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## Longevity and Economic Growth: three Essays

Laurent Brembilla

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# THÈSE DE DOCTORAT

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Longevity and Economic Growth: Three Essays.

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Spécialité Sciences économiques

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Qui veut s'élever au-dessus des hommes  
doit se préparer à une lutte, ne reculer  
devant aucune difficulté. Un grand  
écrivain est un martyr qui ne mourra  
pas, voilà tout.

---

Honoré de Balzac, *Illusions Perdues*



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# INTRODUCTION

Current income per capita in the US is about 15 fold that of Sub-Saharan countries. Understanding income disparities across countries has been one of the main question addressed by economists since *The Wealth of Nations* by Adam Smith (1776). The possibility for longevity to be one important factor shaping these disparities has only started to be considered in the last fifteen years. This contrasts with another demographic variable, fertility, which has played a central role in the theory of economic growth since the essays of Malthus (1807). Indeed, for most of human history, countries have been trapped in the Malthusian regime, in which the level of technological advancement is not correlated with the income level, yet with the population density : further technological progress allows individuals to have more children, which annihilates income per capita gains. The break-up of this mechanism is a necessary condition for countries to enter the modern growth regime. Weil and Galor (1999) argue that this is made possible by a change in the nature of technological progress that becomes more and more skill-demanding. This creates an incentive for individuals to have fewer children and to invest more in their education, which marks the onset of standards of living improvements.

More recent contributions underline the role of life expectancy improvement in this change of regime and as an important dimension to take into account in the economic growth theory. Boucekkine et al. (2004) advance that the exit of the Malthusian regime is not a consequence of a greater demand for human capital, as suggested by Weil and Galor (1999), but is due to a greater human capital supply caused by more favourable climatic conditions that improve life expectancy. At the heart of their explanation is the so-called Ben-Porath mechanism : as longevity increases, the time period during which individuals can benefit from their schooling investments increases, which stimulates human capital supply. To gain intuition, consider the case of individuals facing low survival chances. Oster et al. (2013) study the college attendance decision of a sample of individuals who have a parent with Huntington disease. This illness, which has a one-half probability to be transmitted to child, limits the life expectancy of an infected individual to 20 years after symptoms begin. Oster et al. (2013) show that individuals discovering their genetic mutation during high school are 30% less likely to attend college than those who do not. Furthermore, the earlier the symptoms begin, the earlier individuals stop their schooling as the Ben-Porath mechanism suggests. Note that this effect directly arises from the lifetime budget constraint of individuals : with an increasing utility function, that does not depend on schooling, the best schooling time is the one that maximizes lifetime income and it is increasing with the number of years the indivi-

dual can work. However, for agents that make their decisions by maximizing their lifetime utility function, longevity also directly enters their preferences. To see how this interacts with their decisions, consider again the case of individuals facing a limited life expectancy. Lorentzen et al. (2008) study their behavior on a sample composed of people living near the power plant of Tchernobyl during the catastrophe and individuals infected by HIV. They show that these individuals engage into more risky behaviors such as : smoking, drinking, unsafe sexual relationships. Indeed, these individuals know that they cannot enjoy utility in the future, so they dedicate all their resources, here particularly their health capital, to maximize their present utility which merges into their lifetime utility. However, this is unlikely to compensate the welfare loss due to their premature death because they still have a decreasing marginal utility. On the contrary, individuals who do not face low survival chances can get a greater lifetime welfare by choosing to channel resources in the future. Furthermore, individuals can decide to spend resources to increase their future survival chances to enjoy consumption utility for a longer period. In other words, physical capital supply is also driven by health conditions. This is a second link, first highlighted by Bloom et al. (2003) and Chakraborty (2004), through which longevity and economic growth interact. As human and physical capital are the main production factors, this helps to understand the results of Weil (2007) according to whom up to 20% of across countries income per capita dispersion can be explained by differences of adult survival rates. Thus, studying the role played by health in economics contributes to better understand the mechanics of economic development. This PhD dissertation proposes three essays on the links between health and economic development. More specifically, I will address the three following questions : (i) Can a country experience a higher economic growth rate by spending more on health ? (ii) Do health expenditures endanger economic growth ? (iii) How does the aging process affect the income level and the sectorial labor allocation of a multi-sector economy ?

In addition to the theoretical insights quoted previously, a link between life expectancy and income per capita can be sensed from the observation of the trajectories of the two variables. For most countries, they both fluctuate around a nearly constant value during most of human history before entering a positive growth regime at different dates. More precisely, consider the case of England, the first country to enter the modern growth regime. Evolution of life expectancy has paralleled that of income with some lags. Life expectancy was 39,24 in England in 1820. It was still close to this value in 1850 (40,6), while during the same period income per capita has increased by 41%.<sup>1</sup> Then, except during war periods, both life expectancy and income have been increasing.<sup>2</sup> Life expectancy is now more than the double of its value in 1820, while income has been multiplied by more than 13. The picture

---

1. Historical data on life expectancy are from de la Croix and Sommacal (2009), while those on income are from Fouquet and Broadberry (2015). Any other data used in the introduction, whose source is not mentioned, are obtained from the World bank database. Income is GDP per capita in current US dollars.

2. There are exceptions to this in developed countries. In Russia, life expectancy has decreased after the fall of the soviet union. Case and Deaton (2015) document a rising mortality in midlife among white non-Hispanic Americans during the period 1999-2014.

of developing countries is more contrasted. Overall, thanks to the eradication of smallpox, which was responsible for 2 million deaths a year until the end of 1960, health conditions have improved (Bloom et al. 2005).<sup>3</sup> This is confirmed by the world life expectancy evolution which increased from 52 in 1960 to 71 in 2013. In East Asia, life expectancy stands at 76, while it was 46,4 just after WWII. In Sub-Saharan countries, average life expectancy has increased from 51 to 58,1 over the last decade. However, in terms of income evolution, large disparities exist. In 1962, income per capita of East Asia was 4.3% that of the US, while income per capita of Subsaharan Africa was 4.1% that of the US. In 2014, income per capita of East Asia was 18% that of the US, while income per capita of Subsaharan Africa was 3% that of the US. Regarding Sub-saharan Africa, current life expectancy in the region corresponds to the one of England in 1925. Yet, income per capita at that time exceeded by more than 60% the current level in Sub-saharan countries.

These empirical facts are well captured by the Preston curve (see Figure 1). Preston (1975) estimates the cross-sectional relationship between income per capita and life expectancy. He observes a concave and increasing relationship between the two variables. The concavity of the curve is a consequence of the quasi complete convergence of some developing countries in terms of life expectancy while their convergence in terms of income is still incomplete. For example, life expectancies of US and China, respectively equal to 79 and 75, only differ by 5%, while the income per capita in US is more than sixfold the one of China. The fact that current Sub-saharan individuals live as long as European did in 1925, while their income is only 60% of the English level in 1925 is an illustration of the upward shift with time of the curve.

Understanding the origin of the Preston curve is a mirror problem of understanding the link between life expectancy and income. The curve reflects the causal impact of life expectancy on income, the reverse causality as well as the influence of other correlates. Dalgaard and Strulik (2014) argue that about 80% of the Preston curve can be attributed to the causal effect of income on life expectancy : richer countries can spend more resources on health, for example, by implementing better medical facilities and public infrastructure such as raw sewage disposals that reduce the propagation of diseases. A contrario, empirical evidence on the causal impact of life expectancy on income are contrasted, leaving 20% of the relationship between income and life expectancy unexplained. Acemoglu and Johnson (2007) and Ashraf et al. (2008) find no significant effect of life expectancy on the growth rate of income per capita, while Aghion et al. (2011) and Sunde and Strittmatter (2013) find a positive effect of life expectancy on economic growth. These results are unexpected from two perspectives. First, microeconomic studies unambiguously claim that healthier individuals get a higher income.<sup>4</sup> Second, it is unexpected from life cycle theory, as I mentioned it previously, these

---

3. As for developed countries, some countries have undergone a decrease of life expectancy particularly because of HIV epidemics. This is the case for South-Africa and Namibia. In South-Africa, life expectancy was 62 in 1992, while it was 52 in 2005. In Namibia, life expectancy was 61 in 1991, while it was 54 in 2003.

4. See among others : Duncan and Strauss (1997), Bleakley (2003) and Miguel and Kremer (2004).

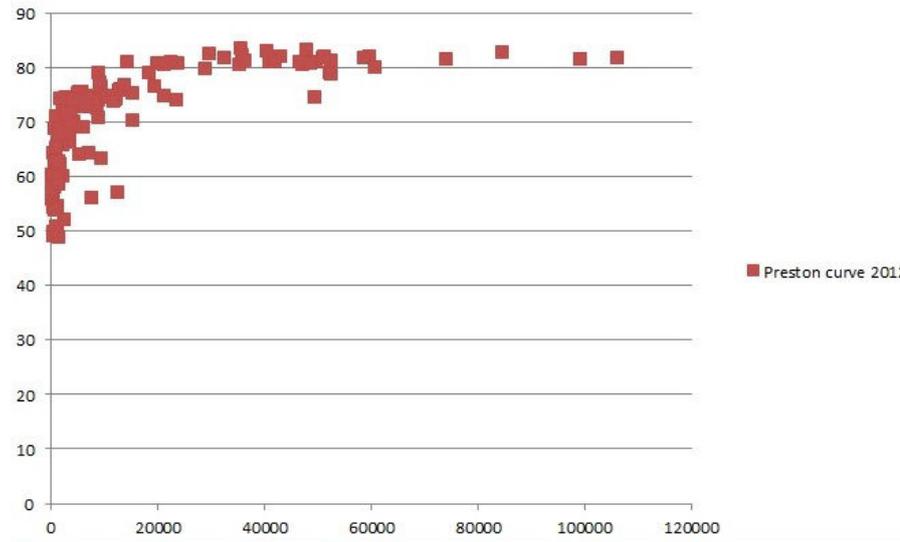


FIGURE 1 – *GDP per capita (x-axis) and life expectancy (y-axis) across countries (2012).*  
*Source : World Bank database.*

models predict that a greater longevity should stimulate both human and physical capital supply. As pointed out by Acemoglu (2010), this means that there are general equilibrium effects at stake.<sup>5</sup> Thus, in this PhD dissertation, I will study theoretically how life expectancy interferes with income from dynamic general equilibrium models. Not only does the understanding of the causal impact of life expectancy on income enables to understand how previous health improvements have contributed to shape the wealth of Nations, but it also enables to better understand how income will evolve in the future as life expectancy will pursue its upward trend. Indeed, evolution of life expectancy in developed countries, coupled with that of fertility, is such that they enter the aging phase : the ratio of workers to people aged more than 65 is increasing. Indeed, the dependency ratio, the ratio of people aged less than 20 or more than 65 to those aged between 20 and 65, currently at 68,35% in OECD countries, is expected to reach 86,7% in 2050 according to OECD.

In various dynamic general equilibrium settings, I will study the consequences on the economic outcome of an exogenous shock on the longevity parameter or I will study the possibility for the longevity to be itself an endogenous variable. There are two types of economic models that I will use extensively for the analysis. The first step, at the microeconomic level, is to derive the behavior of agents, particularly with respect to their longevity, through the life cycle hypothesis. Pioneered by Modigliani and Brumberg (1954), this theory hypothesizes that individuals allocate their lifetime resources in order to maximize their lifetime utility function. Thus, longevity naturally enters the individual decisions as opposed to the infinite horizon framework used in standard growth models. The second step consists in the aggregation of

5. d’Albis et al. (2012) and Sunde and Cervellati (2013) suggest that a life expectancy increase can be the result of an increase of survival probabilities at different ages, which can activate life-cycle mechanisms differently. For example, schooling time responds to increases in survival probabilities in adulthood and not to survival probabilities in retirement period.

the behavior of agents differing with respect to their birth dates to determine the equilibrium of the economy. This is precisely what overlapping generations (henceforth OLG) models allow to do. Pioneered by Allais (1947), Samuelson (1958) and Diamond (1965), OLG models are microfounded macroeconomic models in which agents may differ with respect to their age. They represent a natural tool to introduce demographic variables into macroeconomic studies. Blanchard (1985) is the first who explicitly introduced longevity as an exogenous variable in an OLG model using a version of the life-cycle model of Yaari (1965). There are numerous declinations of this type of work in growth models with different growth engines. For instance, Boucekkine et al. (2003) translate a life-cycle model à la Ben-Porath (1967) into an OLG model, while Prettnner (2013) examines the consequences of a longevity shift in an endogenous growth framework. Chapter 3 places itself in this kind of analysis, that I extend to multi-sector growth models. Indeed, in a multi-sector Diamond model, I study how the labor allocation of the economy is impacted by a longevity shock and I show that these labor reallocations can produce negative effects on the income per worker level. The second type of studies of longevity in OLG economies is due to Cipriani and Blackburn (2002) and Chakraborty (2004). They formulate OLG models in which the longevity is no more a parameter, but an endogenous variable. In Cipriani and Blackburn (2002), human capital investments exert a positive externality on longevity, while in Chakraborty (2004) there is a government that taxes at a constant rate the wage of individuals to finance public health expenditures that increase longevity. In the first chapter, this paper is extensively quoted as I argue that the potential of his framework has not been fully exploited. Indeed, I will study how the tax rate influences the transitional dynamics as well as the steady state of the economy. In the second chapter, I will propose an OLG model with endogenous lifetime in which the agents choose their level of health expenditures. More specifically, the dissertation is organized into three chapters which can be summarized as follows.

In the first chapter, I study the impact of health expenditures on economic growth and on welfare. For this, I draw on the seminal contribution of Chakraborty (2004). In his two-period overlapping generations model with a lifetime depending on public health expenditures, I study the influence of the tax rate, which is an exogenous parameter in Chakraborty (2004), on the economy. First, I determine the growth-maximizing tax rate, which is shown to be 0 in low-income countries. Second, I show that the steady-state income level is an inverted U-shaped function or a decreasing function of the tax rate. Third, I study the tax rate that maximizes the steady-state welfare level.

In the second chapter, I propose a theoretical model to study the growth impacts of health expenditures chosen by the agents. Indeed, I develop a Diamond model with endogenous growth in which young individuals can spend resources to increase their longevity in retirement period. First, I derive the demand for health and show that the income share spent on health is an inverted U-shaped function of income. Second, I fully characterize the dynamic general equilibrium and determine the growth impacts of the health expenditures. Several cases can occur. Health expenditures can speed up or slow down economic growth. They can be a barrier to growth or they can be a necessity for growth to take place. A simple

calibration of the model to OECD countries suggests that the latter case is the most likely one.

Finally, the third chapter determines the theoretical impact of the aging process on the sectorial allocation of labor. To this aim, I build a multi-sector two-period overlapping generations model in which I examine the consequences of both a longevity shift and a fertility shift on the labor allocation of the economy. There are three effects at stake : (i) Given prices, a longevity shift directly affects consumption levels of the individuals. (ii) Given the consumption levels of young and old individuals, aging increases the ratio of old to young individuals. (iii) Aging affects the price vector of an economy through its impact on the accumulation of production factors. These effects change the relative demand between sectors, which modifies the labor allocation. I first state necessary and sufficient conditions for aging to create intratemporal reallocation of resources at the individual as well as at the aggregate level in partial equilibrium. Then, I study the dependence of the labor ratios between sectors with respect to demographic variables along a path satisfying the Kaldor facts.

# ENDOGENOUS LIFETIME AND ECONOMIC GROWTH : THE ROLE OF THE TAX RATE

## 1.1 INTRODUCTION

The last decade has been the stage of a vivid academic debate on the health-growth nexus. This was stimulated by reports from International Organizations advocating health enhancing policies (see Acemoglu and Johnson (2006)). According to their views, the benefits of these policies are two-fold : (i) Improving health has positive welfare impacts. (ii) Improving health spurs economic growth. For example, Weil (2014) quotes the following passage of the WHO commission report :

Improving the health and longevity of the poor is an end in itself, a fundamental goal of economic development. But it is also a means to achieving the other development goals relating to poverty reduction. The linkages of health to poverty reduction and to long-term economic growth are powerful, much stronger than is generally understood. The burden of disease in some low-income regions, especially sub-Saharan Africa, stands as a stark barrier to economic growth and therefore must be addressed frontally and centrally in any comprehensive development strategy.

Over the period 1950-1990, the life expectancy increase in developing countries is well-documented (Bourguignon and Morrison 2002 ; Becker et al. 2005). Even though HIV caused life expectancy reversals in some countries such as in South-Africa or Namibia, overall progresses can be observed from the increase of the world life expectancy average from 52 in 1960 to 71 in 2013.<sup>1</sup> This is particularly due to reductions in infant mortality rates (Cutler et al. 2006). Although survival rates of children in developing countries have not reached those of developed countries, future gains in life expectancy in developing countries will mainly pass through health improvements at older ages. While Acemoglu and Johnson (2006) attribute

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1. These numbers and the others used in the introduction are all obtained from the World Bank database.

the past health improvements to the diffusion of new drugs and new medical practises, survival gains at older ages will depend on domestic health expenditures (Cutler et al. 2006). This motivates us to ask whether statements (i) and (ii) are valid when health improvements are costly.

In Figure 1.1, I plot the ratio of total health expenditures to GDP as a function of GDP per capita across countries. While the share of resources spent on health seems to increase with income when income is not too low, there is no clear relation between the two variables when income is low. Sierra Leone spends almost 12% of its income on health, which is almost the share spent by France, while the income per capita of France is 25 times that of Sierra Leone. On the contrary, in Lao or Pakistan, this ratio is less than 2.75%. This raises several questions : Can a low-income country spur economic growth by spending more resources on health ? From a welfare point of view, can it be optimal for a country not to spend resources on health ? This paper uses theory to answer these questions. Otherwise said, I determine analytically if statements (i) and (ii) remain valid when health improvements are not free of cost.

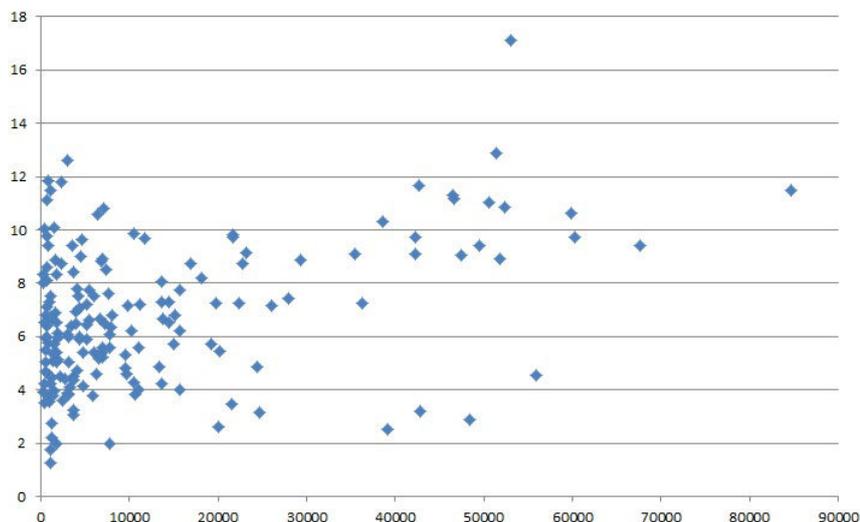


FIGURE 1.1 – *GDP per capita and ratio of total health expenditures to GDP across countries (2012).*  
Source : *World Bank database.*

More precisely, I extend the results of Chakraborty (2004), who builds a Diamond model with a survival probability into second period that depends on public health expenditures. Indeed, I study the impact of the tax rate, which is an exogenous parameter in Chakraborty (2004), on the income level and on the welfare in the steady state (hereafter SS) to answer the two following questions : How does the income level vary with respect to the tax rate ? Is the welfare-maximizing tax rate always positive ? Doing this, I complement a large theoretical literature on the health-growth nexus that has got interested into statements (i) and (ii).

Statement (ii) has received much attention in the literature. On the empirical side, the results are contrasted. Aghion et al.(2011) observe a positive impact of life expectancy growth rate on GDP per capita growth rate. Instrumenting life expectancy by the introduction date

of a public health care system, Strittmatter and Sunde (2013) also find a positive effect of life expectancy on GDP per capita growth rate. On the other hand, Acemoglu and Johnson (2007) argue that the positive impact of health on GDP growth is counteracted by a population increase so that GDP per capita of health improvements become non-significant. Using a simulation approach, Ashraf et al. (2008) also conclude that the income benefits of health improvements are negligible. On the theoretical side, authors have examined the impact of the life expectancy parameter on the income level in various dynamic general equilibrium frameworks. In a R&D based-growth model, Prettner (2013) shows that a longevity increase has a positive effect on the income per capita growth rate. In a growth model with human capital investments, de la Croix and Licandro (1999) show that a longevity increase induces two counteracting effects on the growth rate. On the one hand, the Ben Porath effect increases human capital supply. On the other hand, this leaves more retirees and more people educated a long time ago. Finally, when capital accumulation is the growth engine, longevity increases are seen as positive because they increase the propensity to save of the individuals and then economic growth (Bloom et al. (2003) and Chakraborty (2004)). These results are causal statements that neglect the possible costs of longevity improvements.

My contribution to the literature that has studied statement (ii) is to determine the impact on economic growth of *costly* longevity improvements. Closely related to my work are Chakraborty (2004) and Bhattacharya and Qiao (2007). Both papers are based on Diamond model with an endogenous survival probability into second period. In Bhattacharya and Qiao (2007), the survival probability depends on both private health expenditures, whose level is chosen by the agent, and public health expenditures, whose financing tax rate is an exogenous parameter.<sup>2</sup> The authors show that the SS income per worker level is an inverted U-shaped function of the tax rate. Here I aim to determine how the income per worker level depends on the *total* level of health expenditures. To do this, I abstract from the financing source of health expenditures and directly build on the model of Chakraborty (2004) by considering that all health expenditures are public. Then, I assess how the SS income level depends on these health expenditures. The question is not trivial because health expenditures create a trade-off on savings, hence on economic growth as capital accumulation is the growth engine in our specification. On the one hand, health expenditures increase longevity and so the propensity to save (Bloom et al. (2003) and Chakraborty (2004)). On the other hand, health expenditures reduce the disposable income and so savings. From this framework, I can also provide results on the welfare impact of health expenditures in SS, hence I can assess if the theory is also in line with statement (i).

Statement (i) has been much less examined by economists. Becker et al. (2005) argue that life expectancy improvements in developing countries in the XX<sup>th</sup> century significantly improved welfare. Murphy and Topel (2006) assess the social value of longevity gains in US on the previous century and show that it is potentially large. Theoretically, the literature has also discussed normative aspects of health expenditures particularly the decentralization

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2. The authors also make a simplifying assumption, that I do not follow here, according to which only old agents derive utility from consumption.

of social optimum in the context of health related externalities.<sup>3</sup> For example, Jouvét et al.(2010) and Ponthière (2016) discuss the decentralization of social optimum in economies in which pollution exerts negative externalities on the longevity of individuals. To the best of my knowledge, the literature has not discussed whether positive health expenditures maximize welfare in an environment free of externalities. However, there are reasons to believe that the level of health expenditures that maximizes welfare is not necessarily positive. Even though marginal utility of longevity is positive, longevity improvements can decrease welfare as they can diminish the level of resources per period. Second, health expenditures reduce the disposable income of individuals. Third, it is possible that health expenditures decrease the SS income level. This leaves three possible negative forces on welfare that health expenditures can exert. This justifies to investigate carefully whether the welfare-maximizing level of health expenditures is necessarily positive.

The rest of the paper proceeds as follows. Section 1.2 outlines the model of Chakraborty (2004) with two slight modifications. Section 1.3 studies the growth-maximizing tax rate and the tax rate that maximizes the steady-state income level. Section 1.4 studies the welfare-maximizing tax rate. Section 1.5 concludes.

## 1.2 THE MODEL

The model follows the one outlined by Chakraborty (2004). Consider a two-period OLG model in which the survival into second period occurs with probability  $p_t$  for an individual born at time  $t$ .  $p_t$  is taken as exogenous by the individuals. The number of young agents is constant and normalized to 1. Agents work during the first period and retire in second period. The consumption plan of a cohort- $t$  individual is chosen by maximizing the following lifetime expected utility function :

$$U_t = u(c_{1t}) + p_t u(c_{2t+1}) \quad (1.1)$$

Subject to the budget constraints :

$$c_{1t} + s_t \leq (1 - \tau)w_t \text{ and } c_{2t+1} \leq \frac{1+r_{t+1}}{p_t} s_t.$$

Where  $c_{1t}$  is the first period consumption,  $c_{2t+1}$  the second period consumption,  $r_{t+1}$  the interest rate.  $\tau$  is the tax rate on wages imposed by the government.  $s_t$  are the savings that are invested in capital by mutual funds. Assuming perfect competition among mutual funds, as in Yaari (1965), implies that the rate of return is  $\frac{1+r_{t+1}}{p_t}$

$u$  is the utility per period function. I will consider two different specifications for  $u$  :  $u(c) = \ln(c)$  (case (A)) and  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ , with  $\sigma < 1$  (case (B)). Case (A) is the one considered by Chakraborty (2004). In this case, utility becomes negative when the income level is low, which yields a negative marginal utility of longevity. As long as longevity is not chosen by the agent, this has no consequences, however in section 1.4 when I determine the welfare-maximizing tax rate, this property is crucial. To determine if my results are sensitive to the

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3. de la Croix and Ponthière (2010) derive the golden rule in a growth model with endogenous lifetime.

positiveness of the utility function, I also consider case (B), in which utility level is positive for any income levels. Hence, marginal utility of longevity is always positive. Case (B) is used in growth models in which health expenditures are chosen by individuals such as Chakraborty and Das (2005) and Bhattacharya and Qiao (2007).

It follows that savings are given by :

$$s_t = \frac{p_t}{(1 + r_{t+1})^{\frac{\sigma-1}{\sigma}} + p_t} (1 - \tau)w_t \quad (1.2)$$

Where  $\sigma = 1$  is case (A), while  $\sigma < 1$  is case (B).

The final good is produced with labor,  $L_t$ , and capital,  $K_t$ . It is consumed or invested in physical capital or used by the government to increase longevity. The production function is Cobb-Douglas :  $Y_t = AK_t^\alpha L_t^{1-\alpha}$  with  $A > 0$  and  $\alpha \in (0, 1)$ . Inputs are paid at their marginal productivity. Using the fact that the workforce size is 1, I get that :

$$1 + r_t = A\alpha K_t^{\alpha-1} \quad (1.3)$$

and

$$w_t = A(1 - \alpha)K_t^\alpha \quad (1.4)$$

Where I have assumed that capital fully depreciates at each period.  $p_t$  has yet to be specified. The government finances public health expenditures with a balanced budget and  $p_t$  is an increasing and strictly concave function of health expenditures per young person :  $p_t = p(\tau w_t)$ .

Where  $p$  satisfies :

$$p(0) = \underline{p} > 0, \lim_{x \rightarrow \infty} (p(x)) = \bar{p} \leq 1, p'(0) = \gamma < \infty \quad (1.5)$$

I will also consider the limit cases  $\underline{p} = 0$  and  $\gamma = \infty$  to see how they change my results.

$\tau$  is taken as constant in this section. I will study the influence of  $\tau$  in the next two sections. Finally, the dynamics of the economy is obtained by imposing the equilibrium on the capital market :

$$K_{t+1} = \frac{p(\tau A(1 - \alpha)K_t^\alpha)}{(A\alpha)^{\frac{\sigma-1}{\sigma}} K_{t+1}^{\frac{1-\sigma}{\sigma}(1-\alpha)} + p(\tau A(1 - \alpha)K_t^\alpha)} (1 - \tau)A(1 - \alpha)K_t^\alpha \quad (1.6)$$

The study and the interpretation of (1.6) in case (A) can be found in Chakraborty (2004). Here I add a sufficient condition for the SS of (1.6) to be unique. Moreover, I study the dynamics in case (B) and provide a sufficient condition for the SS uniqueness. Taking care of the uniqueness of the SS is necessary for the following sections in order to define economic variables in SS as functions of the tax rate.

**Proposition 1.1.** *Assume  $\sigma = 1$ . If  $\alpha \leq 1/3$  or if  $\alpha \in (1/3, 1/2]$  and  $x \rightarrow \frac{(-p''(x))x}{p'(x)}$  is increasing, then (1.6) has a unique positive steady state, which is stable.*

**Proposition 1.2.** *Assume  $\sigma < 1$ . (1.6) has a positive steady state. If  $\alpha < 1/2$  and  $x \rightarrow \frac{(-p''(x))x}{p'(x)}$  is increasing, then (1.6) has a unique positive steady-state, which is stable.*

*Proof.* See Appendix A ■

I will note  $K(\tau)$  the unique SS of (1.6), as in the following sections I want to emphasize the impact of  $\tau$  on the SS of the economy. Propositions 1.1 and 1.2 command some technical remarks. In case (A), when  $\alpha \in (1/3, 1/2]$ , an additional condition is required to insure SS uniqueness. The same condition is required for the case (B). This condition is not restrictive for three reasons. First, as  $x \rightarrow \frac{x}{p'(x)}$  is increasing, the assumption only requires that  $x \rightarrow -p''(x)$  does not decrease too much. Second,  $x \rightarrow \frac{(-p''(x))x}{p'(x)}$  takes greater values for large  $x$  than for small  $x$ . Third, the survival functions used in the literature satisfy this condition. Following an example given by Chakraborty (2004), Raffin and Seegmuller (2014) use a survival function of the form :  $p(x) = \frac{p+\bar{p}x}{1+x}$ . This function satisfies the condition of Proposition 1.1 and 1.2. This is also true for  $p(x) = \frac{p+\bar{p}x^\beta}{1+x^\beta}$  with  $\beta \in (0, 1]$ . Consider the following logistic function :  $p(x) = \bar{p} \frac{\frac{p}{\bar{p}-p}}{e^{-kx} + \frac{p}{\bar{p}-p}}$  with  $k > 0$ . For this function to be an admissible survival function,  $\frac{p}{\bar{p}}$  must be greater than  $\frac{1}{2}$ . Then, it also satisfies the condition. A simple way to build survival functions is to consider a probability density,  $f$ , on  $[0, \infty)$  and to define  $p(x) = \bar{p} \int_0^x f(a) da + \underline{p}$ . Then,  $p$  is an admissible survival function if and only if  $f$  is decreasing. For usual decreasing density distributions (Gaussian, exponential, Weibull), the condition is also satisfied. Then, for the rest of the paper, I will assume that  $x \rightarrow \frac{(-p''(x))x}{p'(x)}$  is increasing and  $\alpha < 1/2$ .

### 1.3 THE INFLUENCE OF THE TAX RATE ON THE GROWTH RATE AND ON THE SS INCOME LEVEL

In this section, I examine two questions linked to statement (ii). Is the growth-maximizing tax rate positive? How does the SS income level vary with the tax rate?

#### 1.3.1 The growth-maximizing tax rate

I first determine how the exogenous variable  $\tau$  influences the transitional dynamics of the economy. At time  $t$ , given that capital stock is  $K_t$ , is the tax rate that maximizes next period capital stock positive? Define :

$$\tau_t^* = \arg \max_{\tau \in [0,1]} \left( \frac{K_{t+1}}{K_t} - 1 \right) = \arg \max_{\tau \in [0,1]} \left( \frac{p(\tau A(1-\alpha)K_t^\alpha)}{(A\alpha)^{\frac{\sigma-1}{\sigma}} K_{t+1}^{\frac{1-\sigma}{\sigma}(1-\alpha)} + p(\tau A(1-\alpha)K_t^\alpha)} (1-\tau)A(1-\alpha)K_t^\alpha \right) \quad (1.7)$$

Subject to (1.6).

$\tau_t^*$  maximizes the growth rate of income per worker, the growth rate of output as well as the growth rate of income per capita at time  $t$ . Increasing the tax rate  $\tau$  creates opposite effects on the growth rate. First, it increases the longevity and so the propensity to save  $\frac{p(\tau A(1-\alpha)K_t^\alpha)}{(A\alpha)^{\frac{\sigma-1}{\sigma}} K_{t+1}^{\frac{1-\sigma}{\sigma}(1-\alpha)} + p(\tau A(1-\alpha)K_t^\alpha)}$ . This increases next period capital stock,  $K_{t+1}$ , which decreases

the interest rate and decreases savings when the IES is strictly greater than 1. Moreover, increasing the tax rate decreases the disposable income  $(1 - \tau)A(1 - \alpha)K_t^\alpha$ . This leaves a priori ambiguous the total effect of the tax rate on the growth rate. It is useful to define  $C$ , the unique positive real number that satisfies :

$$C = \frac{\underline{p}}{\alpha^{\frac{\sigma-1}{\sigma}} C^{\frac{1-\sigma}{\sigma}} + \underline{p}} (1 - \alpha) \quad (1.8)$$

Then, the following proposition characterizes  $\tau_t^*$  :

**Proposition 1.3.** *Note  $a(K_t, \tau_t^*)$  the growth rate of the economy at time  $t$  when the tax rate  $\tau_t^*$  is applied. There exists  $\widehat{K} > 0$  such that :*

- I) If  $\gamma A^{\frac{1}{1-\alpha}} C^{\frac{1}{1-\alpha} + \frac{1-\sigma}{\sigma}} < \frac{\underline{p}^2}{\alpha^{\frac{\sigma-1}{\sigma}}}$ , then :*
  - (i) If  $K_t < K(0)$ , then  $\tau_t^* = 0$  and  $a(K_t, \tau_t^*) > 0$ .*
  - (ii) If  $K(0) < K_t < \widehat{K}$ , then  $\tau_t^* = 0$  and  $a(K_t, \tau_t^*) < 0$ .*
  - (iii) If  $K_t > \widehat{K}$ , then  $\tau_t^* > 0$  and  $a(K_t, \tau_t^*) < 0$ .*
- (II) If  $\gamma A^{\frac{1}{1-\alpha}} C^{\frac{1}{1-\alpha} + \frac{1-\sigma}{\sigma}} > \frac{\underline{p}^2}{\alpha^{\frac{\sigma-1}{\sigma}}}$ , then there exists  $K_1 > \widehat{K}$  such that :*
  - (i) If  $K_t < \widehat{K}$ , then  $\tau_t^* = 0$  and  $a(K_t, \tau_t^*) > 0$ .*
  - (ii) If  $\widehat{K} < K_t < K_1$ , then  $\tau_t^* > 0$  and  $a(K_t, \tau_t^*) > 0$ .*
  - (iii) If  $K_t > K_1$ , then  $\tau_t^* > 0$  and  $a(K_t, \tau_t^*) < 0$ .*
- (III) If  $\gamma = \infty$  or  $\underline{p} = 0$ , then  $\widehat{K} = 0$ .*

*Proof.* See Appendix B ■

Proposition 1.3 goes at odds with the claim that a positive level of health expenditures is growth-maximizing and yields a positive growth rate. Indeed, Proposition 1.3 shows that there is only one case in which such a scenario happens. Moreover, a sufficiently high level of capital is required for the optimal tax rate to be positive, which implies that increasing health expenditures is detrimental to economic growth in low-income economies. This is due to my assumption that the marginal productivity of health expenditures in 0 (hence  $\gamma$ ) is finite, otherwise as implied by point (III), the optimal tax rate is initially positive and yields a positive growth rate. Indeed, when the capital stock is low, the wage is low, so the marginal gain in longevity of increasing the tax rate is low. Thus, the increase of the propensity to save is low and it is smaller than the decrease of the disposable income due to the increase of the tax rate. This implies that savings decrease with the tax rate. The role of the marginal productivity of health expenditures in 0 also appears for larger levels of the capital stock. When  $\gamma$  is low such that the economy is in case (I), then the marginal gain in longevity by increasing the tax rate is low for any capital stock levels, which implies that health expenditures are always detrimental to growth. On the contrary, when  $\gamma$  is large enough for the economy to be in case (II), then the government can spur economic growth by setting a positive level of health expenditures for a large enough capital stock.

### 1.3.2 The impact of the tax rate on the SS income level

In this subsection, I want to determine the variations of the function  $\tau \rightarrow K(\tau)$  to assess the long-run economic consequences of health expenditures. The result is in the following proposition, which is one the main results of the paper :

**Proposition 1.4.** (I) If  $\gamma A^{\frac{1}{1-\alpha}} C^{\frac{1}{1-\alpha} + \frac{1-\sigma}{\sigma}} \leq \frac{p^2}{\alpha^{\frac{\sigma-1}{\sigma}}}$ , then  $\tau \rightarrow K(\tau)$  is decreasing.

(II) If  $\gamma A^{\frac{1}{1-\alpha}} C^{\frac{1}{1-\alpha} + \frac{1-\sigma}{\sigma}} > \frac{p^2}{\alpha^{\frac{\sigma-1}{\sigma}}}$ , then  $\tau \rightarrow K(\tau)$  is inverted U-shaped.

(III) If  $\gamma = \infty$  or  $\underline{p} = 0$ , then  $\tau \rightarrow K(\tau)$  is inverted U-shaped.

*Proof.* See Appendix C ■

Proposition 1.4 shows that taking into account the cost of longevity improvements completely modifies the consequences in terms of economic development of health improvements. Consider first the same Diamond economy, yet with a constant survival function equal to  $\underline{p}$  and a tax rate equal to 0. The SS income level of this economy is an increasing function of  $\underline{p}$ . Hence if there are exogenous shocks that increase  $\underline{p}$ , then the SS income level increases. This confirms the statement (ii) of the introduction. However, if increasing longevity is costly, such as specified in our framework, then according to Proposition 1.4, it is possible that increasing longevity can only be realized at the expense of the income level. There are three important parameters that determine the occurrence of this case (case (I)). It happens for a low technology level  $A$ , a low initial marginal productivity of health expenditures  $\gamma$  and a high initial longevity  $\underline{p}$ .<sup>4</sup> These three parameters influence the impact of the tax rate on the propensity to save, hence the benefits in terms of income of health expenditures. As previously argued, when  $\gamma$  is low, increasing health expenditures does not increase by much the longevity and so the propensity to save. This is also the case for a low value of  $A$ , as it implies a low wage, so much that increasing the tax rate does not change by much the longevity. When the initial longevity is already large, health expenditures cannot increase the longevity by much, which also implies low benefits in terms of income of health expenditures. In the contrary case, case (II), the tax rate that maximizes the SS income level is positive, hence the benefits of increasing the tax rate, a higher propensity to save, are initially higher than the costs, a reduced disposable income.

Point (iii) shows that the violations of the Inada conditions by the survival function are determinant ingredients of my results. If  $\gamma = \infty$ , then, as previously argued, increasing initially health expenditures increase by a large amount the longevity, which implies that the propensity to save increases more than the disposable income reduction. Thus, health expenditures initially increase income. When  $\underline{p} = 0$ , if the tax rate is null, then individuals do not live in second period, so they do not save. As capital is an essential input, the income level is null. Thus, the tax rate that maximizes the SS income level is necessarily positive.

The inverted U-shaped curve can also be viewed from another perspective. There are income levels that are obtained by two economies that are similar except their tax rate. Even

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4. I prove that  $\underline{p} \rightarrow \frac{C^{\frac{1}{1-\alpha} + \frac{1-\sigma}{\sigma}}}{p^2}$  is decreasing in Appendix C.

though they achieve the same economic outcome, the economy that spends more on health achieves a higher life expectancy.

Which of these two cases corresponds to low-income or developed countries? Case (I) occurs for economies with a low level of technology and for which the effect of health spending on longevity is small. Thus, this case is more likely to correspond to low-income countries, while case (II) is more likely to correspond to developed countries. This implies that health expenditures spur economic development in low-income countries only if they adopt effective medical technologies.

The results also give some clues on the impacts on growth of the health policies applied across the world, which roughly correspond to the political-economic equilibrium of the model. In this equilibrium, the tax rate applied at each period is that chosen by the young because the old do not vote. This happens because the old realize their mortality shock at the end of the previous period so much that they do not benefit from the current health expenditures. Moreover, as the tax is applied on the wages, old individuals are not concerned by the financing of these expenditures. Thus, the tax rate applied is the one that maximizes (1.1) subject to the budget constraints. Then, when income is low, individuals choose a tax rate equal to 0 because the survival function  $p$  does not satisfy Inada conditions. According to Proposition 1.3, this policy is likely to maximize the economic growth rate. As income grows, individuals decide to spend a positive amount on health expenditures and the tax rate increases, which is in accordance with the situation of developed countries.<sup>5</sup> According to Proposition 1.3, this policy can spur or harm economic growth depending on the effectiveness of the health expenditures.

## 1.4 IMPACT OF THE TAX RATE ON THE SS WELFARE

In this section, I study how the tax rate influences the welfare in SS. Write  $U(\tau)$  the lifetime welfare in SS and define :

$$\tau^* = \arg \max_{\tau \in [0,1]} (U(\tau)) \quad (1.9)$$

I do not provide a complete characterization of the cases in which  $\tau^*$  is positive or not. In light of the results of the previous section, I rather answer the two following questions : Is  $\tau^*$  necessarily positive in an economy in which the tax rate that maximizes the SS income level is positive? Is it possible to have  $\tau^*$  positive in economies in which the tax rate that maximizes the SS income level is 0? It is convenient to separate the case  $\sigma = 1$  from the case  $\sigma < 1$ .

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5. The fact that the tax rate of the political-economic equilibrium increases with income is proved in the appendix B of Chakraborty (2004).

### 1.4.1 $u(c) = \ln(c)$ case :

I first highlight the channels through which the tax rate influences welfare in SS. For any variable  $X$ , I will write  $X(\tau)$  its SS value.  $U(\tau)$  can be written as follows :

$$U(\tau) = \ln\left(\frac{w(\tau)(1-\tau)}{1+p(\tau w(\tau))}\right) + p(\tau A(1-\alpha)K(\tau)^\alpha) \ln\left(\frac{(1+r(\tau))w(\tau)(1-\tau)}{1+p(\tau w(\tau))}\right) \quad (1.10)$$

Let us ignore first the general equilibrium effects of  $\tau$ . Increasing  $\tau$  reduces the disposable income, which reduces welfare in first and second period. Increasing  $\tau$  also increases the longevity, which has two consequences on welfare. First, as the length of the working period is fixed, it implies that individuals must diminish their resources spent per period, hence the consumption levels decrease, which decreases welfare in first and second period. Second, individuals enjoy consumption utility during a longer period. This increases welfare if the utility in second period is positive, otherwise this decreases welfare. Indeed, the logarithm utility case implies that for low income levels, the marginal utility of longevity is negative. Overall, the tax rate produces several counteracting forces on welfare. The total impact of the tax rate also includes its impact on the prices. For example, if the tax rate reduces the SS income level, then increasing  $\tau$  creates an additional negative force on welfare as this reduces the wage.

**Proposition 1.5.** (i) *There exists  $\underline{\alpha} < \frac{1}{2}$ ,  $\bar{A} > 0$  and  $\hat{\gamma} > 0$  such that if  $\alpha < \underline{\alpha}$ ,  $A < \bar{A}$  and  $\gamma \in (\hat{\gamma}, \infty]$ , then  $\tau \rightarrow K(\tau)$  is inverted U-shaped and  $\tau^* = 0$ .*

(ii) *There exists  $\tilde{A} > 0$  such that for all  $A > 0$ , there exists  $0 < \gamma(A) < \frac{\underline{p}^{2+\frac{1}{1-\alpha}}}{(\underline{p}A(1-\alpha))^{1-\alpha}}$*

*such that if  $A > \tilde{A}$  and  $\gamma(A) < \gamma < \frac{\underline{p}^{2+\frac{1}{1-\alpha}}}{(\underline{p}A(1-\alpha))^{1-\alpha}}$ , then  $\tau \rightarrow K(\tau)$  is decreasing and  $\tau^* > 0$ .*

(iii) *If  $\underline{p} = 0$ , then  $\tau^* > 0$ .*

*Proof.* See Appendix D ■

Proposition 1.5 shows that the SS welfare effects of health expenditures may be completely different from their effect on the SS income level.

Proposition 1.5 also clearly goes at odds with statement (i) according to which health improving policies always increase welfare. It further states that even in economies in which health expenditures stimulate the SS income level, a positive level of health expenditures can produce a negative effect on welfare. There are two important parameters restrictions that insure the occurrence of such a scenario. The first one is a lower-bound on  $\gamma$  which insures that the SS income level is an inverted U-shaped function of the tax rate according to Proposition 1.4. The second one is an upper-bound on the technology level  $A$ . The higher  $A$ , the higher income, the higher second period utility level. Hence a low  $A$  implies a low marginal utility of longevity, which implies that welfare can decrease with longevity. Thus, there remains only one channel through which the tax rate positively influences welfare : through its (initial)

positive effect on the income level. Under the parameters restriction of (i), this positive channel is offset by the negative ones, which implies that the tax rate that maximizes SS welfare is equal to 0. Note that this happens for arbitrarily large values of  $\gamma$  (even when  $\gamma = \infty$ ). If  $\gamma$  is very large, health expenditures are very effective to increase longevity initially, however as welfare decreases with longevity, there is still only one channel through which the tax rate positively influences welfare, which is offset by the other negative channels. To which extent does this result depend on the utility specification? I have previously highlighted three channels through which health expenditures negatively impact welfare. First, the disposable income reduction is always present once longevity improvements are costly. Second, the diminution of per period resources also negatively affects welfare independently on the utility function. However, it can be cancelled by an increase of the retirement legal age. Third, for low income levels, the marginal utility of longevity is negative. This is a direct consequence of the logarithmic specification. In the next section, I determine if the result holds with a CES utility function with IES strictly greater than 1, which insures a positive marginal utility of longevity for any income levels.

On the contrary, in case (ii), SS welfare is maximized with a positive level of health expenditures, even though the SS income is diminished by these health expenditures. This happens for a large enough value of the technology parameter. As previously said, the higher  $A$ , the higher the marginal utility of longevity. Thus, if  $A$  is large enough, then welfare increases with longevity and it is possible to find parameters such that this positive force on welfare offsets the negative ones.

Finally, the point (iii) shows that contrary to the study of the SS income level, the origin of the violation of the Inada conditions by the survival function is important to take into account. As previously argued, when  $\gamma = \infty$ ,  $\tau^*$  can be null or positive, while with  $\underline{p} = 0$ ,  $\tau^*$  is always null. The result is obvious as the income level is null in this case if the tax rate is 0, which implies that welfare is  $-\infty$ .

#### 1.4.2 $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ case :

In this case, the SS welfare is given by :

$$U(\tau) = \frac{1}{1-\sigma} ((1-\tau)w(\tau))^{1-\sigma} (1+r(\tau))^{1-\sigma} (p(\tau w(\tau)) + (1+r(\tau))^{\frac{\sigma-1}{\sigma}})^{\sigma} \quad (1.11)$$

Let us consider first the partial equilibrium effects of increasing  $\tau$  on welfare, hence let us maintain  $w(\tau)$  and  $r(\tau)$  fixed in (1.11). This reduces the disposable income, which decreases welfare. This also increases longevity. Recall that a longevity increase has two consequences on welfare : it decreases the level of resources spent per period, which negatively impacts welfare and it increases the period length during which individuals can enjoy consumption utility, which increases welfare as the marginal utility of longevity is always positive in this case. From (1.11), we see that the total effect of a longevity increase on welfare is always positive with our utility specification. This contrasts with the logarithm utility case, in which a longevity increase can decrease welfare. This shuts-off a channel through which health

expenditures can negatively impact welfare. In the following proposition, I show that this does not impede the existence of parameters for which  $\tau^* = 0$  while  $\tau \rightarrow K(\tau)$  is inverted U-shaped.

**Proposition 1.6.** (i) *The set of parameters that imply that  $\tau^* = 0$  and  $\tau \rightarrow K(\tau)$  is inverted U-shaped has positive measure.*

(ii) *If  $\underline{p} = 0$ , then  $\tau^* > 0$ .*

*Proof.* See Appendix E ■

Proposition 1.6 shows that my results obtained with a logarithm utility specification remain valid with a CES utility function with IES strictly greater than 1. Hence a positive marginal utility of longevity does not suffice to make the welfare-maximizing level of health expenditures necessarily positive. This is true even though the SS income level is maximized for a positive level of health expenditures.

## 1.5 CONCLUSION

In this paper, I assessed theoretically the consequences in terms of economic performance and welfare of increasing longevity when these health improvements are costly. To this aim, I extended the results of Chakraborty (2004) by studying the influence of the tax rate on the economy. First, I studied the influence of the tax rate on the transitional dynamics of the economy and I showed that the growth-maximizing tax rate is 0 in low-income countries. Second, I studied how the SS income level varies with the tax rate. I found that the curve is decreasing or inverted U-shaped, hence the level of health expenditures that maximizes the income level is not necessarily positive. Third, I studied the tax rate that maximizes the SS welfare. I observed that this tax rate can be positive in economies in which health expenditures reduce the SS income level, while it can be null in economies in which a positive level of health expenditures maximizes the SS income level.

These results cast doubt on the views that consider health-improving policies as necessarily positive for economic development as well as for welfare. While the literature has already pointed out that a life expectancy increase can be detrimental to economic development, this paper has underlined the costs of the longevity improvements as a cause of this negative nexus. Relative to my results on welfare, my contribution is to shed light on negative channels through which health expenditures affect welfare. Simulation studies could be useful to highlight countries in which increasing health expenditures can increase or not welfare.

## 1.6 APPENDIX A

In the various proofs, I will use the following result :

**Lemma 1.7.** *If  $x \rightarrow \frac{(-p''(x))x}{p'(x)}$  is increasing, then  $x \rightarrow e(x) := \frac{p'(x)x}{p(x)}$  is upper-bounded by 1.*

*Proof.* The sign of  $e'(x)$  is the one of  $1 - \frac{(-p''(x))x}{p'(x)} - e(x)$ . Define  $g(x) = 1 - \frac{(-p''(x))x}{p'(x)}$ .  $g(\cdot)$  is decreasing from a positive value to a negative value. If  $e(x)$  is in the set  $A = \{y \geq 0, y < g(x)\}$ , then  $e'(x) > 0$ , while  $e'(x) \leq 0$  if  $e(x)$  is in the set  $B = \{y \geq 0, y \geq g(x)\}$ . Initially  $e(x) \in A$  because  $e(0) = 0 < g(0)$ . So  $e$  initially increases and must hit the boundary of  $B$ , where its slope is null, while the slope of the boundary of  $B$  is negative. Thus,  $e$  enters the set  $B$ , where it is trapped because if it hits the boundary its slope is null, while the slope of the boundary is negative. Thus,  $x \rightarrow e(x)$  is inverted U- shaped. And  $e(x)$  is upper-bounded by  $g(0)$ , which is strictly smaller than 1 ■

I first show that the dynamical system (1.6) defines a unique trajectory  $(K_t)_{t \geq 0}$  for any initial condition. Rewrite (1.6) as :

$$K_{t+1}[(A\alpha)^{\frac{\sigma-1}{\sigma}} K_{t+1}^{\frac{1-\sigma}{\sigma}(1-\alpha)} + p(\tau A(1-\alpha)K_t^\alpha)] = p(\tau A(1-\alpha)K_t^\alpha)(1-\tau)A(1-\alpha)K_t^\alpha \quad (1.12)$$

The LHS of (1.12) increases with  $K_{t+1}$  from 0 to  $\infty$ . Thus for any  $K_t \geq 0$  and any  $\tau \in [0, 1]$ , there exists a unique  $K_{t+1}$  that solves (1.12). Write this solution as  $K_{t+1} = g(K_t, \tau)$ . The implicit function theorem insures the differentiability of  $g$ . Its partial derivative with respect to  $K$  is given by :

$$\frac{\partial g}{\partial K}(K, \tau) = A(1-\alpha)\alpha K^{\alpha-1} \frac{(1-\tau)p(\tau A(1-\alpha)K^\alpha) + p'(\tau A(1-\alpha)K^\alpha)\tau[(1-\tau)A(1-\alpha)K^\alpha - g(K, \tau)]}{(A\alpha)^{\frac{\sigma-1}{\sigma}}(1 + \frac{1-\sigma}{\sigma}(1-\alpha))g(K, \tau)^{\frac{1-\sigma}{\sigma}(1-\alpha)} + p(\tau A(1-\alpha)K^\alpha)} \quad (1.13)$$

Note that :

$$(1-\tau)A(1-\alpha)K^\alpha - g(K, \tau) = \frac{(A\alpha)^{\frac{\sigma-1}{\sigma}}g(K, \tau)^{\frac{1-\sigma}{\sigma}(1-\alpha)}}{(A\alpha)^{\frac{\sigma-1}{\sigma}}g(K, \tau)^{\frac{1-\sigma}{\sigma}(1-\alpha)} + p(\tau A(1-\alpha)K^\alpha)}(1-\tau)A(1-\alpha)K^\alpha > 0.$$

Thus,  $\frac{\partial g}{\partial K}(K, \tau) > 0$ .

Moreover, from (1.13),  $\frac{\partial g}{\partial K}(K, \tau) \underset{K \rightarrow 0}{\sim} K^{\alpha-1}(1-\tau)A(1-\alpha)\alpha$ . Hence  $\lim_{K \rightarrow 0} \frac{\partial g}{\partial K}(K, \tau) = \infty$ .

Hence if (1.6) has positive steady states, then the first one is stable.

For  $K > 0$ , the fixed point equation writes :

$$K = \frac{p(\tau A(1-\alpha)K^\alpha)}{(A\alpha)^{\frac{\sigma-1}{\sigma}}K^{\frac{1-\sigma}{\sigma}(1-\alpha)} + p(\tau A(1-\alpha)K^\alpha)}(1-\tau)A(1-\alpha)K^\alpha \quad (1.14)$$

$$\Leftrightarrow (A\alpha)^{\frac{\sigma-1}{\sigma}}K^{\frac{1-\sigma}{\sigma}(1-\alpha)} + p(\tau A(1-\alpha)K^\alpha) = p(\tau A(1-\alpha)K^\alpha)(1-\tau)A(1-\alpha)K^{\alpha-1} \quad (1.15)$$

The LHS of (1.15) is increasing with  $K$  from  $\underline{p}$  to  $\infty$ . The sign of the derivative of the RHS with respect to  $K$  is given by :

$$(\alpha-1)K^{\alpha-2}p(\tau A(1-\alpha)K^\alpha) + \tau A(1-\alpha)\alpha K^{2\alpha-2}p'(\tau A(1-\alpha)K^\alpha)$$

$$= K^{\alpha-2}(1-\alpha)p(\tau A(1-\alpha)K^\alpha) \left[ \frac{\tau A \alpha K^\alpha p'(\tau A(1-\alpha)K^\alpha)}{p(\tau A(1-\alpha)K^\alpha)} - 1 \right]$$

As  $\alpha < \frac{1}{2}$  :

$$\frac{\tau A \alpha K^\alpha p'(\tau A(1-\alpha)K^\alpha)}{p(\tau A(1-\alpha)K^\alpha)} < \frac{\tau A(1-\alpha)K^\alpha p'(\tau A(1-\alpha)K^\alpha)}{p(\tau A(1-\alpha)K^\alpha)} < 1$$

Where the last inequality follows from Lemma 1.7. This shows that the RHS of (1.15) is decreasing with  $K$  from  $\infty$  to 0. This proves that the fixed point equation has a unique positive solution.

## 1.7 APPENDIX B

$\tau_t^* = \arg \max_{\tau \in [0,1]} \left( \frac{K_{t+1}}{K_t} - 1 \right) = \arg \max_{\tau \in [0,1]} (g(K_t, \tau))$ . Apply the implicit function theorem to (1.12) to obtain the partial derivative of  $g$  with respect to  $\tau$  :

$$\frac{\partial g}{\partial \tau}(K_t, \tau) = A(1-\alpha)K^\alpha \frac{(1-\tau)A(1-\alpha)K_t^\alpha p' - p - gp'}{(1 + \frac{1-\sigma}{\sigma}(1-\alpha))g^{\frac{1-\sigma}{\sigma}(1-\alpha)}(A\alpha)^{\frac{\sigma-1}{\sigma}} + p}$$

Thus,  $\frac{\partial g}{\partial \tau}(K_t, \tau) > 0$  is equivalent to :

$$(1-\tau)A(1-\alpha)K_t^\alpha - \frac{p(\tau A(1-\alpha)K_t^\alpha)}{p'(\tau A(1-\alpha)K_t^\alpha)} > g(K_t, \tau) \quad (1.16)$$

Note that the LHS of (1.16) is decreasing with  $\tau$ . If (1.16) is satisfied in  $\tau = 0$ , then  $\tau \rightarrow g(K_t, \tau)$  is initially increasing. As the LHS decreases with  $\tau$ , there necessarily exists  $\tau^* \in [0, 1]$  such that  $(1-\tau^*)A(1-\alpha)K_t^\alpha - \frac{p(\tau^* A(1-\alpha)K_t^\alpha)}{p'(\tau^* A(1-\alpha)K_t^\alpha)} = g(K_t, \tau^*)$ , hence  $\frac{\partial g}{\partial \tau}(K_t, \tau^*) = 0$ . The derivative of the LHS of (1.16) is still negative at  $\tau^*$ , this implies that  $(1-\tau)A(1-\alpha)K_t^\alpha - \frac{p(\tau A(1-\alpha)K_t^\alpha)}{p'(\tau A(1-\alpha)K_t^\alpha)} < g(K_t, \tau)$  in the right-neighborhood of  $\tau^*$ . Then, for all  $\tau > \tau^*$ ,  $(1-\tau)A(1-\alpha)K_t^\alpha - \frac{p(\tau A(1-\alpha)K_t^\alpha)}{p'(\tau A(1-\alpha)K_t^\alpha)} \leq g(K_t, \tau)$ , because when  $g$  is equal to the LHS of (1.16), its derivative is null, while the derivative of the LHS is negative. Hence  $\tau \rightarrow g(K_t, \tau)$  is inverted U-shaped in this case. With the same argument, I can show that  $\tau \rightarrow g(K_t, \tau)$  is decreasing if  $A(1-\alpha)K_t^\alpha - \frac{p}{\gamma} < g(K_t, 0)$ .

Use (1.12) to write the condition  $A(1-\alpha)K_t^\alpha - \frac{p}{\gamma} < g(K_t, 0)$  as :

$(A\alpha)^{\frac{\sigma-1}{\sigma}} g(K_t, 0)^{1+\frac{1-\sigma}{\sigma}(1-\alpha)} < \frac{p^2}{\gamma}$ . Note that  $g(0, 0) = 0$  and  $\lim_{K \rightarrow \infty} g(K, 0) = \infty$ . Moreover, as  $\frac{\partial g}{\partial K}(K, 0) > 0$  for all  $K > 0$ , there exists  $\widehat{K}$  such that  $A(1-\alpha)K_t^\alpha - \frac{p}{\gamma} < g(K_t, 0)$  if and only if  $K_t < \widehat{K}$ . This shows that  $\tau_t^* = 0$  (respectively  $\tau_t^* > 0$ ) if  $K_t < \widehat{K}$  (resp.  $K_t > \widehat{K}$ ). I now determine if the economy grows or declines when the tax rate  $\tau_t^*$  is applied. For this, I need to sign  $\frac{g(K_t, \tau_t^*)}{K_t} - 1$ . There are two cases to consider. First, if  $K(0) < \widehat{K}$ .

For  $K_t < \widehat{K}$ ,  $\tau_t^* = 0$ , so  $\frac{g(K_t, \tau_t^*)}{K_t} - 1 > 0$  is equivalent to :

$$\begin{aligned} & \frac{g(K_t, 0)}{K_t} - 1 > 0 \\ \Leftrightarrow & \frac{g(K_t, 0)}{K_t} > \frac{g(K(0), 0)}{K(0)} \end{aligned} \quad (1.17)$$

Write the LHS of (1.17) as :

$$\frac{g(K_t, 0)}{K_t} = \frac{A(1-\alpha)K_t^{\alpha-1}}{1 + g(K_t, 0)^{\frac{1-\sigma}{\sigma}(1-\alpha)} \frac{(A\alpha)^{\frac{\sigma-1}{\sigma}}}{p}} \quad (1.18)$$

This shows that  $K_t \rightarrow \frac{g(K_t, 0)}{K_t}$  is decreasing. Hence (1.17) is equivalent to  $K_t < K(0)$ . Thus, when  $\tau_t^*$  is applied, the economy grows if  $K_t < K(0)$  and declines if  $K(0) < K_t < \widehat{K}$ . For  $K_t > \widehat{K}$ , write  $\frac{g(K_t, \tau)}{K_t}$  as :

$$\frac{g(K_t, \tau)}{K_t} = \frac{p(\tau A(1-\alpha)K_t^\alpha)}{(A\alpha)^{\frac{\sigma-1}{\sigma}} g(K_t, \tau)^{\frac{1-\sigma}{\sigma}(1-\alpha)} + p(\tau A(1-\alpha)K_t^\alpha)} (1-\tau)A(1-\alpha)K_t^{\alpha-1} \quad (1.19)$$

Note that the denominator of the RHS of (1.19) is increasing with  $K_t$ . The derivative of the numerator is :

$$(1-\alpha)K_t^{\alpha-2} p(\tau A(1-\alpha)K_t^\alpha) \left[ \frac{p'(\tau A(1-\alpha)K_t^\alpha) K_t^\alpha \tau A \alpha}{p(\tau A(1-\alpha)K_t^\alpha)} - 1 \right]$$

Note now that  $\frac{p'(\tau A(1-\alpha)K_t^\alpha) K_t^\alpha \tau A \alpha}{p(\tau A(1-\alpha)K_t^\alpha)} < \frac{p'(\tau A(1-\alpha)K_t^\alpha) K_t^\alpha \tau A(1-\alpha)}{p(\tau A(1-\alpha)K_t^\alpha)} < 1$  according to Lemma 1.7.

This implies that  $K_t \rightarrow \frac{g(K_t, \tau)}{K_t}$  is decreasing. Thus, for all  $K_t > \widehat{K}$ ,  $\frac{g(K_t, \tau_t^*)}{K_t} < \frac{g(\widehat{K}, \tau_t^*)}{\widehat{K}} < \frac{g(\widehat{K}, 0)}{\widehat{K}} < 1$ .

Consider now the case  $\widehat{K} < K(0)$ . As  $K_t \rightarrow \frac{g(K_t, 0)}{K_t}$  is decreasing and is worth 1 at  $K_t = K(0)$ , it must be that  $\frac{g(K_t, 0)}{K_t} > 1$  for all  $K_t < \widehat{K}$ . Hence the economy grows if  $K_t < \widehat{K}$  when  $\tau_t^*$  is applied.

Define  $m(K_t) = \max_{\tau \in [0,1]} \left( \frac{g(K_t, \tau)}{K_t} \right)$ .  $m(\widehat{K}) > 1$  and this function is decreasing as  $K_t \rightarrow \frac{g(K_t, \tau)}{K_t}$  is decreasing. Moreover, from (1.12) :

$$\frac{g(K_t, \tau)}{K_t} < A(1-\alpha)K_t^{\alpha-1}$$

This implies that  $\lim_{K_t \rightarrow \infty} (m(K_t)) = 0$ . Thus, there exists a unique  $K_1 > \widehat{K}$  such that  $\frac{g(K_t, \tau_t^*)}{K_t} > 1$  if and only if  $K_t < K_1$ . Finally, to obtain the condition of Proposition 1.3, note that the condition  $K(0) < \widehat{K}$  is equivalent to the condition  $(A\alpha)^{\frac{\sigma-1}{\sigma}} g(K(0), 0)^{1+\frac{1-\sigma}{\sigma}(1-\alpha)} < \frac{p^2}{\gamma}$ . Note that  $g(K(0), 0) = K(0)$  and that  $K(0) = (AC)^{\frac{1}{1-\alpha}}$ , where  $C$  is defined by (1.8). This completes the proof of Proposition 1.3.

## 1.8 APPENDIX C

Define :

$$G(K, \tau) = \frac{p(\tau A(1-\alpha)K^\alpha)}{(A\alpha)^{\frac{\sigma-1}{\sigma}} K^{\frac{1-\sigma}{\sigma}(1-\alpha)} + p(\tau A(1-\alpha)K^\alpha)} (1-\tau)A(1-\alpha)K^{\alpha-1}$$

According to Proposition 1.1 and 1.2, for each  $\tau \in [0, 1]$ , I can define  $K(\tau)$  the unique root to the equation  $G(K, \tau) = 1$  and  $K(\tau)$  is the unique SS of the dynamical system (1.6).

Note that :

$$\frac{(1-\tau)A(1-\alpha)}{G(K, \tau)} = \frac{(A\alpha)^{\frac{\sigma-1}{\sigma}} K^{\frac{1-\alpha}{\sigma}}}{p(\tau A(1-\alpha)K^\alpha)} + K^{1-\alpha} \quad (1.20)$$

The derivative with respect to  $K$  of the first member of the RHS of (1.20) is :

$$\frac{(A\alpha)^{\frac{\sigma-1}{\sigma}} K^{\frac{1-\alpha}{\sigma}-1} (1-\alpha)}{p(\tau A(1-\alpha)K^\alpha)} \left[ \frac{1}{\sigma} - \frac{\tau A\alpha K^\alpha p'(\tau A(1-\alpha)K^\alpha)}{p(\tau A(1-\alpha)K^\alpha)} \right]$$

As  $\frac{\tau A\alpha K^\alpha p'(\tau A(1-\alpha)K^\alpha)}{p(\tau A(1-\alpha)K^\alpha)} < 1$  and  $\frac{1}{\sigma} > 1$ , I obtain that  $K \rightarrow G(K, \tau)$  is decreasing.

$$\begin{aligned} \frac{\partial G}{\partial \tau}(K, \tau) &= \frac{(-1)p(\tau A(1-\alpha)K^\alpha)}{(A\alpha)^{\frac{\sigma-1}{\sigma}} K^{\frac{1-\sigma}{\sigma}(1-\alpha)} + p(\tau A(1-\alpha)K^\alpha)} A(1-\alpha)K^{\alpha-1} + \\ &\quad (1-\tau)A(1-\alpha)K^{\alpha-1} \frac{A(1-\alpha)p'(\tau A(1-\alpha)K^\alpha)(A\alpha)^{\frac{\sigma-1}{\sigma}} K^{\alpha+\frac{1-\sigma}{\sigma}(1-\alpha)}}{((A\alpha)^{\frac{\sigma-1}{\sigma}} K^{\frac{1-\sigma}{\sigma}(1-\alpha)} + p(\tau A(1-\alpha)K^\alpha))^2} \end{aligned} \quad (1.21)$$

Note that  $\frac{\partial^2 G}{\partial \tau^2}(K, \tau) < 0$ , hence  $\tau \rightarrow G(K, \tau)$  is concave. Thus,  $\tau \rightarrow G(K, \tau)$  is decreasing (respectively U-shaped) if  $\frac{\partial G}{\partial \tau}(K, 0) < 0$  (resp.  $\frac{\partial G}{\partial \tau}(K, 0) > 0$ ). Moreover,

$$\frac{\partial G}{\partial \tau}(K, 0) < 0 \Leftrightarrow A(1-\alpha)\gamma K^{\alpha+\frac{1-\sigma}{\sigma}(1-\alpha)} - \underline{p}K^{\frac{1-\sigma}{\sigma}(1-\alpha)} < \frac{\underline{p}^2}{(\alpha A)^{\frac{\sigma-1}{\sigma}}}$$

Assume that this condition holds for  $K(0)$ . Then it must be that  $\tau \rightarrow G(K(0), \tau)$  is decreasing. By definition of  $K(0)$ ,  $G(K(0), 0) = 1$ . Thus  $G(K(0), \tau) < 1$  for all  $\tau > 0$ . As  $K \rightarrow G(K, \tau)$  is decreasing, it must be that  $G(K, \tau) < 1$  for all  $K > K(0)$  and  $\tau \geq 0$ . Hence for  $K > K(0)$ , the equation  $G(K, \tau) = 1$  has no root.

Note now that if  $A(1-\alpha)\gamma K(0)^{\alpha+\frac{1-\sigma}{\sigma}(1-\alpha)} - \underline{p}K(0)^{\frac{1-\sigma}{\sigma}(1-\alpha)} < \frac{\underline{p}^2}{(\alpha A)^{\frac{\sigma-1}{\sigma}}}$ , then this inequality is true for all  $K < K(0)$ . Hence  $\tau \rightarrow G(K, \tau)$  is decreasing for all  $K < K(0)$ . And for all  $K < K(0)$ ,  $G(K, 0) > 1 = G(K(0), 0)$  as  $K \rightarrow G(K, 0)$  is decreasing. This means that the equation  $G(K, \tau) = 1$  has exactly one root for all  $K < K(0)$ , write it  $\tau(K)$ . Note that we can extend the function to  $K = 0$  with  $\tau(K) = 1$ . By the implicit function theorem :

$$\tau'(K) \frac{\partial G}{\partial \tau}(K, \tau(K)) = - \frac{\partial G}{\partial K}(K, \tau(K))$$

Hence  $K \rightarrow \tau(K)$  is a decreasing one-to-one map from  $[0, K(0)]$  to  $[0, 1]$ . Hence  $\tau \rightarrow K(\tau)$  which is the inverse of  $K \rightarrow \tau(K)$ , is decreasing on  $[0, 1]$ .

Assume now that  $\frac{\partial G}{\partial \tau}(K(0), 0) > 0$ . Hence,  $\tau \rightarrow G(K(0), \tau)$  is inverted U-shaped and it is also true for any  $K > K(0)$ . For any  $K > K(0)$ , as  $K \rightarrow G(K, 0)$  is decreasing,  $G(K, 0) < G(K(0), 0) = 1$ .

Because  $\tau \rightarrow G(K(0), \tau)$  is inverted U-shaped, I can define  $m(K) = \max_{\tau \in [0, 1]} G(K, \tau)$  for  $K > K(0)$ . This function is decreasing because  $K \rightarrow G(K, \tau)$  is decreasing for all  $\tau \in [0, 1]$ . Moreover :

$$G(K, \tau) < A(1 - \alpha)K^{\alpha-1}$$

Thus,  $\lim_{K \rightarrow \infty} G(K, \tau) = 0$ . Moreover  $m(K(0)) > 1$ . So there exists  $K^* > K(0)$  such that  $m(K^*) = 1$  and  $m(K) < 1$  if and only if  $K > K^*$ .

So for all  $K > K^*$ , the equation  $G(K, \tau) = 1$  has no root. Define  $\widehat{K}$  as the unique solution to the equation :  $A(1 - \alpha)\gamma\widehat{K}^{\alpha + \frac{1-\sigma}{\sigma}(1-\alpha)} - \underline{p}\widehat{K}^{\frac{1-\sigma}{\sigma}(1-\alpha)} = \frac{\underline{p}^2}{(\alpha A)^{\frac{\sigma-1}{\sigma}}}$ . By assumption  $K(0) > \widehat{K}$ .

For  $K(0) < K < K^*$ , as  $\tau \rightarrow G(K, \tau)$  is inverted U-shaped and takes values strictly greater than 1, the equation  $G(K, \tau) = 1$  has exactly two roots :  $\beta(K)$  and  $\lambda(K)$ , with  $\beta(K) < \lambda(K)$ . As  $\tau \rightarrow G(K, \tau)$  is inverted U-shaped and  $K \rightarrow G(K, \tau)$  is decreasing,  $\beta(K)$  is increasing while  $\lambda(K)$  is decreasing. For  $\widehat{K} < K < K(0)$ ,  $\tau \rightarrow G(K, \tau)$  is still inverted U-shaped, however  $G(K, 0) > 1$ . Thus, the equation  $G(K, \tau) = 1$  has exactly one root, which is decreasing with  $K$ . For  $K < \widehat{K}$ ,  $\tau \rightarrow G(K, \tau)$  is now decreasing, with  $G(K, 0) > 1$  and  $G(K, 1) = 0$ , so the equation  $G(K, \tau) = 1$  has exactly one root, which is decreasing with  $K$ . Extend  $\lambda(K)$  on  $[0, K^*]$ . Set  $\tau^* = \lambda(K^*)$ . It is a decreasing one-to-one map from  $[0, K^*]$  to  $[\tau^*, 1]$ . Consequently, the restriction of  $\tau \rightarrow K(\tau)$  to  $[\tau^*, 1]$  which is the inverse of  $K \rightarrow \lambda(K)$ , is decreasing.  $K \rightarrow \beta(K)$  is an increasing one-to-one map from  $[K(0), K^*]$  to  $[0, \tau^*]$ . This means that the restriction of  $\tau \rightarrow K(\tau)$  to  $[0, \tau^*]$  which is the inverse of  $K \rightarrow \beta(K)$ , is increasing. This proves that  $\tau \rightarrow K(\tau)$  is inverted U-shaped on  $[0, 1]$ .

To get the conditions (I) and (II) of Proposition 1.4, note that  $K(0)$  satisfies the following equation :

$$\frac{K(0)^{1-\alpha}}{A} = \frac{\underline{p}}{\alpha^{\frac{\sigma-1}{\sigma}} \left( \frac{K(0)^{1-\alpha}}{A} \right)^{\frac{1-\sigma}{\sigma}} + \underline{p}} (1 - \alpha)$$

Thus, there exists  $C > 0$  independent of  $A$  and  $\gamma$  such that  $\frac{K(0)^{1-\alpha}}{A} = C$ . Thus, I can rewrite the condition  $\frac{\partial G}{\partial \tau}(K(0), 0) < 0$  as :

$$A^{\frac{1}{1-\alpha}} \gamma (1 - \alpha) C^{\frac{\alpha}{1-\alpha} + \frac{1-\sigma}{\sigma}} < \frac{\underline{p}^2}{\alpha^{\frac{\sigma-1}{\sigma}}} + \underline{p} C^{\frac{1-\sigma}{\sigma}}$$

By using (1.8), I obtain the inequalities in (I) and (II) from the previous inequality. To prove (III), I find an equivalent of  $C$  when  $\underline{p}$  tends to 0. Note first from (1.8) that  $\lim_{\underline{p} \rightarrow 0} C = 0$ .

Then, rewrite (1.8) as :

$$\frac{\alpha^{\frac{\sigma-1}{\sigma}} C^{\frac{1}{\sigma}}}{\underline{p}} = 1 - \alpha - C$$

Thus,  $C \underset{p \rightarrow 0}{\sim} \left( \frac{1-\alpha}{\alpha^{\frac{\sigma-1}{\sigma}}} p \right)^\sigma$ .

This implies that  $\frac{C^{\frac{1}{1-\alpha} + \frac{1-\sigma}{\sigma}}}{p^2} \underset{p \rightarrow 0}{\sim} \frac{\left( \frac{1-\alpha}{\alpha^{\frac{\sigma-1}{\sigma}}} \right)^{1-\sigma + \frac{1-\sigma}{1-\alpha}}}{p^{\frac{1-\alpha(1+\sigma)}{1-\alpha}}}$ . As  $\alpha < \frac{1}{2}$  and  $\sigma < 1$ ,  $\alpha(1+\sigma) < 1$ . Thus,  $\lim_{p \rightarrow 0} \frac{C^{\frac{1}{1-\alpha} + \frac{1-\sigma}{\sigma}}}{p^2} = \infty$ . This proves (III) and completes the proof of Proposition 1.4.

In the interpretation of Proposition 1.4, I claim that  $\underline{p} \rightarrow \frac{C^{\frac{1}{1-\alpha} + \frac{1-\sigma}{\sigma}}}{p^2}$  is decreasing. To prove this, I implicitly differentiate (1.8) to get that :

$$\frac{dC}{d\underline{p}} = \frac{\underline{p}}{C} \frac{\alpha^{\frac{\sigma-1}{\sigma}} C^{\frac{1-\sigma}{\sigma}}}{\underline{p} + \alpha^{\frac{\sigma-1}{\sigma}} \left(1 + \frac{1-\sigma}{\sigma}\right) C^{\frac{1-\sigma}{\sigma}}}$$

Then I can compute the derivative of  $\frac{C^{\frac{1}{1-\alpha} + \frac{1-\sigma}{\sigma}}}{p^2}$  with respect to  $\underline{p}$ . This derivative is positive if and only if :

$$\left( \frac{1}{1-\alpha} + \frac{1-\sigma}{\sigma} \right) \frac{\alpha^{\frac{\sigma-1}{\sigma}} C^{\frac{1-\sigma}{\sigma}}}{\underline{p} + \alpha^{\frac{\sigma-1}{\sigma}} C^{\frac{1-\sigma}{\sigma}} \left(1 + \frac{1-\sigma}{\sigma}\right)} > 2$$

Note that the LHS of (1.8) is upper-bounded by  $\frac{\frac{1}{1-\alpha} + \frac{1-\sigma}{\sigma}}{1 + \frac{1-\sigma}{\sigma}}$ . Note finally that this upper-bound cannot be greater than 2 when  $\alpha < \frac{1}{2}$ . Hence  $\underline{p} \rightarrow \frac{C^{\frac{1}{1-\alpha} + \frac{1-\sigma}{\sigma}}}{p^2}$  is decreasing.

## 1.9 APPENDIX D

Use (1.1) to (1.4) to write :

$$U(\tau) = \ln\left(\frac{A(1-\alpha)K(\tau)^\alpha(1-\tau)}{1+p(A(1-\alpha)K(\tau)^\alpha\tau)}\right) + p(A(1-\alpha)K(\tau)^\alpha\tau) \ln\left(\frac{A^2\alpha(1-\alpha)K(\tau)^{2\alpha-1}(1-\tau)}{1+p(A(1-\alpha)K(\tau)^\alpha\tau)}\right) \quad (1.22)$$

So :

$$U(\tau) \leq \ln\left(\frac{A(1-\alpha)K(\tau)^\alpha}{1+p(A(1-\alpha)K(\tau)^\alpha\tau)}\right) + p(A(1-\alpha)K(\tau)^\alpha\tau) \ln\left(\frac{A^2\alpha(1-\alpha)K(\tau)^\alpha}{1+p(A(1-\alpha)K(\tau)^\alpha\tau)}\right) \quad (1.23)$$

Write the RHS of (1.23) as  $Z(K(\tau), p(A(1-\alpha)K(\tau)^\alpha\tau))$ . Note that :

$$\frac{\partial Z}{\partial K}(K, p) = \frac{\alpha - (1-2\alpha)p}{K}$$

If I impose  $\alpha < (1-2\alpha)\underline{p}$ , then  $\frac{\partial Z}{\partial K}(K(\tau), p(A(1-\alpha)K(\tau)^\alpha\tau)) < 0$  for all  $\tau \in [0, 1]$ . Moreover,

$$\frac{\partial Z}{\partial p}(K, p) = \ln(A^2\alpha(1-\alpha)) + (2\alpha-1)\ln(K) - 1 - \ln(1+p)$$

Then,  $\frac{\partial Z}{\partial p}(K, p) < 0$  is equivalent to :

$$\ln(A^2\alpha(1-\alpha)) < (1-2\alpha)\ln(K) + 1 + \ln(1+p)$$

Note  $\tau^* = \arg \max_{\tau \in [0,1]} (K(\tau))$  and  $\hat{\tau}$  the positive root to the equation  $K(\tau) = K(0)$ . Note that  $K(\tau) > K(0) = \left(\frac{A(1-\alpha)p}{1+p}\right)^{\frac{1}{1-\alpha}}$  on  $[0, \hat{\tau}]$ . So, if I impose :

$$\begin{aligned} \ln(A^2\alpha(1-\alpha)) &< (1-2\alpha) \ln\left(\left(\frac{A(1-\alpha)p}{1+p}\right)^{\frac{1}{1-\alpha}}\right) + 1 + \ln(1+p) \\ \Leftrightarrow \ln(A^{\frac{1}{1-\alpha}}\alpha(1-\alpha)) &< (1-2\alpha) \ln\left(\left(\frac{(1-\alpha)p}{1+p}\right)^{\frac{1}{1-\alpha}}\right) + 1 + \ln(1+p) \end{aligned}$$

Then,  $\frac{\partial Z}{\partial p}(K(\tau), p(A(1-\alpha)K(\tau)^\alpha\tau)) < 0$  for all  $\tau \in [0, \hat{\tau}]$ . For all  $\tau \in [0, \hat{\tau}]$ ,  $K(\tau) \geq K(0)$  and  $p(A(1-\alpha)K(\tau)^\alpha\tau) \geq \underline{p}$ , thus  $Z(K(\tau), p(A(1-\alpha)K(\tau)^\alpha\tau)) \leq Z(K(0), \underline{p}) = U(0)$ .

There remains to show that the maximum cannot be reached on  $[\hat{\tau}, 1]$ . For this, use the fixed point equation to write  $U(\tau)$  as :

$$U(\tau) = \ln\left(\frac{K(\tau)}{p(A(1-\alpha)K(\tau)^\alpha\tau)}\right) + p(A(1-\alpha)K(\tau)^\alpha\tau) \ln\left(\frac{A\alpha K(\tau)^\alpha}{p(A(1-\alpha)K(\tau)^\alpha\tau)}\right)$$

On  $[\hat{\tau}, 1]$ ,  $\tau \rightarrow K(\tau)$  is decreasing, thus  $K(\tau)$  is upper-bounded by  $K(\hat{\tau}) = K(0)$ . Thus, the LHS is upper-bounded by :

$\ln\left(\frac{K(0)}{p(A(1-\alpha)K(\tau)^\alpha\tau)}\right) + p(A(1-\alpha)K(\tau)^\alpha\tau) \ln\left(\frac{A\alpha K(0)^\alpha}{p(A(1-\alpha)K(\tau)^\alpha\tau)}\right) := M(p(A(1-\alpha)K(\tau)^\alpha\tau))$  on  $[\hat{\tau}, 1]$ .

$$M'(p) = -\frac{1}{p} + \ln(A\alpha K(0)^\alpha) - 1 - \ln(p)$$

So  $M'(p) < 0$  is equivalent to :

$$\ln(A\alpha K(0)^\alpha) < 1 + \ln(p) + \frac{1}{p} \quad (1.24)$$

The RHS of (1.24) is lower-bounded by :  $2 + \ln(p)$ . Thus, if I impose  $\ln(A^{\frac{1}{1-\alpha}}\alpha(1-\alpha)^{\frac{\alpha}{1-\alpha}}) < 2 + \ln(p)$ , then  $M'(p) < 0$  for all  $p \geq \underline{p}$ . For all  $\tau \in [\hat{\tau}, 1]$ ,  $p(A(1-\alpha)K(\tau)^\alpha\tau) \geq \underline{p}$ , thus  $M(p(A(1-\alpha)K(\tau)^\alpha\tau)) \leq M(\underline{p}) = U(0)$ . Hence  $\tau^* = 0$ . The conditions that I find for this to happen are :

$$\gamma\left(\frac{A(1-\alpha)p}{1+p}\right)^{\frac{1}{1-\alpha}} > \underline{p}^2 \quad (1.25)$$

$$\alpha < (1-2\alpha)\underline{p} \quad (1.26)$$

$$\ln(A^{\frac{1}{1-\alpha}}\alpha(1-\alpha)^{\frac{\alpha}{1-\alpha}}) < 2 + \ln(\underline{p}) \quad (1.27)$$

$$\ln(A^{\frac{1}{1-\alpha}}\alpha(1-\alpha)) < (1-2\alpha) \ln\left(\left(\frac{(1-\alpha)p}{1+p}\right)^{\frac{1}{1-\alpha}}\right) + 1 + \ln(1+p) \quad (1.28)$$

Where (1.25) insures that  $\tau \rightarrow K(\tau)$  is inverted U-shaped. Choose first  $\alpha$  low enough for (1.26) to be satisfied. Then, choose  $A$  low enough for (1.27) and (1.28) to be satisfied. Then,

choose  $\gamma$  large enough for (1.25) to be satisfied. This completes the proof of the point (i) of Proposition 1.5.

To prove the point (ii), I need to compute  $\frac{dU}{d\tau}(0)$ . For this, I need to compute  $\frac{dK}{d\tau}(\tau)$  and  $\frac{d(K(\tau)^\alpha \tau)}{d\tau}$ . I use the implicit function theorem to compute  $\frac{dK}{d\tau}(\tau)$  from the fixed point equation. I find that :

$$\frac{dK}{d\tau}(\tau) = AK(\tau)^\alpha \frac{\frac{p'K(\tau)}{p} - p}{p + 1 - \frac{A\alpha K(\tau)^\alpha \tau p'}{p}} \quad (1.29)$$

Thus,

$$\frac{dK}{d\tau}(0) = AK(0)^\alpha \frac{\frac{\gamma K(0)}{p} - p}{p + 1} = A^{\frac{1}{1-\alpha}} \left( \frac{(1-\alpha)p}{1+p} \right)^{\frac{\alpha}{1-\alpha}} \frac{\gamma \left( A \left( \frac{(1-\alpha)p}{1+p} \right)^{\frac{1}{1-\alpha}} - p^2 \right)}{p(p+1)}$$

Then, I compute  $\frac{d(K(\tau)^\alpha \tau)}{d\tau}$  from (1.29) :

$$\frac{d(K(\tau)^\alpha \tau)}{d\tau} = AK(\tau)^\alpha (1+p) \frac{1 - \frac{\tau}{1-\tau} \frac{\alpha}{1-\alpha}}{p + 1 - \frac{A\alpha K(\tau)^\alpha \tau p'}{p}}$$

<sup>6</sup>Thus,

$$\left. \frac{d(K(\tau)^\alpha \tau)}{d\tau} \right)_{\tau=0} = AK(0)^\alpha \quad (1.30)$$

Thus, I can compute  $\frac{dU}{d\tau}(0)$ . From (1.30) :

$$\begin{aligned} \frac{dU}{d\tau}(0) &= \frac{dK}{d\tau}(0) \frac{1+p\alpha}{K(0)} + \left. \frac{d(K(\tau)^\alpha \tau)}{d\tau} \right)_{\tau=0} \left[ \ln \left( \frac{A\alpha K(0)^\alpha}{p} \right) - \frac{1+p}{p} \right] \gamma \\ &= \left( \frac{(1-\alpha)p}{1+p} \right)^{\frac{\alpha}{1-\alpha}} \frac{\gamma \left( A \left( \frac{(1-\alpha)p}{1+p} \right)^{\frac{1}{1-\alpha}} - p^2 \right)}{p(p+1)} \frac{1+p\alpha}{\left( \frac{(1-\alpha)p}{1+p} \right)^{\frac{1}{1-\alpha}}} + A^{\frac{1}{1-\alpha}} \left( \frac{(1-\alpha)p}{1+p} \right)^{\frac{\alpha}{1-\alpha}} \left[ \ln \left( \frac{A^{\frac{1}{1-\alpha}} \alpha \left( \frac{(1-\alpha)p}{1+p} \right)^{\frac{\alpha}{1-\alpha}}}{p} \right) - \frac{1+p}{p} \right] \gamma \end{aligned}$$

Note now that it suffices to have  $\frac{dU}{d\tau}(0) > 0$  for  $\tau^*$  to be positive. The condition  $\frac{dU}{d\tau}(0) > 0$  is equivalent to :

$$A^{\frac{1}{1-\alpha}} \left[ \ln \left( \frac{A^{\frac{1}{1-\alpha}} \alpha \left( \frac{(1-\alpha)p}{1+p} \right)^{\frac{\alpha}{1-\alpha}}}{p} \right) - \frac{1+p}{p} + \frac{1+p\alpha}{p(p+1)} \right] \gamma > \frac{p}{p+1} \frac{1+p\alpha}{\left( \frac{(1-\alpha)p}{1+p} \right)^{\frac{1}{1-\alpha}}} \quad (1.31)$$

I now examine if this condition can be compatible with the one that guarantees that  $\tau \rightarrow K(\tau)$  is decreasing, hence  $\gamma A^{\frac{1}{1-\alpha}} < \frac{p^2}{\left( \frac{(1-\alpha)p}{1+p} \right)^{\frac{1}{1-\alpha}}}$ . Choose first  $A$  large enough for

$\ln \left( \frac{A^{\frac{1}{1-\alpha}} \alpha \left( \frac{(1-\alpha)p}{1+p} \right)^{\frac{\alpha}{1-\alpha}}}{p} \right) - \frac{1+p}{p} + \frac{1+p\alpha}{p(p+1)}$  to be positive. Then, (1.33) is equivalent to :

6. From this, we can observe that the Laffer rate is  $1 - \alpha$ .

$$\gamma A^{\frac{1}{1-\alpha}} > \frac{\frac{p}{p+1} \frac{1+p\alpha}{\left(\frac{(1-\alpha)p}{1+p}\right)^{\frac{1}{1-\alpha}}}}{\ln\left(\frac{A^{\frac{1}{1-\alpha}} \alpha \left(\frac{(1-\alpha)p}{1+p}\right)^{\frac{\alpha}{1-\alpha}}}{p}\right) - \frac{1+p}{p} + \frac{1+p\alpha}{p(p+1)}}$$

Note now that for  $A$  large enough :

$$\frac{\frac{p}{p+1} \frac{1+p\alpha}{\left(\frac{(1-\alpha)p}{1+p}\right)^{\frac{1}{1-\alpha}}}}{\ln\left(\frac{A^{\frac{1}{1-\alpha}} \alpha \left(\frac{(1-\alpha)p}{1+p}\right)^{\frac{\alpha}{1-\alpha}}}{p}\right) - \frac{1+p}{p} + \frac{1+p\alpha}{p(p+1)}} < \frac{p^2}{\left(\frac{(1-\alpha)p}{1+p}\right)^{\frac{1}{1-\alpha}}}$$

Thus, I choose  $\gamma$  such that :

$$\frac{1}{A^{\frac{1}{1-\alpha}}} \frac{\frac{p}{p+1} \frac{1+p\alpha}{\left(\frac{(1-\alpha)p}{1+p}\right)^{\frac{1}{1-\alpha}}}}{\ln\left(\frac{A^{\frac{1}{1-\alpha}} \alpha \left(\frac{(1-\alpha)p}{1+p}\right)^{\frac{\alpha}{1-\alpha}}}{p}\right) - \frac{1+p}{p} + \frac{1+p\alpha}{p(p+1)}} < \gamma < \frac{p^2}{\left(A \frac{(1-\alpha)p}{1+p}\right)^{\frac{1}{1-\alpha}}}$$

Under this parametric restriction  $\tau \rightarrow K(\tau)$  is decreasing and  $\frac{dU}{d\tau}(0) > 0$ . This completes the proof of point (ii).

## 1.10 APPENDIX E

Use (1.1) and (1.2) to get that utility in SS is :

$$U(\tau) = \frac{1}{1-\sigma} \left( (1-\tau)w \right)^{1-\sigma} R^{1-\sigma} (p(\tau w) + R^{\frac{\sigma-1}{\sigma}})^{\sigma}$$

Use now (1.4) and (1.3) to write  $U(\tau)$  as :

$$U(\tau) = V(K(\tau), \tau) = \frac{1}{1-\sigma} (1-\tau)^{1-\sigma} \left( K^{\frac{(2\alpha-1)(1-\sigma)}{\sigma}} p(\tau A(1-\alpha)K^{\alpha}) + (\alpha A)^{\frac{\sigma-1}{\sigma}} K^{\alpha \frac{1-\sigma}{\sigma}} \right)^{\sigma}$$

I first find a condition to have  $\frac{\partial V}{\partial K}(K(\tau), \tau) < 0$  for all  $\tau \in [0, 1]$ . Note first that  $V(0, \tau) = V(\infty, \tau)$ , hence the equation  $\frac{\partial V}{\partial K}(K, \tau) = 0$  has at least one root for each  $\tau$ . To lower-bound the smallest root to this equation, note that :

$$\frac{\partial V}{\partial K}(K, \tau) = 0$$

$$\begin{aligned} \iff \frac{\sigma}{1-\sigma} \left( \frac{1-\tau}{K^{\frac{(2\alpha-1)(1-\sigma)}{\sigma}} p + (\alpha A)^{\frac{\sigma-1}{\sigma}} K^{\alpha \frac{1-\sigma}{\sigma}}} \right)^{1-\sigma} \left[ \frac{(2\alpha-1)(1-\sigma)}{\sigma} K^{\frac{(2\alpha-1)(1-\sigma)}{\sigma} - 1} p + \right. \\ \left. A\tau(1-\alpha)\alpha K^{\alpha-1 + \frac{(2\alpha-1)(1-\sigma)}{\sigma}} p' + (\alpha A)^{\frac{\sigma-1}{\sigma}} \alpha \frac{1-\sigma}{\sigma} K^{\alpha \frac{1-\sigma}{\sigma} - 1} \right] = 0 \end{aligned} \quad (1.32)$$

$$\iff \frac{(1-2\alpha)(1-\sigma)}{\sigma} p = A\tau(1-\alpha)\alpha K^{\alpha} p' + (\alpha A)^{\frac{\sigma-1}{\sigma}} \alpha \frac{1-\sigma}{\sigma} K^{(1-\alpha)\frac{1-\sigma}{\sigma}} \quad (1.33)$$

Note that the LHS of (1.33) is initially greater than the RHS, thus to lower-bound the smallest root of this equation, I upper-bound the RHS by  $A(1-\alpha)\alpha K^\alpha \gamma + (\alpha A)^{\frac{\sigma-1}{\sigma}} \alpha \frac{1-\sigma}{\sigma} K^{(1-\alpha)\frac{1-\sigma}{\sigma}}$  and I lower-bound the LHS by  $\frac{(1-2\alpha)(1-\sigma)}{\sigma} \underline{p}$ . This means that any root of the equation  $\frac{\partial V}{\partial K}(K, \tau) = 0$  is lower-bounded by  $\widehat{K}$  which solves the following equation :

$$\frac{(1-2\alpha)(1-\sigma)}{\sigma} \underline{p} = A(1-\alpha)\alpha \widehat{K}^\alpha \gamma + (\alpha A)^{\frac{\sigma-1}{\sigma}} \alpha \frac{1-\sigma}{\sigma} \widehat{K}^{(1-\alpha)\frac{1-\sigma}{\sigma}}$$

I now search for a condition that insures that  $K(\tau) < \widehat{K}$  for all  $\tau \in [0, 1]$ . Note that  $K(\tau) < A^{\frac{1}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}}$ . Thus,  $A^{\frac{1}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} < \widehat{K}$  is sufficient to have  $K(\tau) < \widehat{K}$  for all  $\tau \in [0, 1]$ . The condition is equivalent to :

$$A(1-\alpha)\alpha (A^{\frac{1}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}})^\alpha \gamma + (\alpha A)^{\frac{\sigma-1}{\sigma}} \alpha \frac{1-\sigma}{\sigma} (A^{\frac{1}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}})^{(1-\alpha)\frac{1-\sigma}{\sigma}} < \frac{(1-2\alpha)(1-\sigma)}{\sigma} \underline{p} \quad (1.34)$$

$$\iff A^{\frac{1}{1-\alpha}} \gamma < \frac{1-\sigma}{\sigma} \frac{\underline{p}(1-2\alpha) - \alpha^{1+\frac{\sigma-1}{\sigma}}}{(1-\alpha)^{\frac{1}{1-\alpha}} \alpha}$$

I now check if this inequality can be made compatible with the one that insures that  $\tau \rightarrow K(\tau)$  is inverted U-shaped. For the two inequalities to be compatible, it must be that :

$$\begin{aligned} \frac{\underline{p} + \alpha^{\frac{\sigma-1}{\sigma}} C^{\frac{1-\sigma}{\sigma}}}{(1-\alpha)\alpha^{\frac{\sigma-1}{\sigma}} C^{\frac{1-\sigma}{\sigma} + \frac{1}{1-\alpha}}} &< \frac{1-\sigma}{\sigma} \frac{\underline{p}(1-2\alpha) - \alpha^{1+\frac{\sigma-1}{\sigma}}}{(1-\alpha)^{\frac{1}{1-\alpha}} \alpha} \\ \iff \underline{p} \alpha^{\frac{1}{\sigma}} \frac{\underline{p} + \alpha^{\frac{\sigma-1}{\sigma}} C^{\frac{1-\sigma}{\sigma}}}{(1-\alpha)C^{\frac{1-\sigma}{\sigma} + \frac{1}{1-\alpha}}} &< \frac{1-\sigma}{\sigma} \frac{\underline{p}(1-2\alpha) - \alpha^{1+\frac{\sigma-1}{\sigma}}}{(1-\alpha)^{\frac{1}{1-\alpha}}} \end{aligned} \quad (1.35)$$

Note first that the RHS of (1.35) has a positive limit as  $\alpha$  tends to 0. To obtain the limit of the LHS, use (1.8) :

$$C = \frac{\underline{p}(1-\alpha)}{\underline{p} + \alpha^{\frac{\sigma-1}{\sigma}} C^{\frac{1-\sigma}{\sigma}}}$$

For  $C$  fixed, the RHS of (1.8) tends to 0 as  $\alpha$  tends to 0, while the LHS of ((1.8) tends to  $C$ . Thus it must be that  $\lim_{\alpha \rightarrow 0} C = 0$ . This implies from (1.8) that  $\lim_{\alpha \rightarrow 0} \alpha^{\frac{\sigma-1}{\sigma}} C^{\frac{1-\sigma}{\sigma}} = \infty$ . Then, as  $\alpha$  tends to 0, the LHS of (1.35) is equivalent to  $\underline{p} \frac{\alpha}{C^{\frac{1}{1-\alpha}}}$ .

Because  $\lim_{\alpha \rightarrow 0} (C) = 0$ ,  $\frac{1}{C^{\frac{1}{1-\alpha}}} \leq \frac{1}{C}$  as  $\alpha$  is close to 0. Hence  $0 \leq \lim_{\alpha \rightarrow 0} (\underline{p} \frac{\alpha}{C^{\frac{1}{1-\alpha}}}) \leq \lim_{\alpha \rightarrow 0} (\underline{p} \frac{\alpha}{C}) = 0$ . Hence there exists  $\underline{\alpha} > 0$  such that (1.35) is satisfied for all  $\alpha < \underline{\alpha}$ .

I now search for a condition that insures  $\frac{\partial V}{\partial \tau}(K(\tau), \tau) < 0$  for all  $\tau \in [0, 1]$ .

$$\begin{aligned} \frac{\partial V}{\partial \tau}(K, \tau) &= -(1-\tau)^{-\sigma} (1-\sigma) (K^{\frac{(2\alpha-1)(1-\sigma)}{\sigma}} \underline{p} + (\alpha A)^{\frac{\sigma-1}{\sigma}} K^{\alpha \frac{1-\sigma}{\sigma}})^\sigma + \\ &\quad (1-\tau)^{1-\sigma} A(1-\alpha) K^{\alpha + \frac{(2\alpha-1)(1-\sigma)}{\sigma}} \underline{p}' \sigma (K^{\frac{(2\alpha-1)(1-\sigma)}{\sigma}} \underline{p} + (\alpha A)^{\frac{\sigma-1}{\sigma}} K^{\alpha \frac{1-\sigma}{\sigma}})^{\sigma-1} \end{aligned} \quad (1.36)$$

Note that  $\frac{\partial^2 V}{\partial \tau^2}(K(\tau), \tau) < 0$ . Hence  $\frac{\partial V}{\partial \tau}(K, \tau) < 0$  for all  $\tau \in [0, 1]$  is equivalent to  $\frac{\partial V}{\partial \tau}(K, 0) < 0$ , which is equivalent to :

$$A(1-\alpha) K^\alpha \gamma < \frac{1-\sigma}{\sigma} (\underline{p} + (\alpha A)^{\frac{\sigma-1}{\sigma}} K^{(1-\alpha)\frac{1-\sigma}{\sigma}})$$

$$\Leftrightarrow A(1-\alpha)K^\alpha\gamma - \frac{1-\sigma}{\sigma}(\alpha A)^{\frac{\sigma-1}{\sigma}}K^{(1-\alpha)\frac{1-\sigma}{\sigma}} < \frac{1-\sigma}{\sigma}\underline{p} \quad (1.37)$$

To study the LHS of (1.37) with respect to  $K$ , note that there are two cases to consider :  $\alpha > (1-\alpha)\frac{1-\sigma}{\sigma}$  and  $\alpha < (1-\alpha)\frac{1-\sigma}{\sigma}$ . I only consider the second case because my previous conditions require  $\alpha$  to be small.

Define  $G(K) = A(1-\alpha)K^\alpha\gamma - \frac{1-\sigma}{\sigma}(\alpha A)^{\frac{\sigma-1}{\sigma}}K^{(1-\alpha)\frac{1-\sigma}{\sigma}}$ . This function is inverted U-shaped. Note  $\tilde{K} = \arg \max_{K \geq 0}(G(K))$ . If I impose  $G(\tilde{K}) < \frac{1-\sigma}{\sigma}\underline{p}$ , then  $\frac{\partial V}{\partial \tau}(K(\tau), \tau) < 0$  for all  $\tau \in [0, 1]$ .

I find that :

$$G(\tilde{K}) = \left(1 - \frac{\alpha\sigma}{(1-\alpha)(1-\sigma)}\right) \left(\frac{\alpha\sigma}{(1-\alpha)(1-\sigma)}\right)^{\frac{1}{\frac{(1-\alpha)(1-\sigma)}{\alpha\sigma} - 1}} \left(\frac{\alpha(1-\alpha)^{\frac{1-\alpha}{\sigma}}}{\left(\frac{1-\sigma}{\sigma}\right)^{\frac{1-\sigma}{\sigma}}}\right)^{\frac{1-\sigma}{\frac{(1-\alpha)(1-\sigma)}{\alpha\sigma} - 1}} (\gamma^{1-\alpha}A)^{\frac{1-\sigma}{\frac{(1-\alpha)(1-\sigma)}{\alpha\sigma} - 1}}.$$

Thus, I can write the inequality  $G(\tilde{K}) < \frac{1-\sigma}{\sigma}\underline{p}$  as :

$$\gamma A^{\frac{1}{1-\alpha}} < \frac{1}{1-\alpha} \left(\frac{(1-\alpha)(1-\sigma)^2}{\alpha^{\frac{1}{\sigma}}\sigma^2}\right)^{\frac{\alpha}{1-\alpha}} \left(\frac{\frac{1-\sigma}{\sigma}\underline{p}}{1 - \frac{\alpha\sigma}{(1-\alpha)(1-\sigma)}}\right)^{1 - \frac{\alpha\sigma}{(1-\alpha)(1-\sigma)}} \quad (1.38)$$

I have now to check if this inequality is compatible with the one that insures that  $\tau \rightarrow K(\tau)$  is inverted U-shaped when  $\alpha$  tends towards 0. As  $\alpha$  tends towards 0, the RHS of (1.38) tends to :  $\frac{1-\sigma}{\sigma}\underline{p}$ . The LHS of (1.38) is equivalent to  $\frac{p}{C^{1-\alpha}}$ . Hence as  $\alpha$  tends towards 0, the ratio of the two quantities is equivalent to :

$$\frac{1-\sigma}{\sigma}e^{\frac{\alpha}{1-\alpha}\ln(C)} > \frac{1-\sigma}{\sigma}e^{\frac{\alpha}{1-\alpha}\ln(\alpha)}$$

As the RHS tends to  $\frac{1-\sigma}{\sigma}$  as  $\alpha$  tends towards 0, if I impose  $\frac{1-\sigma}{\sigma} > 1$ , hence  $\sigma < \frac{1}{2}$ , then (1.38) is true for  $\alpha$  small enough.

I now impose the three previous conditions. Recall that  $\tau^* = \arg \max_{\tau \in [0,1]}(K(\tau))$  and  $\hat{\tau}$  is the positive root to the equation  $K(\tau) = K(0)$ .

$\tau \rightarrow V(K(\tau), \tau)$  is decreasing on  $[0, \tau^*]$ . Thus, it cannot reach its maximum on  $[0, \tau^*]$ .

For all  $\tau \in [\tau^*, \hat{\tau}]$ ,  $V(K(\tau), \tau) < V(K(0), 0)$  because  $K(\tau) > K(0)$ .

There remains to prove that the maximum is not reached on  $[\hat{\tau}, 1]$ . To do this, use the fixed point equation to write welfare as :

$$U(\tau) = \frac{1}{1-\sigma} \left(\frac{(1-\tau)w}{p(\tau w) + R^{\frac{\sigma-1}{\sigma}}}\right)^{1-\sigma} R^{1-\sigma} (p(\tau w) + R^{\frac{\sigma-1}{\sigma}}) = \frac{1}{1-\sigma} \left(\frac{K}{p(\tau w)}\right)^{1-\sigma} R^{1-\sigma} (p(\tau w) + R^{\frac{\sigma-1}{\sigma}})$$

$$U(\tau) = K^{\alpha(1-\sigma)} \left(\frac{(\alpha A)^{\frac{\sigma-1}{\sigma}} K^{(1-\alpha)\frac{1-\sigma}{\sigma}}}{p(\tau w)} + 1\right) p(\tau w)^\sigma$$

For  $\tau \in [\hat{\tau}, 1]$ ,  $K(\tau) < K(0)$ . Thus,  $U < K(0)^{\alpha(1-\sigma)} \left(\frac{(\alpha A)^{\frac{\sigma-1}{\sigma}} K(0)^{(1-\alpha)\frac{1-\sigma}{\sigma}}}{p} + 1\right) p^\sigma$

Define  $Z(p) = K(0)^{\alpha(1-\sigma)} \left(\frac{(\alpha A)^{\frac{\sigma-1}{\sigma}} K(0)^{(1-\alpha)\frac{1-\sigma}{\sigma}}}{p} + 1\right) p^\sigma$ . I search for a condition that insures that  $p \rightarrow Z(p)$  is decreasing on  $[p, \bar{p}]$ .  $Z'(p) < 0$  is equivalent to :

$$p \frac{\sigma}{1-\sigma} < (\alpha A)^{\frac{\sigma-1}{\sigma}} K(0)^{(1-\alpha)\frac{1-\sigma}{\sigma}}$$

Consequently, if I impose  $\frac{\sigma}{1-\sigma} < (\alpha A)^{\frac{\sigma-1}{\sigma}} K(0)^{(1-\alpha)\frac{1-\sigma}{\sigma}} = \alpha^{\frac{\sigma-1}{\sigma}} C^{\frac{1-\sigma}{\sigma}}$ , then  $p \rightarrow Z(p)$  is decreasing on  $[\underline{p}, \bar{p}]$ .

Note that this condition does not contradict the previous ones when  $\alpha$  is small because  $\lim_{\alpha \rightarrow 0} (\alpha^{\frac{\sigma-1}{\sigma}} C^{\frac{1-\sigma}{\sigma}}) = \infty$ .

Under this condition  $U < Z(\underline{p}) = V(K(0), 0)$ . This proves that the maximum is not reached on  $[\hat{\tau}, 1]$ . This completes the proof of Proposition 1.6.

# GROWTH, LONGEVITY AND ENDOGENOUS HEALTH EXPENDITURES

## 2.1 INTRODUCTION

The share of health expenditures in GDP is on the rise in developed countries. In US, which is presently the country with the highest share of total health expenditures in GDP, the ratio has increased from 3,2% in 1950 to 17,6% in 2015 (Chernew and Newhouse 2011). The same trend is observed in all other OECD countries (see Figure 2.1). Over the period 2000-2013, the income per capita of the OECD area grew at an average annual rate of 2.9%, while the total level of health expenditures per capita grew at an average annual rate of 4.75%.<sup>1</sup> Concerns have been raised according to which devoting so much resources to the health sector could endanger economic growth (see Kuhn and Prettner (2016)). This paper proposes a simple theoretical framework to assess such concerns. In a standard growth model augmented with endogenous health expenditures, I determine if individuals can voluntarily choose to spend a level of resources on health that harms or even impedes long-run economic growth.

More precisely, let us consider first a Diamond model with a AK technology. Because of constant returns to reproducible factors, there exists a mild condition on the parameters under which the economy perpetually grows. Second, let us add to this standard framework the possibility for young agents to make expenditures to live longer in second period. Does this economy perpetually grow under the mild condition of the AK model? In other words, can the possibility to spend resources to live longer impede long-run economic growth? If the economy perpetually grows, then does it grow faster than the AK economy? Conversely, if the mild condition fails, can the economy grow? These questions are not trivial because health expenditures produce both positive and negative effects on economic growth. In a AK framework, the growth engine is the physical capital accumulation.<sup>2</sup> When agents increase

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1. These numbers and Figure 2.1 are obtained from the OECD database.

2. It could be argued that human capital is also an important growth engine to take into account to study the health-growth nexus. However, as I consider longevity improvements in retirement period, the Ben-Porath mechanism does not operate (Cervellati and Sunde 2013).

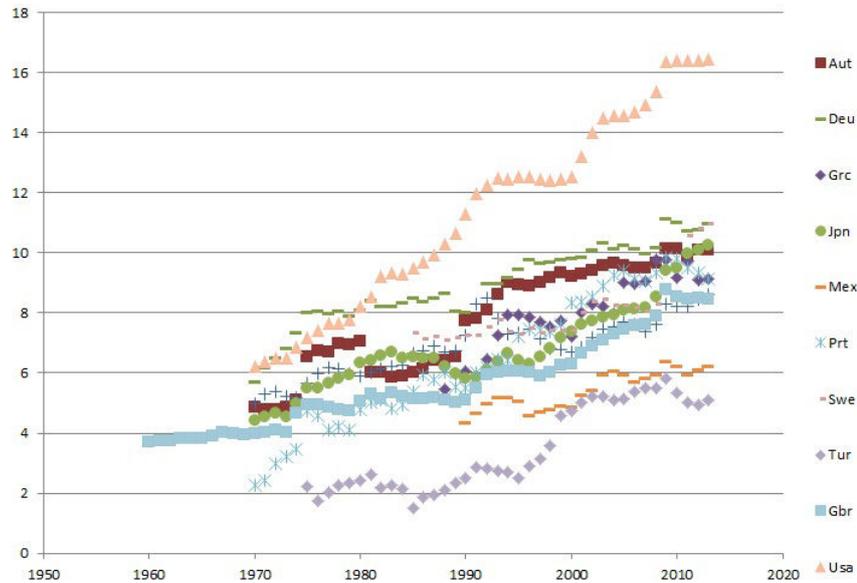


FIGURE 2.1 – Ratio of total health expenditures to GDP in 10 OECD countries

their level of health expenditures, this creates two opposite effects on savings, hence on economic growth. On the one hand, this increases their longevity which positively impacts their propensity to save (Bloom et al. 2003 ; Chakraborty 2004). On the other hand, in the previous chapter, I underline that health expenditures also decrease their disposable income which negatively impacts their savings. Particularly, if young agents were to spend all their resources on health, then there would be no capital stock for the next period and the economy would be trapped to a null income perpetually. Thus, when young agents spend a too large fraction of their income on health, they force a growing economy to stop its expansion. Otherwise said, a high level of health expenditures can be optimal for the current generation, while exerting strong negative *intergenerational externalities* on future generations who could be trapped to a constant income level. This paper focuses on this trade-off that health expenditures create on economic growth and omits other possible channels, such as the impact of health on productivity, to analyze analytically the occurrence of such an event. This enables to shed light on key parameters to take into account for the introduction of a health system to be a growth success.

Doing this, the paper connects two strands of the literature. The first one is a rich theoretical literature on the health-growth nexus, which has investigated the role of longevity in various dynamic general equilibrium settings. This literature first focused on the causal impact of longevity on economic growth by applying shocks on the longevity parameter in different growth models. For example, Boucekkine et al. (2002) and de la Croix and Licandro (1999) study the effect of a longevity increase on economic growth in a model with human capital investments. On the one hand, the human capital supply is stimulated through the Ben-Porath effect following a longevity increase. On the other hand, this creates more retirees and more people educated a long time before, leaving the total impact on economic growth ambiguous. This type of analysis has also been performed in models with a different

growth engine. Prettner (2013) examines the consequences of an exogenous longevity shift in a R&D based growth model, while Chakraborty (2004) shows that the higher the longevity in retirement period, the higher the propensity to save and then economic growth when it is driven by physical capital accumulation. Second, this literature has studied the joint dynamics of income and longevity, when the latter is determined by health expenditures.<sup>3</sup> In an influential paper, Chakraborty (2004) proposes a Diamond model in which the survival probability into second period depends on public health expenditures. Bhattacharya and Qiao (2007) propose an OLG model in which there are public and private health expenditures that affect the longevity. Kuhn and Prettner (2016) introduce a health sector into an endogenous growth model and examine the impact of its size on the growth rate and on the welfare of the individuals. However, in these papers, individuals do not control completely the level of their health expenditures. Indeed, the tax rate that finances the public health expenditures is exogenously fixed in these papers.<sup>4</sup> To rationalize the upward trend of the ratio of health expenditures to GDP and to analyze its consequences in terms of economic growth, I need a framework in which total health expenditures arise from the maximization of lifetime utility by agents. To achieve this, I abstract from the financing source of health and I follow Chakraborty (2004) and Bhattacharya and Qiao (2007) by letting second period longevity to depend on health expenditures, where, contrary to these two papers, the level of these expenditures is chosen by the agent by maximizing his lifetime utility under the budget constraints.

Thus, the paper also sheds light on the determinants of the demand for health.<sup>5</sup> This literature pioneered by Grossman (1972) has proposed various modelling strategies to incorporate health decisions into life-cycle models. In Grossman (1972) or Ehrlich and Chuma (1990), individuals live until their health capital, that depreciates each period and that can be increased through investment, falls under a threshold value. Dalgaard and Strulik (2014), criticize the possibility for individuals to increase their health stock and propose a framework based on research in natural sciences, in which individuals make expenditures to slow the accumulation of deficits caused by aging. Here, I do not follow these approaches as they are not suitable for dynamic general equilibrium. I rather consider a framework in which health expenditures only allow to live longer to enjoy consumption utility.<sup>6</sup> In addition to its analytical convenience for the study of the dynamics of the economy, the formulation allows to establish new results on the income elasticity of health expenditures. Following Jones and Hall (2007), this elasticity is believed to be positively driven by the ratio of health elasticity to consumption elasticity (see also Acemoglu et al. 2013). The first contribution of the paper

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3. Some authors also study growth models in which longevity is determined by various externalities. Cipriani and Blackburn (2002) and Cervellati and Sunde (2005) both examine a model in which investments in education exert a positive externality on the longevity of individuals. In Mariani et al. (2010) and Raffin and Seegmuller (2014), pollution exerts a negative effect on the individuals' longevity. In Ponthiere (2011), good consumption influences the longevity.

4. In Chakraborty and Das (2005), individuals fully control their longevity. Yet the authors focus on the transmission of inequalities in a small open economy framework that does not allow to study the growth consequences of these health expenditures.

5. In line with this literature, I do not study supply side effects of the health sector.

6. See Azomahou et al. (2015) for a discussion on the inclusion of health in individuals' preferences.

is to prove and explain why the ratio of these two elasticities is an imperfect picture of the income elasticity of health expenditures. The second one is to provide a complete characterization of the dynamics of the economy to assess the growth impacts of health expenditures.

The rest of the paper is as follows. Section 2.2 presents the model and characterizes the level of health expenditures. Section 2.3 studies the dynamics of the economy. Section 2.4 discusses alternative preferences and proposes a numerical illustration for OECD countries. Section 2.5 concludes.

## 2.2 THE MODEL

### 2.2.1 Outline

Individuals live for two periods. The young work while the old are retired. For a cohort- $t$  individual, the length of the first and the second period are respectively 1 and  $p_t$ , with  $p_t \leq 1$ . There is a single good in the economy which is produced competitively. This good can be consumed or invested in physical capital or used to increase  $p_t$ . The size of each new cohort is constant and normalized to 1.

### 2.2.2 Firms

Here I follow Raffin and Seegmuller (2014) to introduce a AK technology for the production sector. There is a representative firm which uses labor and capital to produce the unique good of this economy. The production function  $F$  is homogeneous of degree 1 and satisfies Inada conditions :

$$Y_t = F(K_t, B_t L_t) \quad (2.1)$$

Where  $Y_t$  is output at time  $t$ ,  $K_t$  the capital stock,  $L_t$  labor and  $B_t$  is the labor augmenting technological progress. There are positive externalities in the use of capital that linearly increase the productivity of workers :  $B_t = K_t$ . Factors are paid at their marginal productivity. At the equilibrium,

$$1 + r_t = F_1(1, 1) \quad (2.2)$$

$$w_t = K_t F_2(1, 1) \quad (2.3)$$

Where I have assumed that the capital fully depreciates at each period. It is convenient to define  $A = F(1, 1)$  and  $\alpha = \frac{F_1(1,1)}{F(1,1)}$  to write (2.2) and (2.3) as :

$$1 + r_t = A\alpha = 1 + r \quad (2.4)$$

$$w_t = K_t(1 - \alpha)A \quad (2.5)$$

### 2.2.3 Preferences

Individuals choose their consumption levels in both periods. They can also spend resources during their first period to increase their longevity  $p_t$  in second period. More precisely, each cohort- $t$  member maximizes the following lifetime utility function :

$$U_t = u(c_{t,t}) + p(e_t)u(c_{t,t+1}) \quad (2.6)$$

With respect to  $c_{t,t}$ ,  $c_{t,t+1}$  and  $e_t$  subject to the budget constraints :

$$c_{t,t} + s_t + e_t = w_t \quad (2.7)$$

$$p(e_t)c_{t,t+1} = (1 + r)s_t \quad (2.8)$$

Where  $c_{t,t}$  is the consumption level per unit of time in first period of a cohort- $t$  member,  $c_{t,t+1}$  the consumption level per unit of time in second period,  $e_t$  the level of health expenditures,  $s_t$  the savings. The function  $p$  specifies the relationship between the level of health expenditures and the longevity :  $p_t = p(e_t)$ . The function  $p$  is twice differentiable, increasing and strictly concave with :

$$p(0) = \underline{p} \geq 0, \lim_{e \rightarrow \infty} p(e) = \bar{p} \leq 1, p'(0) = \gamma \in (0, +\infty] \quad (2.9)$$

The set of survival functions  $p$  includes the ones that satisfy Inada conditions (this happens if  $p(0) = 0$  and  $\gamma = \infty$ ). Assuming that  $p$  is increasing and strictly concave is usual in the literature, however there is no consensus on the values of  $p(0)$  (null or positive) and  $p'(0)$  (finite or not). In Chakraborty (2004),  $p(0)$  is null and  $p'(0)$  is finite. In Boucekkine and Laffargue (2010),  $p(0)$  is positive and  $p'(0)$  is finite. Finally, in Chakraborty and Das (2005),  $p(0)$  is null and  $p'(0)$  is non-finite. I will consider both possibilities for  $p(0)$  and  $p'(0)$  in the analysis to determine how these assumptions change the results.

As argued by Hall and Jones (2007), the shape of the function  $u$  is crucial for the health spending decision. I will pursue the analysis with the following functional form :

**Assumption 1.**  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ ,  $\sigma < 1$ .

Assumption 1 is a standard assumption in growth models incorporating health decisions. It can be found in Chakraborty and Das (2005) and Bhattacharya and Qiao (2007). It has the advantage to insure a positive flow utility and hence a positive marginal utility of longevity. It also implies that utility from consumption has a constant elasticity, which, as we will see in the next section, has strong implications concerning the health decision of agents. Thus, I will discuss and study numerically other forms of preferences in section 2.4.

### 2.2.4 Partial equilibrium results

In this subsection, I study the solution to the maximization problem of the consumer. I use (2.7) and (2.8) to eliminate the consumption levels in (2.6). Then the problem is reduced to maximize (2.6) with respect to  $s_t$  and  $e_t$ . The First-Order-Condition (FOC) on  $s_t$  yields :

$$s_t = \frac{p(e_t)}{p(e_t) + (1+r)^{\frac{\sigma-1}{\sigma}}}(w_t - e_t) \quad (2.10)$$

From (2.10), we can observe the trade-off that health expenditures create on savings : the propensity to save  $\frac{p(e_t)}{p(e_t)+(1+r)^{\frac{\sigma-1}{\sigma}}}$  increases with  $e_t$ , while the disposable income  $w_t - e_t$  decreases with  $e_t$ .

For an interior solution, the FOC on  $e_t$  yields :

$$c_{t,t}^{-\sigma} + p'(e_t)c_{t,t+1}^{1-\sigma} = p'(e_t)\frac{c_{t,t+1}^{1-\sigma}}{1-\sigma} \quad (2.11)$$

The Left-Hand-Side (LHS) of (2.11) is the marginal cost of health expenditures. It is composed of two terms. The first one is the loss of first period utility from foregone consumption. The second one is the loss of second period utility from diminishing per period resources due to longevity extension. The Right-Hand-Side (RHS), the marginal benefit of health expenditures, is the total second period utility gain due to longevity extension. At the optimal level of health expenditures, the marginal cost of health expenditures must equate its marginal benefit. It is useful to rewrite (2.11) as :

$$c_{t,t}^{-\sigma} = \sigma p'(e_t)\frac{c_{t,t+1}^{1-\sigma}}{1-\sigma} \quad (2.12)$$

Where the RHS of (2.12) is the net marginal benefit of longevity extension, which is positive. This means that despite the reduction of per period resources in retirement period, a longevity extension always increases welfare. Finally, (2.10) and (2.11) imply that an interior solution for the level of health expenditures must solve the following equation :

$$\frac{\sigma}{1-\sigma}\frac{p'(e_t)}{p(e_t) + (1+r)^{\frac{\sigma-1}{\sigma}}}(w_t - e_t) = 1 \quad (2.13)$$

The following proposition characterizes the solution of (2.13) :

**Proposition 2.1.** *The optimal level of health expenditures is unique. Note it  $e(w_t)$ .*

- (i) If  $w_t \leq \frac{1-\sigma}{\sigma}\frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ , then  $e(w_t) = 0$ .
- (ii) If  $w_t > \frac{1-\sigma}{\sigma}\frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ , then  $0 < e(w_t) < w_t$ .
- (iii)  $w_t \rightarrow e(w_t)$  is increasing on  $[\frac{1-\sigma}{\sigma}\frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}, \infty)$ .<sup>7</sup>

*Proof.* See Appendix A ■

When the initial marginal productivity of health expenditures  $\gamma$  is finite, Proposition 2.1 shows that individuals spend resources on health only if their income is above a certain threshold. As the marginal productivity of health expenditures in 0 is finite, the marginal utility of health expenditures in 0 is finite. On the contrary, the marginal utility of consumption is non-finite in 0. Thus, low-income individuals choose to spend their resources only on consumption.

7. When  $\gamma = \infty$ ,  $[\frac{1-\sigma}{\sigma}\frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}, \infty)$  is simply read as  $[0, \infty)$ .

When  $\gamma$  is non-finite, the corner solution for health expenditures disappears and individuals spend resources on health for all positive income levels.

**Remark 2.2.** Consider the alternative assumption found in the literature according to which  $p_t$  is the probability to reach the second period. Assume as in Yaari (1965) that there is a perfect annuity market. If agents internalize the effect of health expenditures on the return of annuities, then the problem of the consumer is identical to the one studied in this section. If agents do not internalize the effect of health expenditures on the return of annuities, then the optimal level of health expenditures,  $e_t^Y$ , would be the solution of the following equation :

$$\frac{p'(e_t^Y)}{p(e_t^Y) + (1+r)^{\frac{\sigma-1}{\sigma}}}(w_t - e_t^Y) = 1 \quad (2.14)$$

Thus,  $e_t < e_t^Y$  if and only if  $\sigma < \frac{1}{2}$ . And all the propositions of the paper can be adapted to this alternative assumption on  $p_t$  given the similarity of the equations (2.13) and (2.14).

Proposition 2.1 also states that health is a normal good. I now sharpen this result by studying the income share spent on health,  $x(w_t) := \frac{e(w_t)}{w_t}$ . To get the exact shape of the function  $x(\cdot)$ , I will need an additional assumption on the function  $p$  which has already been used in the previous chapter :

**Assumption 2.**  $e \rightarrow \frac{(-p''(e))e}{p'(e)}$  is increasing.

In the previous chapter I provide three reasons to believe that Assumption 2 is harmless, the most important one being that usual survival functions satisfy this condition. I am now able to prove the first main result of the paper.

**Proposition 2.3.** (i)  $w_t \rightarrow x(w_t)$  is inverted U-shaped on  $[\frac{1-\sigma}{\sigma} \frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}, \infty)$ .

$$(ii) \quad \lim_{w_t \rightarrow \frac{1-\sigma}{\sigma} \frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}} x(w_t) = \lim_{w_t \rightarrow \infty} x(w_t) = 0.$$

*Proof.* See Appendix B ■

Proposition 2.3 can be restated as follows :

**Corollary 2.4.** The ratio of health expenditures to GDP can be written as a function of GDP,  $Y_t \rightarrow g(Y_t)$ , which is as follows :

$$(i) \quad g(Y_t) = 0 \text{ for all } Y_t \text{ in } [0, \frac{1-\sigma}{\sigma} \frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{(1-\alpha)\gamma}]$$

$$(ii) \quad Y_t \rightarrow g(Y_t) \text{ increases on } [\frac{1-\sigma}{\sigma} \frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{(1-\alpha)\gamma}, \frac{w^*}{1-\alpha}], \text{ where } w^* = \arg \max_{w \geq 0} (x(w)).$$

$$(iii) \quad Y_t \rightarrow g(Y_t) \text{ decreases on } [\frac{w^*}{1-\alpha}, \infty).$$

*Proof.* Note that  $w_t = (1-\alpha)Y_t$ . Then  $g(Y_t) = \frac{x((1-\alpha)Y_t)}{1-\alpha}$  and the result follows from Proposition 2.3 ■

Proposition 2.3 shows that health is a luxury good as individuals start spending resources on health. Thus, our simple framework is consistent with the joint increase of income and

income share of health expenditures that OECD countries have known over the last forty years.<sup>8</sup> It also predicts that this income share should not keep rising perpetually with income. Before interpreting the result, notice that consumption in first and second periods are both normal goods so that the result is not driven by an undesirable feature of the preferences.<sup>9</sup>

As income increases, there are three effects that induce a change of the income share spent on health (see equation (2.12)) : (i) The marginal cost in terms of first period utility loss falls. (ii) The marginal benefits of longevity increases because second period utility level increases. (iii) The marginal effect of health expenditures on longevity decreases. When income is low, the effect (i) is large because the first period marginal utility is convex. The effect (ii) is also large because the second period utility function is concave. Thus, maintaining the effect (iii) as fixed, the income fall reduces the marginal cost of health expenditures and increase their marginal benefit both by a large amount. Thus, health expenditures have to increase for the marginal costs and benefits to equate. Recall that under Assumption 2, the derivative of the survival function cannot be too convex, hence health expenditures must increase by a large amount, superior to income, for the marginal costs and benefits to equate. Thus, the income share spent on health increases. For high income levels, the contrary holds : effect (i) and effect (ii) are small. This is also a consequence of the convexity of the marginal utility and of the concavity of the utility per period. Thus, health expenditures can increase by a small amount for the marginal costs and benefits to equate. One could argue that because the survival function is bounded, the marginal impact of health expenditures on longevity rapidly falls and then that health expenditures still have to increase by a large amount to correct the differences between the marginal costs and benefits. However, for high income levels  $w$ , the ratio of these quantities is of order  $\frac{1}{w}$ , which is shown to always fall faster than the survival function derivative. Hence, the income share spent on health decreases.

To which extent do the results depend on the utility per period specification ? For low income levels, the convexity of the marginal utility and the concavity of the utility apply for more general utility specifications. Hence, the result that health is a luxury good for low income levels should extend to more general utility specifications. For high income levels, my result relies on the ratio of marginal utility to utility, which in my specification is of order  $\frac{1}{w}$ . However, there exist utility specifications in the literature for which this ratio does not fall faster than the survival function derivative. Following Hall and Jones (2007), consider the utility function  $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + b$ ,  $\sigma > 1, b > 0$ . For high income levels, the ratio of marginal costs to benefits is of order  $\frac{1}{w^\sigma}$ , which does not necessarily fall faster than the survival function derivative, leaving the possibility for the income share spent on health to perpetually increase with income.<sup>10</sup>

The result is linked to Hall and Jones (2007). In a static framework in which agents

8. In section 2.4, I plot the cross-sectional relationship between income and income share of health expenditures for OECD countries for 2012. The curve is upward-sloping (see Figure 2.5).

9. To see this, note that  $c_{t,t} = \frac{w_t - e(w_t)}{p(e(w_t)) + (1+r)\frac{1}{\sigma}}$ . Using (2.13),  $c_{t,t} = \frac{1-\sigma}{\sigma} \frac{1}{p'(e(w_t))}$ . As  $w_t \rightarrow e(w_t)$  is increasing,  $c_{t,t}$  is increasing with  $w_t$ . The same proof applies for second period consumption.

10. The result is stated formally in section 2.4

choose their consumption levels as well as their health spending, they show that the ratio of health spending to consumption is an increasing function of the ratio of the elasticity of the survival function to the elasticity of the consumption utility. Hence the income share of health spending is driven by the ratio of these two elasticities. For large income levels  $w$ , our discussion is indeed equivalent to a discussion on the ratio of the two elasticities. With  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ ,  $\sigma < 1$ , this ratio behaves as  $wp'(w)$ , while with  $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + b$ ,  $\sigma > 1, b > 0$ , this ratio behaves as  $w^\sigma p'(w)$ . However, the ratio of the two elasticities does not completely govern the income share spent on health for all income levels. In Figures 2.2 and 2.3, I plot, for the survival function  $p(e) = \frac{e^{0.5}}{1+e^{0.5}}$ , the ratio of the two elasticities, which is decreasing in this case, and  $w \rightarrow x(w)$  which is inverted U-shaped according to Proposition 2.3. Hence, the ratio of the two elasticities is an imperfect picture of the income share spent on health.

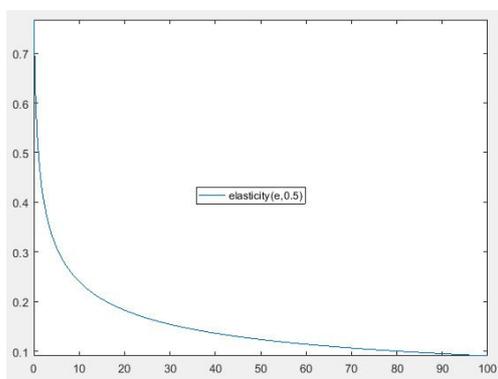


FIGURE 2.2 – Health elasticity to consumption elasticity ratio (i.e.  $e \rightarrow \frac{p'(e)e}{p(e)}(1-\sigma)$ ).  $p(e) = \frac{e^{0.5}}{1+e^{0.5}}$ ,  $\sigma = 0.5$ .

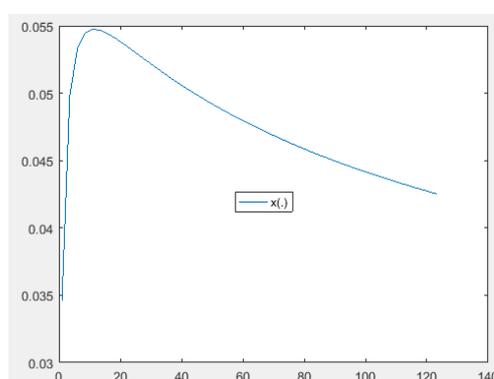


FIGURE 2.3 – Income share spent on health as a function of income (i.e.  $w \rightarrow x(w)$ ).  $p(e) = \frac{e^{0.5}}{1+e^{0.5}}$ ,  $\sigma = 0.5$ ,  $R = 4.801$ .

For the next section, I will maintain Assumption 1. Despite the existence of income levels for which health is a luxury good, there is no perpetual growth of the share of resources spent on health to 1. Otherwise, we could have concluded from the partial equilibrium analysis that the economy would not grow perpetually. Thus, in the next section, I study the dynamic general equilibrium to answer this question. In section 2.4, I rediscuss the alternative specification  $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + b$ ,  $\sigma > 1, b > 0$ .

## 2.3 THE DYNAMIC GENERAL EQUILIBRIUM

In this section, I study the dynamics of the economy of section 2.2, which is obtained by imposing the capital market clearing condition :

$$K_{t+1} = s_t \quad (2.15)$$

Using (2.10) and (2.5), (2.15) is equivalent to :

$$w_{t+1} = \frac{p(e(w_t))}{p(e(w_t)) + (\alpha A)^{\frac{\sigma-1}{\sigma}}} (w_t - e(w_t)) A (1 - \alpha) \quad (2.16)$$

It is useful to study equation (2.16) separately according to the value of  $\underline{p}$ .

### 2.3.1 Dynamics in the case $\underline{p} = 0$ .

In this case, health expenditures are necessary for the economy not to collapse to a null income. Otherwise, without health expenditures, individuals do not live in second period, which implies that they do not save. As capital is an essential input, there is no production. There remains to determine how the economy behaves when individuals spend a positive amount of resources on health. The result is in the following proposition :

**Proposition 2.5.** (i) If  $\frac{\bar{p}}{\bar{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}A(1-\alpha) \leq 1$ , then for all  $w_0 \geq 0$ , the economy converges to a null income.

(ii) If  $\frac{\bar{p}}{\bar{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}A(1-\alpha) > 1$ , then (2.16) has a unique positive steady state,  $w^* > \frac{1-\sigma}{\sigma} \frac{\bar{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ , which is unstable. Hence, if  $w_0 > w^*$ , then the economy perpetually grows, while if  $w_0 < w^*$ , then the economy converges to a null income.

(iii) If the economy perpetually grows, hence if  $\frac{\bar{p}}{\bar{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}A(1-\alpha) > 1$  and  $w_0 > w^*$ , then the growth rate monotonically increases along the trajectory.

*Proof.* See Appendix C ■

Proposition 2.5 shows that the possibility to spend resources on health can have large benefits in terms of economic growth as it can allow the economy to perpetually grow instead of being trapped to a null income. For this to happen, the maximal longevity and the initial income must be high enough. These conditions yield a large enough initial longevity for the individuals to save a level of resources that allows the economy to grow. In this case, according to point (iii), health expenditures create a virtuous cycle : as income increases, health expenditures and then longevity increase, and the propensity to save increases more than the possible disposable income reduction due to greater health expenditures. According to Proposition 2.5, this happens when the maximal longevity  $\bar{p}$  is large. This implies that health expenditures can increase longevity by a large amount and so that the benefits of health expenditures are large and larger than their costs. Thus, savings and the growth rate increase with income.

### 2.3.2 Dynamics in the case $\underline{p} > 0$ .

In this case, health expenditures are no more necessary for this economy to grow. Consider the same economy as the one outlined in section 2.2 without the possibility for the individuals to spend resources on health. It is a standard AK economy, whose dynamics is governed by the following equation :

$$w_{t+1} = \frac{\underline{p}}{\underline{p} + (\alpha A)^{\frac{\sigma-1}{\sigma}}} A(1-\alpha)w_t \quad (2.17)$$

I easily get that the solution to (2.17) perpetually grows if and only if  $\frac{p}{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}} A(1-\alpha) > 1$ . I will write this condition as  $A > \underline{A}$  where  $\underline{A}$  satisfies  $\frac{p}{p+(\alpha \underline{A})^{\frac{\sigma-1}{\sigma}}} \underline{A}(1-\alpha) = 1$ . In the first part of this section, I will maintain this condition and determine if it is sufficient for the economy governed by (2.16) to perpetually grow. Alternatively, we can formulate the problem as follows. The economy perpetually grows if and only if  $G(e(w_t), w_t) := \frac{p(e(w_t))}{p(e(w_t))+(\alpha A)^{\frac{\sigma-1}{\sigma}}}(w_t - e(w_t))A(1-\alpha) > w_t$  for all income levels. Under the condition  $A > \underline{A}$ ,  $e \rightarrow G(e, w_t)$  is decreasing or inverted U-shaped and the equation  $G(e, w_t) = w_t$  has a unique root  $\widehat{e}(w_t)$  such that if for some income levels  $e(w_t) > \widehat{e}(w_t)$ , then the economy does not perpetually grow. Otherwise said, if health expenditures are too large, then the disposable income reduction is greater than the increase of the propensity to save and savings become too low for economic growth to occur. The following proposition, which is the main result of the paper, gives necessary and sufficient conditions for this scenario to happen :

**Proposition 2.6.** *Note  $a := \lim_{e \rightarrow 0} \left( \frac{-p''(e)e}{p'(e)} \right) < 1$ . The three following statements are equivalent :*

(i)  $A > \underline{A}$  and the solution to (2.16) does not perpetually grow.

(ii)  $\alpha > \frac{(p_{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}{\frac{1-a\sigma}{1-\sigma} + (p_{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}$  and  $A \in (\underline{A}, A^*)$ , where  $A^* > \underline{A}$ .

(iii) (2.16) has two steady states  $w_1$  and  $w_2$ , with  $\frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma} < w_1 < w_2$ .  $w_1$  is stable, while  $w_2$  is unstable.

*Proof.* See Appendix D ■

Proposition 2.6 states that the mild condition that insures perpetual growth in the standard AK model ( $A > \underline{A}$ ) is not sufficient to insure perpetual growth when individuals can spend resources to increase their longevity. This means that implementing a health system can have important repercussions on the long-term development of an economy. Assume that  $\gamma$  is finite and consider an economy that does not initially spend resources on health (hence  $w_0 < \frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ ). Its trajectory is initially governed by the evolution equation of the AK model (2.17). Hence, under the condition  $A > \underline{A}$ , the economy grows at a positive rate. When income exceeds the threshold  $\frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ , individuals start spending resources on health. Under the conditions (ii), the economy still grows at a positive rate, yet the growth rate declines until income is trapped to the middle-income level  $w_1$ . Otherwise said, health expenditures create a strong negative intergenerational externality in this case, as they impede any possibility of growth for future generations.

According to conditions (ii), this scenario occurs in economies with a not too large technology level  $A$  and a high initial longevity  $p$ . A high initial longevity implies that health expenditures cannot increase longevity by much since it is already large. This implies that the benefits of health expenditures, a greater propensity to save, are not important in this case.  $A$  influences the growth rate through three channels. First, the greater  $A$ , the greater the interest rate, which increases both savings and health expenditures because the inverse of the intertemporal elasticity of substitution (IES) is strictly smaller than 1. Second, an

increase of  $A$  also acts as an increase of the disposable income (see the linear term in  $A$  in  $G(e(w), w)$ ). The increase of savings and of the disposable income both increase the growth rate. The increase of health expenditures has the two opposite consequences on economic growth previously mentioned : it increases the propensity to save and it decreases the disposable income. In appendix D, I prove that the total effect of an increase of  $A$  on the growth rate is always positive. Hence a greater  $A$  can compensate negative growth effects of health expenditures. This means that once  $A$  is large enough (strictly greater than  $A^*$ ), the economy perpetually grows.

To gain intuition, I reconsider the alternative formulation of the problem : are there income levels for which  $G(e(w_t), w_t) < w_t$ ? I use (2.13) to rewrite this inequality as  $\frac{1-\sigma}{\sigma}A(1-\alpha) < \frac{p'(e(w_t))w_t}{p(e(w_t))}$  and I note that it is sufficient to have  $\frac{1-\sigma}{\sigma}A(1-\alpha) < \frac{p'(w_t)w_t}{p(w_t)}$  for this inequality to be true. Hence when the elasticity of the survival function is high (greater than  $\frac{1-\sigma}{\sigma}A(1-\alpha)$ ), individuals choose to devote a large share of their resources to their health because health expenditures have a strong positive impact on their longevity and then on their welfare.

I now study the dynamics of the economy when it perpetually grows. Does this economy grow faster than the economy that does not spend resources on health? The following proposition gives necessary and sufficient conditions for the economy to grow perpetually and examines if the growth rate is greater than the AK-growth rate :

**Proposition 2.7.** *Note  $g(w_t)$  the growth rate of the economy governed by (2.16) and  $g_{AK} = \frac{p}{p+(1+r)\frac{\sigma-1}{\sigma}}A(1-\alpha) - 1$ . Note also  $\hat{A} = \frac{(p(\frac{1-\sigma}{1-\sigma}-1))^{\frac{\sigma}{\sigma-1}}}{\alpha}$ . Assume  $A > \underline{A}$ .*

(i) *The economy perpetually grows if and only if  $\alpha < \frac{(p_{\frac{1-\sigma}{1-\sigma}}(1-a))^{\frac{\sigma}{\sigma-1}}}{\frac{1-\sigma}{1-\sigma} + (p_{\frac{1-\sigma}{1-\sigma}}(1-a))^{\frac{\sigma}{\sigma-1}}}$  or  $A > A^*$ .*

(ii) *If  $\alpha < \frac{(p_{\frac{1-\sigma}{1-\sigma}}(1-a))^{\frac{\sigma}{\sigma-1}}}{\frac{1-\sigma}{1-\sigma} + (p_{\frac{1-\sigma}{1-\sigma}}(1-a))^{\frac{\sigma}{\sigma-1}}}$  and if  $A \in [\underline{A}, \hat{A}]$ , then :*

$$g(w_t) \begin{cases} = g_{AK} \text{ while } w_t \leq \frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma} \\ > g_{AK} \text{ and } \frac{dg(w_t)}{dt} > 0 \text{ for } w_t > \frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma} \end{cases}$$

(iii) *If  $\alpha < \frac{(p_{\frac{1-\sigma}{1-\sigma}}(1-a))^{\frac{\sigma}{\sigma-1}}}{\frac{1-\sigma}{1-\sigma} + (p_{\frac{1-\sigma}{1-\sigma}}(1-a))^{\frac{\sigma}{\sigma-1}}}$  and if  $A > \hat{A}$  or if  $A > A^*$  and  $\alpha > \frac{(p_{\frac{1-\sigma}{1-\sigma}}(1-a))^{\frac{\sigma}{\sigma-1}}}{\frac{1-\sigma}{1-\sigma} + (p_{\frac{1-\sigma}{1-\sigma}}(1-a))^{\frac{\sigma}{\sigma-1}}}$ ,*

*then there exists  $\tilde{w} > \frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$  such that :*

$$g(w_t) \begin{cases} = g_{AK} \text{ while } w_t \leq \frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma} \\ < g_{AK} \text{ for } w_t \in \left( \frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}, \tilde{w} \right) \\ > g_{AK} \text{ and } \frac{dg(w_t)}{dt} > 0 \text{ for } w_t > \tilde{w} \end{cases}$$

*Proof.* See Appendix E ■

Proposition 2.7 states that there are two types of trajectories for an economy that perpetually grows. Along the first one, health expenditures create a virtuous cycle : as income grows, health expenditures increase and they have a positive impact on the growth rate. Then, at each period, the economy grows faster than the AK-economy. Along the second one,

as individuals start spending resources on health, the growth rate is reduced compared to the one of the AK-economy. Yet, after an income threshold is reached, the growth rate increases and eventually exceeds the one of the AK-economy.

The reason why the economy finally grows faster than the AK-economy in all cases is that the income share spent on health ends decreasing and tends to 0 for large income levels. Thus, the negative effects of health expenditures, a reduced disposable income, vanish, while the positive effects of health expenditures, a higher propensity to save, increase because health expenditures keep rising.

There are two combinations of parameters under which the economy initially grows more slowly than the AK-economy. The first one requires that the initial longevity is high enough and the technology level is high enough to avoid the middle income trap. As previously explained, a high initial longevity implies that health expenditures cannot increase by much the longevity. Then, the benefits of health expenditures are low and are smaller than their costs. The second combination of parameters shows that a high initial longevity is sufficient but not necessary for the economy to initially grow more slowly than the AK-economy. Indeed, this happens also if the initial longevity is low and the technology level is large enough. As previously said, the growth rate of the economy increases with the technology level, however the growth rate of the AK-economy increases too. Recall that there are four channels through which  $A$  impacts the growth rate (interest rate, linear term, longevity, disposable income). The first two are common to both economies, while the last two are absent in the AK-economy. Hence the result is driven by an inverted U-shaped relationship between health expenditures and the growth rate. The increase of  $A$  initially increases more the propensity to save (which is concave in health expenditures) than it reduces the disposable income (which is linear in health expenditures). This implies that the increase of  $A$  initially increases more the growth rate of the economy than the one of the AK economy. Then the contrary happens.

Finally, I determine the dynamics of the economy under the condition  $A < \underline{A}$ . Can the economy perpetually grow under this condition? The following proposition provides the answer :

**Proposition 2.8.** *Assume  $A < \underline{A}$ .*

(i) *If  $A(1 - \alpha) \frac{\bar{p}}{\bar{p} + (\alpha A)^{\frac{\sigma-1}{\sigma}}} \leq 1$ , then the economy converges to a null income.*

(ii) *If  $A(1 - \alpha) \frac{\bar{p}}{\bar{p} + (\alpha A)^{\frac{\sigma-1}{\sigma}}} > 1$ , then the dynamical system (2.16) has a unique unstable steady state,  $w^*$ . If  $w_0 < w^*$ , then the economy converges to a null income. If  $w_0 > w^*$ , then the economy perpetually grows and its growth rate increases along the trajectory.*

*Proof.* See Appendix F ■

Proposition 2.8 shows that health expenditures can be necessary for economic growth to take place. Indeed, when the condition that insures perpetual growth in the AK model fails, it is possible for the economy to perpetually grow. For this to happen, the survival function must not take too small values, otherwise the propensity to save is too low for the economy to grow. Moreover, the initial income level must be high enough. When  $\gamma$  is finite, this poverty trap is due to the fact that individuals initially choose not to spend on health,

which means that the economy behaves exactly as the AK economy, which cannot grow by assumption. Overall, my characterization of the dynamics of the economy shows that the introduction of health expenditures in the AK model yields completely different trajectories depending on the values of the parameters. Indeed, as suggested by Proposition 2.8, health expenditures can be necessary to perpetual growth, while as suggested by Proposition 2.6, health expenditures can annihilate the perpetual growth of the AK economy. This motivates the following section in which I provide a simple calibration of the model.

## 2.4 DISCUSSION AND NUMERICAL ILLUSTRATION

### 2.4.1 Alternative preferences

In the previous sections, utility per period is such that for any survival function, the income share spent on health falls to 0 for large income levels. As previously argued, this is a consequence of assuming an inverse of the IES strictly smaller than 1. There are both types of empirical evidence suggesting values above or below 1 for the IES. Thus, in this section, I briefly discuss my results when utility per period is  $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + b$ ,  $\sigma > 1$ ,  $b > 0$ . In this case, the positive intercept  $b$  is required to avoid that agents unrealistically choose not to spend resources on health for any income level.

With the same notations as before, the FOC on health expenditures now writes :

$$c_{t,t}^{-\sigma} + p'(e_t)c_{t,t+1}^{1-\sigma} = p'(e_t)\left(\frac{c_{t,t+1}^{1-\sigma}}{1-\sigma} + b\right) \quad (2.18)$$

The LHS of (2.18) is the marginal cost of health expenditures : first period utility loss plus second period utility loss due to diminishing per period resources. The RHS of (2.18) is the marginal benefit of health expenditures, which is the total second period utility gain due to longevity extension. Note first that the marginal benefit of health expenditures is negative for low income levels. Hence, the marginal utility of longevity is negative for poor individuals, which implies that they do not spend resources on health. This means that contrary to the case  $\sigma < 1$ , the corner solution for health expenditures is unrelated to the finiteness of the initial marginal productivity of health expenditures  $\gamma$ . Rewrite (2.18) as :

$$c_{t,t}^{-\sigma} = p'(e_t)\left(\frac{\sigma c_{t,t+1}^{1-\sigma}}{1-\sigma} + b\right)$$

For low income levels, as income increases, the first period marginal utility decreases and the second period utility increases by a large amount due to the convexity of the marginal utility and the concavity of the utility per period function. This requires health expenditures to increase by a large amount, more than income, for the marginal cost and the marginal benefit to equate. For large income levels, the second period utility is equivalent to the intercept  $b$ , which does not depend on income. The first period marginal utility decreases by a small amount. Then, the adjustment of health expenditures to equal the marginal cost and the

marginal benefit depends on how the marginal productivity of health expenditures behaves asymptotically. The result is in the following proposition :

**Proposition 2.9.** *The problem of the consumer has a unique solution. There exists  $\underline{w} > 0$  such that  $e(w) = 0$  for  $w \leq \underline{w}$ , while  $w \rightarrow e(w)$  is increasing on  $[\underline{w}, \infty)$ .*

(i)  $w \rightarrow x(w)$  is initially increasing on  $[\underline{w}, \infty)$ .

(ii) If  $\lim_{w \rightarrow \infty} p'(w)w^\sigma = 0$ , then  $\lim_{w \rightarrow \infty} x(w) = 0$ .

(iii) If  $\lim_{w \rightarrow \infty} p'(w)w^\sigma = l < \infty$ , then  $\lim_{w \rightarrow \infty} x(w) = \frac{\frac{(bl)^{\frac{1}{\sigma}}}{(1+r)^{\frac{1-\sigma}{\sigma}} \bar{p}+1}}{1 + \frac{(bl)^{\frac{1}{\sigma}}}{(1+r)^{\frac{1-\sigma}{\sigma}} \bar{p}+1}}$ .

(iv) If  $\lim_{w \rightarrow \infty} p'(w)w^\sigma = \infty$ , then  $\lim_{w \rightarrow \infty} x(w) = 1$ .

*Proof.* See Appendix G ■

Point (i) of Proposition 2.9 confirms our intuition that health is initially a luxury good. The proposition also shows that the parametric specification of the survival function is crucial to determine the shape of the function  $w \rightarrow x(w)$ , which can perpetually increase with income towards 1 or falls to 0. This justifies to have worked with a general survival function. For example, with  $p(e) = \frac{p+\bar{p}e^\epsilon}{1+e^\epsilon}$ ,  $\epsilon \in (0, 1]$ , case (ii) occurs if  $\sigma < 1 + \epsilon$ , case (iii) occurs if  $\sigma = 1 + \epsilon$ , while case (iv) occurs if  $\sigma > 1 + \epsilon$ . With  $p(e) = \frac{2}{\sqrt{\Pi}} \bar{p} \int_0^e e^{-\frac{u^2}{2}} du + \underline{p}$ , only case (ii) occurs.

As an immediate corollary, we see that in case (iv) the economy does not perpetually grow under the mild condition that insures perpetual growth in the AK model. With this utility specification, I cannot provide a complete characterization of the dynamics of the economy as in section 2.3, so I calibrate the model to study the dynamical general equilibrium.

## 2.4.2 Numerical application

In this subsection, I propose a numerical calibration of the model for OECD countries, when  $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + b$ ,  $\sigma > 1, b > 0$ , to answer two questions : Will the income share spent on health continue to increase? How health expenditures modify the trajectory of these economies ?

I consider that a period is equal to 40 years and that individuals enter first period at the age of 25.  $r$  and  $\alpha$  and  $\sigma$  are first set to standard values. The annual interest is 4%, which yields a value of 4.801 for  $R$ .  $\alpha$  is set to 0.3.  $\sigma$  is set to 1.5, which yields a risk aversion coefficient in line with empirical estimates (Chetty 2006). The remaining unknowns are  $b$  and the survival function.

Usually in such models, the unknown parameters are calibrated from the equation that governs the dynamics of the economy (equation (2.16)) to match time series data. However, the two-period framework requires long time series to have a sufficient number of points to match. Here I rather notice that all the remaining unknowns enter the equation that determines the income share spent on health  $x(w)$ .<sup>11</sup> This equation links health expenditures

11. When  $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + b$ ,  $\sigma > 1, b > 0$ , this is the equation (2.32) in Appendix G

to income, thus it can be estimated from cross-sectional data. I then choose my unknowns by minimizing the distance between the income share spent on health generated by the model and the true ones. Regarding first the survival function, I have tried all the examples cited in the paper, yet the best fit of the model is obtained with the following specification :

$$p(e) = \frac{p + \bar{p} \ln(1 + \frac{e}{C})}{1 + \ln(1 + \frac{e}{C})} \quad (2.19)$$

Where  $C$  is a positive constant to be estimated. A bounded and concave function becomes rapidly flat. The parameter  $C$  allows not to have our health expenditures levels on the flat part of the curve, which could not yield a good fit of the cross-sectional variation of longevity. However, the higher the scaling parameter  $C$ , the lower the dispersion of the  $\frac{e_i}{C}$ , which also impedes a good fit of the cross-sectional variation of longevity. Thus, I need a survival function that does not become flat too fast in order not to have to use a too large scaling parameter. This explains why I obtain my best fit with the function (2.19). Thus, there are four parameters to be estimated :  $(b, C, \underline{p}, \bar{p})$ . Note  $(w_i, x_i, p_i)_{i=1..33}$  our data, where  $w_i$  is wage in country  $i$ ,  $x_i$  the income share of health expenditures and  $p_i$  the longevity.<sup>12</sup> Then, my parameters choice  $(b^*, C^*, \underline{p}^*, \bar{p}^*)$  is :

$$\arg \min_{(b, C, \underline{p}, \bar{p})} \left( \sum_{i=1}^{33} (x(w_i) - x_i)^2 + (p_m - p(w_m x(w_m)))^2 + (p_M - p(w_M x(w_M)))^2 \right) \quad (2.20)$$

Where  $m = \arg \min_{i=1..33} (p_i)$  and  $M = \arg \max_{i=1..33} (p_i)$ . I obtain  $(b^*, C^*, \underline{p}^*, \bar{p}^*) =$

$(0.0206, 39991.876, 0.02, 0.65)$ . Figure 2.5 plots  $(w_i, x_i)_{i=1..33}$  and  $(w_i, x(w_i))_{i=1..33}$ . Figure 2.4 plots  $(w_i x_i, p_i)_{i=1..33}$  and  $(w_i x(w_i), p(w_i x(w_i)))_{i=1..33}$ . The two graphs suggest that the model can replicate reasonably well the relationship between income and income share spent on health and health expenditures and longevity. There is one country, US, which is not well captured by the model : its income share on health is much larger than other countries, while its life expectancy is only in the middle of the distribution.

My calibration parameters have two direct consequences. First, without health expenditures, the economy does not grow, hence the condition that insures perpetual growth in the AK model fails. This means that there is a poverty trap. Second, the economy is in the case (iv) of Proposition 2.9, hence the income share spent on health tends to 1 for large income levels. This means that there is a positive stable steady state. I first compute the values of these steady states to assess the position of my sample with respect to these points. I find that the first steady-state income level is worth 12149. Two countries of the sample are below this level, which means that they are trapped. This suggests that health expenditures could

12. The data are obtained from the OECD database for the year 2012. The sample includes all OECD members except Luxembourg (33 countries). I use GDP per capita in current US dollars to compute the corresponding GDP per worker and then the wage. Total health expenditures are also in current US dollars. For the estimation, I multiply these quantities by 40 to get their value on the model period. When reporting the results, I use annual values.  $p_i$  is computed from life expectancy at birth,  $E_i : E_i = 65 + 40p_i$ .

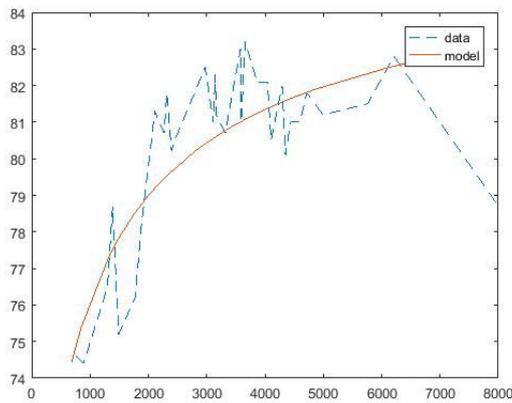


FIGURE 2.4 – *Health expenditures-Life expectancy*

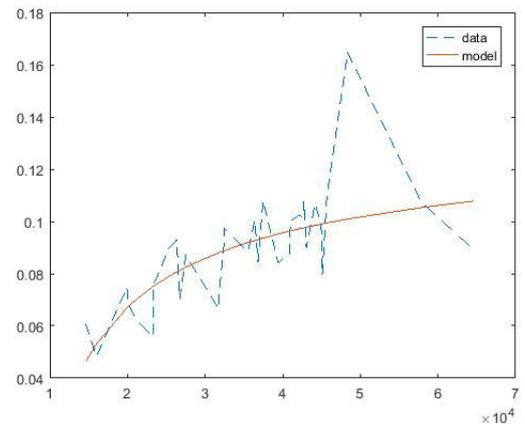


FIGURE 2.5 – *Income-Income share spent on health*

be a barrier to convergence in income levels across countries. The second steady-state income level is greater than  $1.10^9$ , which is well above the income levels of the sample. This means that the growing income share spent on health should not be an obstacle to economic growth in a not too far future for the countries which are not trapped. I now simulate the trajectory over 10 periods for the economy with the median income. Figure 2.6 reports the dynamics of the wage, while Figure 2.7 reports the dynamics of the income share spent on health. Over this period, the economy grows at an accelerating rate and its income share spent on health is increasing. After 10 periods, the income share spent on health reaches 0.33, which is more than the triple of its initial value. This does not prevent the economy from growing. This means that economic growth can take place despite a large amount of resources spent for health. The overall conclusion of this numerical analysis is that health expenditures are more necessary than detrimental to economic growth and that health expenditures endanger economic growth only if their level becomes much larger than their current level in OECD countries.

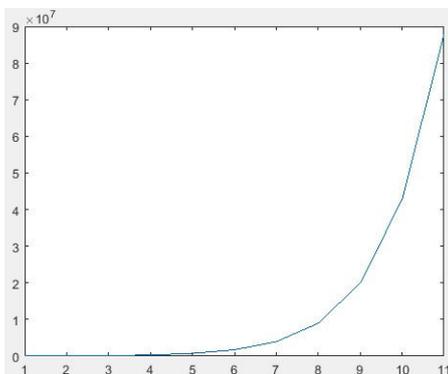


FIGURE 2.6 – *Wage dynamics over 10 periods*

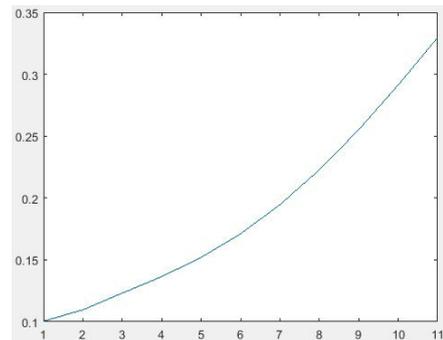


FIGURE 2.7 – *Income share spent on health dynamics over 10 periods*

## 2.5 CONCLUSION

In a two-period OLG model with endogenous growth, I studied the consequences of allowing individuals to choose the level of health expenditures that increase their longevity in retirement period. I presented several results. With a CES utility function, with an IES strictly greater than 1, and a general survival function, I proved that the income share spent on health is an inverted U-shaped function of income. This implies that an increasing ratio of health elasticity to consumption elasticity is neither necessary nor sufficient for health to be a luxury good. Then, I gave a complete characterization of the dynamics of the economy. Under the condition that insures perpetual growth in the same AK economy except that health expenditures are constrained to be null, there are three types of trajectories. Along the first one, the economy perpetually grows and grows at each period faster than the AK economy. Along the second one, the economy perpetually grows, however its growth rate is initially reduced compared to the one of the AK economy, before growing faster than the AK economy. Finally, along the third trajectory, the economy is trapped to a middle income level and does not experience perpetual growth. This means that health expenditures create a strong negative intergenerational externality in this case by impeding any possibility of growth for future generations. I also found that when the condition that insures perpetual growth in the AK model fails, the economy can all the same experience perpetual growth. A simple calibration of the model to OECD countries suggests that this case might be the most likely one, hence that health expenditures are more necessary than detrimental to growth.

Acemoglu and Johnson (2007) conclude their study by noting that the decision to implement a health system in a country can be considered as orthogonal to its development policy given the weak impacts of longevity on economic growth they find. In this paper, I reach a different conclusion as I have shown theoretically that the implementation of a health system is not neutral for economic growth. Indeed, in the framework used, a standard Diamond model with health expenditures, there are economies in which the presence of these health expenditures produce drastic negative consequences by impeding any possibility of long-run economic growth. This should stimulate future empirical research on the health-growth nexus. Given its theoretical focus, the present analysis has omitted possibly important channels through which health expenditures can modify the growth path of an economy such as the impact of health on productivity or the introduction of a social security system. They could be included in future simulation studies.

## 2.6 APPENDIX A

Assume first that  $\underline{p} > 0$  and  $\gamma < \infty$ . The problem of the consumer is equivalent to maximize :

$$V(s_t, e_t) = \frac{(w_t - s_t - e_t)^{1-\sigma}}{1-\sigma} + p(e_t)^\sigma \frac{s_t^{1-\sigma} (1+r)^{1-\sigma}}{1-\sigma}$$

Subject to the constraints  $s_t + e_t \leq w_t$  and  $0 \leq e_t$  and  $0 \leq s_t$ . The Lagrangian associated to this problem writes :

$$L(s_t, e_t, \chi_1, \chi_2, \chi_3) = V(s_t, e_t) + \chi_1(w_t - s_t - e_t) + \chi_2 e_t + \chi_3 s_t$$

Where  $\chi_1$  to  $\chi_3$  are the Lagrange multipliers. Note first that  $\frac{\partial V}{\partial s}(0, e_t) = \infty$  for all  $e_t \in [0, w_t)$ . Thus,  $s_t = 0$  is never optimal and  $\chi_3 = 0$ . Third,  $\frac{\partial V}{\partial s}(w_t - e_t, e_t) = -\infty$  for all  $e_t \in [0, w_t)$ . Thus,  $\chi_1$  is equal to 0 and the KKT conditions for a point  $(s_t, e_t)$  to be an optimum can be written as :

$$(i) \quad \frac{\partial V}{\partial s}(s_t, e_t) = -(w_t - e_t - s_t)^{-\sigma} + p(e_t)^\sigma s_t^{-\sigma} (1+r)^{1-\sigma} = 0$$

$$(ii) \quad \frac{\partial V}{\partial e}(s_t, e_t) = -(w_t - e_t - s_t)^{-\sigma} + \sigma p'(e_t) p(e_t)^{\sigma-1} \frac{s_t^{1-\sigma} (1+r)^{1-\sigma}}{1-\sigma} = -\chi_2$$

$$(iii) \quad \min(\chi_2, e_t) = 0$$

Consider now the possibility that  $e_t = 0$ . From (i), I get the optimal saving,  $s_t = \frac{\underline{p}}{p+(1+r)^{\frac{\sigma-1}{\sigma}}} w_t$ . And  $\chi_2$  must be non-negative. Thus, (ii) writes  $\frac{\partial V}{\partial e}(\frac{\underline{p}}{p+(1+r)^{\frac{\sigma-1}{\sigma}}} w_t, 0) \leq 0$  which is equivalent to  $w_t \leq \frac{1-\sigma}{\sigma} \frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ . Thus, for  $w_t \leq \frac{1-\sigma}{\sigma} \frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ ,  $(\frac{\underline{p}}{p+(1+r)^{\frac{\sigma-1}{\sigma}}} w_t, 0)$  satisfies the KKT conditions and is a possible solution. Consider now the case  $e_t > 0$ . Then,  $\chi_2 = 0$  and the conditions (i) and (ii) imply the equations (2.10) and (2.13) of the text. Note that the left-hand-side (LHS) of (2.13) decreases and is worth 0 at  $e_t = w_t$ . Thus, (2.13) has a unique positive solution if and only if the LHS of (2.13) takes a value strictly greater than 1 at  $e_t = 0$ . This condition is equivalent to  $w_t > \frac{1-\sigma}{\sigma} \frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ . Thus, for  $w_t > \frac{1-\sigma}{\sigma} \frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ , there exists a unique pair  $(s_t, e_t)$  that satisfies the KKT conditions. Note finally that the problem of the consumer has always at least one solution because  $(s, e) \rightarrow V(s, e)$  is continuous on the maximization domain, which is compact. Consequently, the unique pair satisfying the KKT conditions in the two cases  $w_t \leq \frac{1-\sigma}{\sigma} \frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$  and  $w_t > \frac{1-\sigma}{\sigma} \frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$  is the unique solution to the problem of the consumer. In the first case, the optimal level of health expenditures is equal to 0, while it is positive in the second case. The case  $\underline{p} = 0$  follows by continuity. When  $\gamma = \infty$ , notice that the KKT conditions for a corner solution to exist cannot be satisfied. This completes the proof of Proposition 2.1.

## 2.7 APPENDIX B

Set  $\underline{w} = \frac{1-\sigma}{\sigma} \frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ , which is possibly equal to 0 when  $\gamma = \infty$ . I first need to compute  $\lim_{w \rightarrow \underline{w}} x(w)$ . If  $\gamma < \infty$ , then  $e(\underline{w}) = x(\underline{w})\underline{w} = 0$ , then  $\lim_{w \rightarrow \underline{w}} x(w) = 0$ . If  $\gamma = \infty$ , then  $\underline{w} = 0$ . I will need the following lemma :

**Lemma 2.10.**  $\lim_{w \rightarrow 0} (wp'(w)) = 0$

*Proof.* The first derivative of  $w \rightarrow wp'(w)$  is  $p'(w)(1 - \frac{(-p''(w)w)}{p'(w)})$ . The properties of the function  $p$  imply that  $a := \lim_{w \rightarrow 0} (\frac{(-p''(w)w)}{p'(w)}) < 1$ . This implies that  $w \rightarrow wp'(w)$  increases in the neighborhood of 0. Consequently  $\lim_{w \rightarrow 0} (wp'(w))$  exists. It must be finite because  $w \rightarrow wp'(w)$  is initially increasing. Assume  $\lim_{w \rightarrow 0} (wp'(w)) = l > 0$ . Then there must exist  $W > 0$  such that :

$$u < W \Rightarrow up'(u) > \frac{l}{2} \quad (2.21)$$

Consider  $w < W$ . Divide by  $u$  both terms of the previous inequality and integrate it from  $w$  to  $W$  to get :

$$\frac{l}{2} \ln\left(\frac{W}{w}\right) < p(W) - p(w)$$

$$\Leftrightarrow p(w) + \frac{l}{2} \ln(W) < p(W) + \frac{l}{2} \ln(w)$$

This contradicts the fact that  $\lim_{w \rightarrow 0} p(w) > -\infty$ . Then, it must be that  $\lim_{w \rightarrow 0} (wp'(w)) = 0$

■

Then use (2.13) :

$$\frac{\sigma}{1-\sigma} \frac{p'(e(w))e(w)}{p(e(w)) + (1+r)^{\frac{\sigma-1}{\sigma}}} = \frac{x(w)}{1-x(w)}$$

Thus,  $\lim_{w \rightarrow 0} \left(\frac{x(w)}{1-x(w)}\right) = \lim_{w \rightarrow 0} \left(\frac{\sigma}{1-\sigma} \frac{p'(e(w))e(w)}{p(e(w)) + (1+r)^{\frac{\sigma-1}{\sigma}}}\right) = \frac{\sigma}{1-\sigma} \frac{\lim_{w \rightarrow 0} (wp'(w))}{p+(1+r)^{\frac{\sigma-1}{\sigma}}} = 0$  according to Lemma 2.10. This implies that  $\lim_{w \rightarrow 0} (x(w))$  exists and is equal to 0.

Apply now the implicit function theorem to (2.13) to get that :

$$x'(w_t) = \frac{1}{w_t} \frac{p'(x(w_t)w_t)(\sigma - x(w_t)) + x(w_t)p''(x(w_t)w_t)\sigma w_t(1 - x(w_t))}{p'(x(w_t)w_t) + (-p''(x(w_t)w_t))\sigma w_t(1 - x(w_t))} \quad (2.22)$$

Thus,  $x'(w_t) > 0$  is equivalent to :

$$\frac{(-p''(x(w_t)w_t))x(w_t)w_t}{p'(x(w_t)w_t)} < \frac{\sigma - x(w_t)}{\sigma(1 - x(w_t))} \quad (2.23)$$

Define  $m(x, w_t) \equiv \frac{(-p''(xw_t))xw_t}{p'(xw_t)}$  and  $g(x) \equiv \frac{\sigma - x}{\sigma(1 - x)}$ .

By Assumption 2,  $x \rightarrow m(x, w_t)$  increases from a value strictly smaller than 1 on  $[0, 1]$ .  $x \rightarrow g(x)$  decreases from 1 to  $-\infty$  on  $[0, 1]$ . Thus, for any  $w_t$  there exists a unique root on  $[0, 1]$

to the equation  $m(x, w_t) = g(x)$ . Note it  $\Sigma(w_t)$ .  $w_t \rightarrow \Sigma(w_t)$  decreases because  $\frac{\partial m}{\partial w_t}(x, w_t) > 0$  and  $\lim_{w_t \rightarrow \infty}(\Sigma(w_t)) = 0$ . Draw the curve  $w_t \rightarrow \Sigma(w_t)$  on  $[\underline{w}, \infty)$ . Note that  $x'(w_t) > 0$  if and only if  $x(w_t) \in \{y \geq 0, y < \Sigma(w_t)\} = \Delta$ . Initially  $x(w_t) \in \Delta$  because  $\lim_{w_t \rightarrow \underline{w}} x(w_t) = 0$ . There necessarily exists  $w^*$  such that  $x(w^*) = \Sigma(w^*)$  because  $\Sigma$  decreases towards 0. By definition of  $\Sigma$ ,  $x'(w^*) = 0$ , while  $\Sigma'(w^*) < 0$ . Thus,  $w_t \rightarrow x(w_t)$  enters  $\Delta$  at  $w^*$  and  $x(w_t)$  is trapped in  $\Delta$  because wherever it hits the boundary of  $\Delta$ , it has a greater slope than the boundary. Thus  $w \rightarrow x(w)$  is inverted U-shaped.

There remains to compute the limit of  $x(w_t)$  as  $w_t$  tends towards  $\infty$ . This limit exists as  $w_t \rightarrow x(w_t)$  ends decreasing.

**Lemma 2.11.**  $\lim_{w \rightarrow \infty} (wp'(w)) = 0$

*Proof.* I first show that  $w \rightarrow wp'(w)$  ends decreasing. Its first derivative is  $p'(w)(1 - \frac{(-p''(w))w}{p'(w)})$  which is non-positive when  $w$  gets large as  $\lim_{w \rightarrow \infty} \frac{(-p''(w))w}{p'(w)} > 1$ . Therefore,  $\lim_{w \rightarrow \infty} (wp'(w))$  exists and it must be finite because  $w \rightarrow wp'(w)$  ends decreasing. Note this limit  $l$  and assume that  $l$  is positive. Then, there exists  $M > 0$  such that  $wp'(w) > \frac{l}{2}$  for all  $w$  greater than  $M$ . Integrate the previous inequality from  $M$  to  $a > M$  to get :

$$p(a) - p(M) > \frac{l}{2} \ln\left(\frac{a}{M}\right) \quad (2.24)$$

As  $a$  tends towards  $\infty$ , the RHS of (2.24) tends towards  $\infty$ . This contradicts the fact that  $p$  is upper-bounded. Thus,  $l = 0$  ■

From (2.13) :

$$\lim_{w \rightarrow \infty} \left( \frac{x(w)}{1 - x(w)} \right) = \lim_{w \rightarrow \infty} \left( \frac{\sigma}{1 - \sigma} \frac{p'(e(w))e(w)}{p(e(w)) + (1+r)^{\frac{\sigma-1}{\sigma}}} \right) = \frac{\sigma}{1 - \sigma} \frac{\lim_{w \rightarrow \infty} (wp'(w))}{\bar{p} + (1+r)^{\frac{\sigma-1}{\sigma}}} = 0 \quad (2.25)$$

This implies that  $\lim_{w \rightarrow \infty} (x(w)) = 0$ . This completes the proof of Proposition 2.3.

## 2.8 APPENDIX C

If  $\frac{\bar{p}}{\bar{p} + (1+r)^{\frac{\sigma-1}{\sigma}}} A(1 - \alpha) < 1$ , then  $\frac{G(e(w_t), w_t)}{w_t} = \frac{p(e(A, w))}{p(e(A, w)) + (\alpha A)^{\frac{\sigma-1}{\sigma}}} \frac{(w - e(A, w))}{w} A(1 - \alpha) < \frac{\bar{p}}{\bar{p} + (1+r)^{\frac{\sigma-1}{\sigma}}} A(1 - \alpha) < 1$ . Hence, the propagator of (2.16) is strictly below the 45° line for any income levels, which implies that the economy converges to 0.

If  $\frac{\bar{p}}{\bar{p} + (1+r)^{\frac{\sigma-1}{\sigma}}} A(1 - \alpha) > 1$  :

$$\underline{p} = 0 \text{ implies that } \frac{G(e(\frac{1-\sigma}{\sigma}(\alpha A)^{\frac{\sigma-1}{\sigma}}), \frac{1-\sigma}{\sigma}(\alpha A)^{\frac{\sigma-1}{\sigma}})}{\frac{1-\sigma}{\sigma}(\alpha A)^{\frac{\sigma-1}{\sigma}}} = 0 \text{ for all } \gamma \in (0, \infty].$$

Use (2.13) to write  $\frac{G(e(w), w)}{w}$  as :

$$\frac{G(e(w), w)}{w} = \frac{1 - \sigma}{\sigma} \frac{p(e(w))}{wp'(e(w))} A(1 - \alpha) \quad (2.26)$$

Note first from (2.22) that :

$$e'(w) = \frac{\sigma p'(e(w))}{p'(e(w)) + (-p''(e(w)))\sigma(w - e(w))} \quad (2.27)$$

Then I can compute the derivative of  $\frac{G(e(w),w)}{w}$  with respect to  $w$ . I find that :

$$\frac{d\left(\frac{G(e(w),w)}{w}\right)}{dw} = \frac{(1-\sigma)A(1-\alpha)p(e(w))}{w^2} \left( \frac{1}{\frac{G(e(w),w)}{w}} - \frac{(1-\sigma)\frac{(-p''(e(w))e(w))}{p'(e(w))}}{\sigma} \right) \quad (2.28)$$

Thus, I get that :

$$\frac{d\left(\frac{G(e(w),w)}{w}\right)}{dw} > 0 \Leftrightarrow \frac{(1-\sigma)A(1-\alpha)}{\frac{G(e(w),w)}{w}} > \frac{(1-\sigma)\frac{(-p''(e(w))e(w))}{p'(e(w))}}{\sigma} \quad (2.29)$$

At  $w = \frac{1-\sigma}{\sigma} \frac{(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ , this condition writes :

$$\infty > \frac{1-\sigma\alpha}{\sigma}$$

As this condition holds, it must be that  $w \rightarrow \frac{1}{\frac{G(e(w),w)}{w}}$  is initially decreasing. Note also that  $w \rightarrow \frac{(1-\sigma)\frac{(-p''(e(w))e(w))}{p'(e(w))}}{\sigma}$  is decreasing. If  $\frac{1}{\frac{G(e(w),w)}{w}}$  stays below  $\frac{(1-\sigma)\frac{(-p''(e(w))e(w))}{p'(e(w))}}{\sigma}$ , then  $w \rightarrow \frac{1}{\frac{G(e(w),w)}{w}}$  is decreasing. If for some  $y > \frac{1-\sigma}{\sigma} \frac{(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ ,  $\frac{1}{\frac{G(e(y),y)}{w}}$  is equal to  $\frac{(1-\sigma)\frac{(-p''(e(y))e(y))}{p'(e(y))}}{\sigma}$ , then  $\frac{d\left(\frac{1}{\frac{G(e(w),w)}{w}}\right)}{dw}\Big|_{w=y} = 0$ , while the derivative of  $\frac{(1-\sigma)\frac{(-p''(e(w))e(w))}{p'(e(w))}}{\sigma}$  at  $w = y$  is negative. Thus,  $\frac{1}{\frac{G(e(w),w)}{w}} > \frac{(1-\sigma)\frac{(-p''(e(w))e(w))}{p'(e(w))}}{\sigma}$  in the right neighborhood of  $y$ . This proves that  $\frac{d\left(\frac{G(e(w),w)}{w}\right)}{dw} \geq 0$  for all  $w > \frac{1-\sigma}{\sigma} \frac{(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ .

Finally, recall that  $\lim_{w \rightarrow \infty} (x(w)) = 0$ , which implies that  $\lim_{w \rightarrow \infty} \left(\frac{G(e(w),w)}{w}\right) = \frac{\bar{p}}{\bar{p}+(1+r)\frac{\sigma-1}{\sigma}} A(1-\alpha) > 1$ . This proves that the propagator of (2.16) is equal to 0 while  $w \leq \frac{1-\sigma}{\sigma} \frac{(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ , then increases and crosses the 45° line exactly one time. This completes the proof of the point (ii).

Point (iii) follows from the fact that  $w \rightarrow \frac{G(e(w),w)}{w}$  increases on  $\left[\frac{1-\sigma}{\sigma} \frac{(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}, \infty\right)$ .

For the rest of the paper it will be useful to get the variations of  $w \rightarrow \frac{G(e(w),w)}{w}$  when  $\underline{p} > 0$ . Use first (2.29) at  $w = \frac{1-\sigma}{\sigma} \frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$  to see that  $w \rightarrow \frac{G(e(w),w)}{w}$  is initially increasing if and only if  $\frac{1-\sigma}{\sigma} \frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\underline{p}} > \frac{1-\sigma\alpha}{\sigma}$ . Use the same argument as in the case  $\underline{p} = 0$  to get that  $w \rightarrow \frac{G(e(w),w)}{w}$  is increasing. If  $\frac{1-\sigma}{\sigma} \frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\underline{p}} < \frac{1-\sigma\alpha}{\sigma}$ , then  $w \rightarrow \frac{G(e(w),w)}{w}$  is initially decreasing. Hence  $w \rightarrow \frac{1}{\frac{G(e(w),w)}{w}}$  is initially increasing, while  $w \rightarrow \frac{(1-\sigma)\frac{(-p''(e(w))e(w))}{p'(e(w))}}{\sigma}$  is decreasing. Thus, there exists  $y > \frac{1-\sigma}{\sigma} \frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ , such that  $\frac{1}{\frac{G(e(y),y)}{w}} = \frac{(1-\sigma)\frac{(-p''(e(y))e(y))}{p'(e(y))}}{\sigma} \cdot \frac{d\left(\frac{1}{\frac{G(e(w),w)}{w}}\right)}{dw}\Big|_{w=y} = 0$ , while

the derivative of  $\frac{(1-\sigma)\frac{-p''(e(w))e(w)}{p'(e(w))}}{\sigma}$  at  $w = y$  is negative. Thus,  $\frac{1}{\frac{G(e(w),w)}{w}} > \frac{(1-\sigma)\frac{-p''(e(w))e(w)}{p'(e(w))}}{\sigma}$  in the right neighborhood of  $y$  and  $w \rightarrow \frac{G(e(w),w)}{w}$  remains increasing according to the previous argument. Thus,  $w \rightarrow \frac{G(e(w),w)}{w}$  is U-shaped if  $\frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{p} < \frac{1-\sigma a}{\sigma}$ .

## 2.9 APPENDIX D

I will assume that  $\gamma$  is positive, yet by following the same steps, the proof can be adapted to the case  $\gamma = \infty$ .

It will be necessary here to write explicitly the dependence of  $e(w_t)$  with respect to  $A$ , due to the dependence of the interest rate on  $A$ . Then define :

$$H(A, w) = \frac{p(e(A, w))}{p(e(A, w)) + (\alpha A)^{\frac{\sigma-1}{\sigma}}} \frac{(w - e(A, w))}{w} A(1 - \alpha) \quad (2.30)$$

**Step 1 :** I prove that  $\frac{\partial H}{\partial A}(A, w) > 0$  for any pair  $(A, w) \in (0, \infty)^2$ .

For  $(A, w) \in (0, \infty)^2$  such that  $w \leq \frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ ,  $H(A, w) = \frac{p}{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}} A(1 - \alpha)$  and the result follows.

For  $(A, w) \in (0, \infty)^2$  such that  $w > \frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ , apply the implicit function theorem to (2.13) to get that  $\frac{\partial e}{\partial A}(A, w) > 0$ .

Rewrite (2.30) as :

$$H(A, w) = \frac{1 - \sigma}{\sigma} \frac{p(e(A, w))}{w p'(e(A, w))} A(1 - \alpha) \quad (2.31)$$

Thus,  $\frac{\partial H}{\partial A} > 0$ .

**Step 2 :** I prove that for any  $w > 0$ , there exists a unique  $Z(w) > 0$  such that :  $A < Z(w) \iff H(A, w) < 1$ .

This follows from the fact that  $A \rightarrow H(A, w)$  increases (step 1) from 0 to  $\infty$  on  $[0, \infty)$ .

Define now for each  $w > \frac{1-\sigma}{\sigma} \frac{p}{\gamma}$ ,  $A(w)$  as  $w = \frac{1-\sigma}{\sigma} \frac{p+(\alpha A(w))^{\frac{\sigma-1}{\sigma}}}{\gamma}$  and reciprocally for each  $A > 0$ ,  $w(A)$  as  $w(A) = \frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ .

**Step 3 :** I prove that for  $w \leq w(\underline{A})$ ,  $Z(w) = \underline{A}$ .

By definition,  $H(\underline{A}, w(\underline{A})) = 1$ . Thus,  $Z(w(\underline{A})) = \underline{A}$ .

If  $w < w(\underline{A})$ , then  $A(w) > \underline{A}$ . Thus  $H(A(w), w) = \frac{p}{p+(\alpha A(w))^{\frac{\sigma-1}{\sigma}}} A(w)(1 - \alpha)$  is strictly greater than 1. Thus,  $Z(w) < A(w)$ .

So,  $H(Z(w), w) = \frac{p}{p+(\alpha Z(w))^{\frac{\sigma-1}{\sigma}}} Z(w)(1 - \alpha)$ . As  $H(Z(w), w) = 1$ , it must be that  $Z(w) = \underline{A}$ .

This means that under the condition  $A > \underline{A}$ , the propagator of (2.16) is strictly above the 45° line for  $w < w(\underline{A})$ . Thus, any steady state of (2.16) is necessarily greater than  $w(\underline{A})$  under the condition  $A > \underline{A}$ .

**Step 4 :** I prove that  $\max_{w \geq w(\underline{A})} (Z(w))$  exists. And  $\max_{w \geq w(\underline{A})} (Z(w)) > \underline{A} \Leftrightarrow \frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma} \frac{p+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{p}$ .

The continuity of  $(A, w) \rightarrow H(A, w)$  implies the one of  $w \rightarrow Z(w)$ . Moreover,  $Z(w(\underline{A})) = \underline{A}$ .  $Z(w)$  has also a limit in  $\infty$ , noted  $Z(\infty)$ , which satisfies :

$$\frac{\bar{p}}{\bar{p} + (\alpha Z(\infty))^{\frac{\sigma-1}{\sigma}}} Z(\infty)(1 - \alpha) = 1$$

The fact that  $\underline{p} < \bar{p}$  implies that  $Z(\infty) < \underline{A}$ . This proves that  $w \rightarrow Z(w)$  has a maximum on  $[w(\underline{A}), \infty)$ .

Consider now the function  $w \rightarrow H(\underline{A}, w)$  on  $[w(\underline{A}), \infty)$ . From the proof of Proposition 2.5. (Appendix C), this function is increasing (if  $\frac{1-a\sigma}{\sigma} < \frac{w(\underline{A})p'(e(w(\underline{A})))}{p(e(w(\underline{A})))} = \frac{1-\sigma}{\sigma} \frac{p+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{p}$ ) or U-shaped (otherwise).

If  $w \rightarrow H(\underline{A}, w)$  is increasing, then  $H(\underline{A}, w) > 1$  for all  $w \geq w(\underline{A})$ . Thus,  $Z(w) < \underline{A}$  for all  $w \geq w(\underline{A})$ . Hence,  $\max_{w \geq w(\underline{A})} (Z(w))$  is exactly  $\underline{A}$  in this case.

If  $w \rightarrow H(\underline{A}, w)$  is U-shaped, then there exists  $z(\underline{A}) > w(\underline{A})$  such that  $H(\underline{A}, w) < 1$  for all  $w \in (w(\underline{A}), z(\underline{A}))$ . Thus,  $Z(w) > \underline{A}$  for all  $w \in (w(\underline{A}), z(\underline{A}))$ . Hence,  $\max_{w \geq w(\underline{A})} (Z(w))$  is strictly greater than  $\underline{A}$  in this case.

Thus, it must be that  $\max_{w \geq w(\underline{A})} (Z(w)) > \underline{A} \Leftrightarrow \frac{1-a\sigma}{\sigma} < \frac{1-\sigma}{\sigma} \frac{p+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{p}$ .

Write  $A^* = \max_{w \geq w(\underline{A})} (Z(w))$ .

**Step 5 :** I prove that  $\underline{A} < A^*$  and  $A \in (\underline{A}, A^*) \Leftrightarrow (2.16)$  has exactly two positive steady states under the condition  $A > \underline{A}$ .

Assume first that (2.16) has exactly two positive steady states and  $A > \underline{A}$ . It means that  $Z(w)$  takes values strictly greater than  $\underline{A}$  on  $(w(\underline{A}), \infty)$ . Thus,  $\underline{A} < A^*$ . Note also that  $A < A^*$  otherwise  $H(A, w)$  would be strictly greater than 1 for all  $w \geq 0$  which would contradict the fact that (2.16) has two steady states.

Assume now that  $\underline{A} < A^*$  and  $A \in (\underline{A}, A^*)$ . Then,  $w \rightarrow H(A, w)$  takes values strictly smaller than 1 on  $(w(\underline{A}), \infty)$ ,  $H(A, w) > 1$  if  $w \leq w(\underline{A})$  and  $\lim_{w \rightarrow \infty} (H(A, w)) > 1$  because  $A > \underline{A}$ . Thus, (2.16) has at least two steady states. Denote  $w_1(A)$  the smallest one, which is necessarily stable, and  $w_2(A)$ , the highest one, which is necessarily unstable.

From step 4, the condition  $\underline{A} < \max_{w \geq w(\underline{A})} (Z(w))$  is equivalent to  $\frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma} \frac{p+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{p}$ .

Then, it must be that  $\frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma} \frac{p+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{p}$  and according to the proof of Proposition 2.5, it must be that  $w \rightarrow H(A, w)$  is U-shaped. Consequently,  $w_1(A)$  and  $w_2(A)$  are the two only steady states of (2.16).

To complete the proof, I rearrange the condition  $\frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma} \frac{p+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{p}$  to obtain condition (ii).

**Step 6 :** I prove that  $\frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma} \frac{p+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{p} \Leftrightarrow \alpha > \frac{(p \frac{\sigma}{1-\sigma} (1-a))^{\frac{\sigma}{\sigma-1}}}{\frac{1-a\sigma}{1-\sigma} + (p \frac{\sigma}{1-\sigma} (1-a))^{\frac{\sigma}{\sigma-1}}}$ .

By definition of  $\underline{A}$ ,  $\frac{p\underline{A}(1-\alpha)}{p+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}} = 1$ .

Then,  $\frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma} \frac{p+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{p} \Leftrightarrow \frac{p}{p+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}} > \frac{1-\sigma}{1-a\sigma}$  where  $\frac{p}{p+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}} = \frac{1}{\underline{A}(1-\alpha)}$

Set  $m(A) = \frac{p}{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}$  and  $n(A) = \frac{1}{A(1-\alpha)}$ .  $m$  increases, while  $n$  decreases.

Then the previous statement is true if and only if  $m$  is strictly greater than  $n$  at the point at which  $n$  is worth  $\frac{1-\sigma}{1-a\sigma}$ . As this point is  $\frac{1-a\sigma}{(1-\alpha)(1-\sigma)}$ , this condition is  $\frac{p}{p+(\alpha \frac{1-a\sigma}{(1-\alpha)(1-\sigma)})^{\frac{\sigma-1}{\sigma}}} > \frac{1}{\frac{1-a\sigma}{(1-\alpha)(1-\sigma)}(1-\alpha)}$ .

This completes the proof of Proposition 2.6.

## 2.10 APPENDIX E

(i) is a corollary of Proposition 2.6.

Recall first from the proof of Proposition 2.5 that  $w \rightarrow G(e(w), w)$  is increasing (if  $\frac{1-a\sigma}{\sigma} < \frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma} p'(e(\frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma})) = \frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{p}$ ) or U-shaped (otherwise).

If  $A > \underline{A}$  and  $\alpha < \frac{(p \frac{\sigma}{1-\sigma} (1-a))^{\frac{\sigma-1}{\sigma}}}{\frac{1-a\sigma}{1-\sigma} + (p \frac{\sigma}{1-\sigma} (1-a))^{\frac{\sigma-1}{\sigma}}}$ , then according to the proof of Proposition 2.6,  $\frac{1-a\sigma}{\sigma} < \frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{p}$ . Thus for  $A \in (\underline{A}, \hat{A})$ ,  $w \rightarrow G(e(w), w)$  is increasing, while the function is U-shaped for  $A > \hat{A}$ .

If  $A > A^*$  and  $\alpha > \frac{(p \frac{\sigma}{1-\sigma} (1-a))^{\frac{\sigma-1}{\sigma}}}{\frac{1-a\sigma}{1-\sigma} + (p \frac{\sigma}{1-\sigma} (1-a))^{\frac{\sigma-1}{\sigma}}}$ , then according to the proof of Proposition 2.6,  $\frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{p}$  and then for all  $A > A^* > \underline{A}$ ,  $\frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma} \frac{p+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{p}$  and  $w \rightarrow G(e(w), w)$  is U-shaped.

This completes the proof of Proposition 2.7.

## 2.11 APPENDIX F

$w \rightarrow G(e(w), w)$  is an increasing or U-shaped function which is below 1 for  $w \leq \frac{1-\sigma}{\sigma} \frac{p+(\alpha A(w))^{\frac{\sigma-1}{\sigma}}}{\gamma}$ . This means that the equation  $G(e(w), w) = 1$  has a unique solution if and only if  $\lim_{w \rightarrow \infty} G(e(w), w) = \frac{\bar{p}A(1-\alpha)}{\bar{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}} > 1$ . In this case, the unique solution to the fixed point equation is an unstable steady state. If  $\frac{\bar{p}A(1-\alpha)}{\bar{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}} < 1$ , then  $G(e(w), w)$  is strictly smaller than 1 for all income levels, which implies that the economy converges to a null income.

## 2.12 APPENDIX G

I follow the proof of Proposition 2.1. Assume first that  $\underline{p} > 0$  and  $\gamma < \infty$ . The agent maximizes :

$$V(s_t, e_t) = \frac{(w_t - e_t - s_t)^{1-\sigma}}{1-\sigma} + p(e_t)^\sigma \frac{s_t^{1-\sigma}(1+r)^{1-\sigma}}{1-\sigma} + bp(e_t)$$

Subject to the constraints  $s_t + e_t \leq w_t$  and  $0 \leq e_t$  and  $0 \leq s_t$ . The Inada condition of the utility function implies that the Lagrangian writes :

$$L(s_t, e_t, \chi_1) = V(s_t, e_t) + \chi_1 e_t$$

The KKT conditions for a point  $(s_t, e_t)$  to be an optimum can be written as :

$$(i) \frac{\partial V}{\partial s}(s_t, \tau_t) = 0$$

$$(ii) \frac{\partial V}{\partial e}(s_t, e_t) = -(w_t - e_t - s_t)^{-\sigma} + p'(e_t) \left[ \frac{\sigma}{1-\sigma} \left( \frac{s_t(1+r)}{p(e_t)} \right)^{1-\sigma} + b \right] = -\chi_2$$

$$(iii) \min(\chi_2, e_t) = 0$$

Consider now the possibility that  $e_t = 0$ . From (i), I get the optimal saving,  $s_t = \frac{p}{p+(1+r)^{\frac{\sigma-1}{\sigma}}} w_t$ . And  $\chi_2$  must be non-negative. Thus, (ii) writes  $\frac{\partial V}{\partial e} \left( \frac{p}{p+(1+r)^{\frac{\sigma-1}{\sigma}}} w_t, 0 \right) \leq 0$  which is equivalent to  $\gamma \left( b - \frac{\sigma}{1-\sigma} \frac{w_t^{1-\sigma}(1+r)^{1-\sigma}}{(p+(1+r)^{\frac{\sigma-1}{\sigma}})^{1-\sigma}} \right) \leq \frac{w_t^{-\sigma}(1+r)^{1-\sigma}}{(p+(1+r)^{\frac{\sigma-1}{\sigma}})^{-\sigma}}$ . The LHS of this inequality increases from  $-\infty$  to  $b$  on  $[0, \infty)$ , while the RHS decreases from  $\infty$  to 0 on  $[0, \infty)$ . Thus, there exists  $\underline{w} \in (0, \infty)$  such that the previous inequality is satisfied if and only if  $w_t < \underline{w}$ . Thus, for  $w_t < \underline{w}$ ,  $\left( \frac{p}{p+(1+r)^{\frac{\sigma-1}{\sigma}}} w_t, 0 \right)$  satisfies the KKT conditions and is a possible solution. Consider now the case  $e_t > 0$ . Then,  $\chi_2 = 0$  and the conditions (i) and (ii) imply the equation (2.10) and :

$$\frac{1}{p'(e_t)} = \left( \frac{w_t - e_t}{p(e_t) + (1+r)^{\frac{\sigma-1}{\sigma}}} \right)^\sigma R^{\sigma-1} b - \frac{\sigma}{\sigma-1} \frac{w_t - e_t}{p(e_t) + (1+r)^{\frac{\sigma-1}{\sigma}}} \quad (2.32)$$

Note the RHS of (2.32) as  $h\left(\frac{w_t - e_t}{p(e_t) + (1+r)^{\frac{\sigma-1}{\sigma}}}\right)$ , where  $h(x) = x^\sigma R^{\sigma-1} b - \frac{\sigma}{\sigma-1} x$ . Note that  $h$  is increasing where it is non-negative and so  $e_t \rightarrow h\left(\frac{w_t - e_t}{p(e_t) + (1+r)^{\frac{\sigma-1}{\sigma}}}\right)$  is decreasing on  $[0, w_t]$ . Moreover, the RHS of (2.32) is equal to 0 at  $e_t = w_t$ . As the LHS of (2.32) is positive and increasing, (2.32) has a solution, which is also unique, if and only if  $h\left(\frac{w_t}{p+(1+r)^{\frac{\sigma-1}{\sigma}}}\right) > \frac{1}{\gamma}$  which is equivalent to  $\gamma \left( b - \frac{\sigma}{1-\sigma} \frac{w_t^{1-\sigma}(1+r)^{1-\sigma}}{(p+(1+r)^{\frac{\sigma-1}{\sigma}})^{1-\sigma}} \right) > \frac{w_t^{-\sigma}(1+r)^{1-\sigma}}{(p+(1+r)^{\frac{\sigma-1}{\sigma}})^{-\sigma}}$  and so to  $w_t > \underline{w}$ . Thus, for  $w_t > \underline{w}$ , there exists a unique pair  $(s_t, e_t)$  that satisfies the KKT conditions. Note finally that the problem of the consumer has always at least one solution because  $(s, e) \rightarrow V(s, \tau)$  is continuous on the maximization domain, which is compact. Consequently, the unique pair satisfying the KKT conditions in the two cases  $w_t \leq \underline{w}$  and  $w_t > \underline{w}$  is the unique solution to the problem of the consumer. The case  $\underline{p} = 0$  follows by continuity. When

$\gamma = \infty$ , the condition for  $e_t = 0$  to satisfy the KKT conditions write now  $s_t = \frac{p}{p+(1+r)^{\frac{\sigma-1}{\sigma}}} w_t$  and  $\lim_{e_t \rightarrow 0} p'(e_t) \left( b - \frac{\sigma}{1-\sigma} \frac{w_t^{1-\sigma} (1+r)^{1-\sigma}}{(p+(1+r)^{\frac{\sigma-1}{\sigma}})^{1-\sigma}} \right) \leq \frac{w_t^{-\sigma} (1+r)^{1-\sigma}}{(p+(1+r)^{\frac{\sigma-1}{\sigma}})^{-\sigma}}$ . The second inequality is true if and only if  $w_t < (b \frac{1-\sigma}{\sigma})^{\frac{1}{1-\sigma}} \left( \frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{1+r} \right)^{1-\sigma}$ . Thus, for  $w_t < (b \frac{1-\sigma}{\sigma})^{\frac{1}{1-\sigma}} \left( \frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{1+r} \right)^{1-\sigma}$ ,  $(\frac{p}{p+(1+r)^{\frac{\sigma-1}{\sigma}}} w_t, 0)$  satisfies the KKT conditions. Consider now an interior solution. It satisfies the equations (2.10) and (2.32). For (2.32) to have a solution, the necessary and sufficient condition is now :  $\lim_{e_t \rightarrow 0} \frac{1}{p'(e_t)} < \lim_{e_t \rightarrow 0} \left( \frac{w_t - e_t}{p(e_t) + (1+r)^{\frac{\sigma-1}{\sigma}}} \right)^{\sigma} R^{\sigma-1} b - \frac{\sigma}{\sigma-1} \frac{w_t - e_t}{p(e_t) + (1+r)^{\frac{\sigma-1}{\sigma}}}$ , which is equivalent to  $w_t > (b \frac{1-\sigma}{\sigma})^{\frac{1}{1-\sigma}} \left( \frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{1+r} \right)^{1-\sigma}$ . Hence for  $w_t > (b \frac{1-\sigma}{\sigma})^{\frac{1}{1-\sigma}} \left( \frac{p+(1+r)^{\frac{\sigma-1}{\sigma}}}{1+r} \right)^{1-\sigma}$ , there is a unique pair  $(s_t, e_t)$  that satisfies the KKT conditions.

Note that  $h\left(\frac{w_t - e_t}{p(e_t) + (1+r)^{\frac{\sigma-1}{\sigma}}}\right)$  increases with  $w_t$ , which shows that  $e(w_t)$  is increasing.

To see that  $w \rightarrow x(w)$  is initially increasing on  $[\underline{w}, \infty)$ , note that  $x(\underline{w}) = 0$ , while  $x(w) > 0$  on  $[\underline{w}, \infty)$ .

To get  $\lim_{w_t \rightarrow \infty} x(w_t)$ , rewrite first (2.32) at the optimum :

$$\frac{1}{p'(e(w_t))} = \left( \frac{w_t - e(w_t)}{p(e(w_t)) + (1+r)^{\frac{\sigma-1}{\sigma}}} \right)^{\sigma} R^{\sigma-1} b - \frac{\sigma}{\sigma-1} \frac{w_t - e(w_t)}{p(e(w_t)) + (1+r)^{\frac{\sigma-1}{\sigma}}}$$

Note that  $\lim_{w_t \rightarrow \infty} \frac{1}{p'(e(w_t))} = \infty$ , so it must be that  $\lim_{w_t \rightarrow \infty} \frac{w_t - e(w_t)}{p(e(w_t)) + (1+r)^{\frac{\sigma-1}{\sigma}}} = \infty$ . Thus,  $h\left(\frac{w_t - e_t}{p(e_t) + (1+r)^{\frac{\sigma-1}{\sigma}}}\right) \underset{w_t \rightarrow \infty}{\sim} \left( \frac{w_t - e(w_t)}{p(e(w_t)) + (1+r)^{\frac{\sigma-1}{\sigma}}} \right)^{\sigma} R^{\sigma-1} b$ . And then  $\left( \frac{x(w_t)}{1-x(w_t)} \right)^{\sigma} \underset{w_t \rightarrow \infty}{\sim} \frac{p'(e(w_t))e(w_t)^{\sigma}}{(p+(1+r)^{\frac{\sigma-1}{\sigma}})^{\sigma}} R^{\sigma-1} b$ . This gives the three possible cases of Proposition 2.9.



# AGING AND SECTORIAL LABOR ALLOCATION

## 3.1 INTRODUCTION

The aging process that developed countries currently undergo is well summarized by two strong demographic facts : a low fertility rate below the replacement level and increased survival probabilities at old ages. Indeed, the fertility rate in OECD countries decreased from 3.2 in 1970 to 1.76 in 2012. Over this same period, the life expectancy conditional at being alive at 65 increased from 12.5 in 1970 to 20.7 in 2012.<sup>1</sup> These large life expectancy gains translated into life gains during retirement period as the retirement legal age increased much slower than life expectancy did (Prettner and Canning (2014)). Taking into account that old individuals value healthcare more than young individuals do, Hashimoto and Tabata (2010) argue that aging creates a positive shock on the demand for healthcare that spurs labor reallocation towards the health sector. Aisa and Pueyo (2014) note that this result depends on the elasticity of substitution between labor and capital of the non-health sector. As aging stimulates capital accumulation, aging may redirect labor towards the non-health sector if capital and labor are complementary factors of production. On the empirical side, Moreno-Galbis and Sopraseuth (2014) find that aging is a driving force of the increased demand for labor in services sector because aging increases the demand for healthcare. These results suggest two channels through which aging affects the labor allocation of a multi-sector economy. First, the increased demand for healthcare is a consequence of the heterogeneity of preferences between old and young individuals. Second, the fact that aging stimulates capital accumulation, which is a well-known result in one-sector models, implies that aging creates a positive demand shock to the investment sector.<sup>2</sup> In this chapter, I study theoretically how aging affects the labor allocation of a stylized multi-sector economy. I shed light on the possible channels through which aging affects the labor allocation of the economy and I examine how aging redirects labor between sectors along a path satisfying the Kaldor facts.

To this aim, I formulate a two-period overlapping generations (hereafter OLG) model with two consumption goods and one pure capital good. As previously said, aging implies

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1. These values are all computed from the OECD database.

2. See Bloom et al.(2003), Chakraborty (2004), Zhang and Zhang (2005) or El Mekkaoui de Freitas and Oliveira-Martins (2014).

an increased demand for the investment sector, which justifies a separate treatment of this sector. The OLG structure enables to introduce the demographic variables at stake in the aging process (i.e. fertility and the longevity in retirement period) and to make exogenous shocks on them in a simple and tractable manner. I classify the labor reallocations caused by aging into three types. The first two types of reallocation are due to two partial equilibrium effects of aging. Given prices, savings and consumption levels of the individuals are directly affected by a longevity shift. I show that the *intertemporal* reallocation of resources, hence the increase of savings, caused by aging in one-sector models creates *intratemporal* reallocations of resources in a multi-goods framework once intratemporal preferences are non-homothetic. Then, I define the "allocation effect" as the partial equilibrium change of aggregate relative demand between sectors due to aging when the population structure is held fixed. The second type of labor reallocations are those due to pure "demographic effects" : aging diminishes the ratio of young to old people or in other words the ratio of savers to consumers. The "population effect" is then defined as the partial equilibrium change of aggregate relative demand between sectors due to aging with fixed individual allocations. I show that there are two alternative assumptions on the individuals' preferences that make the aggregate relative demand between consumption goods dependent on demographic variables and I explain how differently the "population effect" and the "allocation effect" occur under these two different assumptions. The third type of labor reallocations are those due to the general equilibrium effects of aging. They manifest themselves under two forms. Firstly, if sectors use different production functions, the surplus of capital following a longevity increase creates labor reallocations *ceteris paribus*. Secondly, aging changes the aggregate relative demand through its indirect impact on the price vector. After specifying functional forms for preferences and production functions, I study how these effects translate in general equilibrium. I examine how the relative labor shares vary with longevity and the cohort growth rate along a path satisfying the Kaldor facts. Finally, I provide a simple calibration of the model on the US economy to assess the importance of the labor flows that the aging process could engender.

To build my multi-sector economy, I draw on the vast literature on the process of structural change, which analyzes the inputs reallocation that occur between sectors along the development path.<sup>3</sup> Firstly pointed by Kuznets (1957), two main explanations of structural change have been highlighted. Demand side explanations shed the light on the role of the non-homotheticity of preferences in this process.<sup>4</sup> As income grows, because of technological progress and capital deepening, the consumption share of necessity goods shrinks relatively to the consumption share of superior goods. This creates labor flows from the production of goods with low income elasticity to the production of goods with high income elasticity. Supply side explanations point out the differences of production functions across sectors. Ngai and Pissarides (2007) develop a model where sectors have the same production functions, yet different TFP growth rates. They show that on the unique path along which Kaldor facts

3. The literature is reviewed by Herrendorf et al. (2014).

4. See among others : Matsuyama (1992), Echevarria (1997), Gollin, Parente, Rogerson (2002).

are satisfied, labor flows towards the sector that has the lowest TFP growth rate among consumption goods producing sectors, while the investment sector does not encounter any labor reallocation. In the model of Acemoglu and Guerrieri (2008), there are two sectors that use Cobb-Douglas production functions with different capital intensities. They find that capital deepening implies a labor reallocation towards the less capital intensive sector. Finally, Boppart (2014) proposes an analytical model that incorporates the two types of labor reallocations. Here I specify a model of structural change close to the one of Boppart (2014) before making exogenous shocks on the longevity and fertility parameters. The literature on structural change has long remained mute on the role of the demographic structure. There are a few notable exceptions. In an infinite horizon representative agent model, Leukhina and Turnovsky (2015) characterize the impact of a change of the population size on the process of structural change before applying their model to Britain from which they conclude that population growth has a significant and positive effect on the British industrialization. Bar and Leukhina (2010) examine quantitatively the role of child mortality in a structural change model with fertility choice. In other words, the literature has focused on the interaction between the demographic transition and the structural change process. In this paper, I argue that the aging process must be examined distinctly as I show that fertility and the longevity in retirement period interact differently with the sectorial allocation of labor.

The paper differs from those from Hashimoto and Tabata (2010) and Aisa and Pueyo (2014) with respect to three main aspects. First, these authors focus on the labor reallocation between the health sector and the non-health sector. Here I argue that this masks labor reallocations due to aging inside the non-health sector. Second, contrary to these two papers, I choose not to endogenize fertility to also examine the impact of an exogenous fertility decrease, which is also a pillar of the aging process, on the labor allocation. Third, my use of a stylized multi-sector economy in line with the structural change literature allows to connect more easily the model to data to assess the importance of the effects at stake. Finally, the paper also complements a large literature that studies the exogenous impact of demographic variables on economic growth in one-sector models. In a one-sector Diamond model, a longevity increase or a fertility decrease both increase income per worker. In a one-sector OLG model with a demographic structure à la Yaari (1965), d'Albis (2007) shows that the impact of the population growth rate on income per worker is non-monotonic. Boucekkine et al. (2002) also find a non-monotonic relationship between longevity and income per worker in a one-sector model with human capital investments and endogenous retirement. This chapter complements these findings by showing that a multi-sector framework can also create non-monotonicities between demographic variables and income per worker.

The rest of this paper is organized as follows. Section 3.2 examines the channels through which aging affects the partial equilibrium of a multi-goods economy. Section 3.3 specifies functional forms for utility and production functions to investigate the impact of aging on the sectorial labor allocation in general equilibrium. Section 3.4 concludes.

## 3.2 LONGEVITY IN A MULTI-SECTOR DIAMOND ECONOMY : PARTIAL EQUILIBRIUM ANALYSIS

### 3.2.1 Outline of the model

Individuals live for two periods. They work during the first period and retire during the second period, whose length is set to  $q \in (0, 1]$ . Cohort- $t$  is populated by  $L_t$  homogenous individuals, with  $L_{t+1} = (1 + n)L_t$  and  $n > -1$ . There are three sectors in this economy. Sectors 1 and 2 produce two different consumption goods, while the third sector, sector  $I$ , produces capital good, which is taken as the numéraire. Capital fully depreciates each period.

### 3.2.2 Intratemporal reallocation of resources at the individual level

The preferences of each young individual born at time  $t$  are represented by the following lifetime utility function :

$$U_t = u(c_{1t}, c_{2t}) + qv(d_{1t}, d_{2t}) \quad (3.1)$$

Where  $c_{it}$  is the consumption per unit of time of good  $i$  in first period of a cohort- $t$  individual,  $d_{it}$  the consumption per unit of time in second period.  $u$  and  $v$  are twice differentiable increasing and strictly concave functions. Note that young individuals and old individuals are allowed to have different utility per period functions.

Each agent maximizes (3.1) subject to the budget constraints :

$$P_{1t}c_{1t} + P_{2t}c_{2t} + s_t = w_t \quad (3.2)$$

$$q(d_{1t}P_{1t+1} + d_{2t}P_{2t+1}) = R_{t+1}s_t \quad (3.3)$$

Where  $P_{it}$  is the price of good  $i$  at time  $t$ ,  $R_{t+1}$  the interest rate,  $w_t$  the wage,  $s_t$  the savings.

By solving this optimization problem, I get the following preliminary result :

**Lemma 3.1.**  $s_t = s(w_t, R_{t+1}, q)$  is unique.  $\frac{\partial s_t}{\partial q}$  and  $\frac{\partial(\frac{s_t}{q})}{\partial q}$  are respectively positive and negative.

*Proof.* See Appendix A ■

Lemma 3.1 is a well-known result of the one sector Diamond model : savings increase with longevity in retirement period given prices. Two effects are at stake in this process. First, a greater  $q$  means that the individual discounts less the future so much that with fixed lifetime resources, the individual channels more resources into second period. Second, an increase of  $q$  also acts as a decrease of the interest rate (see (3.3)), whose effect on savings depends on the intertemporal elasticity of substitution. Lemma 3.1 states that the total effect on savings is always positive.<sup>5</sup>In a one-sector model, a shock on  $q$  can only cause *intertemporal*

5. Note that in one sector models, authors often use the specification of Yaari (1965) in which  $q$  is the probability to reach the second period, whose length is equal to 1 and there is a perfect annuity market so much that the return on savings is  $\hat{R} = \frac{R}{q}$ . Then, they make a shock on  $q$  given  $\hat{R}$ , which implies that the second effect is neutralized in this type of partial equilibrium analysis.

reallocation of resources, because there is no *intra-temporal* decision to make. However, in the present specification,  $q$  can also have an impact on the intra-temporal allocation of resources, i.e. on the consumption shares  $\frac{c_{1t}}{c_{2t}}$  and  $\frac{d_{1t}}{d_{2t}}$ . To investigate this, the following corollary of lemma 3.1 is useful :

**Lemma 3.2.** *Let  $e_{1t} = P_{1t}c_1 + P_{2t}c_{2t}$  and  $e_{2t} = d_{1t}P_{1t+1} + d_{2t}P_{2t+1}$  be respectively the per unit of time expenditure levels in first and second periods of a cohort- $t$  individual. Given prices,  $e_{1t}$  and  $e_{2t}$  decrease with  $q$ .*

*Proof.* Immediate from Lemma 3.1 ■

Lemma 3.2 is a reformulation of Lemma 3.1 in terms of expenditure functions. Its interpretation comes from the lifetime budget constraint :

$$e_{1t} + e_{2t} \frac{q}{R_{t+1}} = w_t \quad (3.4)$$

Given prices, the right-hand-side (RHS) of (3.4) is independent of  $q$ . Given expenditures levels, the left-hand-side (LHS) of (3.4) increases with  $q$  because the duration of the second period increases. Then, for the constraint to be fulfilled, the expenditure levels must decrease. This reformulation is useful because expenditure levels summarize all the dependence of the intra-temporal allocation with respect to  $q$ . Thus, the dependence of consumption ratios with respect to  $q$  is obtained in the following proposition :

**Proposition 3.3.** *For  $i \in \{1;2\}$ , note  $\varepsilon_{it}^y = \frac{\partial \ln(c_{it})}{\partial \ln(e_{it})}$  and  $\varepsilon_{it}^o = \frac{\partial \ln(d_{it})}{\partial \ln(e_{it})}$ . Given prices :*

$$(i) \quad \frac{\partial(\frac{c_{1t}}{c_{2t}})}{\partial q} > 0 \text{ if and only if } \varepsilon_{1t}^y < \varepsilon_{2t}^y, \text{ with equality if and only if } \varepsilon_{1t}^y = \varepsilon_{2t}^y.$$

$$(ii) \quad \frac{\partial(\frac{d_{1t}}{d_{2t}})}{\partial q} > 0 \text{ if and only if } \varepsilon_{1t}^o < \varepsilon_{2t}^o, \text{ with equality if and only if } \varepsilon_{1t}^o = \varepsilon_{2t}^o.$$

*Proof.* (i) Because lifetime utility is separable,  $\frac{c_{1t}}{c_{2t}}$  only depends on  $e_{1t}$  and  $\frac{P_{1t}}{P_{2t}}$ . Then,  $\frac{\partial(\frac{c_{1t}}{c_{2t}})}{\partial q} = \frac{\partial e_{1t}}{\partial q} \frac{\partial(\frac{c_{1t}}{c_{2t}})}{\partial e_{1t}}$ . The result follows by using Lemma 3.2.

(ii) Proceed as in (i) ■

Proposition 3.3 is the first main result of the paper. It gives the impact of a longevity increase on the intra-temporal consumption ratios in partial equilibrium. This information is obtained by comparing the expenditure elasticity of the two goods. A longevity increase creates an *intra-temporal* reallocation of resources towards the good with the smaller expenditure elasticity. Indeed, the (intra-temporal) consumption ratios only depend on  $q$  through the expenditure level which is a decreasing function of  $q$  from Lemma 3.2. Then, a decrease of expenditure implies an increase of the consumption of the good with the smaller expenditure elasticity. This result shows that in a multi-goods economy, a longevity increase produces both intertemporal and intra-temporal reallocation of resources at the individual level. Note finally that if the intra-temporal preferences are homothetic, then longevity does not impact the intra-temporal allocation of resources at the individual level.

### 3.2.3 The allocation and the population effect

In this subsection, I examine if a longevity or a fertility shift implies a change of the *aggregate* relative demand between goods in partial equilibrium. For notational convenience, I will write explicitly the dependence of consumption levels with respect to demographic variables only when needed. Their dependence with respect to prices is omitted as they are maintained fixed in this partial equilibrium analysis. I first focus on the aggregate relative demand between consumption goods. The aggregate consumption ratio at time  $t$ ,  $\frac{C_{1t}}{C_{2t}}$ , is given by :

$$\frac{C_{1t}}{C_{2t}} = \frac{c_{1t} + \frac{q}{1+n}d_{1t-1}}{c_{2t} + \frac{q}{1+n}d_{2t-1}} \quad (3.5)$$

This ratio may be impacted by  $q$  through two channels. As highlighted in Lemma 3.1 and Proposition 3.3, the quantities  $c_{1t}$ ,  $c_{2t}$ ,  $d_{1t-1}$ ,  $d_{2t-1}$  depend on  $q$  and then impact  $\frac{C_{1t}}{C_{2t}}$  through an "allocation effect". Also, this ratio directly depends on  $q$  through a "population effect" : given consumption levels, an increase of  $q$  implies that the total consumption in second period of the two goods increases. I will say that :

**Definition 3.4.** Set  $DC(q, n, a, b) = \frac{c_{1t}(q, n) + \frac{a}{1+b}d_{1t-1}(q, n)}{c_{2t}(q, n) + \frac{a}{1+b}d_{2t-1}(q, n)}$ .

(i) Longevity (respectively fertility) is said to be "allocation-neutral" if  $\frac{\partial DC}{\partial q}(q, n, q, n) = 0$  (respectively  $\frac{\partial DC}{\partial n}(q, n, q, n) = 0$ ) for all  $(q, n) \in [0, 1) \times (0, \infty)$  and for any price vector.

(ii) Longevity (respectively fertility) is said to be "population-neutral" if  $\frac{\partial DC}{\partial a}(q, n, q, n) = 0$  (respectively  $\frac{\partial DC}{\partial b}(q, n, q, n) = 0$ ) for all  $(q, n) \in [0, 1) \times (0, \infty)$  and for any price vector.

For the aggregate relative demand between the capital good and the consumption goods, I will use the same definition except that instead of using the function  $DC$ , I will use the function  $DI$  defined by :

$$DI(q, n, a, b) = \frac{s_t(q, n)}{c_{1t}(q, n) + c_{2t}(q, n) + \frac{a}{1+b}(d_{1t-1}(q, n) + d_{2t-1}(q, n))} \quad (3.6)$$

This definition allows to disentangle the two effects of aging on the aggregate relative demand in partial equilibrium. When longevity is "allocation-neutral", the changes of the consumption levels due to aging do not cause a change of the aggregate consumption ratio for a fixed population. When longevity is "population neutral", the composition change of the population due to aging does not affect the aggregate consumption ratio for fixed consumption levels of the population.

Note that Definition 3.4 incorporates the two channels previously studied by the literature. First, Hashimoto and Tabata (2010) and Aisa and Pueyo (2014) argue that aging increases the relative number of individuals valuing healthcare. This is a "population effect" between two consumption sectors, the health sector and the non-health sector. Second, aging increases the savings of individuals or, in other words, the relative demand for capital good. This is an "allocation effect" between the consumption sectors and the investment sector. Definition 3.4 opens the way to other channels through which aging can affect aggregate relative

demand : Can there be allocation effects between consumption goods ? Are there population effects between the investment sector and the consumption sectors ? Moreover, by studying the occurrence of the "population effect" or the "allocation effect", I can determine if aging creates aggregate relative demand changes only between the health and the non-health sector and between the investment and the consumption sectors.

Note first that the population growth rate can only affect  $\frac{C_{1t}}{C_{2t}}$  through a "population effect" since the consumption levels do not depend on  $n$  in partial equilibrium (equivalently, fertility is always "allocation neutral"). To see how different assumptions on preferences drive the dependence of  $\frac{C_{1t}}{C_{2t}}$  with respect to  $q$ , it is useful to consider the case in which the intratemporal preferences are homothetic and identical for young and old individuals. Intratemporal preferences of the young (respectively of the old) are said to be homothetic if  $u$  (respectively  $v$ ) is a monotonic transformation of a linearly homogenous function. Intratemporal preferences of the young and of the old are said to be identical if  $u$  and  $v$  are monotonic transformations of the same linearly homogenous functions. The first assumption implies that there are functions  $\gamma$  and  $\beta$  such that :  $\frac{c_{1t}}{c_{2t}} = \gamma\left(\frac{P_{1t}}{P_{2t}}\right)$  and  $\frac{d_{1t}}{d_{2t}} = \beta\left(\frac{P_{1t+1}}{P_{2t+1}}\right)$ , hence consumption ratios only depend on current relative prices. The second one implies that  $\beta(\cdot) = \gamma(\cdot)$ , hence the current young and the current old do the same intratemporal choice. The aggregate share (3.5) can be written as :

$$\frac{C_{1t}}{C_{2t}} = \frac{c_{1t}}{c_{2t}} \frac{1 + \frac{q}{1+n} \frac{d_{1t-1}}{c_{1t}}}{1 + \frac{q}{1+n} \frac{d_{2t-1}}{c_{2t}}} = \gamma\left(\frac{P_{1t}}{P_{2t}}\right) \quad (3.7)$$

This shows that in this case, the aggregate consumption ratio does not depend on  $q$  in partial equilibrium. This ratio will be affected by a longevity change in general equilibrium only through the impact of  $q$  on the relative price between consumption goods. Two ingredients explain this result. First, the homotheticity of preferences implies that the consumption ratio of each individual at each period only depends on current relative prices. Second, as young and old agents have the same utility per period, the dependence of their current consumption ratio to the relative price is identical. Finally, as the current old and the current young face the same prices,  $\frac{c_{1t}}{c_{2t}} = \frac{d_{1t-1}}{d_{2t-1}}$ . Alternatively, this result can be obtained by analyzing the "allocation effect" and the "population effect" defined previously. Differentiate first (3.5) with respect to  $q$  while maintaining  $c_{1t}$ ,  $c_{2t}$ ,  $d_{1t-1}$  and  $d_{2t-1}$  as constant to get that longevity is "population neutral" if and only if for all  $(q, n) \in [0, 1) \times (0, \infty)$  and any price vector :

$$\frac{d_{1t-1}}{c_{1t}} = \frac{d_{2t-1}}{c_{2t}} \quad (3.8)$$

If we change the composition of the population, while maintaining fixed the consumption levels, then  $\frac{C_{1t}}{C_{2t}}$  changes if and only if the old and the young have different consumption bundles. Since condition (3.8) is fulfilled and  $\frac{C_{1t}}{C_{2t}}$  does not depend on  $q$  when the intratemporal preferences of young and old individuals are identical and homothetic, it must be that under this specification of preferences, longevity is both "allocation-neutral" and "population-neutral". From (3.8), longevity is not "population-neutral" if and only if the consumption

shares of the old and of the young differ for at least one pair  $(q, n) \in [0, 1) \times (0, \infty)$ . There are two ways for this to happen. If intratemporal preferences are homothetic but not identical, the two consumption shares will be two different functions of the current relative price. Then, the two consumption shares will be different for all  $(q, n) \in [0, 1) \times (0, \infty)$ . The alternative condition for the "population effect" to occur is to consider non-homothetic intratemporal preferences. In this case, even though  $u$  and  $v$  are identical, the two consumption shares depend on expenditure levels of different cohorts. Thus, except possibly for some particular pairs  $(q, n)$ , the consumption shares differ and there is a "population effect".

As previously said, longevity is "allocation neutral" if preferences are homothetic and identical. I will show that intratemporal homothetic preferences are needed for longevity to be "allocation neutral", while identical preferences are not. Indeed, if preferences are non-homothetic, then a longevity increase changes the consumption bundles of the young and the old differently because their expenditure levels depend on different prices, which implies a change of the aggregate relative demand. Then, if preferences are homothetic, yet not identical, then the change of the expenditure level of the old and the young can change the total consumption share of each type of population. For example, if after the longevity increase, the old reduce less their expenditure level than the young do, then their total consumption increase with respect to the young's, which affects the aggregate relative demand because the old and the young have different consumption levels. Hence, if preferences are such that the old and the young change their expenditure levels equally after a longevity change, then longevity is "allocation neutral". The condition is in the next theorem, which also collects and complements my previous results :

**Theorem 3.5.** (i) Longevity is "allocation neutral" if and only if intratemporal preferences

are homothetic and  $\left\{ \begin{array}{l} \text{intratemporal preferences are identical} \\ \text{or} \\ \text{the IES is equal to 1} \end{array} \right.$ .

(ii) Longevity (resp. fertility) is "population neutral" if and only if intratemporal preferences are homothetic and identical.

(iii)  $\frac{\partial(\frac{C_{1t}}{C_{2t}})}{\partial q} = \frac{\partial(\frac{C_{1t}}{C_{2t}})}{\partial n} = 0$  for all  $(q, n) \in [0, 1) \times (0, \infty)$  and any price vector if and only if intratemporal preferences are identical and homothetic.

*Proof.* See Appendix B ■

Theorem 3.5 is an important result of this paper stating necessary and sufficient conditions for the aggregate consumption ratio to be invariant with respect to demographic variables in partial equilibrium. Equivalently, Theorem 3.5 states that the aggregate consumption ratio is affected by demographic variables in partial equilibrium if and only if the intratemporal preferences are heterogenous between young and old individuals or if the intratemporal preferences are non-homothetic. This shows that the heterogeneity of preferences between young and old individuals is not necessary for this ratio to depend on  $q$  or  $n$ . This can also be achieved if intratemporal preferences are non-homothetic. The non-homotheticity of preferences between consumption goods is well-documented (between manufacturing goods

and services for example), which implies that aging is susceptible to create labor reallocations inside the non-health sector.

It is useful to compare this result with that of Hashimoto, Tabata (2010) and Aisa and Pueyo (2014). In their framework, individuals enjoy utility from good consumption and healthcare, yet only old individuals enjoy utility from healthcare, which implies that the preferences of the old and the young are heterogeneous. They are also homothetic and the IES is equal to 1. According to Theorem 3.5, these assumptions neutralize the "allocation effect", and the aggregate consumption ratio only depends on aging through the "population effect".

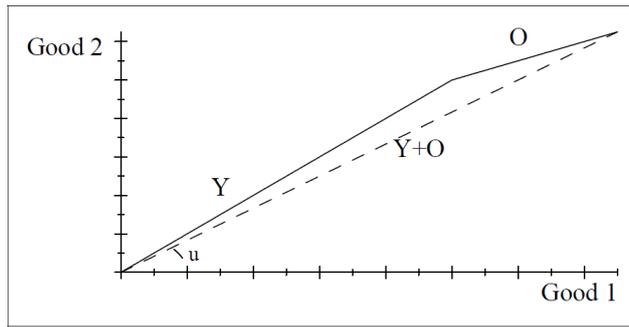
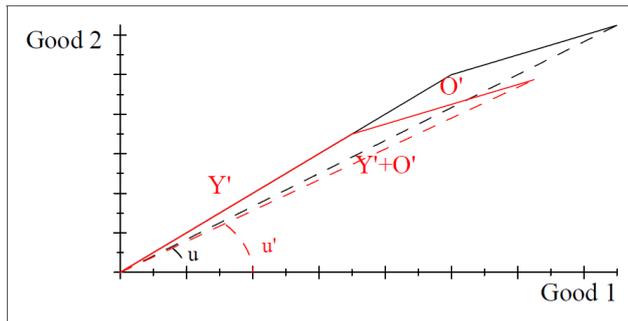
I study now how the "population effect" and the "allocation effect" work under these two different specifications of preferences. First, as previously said, the "population effect" occurs when the young and the old have different consumption ratios. With homothetic yet not identical intratemporal preferences (henceforth case 1), the consumption ratios of the two generations are different functions of the same current relative price. Thus, the "population effect" is monotonic with respect to longevity and fertility in case 1. When intratemporal preferences are identical and non-homothetic (henceforth case 2), the consumption ratios are the same function of different expenditure levels. As the expenditure levels depend on longevity, the "population effect" may be non-monotonic with respect to longevity while it remains monotonic with respect to fertility.

There are two ingredients at work in the "allocation effect". The movement of the four consumption levels is captured by looking at the movement of the consumption ratios of the young and of the old, and the absolute change of the consumption of one good by the young and the old. Hence, the "allocation effect" occurs because the young and the old change their consumption bundle and because the consumption share of each generation in the total consumption changes. In case 1, only the second effect occurs. Then, for the longevity increase to increase  $\frac{C_{1t}}{C_{2t}}$  through the "allocation effect", the expenditure level of the individuals with the greater share of good 1 in their consumption bundle must decrease less. Indeed, if the old have more good 1 in their consumption bundle,  $\frac{C_{1t}}{C_{2t}}$  increases if the total consumption share of the old increases. On the contrary, if the young have more good 1 in their consumption bundle, then their total consumption share must increase for  $\frac{C_{1t}}{C_{2t}}$  to increase. In case 2, both effects work. Even though both young and old individuals increase their consumption share of good 1,  $\frac{C_{1t}}{C_{2t}}$  may decrease if the consumption share of the individuals with the smaller share of good 1 in their consumption bundle increases.

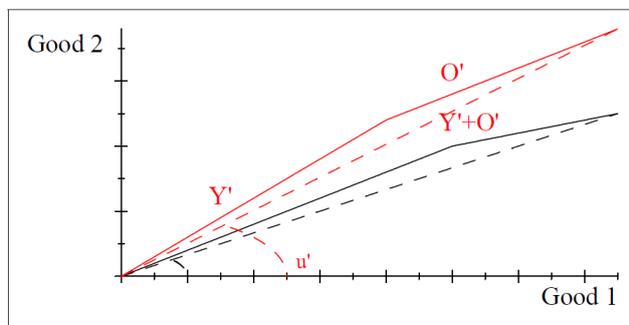
I can illustrate the difference between the two cases in a simple geometric manner. Note  $Y = (y_1, y_2)$  the vector of consumption of the young (i.e.  $y_i$  is the total consumption of good  $i$  by the young) and  $O = (o_1, o_2)$  the vector of consumption of the old. In Figure 3.1, I represent  $Y$  and  $O$ .

Then, the aggregate consumption ratio is given by the tangent of the angle  $u$ . Note that when preferences are homothetic, a longevity shift translates  $Y$  and  $O$ . Noting  $Y'$  and  $O'$  the new value of  $Y$  and  $O$  after a longevity shift, I get that  $u$  also changes to  $u'$  :

Moreover, if intratemporal homothetic preferences are also identical across cohorts, then  $Y$  and  $O$  are colinear, so much that  $Y + O$  and  $Y' + O'$  are colinear and  $u$  is unchanged. In

FIGURE 3.1 –  $Y$ ,  $O$  and  $Y + O$ .FIGURE 3.2 – Change of  $\frac{C_{1t}}{C_{2t}}$  after a longevity shift with homothetic and non-identical preferences.

the non-homothetic case, the longevity shift is no more a translation of  $Y$  and  $O$  (see Figure 3.3).

FIGURE 3.3 – Change of  $\frac{C_{1t}}{C_{2t}}$  after a longevity shift with non-homothetic preferences.

It is important to recall that this result holds for fixed prices. In general equilibrium, as longevity affects the supply of inputs, it can change relative prices, and then aggregate consumption shares even though intratemporal preferences are identical and homothetic. These inputs effects are illustrated in the next section where I specify functional forms for the preferences to study the general equilibrium of the economy. Before turning to the general equilibrium effects of aging, I briefly examine the aggregate relative demand between the capital good and the consumption goods. The aggregate saving to total consumption ratio,  $\frac{S_t}{C_t}$  is given by :

$$\frac{S_t}{C_t} = \frac{s_t}{c_{1t} + c_{2t} + \frac{q}{1+n}(d_{1t-1} + d_{2t-1})} \quad (3.9)$$

Holding  $c_{1t}$ ,  $c_{2t}$ ,  $d_{1t-1}$ ,  $d_{2t-1}$  and  $s_t$  as constant in (3.9), we observe that aging creates a "population effect" that increases the demand for consumption goods relatively to the demand for the capital good. The reason is that aging increases the number of consumers relatively to the number of savers. Aging also creates an "allocation effect" in partial equilibrium between these two sectors : treating  $\frac{q}{1+n}$  as a constant in (3.9), this ratio increases with  $q$  when the two consumption goods are normal for both the young and the old. The total effect of a longevity increase on the relative demand is ambiguous in partial equilibrium : the number of consumers increases relatively to the number of savers ("population effect"), however the consumers consume less, while the savers save more ("allocation effect"). Otherwise said, the demand for the capital good increases because the saving of the young increases, yet the demand for the consumption goods can also increase if the consumption increase of the old overcompensates the consumption decrease of the young. This creates a fundamental distinction between  $\frac{S_t}{C_t}$  and  $\frac{C_{1t}}{C_{2t}}$ , which is summarized in the following proposition :

**Proposition 3.6.** (i) *Aging always creates both an "allocation effect" and a "population effect" between the investment sector and the consumption sectors.*

(ii) *There does not exist utility per period functions  $u$  and  $v$  such that  $\frac{S_t}{C_t}$  is invariant with respect to demographic variables for any price vector.*

*Proof.* See Appendix C ■

Note finally that a fertility decrease always decreases  $\frac{S_t}{C_t}$  in partial equilibrium since it only acts through the "population effect".

### 3.3 GENERAL EQUILIBRIUM

The previous section suggests that in a standard multi-sector economy, aging affects the sectorial labor allocation. In this section, I specify functional forms for preferences and production functions to examine how aging affects the sectorial labor allocation in general equilibrium. First, I interpret good 1 as manufacturing good and good 2 as services. In line with the structural change literature, particularly Boppart (2014), I assume identical yet non-homothetic preferences : manufacturing good is a normal good, while services are a luxury good.

Second, agricultural goods are excluded from the analysis. In developed countries, the labor share of the farming sector is such that aging is unlikely to spur significant labor reallocations with this sector. Technically, the production of such goods involves other inputs such as land and uses capital and labor with different factor proportions than the ones of manufacturing and services sector, which excludes the existence of a path along which Kaldor facts are satisfied. Acemoglu and Guerrieri (2008) and Boppart (2014) also exclude agricultural

goods from their study.<sup>6</sup> Finally, the same structure as in section 3.2 is used and the variable names are conserved.

### 3.3.1 Firms

The three sectors use labor and capital as inputs. The production function in the sectors 1 and 2 is Cobb-Douglas with capital intensity  $\alpha \in (0, 1)$ .<sup>7</sup> The investment sector also uses a Cobb-Douglas production function, yet with a different capital intensity  $\alpha_I \in (0, 1)$ . Factors of production are perfectly mobile across sectors and are paid at their marginal product :

$$w_t = P_{1t}A_{1t}(1 - \alpha)k_{1t}^\alpha = P_{2t}A_{2t}(1 - \alpha)k_{2t}^\alpha = A_{It}(1 - \alpha_I)k_{It}^{\alpha_I} \quad (3.10)$$

$$1 + r_t = P_{1t}A_{1t}\alpha k_{1t}^{\alpha-1} = P_{2t}A_{2t}\alpha k_{2t}^{\alpha-1} = A_{It}\alpha_I k_{It}^{\alpha_I-1} \quad (3.11)$$

Where  $k_{it}$  is the capital to labor ratio of sector  $i$ ,  $A_{it}$  is the labor-augmenting technological progress, whose growth rate,  $g_i$ , is taken as exogenous and constant :

$$A_{it+1} = (1 + g_i)A_{it} \quad (3.12)$$

(3.10) and (3.11) imply that the capital to labor ratios are equal across the two consumption sectors (hence  $k_{1t} = k_{2t}$ ). Moreover, the relative price of consumption goods is equal to the ratio of TFP levels :  $\frac{P_{1t}}{P_{2t}} = \frac{A_{2t}}{A_{1t}}$ .

### 3.3.2 Preferences

In structural change models with non-homothetic preferences, the most commonly used utility per period function,  $u$ , in the literature is the Stone-Geary utility function :

$$u(c_{1t}, c_{2t}) = \frac{m(c_{1t} - c, c_{2t} + d)^{1-\sigma} - 1}{1 - \sigma} \quad (3.13)$$

Where  $\sigma > 0$  is the inverse of the IES,  $c$  and  $d$  are non-negative constants and  $m$  is linearly homogeneous. Thus, with  $c = d = 0$ , preferences are homothetic, while they are non-homothetic if  $c$  or  $d$  is positive. A knife-edge-condition relating preferences and technology must be imposed to insure the existence of an equilibrium with a constant interest rate. To avoid this issue, Boppart (2014) convincingly proposes to use a subclass of "price independent generalized linearity" (PIGL) preferences. PIGL preferences are defined by their indirect utility functions and are a natural starting point for structural change models because all

6. In addition to its Diamond structure, the outlined model differs with that of Boppart (2014) with respect to several aspects. First, the investment sector uses labor as input and experiences technological progress (i.e.  $\alpha_I = 1$  and  $g_I = 0$  in Boppart (2014)). Otherwise, there would be no labor reallocations towards the investment sector. Second, the asymptotic labor share of sector 1 is positive ( $\theta = 0$  in Boppart (2014)). Third, there is no intratemporal heterogeneity across households (in Boppart (2014), individuals differ in their exogenous labor endowment).

7. Herrendorf and Valentinyi (2008) provide evidence on the equality of capital intensities between manufacturing sector and service sector.

preferences with a constant intertemporal elasticity of substitution belong to this class of preferences. To parametrize the indirect utility function per period, I use one specification proposed by Boppart (2014) :

$$V(P_1, P_2, e) = \frac{1}{1-\sigma} \left( \frac{e}{P_1^\theta P_2^{1-\theta}} \right)^{1-\sigma} - \frac{\beta}{\gamma} \left( \frac{P_1}{P_2} \right)^\gamma - \frac{1}{1-\sigma} + \frac{\beta}{\gamma} \quad (3.14)$$

Where  $V$  is the indirect utility function per period,  $P_i$  is the price of good  $i$ ,  $e$  the expenditure level for the period. I impose  $\sigma < 1$ ,  $\gamma < 1$ ,  $\beta > 0$  and  $\theta \in [0, 1]$ . The restriction on  $\sigma$  is made for capturing the empirically relevant cases.<sup>8</sup> A restriction on the expenditure level  $e$ , stated below, must be imposed for  $V$  to be a valid indirect utility specification.<sup>9</sup> When  $1 - \sigma > 0$  and  $\beta > 0$ , the preferences are non-homothetic.

The demand functions are obtained as follows. Given an expenditure level  $e$ , the consumption level of each good is given by the Roy's identity. The expenditure in first period,  $e_{1t}$ , and the expenditure in second period,  $e_{2t}$ , of a cohort- $t$  individual are obtained by maximizing the following lifetime indirect utility function :

$$V(P_{1t}, P_{2t}, e_{1t}) + qV(P_{1t+1}, P_{2t+1}, e_{2t}) \quad (3.15)$$

Subject to the budget constraints (3.2).

The consumption levels and the expenditures levels are :

$$c_{1t} = \frac{\theta e_{1t}}{P_{1t}} + \frac{\beta}{1-\sigma} \left( \frac{e_{1t}}{P_{2t}} \right)^\sigma \left( \frac{P_{1t}}{P_{2t}} \right)^{\gamma-1+\theta(1-\sigma)}; d_{1t} = \frac{\theta e_{2t}}{P_{1t+1}} + \frac{\beta}{1-\sigma} \left( \frac{e_{2t}}{P_{2t+1}} \right)^\sigma \left( \frac{P_{1t+1}}{P_{2t+1}} \right)^{\gamma-1+\theta(1-\sigma)} \quad (3.16)$$

$$c_{2t} = \frac{(1-\theta)e_{1t}}{P_{2t}} - \frac{\beta}{1-\sigma} \left( \frac{e_{2t}}{P_{2t}} \right)^\sigma \left( \frac{P_{1t}}{P_{2t}} \right)^{\gamma+\theta(1-\sigma)}; d_{2t} = \frac{(1-\theta)e_{2t}}{P_{2t+1}} - \frac{\beta}{1-\sigma} \left( \frac{e_{2t}}{P_{2t+1}} \right)^\sigma \left( \frac{P_{1t+1}}{P_{2t+1}} \right)^{\gamma+\theta(1-\sigma)} \quad (3.17)$$

$$e_{1t} = w_t \frac{\frac{(1+r_{t+1})^{\frac{\sigma-1}{\sigma}}}{q} \left( \frac{P_{1t+1}^\theta P_{2t+1}^{1-\theta}}{P_{1t}^\theta P_{2t}^{1-\theta}} \right)^{\frac{1-\sigma}{\sigma}}}{1 + \frac{(1+r_{t+1})^{\frac{\sigma-1}{\sigma}}}{q} \left( \frac{P_{1t+1}^\theta P_{2t+1}^{1-\theta}}{P_{1t}^\theta P_{2t}^{1-\theta}} \right)^{\frac{1-\sigma}{\sigma}}} \quad (3.18)$$

$$e_{2t} = \frac{w_t}{1 + \frac{(1+r_{t+1})^{\frac{\sigma-1}{\sigma}}}{q} \left( \frac{P_{1t+1}^\theta P_{2t+1}^{1-\theta}}{P_{1t}^\theta P_{2t}^{1-\theta}} \right)^{\frac{1-\sigma}{\sigma}}} \frac{1+r_{t+1}}{q} \quad (3.19)$$

And the two following inequalities must be fulfilled :

$$\frac{\sigma\beta^2}{(1-\sigma)^2} \leq (1-\theta)\theta \left( \left( \frac{P_{2t}}{P_{1t}} \right)^\gamma \left( \frac{e_{1t}}{P_{1t}^\theta P_{2t}^{1-\theta}} \right)^{1-\sigma} \right)^2 + \frac{\beta(1-2\theta-\gamma)}{1-\sigma} \left( \frac{P_{2t}}{P_{1t}} \right)^\gamma \left( \frac{e_{1t}}{P_{1t}^\theta P_{2t}^{1-\theta}} \right)^{1-\sigma} \quad (3.20)$$

8. Boppart (2014) estimates  $\sigma$  to 0.78. In section 3.3.4, I find a value of 0.65.

9. The Slutsky matrix must be negative semidefinite for  $V(P_1, P_2, e)$  to be a valid indirect utility specification (see Boppart (2014) for more details). (3.20) and (3.21) insure this condition for young and old individuals.

$$\frac{\sigma\beta^2}{(1-\sigma)^2} \leq (1-\theta)\theta\left(\frac{P_{2t+1}}{P_{1t+1}}\right)^\gamma \left(\frac{e_{2t}}{P_{1t+1}^\theta P_{2t+1}^{1-\theta}}\right)^{1-\sigma} + \frac{\beta(1-2\theta-\gamma)}{1-\sigma} \left(\frac{P_{2t+1}}{P_{1t+1}}\right)^\gamma \left(\frac{e_{2t}}{P_{1t+1}^\theta P_{2t+1}^{1-\theta}}\right)^{1-\sigma} \quad (3.21)$$

The non-homotheticity of preferences is visible in the fact that Engel curves are not linear and the consumption ratios  $(\frac{c_{1t}}{c_{2t}}$  and  $\frac{d_{1t}}{d_{2t}})$  depend on relative prices and expenditure levels.

### 3.3.3 Equilibrium

Noting  $l_{it}$  the labor share of sector  $i \in \{1; 2; I\}$ , the labor market clearing condition reads :

$$l_{1t} + l_{2t} + l_{It} = 1 \quad (3.22)$$

The capital market clearing condition writes :

$$L_t s_t = Y_{It} = (l_{1t+1}k_{1t+1} + l_{2t+1}k_{2t+1} + l_{It+1}k_{It+1})L_{t+1} \quad (3.23)$$

Where  $Y_{it}$  is the output of sector  $i$ . For  $i \in \{1; 2\}$ , the market clearing condition for good  $i$  writes :

$$Y_{it} = L_t c_{it} + q L_{t-1} d_{it-1} \quad (3.24)$$

Then, a dynamic competitive equilibrium is defined as follows :

**Definition 3.7.** *A dynamic competitive equilibrium consists of allocations*

$(l_{1t}, l_{2t}, l_{It}, k_{1t}, k_{2t}, k_{It}, c_{1t}, c_{2t}, s_t, d_{1t-1}, d_{2t-1})_{t=1..∞}$  and prices  $(P_{1t}, P_{2t}, r_t, w_t)_{t=1..∞}$  such that :

(i) *Agents maximize their lifetime indirect utility function (3.15) subject to the budget constraints (3.2) and (3.3).*

(ii) *Firms maximize their profits.*

(iii) *All markets clear (i.e. equations (3.22) to (3.24) are satisfied).*

(iv) *Inequalities (3.20) and (3.21) are satisfied.*

I will restrict the analysis to dynamic competitive equilibrium along which the interest rate is constant (henceforth CGP). Then, along this path,  $w_t$ ,  $P_{1t}$  and  $P_{2t}$  grow respectively at the constant rates  $(1+g_I)^{\frac{1-\alpha}{1-\alpha_I}}$ ,  $\frac{(1+g_I)^{\frac{1-\alpha}{1-\alpha_I}}}{1+g_1}$  and  $\frac{(1+g_I)^{\frac{1-\alpha}{1-\alpha_I}}}{1+g_2}$  according to (3.10) and (3.11).

The market clearing condition of the capital good gives the share of workers in the investment sector :

$$l_{It} = l_I = \frac{1 - \alpha_I}{1 + \frac{R^{\frac{\sigma-1}{\sigma}}}{q} x} \quad (3.25)$$

Where  $x = \left(\frac{1+g_2}{1+g_1}\right)^\theta \frac{(1+g_I)^{\frac{1-\alpha}{1-\alpha_I}}}{1+g_2} \frac{1-\sigma}{\sigma}$  and  $R$  is the interest rate level on the CGP. Equilibrium

on the consumption goods markets give  $l_{1t}$  and  $l_{2t}$  :

$$l_{1t} = \frac{\theta(1-\alpha)\left(\frac{R^{\frac{\sigma-1}{\sigma}}}{q}x + \frac{R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}}\right)}{1 + \frac{R^{\frac{\sigma-1}{\sigma}}}{q}x} + \frac{d\left(\frac{R^{\sigma-1}}{q^\sigma}x^\sigma + \frac{q}{1+n}\left(\frac{R}{q(1+g_I)^{\frac{1}{1-\alpha_I}}}\right)^\sigma\right)R^{\frac{\alpha(1-\sigma)}{1-\alpha_I}}}{\left(1 + \frac{R^{\frac{\sigma-1}{\sigma}}}{q}x\right)^\sigma} \frac{A_{2t}^\sigma\left(\frac{A_{2t}}{A_{1t}}\right)^{\gamma-1+\theta(1-\sigma)}}{A_{1t}A_{It}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}}} \quad (3.26)$$

$$l_{2t} = \frac{(1-\theta)(1-\alpha)\left(\frac{R^{\frac{\sigma-1}{\sigma}}}{q}x + \frac{R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}}\right)}{1 + \frac{R^{\frac{\sigma-1}{\sigma}}}{q}x} - \frac{d\left(\frac{R^{\sigma-1}}{q^\sigma}x^\sigma + \frac{q}{1+n}\left(\frac{R}{q(1+g_I)^{\frac{1}{1-\alpha_I}}}\right)^\sigma\right)R^{\frac{\alpha(1-\sigma)}{1-\alpha_I}}}{\left(1 + \frac{R^{\frac{\sigma-1}{\sigma}}}{q}x\right)^\sigma} \frac{A_{2t}^\sigma\left(\frac{A_{2t}}{A_{1t}}\right)^{\gamma-1+\theta(1-\sigma)}}{A_{1t}A_{It}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}}} \quad (3.27)$$

$$\text{Where } d = \frac{\beta(1-\alpha)^\sigma}{\alpha_I^{\frac{\alpha(1-\sigma)}{1-\alpha_I}}(1-\sigma)} \left(\frac{\alpha(1-\alpha_I)}{\alpha_I(1-\alpha)}\right)^{\alpha(\sigma-1)}.$$

Finally, the labor market clearing condition pins down the value of  $R$ , which depends on both demographic variables. The result is in the next proposition :

**Proposition 3.8.** Assume  $\frac{(\frac{1+g_2}{1+g_1})^{\gamma+\theta(1-\sigma)}}{(1+g_2)^{1-\sigma}(1+g_1)^{\frac{\alpha(1-\sigma)}{1-\alpha_I}}} < 1$  :

(i) Fix  $n \in (-1, \infty)$ . For every  $q \in (0, 1)$ , there exist values for  $A_{10}, A_{20}$  and  $A_{10}$  such that for all  $q$  greater than  $\underline{q}$ , there exists a unique CGP.

(ii) Fix  $q \in (0, 1]$ . For every  $(\underline{n}, \bar{n}) \in (-1, \infty)^2$  with  $\underline{n} < \bar{n}$ , there exist values for  $A_{10}, A_{20}$  and  $A_{10}$  such that for all  $n \in [\underline{n}, \bar{n}]$ , there exists a unique CGP.

(iii) Along a CGP, the interest rate  $R$  is the unique solution of the following equation :

$$(1-\alpha) \frac{R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}} = \alpha_I + \alpha \frac{R^{\frac{\sigma-1}{\sigma}}}{q} \quad (3.28)$$

(iv) For every  $(q, \underline{n}, \bar{n}) \in (0, 1) \times (-1, \infty)^2$ ,  $n \rightarrow R(q, n)$  is increasing on  $[\underline{n}, \bar{n}]$ .

(v) For every  $(q, n) \in (0, 1) \times (-1, \infty)$ ,  $q \rightarrow R(q, n)$  is decreasing on  $[q, 1]$ .

*Proof.* See Appendix D ■

Proposition 3.8 shows that along the CGP, the capital stock per worker and the interest rate behave with longevity and fertility as in the one-sector model. As saving increases with longevity, the capital per worker level increases and the interest rate decreases. The proof of Proposition 3.8 makes clear that to insure the existence of a CGP, the inequalities (3.20) and (3.21) must be dealt carefully. These inequalities depend on both longevity and fertility. Therefore, when I make a shock on a demographic variable, I must prevent the violation of these inequalities. (i) and (ii) are two ways to do this. As the goal of this section is to draw various endogenous variables as functions of longevity and fertility, I will use the point (i) and (ii) to choose the technology levels so as to draw the various curves on the "largest" possible set. More precisely :

**Definition 3.9.** Fix  $q \in (0, 1]$ . Consider  $-1 < \underline{n} < \bar{n}$  and  $(A_{10}, A_{20}, A_{10}) \in (0, \infty)^3$  technology levels that insure the existence of a CGP for all  $n \in (\underline{n}, \bar{n})$ . Let

$g(q, n, t, A_{10}, A_{20}, A_{10})$  be the CGP value of an endogenous variable of the model outlined in section 3.3 where  $(n, t) \in [\underline{n}, \bar{n}] \times [0, \infty)$ . For all time  $t$ , assume that you can define the function  $n \rightarrow h(q, n, t, A_{10}, A_{20}, A_{10})$  on  $(-1, \infty)$  whose restriction to  $[\underline{n}, \bar{n}]$  is equal to  $g(q, n, t, A_{10}, A_{20}, A_{10})$ . Let  $\underline{n}(q, t, A_{10}, A_{20}, A_{10})$  and  $\bar{n}(q, t, A_{10}, A_{20}, A_{10})$  be respectively the smallest and the largest solution to the following equation on  $(0, \infty)$  :

$$\frac{\partial h}{\partial n}(q, n, t, A_{10}, A_{20}, A_{10}) = 0 \quad (3.29)$$

With  $\underline{n}(q, t, A_{10}, A_{20}, A_{10}) = a$  and  $\bar{n}(q, t, A_{10}, A_{20}, A_{10}) = b$  if (3.29) has no solution, with  $0 < a < b$ . Assume further that  $\frac{\partial \underline{n}}{\partial t} = \frac{\partial \bar{n}}{\partial t} = \frac{\partial \underline{n}}{\partial A_{10}} = \frac{\partial \bar{n}}{\partial A_{10}} = 0$  for all  $(t, i) \in [0, \infty) \times \{1; 2; I\}$  so much that we can write  $\underline{n}(q, t, A_{10}, A_{20}, A_{10}) = \underline{n}(q)$  and  $\bar{n}(q, t, A_{10}, A_{20}, A_{10}) = \bar{n}(q)$ .

Then, we can define  $A_g(q)$  the set of initial technology levels whose fertility thresholds  $\underline{n}$  and  $\bar{n}$  are such that  $[\underline{n}(q), \bar{n}(q)] \subset [\underline{n}, \bar{n}]$ . When  $A_g(q) \neq \emptyset$ , we define  $[\underline{n}_g(q), \bar{n}_g(q)]$  an interval that contains all the roots of the partial derivative of  $h$ . If (3.29) has no solution, choose  $[\underline{n}_g(q), \bar{n}_g(q)] = [a, b]$ .

Proceed similarly to define for fixed  $n$ ,  $B_g(n)$  the set of initial technology levels such that the longevity threshold  $q$  is such that the first partial derivative of the extension of  $g(q, n, t, A_{10}, A_{20}, A_{10})$  to  $(0, 1]$  (if it exists) does not cancel on  $(0, q]$ . And define similarly  $[q_g(n), \bar{q}_g(n)]$  the interval that contains all the roots of the first partial derivative of the extension of  $g(q, n, t, A_{10}, A_{20}, A_{10})$ .

Definition 3.9 states formally how to choose initial technology levels such that I can draw  $q \rightarrow g(q, n, t, A_{10}, A_{20}, A_{10})$  or  $n \rightarrow g(q, n, t, A_{10}, A_{20}, A_{10})$  on intervals that contain all change of monotonicity points of the functions without violating the point (iv) of Definition 3.7. Note finally that for a variable  $g$  and a longevity level  $q \in (0, 1]$  (respectively  $n \in (-1, \infty)$ ), the non-emptiness of  $A_g(q)$  (respectively  $B_g(n)$ ) is not guaranteed.

As a first step, I determine how aging affects the labor allocation when preferences are homothetic (i.e.  $\beta = 0$ ). Equivalently, this amounts to study how aging affects the asymptotic labor allocation of the economy.

**Proposition 3.10.** Note  $l_i^*$  the asymptotic labor share of sector  $i \in \{1; 2; I\}$ .

- (i)  $q \rightarrow \frac{l_i^*}{l_i^*}$  and  $n \rightarrow \frac{l_i^*}{l_i^*}$  increases for  $i \in \{1; 2\}$  respectively on  $(0, 1]$  and  $(-1, \infty)$ .
- (ii)  $\frac{l_i^*}{l_i^*}$  does not depend on  $q$ , nor on  $n$ .

*Proof.* See Appendix E ■

In our framework, the old and the young have identical intratemporal preferences. If we also assume that they are homothetic, then, according to Theorem 3.5, the aggregate consumption share only depends on the relative price between the two goods. This relative price only depends on the exogenous technology levels because the two sectors have the same production functions. Thus,  $\frac{l_i^*}{l_i^*}$  is unaffected by the aging process (point (ii)). Point (i) states that a longevity increase spurs labor reallocation from the consumption goods sectors towards the investment sector. There are three effects at stake in this process : the "population effect", the "allocation effect" and the effects of aging on the capital accumulation. From

section 3.2, we know that the first two effects have opposite impacts on  $\frac{S_t}{C_{1t}}$ . In Appendix E, I show that the "allocation effect" always dominates the "population effect". The effects on the capital accumulation manifest themselves under two forms. First, as the consumption sector and the investment sector use capital with different proportions, the capital deepening caused by a longevity increase creates labor reallocation. Given the relative demand, if the investment sector is less (respectively more) capital intensive, then capital deepening increases more (resp. less) the production of the consumption sector. Thus, labor reallocates towards the investment sector (resp. the consumption sector) to compensate the difference of production changes. Second, aging affects the wage, the interest rate levels and the price of consumption goods, which directly impact  $\frac{S_t}{C_{1t}}$ . Consumption levels are linear in the wage because intratemporal preferences are homothetic and saving is linear in the wage because the intertemporal elasticity of substitution is constant. Then,  $\frac{S_t}{C_{1t}}$  does not depend on the wage. The change of the interest rate due to a longevity increase has an ambiguous effect on  $\frac{S_t}{C_{1t}}$ . On the one hand, when the interest rate decreases, the young save less because the intertemporal elasticity of substitution is smaller than 1. Then,  $s_t$  decreases, while  $c_{1t}$  increases. On the other hand, for the same reason, the old spend less :  $d_{1t-1}$  decreases. The capital deepening due to a longevity increase increases (respectively decreases) the consumption goods price if consumption sectors are less (resp. more) capital intensive than the investment sector. As  $\frac{S_t}{C_{1t}}$  positively depends on the consumption good prices, the indirect effect of aging on  $\frac{S_t}{C_{1t}}$  through the consumption price is positive (resp. negative) if  $\alpha < \alpha_I$  (resp.  $\alpha > \alpha_I$ ). The total effect of a longevity increase on  $\frac{l_t^*}{l_i^*}$  is independent of these parameters and is positive. Point (i) also gives the impact of a fertility decrease on  $\frac{l_t^*}{l_i^*}$ , which is the opposite of that of a longevity increase. Fertility acts on  $\frac{l_t^*}{l_i^*}$  as the longevity does, except that a fertility shock does not create an "allocation effect". The absence of the "allocation effect" explains the difference of result between the fertility decrease and the longevity increase. This underlines that the source of aging is crucial to determine the impact of this process on the allocation of labor. The total effect of aging on  $\frac{l_t^*}{l_i^*}$  is ambiguous and depends on the relative magnitude of the fertility and the longevity shocks.

I now examine the labor reallocations caused by aging on the CGP when preferences are non-homothetic. The labor reallocation between the consumption sectors and the investment sector is easily obtained as a corollary of Proposition 3.10 :

**Proposition 3.11.** *For any  $(q, n) \in (0, 1] \times (-1, \infty)$ ,  $A_{\frac{l_t}{l_1+l_2}}(q)$  and  $B_{\frac{l_t}{l_1+l_2}}(n)$  are non-empty.  $q \rightarrow \frac{l_{1t}}{l_{1t}+l_{2t}}$  and  $n \rightarrow \frac{l_{1t}}{l_{1t}+l_{2t}}$  increase for  $i \in \{1; 2\}$  respectively on any closed subinterval of  $(0, 1]$  and  $(-1, \infty)$ .*

*Proof.*  $\frac{l_{1t}}{l_{1t}+l_{2t}} = \frac{1}{\theta} \frac{l_t^*}{l_1^*}$ . Then, according to Proposition 3.10, the functions  $q \rightarrow \frac{l_{1t}}{l_{1t}+l_{2t}}$  and  $n \rightarrow \frac{l_{1t}}{l_{1t}+l_{2t}}$  are monotonic. Thus,  $A_{\frac{l_t}{l_1+l_2}}(q)$  and  $B_{\frac{l_t}{l_1+l_2}}(n)$  are non-empty and the variation of the two functions are obtained from the point (i) of Proposition 3.10. ■

The non-homotheticity of preferences does not impact the labor reallocation between the consumption sectors and the investment sector because the two consumption sectors use the

same production. Thus, the study of  $\frac{l_{It}}{l_{1t}+l_{2t}}$  is identical to that of  $\frac{l_t^*}{l_t^*}$  and the interpretation of Proposition 3.11 is similar to that of Proposition 3.10. In the next subsection, I provide a numerical example of a simultaneous shock on fertility and longevity on  $\frac{l_{It}}{l_{1t}+l_{2t}}$ .

The result suggests to study how aging affects the income per worker level. In a one-sector model, the fact that aging stimulates the capital stock per worker implies that aging increases the income per worker level. However, in a multi-sector framework, this implication is not necessarily true. The income per worker writes :

$$y_t = P_{1t}A_{1t}k_{1t}^\alpha l_{1t} + P_{2t}A_{2t}k_{2t}^\alpha l_{2t} + A_{It}k_{It}^\alpha l_{It} \quad (3.30)$$

Using (3.10) and (3.11) and the labor-market clearing condition (3.22), this can be written as :

$$y_t = P_{2t}A_{2t}k_{2t}^\alpha (1 - l_{It}) + A_{It}k_{It}^\alpha l_{It} \quad (3.31)$$

From (3.31), we see that the capital stock per worker increase due to aging increases income per worker. However, aging also affects  $l_{It}$ . Rewrite (3.31) as :

$$y_t = P_{2t}A_{2t}k_{2t}^\alpha + l_{It}(A_{It}k_{It}^\alpha - P_{2t}A_{2t}k_{2t}^\alpha) \quad (3.32)$$

From (3.32), we see that the impact of aging on  $l_{It}$  can negatively impact the income per worker level. For example, if the output per worker of the consumption sector is greater than that of the investment sector, then a longevity increase implies, according to Propostion 3.11, that labor flows towards the sector with the smaller output per worker, which reduces the aggregate output per worker. The complete result is in the following proposition :

**Proposition 3.12.** *Note  $y(n, q)$  the level of the aggregate output per worker on the CGP. Fix  $q \in (0, 1]$ .  $A_y(q) \neq \emptyset$ . The mapping  $n \rightarrow y(n, q)$  is as follows on  $[\underline{n}_y(q), \bar{n}_y(q)]$  :*

(I) *If  $\alpha > \alpha_I$ , then  $n \rightarrow y(n, q)$  is decreasing.*

(II) *If  $\alpha < \alpha_I$  :*

(i) *If  $\frac{1-\sigma}{\sigma} > \left(\frac{(2-\alpha+\alpha_I)(1-\alpha_I)+2\sqrt{1-\alpha+\alpha_I}}{\alpha_I-\alpha}\right) \frac{\alpha_I}{1-\alpha_I}$ , then  $n \rightarrow y(n, q)$  starts decreasing, ends decreasing and has exactly one local minimum.*

(ii) *If  $\frac{1-\sigma}{\sigma} < \left(\frac{(2-\alpha+\alpha_I)(1-\alpha_I)+2\sqrt{1-\alpha+\alpha_I}}{\alpha_I-\alpha}\right) \frac{\alpha_I}{1-\alpha_I}$ , then  $n \rightarrow y(n, q)$  is decreasing.*

**Proposition 3.13.** *Fix  $n \in (-1, \infty)$ .  $B_y(n) \neq \emptyset$ . The mapping  $q \rightarrow y(n, q)$  is as follows on  $[\underline{q}_y(n), \bar{q}_y(n)]$  :*

(I) *If  $\alpha_I < 1/4$  and  $\alpha \in [4\alpha_I(1 - \alpha_I), \alpha_I + \sqrt{\alpha_I}]$ , then there exists  $-1 < n_1 < n_2$  such that :*

(i) *If  $n \in (-1, n_1]$ , then  $q \rightarrow y(n, q)$  starts increasing, ends increasing and has exactly one local minimum.*

(ii) *If  $n \in [n_1, n_2]$ , then  $q \rightarrow y(n, q)$  is U-shaped.*

(II) *If  $\alpha > \alpha_I$  and  $\alpha_I > 1/4$  or  $\alpha \notin [4\alpha_I(1 - \alpha_I), \alpha_I + \sqrt{\alpha_I}]$  or if  $\alpha_I > 1/4$  and  $\alpha \in [\max(2\alpha_I, 4\alpha_I(1 - \alpha_I), 3\alpha_I)]$ , then there exists  $n_3 > -1$  such that : if  $n \in (-1, n_3)$ , then  $q \rightarrow y(n, q)$  is U-shaped.*

(III) *Otherwise,  $q \rightarrow y(n, q)$  is increasing.*

*Proof.* See Appendix F ■

Propositions 3.12 and 3.13 show that contrary to the one-sector Diamond model, a longevity increase or a fertility decrease does not necessarily increase the aggregate output per worker level. I have highlighted two channels through which aging acts on  $y(n, q)$ . First, aging implies a greater capital stock per worker in each sector. This increases  $y(n, q)$  and it is the only channel at stake in the one-sector model. Second, aging changes the allocation of workers across sectors. This effect increases (resp. decreases)  $y(n, q)$  if aging redirects labor towards the sector with the greater (resp. smaller) output per worker. In our case, as sectors 1 and 2 have the same production function, they have the same output per worker. However, the output per worker of these two sectors differs with that of the investment sector. Therefore, the heterogeneity of production functions across sectors can create negative effects of aging on  $y(n, q)$  if aging creates labor flows towards the sector with the smaller output per worker. According to Proposition 3.11, a longevity increase redirects labor from the consumption sectors towards the investment sector, while a fertility decrease causes the opposite labor flow. Moreover, the more capital intensive sector is also the sector with the greater output per worker. Therefore, if  $\alpha_I > \alpha$ , a longevity increase redirects labor towards the more productive sector (sector I), which unambiguously increases  $y(n, q)$  (note that the case  $\alpha_I > \alpha$  is in the case (III) of Proposition 3.13). If  $\alpha_I < \alpha$ , the two effects of a longevity increase on  $y(n, q)$  are opposite. The contrary holds for a fertility decrease. If  $\alpha_I < \alpha$ , the fertility decrease redirects labor towards the more productive sector (sectors 1 and 2), which unambiguously increases  $y(n, q)$  (case (I) of Proposition 3.12). However, when  $\alpha_I > \alpha$ , the two effects of a fertility decrease on  $y(n, q)$  are opposite.

Non-monotonic relationships between demographic variables and income per worker have been highlighted in previous works. In a one-sector OLG model à la Yaari (1965), d'Albis (2007) shows that the steady-state income per worker level is not monotonic with respect to the population growth rate. Propositions 3.12 and 3.13 strongly echo back this result given that multi-period one-sector OLG models roughly behave as two-period multi-sector OLG models (Balasko et al.(1980)).

I now examine the impact of aging on the labor allocation between consumption sectors. Hence I study the variations of  $\frac{l_{1t}}{l_{2t}}$  with respect to  $q$  and  $n$  along the CGP. (3.26) and (3.27) imply that :

$$\frac{l_{1t}}{l_{2t}} = \frac{\theta + d \frac{\left(\frac{R^{\sigma-1}}{q^\sigma} x^\sigma + \frac{q}{1+n} \left(\frac{R}{1+g_I}\right)^\sigma\right) R^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} \left(1 + \frac{R^{\frac{\sigma-1}{q}}}{q} x\right)^{1-\sigma}}{(1-\alpha) \left(\frac{R^{\frac{\sigma-1}{q}}}{q} x + \frac{R}{(1+n)g_I}\right)} \frac{A_{2t}^\sigma}{A_{1t} A_{It}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}}} \left(\frac{A_{2t}}{A_{1t}}\right)^{\gamma-1+\theta(1-\sigma)}}{1 - \theta - d \frac{\left(\frac{R^{\sigma-1}}{q^\sigma} x^\sigma + \frac{q}{1+n} \left(\frac{R}{1+g_I}\right)^\sigma\right) R^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} \left(1 + \frac{R^{\frac{\sigma-1}{q}}}{q} x\right)^{1-\sigma}}{qg_I \left(\frac{R^{\frac{\sigma-1}{q}}}{q} x + \frac{R}{(1+n)g_I}\right)} \frac{A_{2t}^\sigma}{A_{1t} A_{It}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}}} \left(\frac{A_{2t}}{A_{1t}}\right)^{\gamma-1+\theta(1-\sigma)}}{\quad} \quad (3.33)$$

This ratio decreases with time because the expenditure elasticity of the good 2 is strictly

greater than that of the good 1. I first analyze how it is impacted by  $n$ . As for  $\frac{l_t^*}{l_t}$ , there are two channels at stake : the "population effect" and the effects of  $n$  on the capital accumulation. The direction of the "population effect" is provided in the following lemma :

**Lemma 3.14.** *Fix  $q \in (0, 1]$ . There exists  $\mu(q) \in (-1, \infty)$  such that a decrease of  $n$  implies : An increase (respectively a decrease) of  $\frac{l_t}{l_{2t}}$  through the "population effect" if  $n < \mu(q)$  (resp. if  $n > \mu(q)$ ).*

*Proof.* See Appendix G ■

Point (i) of Lemma 3.14 shows that the direction of the "population effect" changes with  $n$  in general equilibrium.<sup>10</sup> This means that the young consumption share of good 1 overcomes the one of the old as fertility increases or equivalently that the expenditure level of the old overcomes that of the young as fertility increases.

The effects of a fertility decrease on  $\frac{l_t}{l_{2t}}$  through the capital accumulation limit themselves to the change of the real wage and the interest rate, because the two sectors use the same production function. The aggregate relative demand does now depend on the real wage because the intratemporal preferences are non-homothetic. When the real wage increases, the old and the young spend more, which implies that they both reduce their consumption share of good 1. Recall from the discussion of section 3.2 that this is not sufficient to state that an increase of the real wage decreases  $\frac{C_{1t}}{C_{2t}}$  as we also have to check how the consumption shares of each generation respond to a change of the real wage. In the present specification, the consumption share of each generation does not depend on the real wage. This happens for two reasons. The *intertemporal* preferences are homothetic, which implies that expenditure levels of the young and the old are both proportional to their wage. Moreover, the wage levels of the young and the old are proportional along the CGP. Then, the ratio of the two expenditure levels does not depend on the real wage. This implies that the consumption share of each generation does not depend on the real wage. Then, as a fertility decrease increases the real wage, this creates a negative impact on  $\frac{l_t}{l_{2t}}$ . The role of the change of the interest rate is less transparent as the expenditure levels of the young and the old vary with the interest rate in opposite way. A decrease of the interest rate increases the expenditure level of the young and decreases the one of the old *ceteris paribus*. This happens because the intertemporal elasticity of substitution is smaller than 1, then the young decrease their expenditures to save more, while the old can spend more. This means that the interest rate change implies that young and old change their consumption ratios in opposite direction and that the consumption share of each generation changes. Then, the total effect of a change of the interest rate is ambiguous.

The following proposition gives some clues on the shape of the function  $n \rightarrow \frac{l_t}{l_{2t}}$  :

**Proposition 3.15.** *Fix  $q \in (0, 1]$ .  $A_{\frac{1}{2}}(q) \neq \emptyset$ . Let  $(A_{10}, A_{20}, A_{10}) \in A_{\frac{1}{2}}(q)$  and note  $[\underline{n}_{\frac{1}{2}}(q), \bar{n}_{\frac{1}{2}}(q)]$  its associated interval. For all time  $t \geq 0$  :*

(i) *If  $\frac{2}{\sigma} \leq \frac{\alpha}{1-\alpha_1}$ , then  $n \rightarrow \frac{l_t}{l_{2t}}$  is increasing on  $[\underline{n}_{\frac{1}{2}}(q), \bar{n}_{\frac{1}{2}}(q)]$ .*

10. This does not contradict the results of section 3.2 in which I claimed that for fixed prices, the direction of the "population effect" is monotonic with respect to  $n$ .

(ii) If  $\frac{1}{\sigma} \leq \frac{\alpha}{1-\alpha_I} \leq \frac{2}{\sigma}$ , then  $n \rightarrow \frac{l_{1t}}{l_{2t}}$  is initially increasing, ends increasing and has one or none interior maximum on  $[\underline{n}_{\frac{1}{2}}(q), \bar{n}_{\frac{1}{2}}(q)]$ .

(iii) If  $1 \leq \frac{\alpha}{1-\alpha_I} \leq \frac{1}{\sigma}$ , then  $n \rightarrow \frac{l_{1t}}{l_{2t}}$  is initially decreasing, ends increasing and has one or none local maximum on  $[\underline{n}_{\frac{1}{2}}(q), \bar{n}_{\frac{1}{2}}(q)]$ .

(iv) If  $\frac{\alpha}{1-\alpha_I} \leq 1$ , then  $n \rightarrow \frac{l_{1t}}{l_{2t}}$  is initially decreasing, ends decreasing and has one or none local interior maximum on  $[\underline{n}_{\frac{1}{2}}(q), \bar{n}_{\frac{1}{2}}(q)]$ .

*Proof.* See Appendix H ■

The main message of Proposition 3.15 is that the curve  $n \rightarrow \frac{l_{1t}}{l_{2t}}$  can take very different shapes.<sup>11</sup> The main drivers of these shapes are the parameters  $\frac{\alpha}{1-\alpha_I}$  and  $\frac{1}{\sigma} \cdot \frac{\alpha}{1-\alpha_I}$  is the absolute value of the elasticity of the real wage with respect to  $R$ , while  $\sigma$  controls the variation of the expenditure levels with respect to the interest rate. When fertility increases, the greater is  $\frac{\alpha}{1-\alpha_I}$  with respect to  $\frac{1}{\sigma}$ , the greater is the effect of the decrease of the real wage relatively to the direct effect of the interest rate.

I now turn to the study of  $q \rightarrow \frac{l_{1t}}{l_{2t}}$ . Longevity influences  $\frac{l_{1t}}{l_{2t}}$  through the same two channels as fertility does. However, it also creates an "allocation effect". Thus, I can decompose the effect of  $q$  on  $\frac{l_{1t}}{l_{2t}}$  into three effects : the "allocation effect", the "population effect" and the effects of a longevity increase on the capital accumulation. These two latter are qualitatively identical to the ones due to a fertility decrease. The direction of the "allocation effect" and the "population effect" are obtained in the following lemma :

**Lemma 3.16.** (I) A longevity increase increases  $\frac{l_{1t}}{l_{2t}}$  through the "allocation effect".

(II) There exists a fertility threshold  $\kappa$  such that :

(i) If  $n < \kappa$ , there exists  $q^*(n) \in (0, 1)$  such that a longevity increase decreases (resp. increases)  $\frac{l_{1t}}{l_{2t}}$  through the "population effect" if  $q < q^*(n)$  (resp. if  $q > q^*(n)$ ).

(ii) If  $n \geq \kappa$ , a longevity increase decreases  $\frac{l_{1t}}{l_{2t}}$  through the "population effect".

*Proof.* See Appendix I ■

As the good 1 has a smaller expenditure elasticity than good 2, the increase of  $q$  implies that both the young and the old increase their consumption share of good 1. Point (I) shows that this positive force on  $\frac{l_{1t}}{l_{2t}}$  is never counteracted by a change of the consumption share of the old or the young in the total consumption. Thus, we expect that a longevity increase increases more  $\frac{l_{1t}}{l_{2t}}$  than a fertility decrease. To study  $q \rightarrow \frac{l_{1t}}{l_{2t}}$ , I use (3.28) to eliminate the variable  $q$  in (3.33). Then, the problem is reduced to draw the function  $q \rightarrow f(R(q, n))$  for fixed  $n \in (0, \infty)$ , where  $f$  is given by :

$$f(R) = R^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} \frac{\left(1 - \frac{\alpha_I}{\alpha} + \frac{1-\alpha}{\alpha} \frac{R}{n(1+g_I)^{\frac{1}{1-\alpha_I}}}\right)^{1-\sigma} \left(\frac{1-\alpha}{\alpha} \frac{R}{n(1+g_I)^{\frac{1}{1-\alpha_I}}} - \frac{\alpha_I}{\alpha} + \frac{R^{\frac{\sigma-1}{\sigma}+1} x^{1-\sigma}}{n(1+g_I)^{\frac{1-\sigma}{1-\alpha_I}}}\right)}{\left(\frac{1-\alpha}{\alpha} \frac{R}{n(1+g_I)^{\frac{1}{1-\alpha_I}}} - \frac{\alpha_I}{\alpha}\right)^{1-\sigma} \left(\frac{R}{\alpha n(1+g_I)^{\frac{1}{1-\alpha_I}}} - \frac{\alpha_I}{\alpha}\right)} \quad (3.34)$$

11. From the proof of Proposition 3.15, the sign of the derivative of  $\frac{l_{1t}}{l_{2t}}$  with respect to  $n$  is the one of a degree 3 polynomial. Thus, it is possible to state exact existence conditions for the local extrema. However, I choose not to report them because their complexity make them non-informative.

The complexity of the expression of  $f$  makes impossible the study of  $q \rightarrow \frac{l_{1t}}{l_{2t}}$  for all parameters combination. However, the following proposition gives some clues on the shape of the curve :

**Proposition 3.17.** *Fix  $n \in (-1, \infty)$ .  $B_{\frac{1}{2}}(n) \neq \emptyset$ . Note  $[q_{\frac{1}{2}}(n), 1]$  its associated interval.*

- (I)  $q \rightarrow \frac{l_{1t}}{l_{2t}}$  is initially decreasing on  $[q_{\frac{1}{2}}(n), 1]$ .
- (II) There exists  $\chi \in (0, \infty)$ , independent of  $n$ , such that if  $n \leq \chi$  :
  - (i)  $q \rightarrow \frac{l_{1t}}{l_{2t}}$  ends decreasing if  $\alpha(1 - \alpha_I) > \frac{(1-\sigma)^2}{\sigma}$ .
  - (ii)  $q \rightarrow \frac{l_{1t}}{l_{2t}}$  ends increasing if  $\alpha(1 - \alpha_I) < \frac{(1-\sigma)^2}{\sigma}$ .
  - (iii) If  $\sigma \in [\frac{1}{2}, 1 - \frac{1-\alpha_I}{\alpha})$ ,  $q \rightarrow \frac{l_{1t}}{l_{2t}}$  is decreasing on  $[q_{\frac{1}{2}}(n), 1]$ .

*Proof.* See Appendix J ■

Despite the absence of results on all the interval  $[q_{\frac{1}{2}}(n), 1]$ , Proposition 3.17 shows an interesting point : for any values of the model parameters,  $q \rightarrow \frac{l_{1t}}{l_{2t}}$  is initially decreasing. This contrasts with the results of Proposition 3.15, according to which the curve  $n \rightarrow \frac{l_{1t}}{l_{2t}}$  is initially increasing or initially decreasing depending on the values of  $\alpha_I$ ,  $\alpha$  and  $\sigma$ . This echoes back the results on the impact of demographic variables on  $\frac{l_{1t}}{l_{1t}+l_{2t}}$  (Proposition 3.11) that underline that the source of aging is crucial to examine the labor reallocations caused by the aging process. This is also unexpected from the result of Lemma 3.16 according to which a longevity increase increases  $\frac{l_{1t}}{l_{2t}}$  through the "allocation effect" as this creates an additional positive force on  $\frac{l_{1t}}{l_{2t}}$  in comparison with a fertility decrease. This shows that the two types of demographic shocks do not only differ with respect to the presence or not of the "allocation effect" but also with respect to the way they affect the accumulation of inputs and with respect to the "population effect". In the next subsection, I numerically complete the results of Proposition 3.17 by assigning values to the different parameters.

### 3.3.4 Numerical analysis

In this subsection, I assign numerical values to the various parameters of the model to determine the impact of a longevity and a fertility shift on the CGP labor allocation of the US economy. I consider that a period of the model equals to 40 years and that young agents enter the workforce at the age of 25. I assume that the US economy is on the CGP on the period 1970-2010 and I calibrate the model on this period.

Herrendorf et al.(2014) are precious guides for the numerical calibration of multi-sector models. They underline that we have to decide whether we interpret consumption and production functions as value added components of final consumption and value added production functions or respectively as final expenditure consumption and final consumption production functions. Here I follow a final expenditure approach. It means that I use consumption measures data to calibrate the model. Then, the production side at the sectorial level of the model does not have a direct empirical counterpart. More precisely, I use NIPA data of the Bureau of Economic Analysis (BEA), which collects data on manufacturing and services

consumption as well as purchasing prices of each good on the period 1970-2010. Consistent with Boppart (2014), I observe that the consumption share of manufacturing goods grows at a constant rate, which implies that  $\theta = 0$ . I set  $\beta$  equal to 1. For  $\alpha$  and  $\alpha_I$ , I use the estimates of Herrendorf and Valentinyi (2008). The authors underline that estimating production functions in multi-sector models is much more challenging than estimating an aggregate production function. The reason is that sectors do not only use capital and labor as inputs but also other goods produced by the other sectors as inputs. One-sector models do not face this issue since all the intermediate goods are ultimately produced from capital and labor. Herrendorf and Valentinyi (2008) propose to estimate sectoral production functions as if all the intermediate goods used by a sector are produced by this sector with capital and labor. Thus, I must warn the reader that the production functions used in the calibration are not the actual production functions used by the US manufacturing or services sectors, but fictive production functions that consider that each sector produces intermediate goods itself with capital and labor. Put in other words,  $l_{it}$  represents the share of workers involved in the production of good  $i$  and not the share of workers employed by sector  $i$ .

The remaining parameters to estimate are  $\sigma, \gamma, g_1, g_2, g_I, A_{10}, A_{20}, A_{I0}, q, n$ . I regress the log of the relative price between consumption goods on time to find  $\frac{1+g_2}{1+g_1}$  and  $\frac{A_{20}}{A_{10}}$ . Then, I regress the log of income per worker on time to find  $(1+g_I)^{\frac{1}{1-\alpha_I}}$  and  $A_{I0}^{\frac{1}{1-\alpha_I}}$ . Relative to demographic variables,  $q$  is identified as the life expectancy at age 65 of the cohort born in 1905, which is obtained in Bell et al.(1992). The population growth rate on the period is computed from Table 7 of the BEA, from which I find  $n = 0.52$ . Then, I compute  $\sigma$  from (3.28) for the annual interest rate to equate 4.34% (World Bank database). Then, I regress the log of the ratio of total expenditures to the price of services to find  $(1+g_2)(1+g_I)^{\frac{\alpha}{1-\alpha_I}}$ . This allows to deduce  $(1+g_2)$  and  $(1+g_1)$ . Finally, I regress the log of the consumption share of manufacturing good to obtain  $\gamma$  and  $\frac{A_{20}^\sigma}{A_{10}}$ . This allows to obtain  $A_{10}$  and  $A_{20}$ .

Table 3.1 summarizes the values of the parameters.<sup>12</sup>

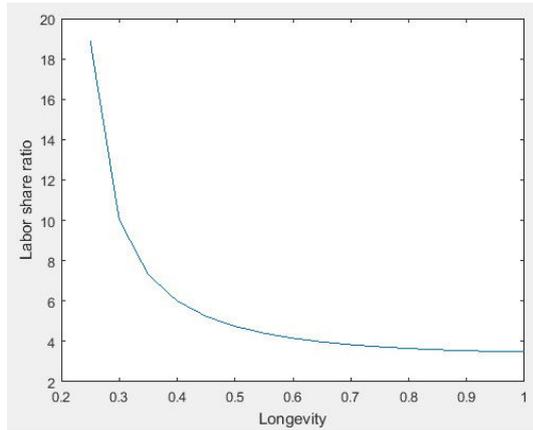
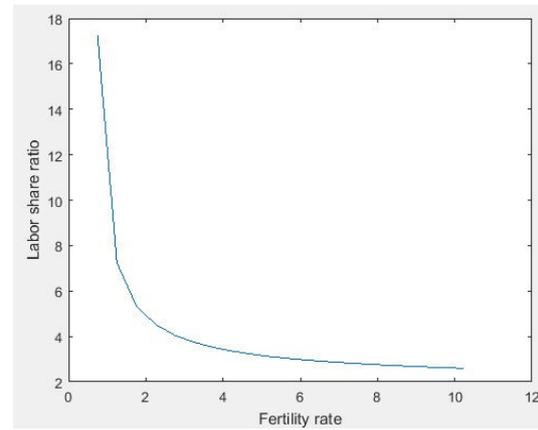
I first use these values to draw the curves  $q \rightarrow \frac{l_{1t}}{l_{2t}}$  and  $n \rightarrow \frac{l_{1t}}{l_{2t}}$  to complete my theoretical results of the previous section.

I find that both curves are decreasing. Therefore a longevity increase and a fertility decrease redirects labor between consumption sectors in opposite direction. I use these curves to determine the magnitude of these labor flows. For example, if the life expectancy at 65 increases from 16 to 20, then the ratio of labor in manufacturing to labor in services decreases by 20.9%. If the life expectancy at 65 increases from 16 to 25, then the ratio decreases by 32.4%. A contrario, if the fertility rate decreases from 1.52 to 1.25, then the ratio increases by 20.9%. If the fertility rate decreases from 1.52 to 1, then the ratio increases by 60.7%. Hence realistic demographic changes produce significant labor flows between consumption sectors. However, as a longevity increase and a fertility decrease have an opposite impact on  $\frac{l_{1t}}{l_{2t}}$ , aging does not necessarily create large labor flows between these two sectors. For instance, if the

12. For these parameters, the general solution produces a non-realistic path, hence to analyze the transitional dynamics, I would have to recalibrate the model. Thus, I focus here on the CGP.

Parameter	Value
$\beta$	1
$\theta$	0
$\alpha$	0.35
$\alpha_I$	0.28
$\sigma$	0.65
$\gamma$	0.29
$1 + g_1$	2.62
$1 + g_2$	1.21
$(1 + g_I)^{\frac{1}{1-\alpha_I}}$	3.12
$A_{10}$	97.05
$A_{20}$	195
$A_{10}^{\frac{1}{1-\alpha_I}}$	483

TABLE 3.1 – Parameter values used in the numerical analysis

FIGURE 3.4 –  $q \rightarrow \frac{l_{1t}}{l_{2t}}$  for  $n = 1.52$ ,  $t = 2010$ FIGURE 3.5 –  $n \rightarrow \frac{l_{1t}}{l_{2t}}$  for  $q = 0.4$ ,  $t = 2010$ 

life expectancy at 65 increases from 16 to 25 and if the fertility rate decreases from 1.52 to 1, then  $\frac{l_{1t}}{l_{2t}}$  decreases by 9%. Note that these magnitudes depend on the period at which the labor shares are computed. In Figure 3.6, I plot the time evolution of  $\frac{l_{1t}}{l_{2t}}$  along the CGP for different demographic parameters. As  $\frac{l_{1t}}{l_{2t}}$  converges with time to 0, these shocks dissipate over time.

I also plot the curves  $q \rightarrow \frac{l_{1t}}{l_{1t}+l_{2t}}$  and  $n \rightarrow \frac{l_{1t}}{l_{1t}+l_{2t}}$ . They confirm my theoretical results and they allow to assess the magnitude of the labor reallocations between the investment sector and the consumption sectors. If life expectancy at 65 increases from 16 to 20, then  $\frac{l_{1t}}{l_{1t}+l_{2t}}$  increases by 14.7%. If life expectancy at 65 increases from 16 to 25, then  $\frac{l_{1t}}{l_{1t}+l_{2t}}$  increases by 31.3%. If the fertility rate decreases from 1.52 to 1, then  $\frac{l_{1t}}{l_{1t}+l_{2t}}$  decreases by 13%. Hence realistic demographic shocks also create significant labor reallocations between the investment sector and the consumption sectors. Finally, if the life expectancy at 65 increases from 16 to 25 and if the fertility rate decreases from 1.52 to 1, then  $\frac{l_{1t}}{l_{1t}+l_{2t}}$  increases by 14.5%. Note

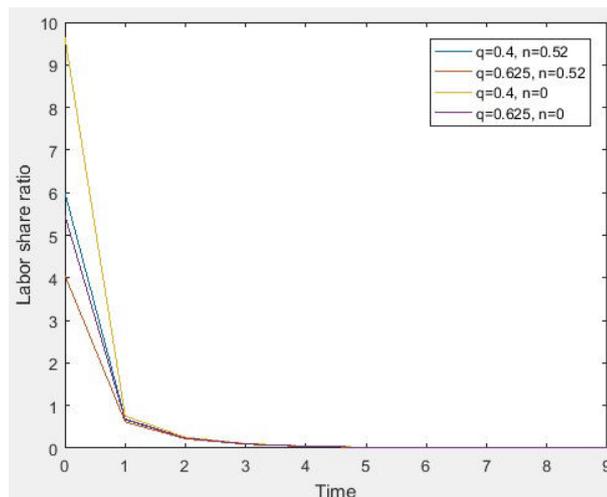


FIGURE 3.6 – Time evolution of  $\frac{l_{1t}}{l_{2t}}$  for different demographic parameters

that along the CGP,  $\frac{l_{1t}}{l_{1t}+l_{2t}}$  is time-invariant, hence these changes hold for all future periods provided that the economy stays on the CGP.

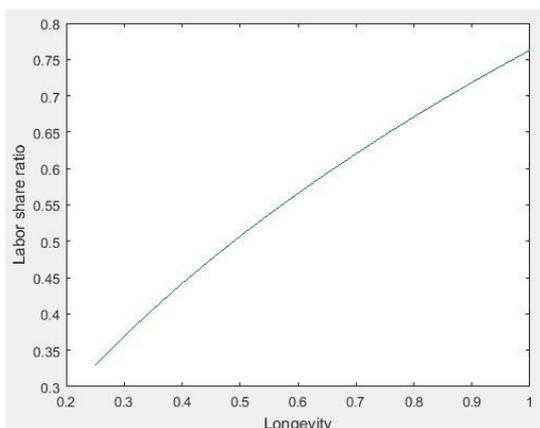


FIGURE 3.7 –  $q \rightarrow \frac{l_{1t}}{l_{1t}+l_{2t}}$  for  $n = 1.52$

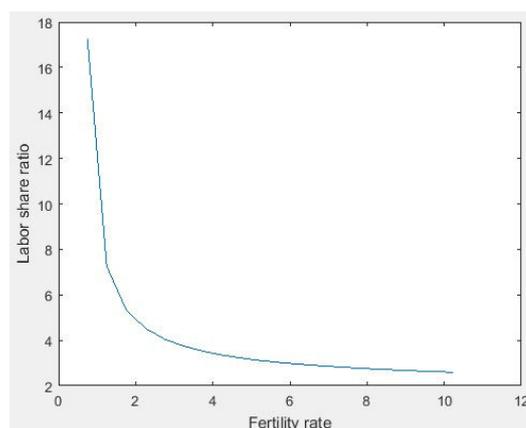


FIGURE 3.8 –  $n \rightarrow \frac{l_{1t}}{l_{1t}+l_{2t}}$  for  $q = 0.4$

Second, I determine the impact of demographic variables on income per worker level. I find that aging unambiguously increases income per worker. Table 3.2 reports the impact on the income per worker level of different demographic shocks.

Lastly, I assess the magnitude of the three channels through which aging affects the labor ratios. Rewrite (3.33) as :

$$\frac{l_{1t}}{l_{2t}} = \frac{d \frac{(\frac{R^{\sigma-1}}{q^{\sigma-1}} x^{\sigma} + \frac{q}{1+n} (\frac{R}{1})^{\sigma}) R^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} (1 + \frac{R^{\frac{\sigma-1}{q}} x)^{1-\sigma}}{q(1+g_I)^{\frac{1-\alpha_I}{1-\alpha_I}})}{(1-\alpha) (\frac{R^{\frac{\sigma-1}{q}} x + \frac{q}{1+n} \frac{R}{1})^{\frac{1-\alpha_I}{1-\alpha_I}}}}{1 - d \frac{(\frac{R^{\sigma-1}}{q^{\sigma-1}} x^{\sigma} + \frac{q}{1+n} (\frac{R}{1})^{\sigma}) R^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} (1 + \frac{R^{\frac{\sigma-1}{q}} x)^{1-\sigma}}{q g_I^{\frac{1-\alpha_I}{1-\alpha_I}}}}{(1-\alpha) (\frac{R^{\frac{\sigma-1}{q}} x + \frac{q}{1+n} \frac{R}{1})^{\frac{1-\alpha_I}{1-\alpha_I}}}} \frac{A_{2t}^{\sigma}}{A_{1t} A_{It}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}}} (\frac{A_{2t}}{A_{1t}})^{\gamma-1+\theta(1-\sigma)}}{(3.35)}$$

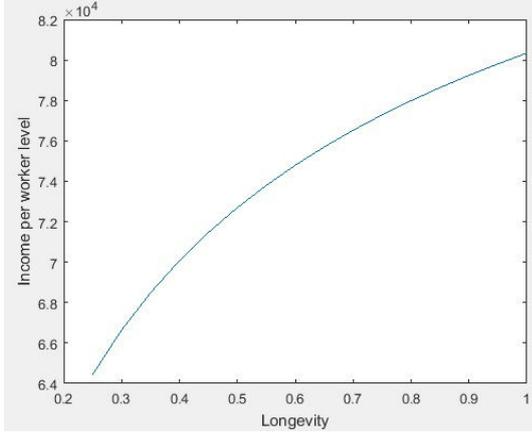


FIGURE 3.9 –  $q \rightarrow y_t$  for  $n = 1.52$

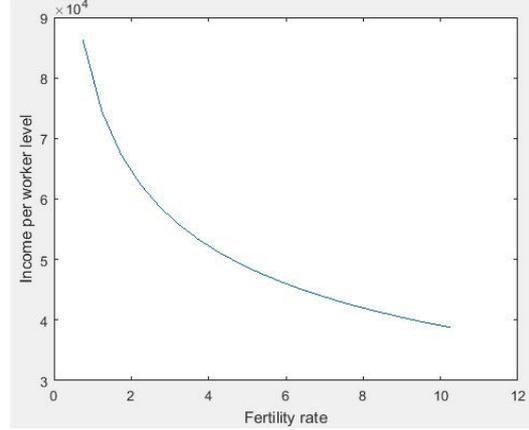


FIGURE 3.10 –  $n \rightarrow y_t$  for  $q = 0.4$

	$\frac{l_{1t}}{l_{2t}}$	$\frac{l_{1t}}{l_{1t}+l_{2t}}$	$y_t$
$q = 0.4, n = 0.25$	20.9%	-6.3%	6%
$q = 0.4, n = 0$	60.7%	-13%	13.2%
$q = 0.5, n = 0.52$	-20.9%	14.7%	3.7%
$q = 0.625, n = 0.52$	-32.4%	31.3%	7.4%
$q = 0.625, n = 0$	-9%	14.5%	22%

TABLE 3.2 – Percentage change of  $\frac{l_{1t}}{l_{2t}}, \frac{l_{1t}}{l_{1t}+l_{2t}}, y_t$  when  $(q, n)$  changes from  $(0.4, 0.52)$ .

To neutralize the general equilibrium effects, I maintain constant the interest rate  $R$  in (3.35). To neutralize the population effect I maintain constant the term  $\frac{q}{1+n}$  in (3.35). Finally, the remaining dependence with respect to  $q$  of  $\frac{l_{1t}}{l_{2t}}$  is the allocation effect. I proceed similarly for  $\frac{l_{1t}}{l_{1t}+l_{2t}}$ . Table 3.3 reports the impact of each effect on the labor ratios when the two others are neutralized when the life expectancy at 65 increases to 25 and the fertility rate decreases to 1.

Relative to the impact of aging on  $\frac{l_{1t}}{l_{1t}+l_{2t}}$ , we observe that it is mainly driven by a competition between the allocation effect and the population effect. As predicted by the theory, the allocation effect diminishes  $\frac{l_{1t}}{l_{1t}+l_{2t}}$ , while the population effect increases  $\frac{l_{1t}}{l_{1t}+l_{2t}}$ . These two effects are of non-negligible order. Relative to the impact of aging on  $\frac{l_{1t}}{l_{2t}}$ , the three effects are significant. Both the allocation effect and the population effect decrease  $\frac{l_{1t}}{l_{2t}}$ , yet general equilibrium effects increase  $\frac{l_{1t}}{l_{2t}}$  by a large amount, which leaves the total effect as relatively small.

	$\frac{l_{1t}}{l_{2t}}$	$\frac{l_{1t}}{l_{1t}+l_{2t}}$
Population effect	-39.1%	-38.8%
Allocation effect	-35.4%	56.3%
General equilibrium effects	58%	1.36%

TABLE 3.3 – Percentage change of  $\frac{l_{1t}}{l_{2t}}$  and  $\frac{l_{1t}}{l_{1t}+l_{2t}}$  due to each effect

## 3.4 CONCLUSION

This paper examined the impact of aging on the labor allocation of a stylized multi-sector economy. There are three effects at stake in this process. A longevity shift directly modifies the consumption levels of individuals and changes the ratio of young to old people. In partial equilibrium, I found that these two effects change the aggregate relative demand between consumption goods if and only if the intratemporal preferences are non-homothetic or not identical between young and old individuals and that these two effects have opposite consequences on the aggregate relative demand between capital good and consumption good. Finally, as aging affects the accumulation of production factors, there are general equilibrium effects that create labor reallocations. In a two-period OLG model with two consumption goods and one capital good, I studied the direction of the labor flows between sectors caused by aging on a path satisfying the Kaldor facts. Concerning the labor flows between consumption sectors and the investment sector, I argued that, depending to its source, aging may have a different impact : a longevity increase spurs labor reallocation towards the investment sector, whereas a fertility decrease spurs labor reallocation towards the consumption sectors. Concerning the labor flows between consumption sectors, I also showed that longevity and the population growth rate do not act similarly. For example, for low longevity values, increasing longevity always redirects labor from the production of services towards the production of manufacturing goods while under certain parameters specifications, a fertility decrease creates an opposite labor reallocation. I then calibrated the model on the US economy to examine the impact of realistic demographic shocks on the CGP labor allocation between sectors. I found that both a longevity increase and a fertility decrease significantly affect the CGP labor allocation. Yet, because a longevity increase and a fertility decrease act in opposite direction, the impact of aging on the labor allocation is not necessarily of high magnitude. For example, if life expectancy at 65 increases from 16 years to 25 years and the population growth rate decreases from 1.52 to 1, then the ratio of labor in manufacturing to labor in services decreases by 9%, while the ratio of labor in investment sector to labor in consumption sectors increases by 14.5%.

The main contribution of the paper is theoretical : it explains and characterizes how different assumptions on preferences and production functions create channels through which aging affects the labor allocation of a stylized multi-sector economy. This should stimulate and guide future empirical research on the labor allocation impacts of aging. A more quantitative analysis could be pursued. In my framework, old and young agents have the same preferences. This could be relaxed by allowing old individuals to home produce some goods as documented by Hurst and Aguiar (2005) to see how this affects the results.

### 3.5 APPENDIX A

The existence and uniqueness of  $s_t$  is guaranteed by the strict concavity of the functions  $u$  and  $v$ . Note  $L_t$  the Lagrangian of the maximization problem solved by a cohort- $t$  individual.

$$L_t = u(c_{1t}, c_{2t}) + qv(d_{1t}, d_{2t}) + \lambda_t(w_t - P_{1t}c_{1t} - P_{2t}c_{2t} - \frac{q}{R_{t+1}}(d_{1t}P_{1t+1} + d_{2t}P_{2t+1})) \quad (3.36)$$

Where  $\lambda_t$  is the Lagrange multiplier associated to the lifetime budget constraint. Use the first-order condition (FOC) for  $c_{1t}$  and  $c_{2t}$  and the first period budget constraint to get :

$$u_1(c_{1t}, \frac{w_t - P_{1t}c_{1t} - s_t}{P_{2t}}) - \frac{P_{1t}}{P_{2t}}u_2(c_{1t}, \frac{w_t - P_{1t}c_{1t} - s_t}{P_{2t}}) = 0 \quad (3.37)$$

Use Inada conditions and the strict concavity of  $u$  to prove that the LHS of (3.37) is a decreasing function of  $c_{1t}$  from  $\infty$  to  $-\infty$  on  $[0, \frac{w_t - s_t}{P_{1t}}]$ . Then, I can define  $c_1(s_t)$  the unique solution to (3.37), whose derivative is given by the implicit function theorem :

$$c_1'(s_t) = \frac{1}{P_{2t}} \frac{u_{12} - \frac{P_{1t}}{P_{2t}}u_{22}}{u_{11} - 2\frac{P_{1t}}{P_{2t}}u_{12} + (\frac{P_{1t}}{P_{2t}})^2u_{22}} \quad (3.38)$$

Where the arguments of the various functions have been omitted for notational convenience. Use now the first-order condition (FOC) for  $d_{1t}$  and  $d_{2t}$  and the second period budget constraint to get :

$$v_1(d_{1t}, \frac{\frac{R_{t+1}}{q}s_t - \frac{d_{1t}}{P_{1t+1}}}{P_{2t+1}}) - \frac{P_{1t+1}}{P_{2t+1}}v_2(d_{1t}, \frac{\frac{R_{t+1}}{q}s_t - \frac{d_{1t}}{P_{1t+1}}}{P_{2t+1}}) = 0 \quad (3.39)$$

Proceed as for  $c_{1t}$  to prove that (3.39) has a unique solution  $d_1(s_t, q)$ , whose partial derivatives are given by :

$$\frac{\partial d_1}{\partial s_t}(s_t, q) = \frac{(-R_{t+1})}{P_{2t+1}q} \frac{v_{12} - \frac{P_{1t+1}}{P_{2t+1}}v_{22}}{v_{11} - 2\frac{P_{1t+1}}{P_{2t+1}}v_{12} + (\frac{P_{1t+1}}{P_{2t+1}})^2v_{22}} \quad (3.40)$$

$$\frac{\partial d_1}{\partial q}(s_t, q) = \frac{R_{t+1}}{P_{2t+1}q^2} s_t \frac{v_{12} - \frac{P_{1t+1}}{P_{2t+1}}v_{22}}{v_{11} - 2\frac{P_{1t+1}}{P_{2t+1}}v_{12} + (\frac{P_{1t+1}}{P_{2t+1}})^2v_{22}} \quad (3.41)$$

Use now the FOCs for  $c_{1t}$  and  $d_{1t}$  to get :

$$u_1(c_1(s_t), \frac{w_t - P_{1t}c_1(s_t) - s_t}{P_{2t}}) = \frac{R_{t+1}P_{1t}}{P_{1t+1}}v_1(d_1(s_t, q), \frac{\frac{R_{t+1}}{q}s_t - \frac{d_1(s_t, q)}{P_{1t+1}}}{P_{2t+1}}) \quad (3.42)$$

(3.42) has a unique solution because  $s_t$  is unique. This defines implicitly  $s_t$  as a function of  $q$ . Differentiate (3.42) with respect to  $q$  and use (3.38), (3.41) and (3.40) to get :

$$\frac{\partial s_t}{\partial q} = \frac{R_{t+1}^2}{P_{2t+1}^2} s_t \frac{\frac{v_{21}^2 - v_{22}v_{11}}{v_{11} - 2\frac{P_{1t+1}}{P_{2t+1}}v_{12} + (\frac{P_{1t+1}}{P_{2t+1}})^2v_{22}}}{\frac{1}{P_{2t}^2} \frac{u_{21}^2 - u_{22}u_{11}}{u_{11} - 2\frac{P_{1t}}{P_{2t}}u_{12} + (\frac{P_{1t}}{P_{2t}})^2u_{22}} + \frac{R_{t+1}^2}{q^2 P_{2t+1}^2} \frac{v_{21}^2 - v_{22}v_{11}}{v_{11} - 2\frac{P_{1t+1}}{P_{2t+1}}v_{12} + (\frac{P_{1t+1}}{P_{2t+1}})^2v_{22}}} \quad (3.43)$$

The concavity of  $u$  and  $v$  implies the following inequalities :

$$v_{21}^2 - v_{22}v_{11} < 0 \quad (3.44)$$

$$u_{21}^2 - u_{22}u_{11} < 0 \quad (3.45)$$

$$v_{11} - 2\frac{P_{1t+1}}{P_{2t+1}}v_{12} + \left(\frac{P_{1t+1}}{P_{2t+1}}\right)^2v_{22} < 0 \quad (3.46)$$

$$u_{11} - 2\frac{P_{1t}}{P_{2t}}u_{12} + \left(\frac{P_{1t}}{P_{2t}}\right)^2u_{22} < 0 \quad (3.47)$$

This proves that  $\frac{\partial s_t}{\partial q} > 0$ . Rewrite  $\frac{\partial s_t}{\partial q}$  as :

$$\frac{\partial s_t}{\partial q} = \frac{s_t}{q} \frac{1}{\frac{\frac{u_{21}^2 - u_{22}u_{11}}{u_{11} - 2\frac{P_{1t}}{P_{2t}}u_{12} + \left(\frac{P_{1t}}{P_{2t}}\right)^2u_{22}}}{\frac{P_{2t+1}^2 q}{R_{t+1}^2 P_{2t}^2} \frac{v_{21}^2 - v_{22}v_{11}}{v_{11} - 2\frac{P_{1t+1}}{P_{2t+1}}v_{12} + \left(\frac{P_{1t+1}}{P_{2t+1}}\right)^2v_{22}}} + 1}} < \frac{s_t}{q} \quad (3.48)$$

This proves that  $\frac{\partial(\frac{s_t}{q})}{\partial q}$  is negative.

### 3.6 APPENDIX B

(i) Longevity is allocation neutral if and only the following equalities are true for all  $(q, n)$  and any price vector :

$$\left(\frac{\partial c_{1t}}{\partial q} + \frac{q}{1+n} \frac{\partial d_{1t-1}}{\partial q}\right)(c_{2t} + \frac{q}{1+n} d_{2t-1}) = \left(\frac{\partial c_{2t}}{\partial q} + \frac{q}{1+n} \frac{\partial d_{2t-1}}{\partial q}\right)(c_{1t} + \frac{q}{1+n} d_{1t-1}) \quad (3.49)$$

$$\Leftrightarrow (1+n)^2(c_{2t} \frac{\partial c_{1t}}{\partial q} - c_{1t} \frac{\partial c_{2t}}{\partial q}) + q(1+n)(c_{2t} \frac{\partial d_{1t-1}}{\partial q} + d_{2t-1} \frac{\partial c_{1t}}{\partial q} - d_{1t-1} \frac{\partial c_{2t}}{\partial q} - c_{1t} \frac{\partial d_{2t-1}}{\partial q}) + q^2(d_{2t-1} \frac{\partial d_{1t-1}}{\partial q} - d_{1t-1} \frac{\partial d_{2t-1}}{\partial q}) = 0 \quad (3.50)$$

$$\Leftrightarrow \begin{cases} c_{2t} \frac{\partial c_{1t}}{\partial q} = c_{1t} \frac{\partial c_{2t}}{\partial q} \\ d_{2t-1} \frac{\partial d_{1t-1}}{\partial q} = d_{1t-1} \frac{\partial d_{2t-1}}{\partial q} \\ c_{2t} \frac{\partial d_{1t-1}}{\partial q} + d_{2t-1} \frac{\partial c_{1t}}{\partial q} = d_{1t-1} \frac{\partial c_{2t}}{\partial q} + c_{1t} \frac{\partial d_{2t-1}}{\partial q} \end{cases} \quad (3.51)$$

The two first equalities imply that intratemporal preferences are homothetic. Thus, there exists  $\mu_1$  and  $\mu_2$  which only depend on good prices such that :  $c_{1t} = \mu_1 c_{2t}$  and  $d_{1t-1} = \mu_2 d_{2t-1}$ . Thus the third equality can be written as :

$$c_{2t} \frac{\partial \mu_2 d_{2t-1}}{\partial q} + d_{2t-1} \frac{\partial \mu_1 c_{2t}}{\partial q} = \mu_2 d_{2t-1} \frac{\partial c_{2t}}{\partial q} + \mu_1 c_{2t} \frac{\partial d_{2t-1}}{\partial q} \quad (3.52)$$

$$\Leftrightarrow c_{2t} \frac{\partial d_{2t-1}}{\partial q} (\mu_2 - \mu_1) = d_{2t-1} \frac{\partial c_{2t}}{\partial q} (\mu_2 - \mu_1) \quad (3.53)$$

$$\Leftrightarrow \mu_2 = \mu_1 \text{ or } c_{2t} \frac{\partial d_{2t-1}}{\partial q} = d_{2t-1} \frac{\partial c_{2t}}{\partial q} \text{ and } \mu_2 \neq \mu_1$$

When  $\mu_2 = \mu_1$  the preferences of the young and the old are identical. Consider now the case  $c_{2t} \frac{\partial d_{2t-1}}{\partial q} = d_{2t-1} \frac{\partial c_{2t}}{\partial q}$  and  $\mu_2 \neq \mu_1$ .

As intratemporal preferences are homothetic, consumption levels are linear functions of expenditure levels and the slope is independent of  $q$ . Thus,  $c_{2t} \frac{\partial d_{2t-1}}{\partial q} = d_{2t-1} \frac{\partial c_{2t}}{\partial q}$  is equivalent to  $\frac{\partial(\frac{e_{1t}}{e_{2t}})}{\partial q} = 0$ . I now show that if this condition holds for all  $q$  and any price vector, then the intertemporal elasticity of substitution (IES) is constant and equal to 1. Because intratemporal preference are homothetic, I can write utility functions  $u$  and  $v$  as  $u(.,.) = G(g(.,.))$  and  $v(.) = H(h(.,.))$  where  $g$  and  $h$  are homogeneous functions of degree 1 and  $G$  and  $H$  are increasing. Thus, for a cohort- $t$  individual, the lifetime utility function can be written as follows :

$$U_t = G(g(c_{1t}, c_{2t})) + qH(h(d_{1t}, d_{2t})) \quad (3.54)$$

$$U_t = G(e_{1t}z_{1t}) + qH(e_{2t}z_{2t}) \quad (3.55)$$

Where  $z_{it}$  only depends on the good prices. The Euler equation yields :

$$\frac{G'(e_{1t}z_{1t})}{H'(e_{2t}z_{2t})} = \frac{R_{t+1}z_{2t}}{z_{1t}} \quad (3.56)$$

Which can be written as  $e_{2t} = \frac{1}{z_{2t}} (H')^{-1}(\frac{z_{1t}}{R_{t+1}z_{2t}} G'(e_{1t}z_{1t}))$

Using the lifetime budget constraint, I get that  $e_{1t}$  solves :

$$e_{1t} + \frac{q}{R_{t+1}} \frac{1}{z_{2t}} (H')^{-1}(\frac{z_{1t}}{R_{t+1}z_{2t}} G'(e_{1t}z_{1t})) = w_t \quad (3.57)$$

I implicitly differentiate (3.57) to obtain :

$$\frac{\partial e_{1t}}{\partial q} + \frac{e_{2t}}{R_{t+1}} + \frac{qz_{1t}^2 G''(e_{1t}z_{1t})}{R_{t+1}^2 z_{2t}^2} ((H')^{-1})'(H'(e_{2t}z_{2t})) \frac{\partial e_{1t}}{\partial q} = 0 \quad (3.58)$$

Note that  $((H')^{-1})'(x) = \frac{1}{H''((H')^{-1}(x))}$

Thus,

$$\frac{\partial e_{1t}}{\partial q} = \frac{(-\frac{e_{2t}}{R_{t+1}})}{1 + \frac{qz_{1t}^2}{R_{t+1}^2 z_{2t}^2} \frac{G''(e_{1t}z_{1t})}{H''(H'(e_{2t}z_{2t}))}} \quad (3.59)$$

I proceed similarly to obtain :

$$\frac{\partial e_{2t-1}}{\partial q} = \frac{(-\frac{e_{2t-1}}{R_t})}{\frac{q}{R_t} + \frac{R_t z_{1t-1}^2}{z_{2t-1}^2} \frac{H''(e_{2t-1}z_{2t-1})}{G''(G'(e_{1t-1}z_{1t-1}))}} \quad (3.60)$$

Using (3.59) and (3.60), the condition  $\frac{\partial(\frac{e_{1t}}{e_{2t}})}{\partial q} = 0$  can be written as :

$$e_{1t} \frac{\partial e_{2t-1}}{\partial q} = e_{2t-1} \frac{\partial e_{1t}}{\partial q} \quad (3.61)$$

$$\Leftrightarrow \frac{\frac{e_{1t}}{R_t}}{\frac{q}{R_t} + \frac{R_t z_{1t-1}^2}{z_{2t-1}^2} \frac{H''(e_{2t-1} z_{2t-1})}{G''(G'(e_{1t-1} z_{1t-1}))}} = \frac{\frac{e_{2t}}{R_{t+1}}}{1 + \frac{q z_{1t}^2}{R_{t+1}^2 z_{2t}^2} \frac{G''(e_{1t} z_{1t})}{H''(H'(e_{2t} z_{2t}))}} \quad (3.62)$$

$$\Leftrightarrow \frac{R_{t+1} e_{1t}}{e_{2t}} \left[ 1 + \frac{q z_{1t}^2}{R_{t+1}^2 z_{2t}^2} \frac{G''(e_{1t} z_{1t})}{H''(H'(e_{2t} z_{2t}))} \right] = R_t \left[ \frac{q}{R_t} + \frac{R_t z_{1t-1}^2}{z_{2t-1}^2} \frac{H''(e_{2t-1} z_{2t-1})}{G''(G'(e_{1t-1} z_{1t-1}))} \right] \quad (3.63)$$

$$\frac{R_{t+1} e_{1t}}{e_{2t}} \left[ 1 + \frac{q z_{1t}^2}{R_{t+1}^2 z_{2t}^2} \frac{G''(e_{1t} z_{1t})}{H''(H'(e_{2t} z_{2t}))} \right] = q + \frac{R_t z_{1t-1}^2}{z_{2t-1}^2} \frac{H''(e_{2t-1} z_{2t-1})}{G''(G'(e_{1t-1} z_{1t-1}))} \quad (3.64)$$

Note that the RHS of (3.64) depends on  $R_t$  while the LHS does not. So for (3.64) to hold for all  $q$  and any price level, it must be that  $R_t^2 \frac{H''(e_{2t-1} z_{2t-1})}{G''(G'(e_{1t-1} z_{1t-1}))}$  does not depend on  $R_t$  for any price vector. Particularly for the price vector faced by cohort- $t$ , thus  $\frac{G''(e_{1t} z_{1t})}{R_{t+1}^2 H''(H'(e_{2t} z_{2t}))}$  does not depend on  $R_{t+1}$ . As the RHS of (3.64) does not depend on  $R_{t+1}$ , the LHS must not depend on  $R_{t+1}$  for (3.64) to be satisfied. This implies that  $\frac{R_{t+1} e_{1t}}{e_{2t}}$  does not depend on  $R_{t+1}$ . Rewrite the lifetime budget constraint of a cohort- $t$  individual as :

$$1 + q \frac{e_{2t}}{R_{t+1} e_{1t}} = \frac{w_t}{e_{1t}} \quad (3.65)$$

As  $\frac{e_{2t}}{R_{t+1} e_{1t}}$  does not depend on  $R_{t+1}$ ,  $e_{1t}$  does not depend on  $R_{t+1}$ . Thus, I can write  $e_{2t}$  as  $e_{2t} = D R_{t+1}$  where  $D$  does not depend on  $R_{t+1}$ . Use now the Euler equation :

$$G'(e_{1t} z_{1t}) = \frac{R_{t+1} z_{2t}}{z_{1t}} H'(D R_{t+1} z_{2t}) \quad (3.66)$$

As  $e_{1t}$  does not depend on  $R_{t+1}$ , I can integrate (3.66) with respect to  $R_{t+1}$  to get that the function  $H(\cdot)$  has the following form :

$$H(x) = A \ln(x) + B \quad (3.67)$$

Then, the Euler equation implies that  $G(\cdot)$  has the same form. Thus, the IES is constant and equal to 1. Reciprocally, if the IES is constant and equal to 1, then  $\frac{\partial(\frac{e_{1t}}{e_{2t}})}{\partial q} = 0$ . This proves (i).

(ii) Sufficiency is proved in the text. From the text, longevity is population neutral if and only if for all  $q$  and any price vector  $\frac{d_{1t-1}}{c_{1t}} = \frac{d_{2t-1}}{c_{2t}}$ , which can be written as  $\frac{c_{2t}}{c_{1t}} = \frac{d_{2t-1}}{d_{1t-1}}$ . Generally  $\frac{c_{2t}}{c_{1t}}$  depends on  $R_{t+1}$  while  $\frac{d_{2t-1}}{d_{1t-1}}$  depends on  $R_t$ . Thus for the equality to be true for any price vector, it must be that the consumption shares do not depend on the expenditure levels, hence intratemporal preferences must be homothetic. They must also be identical for

the consumption shares to equal. This proves (ii).

(iii) Sufficiency is proved in the text. To prove necessity, note that :

$$\frac{\partial(\frac{c_{1t}}{c_{2t}})}{\partial q} = 0 \quad (3.68)$$

$$\Rightarrow \left( \frac{\partial c_{1t}}{\partial q} + \frac{q}{1+n} \frac{\partial d_{1t-1}}{\partial q} + \frac{d_{1t-1}}{1+n} \right) (c_{2t} + \frac{q}{1+n} d_{2t-1}) = \left( \frac{\partial c_{2t}}{\partial q} + \frac{q}{1+n} \frac{\partial d_{2t-1}}{\partial q} + \frac{d_{2t-1}}{1+n} \right) (c_{1t} + \frac{q}{1+n} d_{1t-1}) \quad (3.69)$$

$$\begin{aligned} \Rightarrow & (1+n)^2 \left( c_{2t} \frac{\partial c_{1t}}{\partial q} - c_{1t} \frac{\partial c_{2t}}{\partial q} \right) + q^2 \left( d_{2t-1} \frac{\partial d_{1t-1}}{\partial q} - d_{1t-1} \frac{\partial d_{2t-1}}{\partial q} \right) \\ & + (1+n) \left( c_{2t} d_{1t-1} + q \left( c_{2t} \frac{\partial d_{1t-1}}{\partial q} + d_{2t-1} \frac{\partial c_{1t}}{\partial q} \right) - c_{1t} d_{2t-1} - q \left( d_{1t-1} \frac{\partial c_{2t}}{\partial q} + c_{1t} \frac{\partial d_{2t-1}}{\partial q} \right) \right) = 0 \end{aligned} \quad (3.70)$$

$$\Rightarrow \begin{cases} c_{2t} \frac{\partial c_{1t}}{\partial q} = c_{1t} \frac{\partial c_{2t}}{\partial q} \\ d_{2t-1} \frac{\partial d_{1t-1}}{\partial q} = d_{1t-1} \frac{\partial d_{2t-1}}{\partial q} \\ q \left( c_{2t} \frac{\partial d_{1t-1}}{\partial q} + d_{2t-1} \frac{\partial c_{1t}}{\partial q} \right) + c_{2t} d_{1t-1} = q \left( d_{1t-1} \frac{\partial c_{2t}}{\partial q} + c_{1t} \frac{\partial d_{2t-1}}{\partial q} \right) + c_{1t} d_{2t-1} \end{cases} \quad (3.71)$$

The two first equalities imply that intratemporal preferences are homothetic. Thus, there exists  $\mu_1$  and  $\mu_2$  which only depend on good prices such that :  $c_{1t} = \mu_1 c_{2t}$  and  $d_{1t-1} = \mu_2 d_{2t-1}$ . Thus, the third equality can be written as :

$$\mu_2 c_{2t} d_{2t-1} + q \left( c_{2t} \frac{\partial \mu_2 d_{2t-1}}{\partial q} + d_{2t-1} \frac{\partial \mu_1 c_{2t}}{\partial q} \right) = q \left( \mu_2 d_{2t-1} \frac{\partial c_{2t}}{\partial q} + \mu_1 c_{2t} \frac{\partial d_{2t-1}}{\partial q} \right) + \mu_1 c_{2t} d_{2t-1} \quad (3.72)$$

$$\Rightarrow (\mu_2 - \mu_1) c_{2t} d_{2t-1} + c_{2t} \frac{\partial d_{2t-1}}{\partial q} (\mu_2 - \mu_1) = d_{2t-1} \frac{\partial c_{2t}}{\partial q} (\mu_2 - \mu_1) \quad (3.73)$$

$$\Rightarrow \mu_2 = \mu_1 \text{ or } c_{2t} d_{2t-1} + q c_{2t} \frac{\partial d_{2t-1}}{\partial q} = q d_{2t-1} \frac{\partial c_{2t}}{\partial q} \text{ and } \mu_2 \neq \mu_1$$

When  $\mu_2 = \mu_1$  the preferences of the young and the old are identical. Consider now the case  $c_{2t} d_{2t-1} + q c_{2t} \frac{\partial d_{2t-1}}{\partial q} = q d_{2t-1} \frac{\partial c_{2t}}{\partial q}$  and  $\mu_2 \neq \mu_1$ . The equality can be written as :

$$\frac{1}{q} + \frac{\partial \ln(d_{2t-1})}{\partial q} = \frac{\partial \ln(c_{2t})}{\partial q} \quad (3.74)$$

$$\Rightarrow \frac{1}{q} = \frac{\partial \ln(\frac{c_{2t}}{d_{2t-1}})}{\partial q} = \frac{\partial \ln(\frac{e_{1t}}{e_{2t-1}})}{\partial q} \quad (3.75)$$

Where the last equality follows from the fact that the consumption ratio only depends on  $q$  through the ratio of the expenditure levels, as the intratemporal preferences are homothetic. I compute  $\frac{\partial \ln(\frac{e_{1t}}{e_{2t-1}})}{\partial q}$  from the proof of (i) :

$$\frac{\partial \ln(\frac{e_{1t}}{e_{2t-1}})}{\partial q} = \frac{1}{e_{1t}} \frac{\partial e_{1t}}{\partial q} - \frac{1}{e_{2t-1}} \frac{\partial e_{2t-1}}{\partial q} \quad (3.76)$$

And with (3.59) and (3.60) :

$$\frac{\partial \ln(\frac{e_{1t}}{e_{2t-1}})}{\partial q} = \frac{(-\frac{e_{2t}}{e_{1t}R_{t+1}})}{1 + \frac{qz_{1t}^2}{R_{t+1}^2 z_{2t}^2} \frac{G''(e_{1t}z_{1t})}{H''(H'(e_{2t}z_{2t}))}} + \frac{1}{q + \frac{R_t^2 z_{1t-1}^2}{z_{2t-1}^2} \frac{H''(e_{2t-1}z_{2t-1})}{G''(G'(e_{1t-1}z_{1t-1}))}} \quad (3.77)$$

Thus, (3.77) writes :

$$\frac{1}{q} = \frac{(-\frac{e_{2t}}{e_{1t}R_{t+1}})}{1 + \frac{qz_{1t}^2}{R_{t+1}^2 z_{2t}^2} \frac{G''(e_{1t}z_{1t})}{H''(H'(e_{2t}z_{2t}))}} + \frac{1}{q + \frac{R_t^2 z_{1t-1}^2}{z_{2t-1}^2} \frac{H''(e_{2t-1}z_{2t-1})}{G''(G'(e_{1t-1}z_{1t-1}))}} \quad (3.78)$$

I now use the same argument as in the proof of (i) to show that  $R_t^2 \frac{H''(e_{2t-1}z_{2t-1})}{G''(G'(e_{1t-1}z_{1t-1}))}$  does not depend on  $R_t$ . Thus  $\frac{1}{R_{t+1}^2} \frac{G''(e_{1t}z_{1t})}{H''(H'(e_{2t}z_{2t}))}$  does not depend on  $R_{t+1}$ , which implies from (3.78) that  $\frac{e_{2t}}{e_{1t}R_{t+1}}$  does not depend on  $R_{t+1}$ . From the proof of (i), it implies that the IES is equal to 1. This implies that  $\frac{\partial \ln(\frac{e_{1t}}{e_{2t-1}})}{\partial q} = 0$ , hence (3.75) is not satisfied. Therefore, the only possibility for  $\frac{\partial(\frac{C_{1t}}{C_{2t}})}{\partial q}$  to equal 0 for all  $(q, n)$  and any price vector is that preferences are homothetic and identical. This proves (iii).

### 3.7 APPENDIX C

The fact that  $\frac{S_t}{C_t}$  is non-constant with respect to  $n$  is obvious from (3.9). Assume that preferences exist such that  $\frac{S_t}{C_t}$  does not depend on  $q$  for any price vector. This implies that for every  $q, n$  and any price vector :

$$\frac{d}{dq} \left( \frac{S_t}{c_{1t} + c_{2t} + \frac{q}{1+n}(d_{1t-1} + d_{2t-1})} \right) = 0 \quad (3.79)$$

$$\Leftrightarrow \frac{\frac{dS_t}{dq}}{S_t} = \frac{(1+n)(\frac{dc_{1t}}{dq} + \frac{dc_{2t}}{dq}) + q(\frac{dd_{1t-1}}{dq} + \frac{dd_{2t-1}}{dq}) + d_{1t-1} + d_{2t-1}}{(1+n)(c_{1t} + c_{2t}) + q(d_{1t-1} + d_{2t-1})} \quad (3.80)$$

Note that the LHS of (3.80) does not depend on  $n$ . Thus, for (3.80) to hold, it must be that the RHS does not depend on  $n$ . This is true if for every  $q$  and any price vector :

$$\left( \frac{dc_{1t}}{dq} + \frac{dc_{2t}}{dq} \right) q (d_{1t-1} + d_{2t-1}) = (c_{1t} + c_{2t}) \left( q \left( \frac{dd_{1t-1}}{dq} + \frac{dd_{2t-1}}{dq} \right) + d_{1t-1} + d_{2t-1} \right) \quad (3.81)$$

Under the equality (3.81), the RHS of (3.80) rewrites :

$$\frac{ds_t}{dq} = \frac{\frac{dc_{1t}}{dq} + \frac{dc_{2t}}{dq}}{c_{1t} + c_{2t}} \quad (3.82)$$

From Lemma 3.1 and 3.2,  $\frac{ds_t}{dq} > 0$  and  $\frac{dc_{1t}}{dq} = P_{1t} \frac{dc_{1t}}{dq} + P_{2t} \frac{dc_{2t}}{dq} < 0$ . Therefore, (3.82) does not hold when  $P_{1t} = P_{2t}$ . This contradicts the fact that  $\frac{S_t}{C_t}$  does not depend on  $q$  for any price vector.

### 3.8 APPENDIX D

Fix  $(q, n)$  and the initial technology levels. Consider the allocation of the text defined by :

(a) the consumption levels obtained by the maximization of (3.15) subject to (3.2) and (3.3).

(b) the capital to labor ratio obtained by firms' profit maximization.

(c) the labor shares (3.26), (3.27) and (3.25) obtained by imposing the equilibrium on the capital good market, the good 1 market and the good 2 market.

(d) a constant interest rate equal to  $R$ .

According to Definition 3.7, this will be the allocation of a dynamic competitive equilibrium if it clears the labor market and if the two inequalities (3.20) and (3.21) are satisfied.

The labor market clearing condition writes :

$$l_{1t} + l_{2t} + l_{It} = 1 \quad (3.83)$$

$$\Leftrightarrow \frac{(1 - \alpha) \left( \frac{R^{\frac{\sigma-1}{\sigma}}}{q} x + \frac{R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}} \right)}{1 + \frac{R^{\frac{\sigma-1}{\sigma}}}{q} x} + \frac{1 - \alpha_I}{1 + \frac{R^{\frac{\sigma-1}{\sigma}}}{q} x} = 1 \quad (3.84)$$

$$\Leftrightarrow \alpha_I + \alpha \frac{R^{\frac{\sigma-1}{\sigma}}}{q} x = \frac{(1 - \alpha) R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}} \quad (3.85)$$

The LHS of (3.85) decreases with  $R$  from  $\infty$  to  $\alpha_I$ , while the RHS of (3.85) increases from 0 to  $\infty$ . Thus, there exists a unique admissible value for  $R$  (noted  $R(q, n)$ ) for the allocation defined previously to be a dynamic competitive equilibrium. This proves (iii) and the uniqueness of the CGP upon existence. (iv) and (v) are a direct application of the implicit function theorem. For the allocation defined to be a dynamic competitive equilibrium, there remains to check that (3.20) and (3.21) are satisfied. (3.20) writes :

$$\frac{\sigma \beta^2}{(1 - \sigma)^2} \leq (1 - \theta) \theta (X(n, q, t))^2 + \frac{\beta(1 - 2\theta - \gamma)}{1 - \sigma} X(n, q, t) \quad (3.86)$$

Where

$$X(n, q, t) = A_{2t}^{1-\sigma} A_{1t}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} \left( \frac{A_{1t}}{A_{2t}} \right)^{\gamma + \theta(1-\sigma)} y \left( \frac{\frac{xR(q, n)^{\frac{\sigma-1}{\sigma} - \frac{\alpha}{1-\alpha_I}}}{q}}{1 + \frac{xR(q, n)^{\frac{\sigma-1}{\sigma}}}{q}} \right)^{1-\sigma} \quad (3.87)$$

And  $y = (\alpha_I^{\frac{\alpha}{1-\alpha_I}} (1-\alpha) (\frac{\alpha(1-\alpha_I)}{(1-\alpha)\alpha_I})^\alpha)^{1-\sigma}$ .

If  $\frac{(1+g_2)^{\gamma+\theta(1-\sigma)}}{(1+g_1)^{\gamma+\theta(1-\sigma)}} < 1$ ,  $X(n, q, t)$  increases with time. Moreover,  $X \rightarrow (1-\theta)\theta X^2 + \frac{\beta(1-2\theta-\gamma)}{1-\sigma} X$  is always increasing when it takes positive values. Thus, if the inequality (3.86) is satisfied at time 0, it will be satisfied at all periods. Consider now  $q \in (0, 1)$ . I

have that  $\Omega_n = \min_{q \in [q, 1]} \left( \frac{\frac{xR(q, n)^{\frac{\sigma-1}{\sigma} - \frac{\alpha}{1-\alpha_I}}}{q}}{1 + \frac{xR(q, n)^{\frac{\sigma-1}{\sigma}}}{q}} \right) > 0$ . Then, choose  $A_{20}, A_{10}$  and  $A_{I0}$  such that  $A_{20}^{1-\sigma} A_{I0}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} (\frac{A_{10}}{A_{20}})^{\gamma+\theta(1-\sigma)}$  is high enough in order for the following inequality to be satisfied :

$$\frac{\sigma\beta^2}{(1-\sigma)^2} \leq (1-\theta)\theta(A_{20}^{1-\sigma} A_{I0}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} (\frac{A_{10}}{A_{20}})^{\gamma+\theta(1-\sigma)} y \Omega_n)^2 + \frac{\beta(1-2\theta-\gamma)}{1-\sigma} A_{20}^{1-\sigma} A_{I0}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} (\frac{A_{10}}{A_{20}})^{\gamma+\theta(1-\sigma)} y \Omega_n \quad (3.88)$$

Then, for all  $q$  greater than  $\underline{q}$ , (3.86) is satisfied. Proceed similarly for the inequality (3.21), which can be written as :

$$\frac{\sigma\beta^2}{(1-\sigma)^2} \leq (1-\theta)\theta(Y(n, q, t))^2 + \frac{\beta(1-2\theta-\gamma)}{1-\sigma} Y(n, q, t) \quad (3.89)$$

Where

$$Y(n, q, t) = A_{2t}^{1-\sigma} A_{It}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} (\frac{A_{1t}}{A_{2t}})^{\gamma+\theta(1-\sigma)} y \left( \frac{\frac{xR(q, n)^{1-\frac{\alpha}{1-\alpha_I}}}{q}}{1 + \frac{xR(q, n)^{\frac{\sigma-1}{\sigma}}}{q}} \right)^{1-\sigma} \quad (3.90)$$

$\Psi_n = \min_{q \in [q, 1]} \left( \frac{\frac{xR(q, n)^{1-\frac{\alpha}{1-\alpha_I}}}{q}}{1 + \frac{xR(q, n)^{\frac{\sigma-1}{\sigma}}}{q}} \right) > 0$ . Then, choose  $A_{20}, A_{10}$  and  $A_{I0}$  such that  $A_{20}^{1-\sigma} A_{I0}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} (\frac{A_{10}}{A_{20}})^{\gamma+\theta(1-\sigma)}$  is high enough in order for the following inequality to be satisfied :

$$\frac{\sigma\beta^2}{(1-\sigma)^2} \leq (1-\theta)\theta(A_{20}^{1-\sigma} A_{I0}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} (\frac{A_{10}}{A_{20}})^{\gamma+\theta(1-\sigma)} y \Psi_n)^2 + \frac{\beta(1-2\theta-\gamma)}{1-\sigma} A_{20}^{1-\sigma} A_{I0}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} (\frac{A_{10}}{A_{20}})^{\gamma+\theta(1-\sigma)} y \Psi_n \quad (3.91)$$

Among the two pairs of initial conditions, choose the one that yields the highest value for  $A_{20}^{1-\sigma} A_{I0}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} (\frac{A_{10}}{A_{20}})^{\gamma+\theta(1-\sigma)}$ . Then (i) holds.

To prove (ii), fix now  $q \in (0, 1]$ . Consider  $(\underline{n}, \bar{n}) \in (-1, \infty)^2$ .  $\Omega_q = \min_{n \in [\underline{n}, \bar{n}]} \left( \frac{\frac{xR(q, n)^{\frac{\sigma-1}{\sigma} - \frac{\alpha}{1-\alpha_I}}}{q}}{1 + \frac{xR(q, n)^{\frac{\sigma-1}{\sigma}}}{q}} \right)$

and  $\Psi_q = \min_{n \in [\underline{n}, \bar{n}]} \left( \frac{\frac{xR(q, n)^{1-\frac{\alpha}{1-\alpha_I}}}{q}}{1 + \frac{xR(q, n)^{\frac{\sigma-1}{\sigma}}}{q}} \right)$  are both positive. Then choose  $A_{20}, A_{10}$  and  $A_{I0}$  such that the two following inequalities are satisfied :

$$\frac{\sigma\beta^2}{(1-\sigma)^2} \leq (1-\theta)\theta(A_{20}^{1-\sigma}A_{I0}^{\frac{\alpha(1-\sigma)}{1-\alpha I}}(\frac{A_{10}}{A_{20}})^{\gamma+\theta(1-\sigma)}y\Omega_q)^2 + \frac{\beta(1-2\theta-\gamma)}{1-\sigma}A_{20}^{1-\sigma}A_{I0}^{\frac{\alpha(1-\sigma)}{1-\alpha I}}(\frac{A_{10}}{A_{20}})^{\gamma+\theta(1-\sigma)}y\Omega_q \quad (3.92)$$

$$\frac{\sigma\beta^2}{(1-\sigma)^2} \leq (1-\theta)\theta(A_{20}^{1-\sigma}A_{I0}^{\frac{\alpha(1-\sigma)}{1-\alpha I}}(\frac{A_{10}}{A_{20}})^{\gamma+\theta(1-\sigma)}y\Psi_q)^2 + \frac{\beta(1-2\theta-\gamma)}{1-\sigma}A_{20}^{1-\sigma}A_{I0}^{\frac{\alpha(1-\sigma)}{1-\alpha I}}(\frac{A_{10}}{A_{20}})^{\gamma+\theta(1-\sigma)}y\Psi_q \quad (3.93)$$

Then (ii) holds.

### 3.9 APPENDIX E

The asymptotic labor share in sector 1 and 2 is obtained by taking the limit of the expressions (3.26) and (3.27) when  $t$  tends to  $\infty$ . Then,

$$l_1^* = \frac{\theta(1-\alpha)(\frac{R^{\frac{\sigma-1}{\sigma}}}{q}x + \frac{R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha I}}})}{1 + \frac{R^{\frac{\sigma-1}{\sigma}}}{q}x} \quad (3.94)$$

$$l_2^* = \frac{(1-\theta)(1-\alpha)(\frac{R^{\frac{\sigma-1}{\sigma}}}{q}x + \frac{R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha I}}})}{1 + \frac{R^{\frac{\sigma-1}{\sigma}}}{q}x} = \frac{1-\theta}{\theta}l_1^* \quad (3.95)$$

Therefore, the ratio  $\frac{l_1^*}{l_2^*} = \frac{\theta}{1-\theta}$  does not depend on demographic variables. This proves (ii). The asymptotic labor share of the investment sector is given by (3.25). Then,

$$\frac{l_I^*}{l_1^*} = \theta \frac{1-\alpha}{1-\alpha I} \frac{1}{(\frac{R^{\frac{\sigma-1}{\sigma}}}{q}x + \frac{R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha I}}})} \quad (3.96)$$

Note first that if  $R$  is held constant in (3.96),  $\frac{l_I^*}{l_1^*}$  increases with  $q$ , which shows that the "allocation effect" dominates the "population effect". To study  $q \rightarrow \frac{l_I^*}{l_1^*}$ , fix  $n \in (-1, \infty)$  and eliminate  $q$  in the last expression from (3.28) :

$$\frac{l_I^*}{l_1^*} = \theta \frac{1-\alpha}{1-\alpha I} \frac{1}{\frac{R}{(1+n)\alpha(1+g_I)^{\frac{1}{1-\alpha I}}} - \frac{\alpha I}{\alpha}} \quad (3.97)$$

As  $R$  is a decreasing function of  $q$ , I get that  $q \rightarrow \frac{l_I^*}{l_1^*}$  is increasing.

To study  $n \rightarrow \frac{l_I^*}{l_1^*}$ , fix  $q \in (0, 1]$  and eliminate  $n$  in the expression (3.96) from (3.28) :

$$\frac{l_I^*}{l_1^*} = \theta \frac{1-\alpha}{1-\alpha I} \frac{1}{\frac{R^{\frac{\sigma-1}{\sigma}}}{q}x(1 + \frac{1}{1-\alpha}) + \frac{\alpha I}{1-\alpha}} \quad (3.98)$$

As  $R$  increase with  $n$  and  $\sigma < 1$ ,  $n \rightarrow \frac{l_I^*}{l_1^*}$  is increasing.

## 3.10 APPENDIX F

I first have to compute  $y(n, q)$ . The income per worker  $y_t$  is given by :

$$y_t = P_{1t}A_{1t}k_{1t}^\alpha l_{1t} + P_{2t}A_{2t}k_{2t}^\alpha l_{2t} + A_{It}k_{It}^{\alpha_I}l_{It} \quad (3.99)$$

Using that  $k_{1t} = k_{2t}$  and the formula of the relative price, I get :

$$y_t = P_{1t}A_{1t}k_{1t}^\alpha (l_{1t} + l_{2t}) + A_{It}k_{It}^{\alpha_I}l_{It} \quad (3.100)$$

And use now (3.10) and (3.11) to eliminate  $P_{1t}$ ,  $k_{1t}$  and  $k_{It}$ . Use also the expressions of the labor shares (3.25), (3.26) and (3.27) to get :

$$y_t = A_{It}^{\frac{1}{1-\alpha_I}} \left(\frac{\alpha_I}{R}\right)^{\frac{\alpha_I}{1-\alpha_I}} \frac{1 + \frac{\alpha_I}{1-\alpha} + \frac{1}{1-\alpha} \frac{R^{\frac{\sigma-1}{\sigma}}}{q} x}{1 + \frac{R^{\frac{\sigma-1}{\sigma}}}{q} x} \quad (3.101)$$

Then,  $y(n, q) = A_{I0}^{\frac{1}{1-\alpha_I}} \left(\frac{\alpha_I}{R}\right)^{\frac{\alpha_I}{1-\alpha_I}} \frac{1 + \frac{\alpha_I}{1-\alpha} + \frac{1}{1-\alpha} \frac{R^{\frac{\sigma-1}{\sigma}}}{q} x}{1 + \frac{R^{\frac{\sigma-1}{\sigma}}}{q} x}$ . Fix  $q \in (0, 1]$ . The dependence of  $y(n, q)$  on  $n$  is only due to the dependence of  $y(n, q)$  on  $R$ . The derivative of  $y(n, q)$  with respect to  $R$  has the sign of the following polynomial :

$$\begin{aligned} J\left(\frac{R^{\frac{\sigma-1}{\sigma}}}{q} x\right) &= -\left(\frac{R^{\frac{\sigma-1}{\sigma}}}{q} x\right)^2 \frac{\alpha_I}{(1-\alpha_I)} \\ &\quad + \frac{R^{\frac{\sigma-1}{\sigma}}}{q} x \left[ \left(\frac{1-\sigma}{\sigma}\right)(\alpha_I - \alpha) - \alpha_I(2 - \alpha + \alpha_I) \right] - \frac{\alpha_I(1 + \alpha_I - \alpha)}{(1-\alpha_I)} \end{aligned} \quad (3.102)$$

This suffices to prove that  $A_y(q) \neq \emptyset$ . Note first that if  $\alpha_I < \alpha$ , then  $J$  is always negative. This proves the point (I) of Proposition 3.12.

The determinant of  $J$  is positive if  $\frac{1-\sigma}{\sigma}$  is *not* in the following interval :

$$\left[ \frac{(2 - \alpha + \alpha_I)(1 - \alpha_I) - 2\sqrt{1 - \alpha + \alpha_I}}{\alpha_I - \alpha} \frac{\alpha_I}{1 - \alpha_I}, \frac{(2 - \alpha + \alpha_I)(1 - \alpha_I) + 2\sqrt{1 - \alpha + \alpha_I}}{\alpha_I - \alpha} \frac{\alpha_I}{1 - \alpha_I} \right] \quad (3.103)$$

And for the two roots to be positive,  $\frac{1-\sigma}{\sigma}$  must be greater than  $\frac{(2-\alpha+\alpha_I)(1-\alpha_I)+2\sqrt{1-\alpha+\alpha_I}}{\alpha_I-\alpha} \frac{\alpha_I}{1-\alpha_I}$ . This proves (II).

Fix now  $n \in (-1, \infty)$ . Use (3.28) to rewrite  $y(n, q)$  as :

$$y(n, q) = A_{I0}^{\frac{1}{1-\alpha_I}} \left(\frac{\alpha_I}{R}\right)^{\frac{\alpha_I}{1-\alpha_I}} \frac{\alpha - \alpha_I + \frac{R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}}}{\alpha - \alpha_I + (1-\alpha) \frac{R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}}} \quad (3.104)$$

$y(n, q)$  depends on  $q$  only through  $R$ . The sign of the derivative of  $y(n, q)$  with respect to  $R$  is the one of the following polynomial :

$$K\left(\frac{R}{n(1+g_I)^{\frac{1}{1-\alpha_I}}}\right) = -\left(\frac{R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}}\right)^2 \alpha_I(1-\alpha) + \frac{R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}} (\alpha - \alpha_I)(\alpha - 2\alpha_I) - \alpha_I(\alpha - \alpha_I)^2 \quad (3.105)$$

This proves that  $B_y(n) \neq \emptyset$ . The determinant of  $K$  is positive if  $\alpha > 4\alpha_I(1 - \alpha_I)$ . And the two roots are positive if and only if  $(\alpha - \alpha_I)(\alpha - 2\alpha_I) > 0$ .

Assume now that  $\alpha > 4\alpha_I(1 - \alpha_I)$  and  $(\alpha - \alpha_I)(\alpha - 2\alpha_I) > 0$ .  $K$  has two positive roots  $\beta_1$  and  $\beta_2$  with  $\beta_1 < \beta_2$ . To draw the curve  $q \rightarrow y(n, q)$ , we need to determine the position of  $R(1, n)$  with respect to  $\beta_1(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}$  and  $\beta_2(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}$ .

$$R(1, n) < \beta_2(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}$$

$$\iff 2\alpha\alpha_I x R(1, n)^{\frac{\sigma-1}{\sigma}} < \alpha^2 - 3\alpha_I\alpha + \sqrt{\Delta_K} \quad (3.106)$$

Where  $\Delta_K$  is the discriminant of  $K$ . The LHS of (3.106) decreases with  $n$  on  $(-1, \infty)$  from  $\infty$  to 0. Hence (3.106) is possibly true if and only if the RHS of (3.106) is positive. This is true if  $\alpha - 3\alpha_I > 0$ . Otherwise, this is equivalent to :

$$\sqrt{\Delta_K} > \alpha(3\alpha_I - \alpha) \iff \Delta_K > \alpha^2(3\alpha_I - \alpha)^2 \quad (3.107)$$

$$\iff \alpha^2 - 2\alpha\alpha_I - \alpha_I(1 - \alpha_I) > 0 \quad (3.108)$$

$$\iff \alpha > \alpha_I + \sqrt{\alpha_I} \quad (3.109)$$

I need to check if this condition is compatible with  $\alpha > 4\alpha_I(1 - \alpha_I)$  and  $(\alpha - \alpha_I)(\alpha - 2\alpha_I) > 0$  and  $\alpha < 3\alpha_I$ . Necessarily for  $(\alpha - \alpha_I)(\alpha - 2\alpha_I) > 0$  to be true, it must be that  $\alpha > 2\alpha_I$ . For  $3\alpha_I$  to be greater than  $\alpha_I + \sqrt{\alpha_I}$ , it must be that  $\alpha_I > \frac{1}{4}$ . Then  $3\alpha_I > 4\alpha_I(1 - \alpha_I)$ . Hence I need to choose  $\alpha_I > \frac{1}{4}$  and  $\alpha \in [\max(2\alpha_I, 4\alpha_I(1 - \alpha_I), \alpha_I + \sqrt{\alpha_I}), 3\alpha_I]$ . Note that for  $\alpha_I > \frac{1}{4}$ ,  $\max(2\alpha_I, 4\alpha_I(1 - \alpha_I), \alpha_I + \sqrt{\alpha_I}) = \max(2\alpha_I, 4\alpha_I(1 - \alpha_I))$ . Then, under these conditions, (3.106) is true for  $n$  large enough. This proves the second set of conditions of (II).

I now determine the position of  $R(1, n)$  with respect to  $\beta_1(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}$  :

$$R(1, n) < \beta_1(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}$$

$$\iff 2\alpha\alpha_I x R(1, n)^{\frac{\sigma-1}{\sigma}} < \alpha^2 - 3\alpha_I\alpha - \sqrt{\Delta_K} \quad (3.110)$$

As previously, I only need to know the sign of the RHS of (3.110) :

$$\alpha^2 - 3\alpha_I\alpha - \sqrt{\Delta_K} > 0$$

$$\iff \begin{cases} -\alpha^2 + 2\alpha\alpha_I + \alpha_I(1 - \alpha_I) > 0 \\ \alpha > 3\alpha_I \end{cases}$$

Which is equivalent to  $\alpha > 3\alpha_I$  and  $\alpha < \alpha_I + \sqrt{\alpha_I}$ . These conditions must be compatible with  $\alpha > 4\alpha_I(1 - \alpha_I)$  and  $(\alpha - \alpha_I)(\alpha - 2\alpha_I) > 0$ . These four conditions are equivalent to  $\alpha > 3\alpha_I$  and  $\alpha < \alpha_I + \sqrt{\alpha_I}$  and  $\alpha > 4\alpha_I(1 - \alpha_I)$ . The first two are only possible together if  $\alpha_I < \frac{1}{4}$ . Then the conditions can be written as  $\alpha_I < \frac{1}{4}$  and  $\alpha \in [\max(3\alpha_I, 4\alpha_I(1 - \alpha_I)), \alpha_I + \sqrt{\alpha_I}]$ . Note finally that for  $\alpha_I < \frac{1}{4}$ ,  $\max(3\alpha_I, 4\alpha_I(1 - \alpha_I)) = 4\alpha_I(1 - \alpha_I)$ . Then, under these conditions, I can choose  $n$  such that (3.110) is satisfied.

This completes the proof of Proposition 3.13.

### 3.11 APPENDIX G

Fix  $q \in (0, 1]$ . A decrease of  $n$  increases (respectively decreases)  $\frac{l_{1t}}{l_{2t}}$  through the "population effect" if  $\frac{d_{1t-1}}{d_{2t-1}} > \frac{c_{1t}}{c_{2t}}$  (respectively  $\frac{d_{1t-1}}{d_{2t-1}} < \frac{c_{1t}}{c_{2t}}$ ). Because good 1 has smaller expenditure-elasticity than good 2 and because old and young have the same intratemporal preferences :

$$\frac{d_{1t-1}}{d_{2t-1}} > \frac{c_{1t}}{c_{2t}} \iff e_{2t-1} < e_{1t} \quad (3.111)$$

From (3.18) and (3.19), this is equivalent to  $R < (1 + g_I)^{\frac{1}{\sigma(1-\alpha_I)}} x$ .  $n \rightarrow R(q, n)^{\frac{1}{\sigma}}$  is increasing from 0 to  $\infty$  on  $(-1, \infty)$ . Therefore, there exists a threshold  $\mu(q)$  such that if  $n < \mu(q)$  (respectively  $n > \mu(q)$ ), then  $\frac{d_{1t-1}}{d_{2t-1}} > \frac{c_{1t}}{c_{2t}}$  (resp  $\frac{d_{1t-1}}{d_{2t-1}} < \frac{c_{1t}}{c_{2t}}$ ). This proves that Lemma 3.14 holds.

### 3.12 APPENDIX H

Fix  $q \in (0, 1]$ . From (3.33), the variation of  $\frac{C_{1t}}{C_{2t}}$  with respect to  $n$  are the ones of  $\frac{(\frac{R^{\sigma-1}}{q^{\sigma-1}} x^{\sigma} + \frac{q}{1+n} (\frac{R}{1-\alpha_I})^{\sigma}) R^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} (1 + \frac{R^{\sigma-1}}{q} x)}{q(1+g_I)^{\frac{1-\alpha_I}{\sigma}}}$  =  $H(n)$ . Eliminate the direct dependence with respect to  $n$  in the expression of  $H$  by using (3.28) :

$$H(n) = G(R(q, n)) = \frac{(\frac{x^{\sigma}}{q^{\sigma}} + \frac{\alpha_I q^{1-\sigma} (1+g_I)^{\frac{1-\sigma}{1-\alpha_I}}}{1-\alpha} + \frac{\alpha(1+g_I)^{\frac{1-\sigma}{1-\alpha_I}} R^{\frac{\sigma-1}{\sigma}}}{(1-\alpha)q^{\sigma}}) R^{(1-\sigma)(\frac{\alpha}{1-\alpha_I}-1)} (1 + \frac{R^{\frac{\sigma-1}{\sigma}}}{q} x)^{1-\sigma}}{(\frac{1}{1-\alpha} \frac{R^{\frac{\sigma-1}{\sigma}}}{q} x + \frac{\alpha_I}{1-\alpha})} \quad (3.112)$$

Rewrite  $G(R)$  as follows :

$$G(R) = \frac{(A + BR^{\frac{\sigma-1}{\sigma}})(R^{\frac{\alpha}{1-\alpha_I}-1} + \frac{R^{\frac{\alpha}{1-\alpha_I}-\frac{1}{\sigma}}}{q} x)^{1-\sigma}}{(CR^{\frac{\sigma-1}{\sigma}} + D)} \quad (3.113)$$

The sign of the first derivative of  $G$  is the one of the following function :

$$\begin{aligned}
P(R^{\frac{\sigma-1}{\sigma}}) &= BC \frac{x}{q} \left( \sigma \frac{\alpha}{1-\alpha_I} - 1 \right) R^{3\frac{\sigma-1}{\sigma}} \\
&\quad + R^{2\frac{\sigma-1}{\sigma}} \left[ BC \sigma \left( \frac{\alpha}{1-\alpha_I} - 1 \right) + BD \frac{x}{q} \left( \sigma \frac{\alpha}{1-\alpha_I} - 2 \right) + \frac{x}{q} AC \sigma \frac{\alpha}{1-\alpha_I} \right. \\
&+ R^{\frac{\sigma-1}{\sigma}} \left[ AD \frac{x}{q} \left( \sigma \frac{\alpha}{1-\alpha_I} - 1 \right) + AC \left( \sigma \frac{\alpha}{1-\alpha_I} + 1 - \sigma \right) + BD \left( \sigma \frac{\alpha}{1-\alpha_I} - \sigma - 1 \right) \right] + AD \sigma \left( \frac{\alpha}{1-\alpha_I} - 1 \right)
\end{aligned} \tag{3.114}$$

Rewrite the polynomial  $P(X)$  as  $P(X) = a_3X^3 + a_2X^2 + a_1X + a_0$ .

Note that the first derivative of  $G$  cancels at most three times. Thus, you can apply the procedure outlined in Definition 3.9 and the set  $A_{\frac{1}{2}}(q)$  is not the empty set.

(i) If  $\frac{2}{\sigma} \leq \frac{\alpha}{1-\alpha_I}$ , then  $a_3, a_2, a_1, a_0$  are all positive. Then,  $n \rightarrow G(R(q, n))$  is increasing.

(ii) If  $\frac{1}{\sigma} \leq \frac{\alpha}{1-\alpha_I} \leq \frac{2}{\sigma}$ , then  $a_3$  and  $a_0$  are positive. Thus  $P$  is initially positive, ends positive and has at most two roots on  $[0, \infty)$ . This proves (ii).

(iii) If  $1 \leq \frac{\alpha}{1-\alpha_I} \leq \frac{1}{\sigma}$ , then  $a_3$  is negative, while  $a_0$  is positive. Thus,  $P$  is initially positive, ends negative and has 1 or 3 roots on  $[0, \infty)$ . This proves (iii).

(iv) If  $\frac{\alpha}{1-\alpha_I} \leq 1$ , then  $a_3$  and  $a_0$  are negative. Thus  $P$  is initially negative, ends negative and has at most two roots on  $[0, \infty)$ . This proves (iv).

### 3.13 APPENDIX I

(I) Set  $a = \frac{q}{1+n}$ , write (3.33) as :

$$\begin{aligned}
\frac{l_{1t}}{l_{2t}} &= \frac{d \frac{\left( \frac{R^{\sigma-1}}{q^\sigma} x^\sigma + a \left( \frac{R}{1} \right)^\sigma \right) R^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} \left( 1 + \frac{R^{\sigma-1}}{q} x \right)^{1-\sigma}}{q(1+g_I)^{\frac{1-\alpha_I}{1-\alpha_I}} (1-\alpha) \left( \frac{R^{\sigma-1}}{q} x + a \frac{R}{1} \right)^{\frac{1-\alpha_I}{1-\alpha_I}}} \frac{A_{2t}^\sigma}{A_{1t} A_{It}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}}} \left( \frac{A_{2t}}{A_{1t}} \right)^{\gamma-1+\theta(1-\sigma)}}{1 - d \frac{\left( \frac{R^{\sigma-1}}{q^\sigma} x^\sigma + a \left( \frac{R}{1} \right)^\sigma \right) R^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} \left( 1 + \frac{R^{\sigma-1}}{q} x \right)^{1-\sigma}}{qg_I^{\frac{1-\alpha_I}{1-\alpha_I}} (1-\alpha) \left( \frac{R^{\sigma-1}}{q} x + a \frac{R}{1} \right)^{\frac{1-\alpha_I}{1-\alpha_I}}} \frac{A_{2t}^\sigma}{A_{1t} A_{It}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}}} \left( \frac{A_{2t}}{A_{1t}} \right)^{\gamma-1+\theta(1-\sigma)}} \tag{3.115}
\end{aligned}$$

$$\begin{aligned}
\frac{l_{1t}}{l_{2t}} &= \frac{d \frac{\left( R^{\sigma-1} x^\sigma + a \left( \frac{R}{1} \right)^\sigma \right) R^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} \left( q + R^{\frac{\sigma-1}{\sigma}} x \right)^{1-\sigma}}{(1+g_I)^{\frac{1-\alpha_I}{1-\alpha_I}} (1-\alpha) \left( R^{\frac{\sigma-1}{\sigma}} x + a \frac{R}{1} \right)^{\frac{1-\alpha_I}{1-\alpha_I}}} \frac{A_{2t}^\sigma}{A_{1t} A_{It}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}}} \left( \frac{A_{2t}}{A_{1t}} \right)^{\gamma-1+\theta(1-\sigma)}}{1 - d \frac{\left( R^{\sigma-1} x^\sigma + a \left( \frac{R}{1} \right)^\sigma \right) R^{\frac{\alpha(1-\sigma)}{1-\alpha_I}} \left( q + R^{\frac{\sigma-1}{\sigma}} x \right)^{1-\sigma}}{(1+g_I)^{\frac{1-\alpha_I}{1-\alpha_I}} (1-\alpha) \left( R^{\frac{\sigma-1}{\sigma}} x + a \frac{R}{1} \right)^{\frac{1-\alpha_I}{1-\alpha_I}}} \frac{A_{2t}^\sigma}{A_{1t} A_{It}^{\frac{\alpha(1-\sigma)}{1-\alpha_I}}} \left( \frac{A_{2t}}{A_{1t}} \right)^{\gamma-1+\theta(1-\sigma)}} \tag{3.116}
\end{aligned}$$

To get the direction of the allocation effect, differentiate the previous expression with respect to  $q$ , while  $a$  and  $R$  are maintained fixed. Thus, an increase of  $q$  increases  $\frac{l_{1t}}{l_{2t}}$  through the "allocation effect".

(II) Fix  $n \in (-1, \infty)$ . From the proof of Lemma 3.14, a longevity increase  $\frac{I_t}{I_2}$  through the "population effect" if  $R < (1 + g_I)^{\frac{1}{\sigma(1-\alpha_I)}} x$ .  $q \rightarrow R(q, n)$  is decreasing from  $\infty$  to  $R(1, n)$  on  $(0, 1]$ . Thus, if  $R(1, n) < (1 + g_I)^{\frac{1}{\sigma(1-\alpha_I)}} x$  (respectively  $R(1, n) > (1 + g_I)^{\frac{1}{\sigma(1-\alpha_I)}} x$ ), then the curve is inverted-U shape (resp. decreasing). As  $n \rightarrow R(1, n)$  increases from 0 to  $\infty$  on  $(-1, \infty)$ , I can define  $\kappa$  the fertility level such that  $R(1, n) = (1 + g_I)^{\frac{1}{\sigma(1-\alpha_I)}} x$ . This proves (II).

### 3.14 APPENDIX J

A tedious calculation tells us that the extrema of  $f$  must solve the following equation :

$$\begin{aligned} & \frac{1-\alpha}{\alpha} \frac{R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}} + \left(\frac{\sigma-1}{\sigma} + 1\right) \frac{R^{\frac{\sigma-1}{\sigma}+1} x^{1-\sigma}}{(1+n)(1+g_I)^{\frac{\sigma}{1-\alpha_I}}} \\ & \frac{1-\alpha}{\alpha} \frac{R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}} - \frac{\alpha_I}{\alpha} + \frac{R^{\frac{\sigma-1}{\sigma}+1} x^{1-\sigma}}{(1+n)(1+g_I)^{\frac{\sigma}{1-\alpha_I}}} \\ & = \frac{\frac{1-\alpha}{\alpha} \frac{(1-\sigma)R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}}}{\frac{1-\alpha}{\alpha} \frac{R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}} - \frac{\alpha_I}{\alpha}} - \frac{\alpha(1-\sigma)}{1-\alpha_I} \\ & - \frac{\frac{1-\alpha}{\alpha} \frac{R(1-\sigma)}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}}}{1 - \frac{\alpha_I}{\alpha} + \frac{1-\alpha}{\alpha} \frac{R}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}}} + \frac{\frac{R}{\alpha(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}}}{\frac{R}{\alpha(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}} - \frac{\alpha_I}{\alpha}} \quad (3.117) \end{aligned}$$

As  $R$  goes to  $\infty$ , the LHS tends to 1, while the RHS tends to  $1 - \frac{\alpha(1-\sigma)}{1-\alpha_I}$ . This proves that the extrema of  $f$  are upper-bounded and that  $B_{\frac{I_t}{I_2}}(n)$  is not the empty set. Moreover, as  $R \rightarrow f(R)$  ends increasing,  $q \rightarrow \frac{I_t}{I_2}$  is initially decreasing. This proves (I). To prove (II), evaluate the derivative of  $f$  at  $R(1, n)$ . Use (3.28) to eliminate the terms  $R(1, n)^{\frac{\sigma-1}{\sigma}}$  in the expression. I get that  $f'(R(1, n)) > 0$  if and only if :

$$\begin{aligned} & \frac{R(1, n)}{1+n} \left[ \frac{1-\alpha}{\alpha(1+g_I)^{\frac{1}{1-\alpha_I}}} + \left(\frac{\sigma-1}{\sigma} + 1\right) \frac{x^{-\sigma}}{(1+g_I)^{\frac{\sigma}{1-\alpha_I}}} \left( \frac{(1-\alpha)R(1, n)}{\alpha(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}} - \frac{\alpha_I}{\alpha} \right) \right] \\ & > \frac{\left(1 + \frac{R(1, n)x^{-\sigma}}{(1+n)(1+g_I)^{\frac{\sigma}{1-\alpha_I}}}\right) Q\left(\frac{R(1, n)}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}}\right)}{\left(1 - \frac{\alpha_I}{\alpha} + \frac{1-\alpha}{\alpha} \frac{R(1, n)}{(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}}\right) \left(\frac{R(1, n)}{\alpha(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}} - \frac{\alpha_I}{\alpha}\right)} \quad (3.118) \end{aligned}$$

Where

$$Q(X) = \frac{1-\alpha}{\alpha} (1-\sigma) X \left(\frac{X}{\alpha} - \frac{\alpha_I}{\alpha}\right) - \left[\frac{\alpha(1-\sigma)}{1-\alpha_I} \left(\frac{X}{\alpha} - \frac{\alpha_I}{\alpha}\right) - \frac{X}{\alpha}\right] \left(1 - \frac{\alpha_I}{\alpha} + \frac{1-\alpha}{\alpha} X\right) \left(\frac{1-\alpha}{\alpha} X - \frac{\alpha_I}{\alpha}\right) \quad (3.119)$$

(3.118) can be written as  $H\left(\frac{R(1, n)}{1+n}\right) > 0$ , where  $H$  is a fourth degree polynomial, which is independent on  $n$ . The coefficient of the monomial of degree 4 is  $\frac{x^{-\sigma}}{(1+g_I)^{\frac{\sigma}{1-\alpha_I}}} \left(\frac{\alpha(1-\alpha_I)}{1-\sigma} + \frac{\sigma-1}{\sigma}\right)$ .

When this coefficient is positive (resp. negative), there exists  $\varrho_1 > \frac{(1+g_I)^{\frac{1}{1-\alpha_I}} \alpha_I}{(1-\alpha)}$  (resp.  $\varrho_2 > \frac{(1+g_I)^{\frac{1}{1-\alpha_I}} \alpha_I}{(1-\alpha)}$ ), such that  $H$  is positive (resp. negative) above  $\varrho_1$  (resp.  $\varrho_2$ ). Define  $\chi_1$  as the threshold for  $n$  such that for  $n \leq \chi_1$ ,  $\frac{R(1,n)}{n} \geq \max(\varrho_1, \varrho_2)$ .  $\chi_1$  exists because  $n \rightarrow \frac{R(1,n)}{n}$  is decreasing from  $\infty$  to  $\frac{(1+g_I)^{\frac{1}{1-\alpha_I}} \alpha_I}{(1-\alpha)}$  according to (3.28). This proves (II)(i) and (II)(ii). To prove (II)(iii), note first that when  $\sigma > \frac{1}{2}$ , the LHS of (3.117) is positive for  $R > \frac{\alpha_I}{1-\alpha}(1+n)(1+g_I)^{\frac{1}{1-\alpha_I}}$ . Moreover, when  $1 - \frac{\alpha(1-\sigma)}{1-\alpha_I} < 0$ , the RHS of (3.117) has a negative limit in  $\infty$ . As the RHS of (3.117) is a function, independent of  $n$ , of  $\frac{R}{1+n}$ , there exists  $\phi > \frac{(1+g_I)^{\frac{1}{1-\alpha_I}} \alpha_I}{(1-\alpha)}$  such that the RHS of (3.117) is negative for all  $R > (1+n)\phi$ . This means that (3.117) has no solution greater than  $\phi(1+n)$ . Note now that if  $n$  is smaller than a threshold  $\chi_2$ ,  $R(1,n)$  is greater than  $\phi(1+n)$ , which proves that  $q \rightarrow \frac{l_t}{l_t}$  has no extrema on  $[\underline{q}(n), 1]$ . Take  $\chi = \min(\chi_1, \chi_2)$  to complete the proof.

# CONCLUSION

In this dissertation, I studied the link between longevity and economic growth from OLG models. In the first chapter, I used the framework of Chakraborty (2004), in which longevity depends on public health expenditures, to determine the impact of the health policy on economic growth and welfare. In the second chapter, I proposed an endogenous growth model in which agents can spend resources to live longer in retirement period to analyze the joint dynamics of health expenditures and income and to assess the growth impacts of health expenditures. In the third chapter, longevity is an exogenous variable and I assessed its impact on the income level and the labor allocation of a multi-sector economy. My results of the first two chapters highlight that health expenditures exert both positive and negative effects on economic growth when it is driven by capital accumulation. On the one hand, health expenditures increase longevity which increases the propensity to save (Bloom et al. (2003), Chakraborty (2004)). On the other hand, they decrease the disposable income, which decreases savings. This explains why in the first chapter I found that the violation of Inada conditions by the survival function implies that health expenditures reduce the growth rate in low-income countries. I also characterized the steady-state income level as a function of the tax rate that finances health expenditures : it is decreasing or inverted U-shaped. Hence health expenditures can be detrimental to economic development. Finally, I showed that a positive impact of health expenditures on income is neither sufficient nor necessary for health expenditures to improve welfare. In the second chapter, when individuals choose the level of their health expenditures, I noticed the possibility for the negative effects of health expenditures to always overcome their positive effects. This prevents the economy from perpetually growing. More importantly, the condition for this to happen is not always contradictory with that that insures perpetual growth in the same economy except that health expenditures are not possible. Put in other words, health expenditures create a strong negative intergenerational externality in this case by annihilating any possibility of growth for future generations. I also found that health expenditures can increase the growth rate of the economy, hence they can create a positive intergenerational externality. A calibration of the model to OECD countries suggests that this case is the most likely one. In the third chapter, as I considered exogenous longevity improvements, this only creates a greater propensity to save. While in a one-sector Diamond model this is sufficient to state that a longevity increase increases income per worker, I underlined here that it is no more sufficient in a multi-sector economy. The reason is that the longevity improvement can redirect labor towards sectors with a smaller output per worker. I also decomposed and characterized the impact of a longevity increase on the labor allocation. A calibration of the model on the US economy suggests that these

effects are significant.

To keep the models analytically tractable, I used simple frameworks. They could be enriched to obtain more quantitative answers to our various questions. Particularly, Chapter 3 could be pursued by introducing social security and endogenous retirement. This would give a better picture of the economic consequences of the aging process. The analysis of the first two chapters could also be pursued econometrically. As previously reported, there is no empirical consensus on the causal impact of life expectancy on economic growth. My results suggests that the impact of a life expectancy improvement on economic growth depends on whether it is costly or not. Disentangling the costly longevity improvements from the exogenous ones could reconcile some empirical findings on the health-growth nexus. Theoretically, drawing on Chapter 2 and Chapter 3 to build a growth model in which there exists a health sector could be promising. Given the specificity of health in terms of demand and production, this could give further insights on the links between health expenditures and economic growth.

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## Résumé

Cette thèse s'intéresse à la relation de long-terme entre la longévité et le développement économique. Dans le premier chapitre, j'analyse l'impact des dépenses de santé sur la croissance économique et le bien-être. Pour cela, j'étudie l'influence du taux d'imposition dans une économie avec temps de vie endogène (Chakraborty (2004)). Je détermine le taux d'imposition qui maximise le taux de croissance. Je montre notamment que celui-ci est nul dans les économies à faible revenu. Puis je m'intéresse aux variations du niveau de production par travailleur dans l'état stationnaire par rapport au taux d'imposition. Je montre que cette fonction est décroissante ou en U inversé. Enfin, j'étudie le taux d'imposition qui maximise le bien-être dans l'état stationnaire. Dans le second chapitre, je propose un modèle pour étudier l'impact sur la croissance de dépenses de santé choisies par les agents. En effet, je développe un modèle à générations imbriquées avec croissance endogène dans lequel les individus peuvent dépenser des ressources pour vivre plus longtemps dans leur période de retraite. Je donne une caractérisation complète de l'équilibre général dynamique puis je détermine l'impact sur la croissance économique des dépenses de santé. Celles-ci peuvent empêcher la croissance ou lui être nécessaires. Une calibration du modèle aux pays de l'OCDE suggère que cette dernière possibilité est la plus probable. Enfin, le troisième chapitre étudie l'impact théorique du vieillissement sur l'allocation sectorielle des travailleurs. Pour cela, je développe un modèle multisectoriel dans lequel j'examine les conséquences sur le revenu par travailleur et l'allocation sectorielle des travailleurs d'un choc de longévité et de fertilité. Je montre que contrairement aux modèles unisectoriels, le revenu par travailleur n'est pas forcément monotone par rapport aux variables démographiques. Des chocs démographiques réalistes produisent des mouvements non-négligeables de travailleurs.

## Mots Clés

croissance économique, longévité, générations imbriquées, changement structurel, vieillissement, temps de vie endogène, santé et développement

## Abstract

This dissertation is interested in the long-run relationship between longevity and economic development. In the first chapter, I analyse the impact of health expenditures on economic growth and welfare. For this, I study the influence of the tax rate in an economy à la Chakraborty (2004). I first determine the growth-maximizing tax rate, which is shown to be 0 in low-income countries. Second, I show that the steady-state income level is an inverted U-shaped function or a decreasing function of the tax rate. Third, I study the tax rate that maximizes the steady-state welfare level. In the second chapter, I propose a theoretical model to study the growth impacts of health expenditures chosen by the agents. Indeed, I develop a Diamond model with endogenous growth in which young individuals can spend resources to increase their longevity in retirement period. I give a full characterization of the dynamic general equilibrium and determine the growth impacts of health expenditures. They can speed up or slow down economic growth. They can be a barrier or a necessity for growth to take place. A calibration to OECD countries suggests that the latter case is the most likely one. Finally, the third chapter studies the theoretical impact of the aging process on the sectorial labor allocation. To this aim, I develop a multi-sector two-period overlapping generations model in which I examine the consequences of both a longevity shift and a fertility shift on the labor allocation of the economy and on the income per worker level. I show that contrary to one-sector models, the income per worker level is not necessarily monotonic with respect to demographic variables. Realistic demographic shocks are also shown to create significant labor reallocation across sectors.

## Keywords

economic growth, longevity, overlapping generations, structural change, aging, endogenous lifetime, health and development