A) = C_1C_2(A+1)|x-y|(\phi(x) + \phi(y)).$$

This ends the proof of the local case (i).

9. **Nonlocal case:** *Hypothesis ii. holds, i.e., $\mathcal{F}(x, [u^\lambda]) = \mathcal{I}(x, u^\lambda, Du^\lambda)$.* We come back to the end of Step 7 and:

- We give some estimate for the difference of the nonlocal terms which are quite different of those for local term.
- The concave test function will be different and we need a preliminary step (interesting by itself) which consists in proving first Hölder regularity $\forall \tau \in (0,1)$ and using this regularity to reach Lipschitz regularity.

We need first to give the estimates for the nonlocal terms with general concave test function ψ. They are stated in the following Proposition the proof of which is given in Section 4.3.3.

Proposition 4.1.1. ***(Concave estimates - general nonlocal operators).*** *Let $u \in C(I\!R^N) \cap \mathcal{E}_\mu(I\!R^N)$ and (M1), (M2) hold with $\beta \in (0,2)$. Define*

$$\Psi(x,y) = u(x) - u(y) - \varphi(x,y), \tag{4.34}$$

where φ is defined in (4.9), and assume the maximum of Ψ is positive and reached at (x,y), with $x \neq y$. Let

$$a \;=\; x - y, \qquad \hat{a} = a/|a|.$$

For any $a_0 > 0$, the followings hold

(i) ***(Rough estimate for big $|a|$).*** *For all $|a| \geq a_0$,*

$$\mathcal{I}(x,u,D_x\varphi) - \mathcal{I}(y,u,-D_y\varphi) \leq C_1(\psi(|a|) + \delta)\left(\mathcal{I}(x,\phi,D\phi) + \mathcal{I}(y,\phi,D\phi)\right) \tag{4.35}$$

(ii) ***(More precise estimate for small $|a|$).*** *For all $|a| \leq a_0$,*

$$\begin{aligned} & \mathcal{I}(x,u,D_x\varphi) - \mathcal{I}(y,u,-D_y\varphi) \\ \leq\; & C_1(\psi(|a|) + \delta)\left(\mathcal{I}(x,\phi,D\phi) + \mathcal{I}(y,\phi,D\phi)\right) \\ & + C_1\mu\psi'(|a|)(\phi(x) + \phi(y)) \int_{\mathcal{C}_{\eta,\gamma}(a)} \phi(z)|z|^2\nu(dz) \\ & + \frac{1}{2}\Phi(x,y) \int_{\mathcal{C}_{\eta,\gamma}(a)} \sup_{|s|\leq 1} \left((1-\tilde{\eta}^2)\frac{\psi'(|a+sz|)}{|a+sz|} + \tilde{\eta}^2\psi''(|a+sz|) \right) |z|^2\nu(dz) \end{aligned} \tag{4.36}$$

where

$$\mathcal{C}_{\eta,\gamma}(a) = \{z \in B_\gamma; (1-\eta)|z||a| \le |a.z|\},$$

and $\gamma = |a|\gamma_0$, $\tilde{\eta} = \frac{1-\eta-\gamma_0}{1+\gamma_0} > 0$ *with* $\gamma_0 \in (0,1)$, $\eta \in (0,1)$ *small enough.*

9.1. Hölder continuity.

Lemma 4.1.3. **(Hölder estimates)** *Let* $\beta \in (1,2)$ *in (M2), consider* (4.1) *with* $\mathcal{F}(x,[u^\lambda]) = \mathcal{I}(x,u^\lambda,Du^\lambda)$. *Let* $\mu > 0$ *and* $u^\lambda \in C(I\!R^N) \cap \mathcal{E}_\mu(I\!R^N)$ *be a solution of* (4.1). *Assume that* (4.3), (4.4) *and* (4.6) *hold. Then for all* $0 < \tau < 1$, *there exists a constant* $C = C_\tau > 0$ *independent of* λ *such that*

$$|u^\lambda(x) - u^\lambda(y)| \le C|x-y|^\tau(\phi(x)+\phi(y)), \quad x,y \in I\!R^N,\ \lambda \in (0,1). \tag{4.37}$$

Remark 4.1.1. For $\beta \in (0,1]$, the ellipticity combined with the Ornstein-Uhlenbeck operator seems not powerful enough to control bad terms of the equation as in the local case. Therefore, in this case, we should need an additional condition on the size of α, see Theorem 4.1.2.

Proof of Lemma 4.1.3. The beginning of the proof follows the lines of Step 1 to Step 7. We only need to construct a suitable concave increasing function (different from the one of Step 8.3) to get a contradiction in (4.16).

9.1.1. Construction of the concave test-function ψ. For $r_0, C_2 > 0$ to be fixed later, let $\tau \in (0,1)$ and define the C^2 concave increasing function $\psi : [0,\infty) \to [0,\infty)$ as follows:

$$\psi(r) = 1 - e^{-C_2 r^\tau} \quad \text{for } r \in [0, r_0], \tag{4.38}$$

$\psi(r)$ is linear on $[r_0+1, +\infty)$ with derivative $\psi'(r) = C_2\tau(r_0+1)^{\tau-1}e^{-C_2(r_0+1)^\tau}$, and ψ is extended in a smooth way on $[r_0, r_0+1]$ such that, for all $r \ge 0$,

$$\psi'(r) \ge \psi'_{\min} := C_2\tau(r_0+1)^{\tau-1}e^{-C_2(r_0+1)^\tau}. \tag{4.39}$$

9.1.2. Global estimate to get a contradiction in (4.16). As in Step 8 of the Theorem, to estimate and reach a contradiction for (4.16) we need to separate the proof into two cases. For $r = |x-y|$ small we use the ellipticity coming from the nonlocal operator to control the other terms and for $r = |x-y|$ big enough, we can take the benefit of the Ornstein-Uhlenbeck term to control everything.

9.1.3. Reach a contradiction in (4.16) *when* $|x-y| = r \ge r_0$ *for a suitable choice of* r_0. We first take $a_0 = r_0$ in Proposition 4.1.1 and use (4.35) to estimate the difference of nonlocal terms in (4.16).

Then, plugging (4.17), (4.18), (4.19), (4.20) and (4.35) into (4.16) we obtain

$$\begin{aligned}
& \lambda\sqrt{\delta} + \Phi(x,y)\{\alpha\psi'(r)r - 2C_H\psi'(r)\} - 2C_H \\
& + C_1(\psi(r)+\delta)\left(-\mathcal{L}_L[\phi](x,y) + \langle b(x), D\phi(x)\rangle + \langle b(y), D\phi(y)\rangle\right) \\
\le\ & C_f(\phi(x)+\phi(y))r,
\end{aligned}$$

where $\mathcal{L}_L$ is the operator introduced by (4.21) with $L(x,y) = C_H$ in the case of $\mathcal{F}$ is the nonlocal operator in (2.27).

We fix all of constants in the same way that we did in Step 8 of the Theorem. More precisely, we fix

$$\begin{aligned} r_0 &= \max\{\frac{3(2C_H + \mu\hat{C}(\nu))}{\alpha}; 2R\}, \\ C_1 &= \frac{3C_f}{\alpha\psi'_{\min}} + 1, &(4.40) \\ A &\geq \frac{2C_H}{C_f r_0}, &(4.41) \end{aligned}$$

where R is a constant coming from (4.23).

We use these choices of constants and the same arguments as those of Step 8.5 we get that (4.16) leads to a contradiction.

9.1.4. Reach a contradiction in (4.16) *when* $|x-y| = r \leq r_0$. In this case, we use the construction of ψ in (4.38) applying to Proposition 4.1.1 (4.36) to estimate the difference of the two nonlocal terms in (4.16). This estimate is presented in the following lemma the proof of which is given in Section 4.3.3.

Lemma 4.1.4. *Under the assumptions of Proposition 4.1.1, let ψ be a concave function defined by* (4.38)*. Then for $0 < |x-y| = r \leq r_0$, there exist constants $C^1_\nu, C(\nu,\tau) > 0$ such that*

$$\begin{aligned} \mathcal{I}(x,u,D_x\varphi) - \mathcal{I}(y,u,-D_y\varphi) \;\leq\; & C_1(\psi(r)+\delta)\left(\mathcal{I}(x,\phi,D\phi) + \mathcal{I}(y,\phi,D\phi)\right) &(4.42) \\ & +\Phi(x,y)\left(\mu C^1_\nu - C(\nu,\tau)(1+C_2 r^\tau) r^{1-\beta}\right)\psi'(r). \end{aligned}$$

Plugging (4.17), (4.18), (4.19), (4.20) and Lemma 4.1.4 (4.42) into (4.16) and introducing $\mathcal{L}_L$ given by (4.21) with $L(x,y) = C_H$ and $\mathcal{F} = \mathcal{I}$, we obtain

$$\begin{aligned} & \lambda\sqrt{\delta} + C_1(\phi(x)+\phi(y)+A)\{C(\nu,\tau)\psi'(r)(1+C_2r^\tau)r^{1-\beta} + \alpha\psi'(r)r\} &(4.43) \\ \leq\; & C_1(\phi(x)+\phi(y)+A)\{2C_H + \mu C^1_\nu\}\psi'(r) \\ & +C_1(\psi(r)+\delta)(\mathcal{L}_L[\phi](x,y) - \langle b(x), D\phi(x)\rangle - \langle b(y), D\phi(y)\rangle) \\ & +2C_H + C_f(\phi(x)+\phi(y))r. \end{aligned}$$

Since both $|x|$ and $|y|$ may be smaller than R, we cannot estimate $\mathcal{L}_L[\phi](x,y) - \langle b(x), D\phi(x)\rangle - \langle b(y), D\phi(y)\rangle$ from above in a better way than $2K$. Taking into account (4.28) and setting $C(H,\nu,\mu) = 2C_H + \mu C^1_\nu$, inequality (4.43) reduces to

$$\begin{aligned} & \lambda\sqrt{\delta} + C_1(\phi(x)+\phi(y)+A)C(\nu,\tau)\psi'(r)(1+C_2r^\tau)r^{1-\beta} \\ \leq\; & C_1(\phi(x)+\phi(y)+A)C(H,\nu,\mu)\psi'(r) + C_1(\psi(r)+\delta)2K + 2C_H. \end{aligned}$$

Since $\tau < 1 < \beta$ then $\tau - \beta < 0$ and $r \leq r_0$, using that $\psi(r) \leq 1$ and $C_1 \geq 1$, we then increase A from (4.41) in order that

$$\frac{1}{2}AC(\nu,\tau)\psi'(r)r^{1-\beta} = \frac{1}{2}AC(\nu,\tau)C_2\tau r^{\tau-\beta}e^{-C_2r^\tau} \geq 2(K+C_H) \geq 2K\psi(r) + \frac{2C_H}{C_1},$$

which leads to the choice

$$A \geq \max\left\{\frac{2C_H}{C_f r_0}, \frac{4(K+C_H)r_0^{\beta-\tau}e^{C_2 r_0^\tau}}{C(\nu,\tau)C_2\tau}\right\}. \tag{4.44}$$

Therefore, inequality (4.43) now becomes

$$\lambda\sqrt{\delta} + \frac{1}{2}\Phi(x,y)C(\nu,\tau)\psi'(r)(1+C_2r^\tau)r^{1-\beta} \leq C(H,\nu,\mu)\psi'(r)\Phi(x,y) + 2C_1\delta K. \tag{4.45}$$

The rest of the proof is only to fix parameters in order to reach a contradiction in (4.45). It is now played with the main role of the ellipticity.

Recalling that $\beta > 1$, we fix

$$r_s = \min\left\{\left(\frac{C(\nu,\tau)}{2C(H,\nu,\mu)}\right)^{\frac{1}{\beta-1}}; r_0\right\}; \qquad C_2 = \frac{2C(H,\nu,\mu)}{C(\nu,\tau)}\max\{r_s^{\beta-1-\tau}; r_0^{\beta-1-\tau}\}.$$

If $r \leq r_s$ then from the choice of r_s, we have

$$\frac{1}{2}C(\nu,\tau)(1+C_2r^\tau)r^{1-\beta} \geq \frac{1}{2}C(\nu,\tau)r^{1-\beta} \geq \frac{1}{2}C(\nu,\tau)r_s^{1-\beta} \geq C(H,\nu,\mu).$$

If $r_s \leq r \leq r_0$, we consider two cases

- If $1+\tau-\beta \geq 0$, because of the choice of C_2 from above we have

$$\frac{1}{2}C(\nu,\tau)(1+C_2r^\tau)r^{1-\beta} \geq \frac{1}{2}C(\nu,\tau)C_2r^{1+\tau-\beta} \geq \frac{1}{2}C(\nu,\tau)C_2r_s^{1+\tau-\beta} \geq C(H,\nu,\mu).$$

- If $1+\tau-\beta \leq 0$, because of the choice of C_2 from above we have

$$\frac{1}{2}C(\nu,\tau)(1+C_2r^\tau)r^{1-\beta} \geq \frac{1}{2}C(\nu,\tau)C_2r^{1+\tau-\beta} \geq \frac{1}{2}C(\nu,\tau)C_2r_0^{1+\tau-\beta} \geq C(H,\nu,\mu).$$

Therefore, in any case, due to the choice of the constants, inequality (4.43) reduces to

$$\lambda\sqrt{\delta} \leq 2C_1\delta K.$$

This is not possible for δ small enough. Then we get a contradiction in (4.16).

9.1.5. Conclusion. For every $x, y \in \mathbb{R}^N$, we have proved that

$$|u(x)-u(y)| \leq \sqrt{\delta} + (\psi(|x-y|)+\delta)C_1(\phi(x)+\phi(y)+A).$$

Since A, ψ do not depend on δ, letting $\delta \to 0$ and recalling that $\psi(r) = 1-e^{-C_2r^\tau} \leq C_2r^\tau$, for $\tau \in (0,1)$. Hence we get

$$|u(x)-u(y)| \leq C_1C_2|x-y|^\tau(\phi(x)+\phi(y)+A) = C_1C_2(1+A)|x-y|^\tau(\phi(x)+\phi(y))$$

and finally, (4.37) holds with $C = C_1C_2(1+A)$. □

9.2. Improvement of $\tau-$Holder continuity to Lipschitz continuity.

9.2.1. Construction of the concave test-function ψ. Let $\theta \in (0, \frac{\beta-1}{N+2-\beta})$, $\varrho = \varrho(\theta)$ be a constant such that $\varrho > \frac{2^{1-\theta}}{\theta}$, $\epsilon = \epsilon(\theta) > 0$ small such that

$$\epsilon = \frac{1}{2}\left(\frac{1}{2\varrho(1+\theta)}\right)^{\frac{1}{\theta}}.$$

Let $r_0 = r_0(\theta) > 0$ be a small constant to be fixed later such that

$$r_0 \leq \frac{1}{2}\left(\frac{1}{2\varrho(1+\theta)}\right)^{\frac{1}{\theta}}. \tag{4.46}$$

We then define a C^2 concave increasing function $\psi : [0, +\infty) \to [0, \infty)$ such that

$$\psi(r) = r - \varrho r^{1+\theta} \quad \text{for } r \in [0, r_0], \tag{4.47}$$

$\psi(r)$ is linear on $[r_0 + \epsilon, +\infty)$ with derivative $\psi'(r) = 1 - \varrho(1+\theta)(r_0+\epsilon)^\theta$, and ψ is extended in a smooth way on $[r_0, r_0+\epsilon]$ such that, for all $r \geq 0$,

$$\frac{1}{2} \leq \psi'_{\min} := 1 - \varrho(1+\theta)(r_0+\epsilon)^\theta \leq \psi'(r) \leq 1. \tag{4.48}$$

We continue the proof by giving it in two cases. For $|x-y| = r \geq r_0$, we use the $\tau-$Hölder continuity of u to get a contradiction directly without using the equation. For $|x-y| = r \leq r_0$ with r_0 small and fixed in (4.46), we use the benefit of the ellipticity which comes from the nonlocal operator combined with the Ornstein-Uhlenbeck term to get a contradiction.

9.2.2. Reach a contradiction when $r = |x-y| \geq r_0$. Recalling that for all $\delta > 0$ small we have

$$u(x) - u(y) > (\psi(r) + \delta)\Phi(x,y).$$

By the concavity of ψ and the $\tau-$Hölder continuity of u, we have

$$r\psi'(r)\Phi(x,y) \leq \psi(r)\Phi(x,y) < u(x) - u(y) \leq C_\tau r^\tau(\phi(x) + \phi(y)), \quad \tau \in (0,1).$$

Recalling that $\Phi(x,y) = C_1(\phi(x) + \phi(y) + A)$ and dividing the above inequalities by $\phi(x) + \phi(y) + A$, we have

$$C_1\psi'(r) < C_\tau r^{\tau-1} \leq C_\tau r_0^{\tau-1}, \quad \tau \in (0,1), \ r \geq r_0.$$

Moreover, from (4.48) we know that $\psi'(r) \geq \psi'_{\min} \geq \frac{1}{2}$. Thus we only need to fix

$$C_1 \geq 2C_\tau r_0^{\tau-1}. \tag{4.49}$$

Then we get a contradiction for all $r \geq r_0$.

9.2.3 Reach a contradiction when $r = |x-y| \leq r_0$. Using the concave test function defined in (4.47) to Proposition 4.1.1 we get the following estimate for the difference of two nonlocal terms in (4.16).

Lemma 4.1.5. *Under the assumptions of Proposition 4.1.1, let ψ be defined by (4.47), $a_0 = r_0$ and $r = |x-y| \leq r_0$. There exist $C(\nu), C^1_\nu > 0$ such that for $\Lambda = \Lambda(\nu) = C(\nu)[\varrho\theta 2^{\theta-1} - 1] > 0$ we have*

$$\begin{aligned} & \mathcal{I}(x,u,D_x\varphi) - \mathcal{I}(y,u,-D_y\varphi) \\ \leq\ & C_1(\psi(r)+\delta)\left(\mathcal{I}(x,\phi,D\phi)+\mathcal{I}(y,\phi,D\phi)\right) - \left(\Lambda r^{-\tilde{\theta}} - \mu C^1_\nu \psi'(r)\right)\Phi(x,y), \end{aligned} \tag{4.50}$$

where $\tilde{\theta} = \beta - 1 - \theta(N+2-\beta) > 0$.

The proof of this Lemma is given in Section 4.3.3.

Plugging (4.17), (4.18), (4.19), (4.20) and Lemma 4.1.5 (4.50) into (4.16), introducing $\mathcal{L}_L$ given by (4.21) with $L(x,y) = C_H$, $\mathcal{F}$ is the nonlocal operator in (2.27) and applying (4.23), we obtain

$$\begin{aligned} & \lambda\sqrt{\delta} + C_1(\phi(x)+\phi(y)+A)(\Lambda r^{-\tilde{\theta}} + \alpha\psi'(r)r) \\ \leq\ & C_1(\phi(x)+\phi(y)+A)(2C_H + \mu C^1_\nu)\psi'(r) \\ & + C_1(\psi(r)+\delta)2K + 2C_H + C_f(\phi(x)+\phi(y))r, \end{aligned} \tag{4.51}$$

where

$$\Lambda = C(\nu)(\varrho\theta 2^{\theta-1} - 1) > 0, \quad \tilde{\theta} = \beta - 1 - \theta(N+2-\beta) > 0.$$

We first increase C_1 in (4.49) as

$$C_1 = \max\{\frac{C_f}{\alpha\psi'_{\min}} + 1,\ 2C_\tau r_0^{\tau-1}\} \tag{4.52}$$

in order to get rid of the f-terms. Indeed, for every $r \in [0,+\infty)$,

$$\begin{aligned} C_1(\phi(x)+\phi(y)+A)\alpha\psi'(r)r &\geq \alpha C_1\psi'_{\min}(\phi(x)+\phi(y))r \\ &\geq C_f(\phi(x)+\phi(y))r. \end{aligned} \tag{4.53}$$

Then taking into account (4.53) we get

$$\begin{aligned} & \lambda\sqrt{\delta} + C_1(\phi(x)+\phi(y)+A)\Lambda r^{-\tilde{\theta}} \\ \leq\ & C_1(\phi(x)+\phi(y)+A)\bar{C}\psi'(r) + C_1(\psi(r)+\delta)2K + 2C_H, \end{aligned}$$

where $\bar{C} = 2C_H + \mu C^1_\nu$.

Moreover, since $\psi(r) \leq r \leq r_0$ and $C_1 \geq 1$, then we fix $A > 0$ in order that

$$\frac{1}{2}A\Lambda r^{-\tilde{\theta}} \geq \frac{1}{2}A\Lambda r_0^{-\tilde{\theta}} \geq 2(Kr_0 + C_H) \geq 2K\psi(r) + \frac{2C_H}{C_1},$$

which leads to the choice

$$A \geq \frac{4(Kr_0 + C_H)r_0^{\tilde{\theta}}}{\Lambda}.$$

Then, inequality (4.51) becomes

$$\begin{aligned} & \lambda\sqrt{\delta} + C_1(\phi(x)+\phi(y)+A)\frac{1}{2}\Lambda r^{-\tilde{\theta}} \\ \leq \quad & C_1(\phi(x)+\phi(y)+A)\bar{C}\psi'(r) + 2KC_1\delta. \end{aligned}$$

Now we fix r_0 satisfying (4.46) such that

$$r_0 = \min\{\left(\frac{\Lambda}{2\bar{C}}\right)^{\frac{1}{\theta}}, \ \frac{1}{2}\left(\frac{1}{2\varrho(1+\theta)}\right)^{\frac{1}{\theta}}\}.$$

Then with the definition of r_0 as above, we get rid of $\bar{C}\psi'(r)$ term.
Finally, inequality (4.51) becomes

$$0 < \lambda\sqrt{\delta} \leq 2C_1K\delta,$$

which is a contradiction for δ small enough.

9.2.4. Conclusion: We have proved that $M_{\delta,A,C_1} \leq 0$, for δ small enough. For every $x, y \in I\!R^N$,

$$|u(x)-u(y)| \leq \sqrt{\delta} + (\psi(|x-y|)+\delta)C_1(\phi(x)+\phi(y)+A).$$

Since A and ψ do not depend on δ, letting $\delta \to 0$ and using the fact that $\psi(r) \leq r$, we get

$$|u(x)-u(y)| \leq C_1|x-y|(\phi(x)+\phi(y)+A) \leq C_1(A+1)|x-y|(\phi(x)+\phi(y)).$$

Since $\phi \geq 1$, (4.7) holds with $C = C_1(A+1)$. This concludes Step 9 and the proof of Theorem 4.1.1 in the nonlocal case (ii). $\square$

4.1.2 Regularity of solutions for degenerate elliptic equations.

We now consider degenerate elliptic equations, i.e., (4.5) does not necessarily hold (for the local case) and β can be in the whole interval $(0,2)$ (for the nonlocal case). In this degenerate case, we need to strengthen the assumption on H to obtain the Lipschitz regularity results.

$$\begin{cases} \text{There exist } L_{1H}, L_{2H} > 0 \text{ such that for all } x, y, p, q \in I\!R^N \\ |H(x,p)-H(y,p)| \leq L_{1H}|x-y|(1+|p|), \\ |H(x,p)-H(x,q)| \leq L_{2H}|p-q|(1+|x|). \end{cases} \tag{4.54}$$

Theorem 4.1.2. *Let $\mu > 0$, $u^\lambda \in C(I\!R^N) \cap \mathcal{E}_\mu(I\!R^N)$ be a solution of (4.1). Assume that (4.3), (4.4) and (4.54) hold. If one of two following holds*

(i) $\mathcal{F}(x,[u^\lambda]) = \mathrm{tr}(A(x)D^2u^\lambda(x))$ *and (4.2) holds,*

(ii) $\mathcal{F}(x,[u^\lambda]) = \mathcal{I}(x,u^\lambda,Du^\lambda)$ *and (M1), (M2) hold with $\beta \in (0,2)$,*

then, there exists

$$C(\mathcal{F}, H) = \begin{cases} C(\sigma, H) \text{ in the local case } (i) \\ C(H) \text{ in the nonlocal case } (ii) \end{cases}$$

such that, for any $\alpha > C(\mathcal{F}, H)$, there exists a constant C independent of λ such that

$$|u^\lambda(x) - u^\lambda(y)| \le C|x-y|(\phi_\mu(x) + \phi_\mu(y)), \quad x, y \in I\!R^N,\ \lambda \in (0,1). \tag{4.55}$$

Remark 4.1.2. To obtain the local uniform Lipschitz estimate for the solutions of (4.1) in the degenerate case, we need an additional condition on α which is the "size" of the Ornstein-Uhlenbeck operator, see (4.3).

Proof. The beginning of the proof follows line to line from Step 1 to Step 6 excepting Step 5 (estimate for $H-$terms).

Recalling the viscosity inequality that we have to estimate in order to get a contradiction is

$$\begin{aligned} & \lambda(u(x) - u(y)) - (\mathcal{F}(x,[u]) - \mathcal{F}(y,[u])) \\ & +\langle b(x), D_x\varphi(x,y)\rangle - \langle b(y), -D_y\varphi(x,y)\rangle \\ & +H(x, D_x\varphi(x,y)) - H(y, -D_y\varphi(x,y)) \\ \le \ & f(x) - f(y), \end{aligned} \tag{4.56}$$

where φ is defined in (4.9), $D_x\varphi$ and $D_y\varphi$ are given by (4.11) and (4.12) respectively, $\mathcal{F}(x,[u]) = \operatorname{tr}(A(x)X)$ and $\mathcal{F}(y,[u]) = \operatorname{tr}(A(y)Y)$ in the local case and $\mathcal{F}(x,[u]) = \mathcal{I}(x,u,D_x\varphi)$ and $\mathcal{F}(y,[u]) = \mathcal{I}(y,u,-D_y\varphi)$ in the nonlocal one.

Since we are doing the proof for degenerate equation, we do not need to construct very complicated concave test functions as the one we built in Theorem 4.1.1 in order to get the ellipticity. We only need to take

$$\psi(r) = r, \quad \forall r \in [0,\infty) \tag{4.57}$$

in both local and nonlocal case.

Now using (4.11), (4.12) and (4.54) to estimate for H-terms in (4.56), we have

$$\begin{aligned} & H(x, D_x\varphi) - H(y, -D_y\varphi) \\ = \ & H(x, D_x\varphi) - H(x, -D_y\varphi) + H(x, -D_y\varphi) - H(y, -D_y\varphi) \\ \ge \ & -C_1L_{2H}(\psi(|x-y|)+\delta)(|D\phi(x)| + |D\phi(y)|)(1+|x|) \\ & -L_{1H}|x-y| - C_1L_{1H}|x-y|(\psi(|x-y|)+\delta)|D\phi(y)| \\ & -L_{1H}|x-y|\psi'(|x-y|)\Phi(x,y) \\ \ge \ & -C_1L_H(\psi(|x-y|)+\delta)(|D\phi(x)| + |D\phi(y)|)(1+|x|+|x-y|) \\ & -L_{1H}|x-y| - L_{1H}|x-y|\psi'(|x-y|)\Phi(x,y), \end{aligned} \tag{4.58}$$

here $L_H = \max\{L_{1H}, L_{2H}\}$.

1. Proof in the case (i) We first estimate the $\mathcal{F}$ term. Recalling that $\mathcal{F}(x,[u]) = \operatorname{tr}(A(x)D^2u(x))$, using assumption (4.2) and applying Lemma 4.1.2 we get

$$\begin{aligned} & -\operatorname{tr}(A(x)X - A(y)Y) \\ \geq\ & -C_1(\psi(|x-y|)+\delta)\{\operatorname{tr}(A(x)D^2\phi(x)) + \operatorname{tr}(A(y)D^2\phi(y)) \\ & +\mathcal{C}_\sigma(|D\phi(x)| + |D\phi(y)|)\} - \mathcal{C}_\sigma|x-y|\psi'(|x-y|)\Phi(x,y) + O(\zeta), \end{aligned} \tag{4.59}$$

where $\mathcal{C}_\sigma$ is defined in (4.89).

We recall that the u, b and f terms can be estimated as (4.17), (4.18) and (4.20) respectively.

Plugging (4.17), (4.18), (4.20), (4.58) and (4.59) into (4.56), letting $\zeta \to 0$ and using (4.57), we obtain

$$\begin{aligned} & \lambda\sqrt{\delta} + C_1\alpha(\phi(x)+\phi(y)+A)|x-y| \\ \leq\ & C_1(\mathcal{C}_\sigma + L_{1H})(\phi(x)+\phi(y)+A)|x-y| \\ & +C_1(|x-y|+\delta)\left(\mathcal{L}_L[\phi](x,y) - \langle b(x), D\phi(x)\rangle - \langle b(y), D\phi(y)\rangle\right) \\ & +L_{1H}|x-y| + C_f(\phi(x)+\phi(y))|x-y|. \end{aligned} \tag{4.60}$$

where $\mathcal{L}_L$ is introduced in (4.21) with $L(x,y) = 2(\mathcal{C}_\sigma + L_H)(1+|x|+|y|)$ and $\mathcal{F}$ is the local operator in (2.27).

Taking $\alpha > C(\sigma, H) := 4(\mathcal{C}_\sigma + L_H)$, applying Lemma 4.1.1 (4.22) and since $L_{1H} \leq L_H$, inequality (4.60) reduces to

$$\begin{aligned} & \lambda\sqrt{\delta} + C_1\alpha(\phi(x)+\phi(y)+A)|x-y| \\ \leq\ & C_1(\mathcal{C}_\sigma + L_H)(\phi(x)+\phi(y)+A)|x-y| \\ & +C_1(|x-y|+\delta)2K + L_H|x-y| + C_f(\phi(x)+\phi(y))|x-y|. \end{aligned}$$

Now we fix

$$C_1 \geq \frac{4C_f}{3\alpha} + 1; \qquad A \geq \frac{4}{3\alpha}(K + L_H). \tag{4.61}$$

By these choices and noticing that $\psi' = 1$, $C_1 \geq 1$, we obtain

$$\begin{aligned} & \frac{3}{4}C_1(\phi(x)+\phi(y)+A)\alpha|x-y| \\ =\ & \frac{3}{4}C_1(\phi(x)+\phi(y))\alpha|x-y| + \frac{3}{4}C_1A\alpha|x-y| \\ \geq\ & C_f(\phi(x)+\phi(y))|x-y| + C_1K|x-y| + L_H|x-y|. \end{aligned} \tag{4.62}$$

Taking into account (4.62) and noticing that $\alpha > 4(\mathcal{C}_\sigma + L_H)$, inequality (4.60) now becomes

$$\lambda\sqrt{\delta} \leq 2C_1K\delta.$$

This is not possible for δ small enough, hence we reach a contradiction.

2. Proof in the case (ii) When $\mathcal{F} = \mathcal{I}$ we use Proposition 4.1.1 (4.35) to estimate the difference of two nonlocal terms in (4.56).

Plugging (4.17), (4.18), (4.20), (4.35) and (4.58) into (4.56) and using (4.57) we obtain

$$\begin{aligned} & \lambda\sqrt{\delta} + C_1\alpha(\phi(x) + \phi(y) + A)|x-y| \\ \leq\ & C_1 L_{1H}(\phi(x) + \phi(y) + A)|x-y| \\ & + C_1(|x-y| + \delta)\left(\mathcal{L}_L[\phi](x,y) - \langle b(x), D\phi(x)\rangle - \langle b(y), D\phi(y)\rangle\right) \\ & + C_H|x-y| + C_f(\phi(x) + \phi(y))|x-y|. \end{aligned} \tag{4.63}$$

where $\mathcal{L}_L$ is introduced in (4.21) with $\mathcal{F}$ is the nonlocal operator in (2.27) and $L(x,y) = 2L_H(1 + |x| + |y|)$

Taking $\alpha > 4L_H$ and using the same arguments as in the local case (i) we get also a contradiction.

3. Conclusion. For any

$$\alpha > C(\mathcal{F}, H) = \begin{cases} C(\sigma, H) \text{ in the local case } (i) \\ C(H) \text{ in the nonlocal case } (ii), \end{cases}$$

we have proved that $M \leq 0$ in both the local and the nonlocal case. It follows that, for every $x, y \in I\!R^N$,

$$|u(x) - u(y)| \leq \sqrt{\delta} + (\psi(|x-y|) + \delta)C_1(\phi(x) + \phi(y) + A).$$

Since A, ψ do not depend on δ, letting $\delta \to 0$ we get

$$|u(x) - u(y)| \leq C_1|x-y|(\phi(x) + \phi(y) + A) = C_1(A+1)|x-y|(\phi(x) + \phi(y)).$$

Here ϕ stands for ϕ_μ and since $\phi \geq 1$, (4.55) holds with $C = C_1(A+1)$. $\square$

Remark 4.1.3. If σ is a constant matrix, i.e. $L_\sigma = 0$ in (4.2) then (4.59) reduces to

$$\begin{aligned} & -\mathrm{tr}(A(x)X - A(y)Y) \\ \geq\ & -C_1(\psi(|x-y|) + \delta)(\mathrm{tr}(A(x)D^2\phi(x)) + \mathrm{tr}(A(y)D^2\phi(y))) + O(\zeta). \end{aligned} \tag{4.64}$$

Therefore using similarly arguments with the proof of the theorem we can prove that (4.55) holds for any $\alpha > C(H)$ (constant depends only on H).

Remark 4.1.4. The natural extension (regarding regularity with respect to x of H) is

$$\begin{cases} \text{There exist } L_{1H}, L_{2H} > 0 \text{ such that for all } x, y, p, q \in I\!R^N \\ |H(x,p) - H(y,p)| \leq L_{1H}|x-y|(1 + |p|), \\ |H(x,p) - H(x,q)| \leq L_{2H}|p-q|. \end{cases} \tag{4.65}$$

(i). Local case: If σ is a constant matrix, i.e. $L_\sigma = 0$ in (4.2) and $H(x,p) = H(p)$ is Lipschitz continuous ($L_{1H} = 0$ in (4.65)) (as in [42]), then $C(\sigma, H) = 0$ in the proof, meaning that Theorem 4.1.2 $(i.)$ holds with further assumption on the size of the Ornstein-Uhlenbeck operator. Then we can prove (4.55) for any $\alpha > 0$.

Indeed, using (4.64) instead of (4.59) to estimate the local terms. Moreover, using (4.65) to estimate the H-terms, (4.58) is replaced by

$$H(x, D_x\varphi) - H(y, -D_y\varphi) \geq -C_1 L_{2H}(\psi + \delta)(|D\phi(x)| + |D\phi(y)|). \tag{4.66}$$

Then replacing (4.59) and (4.58) by (4.64) and (4.66) respectively in (4.60) we obtain

$$\begin{array}{rl} & \lambda\sqrt{\delta} + C_1\alpha(\phi(x) + \phi(y) + A)|x-y| \\ \leq & C_1(|x-y| + \delta)\left(\mathrm{tr}(A(x)D^2\phi(x)) - \langle b(x), D\phi(x)\rangle + L_{2H}|D\phi(x)|\right) \\ & +C_1(|x-y| + \delta)\left(\mathrm{tr}(A(y)D^2\phi(y)) - \langle b(y), D\phi(y)\rangle + L_{2H}|D\phi(y)|\right) \\ & +C_f(\phi(x) + \phi(y))|x-y| + +O(\zeta). \end{array}$$

Sending $\zeta \to 0$, introducing $\mathcal{L}_L$ given by (4.21) with $L(x,y) = L_{2H}$ and applying (4.23) in the case of $\mathcal{F}$ is the local operator in (2.27), for any $\alpha > 0$ we get

$$\begin{array}{rl} & \lambda\sqrt{\delta} + C_1\alpha(\phi(x) + \phi(y) + A)|x-y| \\ \leq & C_1(|x-y| + \delta)2K + C_f(\phi(x) + \phi(y))|x-y|. \end{array} \tag{4.67}$$

We fix δ small enough such that $\lambda\sqrt{\delta} > 2C_1K\delta$ and

$$C_1 \geq \frac{C_f}{\alpha}; \quad A \geq \frac{2K}{\alpha}.$$

Therefore (4.67) can not hold and we conclude as in Step 3 for any $\alpha > 0$.

(ii). Nonlocal case: If we use (4.66) instead of (4.58) in the global estimate for nonlocal case (4.63), we get

$$\begin{array}{rl} & \lambda\sqrt{\delta} + C_1\alpha(\phi(x) + \phi(y) + A)|x-y| \\ \leq & C_1(|x-y| + \delta)\left(\mathcal{I}(x, \phi, D\phi) - \langle b(x), D\phi(x)\rangle + L_{2H}|D\phi(x)|\right) \\ & +C_1(|x-y| + \delta)\left(\mathcal{I}(y, \phi, D\phi) - \langle b(y), D\phi(y)\rangle + L_{2H}|D\phi(y)|\right) \\ & +C_f(\phi(x) + \phi(y))|x-y|. \end{array}$$

Then we argue as in the local case and have the same conclusion

Remark 4.1.5. If σ is any bounded lipschitz continuous symmetric matrix, i.e. (4.2) holds. Assume (4.65) with $L_{1H} = 0$, then the conclusion of Theorem 4.1.2 should be there exists $C(\mathcal{F}) = C(\sigma)$ (in the local case only) such that for any $\alpha > C(\mathcal{F})$, (4.55) holds. The condition on α here is to compensate the degeneracy only.

4.2 Regularity of the solutions of the evolution equation

In this section we establish local uniform (in time) Lipschitz estimates for the solutions of Cauchy problem

$$\begin{cases} u_t(x,t) - \mathcal{F}(x, [u]) + \langle b(x), Du(x,t)\rangle + H(x, Du(x,t)) = f(x) & \text{in } Q \\ u(\cdot, 0) = u_0 \quad \text{on } I\!R^N. \end{cases} \tag{4.68}$$

4.2.1 Regularity in the uniformly parabolic case

Theorem 4.2.1. *Let $u \in C(R_T) \cap \mathcal{E}_\mu(R_T)$ be a solution of (4.68). In addition to the hypotheses of Theorem 4.1.1, assume that*

$$|u_0(x) - u_0(y)| \leq C_0|x-y|(\phi_\mu(x) + \phi_\mu(y)) \text{ for } x, y \in I\!R^N. \tag{4.69}$$

Then there exists a constant $C > 0$ independent of T such that

$$|u(x,t) - u(y,t)| \leq C|x-y|(\phi_\mu(x) + \phi_\mu(y)) \text{ for } x, y \in I\!R^N,\ t \in [0,T). \tag{4.70}$$

Proof. We only give a sketch of proof since it is close to the proof of Theorem 4.1.1.

Let $\epsilon, \delta, A, C_1 > 0$ and a C^2 concave and increasing function $\psi : I\!R_+ \to I\!R_+$ with $\psi(0) = 0$ which is defined as in the proof of Theorem 4.1.1 depending on the local or nonlocal case such that

$$C_1\psi'_{\min} \geq C_0. \tag{4.71}$$

We consider

$$M = \sup_{(I\!R^N)^2 \times [0,T)} \left\{ u(x,t) - u(y,t) - (\psi(|x-y|) + \delta)\Phi(x,y) - \frac{\epsilon}{T-t} \right\}, \tag{4.72}$$

where $\Phi(x,y) = C_1(\phi(x) + \phi(y) + A)$, $\phi = \phi_\mu$ defined by (2.42). Set $\varphi(x,y,t) = (\psi(|x-y|) + \delta)\Phi(x,y) - \frac{\epsilon}{T-t}$.

If $M \leq 0$ for some good choice of A, C_1 ψ independent of $\delta, \epsilon > 0$, then we get some locally uniform estimates by letting $\delta, \epsilon \to 0$. So we argue by contradiction, assuming that $M > 0$.

Since $u \in \mathcal{E}_\mu(I\!R^N) \cap C(R_T)$, the supremum is achieved at some point (x,y,t) with $x \neq y$ thanks to $\delta, \epsilon > 0$ and the continuity of u.

Since $M > 0$, if $t = 0$, from (4.69) and by concavity of ψ, i.e., $\psi(|x-y|) \geq \psi'(|x-y|)|x-y| \geq \psi'_{\min}|x-y|$, we have

$$\begin{aligned}
0 < M &= u(x,0) - u(y,0) - C_1(\psi(|x-y|) + \delta)(\phi(x) + \phi(y) + A) - \frac{\epsilon}{T} \\
&\leq C_0|x-y|(\phi(x) + \phi(y)) - C_1\psi(|x-y|)(\phi(x) + \phi(y) + A) - \frac{\epsilon}{T} \\
&\leq C_0|x-y|(\phi(x) + \phi(y) + A) - C_1\psi'_{\min}|x-y|(\phi(x) + \phi(y) + A) - \frac{\epsilon}{T} \\
&\leq -\frac{\epsilon}{T}.
\end{aligned}$$

The last inequality is obtained thanks to (4.71). Therefore $t > 0$.

Now we can apply [30, Theorem 8.3] in the local case and [15, Corollary 2] in the nonlocal one to learn that, for any $\varrho > 0$, there exist $a, b \in I\!R$ and $X, Y \in \mathcal{S}^N$ such that

$$(a, D_x\varphi, X) \in \overline{\mathcal{P}}^{2,+}u(x,t); \qquad (b, -D_y\varphi, Y) \in \overline{\mathcal{P}}^{2,-}u(y,t),$$

$$a - b = \varphi_t(x,y,t) = \frac{\epsilon}{(T-t)^2} \geq \frac{\epsilon}{T^2}$$

and

$$\begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq A + \varrho A^2.$$

where $A = D^2\varphi(x,y,t)$ and $\varrho A^2 = O(\varrho)$ (ϱ will be sent to 0 first).

Writing the viscosity inequalities at (x,y,t) in the local and nonlocal case we have

$$\begin{aligned} &a - \mathcal{F}(x,[u]) + \langle b(x), D_x\varphi\rangle + H(x, D_x\varphi) \leq f(x) \\ &b - \mathcal{F}(y,[u]) + \langle b(y), -D_y\varphi\rangle + H(y, -D_y\varphi) \geq f(y), \end{aligned}$$

where $\mathcal{F}(x,[u]) = \mathrm{tr}(A(x)X)$ and $\mathcal{F}(y,[u]) = \mathrm{tr}(A(y)Y)$ in the local case and $\mathcal{F}(x,[u]) = \mathcal{I}(x,u,D_x\varphi)$ and $\mathcal{F}(y,[u]) = \mathcal{I}(y,u,-D_y\varphi)$ in the nonlocal one.

Subtracting the viscosity inequalities, we have

$$\begin{aligned} & \frac{\epsilon}{T^2} - (\mathcal{F}(x,[u]) - \mathcal{F}(y,[u])) + \langle b(x), D_x\varphi\rangle - \langle b(y), -D_y\varphi\rangle \\ & +H(x, D_x\varphi) - H(y, -D_y\varphi) \\ \leq\ & f(x) - f(y). \end{aligned} \tag{4.73}$$

All of the different terms in (4.73) are estimated as in the proof of Theorem 4.1.1. We only need to fix

$$\delta = \epsilon \min\{1, \frac{1}{3C_1KT^2}\}$$

small, then

$$\frac{\epsilon}{T^2} > 2C_1\delta K, \tag{4.74}$$

where $K > 0$ is a constant coming from (4.23). Therefore we reach a contradiction as in the proof of Theorem 4.1.1.

We have proved that $M \leq 0$ for δ small enough. It follows that, for every $x, y \in I\!R^N$, $t \in [0,T)$,

$$|u(x,t) - u(y,t)| \leq C_1(\psi(|x-y|) + \delta)(\phi(x) + \phi(y) + A) + \frac{\epsilon}{T-t}.$$

Since A, C_1, ψ do not depend on δ, ϵ, letting $\delta, \epsilon \to 0$ and using the fact that ψ is a concave increasing function, we have $\psi(r) \leq \psi'(0)r$. Finally, for all $x, y \in I\!R^N$, $t \in [0,T)$ we get

$$\begin{aligned} |u(x,t) - u(y,t)| &\leq C_1\psi'(0)|x-y|(\phi(x) + \phi(y) + A) \\ &\leq C_1\psi'(0)(A+1)|x-y|(\phi(x) + \phi(y)). \end{aligned}$$

Since $\phi \geq 1$, (4.70) holds with $C = C_1\psi'(0)(A+1)$. □

4.2.2 Regularity of solutions for degenerate parabolic equation

Theorem 4.2.2. *Let $u \in C(R_T) \cap \mathcal{E}_\mu(R_T)$ be a solution of (4.68). In addition to the hypotheses of Theorem 4.1.2, assume that*

$$|u_0(x) - u_0(y)| \leq C_0|x-y|(\phi_\mu(x) + \phi_\mu(y)) \text{ for } x, y \in I\!R^N.$$

There exists

$$C(\mathcal{F}, H) = \begin{cases} C(\sigma, H) \text{ in the local case,} \\ C(H) \text{ in the nonlocal case} \end{cases}$$

such that, for any $\alpha > C(\mathcal{F}, H)$, there exists a constant $C > 0$ independent of T such that

$$|u(x,t) - u(y,t)| \leq C|x-y|(\phi_\mu(x) + \phi_\mu(y)) \text{ for } x, y \in I\!R^N,\ t \in [0,T).$$

The proof of this Theorem is an adaptation of the one of Theorem 4.1.2 using the same extension to the parabolic case as explained in the proof of Theorem 4.2.1, so we omit here.

We have the same Remarks as presented for the elliptic equation.

4.3 Estimates for local, nonlocal operator and growth function.

4.3.1 Estimates for exponential growth function

Proof of Lemma 4.1.1 . *1. Estimates on ϕ.* Recalling that $\phi(x) = e^{\mu\sqrt{|x|^2+1}}$ and setting $\langle x \rangle = \sqrt{|x|^2+1}$, for $x \in I\!R^N$, we have

$$D\phi = \frac{\mu x}{\langle x \rangle}\phi(x), \quad D^2\phi = \frac{\mu\phi(x)}{\langle x \rangle}\Big[I - (1-\mu\langle x \rangle)\frac{x}{\langle x \rangle} \otimes \frac{x}{\langle x \rangle}\Big]. \tag{4.75}$$

2. Estimates on the Ornstein-Uhlenbeck operator. From (4.3) and (4.75) we have the estimate

$$\begin{aligned} \langle b(x), D\phi(x) \rangle &= \langle b(x) - b(0), D\phi(x) \rangle + \langle b(0), D\phi(x) \rangle \\ &= \langle b(x) - b(0), \mu\frac{x}{\langle x \rangle}\phi(x) \rangle + \langle b(0), \mu\frac{x}{\langle x \rangle}\phi(x) \rangle \\ &\geq \mu\frac{\phi(x)}{\langle x \rangle}[\alpha|x|^2 + \langle b(0), x \rangle] \\ &\geq \mu\phi(x)\left(\frac{\alpha|x|^2}{\langle x \rangle} - |b(0)|\right). \end{aligned} \tag{4.76}$$

3. Estimates for the local term. We compute that

$$
\begin{aligned}
\operatorname{tr}(A(x)D^2\phi(x)) &= \frac{\mu\phi(x)}{\langle x\rangle}\Big[\operatorname{tr}(A(x)) - (1-\mu\langle x\rangle)\operatorname{tr}\Big(A(x)\frac{x}{\langle x\rangle}\otimes\frac{x}{\langle x\rangle}\Big)\Big] \\
&\le \frac{\mu\phi(x)}{\langle x\rangle}\Big[|\sigma|^2 + \mu\langle x\rangle|\sigma|^2\Big] \\
&\le \mu\phi(x)|\sigma|^2\Big(1+\mu\Big) \qquad (4.77) \\
&= C(\mu,|\sigma|)\phi(x).
\end{aligned}
$$

4. Estimates for the nonlocal term. We have

$$
\begin{aligned}
&\mathcal{I}(x,\phi,D\phi) \qquad (4.78) \\
= &\int_B(\phi(x+z)-\phi(x)-D\phi(x)\cdot z)\nu(dz) + \int_{B^c}(\phi(x+z)-\phi(x))\nu(dz) \\
= &\int_B\int_0^1(1-s)\langle D^2\phi(x+sz)z,z\rangle ds\nu(dz) + \phi(x)\int_{B^c}\left(\frac{\phi(x+z)}{\phi(x)}-1\right)\nu(dz) \\
\le &\int_B\int_0^1|D^2\phi(x+sz)||z|^2 ds\nu(dz) + \phi(x)\int_{B^c}\left(\frac{\phi(x+z)}{\phi(x)}-1\right)\nu(dz)
\end{aligned}
$$

Recalling that $\langle x\rangle = \sqrt{|x|^2+1}$, from (4.75) we have

$$
\begin{aligned}
|D^2\phi(x)| &\le \frac{\mu\phi(x)}{\langle x\rangle}\Big[|I| + \Big|\frac{x}{\langle x\rangle}\otimes\frac{x}{\langle x\rangle}\Big| + \mu\langle x\rangle\Big|\frac{x}{\langle x\rangle}\otimes\frac{x}{\langle x\rangle}\Big|\Big] \qquad (4.79) \\
&\le \frac{\mu\phi(x)}{\langle x\rangle}\left(\sqrt{N} + \frac{|x|^2}{|x|^2+1} + \mu\frac{|x|^2}{\langle x\rangle}\right) \\
&\le \mu(\sqrt{N}+1+\mu)\phi(x) \\
&= C(\mu,N)\phi(x)
\end{aligned}
$$

On the other hand, for any a, b we have

$$1+(a+b)^2 \le 1+a^2+2\sqrt{1+a^2}\sqrt{1+b^2}+b^2 \le \left(\sqrt{1+a^2}+\sqrt{1+b^2}\right)^2.$$

This implies $\sqrt{1+(a+b)^2} \le \sqrt{1+a^2}+\sqrt{1+b^2}$ and therefore

$$\phi(x+z) = e^{\mu\sqrt{1+|x+z|^2}} \le \phi(x)\phi(z), \quad \forall x,z\in I\!R^N. \qquad (4.80)$$

Using (4.79) and (4.80) we have

$$
\begin{aligned}
|D^2\phi(x+sz)| &\le C(\mu,N)\phi(x+sz) \qquad \forall s\in[0,1] \\
&\le C(\mu,N)\phi(x)\phi(z).
\end{aligned}
$$

Then inequality (4.78) becomes

$$\mathcal{I}(x,\phi,D\phi) \;\le\; C(\mu,N)\phi(x)\int_B\phi(z)|z|^2\nu(dz) + \phi(x)\int_{B^c}(\phi(z)-1)\nu(dz).$$

Using $(M1)$, there exists a constant $C^1_\nu > 0$ such that

$$\mathcal{I}(x,\phi,D\phi) \leq C(\mu,N)C^1_\nu\phi(x) + C^1_\nu\phi(x) = C(\mu,\nu)\phi(x). \tag{4.81}$$

5. Estimate of Lemma and end the computations for both cases. Let $L>0$, $L(x,y) := L(1+|x|+|y|)$ and set

$$\mathcal{L}_L[\phi](x,y) := \mathcal{F}(x,[\phi]) + \mathcal{F}(y,[\phi]) + L(x,y)(|D\phi(x)|+|D\phi(y)|).$$

Set

$$C(\mathcal{F}) = \begin{cases} C(\mu,|\sigma|), & \text{if } \mathcal{F}(x,[\phi]) = \mathrm{tr}(\sigma(x)\sigma^T(x)D^2\phi(x)) \\ C(\mu,\nu), & \text{if } \mathcal{F}(x,[\phi]) = \mathcal{I}(x,\phi,D\phi). \end{cases}$$

Since $|D\phi(x)| \leq \mu\phi(x)$, from (4.76), (4.77) and (4.81) we have

$$\begin{aligned} & -\mathcal{L}_L[\phi](x,y) + \langle b(x), D\phi(x)\rangle + \langle b(y), D\phi(y)\rangle \\ = \; & -\mathcal{F}(x,[\phi]) + \langle b(x), D\phi(x)\rangle - L(1+|x|+|y|)|D\phi(x)| \\ & -\mathcal{F}(y,[\phi]) + \langle b(y), D\phi(y)\rangle - L(1+|x|+|y|)|D\phi(y)| \\ \geq \; & \mu\phi(x)\left(\alpha\frac{|x|^2}{\langle x\rangle} - |b(0)| - C(\mathcal{F}) - L - L|x| - L|y|\right), \\ & +\mu\phi(y)\left(\alpha\frac{|y|^2}{\langle y\rangle} - |b(0)| - C(\mathcal{F}) - L - L|x| - L|y|\right). \end{aligned} \tag{4.82}$$

We take $\alpha > 2L$, if

$$|x| \geq R_y := \frac{2}{\alpha - 2L}\left(\frac{2}{\mu} + |b(0)| + C(\mathcal{F}) + L + L|y|\right)$$

and

$$|y| \geq R_x := \frac{2}{\alpha - 2L}\left(\frac{2}{\mu} + |b(0)| + C(\mathcal{F}) + L + L|x|\right)$$

then $-\mathcal{L}_L[\phi](x,y) + \langle b(x), D\phi(x)\rangle + \langle b(y), D\phi(y)\rangle \geq 2\phi(x) + 2\phi(y)$. Setting

$$\sup_{x\in B(0,R_y)}\{\mu\phi(x)\left(-\alpha\frac{|x|^2}{\langle x\rangle} + |b(0)| + C(\mathcal{F}) + L + L|x| + L|y|\right)\} =: K_y,$$
$$\sup_{y\in B(0,R_x)}\{\mu\phi(y)\left(-\alpha\frac{|y|^2}{\langle y\rangle} + |b(0)| + C(\mathcal{F}) + L + L|x| + L|y|\right)\} =: K_x$$

we obtain that, for all $x, y \in I\!R^N$,

$$-\mathcal{L}_L[\phi](x,y) + \langle b(x), D\phi(x)\rangle + \langle b(y), D\phi(y)\rangle \geq \phi(x) + \phi(y) + \phi(x) - K_x + \phi(y) - K_y.$$

Since $\alpha > 2L$, we have $\sup_{I\!R^N}\{-\phi(x)+K_x\}, \sup_{I\!R^N}\{-\phi(y)+K_y\} < +\infty$. Hence, set

$$K := \sup\{\sup_{I\!R^N}\{-\phi(x)+K_x\}, \sup_{I\!R^N}\{-\phi(y)+K_y\}\}$$

we obtain

$$-\mathcal{L}_L[\phi](x,y) + \langle b(x), D\phi(x)\rangle + \langle b(y), D\phi(y)\rangle \geq \phi(x) + \phi(y) - 2K. \tag{4.83}$$

Suppose that $L(x,y) = L$ independent of x, y. Setting

$$R_\phi := \frac{2}{\alpha}\left(\frac{1}{\mu} + |b(0)| + C(\mathcal{F}) + L\right),$$

$$K := \sup_{x\in B(0,R_\phi)} \{\mu\phi(x)\left(-\alpha\frac{|x|^2}{\langle x\rangle} + |b(0)| + C(\mathcal{F}) + L\right)\}$$

and using the same arguments as above we obtain that (4.83) holds for any $\alpha > 0$.

Since $\phi(x) \to +\infty$ as $x \to +\infty$, there exists $R \geq R_\phi$ such that

$$-\mathcal{F}(x,[\phi]) + \langle b(x), D\phi(x)\rangle \geq \phi(x) - K \leq \begin{cases} -K & \text{for } |x| \leq R, \\ K & \text{for } |x| \geq R. \end{cases}$$

Notice that K and R depend only on $\mathcal{F}, b, L, \mu$. □

4.3.2 Estimates for local operator

Proof of Lemma 4.1.2. *1. Using the matrix inequality* (4.15). From (4.15), setting

$$X_x = X - C_1(\psi+\delta)D^2\phi(x) \quad \text{and} \quad Y_y = Y + C_1(\psi+\delta)D^2\phi(y). \tag{4.84}$$

We have, for every $\zeta, \xi \in I\!R^N$,

$$\begin{aligned}\langle X_x\zeta,\zeta\rangle - \langle Y_y\xi,\xi\rangle + O(\varrho) \quad \leq \quad & \psi''\Phi\langle\zeta-\xi, p\otimes p(\zeta-\xi)\rangle + \psi'\Phi\langle\zeta-\xi, \mathcal{C}(\zeta-\xi)\rangle \\ & +C_1\psi'[\langle p\otimes D\phi(x) + D\phi(x)\otimes p\zeta,\zeta\rangle \\ & +\langle(p\otimes D\phi(y) - D\phi(x)\otimes p)\xi,\zeta\rangle \\ & +\langle(D\phi(y)\otimes p - p\otimes(D\phi(x))\zeta,\xi\rangle \\ & -\langle(p\otimes D\phi(y) + D\phi(y)\otimes p)\xi,\xi\rangle],\end{aligned}$$

where p and $\mathcal{C}$ are given by (4.10).

2. Computing the trace with suitable orthonormal basis. Following Ishii-Lions and Barles, we choose an orthonormal basis $(e_i)_{1\leq i\leq N}$ to compute $\mathrm{tr}(\sigma_x\sigma_x^T X)$ and another one, $(\tilde{e}_i)_{1\leq i\leq N}$ to compute $\mathrm{tr}(\sigma_x\sigma_x^T Y)$.

Now we estimate $\mathrm{tr}(A(x)X_x - A(y)Y_y)$ with $A = \sigma\sigma^T$ in the following way:

$$\begin{aligned}
\mathrm{tr}(A(x)X_x - A(y)Y_y) &= \sum_{i=1}^{N}\langle X_x\sigma_x e_i, \sigma_x e_i\rangle - \langle Y_y\sigma_y\tilde{e}_i, \sigma_y\tilde{e}_i\rangle \\
&\leq \sum_{i=1}^{N}\psi''\Phi\langle p\otimes pQ_i, Q_i\rangle + \psi'\Phi\langle \mathcal{C}Q_i, Q_i\rangle \\
&\quad +C_1\psi'[\langle p\otimes D\phi(x)\sigma_x e_i, Q_i\rangle + \langle D\phi(x)\otimes pQ_i, \sigma_x e_i\rangle \\
&\quad +\langle p\otimes D\phi(y)\sigma_y\tilde{e}_i, Q_i\rangle + \langle D\phi(y)\otimes pQ_i, \sigma_y\tilde{e}_i\rangle] + O(\varrho) \\
&\leq \psi''\Phi\langle p, Q_1\rangle^2 + \sum_{i=1}^{N}\psi'\Phi\langle \mathcal{C}Q_i, Q_i\rangle \\
&\quad +C_1\psi'[\langle p\otimes D\phi(x)\sigma_x e_i, Q_i\rangle + \langle D\phi(x)\otimes pQ_i, \sigma_x e_i\rangle \\
&\quad +\langle p\otimes D\phi(y)\sigma_y\tilde{e}_i, Q_i\rangle + \langle D\phi(y)\otimes pQ_i, \sigma_y\tilde{e}_i\rangle] + O(\varrho).
\end{aligned}$$

where we set $Q_i = \sigma_x e_i - \sigma_y\tilde{e}_i$ and noticing that $\psi''\Phi\langle p\otimes pQ_i, Q_i\rangle = \psi''\Phi\langle p, Q_i\rangle^2 \leq 0$ since ψ is concave function.

We now set up suitable basis in two following cases.

2.1. Estimates for the trace when σ is degenerate, i.e., (4.2) holds only. We choose any orthonormal basis such that $e_i = \tilde{e}_i$. It follows

$$\begin{aligned}
&\mathrm{tr}(A(x)X_x - A(y)Y_y) \\
\leq\ & \sum_{i=1}^{N}\psi'\Phi\langle \mathcal{C}(\sigma_x-\sigma_y)e_i, (\sigma_x-\sigma_y)e_i\rangle \\
&+C_1\psi'[\langle p\otimes D\phi(x)\sigma_x e_i, (\sigma_x-\sigma_y)e_i\rangle + \langle D\phi(x)\otimes p(\sigma_x-\sigma_y)e_i, \sigma_x e_i\rangle \\
&+\langle p\otimes D\phi(y)\sigma_y e_i, (\sigma_x-\sigma_y)e_i\rangle + \langle D\phi(y)\otimes p(\sigma_x-\sigma_y)e_i, \sigma_y e_i\rangle] + O(\varrho) \\
\leq\ & N\psi'\Phi|\sigma_x-\sigma_y|^2|\mathcal{C}| + NC_1\psi'[|p\otimes D\phi(x)||\sigma_x||\sigma_x-\sigma_y| + |D\phi(x)\otimes p||\sigma_x-\sigma_y||\sigma_x| \\
&+|p\otimes D\phi(y)||\sigma_y||\sigma_x-\sigma_y| + |D\phi(y)\otimes p||\sigma_x-\sigma_y||\sigma_y|] + O(\varrho).
\end{aligned}$$

By (4.10) we first have $|\mathcal{C}| \leq 1/|x-y|$, and $|D\phi\otimes p|, |p\otimes D\phi| \leq |D\phi|$. Using the fact that σ is a Lipschitz and bounded function, i.e., (4.2) holds, we obtain

$$\mathrm{tr}(A(x)X_x - A(y)Y_y) \leq NL_\sigma^2|x-y|\psi'\Phi + 2NC_\sigma L_\sigma C_1(|D\phi(x)| + |D\phi(y)|)|x-y|\psi'.$$

By the concavity of ψ, i.e., $\psi'(|x-y|)|x-y| \leq \psi(|x-y|)$ and from (4.84), we get

$$\begin{aligned}
&-\mathrm{tr}(A(x)X - A(y)Y) \\
\geq\ & -\tilde{C}_\sigma|x-y|\psi'\Phi - C_1(\psi+\delta)[\mathrm{tr}(A(x)D^2\phi(x)) + \mathrm{tr}(A(y)D^2\phi(y)) \\
&\quad +\tilde{C}_\sigma(|D\phi(x)| + |D\phi(y)|)] + O(\varrho),
\end{aligned} \tag{4.85}$$

where

$$\tilde{C}_\sigma = \max\{NL_\sigma^2, 2NC_\sigma L_\sigma\}. \tag{4.86}$$

2.2. More precise estimate when σ is uniformly ellipticity, i.e., (4.5) holds. Since σ is uniformly invertible, we can choose

$$e_1 = \frac{\sigma_x^{-1}p}{|\sigma_x^{-1}p|}\ , \tilde{e}_1 = -\frac{\sigma_y^{-1}p}{|\sigma_y^{-1}p|}$$

If e_1 and $\tilde{e_1}$ are collinear, then we can complete the basis with orthonormal unit vectors $e_i = \tilde{e}_i \in e_1^{\perp},\ 2 \le i \le N$. Otherwise, in the plane span $\{e_1, \tilde{e}_1\}$, we consider a rotation $\mathcal{R}$ of angle $\frac{\pi}{2}$ and define

$$e_2 = \mathcal{R}e_1,\ \ \tilde{e}_2 = -\mathcal{R}\tilde{e}_1.$$

Finally, noticing that span $\{e_1, e_2\}^{\perp} = \text{span}\{\tilde{e}_1, \tilde{e}_2\}^{\perp}$, we can complete the orthonormal bases with unit vectors $e_i = \tilde{e}_i \in \text{span}\{e_1, e_2\}^{\perp},\ 3 \le i \le N$.

For $i = 1$, we compute

$$Q_1 = \sigma_x e_1 - \sigma_y \tilde{e}_1 = [\frac{1}{|\sigma_x^{-1}p|} + \frac{1}{|\sigma_y^{-1}p|}]p$$

This implies

$$|\langle p, Q_1\rangle| = |\langle [\frac{1}{|\sigma_x^{-1}p|} + \frac{1}{|\sigma_y^{-1}p|}]p, p\rangle| = \frac{1}{|\sigma_x^{-1}p|} + \frac{1}{|\sigma_y^{-1}p|}$$

We have

$$|\sigma^{-1}p|^2 = \langle \sigma^{-1}p, \sigma^{-1}p\rangle = \langle (\sigma^{-1})^T\sigma^{-1}p, p\rangle = \langle (\sigma\sigma^T)^{-1}p, p\rangle = \langle A^{-1}p, p\rangle \le |A^{-1}p|.$$

From (4.5) , we have $\langle Ap, p\rangle \ge \rho|p|^2 \Rightarrow \langle AA^{-1}p, A^{-1}p\rangle \ge \rho|A^{-1}p|^2 \Rightarrow \langle p, A^{-1}p\rangle \ge \rho|A^{-1}p|^2$.

This implies $|A^{-1}p| \le \frac{1}{\rho}$, so $|\langle p, Q_1\rangle| \ge 2\sqrt{\rho}$ and notice that $|Q_1| - |\langle p, Q_1\rangle| = 0$.

Moreover,

$$\begin{aligned}
\langle p \otimes D\phi(x)\sigma_x e_1, Q_1\rangle &= \frac{1}{|\sigma_x^{-1}p|}[\frac{1}{|\sigma_x^{-1}p|} + \frac{1}{|\sigma_y^{-1}p|}]\langle p \otimes D\phi(x)p, p\rangle \\
&= \frac{1}{|\sigma_x^{-1}p|}[\frac{1}{|\sigma_x^{-1}p|} + \frac{1}{|\sigma_y^{-1}p|}]\langle D\phi(x), p\rangle
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle D\phi(x) \otimes pQ_1, \sigma_x e_1\rangle &= \frac{1}{|\sigma_x^{-1}p|}[\frac{1}{|\sigma_x^{-1}p|} + \frac{1}{|\sigma_y^{-1}p|}]\langle D\phi(x), p\rangle \\
\langle p \otimes D\phi(y)\sigma_y \tilde{e}_1, Q_1\rangle &= -\frac{1}{|\sigma_y^{-1}p|}[\frac{1}{|\sigma_x^{-1}p|} + \frac{1}{|\sigma_y^{-1}p|}]\langle D\phi(y), p\rangle \\
\langle D\phi(y) \otimes pQ_1, \sigma_y \tilde{e}_1\rangle &= -\frac{1}{|\sigma_y^{-1}p|}[\frac{1}{|\sigma_x^{-1}p|} + \frac{1}{|\sigma_y^{-1}p|}]\langle D\phi(y), p\rangle
\end{aligned}$$

Since

$$C_\sigma|\sigma_x^{-1}p| \geq |\sigma_x||\sigma_x^{-1}p| = |\sigma_x\sigma_x^{-1}p| = 1 \Rightarrow \frac{1}{|\sigma_x^{-1}p|} \leq C_\sigma$$

So

$$\begin{aligned}
&\langle p\otimes D\phi(x)\sigma_x e_1, Q_1\rangle + \langle D\phi(x)\otimes pQ_1, \sigma_x e_1\rangle + \langle p\otimes D\phi(y)\sigma_y\tilde{e}_1, Q_1\rangle \\
&+\langle D\phi(y)\otimes pQ_1, \sigma_y\tilde{e}_1\rangle \\
=\ & \frac{2}{|\sigma_x^{-1}p|}[\frac{1}{|\sigma_x^{-1}p|}+\frac{1}{|\sigma_y^{-1}p|}]\langle D\phi(x), p\rangle - \frac{2}{|\sigma_y^{-1}p|}[\frac{1}{|\sigma_x^{-1}p|}+\frac{1}{|\sigma_y^{-1}p|}]\langle D\phi(y), p\rangle \\
\leq\ & 4C_\sigma^2(|\langle D\phi(x), p\rangle| + |\langle D\phi(y), p\rangle|) \\
\leq\ & 4C_\sigma^2(|D\phi(x)| + |D\phi(y)|)
\end{aligned}$$

Next, using the concavity of ψ and the above estimates, we obtain

$$\begin{aligned}
\operatorname{tr}(A(x)X_x - A(y)Y_y) \ \leq\ & 4\rho\psi''\Phi + 4C_1C_\sigma^2\psi'(|D\phi(x)|+|D\phi(y)|) + \sum_{i=2}^{N}\frac{\psi'\Phi}{|x-y|}|Q_i|^2 \\
&+C_1\psi'[\langle p\otimes D\phi(x)\sigma_x e_i, Q_i\rangle + \langle D\phi(x)\otimes pQ_i, \sigma_x e_i\rangle \\
&+\langle p\otimes D\phi(y)\sigma_y\tilde{e}_i, Q_i\rangle + \langle D\phi(y)\otimes pQ_i, \sigma_y\tilde{e}_i\rangle] + O(\varrho).
\end{aligned}$$

For $i = 2$, we compute

$$Q_2 = \sigma_x e_2 - \sigma_y\tilde{e}_2 = \sigma_x\mathcal{R}e_1 + \sigma_y\mathcal{R}\tilde{e}_1 = (\sigma_x - \sigma_y)\mathcal{R}e_1 + \sigma_y(\mathcal{R}e_1 + \mathcal{R}\tilde{e}_1) \tag{4.87}$$

- $|x-y| \leq 1$, we use the fact that σ is lipschitz in (4.87) to obtain

$$|Q_2| \leq L_\sigma|x-y| + C_\sigma|\mathcal{R}e_1 + \mathcal{R}\tilde{e}_1| = L_\sigma|x-y| + C_\sigma|e_1+\tilde{e}_1|.$$

And

$$\begin{aligned}
|e_1+\tilde{e}_1| \ =\ & |\frac{\sigma_x^{-1}p}{|\sigma_x^{-1}p|} - \frac{\sigma_y^{-1}p}{|\sigma_y^{-1}p|}| \\
=\ & |\frac{\sigma_x^{-1}p}{|\sigma_x^{-1}p|} - \frac{\sigma_y^{-1}p}{|\sigma_x^{-1}p|} + \frac{\sigma_y^{-1}p}{|\sigma_x^{-1}p|} - \frac{\sigma_y^{-1}p}{|\sigma_y^{-1}p|}| \\
\leq\ & \frac{1}{|\sigma_x^{-1}p|}|\sigma_x^{-1}p - \sigma_y^{-1}p| + |\sigma_y^{-1}p||\frac{1}{|\sigma_x^{-1}p|} - \frac{1}{|\sigma_y^{-1}p|}| \\
\leq\ & \frac{1}{|\sigma_x^{-1}p|}|\sigma_x^{-1}p - \sigma_y^{-1}p| + \frac{1}{|\sigma_x^{-1}p|}|\sigma_x^{-1}p - \sigma_y^{-1}p| \\
\leq\ & \frac{2}{|\sigma_x^{-1}p|}|\sigma_x^{-1} - \sigma_y^{-1}|.
\end{aligned}$$

We easily have

$$\begin{aligned}
|\sigma_x^{-1} - \sigma_y^{-1}| \ =\ & |\sigma_y^{-1}[\sigma_y\sigma_x^{-1} - I]| = |\sigma_y^{-1}[\sigma_y - \sigma_x]\sigma_x^{-1}| \\
\leq\ & |A^{-1}|_\infty L_\sigma|x-y| \leq \frac{L_\sigma|x-y|}{\rho}
\end{aligned}$$

Therefore

$$|Q_2| \leq L_\sigma|x-y| + \frac{2{C_\sigma}^2 L_\sigma|x-y|}{\rho} = (1+\frac{2{C_\sigma}^2}{\rho})L_\sigma|x-y|.$$

- $|x-y|\geq 1$, by using the property that σ is bounded in (4.87), we get

$$\begin{aligned} |Q_2| &\leq |\sigma_x - \sigma_y| + |\sigma_y||e_1 + \tilde{e}_1| \\ &\leq 2C_\sigma + C_\sigma \frac{2}{|\sigma_x^{-1}p|}|\sigma_x^{-1} - \sigma_y^{-1}| \\ &\leq 2C_\sigma\left(1+\frac{2C_\sigma^2}{\rho}\right). \end{aligned}$$

So we get

$$|Q_2| \quad \leq \quad \max\{L_\sigma, 2C_\sigma\}\left(1+\frac{2C_\sigma^2}{\rho}\right)\min\{1,|x-y|\}.$$

Moreover

$$\begin{aligned} & \langle p\otimes D\phi(x)\sigma_x e_2, Q_2\rangle + \langle D\phi(x)\otimes pQ_2, \sigma_x e_2\rangle + \langle p\otimes D\phi(y)\sigma_y\tilde{e}_2, Q_2\rangle \\ & +\langle D\phi(y)\otimes pQ_2, \sigma_y\tilde{e}_2\rangle \\ = \; & \langle p\otimes D\phi(x)\sigma_x\mathcal{R}e_1, Q_2\rangle + \langle D\phi(x)\otimes pQ_2, \sigma_x\mathcal{R}e_1\rangle - \langle p\otimes D\phi(y)\sigma_y\mathcal{R}\tilde{e}_1, Q_2\rangle \\ & -\langle D\phi(y)\otimes pQ_2, \sigma_y\mathcal{R}\tilde{e}_1\rangle \\ \leq \; & |Q_2|[|\sigma_x|(|p\otimes D\phi(x)| + |D\phi(x)\otimes p|) + |\sigma_y|(|p\otimes D\phi(y)| + |D\phi(y)\otimes p|)] \\ \leq \; & 2C_\sigma L_\sigma(1+\frac{2{C_\sigma}^2}{\rho})|x-y|(|D\phi(x)|+|D\phi(y)|) \end{aligned}$$

Hence, for $i=1,2$, we have:

$$\begin{aligned} & \operatorname{tr}(A(x)X_x - A(y)Y_y) \\ \leq \; & 4\rho\psi''\Phi + 4C_1C_\sigma^2\psi'(|D\phi(x)|+|D\phi(y)|) + \max{}^2\{L_\sigma, 2C_\sigma\}\left(1+\frac{2C_\sigma^2}{\rho}\right)^2\psi'\Phi \\ & +2C_1C_\sigma L_\sigma(1+\frac{2{C_\sigma}^2}{\rho})\psi'|x-y|(|D\phi(x)|+|D\phi(y)|) \\ & +\sum_{i=3}^N \frac{\psi'\Phi}{|x-y|}|Q_i|^2 + C_1\psi'[\langle p\otimes D\phi(x)\sigma_x e_i, Q_i\rangle + \langle D\phi(x)\otimes pQ_i, \sigma_x e_i\rangle \\ & +\langle p\otimes D\phi(y)\sigma_y\tilde{e}_i, Q_i\rangle + \langle D\phi(y)\otimes pQ_i, \sigma_y\tilde{e}_i\rangle] + O(\varrho). \end{aligned}$$

Recall that $e_i = \tilde{e}_i,\ 3\leq i\leq N$, similarly with the above estimates we have

$$|Q_i| \leq \max\{L_\sigma, 2C_\sigma\}\min\{1,|x-y|\}, \text{ for } i\geq 3$$

and

$$\begin{aligned}
& \langle p \otimes D\phi(x)\sigma_x e_i, Q_i\rangle + \langle D\phi(x) \otimes pQ_i, \sigma_x e_i\rangle \\
& +\langle p \otimes D\phi(y)\sigma_y \tilde{e}_i, Q_i\rangle + \langle D\phi(y) \otimes pQ_i, \sigma_y \tilde{e}_i\rangle \\
\leq\ & 2C_\sigma L_\sigma |x-y|(|D\phi(x)| + |D\phi(y)|)
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
& \mathrm{tr}(A(x)X_x - A(y)Y_y) \qquad (4.88)\\
\leq\ & 4\rho\psi''\Phi + 4C_1{C_\sigma}^2\psi'(|D\phi(x)| + |D\phi(y)|) + \max{}^2\{L_\sigma, 2C_\sigma\}\left(1 + \frac{2C_\sigma^2}{\rho}\right)^2\psi'\Phi \\
& +2C_1C_\sigma L_\sigma(1 + \frac{2{C_\sigma}^2}{\rho})|x-y|\psi'(|D\phi(x)| + |D\phi(y)|) \\
& +(N-2)\max{}^2\{L_\sigma, 2C_\sigma\}\psi'\Phi + (N-2)2C_1C_\sigma L_\sigma|x-y|\psi'(|D\phi(x)| + |D\phi(y)|) \\
\leq\ & 4\rho\psi''\Phi + 4C_1C_\sigma^2\psi'(|D\phi(x)| + |D\phi(y)|) + C_{\sigma 1}\psi'\Phi \\
& +C_1C_{\sigma 2}|x-y|\psi'(|D\phi(x)| + |D\phi(y)|),
\end{aligned}$$

where

$$C_{\sigma 1} = \max{}^2\{L_\sigma, 2C_\sigma\}\left(\left(1 + \frac{2C_\sigma^2}{\rho}\right)^2 + N - 2\right); \qquad C_{\sigma 2} = 2C_\sigma L_\sigma(N - 1 + \frac{2{C_\sigma}^2}{\rho}).$$

3. Conclusion. Using the concavity of ψ and choosing

$$\tilde{\tilde{C}}_\sigma(N,\rho,\sigma) := \max\{C_{\sigma 1}, C_{\sigma 2}, 4C_\sigma^2\}, \qquad \mathcal{C}_\sigma = \max\{\tilde{C}_\sigma, \tilde{\tilde{C}}_\sigma\}, \qquad (4.89)$$

where $\tilde{C}_\sigma$ is defined by (4.86).

Then from (4.85) and (4.88) we get:

- Under assumption (4.2)

$$\begin{aligned}
& -\mathrm{tr}(A(x)X - A(y)Y) \\
\geq\ & -\mathcal{C}_\sigma|x-y|\psi'(|x-y|)\Phi(x,y) - C_1(\psi(|x-y|) + \delta)\{\mathrm{tr}(A(x)D^2\phi(x)) \\
& \quad +\mathrm{tr}(A(y)D^2\phi(y)) + \mathcal{C}_\sigma(|D\phi(x)| + |D\phi(y)|)\} + O(\varrho).
\end{aligned}$$

- If in addition (4.5) holds

$$\begin{aligned}
& -\mathrm{tr}(A(x)X - A(y)Y) \\
\geq\ & -[4\rho\psi''(|x-y|) + \mathcal{C}_\sigma\psi'(|x-y|)]\Phi(x,y) \\
& -C_1\mathcal{C}_\sigma\psi'(|x-y|)(|D\phi(x)| + |D\phi(y)|) \\
& -C_1(\psi(|x-y|) + \delta)\{\mathrm{tr}(A(x)D^2\phi(x)) + \mathrm{tr}(A(y)D^2\phi(y)) \\
& +\mathcal{C}_\sigma(|D\phi(x)| + |D\phi(y)|)\} + O(\varrho).
\end{aligned}$$

$\square$

4.3.3 Estimates for nonlocal operator

Proof of Proposition 4.1.1. Several parts of the proof are inspired by [11]. Let $a_0 > 0$, we separate the proof into two parts:

a). Estimate for the nonlocal terms when $|a| \le a_0$. We split the domain of integration into three pieces and take the integrals on each of these domains. Namely we part the ball B_γ of radius γ into the subset $\mathcal{C}_{\eta,\gamma}(a)$ with $\eta = \eta(|a|)$ and $\gamma = \gamma(|a|)$ and its complementary $B_\gamma \backslash \mathcal{C}_{\eta,\gamma}(a)$. We write the difference of the nonlocal terms, corresponding to the maximum point (x, y), as the sum

$$\mathcal{I}(x, u, D_x\varphi) - \mathcal{I}(y, u, -D_y\varphi) = \mathcal{T}^1(x, y) + \mathcal{T}^2(x, y) + \mathcal{T}^3(x, y) = \mathcal{T}(x, y)$$

where

$$\begin{aligned}
\mathcal{T}^1(x,y) &= \int_{I\!R^N\backslash B} [u(x+z) - u(x) - (u(y+z) - u(y))]\nu(dz) \\
\mathcal{T}^2(x,y) &= \int_{B\backslash \mathcal{C}_{\eta,\gamma}(a)} [u(x+z) - u(x) - D_x\varphi(x,y)\cdot z \\
&\qquad - (u(y+z) - u(y) + D_y\varphi(x,y)\cdot z)]\nu(dz) \\
\mathcal{T}^3(x,y) &= \int_{\mathcal{C}_{\eta,\gamma}(a)} [u(x+z) - u(x) - D_x\varphi(x,y)\cdot z \\
&\qquad - (u(y+z) - u(y) + D_y\varphi(x,y)\cdot z)]\nu(dz).
\end{aligned}$$

Notice that $D_x\varphi$ and $D_y\varphi$ are given by (4.11) and (4.12) respectively.

Since (x, y) is a maximum point of $\Psi(\cdot,\cdot)$, we have

$$\begin{aligned}
& u(x+z) - u(x) - D_x\varphi(x,y)\cdot z \\
\le\ & u(y+z') - u(y) + D_y\varphi(x,y)\cdot z' + \varphi(x+z, y+z') - \varphi(x,y) \\
& + D_y\varphi(x,y)\cdot(z - z') - (D_x\varphi(x,y) + D_y\varphi(x,y))\cdot z.
\end{aligned} \tag{4.90}$$

In the following we give estimates for each of these integral terms, using inequality (4.90) and properties of the Lévy measure ν.

1. Estimate for $\mathcal{T}^1$. Since (x, y) is a maximum point of $\Psi(\cdot,\cdot)$, we have

$$\begin{aligned}
& u(x+z) - u(y+z) - (u(x) - u(y)) \\
\le\ & C_1(\psi(|a|) + \delta)[\phi(x+z) - \phi(x) + \phi(y+z) - \phi(y)]
\end{aligned}$$

Therefore,

$$\mathcal{T}^1(x,y) \le C_1(\psi(|a|) + \delta)\left(\mathcal{I}[I\!R^N\backslash B](x, \phi, D\phi) + \mathcal{I}[I\!R^N\backslash B](y, \phi, D\phi)\right). \tag{4.91}$$

2. Estimate for $\mathcal{T}^2$. Taking $z' = z$ in inequality (4.90) we have

$$\begin{aligned}
& u(x+z) - u(x) - D_x\varphi(x,y).z - (u(y+z) - u(y) + D_y\varphi(x,y)\cdot z) \\
\le\ & \varphi(x+z, y+z) - \varphi(x,y) - (D_x\varphi(x,y) + D_y\varphi(x,y))\cdot z \\
\le\ & (\psi(|a|) + \delta)[\Phi(x+z, y+z) - \Phi(x,y)] - C_1(\psi(|a|) + \delta)(D\phi(x) + D\phi(y))\cdot z \\
\le\ & C_1(\psi(|a|) + \delta)[\phi(x+z) - \phi(x) - D\phi(x)\cdot z + \phi(y+z) - \phi(y) - D\phi(y)\cdot z].
\end{aligned} \tag{4.92}$$

Then

$$\begin{aligned}\mathcal{T}^2(x,y) &\le C_1(\psi(|a|)+\delta)\int_{B\backslash\mathcal{C}_{\eta,\gamma}(a)}[\phi(x+z)-\phi(x)-D\phi(x)\cdot z \\ &\qquad +\phi(y+z)-\phi(y)-D\phi(y)\cdot z]\nu(dz) \\ &= C_1(\psi(|a|)+\delta)\left(\mathcal{I}[B\backslash\mathcal{C}_{\eta,\gamma}(a)](x,\phi,D\phi)+\mathcal{I}[B\backslash\mathcal{C}_{\eta,\gamma}(a)](y,\phi,D\phi)\right).\end{aligned} \tag{4.93}$$

3. Estimate for $\mathcal{T}^3$. Taking $z'=0$ and $z=0$ in inequality (4.90) we have

$$\begin{aligned}u(x+z)-u(x)-D_x\varphi(x,y)\cdot z &\le \varphi(x+z,y)-\varphi(x,y)-D_x\varphi(x,y)\cdot z \\ -(u(y+z)-u(y)+D_y\varphi(x,y)\cdot z) &\le \varphi(x,y+z')-\varphi(x,y)-D_y\varphi(x,y)\cdot z'.\end{aligned}$$

Therefore

$$\mathcal{T}^3(x,y) \le \int_{\mathcal{C}_{\eta,\gamma}(a)}[\varphi^1(x,y,z)+\varphi^2(x,y,z)]\nu(dz),$$

where

$$\begin{aligned}\varphi^1(x,y,z)&=\varphi(x+z,y)-\varphi(x,y)-D_x\varphi(x,y)\cdot z,\\ \varphi^2(x,y,z)&=\varphi(x,y+z)-\varphi(x,y)-D_y\varphi(x,y)\cdot z.\end{aligned}$$

We estimate separately for each of these functions.

From (4.9) and (4.11) we have

$$\begin{aligned}&\varphi^1(x,y,z)\\ =\ &(\psi(|a+z|)+\delta)\Phi(x+z,y)-(\psi(|a|)+\delta)\Phi(x,y)-\psi'(|a|)\hat{a}\cdot z\Phi(x,y)\\ &-C_1(\psi(|a|)+\delta)D\phi(x)\cdot z\\ =\ &(\psi(|a+z|)+\delta)[\Phi(x+z,y)-\Phi(x,y)]-C_1(\psi(|a|)+\delta)D\phi(x)\cdot z\\ &+[\psi(|a+z|)-\psi(|a|)-\psi'(|a|)\hat{a}\cdot z]\Phi(x,y)\\ =\ &C_1(\psi(|a+z|)-\psi(|a|))[\phi(x+z)-\phi(x)]\\ &+C_1(\psi(|a|)+\delta)(\phi(x+z)-\phi(x)-D\phi(x)\cdot z)\\ &+[\psi(|a+z|)-\psi(|a|)-\psi'(|a|)\hat{a}\cdot z]\Phi(x,y)\end{aligned} \tag{4.94}$$

Similarly, from (4.9) and (4.12) we have

$$\begin{aligned}\varphi^2(x,y,z) &= C_1(\psi(|a-z|)-\psi(|a|))[\phi(y+z)-\phi(y)]\\ &\quad +C_1(\psi(|a|)+\delta)(\phi(y+z)-\phi(y)-D\phi(y)\cdot z)\\ &\quad +[\psi(|a-z|)-\psi(|a|)+\psi'(|a|)\hat{a}\cdot z]\Phi(x,y)\end{aligned} \tag{4.95}$$

Then from (4.94), (4.94) and (4.95) we obtain

$$\begin{aligned}\mathcal{T}^3(x,y) \le\ & C_1\int_{\mathcal{C}_{\eta,\gamma}(a)}(\psi(|a+z|)-\psi(|a|))[\phi(x+z)-\phi(x)]\\ &\qquad +(\psi(|a-z|)-\psi(|a|))[\phi(y+z)-\phi(y)]\nu(dz)\\ &+C_1(\psi(|a|)+\delta)(\mathcal{I}[\mathcal{C}_{\eta,\gamma}(a)](x,\phi,D\phi)+\mathcal{I}[\mathcal{C}_{\eta,\gamma}(a)](y,\phi,D\phi))\\ &+\Phi(x,y)\int_{\mathcal{C}_{\eta,\gamma}(a)}(\psi(|a+z|)-\psi(|a|)-\psi'(|a|)\hat{a}\cdot z)\\ &\qquad +\left(\psi(|a-z|)-\psi(|a|)+\psi'(|a|)\hat{a}\cdot z\right)\nu(dz)\end{aligned}$$

Because of the monotonicity and the concavity of ψ we have

$$\psi(|a+z|) - \psi(|a|) \quad \leq \quad \psi(|a|+|z|) - \psi(|a|) \leq \psi'(|a|)|z|. \tag{4.96}$$

Since $\phi \in C^\infty(I\!R^N)$ is a convex function, recalling (4.75) and using (4.80) we have

$$\phi(x+z) - \phi(x) \leq |D\phi(x+z)||z| \leq \mu\phi(x+z)|z| \leq \mu\phi(x)\phi(z)|z|, \quad \forall x, z \in I\!R^N. \tag{4.97}$$

Using (4.96) and (4.97) to estimate for (4.96) we obtain

$$\begin{aligned} \mathcal{T}^3(x,y) \quad \leq \quad & C_1\mu\psi'(|a|)(\phi(x)+\phi(y)) \int_{\mathcal{C}_{\eta,\gamma}(a)} \phi(z)|z|^2\nu(dz) \\ & + C_1(\psi(|a|)+\delta)(\mathcal{I}[\mathcal{C}_{\eta,\gamma}(a)](x,\phi,D\phi) + \mathcal{I}[\mathcal{C}_{\eta,\gamma}(a)](y,\phi,D\phi)) \\ & + \Phi(x,y) \int_{\mathcal{C}_{\eta,\gamma}(a)} [\psi(|a+z|) - \psi(|a|) - \psi'(|a|)\hat{a}\cdot z \\ & \qquad + \psi(|a-z|) - \psi(|a|) + \psi'(|a|)\hat{a}\cdot z]\nu(dz). \end{aligned}$$

Then we apply [11, Lemma 12] to get the estimate for $\mathcal{T}^3$

$$\begin{aligned} \mathcal{T}^3(x,y) \quad \leq \quad & C_1\mu\psi'(|a|)(\phi(x)+\phi(y)) \int_{\mathcal{C}_{\eta,\gamma}(a)} \phi(z)|z|^2\nu(dz) \\ & + C_1(\psi(|a|)+\delta)(\mathcal{I}[\mathcal{C}_{\eta,\gamma}(a)](x,\phi,D\phi) + \mathcal{I}[\mathcal{C}_{\eta,\gamma}(a)](y,\phi,D\phi)) \\ & + \Phi(x,y)\frac{1}{2} \int_{\mathcal{C}_{\eta,\gamma}(a)} \sup_{|s|\leq 1} \left((1-\tilde{\eta}^2)\frac{\psi'(|a+sz|)}{|a+sz|} + \tilde{\eta}^2\psi''(|a+sz|) \right) |z|^2\nu(dz), \end{aligned} \tag{4.98}$$

where $\tilde{\eta} = \frac{1-\eta-\gamma_0}{1+\gamma_0}$, $\eta \in (0,1)$ small enough and $\gamma = \gamma_0|a|$ with $\gamma_0 \in (0,1)$

4. Estimate for $\mathcal{T}$. Combine (4.91), (4.93) and (4.98) the difference of two nonlocal terms becomes

$$\begin{aligned} \mathcal{T}(x,y) \quad = \quad & \mathcal{I}(x,u,D_x\varphi) - \mathcal{I}(y,u,-D_y\varphi) \\ \leq \quad & C_1(\psi(|a|)+\delta)\left(\mathcal{I}(x,\phi,D\phi) + \mathcal{I}(y,\phi,D\phi)\right) \\ & + C_1\mu\psi'(|a|)(\phi(x)+\phi(y)) \int_{\mathcal{C}_{\eta,\gamma}(a)} \phi(z)|z|^2\nu(dz) \\ & + \frac{1}{2}\Phi(x,y) \int_{\mathcal{C}_{\eta,\gamma}(a)} \sup_{|s|\leq 1} \left((1-\tilde{\eta}^2)\frac{\psi'(|a+sz|)}{|a+sz|} + \tilde{\eta}^2\psi''(|a+sz|) \right) |z|^2\nu(dz). \end{aligned} \tag{4.99}$$

b). Estimate for the nonlocal terms when $|a| \geq a_0$. In this part, we just split the domain into two pieces, one is the ball B and its complement is $I\!R^N \backslash B$. Using the same arguments as in Part a) (estimate for $\mathcal{T}^1$ and $\mathcal{T}^2$) we get

$$\begin{aligned} & \mathcal{I}(x,u,D_x\varphi) - \mathcal{I}(y,u,-D_y\varphi) \\ \leq \quad & C_1(\psi(|a|)+\delta) \int_{I\!R^N\backslash B} [\phi(x+z) - \phi(x) + \phi(y+z) - \phi(y)]\nu(dz) \\ & + C_1(\psi(|a|)+\delta) \int_B \{\phi(x+z) - \phi(x) - D\phi(x)\cdot z \\ & \qquad + \phi(y+z) - \phi(y) - D\phi(y)\cdot z\}\nu(dz) \\ = \quad & C_1(\psi(|a|)+\delta)(\mathcal{I}(x,\phi,D\phi) + \mathcal{I}(y,\phi,D\phi). \end{aligned}$$

□

Proof of Lemma 4.1.4. Recalling from Proposition 4.1.1, let $a_0 > 0$, $|a| = |x-y| \le a_0$, we have

$$\begin{aligned}
& \mathcal{I}(x,u,D_x\varphi) - \mathcal{I}(y,u,-D_y\varphi) \\
\le\ & C_1(\psi(|a|)+\delta)\left(\mathcal{I}(x,\phi,D\phi)+\mathcal{I}(y,\phi,D\phi)\right) \\
& + C_1\mu\psi'(|a|)(\phi(x)+\phi(y))\int_{\mathcal{C}_{\eta,\gamma}(a)}\phi(z)|z|^2\nu(dz) \\
& + \frac{1}{2}\Phi(x,y)\int_{\mathcal{C}_{\eta,\gamma}(a)}\sup_{|s|\le 1}\left((1-\tilde{\eta}^2)\frac{\psi'(|a+sz|)}{|a+sz|}+\tilde{\eta}^2\psi''(|a+sz|)\right)|z|^2\nu(dz)
\end{aligned}$$

Using $(M1)$ we first get

$$\begin{aligned}
& C_1\mu\psi'(|a|)(\phi(x)+\phi(y))\int_{\mathcal{C}_{\eta,\gamma}(a)}\phi(z)|z|^2\nu(dz) \\
\le\ & C_1\mu\psi'(|a|)(\phi(x)+\phi(y))C^1_\nu \\
\le\ & \mu C^1_\nu\psi'(|a|)\Phi(x,y).
\end{aligned} \tag{4.100}$$

Recall that $\psi(r) = 1 - e^{-C_2 r^\tau}$ for $r \le r_0$, $\tau \in (0,1)$, we have the derivatives

$$\begin{aligned}
\psi'(r) &= C_2\tau r^{\tau-1}e^{-C_2 r^\tau}, \\
\psi''(r) &= C_2\tau(\tau-1)r^{\tau-2}e^{-C_2r^\tau} - (C_2\tau r^{\tau-1})^2e^{-C_2r^\tau}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
l(|a|) &:= (1-\tilde{\eta}^2)\frac{\psi'(|a+sz|)}{|a+sz|}+\tilde{\eta}^2\psi''(|a+sz|) \\
&= (1-\tilde{\eta}^2)C_2\tau e^{-C_2|a+sz|^\tau}|a+sz|^{\tau-2}+\tilde{\eta}^2C_2\tau(\tau-1)\tau e^{-C_2|a+sz|^\tau}|a+sz|^{\tau-2} \\
&\quad -\tilde{\eta}^2(C_2\tau)^2e^{-C_2|a+sz|^\tau}|a+sz|^{2(\tau-1)} \\
&= C_2\tau e^{-C_2|a+sz|^\tau}|a+sz|^{\tau-2}\left(1-\tilde{\eta}^2(2-\tau)\right)-\tilde{\eta}^2(C_2\tau)^2e^{-C_2|a+sz|^\tau}|a+sz|^{2(\tau-1)}.
\end{aligned}$$

Note that, on the set $\mathcal{C}_{\eta,\gamma}(|a|)$ we have the following upper bound

$$|a+sz| \le |a|+|s||z| \le |a|+\gamma = |a|(1+\gamma_0).$$

Taking $\tau \in (0,1)$ (possibly arbitrary close to 1) then $2-\tau > 1$. So we can choose η and γ_0 sufficiently small enough such that

$$(2-\tau)\tilde{\eta}^2 = (2-\tau)\left(\frac{1-\eta-\gamma_0}{1+\gamma_0}\right)^2 > \theta > 1 \ \text{ for some } \theta \in (1, 2-\tau).$$

Therefore we obtain

$$\begin{aligned}
l(|a|) &\le -C_2\tau(\theta-1)e^{-C_2|a+sz|^\tau}|a+sz|^{\tau-2}-(C_2\tau)^2\frac{\theta}{2-\tau}e^{-C_2|a+sz|^\tau}|a+sz|^{2(\tau-1)} \\
&\le -C_2\tau(\theta-1)e^{-C_2|a|^\tau}e^{-C_2|z|^\tau}(1+\gamma_0)^{\tau-2}|a|^{\tau-2} \\
&\qquad -(C_2\tau)^2\frac{\theta}{2-\tau}e^{-C_2|a|^\tau}e^{-C_2|z|^\theta}(1+\gamma_0)^{2(\tau-1)}|a|^{2(\tau-1)} \\
&= -\psi'(|a|)\left(C^1(\nu,\tau)|a|^{-1}+C_2C^2(\nu,\tau)|a|^{\tau-1}\right)e^{-C_2|z|^\tau}.
\end{aligned}$$

Remark that η, γ_0 do not depend on $|a|$. Taking the integral for $l(|a|)$ over the cone $\mathcal{C}_{\eta,\gamma}(a)$ for $\gamma = \gamma_0|a|$ and using $(M2)$ we obtain

$$\begin{aligned}
& \frac{1}{2}\int_{\mathcal{C}_{\eta,\gamma}(a)} \sup_{|s|\leq 1} l(|a|)|z|^2\nu(dz) \qquad (4.101)\\
\leq\ & -\frac{1}{2}\psi'(|a|)\left(C^1(\nu,\tau)|a|^{-1} + C_2C^2(\nu,\tau)|a|^{\tau-1}\right)\int_{\mathcal{C}_{\eta,\gamma}(a)} e^{-C_2|z|^\tau}|z|^2\nu(dz)\\
\leq\ & -C(\nu,\tau)\psi'(|a|)(|a|^{-1} + C_2|a|^{\tau-1})|a|^{2-\beta}\\
=\ & -C(\nu,\tau)\psi'(|a|)(1 + C_2|a|^{\tau})|a|^{1-\beta},
\end{aligned}$$

where $C(\nu,\tau) = \min\{C^1(\nu,\tau), C^2(\nu,\tau)\}C^2_\nu$, C^2_ν is in $(M2)$.

Finally, combine (4.100), and (4.101) we obtain

$$\begin{aligned}
\mathcal{I}(x,u,D_x\varphi) - \mathcal{I}(y,u,-D_y\varphi) \ \leq\ & C_1(\psi(r)+\delta)\left(\mathcal{I}(x,\phi,D\phi) + \mathcal{I}(y,\phi,D\phi)\right)\\
& +\Phi(x,y)\left(\mu C^1_\nu - C(\nu,\tau)(1+C_2r^\tau)r^{1-\beta}\right)\psi'(r)
\end{aligned}$$

for $r = |a| \leq r_0 = a_0$. □

Proof of Lemma 4.1.5. Let ψ be the concave function defined in (4.47). Let $0 < |x-y| = a \leq r_0$, it follows from [11, Corollary 9] that there exists a constant $C = C(\nu) > 0$ such that for $\Lambda(\nu) = C(\varrho\theta 2^{\theta-1} - 1) > 0$ we have

$$\begin{aligned}
& \frac{1}{2}\int_{\mathcal{C}_{\eta,\gamma}(a)} \sup_{|s|\leq 1}\left((1-\tilde{\eta}^2)\frac{\psi'(|a+sz|)}{|a+sz|} + \tilde{\eta}^2\psi''(|a+sz|)\right)|z|^2\nu(dz) \qquad (4.102)\\
\leq\ & -\Lambda|a|^{1-\beta+\theta(N+2-\beta)}.
\end{aligned}$$

Taking $a_0 = r_0$ and applying to Proposition 4.1.1 (4.36), we have

$$\begin{aligned}
& \mathcal{I}(x,u,D_x\varphi) - \mathcal{I}(y,u,-D_y\varphi)\\
\leq\ & C_1(\psi(|a|)+\delta)\left(\mathcal{I}(x,\phi,D\phi) + \mathcal{I}(y,\phi,D\phi)\right)\\
& + C_1\mu\psi'(|a|)(\phi(x)+\phi(y))\int_{\mathcal{C}_{\eta,\gamma}(a)}\phi(z)|z|^2\nu(dz)\\
& +\frac{1}{2}\Phi(x,y)\int_{\mathcal{C}_{\eta,\gamma}(a)} \sup_{|s|\leq 1}\left((1-\tilde{\eta}^2)\frac{\psi'(|a+sz|)}{|a+sz|} + \tilde{\eta}^2\psi''(|a+sz|)\right)|z|^2\nu(dz).
\end{aligned}$$

Combining (4.100) and (4.102) we obtain

$$\begin{aligned}
\mathcal{I}(x,u,D_x\varphi) - \mathcal{I}(y,u,-D_y\varphi) \ \leq\ & C_1(\psi(r)+\delta)\left(\mathcal{I}(x,\phi,D\phi) + \mathcal{I}(y,\phi,D\phi)\right)\\
& -\left(\Lambda r^{-\tilde{\theta}} - \mu C^1_\nu\psi'(r)\right)\Phi(x,y),
\end{aligned}$$

where $\tilde{\theta} = \beta - 1 - \theta(N+2-\beta) > 0$. □

4.4 Overview of the Lipschitz regularity results for the viscous Ornstein-Uhlenbeck equation

Table 4.1: Lipschitz regularity results for the second order Hamilton-Jacobi equation with Ornstein-Uhlenbeck operator in the local and nonlocal case

Diffusion	Hamiltonian	Size α of the Ornstein-Uhlenbeck term	Results
Nondegenerate local: elliptic nonlocal: $\beta \in (1,2)$	H sublinear $\lvert H(x,p)\rvert \leq C(1+\lvert p\rvert)$	$\forall \alpha > 0$	Theorem 4.1.1
Can be degenerate local: A constant nonlocal: $\beta \in (0,2)$	$H(p) \in \mathrm{Lip}(I\!R^N)$	$\forall \alpha > 0$	Ishii et al. [42] & Remark 4.1.4
Can be degenerate local: A constant nonlocal: $\beta \in (0,2)$	$\lvert H(x,p) - H(y,p)\rvert \leq C(1+\lvert p\rvert)\lvert x-y\rvert$ $\lvert H(x,p) - H(x,q)\rvert \leq C(1+\lvert x\rvert)\lvert p-q\rvert$	for $\alpha > C(H)$	Theorem 4.1.2 & Remark 4.1.3
local: A degenerate	$H(p) \in \mathrm{Lip}(I\!R^N)$	$\alpha > C(A)$	Remark 4.1.5
local: A degenerate	$\lvert H(x,p) - H(y,p)\rvert \leq C(1+\lvert p\rvert)\lvert x-y\rvert$ $\lvert H(x,p) - H(x,q)\rvert \leq C(1+\lvert x\rvert)\lvert p-q\rvert$	$\alpha > C(A,H)$	Theorem 4.1.2
Bounded solutions local: A elliptic constant	$\lvert H(x,p\rvert \leq C(\lvert p\rvert + \lvert p\rvert^2)$	$\forall \alpha > 0$	Ghilli [45]

Chapter 5

Well-posedness of the stationary and evolution problem

In this chapter we first build continuous viscosity solutions for (4.1) and (4.68). As usual, we use Perron's method for viscosity solutions (see [10, 30, 50]) to build discontinuous viscosity solutions for (4.1) and (4.68). The continuity of solutions is then proved by strong comparison principles for (4.1) and (4.68). But strong comparison results may not hold for the equations when using the very general assumption (4.6) on the Hamiltonian. We only have comparison results (Theorem 5.1.1, Theorem 5.2.1) between a discontinuous sub or supersolution with a Lipschitz continuous solution. Therefore, we need another approach to build continuous solutions of (4.1) and (4.68). It is based on a truncation of the Hamiltonian and the unbounded f term.

Next we show the uniqueness for solutions of (4.1) and (4.68). It is a direct consequence of the comparison principle. But to obtain this result we need a stronger assumption on the Hamiltonian

$$\begin{cases} \text{There exists } L_H > 0 \text{ such that} \\ |H(x,p) - H(x,q)| \le L_H |p-q|, \quad \forall x, p, q \in I\!R^N. \end{cases} \tag{5.1}$$

Throughout this Chapter, we use ϕ standing for ϕ_μ, $\mu > 0$ defined by (2.42).

5.1 Well-posedness of the stationary problem

Theorem 5.1.1. *Suppose that (4.3), (5.1), $f \in C(I\!R^N)$ and either (4.2) or (M1) hold. Let $u \in USC(I\!R^N) \cap \mathcal{E}_\mu^+(I\!R^N)$ and $v \in LSC(I\!R^N) \cap \mathcal{E}_\mu^-(I\!R^N)$ be a viscosity subsolution and a viscosity supersolution of (4.1), respectively. Assume that either u or v is locally Lipschitz continuous in $I\!R^N$. Then $u \le v$ in $I\!R^N$.*

Proof. We argue by contradiction assuming that $u(z) - v(z) \ge 2\eta > 0$ for some $z \in I\!R^N$. We consider

$$\Psi(x,y) = u(x) - v(y) - \frac{|x-y|^2}{2\epsilon^2} - \beta(\phi(x) + \phi(y)),$$

where ϵ, β are positive parameters. For small β we have $\Psi(z,z) \ge \eta$. Since $u \in \mathcal{E}_\mu^+(I\!R^N)$, $v \in \mathcal{E}_\mu^-(I\!R^N)$, Ψ will attain a maximum at $(\bar{x}, \bar{y}) \in B(0, R_\beta) \times B(0, R_\beta)$, where R_β

does not depend on ϵ. It follows that $u(x) - v(y) - \beta(\phi(x) + \phi(y))$ are bounded in $B(0, R_\beta) \times B(0, R_\beta)$, so the following classical properties hold (see [10]) up to some subsequences,

$$\frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} \to 0, \qquad \bar{x}, \bar{y} \to \hat{x} \in B(0, R_\beta) \text{ as } \epsilon \to 0, \ \beta \text{ is fixed.} \tag{5.2}$$

Assuming that v for instance is locally Lipschitz continuous, since $\Psi(\bar{x}, \bar{x}) \leq \Psi(\bar{x}, \bar{y})$ then we have

$$\begin{aligned} \frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} &\leq v(\bar{x}) - v(\bar{y}) + \beta(\phi(\bar{x}) - \phi(\bar{y})) \\ &\leq C|\bar{x} - \bar{y}|(\phi(\bar{x}) + \phi(\bar{y})) + \beta\mu|\bar{x} - \bar{y}|(\phi(\bar{x}) + \phi(\bar{y})). \end{aligned}$$

This implies, up to some subsequence, $p_\epsilon := \frac{\bar{x} - \bar{y}}{\epsilon^2}$ remains bounded when $\epsilon \to 0$ and moreover $p_\epsilon \to \hat{p}$, for some $\hat{p} \in I\!R^N$.

By [30, Theorem 3.2] in the local case and [15, Corollary 1] in the nonlocal one to learn that, for every $\varrho > 0$, there exist $(p_\epsilon + \beta D\phi(\bar{x}), X) \in \bar{J}^{2,+}u(\bar{x})$, $(p_\epsilon - \beta D\phi(\bar{y}), Y) \in \bar{J}^{2,-}v(\bar{y})$ such that

$$\begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq A + \varrho A^2, \tag{5.3}$$

where

$$A = \frac{2}{\epsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \beta \begin{pmatrix} D^2\phi(\bar{x}) & 0 \\ 0 & D^2\phi(\bar{y}) \end{pmatrix},$$

and $\varrho A^2 = O(\varrho)$ (ϱ will be sent to 0 first).

Writing the viscosity inequalities at $(\bar{x}, \bar{y})$, we have

$$\begin{aligned} &\lambda u(\bar{x}) - \mathcal{F}(\bar{x}, [u]) + \langle b(\bar{x}), p_\epsilon + \beta D\phi(\bar{x})\rangle + H(\bar{x}, p_\epsilon + \beta D\phi(\bar{x})) \leq f(\bar{x}), \\ &\lambda v(\bar{y}) - \mathcal{F}(\bar{y}, [v]) + \langle b(\bar{y}), p_\epsilon - \beta D\phi(\bar{y})\rangle + H(\bar{y}, p_\epsilon - \beta D\phi(\bar{y})) \geq f(\bar{y}). \end{aligned}$$

Subtracting the viscosity inequalities, we obtain

$$\begin{aligned} &\lambda(u(\bar{x}) - v(\bar{y})) - (\mathcal{F}(\bar{x}, [u]) - \mathcal{F}(\bar{y}, [v])) \\ &+\langle b(\bar{x}) - b(\bar{y}), p_\epsilon\rangle + \beta\langle b(\bar{x}), D\phi(\bar{x})\rangle + \beta\langle b(\bar{y}), D\phi(\bar{y})\rangle \\ &+H(\bar{x}, p_\epsilon + \beta D\phi(\bar{x})) - H(\bar{y}, p_\epsilon - \beta D\phi(\bar{y})) \\ \leq\ & f(\bar{x}) - f(\bar{y}), \end{aligned} \tag{5.4}$$

where $\mathcal{F}(\bar{x}, [u]) = \mathrm{tr}(A(\bar{x})X)$ and $\mathcal{F}(\bar{y}, [u]) = \mathrm{tr}(A(\bar{y})Y)$ in the local case and $\mathcal{F}(\bar{x}, [u]) = \mathcal{I}(\bar{x}, u, p_\epsilon + \beta D\phi(\bar{x}))$ and $\mathcal{F}(\bar{y}, [u]) = \mathcal{I}(\bar{y}, u, p_\epsilon - \beta D\phi(\bar{y}))$ in the nonlocal one.

In the case $\mathcal{F}$ is the local operator in (2.27) and (4.2) holds only, applying Lemma 4.1.2 for the classical test function $\frac{|x-y|^2}{2\epsilon^2} + \beta(\phi(x) + \phi(y))$ instead of (4.9), we obtain

$$\mathrm{tr}(A(\bar{x})X - A(\bar{y})Y) \leq L_\sigma^2 \frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} + \beta \mathrm{tr}(A(\bar{x})D^2\phi(\bar{x})) + \beta \mathrm{tr}(A(\bar{y})D^2\phi(\bar{y})).$$

On the other hand, if $\mathcal{F}$ is the nonlocal operator defined by (2.27) then applying Proposition 4.1.1 (4.35) by using $\varphi(x,y) = \frac{|x-y|^2}{2\epsilon^2} + \beta(\phi(x)+\phi(y))$, we get

$$\mathcal{I}(\bar{x}, u, p_\epsilon + \beta D\phi(\bar{x})) - \mathcal{I}(\bar{y}, v, p_\epsilon - D\phi(\bar{y})) \leq \beta\mathcal{I}(\bar{x}, \phi, D\phi) + \beta\mathcal{I}(\bar{y}, \phi, D\phi).$$

Therefore, in any cases we have

$$\mathcal{F} \leq \beta\mathcal{F}(\bar{x}, \phi) + \beta\mathcal{F}(\bar{y}, \phi) + L_\sigma^2 \frac{|\bar{x}-\bar{y}|^2}{2\epsilon^2}. \tag{5.5}$$

Since $\Psi(\bar{x}, \bar{y}) \geq \Psi(z, z) \geq \eta$, we have $u(\bar{x}) - v(\bar{y}) \geq \eta$. Using (4.3) and taking into account (5.5), inequality (5.4) can be estimated as

$$\begin{aligned}
& \lambda\eta - \beta\mathcal{F}(\bar{x}, [\phi]) - \beta\mathcal{F}(\bar{y}, [\phi]) - L_\sigma^2 \frac{|\bar{x}-\bar{y}|^2}{2\epsilon^2} + \alpha\frac{|\bar{x}-\bar{y}|^2}{2\epsilon^2} + \beta\langle b(\bar{x}), D\phi(\bar{x})\rangle \\
& + \beta\langle b(\bar{y}), D\phi(\bar{y})\rangle + H(\bar{x}, p_\epsilon + \beta D\phi(\bar{x})) - H(\bar{y}, p_\epsilon - \beta D\phi(\bar{y})) \\
\leq \ & f(\bar{x}) - f(\bar{y}).
\end{aligned}$$

Now sending ϵ to 0, using (5.2) and since $f \in C(I\!R^N)$ we obtain

$$\lambda\eta - 2\beta\mathcal{F}(\hat{x}, [\phi]) + 2\beta\langle b(\hat{x}), D\phi(\hat{x})\rangle + H(\hat{x}, \hat{p} + \beta D\phi(\hat{x})) - H(\hat{x}, \hat{p} - \beta D\phi(\hat{x})) \leq 0.$$

Since $H(x,p)$ is lipschitz in p uniformly in x, i.e., (5.1) holds, we get

$$\lambda\eta - 2\beta\mathcal{F}(\hat{x}, [\phi]) + 2\beta\langle b(\hat{x}), D\phi(\hat{x})\rangle - 2\beta L_H |D\phi(\hat{x})| \leq 0.$$

Recalling from Lemma 4.1.1, there exists a constant $K(L_H, \mathcal{F}) > 0$ such that

$$-\mathcal{F}(x, [\phi]) + \langle b(x), D\phi(x)\rangle - L_H|D\phi(x)| \geq \phi(x) - K \qquad \forall x \in I\!R^N.$$

Therefore, we have

$$\lambda\eta + 2\beta\phi(\hat{x}) - 2\beta K \leq 0.$$

Since $\phi > 0$, sending β to 0, we get a contradiction. $\square$

Theorem 5.1.2. *Suppose that (4.3), (4.4) and (4.6) hold. Assume either (4.2), (4.5) or (M1) holds. For all $\lambda \in (0,1)$, there exists a continuous viscosity solution u^λ of (4.1) in the class of functions*

$$u^\lambda \in \mathcal{E}_\mu(I\!R^N), \tag{5.6}$$
$$|u^\lambda(x) - u^\lambda(y)| \leq C(\phi(x) + \phi(y))|x-y|, \quad x, y \in I\!R^N, \tag{5.7}$$

where $C > 0$ is a constant independent of λ.
In addition, if (5.1) holds then the solution is unique in $C(I\!R^N) \cap \mathcal{E}_\mu(I\!R^N)$.

Proof. 1. Construction of a continuous viscosity solution to a truncated equation. In order to recover the classical framework of viscosity solutions, we first truncate the datas on the equations.

Recall that $\phi(x) = e^{\mu\sqrt{|x|^2+1}}$ and $f \in \mathcal{E}_\mu(I\!R^N)$. We have, $\forall \epsilon > 0$, $\exists M(\epsilon) > 0$ such that

$$|f(x)| \le \epsilon\phi(x) + M(\epsilon) \quad \forall x \in I\!R^N.$$

Let $m \ge 1$, there exists $C(m) > 0$ such that

$$f(x) \ge -\frac{1}{2m}\phi(x) - C(m).$$

Therefore, there exists $R_m > 0$ such that

$$f(x) + \frac{1}{m}\phi(x) \ge m, \quad \text{for } |x| \ge R_m.$$

We then define

$$f_m(x) = \min\{f(x) + \frac{1}{m}\phi(x), m\}. \tag{5.8}$$

The function f_m is bounded by some constant C_m, still satisfies (4.4) with the constant $C_f + \frac{\mu}{m}$ and is Lipschitz continuous in $I\!R^N$ with a constant $L_m = (C_f + \frac{\mu}{m})\sup_{B(0,R_m)} 2\phi$. Indeed, from (4.4) we have

$$\begin{aligned}
|f_m(x) - f_m(y)| &= \min\{f(x) + \frac{1}{m}\phi(x), m\} - \min\{f(y) + \frac{1}{m}\phi(y), m\} \\
&\le f(x) - f(y) + \frac{1}{m}(\phi(x) - \phi(y)) \\
&\le C_f|x-y|(\phi(x) + \phi(y)) + \frac{1}{m}D\phi(z)(x-y), \quad z \in (x,y) \\
&\le C_f|x-y|(\phi(x) + \phi(y)) + \frac{\mu}{m}\phi(z)|x-y| \\
&\le C_f|x-y|(\phi(x) + \phi(y)) + \frac{\mu}{m}|x-y|(\phi(x) + \phi(y)) \qquad (5.9) \\
&\le L_m|x-y|. \qquad (5.10)
\end{aligned}$$

since ϕ is convex and $D\phi(z) = \mu\frac{z}{\langle z\rangle}\phi(z) \le \mu\phi(z)$. Moreover, $f_m \to f$ locally uniformly in $I\!R^N$.

Let $n \ge 1$, we now truncate the Hamiltonian by defining an Hamiltonian H_{mn} such that

$$H_{mn}(x,p) = \begin{cases} H_m(x,p) & \text{if } |p| \le n \\ H_m(x, n\frac{p}{|p|}) & \text{if } |p| \ge n, \end{cases} \tag{5.11}$$

with

$$H_m(x,p) = \begin{cases} H(x,p) & \text{if } |x| \le m \\ H(m\frac{x}{|x|}, p) & \text{if } |x| \ge m. \end{cases}$$

It is easy to verify that $H_{mn} \in BUC(I\!R^N \times I\!R^N)$ with a modulus of continuity depending on m, n and satisfies (4.6) with the same constant C_H. Indeed,

- for $|p| \geq n$,

$$|H_{mn}(x,p)| = |H_m(x, n\frac{p}{|p|})| = \begin{cases} H(x, n\frac{p}{|p|}), |x| \leq m \\ H(m\frac{x}{|x|}, n\frac{p}{|p|}), |x| \geq m \end{cases} \leq C_H(1+n) \leq C_H(1+|p|),$$

- for $|p| \leq n$,

$$|H_{mn}(x,p)| = |H_m(x,p)| = \begin{cases} H(x,p), |x| \leq m \\ H(m\frac{x}{|x|}, p), |x| \geq m \end{cases} \leq C_H(1+|p|).$$

Moreover, we can check easily that $H_m(.,p)$ converge locally uniformly to $H(.,p)$ in $I\!R^N$. Indeed, fix $p \in I\!R^N$, $\forall x \in \Omega \subset I\!R^N$, $\forall \epsilon > 0$ we can take $N = |x| + diam(\Omega)$ then $\forall y \in B(x, diam(\Omega))$, $\forall m \geq N$ we have

$$|H_m(y,p) - H(y,p)| = |H(y,p) - H(y,p)| = 0 < \epsilon.$$

Therefore $H_m(.,p)$ converge locally uniformly to $H(.,p)$ in $I\!R^N$.

Similarly, we have $H_{mn}(x,.)$ converge locally uniformly to $H_m(x,.)$ in $I\!R^N$. Therefore, $H_{mn} \to H$ locally uniformly in $I\!R^N \times I\!R^N$.

We then consider the new equation

$$\lambda u - \mathcal{F}(x,[u]) + \langle b(x), Du\rangle + H_{mn}(x,Du) = f_m(x) \quad \text{in } I\!R^N. \tag{5.12}$$

The classical strong comparison principle holds for bounded discontinuous viscosity sub and supersolutions (see Theorem 5.4.1 in the Appendix). Noticing that $u^{\pm}_{\lambda,mn}(x) = \pm\lambda^{-1}(C_m + C_H)$ are respectively a super and a subsolution of (5.12), we obtain by means of Perron's method, the existence and uniqueness of a continuous viscosity solution $u_{\lambda,mn}$ of (5.12) such that $|u_{\lambda,mn}| \leq \tilde{C}_m$ independent of n. We refer to classical references [30] for the details.

2. Convergence of the solution of the approximate equation to a continuous solution of (4.1). Recall that H_{mn} satisfies (4.6) with constants C_H independent of m, n. Moreover, from (5.10) we have f_m is L_m-lipschitz. Then by applying Theorem 4.1.1 for bounded solutions $u_{\lambda,mn}$ we obtain that $u_{\lambda,mn}$ is K_m-lipschitz continuous, i.e.,

$$|u_{\lambda,mn}(x) - u_{\lambda,mn}(y)| \quad \leq \quad K_m|x-y| \quad \forall x, y \in I\!R^N.$$

Therefore, the family $(u_{\lambda,mn})_{n\geq 1}$ is uniformly equicontinuous in $I\!R^N$. By Arzelà Ascoli Theorem, it follows that, up to some subsequence,

$$u_{\lambda,mn} \to u_{\lambda,m} \quad \text{as } n \to +\infty \text{ locally uniformly in } I\!R^N.$$

By stability ([30]), $u_{\lambda,m}$ is a continuous viscosity solution of (5.12) with H_m in place of H_{mn}.

Similarly H_m (respectively f_m) satisfies (4.6) (respectively (4.4)) with constants C_H and $C_f + \mu$ independent of $m \geq 1$. By applying Theorem 4.1.1 again, we obtain that $u_{\lambda,m}$ satisfies (5.7) with C independent of λ, m.

To apply Arzelà Ascoli Theorem sending $m \to \infty$, we need some local L^∞ bound for $u_{\lambda,m}$ independent of m. Hence, we need the following lemma

Lemma 5.1.1. *For all $R > 0$, there exists a constant $C_R > 0$ independent of m and $\lambda \in (0,1)$ such that*

$$|\lambda u_{\lambda,m}(x)| \le C_R, \quad \forall x \in B(0,R).$$

Proof of Lemma 5.1.1. Let $\epsilon > 0$, we consider

$$\max_{\mathbb{R}^N}\{u_{\lambda,m}(x) - \epsilon\phi(x)\}.$$

Since $u_{\lambda,m}$ is a bounded viscosity solution of (5.12), the maximum is achieved at some point $y \in \mathbb{R}^N$ and

$$\lambda u_{\lambda,m}(y) - \mathcal{F}(y,[\epsilon\phi]) + \langle b(y), \epsilon D\phi(y)\rangle + H_m(y, \epsilon D\phi(y)) \le f_m(y).$$

Recall that H_m satisfies (4.6) with constant C_H independent of m. Hence, we have

$$\lambda u_{\lambda,m}(y) - \epsilon\mathcal{F}(y,[\phi]) + \epsilon\langle b(y), D\phi(y)\rangle - C_H - \epsilon C_H|D\phi(y)| \le f_m(y),$$

Now using Lemma 4.1.1 (4.22) with $x = y$ and $L(x,y) = L_0 = C_H$ to get that there exists a constants $K(\mathcal{F}, b, C_H) > 0$ such that

$$-\mathcal{F}(y,[\phi]) + \langle b(y), D\phi(y)\rangle - C_H|D\phi(y)| \ge \phi(y) - K.$$

Therefore

$$\lambda u_{\lambda,m}(y) \le f_m(y) - \epsilon\phi(y) + \epsilon K + C_H. \tag{5.13}$$

Fix $\epsilon > 0$ and take $m \ge \frac{3}{\epsilon}$. Since $f \in \mathcal{E}_\mu(\mathbb{R}^N)$, we have $\exists M(\epsilon) > 0$ such that

$$f(y) \le \frac{\epsilon}{3}\phi(y) + M(\epsilon).$$

Therefore, from (5.13) and by the definition of f_m we have

$$\begin{aligned}
\lambda u_{\lambda,m}(y) &\le f(y) + \frac{1}{m}\phi(y) - \epsilon\phi(y) + \epsilon K + C_H \\
&\le \frac{\epsilon}{3}\phi(y) + M(\epsilon) + \frac{\epsilon}{3}\phi(y) - \epsilon\phi(y) + \epsilon K + C_H \\
&\le M(\epsilon) + \epsilon K + C_H.
\end{aligned}$$

On the other hand, since y is a maximum point of $u(x) - \epsilon\phi(x)$ for $x \in \mathbb{R}^N$, then using (5.13) we have, $\forall \lambda \in (0,1)$,

$$\begin{aligned}
\lambda u_{\lambda,m}(x) &\le \lambda\epsilon\phi(x) + \lambda u_{\lambda,m}(y) - \lambda\epsilon\phi(y) \quad \forall x \in \mathbb{R}^N \\
&\le \lambda\epsilon\phi(x) - \lambda\epsilon\phi(y) + M(\epsilon) + \epsilon K + C_H \\
&\le \epsilon\phi(x) + M(\epsilon) + \epsilon K + C_H \\
&\le C_R \qquad \forall x \in B(0,R),
\end{aligned}$$

where $C_R = \sup_{B(0,R)}\{\epsilon\phi(x) + M(\epsilon) + \epsilon K + C_H\}$ is independent of m and λ.

The proof for the opposite inequality is the same by considering $\min_{\mathbb{R}^N}\{u_{\lambda,m}(x) + \epsilon\phi(x)\}$. □

Now we can apply Arzelà Ascoli Theorem to get, up to some subsequence,

$$u_{\lambda,m} \to u_\lambda \text{ as } m \to \infty \text{ locally uniformly in } I\!R^N.$$

By stability [30], u_λ is a continuous viscosity solution of (4.1).

Next we need to prove that $u_\lambda \in \mathcal{E}_\mu(I\!R^N)$. As in the proof of previous lemma, we have

$$|\lambda u_{\lambda,m}(x)| \leq \epsilon\phi(x) + M(\epsilon) + \epsilon K + C_H.$$

Since $u_{\lambda,m} \to u_\lambda$ as $m \to \infty$, then we get

$$|\lambda u_\lambda(x)| \leq \epsilon\phi(x) + M(\epsilon) + \epsilon K + C_H.$$

This holds for any $\epsilon > 0$, it means that $u_\lambda \in \mathcal{E}_\mu(I\!R^N)$.

Applying Theorem 4.1.1 for $u_\lambda \in C(I\!R^N) \cap \mathcal{E}_\mu(I\!R^N)$ we find that u_λ belongs to class of function (5.7).

We conclude to the existence of a continuous solution u of (4.1) belonging to the class (5.6)-(5.7).

3. Uniqueness of the solution of (4.1) *in* $C(I\!R^N) \cap \mathcal{E}_\mu(I\!R^N)$. It is a direct consequence of comparison principle (see Theorem 5.1.1) if we assume in addition (5.1). □

5.2 Well-posedness of the evolution problem

Theorem 5.2.1. *Suppose that* (4.3), (5.1) *and either* (4.2) *or (M1) hold. Let* $u \in USC(R_T) \cap \mathcal{E}^+_\mu(R_T)$ *and* $v \in LSC(R_T) \cap \mathcal{E}^-_\mu(R_T)$ *be a viscosity subsolution and a viscosity supersolution of* (4.68) *with* $u(\cdot,0) = u_0(\cdot)$, $f = f_1 \in C(I\!R^N)$ *and* $v(\cdot,0) = v_0(\cdot)$, $f = f_2 \in C(I\!R^N)$, *respectively. Assume that either* $u(\cdot,t)$ *or* $v(\cdot,t)$ *is locally Lipschitz continuous in* $I\!R^N$ *and* $\sup_{I\!R^N}\{u_0(x) - v_0(x)\} < +\infty$. *Then*

$$u(x,t) - v(x,t) \leq \sup_{I\!R^N}\{u_0(y) - v_0(y)\} + t|(f_1 - f_2)^+|_\infty, \quad \forall (x,t) \in R_T.$$

Proof. Set $U = \sup_{I\!R^N}\{u_0(x) - v_0(x)\} < +\infty$, we are going to prove that for all $\varrho > 0$,

$$u(x,t) - v(x,t) - \frac{\varrho}{T-t} - |(f_1 - f_2)^+|_\infty t - U \leq 0 \;\; \forall (x,t) \in R_T.$$

We argue by contradiction assuming that there exists some $(z,s) \in R_T$ such that

$$u(z,s) - v(z,s) - \frac{\varrho}{T-s} - |(f_1 - f_2)^+|_\infty s - U \geq 2\gamma > 0.$$

Let $\epsilon, \beta > 0$, we consider

$$\Psi(x,y,t) = u(x,t) - v(y,t) - \frac{|x-y|^2}{\epsilon^2} - \frac{\varrho}{T-t} - t|(f_1 - f_2)^+|_\infty - U - \beta(\phi(x) + \phi(y)),$$

and set

$$\varphi(x,y,t) = \frac{|x-y|^2}{\epsilon^2} + \frac{\varrho}{T-t} + t|(f_1 - f_2)^+|_\infty + U + \beta(\phi(x) + \phi(y)). \tag{5.14}$$

For small β we have $\Psi(z,z,s) \geq \gamma$. Since $u \in \mathcal{E}_\mu^+(R_T)$ and $v \in \mathcal{E}_\mu^-(R_T)$, the maximum of Ψ is achieved at some point $(\bar{x},\bar{y},\bar{t}) \in B(0,R_\beta) \times B(0,R_\beta) \times [0,T)$, where R_β does not depend on ϵ. It follows that $u(x,t) - v(y,t) - \frac{\varrho}{T-t} - t|(f_1-f_2)^+|_\infty - U - \beta(\phi(x)+\phi(y))$ are bounded in $B(0,R_\beta) \times B(0,R_\beta) \times [0,T)$, so the following classical properties hold (see [10]) up to some subsequences,

$$\frac{|\bar{x}-\bar{y}|^2}{\epsilon^2} \to 0, \qquad \bar{x},\bar{y} \to \hat{x} \in B(0,R_\beta) \text{ as } \epsilon \to 0,\ \beta \text{ is fixed.} \tag{5.15}$$

Assuming that $v(\cdot,\bar{t})$ for instance is locally Lipschitz continuous in $I\!R^N$, since $\Psi(\bar{x},\bar{x},\bar{t}) \leq \Psi(\bar{x},\bar{y},\bar{t})$ then we have

$$\begin{aligned}
\frac{|\bar{x}-\bar{y}|^2}{\epsilon^2} &\leq v(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) + \beta(\phi(\bar{x})-\phi(\bar{y})) \\
&\leq C|\bar{x}-\bar{y}|(\phi(\bar{x})+\phi(\bar{y})) + \beta\mu|\bar{x}-\bar{y}|(\phi(\bar{x})+\phi(\bar{y})).
\end{aligned}$$

This implies, up to some subsequence, $p_\epsilon := 2\frac{\bar{x}-\bar{y}}{\epsilon^2}$ remains bounded when $\epsilon \to 0$ and moreover $p_\epsilon \to \hat{p}$, for some $\hat{p} \in I\!R^N$.

If $\bar{t} = 0$, we know that for small β, $\Psi(\bar{x},\bar{y},0) > 0$, hence

$$\begin{aligned}
0 &< u(\bar{x},0) - v(\bar{y},0) - \frac{|\bar{x}-\bar{y}|^2}{\epsilon^2} - U - \frac{\varrho}{T} - \beta(\phi(\bar{x})+\phi(\bar{y})) \\
&\leq \limsup_{\epsilon\to 0}\{u(\bar{x},0) - v(\bar{y},0) - U - \frac{\varrho}{T} - \beta(\phi(\bar{x})+\phi(\bar{y}))\} \\
&\leq -\frac{\varrho}{T} - 2\beta\phi(\hat{x}).
\end{aligned}$$

Therefore $\bar{t} > 0$, now we can apply [30, Theorem 8.3] in the local case and [15, Corollary 2] in the nonlocal one to learn that, for any $\zeta > 0$, there exist $r,s \in I\!R$ and $X,Y \in \mathcal{S}^N$ such that

$$(r, D_x\varphi, X) \in \overline{\mathcal{P}}^{2,+}u(\bar{x},\bar{t}); \qquad (s, -D_y\varphi, Y) \in \overline{\mathcal{P}}^{2,-}v(\bar{y},\bar{t}),$$

$$r - s = \varphi_t(\bar{x},\bar{y},\bar{t}) = \frac{\epsilon}{(T-\bar{t})^2} + |(f_1-f_2)^+|_\infty \geq \frac{\epsilon}{T^2} + |(f_1-f_2)^+|_\infty$$

and

$$\begin{aligned}
D_x\varphi(\bar{x},\bar{y},\bar{t}) &= p_\epsilon + \beta D\phi(\bar{x}), \\
-D_y\varphi(\bar{x},\bar{y},\bar{t}) &= p_\epsilon - \beta D\phi(\bar{y}),
\end{aligned}$$

$$\begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq A + \zeta A^2, \qquad \text{where } A = D^2\varphi(\bar{x},\bar{y},\bar{t})$$

and $\zeta A^2 = O(\zeta)$ (ζ will be sent to 0 first).

Writing the viscosity inequalities at $(\bar{x},\bar{y},\bar{t})$ we have

$$\begin{aligned}
&r - \mathcal{F}(\bar{x},[u]) + \langle b(\bar{x}), D_x\varphi\rangle + H(\bar{x}, p_\epsilon + \beta D\phi(\bar{x})) \leq f_1(\bar{x}) \\
&s - \mathcal{F}(\bar{y},[v]) + \langle b(\bar{y}), -D_y\varphi\rangle + H(\bar{y}, p_\epsilon - \beta D\phi(\bar{y})) \geq f_2(\bar{y}).
\end{aligned}$$

Subtracting the viscosity inequalities, we obtain

$$\begin{array}{ll}
& \frac{\varrho}{T^2} + |(f_1-f_2)^+|_\infty - (\mathcal{F}(\bar{x},[u]) - \mathcal{F}(\bar{y},[v])) + \langle b(\bar{x}) - b(\bar{y}), \frac{2(\bar{x}-\bar{y})}{\epsilon^2}\rangle \\
& +\beta\langle b(\bar{x}), D\phi(\bar{x})\rangle + \beta\langle b(\bar{y}), D\phi(\bar{y})\rangle + H(\bar{x}, p_\epsilon + \beta D\phi(\bar{x})) - H(\bar{y}, p_\epsilon - \beta D\phi(\bar{y})) \\
\leq & f_1(\bar{x}) - f_2(\bar{y}),
\end{array} \tag{5.16}$$

where $\mathcal{F}(\bar{x},[u]) = \mathrm{tr}(A(\bar{x})X)$ and $\mathcal{F}(\bar{y},[u]) = \mathrm{tr}(A(\bar{y})Y)$ in the local case and $\mathcal{F}(\bar{x},[u]) = \mathcal{I}(\bar{x}, u, p_\epsilon + \beta D\phi(\bar{x}))$ and $\mathcal{F}(y,[u]) = \mathcal{I}(\bar{y}, u, p_\epsilon - \beta D\phi(\bar{y}))$ in the nonlocal one.

In the case $\mathcal{F}$ is the local operator in (2.27) and (4.2) holds only, applying Lemma 4.1.2 for the classical test function (5.14) instead of (4.9), we obtain

$$\mathrm{tr}(A(\bar{x})X - A(\bar{y})Y) \leq L_\sigma^2 \frac{|\bar{x}-\bar{y}|^2}{\epsilon^2} + \beta \mathrm{tr}(A(\bar{x})D^2\phi(\bar{x})) + \beta\mathrm{tr}(A(\bar{y})D^2\phi(\bar{y})).$$

On the other hand, if $\mathcal{F} = \mathcal{I}$ then applying Proposition 4.1.1 (4.35) by using (5.14), we get

$$\mathcal{I}(\bar{x}, u, p_\epsilon + \beta D\phi(\bar{x})) - \mathcal{I}(\bar{y}, v, p_\epsilon - D\phi(\bar{y})) \leq \beta\mathcal{I}(\bar{x}, \phi, D\phi) + \beta\mathcal{I}(\bar{y}, \phi, D\phi).$$

Therefore, in any cases we have

$$\mathcal{F} \leq \beta\mathcal{F}(\bar{x},[\phi]) + \beta\mathcal{F}(\bar{y},[\phi]) + L_\sigma^2\frac{|\bar{x}-\bar{y}|^2}{\epsilon^2}. \tag{5.17}$$

Using (4.3) to estimate for b-terms in (5.16) and taking into account (5.17), we get

$$\begin{array}{ll}
& \frac{\varrho}{T^2} + |(f_1-f_2)^+|_\infty - L_\sigma^2\frac{|\bar{x}-\bar{y}|^2}{\epsilon^2} + \frac{2}{\epsilon^2}\alpha|\bar{x}-\bar{y}|^2 \\
& +\beta(-\mathcal{F}(\bar{x},[\phi]) - \mathcal{F}(\bar{y},[\phi]) + \langle b(\bar{x}), D\phi(\bar{x})\rangle + \langle b(\bar{y}), D\phi(\bar{y})\rangle) \\
& +H(\bar{x}, p_\epsilon + \beta D\phi(\bar{x})) - H(\bar{y}, p_\epsilon - \beta D\phi(\bar{y})) \\
\leq & f_1(\bar{x}) - f_2(\bar{y}).
\end{array}$$

Now sending ϵ to 0, using (5.15) and since $f_1, f_2 \in C(I\!R^N)$, we obtain

$$\begin{array}{ll}
& \frac{\varrho}{T^2} + |(f_1-f_2)^+|_\infty + 2\beta(-\mathcal{F}(\hat{x},[\phi]) + \langle b(\hat{x}), D\phi(\hat{x})\rangle) \\
& +H(\hat{x}, \hat{p} + \beta D\phi(\hat{x})) - H(\hat{x}, \hat{p} - \beta D\phi(\hat{x})) \\
\leq & f_1(\hat{x}) - f_2(\hat{x}), \quad \text{for some } \hat{x} \in B(0, R_\beta).
\end{array}$$

Since $H(x,.)$ satisfies (5.1) then we get

$$\frac{\varrho}{T^2} - 2\beta\mathcal{F}(\hat{x},[\phi]) + 2\beta\langle b(\hat{x}), D\phi(\hat{x})\rangle - 2\beta L_H|D\phi(\hat{x})| \leq 0.$$

Recalling from Lemma 4.1.1, there exists a constant $K(L_H, \mathcal{F}) > 0$ such that

$$-\mathcal{F}(x,[\phi]) + \langle b(x), D\phi(x)\rangle - L_H|D\phi(x)| \geq \phi(x) - K \qquad \forall x \in I\!R^N.$$

Therefore, we have

$$\frac{\varrho}{T^2} + 2\beta\phi(\hat{x}) - 2\beta K \leq 0.$$

Since $\phi > 0$, sending β to 0, we get a contradiction.

We have proved that

$$u(x,t) - v(x,t) - \frac{\varrho}{T-t} - t|(f_1 - f_2)^+|_\infty\} - \max_{I\!R^N}\{u(y,0) - v(y,0)\} \leq 0, \quad \forall (x,t) \in R_T.$$

Sending ϱ to 0 we get the conclusion. □

Theorem 5.2.2. *Suppose (4.3), (4.4), (4.6), (4.69) and $u_0 \in \mathcal{E}_\mu(I\!R^N) \cap C(I\!R^N)$. Assume either (4.2), (4.5) or (M1) holds. There exists a continuous viscosity solution u of (4.68) in the class of functions*

$$u \in \mathcal{E}_\mu(\bar{Q}), \tag{5.18}$$
$$|u(x,t) - u(y,t)| \leq C|x-y|(\phi(x) + \phi(y)), \quad x, y \in I\!R^N,\ t \in [0,T), \tag{5.19}$$

where $C > 0$ is a constant independent of T.
In addition, if (5.1) holds then the solution is unique in $C(\bar{Q}) \cap \mathcal{E}_\mu(\bar{Q})$.

Proof. We only give a sketch of proof since it is similar with the proof of Theorem 5.1.2.

1. Construction of a continuous viscosity solution to a truncated equation. Let $m \geq 1$, we first truncate the initial data

$$u_{0m}(x) = \min\{u_0(x) + \frac{1}{m}\phi(x), m\}. \tag{5.20}$$

Since $u_0 \in \mathcal{E}_\mu(I\!R^N)$, similarly with the arguments for f_m in the proof of Theorem 5.1.2, we get

$$\begin{aligned} &|u_{0m}(x)| \leq C_m, \quad \forall x \in I\!R^N \\ &|u_{0m}(x) - u_{0m}(y)| \leq L_m|x-y|. \end{aligned} \tag{5.21}$$

Moreover, u_{0m} still satisfies (4.69) with the constant $C_0 + \mu$ and $u_{0m} \to u_0$ locally uniformly in $I\!R^N$.

We then introduce the truncated evolution problem (4.68) with H_{mn} (respectively f_m) defined by (5.11) (respectively (5.8)) for $m, n \geq 1$ and with the initial data defined by (5.20). The classical comparison principle (Theorem 5.4.2) holds for bounded discontinuous viscosity sub and supersolutions of

$$u_t - \mathcal{F}(x,[u]) + \langle b(x), Du\rangle + H_{mn}(x, Du) = f_m(x) \quad \text{in } I\!R^N, \tag{5.22}$$

with the initial data $u_{mn}(x,0) = u_0(x)$.

Noticing that $u^{\pm}_{mn}(x,t) = \pm(C_m + (C_m + C_H)t)$ are respectively a super and a subsolution of (5.22). Moreover

$$u^-_{mn}(x,0) = -C_m \leq u_{0m}(x) \leq C_m = u^+_{mn}(x,0).$$

Then by means of Perron's method, we obtain the existence and uniqueness of a continuous viscosity solution u_{mn} of (5.22) such that $|u_{mn}| \leq \tilde{C}_m$ independent of n. We refer to classical references [30] for the details.

2. Convergence of the solution of the truncated equation to a continuous solution of (4.68). Recall that H_{mn} satisfies (4.6) with constants C_H independent of $m, n,$. Moreover, from (5.10) and (5.21) we have f_m and u_{0m} are L_m-lipschitz. Then by applying Theorem 4.2.1 for bounded solutions - u_{mn} we obtain that u_{mn} is K_m-lipschitz continuous, i.e.,

$$|u_{mn}(x,t) - u_{mn}(y,t)| \quad \leq \quad K_m|x-y| \quad \forall x, y \in I\!R^N, t \in [0,T).$$

Therefore, the family $(u_{mn})_{n\geq 1}$ is uniformly equicontinuous and bounded in $\bar{Q}$. It follows that, up to some subsequence,

$$u_{mn} \to u_m \quad \text{as } n \to +\infty \text{ locally uniformly in } \bar{Q}.$$

By stability [30], u_m is a continuous viscosity solution of (5.12) with H_m in place of H_{mn}.

Similarly H_m (respectively f_m) satisfies (4.6) (respectively (4.4)) with constants C_H and $C_f + \mu$, u_{0m} satisfies (4.69) with constant $C_0 + \mu$ independent of m. By applying Theorem 4.1.1 again, we obtain that u_m satisfies (5.19) with C independent of m.

To apply Arzelà Ascoli Theorem sending $m \to \infty$, we need some local bound for u_m. Therefore we need to use following Lemma the proof of which is given in the Appendix

Lemma 5.2.1. *Let $T > 0$, for all $R > 0$, there exists a constant $C_{RT} > 0$ independent of m such that*

$$|u_m(x,t)| \leq C_{RT}, \quad \forall x \in B(0,R), \ \forall t \in [0,T).$$

Moreover $u_m \in \mathcal{E}_\mu(\bar{Q})$.

From Lemma 5.2.1, there is a constant $C_{RT} > 0$ independent of m such that

$$|u_m(x,t)| \leq C_{RT}, \quad \forall x \in B(0,R), \ \forall t \in [0,T).$$

Therefore, the family $(u_m)_{m\geq 1}$ is uniformly equicontinuous and bounded on compact subsets of $\bar{Q}$. By Arzelà Ascoli Theorem, it follows that, up to some subsequence,

$$u_m \to u \quad \text{as } m \to +\infty \text{ locally uniformly in } \bar{Q}.$$

By stability [30], u is a continuous viscosity solution of (4.68). Using Lemma 5.2.1 again, we get $u \in \mathcal{E}_\mu(\bar{Q})$.

We conclude to the existence of a continuous solution u of (4.68) belonging to the class (5.18)-(5.19).

3. Uniqueness of the solution of (4.68) *in the class $C(\bar{Q}) \cap \mathcal{E}_\mu(\bar{Q})$.* It is a direct consequence of comparison principle (see Theorem 5.2.1) if we assume in addition (5.1). □

5.3 Well-posedness of the stationary and evolution problem with different assumptions on the Hamiltonian.

In this section, we study well-posedness for both stationary and evolution problem by using different assumptions on the Hamiltonian. They are

$$\begin{cases} \text{There exists } L_{H_1}, L_{H_2} > 0 \text{ such that} \\ |H(x,p) - H(y,p)| \le L_{H_1}|x-y|(1+|p|), \quad x,y,p,q \in I\!R^N \\ |H(x,p) - H(x,q)| \le L_{H_2}|p-q| \\ |H(x,0)| \le H_0, \end{cases} \tag{5.23}$$

and weaker assumption

$$\begin{cases} \text{There exists } C_H > 0 \text{ such that} \\ |H(x,p) - H(y,p)| \le C_H|x-y|(1+|p|), \quad x,y,p,q \in I\!R^N \\ |H(x,p) - H(x,q)| \le C_H|p-q|(1+|x|), \\ |H(x,0)| \le H_0. \end{cases} \tag{5.24}$$

From (5.23), we have

$$H(x,p) \le H(x,0) + L_{H_2}|p| \le H_0 + L_{H_2}|p| \le (H_0 + L_{H_2})(1+|p|).$$

- $|x-y| \le 1 : |H(x,p) - H(y,p)| \le L_{H_1}(1+|p|)|x-y|.$
- $|x-y| \ge 1 : |H(x,p) - H(y,p)| \le |H(x,p)| + |H(y,p)| \le 2(H_0 + L_{H_2})(1+|p|).$

Put

$$L_H = \max\{L_{H_1}, 2(H_0 + L_{H_2})\}.$$

So we have the new condition on H

$$|H(x,p) - H(y,p)| \le L_H(1+|p|)(1 \wedge |x-y|).$$

Therefore, hypothesis (5.23) now can be rewritten as

$$\begin{cases} |H(x,p) - H(y,p)| \le L_H(1+|p|)(1 \wedge |x-y|), \quad x,y,p,q \in I\!R^N, \\ |H(x,p) - H(x,q)| \le L_{H_2}|p-q|. \end{cases} \tag{5.25}$$

5.3.1 Results for stationary problem

Theorem 5.3.1. *Let $u \in USC(I\!R^N) \cap \mathcal{E}_\mu^+(I\!R^N)$ and $v \in LSC(I\!R^N) \cap \mathcal{E}_\mu^-(I\!R^N)$ be a viscosity subsolution and a viscosity supersolution of (4.1), respectively. Suppose that (4.3), (5.25), $f \in C(I\!R^N)$ and either (4.2) or (M1) hold. Then $u \le v$ in $I\!R^N$.*

Theorem 5.3.2. *For any $\alpha > C(H)$, under the assumptions of Theorem 5.3.1 with (5.25) replaced by (5.24). Then $u \le v$ in $I\!R^N$.*

Theorem 5.3.3. *Under the assumptions of Theorem 5.3.1, there is a unique solution $u^\lambda \in C(I\!R^N) \cap \mathcal{E}_\mu(I\!R^N)$ of (4.1).*

Theorem 5.3.4. *Under the assumptions of Theorem 5.3.2, there is a unique solution $u^\lambda \in C(I\!R^N) \cap \mathcal{E}_\mu(I\!R^N)$ of (4.1).*

Proof of Theorem 5.3.1. Recalling from Lemma 4.1.1 (4.23) with $L(x,y) = L_{H_2}$ and fix a constant $K > 0$ such that

$$-\mathcal{F}(x,[\phi]) + \langle b(x), D\phi(x)\rangle - L_{H_2}|D\phi(x)| \geq \phi(x) - K \quad \text{in } I\!R^N. \tag{5.26}$$

Fix such a constant K and, for $\epsilon \in (0,1)$, we define the functions u_ϵ, v_ϵ on $I\!R^N$ as following

$$u_\epsilon(x) = u(x) - \epsilon(\phi(x) + \lambda^{-1}K) \tag{5.27}$$

$$v_\epsilon(x) = v(x) - \epsilon(\phi(x) + \lambda^{-1}K) \tag{5.28}$$

Observe that $u_\epsilon, -v_\epsilon \in USC(I\!R^N)$ and using (5.26), we easily verify that u_ϵ and v_ϵ are respectively a viscosity subsolution and a viscosity supersolution of (4.1).

Now we prove that $u_\epsilon \leq v_\epsilon$ in $I\!R^N$. We argue by contradiction assuming that $u_\epsilon(z) - v_\epsilon(z) \geq 2\delta > 0$ for some $z \in I\!R^N$.

We consider,

$$\Psi(x,y) = \sup_{I\!R^N \times I\!R^N} \{u_\epsilon(x) - v_\epsilon(y) - \frac{|x-y|^2}{2\eta^2} - \beta(|x|^2 + |y|^2)\}.$$

where η, β are small parameters. For small β we have $\Psi(z,z) \geq \delta$. Since $u \in \mathcal{E}_\mu^+(I\!R^N), v \in \mathcal{E}_\mu^-(I\!R^N)$, we suppose that Ψ attains its maximum at some points $(x,y) \in B(0,R_\beta) \times B(0,R_\beta)$, where R_β does not depend on η. The following classical properties hold (see [10]) up to some subsequences,

$$\frac{|x-y|^2}{2\eta^2} \to 0 \ \text{ as } \eta \to 0 \text{ and } \beta|x|, \beta|y| \to 0 \text{ as } \beta \to 0. \tag{5.29}$$

Setting $p_\eta = \frac{x-y}{\eta^2}$. Applying [30, Theorem 3.2] in the local case and [15, Corollary 1] in the nonlocal one to learn that, for any $\varrho > 0$, there exist $X, Y \in \mathcal{S}^N$ such that $(p_\eta + 2\beta x, X) \in \bar{J}^{2,+}u_\epsilon(x)$, $(p_\eta - 2\beta y, Y) \in \bar{J}^{2,-}v_\epsilon(y)$, and

$$\begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq E + \varrho E^2,$$

where

$$E = \frac{1}{\eta^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\beta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

and $\varrho E^2 = O(\varrho)$ (ϱ will be sent to 0 first).

Writing the viscosity inequalities at (x,y), we have

$$\lambda u_\epsilon(x) - \mathcal{F}(x,[u]) + \langle b(x), p_\eta + 2\beta x\rangle + H(x, p_\eta + 2\beta x) \le f(x),$$
$$\lambda v_\epsilon(y) - \mathcal{F}(y,[v]) + \langle b(y), p_\eta - 2\beta y\rangle + H(y, p_\eta - 2\beta y) \ge f(y).$$

Subtracting the viscosity inequalities, we obtain

$$\begin{aligned} & \lambda(u_\epsilon(x) - v_\epsilon(y)) - (\mathcal{F}(x,[u]) - \mathcal{F}(y,[v])) \\ & + \langle b(x) - b(y), p_\eta\rangle + 2\beta\langle b(x), x\rangle + 2\beta\langle b(y), y\rangle \\ & + H(x, p_\eta + 2\beta x) - H(y, p_\eta - 2\beta y) \\ \le\ & f(x) - f(y), \end{aligned} \tag{5.30}$$

where $\mathcal{F}(x,[u]) = \mathrm{tr}(A(x)X)$ and $\mathcal{F}(y,[u]) = \mathrm{tr}(A(y)Y)$ in the local case and $\mathcal{F}(x,[u]) = \mathcal{I}(x,u,p_\eta + 2\beta x)$ and $\mathcal{F}(y,[u]) = \mathcal{I}(y,u,p_\eta - 2\beta y)$ in the nonlocal one.

In the case $\mathcal{F}$ is the local operator in (2.27) and (4.2) holds only, applying Lemma 4.1.2 for the classical test function $\frac{|x-y|^2}{2\eta^2} + \beta(|x|^2 + |y|^2)$ instead of (4.9), we obtain

$$\begin{aligned} \mathrm{tr}(A(x)X - A(y)Y) &\le L_\sigma^2 \frac{|x-y|^2}{\eta^2} + 2\beta \mathrm{tr}(A(x)) + 2\beta \mathrm{tr}(A(y)) \\ &\le L_\sigma^2 \frac{|x-y|^2}{\eta^2} + 4\beta|\sigma|^2. \end{aligned}$$

On the other hand, if $\mathcal{F}$ is the nonlocal operator defined by (2.27) then applying Proposition 4.1.1 (4.35) by using $\varphi(x,y) = \frac{|x-y|^2}{2\eta^2} + \beta(|x|^2 + |y|^2)$, we get

$$\mathcal{I}(x,u,p_\eta + 2\beta x) - \mathcal{I}(y,v,p_\eta - 2\beta y) \le 2\beta\mathcal{I}(x,|x|^2,2\beta x) + 2\beta\mathcal{I}(y,|y|^2,2\beta y).$$

Moreover, by the definition of $\mathcal{I}$ in (2.23) and since (2.25) we have

$$\begin{aligned} \mathcal{I}(x,|x|^2,2\beta x) &= \int_{B^c} (|x+z|^2 - |x|^2)\nu(dz) + \int_B (|x+z|^2 - |x|^2 - 2x\cdot z)\nu(dz) \\ &= \int_{B^c} (|x+z|^2 - |x|^2)\nu(dz) + \int_B \int_0^1 (1-s)2|z|^2 \nu(dz) \\ &\le C(\nu), \end{aligned}$$

here we used the property (M1) of ν. Then we get

$$\mathcal{I}(x,u,p_\eta + 2\beta x) - \mathcal{I}(y,v,p_\eta - 2\beta y) \le 4\beta C(\nu).$$

Therefore, in any cases we have

$$\mathcal{F} \le 4\beta C(\mathcal{F}) + L_\sigma^2 \frac{|x-y|^2}{\eta^2}. \tag{5.31}$$

From (4.3) we have

$$\begin{aligned} & \langle b(x) - b(y), p_\eta\rangle + 2\beta(\langle b(x), x\rangle + \langle b(y), y\rangle) \\ =\ & \langle b(x) - b(y), p_\eta\rangle + 2\beta\langle b(x) - b(0), x\rangle + 2\beta\langle b(y) - b(0), y\rangle + 2\beta\langle b(0), x+y\rangle \\ \ge\ & \alpha\frac{|x-y|^2}{\eta^2} + 2\beta\alpha(|x|^2 + |y|^2) - 2\beta|b(0)|(|x| + |y|). \end{aligned} \tag{5.32}$$

To estimate for H term, using (5.25), we have

$$
\begin{aligned}
& |H(x, p_\eta + 2\beta x) - H(y, p_\eta - 2\beta y)| \\
\leq \;& L_{H_1}|x-y| + L_{H_1}\frac{|x-y|^2}{2\eta^2} + 2\beta L_{H_1}|x-y||x| + 2\beta L_{H_2}(|x|+|y|).
\end{aligned} \tag{5.33}
$$

Since $\delta \leq \Psi(z,z) \leq u_\epsilon(x) - v_\epsilon(y)$, from (5.31), (5.32), (5.33) we have

$$
\begin{aligned}
& \lambda\delta + \alpha\frac{|x-y|^2}{\eta^2} + 2\beta\alpha(|x|^2+|y|^2) \\
\leq \;& 4\beta C(\mathcal{F}) + L_\sigma^2\frac{|x-y|^2}{\eta^2} + L_{H_1}|x-y| + L_{H_1}\frac{|x-y|^2}{2\eta^2} \\
& +2\beta L_{H_1}|x-y||x| + 2\beta L_{H_2}(|x|+|y|) + f(x) - f(y).
\end{aligned} \tag{5.34}
$$

Since $f \in C(I\!R^N)$, sending η to 0 then β to 0 and using (5.29) we get a contradiction. We have proved that $u_\epsilon \leq v_\epsilon$ $\forall\epsilon \in (0,1)$, i.e.,

$$
u(x) - v(x) - 2\epsilon(\phi(x) + \lambda^{-1}K) \leq 0 \qquad \forall x \in I\!R^N, \;\; \forall \epsilon \in (0, 1.)
$$

Sending ϵ to 0 we conclude that $u \leq v$ in $I\!R^N$. $\square$

Proof of Theorem 5.3.2. Recalling Lemma 4.1.1, for any $\alpha > 2C_H$, there exists a constant $K > 0$ such that

$$
-\mathcal{F}(x, [\phi]) + \langle b(x), D\phi(x)\rangle - C_H|D\phi(x)|(1+|x|) \geq \phi(x) - K \quad \text{in } I\!R^N.
$$

Then u_ϵ and v_ϵ defined in (5.27) and (5.28) are respectively a viscosity subsolution and a viscosity supersolution of (4.1).

Hence, we do the same argument with the proof of Theorem 5.3.1 with the estimate of H in (5.33) replaced by

$$
\begin{aligned}
& H(x, p_\eta + 2\beta x) - H(y, p_\eta - 2\beta y) \\
\leq \;& C_H|x-y|(1+|p_\eta|+2\beta|x|) + 2\beta C_H(|x|+|y|)(1+|x|),
\end{aligned}
$$

where (5.24) is used.

Therefore, instead of (5.34), we have

$$
\begin{aligned}
& \lambda\delta + \alpha\frac{|x-y|^2}{\eta^2} + 2\beta\alpha(|x|^2+|y|^2) \\
\leq \;& 4\beta C(\mathcal{F}) + L_\sigma^2\frac{|x-y|^2}{\eta^2} + C_H|x-y|(1+|p_\eta|+2\beta|x|) \\
& +2\beta C_H(|x|+|y|)(1+|x|)) + f(x) - f(y).
\end{aligned}
$$

Since $f \in C(I\!R^N)$, for $\alpha > 2C_H$, sending η to 0 and then β to 0 we reach contradiction and get a conclusion as in Theorem 5.3.1. $\square$

The proof of Theorem 5.3.3 and Theorem 5.3.4 are easily adapted thanks to [42] and by Theorem 5.3.1 and Theorem 5.3.2 respectively.

5.3.2 Results for evolution problem

Theorem 5.3.5. *Let $u \in USC(R_T) \cap \mathcal{E}_\mu^+(R_T)$ and $v \in LSC(R_T) \cap \mathcal{E}_\mu^-(R_T)$ be a viscosity subsolution and a viscosity supersolution of (4.68), respectively. Suppose that (4.3), (5.25) and either (4.2) or (2.25) hold. Assume that $u(x,0) \le v(x,0)$ for all $x \in \mathbb{R}^N$, then $u \le v$ on R_T.*

Theorem 5.3.6. *For any $\alpha > C(H)$, under the assumptions of Theorem 5.3.5 with (5.25) replaced by (5.24). Then $u \le v$ on R_T.*

Theorem 5.3.7. *Under the assumptions of Theorem 5.3.5, there is a unique solution $u \in C(R_T) \cap \mathcal{E}_\mu(R_T)$.*

Theorem 5.3.8. *Under the assumptions of Theorem 5.3.6, there is a unique solution $u \in C(R_T) \cap \mathcal{E}_\mu(R_T)$.*

Proof of Theorem 5.3.5. We first recall from Lemma 4.1.1 that there exists a constant $K > 0$ so that (4.23) holds. Fix any $\epsilon > 0$, and define the function $u_\epsilon \in USC(R_T)$ and $v_\epsilon \in LSC(R_T)$ by

$$\begin{aligned} u_\epsilon(x,t) &= u(x,t) - \epsilon\phi(x) - \epsilon K t, \\ v_\epsilon(x,t) &= v(x,t) + \epsilon\phi(x) + \epsilon K t. \end{aligned}$$

Observe that $u_\epsilon, -v_\epsilon \in USC(R_T)$ and that u_ϵ and v_ϵ are, respectively, a viscosity subsolution and a viscosity supersolution of (4.68) in Q_T.

Next we use the following lemma the proof of which is similar with the proof of Theorem 5.2.1 in case of bounded discontinuous viscosity solutions where (5.25) is used instead of (5.1).

Lemma 5.3.1. *Under the assumptions of Theorem 5.3.5, if u is viscosity subsolution of*

$$v_t - \mathcal{F}(x,[v]) + \langle b(x), Dv(x,t)\rangle + H(x, Dv(x,t)) = f_1(x) \qquad \text{in } \mathbb{R}^N \times (0,T), \quad (5.35)$$

and v is viscosity supersolution of

$$v_t - \mathcal{F}(x,[v]) + \langle b(x), Dv(x,t)\rangle + H(x, Dv(x,t)) = f_2(x) \qquad \text{in } \mathbb{R}^N \times (0,T). \quad (5.36)$$

Suppose that $f_1, f_2 \in C(\mathbb{R}^N)$, $u, -v \in USC(R_T)$ are bounded, then

$$u(x,t) - v(x,t) \le \sup_{\mathbb{R}^N}\{u(y,0) - v(y,0)\} + t|(f_1 - f_2)^+|_\infty. \quad (5.37)$$

Apply the above lemma for u_ϵ and v_ϵ with $f_1 = f_2 = f$ to get that

$$\begin{aligned} u_\epsilon(x,t) - v_\epsilon(x,t) &\le \sup_{\mathbb{R}^N}\{u_\epsilon(y,0) - v_\epsilon(y,0)\} \\ &\le \sup_{\mathbb{R}^N}\{u(y,0) - v(y,0) - 2\epsilon\phi(y)\} \\ &\le 0 \qquad \text{for any } \epsilon > 0. \end{aligned}$$

Sending $\epsilon \to 0$ allows us to conclude that $u \le v$ on R_T. □

Proof of Theorem 5.3.6. The proof of this Theorem is an adaptation of the one of Theorem 5.3.2 using the same extension to the parabolic case as explained in the proof of Theorem 5.3.5. More precisely, we suppose in addition that $\alpha > C(H)$, where α comes from (4.3) and use (5.24) instead of (5.25). $\square$

For the proof of Theorem 5.3.7 and 5.3.8, the existences are done thanks to Perron's method as [42], the uniquenesses of solution are direct consequence of Theorem 5.3.5 and 5.3.6 respectively.

5.4 Appendix

Theorem 5.4.1. *Let $u \in BUSC(I\!R^N)$ and $v \in BLSC(I\!R^N)$ be a viscosity subsolution and a viscosity supersolution of (5.12), respectively. Assume that $f \in BC(I\!R^N)$, $H \in BUC(I\!R^N \times I\!R^N)$, (4.3) and either (4.2) or (2.25) hold. Then $u \leq v$ in $I\!R^N$.*

Proof. We use the same arguments with the proof of Theorem 5.3.1 for the bounded case. The only difference here is $H \in BUC(I\!R^N \times I\!R^N)$, hence instead of using (5.33) we use

$$|H(x, p_\eta + 2\beta x) - H(y, p_\eta - 2\beta y)| \leq \omega(|x-y| + 2\beta|x+y|).$$

Since $f \in BC(I\!R^N)$, then (5.34) is replaced by

$$\begin{aligned} & \lambda\delta + \alpha\frac{|x-y|^2}{\eta^2} + 2\beta\alpha(|x|^2 + |y|^2) \\ \leq \quad & L_\sigma^2 + 4\beta C(\mathcal{F}) + \frac{|x-y|^2}{\eta^2} + \omega_H(|x-y| + 2\beta|x+y|) + \omega_f(|x-y|). \end{aligned}$$

Sending η to 0 then β to 0 and using (5.29) we get a contradiction. $\square$

Theorem 5.4.2. *Let $u \in BUSC(I\!R^N \times [0,T))$ and $v \in BLSC(I\!R^N \times [0,T))$ be a viscosity subsolution and a viscosity supersolution of (5.12), respectively. Assume that $f \in BC(I\!R^N)$, $H \in BUC(I\!R^N \times I\!R^N)$, (4.3) and either (4.2) or (2.25) hold. Suppose that $u(x,0) \leq v(x,0)$ for all $x \in I\!R^N$. Then $u \leq v$ in $I\!R^N$.*

The proof is easily adapted by the proof of Theorem 5.4.1.

Proof of Lemma 5.2.1. The proof is inspired by [42]. By the definition of u_{0m} in (5.20), we have $u_{0m} \in \mathcal{E}_\mu(I\!R^N) \cap C(I\!R^N)$. Hence, by [42], for any $\epsilon \in (0,1)$ there is a constant $M(\epsilon) > 0$ such that for all $x, y \in I\!R^N$,

$$|u_{0m}(x) - u_{0m}(y)| \leq \epsilon(\phi(x) - \phi(y)) + M(\epsilon)|x-y|.$$

Fix such a collection $\{M(\epsilon) | \epsilon \in (0,1)\}$ of positive numbers.

Let $y \in I\!R^N, \epsilon \in (0,1)$, and $A > 0$, and set

$$g(x,t) = u_{0m}(y) + \epsilon(\phi(x) + \phi(y)) + M(\epsilon)\langle x-y\rangle + At \qquad \text{for } (x,t) \in \overline{Q}.$$

We compute that for $(x,t) \in Q$,

$$Dg(x,t) = D_x g(x,t) = \epsilon D\phi(x) + M(\epsilon)\frac{x-y}{\langle x-y\rangle}; \tag{5.38}$$

$$D^2 g(x,t) = D_x^2 g(x,t) = \epsilon D^2\phi(x) + \frac{M(\epsilon)}{\langle x-y\rangle}\left[I - \frac{x-y}{\langle x-y\rangle} \otimes \frac{x-y}{\langle x-y\rangle}\right] \tag{5.39}$$

Set $q = \frac{x-y}{\langle x-y\rangle}$. If $\mathcal{F}$ is the local operator defined by (2.27). Then from (5.39) we have

$$\begin{aligned}
& \mathcal{F}(x,[g]) = \mathrm{tr}(A(x)D^2 g(x,t)) \qquad (5.40)\\
= \;& \epsilon \mathrm{tr}(A(x)D^2\phi(x)) + \frac{M(\epsilon)}{\langle x-y\rangle}\left[\mathrm{tr}(A(x)) - \mathrm{tr}(A(x)q\otimes q)\right]\\
\geq \;& -\epsilon \mathrm{tr}(A(x)D^2\phi(x)) - \frac{M(\epsilon)}{\langle x-y\rangle}|\sigma|_F^2\\
\geq \;& -\epsilon \mathrm{tr}(A(x)D^2\phi(x)) - M(\epsilon)|\sigma|^2.
\end{aligned}$$

On the other hand, if $\mathcal{F}$ is the nonlocal operator defined by (2.27), then from (5.38) we have

$$\begin{aligned}
\mathcal{F}(x,[g]) &= \int_{\mathbb{R}^N}(g(x+z,t) - g(x,t) - Dg(x,t)\cdot z\mathbb{I}_B(z))d\nu(z)\\
&= \epsilon\int_{\mathbb{R}^N}(\phi(x+z) - \phi(x) - D\phi(x)\cdot z\mathbb{I}_B(z))d\nu(z)\\
&\quad + M(\epsilon)\int_{\mathbb{R}^N}\langle x+z-y\rangle - \langle x-y\rangle - D(\langle x-y\rangle)\cdot z\mathbb{I}_B(z))d\nu(z)\\
&= \epsilon\mathcal{I}(x,\phi,D\phi) + M(\epsilon)\int_{B^c}(\langle x+z-y\rangle - \langle x-y\rangle)\nu(dz)\\
&\quad + M(\epsilon)\int_B(\langle x+z-y\rangle - \langle x-y\rangle) - D(\langle x-y\rangle)\cdot z)d\nu(z)\\
&= \epsilon\mathcal{I}(x,\phi,D\phi) + M(\epsilon)\int_{B^c}\int_0^1\frac{x-y+sz}{\sqrt{|x-y+sz|^2+1}}\cdot z ds\nu(dz)\\
&\quad + \int_B\int_0^1(1-s)\left(|z|^2 - |\frac{x-y+sz}{\sqrt{|x-y+sz|^2+1}}\cdot z|^2\right)ds\nu(dz)\\
&\leq \epsilon\mathcal{I}(x,\phi,D\phi) + M(\epsilon)\left(\int_{B^c}|z|\nu(dz) + \int_B|z|^2\nu(dz)\right)\\
&\leq \epsilon\mathcal{I}(x,\phi,D\phi) + M(\epsilon)C(\nu).
\end{aligned}$$

Hence, from (5.40) and (5.41) we have

$$\mathcal{F}(x,[g]) \leq \epsilon\mathcal{F}(x,[\phi]) + M(\epsilon)C(\mathcal{F}). \tag{5.41}$$

From (5.38) and (4.3), we have

$$\begin{aligned}\langle b(x), Dg(x,t)\rangle &= \langle b(x), \epsilon D\phi(x) + M(\epsilon)\frac{x-y}{\langle x-y\rangle}\rangle \\ &= \epsilon\langle b(x), D\phi(x)\rangle + \frac{M(\epsilon)}{\langle x-y\rangle}(\langle b(x)-b(y), x-y\rangle + \langle b(y), x-y\rangle) \\ &\geq \epsilon\langle b(x), D\phi(x)\rangle + M(\epsilon)\frac{\alpha|x-y|^2}{\langle x-y\rangle} - M(\epsilon)|b(y)|\frac{|x-y|}{\langle x-y\rangle} \\ &\geq \epsilon\langle b(x), D\phi(x)\rangle - M(\epsilon)|b(y)|.\end{aligned} \tag{5.42}$$

And from (5.38), (4.6) we have

$$\begin{aligned}H(x, Dg(x,t)) &= H(x, \epsilon D\phi(x) + M(\epsilon)q) \\ &\geq -C_H - \epsilon C_H|D\phi(x)| - M(\epsilon)C_H.\end{aligned} \tag{5.43}$$

From (5.41),(5.42), and (5.43) we have

$$\begin{aligned}& g_t - \mathcal{F}(x,[g]) + \langle b(x), Dg(x,t)\rangle + H(x, Dg(x,t)) \\ \geq\ & A + \epsilon\left(-\mathcal{F}(x,[\phi]) + \langle b(x), D\phi(x)\rangle - C_H|D\phi(x)|\right) \\ & -M(\epsilon)\left[C(\mathcal{F}) + |b(y)| + C_H\right] - C_H.\end{aligned}$$

We choose a constant $K > 0$ so that (4.23) holds. Notice that $f_m \in \mathcal{E}_\mu(I\!R^N)$, for each $\epsilon \in (0,1)$ we may choose a constant $C(\epsilon) > 0$ so that

$$|f_m(x)| \leq \epsilon\phi(x) + C(\epsilon) \text{ for } x \in I\!R^N, \quad \forall m \geq 1. \tag{5.44}$$

For each $y \in I\!R^N$ and $\epsilon \in (0,1)$ we set

$$A(y,\epsilon) = M(\epsilon)\left[C(\mathcal{F}) + |b(y)| + C_H\right] + C_H + \epsilon K + C(\epsilon),$$

and define the functions $\psi^+ \in C^\infty(\overline{Q})$, parametrized by y, ϵ, by

$$\psi^+(x,t;y,\epsilon) = u_{0m}(y) + \epsilon(\phi(x) + \phi(y)) + M(\epsilon)\langle x-y\rangle + A(y,\epsilon)t.$$

Observe that, for any $y \in I\!R^N$ and $\epsilon \in (0,1)$, the function $h(x,t) := \psi^+(x,t;y,\epsilon)$ satisfies

$$\begin{aligned}& h_t - \mathcal{F}(x,[h]) + \langle b(x), Dh(x,t)\rangle + H(x, Dh(x,t)) \\ &\quad \geq A(y,\epsilon) + \epsilon(\phi(x) - K) - M(\epsilon)\left[C(\mathcal{F}) + |b(y)| + C_H\right] - C_H \\ &\quad \geq \epsilon\phi(x) + C(\epsilon) \\ &\quad \geq f_m(x) \text{ for } (x,t) \in Q.\end{aligned}$$

That is, the functions $h(x,t)$ is classical (and hence viscosity) supersolution of (5.22). Observe also that

$$h(x,0) \geq u_{0m}(y) + \epsilon(\phi(x) + \phi(y)) + M(\epsilon)\langle x-y\rangle \geq u_0(x) \text{ for } (x,t) \in \overline{Q}.$$

Similarly we define the function

$$k(x,t) := \psi^-(x,t;y,\epsilon) = u_{0m}(y) - \epsilon(\phi(x)+\phi(y)) - M(\epsilon)\langle x-y\rangle - A(y,\epsilon)t,$$

and then observe as before that $k(x,t)$ is a viscosity subsolution of (5.22) and that for $(x,t) \in \overline{Q}$,

$$u_{0m}(x) \geq u_{0m}(y) - \epsilon(\phi(x)+\phi(y)) - M(\epsilon)\langle x-y\rangle = k(x,0).$$

By comparison principle (Theorem 5.4.2) we obtain, for any $y \in I\!R^N$,

$$\begin{aligned} & u_{0m}(y) - \epsilon(\phi(x)+\phi(y)) - M(\epsilon)\langle x-y\rangle - A(y,\epsilon)t \\ \leq\ & u_{mn}(x,t) \\ \leq\ & u_{0m}(y) + \epsilon(\phi(x)+\phi(y)) + M(\epsilon)\langle x-y\rangle + A(y,\epsilon)t. \end{aligned}$$

Hence,

$$|u_{mn}(x,t) - u_{0m}(0)| \leq \epsilon(\phi(x)+\phi(0)) + M(\epsilon)\langle x\rangle + A(0,\epsilon)t \text{ for } (x,t) \in \overline{Q}, \epsilon \in (0,1).$$

Therefore,

$$|u_{mn}(x,t)| \leq C_{RT}, \quad \forall (x,t) \in B(0,R)\times[0,T),$$

where $C_{RT} = \sup_{B(0,R)\times[0,T)}\{|u_{0m}(0)| + \epsilon(\phi(x)+\phi(0)) + M(\epsilon)\langle x\rangle + A(0,\epsilon)t\}$ independent of m and n, since $|u_{0m}(0)| \leq |u_0(0) + \frac{1}{m}\phi(0)| \leq |u_0(0)+\phi(0)|, \forall m \geq 1$.

Moreover,

$$\begin{aligned} \lim_{|x|\to\infty} \sup_{0\leq t<T} \frac{|u_{mn}(x,t)|}{\phi(x)} &\leq \epsilon + \lim_{|x|\to\infty} \sup_{0\leq t<T} \frac{|u_0(0) + \epsilon\phi(0) + M(\epsilon)\langle x\rangle + A(0,\epsilon)t|}{\phi(x)} \\ &\leq \epsilon, \quad \forall \epsilon > 0, \end{aligned}$$

which guarantees that $u_{mn} \in \mathcal{E}_\mu(\overline{Q})$. Since $u_{mn} \to u_m$ as $n \to \infty$ locally uniformly in $\bar{Q}$. Then we obtain

$$\begin{aligned} & |u_m(x,t)| \leq C_{RT}, \quad \forall (x,t) \in B(0,R)\times[0,T), \\ & u_m \in \mathcal{E}_\mu(\overline{Q}). \end{aligned}$$

□

Chapter 6

Application to ergodic problem and long time behavior of solutions

In this Chapter we will use some local uniform bounds that we obtained in Chapter 3 to solve the ergodic problem and then study long time behavior.

In order that the nonlocal operator is well-defined for unbounded functions with exponential growth, we have to consider only functions with a growth less than the fixed one which is taken into account in the measure ν. More precisely, we fix $\mu > 0$ and consider $\mathcal{I}$ given by (2.23) satisfying (M1) and (M2). So in the following we will assume

$$\begin{cases} f \in \mathcal{E}_\theta(I\!\!R^N) \cap C(I\!\!R^N) \\ |f(x) - f(y)| \le C_f |x-y|(\phi_\theta(x) + \phi_\theta(y)), \end{cases} \tag{6.1}$$

where $\phi_\theta(x) = e^{\theta\sqrt{|x|^2+1}}$, $x \in I\!\!R^N$ and we will work with solutions in the class $\mathcal{E}_\theta$ with $\theta < \mu$.

6.1 Application to ergodic problem

In this section we study the ergodic control problem:

$$c - \mathcal{F}(x,[v]) + \langle b(x), Dv(x)\rangle + H(x, Dv(x)) = f(x) \qquad \text{in } I\!\!R^N, \tag{6.2}$$

where $\mathcal{F}$ is defined in (2.27), the unknown is a pair of a constant c and a function v.

Recall that in order to solve (6.2), we first solve the approximate problem

$$\lambda u^\lambda(x) - \mathcal{F}(x,[u^\lambda]) + \langle b(x), Du^\lambda(x)\rangle + H(x, Du^\lambda(x)) = f(x), \text{ in } I\!\!R^N, \ \lambda \in (0,1). \tag{6.3}$$

With (6.1) and thanks to Theorem 5.1.1, Theorem 5.1.2 and Theorem 4.1.1, there exists a unique viscosity solution u^λ of (6.3) such that

$$\begin{aligned} &u^\lambda \in \mathcal{E}_\theta(I\!\!R^N) \cap C(I\!\!R^N), \\ &|u^\lambda(x) - u^\lambda(y)| \le C|x-y|(\phi_\theta(x) + \phi_\theta(y)), \quad \forall x, y \in I\!\!R^N, \end{aligned} \tag{6.4}$$

where $C > 0$ independent of λ.

Now we are ready to find a solution for (6.2).

Theorem 6.1.1. *Let $\mu > 0$, $0 < \theta < \mu$, assume that (6.1) holds for some constant. Then there is a solution $(c, v) \in I\!R \times C(I\!R^N)$ of (6.2) such that*

$$v \in \bigcap_{\theta<\gamma<\mu} \mathcal{E}_\gamma(I\!R^N). \tag{6.5}$$

Proof. Let $v^\lambda \in C(I\!R^N) \cap \mathcal{E}_\theta(I\!R^N)$, with $\lambda \in (0,1)$ be the unique solution of (6.3). In view of (6.4) and Lemma 5.1.1 there is a constant $C > 0$ independent of λ such that

$$\begin{array}{rcl} \lambda|v^\lambda(0)| & \le & C, \\ |v^\lambda(x) - v^\lambda(y)| & \le & C|x-y|(\phi_\theta(x) + \phi_\theta(y)) \qquad \text{for } \ x, y \in I\!R^N. \end{array} \tag{6.6}$$

Define $\omega^\lambda, z^\lambda \in C(I\!R^N)$ by $\omega^\lambda(x) := v^\lambda(x) - v^\lambda(0)$ and $z^\lambda(x) := \lambda v^\lambda(x)$, respectively. Then for all $x, y \in I\!R^N$,

$$\begin{array}{rcll} |z^\lambda(0)| & \le & C, & \\ |z^\lambda(x) - z^\lambda(0)| & = & |\lambda v^\lambda(x) - \lambda v^\lambda(0)| & \\ & \le & \lambda C|x|(\phi_\theta(x) + e^\theta), & (6.7) \\ |\omega^\lambda(x)| & \le & C|x|(\phi_\theta(x) + e^\theta), & (6.8) \\ |\omega^\lambda(x) - \omega^\lambda(y)| & \le & C|x-y|(\phi_\theta(x) + \phi_\theta(y)). & \end{array}$$

Therefore, $\{\omega^\lambda\}_{\lambda\in(0,1)}$ is a uniformly bounded and equi-continuous family on any balls of $I\!R^N$. By Ascoli's theorem we can choose a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0,1)$ such that as $j \to \infty$,

$$\begin{array}{c} \lambda_j \to 0, \qquad z^{\lambda_j} \to c, \\ \omega^{\lambda_j} \to v \ \text{ in } \ C(I\!R^N) \end{array}$$

for some $c \in I\!R$ and $v \in C(I\!R^N)$. By (6.7) we have, as $j \to \infty$,

$$z^{\lambda_j}(x) \to c \qquad \text{uniformly on balls of } I\!R^N.$$

By the stability of viscosity solutions (see [2], [15] and [30]), we find that v satisfies (6.2) in the viscosity sense. Let $\theta < \gamma < \mu$. Since

$$\lim_{|x|\to\infty} \frac{|x|\phi_\theta(x)}{\phi_\gamma(x)} = \lim_{|x|\to\infty} \frac{|x|e^{\theta\sqrt{|x|^2+1}}}{e^{\gamma\sqrt{|x|^2+1}}} = \lim_{|x|\to\infty} \frac{|x|}{e^{(\gamma-\theta)\sqrt{|x|^2+1}}} = 0,$$

we see from (6.8) that $v \in \mathcal{E}_\gamma(I\!R^N)$. $\square$

To prove the uniqueness of solution for (6.2) we need to linearize the equation in order to apply the strong maximum principle. To do that, we need to assume that (5.25) holds.

Theorem 6.1.2. *Assume that (4.3), (4.4), (5.1) and either (4.2), (4.5) or (M1), (M2) hold. Let $\mu > 0$, $0 < \theta < \mu$, (c, v), $(d, \omega) \in I\!R \times C(I\!R^N)$ be solutions of ergodic problem (6.2) such that $v, \omega \in \mathcal{E}_\gamma(I\!R^N)$, $\theta < \gamma < \mu$. Then $c = d$ and if, in addition (5.25) holds, there is a constant $C \in I\!R$ such that $v - \omega = C$ in $I\!R^N$.*

Proof. 1. To show that $c = d$, we argue by contradiction. Thus we suppose that $c \neq d$. Without loss of generality we may assume that $c > d$.

Recall from Lemma 4.1.1 (4.23) that there is a constant $K_\gamma > 0$ such that

$$-\mathcal{F}(x, [\phi_\gamma]) + \langle b(x), D\phi_\gamma\rangle - L_H|D\phi_\gamma| \geq \phi_\gamma - K_\gamma \qquad \text{in } I\!R^N. \tag{6.9}$$

Choose a constant $\epsilon > 0$ small enough so that $2\epsilon K_\gamma < c - d$.

For simplicity, we write ϕ instead of ϕ_γ. Since $(c, v), (d, \omega)$ are two solutions of ergodic problem (6.2) and because of (6.9), we can easily verify that $\tilde{v}(x, t) = v(x) - \epsilon\phi(x) + ct$ is viscosity subsolution of

$$v_t - \mathcal{F}(x, [v]) + \langle b(x), Dv(x, t)\rangle + H(x, Dv(x, t)) = f_1(x) \qquad \text{in } I\!R^N \times (0, T),$$

and $\tilde{\omega}(x, t) = \omega(x) + \epsilon\phi(x) + dt$ is viscosity supersolution of

$$v_t - \mathcal{F}(x, [v]) + \langle b(x), Dv(x, t)\rangle + H(x, Dv(x, t)) = f_2(x) \qquad \text{in } I\!R^N \times (0, T),$$

where $f_1(x) = f(x) - \epsilon\phi_\gamma(x) + \epsilon K_\gamma$, $f_2(x) = f(x) + \epsilon\phi_\gamma(x) - \epsilon K_\gamma$. Applying Theorem 5.2.1 for $\tilde{v}$ and $\tilde{\omega}$, we have for all $(x, t) \in I\!R^N \times (0, T)$

$$v(x) - \omega(x) - 2\epsilon\phi(x) + (c - d)t \leq \sup_{I\!R^N}\{v(y) - \omega(y) - 2\epsilon\phi(y)\} + t \parallel (2\epsilon K_\gamma - 2\epsilon\phi)^+ \parallel_\infty .$$

This implies

$$(c - d)t \leq 2\epsilon K_\gamma t,$$

which is a contradiction. Thus $c = d$.

2. We next show that for some constant $C \in I\!R$,

$$v - \omega = C \qquad \text{in } I\!R^N. \tag{6.10}$$

We first define $u \in C(I\!R^N)$ by $u(x) = v(x) - \omega(x)$. Then we use the following Lemma to get that u is viscosity subsolution of (6.11).

Lemma 6.1.1. *Assume that (4.3), (5.25), $f \in \mathcal{E}_\theta(I\!R^N) \cap C(I\!R^N)$ and either (4.2) or (M1),(M2) hold. Let $\mu > 0$, $0 < \theta < \mu$, $\theta < \gamma < \mu$, $v_1, v_2 \in C(I\!R^N) \cap \mathcal{E}_\gamma(I\!R^N)$ be a viscosity subsolution and a viscosity supersolution of the ergodic problem (6.2) respectively, with the same ergodic constant c. Set $\omega = v_1 - v_2$ then ω is a continuous viscosity subsolution of*

$$-\mathcal{F}(x, [\omega]) + \langle b(x), D\omega(x)\rangle - L_{H_2}|D\omega| = 0. \tag{6.11}$$

The proof of this lemma is classical and given in the Appendix.

We now comeback to the proof of Theorem 6.1.2.

For $\epsilon > 0$, since $u = v - \omega \in \mathcal{E}_\gamma$, hence $u - \epsilon\phi_\gamma$ has a maximum point. Let $\bar{x} \in I\!R^N$ be a maximum point of $u - \epsilon\phi_\gamma$. From Lemma 6.1.1, u is a viscosity subsolution of (6.11), so by the definition of viscosity subsolution, we have

$$-\mathcal{F}(\bar{x}, [\epsilon\phi_\gamma]) + \langle b(\bar{x}), D\epsilon\phi_\gamma(\bar{x})\rangle - L_{H_2}|D\epsilon\phi_\gamma(\bar{x})| \leq 0 \qquad \text{at } \bar{x} \in I\!R^N.$$

This implies

$$-\mathcal{F}(\bar{x}, [\phi_\gamma]) + \langle b(\bar{x}), D\phi_\gamma(\bar{x})\rangle - L_{H_2}|D\phi_\gamma(\bar{x})| \leq 0 \qquad \text{at } \bar{x} \in I\!R^N. \tag{6.12}$$

On the other hand, recall that there is a constant $K_\gamma > 0$ such that

$$-\mathcal{F}(x, [\phi_\gamma]) + \langle b(x), D\phi_\gamma(x)\rangle - L_{H_2}|D\phi_\gamma(x)| \geq \phi_\gamma(x) - K_\gamma \qquad \text{for } x \in I\!R^N.$$

Therefore, there is a constant $R > 0$ such that

$$-\mathcal{F}(x, [\phi_\gamma]) + \langle b(x), D\phi_\gamma(x)\rangle - L_{H_2}|D\phi_\gamma(x)| > 0 \qquad \text{for } x \in I\!R^N \backslash \mathrm{int}B(0, R). \tag{6.13}$$

From (6.12) and (6.13) we deduce that $u - \epsilon\phi_\gamma$ can only attain a maximum at $\bar{x} \in \mathrm{int}B(0, R)$. It means that

$$\sup_{I\!R^N}(u - \epsilon\phi_\gamma) = \max_{B(0,R)}(u - \epsilon\phi_\gamma)$$

Hence, easy to see that

$$\sup_{I\!R^N} u = \max_{B(0,R)} u \tag{6.14}$$

Therefore we have

$$\sup_{I\!R^N} u = \max_{B(0,r)} u \qquad \text{for any } r > R.$$

We apply the strong maximum principle to u on $B(0, r)$ (see [8, 32] in the local case and [28, 27] in the nonlocal one), with arbitrary $r > R$, to conclude that u is a constant function on $B(0, r)$, with $r > R$, which clearly guarantees that $u(x) = u(0)$ for all $x \in I\!R^N$. This completes the proof. □

6.2 Application to long time behavior of solutions

In this section we study the long time behavior of solutions of (4.68). The main result in this section is stated as follows.

Theorem 6.2.1. *Let $\mu > 0$, $0 < \theta < \mu$, assume that (4.3), (5.25), (6.1) and either*

(i) $\mathcal{F}(x, [u^\lambda]) = \mathrm{tr}(A(x)D^2u^\lambda(x))$ and (4.2), (4.5) hold

or

(ii) $\mathcal{F}(x, [u^\lambda]) = \mathcal{I}(x, u^\lambda, Du^\lambda)$ and suppose that (M1), (M2) holds with $\beta \in (1, 2)$.

Let γ be a constant satisfying $\theta < \gamma < \mu$. Let $u \in \mathcal{E}_\theta(\overline{Q}) \cap C(\overline{Q})$ be the unique solution of (4.68) and $(c, v) \in I\!R \times (C(I\!R^N) \cap \mathcal{E}_\gamma(I\!R^N))$ a solution of (6.2). Then there is a constant $a \in I\!R$ such that

$$\lim_{t\to\infty} \max_{B(0,R)} |u(x, t) - (ct + v(x) + a)| = 0 \qquad \textit{for all } R > 0. \tag{6.15}$$

In this section we devote ourselves to proving Theorem 6.2.1, and we henceforth assume the hypotheses of the theorem.

We set

$$\begin{aligned} f_c(x) &= f(x) - c \qquad \text{for } x \in I\!R^N, \\ u_c(x,t) &= u(x,t) - ct \qquad \text{for } (x,t) \in \overline{Q}. \end{aligned}$$

Then u_c and (c, v) solve, respectively, the Cauchy problem (4.68) and the ergodic problem (6.2), with f_c in place of f. Assertions (6.15) now reads

$$\lim_{t\to\infty} \max_{B(0,R)} |u_c(x,t) - (v(x) + a)| = 0 \qquad \text{for all } R > 0.$$

Replacement of u and f by u_c and f_c, respectively, reduces the proof of Theorem 5.1 to the case of $c = 0$. Therefore we assume in the remainder of this section that $c = 0$.

In what follows we denote by η the function defined by

$$\eta(x,t) = u(x,t) - v(x) \qquad \text{on } \overline{Q}.$$

Since v is viscosity solution (and hence supersolution) and u is viscosity solution (and hence subsolution) of

$$-\mathcal{F}(x,[v]) + \langle b(x), Dv(x)\rangle + H(x, Dv(x)) = f(x) \qquad \text{in } I\!R^N. \tag{6.16}$$

and

$$u_t - \mathcal{F}(x,[u]) + \langle b(x), Du(x,t)\rangle + H(x, Du(x,t)) = f(x) \qquad \text{in } I\!R^N.$$

respectively, then applying Lemma 6.1.1 (in the case of parabolic equation) for $\eta = u - v$ we obtain that η is a viscosity subsolution of

$$\eta_t - \mathcal{F}(x,[\eta]) + \langle b(x), D\eta\rangle - L_{H_2}|D\eta| = 0 \qquad \text{in } Q. \tag{6.17}$$

We write ϕ and ψ for ϕ_μ and ϕ_γ, respectively. Fix a constant $K > e^\gamma$ so that

$$-\mathcal{F}(x,[\psi]) + \langle b(x), D\psi(x)\rangle - L_{H_2}|D\psi(x)| \ge \psi(x) - K \qquad \text{for } x \in I\!R^N.$$

Let φ denote the function defined by $\varphi(x,t) = (\psi(x) - K)e^{-t}$ on $\overline{Q}$. It follows that

$$\varphi_t - \mathcal{F}(x,[\varphi]) + \langle b(x), D\varphi\rangle - L_{H_2}|D\varphi| \ge 0 \qquad \text{in } Q. \tag{6.18}$$

We divide the proof of Theorem 6.2.1 into several lemmas which proofs are given in [42].

Lemma 6.2.1. *There is a function $L : (0,1] \to (0,\infty)$ and for each $R > 0$ a modulus σ_R such that*

$$\begin{aligned} |u(x,t)| &\le \epsilon\psi(x) + L(\epsilon) \qquad \textit{for } (x,t) \in \overline{Q}, \epsilon \in (0,1], \qquad (6.19)\\ |u(x,t) - u(x,s)| &\le \sigma_R(|t-s|) \qquad \textit{for } x \in B(0,R), t,s \in [0,\infty), R > 0. \qquad (6.20) \end{aligned}$$

Lemma 6.2.2. *The sets $\{u(\cdot,t)|t \geq 0\}$ and $\{u(\cdot,\cdot+t)|t \geq 0\}$ are precompact in $C(\mathbb{R}^N)$ and $C(\overline{Q})$, respectively.*

We define the functions $v^+, v^- : \mathbb{R}^N \to \mathbb{R}$ by

$$\begin{aligned} v^+(x) &= \limsup_{t\to\infty} u(x,t), \\ v^-(x) &= \liminf_{t\to\infty} u(x,t). \end{aligned}$$

Lemma 6.2.3. *The functions v^+ and v^- are solutions in $C(\mathbb{R}^N)$ of (6.16).*

We introduce the $\omega-$*limit set* $\Omega(u)$ as the set of those functions $\omega \in C(\overline{Q})$ for which there is a sequence $\{t_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ such that, as $j \to \infty, t_j \to \infty$ and $u(x,t+t_j) \to \omega(x,t)$ uniformly on bounded subsets of $\overline{Q}$. By the stability of viscosity solutions of (4.68) in $C(Q)$, we see that any $\omega \in \Omega(u)$ is a viscosity solution of (4.68). By the definition, it is obvious that

$$v^-(x) \leq \omega(x,t) \leq v^+(x) \qquad \text{for } (x,t) \in \overline{Q} \text{ and } \omega \in \Omega(u).$$

Lemma 6.2.4. *There is a function $\omega^- \in \Omega(u)$ such that $\omega^-(0,1) = v^-(0)$.*

Now, we are ready to prove Theorem 6.2.1.

Proof of Theorem 6.2.1. By Lemma 6.2.3, the function v^+ and v^- are a viscosity subsolution and a viscosity supersolution of (6.16), respectively. By the strong maximum principle, there are constants $a, b \in \mathbb{R}$ such that $v^+(x) = v(x)+a$ and $v^-(x) = v(x)+b$ for all $x \in \mathbb{R}^N$. Since $v^+(x) \geq v^-(x)$ for all $x \in \mathbb{R}^N$, we have $a \geq b$. If $a = b$, then we find by the definition of $v^\pm$ and by the precompactness of $\{u(.,t)|t \geq 0\}$ in $C(\mathbb{R}^N)$ (due to Lemma 6.2.2) that $u(.,t) \to v+a$ uniformly in $C(\mathbb{R}^N)$ as $t \to \infty$.

So we only need to prove that $a = b$ (or $a \leq b$).

Now, we prove that there is a sequence $\{t_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ such that

$$u(\cdot,t_j) \to v^- \qquad \text{in } C(\mathbb{R}^N) \qquad \text{as } j \to \infty. \tag{6.21}$$

By Lemma 6.2.4, there is a function $\omega^- \in \Omega(u)$ such that $\omega^-(0,1) = v^-(0)$. Since $v^-(x) \leq \omega^-(x,t)$ for all $(x,t) \in \overline{Q}$, the function $\zeta \in C(\overline{Q})$ defined by $\zeta(x,t) = v^-(x) - \omega^-(x,t)$ attains a maximum at the point $(0,1) \in Q$. Moreover, we know that v^- is a viscosity subsolution of (6.16) and ω^- is a viscosity supersolution of (4.68). Thanks to Lemma 6.1.1, we get ζ is a viscosity subsolution of (6.17) in the case of parabolic equation. By apply the strong maximum principle to ζ (see [8, 32] in the local case and [28, 27] in the nonlocal one), we find that

$$\zeta(x,t) = \zeta(0,1) = 0 \qquad \text{for } (x,t) \in \mathbb{R}^N \times [0,1).$$

That is, $\omega^-(x,t) = v^-(x)$ for all $(x,t) \in \mathbb{R}^N \times [0,1)$. By the definition of $\Omega(u)$, there is a sequence $\{t_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ such that as $j \to \infty, t_j \to \infty$ and

$$u(\cdot,t_j) \to \omega^-(\cdot,0) = v^- \qquad \text{in } C(\mathbb{R}^N),$$

which shows (6.21).

For $T > 0$ we define the function $\eta^T \in C(\overline{Q})$ by $\eta^T(x,t) = u(x,t+T) - v(x) - b$. It follows from Lemma 6.1.1 that η^T is a viscosity subsolution of (6.17).

For $\epsilon > 0$, we define the function g_ϵ by $g_\epsilon(x,t) = \epsilon\varphi(x,t) + \epsilon(1+B)$. Hence, from (6.18), g_ϵ is viscosity supersolution of (6.17).

The rest of the proof is to use the comparison principle to obtain that $\eta^T \le g_\epsilon$ then send t to infinity and finally ϵ to zero to get $a \le b$.

Fix any $\epsilon > 0$ and note that

$$\epsilon\varphi(x,0) + \epsilon K \ge 0 \qquad \text{for } x \in I\!R^N.$$

Recalling (6.19), we may choose a constant $C \equiv C(\epsilon) > 0$ such that

$$|u(x,t)| \le \frac{\epsilon}{2}\psi(x) + C \qquad \text{for } (x,t) \in \overline{Q}$$

.

Since $v \in \mathcal{E}_\gamma(I\!R^N)$, we may choose a constant $R \equiv R(\epsilon) > 0$ such that

$$\frac{\epsilon}{2}\psi(x) + C - v(x) - b \le \epsilon\varphi(x,0) \qquad \text{for } x \in I\!R^N \backslash B(0,R).$$

Then we get

$$\eta^T(x,0) \le \epsilon\varphi(x,0) \text{ for } x \in I\!R^N \backslash B(0,R), T > 0.$$

According to (6.21), we may fix a constant $T \equiv T(\epsilon) > 0$ so that

$$\eta^T(x,0) \le \epsilon \qquad \text{for } x \in B(0,R).$$

We now have

$$\eta^T(x,0) \le \epsilon\varphi(x,0) + \epsilon K + \epsilon = g_\epsilon(x,0) \qquad \text{for } x \in I\!R^N.$$

By (6.19), we have

$$\lim_{|x|\to\infty} \sup_{0 \le t \le S} (\eta^T(x,t) - g_\epsilon(x,t)) = -\infty \qquad \text{for } S > 0.$$

So we apply the comparison principle to η^T and g_ϵ as usual, to conclude that

$$\begin{aligned} \eta^T(x,t) &\le g_\epsilon(x,t) \\ &= \epsilon\varphi(x,t) + \epsilon(1+K) \qquad \text{for } (x,t) \in \overline{Q}. \end{aligned}$$

Taking $x = 0$, we have

$$u(0,t+T) \le v(0) + b + \epsilon\varphi(0,t) + \epsilon(1+K) \qquad \text{for } t \ge 0.$$

Here we use the fact that $\varphi(0,t) < 0$. Sending $t \to \infty$ along a sequence yields

$$v^+(0) \le v(0) + b + \epsilon(1+K),$$

which guarantees that $a \le b$ as $\epsilon > 0$ is arbitrary. This completes the proof. $\square$

6.3 Appendix

Proof of Lemma 6.1.1 . We divide the proof in several steps.

Step 1. Viscosity inequalities for v_1 and v_2. This step is classical in viscosity theory. Let $\varphi \in C^2(I\!R^N)$ and $\overline{x} \in I\!R^N$ be a local maximum point of $\omega - \varphi$. We can assume that this maximum is strict in the same ball $\overline{B}(\overline{x}, R)$. Let

$$\Theta(x,y) = \varphi(x) + \frac{|x-y|^2}{\epsilon^2}$$

and consider

$$M_\epsilon := \max_{x,y \in \overline{B}(\overline{x},R)} \{v_1(x) - v_2(y) - \Theta(x,y)\}$$

This maximum is achieved at a point (x_ϵ, y_ϵ) and, since the maximum is strict, we know [10] that

$$\begin{array}{l} x_\epsilon, y_\epsilon \to \bar{x}, \quad \frac{|x_\epsilon - y_\epsilon|^2}{\epsilon^2} \to 0 \qquad \text{as } \epsilon \to 0 \\ M_\epsilon = v_1(x_\epsilon) - v_2(y_\epsilon) - \Theta(x_\epsilon, y_\epsilon) \to v_1(\overline{x}) - v_2(\overline{x}) - \varphi(\overline{x}) = \omega(\overline{x}) - \varphi(\overline{x}). \end{array} \tag{6.22}$$

This means that, at the limit $\epsilon \to 0$, we obtain some information on $\omega - \varphi$ at $\overline{x}$ which will provide the new equation for ω. Before that, we can take Θ as a test-function to use the fact that v_1 is a subsolution and v_2 is a supersolution. Indeed $x \in \overline{B}(\overline{x}, R) \mapsto v_1(x) - v_2(y_\epsilon) - \Theta(x, y_\epsilon)$ achieves its maximum at x_ϵ and $y \in \overline{B}(\overline{x}, R) \mapsto -v_1(x_\epsilon) - v_2(y) + \Theta(x_\epsilon, y)$ achieves its minimum at y_ϵ. Thus, applying [30, Theorem 3.2] in the local case and [15, Corollary 1] in the nonlocal one to learn that, for any $\varrho > 0$, there exist $X, Y \in \mathcal{S}^N$ such that $(D_x\Theta(x_\epsilon, y_\epsilon), X) \in \overline{J}^{2,+} v_1(x_\epsilon)$, $(-D_y\Theta(x_\epsilon, y_\epsilon), Y) \in \overline{J}^{2,-} v_2(y_\epsilon)$ and

$$\begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq M + \varrho M^2 \qquad \text{where } \ M = D^2\Theta(x_\epsilon, y_\epsilon). \tag{6.23}$$

Setting $p_\epsilon = 2\frac{x_\epsilon - y_\epsilon}{\epsilon^2}$, we have

$$D_x\Theta(x_\epsilon, y_\epsilon) = p_\epsilon + D\varphi(x_\epsilon) \ \text{ and } \ D_y\Theta(x_\epsilon, y_\epsilon) = -p_\epsilon, \tag{6.24}$$

hence

$$M \;\; = \;\; \begin{pmatrix} D^2\varphi(x_\epsilon) + 2I/\epsilon^2 & -2I/\epsilon^2 \\ -2I/\epsilon^2 & 2I/\epsilon^2 \end{pmatrix}$$

and $\varrho M^2 = O(\varrho)$ (ϱ will be sent to 0 first).

Writing the subsolution viscosity inequality for v_1 and the supersolution viscosity inequality for v_2 at (x_ϵ, y_ϵ) we have

$$\begin{array}{c} c - \mathcal{F}(x_\epsilon, [v_1]) + \langle b(x_\epsilon), D_x\Theta(x_\epsilon, y_\epsilon)\rangle + H(x_\epsilon, D_x\Theta(x_\epsilon, y_\epsilon)) \leq f(x_\epsilon) \\ c - \mathcal{F}(y_\epsilon, [v_2]) + \langle b(y_\epsilon), -D_y\Theta(x_\epsilon, y_\epsilon)\rangle + H(y_\epsilon, -D_y\Theta(x_\epsilon, y_\epsilon)) \geq f(y_\epsilon). \end{array}$$

Subtracting the viscosity inequalities, we have

$$
\begin{aligned}
& -(\mathcal{F}(x_\epsilon,[v_1])-\mathcal{F}(y_\epsilon,[v_2]))+\langle b(x_\epsilon),D_x\Theta\rangle-\langle b(y_\epsilon),-D_y\Theta\rangle \\
& +H(x_\epsilon,D_x\Theta)-H(y_\epsilon,-D_y\Theta) \\
\leq\ & f(x_\epsilon)-f(y_\epsilon).
\end{aligned}
\tag{6.25}
$$

Step 2. Estimate of $\mathcal{S}:=\mathcal{F}(x_\epsilon,[v_1])-\mathcal{F}(y_\epsilon,[v_2])$.

Step 2.1. When $\mathcal{F}$ is the local operator defined in (2.27). From (6.23), it follows

$$
\langle Xp,p\rangle-\langle Yq,q\rangle\leq\langle D^2\varphi(x_\epsilon)p,p\rangle+\frac{2}{\epsilon^2}|p-q|^2.
$$

Using (4.2), we estimate $\mathcal{S}$ by using an orthonormal basis $(e_i)_{1\leq i\leq N}$ of $I\!R^N$ in the following way

$$
\begin{aligned}
\mathcal{S} &= \operatorname{tr}(\sigma(x_\epsilon)\sigma^T(x_\epsilon)X-\sigma(y_\epsilon)\sigma^T(y_\epsilon)Y) \\
&= \sum_{i=1}^N\langle X\sigma(x_\epsilon)e_i,\sigma(x_\epsilon)e_i\rangle-\langle Y\sigma(y_\epsilon)e_i,\sigma(y_\epsilon)e_i\rangle \\
&\leq \sum_{i=1}^N\langle D^2\varphi(x_\epsilon)\sigma(x_\epsilon)e_i,\sigma(x_\epsilon)e_i\rangle+\frac{2}{\epsilon^2}|\sigma(x_\epsilon)-\sigma(y_\epsilon)|^2 \\
&\leq \operatorname{tr}(\sigma(x_\epsilon)\sigma(x_\epsilon)^TD^2\varphi(x_\epsilon))+\frac{2L_\sigma^2}{\epsilon^2}|x_\epsilon-y_\epsilon|^2 \\
&\to \operatorname{tr}(\sigma(\bar{x})\sigma(\bar{x})^TD^2\varphi(\bar{x}))\quad\text{as }\epsilon\to0.
\end{aligned}
\tag{6.26}
$$

Step 2.2. When $\mathcal{F}$ is the nonlocal operator defined in (2.27). For each $\delta>0$, we have

$$
\begin{aligned}
\mathcal{S} &= \mathcal{I}(x_\epsilon,v_1,D_x\Theta)-\mathcal{I}(y_\epsilon,v_2,-D_y\Theta) \\
&= \mathcal{I}[B_\delta](x_\epsilon,\Theta,D_x\Theta)+\mathcal{I}[B^\delta](x_\epsilon,v_1,D_x\Theta) \\
&\qquad -\mathcal{I}[B_\delta](y_\epsilon,\Theta,-D_y\Theta)-\mathcal{I}[B^\delta](y_\epsilon,v_2,-D_y\Theta).
\end{aligned}
$$

From (6.24), we first estimate

$$
\begin{aligned}
I_1 &:= \mathcal{I}[B_\delta](x_\epsilon,\Theta,D_x\Theta)-\mathcal{I}[B_\delta](y_\epsilon,\Theta,-D_y\Theta) \\
&= \int_{B_\delta}\{(\Theta(x_\epsilon+z,y_\epsilon)-\Theta(x_\epsilon,y_\epsilon)-D_x\Theta(x_\epsilon,y_\epsilon)\cdot z) \\
&\qquad -(\Theta(x_\epsilon,y_\epsilon+z)-\Theta(x_\epsilon,y_\epsilon)+D_y\Theta(x_\epsilon,y_\epsilon)\cdot z)\}\nu(dz) \\
&= \int_{B_\delta}\{\Theta(x_\epsilon+z,y_\epsilon)-\Theta(x_\epsilon,y_\epsilon+z)-(D_x\Theta(x_\epsilon,y_\epsilon)+D_y\Theta(x_\epsilon,y_\epsilon))\cdot z\}\nu(dz) \\
&= \int_{B_\delta}\{\varphi(x_\epsilon+z)-\varphi(x_\epsilon)+\frac{|x-y+z|^2}{\epsilon^2}-\frac{|x-y-z|^2}{\epsilon^2}-D\varphi(x_\epsilon)\cdot z\}\nu(dz) \\
&= \mathcal{I}[B_\delta](x_\epsilon,\varphi,D\varphi)+\frac{1}{\epsilon^2}o_\delta(1).
\end{aligned}
\tag{6.27}
$$

On the other hand, at the maximum point (x_ϵ, y_ϵ) we have

$$v_1(x_\epsilon + z) - v_2(y_\epsilon + z) - (v_1(x_\epsilon) - v_2(y_\epsilon)) \leq \varphi(x_\epsilon + z) - \varphi(x_\epsilon),$$

for each $z \in B$. Hence, for each $0 < \delta < \kappa < 1$, using this inequality we obtain

$$\begin{aligned} I_2 \;&:= \; = \mathcal{I}[B^\delta](x_\epsilon, v_1, D_x\Theta) - \mathcal{I}[B^\delta](y_\epsilon, v_2, -D_y\Theta) \\ &= \int_{B_\delta^c} \{v_1(x_\epsilon + z) - v_1(x_\epsilon) - D_x\Theta(x_\epsilon, y_\epsilon) \cdot z\mathbb{I}_B(z) \\ &\qquad - [v_2(y_\epsilon + z) - v_2(y_\epsilon) + D_y\Theta(x_\epsilon, y_\epsilon) \cdot z\mathbb{I}_B(z)]\} d\nu(z)\} \\ &\leq J^\kappa + \mathcal{I}[B_\kappa \setminus B_\delta](x_\epsilon, \varphi, D\varphi), \end{aligned} \tag{6.28}$$

where

$$J^\kappa = \int_{B_\kappa^c} \{v_1(x_\epsilon + z) - v_2(y_\epsilon + z) - (v_1(x_\epsilon) - v_2(y_\epsilon)) - D\varphi(x_\epsilon) \cdot z\mathbb{I}_B(z)\}\nu(dz).$$

Therefore from (6.27) and (6.28), we conclude that for all $0 < \delta < \kappa < 1$

$$\mathcal{S} = I_1 + I_2 \leq J^\kappa + \mathcal{I}[B_\kappa](x_\epsilon, \varphi, D\varphi) + \frac{1}{\epsilon^2} o_\delta(1). \tag{6.29}$$

Since $v_1, v_2 \in \mathcal{E}_\gamma(I\!R^N)$ hence there exists $C > 0$ such that

$$|v_i(x)| \leq C\phi_\gamma(x) \quad \forall i = 1, 2, \quad \forall x \in I\!R^N.$$

Let $\gamma < \mu$, thanks to (M1) we have $\int_{B^c} \phi_\gamma(z)\nu(dz) \leq \int_{B^c} \phi_\mu(z)\nu(dz) < +\infty$. Hence, using (6.22) and applying Dominated convergence Theorem we get that for each $\kappa > 0$ fixed,

$$\lim_{\epsilon \to 0} J^\kappa \leq \mathcal{I}[B_\kappa^c](\bar{x}, \omega, D\varphi).$$

Therefore, letting δ and $\epsilon \to 0$ in (6.29) and using (6.22) we obtain

$$\lim_{\epsilon \to 0} \mathcal{S} \leq \mathcal{I}(\bar{x}, \omega, D\varphi). \tag{6.30}$$

Step 3. Estimate of $\mathcal{B} := \langle b(x_\epsilon), D_x\Theta(x_\epsilon, y_\epsilon)\rangle - \langle b(y_\epsilon), -D_y\Theta(x_\epsilon, y_\epsilon)\rangle$

From (6.24), using (4.3) we have

$$\begin{aligned} \mathcal{B} &= \langle b(x_\epsilon), p_\epsilon + D\varphi(x_\epsilon)\rangle - \langle b(y_\epsilon), p_\epsilon\rangle \\ &= \langle b(x_\epsilon), p_\epsilon\rangle + \langle b(x_\epsilon), D\varphi(x_\epsilon)\rangle - \langle b(y_\epsilon), p_\epsilon\rangle \\ &= \langle b(x_\epsilon) - b(y_\epsilon), p_\epsilon\rangle + \langle b(x_\epsilon), D\varphi(x_\epsilon)\rangle \\ &\geq 2\alpha \frac{|x_\epsilon - y_\epsilon|^2}{\epsilon^2} + \langle b(x_\epsilon), D\varphi(x_\epsilon)\rangle. \end{aligned} \tag{6.31}$$

Step 4. Estimate of $\mathcal{H} := H(x_\epsilon, D_x\Theta(x_\epsilon, y_\epsilon)) - H(y_\epsilon, -D_y\Theta(x_\epsilon, y_\epsilon))$.

From (6.24) and (5.23), we have

$$\begin{aligned}
\mathcal{H} &\geq -L_{H_2}|D_x\Theta(x_\epsilon,y_\epsilon)+D_y\Theta(x_\epsilon,y_\epsilon)| - L_{H_1}(1+|D_y\Theta(x_\epsilon,y_\epsilon)|)|x_\epsilon - y_\epsilon| \\
&= -L_{H_2}|p_\epsilon + D\varphi(x_\epsilon) - p_\epsilon| - L_{H_1}(1+|p_\epsilon|)|x_\epsilon - y_\epsilon| \\
&= -L_{H_2}|D\varphi(x_\epsilon)| - L_{H_1}|x_\epsilon - y_\epsilon| - 2L_{H_1}\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon^2}.
\end{aligned} \tag{6.32}$$

Step 5. Estimate of $\mathcal{F} := f(y_\epsilon) - f(x_\epsilon)$.

Since $f \in C(I\!R^N)$, hence we have

$$\mathcal{F} \leq o_\epsilon(1). \tag{6.33}$$

Step 6. Finally, by sending ϵ to 0 and combining (6.26), (6.30), (6.31), (6.32) and (6.33) to (6.25), we obtain

$$-\mathcal{F}(\bar{x},[\omega]) + \langle b(\bar{x}), D\varphi(\bar{x})\rangle - L_{H_2}|D\varphi(\bar{x})| \leq 0$$

which means exactly that ω is a subsolution of (6.11). $\square$

Chapter 7

Long time behavior of unbounded solutions of first order Hamilton-Jacobi equations in $I\!R^N$

.

7.1 Introduction

This work is a first attempt to obtain the large time behavior for the solutions of the first-order Hamilton-Jacobi equation

$$\begin{cases} u_t(x,t) + |Du(x,t)| = l(x), & \text{in } I\!R^N \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } I\!R^N, \end{cases} \tag{7.1}$$

for unbounded solutions in the whole space $I\!R^N$.

The additional difficulties with respect to the periodic setting is the lack of compactness. The ergodic constant may be not unique and the convergence of solutions, as $t \to +\infty$, may fail or/and depend on the initial data.

In this work, we work with functions having at most quadratic growth with suitable local Lipschitz regularity assumptions. To state the assumptions, we introduce the following classes of functions,

$$\begin{aligned} \mathcal{Q} := \{u : I\!R^N \times [0,+\infty) \to I\!R :\ & \text{for all } t \geq 0 \text{ there exists } C > 0 \text{ such that} \\ & \text{for all } x \in I\!R^N, |u(x,t)| \leq C(1+|x|^2)\}, \end{aligned} \tag{7.2}$$

and

$$\begin{aligned} \mathcal{L} := \{u : I\!R^N \times [0,+\infty) \to I\!R :\ & \text{for all } t \geq 0 \text{ there exists } C > 0 \text{ such that} \\ & \text{for all } x,y \in I\!R^N, |u(x,t) - u(y,t)| \leq C(1+|x|^2+|y|^2)|x-y|\}. \end{aligned} \tag{7.3}$$

This is a the first result of a work in collaboration with Olivier Ley (IRMAR, Rennes, France) and Thanh Viet Phan (Ton Duc Thang University, Vietnam)

We suppose that both l and u_0 have quadratic growth and are locally Lipschitz continuous,

$$l, u_0 \in \mathcal{Q} \cap \mathcal{L}. \tag{7.4}$$

Moreover, we need some additional assumptions on l to overcome the lack of compactness of the problem,

$$\text{For all } x \in I\!R^N,\ l(x) \geq 0, \tag{7.5}$$

and

$$\liminf_{|x|\to+\infty} l(x) > \inf_{I\!R^N} l. \tag{7.6}$$

Notice that this latter assumption implies that

$$\mathcal{K} := \operatorname{argmin} l \quad \text{is a compact subset of } I\!R^N. \tag{7.7}$$

It is then possible to solve the approximate stationary problem. For a given constant $\lambda > 0$ (to be sent to zero), we consider

$$\lambda v^\lambda(x) + |Dv^\lambda(x)| = l(x) \qquad \text{in } I\!R^N. \tag{7.8}$$

We prove in Theorem 7.2.1 that, if $l \in \mathcal{Q} \cap \mathcal{L}$ satisfies (7.5), then there exists a unique solution $u^\lambda \in \mathcal{Q} \cap \mathcal{L}$. The constant C in (7.2) depends on λ in general but C in (7.3) is *independent* of λ, which is the crucial point. It allows to solve the ergodic problem

$$|Dv(x)| = l(x) + c \qquad \text{in } I\!R^N, \tag{7.9}$$

where the unknown is both the constant c and the function v.

Theorem 7.1.1. *Assume $l \in \mathcal{Q} \cap \mathcal{L}$ satisfies (7.5). Then there exists a solution $(c, v) \in I\!R^- \times \mathcal{L}$ of (7.9). If (c, v) and (d, w) are two solutions of (7.9) such that $\sup_{I\!R^N} |v - w| < +\infty$ then $c = d$. Moreover, there exists a solution (c, v) with $c = -\inf_{I\!R^N} l$.*

Coming back to (7.1), it is not difficult ([49, 55]) to see that, assuming merely (7.4), there exists a unique solution $u \in \mathcal{Q} \cap \mathcal{L}$ but C *depends* on t both in (7.2) and (7.3). Under additional assumptions, we can drop these dependence,

Theorem 7.1.2. *Assume (7.4)-(7.5) and choose u_0 such that there exists $L_1, L_2 > 0$,*

$$\sup_{I\!R^N} |u_0 - v| \leq L_1, \tag{7.10}$$

$$|D(u_0 - v)| \leq L_2,\ \text{a.e. in } I\!R^N \ (\text{i.e., } u_0 - v \text{ is Lispchitz continuous}), \tag{7.11}$$

where (c, v) is one solution of the ergodic problem given by Theorem 7.1.1. Then, for all $x, y \in I\!R^N$, $t \geq 0$,

$$v(x) - L_1 \leq u(x,t) + ct \leq v(x) + L_1, \tag{7.12}$$

$$|(u(x,t) + ct)_t| \leq L_2 \ \text{a.e.} \tag{7.13}$$

$$|Du(x,t)| \leq L_2 + |c| + l(x) \ \text{a.e.} \tag{7.14}$$

In this chapter we write the ergodic equation $H = c$ instead of $c + H = 0$ as before in the thesis. It changes c to $-c$ with respect to the Introduction and previous Chapters.

Our convergence result follows

Theorem 7.1.3. *Assume (7.4), (7.5), (7.6) and suppose that the ergodic problem (7.9) has a solution (c, v) which is bounded from below with $c = -\inf_{I\!R^N} l$. If the initial data u_0 is such that (7.10)-(7.11) hold, then the solution u of (7.1) satisfies*

$$u(x,t) + ct \underset{t\to+\infty}{\to} w(x) \quad \text{locally uniformly in } I\!R^N, \tag{7.15}$$

where (c, w) is a solution to the ergodic problem (7.9) such that $\sup_{I\!R^N} |v - w| < \infty$.

This result means that, up to choose the initial data "sufficiently close" to some particular solutions v of the ergodic problem, the solution $u(\cdot, t)$ is attracted by v. Several issues remain. At first, is it always possible to build a bounded from below solution of (7.9)? Is it possible to build a solution (c, v) to the ergodic problem for all $c \geq -\min l$? Is it possible to obtain some convergence results if u_0 is not bounded from below? In Section 7.6, we give some basic examples and interpretations in terms of the underlying optimal control problem. We show that it is possible to build a bounded from below solution of (7.9) and the theorem applies. But convergence seems to fail when u_0 satisfies (7.10)-(7.11) and the solution v of the ergodic problem is not bounded from below.

This problem has been little studied comparing to the periodic case ([38, 69, 20, 40, 34, 16],...). A work close to our framework is due to Barles & Roquejoffre [19] who generalize Namah & Roquejoffre [69] in the unbounded setting. Contrary to [19], we do not assume the Hamiltonian $H(x, p) = |Du| - l(x)$ to be bounded uniformly continuous with respect to x in $I\!R^N$, which brings additional difficulties. In our work, the solutions of (7.1) are naturally unbounded both from above and below with a large growth. Moreover, we do not require in the convergence result the initial data and the solution of the ergodic to be asymptotic at infinity ((7.10) is enough). Another important work in the unbounded setting is the one of Ishii [52]. One of the main difference with his result is that we do not need the strict convexity of the Hamiltonian. Moreover, the proofs in [52] are based to the interpretation of (7.1)-(7.8) in terms of a control problem and uses weak KAM theory. Instead, we produce a pure PDE proof. Let us mention another contributions to the large time behavior of solutions of (7.1) in $I\!R^N$. There are the works of Ichihara & Ishii [53, 48] where the assumptions are given in terms of existence of particular sub or supersolutions.

The paper is organized as follows. We start by studying the approximate stationary problem (7.8) in $\mathcal{Q} \cap \mathcal{L}$ in Section 7.2. It allows us to solve the ergodic problem (7.9), see Section 7.3. Then, we consider the evolution problem (7.1) in Section 7.4 and prove that there exists a unique solution in $\mathcal{Q} \cap \mathcal{L}$. Section 7.5 is devoted to the proof of the main theorem of convergence. Finally, Section 7.6 studies an example in details using both the Hamilton-Jacobi equation (7.1) and the associated optimal control problem.

7.2 The approximate stationary problem (7.8)

Proposition 7.2.1. *Let $l \in \mathcal{Q} \cap \mathcal{L}$. Let $u \in USC(I\!R^N) \cap \mathcal{Q}$ and $v \in LSC(I\!R^N) \cap \mathcal{Q}$ be a viscosity subsolution and supersolution of (7.8) respectively. Then $u \leq v$ in $I\!R^N$.*

Proof of Proposition 7.2.1. For $\epsilon, \alpha, \mu > 0$, we consider

$$M = \sup_{\mathbb{R}^N \times \mathbb{R}^N} \{u(x) - v(y) - \frac{|x-y|^2}{\epsilon^2} - \alpha(1 + |x|^4 + |y|^4) - \mu\}.$$

Since $u, v \in \mathcal{Q}$, the supremum is achieved at a point $(\bar{x}, \bar{y})$ for all $\epsilon, \alpha, \mu > 0$.

Suppose by contradiction that there exists $\hat{x} \in \mathbb{R}^N$ such that $u(\hat{x}) > v(\hat{x})$. Then $M > 0$ for α, μ small enough.

We set $p_\epsilon = 2\frac{\bar{x}-\bar{y}}{\epsilon^2}$. Writing that u is a viscosity subsolution of (7.8), we have

$$\lambda u(\bar{x}) + |p + 4\alpha|\bar{x}|^2\bar{x}| \leq l(\bar{x})$$

and writing that v is a viscosity supersolution of (7.8), we have

$$\lambda v(\bar{y}) + |p - 4\alpha|\bar{y}|^2\bar{y}| \geq l(\bar{y}).$$

Subtracting the viscosity inequalities and using that $l \in \mathcal{L}$, we get

$$\begin{aligned} \lambda(u(\bar{x}) - v(\bar{y})) &\leq |p - 4\alpha|\bar{y}|^2\bar{y}| - |p + 4\alpha|\bar{x}|^2\bar{x}| + l(\bar{x}) - l(\bar{y}) \\ &\leq 4\alpha(|\bar{x}|^3 + |\bar{y}|^3) + C(1 + |\bar{x}|^2 + |\bar{y}^2|)|\bar{x} - \bar{y}|. \end{aligned}$$

Since $M > 0$, we have

$$\lambda(u(\bar{x}) - v(\bar{y})) > \lambda\left(\frac{|\bar{x}-\bar{y}|^2}{\epsilon^2} + \alpha(1 + |\bar{x}|^4 + |\bar{y}|^4) + \mu\right).$$

Therefore

$$\begin{aligned} &\lambda\left(\frac{|\bar{x}-\bar{y}|^2}{\epsilon^2} + \alpha(1 + |\bar{x}|^4 + |\bar{y}|^4) + \mu\right) \\ \leq\ & 4\alpha(|\bar{x}|^3 + |\bar{y}|^3) + C(1 + |\bar{x}|^2 + |\bar{y}^2|)|\bar{x} - \bar{y}|. \end{aligned} \tag{7.16}$$

By Young's inequality, there exists a constant $\hat{C} > 0$ such that

$$\begin{aligned} 4\alpha(|\bar{x}|^3 + |\bar{y}|^3) &\leq \frac{\hat{C}\alpha}{\lambda^3} + \frac{\lambda\alpha}{2}(1 + |\bar{x}|^4 + |\bar{y}|^4), \\ C(1 + |\bar{x}|^2 + |\bar{y}^2|)|\bar{x} - \bar{y}| &\leq \lambda\frac{|\bar{x}-\bar{y}|^2}{\epsilon^2} + \frac{\epsilon^2}{\lambda}\hat{C}(1 + |\bar{x}|^4 + |\bar{y}|^4). \end{aligned}$$

From (7.16), we get

$$\frac{\lambda\alpha}{2}(1 + |\bar{x}|^4 + |\bar{y}|^4) + \lambda\mu \leq \frac{\hat{C}\alpha}{\lambda^3} + \frac{\epsilon^2}{\lambda}\hat{C}(1 + |\bar{x}|^4 + |\bar{y}|^4),$$

which leads to a contradiction choosing $\alpha < \lambda^4\mu/\hat{C}$ and $\epsilon < (\lambda\sqrt{\alpha})/\sqrt{4\hat{C}}$. $\square$

Proposition 7.2.2. *Assume that $l \in \mathcal{Q} \cap \mathcal{L}$ satisfies (7.5). Then there exists a unique viscosity solution $v^\lambda \in C(\mathbb{R}^N) \cap \mathcal{Q}$ of (7.8).*

Proof of Proposition 7.2.2. We apply Perron method to find a viscosity solution of (7.8). For $\lambda > 0$, we define two functions by setting

$$\underline{v}(x) = 0 \qquad \text{and} \qquad \overline{v}(x) = \frac{l(x)}{\lambda}.$$

We can easily verify that $\underline{v} \in C(I\!R^N) \cap \mathcal{Q}$ and $\overline{v} \in C(I\!R^N) \cap \mathcal{Q}$ are respectively sub and supersolution of (7.8) (we need (7.5) for the subsolution property). We consider the set $\mathcal{F}$ of subsolutions w of (7.8) such that $\underline{v} \le w \le \overline{v}$. The set $\mathcal{F}$ is nonempty and $v(x) := \sup_{w\in\mathcal{F}} w(x)$ is well-defined. Then, in view of Perron method, v^* and v_* are a viscosity sub and supersolution of (7.8), respectively. Notice that

$$0 \le v_*(x) \le v(x) \le v^*(x) \le \frac{l(x)}{\lambda} \le C_\lambda(1+|x|^2) \quad \text{ for all } x \in I\!R^N,$$

where v_*, v^* are the upper-semicontinuous envelope of v. Applying Proposition 7.2.1, we obtain $v^* \le v_*$ in $I\!R^N$, that is, $v \in C(I\!R^N)$. Uniqueness of solutions of (7.8) is a direct consequence of Proposition 7.2.1. □

Theorem 7.2.1. *Assume that $l \in \mathcal{Q} \cap \mathcal{L}$ satisfies (7.5). Then the unique solution v^λ of (7.8) given by Proposition 7.2.2 satisfies*

$$|v^\lambda(x) - v^\lambda(y)| \le C(1+|x|^2+|y|^2)|x-y| \qquad \textit{for all } x, y \in I\!R^N, \tag{7.17}$$

where C is independent of λ.

In other words, this result means that the unique solution of (7.8) lies in $\mathcal{Q}\cap\mathcal{L}$ with a constant C independent of λ in (7.3).

Proof of Theorem 7.2.1.
Step 1. In this step we show that v^λ is locally Lipschitz with a constant C depending on λ. Recall that, from Proposition 7.2.2, we have

$$0 \le v^\lambda(x) \le \frac{l(x)}{\lambda} \le C_\lambda(1+|x|^2), \quad \text{for all } x \in I\!R^N, \tag{7.18}$$

where C_λ depends on λ.

Fix $x \in I\!R^N$ and $K > 0$ and consider the function

$$g(y) = v^\lambda(y) - K|x-y|(1+|x|^4+|y|^4). \tag{7.19}$$

We prove that g attains global maximum at a point y_0. Indeed, there exist $r_\lambda > 0$ such that

$$C_\lambda(1+|y|^2) \le K|x-y|(1+|x|^4+|y|^4), \quad \text{for all } y \notin B(x, r_\lambda).$$

Combining the above inequality with (7.18) we have

$$g(y) \le 0, \quad \text{for all } y \notin B(x, r_\lambda).$$

Moreover, since g is continuous and $\overline{B(x,r_\lambda)}$ is compact there exist $y_0 \in \overline{B(x,r_\lambda)}$ such that

$$g(y_0) = \max_{y\in\overline{B(x,r_\lambda)}} g(y) \geq g(x) = v^\lambda(x) \geq 0.$$

Therefore g attains its global maximum at some point y_0.

Suppose by contradiction that $y_0 \neq x$. Since v^λ is a subsolution of (7.8) and g attains its global maximum at $y_0 \neq x$, we can write the viscosity inequality for the subsolution at y_0 yielding

$$\lambda v^\lambda(y_0) + \left| K\frac{y_0 - x}{|y_0 - x|}(1 + |x|^4 + |y_0|^4) + K|y_0 - x|4y_0|y_0|^2 \right| \leq l(y_0). \tag{7.20}$$

Using that $v^\lambda \geq 0$, we obtain

$$K(1 + |x|^4 + |y_0|^4) \leq l(y_0) + 4K|y_0 - x||y_0|^3. \tag{7.21}$$

Since g attains its global maximum at y_0, we have

$$v^\lambda(y_0) - K|x - y_0|(1 + |x|^4 + |y_0|^4) \geq v^\lambda(x) \geq 0.$$

Hence, using (7.20),

$$K|x - y_0|(1 + |x|^4 + |y_0|^4) \leq v^\lambda(y_0) \leq \frac{l(y_0)}{\lambda}. \tag{7.22}$$

Combining (7.21) with (7.22), we obtain

$$\begin{aligned} K(1 + |x|^4 + |y_0|^4) \quad &\leq \quad l(y_0) + \frac{4|y_0|^3}{1 + |x|^4 + |y_0|^4}\frac{l(y_0)}{\lambda} \\ &\leq \quad C(1 + |y_0|^2) + \frac{4C(|y_0|^3 + |y_0|^5)}{\lambda(1 + |x|^4 + |y_0|^4)}. \end{aligned} \tag{7.23}$$

The above inequality is not possible for $K = K_\lambda$ large enough. This implies that, if we choose $K = K_\lambda$ large enough, then necessarily $y_0 = x$. Writing that $g(y) \leq g(x)$ for all $y \in I\!R^N$, we get

$$v^\lambda(y) - v^\lambda(x) \leq K_\lambda|x - y|(1 + |x|^4 + |y|^4),$$

which gives the result since the choice of K_λ does not depend on the choice of x at the beginning.

Step 2. We refine Step 1 proving that the constant of Lipschitz K_λ may be chosen independent of λ. From step 1, v^λ is locally Lipschitz. Thus by Rademacher Theorem, v^λ is differentiable almost everywhere in $I\!R^n$. Therefore,

$$\lambda v^\lambda(x) + |Dv^\lambda(x)| = l(x), \quad \text{a.e } x \in I\!R^n.$$

Since $\lambda v^\lambda(x) \geq 0$, we have

$$|Dv^\lambda(x)| \leq l(x), \quad \text{a.e } x \in I\!R^n.$$

Let $\Sigma = \{x \in I\!R^N : v^\lambda \text{ is not differentiable at } x\}$. We have $\mathcal{L}^N(\Sigma) = 0$, where $\mathcal{L}^N$ is the Lebesgue measure on $I\!R^N$. Using Fubini Theorem, It follows that

$$\mathcal{L}^1([x,y] \cap \Sigma) = 0, \quad \text{a.e } x, y \in I\!R^N,$$

where $[x,y] = \{tx + (1-t)y : t \in [0,1]\}$.

Let $x, y \in I\!R^N$ such that $f(t) = u(tx + (1-t)y)$ is differentiable a.e. on $[0,1]$. We have

$$\begin{aligned}
u(x) - u(y) = \int_0^1 f'(t)dt &= \int_0^1 \langle Du(tx + (1-t)y), x - y\rangle dt \\
&\leq \int_0^1 |Du(tx + (1-t)y)||x - y|dt \\
&\leq \int_0^1 |l(tx + (1-t)y)||x - y|dt \\
&\leq \int_0^1 C(1 + |tx + (1-t)y|^2)|x - y|dt \\
&\leq \int_0^1 C(1 + t|x|^2 + (1-t)|y|^2)|x - y|dt \\
&\leq C(1 + |x|^2 + |y|^2)|x - y|,
\end{aligned}$$

where C is the constant appearing in (7.3) for l (which does not depend on λ). Therefore we get (7.17). □

7.3 The Ergodic problem

We are now able to give the proof of Theorem 7.1.1.

Proof of Theorem 7.1.1.
Part 1. The existence of a solution to (7.9) follows from the classical arguments of [63]. Indeed, let v^λ be the unique solution of (7.8). We set $z^\lambda(x) = \lambda v^\lambda(x)$, and $w^\lambda(x) = v^\lambda(x) - v^\lambda(0)$. Using (7.17) and (7.18), we infer, for all $x, y \in I\!R^N$,

$$\begin{aligned}
&0 \leq z^\lambda(x) \leq l(x), \\
&|z^\lambda(x) - z^\lambda(0)| \leq \lambda|v^\lambda(x) - v^\lambda(0)| \leq \lambda C(1 + |x|^2)|x|, && (7.24)\\
&|w^\lambda(x)| \leq C(1 + |x|^2)|x|, && (7.25)\\
&|w^\lambda(x) - w^\lambda(y)| \leq C(1 + |x|^2 + |y|^2)|x - y|.
\end{aligned}$$

Accordingly, $\{z^\lambda\}_{\lambda \in (0,1)}, \{w^\lambda\}_{\lambda \in (0,1)}$ are uniformly bounded and equicontinuous families in any ball of $I\!R^N$. Thus by Ascoli Theorem, we can extract a subsequence $\lambda_n \to 0$,

$$z^{\lambda_n} \to -c, \quad w^{\lambda_n} \to v \quad \text{locally uniformly in } I\!R^N \tag{7.26}$$

for some $c \in I\!R^-$ and $v \in \mathcal{L}$ grows at most as (7.25).

Since v^λ is solution of (7.8), w^λ satisfies

$$\lambda w^\lambda(x) + |Dw^\lambda(x)| = l(x) - z^\lambda(0).$$

Thus by the stability result of viscosity solutions ([30, 10, 5]), we find that v satisfies (7.9) in the viscosity sense.

Part 2. Let (c, v) and (d, w) be two solutions of (7.9) and set

$$\tilde{v}(x,t) = v(x) - ct$$
$$\tilde{w}(x,t) = w(x) - dt.$$

To show that $c = d$, we argue by contradiction, assuming that $c < d$. Obviously, $\tilde{v}$ is viscosity solution of (7.1) with $u_0 = v$ and $\tilde{w}$ is viscosity solution of (7.1) with $u_0 = w$. By Theorem 7.7.1, we get that

$$\tilde{v}(x,t) - \tilde{w}(x,t) \le \sup_{\mathbb{R}^N}\{v - w\} \qquad \text{for all } (x,t) \in \mathbb{R}^N \times [0,\infty).$$

This means that

$$(d-c)t + v(x) - w(x) \le \sup_{\mathbb{R}^N}\{v - w\} \qquad \text{for all } (x,t) \in \mathbb{R}^N \times [0,\infty).$$

Recalling that $\sup_{\mathbb{R}^N} |v - w| < +\infty$, we get a contradiction for t large enough. By exchanging the roles of v, w, we conclude that $c = d$.

Part 3. We first notice that 0 is a viscosity subsolution of (7.8) so we have $v^\lambda \ge 0$. We consider, for $\alpha > 0$,

$$M_{\lambda,\alpha} = \inf_{\mathbb{R}^N}\{v^\lambda(x) + \alpha|x|^2\}.$$

The infimum is achieved at $x_{\lambda\alpha} \in \mathbb{R}^N$. Since v^λ is a viscosity solution of (7.8), we have

$$\lambda v^\lambda(x_{\lambda\alpha}) + |2\alpha x_{\lambda\alpha}| \ge l(x_{\lambda\alpha}) \ge \inf_{\mathbb{R}^N} l. \tag{7.27}$$

Writing that $x_{\lambda\alpha}$ is a minimum, we have

$$v^\lambda(x_{\lambda\alpha}) + \alpha|x_{\lambda\alpha}|^2 \le v^\lambda(0),$$

from which we infer $\alpha|x_{\lambda\alpha}| \le \sqrt{\alpha v^\lambda(0)}$. From (7.27), we then obtain

$$\lambda v^\lambda(0) + 2\sqrt{\alpha v^\lambda(0)} \ge \inf_{\mathbb{R}^N} l.$$

Letting $\alpha \to 0$ then $\lambda \to 0$, we get from (7.26) that $-c \ge \inf_{\mathbb{R}^N} l$.

We now prove the opposite inequality. Let $\epsilon > 0$ and $\hat{x} \in \mathbb{R}^N$ such that $l(\hat{x}) \le \inf l + \epsilon$. Since the locally Lipschitz continuous function v^λ is differentiable a.e. in $\mathbb{R}^N$, for all $\lambda > 0$, there exists $\hat{x}_\lambda \in B(\hat{x}, 1)$ such that $l(\hat{x}_\lambda) \le \inf l + 2\epsilon$ and v^λ is differentiable at $\hat{x}_\lambda$. Hence, we can write the equation at $\hat{x}_\lambda$ yielding

$$\lambda v^\lambda(\hat{x}_\lambda) \le \lambda v^\lambda(\hat{x}_\lambda) + |Dv^\lambda(\hat{x}_\lambda)| = l(\hat{x}_\lambda) \le \inf l + 2\epsilon.$$

Since $\hat{x}_\lambda$ is bounded, using (7.26) and letting $\lambda \to 0$ along a subsequence, we infer

$$-c \le \inf l + 2\epsilon.$$

and we conclude since ϵ is arbitrary small. $\square$

7.4 The Cauchy problem (7.1)

In this section we study the Cauchy problem (7.1) and start with a first result of existence and uniqueness of a solution.

Proposition 7.4.1. *Assume (7.4). Then there exists a unique continuous viscosity solution $u \in \mathcal{Q}$ of (7.1), which satisfies*

$$|u(x,t)| \leq Ce^{(C+1)t}(1+|x|^2) \quad \textit{for all } (x,t) \in I\!R^N \times [0,+\infty), \tag{7.28}$$

where C is the biggest constant appearing in (7.2) for u_0 and l.

Proof of Proposition 7.4.1. For $K > 0$, we define two functions by setting

$$\underline{u}(x,t) = -Ke^{(K+1)t}(1+|x|^2) \quad \text{and} \quad \overline{u}(x,t) = Ke^{t}(1+|x|^2) \qquad \forall (x,t) \in I\!R^N \times [0,\infty).$$

We claim that $\underline{u}$ and $\overline{u}$ are, respectively, a viscosity subsolution and a viscosity supersolution of (7.1). At first $\underline{u}(x,0) = -K(1+|x|^2) \leq u_0(x)$ and $\overline{u}(x,0) = K(1+|x|^2) \geq u_0(x)$ if K is chosen bigger than the constant in (7.2) for $u_0 \in \mathcal{Q}$. Moreover, we may compute

$$\begin{array}{rcl} \underline{u}_t(x,t) + |D\underline{u}(x,t)| & = & -K(K+1)e^{(K+1)t}(1+|x|^2) + |Ke^{(K+1)t}2x| \\ & = & -Ke^{(K+1)t}\left((K+1)(1+|x|^2) - 2|x|\right) \\ & \leq & l(x), \end{array}$$

if K is chosen bigger than the constant in (7.2) for $l \in \mathcal{Q}$. Similarly, we can check in an easier way that $\overline{u}$ is a viscosity supersolution.

We then apply Perron's method ([30, Theorem 4.1]) to build a discontinuous solution u of (7.1) satisfying

$$\underline{u}(x,t) \leq u(x,t) \leq \overline{u}(x,t) \quad \text{for all } (x,t) \in I\!R^N \times [0,+\infty).$$

Thanks to the comparison principle for (7.1) (see Theorem 7.7.1), we conclude that u is continuous. Uniqueness follows by applying again Theorem 7.7.1. □

We now prove Theorem 7.1.2.

Proof of Theorem 7.1.2.
1. Proof of (7.12). Fix one solution (c,v) of (7.9) given by Theorem 7.1.1. Define

$$U^{\pm}(x,t) = v(x) - ct \pm L_1,$$

where L_1 is defined in (7.10). Observe that $U^{\pm}$ are, respectively, viscosity supersolution and subsolution of (7.1). We sketch the proof for U^-. At first

$$U^-(x,0) = v(x) - L_1 = u_0(x) + v(x) - u_0(x) - L_1 \leq u_0(x)$$

thanks to (7.10). Moreover, for $t > 0$,

$$(U^-)_t + |DU^-| - l(x) = -c + |Dv| - l(x) \leq 0 \quad \text{for all } (x,t) \in I\!R^N \times (0,+\infty),$$

since (c, v) is solution to (7.9). Applying the comparison Theorem 7.7.1 between the subsolution U^-, the solution u and the supersolution U^+, we conclude

$$v(x) - ct - L_1 \leq u(x,t) \leq v(x) - ct + L_1 \quad \text{for all } (x,t) \in I\!R^N \times [0, +\infty).$$

2. Proof of (7.13). We introduce another super and subsolutions of (7.1) respectively, $u^{\pm}(x,t) = u_0(x) \pm (L_2 + c)t$. They satisfy the initial condition by construction. Let us check for instance that u^- is a subsolution in $I\!R^N \times (0, +\infty)$ (the proof for the supersolution u^+ is similar).

Let $\varphi \in C^1(I\!R^N \times (0,\infty))$ and (x_0, t_0) a local maximum point of $u^- - \varphi$. On the one side, the function $t \mapsto u_0(x_0) - (L_2 + c)t - \varphi(x_0, t)$ is smooth and attains its maximum at t_0, so we have

$$\varphi_t(x_0, t_0) = -(L_2 + c). \tag{7.29}$$

On the other side, let $h \in I\!R$, $z = \frac{D\varphi(x_0,t_0)}{|D\varphi(x_0,t_0)|}$. Since (x_0, t_0) is a maximum point of $u^- - \varphi$, we have

$$u^-(x_0 + hz, t_0) - u^-(x_0, t_0) \quad \leq \quad \varphi(x_0 + hz, t_0) - \varphi(x_0, t_0) = h|D\varphi(x_0, t_0)| + o(hz),$$

where $\frac{o(hz)}{|h|} \to 0$ as $h \to 0$. From the definition of u^- and (7.11), we have

$$\begin{aligned} u^-(x_0 + hz, t_0) - u^-(x_0, t_0) &= (u_0 - v)(x_0 + hz) - (u_0 - v)(x_0) + v(x_0 + hz) - v(x_0) \\ &\geq -L_2|h| + v(x_0 + hz) - v(x_0). \end{aligned}$$

Collecting the two previous inequality, dividing the result by $h < 0$ and sending $h \uparrow 0$, we obtain that there exists $p \in I\!R^N$ in the viscosity superdifferential $D^+v(x_0)$ such that

$$|D\varphi(x_0, t_0)| \leq L_2 + \limsup_{h \to 0} \frac{v(x_0 + hz) - v(x_0)}{h} \leq L_2 + \langle p, z \rangle \leq L_2 + |p|.$$

Since v is a viscosity solution so subsolution of (7.9), it follows

$$|D\varphi(x_0, t_0)| \leq L_2 + l(x_0) + c. \tag{7.30}$$

Finally, from (7.29) and (7.30), we obtain

$$\varphi_t(x_0, t_0) + |D\varphi(x_0, t_0)| \leq l(x_0),$$

which proves the subsolution property for u^- at (x_0, t_0).

By the comparison Theorem 7.7.1, we obtain

$$u_0(y) - L_2 h \leq u(y, h) + ch \leq u_0(y) + L_2 h \quad \text{for all } y \in I\!R^N,\ h \geq 0.$$

Let $h \geq 0$. Applying again Theorem 7.7.1 with $u(\cdot, \cdot + h)$ subsolution of (7.1) with initial condition $u(\cdot, h)$ and u supersolution of (7.1) with initial condition u_0, it follows, for all $x \in I\!R^N$, $t \geq 0$,

$$u(x, t+h) + c(t+h) - u(x,t) - ct \leq \sup_{y \in I\!R^N} \{u(y,h) + ch - u_0(y)\} \leq L_2|h|.$$

Exchanging the roles of $u(\cdot, \cdot + h)$ and u, we finally obtain (7.13).

3. Proof of (7.14). Fix $(x,t) \in I\!\!R^N \times (0,+\infty)$. For $\epsilon > 0$, we consider

$$\Psi(y,s) = u(y,s) + cs - K|x-y|(1+|x|^2+|y|^2) - \frac{|s-t|^2}{\epsilon^2},$$

where $K > 0$ is fixed below. Using (7.12) and (7.25), Ψ satisfies

$$\begin{aligned}\Psi(y,s) &\leq v(y) + L_1 - K|x-y|(1+|x|^2+|y|^2) - \frac{|s-t|^2}{\epsilon^2}\\ &\leq C(1+|y|^2)|y| + L_1 - K|x-y|(1+|x|^2+|y|^2) - \frac{|s-t|^2}{\epsilon^2} \underset{|y|\to+\infty, s\to+\infty}{\to} -\infty,\end{aligned}$$

if we choose $K > C$ where C is the constant appearing in the bound for v given by (7.25). Notice that C is independent of the particular choice of the solution (c,v) of the ergodic problem. We infer that Ψ achieves its maximum at some point $(\bar{y},\bar{s})$.

Writing that $\Psi(\bar{y},\bar{s}) \geq \Psi(\bar{y},t)$ and using (7.13), we obtain

$$\frac{|\bar{s}-t|^2}{\epsilon^2} \leq (u(\bar{y},\bar{s}) + c\bar{s}) - (u(\bar{y},t)+ct) \leq L_2|\bar{s}-t|.$$

It follows

$$\frac{|\bar{s}-t|}{\epsilon^2} \leq L_2 \tag{7.31}$$

and, in particular, $\bar{s} \to t$ as $\epsilon \to 0$. Notice that $\bar{s} > t/2 > 0$ for ϵ enough small.

Assume by contradiction that $\bar{y} \neq x$. We can therefore write the viscosity inequality for the subsolution u of (7.1) at $(\bar{y},\bar{s})$ yealding

$$-c + 2\frac{\bar{s}-t}{\epsilon^2} + \left| K\frac{\bar{y}-x}{|\bar{y}-x|}(1+|x|^2+|\bar{y}|^2) + 2K|x-\bar{y}|\bar{y} \right| \leq l(\bar{y}).$$

Using that $l \in \mathcal{Q}$, it follows

$$\begin{aligned}K(1+|x|^2+|\bar{y}|^2+2|x-\bar{y}|^2) &\leq c + 2L_2 + 2K|x-\bar{y}||x| + C(1+|\bar{y}|^2)\\ &\leq c + 2L_2 + 2K|x-\bar{y}|^2 + \frac{K}{2}|x|^2 + C(1+|\bar{y}|^2),\end{aligned}$$

which leads to a contradiction if $K > C + c^+ + 2L_2$.

We conclude that the maximum of Ψ is necessarily achieved for $\bar{y} = x$ for K sufficiently large in terms of the datas of the equation. Writing, for all $y \in I\!\!R^N$, that $\Psi(y,\bar{s}) \leq \Psi(x,\bar{s})$, we obtain

$$u(y,\bar{s}) - u(x,\bar{s}) \leq K|x-y|(1+|x|^2+|y|^2).$$

Sending ϵ to 0, we are leading to

$$u(y,t) - u(x,t) \leq K|x-y|(1+|x|^2+|y|^2), \tag{7.32}$$

which implies that u is Lipschitz continuous with respect to x for any $t > 0$ with a constant independent of t. Notice that, by continuity of u, the same inequality holds for $t = 0$.

To conclude, we notice that we can refine the Lipschitz constant. Since u is locally Lipschitz continuous in $\mathbb{R}^N \times [0, +\infty)$ by (7.13) and (7.32), the equation holds a.e. in $\mathbb{R}^N \times (0, +\infty)$ so

$$|Du(x,t)| = -u_t(x,t) + l(x) \leq L_2 + |c| + l(x) \quad \text{a.e. in } \mathbb{R}^N \times (0, +\infty).$$

This proves (7.14) and completes the proof of the theorem. □

7.5 Long time behavior of some solutions of (7.1)

We are now in position to give the proof of Theorem 7.1.3.

Proof of Theorem 7.1.3. We fix a solution (c, v) to the ergodic problem (7.9) such that v is bounded from below and $c = -\inf l$. The existence of a solution to the ergodic problem is given by Theorem 7.1.1 and the fact that v is bounded from below is an assumption of the theorem.

1. Convergence on $\mathcal{K} = \operatorname{argmin} l$.

Lemma 7.5.1. *There exists a Lipschitz continuous function $\phi : \mathcal{K} \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} u(x,t) + ct = \phi(x), \quad \textit{uniformly for } x \in \mathcal{K}.$$

The idea of the proof of the lemma is very simple: formally, the function $\tilde{u}(x,t) := u(x,t) + ct$ satisfies

$$\tilde{u}_t(x,t) \leq \tilde{u}_t(x,t) + |D\tilde{u}(x,t)| = l(x) + c = l(x) - \inf\, l. \tag{7.33}$$

For $x \in \mathcal{K}$, the right-hand side vanishes so $t \mapsto \tilde{u}(x,t)$ is decreasing. Since it is bounded from below, there is a limit as $t \to +\infty$. The rigorous proof is postponed at the end of the section.

2. Half-relaxed limits. From (7.12), the following half-relaxed limits are well-defined for all $x \in \mathbb{R}^N$,

$$\underline{u}(x) = \liminf_{t\to+\infty}{}_* \, u(x,t) + ct$$
$$\overline{u}(x) = \limsup_{t\to+\infty}{}^* \, u(x,t) + ct.$$

Moreover, both $\underline{u}$ and $\overline{u}$ are still locally Lipschitz continuous in $\mathbb{R}^N$, satisfy (7.12), (7.14) and $\underline{u} \leq \overline{u}$. To conclude to the locally uniform convergence of $u(\cdot, t) + ct$ as $t \to +\infty$, it is sufficient to prove that $\overline{u} \leq \underline{u}$. Notice that, by Lemma 7.5.1, we already know

$$\underline{u}(x) = \overline{u}(x) = \phi(x) \quad \text{for } x \in \mathcal{K}. \tag{7.34}$$

By classical stability results of viscosity solutions [30, 10, 5], $\underline{u}$ and $\overline{u}$ are respectively a super and a subsolution of the ergodic problem (7.9).

3. Proof of $\overline{u} \leq \underline{u}$. We argue by contradiction, assuming that there exists $\hat{x} \in I\!\!R^N$ such that

$$\overline{u}(\hat{x}) - \underline{u}(\hat{x}) > 2\eta > 0. \tag{7.35}$$

For $\mu \in (0,1)$ and $\alpha > 0$, we consider

$$M_{\mu\alpha} := \sup_{I\!\!R^N}\{\mu(\overline{u}(x) - \frac{1}{\sqrt{\mu}}v(x)) - (\underline{u}(x) - v(x)) - \alpha(1+|x|^2)\}. \tag{7.36}$$

Choosing α close enough to 0 and μ close enough to 1 such that

$$\mu(\overline{u}(\hat{x}) - \frac{1}{\sqrt{\mu}}v(\hat{x})) - (\underline{u}(\hat{x}) - v(\hat{x})) - \alpha(1+|\hat{x}|^2) > \eta,$$

we infer that $M_{\mu\alpha} \geq \eta > 0$.

We claim that $M_{\mu\alpha}$ is finite and the supremum is achieved at some $\bar{x} \in I\!\!R^N$. Indeed, from (7.12) and since v is bounded from below (let say by $-C_0$, $C_0 > 0$), we obtain

$$\begin{aligned} & \mu(\overline{u}(x) - \frac{1}{\sqrt{\mu}}v(x)) - (\underline{u}(x) - v(x)) - \alpha(1+|x|^2) \qquad (7.37) \\ = \; & (1-\mu)(v(x) - \overline{u}(x)) + \overline{u}(x) - \underline{u}(x) + (\mu - \sqrt{\mu})v(x) - \alpha(1+|x|^2) \\ \leq \; & (1-\mu)L_1 + 2L_1 + (\sqrt{\mu} - \mu)C_0 - \alpha(1+|x|^2) \underset{|x|\to+\infty}{\to} -\infty, \end{aligned}$$

which proves the claim.

We observe that, when $\alpha \to 0$, $\bar{x}$ cannot converge to a point of $\mathcal{K}$. Indeed, if there exists a subsequence $\alpha \to 0$ and $x_0 \in \mathcal{K}$ such that $\bar{x} \to x_0$ as $\alpha \to 0$, then, from (7.34), we infer

$$\begin{aligned} 0 < \eta \leq \limsup_{\alpha\to 0} M_{\mu\alpha} &= \mu\overline{u}(x_0) - \underline{u}(x_0) + (1-\sqrt{\mu})v(x_0) - \alpha(1+|x_0|^2) \\ &= (\mu - 1)\phi(x_0) + (1-\sqrt{\mu})v(x_0) - \alpha(1+|x_0|^2), \end{aligned}$$

which is absurd for μ close enough to 1.

Since $\mathcal{K}$ is a compact subset of $I\!\!R^N$ by assumption, it follows that there exists $\epsilon_1 > 0$ such that

$$\operatorname{dist}(\bar{x}, \mathcal{K}) \geq \epsilon_1, \quad \text{for all } \alpha > 0.$$

From this inequality and (7.6), we infer that there exists $\epsilon_2 > 0$ such that

$$l(\bar{x}) > \inf_{I\!\!R^N} l + \epsilon_2 = -c + \epsilon_2, \quad \text{for all } \alpha > 0. \tag{7.38}$$

We now take profit of the equation (7.9).

Lemma 7.5.2. *We have*

$$(\sqrt{\mu}-\mu)(l(\bar{x})+c) \leq 2\alpha|\bar{x}|. \tag{7.39}$$

To avoid technicality here and give the main ideas, we prove this lemma by assuming that that $\overline{u}, \underline{u}$ and v are smooth. This is not true in general and the rigorous proof of the lemma is postponed after the proof of the theorem.

Writing the viscosity inequalities at $\bar{x}$ for the subsolution $\overline{u}$ and the supersolution $\underline{u}$ of (7.9), we have

$$\begin{aligned} \mu|D\overline{u}(\bar{x})| &\leq \mu(l(\bar{x})+c), \\ |D\underline{u}(\bar{x})| &\geq l(\bar{x})+c. \end{aligned}$$

Noticing that $D(\mu\overline{u})(\bar{x}) = D\underline{u}(\bar{x}) - (1-\sqrt{\mu})Dv(\bar{x}) + 2\alpha\bar{x}$, subtracting the previous inequalities and using that v is a subsolution of the ergodic problem (7.9), we get

$$\begin{aligned} (\mu-1)(l(\bar{x})+c) &\geq |D\underline{u}(\bar{x}) - (1-\sqrt{\mu})Dv(\bar{x}) + 2\alpha\bar{x}| - |D\underline{u}(\bar{x})| \\ &\geq -(1-\sqrt{\mu})|Dv(\bar{x})| - 2\alpha|\bar{x}| \\ &\geq -(1-\sqrt{\mu})(l(\bar{x})+c) - 2\alpha|\bar{x}| \end{aligned}$$

which gives (7.39).

Writing that the maximum is nonnegative, we obtain that

$$\alpha(1+|\bar{x}|^2) \leq (1-\mu)L_1 + 2L_1 + (\sqrt{\mu}-\mu)C$$

and therefore

$$\alpha|\bar{x}| \to 0 \quad \text{as } \alpha \to 0. \tag{7.40}$$

Therefore, letting $\alpha \to 0$ in (7.39), we obtain

$$(\sqrt{\mu}-\mu)(l(\bar{x})+c) \leq 0.$$

Plugging (7.38) in the previous inequality, we obtain a contradiction since $0<\mu<1$. It completes the proof of the theorem. $\square$

Proof of Lemma 7.5.1. Since $\tilde{u} = u + ct$ is locally Lipschitz continuous by Theorem 7.1.2, the formal computation (7.33) holds a.e. in $I\!R^N \times (0,+\infty)$. Let $x \in \mathcal{K}$, $t, h > 0$ and $r > 0$ we have

$$\begin{aligned} \frac{1}{|B(x,r)|}\int_{B(x,r)} (\tilde{u}(y,t+h) - \tilde{u}(y,t))dy &= \frac{1}{|B(x,r)|}\int_{B(x,r)}\int_t^{t+h} \tilde{u}_t(y,s)dsdy \\ &\leq \frac{1}{|B(x,r)|}\int_{B(x,r)} (l(y) - \inf\, l)dy \int_t^{t+h} ds \end{aligned}$$

Since $\tilde{u}$ and l are continuous and $l(x) = \inf\, l$, letting $r \to 0$, we obtain

$$\tilde{u}(x,t+h) \leq \tilde{u}(x,t) \qquad \text{for all } x \in \mathcal{K},\ t,h \geq 0.$$

Therefore $t \mapsto \tilde{u}(x,t)$ is nonincreasing function on $[0,\infty)$, which is bounded from below according to (7.12) since $\mathcal{K}$ is a compact subset of $I\!R^N$ by the Assumption (7.6). By Dini Theorem, $\tilde{u}(\cdot,t)$ converges uniformly on $\mathcal{K}$ to a function $\phi : \mathcal{K} \to I\!R$ which is still locally Lipschitz continuous thus Lipschitz continuous since $\mathcal{K}$ is bounded. $\square$

Proof of Lemma 7.5.2. The rigorous proof uses the theory of viscosity solutions. In this case we need to "triple" the variables. Recall that $\bar{x} \in I\!\!R^N$ is a maximum point of $M_{\mu\alpha}$ (see (7.36)). Fix such $\bar{x}$, for $\epsilon > 0$, we consider

$$M_{\mu\alpha\epsilon} := \sup_{\overline{\Omega}_{c_\alpha}} \{w_\mu(x, y, z) - \varphi(x, y, z)\},$$

where

$$\Omega_{c_\alpha} = \{(x, y, z) \in B(\bar{x}, r_{c_\alpha}) \times B(\bar{x}, r_{c_\alpha}) \times B(\bar{x}, r_{c_\alpha}) \text{ with } r_{c_\alpha} = \frac{\alpha}{18C}, \\ C > 0 \text{ coming from Theorem 7.2.1}\},$$

$$w_\mu(x, y, z) = \mu(\overline{u}(x) - \frac{1}{\sqrt{\mu}} v(z)) - (\underline{u}(y) - v(z)) = \mu\overline{u}(x) - \underline{u}(y) + (1 - \sqrt{\mu})v(z),$$

$$\varphi(x, y, z) = \frac{|x - y|^2}{2\epsilon^2} + \frac{|x - z|^2}{2\epsilon^2} + \frac{|y - z|^2}{2\epsilon^2} + \frac{\alpha}{3}(|x|^2 + |y|^2 + |z|^2) + \epsilon|x - \bar{x}|^2.$$

The supremum is achieved at some $(\tilde{x}, \tilde{y}, \tilde{z}) \in \overline{\Omega}_{c_\alpha}$.

We proceed as in (7.37). Let $(x, y, z) \in \overline{\Omega}_{c_\alpha}$, we have

$$\begin{aligned}
w_\mu(x, y, z) &= (1 - \mu)(v(z) - \bar{u}(x)) + \bar{u}(x) - \underline{u}(y) + (\mu - \sqrt{\mu})v(z) \\
&= (1 - \mu)(v(x) - \bar{u}(x)) + (1 - \mu)(v(z) - v(x)) \\
&\qquad \bar{u}(x) - \underline{u}(x) + \underline{u}(x) - \underline{u}(y) + (\mu - \sqrt{\mu})v(z) \\
&\leq (1 - \mu)L_1 + (1 - \mu)C|x - z|(1 + |x|^2 + |z|^2) \\
&\qquad 2L_1 + C|x - y|(1 + |x|^2 + |y|^2) + (\sqrt{\mu} - \mu)C_0 \\
&\leq C(\mu, L_1, C_0) + \frac{\alpha}{9}(1 + |x|^2 + |z|^2) + \frac{\alpha}{9}(1 + |x|^2 + |y|^2),
\end{aligned}$$

where $C(\mu, L_1, C_0) = (1 - \mu)L_1 + 2L_1 + (\sqrt{\mu} - \mu)C_0$. Therefore $M_{\mu\alpha\epsilon} \leq C(\mu, L_1, C_0)$ with a bound independent of α. Using that $\bar{x}$ is a maximum of $M_{\mu\alpha}$ (see (7.36)) and $M_{\mu\alpha\epsilon} \geq M_{\mu\alpha} > 0$, we obtain the classical properties ([10, 5]) as $\epsilon \to 0$,

$$\begin{aligned}
&\frac{|\tilde{x} - \tilde{y}|^2}{2\epsilon^2}, \frac{|\tilde{y} - \tilde{z}|^2}{2\epsilon^2}, \frac{|\tilde{y} - \tilde{z}|^2}{2\epsilon^2}, \epsilon|\tilde{x} - \bar{x}|^2 \to 0, \\
&\tilde{x}, \tilde{y}, \tilde{z} \to \bar{x}, \\
&M_{\mu\alpha\epsilon} \to M_{\mu\alpha}.
\end{aligned}$$

We now choose ϵ small enough such that

$$|\tilde{x} - \bar{x}|, |\tilde{y} - \bar{x}|, |\tilde{z} - \bar{x}| < \frac{\alpha}{18C}.$$

It follows that $(\tilde{x}, \tilde{y}, \tilde{z})$ does not lie on the boundary of Ω_{c_α} and we can write the viscosity inequalities. Hence, setting $p_1 = \frac{\tilde{x} - \tilde{y}}{\epsilon^2}$, $p_2 = \frac{\tilde{x} - \tilde{z}}{\epsilon^2}$, $p_3 = \frac{\tilde{y} - \tilde{z}}{\epsilon^2}$, $q = 2\epsilon(\tilde{x} - \bar{x})$ and writing the viscosity inequalities for the subsolutions $\overline{u}, v$ at $\tilde{x}, \tilde{z}$ respectively, and for the supersolution $\underline{u}$ at $\tilde{y}$, we get

$$\begin{aligned} |p_1 + p_2 + q + \frac{2}{3}\alpha\tilde{x}| &\leq \mu(l(\tilde{x}) + c), \\ |-p_2 - p_3 + \frac{2}{3}\alpha\tilde{z}| &\leq (1 - \sqrt{\mu})(l(\tilde{z}) + c), \\ |p_1 - p_3 - \frac{2}{3}\alpha\tilde{y}| &\geq l(\tilde{y}) + c. \end{aligned}$$

Subtracting the inequalities, with a straightforward use of triangle inequalities, we obtain

$$-|\frac{2}{3}\alpha\tilde{x} + \frac{2}{3}\alpha\tilde{y} + \frac{2}{3}\alpha\tilde{z} + q| \leq \mu(l(\tilde{x}) + c) + (1 - \sqrt{\mu})(l(\tilde{z}) + c) - l(\tilde{y}) - c.$$

Sending $\epsilon \to 0$, we conclude that (7.39) holds. □

7.6 Examples and the associated control problem

Consider the one-dimensional Hamilton-Jacobi equation

$$\begin{cases} u_t(x,t) + |u_x(x,t)| = 1 + |x| & \text{in } I\!R \times (0, \infty) \\ u(x,0) = u_0(x), \end{cases} \tag{7.41}$$

where $l(x) = |x| + 1$ and u_0 belong to $\mathcal{Q} \cap \mathcal{L}$, $l \geq 0$ with $\mathcal{K} = \{0\}$ and $\min_{I\!R^N} l = 1$. By Proposition 7.4.1, there exists a unique solution $u(x,t)$ of (7.41).

Moreover, u is the value function of the following associated deterministic optimal control problem. Consider the controlled ordinary differential equation

$$\begin{cases} \dot{X}(s) = \alpha(s), \\ X(0) = x \in I\!R, \end{cases} \tag{7.42}$$

where the control $\alpha(\cdot) \in L^\infty([0,+\infty);[-1,1])$ (i.e., $|\alpha(t)| \leq 1$ a.e. $t \geq 0$). For any given control α, (7.42) has a unique solution $X(t) = X_{x,\alpha(\cdot)} = x + \int_0^t \alpha(s)ds$. We define the cost functional

$$J(x,t,\alpha) = \int_0^t (|X(s)| + 1)ds + u_0(X(t)),$$

and the value function

$$V(x,t) = \inf_{\alpha \in L^\infty([0,+\infty);[-1,1])} J(x,t,\alpha).$$

It is classical to check that $V(x,t) = u(x,t)$ is the unique viscosity solution of (7.41), see [10, 5].

7.6.1 Solutions to the ergodic problem

There are infinitely many essentially different solutions to the ergodic problem with different constants. Define $S(x) = \frac{1}{2}\int_0^x |y|dy$. The following pair (c, v) are solutions to (7.9).

- $(-1, x^2)$ and $(-1, -x^2)$. They are bounded from below (respectively from above) with $c = -\min l$;
- $(-1, S(x))$ and $(-1, -S(x))$. They are neither bounded from below nor from above and $c = -\min l$;
- $(\lambda - 1, \lambda x + S(x))$ and $(\lambda - 1, -\lambda x - S(x))$ for $\lambda > 0$. They are neither bounded from below nor from above and $c \neq -\min l$;

Notice that it is possible to build a bounded from below solution to the ergodic problem for (7.41) and that we can build a solution (c, v) to the ergodic problem for all $c \geq -\min l$.

7.6.2 Equation (7.41) with $u_0(x) = S(x)$

When (c, v) is solution to (7.9), it is obvious that $u(x,t) := -ct + v(x)$ is the unique solution to (7.41) with $u_0(x) = v(x)$ and the convergence holds, i.e., $u(x,t) + ct \to v(x)$ as $t \to +\infty$. In particular, if $u_0(x) = S(x)$, the solution of (7.41) is $u(x,t) = t + S(x)$.

Let us find the solution by computing the value function of the control problem recalled above. Let $t > 0$, we want to compute $V(x,t)$ for any $x \in \mathbb{R}$ by determining the optimal controls and trajectories.

1st case: $x \geq 0$.
There are infinitely many optimal strategies: they consist in going as quickly as possible to 0 ($=$ argmin l), to wait at 0 for a while and to go as quickly as possible to the left. For any $0 \leq \tau \leq t - x$, it corresponds to the optimal controls and trajectories

$$\alpha(s) = \begin{cases} -1, & 0 \leq s \leq x, \\ 0, & x \leq s \leq x + \tau, \\ -1, & x + \tau \leq s \leq t, \end{cases} \qquad X(s) = \begin{cases} x - s, & 0 \leq s \leq x, \\ 0, & x \leq s \leq x + \tau, \\ -(s - x - \tau), & x + \tau \leq s \leq t. \end{cases} \tag{7.43}$$

They lead to $V(x,t) = J(x,t,\alpha) = t + S(x)$. Among these optimal strategies, there are two of particular interest:

- The first one is to go as quickly as possible to 0 and to remain there ($\tau = t - x$). This strategy is typical of what happens in the periodic case: the optimal trajectories are attracted by argmin l.
- The second one is to go as quickly as possible to the left until time t ($\tau = 0$). This situation is very different to the periodic case. Due to the unbounded (from below) final cost u_0, some optimal trajectories are not anymore atttracted by argmin l and are unbounded.

2nd case: $x < 0$.
In this case there is not anymore bounded optimal trajectories. The only optimal strategy is to go as quickly as possible to the left. The optimal control are $\alpha(s) = -1$, $X(s) = x - s$ for $0 \leq s \leq t$ leading to $V(x,t) = J(x,t,-1) = t + S(x)$.

The analysis of this case in terms of control will help us for the following examples.

7.6.3 Equation (7.41) with $u_0(x) = x^2 + b(x)$ with b bounded Lipschitz continuous

When the initial data is a solution to the ergodic problem, we have trivially the convergence. Interesting examples arise when the initial data u_0 is a bounded perturbation of a solution of the ergodic problem. We consider here a bounded Lipschitz continuous perturbation of a bounded from below solution of the ergodic problem as in Theorem 7.1.3. To simplify the computations, we choose a periodic perturbation b.

For any x, an optimal strategy can be chosen among those described in Example 7.6.2. More precisely: go as quickly as possible to 0, wait nearly until time t and move a little to reach the minimum of the periodic perturbation. For t large enough (at least $t > x$), we compute the cost with α, X given by (7.43),

$$J(x,t,\alpha) = t + x^2 + b(-t + x + \tau).$$

For every t large enough, there exists $0 \leq \tau = \tau_t < t - x$ such that $b(-t+x+\tau_t) = \min b$. It leads to

$$V(x,t) = J(x,t,\alpha) = t + x^2 + \min b.$$

Therefore, we have the convergence as announced in Theorem 7.1.3.

7.6.4 Equation (7.41) with $u_0(x) = S(x) + b(x)$ with b bounded Lipschitz continuous

We compute the value function as above. Due to the unboundedness from below of u_0 we need to distinguish the cases $x \geq 0$ and $x < 0$ as in Example 7.6.2.

1st case: $x \geq 0$.
We use the same strategy as in Example 7.6.3 leading to $V(x,t) = J(x,t,\alpha) = t + x^2 + \min b$.

2nd case: $x < 0$.
In this case, the optimal strategy suggested by Examples 7.6.2 and 7.6.3 is to start by waiting a small time τ before going as quickly as possible to the left. The waiting time correspond to an attempt to reach a minimum of b at the left end of the trajectory. It corresponds to the control $\alpha(s) = \left\{ \begin{smallmatrix} 0, & 0 \leq s \leq \tau, \\ -1, & \tau \leq s \leq t, \end{smallmatrix} \right.$ and the trajectory $X(s) = \left\{ \begin{smallmatrix} x, & 0 \leq s \leq \tau, \\ x - (s - \tau), & \tau \leq s \leq t. \end{smallmatrix} \right.$ leading to

$$J(x,t,\alpha) = t + S(x) + \tau|x| + b(x - t + \tau).$$

Due to the boundedness of b, in order to be optimal, we see that necessarily $\tau = O(1/|x|)$ to keep the positive term $\tau|x|$ in J bounded. So, for large $|x|$, $x < 0$, we have $b(x-t+\tau) \approx b(x-t)$. Since $b(x-t)$ has no limit as $t \to +\infty$, the convergence for $V(x,t)$ cannot hold.

In this case, u_0 is a bounded perturbation of a solution $(c, v) = (-1, S(x))$ of the ergodic problem with $c = -\min l$ but v is not bounded from below and the convergence of the value function does not hold. It follows that the assumptions of Theorem 7.1.3 cannot be weakened easily. In particular, the boundedness from below of the solution of the ergodic problem seems to be crucial.

7.6.5 Equation (7.41) with $u_0(x) = S(x) + x + \sin(x)$

When $u_0(x) = S(x)+x+\sin(x)$, the solution of (7.41) is $u(x,t) = S(x)+x+\sin(x-t)$. Clearly, we do not have the convergence. In this case, u_0 is a bounded perturbation of the solution $(0, S(x) + x)$ of the ergodic problem but $c \neq -\min l$ and $S(x) + x$ is not bounded from below.

7.7 Appendix: Comparison principle for the solutions of (7.1)

The comparison result for the unbounded solutions of (7.1) is a consequence of a general comparison result for first-order Hamilton-Jacobi equations which holds without growth restriction at infinity.

Theorem 7.7.1. *([49, 55]) Let $T > 0$ and assume that $u \in USC(I\!R^N \times [0,T])$ and $v \in LSC(I\!R^N \times [0,T])$ are respectively a subsolution of (7.1) with initial data $u_0 \in C(I\!R^N)$ and a supersolution of (7.1) with initial data $v_0 \in C(I\!R^N)$. Then, for every $x_0 \in I\!R^N$ and $r > 0$,*

$$u(x,t) - v(x,t) \le \sup_{\overline{B(x_0,r)}} \{u_0(y) - v_0(y)\} \quad \text{for every } (x,t) \in \bar{\mathcal{D}}(x_0,r), \tag{7.44}$$

where

$$\bar{\mathcal{D}}(x_0,r) = \{(x,t) \in B(x_0,r) \times (0,T) : e^T(1 + |x - x_0|) - 1 \le r\}.$$

When $\sup_{I\!R^N}\{u_0 - v_0\} < +\infty$, a straightforward consequence is

$$u(x,t) - v(x,t) \le \sup_{I\!R^N}\{u_0 - v_0\} \quad \text{for every } (x,t) \in I\!R^N \times [0,+\infty). \tag{7.45}$$

We give the proof for reader's convenience. We start by a lemma, proving it and using it to prove the theorem.

Lemma 7.7.1. *Under the assumptions of Theorem 7.7.1, the function $w = u - v$ is a viscosity subsolution of*

$$w_t(x,t) - |Dw(x,t)| = 0 \qquad \text{in } B(x_0,r) \times (0,T). \tag{7.46}$$

Proof of Lemma 7.7.1. Let $\varphi \in C^1(B(x_0,r)\times(0,T))$ and $(\overline{x},\overline{t}) \in B(x_0,r)\times(0,T)$ be a local maximum point of $w-\varphi$. We can assume that this maximum is strict in $\overline{B(\overline{x},\tau)}\times[\overline{t}-\tau,\overline{t}+\tau]$ for $\tau>0$ small enough. Let

$$\Theta(x,y,t,s) = \varphi(x,t) + \frac{|x-y|^2}{\epsilon^2} + \frac{|t-s|^2}{\eta^2}$$

and consider

$$M_{\epsilon,\eta} := \max_{\overline{B(\overline{x},\tau)}^2\times[\overline{t}-\tau,\overline{t}+\tau]^2} \{u(x,t)-v(y,s)-\Theta(x,y,t,s)\}.$$

Since $(\overline{x},\overline{t})$ is strict maximum, we know that

$$M_{\epsilon,\eta} \to \max_{\overline{B(\overline{x},\tau)}^2\times[\overline{t}-\tau,\overline{t}+\tau]^2} w-\varphi = w(\overline{x},\overline{t})-\varphi(\overline{x},\overline{t}),$$

$x_{\epsilon,\eta}, y_{\epsilon,\eta} \to \overline{x}$ and $t_{\epsilon,\eta}, s_{\epsilon,\eta} \to \overline{t}$ when $\epsilon,\eta\to 0$. As u and v is a viscosity subsolution and viscosity supersolution of (7.1) respectively, by a straightforward calculation, we obtain

$$\begin{aligned}
2\frac{t_{\epsilon,\eta}-s_{\epsilon,\eta}}{\eta^2} + \varphi_t(x_{\epsilon,\eta},t_{\epsilon,\eta}) + |2\frac{x_{\epsilon,\eta}-y_{\epsilon,\eta}}{\epsilon^2} + D\varphi(x_{\epsilon,\eta},t_{\epsilon,\eta})| &\le l(x_{\epsilon,\eta}),\\
2\frac{t_{\epsilon,\eta}-s_{\epsilon,\eta}}{\eta^2} + |2\frac{x_{\epsilon,\eta}-y_{\epsilon,\eta}}{\epsilon^2}| &\ge l(y_{\epsilon,\eta}).
\end{aligned}$$

Subtracting the viscosity inequalities and using that $l\in\mathcal{L}$, we have

$$\varphi_t(x_{\epsilon,\eta},t_{\epsilon,\eta}) - |D\varphi(x_{\epsilon,\eta},t_{\epsilon,\eta})| \le C(1+|x_{\epsilon,\eta}|^2+|y_{\epsilon,\eta}|^2)|x_{\epsilon,\eta}-y_{\epsilon,\eta}|.$$

Letting $\epsilon,\eta\to 0$ we get that $x_{\epsilon,\eta},y_{\epsilon,\eta}\to\overline{x}$, $t_{\epsilon,\eta},s_{\epsilon,\eta}\to\overline{t}$. Hence

$$\varphi_t(\overline{x},\overline{t}) - |D\varphi(\overline{x},\overline{t})| \le 0.$$

Therefore $w=u-v$ is a viscosity subsolution of (7.46). $\square$

Proof of Theorem 7.7.1. Let $0<\epsilon<r$ and $\chi_\epsilon : I\!R^+ \to I\!R^+\cup\{+\infty\}$ be a C^1 function in $[0,r)$ such that $\chi_\epsilon\equiv 0$ in $[0,r-\epsilon]$, χ_ϵ is increasing in $[0,r)$ with limit $+\infty$ at r and $\chi\equiv +\infty$ in $[r,+\infty)$. Let $\eta>0$, the function $w(x,t-\eta t-\frac{\eta}{T-t})-\chi_\epsilon(e^t(1+\sqrt{|x-x_0|^2+\epsilon^2})-1)$ achieves its maximum at a point $(\bar{x},\bar{t})\in B(x_0,r)\times[0,T)$. If $\bar{t}>0$, then we have a local maximum and, using that w is a subsolution of (7.46) in $B(x_0,r)\times(0,T)$, we obtain from Lemma 7.7.1,

$$\eta + \frac{\eta}{(T-\bar{t})^2} + \chi_\epsilon' e^{\bar{t}}\left(1+\sqrt{|\bar{x}-x_0|^2+\epsilon^2} - \frac{|\bar{x}-x_0|^2}{\sqrt{|\bar{x}-x_0|^2+\epsilon^2}}\right) \le 0, \tag{7.47}$$

which is a contradiction, since $\chi_\epsilon'\ge 0$. Thus $\bar{t}=0$ and we obtain that, for every $(x,t)\in B(x_0,r)\times[0,T]$,

$$\begin{aligned}
& w(x,t)-\eta t-\frac{\eta}{T-t}-\chi_\epsilon(e^t(1+\sqrt{|x-x_0|^2+\epsilon^2})-1) \qquad (7.48)\\
\le\;& w(\bar{x},0) \le \sup_{y\in\bar{B}(x_0,r)}\{w(y,0)\}.
\end{aligned}$$

Letting η go to 0, we have the desired comparison (7.44) for every (x, t) such that

$$\chi_\epsilon(e^t(1 + \sqrt{|x - x_0|^2 + \epsilon^2}) - 1) = 0 \Leftrightarrow e^t(1 + \sqrt{|x - x_0|^2 + \epsilon^2}) - 1 \leq r - \epsilon.$$

Letting ϵ go to 0, we get the comparison (7.44) for every $(x, t) \in \bar{\mathcal{D}}(x_0, r)$. $\square$

Chapter 8

Conferences and Poster

- Participation à la 3ème réunion ANR HJnet à Paris 7 (28-29 jan 2014).
- Participation à la conférence SMAI-MODE et j'ai assisté au mini-cours Optimisation Polynomiale (INSA de Rennes 24-25 mars 2014).
- Participation Programme CIMI Opérateurs non locaux, EDPs et processus de Lévy, Toulouse (26-29 mai 2014).
- Participation à la conférence NETCO, Tours (23-27 juin 2014). En particulier, j'ai assisté aux 2 mini-cours qui ont été donnés.
- Participation aux journées doctorales Lebesgues (Rennes 13-15 octobre 2014).
- Participation à la 4ème rencontre ANR HJnet à Paris 7 (18-19 mars 2015).
- Participation au mini-cours Optimisation numérique et applications à Limoges (15-17 mai 2015 (15 heures)).
- Participation au mini-cours Ecole doctorale MATISSE : Nicoletta Tchou – Olivier Ley sur le sujet “ Comportement en temps long pour les équations de Hamilton-Jacobi ” (2015)(8 heures).
- Participation au workshop sur les problèmes de Whitney (19-23 octobre 2015).
- Participation à la 5ème rencontre ANR HJnet à Tours (14-15 janvier 2016).
- Participation au cours en Master 2 à l'université de Rennes 1 : Moyennisation et comportement asymptotique en contrôle optimal (Marc Quincampoix)(2016).
- Participation au mini-cours Optimisation à Toulouse (21-22 mars 2016).
- Participation et présentation orale aux journées SMAI-MODE à Toulouse (23-25 mars 2016).
- Participation et présentation d'un poster à la conférence HJ2016 à Rennes (30 mai – 3 juin 2016).

Regularity results for solutions of the Hamilton-Jacobi equation with Ornstein-Uhlenbeck operator and applications

(Joint work with E. Chasseigne and O. Ley)

Thi Tuyen Nguyen

University of Rennes 1

thi-tuyen.nguyen@univ-rennes1.fr

Abstract

We present some Lipschitz regularity results for unbounded viscosity solutions of uniformly elliptic and nonlocal equations with Ornstein-Uhlenbeck operator in $\mathbb{R}^N$. We generalize the result obtained by Fujita, Ishii and Loreti [4]. Our work is based on Ishii-Lions's method in the local case and Barles, Chasseigne, Ciomaga and Imbert [1] in the nonlocal one. We give also some applications to solve ergodic problem and study long time behavior of solutions of the corresponding Cauchy problem.

Introduction

We study the Lipschitz regularity for solutions of the stationary problem

$$\lambda u^\lambda(x) - \mathcal{F}(x,[u^\lambda]) + \langle b(x), Du^\lambda(x)\rangle + H(x, Du^\lambda(x)) = f(x) \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $\lambda \in (0,1)$, $\mathcal{F}$ denotes $\mathrm{tr}(\sigma(x)\sigma(x)^T D^2u^\lambda(x))$ (local) or $\mathcal{I}(x, u^\lambda, Du^\lambda)$ (nonlocal) operator with

$$\mathcal{I}(x,u^\lambda,Du^\lambda) = \int_{\mathbb{R}^N}\{u^\lambda(x+z) - u^\lambda(x) - Du^\lambda(x).z\mathbb{I}_B(z)\}\nu(dz), \tag{2}$$

where ν is a Lévy measure defined by

$$\nu(dz) = \begin{cases} \frac{dz}{|z|^{N+\beta}}, \ \beta\in(0,2) & \text{on } B\\ e^{-|z|}dz & \text{on } B^c.\end{cases} \tag{3}$$

The diffusion matrix $A(x) = \sigma(x)\sigma(x)^T$ satisfies

$$\begin{cases}\sigma \in C(\mathbb{R}^N, \mathcal{M}_N), \text{ there exists } C_\sigma,\ L_\sigma > 0 \text{ such that}\\ |\sigma(x)| \le C_\sigma, \quad |\sigma(x)-\sigma(y)| \le L_\sigma|x-y| \quad x\in\mathbb{R}^N\end{cases} \tag{4}$$

$$\begin{cases}\text{(Strict ellipticity) There exists } \rho>0 \text{ such that}\\ A(x) \ge \rho I,\ \ \forall x\in\mathbb{R}^N.\end{cases} \tag{5}$$

The $\langle b(x), Du^\lambda(x)\rangle$ term is called Ornstein-Uhlenbeck term and satisfies

$$\begin{cases}\text{There exists } \alpha>0 \text{ such that}\\ \langle b(x)-b(y), x-y\rangle \ge \alpha|x-y|^2 \quad x,y\in\mathbb{R}^N.\end{cases} \tag{6}$$

H and f are continuous functions which satisfy

$$|H(x,p)| \le C_H(1+|p|), \quad x,p\in\mathbb{R}^N \text{ (sublinearity)}, \tag{7}$$

$$\begin{cases}\text{For } f\in\mathcal{E}_\mu(\mathbb{R}^N), \text{ there exists } C_f,\mu>0 \text{ such that}\\ |f(x)-f(y)| \le C_f(\phi_\mu(x)+\phi_\mu(y))|x-y|, \quad x,y\in\mathbb{R}^N,\end{cases} \tag{8}$$

respectively, where $\phi_\mu(x) = e^{\mu\sqrt{|x|^2+1}}$, $\mu>0, x\in\mathbb{R}^N$ and

$$\mathcal{E}_\mu(\mathbb{R}^N) = \{v:\mathbb{R}^N\to\mathbb{R} : \lim_{|x|\to+\infty}\frac{v(x)}{\phi_\mu(x)} = 0\} \text{ (exponential growth)}.$$

Main Objectives

Under the assumptions introduced in the introduction part, we try to obtain local Lipschitz bounds for solutions of (1) which belong to class $\mathcal{E}_\mu(\mathbb{R}^N)$.

We use this result to solve ergodic problem

$$c - \mathcal{F}(x,[v]) + \langle b(x), Dv(x)\rangle + H(x, Dv(x)) = f(x) \quad \text{in } \mathbb{R}^N, \tag{9}$$

where $c\in\mathbb{R}$.

In parallel, we also try to obtain the local Lipschitz bounds for solutions of Cauchy problem

$$\begin{cases} u_t(x,t) - \mathcal{F}(x,[u]) + \langle b(x), Du(x,t)\rangle + H(x,Du(x,t)) = f(x) & \text{in } \mathbb{R}^N\times(0,\infty)\\ u(x,0) = u_0(x) \quad \text{on } \mathbb{R}^N.\end{cases} \tag{10}$$

Here u is studied in the class of function

$$\mathcal{E}_\mu(\Omega) = \left\{ v:\Omega\to\mathbb{R} : \lim_{|x|\to+\infty}\sup_{0\le t<T}\frac{v(x,t)}{\phi_\mu(x)} = 0\right\},$$

where Ω denotes $R_T := \mathbb{R}^N\times[0,T)$ or $\overline{Q} := \mathbb{R}^N\times[0,\infty)$. Then using it to study long time behavior of solutions of (10).

Results for nondegenerate equation

Theorem 1. *Let $u^\lambda \in C(\mathbb{R}^N)\cap\mathcal{E}_\mu(\mathbb{R}^N)$ be a solution of* (1). *Assume that* (6) *holds for any $\alpha>0$,* (7) *and* (8) *hold. If one of two following holds*

i. Let $\mu>0$, $\mathcal{F}(x,[u^\lambda]) = \mathrm{tr}(A(x)D^2u^\lambda(x))$ and suppose that (4), (5) *hold.*

ii. Let $\mu\in(0,1)$, $\mathcal{F}(x,[u^\lambda]) = \mathcal{I}(x,u^\lambda,Du^\lambda)$ and suppose that $\beta\in(1,2)$ in (3).

Then there exists a constant C independent of λ such that

$$|u^\lambda(x) - u^\lambda(y)| \le C|x-y|(\phi(x)+\phi(y)), \quad x,y\in\mathbb{R}^N,\ \lambda\in(0,1). \tag{11}$$

Theorem 2. *Let $u\in C(R_T)\cap\mathcal{E}_\mu(R_T)$ be a solution of* (10). *In addition to the hypotheses of Theorem 1, assume that*

$$|u_0(x) - u_0(y)| \le C_0|x-y|(\phi(x)+\phi(y)) \text{ for } x,y\in\mathbb{R}^N. \tag{12}$$

Then there exists a constant $C>0$, depending only on $\alpha,\mu,C_0,C_f,C_H,\mathcal{F}$ such that

$$|u(x,t)-u(y,t)| \le C|x-y|(\phi(x)+\phi(y)) \text{ for } x,y\in\mathbb{R}^N,\ t\in[0,T). \tag{13}$$

Ideas of proof:

- Local equation: Combine the effects of Ornstein-Uhlenbeck operator and the ellipticity using Ishii-Lions's method.
- Nonlocal equation: We use the same idea as in the local one. The ellipticity in this case is replaced by an estimate proved in [1] which allows to apply Ishii-Lions method's idea when $\beta\in(1,2)$.

Results for degenerate equation

In the degenerate case, we need to strengthen the assumption on H:

$$\begin{cases}\text{There exists } C_H>0 \text{ such that}\\ |H(x,p)-H(y,p)| \le C_H|x-y|(1+|p|), \quad x,y,p,q\in\mathbb{R}^N\\ |H(x,p)-H(x,q)| \le C_H|p-q|(1+|x|).\end{cases} \tag{14}$$

Theorem 3. *Let $u^\lambda\in C(\mathbb{R}^N)\cap\mathcal{E}_\mu(\mathbb{R}^N)$ be a solution of* (1). *Assume that* (6), (8) *and* (14) *hold. If one of two following holds*

i. Let $\mu>0$, $\mathcal{F}(x,[u^\lambda]) = \mathrm{tr}(A(x)D^2u^\lambda(x))$ and suppose that (4) *holds.*

ii. Let $\mu\in(0,1)$, $\mathcal{F}(x,[u^\lambda]) = \mathcal{I}(x,u^\lambda,Du^\lambda)$).

Moreover, let $\alpha > C(\mathcal{F},H)$, then there exists a constant C independent of λ such that (11) *holds.*

Remark. If σ is a constant matrix (typically identity matrix as in [4]), H satisfies (14) and does not depend on x. Then we can obtain (11) without additional condition on the size of α in the Ornstein-Uhlenbeck term.

We obtain the same result for solutions of the Cauchy problem.

Some applications

Theorem 4. ***(Ergodic problem).*** *Suppose that the assumptions of Theorem 1 holds. Then there is a solution $(c,v)\in\mathbb{R}\times C(\mathbb{R}^N)$ of* (9) *such that*

$$v \in \bigcap_{\gamma>\mu}\mathcal{E}_\gamma(\mathbb{R}^N).$$

To study long time behavior of solutions of (10), we have to linearize the equation in order to apply strong maximum principle. Therefore, we need the following assumption on H

$$\begin{cases}\text{There exists } L_{H_1}, L_{H_2}>0 \text{ such that}\\ |H(x,p)-H(y,p)| \le L_{H_1}|x-y|(1+|p|), \quad x,y,p,q\in\mathbb{R}^N\\ |H(x,p)-H(x,q)| \le L_{H_2}|p-q|\\ |H(x,0)| \le H_0,\end{cases} \tag{15}$$

Theorem 5. ***(Long time behavior).*** *Let $\mu\in(0,1)$, $\beta\in(1,2)$, $\alpha>0$. Assume that* (3), (4), (5), (6), (8) *and* (15). *Let $\gamma\in(0,1)$ be a constant satisfying $\mu<\gamma$. Let $u\in\mathcal{E}_\mu(\overline{Q})\cap C(\overline{Q})$ be the unique solution of* (10) *and $(c,v)\in\mathbb{R}\times(C(\mathbb{R}^N)\cap\mathcal{E}_\gamma(\mathbb{R}^N))$ be a solution of* (9). *Then there is a constant $a\in\mathbb{R}$ such that*

$$\lim_{t\to\infty}\max_{B(0,R)}|u(x,t)-(ct+v(x)+a)| = 0 \qquad \textit{for all } R>0.$$

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Résumé

Comportement en temps long des solutions de quelques équations de Hamilton-Jacobi du premier et second ordre, locales et non-locales, dans des cas non-périodiques

La motivation principale de cette thèse est l'étude du comportement en temps grand des solutions non-bornées d'équations de Hamilton-Jacobi visqueuses dans $I\!R^N$ en présence d'un terme de Ornstein-Uhlenbeck. Nous considérons la même question dans le cas d'une équation de Hamilton-Jacobi du premier ordre. Dans le premier cas, qui constitue le coeur de la thèse, nous généralisons les résultats de Fujita, Ishii & Loreti (2006) dans plusieurs directions. La première est de considérer des opérateurs de diffusion plus généraux en remplaçant le Laplacien par une matrice de diffusion quelconque. Nous considérons ensuite des opérateurs non-locaux intégro-différentiels de type Laplacien fractionnaire. Le second type d'extension concerne le Hamiltonien qui peut dépendre de x et est seulement supposé sous-linéaire par rapport au gradient.

Chapitre 3: Nous introduisons la définition et les propriétés de base des processus de Ornstein-Uhlenbeck gouvernés par un mouvement brownien (cas local) ou par un processus de Lévy (cas non-local). De plus, nous expliquons le lien entre des problèmes de contrôle optimal stochastique et nos équations de Hamilton-Jacobi à la fois dans le cas local et non-local. Enfin, nous décrivons quel est l'effet de l'opérateur de Ornstein-Uhlenbeck dans nos équations et comment il permet de retrouver certaines propriétés de compacité qui font défaut dans le cadre non-borné.

Chapitre 4: (Résultats principaux). Nous établissons des bornes a priori de gradient uniformes pour les solutions de l'équation locale et non-locale. Ces résultats sont une étape essentielle pour obtenir les résultats de convergence. Les difficultés des preuves proviennent du caractère non-borné des solutions et des hypothèses faibles sur l'Hamiltonien (H est seulement sous-linéaire). Nous considérons dans un premier temps des équations elliptiques dans le cas local et "non-dégénérées" dans le cas non-local (ce qui signifie que l'ordre dans la mesure du terme intégro-différentiel est $\beta \in (1,2)$). La preuve met à profit l'effet du terme de Ornstein-Uhlenbeck combiné à la non-dégénérescence des équations, ce qui permet d'utiliser des idées de Ishii & Lions (1990) dans le cas local et de Barles, Chasseigne, Ciomaga & Imbert (2008) dans le cas non-local. Nous obtenons également des bornes de gradient uniformes dans le cas dégénéré, c'est-à-dire quand l'équation locale est elliptique ou parabolique dégénérée et quand l'équation non-locale a un opérateur d'ordre $\beta \in (0,2)$, sous des hypothèses plus fortes sur H et une condition sur la magnitude du terme de Ornstein-Uhlenbeck.

Chapitre 5: Nous étudions l'existence et l'unicité de solutions dans le cadre local et non-local. Pour construire une solution continue, la technique habituelle dans la théorie des solutions de viscosité est d'appliquer la méthode de Perron associée à un principe de comparaison fort. Mais pour des équations avec des Hamiltoniens seulement sous-linéaires, il est difficile voir impossible d'obtenir un principe de comparaison. Nous utilisons donc une autre méthode qui est basée sur des troncatures de H et de la donnée initiale comme dans Barles & Souganidis (2001) et qui utilise les estimations a priori du Chapitre 4.

Chapitre 6: Nous appliquons nos résultats de régularité Lipschitz pour résoudre le problème ergodique associé et étudier le comportement en temps grand des solutions dans le cas local uniformément parabolique et dans le cas non-local non-dégénéré (l'ordre β est dans $(1,2)$). Les preuves de convergence utilisent le principe du maximum fort.

Chapitre 7: Le comportement en temps grand des solutions non-bornées d'équations de Hamilton-Jacobi du premier ordre est considéré. Les équations sont de type eikonal avec un coût instantanné l non-borné. La fonction l et la donnée initiale u_0 sont prises à croissance quadratique. Quand $l \geq 0$ et $\operatorname{argmin} l$ est compact, nous établissons l'existence de solutions (c, v) du problème ergodique, où c est une constante ergodique. De plus, s'il existe une solution v bornée inférieurement, nous obtenons le comportement en temps grand des solutions de l'équations de Hamilton-Jacobi quand on démarre d'une donnée initiale u_0 suffisamment proche de v. Des exemples illustrent ces résultats et leur limitation.

Abstract

Long time behavior of solutions of some first and second order, local and nonlocal Hamilton-Jacobi equations in non-periodic settings

The main aim of this thesis is to study large time behavior of unbounded solutions of viscous Hamilton-Jacobi equations in $I\!R^N$ in presence of an Ornstein-Uhlenbeck drift. We also consider the same issue for a first order Hamilton-Jacobi equation. In the first case, which is the core of the thesis, we generalize the results obtained by Fujita, Ishii & Loreti (2006) in several directions. The first one is to consider more general operators. We first replace the Laplacian by a general diffusion matrix and then consider a nonlocal integro-differential operator of fractional Laplacian type. The second kind of extension is to deal with more general Hamiltonians which are merely sublinear.

Chapter 3: We introduce the definition and the fundamental properties of the Ornstein-Uhlenbeck process driven by Brownian Motion (related with the local equation) or by a Lévy process (related with the nonlocal one). Besides, we give the link between stochastic optimal control and the Hamilton-Jacobi equation in both the local and the nonlocal case. Moreover, we explain the effect of the Ornstein-Uhlenbeck term in our equations, which allows to recover some compactness properties in the unbounded setting.

Chapter 4: (Main results of the thesis). We establish a priori local uniform gradient bounds for solutions of both the local and the nonlocal equation. This is the main step to obtain the convergence result. Difficulties come from the unboundedness of the solutions and the weak assumptions on the Hamiltonian (H is sublinear). We first consider elliptic equations in the local case and nondegenerate equations in the nonlocal case (which means that the order in the measure of the integro-differential operator is $\beta \in (1,2)$). The proof takes profit of the effect of the Ornstein-Uhlenbeck operator combined with the nondegeneracy of the equation, following some ideas of Ishii & Lions (1990) in the local case and Barles, Chasseigne, Ciomaga & Imbert (2008) in the nonlocal one. We also obtain the same uniform gradient bounds for degenerate equations, i.e., when the local equation is degenerate elliptic or parabolic and the nonlocal one has an operator of order $\beta \in (0,2)$, under stronger assumptions on the Hamiltonian and a condition on the size of the Ornstein-Uhlenbeck term.

Chapter 5: We study the existence and uniqueness of solutions for both the local and the nonlocal equation. To build a continuous solutions of the equations, classical techniques in viscosity theory usually use Perron's method together with a strong comparison result. But for equations with merely sublinear Hamiltonians, it is difficult to obtain the comparison result. Therefore we use another approach inspiring by Barles & Souganidis (2001). It is based on a truncation on the Hamiltonian and the initial data and takes profit of our the priori estimates of Chapter 4.

Chapter 6: We give applications of the Lipschitz regularity results to solve the associated ergodic problem and study large time behavior for both the local and nonlocal equation when the local equation is elliptic and the nonlocal is nondegenerate (the order β satisfies $1 < \beta < 2$). The proofs are based on the strong maximum principle.

Chapter 7: Large time behavior of unbounded solutions of some first-order Hamilton-Jacobi equations in $I\!R^N$ are investigated. The equation is of Eikonal type with an unbounded running cost l. The function l and the initial data u_0 are assumed to have quadratic growth. When $l \geq 0$ and $\operatorname{argmin} l$ is a compact subset of $I\!R^N$, we establish the existence of solutions (c, v) to the ergodic problem, where c is an ergodic constant. Moreover, if there exists a bounded from below solution v, then we obtain the large time behavior of the solution of the Hamilton-Jacobi equation when starting with u_0 close enough to v. Some examples are studied.