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Effective forms of the Manin-Mumford conjecture and realisations of the abelian polylogarithm

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de Toulouse

THÈSE

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Délivré par : *l'Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)*

Présentée et soutenue le *15/09/2016* par :

Danny SCARPONI

**Formes effectives de la conjecture de Manin-Mumford
et réalisations du polylogarithme abélien.**

JURY

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Contents

Acknowledgements	i
Contents	ii
Introduction	v
1 Preliminaries and presentation of the results	1
1.1 The Manin-Mumford conjecture	1
1.1.1 A brief survey on the Manin-Mumford conjecture	1
1.1.2 Buium's proof	2
1.1.3 Generalization of Buium's proof to higher dimensions	4
1.2 Polylogarithm on abelian schemes and Deligne-Beilinson cohomology	6
1.2.1 Arithmetic intersection theory	6
1.2.2 Analytic-Deligne and Deligne-Beilinson cohomology	13
1.2.3 Our result on the degree zero part of the motivic polylogarithm	14
2 The Manin-Mumford conjecture for subvarieties with ample cotangent bundle	17
2.1 Introduction	17
2.2 Notations	20
2.3 The Greenberg transform and the critical schemes	21
2.4 The geometry of vector bundles in positive characteristic	23
2.5 Sparsity of p -divisible unramified liftings	26
2.6 The Manin-Mumford conjecture for subvarieties with ample cotangent bundle	34
2.7 An explicit bound for the cardinality of the prime-to- p torsion of Debarre's subvarieties	36
2.8 Appendix	44

3 Polylogarithm on abelian schemes and Deligne-Beilinson cohomology	47
3.1 Introduction	47
3.1.1 The degree zero part of the motivic polylogarithm	47
3.1.2 Our main result	49
3.1.3 An outline of the chapter	49
3.2 Notations	50
3.3 Burgos' arithmetic Chow groups	51
3.3.1 Smooth forms with logarithmic singularities along infinity . .	51
3.3.2 Definition of Burgos' arithmetic Chow groups	52
3.3.3 Two important properties	53
3.4 An intermediate result	54
3.5 The case of an abelian scheme over an arithmetic variety	55
3.6 The main result	59
Bibliography	61

Introduction

In this thesis we approach two independent problems in the field of arithmetic geometry, one regarding the torsion points of abelian varieties and the other the motivic polylogarithm on abelian schemes.

The Manin–Mumford conjecture of Yuri Manin and David Mumford states that a curve C of genus greater or equal than 2 in its Jacobian variety J can only contain a finite number of points that are of finite order in J . In 1983, after having proved the conjecture, Raynaud generalized its result to higher dimensions: if A is an abelian variety and X is a subvariety of A not containing any translate of an abelian subvariety of A , then X can only have a finite number of points that are of finite order in A . Various other proofs (sometimes only for the case of curves) later appeared, and in 1985 Coleman was the first to provide an explicit bound for the cardinality of the intersection of a curve with the torsion subgroup of its Jacobian, in the special case in which the Jacobian has complex multiplication.

In 1996, Buium presented both a new proof of the Manin–Mumford conjecture for curves and a new bound which does not require any assumption on J . In this thesis, we show that Buium’s argument can be made applicable in higher dimensions to prove the general Manin–Mumford conjecture for subvarieties with ample cotangent bundle. We also prove a quantitative version of the conjecture for a class of subvarieties with ample cotangent studied by Debarre. Such a result is particularly interesting, since so far we only had bounds in the one dimensional case. Our proof also generalizes to any dimension a result on the sparsity of p -divisible unramified liftings obtained by Raynaud in the case of curves.

In 2014, Kings and Rössler gave a simple axiomatic description of the degree zero part of the motivic polylogarithm on abelian schemes and they showed that its realisation in analytic Deligne cohomology can be described in terms of the Bismut–Köhler higher analytic torsion form of the Poincaré bundle. In this thesis,

we use the arithmetic intersection theory in the sense of Burgos to refine Kings and Rössler's result. More precisely, we prove that if the base scheme is proper, then not only the realisation in analytic Deligne cohomology but also the realisation in Deligne-Beilinson cohomology of the degree zero part of the motivic polylogarithm can be described in terms of the Bismut-Köhler higher analytic torsion form of the Poincaré bundle.

This thesis is composed by three chapters. In the first one we provide the reader with some preliminaries and present our two results. Chapter 2 (resp. chapter 3) contains the proof of our result on torsion points (resp. on the motivic polylogarithm on abelian schemes). Notice that chapters 2 and 3 can be read independently, without reference to chapter 1.

Chapter 1

Preliminaries and presentation of the results

This chapter is divided in two sections. In the first one, we give a brief survey on the Manin-Mumford conjecture for number fields and we describe our generalization of Buium's proof to higher dimensions. Section 2 is mainly devoted to recall some basic definitions and facts about arithmetic intersection theory. We also shortly present our result on the degree zero part of the motivic polylogarithm on abelian schemes.

1.1 The Manin-Mumford conjecture

1.1.1 A brief survey on the Manin-Mumford conjecture

The Manin-Mumford Conjecture for number fields is a deep and important finiteness question (raised independently by Manin and Mumford) regarding the intersection of a curve with the torsion subgroup of its Jacobian:

Theorem 1.1.1. *Let K denote a number field, \overline{K} an algebraic closure of K and let C/K be a curve of genus $g \geq 2$. Denote by J the Jacobian of C and fix an embedding $C \hookrightarrow J$ defined over K . Then the set $C(\overline{K}) \cap \text{Tor}(J(\overline{K}))$ is finite.*

Theorem 1.1.1 was proved by Raynaud in 1983, see [Ray83b]. Some months later, Raynaud generalized his result obtaining the following (see [Ray83c]):

Theorem 1.1.2. *Let K and \overline{K} be as above. Let A be an abelian variety and X an algebraic subvariety, both defined over K . If X does not contain any translation of an abelian subvariety of A of dimension at least one, then $X(\overline{K}) \cap \text{Tor}(A(\overline{K}))$ is finite.*

Various other proofs (sometimes only for the case of curves) later appeared, due to Serre ([Ser85]), Coleman ([Col85]), Hindry ([Hin88]), Buium ([Bui96]), Hrushovski ([Hru01]), Pink-Rössler ([PR02]).

Coleman was the first to provide an explicit bound for the cardinality of the intersection of a curve with the torsion subgroup of its Jacobian. In [Col85], he considered the special case in which the Jacobian J of the curve C/K has complex multiplication. In this situation he proved that

$$\sharp(C(\overline{K}) \cap \text{Tor}(J(\overline{K}))) \leq pg$$

where $g \geq 2$ is the genus of C and p is the smallest prime greater or equal than 5 divisible by a prime \mathfrak{p} of K with the following properties: K/\mathbb{Q} is unramified at \mathfrak{p} and the curve C/K has ordinary reduction at \mathfrak{p} . Coleman's bound is sharp and, as shown by an example in [Col85], it fails in general, in the noncomplex multiplication case.

In 1996 Buium gave a new proof of Theorem 1.1.1 and provided a bound without assuming J has complex multiplication.

Theorem 1.1.3. *In the same hypotheses of Theorem 1.1.1, let \mathfrak{p} be a prime of K above $p > 2g$ such that K/\mathbb{Q} is unramified at \mathfrak{p} and C/K has good reduction at \mathfrak{p} . Then*

$$\sharp(C(\overline{K}) \cap \text{Tor}(J(\overline{K}))) \leq p^{4g} 3^g (p(2g - 2) + 6g)g!$$

1.1.2 Buium's proof

We sketch here Buium's proof.

1. Thanks to a result due to Coleman [Col87], Theorem 1.1.3 is an easy consequence of its "nonramified version":

Theorem 1.1.4. *Let \mathfrak{p} be as in Theorem 1.1.3 and let R be the ring of Witt vectors with coordinates in $k := \overline{k(\mathfrak{p})}$, the algebraic closure of the residue field of \mathfrak{p} . Let C/R be a smooth projective curve of genus $g \geq 2$ possessing an R -point and embedded via this point into its Jacobian J/R . Then $\sharp(C(R) \cap \text{Tor}(J(R)))$ is finite and*

$$\sharp(C(R) \cap \text{Tor}(J(R))) \leq p^{4g} 3^g (p(2g - 2) + 6g)g!$$

2. For any scheme Y of finite type over R , denote by Y_0 the k -scheme $Y \otimes_R k$. Buium associates to Y_0 the first p -jet space Y_0^1 , a scheme over k equipped with

a lifting map $\nabla_0^1 : Y(R) \rightarrow Y_0^1(k)$ and a map $Y_0^1 \rightarrow Y_0$. If Y is smooth along Y_0 , then Y_0^1 is a torsor on Y_0 under the Frobenius tangent bundle

$$\mathrm{FT}(Y_0/k) := \mathrm{Spec}(\mathrm{Sym} F_{Y_0}^* \Omega_{Y_0/k}).$$

3. If Y/R is a smooth projective curve of genus $g \geq 2$, then the torsor $Y_0^1 \rightarrow Y_0$ is not trivial. Therefore, in the situation of Theorem 1.1.4, we get a nontrivial torsor C_0^1 over C_0 which corresponds to an extension of vector bundles

$$0 \rightarrow \mathcal{O}_{C_0} \rightarrow E_C \rightarrow F_{C_0}^* \Omega_{C_0/k} \rightarrow 0.$$

The non-triviality of the torsor implies that this extension is non-split and one can deduce that the vector bundle E_C is ample.

4. C_0^1 identifies with $\mathbb{P}(E_C) \setminus \mathbb{P}(F_{C_0}^* \Omega_{C_0/k})$ which is affine thanks to the ampleness of E_C .
5. If $p > 2$, the map $\nabla_0^1 : J(R) \rightarrow J_0^1(k)$ is injective if restricted to $\mathrm{Tor}(J(R))$, so

$$\sharp(C(R) \cap \mathrm{Tor}(J(R))) = \sharp(\nabla_0^1(C(R) \cap \mathrm{Tor}(J(R)))).$$

6. Let $B := pJ_0^1$ be the maximal abelian subvariety of J_0^1 . Then the image of $\nabla_0^1(\mathrm{Tor}(J(R)))$ under the homomorphism

$$J_0^1(k) \rightarrow J_0^1(k)/B(k)$$

has cardinality at most p^{2g} . This implies that $\nabla_0^1(C(R) \cap \mathrm{Tor}(J(R)))$ is the union of at most p^{2g} sets of the type

$$(B(k) + b) \cap C_0^1(k),$$

where $b \in J_0^1(k)$. Each of these is finite, being B proper and C_0^1 affine.

7. To bound the cardinality of $C(R) \cap \mathrm{Tor}(J(R))$ it is then enough to focus on the intersection $B(k) \cap C_0^1(k)$. Consider the exact sequence of vector bundles corresponding to the torsor $J_0^1 \rightarrow J_0$:

$$0 \rightarrow \mathcal{O}_{J_0} \rightarrow E_J \rightarrow F_{J_0}^* \Omega_{J_0/k} \rightarrow 0.$$

Buium shows that there exists an embedding $\mathbb{P}(E_C) \hookrightarrow \mathbb{P}(E_J)$, so that both C_0^1 and B can be seen inside $\mathbb{P}(E_J)$. He then fixes a suitable embedding of $\mathbb{P}(E_J)$ into some projective space and uses Bezout's theorem in Fulton's form to estimate $\sharp(B(k) \cap \mathbb{P}(E_C)(k))$. The set $B(k) \cap C_0^1(k)$ is clearly contained in $B(k) \cap \mathbb{P}(E_C)(k)$ so we get the desired bound.

1.1.3 Generalization of Buium's proof to higher dimensions

In this thesis we first give a new proof of the “generalized” Manin-Mumford conjecture (Theorem 1.1.2), in the case of subvarieties having ample cotangent bundle, following a similar approach to the one used by Buium. (See Theorem 2.6.2.)

As in the dimensional one case, the main point is proving a non-ramified version of Theorem 1.1.2. The main difficulty in generalizing Buium's work comes from part (3) in the previous subsection. In his situation, Buium proved it in three steps:

- (a) To show that C_0^1 is not trivial as a torsor over C_0 , he used a result by Raynaud stating that no smooth curve of genus $g \geq 2$ over $k := \overline{k(\mathfrak{p})}$ admits a lifting of the Frobenius to $R_1 := R/p^2$ (cf. Lemma 1.5.4 in [Ray83a]).
- (b) The non triviality of C_0^1 is equivalent to: the sequence

$$0 \rightarrow \mathcal{O}_{C_0} \rightarrow E_C \rightarrow F_{C_0}^* \Omega_{C_0/k} \rightarrow 0 \quad (1.1)$$

is nonsplit. Since $F_{C_0}^* \Omega_{C_0/k}$ is ample, one can apply Corollary 3 in [MD84] obtaining that either E_C is ample or the pullback by a suitable power of Frobenius of the sequence (1.1) splits.

- (c) But due to the fact that $\deg F_{C_0}^* \Omega_{C_0/k} > (2g - 2)/p$, the maps induced by Frobenius

$$\mathrm{Ext}^1(F_{C_0}^* \Omega_{C_0/k}, \mathcal{O}_{C_0}) \rightarrow \mathrm{Ext}^1(F_{C_0}^{2,*} \Omega_{C_0/k}, \mathcal{O}_{C_0}) \rightarrow \mathrm{Ext}^1(F_{C_0}^{3,*} \Omega_{C_0/k}, \mathcal{O}_{C_0}) \rightarrow \dots$$

are injective by Tango's criterion (cf. Theorem 15 and definition 11 in [T⁺72]). Now the sequence (1.1) is nonsplit, so its pullback under a power of Frobenius cannot split and E_C is ample.

Let us see what we can do to show that E_X is ample for X as in our hypotheses. By means of the theory of strongly semistable sheaves, we prove the following fundamental result (cf. Lemma 2.5.2 for more details):

Lemma 1.1.5. *Let n be the dimension of X . Then there exists a nonempty open subscheme $\tilde{U} \subseteq \mathrm{Spec}(\mathcal{O}_K)$ such that for any $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$ and any $s \in \mathbb{N}^*$*

$$\mathrm{Hom}_{X_0} \left(F_{X_0}^{s,*} \Omega_{X_0}, \Omega_{X_0} \right) = 0.$$

(Here $\deg(\Omega_X)$ stands for the degree of Ω_X with respect to any fixed very ample line bundle on X .)

An easy consequence of this Lemma is the analogue of Tango's criterion (part (c) above) in higher dimensions (cf. Corollary 2.5.3):

Corollary 1.1.6. *For any \mathfrak{p} as in Lemma 1.1.5, the maps induced by Frobenius*

$$\mathrm{Ext}^1(F_{X_0}^* \Omega_{X_0/k}, \mathcal{O}_{X_0}) \rightarrow \mathrm{Ext}^1(F_{X_0}^{2,*} \Omega_{X_0/k}, \mathcal{O}_{X_0}) \rightarrow \mathrm{Ext}^1(F_{X_0}^{3,*} \Omega_{X_0/k}, \mathcal{O}_{X_0}) \rightarrow \dots$$

are injective.

The Cartier isomorphism and Lemma 1.1.5 imply then that for any \mathfrak{p} as in Lemma 1.1.5, X_0 does not admit a lifting of the Frobenius over R_1 and so $X_0^1 \rightarrow X_0$ is not the trivial torsor (part (a) above). In particular this allows us to generalize to higher dimensions Raynaud's result on the sparsity of p -divisible unramified liftings for curves (see Théorème 4.4.1 in [Ray83b]):

Theorem 1.1.7. *For any \mathfrak{p} as in Lemma 1.1.5, the set*

$$\left\{ P \in X_0(\overline{k(\mathfrak{p})}) \mid P \text{ lifts to an element of } pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1) \right\}$$

is not Zariski dense in X_0 .

Here $A_{\mathfrak{p}^1}$ (resp. $X_{\mathfrak{p}^1}$) is $A \otimes_R R_1$ (resp. $X \otimes_R R_1$).

Now Corollary 3 in [MD84] (used in part (b)) has a slightly weaker analogue in dimension greater than one (cf. Corollary 2 in [MD84]): to apply it we need to suppose the ampleness of the cotangent bundle of our subvariety X and we need to show that the torsor X_0^1 cannot be trivialised by any proper surjective morphism $Y \rightarrow X_0$. This can be derived as a consequence of Lemma 1.1.5 and Corollary 2.8 in [Rös14].

Notice that in our proof we make use of the Greenberg transform rather than of the p -jet spaces defined by Burgos. The two approaches are equivalent (this is shown in Buium paper [Bui96].)

Subvarieties of abelian varieties with ample cotangent bundle have been studied by O. Debarre in his article [Deb05]. In it, he proved for example that the intersection of at least $n/2$ sufficiently ample general hypersurfaces in an abelian variety of dimension n has ample cotangent bundle. This provides us with an entire class of subvarieties of abelian varieties for which our proof of the Manin-Mumford conjecture works. For this class of subvarieties, we provide an explicit bound for the set of prime-to- p torsion points:

Theorem 1.1.8. *Let K be a number field, A/K be an abelian variety of dimension n and $L = \mathcal{O}_A(D)$ be a very ample line bundle on A . Let $c, e \in \mathbb{N}$ with $c \geq n/2$. Let $H_1, H_2, \dots, H_c \in |L^e|$ be general and let e be sufficiently big, so that the subvariety*

$$X := H_1 \cap H_2 \cap \dots \cap H_c$$

has ample cotangent bundle. There exists a nonempty open subset $V \subseteq \text{Spec}(\mathcal{O}_K)$ such that, if \mathfrak{p} is in V and \mathfrak{p} is above a prime $p > (n - c)^2 c e^{c+1} (L^n)$, then the cardinality of $\text{Tor}^p(A(\overline{K})) \cap X(\overline{K})$ is bounded above by

$$p^{2n} \left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-p)^{n-c-h} e^{n-h} \cdot Q_{n,c,h} \right) (L^n)^2.$$

where $\text{Tor}^p(A(\overline{K}))$ is the set of prime-to- p torsion points of $A(\overline{K})$ and $Q_{n,h,c}$ is a natural number depending on n, c and h . (Cf. Theorem 2.7.1 for details about $Q_{n,h,c}$ and V .)

To obtain such a “quantitative version” of the Manin-Mumford conjecture (for the prime-to- p torsion) we use the same technique explained in part (7) of the sketch of Buium’s proof. We embed pA_0^1 and X_0^1 into the same projective space and use Bezout’s Theorem to compute the cardinality of the intersection $pA_0^1(k) \cap X_0^1(k)$.

1.2 Polylogarithm on abelian schemes and Deligne-Beilinson cohomology

1.2.1 Arithmetic intersection theory

We recall briefly some notions in the intersection theory of arithmetic varieties introduced by Gillet and Soulé in their foundational paper [GS90].

Currents and Green currents on complex manifolds Let V be a complex manifold of dimension d ; for simplicity we suppose that all the components of V have the same dimension. The space $E^n(V)$ of smooth complex valued n -forms on V has a topology defined using the sup norms, on compact subsets of coordinate charts of V , of the k -fold partial derivatives of the coefficients of a form, for all $k \geq 0$, see [dR60] Par. 9 for details. Since the topology is defined by a family of seminorms, $E^n(V)$ is a locally convex topological space. Let $E_c^n(V)$ be the subspace of compactly supported forms. We write $D_n(V)$ for the bornological dual of $E_c^n(V)$ and call it the space of currents of dimension n on V . Since X is a complex manifold, we have a decomposition

$$E_c^n(V) = \bigoplus_{p+q=n} E_c^{p,q}(V)$$

from which we get a decomposition

$$D_n(V) = \bigoplus_{p+q=n} D_{p,q}(V).$$

Analogously, the exterior derivative

$$d = \partial + \bar{\partial} : E_c^n(V) \rightarrow E_c^{n+1}(V)$$

induces a dual homomorphism

$$b = b' + b'' : D_{n+1}(V) \rightarrow D_n(V).$$

One of the easiest examples of current is the following: for any closed k -dimensional submanifold $W \subseteq V$, there is a current $\delta_W \in D_{2k}(V)$, called current of integration over W , defined by

$$\delta_W(\alpha) = \int_W i^* \alpha$$

for $\alpha \in E_c^{2k}(V)$ and $i : W \rightarrow V$ the inclusion map. By means of a resolution of singularities, one can also define the current of integration over any k -dimensional analytic subspace of X . Finally, by linearity, this definition extends to any analytic cycle of dimension k .

Another simple class of currents is that of currents associated to forms. More precisely, any form $\alpha \in E^{p,q}(V)$ gives rise to an element $[\alpha] \in D_{d-p,d-q}(V)$ defined by

$$[\alpha](\beta) = \int_V \alpha \wedge \beta$$

for any $\beta \in E^{d-p,d-q}(V)$. Therefore for any pair (p, q) we obtain a map

$$E^{p,q}(V) \rightarrow D_{d-p,d-q}(V).$$

This map is continuous and has dense image (with respect to the natural topology one has on $D_{d-p,d-q}(V)$, see [dR60], Par. 10). This leads us to write

$$D_{d-p,d-q}(V) = D^{p,q}(V)$$

and interpret $D^{p,q}(V)$ as the space of forms of type (p, q) with distribution coefficients ([dR60]). Notice that the map $E^{p,q}(V) \rightarrow D_{d-p,d-q}(V)$ defined above can be extended to all L^1 -forms of type (p, q) on X : in fact if α is such a form, the integral $\int_V \alpha \wedge \beta$ is well-defined, for all $\beta \in E^{d-p,d-q}(V)$.

Stokes theorem shows that for any $\alpha \in E^n(V)$ we have $[d\alpha] = (-1)^{n+1}b[\alpha]$. Therefore, if we define

$$d = (-1)^{n+1}b : D^n(V) \rightarrow D^{n+1}(V),$$

then the inclusion $E^n(V) \rightarrow D^n(V)$ commutes with d . We will write ∂ and $\bar{\partial}$ for $(-1)^{n+1}b'$ and $(-1)^{n+1}b''$. Finally we shall write

$$\tilde{E}^{p,q}(V) = E^{p,q}(V)/(\partial E^{p-1,q}(V) + \bar{\partial} E^{p,q-1}(V))$$

and

$$\tilde{D}^{p,q}(V) = D^{p,q}(V)/(\partial D^{p-1,q}(V) + \bar{\partial} D^{p,q-1}(V)).$$

We recall now under which conditions pullbacks and pushforwards of currents are defined. Suppose $f : V^d \rightarrow W^{d-r}$ is a holomorphic map of complex manifolds. Then, if f is proper, we have maps

$$f^* : E_c^{p,q}(W) \rightarrow E_c^{p,q}(V)$$

and therefore dual maps

$$f_* : D^{p,q}(V) \rightarrow D^{p-r,q-r}(W).$$

On the other hand, if f is smooth, then we have integration over the fibre homomorphisms

$$\int_f : E_c^{p,q}(V) \rightarrow E_c^{p-r,q-r}(W)$$

and therefore dual homomorphisms

$$f^* : D^{p,q}(W) \rightarrow D^{p,q}(V).$$

We are now ready to give the fundamental definition of Green current.

Definition 1.2.1. If V is a complex manifold and $Y = \sum n_i Y_i$ is a codimension p analytic cycle on V , a Green current for Y is an element $g \in \tilde{D}^{p-1,p-1}(V)$, which is the class of a real current, such that $\frac{i}{2\pi} \partial \bar{\partial} g + \delta_Y = \omega$, with ω a smooth form (necessarily of type (p,p)).

One can prove that such a Green current always exists if V is compact and Kähler (cf. pag. 103 in [GS90]).

We conclude this section by recalling that we have an important operation on Green currents called the $*$ -product. Suppose V is a complex manifold, $Y = \sum n_i Y_i$ is a codimension p analytic cycle on V and $Y' = \sum n_i Y'_i$ is a codimension q analytic cycle on V . Suppose also that g is a Green current for Y and g' a Green current for Y' . Starting with g and g' one can define a current, denoted $g * g'$, which is a Green current for the cycle obtained intersecting Y and Y' . Defining the $*$ -product would take us too far afield: see sections 2.1 and 2.2 in [GS90]. All we need is to know that $*$ is associative and commutative.

Arithmetic varieties and arithmetic Chow groups

Definition 1.2.2. An arithmetic ring is a triple (R, Σ, F_∞) consisting of an excellent regular Noetherian integral domain R , a finite nonempty set Σ of monomorphisms $\sigma : R \rightarrow \mathbb{C}$, and an anti-linear involution F_∞ of the \mathbb{C} -algebra $\mathbb{C}^\Sigma := \underbrace{\mathbb{C} \times \dots \times \mathbb{C}}_{|\Sigma|}$, such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{\delta} & \mathbb{C}^\Sigma \\ \text{Id} \downarrow & & \downarrow F_\infty \\ R & \xrightarrow{\delta} & \mathbb{C}^\Sigma \end{array}$$

commutes (here by δ we mean the natural map to the product induced by the family of maps Σ).

Definition 1.2.3. An arithmetic variety over the arithmetic ring (R, Σ, F_∞) is a scheme X which is flat and of finite type over R . If F is the fraction field of R , we write X_F for the generic fibre of X and we suppose that X_F is smooth and quasi-projective.

As usual we write

$$X(\mathbb{C}) := \coprod_{\sigma \in \Sigma} (X \times_{R, \sigma} \mathbb{C})(\mathbb{C}).$$

Since X_F is supposed to be smooth, then $X(\mathbb{C})$ is a complex manifold and the anti-linear automorphism F_∞ of \mathbb{C}^Σ induces a continuous involution $F_\infty : X(\mathbb{C}) \rightarrow X(\mathbb{C})$. For brevity's sake we shall write $E^{p,q}(X)$ for the space $E^{p,q}(X(\mathbb{C}))$ and $D^{p,q}(X)$ for the space $D^{p,q}(X(\mathbb{C}))$. Observe that F_∞ acts on both $E^{*,*}(X)$ and $D^{*,*}(X)$, so we can define the \mathbb{R} -vector space $E^{p,p}(X_{\mathbb{R}})$ (resp. $D^{p,p}(X_{\mathbb{R}})$) to be the subspace of $E^{p,p}(X)$ (resp. $D^{p,p}(X)$) consisting of real forms (resp. currents) satisfying $F_\infty^* \alpha = (-1)^p \alpha$. Similarly we define

$$\tilde{E}^{p,p}(X_{\mathbb{R}}) = E^{p,p}(X_{\mathbb{R}}) / (\text{Im} \partial + \text{Im} \bar{\partial})$$

and

$$\tilde{D}^{p,p}(X_{\mathbb{R}}) = D^{p,p}(X_{\mathbb{R}}) / (\text{Im} \partial + \text{Im} \bar{\partial}).$$

Before giving the definition of arithmetic Chow groups, we briefly recall the definition of (classical) Chow groups. Let us denote by $Z^p(X)$ the group of cycles of codimension p in X , i.e. the free abelian group on the set of codimension p integral subschemes

of X . If T is a codimension p integral subscheme, we write $[T]$ for the associated cycle. If $Y \subseteq X$ is an integral subscheme of codimension $(p - 1)$, with generic point y , then for any $f \in k(y)^*$ we define a codimension p cycle

$$\operatorname{div}(f) = \sum_V \operatorname{ord}_V(f)[V].$$

Here the sum is over all integral subschemes V of Y of codimension p in X and the definition of $\operatorname{ord}_V(\cdot)$ can be found in [Ful97] A.3. The p -th Chow group of X is then

$$\operatorname{CH}^p(X) = Z^p(X)/\operatorname{Rat}^p(X)$$

where $\operatorname{Rat}^p(X) \subseteq Z^p(X)$ is the subgroup generated by all cycles of the form $\operatorname{div}(f)$.

We are now ready to introduce the arithmetic Chow groups. Since for any subscheme Y of X , $Y(\mathbb{C})$ is invariant under F_∞ , then integration over $Y(\mathbb{C})$ defines a current in $D^{p,p}(X_\mathbb{R})$ which we denote by δ_Y . Extending by linearity, we obtain a map

$$Z^p(X) \rightarrow D^{p,p}(X_\mathbb{R}).$$

We define the group $\hat{Z}^p(X)$ of arithmetic cycles of codimension p as the subgroup of

$$Z^p(X) \oplus \tilde{D}^{p,p}(X_\mathbb{R})$$

consisting of pairs $(Z = \sum n_i[Z_i], g)$ such that g is a Green current for Z .

If $Y \subseteq X$ is an integral smooth subscheme of codimension $(p - 1)$ and $f \in k(Y)^*$, f gives rise to a rational function $f(\mathbb{C})$ on $Y(\mathbb{C})$. The function $\log |f(\mathbb{C})|^2$ is real valued and L^1 on $Y(\mathbb{C})$; it therefore defines a current

$$[\log |f(\mathbb{C})|^2] \in D^{0,0}(Y(\mathbb{C})).$$

If $i : Y(\mathbb{C}) \hookrightarrow X(\mathbb{C})$ is the natural inclusion, then

$$i_*([\log |f(\mathbb{C})|^2]) \in D^{p-1,p-1}(X).$$

Since both $Y(\mathbb{C})$ and $f(\mathbb{C})$ are invariant under F_∞ , then $i_*([\log |f(\mathbb{C})|^2])$ actually belongs to $D^{p-1,p-1}(X_\mathbb{R})$. The Poincaré-Lelong Lemma ([GH14]) applied to $X(\mathbb{C})$ tells us that

$$\frac{i}{2\pi} \partial \bar{\partial} i_*([\log |f(\mathbb{C})|^2]) = \delta_{\operatorname{div}(f)},$$

the current associated to the restriction to $X(\mathbb{C})$ of $\operatorname{div}(f)$, viewed as a codimension p cycle on X . In other words,

$$\widehat{\operatorname{div}}(f) := (\operatorname{div}(f), -i_*([\log |f(\mathbb{C})|^2]))$$

is an element of $\hat{Z}^p(X)$.

Definition 1.2.4. We define the p -th arithmetic Chow group of X as

$$\widehat{\text{CH}}^p(X) = \widehat{Z}^p(X) / \widehat{\text{R}}^p(X)$$

where $\widehat{\text{R}}^p(X)$ is the subgroup generated by all $\widehat{\text{div}}(f)$, for some W codimension $p-1$ integral subscheme, as above.

For each pair of non-negative integers (p, q) , there exists a pairing

$$\begin{aligned} \widehat{\text{CH}}^p(X) \otimes \widehat{\text{CH}}^q(X) &\rightarrow \widehat{\text{CH}}^{p+q}(X)_{\mathbb{Q}} \\ \alpha \otimes \beta &\mapsto \alpha \cdot \beta \end{aligned}$$

which is defined by means of the classical intersection theory of cycles and of the $*$ -product of Green currents. Such pairings make

$$\widehat{\text{CH}}^*(X)_{\mathbb{Q}} := \bigoplus_{p \geq 0} \widehat{\text{CH}}^p(X)_{\mathbb{Q}}$$

into a commutative ring called the arithmetic Chow ring of X (cf. section 4.2 in [GS90]).

We can define the following maps involving $\widehat{\text{CH}}^p(X)$:

- $\zeta : \widehat{\text{CH}}^p(X) \rightarrow \text{CH}^p(X)$ sending (Z, g) to Z ;
- $a : \widetilde{E}^{p-1, p-1}(X_{\mathbb{R}}) \rightarrow \widehat{\text{CH}}^p(X)$, sending α to $(0, \alpha)$;
- $\omega : \widehat{\text{CH}}^p(X) \rightarrow E^{p, p}(X_{\mathbb{R}})$ sending (Z, g) to $\omega(Z, g) := \delta_Z + \frac{i}{2\pi} \partial \bar{\partial} g$.

Recall that we have groups

$$\text{CH}^{p, p-1}(X) = \frac{\text{Ker}\{d^{p-1} : \bigoplus_{x \in X^{(p-1)}} k(x)^* \rightarrow \bigoplus_{x \in X^{(p)}} \mathbb{Z}\}}{\text{Im}\{d^{p-2} : \bigoplus_{x \in X^{(p-2)}} K_2 k(x) \rightarrow \bigoplus_{x \in X^{(p-1)}} k(x)^*\}}.$$

Here $X^{(j)} = \{x \in X \mid \mathcal{O}_{X, x} \text{ has Krull dimension } j\}$ for $j = p, p-1, p-2$ and d^{p-1}, d^{p-2} are the differentials in the E_1 term of the spectral sequence of [Qui73] Par. 7, so that $\text{CH}^{p, p-1}$ is the $E_2^{p-1, -p}$ term of that spectral sequence. The differential d^{p-1} sends $f \in k(x)^*$ to $\text{div}(f)$ and d^{p-2} is essentially the tame symbol: see [Qui73] Par. 7. We have the following fundamental exact sequence

$$\text{CH}^{p, p-1}(X) \xrightarrow{\rho_{\text{an}}} \widetilde{E}^{p-1, p-1}(X_{\mathbb{R}}) \xrightarrow{a} \widehat{\text{CH}}^p(X) \xrightarrow{\zeta} \text{CH}^p(X) \longrightarrow 0 \quad (1.2)$$

(see Theorem 3.3.5 in [GS90] for the definition of ρ_{an}).

Let us now recall under which conditions pullbacks and pushforwards of arithmetic Chow groups are defined. Let $f : X \rightarrow Y$ be a morphism between arithmetic varieties over the arithmetic ring (R, Σ, F_∞) . Suppose that f induces a smooth map $X_F \rightarrow Y_F$ between the generic fibers of X and Y . Then (see Theorem 3.6.1 in [GS90]):

1. if f is flat, for all $p \geq 0$ there is a natural homomorphism

$$\begin{aligned} f^* : \widehat{\text{CH}}^p(Y) &\rightarrow \widehat{\text{CH}}^p(X) \\ (Z, g) &\mapsto (f^*Z, f^*g) \end{aligned}$$

2. if f is proper and X and Y are equidimensional, for all $p \geq 0$ there is a map

$$\begin{aligned} f_* : \widehat{\text{CH}}^p(X) &\rightarrow \widehat{\text{CH}}^{p-d}(Y) \\ (Z, g) &\mapsto (f_*Z, f_*g). \end{aligned}$$

where $d = \dim(X) - \dim(Y)$.

Furthermore, if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are two maps inducing smooth maps between generic fibres, then $f^*g^* = (gf)^*$ and $(gf)_* = g_*f_*$ when either composition makes sense. The reader can find more details about f^* and f_* in section 3.6 in [GS90].

In arithmetic intersection theory, one has an equivalent of the classical Chern character, called the arithmetic Chern character. Let E be an algebraic vector bundle on the arithmetic variety X and let h be an hermitian metric on the corresponding holomorphic vector bundle $E(\mathbb{C})$ and $X(\mathbb{C})$. The pair (E, h) , denoted \overline{E} , is by definition an hermitian vector bundle on X . We shall always suppose that h is invariant under complex conjugation, i.e. $F_\infty^*(h) = h$. The arithmetic Chern character $\widehat{\text{ch}}(\cdot)$ is defined as follows.

Theorem 1.2.5. *There exists a unique way to define a characteristic class*

$$\widehat{\text{ch}}(\overline{E}) \in \widehat{\text{CH}}^*(X)_\mathbb{Q}$$

satisfying the following properties:

1. $f^*\widehat{\text{ch}}(\overline{E}) = \widehat{\text{ch}}(f^*\overline{E})$, for every morphism $f : Y \rightarrow X$ of arithmetic varieties;
2. $\widehat{\text{ch}}(\overline{E} \oplus \overline{F}) = \widehat{\text{ch}}(\overline{E}) + \widehat{\text{ch}}(\overline{F})$, for all hermitian vector bundles $\overline{E}, \overline{F}$;
3. $\widehat{\text{ch}}(\overline{E} \otimes \overline{F}) = \widehat{\text{ch}}(\overline{E}) \cdot \widehat{\text{ch}}(\overline{F})$, for all hermitian vector bundles $\overline{E}, \overline{F}$;

4. $\widehat{\text{ch}}(\overline{L}) = \exp(\widehat{c}_1(\overline{L}))$, for every hermitian line bundle \overline{L} , where $\widehat{c}_1(\overline{L})$ is by definition the class of the arithmetic cycle $(\text{div}(s), -\log h(s, s))$, where s is a nonzero rational section of L ;
5. $\omega(\widehat{\text{ch}}(\overline{E})) = \text{ch}(\overline{E})$ is the classical Chern character form.

Cf. IV.4 in [SAB94] for the proof.

1.2.2 Analytic-Deligne and Deligne-Beilinson cohomology

This section is dedicated to recall the definitions and some basic facts on two important cohomology theories: the analytic Deligne cohomology and the Deligne-Beilinson cohomology. The reader can refer to [EV] for a complete treatment.

Let V be a complex manifold and let Ω_V^* be the de Rham complex of holomorphic differential forms on V . We denote by \mathbb{H}^\bullet the hypercohomology functor from the derived category of \mathbb{Z} -sheaves to the derived category of abelian groups. For any complex of \mathbb{Z} -sheaves A^\bullet we shall write $\mathbb{H}^q(A^\bullet)$ for the q -th cohomology of the complex $\mathbb{H}^*(A^\bullet)$.

Definition 1.2.6. We define the analytic real Deligne complex $\mathbb{R}(p)_{D,\text{an}}$ of V to be

$$0 \longrightarrow \mathbb{R}(p) \longrightarrow \mathcal{O}_V \xrightarrow{d} \Omega_V^1 \longrightarrow \dots \longrightarrow \Omega_V^{p-1} \longrightarrow 0 \quad (1.3)$$

where $\mathbb{R}(p) := (2\pi i)^p \mathbb{R}$ is in degree zero. We define the analytic real Deligne cohomology of V as

$$H_{D,\text{an}}^q(V, \mathbb{R}(p)) := \mathbb{H}^q(V, \mathbb{R}(p)_{D,\text{an}}).$$

Now let W be a complex algebraic manifold of dimension n . A good compactification of W is a proper algebraic manifold \overline{W} with an embedding $j : W \hookrightarrow \overline{W}$ such that $D = \overline{W} \setminus W$ is a normal crossing divisor (i.e. locally in the analytic topology D has smooth components intersecting transversally). Let $\Omega_{\overline{W}}^\bullet(\log D)$ be the de Rham complex of meromorphic forms on \overline{W} , holomorphic on W and with at most logarithmic poles along D . We have a filtration of $\Omega_{\overline{W}}^\bullet(\log D)$ by subcomplexes

$$F_D^p = \left(0 \rightarrow \Omega_{\overline{W}}^p(\log D) \rightarrow \Omega_{\overline{W}}^{p+1}(\log D) \rightarrow \dots \rightarrow \Omega_{\overline{W}}^n(\log D) \right)$$

Notice that we have two natural maps $\varepsilon : Rj_*\mathbb{R}(p) \rightarrow Rj_*\Omega_{\overline{W}}^\bullet$ and $i : F_D^p \rightarrow Rj_*\Omega_{\overline{W}}^\bullet$.

Definition 1.2.7. The real Deligne-Beilinson complex of (\overline{W}, W) is

$$\mathbb{R}(p)_D = \mathbb{R}(p)_{D,\overline{W}} = \text{Cone}(\varepsilon - i : Rj_*\mathbb{R}(p) \oplus F_D^p \rightarrow Rj_*\Omega_{\overline{W}}^\bullet)[-1].$$

One can show that $\mathbb{H}^q(\overline{W}, \mathbb{R}(p)_D)$ is independent of the good compactification chosen. Since each manifold over \mathbb{C} allows a good compactification, we can give the following definition.

Definition 1.2.8. The real Deligne-Beilinson cohomology of W is

$$H_D^q(W, \mathbb{R}(p)) := \mathbb{H}^q(\overline{W}, \mathbb{R}(p)_D).$$

Notice that by construction, $H_D^q(W, \mathbb{R}(p)) = H_{D,\text{an}}^q(W, \mathbb{R}(p))$ if W is compact (so that D is empty). More generally, there is a natural “forgetful” morphism of \mathbb{R} -vector spaces

$$H_D^q(W, \mathbb{R}(p)) \rightarrow H_{D,\text{an}}^q(W, \mathbb{R}(p))$$

(what is forgotten is the logarithmic structure, cf. 2.13 in [EV]).

If X is an arithmetic variety over the arithmetic ring (R, Σ, F_∞) , we define

$$H_D^q(X_{\mathbb{R}}, \mathbb{R}(p)) := \{ \gamma \in H_D^q(X(\mathbb{C}), \mathbb{R}(p)) \mid F_\infty^* \gamma = (-1)^p \gamma \}.$$

Similarly we define

$$H_{D,\text{an}}^q(X_{\mathbb{R}}, \mathbb{R}(p)) := \{ \gamma \in H_{D,\text{an}}^q(X(\mathbb{C}), \mathbb{R}(p)) \mid F_\infty^* \gamma = (-1)^p \gamma \}.$$

The map ρ_{an} in the sequence (1.2) is by construction the following composite function

$$\text{CH}^{p,p-1}(X) \xrightarrow{\rho} H_D^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(g)) \xrightarrow{\text{forgetful}} H_{D,\text{an}}^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(g)) \longrightarrow \tilde{E}^{p-1,p-1}(X_{\mathbb{R}})$$

where the third map is a natural inclusion. Indeed we have

$$H_{D,\text{an}}^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(p)) = \{ c \in (2\pi i)^{p-1} E^{p-1,p-1}(X_{\mathbb{R}}) \mid \partial \bar{\partial} c = 0 \} / (\text{Im} \partial + \text{Im} \bar{\partial})$$

and the class of c is sent to the class of $c/(2\pi i)^{p-1}$. Cf. Corollary 6.3 in [Bur97] for the definition of ρ .

1.2.3 Our result on the degree zero part of the motivic polylogarithm

In [Bur97], Burgos introduced a new arithmetic intersection theory in which the role of Green currents is played by Green forms, i.e. classes of forms with logarithmic singularities at infinity (cf. section 3 in Chapter 3). Such forms are particularly interesting since the associated complex naturally computes the Deligne-Beilinson

cohomology. Burgos' approach leads to a new version of arithmetic Chow groups denoted $\widehat{\text{CH}}_{\log}^*(\cdot)$. These groups coincide with the classic arithmetic Chow groups for proper arithmetic varieties but are different in general. In this thesis we use Burgos' theory to give a refinement of a result about the degree zero part of the motivic polylogarithm on abelian schemes proved by Kings and Rössler (cf. [KR14]).

In their paper, Kings and Rössler gave a simple axiomatic description of the degree zero part of the motivic polylogarithm and showed that its realisation in analytic Deligne cohomology can be described in terms of the Bismut-Köhler higher analytic torsion form of the Poincaré bundle. The setting in which they worked is the following. Suppose S is a smooth scheme over a subfield k of the complex numbers. Let $\pi : \mathcal{A} \rightarrow S$ be an abelian scheme of relative dimension g and call $\varepsilon : S \rightarrow \mathcal{A}$ the zero section. Fix $N > 1$ an integer and let $\mathcal{A}[N]$ be the finite group scheme of N -torsion points. For any integer $a > 1$ and any $W \subseteq \mathcal{A}$ open subscheme such that

$$j : [a]^{-1}(W) \hookrightarrow W$$

is an open immersion (here $[a] : \mathcal{A} \rightarrow \mathcal{A}$ is the a -multiplication on \mathcal{A}), the trace map with respect to a is then defined as

$$\text{tr}_{[a]} : H_{\mathcal{M}}(W, *) \xrightarrow{j^*} H_{\mathcal{M}}([a]^{-1}(W), *) \xrightarrow{[a]^*} H_{\mathcal{M}}(W, *) \quad (1.4)$$

where $H_{\mathcal{M}}(W, *)$ refers to the motivic cohomology of W introduced by Soulé in [Sou85]. The generalized eigenspace of $\text{tr}_{[a]}$ of weight zero is defined as

$$H_{\mathcal{M}}(W, *)^{(0)} := \left\{ \psi \in H_{\mathcal{M}}(W, *) \mid (\text{tr}_{[a]} - \text{Id})^k \psi = 0 \text{ for some } k \geq 1 \right\}$$

and the zero step of the motivic polylogarithm is a class in the generalized eigenspace of weight zero

$$\text{pol}^0 \in H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)}.$$

To describe it more precisely, consider the residue map along $\mathcal{A}[N]$

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g) \rightarrow H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \varepsilon(S), 0).$$

This map induces an isomorphism

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)} \cong H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \varepsilon(S), 0)^{(0)}$$

(see Corollary 2.2.2 in [KR14]) and pol^0 is the unique element mapping to the fundamental class of $\mathcal{A}[N] \setminus \varepsilon(S)$.

In [MR], V. Maillot and D. Rössler constructed a canonical class of currents $\mathfrak{g}_{\mathcal{A}^\vee}$ on \mathcal{A} (cf. Theorem 3.5.1) which can be described in term of the Bismut Köhler analytic torsion form of the Poincaré bundle and which gives rise to a class in analytic Deligne cohomology

$$([N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]} \in H_{D,\text{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)).$$

Since $\text{CH}^{p,p-1}(\cdot)_{\mathbb{Q}} \cong H_{\mathcal{M}}^{2p-1}(\cdot, p)$ (see section 1.4 in [BGKK07] for this), to ρ and ρ_{an} correspond maps

$$\text{cyc} : H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g) \rightarrow H_D^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g))$$

and

$$\text{cyc}_{\text{an}} : H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g) \rightarrow H_{D,\text{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)).$$

The main result in [KR14] is the following.

Theorem 1.2.9. *We have*

$$-2 \cdot \text{cyc}_{\text{an}}(\text{pol}^0) = ([N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]}.$$

In this thesis we give a refinement of Theorem 1.2.9 supposing S is proper over k (see Corollary 3.6.1 and Theorem 3.6.2). More precisely, we show that in this case the class of currents $[N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee}$ provides us not only with the realization of pol^0 in real analytic Deligne cohomology but also in Deligne-Beilinson cohomology:

Theorem 1.2.10. *Let S be proper over k . The class of currents $[N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee}$ gives rise to an element*

$$\overline{([N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]}} \in \text{Im} \left(\text{cyc} : H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g) \rightarrow H_D^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)) \right)$$

verifying

$$-2 \cdot \text{cyc}(\text{pol}^0) = \overline{([N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]}}.$$

Notice that necessarily we have

$$\text{forgetful} \left(\overline{([N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]}} \right) = ([N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]}.$$

Chapter 2

The Manin-Mumford conjecture for subvarieties with ample cotangent bundle

Abstract

We give a new proof of the Manin-Mumford conjecture for subvarieties of abelian varieties having ample cotangent bundle, when all data are defined over a number field. Our strategy follows Buium's approach in the case of curves; i.e. we prove an intermediate "non-ramified version" from which the conjecture easily follows. In order to do so, we use the Greenberg transform, which assumes the role of the p -jet spaces in Buium's work, and the theory of strongly semistable sheaves. Furthermore, we provide an explicit bound for the cardinality of the set of prime-to- p torsion points of subvarieties obtained as the intersection of at least $n/2$ sufficiently ample general hypersurfaces in an abelian variety of dimension n .

2.1 Introduction

The Manin-Mumford conjecture is a significant result concerning the intersection of a subvariety X of an abelian variety A with the group of torsion points of A , when all data are defined over a number field. Raynaud first proved the conjecture in 1983, and since then various other proofs (sometimes only for the case of curves) surfaced, due to Serre, Coleman, Hindry, Buium, Hrushovski, Pink-Rössler. In this paper we give a new proof of the Manin-Mumford conjecture in the case of a subvariety X with ample cotangent bundle (cf. Theorem 2.6.2):

Theorem 2.1.1. *Let K be a number field, A be an abelian variety over K and $X \subseteq A$ be a smooth subvariety over K with trivial translation stabilizer and ample cotangent bundle. Then the set*

$$\mathrm{Tor}(A(\overline{K})) \cap X(\overline{K})$$

is finite.

Here \overline{K} is a fixed algebraic closure of K and $\mathrm{Tor}(A(\overline{K}))$ is the subgroup of torsion points of $A(\overline{K})$. See the begin of the next section for the definition of translation stabilizer.

Our strategy for proving Theorem 2.1.1 follows Buium’s approach in the case of curves (cf. [Bui96]). We fix a prime \mathfrak{p} of K above a prime p such that K/\mathbb{Q} is unramified at \mathfrak{p} and X has good reduction at \mathfrak{p} . The key point then is to prove a “non-ramified version” of Theorem 2.1.1: more precisely, we prove that Theorem 2.1.1 holds with \overline{K} replaced by the maximal extension of K contained in \overline{K} which is unramified above \mathfrak{p} and with the torsion replaced by the prime-to- p torsion (see Theorem 2.6.1 and the first part of the proof of Theorem 2.6.2).

This “non-ramified version” is a consequence of a result on the sparsity of p -divisible unramified liftings which holds for general subvarieties, not necessarily having ample cotangent bundle. Let U be an open subscheme of $\mathrm{Spec}(\mathcal{O}_K)$ not containing any ramified prime and such that A/K extends to an abelian scheme \mathcal{A}/U and X extends to a smooth closed integral subscheme \mathcal{X} of \mathcal{A} . For any $\mathfrak{p} \in U$, let R (resp. R_1) be the ring of Witt vectors (resp. of length 2) with coordinates in the algebraic closure $\overline{k(\mathfrak{p})}$ of the residue field of \mathfrak{p} . We denote by $X_{\mathfrak{p}^1}$ (resp. $A_{\mathfrak{p}^1}$) the R_1 -scheme $\mathcal{X} \times_U \mathrm{Spec} R_1$ (resp. $\mathcal{A} \times_U \mathrm{Spec} R_1$) and we denote by $X_{\mathfrak{p}^0}$ the $\overline{k(\mathfrak{p})}$ -scheme $\mathcal{X} \times_U \mathrm{Spec} \overline{k(\mathfrak{p})}$. Define \tilde{U} as the nonempty open subscheme of U consisting of all $\mathfrak{p} \in U$ such that $X_{\mathfrak{p}^0}$ has trivial stabilizer.

Our result on the sparsity of p -divisible unramified liftings is the following (cf. Theorem 2.5.4 for a more precise formulation):

Theorem 2.1.2. *Let n be the dimension of X . Let $\mathfrak{p} \in \tilde{U}$ be above a prime p with $p > n^2 \deg(\Omega_X)$. Then the set*

$$\left\{ P \in X_{\mathfrak{p}^0}(\overline{k(\mathfrak{p})}) \mid P \text{ lifts to an element of } pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1) \right\}$$

is not Zariski dense in $X_{\mathfrak{p}^0}$.

(Here $\deg(\Omega_X)$ refers to the degree of Ω_X computed with respect to any fixed very ample line bundle on X . See Section 4 for its definition.)

Theorem 2.1.2 is an effective form (in the case of a number field) of a result on the sparsity of highly p -divisible unramified liftings given by D. Rössler (see Theorem 4.1 in [Rös13] and the comment right after Theorem 2.5.4 in this paper). Theorem 2.1.2 is also a generalization to higher dimensions of the analogue result obtained by Raynaud in the case of curves (cf. Théorème 4.4.1 in [Ray83b]).

The proof of Theorem 2.1.2 lies on the impossibility of lifting the Frobenius of X_{p^0} over R_1 . This is a well-known fact in the case of smooth curves of genus at least 2: Raynaud proved it (see Lemma I.5.4 in [Ray83a]) by means of the Cartier isomorphism. To prove the impossibility of lifting the Frobenius in higher dimensions (for subvarieties with trivial stabilizer) we make use of the theory of strongly semistable sheaves and of the Cartier isomorphism.

Subvarieties of abelian varieties with ample cotangent bundle have been studied by O. Debarre in his article [Deb05]. In it, he proved for example that the intersection of at least $n/2$ sufficiently ample general hypersurfaces in an abelian variety of dimension n has ample cotangent bundle. This provides us with an entire class of subvarieties of abelian varieties for which our proof of the Manin-Mumford conjecture works. For this class of subvarieties, we give an explicit bound for the set of prime-to- p torsion points:

Theorem 2.1.3. *Let K be a number field, A/K be an abelian variety of dimension n and $L = \mathcal{O}_A(D)$ be a very ample line bundle on A . Let $c, e \in \mathbb{N}$ with $c \geq n/2$. Let $H_1, H_2, \dots, H_c \in |L^e|$ be general and let e be sufficiently big, so that the subvariety*

$$X := H_1 \cap H_2 \cap \dots \cap H_c$$

has ample cotangent bundle. There exists a nonempty open subset $V \subseteq \text{Spec}(\mathcal{O}_X)$ such that, if \mathfrak{p} is in V and \mathfrak{p} is above a prime $p > (n - c)^2 ce^{c+1}(L^n)$, then the cardinality of $\text{Tor}^p(A(\overline{K})) \cap X(\overline{K})$ is bounded above by

$$p^{2n} \left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-p)^{n-c-h} e^{n-h} \cdot Q_{n,c,h} \right) (L^n)^2.$$

where $\text{Tor}^p(A(\overline{K}))$ is the set of prime-to- p torsion points of $A(\overline{K})$ and $Q_{n,h,c}$ is a natural number depending on n, c and h . (Cf. Theorem 2.7.1 for details about $Q_{n,h,c}$ and V .)

To obtain such a “quantitative version” of the Manin-Mumford conjecture (for the prime-to- p torsion) we use the same technique present in Buium’s paper. If $\text{Gr}_1(X_{p^1})$ and $\text{Gr}_1(A_{p^1})$ denote the Greenberg transform of level 1 of X and A (see Section 3), then we embed the two varieties $\text{Gr}_1(X_{p^1})$ and $[p]_*\text{Gr}_1(A_{p^1})$ into the

same projective space and use Bezout's Theorem to compute the cardinality of their intersection.

We conclude this introduction by giving an outline of the chapter:

- in Section 2 we fix those notations which will stay unchanged throughout the paper;
- in Section 3 we recall definitions and basic properties of the Greenberg transform and the critical schemes. Notice that the use of the Greenberg transform in this paper corresponds to the use of p -jet spaces in Buium's proof;
- Section 4 is dedicated to the theory of strongly semistable sheaves;
- Section 5 contains the proof of Theorem 2.1.2;
- in Section 6 we prove Theorem 2.1.1;
- in Section 7 we prove Theorem 2.1.3;
- Section 8 is an Appendix containing the proof of the following fact: the tangent bundle of a smooth subvariety of an abelian variety which has trivial stabilizer has no non-zero global sections (cf. Fact 2.8.1).

2.2 Notations

We fix the following notations:

- K a number field,
- \overline{K} an algebraic closure of K ,
- A/K an abelian variety,
- $X \subseteq A$ a closed integral subscheme, smooth over K ,
- $\text{Stab}_A(X)$ the translation stabilizer of X in A , i.e. the closed subgroup scheme of A characterized uniquely by the fact that for any K -scheme S and any morphism $b : S \rightarrow A$, translation by b on the product $A \times_K S$ maps the subscheme $X \times_K S$ to itself if and only if b factors through $\text{Stab}_A(X)$,
- U an open subscheme of $\text{Spec}(\mathcal{O}_K)$ not containing any ramified prime and such that A/K extends to an abelian scheme \mathcal{A}/U and X extends to a smooth closed integral subscheme \mathcal{X} of \mathcal{A} .

For any prime number p , any $\mathfrak{p} \in U$ above p and any $n \geq 0$, we denote by:

- $k(\mathfrak{p})$ the residue field $\mathcal{O}_K/\mathfrak{p}$ for \mathfrak{p} ,
- $K_{\mathfrak{p}}$ the completion of K with respect to \mathfrak{p} ,
- $\widehat{K_{\mathfrak{p}}^{\text{unr}}}$ the completion of the maximal unramified extension of $K_{\mathfrak{p}}$,
- $R := W(\overline{k(\mathfrak{p})})$ (resp. $R_n := W_n(\overline{k(\mathfrak{p})})$) the ring of Witt vectors (resp. the ring of Witt vectors of length $n + 1$) with coordinates in $\overline{k(\mathfrak{p})}$. We recall that R can be identified with the ring of integers of $\widehat{K_{\mathfrak{p}}^{\text{unr}}}$ and R_0 with $\overline{k(\mathfrak{p})}$,
- $X_{\mathfrak{p}^n}$ the R_n -scheme $\mathcal{X} \times_U \text{Spec } R_n$
 $A_{\mathfrak{p}^n}$ the R_n -scheme $\mathcal{A} \times_U \text{Spec } R_n$,
- $\text{Tor}(A(\overline{K}))$ the torsion points in $A(\overline{K})$,
- $\text{Tor}^p(A(\overline{K})) \subseteq \text{Tor}(A(\overline{K}))$ the prime-to- p torsion,
- $\text{Tor}_p(A(\overline{K})) \subseteq \text{Tor}(A(\overline{K}))$ the p -torsion.

2.3 The Greenberg transform and the critical schemes

Now we recall some basic facts about the Greenberg transform (for more details, see [Gre61], [Gre63] and [[BLR90] p. 276-277]).

Throughout this section, a prime number p and a $\mathfrak{p} \in U$ above p are fixed.

For any $n \geq 0$, the Greenberg transform of level n is a covariant functor Gr_n from the category of R_n -schemes locally of finite type, to the category of $\overline{k(\mathfrak{p})}$ -schemes locally of finite type. If Y_n is an R_n -scheme locally of finite type, $\text{Gr}_n(Y_n)$ is a $\overline{k(\mathfrak{p})}$ -scheme with the property

$$Y_n(R_n) = \text{Gr}_n(Y_n)(\overline{k(\mathfrak{p})}).$$

More precisely, we can interpret R_n as the set of $\overline{k(\mathfrak{p})}$ -valued points of a ring scheme \mathcal{R}_n over $\overline{k(\mathfrak{p})}$. For any $\overline{k(\mathfrak{p})}$ -scheme T , we define $\mathbb{W}_n(T)$ as the ringed space over R_n consisting of T as a topological space and of $\text{Hom}_{\overline{k(\mathfrak{p})}}(T, \mathcal{R}_n)$ as a structure sheaf. By definition $\text{Gr}_n(Y_n)$ represents the functor from the category of schemes over $\overline{k(\mathfrak{p})}$ to the category of sets given by

$$T \mapsto \text{Hom}_{R_n}(\mathbb{W}_n(T), Y_n)$$

where Hom stands for homomorphisms of ringed spaces. In other words, the functor Gr_n is right adjoint to the functor \mathbb{W}_n .

The functor Gr_n respects closed immersions, open immersions, fibre products, smooth and étale morphisms. Furthermore it sends group schemes over R_n to group schemes over $\overline{k(\mathfrak{p})}$. The canonical morphism $R_{n+1} \rightarrow R_n$ gives rise to a functorial transition morphism $\text{Gr}_{n+1} \rightarrow \text{Gr}_n$.

Let Y_n be a scheme over R_n locally of finite type. Then for any $m < n$ we can define

$$Y_m := Y_n \times_{R_n} R_m.$$

Let us call $F_{Y_0} : Y_0 \rightarrow Y_0$ the absolute Frobenius endomorphism of Y_0 and $\Omega_{Y_0/\overline{k(\mathfrak{p})}}$ the sheaf of relative differentials.

For any finite rank locally free sheaf \mathcal{F} over Y_0 we will write

$$V(\mathcal{F}) := \underline{\text{Spec}}(\text{Sym}(\mathcal{F}^\vee))$$

for the vector bundle over $\overline{k(\mathfrak{p})}$ associated to \mathcal{F} .

If Y_n is smooth over R_n , then $\Omega_{Y_0/\overline{k(\mathfrak{p})}}$ is a locally free sheaf. A key result in [Gre63] is the following structure theorem:

$$\text{Gr}_1(Y_1) \rightarrow \text{Gr}_0(Y_0)$$

is a torsor under

$$V\left(F_{Y_0}^* \Omega_{Y_0/\overline{k(\mathfrak{p})}}\right).$$

Let X , A , \mathcal{X} and \mathcal{A} be as fixed in the previous section. For any $n \geq 0$ we define the n -critical scheme

$$\text{Crit}^n(\mathcal{X}, \mathcal{A}) := [p^n]_* \text{Gr}_n(A_{\mathfrak{p}^n}) \cap \text{Gr}_n(X_{\mathfrak{p}^n}),$$

where $[p^n]_* \text{Gr}_n(A_{\mathfrak{p}^n})$ refers to the scheme-theoretic image of $\text{Gr}_n(A_{\mathfrak{p}^n})$ by the multiplication map $[p^n]$. Notice that $\text{Crit}^n(\mathcal{X}, \mathcal{A})$ is a scheme over $\overline{k(\mathfrak{p})}$ and that $\text{Crit}^0(\mathcal{X}, \mathcal{A}) = X_{\mathfrak{p}^0}$, since Gr_0 is the identity.

The natural morphisms $\text{Gr}_{n+1}(A_{\mathfrak{p}^{n+1}}) \rightarrow \text{Gr}_n(A_{\mathfrak{p}^n})$ lead to a projective system of $\overline{k(\mathfrak{p})}$ -schemes:

$$\cdots \rightarrow \text{Crit}^2(\mathcal{X}, \mathcal{A}) \rightarrow \text{Crit}^1(\mathcal{X}, \mathcal{A}) \rightarrow X_{\mathfrak{p}^0}$$

whose connecting morphisms are both affine and proper, hence finite. In fact, transition morphisms are always affine and for any $n \geq 0$ the subscheme $[p^n]_* \text{Gr}_n(A_{\mathfrak{p}^n})$ is proper, being the greatest abelian subvariety of $\text{Gr}_n(A_{\mathfrak{p}^n})$.

We shall write $\text{Exc}^n(\mathcal{X}, \mathcal{A})$ for the scheme theoretic image of the morphism $\text{Crit}^n(\mathcal{X}, \mathcal{A}) \rightarrow X_{\mathfrak{p}^0}$.

2.4 The geometry of vector bundles in positive characteristic

Let us recall some results on vector bundles in positive characteristic we will need later (see paragraph 2 in [Rös14] for all the details and proofs).

Let Y be a smooth projective variety over an algebraically closed field l_0 of positive characteristic. We write as before Ω_{Y/l_0} for the sheaf of differentials of Y over l_0 and $F_Y : Y \rightarrow Y$ for the absolute Frobenius endomorphism of Y . We start with following elementary lemma.

Lemma 2.4.1. *Let*

$$0 \rightarrow V \rightarrow W \rightarrow N \rightarrow 0$$

be an exact sequence of vector bundles on Y . Suppose that $W \simeq \mathcal{O}_Y^l$ for some $l > 0$. Then for any dominant proper morphism $\phi : Y' \rightarrow Y$, where Y' is integral, the morphism

$$\phi^* : H^0(Y, V) \rightarrow H^0(Y', \phi^*V)$$

is an isomorphism.

Proof. We have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(Y, V) & \longrightarrow & H^0(Y, W) & \longrightarrow & H^0(Y, N) \\ & & \downarrow \phi^* & & \downarrow \phi^* & & \downarrow \phi^* \\ 0 & \longrightarrow & H^0(Y', \phi^*V) & \longrightarrow & H^0(Y', \phi^*W) & \longrightarrow & H^0(Y', \phi^*N) \end{array}$$

In this diagram all three vertical arrows are injective thanks to the surjectivity of ϕ . Furthermore, the middle vertical arrow is an isomorphism, since both $H^0(Y, \mathcal{O}_Y)$ and $H^0(Y', \mathcal{O}_{Y'})$ coincide with l_0 and both W and ϕ^*W are trivial. The five lemma now implies that the left vertical arrow is surjective. \square

Let L be a very ample line bundle on Y . If V is a torsion free coherent sheaf on Y , we shall write

$$\mu(V) = \mu_L(V) = \deg_L(V)/\text{rk}(V)$$

for the slope of V (with respect to L). Here $\text{rk}(V)$ is the rank of V , i.e. the dimension of the stalk of V at the generic point of Y . Furthermore,

$$\deg_L(V) := \int_Y c_1(V) \cdot c_1(L)^{\dim(Y)-1}$$

where $c_1(\cdot)$ refers to the first Chern class with values in an arbitrary Weil cohomology theory and the integral \int_Y stands for the push-forward morphism to $\text{Spec}l_0$ in that theory. Recall that V is called semistable (with respect to L) if for every coherent subsheaf W of V , we have $\mu(W) \leq \mu(V)$ and it is called strongly semistable if $F_Y^{n,*}V$ is semistable for all $n \geq 0$.

In general, there exists a filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{r-1} \subseteq V_r = V$$

of V by subsheaves, such that the quotients V_i/V_{i-1} are all semistable and such that the slopes $\mu(V_i/V_{i-1})$ are strictly decreasing for $i \geq 1$. This filtration is unique and is called the Harder-Narasimhan (HN) filtration of V . We will say that V has a strongly semistable HN filtration if all the quotients V_i/V_{i-1} are strongly semistable. We shall write

$$\mu_{\min}(V) := \inf\{\mu(V_i/V_{i-1})\}_{i \geq 1}$$

and

$$\mu_{\max}(V) := \sup\{\mu(V_i/V_{i-1})\}_{i \geq 1}.$$

An important consequence of the definitions is the following fact: if V and W are two torsion free sheaves on Y and $\mu_{\min}(V) > \mu_{\max}(W)$, then $\text{Hom}_Y(V, W) = 0$. We also recall that we have the following equivalences:

$$V \text{ is semistable} \Leftrightarrow \mu_{\min}(V) = \mu_{\max}(V) \Leftrightarrow \mu_{\min}(V) = \mu(V).$$

The first one is clear, the second one is a consequence of:

$$\mu_{\min}(V) = \min\{\mu(Q) \mid Q \text{ is a quotient of } V\}.$$

For more on the theory of semistable sheaves, see the monograph [HL10].

The following two theorems are key results from A. Langer (cf. Theorem 2.7 and Corollary 6.2 in [Lan04]).

Theorem 2.4.2. *If V is a torsion free coherent sheaf on Y , then there exists $n_0 \geq 0$ such that $F_Y^{n,*}V$ has a strongly semistable HN filtration for all $n \geq n_0$.*

If V is a torsion free coherent sheaf on Y , we now define

$$\bar{\mu}_{\min}(V) := \lim_{r \rightarrow \infty} \mu_{\min}(F_Y^{r,*}V) / \text{char}(l_0)^r$$

and

$$\bar{\mu}_{\max}(V) := \lim_{r \rightarrow \infty} \mu_{\max}(F_Y^{r,*}V) / \text{char}(l_0)^r.$$

Note that Theorem 2.4.2 implies that the two sequences $\mu_{\min}(F_Y^{r,*}V)/\text{char}(l_0)^r$ and $\mu_{\max}(F_Y^{r,*}V)/\text{char}(l_0)^r$ become constant when r is sufficiently large, so the above definitions of $\bar{\mu}_{\min}$ and $\bar{\mu}_{\max}$ make sense. Furthermore the sequences $\mu_{\min}(F_Y^{r,*}V)/\text{char}(l_0)^r$ and $\mu_{\max}(F_Y^{r,*}V)/\text{char}(l_0)^r$ are respectively weakly decreasing and weakly increasing, therefore we have

$$\mu_{\min}(V) \geq \bar{\mu}_{\min}(V) \quad \text{and} \quad \bar{\mu}_{\max}(V) \geq \mu_{\max}(V).$$

Let us define

$$\alpha(V) := \max \{ \mu_{\min}(V) - \bar{\mu}_{\min}(V), \bar{\mu}_{\max}(V) - \mu_{\max}(V) \}.$$

Theorem 2.4.3. *If V is of rank r , then*

$$\alpha(V) \leq \frac{r-1}{\text{char}(l_0)} \max \{ \bar{\mu}_{\max}(\Omega_{Y/l_0}), 0 \}.$$

In particular, if $\bar{\mu}_{\max}(\Omega_{Y/l_0}) \geq 0$ and $\text{char}(l_0) \geq d = \dim Y$,

$$\bar{\mu}_{\max}(\Omega_{Y/l_0}) \leq \frac{\text{char}(l_0)}{\text{char}(l_0) + 1 - d} \mu_{\max}(\Omega_{Y/l_0}).$$

The following lemmas will be a key input in the proof of Theorem 2.5.4.

Lemma 2.4.4. *Let V be a vector bundle over Y . Suppose that*

- *for any surjective finite morphism $\phi : Y' \rightarrow Y$, we have $H^0(Y', \phi^*V) = 0$,*
- *V^\vee is globally generated.*

Then

- *for any surjective finite morphism $\phi : Y' \rightarrow Y$, such that Y' is smooth over l_0 , we have $\mu_{\min}(\phi^*V^\vee) > 0$. In particular $\bar{\mu}_{\min}(V^\vee) > 0$;*
- *there is an $n_0 \in \mathbb{N}$ such that $H^0(Y, F_Y^{n,*}V \otimes \Omega_{Y/l_0}) = 0$ for all $n > n_0$.*

Proof. The bundle V^\vee is globally generated, so for any ϕ as in the hypotheses $\phi^*(V^\vee)$ is globally generated. This implies

$$\mu_{\min}(\phi^*(V^\vee)) \geq 0.$$

Actually $\mu_{\min}(\phi^*(V^\vee))$ cannot be zero. In fact, suppose by contradiction that $\phi^*(V^\vee)$ has a non-zero semistable quotient $Q = \phi^*(V^\vee)/Q_0$ of degree zero. Lemma 2.2 in [Rös14] shows that any globally generated torsion free sheaf of degree zero is trivial, so we have

$$\phi^*(V^\vee) \simeq Q_0 \oplus \mathcal{O}_{Y'}^{\oplus d}$$

for some $d > 0$. This implies that $\phi^*(V^\vee)$ has a non-vanishing section, which contradicts the assumptions.

To prove the second assertion it is enough to show that

$$\mu_{\min}(F_Y^{n,*}(V^\vee)) > \mu_{\max}(\Omega_{Y/l_0})$$

for n large enough, since

$$H^0(Y, F_Y^{n,*}V \otimes \Omega_{Y/l_0}) = \text{Hom}_Y(F_Y^{n,*}(V^\vee), \Omega_{Y/l_0}).$$

Now taking $\phi = F_Y^{n,*}$, we obtain that $\mu_{\min}(F_Y^{n,*}(V^\vee)) > 0$ for any n . Since the sequence

$$\mu_{\min}(F_Y^{n,*}(V^\vee)) / \text{char}(l_0)^n$$

becomes constant for n sufficiently large, it follows that $\mu_{\min}(F_Y^{n,*}(V^\vee))$ tends to infinity and therefore

$$\mu_{\min}(F_Y^{n,*}(V^\vee)) > \mu_{\max}(\Omega_{Y/l_0})$$

for n big enough. □

Lemma 2.4.5. *Let V and Y be as in Lemma 2.4.4. Let n_0 be a natural number verifying $H^0(Y, F_Y^{n,*}V \otimes \Omega_{Y/l_0}) = 0$ for all $n > n_0$ and let $T \rightarrow Y$ be a torsor under*

$$V(F_Y^{n_0,*}V) := \underline{\text{Spec}}(\text{Sym}(F_Y^{n_0,*}V^\vee)).$$

Let $\phi : Y' \rightarrow Y$ be a proper surjective morphism and suppose that Y' is irreducible. Then we have the implication:

$$\phi^*T \text{ is a trivial } V(\phi^*(F_Y^{n_0,*}V))\text{-torsor} \implies T \text{ is a trivial } V(F_Y^{n_0,*}V)\text{-torsor}.$$

The proof of this lemma uses Lemma 2.4.4, an argument attributed to Moret-Bailly in order to restrict to the case in which ϕ is generically purely inseparable and a result by Spzuro, Lewin-Ménégaux stating the injectivity of the map

$$H^1(Y, V) \rightarrow H^1(Y, F_Y^*V)$$

whenever $H^0(Y, F_Y^*V \otimes \Omega_{Y/l_0}) = 0$. See Corollary 2.8 in [Rös14] for the actual proof.

2.5 Sparsity of p -divisible unramified liftings

Let K , A and X be as fixed in Notations and let $\text{Stab}_A(X)$ be trivial.

In this section we prove our result on the sparsity of p -divisible unramified liftings (cf. Theorem 2.5.4 below). A fundamental intermediate step to do so will be Lemma 2.5.2.

The construction of the stabilizer commutes with the base change, so we have

$$\mathrm{Stab}_A(X) = \mathrm{Stab}_A(\mathcal{X}) \times_U \mathrm{Spec}K.$$

Since $\mathrm{Stab}_A(X)$ is trivial, by generic flatness and finiteness, we can restrict the map $\pi : \mathrm{Stab}_A(\mathcal{X}) \rightarrow U$ to the inverse image of a non-empty open subscheme $U' \subset U$ to obtain a finite flat commutative group scheme of degree one

$$\pi|_{\pi^{-1}(U')} : \pi^{-1}(U') \rightarrow U'.$$

Corollary 3 in paragraph 4 of [SCA86] implies that $\pi|_{\pi^{-1}(U')}$ is étale and therefore an isomorphism. In particular, for any $\mathfrak{q} \in U'$ we have that $\mathrm{Stab}_{A_{\mathfrak{q}^0}}(X_{\mathfrak{q}^0})$ is trivial. Therefore there exist only finitely many $\mathfrak{q}_1, \dots, \mathfrak{q}_k$ elements in U such that $\mathrm{Stab}_{A_{\mathfrak{q}^0}}(X_{\mathfrak{q}^0})$ is not trivial. We will denote by $\tilde{U} \subseteq U$ the open subscheme

$$\tilde{U} := U \setminus \{\mathfrak{q}_1, \dots, \mathfrak{q}_k\}.$$

For any $\mathfrak{p} \in U$ we denote by $F_{\overline{k(\mathfrak{p})}}$ the Frobenius automorphism on $\overline{k(\mathfrak{p})}$ and by F_{R_1} the automorphism of R_1 induced by $F_{\overline{k(\mathfrak{p})}}$. We define

$$\begin{aligned} X'_{\mathfrak{p}^0} &:= X_{\mathfrak{p}^0} \times_{F_{\overline{k(\mathfrak{p})}}} \overline{k(\mathfrak{p})} \\ X'_{\mathfrak{p}^1} &:= X_{\mathfrak{p}^1} \times_{F_{R_1}} R_1 \end{aligned}$$

and we write

$$F_{X_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})}} : X_{\mathfrak{p}^0} \rightarrow X'_{\mathfrak{p}^0}$$

for the relative Frobenius on $X_{\mathfrak{p}^0}$. For brevity's sake, from now on we will write

$$\Omega_{X_{\mathfrak{p}^0}} \quad \left(\text{resp. } \Omega_{X'_{\mathfrak{p}^0}}, \Omega_{X_{\mathfrak{p}^1}}, \Omega_{X'_{\mathfrak{p}^1}}, \Omega_X \right)$$

instead of

$$\Omega_{X_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})}} \quad \left(\text{resp. } \Omega_{X'_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})}}, \Omega_{X_{\mathfrak{p}^1}/R_1}, \Omega_{X'_{\mathfrak{p}^1}/R_1}, \Omega_{X/K} \right).$$

We need the recall the following fundamental result due to Cartier (see [Kat70], Th. 7.2 for its proof):

Theorem 2.5.1. *There exists a unique homomorphism of $\mathcal{O}_{X'_{p_0}}$ -graded algebras*

$$C^{-1} : \bigoplus_{i \geq 0} \Omega_{X'_{p_0}}^i \rightarrow \bigoplus_{i \geq 0} H^i \left(F_{X_{p_0}/k(\mathfrak{p}),*} \Omega_{X_{p_0}}^\bullet \right)$$

such that $C^{-1}d(x \otimes 1) = \text{class of } x^{p-1}dx$ for all global sections x of \mathcal{O}_X . Furthermore C^{-1} is an isomorphism and its inverse is called the Cartier isomorphism.

Observe now that since U is normal, then \mathcal{A} is projective over U (cf. Th.XI 1.4 in [Ray77]). Therefore there exists a U -very ample line bundle L on \mathcal{X} . For any $\mathfrak{p} \in U$ different from the generic point ξ , let us denote by $L_{\mathfrak{p}}$ the inverse image of L on $X_{\mathfrak{p}_0}$. Similarly we denote by L_{ξ} the inverse image of L on X . From now on, for any vector bundle $G_{\mathfrak{p}}$ over $X_{\mathfrak{p}_0}$, we will write $\deg(G_{\mathfrak{p}})$ for the degree of $G_{\mathfrak{p}}$ with respect to $L_{\mathfrak{p}}$. Analogously, if G_{ξ} is a vector bundle over X , we will write $\deg(G_{\xi})$ for the degree of G_{ξ} with respect to L_{ξ} . Now consider the vector bundle $\Omega_{\mathcal{X}/U}$ over \mathcal{X} . The map from U to \mathbb{Z} defined by

$$\mathfrak{p} \mapsto \chi \left((\Omega_{\mathcal{X}/U} \otimes L^m)_{\mathfrak{p}} \right) = \chi \left(\Omega_{X_{\mathfrak{p}_0}} \otimes L_{\mathfrak{p}}^m \right)$$

and

$$\xi \mapsto \chi \left((\Omega_{\mathcal{X}/U} \otimes L^m)_{\xi} \right) = \chi \left(\Omega_X \otimes L_{\xi}^m \right)$$

(here χ refers to the Euler characteristic) is locally constant on U (cf. Ch.II, Sec.5 in [MRM74]). But $\deg(\Omega_{X_{\mathfrak{p}_0}})$ only depends on the polynomial map $\chi \left(\Omega_{X_{\mathfrak{p}_0}} \otimes L_{\mathfrak{p}}^m \right)$, so for every $\mathfrak{p} \in U$ we have $\deg \left(\Omega_{X_{\mathfrak{p}_0}} \right) = \deg(\Omega_X)$.

Lemma 2.5.2. *Let K , A and X be as fixed in Notations, let $\text{Stab}_A(X)$ be trivial and let n be the dimension of X over K . Then*

$$\text{Hom}_{X_{\mathfrak{p}_0}} \left(F_{X_{\mathfrak{p}_0}}^{k,*} \Omega_{X_{\mathfrak{p}_0}}, \Omega_{X_{\mathfrak{p}_0}} \right) = 0$$

for any $k \geq 1$ and any $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$.

Proof. Let us fix $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$. We know that if

$$\mu_{\min} \left(F_{X_{\mathfrak{p}_0}}^{k,*} \Omega_{X_{\mathfrak{p}_0}} \right) > \mu_{\max} \left(\Omega_{X_{\mathfrak{p}_0}} \right)$$

then $\text{Hom}_{X_{\mathfrak{p}_0}} \left(F_{X_{\mathfrak{p}_0}}^{k,*} \Omega_{X_{\mathfrak{p}_0}}, \Omega_{X_{\mathfrak{p}_0}} \right) = 0$. Since $\mu_{\min} \geq \bar{\mu}_{\min}$ and $\bar{\mu}_{\max} \geq \mu_{\max}$, it is sufficient to show that

$$\bar{\mu}_{\min} \left(F_{X_{\mathfrak{p}_0}}^{k,*} \Omega_{X_{\mathfrak{p}_0}} \right) > \bar{\mu}_{\max} \left(\Omega_{X_{\mathfrak{p}_0}} \right)$$

for every $k \geq 1$.

Recall now that $\text{Stab}_{A_{p^0}}(X_{p^0}) = 0$ implies $H^0(X_{p^0}, \Omega_{X_{p^0}}^\vee) = 0$ (see Appendix for this). The existence of the short exact sequence

$$0 \rightarrow \Omega_{X_{p^0}}^\vee \rightarrow i^* \Omega_{A_{p^0}}^\vee \rightarrow N \rightarrow 0$$

(here N is the normal bundle to X_{p^0} in A_{p^0} and $i : X_{p^0} \hookrightarrow A_{p^0}$ is the closed immersion) assure that we can apply Lemma 2.4.1 with $Y = X_{p^0}$ and $V = \Omega_{X_{p^0}}^\vee$. Therefore for any surjective finite morphism $\phi : Y' \rightarrow X_{p^0}$, we have

$$H^0(Y', \phi^* \Omega_{X_{p^0}}^\vee) \simeq H^0(X_{p^0}, \Omega_{X_{p^0}}^\vee) = 0.$$

This tells us that the first hypothesis of Lemma 2.4.4 is satisfied (again in the case $Y = X_{p^0}$ and $V = \Omega_{X_{p^0}}^\vee$). The second hypothesis is also satisfied: to see that $\Omega_{X_{p^0}}$ is globally generated it is enough to dualize the exact sequence above

$$0 \rightarrow N^\vee \rightarrow i^* \Omega_{A_{p^0}} \rightarrow \Omega_{X_{p^0}} \rightarrow 0$$

and remember that the vector bundle $\Omega_{A_{p^0}}$ is free. Therefore Lemma 2.4.4 implies $\bar{\mu}_{\min}(\Omega_{X_{p^0}}) > 0$. Using this and the equality $\bar{\mu}_{\min}(F_{X_{p^0}}^{k,*} \Omega_{X_{p^0}}) = p^k \bar{\mu}_{\min}(\Omega_{X_{p^0}})$, then all we have to verify is that

$$p \bar{\mu}_{\min}(\Omega_{X_{p^0}}) > \bar{\mu}_{\max}(\Omega_{X_{p^0}}).$$

Theorem 2.4.3 gives us the following inequality

$$\bar{\mu}_{\min}(\Omega_{X_{p^0}}) - \mu_{\min}(\Omega_{X_{p^0}}) \geq \frac{1-n}{p} \bar{\mu}_{\max}(\Omega_{X_{p^0}})$$

so that

$$p \bar{\mu}_{\min}(\Omega_{X_{p^0}}) \geq p \mu_{\min}(\Omega_{X_{p^0}}) + (1-n) \bar{\mu}_{\max}(\Omega_{X_{p^0}}).$$

We are then reduced to prove that

$$p \mu_{\min}(\Omega_{X_{p^0}}) > n \bar{\mu}_{\max}(\Omega_{X_{p^0}}).$$

We make use again of Theorem 2.4.3

$$\bar{\mu}_{\max}(\Omega_{X_{p^0}}) \leq \frac{p}{p+1-n} \mu_{\max}(\Omega_{X_{p^0}})$$

so it is enough to show that

$$(p+1-n) \mu_{\min}(\Omega_{X_{p^0}}) > n \mu_{\max}(\Omega_{X_{p^0}}).$$

If $\Omega_{X_{p_0}}$ is semistable, we obtain $p > 2n - 1$. Otherwise, we can estimate $\mu_{\max}(\Omega_{X_{p_0}})$ and $\mu_{\min}(\Omega_{X_{p_0}})$ in the following way. There exists a subsheaf $0 \neq M \subsetneq \Omega_{X_{p_0}}$ such that

$$\mu_{\max}(\Omega_{X_{p_0}}) = \frac{\deg(M)}{\operatorname{rk}(M)}$$

therefore we have $\mu_{\max}(\Omega_{X_{p_0}}) \leq \deg(M) \leq \deg(\Omega_{X_{p_0}}) - 1$. Similarly,

$$\mu_{\min}(\Omega_{X_{p_0}}) = \frac{\deg(Q)}{\operatorname{rk}(Q)}$$

for some Q quotient of $\Omega_{X_{p_0}}$, so $\mu_{\min}(\Omega_{X_{p_0}}) \geq 1/n$. It is then sufficient to have

$$p > n^2 \deg(\Omega_{X_{p_0}}) + (n - 1 - n^2).$$

Since $n - 1 - n^2$ is always negative, we are reduced to $p > n^2 \deg(\Omega_{X_{p_0}})$. Now $\deg(\Omega_{X_{p_0}})$ is greater or equal to one, so $n^2 \deg(\Omega_{X_{p_0}}) \geq 2n - 1$ for any n . This ensures us that the condition

$$p > n^2 \deg(\Omega_{X_{p_0}})$$

is sufficient to have $\mu_{\min}(F_{X_{p_0}}^{k,*} \Omega_{X_{p_0}}) > \mu_{\max}(\Omega_{X_{p_0}})$ for every $k \geq 1$ whether $\Omega_{X_{p_0}}$ is semistable or not. To conclude it is enough to remember that $\deg(\Omega_{X_{p_0}})$ coincides with $\deg(\Omega_X)$. \square

Corollary 2.5.3. *The map*

$$H^1(X_{p_0}, F_{X_{p_0}}^* \Omega_{X_{p_0}}^\vee) \rightarrow H^1(X_{p_0}, F_{X_{p_0}}^{k,*} \Omega_{X_{p_0}}^\vee)$$

is injective for every $k \geq 1$ and every $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$.

Proof. Lemma 2.5.2 and Spziro, Lewin-Ménégaux result (stated at the end of the previous section) imply that

$$H^1(X_{p_0}, F_{X_{p_0}}^{h,*} \Omega_{X_{p_0}}^\vee) \rightarrow H^1(X_{p_0}, F_{X_{p_0}}^{h+1,*} \Omega_{X_{p_0}}^\vee)$$

is injective for every $h \geq 0$. Therefore the composition

$$H^1(X_{p_0}, F_{X_{p_0}}^* \Omega_{X_{p_0}}^\vee) \hookrightarrow H^1(X_{p_0}, F_{X_{p_0}}^{2,*} \Omega_{X_{p_0}}^\vee) \hookrightarrow \dots \hookrightarrow H^1(X_{p_0}, F_{X_{p_0}}^{k,*} \Omega_{X_{p_0}}^\vee)$$

is an injective map. \square

We are now ready to prove Theorem 2.5.4.

Theorem 2.5.4. *With the same hypotheses as in Lemma 2.5.2, for any $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$, the set*

$$\left\{ P \in X_{\mathfrak{p}^0}(\overline{k(\mathfrak{p})}) \mid P \text{ lifts to an element of } pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1) \right\}$$

is not Zariski dense in $X_{\mathfrak{p}^0}$.

Notice that in [Rös13], D. Rössler proved a result on the sparsity of highly p -divisible unramified liftings implying that, for m big enough, the set

$$\left\{ P \in X_{\mathfrak{p}^0}(\overline{k(\mathfrak{p})}) \mid P \text{ lifts to an element of } p^m A_{\mathfrak{p}^m}(R_m) \cap X_{\mathfrak{p}^m}(R_m) \right\}$$

is not Zariski dense in $X_{\mathfrak{p}^0}$ (cf. Th. 4.1 in [Rös13]). Theorem 2.5.4 can be viewed as an effective form of Rössler's result.

Proof. Since $\text{Crit}^1(\mathcal{X}, \mathcal{A})(\overline{k(\mathfrak{p})}) = pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1)$, we have that

$$\left\{ P \in X_{\mathfrak{p}^0}(\overline{k(\mathfrak{p})}) \mid P \text{ lifts to an element of } pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1) \right\}$$

coincides with the image of $\text{Crit}^1(\mathcal{X}, \mathcal{A})(\overline{k(\mathfrak{p})}) \rightarrow X_{\mathfrak{p}^0}(\overline{k(\mathfrak{p})})$. Therefore, the thesis of our theorem is equivalent to: $\text{Exc}^1(\mathcal{X}, \mathcal{A})$ does not coincide with $X_{\mathfrak{p}^0}$.

Let us suppose by contradiction that $\text{Exc}^1(\mathcal{X}, \mathcal{A}) = X_{\mathfrak{p}^0}$.

Consider the commutative diagram of $\overline{k(\mathfrak{p})}$ -schemes

$$\begin{array}{ccc} \text{Crit}^1(\mathcal{X}, \mathcal{A}) & \longrightarrow & X_{\mathfrak{p}^0} \\ \downarrow & & \downarrow \text{Id} \\ \text{Gr}_1(X_{\mathfrak{p}^1}) & \longrightarrow & X_{\mathfrak{p}^0} \end{array}$$

where the left vertical morphism is a closed immersion. Since we are assuming $\text{Exc}^1(\mathcal{X}, \mathcal{A}) = X_{\mathfrak{p}^0}$, then $\text{Crit}^1(\mathcal{X}, \mathcal{A}) \rightarrow X_{\mathfrak{p}^0}$ is surjective. We choose an irreducible component

$$\text{Crit}^1(\mathcal{X}, \mathcal{A})_0 \hookrightarrow \text{Crit}^1(\mathcal{X}, \mathcal{A})$$

which dominates $X_{\mathfrak{p}^0}$. Now consider the $V\left(F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}\right)$ -torsor $\pi_1 : \text{Gr}_1(X_{\mathfrak{p}^1}) \rightarrow X_{\mathfrak{p}^0}$. Lemma 2.5.2 allows us to apply Lemma 2.4.5 with $T = \text{Gr}_1(X_{\mathfrak{p}^1})$, $Y = X_{\mathfrak{p}^0}$, $n_0 = 1$ and ϕ equal to

$$\text{Crit}^1(\mathcal{X}, \mathcal{A})_0 \rightarrow X_{\mathfrak{p}^0}.$$

We have that $\phi^* \text{Gr}_1(X_{\mathfrak{p}^1})$ is trivial as $V\left(\phi^* F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}\right)$ -torsor, since

$$\text{Crit}^1(\mathcal{X}, \mathcal{A})_0 \subseteq \text{Gr}_1(X_{\mathfrak{p}^1}).$$

Hence we obtain that $\pi_1 : \text{Gr}_1(X_{\mathfrak{p}^1}) \rightarrow X_{\mathfrak{p}^0}$ is trivial as $V\left(F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}\right)$ -torsor. Let us take a section $\sigma : X_{\mathfrak{p}^0} \rightarrow \text{Gr}_1(X_{\mathfrak{p}^1})$. By definition of Greenberg transform, the map σ over $\overline{k(\mathfrak{p})}$ corresponds to a map $\bar{\sigma} : \mathbb{W}_1(X_{\mathfrak{p}^0}) \rightarrow X_{\mathfrak{p}^1}$ over R_1 . We can precompose $\bar{\sigma}$ with the morphism $t : X_{\mathfrak{p}^1} \rightarrow \mathbb{W}_1(X_{\mathfrak{p}^0})$ corresponding to

$$\begin{aligned} W_1(\mathcal{O}_{X_{\mathfrak{p}^0}}) &\rightarrow \mathcal{O}_{X_{\mathfrak{p}^1}} \\ (a_0, a_1) &\mapsto \tilde{a}_0^p + \tilde{a}_1 p \end{aligned}$$

where \tilde{a}_i lifts a_i . Consider now the following diagram

$$\begin{array}{ccccc} X_{\mathfrak{p}^1} & \xrightarrow{t} & \mathbb{W}_1(X_{\mathfrak{p}^0}) & \xrightarrow{\bar{\sigma}} & X_{\mathfrak{p}^1} \\ \uparrow & & \uparrow & & \uparrow \\ X_{\mathfrak{p}^0} & \xrightarrow{F_{X_{\mathfrak{p}^0}}} & X_{\mathfrak{p}^0} & \xrightarrow{\text{Id}} & X_{\mathfrak{p}^0} \end{array}$$

Notice that the composition

$$X_{\mathfrak{p}^0} \longrightarrow X_{\mathfrak{p}^1} \xrightarrow{t} \mathbb{W}_1(X_{\mathfrak{p}^0})$$

simply corresponds to the map

$$\begin{aligned} W_1(\mathcal{O}_{X_{\mathfrak{p}^0}}) &\rightarrow \mathcal{O}_{X_{\mathfrak{p}^0}} \\ (a_0, a_1) &\mapsto \tilde{a}_0^p. \end{aligned}$$

Therefore the left square in the above diagram is commutative. The properties of the Greenberg transform and the equality $\pi_1 \circ \sigma = \text{Id}_{X_{\mathfrak{p}^0}}$ imply that the right square is commutative too. We obtain in this way that $\bar{\sigma} \circ t : X_{\mathfrak{p}^1} \rightarrow X_{\mathfrak{p}^1}$ is a lift of the Frobenius $F_{X_{\mathfrak{p}^0}}$ over R_1 .

The diagram below is also commutative

$$\begin{array}{ccccc} X_{\mathfrak{p}^1} & \xrightarrow{t} & \mathbb{W}_1(X_{\mathfrak{p}^0}) & \xrightarrow{\bar{\sigma}} & X_{\mathfrak{p}^1} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec}(R_1) & \xrightarrow{F_{R_1}} & \text{Spec}(R_1) & \xrightarrow{\text{Id}} & \text{Spec}(R_1) \end{array}$$

In fact, by definition, $\bar{\sigma}$ is a morphism over R_1 , so the right square is commutative. The commutativity of the left square is easy to check, since we know explicitly t and F_{R_1} .

The commutativity of the diagram above implies the existence of a morphism

$$\tilde{F} : X_{p^1} \rightarrow X'_{p^1}$$

over R_1 lifting $F_{X_{p^0}/\overline{k(\mathfrak{p})}}$. Now we use a classical argument involving lifting the Frobenius and the Cartier isomorphism (cf. part (b) of the proof of Théorème 2.1 in [DI87]). Since

$$F_{X_{p^0}/\overline{k(\mathfrak{p})}}^* : \Omega_{X'_{p^0}} \rightarrow F_{X_{p^0}/\overline{k(\mathfrak{p})},*} \Omega_{X_{p^0}}$$

is the zero map, then the image of $\tilde{F}^* : \Omega_{X'_{p^1}} \rightarrow \tilde{F}_* \Omega_{X_{p^1}}$ is contained in $p\tilde{F}_* \Omega_{X_{p^1}}$. Furthermore, the multiplication by p induces an isomorphism

$$p : F_{X_{p^0}/\overline{k(\mathfrak{p})},*} \Omega_{X_{p^0}} \rightarrow p\tilde{F}_* \Omega_{X_{p^1}},$$

so that there exists a unique map

$$f := p^{-1}\tilde{F}^* : \Omega_{X'_{p^0}} \rightarrow F_{X_{p^0}/\overline{k(\mathfrak{p})},*} \Omega_{X_{p^0}}$$

making the diagram below commutative

$$\begin{array}{ccc} \Omega_{X'_{p^1}} & \xrightarrow{\tilde{F}^*} & p\tilde{F}_* \Omega_{X_{p^1}} \\ \downarrow & & \uparrow p \\ \Omega_{X'_{p^0}} & \xrightarrow{f} & F_{X_{p^0}/\overline{k(\mathfrak{p})},*} \Omega_{X_{p^0}} \end{array}$$

If x is a local section of $\mathcal{O}_{X_{p^1}}$ with reduction x_0 modulo p , then

$$\tilde{F}^*(x \otimes 1) = x^p + pu(x)$$

where $u(x)$ is a local section of $\mathcal{O}_{X_{p^1}}$ and

$$f(dx_0 \otimes 1) = x_0^{p-1} dx_0 + d(u(x)).$$

In particular we have that $df = 0$ and the map

$$H^1 \circ f : \Omega_{X'_{p^0}} \rightarrow F_{X_{p^0}/\overline{k(\mathfrak{p})},*} \Omega_{X_{p^0}} \rightarrow H^1 \left(F_{X_{p^0}/\overline{k(\mathfrak{p})},*} \Omega_{X_{p^0}}^\bullet \right)$$

coincides with the inverse of the Cartier isomorphism C^{-1} in degree one (see Theorem 2.5.1). Therefore $H^1 \circ f$ is an isomorphism which implies that f is not the zero map. We can now consider the adjoint of f

$$\bar{f} : F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}} = F_{X_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})}}^* \Omega_{X'_{\mathfrak{p}^0}} \rightarrow \Omega_{X_{\mathfrak{p}^0}}.$$

Being f nonzero, then also \bar{f} is nonzero and this contradicts Lemma 2.5.2. \square

2.6 The Manin-Mumford conjecture for subvarieties with ample cotangent bundle

Let K , X , A be as fixed in Notations and let $\text{Stab}_A(X)$ be trivial. In this section we also suppose that Ω_X is ample. Then by Proposition 4.4 in [Har66] we know that $\Omega_{X_{\mathfrak{p}^0}}$ is ample for all \mathfrak{p} in a nonempty open subscheme W of U . Let us denote by \bar{U} the open subscheme

$$\bar{U} := W \cap \tilde{U} \subseteq U,$$

so that every $\mathfrak{p} \in \bar{U}$ verifies:

- K/\mathbb{Q} is unramified at \mathfrak{p} ,
- X/K has good reduction at \mathfrak{p} ,
- $\text{Stab}_{A_{\mathfrak{p}^0}}(X_{\mathfrak{p}^0})$ is trivial,
- $\Omega_{X_{\mathfrak{p}^0}}$ is ample.

Theorem 2.6.1. *Let K , A and X be as fixed in Notations, let $\text{Stab}_A(X)$ be trivial, let Ω_X be ample and n be the dimension of X over K . For any $\mathfrak{p} \in \bar{U}$ above a prime $p > n^2 \deg(\Omega_X)$ the set*

$$pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1)$$

is finite.

Proof. This is clearly equivalent to show that the scheme

$$\text{Crit}^1(\mathcal{X}, \mathcal{A}) = [p]_* \text{Gr}_1(A_{\mathfrak{p}^1}) \cap \text{Gr}_1(X_{\mathfrak{p}^1})$$

is finite over $\overline{k(\mathfrak{p})}$. We have already observed that $[p]_* \text{Gr}_1(A_{\mathfrak{p}^1})$ is proper over $\overline{k(\mathfrak{p})}$, being the greatest abelian subvariety of $\text{Gr}_1(A_{\mathfrak{p}^1})$. Therefore if we prove that $\text{Gr}_1(X_{\mathfrak{p}^1})$ is affine we are done, since any affine proper morphism is finite. To prove that $\text{Gr}_1(X_{\mathfrak{p}^1})$ is affine, we use exactly the same argument the reader can find in Proposition 1.10 in [Bui96].

2.6. The Manin-Mumford conjecture for subvarieties with ample cotangent bundle

Let us consider the $V\left(F_{X_{p^0}}^* \Omega_{X_{p^0}}\right)$ -torsor $\pi_1 : \text{Gr}_1(X_{p^1}) \rightarrow X_{p^0}$. While proving Theorem 2.5.4, we have seen that π_1 is not trivial. Now let

$$\eta \in H^1\left(X_{p^0}, F_{X_{p^0}}^* \Omega_{X_{p^0}}^\vee\right)$$

be the class defined by π_1 . Under the natural isomorphism

$$H^1\left(X_{p^0}, F_{X_{p^0}}^* \Omega_{X_{p^0}}^\vee\right) \simeq \text{Ext}^1\left(F_{X_{p^0}}^* \Omega_{X_{p^0}}, \mathcal{O}_{X_{p^0}}\right)$$

η corresponds to some extension

$$0 \rightarrow \mathcal{O}_{X_{p^0}} \rightarrow E \rightarrow F_{X_{p^0}}^* \Omega_{X_{p^0}} \rightarrow 0. \quad (2.1)$$

We denote by $\mathbb{P}(E)$ the projective bundle over X_{p^0} associated to E , and by D the divisor $\mathbb{P}\left(F_{X_{p^0}}^* \Omega_{X_{p^0}}\right) \subseteq \mathbb{P}(E)$. Then the torsor π_1 identifies with $\mathbb{P}(E) \setminus D$.

The fact that π_1 is not the trivial torsor implies that the short sequence (2.1) is nonsplit. We can say even more: for any proper surjective morphism $\phi : Y' \rightarrow X_{p^0}$ with Y' irreducible, the pullback of (2.1) through ϕ is nonsplit. This is exactly what Lemma 2.4.5 tells us (notice that we can apply Lemma 2.4.5 to our situation thanks to Lemma 2.5.2). This and our assumption on the ampleness of $\Omega_{X_{p^0}}$ allow us to apply Corollary 2 in Sec. 1 of [MD84]: we obtain that E is ample, i.e. $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample. But D belongs to the linear system of $\mathcal{O}_{\mathbb{P}(E)}(1)$, hence D is ample and $\text{Gr}_1(X_{p^1})$ is affine. \square

It is now easy to deduce a new proof of the Manin-Mumford conjecture in the case of a subvariety with ample cotangent bundle.

Theorem 2.6.2. *Let K , A and X be as fixed in Notations, let $\text{Stab}_A(X)$ be trivial and let Ω_X be ample. The set*

$$\text{Tor}(A(\overline{K})) \cap X(\overline{K})$$

is finite.

Proof. Let us fix $p > n^2 \deg(\Omega_X)$ and write

$$\text{Tor}(A(\overline{K})) = \text{Tor}^p(A(\overline{K})) \oplus \text{Tor}_p(A(\overline{K})).$$

We prove the finiteness of $\text{Tor}^p(A(\overline{K})) \cap X(\overline{K})$ and the finiteness of $\text{Tor}_p(A(\overline{K})) \cap X(\overline{K})$ separately.

If $\mathfrak{p} \in \overline{U}$ is above p , then the field of definition of every prime-to- p torsion point is unramified at \mathfrak{p} . This implies

$$\mathrm{Tor}^p(A(\overline{K})) \cap X(\overline{K}) \subseteq \mathrm{Tor}^p(\mathcal{A}(R)) \cap \mathcal{X}(R)$$

where R is the ring of Witt vectors with coordinates in $\overline{k(\mathfrak{p})}$. By precomposing with the morphism $\mathrm{Spec} R_1 \rightarrow \mathrm{Spec} R$ we obtain a homomorphism $\delta : \mathcal{A}(R) \rightarrow A_{\mathfrak{p}^1}(R_1)$. The restriction of δ to $\mathrm{Tor}^p(\mathcal{A}(R))$ is injective: in fact if $p \geq 3$ then the restriction to the entire torsion is injective and if $p = 2$ the only torsion points contained in the kernel are of order 2 (see [[Sil86], Chapter IV, Theorem 6.1] for this). So we have an injection

$$\mathrm{Tor}^p(\mathcal{A}(R)) \cap \mathcal{X}(R) \hookrightarrow \mathrm{Tor}^p(A_{\mathfrak{p}^1}(R_1)) \cap X_{\mathfrak{p}^1}(R_1).$$

Since $\mathrm{Tor}^p(A_{\mathfrak{p}^1}(R_1)) \subseteq pA_{\mathfrak{p}^1}(R_1)$, we obtain

$$\mathrm{Tor}^p(A(\overline{K})) \cap X(\overline{K}) \subseteq pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1).$$

Therefore $\mathrm{Tor}^p(A(\overline{K})) \cap X(\overline{K})$ is a finite set by Theorem 2.6.1.

Now, for brevity's sake, let us write $Z := \mathrm{Tor}_p(A(\overline{K})) \cap X(\overline{K})$ and let us denote by \overline{Z} the Zariski closure of Z . We use here an argument given by D. Rössler in the last remark of [Rös05]. Let us write $A[p]$ for the p -torsion points of A and K' for the extension of K generated by the points in $A[p]$. Then Z is the union

$$Z = (Z \cap A(K')) \cup (Z \cap A(K')^c).$$

Since Z is Zariski dense in \overline{Z} , then at least one of the above sets has to be dense in Z . But $(Z \cap A(K')^c)$ is not dense in \overline{Z} : this is a consequence of Proposition 3 in [Rös05] and of our assumption $\mathrm{Stab}(X) = 0$. Then $(Z \cap A(K'))$, which is a finite set by the theorem of Mordell-Weil, is dense in \overline{Z} . This implies that \overline{Z} is finite. In particular Z is finite. \square

2.7 An explicit bound for the cardinality of the prime-to- p torsion of Debarre's subvarieties

In this section we provide an upper bound for the number of points in the set $\mathrm{Tor}^p(A(\overline{K})) \cap X(\overline{K})$ in the case in which X is a subvariety with ample cotangent bundle of the type studied by Debarre in [Deb05].

Let A/K be an abelian variety of dimension n and let $L = \mathcal{O}_A(D)$ be a very ample line bundle on A . Let $c \in \mathbb{N}$ be greater than $n/2$ and $e \in \mathbb{N}$. For $H_1, H_2, \dots, H_c \in |L^e|$ general and e sufficiently big, the subvariety $X := H_1 \cap H_2 \cap \dots \cap H_c$ has ample

cotangent bundle (Theorem 8 in [Deb05]). Suppose that X is smooth and has trivial stabilizer.

First of all, let us take a sufficiently small open $V \subseteq \text{Spec}(\mathcal{O}_K)$ such that A extends over V to an abelian scheme \mathcal{A} , L extends to a V -very ample line bundle \mathcal{L} , H_i extends to \mathcal{H}_i for every i and $\mathcal{X} := \mathcal{H}_1 \cap \mathcal{H}_2 \cap \dots \cap \mathcal{H}_c$ is smooth and has ample cotangent bundle. We can restrict V if necessary and suppose that for all $\mathfrak{p} \in V$ the stabilizer $\text{Stab}_{A_{\mathfrak{p}^0}}(X_{\mathfrak{p}^0})$ is trivial and K/\mathbb{Q} is unramified at \mathfrak{p} .

For any $m \in \mathbb{N}$ we define

$$C^m := \left\{ (r_1, r_2, \dots) \mid r_i \in \mathbb{N} \text{ and } \sum ir_i = m \right\}$$

and for any $\underline{\beta} \in C^m$ we define

$$M_{\underline{\beta}} := (-1)^{\sum \beta_i} \binom{\sum \beta_i}{\beta_1, \beta_2, \dots}$$

$$R_{\underline{\beta}}^c := \prod_{j \geq 1} \binom{c}{j}^{\beta_j}.$$

Finally for any $m \in \mathbb{N}$ we write

$$W_{m,c} := \sum_{\underline{\beta} \in C^m} M_{\underline{\beta}} R_{\underline{\beta}}^c.$$

Theorem 2.7.1. *Let K be a number field, A/K be an abelian variety of dimension n and let $L = \mathcal{O}_A(D)$ be a very ample line bundle on A . Let $c, e \in \mathbb{N}$ with $c \geq n/2$. Let $H_1, H_2, \dots, H_c \in |L^e|$ be general and e be sufficiently big, so that the subvariety $X := H_1 \cap H_2 \cap \dots \cap H_c$ has ample cotangent bundle. Suppose that X is smooth and has trivial stabilizer. If \mathfrak{p} is in V (the open subscheme of $\text{Spec}(\mathcal{O}_X)$ defined above) and \mathfrak{p} is above a prime $p > (n-c)^2 ce^{c+1}(L^n)$, then the cardinality of $\text{Tor}^p(A(\overline{K})) \cap X(\overline{K})$ is bounded by*

$$p^{2n} \left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-p)^{n-c-h} e^{n-h} \cdot \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} W_{i,c}^{\beta_i} \right) (L^n)^2.$$

Proof. Fix \mathfrak{p} above a prime $p > (n-c)^2 \deg_{L|_X} \Omega_X$. Then we know that

$$\text{Tor}^p(A(\overline{K})) \cap X(\overline{K}) \subseteq ([p]_* \text{Gr}_1(A_{\mathfrak{p}^1}) \cap \text{Gr}_1(X_{\mathfrak{p}^1}))(\overline{k(\mathfrak{p})})$$

which is finite by Theorem 2.6.1. We have two exact sequences of vector bundles

$$0 \rightarrow \mathcal{O}_{X_{\mathfrak{p}^0}} \rightarrow E_X \rightarrow F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}} \rightarrow 0$$

$$0 \rightarrow \mathcal{O}_{A_{\mathfrak{p}^0}} \rightarrow E_A \rightarrow F_{A_{\mathfrak{p}^0}}^* \Omega_{A_{\mathfrak{p}^0}} \rightarrow 0$$

corresponding to the torsors $\mathrm{Gr}_1(X_{\mathfrak{p}^1}) \rightarrow X_{\mathfrak{p}^0}$ and $\mathrm{Gr}_1(A_{\mathfrak{p}^1}) \rightarrow A_{\mathfrak{p}^0}$. If we denote by D_X the divisor $\mathbb{P}(F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}) \subseteq \mathbb{P}(E_X)$ and D_A the divisor $\mathbb{P}(F_{A_{\mathfrak{p}^0}}^* \Omega_{A_{\mathfrak{p}^0}}) \subseteq \mathbb{P}(E_A)$, then we have two identifications

$$\begin{aligned}\mathrm{Gr}_1(X_{\mathfrak{p}^1}) &\simeq \mathbb{P}(E_X) \setminus D_X \\ \mathrm{Gr}_1(A_{\mathfrak{p}^1}) &\simeq \mathbb{P}(E_A) \setminus D_A.\end{aligned}$$

We shall write i for the closed embedding $i : \Omega_{X_{\mathfrak{p}^0}} \rightarrow \Omega_{A_{\mathfrak{p}^0}}$. It is not difficult to show that there is a natural restriction homomorphism $i^*E_A \rightarrow E_X$ prolonging the homomorphism $i^*\Omega_{A_{\mathfrak{p}^0}} \rightarrow \Omega_{X_{\mathfrak{p}^0}}$. The homomorphism $i^*E_A \rightarrow E_X$ is clearly surjective, so it induces a closed embedding $j : \mathbb{P}(E_X) \hookrightarrow \mathbb{P}(E_A)$ prolonging the embedding $\mathrm{Gr}_1(X_{\mathfrak{p}^1}) \hookrightarrow \mathrm{Gr}_1(A_{\mathfrak{p}^1})$. Therefore we have a commutative diagram

$$\begin{array}{ccc} & & [p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \\ & & \downarrow \\ \mathrm{Gr}_1(X_{\mathfrak{p}^1}) & \hookrightarrow & \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \\ \downarrow & & \downarrow \\ \mathbb{P}(E_X) & \xrightarrow{j} & \mathbb{P}(E_A) \\ \downarrow \pi_X & & \downarrow \pi_A \\ X_{\mathfrak{p}^0} & \xrightarrow{i} & A_{\mathfrak{p}^0} \\ \downarrow & \swarrow & \downarrow \\ \mathrm{Spec}(\overline{k(\mathfrak{p})}) & & \mathrm{Spec}(\overline{k(\mathfrak{p})}) \end{array} \quad \begin{array}{l} \curvearrowright \\ T \end{array}$$

Let us denote by $\mathcal{L}_{\mathfrak{p}}$ the base change of \mathcal{L} to $A_{\mathfrak{p}^0}$. It is standard to prove that the line bundle

$$\mathcal{H} := \pi_A^* \mathcal{L}_{\mathfrak{p}} \otimes \mathcal{O}_{\mathbb{P}(E_A)}(1).$$

is very ample on $\mathbb{P}(E_A)$ (cf. pag 4 in [BV96]). We have

$$\mathcal{H}|_{\mathbb{P}(E_X)} = \pi_X^* i^* \mathcal{L}_{\mathfrak{p}} \otimes \mathcal{O}_{\mathbb{P}(E_X)}(1)$$

$$\mathcal{H}|_{[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1})} = T^* \mathcal{L}_{\mathfrak{p}}$$

since $D_A \in |\mathcal{O}_{\mathbb{P}(E_A)}(1)|$ and $[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \subseteq \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \simeq \mathbb{P}(E_A) \setminus D_A$. We know that $[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1})$ is the maximal abelian subvariety of $\mathrm{Gr}_1(A_{\mathfrak{p}^1})$ and we also know that

the multiplication by p map on $\mathrm{Gr}_1(A_{\mathfrak{p}^1})$ factors through the isogeny T . This implies that T has degree at most p^{2n} , so we have the following estimate

$$\mathrm{deg}_{\mathcal{H}}([p]_*\mathrm{Gr}_1(A_{\mathfrak{p}^1})) \leq p^{2n}(\mathcal{L}_{\mathfrak{p}}^n).$$

Let us now consider $\mathrm{deg}_{\mathcal{H}}(\mathbb{P}(E_X))$. It coincides with

$$\int_{\mathbb{P}(E_X)} c_1(\mathcal{H}|_{\mathbb{P}(E_X)})^{2n-2c} \quad (2.2)$$

where c_1 stands for the first Chern class in the Chow ring and $\int_{\mathbb{P}(E_X)}$ stands for the push-forward morphism to $\mathrm{Spec}(\overline{k(\mathfrak{p})})$ in the Chow theory. Since

$$c_1(\mathcal{H}|_{\mathbb{P}(E_X)}) = c_1(\pi_X^* i^* \mathcal{L}_{\mathfrak{p}}) + c_1(\mathcal{O}_{\mathbb{P}(E_X)}(1))$$

we can re-write (2.2) as

$$\int_{\mathbb{P}(E_X)} \sum_{h=0}^{2n-2c} \binom{2n-2c}{h} c_1(\pi_X^* i^* \mathcal{L}_{\mathfrak{p}})^h \cdot c_1(\mathcal{O}_{\mathbb{P}(E_X)}(1))^{2n-2c-h}.$$

Equivalently

$$\int_{X_{\mathfrak{p}^0}} \sum_{h=0}^{2n-2c} \binom{2n-2c}{h} c_1(i^* \mathcal{L}_{\mathfrak{p}})^h \cdot \pi_{X,*} \left(c_1(\mathcal{O}_{\mathbb{P}(E_X)}(1))^{2n-2c-h} \right)$$

and by definition of Segre class this is

$$\int_{X_{\mathfrak{p}^0}} \sum_{h=0}^{2n-2c} \binom{2n-2c}{h} c_1(i^* \mathcal{L}_{\mathfrak{p}})^h \cdot s_{n-c-h}(E_X).$$

But $s_k = 0$ if $k < 0$, so we end up with

$$\int_{X_{\mathfrak{p}^0}} \sum_{h=0}^{n-c} \binom{2n-2c}{h} c_1(i^* \mathcal{L}_{\mathfrak{p}})^h \cdot s_{n-c-h}(E_X).$$

Now the exact sequence

$$0 \rightarrow \mathcal{O}_{X_{\mathfrak{p}^0}} \rightarrow E_X \rightarrow F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}} \rightarrow 0$$

implies

$$s_{n-c-h}(E_X) = \sum_{i+j=n-c-h} s_i(\mathcal{O}_{X_{\mathfrak{p}^0}}) s_j(F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}) = s_{n-c-h}(F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}})$$

and so

$$s_{n-c-h}(E_X) = s_{n-c-h}(F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}) = p^{n-c-h} s_{n-c-h}(\Omega_{X_{\mathfrak{p}^0}})$$

(here we have used the following fact: the pullback of a cycle η of codimension j through the Frobenius map is $p^j\eta$). Therefore we have to study the following sum

$$\sum_{h=0}^{n-c} \binom{2n-2c}{h} p^{n-c-h} c_1(i^*\mathcal{L}_p)^h \cdot s_{n-c-h}(\Omega_{X_{p_0}}) \quad (2.3)$$

We can rewrite $s_{n-c-h}(\Omega_{X_{p_0}})$ as a function of $c_1(i^*\mathcal{L}_p)$. Recall that if $\sum_{m \geq 0} a_m t^m$ is a formal power series with $a_0 = 1$ then its inverse (for the multiplication) is

$$\left(\sum_{m \geq 0} a_m t^m \right)^{-1} = \sum_{m \geq 0} \left(\sum_{\underline{\beta} \in C^m} M_{\underline{\beta}} \prod_{i \geq 1} a_i^{\beta_i} \right) t^m.$$

Since

$$s_t(\Omega_{X_{p_0}}) = 1 + s_1(\Omega_{X_{p_0}})t + s_2(\Omega_{X_{p_0}})t^2 + \dots$$

is the inverse power series of

$$c_t(\Omega_{X_{p_0}}) = 1 + c_1(\Omega_{X_{p_0}})t + c_2(\Omega_{X_{p_0}})t^2 + \dots$$

we obtain

$$s_{n-c-h}(\Omega_{X_{p_0}}) = \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} c_i(\Omega_{X_{p_0}})^{\beta_i} = \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} (-1)^{i\beta_i} c_i(T_{X_{p_0}})^{\beta_i}$$

The normal exact sequence for i

$$0 \rightarrow T_{X_{p_0}} \rightarrow i^*T_{A_{p_0}} \rightarrow N \rightarrow 0$$

implies

$$c_t(T_{X_{p_0}})c_t(N) = c_t(i^*T_{A_{p_0}}) = 1$$

where N is the normal bundle for i , so we have that $c_t(T_{X_{p_0}})$ is the inverse of $c_t(N)$. Recalling that

$$c_t(N) = (1 + c_1(i^*\mathcal{L}_p^e)t)^c$$

and applying the formula for the inverse of a formal power series to $c_t(T_{X_{p_0}})$ we get

$$\begin{aligned} c_i(T_{X_{p_0}}) &= \sum_{\underline{\beta} \in C^i} M_{\underline{\beta}} \prod_{j \geq 1} c_j(N)^{\beta_j} = \sum_{\underline{\beta} \in C^i} M_{\underline{\beta}} \prod_{j \geq 1} \left(\binom{c}{j} c_1(i^*\mathcal{L}_p^e)^j \right)^{\beta_j} \\ &= \sum_{\underline{\beta} \in C^i} M_{\underline{\beta}} \prod_{j \geq 1} \binom{c}{j}^{\beta_j} c_1(i^*\mathcal{L}_p^e)^{j\beta_j} \\ &= c_1(i^*\mathcal{L}_p^e)^i \sum_{\underline{\beta} \in C^i} M_{\underline{\beta}} R_{\underline{\beta}}^c \\ &= c_1(i^*\mathcal{L}_p^e)^i W_{i,c}. \end{aligned}$$

Therefore we have

$$\begin{aligned}
 s_{n-c-h}(\Omega_{X_{p^0}}) &= \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} (-1)^{i\beta_i} c_1(i^* \mathcal{L}_{\mathbf{p}}^e)^{i\beta_i} W_{i,c}^{\beta_i} \\
 &= \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} (-1)^{n-c-h} c_1(i^* \mathcal{L}_{\mathbf{p}}^e)^{n-c-h} \prod_{i \geq 1} W_{i,c}^{\beta_i} \\
 &= \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} (-e)^{n-c-h} c_1(i^* \mathcal{L}_{\mathbf{p}})^{n-c-h} \prod_{i \geq 1} W_{i,c}^{\beta_i}.
 \end{aligned}$$

Substituting in 2.3, we obtain

$$\left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-pe)^{n-c-h} \cdot \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} W_{i,c}^{\beta_i} \right) c_1(i^* \mathcal{L}_{\mathbf{p}})^{n-c}.$$

Therefore $\deg_{\mathcal{H}}(\mathbb{P}(E_X))$ is

$$\left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-pe)^{n-c-h} \cdot \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} W_{i,c}^{\beta_i} \right) \int_{X_{p^0}} c_1(i^* \mathcal{L}_{\mathbf{p}})^{n-c}$$

Now since $X_{p^0} = H_{1,p} \cap \dots \cap H_{c,p}$ where $H_{1,p}, \dots, H_{c,p}$ belong to $|\mathcal{L}_{\mathbf{p}}^e|$, we have

$$\int_{X_{p^0}} c_1(i^* \mathcal{L}_{\mathbf{p}})^{n-c} = e^c \int_{A_{p^0}} c_1(\mathcal{L}_{\mathbf{p}})^n = e^c (\mathcal{L}_{\mathbf{p}}^n)$$

and

$$\deg_{\mathcal{H}}(\mathbb{P}(E_X)) = \left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-p)^{n-c-h} e^{n-h} \cdot \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} W_{i,c}^{\beta_i} \right) (\mathcal{L}_{\mathbf{p}}^n).$$

Now Bezout's theorem in Fulton's form (cf. [Ful97], p. 148) says that the number of irreducible components in the intersection of two projective varieties of degrees d_1, d_2 cannot exceed $d_1 d_2$. In particular

$$\#(\text{Tor}^p(A(\overline{K})) \cap X(\overline{K})) \leq p^{2n} \left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-p)^{n-c-h} e^{n-h} \cdot \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} W_{i,c}^{\beta_i} \right) (\mathcal{L}_{\mathbf{p}}^n)^2.$$

Notice that $(\mathcal{L}_{\mathbf{p}}^n) = (L^n)$, by the same reasoning done after Theorem 2.5.1. To conclude we need to compute $\deg_{L|_X} \Omega_X$. We have seen that

$$c_1(\Omega_X) = -c_1(T_X) = -c_1(L|_X^e) W_{1,c}$$

and $W_{1,c} = -c$, so $c_1(\Omega_X) = cec_1(L|_X)$ and

$$\begin{aligned} \deg_{L|_X} \Omega_X &= \int_X c_1(\Omega_X)c_1(L|_X)^{n-c-1} = ce \int_X c_1(L|_X)^{n-c} \\ &= ce \int_A c_1(L)^{n-c}c_1(L^e)^c \\ &= ce^{c+1} \int_A c_1(L)^n \\ &= ce^{c+1}(L^n) \end{aligned}$$

□

Remark 2.7.1. 1. We would like to point out that in the case of a simple abelian variety, Theorem 2.7.1 can be refined substituting “e be sufficiently big” with “e be strictly bigger than n” (cf. Thm 7, [Deb05]). Analogously, in the case of an abelian variety of dimension 4, we can substitute “e be sufficiently big” with “e strictly bigger than 4” (cf. Thm 9, [Deb05]).

2. The class of subvarieties with ample cotangent bundle provided by Debarre is bigger than the one we have considered, i.e. one can take any intersection $X := H_1 \cap H_2 \cap \dots \cap H_c$ where $H_i \in |L_i^{e_i}|$ for some very ample line bundle L_i and some e_i big enough. It is easy to make our proof work in the case $L_1 = L_2 = \dots = L_c$ and e_1, e_2, \dots, e_c are arbitrary (and big enough): we have

$$c_j(N) = \sum_{1 \leq i_1 < \dots < i_j \leq c} e_{i_1} \cdots e_{i_j} c_1(i^* \mathcal{L}_p)^j$$

which implies

$$c_i(T_{X_{p,0}}) = c_1(i^* \mathcal{L}_p)^i Z_{i,c,\underline{e}} \quad (2.4)$$

where $\underline{e} = (e_1, \dots, e_c)$ and

$$Z_{i,c,\underline{e}} := \sum_{\underline{\beta} \in C^i} M_{\underline{\beta}} \prod_{j \geq 1} \left(\sum_{1 \leq i_1 < \dots < i_j \leq c} e_{i_1} \cdots e_{i_j} \right)^{\beta_j}.$$

Therefore $\deg_{\mathcal{H}}(\mathbb{P}(E_X))$ is

$$\left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-p)^{n-c-h} \cdot \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} Z_{i,c,\underline{e}}^{\beta_i} \right) \int_{X_{p,0}} c_1(i^* \mathcal{L}_p)^{n-c}.$$

Since

$$\begin{aligned} \int_{X_{\mathfrak{p}_0}} c_1(i^* \mathcal{L}_{\mathfrak{p}})^{n-c} &= \int_{A_{\mathfrak{p}_0}} c_1(\mathcal{L}_{\mathfrak{p}})^{n-c} c_1(\mathcal{L}_{\mathfrak{p}}^{e_1}) \cdots c_1(\mathcal{L}_{\mathfrak{p}}^{e_c}) \\ &= \left(\prod_{i=1}^c e_i \right) \int_{A_{\mathfrak{p}_0}} c_1(\mathcal{L}_{\mathfrak{p}})^n = \left(\prod_{i=1}^c e_i \right) (\mathcal{L}_{\mathfrak{p}}^n) \end{aligned}$$

we get the following bound for the cardinality of $\text{Tor}^p(A(\overline{K})) \cap X(\overline{K})$:

$$p^{2n} \left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-p)^{n-c-h} \cdot \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} Z_{i,c,\underline{e}}^{\beta_i} \right) \left(\prod_{i=1}^c e_i \right) (\mathcal{L}_{\mathfrak{p}}^n)^2.$$

This bound holds for $p > (n-c)^2 \deg_{L|X} \Omega_X$ such that there exists $\mathfrak{p} \in V$ above p . We compute

$$c_1(\Omega_X) = -c_1(T_X) = -c_1(L|_X) Z_{1,c,\underline{e}}$$

and $Z_{1,c,\underline{e}} = -\sum_{i=1}^c e_i$, so

$$\begin{aligned} \deg_{L|X} \Omega_X &= \int_X c_1(\Omega_X) c_1(L|_X)^{n-c-1} \\ &= \left(\sum_{i=1}^c e_i \right) \int_X c_1(L|_X)^{n-c} \\ &= \left(\sum_{i=1}^c e_i \right) \left(\prod_{i=1}^c e_i \right) (L^n). \end{aligned}$$

3. One could be tempted to generalize Theorem 2.7.1 also to the case in which H_1, \dots, H_c belong to different $|L_1^{e_1}|, \dots, |L_c^{e_c}|$. In this situation one would have

$$c_t(N) = \prod_{i=1}^c \left(1 + c_1(i^* \mathcal{L}_{i,\mathfrak{p}}^{e_i}) t \right)$$

and

$$c_j(N) = \sum_{1 \leq i_1 < \dots < i_j \leq c} \prod_{k=i_1}^{i_j} c_1(i^* \mathcal{L}_{k,\mathfrak{p}}^{e_k}).$$

Therefore

$$\begin{aligned} c_i(T_{X_{\mathfrak{p}_0}}) &= \sum_{\underline{\beta} \in C^i} M_{\underline{\beta}} \prod_{j \geq 1} c_j(N)^{\beta_j} \\ &= \sum_{\underline{\beta} \in C^i} M_{\underline{\beta}} \prod_{j \geq 1} \left(\sum_{1 \leq i_1 < \dots < i_j \leq c} \prod_{k=i_1}^{i_j} c_1(i^* \mathcal{L}_{k,\mathfrak{p}}^{e_k}) \right)^{\beta_j}. \end{aligned}$$

This expression for $c_i(T_{X_{p,0}})$ is more complex than the corresponding one in (2.4) in the case $L_1 = L_2 = \dots = L_c$. For this reason, a nice formula for the cardinality of $\text{Tor}^p(A(\overline{K})) \cap X(\overline{K})$ does not seem easily attainable in this general context.

2.8 Appendix

The following fact is well-known, but we were not able to find a published proof of it, so we give one here.

Fact 2.8.1. Let A be an abelian variety over an algebraically closed field k and let X be a closed integral subscheme, smooth over k . If $\text{Stab}_A(X)$ is trivial, then $H^0(X, \Omega_{X/k}^\vee) = 0$.

Proof. Let $k(\varepsilon) := k[\varepsilon]/(\varepsilon^2)$ be the ring of dual numbers and let $A_{k(\varepsilon)}$ (resp. $X_{k(\varepsilon)}$) be the base change of A (resp. X) to $k(\varepsilon)$. Notice that the base change of $A_{k(\varepsilon)}$ (resp. $X_{k(\varepsilon)}$) to k gives A (resp. X). This means that we have two closed immersions $j_A : A \hookrightarrow A_{k(\varepsilon)}$ and $j_X : X \hookrightarrow X_{k(\varepsilon)}$. As usual, we shall say that $h : A_{k(\varepsilon)} \rightarrow A_{k(\varepsilon)}$ (resp. $f : X_{k(\varepsilon)} \rightarrow X_{k(\varepsilon)}$) is a deformation of the identity on A (resp. on X) if

$$h_0 := h \times \text{Id}_k : A_{k(\varepsilon)} \times_{k(\varepsilon)} k \rightarrow A_{k(\varepsilon)} \times_{k(\varepsilon)} k$$

$$\text{(resp. } f_0 := f \times \text{Id}_k : X_{k(\varepsilon)} \times_{k(\varepsilon)} k \rightarrow X_{k(\varepsilon)} \times_{k(\varepsilon)} k)$$

is the identity on A (resp. on X). The set of deformations of the identity on A (resp. on X) is nonempty, since it contains the identity on $A_{k(\varepsilon)}$ (resp. on $X_{k(\varepsilon)}$). Theorem 8.5.9 in [FIK⁺] tells us that the deformations of the identity on A form an affine space under

$$H^0\left(A, \varepsilon \mathcal{O}_{A_{k(\varepsilon)}} \otimes_{\mathcal{O}_A} j_A^* \Omega_{A_{k(\varepsilon)}/k(\varepsilon)}^\vee\right).$$

Here the sheaf of ideals $\varepsilon \mathcal{O}_{A_{k(\varepsilon)}} \subseteq \mathcal{O}_{A_{k(\varepsilon)}}$ is seen as a \mathcal{O}_A -module thanks to the fact $\varepsilon^2 = 0$. We have

$$\begin{aligned} \varepsilon \mathcal{O}_{A_{k(\varepsilon)}} \otimes_{\mathcal{O}_A} j_A^* \Omega_{A_{k(\varepsilon)}/k(\varepsilon)}^\vee &\simeq \varepsilon \mathcal{O}_{A_{k(\varepsilon)}} \otimes_{\mathcal{O}_A} \left(\mathcal{O}_A \otimes_{\mathcal{O}_{A_{k(\varepsilon)}}} \Omega_{A_{k(\varepsilon)}/k(\varepsilon)}^\vee \right) \\ &\simeq \varepsilon \mathcal{O}_{A_{k(\varepsilon)}} \otimes_{\mathcal{O}_{A_{k(\varepsilon)}}} \Omega_{A_{k(\varepsilon)}/k(\varepsilon)}^\vee. \end{aligned}$$

Now notice that the ideal $(\varepsilon) \subseteq k(\varepsilon)$ is isomorphic to k as a $k(\varepsilon)$ -algebra. Tensoring with $\mathcal{O}_{A_{k(\varepsilon)}}$, we obtain that $\varepsilon \mathcal{O}_{A_{k(\varepsilon)}} \simeq \mathcal{O}_A$ as $\mathcal{O}_{A_{k(\varepsilon)}}$ -algebras. This implies

$$\varepsilon \mathcal{O}_{A_{k(\varepsilon)}} \otimes_{\mathcal{O}_A} j_A^* \Omega_{A_{k(\varepsilon)}/k(\varepsilon)}^\vee \simeq \mathcal{O}_A \otimes_{\mathcal{O}_{A_{k(\varepsilon)}}} \Omega_{A_{k(\varepsilon)}/k(\varepsilon)}^\vee \simeq \Omega_{A/k}^\vee.$$

Therefore, the deformations of the identity on A form an affine space under $H^0\left(A, \Omega_{A/k}^\vee\right)$. Similarly, the deformations of the identity on X form an affine space under $H^0\left(X, \Omega_{X/k}^\vee\right)$. Let us consider now the exact sequence

$$0 \rightarrow \Omega_{X/k}^\vee \rightarrow i^* \Omega_{A/k}^\vee \rightarrow N \rightarrow 0$$

where $i : X \hookrightarrow A$ is the closed immersion and N is the normal bundle to X in A . On the global sections we obtain the exact sequence

$$0 \rightarrow H^0\left(X, \Omega_{X/k}^\vee\right) \rightarrow H^0\left(X, i^* \Omega_{A/k}^\vee\right) \rightarrow H^0(X, N).$$

Since $\Omega_{A/k}^\vee \simeq \mathcal{O}_A^{\oplus \dim A}$, we have that $i^* \Omega_{A/k}^\vee \simeq \mathcal{O}_X^{\oplus \dim A}$. This and the irreducibility of X imply that we have an isomorphism

$$H^0\left(X, i^* \Omega_{A/k}^\vee\right) \simeq H^0\left(A, \Omega_{A/k}^\vee\right).$$

We deduce that the set of deformations of the identity on X is a subset of the set of deformations of the identity on A : a deformation of the identity on A comes from a deformation of the identity on X if and only if it corresponds to a vector field of A tangent to X , i.e. there is a bijection between the set of deformations of the identity on A coming from deformations of the identity on X and $H^0\left(X, \Omega_{X/k}^\vee\right)$.

Now let us consider the closed subscheme $\text{Stab}_A(X)$. It represents the functor that associates to any S scheme over k the set

$$\{x \in A(S) \mid X_S + x = X_S\}$$

where X_S stands for $X \times_k S$ and $+x$ is the translation by x on $A_S = A \times_k S$. In particular, if we put $S = \text{Spec}(k(\varepsilon))$ we have

$$\text{Stab}_A(X)(k(\varepsilon)) = \{x \in A(k(\varepsilon)) \mid X_{k(\varepsilon)} + x = X_{k(\varepsilon)}\}.$$

We write $\text{Stab}_A(X)(k(\varepsilon))_0$ for the set of points in $\{x \in A(k(\varepsilon)) \mid X_{k(\varepsilon)} + x = X_{k(\varepsilon)}\}$ which reduce to the zero of A . This is a subset of the tangent space to A in zero

$$\text{Tang}_0(A) = \{x \in A(k(\varepsilon)) \mid x \text{ reduces to zero}\}.$$

Since $\Omega_{A/k}^\vee \simeq \mathcal{O}_A^{\dim A}$, we have an isomorphism $\text{Tang}_0(A) \simeq H^0\left(A, \Omega_{A/k}^\vee\right)$ and hence

$$\text{Stab}_A(X)(k(\varepsilon))_0 \subseteq H^0\left(A, \Omega_{A/k}^\vee\right).$$

Identifying $H^0\left(A, \Omega_{A/k}^\vee\right)$ with the set of deformations of the identity on A , the inclusion above is given by

$$x \mapsto +x : A_{k(\varepsilon)} \rightarrow A_{k(\varepsilon)}.$$

It is clear then that $\text{Stab}_A(X)(k(\varepsilon))_0$ corresponds exactly to the set of deformations of the identity on A coming from deformations of the identity on X . By our hypothesis $\text{Stab}_A(X)$ is trivial, therefore $\text{Stab}_A(X)(k(\varepsilon))_0 = 0$. Equivalently

$$H^0\left(X, \Omega_{X/k}^\vee\right) = 0.$$

□

Chapter 3

Polylogarithm on abelian schemes and Deligne-Beilinson cohomology

Abstract

We use Burgos' theory of arithmetic Chow groups to exhibit a realization of the degree zero part of the polylogarithm on abelian schemes in Deligne-Beilinson cohomology.

3.1 Introduction

3.1.1 The degree zero part of the motivic polylogarithm

In [KR14], G. Kings and D. Rössler have given a simple axiomatic description of the degree zero part of the polylogarithm on abelian schemes. We briefly recall it here.

In [Sou85], C. Soulé has defined motivic cohomology for any variety V over a field

$$H_{\mathcal{M}}^i(V, j) := \mathrm{Gr}_{\gamma}^j K_{2j-i}(V) \otimes \mathbb{Q}.$$

Now let $\pi : \mathcal{A} \rightarrow S$ be an abelian scheme of relative dimension g , let $\varepsilon : S \rightarrow \mathcal{A}$ be the zero section, let $N > 1$ be an integer and let $\mathcal{A}[N]$ be the finite group scheme of N -torsion points. Here S is smooth over a subfield k of the complex numbers. For any integer $a > 1$ and any $W \subseteq \mathcal{A}$ open sub-scheme such that

$$j : [a]^{-1}(W) \hookrightarrow W$$

is an open immersion (here $[a] : \mathcal{A} \rightarrow \mathcal{A}$ is the a -multiplication on \mathcal{A}), the trace map with respect to a is defined as

$$\mathrm{tr}_{[a]} : H_{\mathcal{M}}(W, *) \xrightarrow{j^*} H_{\mathcal{M}}([a]^{-1}(W), *) \xrightarrow{[a]_*} H_{\mathcal{M}}(W, *). \quad (3.1)$$

For any integer r we let

$$H_{\mathcal{M}}(W, *)^{(r)} := \left\{ \psi \in H_{\mathcal{M}}(W, *) \mid (\mathrm{tr}_{[a]} - a^r \mathrm{Id})^k \psi = 0 \text{ for some } k \geq 1 \right\}$$

be the generalized eigenspace of $\mathrm{tr}_{[a]}$ of weight r .

Then the zero step of the motivic polylogarithm is a class in motivic cohomology

$$\mathrm{pol}^0 \in H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)}.$$

To describe it more precisely, consider the residue map along $\mathcal{A}[N]$

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g) \rightarrow H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \varepsilon(S), 0).$$

This map induces an isomorphism

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)} \cong H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \varepsilon(S), 0)^{(0)}$$

(see Corollary 2.2.2 in [KR14]) and pol^0 is the unique element mapping to the fundamental class of $\mathcal{A}[N] \setminus \varepsilon(S)$.

Now let us consider the map $\mathrm{cyc}_{\mathrm{an}}$ defined as the composition

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g) \xrightarrow{\mathrm{cyc}} \mathbf{H}_D^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)) \xrightarrow{\mathrm{forgetful}} \mathbf{H}_{D,\mathrm{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)) \quad (3.2)$$

where cyc is the regulator map into Deligne-Beilinson cohomology and the second map is the forgetful map from Deligne-Beilinson cohomology to analytic real Deligne cohomology (see the end of section 2 for the notations used here).

In [MR], V. Maillot and D. Rössler constructed a canonical class of currents $\mathfrak{g}_{\mathcal{A}^\vee}$ on \mathcal{A} (cf. Theorem 3.5.1 in Section 5) which gives rise to a class in analytic Deligne cohomology

$$([N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]} \in \mathbf{H}_{D,\mathrm{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)).$$

This element is represented by $\frac{\gamma}{(2\pi i)^{1-g}}$, where γ is any smooth form on $\mathcal{A} \setminus \mathcal{A}[N]$ in the class $[N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee}$.

The main result in [KR14] is the following.

Theorem 3.1.1. *We have*

$$-2 \cdot \text{cyc}_{\text{an}}(\text{pol}^0) = ([N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]}.$$

Furthermore the map

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)} \rightarrow H_{D, \text{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g))$$

induced by cyc_{an} is injective.

3.1.2 Our main result

In this paper we give a refinement of Theorem 3.1.1 supposing S is proper over k (see Corollary 3.6.1 and Theorem 3.6.2).

Before stating our result, we recall that in [Bur97], Burgos introduced a complex that naturally computes the Deligne-Beilinson cohomology: this is the complex $E_{\log}^*(\cdot)$ of smooth forms with logarithmic singularities along infinity (see section 3.1 for its definition).

Theorem 3.1.2. *Let S be proper over k . The class of currents $[N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee}$ has a representative which is smooth on $\mathcal{A} \setminus \mathcal{A}[N]$ and has logarithmic singularities along infinity. Any such η defines an element*

$$\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} \in \text{Im} \left(\text{cyc} : H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g) \rightarrow H_D^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)) \right)$$

which does not depend on the choice of η . This element verifies

$$-2 \cdot \text{cyc}(\text{pol}^0) = \frac{\tilde{\eta}}{(2\pi i)^{1-g}}.$$

3.1.3 An outline of the chapter

Let us now give an outline of the contents of each section.

In section 2 we review some notations and definitions coming from Arakelov theory.

In section 3 we recall Burgos' theory of arithmetic Chow groups.

Sections 4, 5 and 6 contain the proof of Theorem 3.1.2. This proof combines two arguments. On one hand we use Burgos' theory in order to prove an interesting intermediate result which relates the classical arithmetic Chow groups to Deligne-Beilinson cohomology (see Theorem 3.4.1). On the other hand we do some calculations using the current $\mathfrak{g}_{\mathcal{A}^\vee}$ in order to prove that the class of

$$T := [N]^*(\varepsilon(S), \mathfrak{g}_{\mathcal{A}^\vee}) - N^{2g}(\varepsilon(S), \mathfrak{g}_{\mathcal{A}^\vee})$$

in $\widehat{\text{CH}}^g(\mathcal{A})_{\mathbb{Q}}$ is zero (cf. Proposition 3.5.2). This proposition will allow us to apply Theorem 3.4.1 to our case (see section 6).

3.2 Notations

We begin with a review of some notations and definitions coming from Arakelov theory (see Sections 1, 2, 3 in [GS90] for a compendium).

Let (R, Σ, F_{∞}) be an arithmetic ring i.e.

- R is an excellent regular Noetherian integral domain,
- Σ is a finite nonempty set of monomorphisms $\sigma : R \rightarrow \mathbb{C}$,
- F_{∞} is an anti-linear involution of the \mathbb{C} -algebra $\mathbb{C}^{\Sigma} := \underbrace{\mathbb{C} \times \dots \times \mathbb{C}}_{|\Sigma|}$, such that

the diagram

$$\begin{array}{ccc} R & \xrightarrow{\delta} & \mathbb{C}^{\Sigma} \\ \text{Id} \downarrow & & \downarrow F_{\infty} \\ R & \xrightarrow{\delta} & \mathbb{C}^{\Sigma} \end{array}$$

commutes (here by δ we mean the natural map to the product induced by the family of maps Σ).

Let X be an arithmetic variety over R , i.e. a scheme of finite type over R , which is flat, quasi-projective and regular. As usual we write

$$X(\mathbb{C}) := \coprod_{\sigma \in \Sigma} (X \times_{R, \sigma} \mathbb{C})(\mathbb{C}).$$

We notice that F_{∞} induces an involution $F_{\infty} : X(\mathbb{C}) \rightarrow X(\mathbb{C})$. Let $p, q \in \mathbb{N}$ and y be any cycle in $Z^p(X)$ with $Y := \text{supp } y$. We denote by:

- $D^{p,p}(X_{\mathbb{R}})$ the \mathbb{R} -vector space of real currents ζ on $X(\mathbb{C})$ of type (p, p) such that $F_{\infty}^* \zeta = (-1)^p \zeta$,
- $\tilde{D}^{p,p}(X_{\mathbb{R}})$ the quotient $D^{p,p}(X_{\mathbb{R}}) / (\text{Im } \partial + \text{Im } \bar{\partial})$,
- $E^{p,p}(X_{\mathbb{R}})$ the \mathbb{R} -vector space of smooth real forms ω on $X(\mathbb{C})$ of type (p, p) such that $F_{\infty}^* \omega = (-1)^p \omega$,

- $\tilde{E}^{p,p}(X_{\mathbb{R}})$ the quotient $E^{p,p}(X_{\mathbb{R}})/(\text{Im}\partial + \text{Im}\bar{\partial})$,
- $\text{CH}^q(X)$ the q -th ordinary Chow group of X ,
- $\widehat{\text{CH}}^q(X)$ the q -th arithmetic Chow group of X .

We also fix the following notations for the analytic real Deligne cohomology and the Deligne-Beilinson cohomology (see [EV] for the definitions):

- $H_{D,\text{an}}^q(X(\mathbb{C}), \mathbb{R}(p))$ the q -th analytic real Deligne cohomology \mathbb{R} -vector space,
- $H_{D,\text{an}}^q(X_{\mathbb{R}}, \mathbb{R}(p))$ the set $\{\gamma \in H_{D,\text{an}}^q(X(\mathbb{C}), \mathbb{R}(p)) \mid F_{\infty}^* \gamma = (-1)^p \gamma\}$,
- $H_D^q(X(\mathbb{C}), \mathbb{R}(p))$ the q -th Deligne-Beilinson cohomology \mathbb{R} -vector space,
- $H_{D,Y}^q(X(\mathbb{C}), \mathbb{R}(p))$ the q -th Deligne-Beilinson cohomology \mathbb{R} -vector space with support in Y ,
- $H_D^q(X_{\mathbb{R}}, \mathbb{R}(p))$ the set $\{\gamma \in H_D^q(X(\mathbb{C}), \mathbb{R}(p)) \mid F_{\infty}^* \gamma = (-1)^p \gamma\}$,
- $H_{D,Y}^q(X_{\mathbb{R}}, \mathbb{R}(p))$ the set $\{\gamma \in H_{D,Y}^q(X(\mathbb{C}), \mathbb{R}(p)) \mid F_{\infty}^* \gamma = (-1)^p \gamma\}$.

3.3 Burgos' arithmetic Chow groups

For the convenience of the reader we recall in this section some definitions and basic facts of Burgos' arithmetic Chow groups (cf. [Bur97] for more details).

3.3.1 Smooth forms with logarithmic singularities along infinity

Let us start with the definition of smooth differential forms with logarithmic singularities along infinity.

Let V be a smooth algebraic variety over \mathbb{C} and let D be a divisor with normal crossings on V . Let us write $W = V \setminus D$ and let $j : W \hookrightarrow V$ be the inclusion. Let \mathcal{E}_V^* be the sheaf of complex C^∞ differential forms on V and let $E^*(V)$ denote $\Gamma(V, \mathcal{E}_V^*)$. The complex of sheaves $\mathcal{E}_V^*(\log D)$ is the sub- \mathcal{E}_V^* algebra of $j_* \mathcal{E}_W^*$ generated locally by the sections

$$\log z_i \bar{z}_i, \frac{dz_i}{z_i}, \frac{d\bar{z}_i}{\bar{z}_i}, \quad \text{for } i = 1, \dots, M,$$

where $z_1 \cdots z_M = 0$ is a local equation of D .

Let us write $E_V^*(\log D) = \Gamma(V, \mathcal{E}_V^*(\log D))$ and let $E_{V,\mathbb{R}}^*(\log D)$ be the subcomplex of real forms.

Let I be the category of all smooth compactifications of V . That is, an element $(\tilde{V}_\alpha, i_\alpha)$ of I is a smooth complex variety \tilde{V}_α and an immersion $i_\alpha : V \hookrightarrow \tilde{V}_\alpha$ such that $D_\alpha = \tilde{V}_\alpha - i_\alpha(V)$ is a normal crossing divisor. The morphisms of I are the maps $f : \tilde{V}_\alpha \rightarrow \tilde{V}_\beta$ such that $f \circ i_\alpha = i_\beta$. The opposed category I° is directed (see [Del71]). The complex of smooth differential forms with logarithmic singularities along infinity $E_{\log}^*(V)$ is defined as

$$E_{\log}^*(V) = \varinjlim_{\alpha \in I^\circ} E_{\tilde{V}_\alpha}^*(\log D_\alpha)$$

This complex is a subcomplex of $E^*(V)$ and we shall denote by $E_{\log, \mathbb{R}}^*(V)$ the corresponding real subcomplex.

The complex $E_{\log}^*(V)$ has a natural bigrading

$$E_{\log}^*(V) = \bigoplus E_{\log}^{p,q}(V).$$

The Hodge filtration of this complex is defined by

$$F^p E_{\log}^n(V) = \bigoplus_{\substack{p' \geq p \\ p' + q' = n}} E_{\log}^{p',q'}(V).$$

We write $E_{\log, \mathbb{R}}^*(V, p) = (2\pi i)^p E_{\log, \mathbb{R}}^*(V) \subseteq E_{\log}^*(V)$.

An important property of the complex $E_{\log}^*(V)$ is that it is strictly related to Deligne-Beilinson cohomology. More precisely, let us consider the following complex

$$E_{\log, \mathbb{R}}^*(V, p)_D := s(u : E_{\log, \mathbb{R}}^*(V, p) \oplus F^p E_{\log}^*(V) \rightarrow E_{\log}^*(V))$$

where $u(a, b) = b - a$ and $s(u)$ stands for the simple complex of the map u . Then

$$H_D^*(V, \mathbb{R}(p)) = H^*(E_{\log, \mathbb{R}}^*(V, p)_D).$$

3.3.2 Definition of Burgos' arithmetic Chow groups

Let X/R be an arithmetic variety and fix $p \in \mathbb{N}^*$. Any cycle $y \in Z^p(X)$ defines a class in

$$\rho(y) \in H_{D,Y}^{2p}(X_{\mathbb{R}}, \mathbb{R}(p))$$

where $Y = \text{supp } y$. Any $g \in E_{\log, \mathbb{R}}^{p-1, p-1}(X(\mathbb{C}) \setminus Y(\mathbb{C}), p-1)$ with $\partial \bar{\partial} g \in E^{2p}(X(\mathbb{C}))$, also defines a class

$$\{-2\partial \bar{\partial} g, g\} \in H_{D,Y}^{2p}(X, \mathbb{R}(p)).$$

The space of Green forms associated with y is then

$$\text{GE}_y^p(X_{\mathbb{R}}) := \left\{ g \in E_{\log, \mathbb{R}}^{p-1, p-1}(X(\mathbb{C}) \setminus Y(\mathbb{C}), p-1) \left| \begin{array}{l} -2\partial \bar{\partial} g \in E^{2p}(X(\mathbb{C})) \\ \{-2\partial \bar{\partial} g, g\} = \rho(y) \\ F_\infty^* g = \bar{g} \end{array} \right. \right\} / (\text{Im } \partial + \text{Im } \bar{\partial}).$$

The group of arithmetic cycles in the sense of Burgos is

$$\hat{Z}^p(X) := \{(y, \tilde{g}) \mid y \in Z^p(X) \text{ and } \tilde{g} \in \text{GE}_y^p(X_{\mathbb{R}})\}.$$

If W is a codimension $p - 1$ irreducible subvariety of X and $f \in k(W)^*$, we have a well-defined subvariety $W(\mathbb{C})$ of $X(\mathbb{C})$ and a well-defined function $f_{\mathbb{C}} \in k(W(\mathbb{C}))^*$. To $f_{\mathbb{C}}$ is associated a class

$$\rho(f_{\mathbb{C}}) \in \mathbb{H}_D^{2p-1}((X \setminus F)_{\mathbb{R}}, \mathbb{R}(p))$$

where $F = \text{supp div } f$. Since $\mathbb{H}_D^{2p-1}((X \setminus F)_{\mathbb{R}}, \mathbb{R}(p))$ is the same as

$$\left\{ g \in E_{\log, \mathbb{R}}^{p-1, p-1}(X(\mathbb{C}) \setminus F(\mathbb{C}), p-1) \mid \partial \bar{\partial} g = 0 \text{ and } F_{\infty}^* g = \bar{g} \right\} / (\text{Im } \partial + \text{Im } \bar{\partial}),$$

then one can check that $\rho(f_{\mathbb{C}})$ defines an element in $\text{GE}_{\text{div } f}^p(X_{\mathbb{R}})$, which we denote by $\text{b}(\rho(f_{\mathbb{C}}))$. Let $\widehat{\text{Rat}}^p(X)$ be the subgroup of $\hat{Z}^p(X)$ generated by the elements of the form

$$\widehat{\text{div}} f = (\text{div } f, -\text{b}(\rho(f_{\mathbb{C}}))).$$

The arithmetic Chow group $\widehat{\text{CH}}_{\log}^p(X)$ in the sense of Burgos is

$$\widehat{\text{CH}}_{\log}^p(X) := \hat{Z}^p(X) / \widehat{\text{Rat}}^p(X).$$

The class of an element $(y, \tilde{g}) \in \hat{Z}^p(X)$ will be denoted by $[y, \tilde{g}]$.

3.3.3 Two important properties

Burgos' arithmetic Chow groups fit in the following exact sequence

$$\text{CH}^{p, p-1}(X) \xrightarrow{\rho} \mathbb{H}_D^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(p)) \xrightarrow{a} \widehat{\text{CH}}_{\log}^p(X) \xrightarrow{(\zeta, -\omega)} \text{CH}^p(X) \oplus \text{ZE}_{\log}^{p, p}(X_{\mathbb{R}}) \quad (3.3)$$

where $\text{CH}^{p, p-1}$ is Gillet-Soulé's version of one of Bloch's higher Chow groups, ρ is defined in the proof of Corollary 6.3 in [Bur97], the map a sends the class of \tilde{f} to $[0, \tilde{f}]$ and $(\zeta, -\omega)([y, \tilde{g}]) = (y, 2\partial\bar{\partial}g)$. Later we will make use of the fact

$$\text{CH}^{p, p-1}(X)_{\mathbb{Q}} \cong H_{\mathcal{M}}^{2p-1}(X, p)$$

(see section 1.4 in [BGKK07] for this).

Furthermore there exists a homomorphism

$$\psi_X : \widehat{\text{CH}}_{\log}^p(X) \rightarrow \widehat{\text{CH}}^p(X)$$

which is compatible with pull-backs and is an isomorphism if X is proper over R . In particular for any $y \in Z^p(X)$, we have a commutative diagram

$$\begin{array}{ccc} \widehat{\mathrm{CH}}_{\log}^p(X) & \xrightarrow{i^*} & \widehat{\mathrm{CH}}_{\log}^p(X \setminus Y) \\ \psi_X \downarrow & & \downarrow \psi_{X \setminus Y} \\ \widehat{\mathrm{CH}}^p(X) & \xrightarrow{i^*} & \widehat{\mathrm{CH}}^p(X \setminus Y) \end{array}$$

where $Y = \mathrm{supp} \, y$ and the map i is the immersion $X \setminus Y \hookrightarrow X$.

3.4 An intermediate result

Theorem 3.4.1. *Let X/R be a proper arithmetic variety, y any cycle in $Z^p(X)$ and h any Green current for y . Then there exists a representative h_0 of h belonging to $E_{\log, \mathbb{R}}^{p-1, p-1}(X(\mathbb{C}) \setminus Y(\mathbb{C}))$, where $Y = \mathrm{supp} \, y$. If $\omega([y, h]) := \delta_y + \mathrm{dd}^c h = 0$, then $\frac{h_0}{2(2\pi i)^{1-p}}$ defines a class*

$$\frac{\widetilde{h}_0}{2(2\pi i)^{1-p}} \in H_D^{2p-1}((X \setminus Y)_{\mathbb{R}}, \mathbb{R}(p))$$

which does not depend on the choice of h_0 and verifies

$$a \left(\frac{\widetilde{h}_0}{2(2\pi i)^{1-p}} \right) = i^* (\psi_X^{-1}([y, h])).$$

Proof. If we denote by $\mathrm{GC}_X(y)$ the space of Green currents for the cycle y , Theorem 5.9 in [Bur97] tells us that there is a natural isomorphism

$$\mathrm{GE}_y^p(X_{\mathbb{R}}) \rightarrow \mathrm{GC}_X(y)$$

which sends \tilde{g} to the class of the current $2(2\pi i)^{1-p}[g]$, where $[g]$ sends a form ω to

$$[g](\omega) = \int_{X(\mathbb{C})} \omega \wedge g.$$

Let $\widetilde{h}_{\log} \in \mathrm{GE}_y^p(X_{\mathbb{R}})$ be the inverse image of h by this isomorphism. Any element in $2(2\pi i)^{1-p}\widetilde{h}_{\log}$ gives a representative of h belonging to $E_{\log, \mathbb{R}}^{p-1, p-1}(X(\mathbb{C}) \setminus Y(\mathbb{C}))$.

If $\delta_y + \mathrm{dd}^c h = 0$, we deduce

$$0 = (\delta_y + \mathrm{dd}^c h)|_{X(\mathbb{C}) \setminus Y(\mathbb{C})} = (\mathrm{dd}^c h)|_{X(\mathbb{C}) \setminus Y(\mathbb{C})} = 2(2\pi i)^{1-p} \mathrm{dd}^c \left(\widetilde{h}_{\log} \right).$$

Therefore we have a well-defined element \widetilde{h}_{\log} in

$$\left\{ g \in E_{\log, \mathbb{R}}^{p-1, p-1}(X(\mathbb{C}) \setminus Y(\mathbb{C}), p-1) \left| \begin{array}{l} \partial \bar{\partial} g = 0 \\ F_{\infty}^* g = \bar{g} \end{array} \right. \right\} / (\mathrm{Im} \partial + \mathrm{Im} \bar{\partial})$$

i.e. in $H_D^{2p-1}((X \setminus Y)_{\mathbb{R}}, \mathbb{R}(p))$.

By definition of \widetilde{h}_{\log} and by definition of the pullback in Burgos' theory, we have

$$i^* (\psi_X^{-1}([y, h])) = i^* \left([y, \widetilde{h}_{\log}] \right) = [0, \widetilde{h}_{\log}] = a \left(\widetilde{h}_{\log} \right).$$

To conclude the proof, take any representative h_0 of h in $E_{\log, \mathbb{R}}^{p-1, p-1}(X(\mathbb{C}) \setminus Y(\mathbb{C}))$. We want to show that $\frac{h_0}{2(2\pi i)^{1-p}}$ defines a class $\frac{\widetilde{h}_0}{2(2\pi i)^{1-p}}$ in $H_D^{2p-1}((X \setminus Y)_{\mathbb{R}}, \mathbb{R}(p))$ and that this class is \widetilde{h}_{\log} . Since $\delta_y + \text{dd}^c h_0 = 0$, Proposition 6.5 in [BGKK07] implies that $\frac{\widetilde{h}_0}{2(2\pi i)^{1-p}}$ is an element in $\text{GE}_y^p(X_{\mathbb{R}})$ (please notice that the definitions of current associated to a cycle and of current associated to a differential form used in Burgos' paper slightly differ from the ones used here). This element must coincide with \widetilde{h}_{\log} , since both have image h in $\text{GC}_X(y)$. \square

Remark 3.4.1. Tensoring with \mathbb{Q} over \mathbb{Z} is an exact operation, so the exact sequence (3.3) shows that

$$\begin{aligned} i^* (\psi_X^{-1}([y, h])) = 0 \in \widehat{\text{CH}}_{\log}^p(X \setminus Y)_{\mathbb{Q}} &\Leftrightarrow \frac{\widetilde{h}_0}{2(2\pi i)^{1-p}} \in \rho \left(\text{CH}^{p, p-1}(X \setminus Y)_{\mathbb{Q}} \right) \\ &\Leftrightarrow \frac{\widetilde{h}_0}{2(2\pi i)^{1-p}} \in \text{cyc} \left(H_{\mathcal{M}}^{2p-1}(X \setminus Y, p) \right) \end{aligned}$$

3.5 The case of an abelian scheme over an arithmetic variety

In this section we explain in detail how the class $\mathfrak{g}_{\mathcal{A}^\vee}$ gives rise to an element

$$([N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]}$$

in $H_{D, \text{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g))$.

Let S be an arithmetic variety over R and let $\pi : \mathcal{A} \rightarrow S$ be an abelian scheme over S of relative dimension g . We shall write as usual $\mathcal{A}^\vee \rightarrow S$ for the dual abelian scheme of \mathcal{A} . We let $\varepsilon_{\mathcal{A}} = \varepsilon$ be the zero-section of $\pi : \mathcal{A} \rightarrow S$. We shall denote by $S_0 = S_{0, \mathcal{A}} = \varepsilon_{\mathcal{A}}(S)$ the reduced closed subscheme of \mathcal{A} , which is the image of $\varepsilon_{\mathcal{A}}$. We write \mathcal{P} for the Poincaré bundle on $\mathcal{A} \times_S \mathcal{A}^\vee$ and $p_1 : \mathcal{A} \times_S \mathcal{A}^\vee \rightarrow \mathcal{A}$ for the first projection. Since \mathcal{A} and S are arithmetic varieties over R , we have two well-defined complex manifolds $\mathcal{A}(\mathbb{C})$ and $S(\mathbb{C})$ and two well-defined \mathbb{R} -vector spaces $D^{g-1, g-1}(\mathcal{A}_{\mathbb{R}})$ and $E^{g-1, g-1}(\mathcal{A}_{\mathbb{R}})$. We endow the Poincaré bundle \mathcal{P} with the unique metric $h_{\mathcal{P}}$ such that the canonical rigidification of \mathcal{P} along the zero-section $\varepsilon \times \text{Id}_{\mathcal{A}^\vee} : \mathcal{A}^\vee \rightarrow \mathcal{A} \times_S \mathcal{A}^\vee$ is an isometry and such that the curvature form of $h_{\mathcal{P}}$ is

translation invariant along the fibres of the map $\mathcal{A}(\mathbb{C}) \times_{S(\mathbb{C})} \mathcal{A}^\vee(\mathbb{C}) \rightarrow \mathcal{A}^\vee(\mathbb{C})$. We write $\overline{\mathcal{P}} = (\mathcal{P}, h_{\mathcal{P}})$ for the resulting hermitian line bundle.

Theorem 3.5.1. *There is a unique class of currents $\mathfrak{g}_{\mathcal{A}^\vee} \in \tilde{D}^{g-1, g-1}(\mathcal{A}_{\mathbb{R}})$ with the following three properties:*

1. Any element of $\mathfrak{g}_{\mathcal{A}^\vee}$ is a Green current for $S_0(\mathbb{C})$.
2. The identity $(S_0, \mathfrak{g}_{\mathcal{A}^\vee}) = (-1)^g p_{1,*} \left(\widehat{\text{ch}}(\overline{\mathcal{P}}) \right)^{(g)}$ holds in $\widehat{\text{CH}}^g(\mathcal{A})_{\mathbb{Q}}$.
3. The identity $\mathfrak{g}_{\mathcal{A}^\vee} = [n]_* \mathfrak{g}_{\mathcal{A}^\vee}$ holds for all $n \geq 2$.

Here $[n] : \mathcal{A} \rightarrow \mathcal{A}$ is the multiplication-by- n morphism and $\widehat{\text{ch}}(\overline{\mathcal{P}})$ is the arithmetic Chern character of the hermitian bundle $\overline{\mathcal{P}}$.

Let us fix N a positive natural number and call

$$T := [N]^*(S_0, \mathfrak{g}_{\mathcal{A}^\vee}) - N^{2g}(S_0, \mathfrak{g}_{\mathcal{A}^\vee}).$$

We will write $[T]$ for its class in $\widehat{\text{CH}}^g(\mathcal{A})$. We denote by $\mathcal{A}[N]$ the N -torsion of \mathcal{A} , by i the immersion $\mathcal{U} := \mathcal{A} \setminus \mathcal{A}[N] \hookrightarrow \mathcal{A}$ and by Γ the restriction of the class of currents

$$[N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee}$$

to $\mathcal{U}(\mathbb{C})$. Then the morphism

$$i^* : \widehat{\text{CH}}^g(\mathcal{A}) \rightarrow \widehat{\text{CH}}^g(\mathcal{U})$$

sends $[T]$ to $[0, \Gamma]$. We recall the fundamental exact sequence

$$\text{CH}^{g, g-1}(\mathcal{U}) \xrightarrow{\rho_{\text{an}}} \tilde{E}^{g-1, g-1}(\mathcal{U}_{\mathbb{R}}) \xrightarrow{a} \widehat{\text{CH}}^g(\mathcal{U}) \longrightarrow \text{CH}^g(\mathcal{U}) \longrightarrow 0 \quad (3.4)$$

(See Theorem 3.3.5 in [GS90]) for this). Here the map a sends the class of ω to $[0, \omega]$. By construction, ρ_{an} is the following composite function

$$\text{CH}^{g, g-1}(\mathcal{U}) \xrightarrow{\rho} \text{H}_D^{2g-1}(\mathcal{U}_{\mathbb{R}}, \mathbb{R}(g)) \xrightarrow{\text{forgetful}} \text{H}_{D, \text{an}}^{2g-1}(\mathcal{U}_{\mathbb{R}}, \mathbb{R}(g)) \longrightarrow \tilde{E}^{g-1, g-1}(\mathcal{U}_{\mathbb{R}})$$

where the third map is a natural inclusion. Indeed we have

$$\text{H}_{D, \text{an}}^{2g-1}(\mathcal{U}_{\mathbb{R}}, \mathbb{R}(g)) = \{c \in (2\pi i)^{g-1} E^{g-1, g-1}(\mathcal{U}_{\mathbb{R}}) \mid \partial \bar{\partial} c = 0\} / (\text{Im} \partial + \text{Im} \bar{\partial})$$

and the class of c is sent to the class of $c/(2\pi i)^{g-1}$.

Since the image of $[0, \Gamma]$ in $\mathrm{CH}^g(\mathcal{U})$ is 0, there exists an element in $\tilde{E}^{g-1, g-1}(\mathcal{U}_{\mathbb{R}})$ sent to $[0, \Gamma]$ by a . We know how to construct such an element: thanks to Theorem 1.3.5 and Theorem 1.2.4 in [GS90], the class of currents $[N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee}$ has a representative γ in $E^{g-1, g-1}(\mathcal{U}_{\mathbb{R}})$. Now if γ' is another representative smooth on $\mathcal{U}(\mathbb{C})$, we have that $\gamma - \gamma'$ is a smooth form on $\mathcal{U}(\mathbb{C})$ and $\gamma - \gamma' = \partial c_1 + \bar{\partial} c_2$ for some currents c_1 and c_2 . Therefore there exist two smooth forms ω_1 and ω_2 such that $\gamma - \gamma' = \partial \omega_1 + \bar{\partial} \omega_2$ (see Theorem 1.2.2 (ii) in [GS90]). This implies that the class $\tilde{\gamma} \in \tilde{E}^{g-1, g-1}(\mathcal{U}_{\mathbb{R}})$ does not depend on γ . We have

$$a(\tilde{\gamma}) = [0, \tilde{\gamma}] = [0, \Gamma].$$

The calculations we do to prove the next proposition are basically the same the reader can find in Lemma 2.4.5 in [KR14].

Proposition 3.5.2. *The element $[T]$ is zero in $\widehat{\mathrm{CH}}^g(\mathcal{A})_{\mathbb{Q}}$.*

Proof. By Theorem 3.5.1, we have

$$[T] = (-1)^g ([N]^* - N^{2g}) \left(p_{1,*} \left(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}) \right)^{(g)} \right)$$

in $\widehat{\mathrm{CH}}^g(\mathcal{A})_{\mathbb{Q}}$. Now consider the following square of schemes over S :

$$\begin{array}{ccc} \mathcal{A} \times_S \mathcal{A}^\vee & \xrightarrow{N \times \mathrm{Id}} & \mathcal{A} \times_S \mathcal{A}^\vee \\ p_1 \downarrow & & \downarrow p_1 \\ \mathcal{A} & \xrightarrow{N} & \mathcal{A} \end{array}$$

For any Q scheme over S and any pair of morphisms of schemes over S

$$\begin{aligned} (\zeta, \sigma) &: Q \rightarrow \mathcal{A} \times_S \mathcal{A}^\vee \\ \eta &: Q \rightarrow \mathcal{A} \end{aligned}$$

we can define $(\eta, \sigma) : Q \rightarrow \mathcal{A} \times_S \mathcal{A}^\vee$. It is easy to see that this is a morphism of schemes over S and that, if $N \circ \eta = \zeta$, it verifies $(N \times \mathrm{Id}) \circ (\eta, \sigma) = (\zeta, \sigma)$ and $p_1 \circ (\eta, \sigma) = \eta$. Furthermore, (η, σ) is the unique morphism of schemes over S with these properties. Therefore the square above is cartesian. Since the direct image in

arithmetic Chow theory is naturally compatible with smooth base change, we have

$$\begin{aligned} (-1)^g [N]^* \left(p_{1,*} \left(\widehat{\text{ch}}(\overline{\mathcal{P}}) \right)^{(g)} \right) &= (-1)^g p_{1,*} \left[\left((N \times \text{Id})^* \widehat{\text{ch}}(\overline{\mathcal{P}}) \right)^{(2g)} \right] \\ &= (-1)^g p_{1,*} \left[\left(\widehat{\text{ch}}((N \times \text{Id})^*(\overline{\mathcal{P}})) \right)^{(2g)} \right]. \end{aligned}$$

From the definition of dual abelian scheme we know that there is an isomorphism between the group $\text{End}(\mathcal{A}, \mathcal{A})$ and the group of isomorphism classes of invertible sheaves \mathcal{L} on $\mathcal{A} \times_S \mathcal{A}^\vee$ with rigidification along $\varepsilon_{\mathcal{A}^\vee}$ such that $\mathcal{L} \otimes k(a)$ is algebraically equivalent to zero for all $a \in \mathcal{A}^\vee$. Via this isomorphism, a map $\phi : \mathcal{A} \rightarrow \mathcal{A}$ is sent to $(\phi \times \text{Id}_{\mathcal{A}^\vee})^*(\mathcal{P})$, so the image of $\text{Id}_{\mathcal{A}}$ is \mathcal{P} . Then the image of $[N]$, i.e $(N \times \text{Id})^*(\mathcal{P})$, must coincide with $\mathcal{P}^{\otimes N}$. Therefore:

$$(-1)^g [N]^* \left(p_{1,*} \left(\widehat{\text{ch}}(\overline{\mathcal{P}}) \right)^{(g)} \right) = (-1)^g p_{1,*} \left[\left(\widehat{\text{ch}}(\overline{\mathcal{P}^{\otimes N}}) \right)^{(2g)} \right]$$

and

$$\begin{aligned} [T] &= (-1)^g \left[p_{1,*} \left[\left(\widehat{\text{ch}}(\overline{\mathcal{P}^{\otimes N}}) \right)^{(2g)} \right] - N^{2g} p_{1,*} \left(\widehat{\text{ch}}(\overline{\mathcal{P}}) \right)^{(g)} \right] \\ &= (-1)^g \left[p_{1,*} \left(\frac{\widehat{c}_1(\overline{\mathcal{P}^{\otimes N}})^{2g}}{(2g)!} \right) - N^{2g} p_{1,*} \left(\frac{\widehat{c}_1(\overline{\mathcal{P}})^{2g}}{(2g)!} \right) \right] \\ &= (-1)^g \left[p_{1,*} \left(\frac{N^{2g} \widehat{c}_1(\overline{\mathcal{P}})^{2g}}{(2g)!} \right) - N^{2g} p_{1,*} \left(\frac{\widehat{c}_1(\overline{\mathcal{P}})^{2g}}{(2g)!} \right) \right] \\ &= 0 \end{aligned}$$

where $\widehat{c}_1(\cdot)$ refers to the first arithmetic Chern class of a hermitian bundle. Notice that we used the multiplicativity of $\widehat{c}_1(\cdot)$. \square

Corollary 3.5.3. *The class $(2\pi i)^{g-1} \tilde{\gamma} \in (2\pi i)^{g-1} \tilde{E}^{g-1, g-1}(\mathcal{U}_{\mathbb{R}})$ is in the image of*

$$\text{cyc}_{\text{an}} : H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g) \rightarrow H_{D, \text{an}}^{2g-1}(\mathcal{U}, \mathbb{R}(g)).$$

Proof. Proposition 3.5.2 implies that

$$0 = i^*([T]) = [0, \Gamma] = a(\tilde{\gamma})$$

in $\widehat{\text{CH}}^g(\mathcal{U})_{\mathbb{Q}}$ and the exactness of sequence (3.4) gives us $\tilde{\gamma} \in \rho_{\text{an}}(\text{CH}^{g,g-1}(\mathcal{U})_{\mathbb{Q}})$. By the definition of ρ_{an} , we obtain that

$$(2\pi i)^{g-1} \tilde{\gamma} \in (\text{forgetful} \circ \rho)(\text{CH}^{g,g-1}(\mathcal{U})_{\mathbb{Q}}).$$

This is exactly what we wanted to prove, once one identifies $H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g)$ with $\text{CH}^{g,g-1}(\mathcal{U})_{\mathbb{Q}}$. \square

3.6 The main result

We are now able to apply Theorem 3.4.1 to our situation. We assume that S is proper over R , so \mathcal{A} is proper over R .

Corollary 3.6.1. *There exists a representative of $[N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee}$ belonging to $E_{\log, \mathbb{R}}^{g-1, g-1}(\mathcal{U}(\mathbb{C}))$. Any such η defines an element*

$$\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} \in \text{Im} \left(\text{cyc} : H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g) \rightarrow H_D^{2g-1}(\mathcal{U}_{\mathbb{R}}, \mathbb{R}(g)) \right)$$

which does not depend on the choice of η and verifies

$$a \left(\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} \right) = i^* (\psi_{\mathcal{A}}^{-1}([y, h])) = 0.$$

Furthermore

$$\text{forgetful} \left(\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} \right) = \frac{\tilde{\gamma}}{2(2\pi i)^{1-g}}.$$

Proof. To prove the first assertion we apply Theorem 3.4.1 and Remark 3.4.1 with X equal to \mathcal{A} and (y, h) equal to T . The hypotheses are satisfied since $[T] = 0$ in $\widehat{\text{CH}}^g(\mathcal{A})_{\mathbb{Q}}$, so $i^* (\psi_{\mathcal{A}}^{-1}([T])) = 0$ in $\widehat{\text{CH}}_{\log}^g(\mathcal{U})_{\mathbb{Q}}$.

To prove the second assertion, it is enough to notice that any representative η of $[N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee}$ belonging to $E_{\log, \mathbb{R}}^{g-1, g-1}(\mathcal{U}(\mathbb{C}))$, also belongs to $E^{g-1, g-1}(\mathcal{U}_{\mathbb{R}})$. \square

Remark 3.6.1. A natural analogue of the operator $\text{tr}_{[a]}$ operates on Deligne-Beilinson cohomology and the map cyc intertwines this operator with $\text{tr}_{[a]}$. Therefore from the existence of the Jordan decomposition, the fact

$$\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} \in \text{Im} \left(\text{cyc} : H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g) \rightarrow H_D^{2g-1}(\mathcal{U}_{\mathbb{R}}, \mathbb{R}(g)) \right)$$

and property (3) in Theorem 3.5.1, we deduce

$$\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} \in \text{Im} \left(\text{cyc} : H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g)^{(0)} \rightarrow H_D^{2g-1}(\mathcal{U}_{\mathbb{R}}, \mathbb{R}(g)) \right).$$

We are ready to prove our main result.

Theorem 3.6.2. *Let $\text{pol}^0 \in H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g)$ be the zero step of the motivic polylogarithm on \mathcal{A} (as defined in the Introduction). Then*

$$-2 \cdot \text{cyc}(\text{pol}^0) = \frac{\tilde{\eta}}{(2\pi i)^{1-g}}$$

in $H_D^{2g-1}(\mathcal{U}_{\mathbb{R}}, \mathbb{R}(g))$.

Proof. We start noticing that

$$\text{cyc}_{\text{an}}(\text{pol}^0) = -\frac{\tilde{\gamma}}{2(2\pi i)^{1-g}} = \text{forgetful} \left(-\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} \right).$$

By Remark 3.6.1 we know that

$$-\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} = \text{cyc}(l)$$

for some $l \in H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g)^{(0)}$. Therefore we have

$$\text{cyc}_{\text{an}}(\text{pol}^0) = \text{cyc}_{\text{an}}(l)$$

and the injectivity of cyc_{an} on $H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g)^{(0)}$ implies that $\text{pol}^0 = l$. We then obtain

$$\text{cyc}(\text{pol}^0) = \text{cyc}(l) = -\frac{\tilde{\eta}}{2(2\pi i)^{1-g}}.$$

□

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Résumé. Dans cette thèse nous étudions deux problèmes dans le domaine de la géométrie arithmétique, concernant respectivement les points de torsion des variétés abéliennes et le polylogarithme motivique sur les schémas abéliens.

La conjecture de Manin-Mumford (démontrée par Raynaud en 1983) affirme que si A est une variété abélienne et X est une sous-variété de A ne contenant aucune translatee d'une sous-variété abélienne de A , alors X ne contient qu'un nombre fini de points de torsion de A . En 1996, Buium présenta une forme effective de la conjecture dans le cas des courbes. Dans cette thèse, nous montrons que l'argument de Buium peut être utilisé aussi en dimension supérieure pour prouver une version quantitative de la conjecture pour une classe de sous-variétés avec fibré cotangent ample étudiée par Debarre. Nous généralisons aussi à toute dimension un résultat sur la dispersion des relèvements p -divisibles non ramifiés obtenu par Raynaud dans le cas des courbes.

En 2014, Kings and Rössler ont montré que la réalisation en cohomologie de Deligne analytique de la part de degré zéro du polylogarithme motivique sur les schémas abéliens peut être reliée aux formes de torsion analytique de Bismut-Köhler du fibré de Poincaré. Dans cette thèse, nous utilisons la théorie de l'intersection arithmétique dans la version de Burgos pour raffiner ce résultat dans le cas où la base du schéma abélien est propre.

Abstract. In this thesis we approach two independent problems in the field of arithmetic geometry, one regarding the torsion points of abelian varieties and the other the motivic polylogarithm on abelian schemes.

The Manin-Mumford conjecture (proved by Raynaud in 1983) states that if A is an abelian variety and X is a subvariety of A not containing any translate of an abelian subvariety of A , then X can only have a finite number of points that are of finite order in A . In 1996, Buium presented an effective form of the conjecture in the case of curves. In this thesis, we show that Buium's argument can be made applicable in higher dimensions to prove a quantitative version of the conjecture for a class of subvarieties with ample cotangent studied by Debarre. Our proof also generalizes to any dimension a result on the sparsity of p -divisible unramified liftings obtained by Raynaud in the case of curves.

In 2014, Kings and Rössler showed that the realisation in analytic Deligne cohomology of the degree zero part of the motivic polylogarithm on abelian schemes can be described in terms of the Bismut-Köhler higher analytic torsion form of the Poincaré bundle. In this thesis, using the arithmetic intersection theory in the sense of Burgos, we give a refinement of Kings and Rössler's result in the case in which the base of the abelian scheme is proper.