Équations paraboliques non linéaires pour des problèmes d'hydrogéologie et de transition de phase

Jana Alkhayal

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Jana ALKHAYAL

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Les rapporteurs :

Monsieur	Philippe	**Laurençot**	Université de Toulouse	Rapporteur
Monsieur	Hiroshi	**Matano**	University of Tokyo	Rapporteur

Les membres du jury :

Monsieur	Antonin	**Chambolle**	Ecole polytechnique	Président
Madame	Danielle	**Hilhorst**	Université Paris-Sud	Directrice de thèse
Monsieur	Mustapha	**Jazar**	Université Libanaise	Directeur de thèse
Monsieur	Philippe	**Laurençot**	Université de Toulouse	Rapporteur
Monsieur	Régis	**Monneau**	Université Paris-Est	Examinateur
Monsieur	Yoshihisa	**Morita**	Université de Ryukoku	Invité
Monsieur	Filippo	**Santambrogio**	Université Paris-Sud	Examinateur

Jana ALKHAYAL
jana.khayal@hotmail.com

Thèse préparée au Département de Mathématiques d'Orsay
Laboratoire de mathématiques (UMR 8628)
Université Paris-Sud 11
91405 Orsay Cedex

À mes parents Amal et Tarek
À mes soeurs Maya et Nour

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Equations paraboliques non linéaires pour des problèmes d'hydrogéologie et de transition de phase

Résumé

L'objet de cette thèse est d'étudier l'existence de solution pour une classe de systèmes d'évolution fortement couplés, ainsi que la limite singulière d'une équation aux dérivées partielles d'advection-réaction-diffusion.
Au chapitre 1, nous décrivons brièvement la dérivation d'un modèle d'intrusion saline pour des aquifères confinés et non confinés. Dans ce but nous nous appuyons sur la loi de Darcy et la loi de conservation de masse en négligeant l'effet de la dimension verticale.
Au chapitre 2, nous considérons un système qui généralise le modèle d'intrusion saline dans des aquifères non confinés. C'est un système non linéaire parabolique dégénéré fortement couplé. Après avoir discrétisé en temps, gelé et tronqué des coefficients et finalement régularisé les équations, nous appliquons le théorème de Lax-Milgram pour prouver l'existence et l'unicité de la solution d'un problème linéaire associé. Nous appliquons ensuite un théorème du point fixe pour démontrer l'existence d'une solution du problème non linéaire approché. Nous obtenons du plus une estimation d'entropie, qui permet en particulier de démontrer la positivité de la solution. Finalement, nous passons à la limite dans le système et dans l'entropie pour prouver l'existence de solution pour le problème initial.
Au chapitre 3, nous montrons l'existence de solution pour un système qui contient en particulier le modèle d'intrusion saline dans des aquifères confinés. Ce système est semblable au système du chapitre 2, mais la pression intervient comme inconnue supplémentaire. Il se rajoute la contrainte que la somme des hauteurs inconnues est une fonction donnée et la pression est en fait un multiplicateur de Lagrange associé à cette contrainte. Nous obtenons de nouveau une inégalité d'entropie et effectuons également une estimation sur le gradient de la pression.
Au chapitre 4, nous nous intéressons à la description d'interfaces abruptes qui se déplacent selon un mouvement donné, par exemple le mouvement par courbure moyenne. Des singularités peuvent apparaître en temps fini ce qui explique la nécessité de définir une nouvelle notion de surface. Dans ce chapitre, on introduit les notions de "varifolds", ou surfaces généralisées, qui étendent les notions de "manifolds". A ces varifolds on associe une courbure moyenne généralisée ainsi qu'une vitesse normale généralisée.
Au chapitre 5, nous considérons une équation d'advection-réaction-diffusion qui intervient dans un système de chimiotaxie-croissance proposé par Mimura et Tsujikawa. L'inconnue est la densité de population qui est soumise aux effets de diffusion et de croissance et qui a tendance à migrer vers des forts gradients de la substance chimiotactique. Quand un petit paramètre tend vers zéro, la solution converge vers une fonction étagée ; l'interface diffuse associée converge vers une interface abrupte qui se déplace selon un mouvement par courbure moyenne perturbé. Nous représentons ces interfaces par des varifolds définis à partir de la fonctionnelle de Lyapunov du problème d'Allen-Cahn. Nous établissons une formule de monotonie et nous montrons une propriété d'équipartition de l'énergie. Nous prouvons de plus que le varifold est rectifiable et que la fonction de multiplicité associée est presque partout entière.

Mots clés. systèmes paraboliques dégénérés fortement couplés, entropie, équations de réaction-diffusion, perturbations singulières, varifolds, intrusion saline, chimiotaxie.

AMS subject classifications. 35K57, 35K65, 35B10, 35D30, 35R35, 49Q20, 82B26, 37N10.

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Nonlinear parabolic equations for hydrogeology and phase transition problems

Abstract

The aim of this thesis is to study the existence of a solution for a class of evolution systems which are strongly coupled, as well as the singular limit of an advection-reaction-diffusion equation.

In chapter 1, we describe briefly the derivation of a seawater intrusion model in confined and unconfined aquifers. For this purpose we combine Darcy's law with a mass conservation law and we neglect the effect of the vertical dimension.

In chapter 2, we consider a system that generalizes the seawater intrusion model in unconfined aquifer. It is a strongly coupled nonlinear degenerate parabolic system. After discretizing in time, freezing and truncating the coefficients and finally regularizing the equations we apply Lax Milgram theorem to prove the existence of a unique solution for the elliptic linear associated system. Then we apply a fixed point theorem to prove the existence of a solution for the nonlinear approximated problem. We obtain in addition an entropy estimate, which allows us in particular to prove the positivity of the solution. Finally, we pass to the limit in the system and the entropy in order to prove the existence of a solution for the initial problem.

In chapter 3, we prove the existence of a solution for a system that contains in particular the seawater intrusion model in confined aquifers. This system is very similar to that introduced in chapter 2, only the pressure is a new unknown and we have the constraint that the sum of the unknown heights is a given function. The pressure is the Lagrange multiplier associated to the constraint. We obtain again an entropy estimate and we establish an estimate on the gradient of the pressure.

In chapter 4, we are intrested in the study of sharp interfaces that moves by a certain flow, by mean curvature flow for example. Singularities may occur in finite time which explains the necessity of having a differnet notion of surfaces. In this chapter, we introduce the notion of "varifolds" or generalized surfaces that extend the notion of manifolds. To these varifolds we associate a generalized mean curvature and a generalized normal velocity.

In chapter 5, we consider an advection-reaction-diffusion equation arising from a chemotaxis-growth system proposed by Mimura and Tsujikawa. The unknown is the population density which is subjected to the effects of diffusion, of growth and to the tendency of migrating toward higher gradients of the chemotactic substance. When a small parameter tends to zero, the solution converges to a step function ; the associated diffuse interface converges to a sharp interface which moves by perturbed mean curvature. We represent these interfaces by varifolds defined by the Lyapunov functional of the Allen-Cahn problem. We establish a monotonicity formula and we prove a property of equipartition of energy. We prove also the rectifiability of the varifold and that the multiplicity function is almost everywhere integer.

Key words. parbolic degenerate strongly coupled system, entropy, reaction-diffusion equation, singular limit, varifolds, seawater intrusion, chemiotaxis.

AMS subject classifications. 35K57, 35K65, 35B10, 35D30, 35R35, 49Q20, 82B26, 37N10.

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Introduction

Cette thèse est consacrée à l'étude des systèmes d'équations aux dérivées partielles non linéaires qui interviennent dans des domaines variés dont en particulier l'hydrogéologie et les transitions de phase. On s'intéresse à des problèmes à frontières libres, ou problèmes d'interfaces, où l'interface s'exprime explicitement ou comme la limite d'une zone de transition de phase.

Nous considérons tout d'abord des problèmes liés à l'intrusion saline dans des aquifères côtiers. Il s'agit de la pénétration de l'eau de mer salée dans les aquifères d'eau douce. Ce comportement est dû au fait que l'eau de mer a une densité et une pression plus élevées que l'eau douce. Certaines activités humaines intensifient l'effet de l'intrusion saline comme le pompage excessif ou la surexploitation de l'eau douce. Par conséquence, la dépression créée par les volumes extraits peut provoquer une avancée du biseau salé dans les aquifères et l'eau salée peut même atteindre les puits de pompage. C'est dans le but de maitriser ce phénomène que l'on établit des modèles mathématiques qui sont étudiés à la fois analytiquement et numériquement.
En combinant la loi de Darcy et la loi de conservation de masse, on peut décrire l'évolution des interfaces entre l'eau salée et l'eau douce et entre l'eau douce et le sol sous la forme de systèmes non linéaires fortement couplés. Les systèmes considérés dans cette thèse sont assez généraux puisqu'ils contiennent à la fois les problèmes d'intrusion saline et d'autres applications telles que l'électrochimie et la dynamique des populations. On démontre l'existence d'une solution pour les systèmes considérés, en appliquant une méthode de point fixe à un système régularisé et en passant ensuite à la limite en s'appuyant en particulier sur une estimation d'entropie. L'unicité des solutions est encore ouverte.

Nous étudions également la limite singulière d'une équation liée à la chimiotaxie. On peut observer que la plupart des espèces biologiques se déplacent par marche aléatoire ou par un mouvement dirigé. Un exemple de mouvement dirigé est donné par la chimiotaxie, où les individus biologiques se concentrent vers les zones de plus fort gradient d'une substance chimique. Ici, la densité de substance chimique q_ε est considérée comme connue. Plus précisément, nous étudions la limite singulière de la solution u^ε de l'équation

$$\partial_t u^\varepsilon = \Delta u^\varepsilon - \nabla \cdot (u^\varepsilon \nabla \chi(q_\varepsilon)) + \frac{1}{\varepsilon^2} f(u^\varepsilon, \varepsilon\alpha),$$

avec des conditions aux limites périodiques et une condition initiale. Le terme de croissance f est donné par $f(r,\alpha) = r(1-r)(r-\frac{1}{2}+\alpha)$. Dans le cas où l'on néglige le terme d'advection et où $\alpha = 0$, cette équation coincide avec l'équation d'Allen-Cahn qui a été très étudiée dans les dernières décennies :

$$\partial_t u^\varepsilon = \Delta u^\varepsilon - \frac{1}{\varepsilon^2} u^\varepsilon (u^\varepsilon - 1)(u^\varepsilon - \frac{1}{2}).$$

Quand ε tend vers zéro, la solution u^ε converge vers une fonction étagée u qui prend les valeurs 0 et 1. Des interfaces diffuses convergent vers des interfaces abruptes entre les régions $\{u = 0\}$ et $\{u = 1\}$. De nombreuses études ont montré que ces interfaces évoluent selon le mouvement par courbure moyenne.
Nous étudions ici la limite singulière au sens des "*Surfaces généralisées*" ou "*Varifolds*", c'est-à-dire des surfaces où des singularités peuvent se produire, pour le problème de chimiotaxie.

Première partie : Problème d'intrusion saline

Cette partie porte sur le problème d'intrusion saline. Après avoir dérivé formellement les modèles, on montre l'existence de solution pour deux systèmes qui généralisent des modèles d'aquifères non confinés et confinés.

Chapitre 1 : Dérivation formelle d'un modèle d'intrusion saline

Ce Chapitre est basé sur un article [43] dû à Jazar et Monneau. On considère un écoulement d'eau douce dans un milieu poreux saturé [9] dans le but de décrire un modèle d'intrusion saline. On suppose constantes les densités de l'eau douce et de l'eau salée et on applique la loi de Darcy dans l'eau douce et dans l'eau salée ; cette loi exprime le fait que la vitesse est proportionnelle à la pression. On suppose aussi que le rapport entre l'épaisseur et la longueur horizontale du milieu poreux tends vers zéro. En conséquence, on obtient des modèles réduits de type Boussinesq. L'étude est faite dans les deux cas d'aquifères confinés et d'aquifères non confinés.

Chapitre 2 : Existence de solution pour un système dégénéré fortement couplé avec application à l'intrusion saline

Ce chapitre fait l'objet d'un article écrit en collaboration avec S. Issa, M. Jazar et R. Monneau, soumis pour publication dans "ESAIM : Control, Optimisation and Calculus of Variations".
Le but de ce chapitre est de montrer l'existence d'une solution pour un système non linéaire parabolique dégńéré et fortement couplé, qui contient en particulier le modèle d'intrusion saline dans des aquifères non confinés ; il est de la forme

$$\partial_t u^i = \operatorname{div}\left(u^i \sum_{j=1}^{m} A_{ij} \nabla u^j\right) \qquad \text{dans } \Omega_T, \quad \text{pour } i = 1, \dots, m. \tag{1}$$

avec la condition initiale

$$u^i(0, x) = u_0^i(x) \geq 0 \qquad \text{p.p. dans } \Omega, \quad \text{pour } i = 1, \dots, m. \tag{2}$$

Le domaine considéré est le tore $\Omega := \mathbb{T}^N = (\mathbb{R}/\mathbb{Z})^N$ avec $N \geq 1$, $\Omega_T := (0, T) \times \Omega$, $T > 0$ et $m \geq 1$ est un entier.
La matrice $A = (A_{ij})_{1 \leq i,j \leq m}$ satisfait une condition de positivité : on suppose qu'il existe deux matrices $m \times m$ diagonales définies positives L et R et une constante $\delta_0 > 0$, telles que

$$\zeta^T LAR\, \zeta \geq \delta_0 |\zeta|^2, \quad \text{pour tout} \quad \zeta \in \mathbb{R}^m. \tag{3}$$

En général, ce type de système est difficile à étudier car en particulier le principe de maximum ne s'applique pas. De nombreux travaux ont porté sur l'existence de solutions pour des systèmes non linéaires fortement couplés, citons parmi eux [75, 74, 53, 47, 18, 17, 22]. Notre preuve d'existence est basée sur une estimation d'entropie qui nous permet de contrôler le gradient de la solution et sa positivité. On montre que la solution u, au sens de distributions, satisfait une estimation d'entropie de la forme

$$\sum_{i=1}^{m}\int_{\Omega}\Psi(u^i(t_2))+\delta_0\sum_{i=1}^{m}\int_{t_1}^{t_2}\int_{\Omega}|\nabla u^i|^2\leq\sum_{i=1}^{m}\int_{\Omega}\Psi(u_0^i), \tag{4}$$

pour presque tout $t_1<t_2$ avec $u^i(t_2)=u^i(t_2,\cdot)$ et

$$\Psi(a)-\frac{1}{e}=\begin{cases} a\ln a & \text{pour} \quad a>0,\\ 0 & \text{pour} \quad a=0,\\ +\infty & \text{pour} \quad a<0,\end{cases} \tag{5}$$

où

$$\sum_{i=1}^{m}\int_{\Omega}\Psi(u_0^i)<+\infty. \tag{6}$$

En fait nous ne disposons pas de suffisamment de régularité, ce qui nous amène à des approximations qui nous permettent d'obtenir un système linéaire discrétisé en temps dont on vise à trouver des solutions points fixes. Finalement, nous démontrons l'existence de solution pour le système initial en passant à la limite pour tous les paramètres ajoutés.

Chapitre 3 : Existence de solution pour un système dégénéré fortement couplé avec contrainte découlant d'un problème d'intrusion saline

Ce chapitre fait l'objet d'un article écrit en collaboration avec M. Jazar et R. Monneau, soumis pour publication dans " Mathematical Models and Methods in Applied Sciences". Nous considérons un système non linéaire parabolique dégénéré et fortement couplé avec contrainte ; le modèle d'intrusion saline dans des aquifères confinés en est un cas particulier. Il est de la forme

$$\begin{cases} \partial_t u^i=\operatorname{div}\left(u^i\nabla\left(p+\sum\limits_{j=1}^{m}A_{ij}u^j\right)\right) & \text{dans } \Omega_T, \quad \text{pour } i=1,\dots,m,\\ \sum\limits_{i=1}^{m}u^i(t,x)=f(x) & \text{dans } \Omega_T, \end{cases} \tag{7}$$

où f est une fonction donnée telle que : il existe $0<\ell_1<\ell_2<\infty$ satisfaisant

$$\ell_1\leq f(x)\leq\ell_2, \quad \text{pour tout } x\in\Omega \text{ et } f\in H^1(\Omega). \tag{8}$$

On impose des conditions aux limites periodiques et les conditions initiales

$$\begin{aligned} u^i(0,x) &= u_0^i(x)\geq 0 \qquad \text{p.p. dans } \Omega, \quad \text{pour } i=1,\dots,m,\\ p(0,x) &= p_0(x) \qquad\qquad \text{dans } \Omega. \end{aligned} \tag{9}$$

Dans le cas confiné (7), p est la pression définie sur Ω_T, qui apparait comme un multiplicateur de Lagrange de la contrainte sur $u = (u^i)_{1\leq i\leq m}$, donnée dans la deuxième ligne de (7).
Mentionnons qu'un différent type de modèle d'intrusion saline dans des aquifères confinés a été étudié dans [61, 71, 23].
Notre but est de montrer l'existence, au sens de distributions, d'une solution $((u^i)_{1\leq i\leq m}, p)$ telle que $u = (u^i)_{1\leq i\leq m}$ est positive presque partout ; de plus u satisfait une estimation d'entropie, pour presque tout $t_1, t_2 \in (0,T)$ avec $u^i(t_2) = u^i(t_2,\cdot)$, de la forme

$$\sum_{i=1}^{m}\int_\Omega \Psi(u^i(t_2)) + \frac{\delta_0}{2}\sum_{i=1}^{m}\left\|\nabla u^i\right\|^2_{L^2(t_1,t_2;L^2(\Omega))} \leq \sum_{i=1}^{m}\int_\Omega \Psi(u_0^i) + C_0\left\|\nabla f\right\|^2_{L^2(\Omega)}, \tag{10}$$

où Ψ est donnée par (5) et $C_0 = \dfrac{m\,\ell_2^2\left\|A\right\|^2}{2\,\ell_1^2\delta_0}$; on montre de plus que p satisfait

$$\left\|\nabla p\right\|^2_{L^2(\Omega)} \leq \frac{2\ell_2^2\left\|A\right\|^2 m}{\ell_1^2}\left\|\nabla u\right\|^2_{(L^2(\Omega))^m}. \tag{11}$$

Pour obtenir ce résultat on suit la stratégie proposé dans le Chapitre 2.

Deuxième Partie : Limites singulières et chimiotaxie

Cette partie de la thèse correspond à des travaux réalisés en collaboration avec D. Hilhorst.

Chapitre 4 : Les Varifolds : propriétés et définitions

Dans ce chapitre, on présente des notions de la théorie de la mesure géométrique [69], qui permet de définir une "surface généralisée" ou "varifold" par une classe de mesures de Radon ; ces mesures de Radon possèdent des bonnes propriétés de compacité que nous exploitons dans l'étude des limites singulières d'équations de réaction-diffusion. Cette théorie associe à ce varifold des outils géométriques comme la "courbure moyenne généralisée".

Chapitre 5 : Limite singulière au sens des varifolds d'un modèle de chimiotaxie-croissance

Dans ce chapitre on étudie la limite singulière d'un modèle d'advection-réaction-diffusion qui décrit l'évolution de densités de bactéries. En 1996, Mimura et Tsujikawa [54] ont proposé un nouveau modèle de chimiotaxie-croissance. Ils supposent que le coefficient de diffusion et le coefficient du terme chimiotactique sont très faibles par rapport au terme de croissance. Ils écrivent le système sous la forme

$$(MT^\varepsilon)\quad \begin{cases} \partial_t u^\varepsilon &= \varepsilon^2\Delta u^\varepsilon - \varepsilon\nabla\cdot(u^\varepsilon\nabla\chi(q_\varepsilon)) + f(u^\varepsilon),\\ \tau\partial_t q_\varepsilon &= \Delta q_\varepsilon + u^\varepsilon - \gamma q_\varepsilon, \end{cases}$$

où $\varepsilon > 0$ et $\tau > 0$ sont des petits paramètres, $\gamma > 0$, u_ε représente une densité de population, q_ε est la densité d'une substance chimiotactique, $\nabla\chi(q_\varepsilon)$ est la vitesse du mouvement de u^ε induit par la chimiotaxie et f est une fonction cubique de zéros stables 0 et 1, telle que

$\int_0^1 f > 0$; quand $\varepsilon \to 0$ une interface abrupte Γ_t se crée entre les régions où la fonction limite u prend les valeurs 0 et 1.
Nous rappelons tout d'abord une dérivation formelle de la loi du mouvement de Γ_t présentée dans [10] ; il s'agit d'un mouvement par courbure moyenne perturbé.
Bonami, Hilhorst, Logak et Mimura [10, 11] et Alfaro [1] ont considéré une variante du système (MT^ε), dans le cas où $\tau = 0$ avec des conditions aux limites de Neumann homogènes ; l'équation pour la substance chimiotactique est alors elliptique. Ils démontrent en particulier l'existence et l'unicité, locales en temps, de la solution classique du problème à frontières libres limite et la convergence de la solution vers celle de ce problème. En l'absence de l'effet chimiotactique, le problème limite se réduit à l'équation de mouvement par courbure moyenne et la convergence vers une solution classique sur de petits intervalles de temps a été largement étudiée, on citera par exemple [16, 27, 28, 19].
En général des singularités peuvent se produire en temps fini ; c'est pourquoi il est nécessaire de considérer des solutions faibles ; la convergence vers une solution de viscosité a été établie dans des divers travaux où le terme chimiotactique n'intervient pas, notamment pour l'équation d'Allen-Cahn, dans [8, 34, 42] et dans le cas d'une équation de chimiotaxie-croissance dans [37]. On suppose ici que la densité de la substance chimiotactique q_ε est donnée. On considère l'équation de réaction-diffusion-advection

$$(P^\varepsilon) \quad \begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon - \nabla \cdot (u^\varepsilon \nabla \chi(q_\varepsilon)) + \dfrac{1}{\varepsilon^2} f(u^\varepsilon, \varepsilon\alpha) & \text{in } Q_T := \Omega \times (0,T), \\ u^\varepsilon(x,0) = u_0^\varepsilon(x) & \text{in } \Omega, \end{cases}$$

où $\Omega := \mathbb{T}^n$ est le tore et $f(r,\alpha) = r(1-r)(r-\frac{1}{2}+\alpha)$. On étudie le comportement d'une famille de solutions régulières $(u^\varepsilon)_{\varepsilon>0}$ de (P^ε) quand $\varepsilon \to 0$ et l'on montre que u^ε converge vers une fonction limite u, qui est une fonction étagée prenant les valeurs 0 et 1 presque partout sur le domaine. L'interface Γ_t, qui sépare les deux régions, suit la loi du mouvement

$$v = h + (\nabla \chi(q) \cdot \nu)\, \nu + \frac{1}{\theta}\sqrt{2}\alpha\nu, \tag{12}$$

où v est la vitesse normale de l'interface, h est le vecteur courbure moyenne de l'interface, θ est la fonction de multiplicité qui est égale à 1 dans le cas d'une interface régulière et ν est le vecteur normal à l'interface.
Notre but est de montrer la convergence vers l'équation de mouvement (12) sur un grand intervalle de temps, tout en gardant des informations tangentielles de l'interface et en évitant les "annulations d'interfaces". Pour cette raison, des notions de "surfaces généralisées" ont été introduites par Young dans [77] puis améliorées par Almgren [6] qui leur a donné le nom de "varifolds" voir aussi [69] et [4].
Différents travaux ont porté sur le mouvement par courbure moyenne au sens des varifolds, citons [13, 41, 55]. Les mouvements par courbure moyenne perturbés ont été également étudiés dans ce sens par Mugnai et Röger [60] en dimensions d'espace 2 et 3, et par Takasao et Tonegawa [70] dans le cas où le terme de perturbation est un terme de transport plus simple.
Nous étendons leurs résultats au problème (P^ε) en dimension d'espace arbitraire.
Dans ce but nous introduisons l'énergie

$$E^\varepsilon(u^\varepsilon) := \int_\Omega \left(\frac{\varepsilon\, |\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx,$$

où $W(r) := \frac{1}{4}r^2(1-r)^2$ est un potentiel double puits. Nous montrons que si cette fonctionnelle est bornée à l'instant initial, elle reste bornée uniformément en ε sur des intervalles de temps arbitraires $[0, T]$.
Nous définissons de plus la fonction de "discrepancy" pour tout $t \in [0, T]$

$$\xi_t^\varepsilon(u^\varepsilon) := \int_\Omega \left(\frac{\varepsilon \left|\nabla u^\varepsilon\right|^2}{2} - \frac{W(u^\varepsilon)}{\varepsilon} \right) dx.$$

Nous établissons une formule de monotonie similaire à celle démontrée par Ilmanen dans [41] et nous montrons que la fonction de "discrepancy" tend vers zéro quand $\varepsilon \to 0$, ce qui traduit une sorte d'équipartition de l'énergie. De plus nous montrons qu'une classe de mesures de Radon associées à l'énergie $E^\varepsilon(u^\varepsilon)$ sont rectifiables, de plus qu' elles possèdent une courbure moyenne généralisée et une vitesse généralisée qui satisfait (12) et que la fonction de multiplicité associée est presque partout entière. Notre travail s'appuie essentiellement sur des méthodes utilisées dans [41, 70, 60, 72].

Chapter 1

Formal derivation of seawater intrusion models

Abstract. Based on [43], we derive briefly two seawater intrusion models in confined and unconfined aquifer. The porous medium is considered saturated and the fresh and saltwater have constant densities. By neglecting the vertical dimension and combining Darcy's law with the mass conservation law we get two reduced model of Boussineq type [12].

1.1 Introduction

Hydrogeology (hydro- meaning water, and -geology meaning the study of the Earth) is the area of geology that deals with the distribution and movement of groundwater in the soil and rocks of the Earth's crust, commonly in aquifers.

The Organization of this chapter is as follows: In Section 1.2 we present some notions and laws in hydrogeology such as Darcy's law and the mass conservation law. In Section 1.3 we present a brief and formal derivation of the seawater intrusion model in unconfined aquifers. Finally, Section 1.4 is devoted to the formal modeling of the seawater intrusion problem in confined aquifers.

1.2 Definitions in hydrogeology

Intrinsic permeability:
The intrinsic permeability k of a medium is a function of the size of pores and the degree of interconnectivity. It only depends on the properties of the medium and does not depend on the fluid. It is represented as a tensor with unit m^2.

Porosity:
The porosity θ is a measure of the empty spaces in a material, and is a fraction of the volume of voids over the total volume; $\theta \in (0, 1]$. In the case of a saturated medium, the volume of empty spaces is equal to the volume of the fluid in the medium.

Dynamic viscosity:
The dynamic viscosity μ is the proportionality coefficient of the force applied between two layers of different speeds. With increased viscosity, fluid's ability to flow decreases. Its unit is Pa.s.

Specific weight:
The specific weight γ of a fluid is the product of the density of the fluid ρ and the acceleration gravity g,

$$\gamma = \rho\, g.$$

Hydraulic conductivity:
The hydraulic conductivity K is a property of both the medium and the fluid. It is expressed as

$$K = \frac{k\rho g}{\mu}.$$

This property indicates the ability of a porous media to transfer a volume of a fluid.

Hydraulic head:
The hydraulic head h is a measure of the potential of the fluid at the measurement point. It is formed of two components: the elevation component and the pressure component. It is expressed as

$$h = z + \frac{p}{\gamma}.$$

Effective velocity of the flow:
The effective velocity of the flow q is expressed as

$$q = \theta v,$$

where v is the velocity of the flow.

Conservation of mass:
The conservation of mass during displacement is given by the following equation

$$\partial_t(\rho\,\theta) + \mathrm{div}(\rho\,q) = 0.$$

Darcy's law [25]:
This law was formulated by Henry Darcy:

$$q = -K \cdot \nabla h.$$

Replacing the hydraulic conductivity K and the hydraulic head h in this equation we get

$$q = -\frac{k}{\mu}\nabla(p + \gamma z).$$

The effective velocity q of the flow is thus related to the gradient of the pressure ∇p through the Darcy's law.

Aquifer:
An aquifer is an underground layer through which water can easily move. Aquifers must be both permeable and porous. From such layers, freshwater can be usefully extracted using a well.

Unconfined aquifer:
Unconfined aquifers are those into which water seeps from the ground surface directly above the aquifer.

Confined aquifer:
Confined aquifers are those in which an impermeable dirt/rock layer exists that prevents water from seeping into the aquifer from the ground surface located directly above. Instead, water seeps into confined aquifers from farther away where the impermeable layer doesn't exist.

Seawater intrusion [9]:
Seawater intrusion is the movement of saltwater into freshwater aquifers. It occurs in coastal freshwater aquifers when the different densities of both the saltwater and freshwater allow the ocean water to intrude into the freshwater aquifer. These areas are usually supporting large populations where the demanding groundwater withdrawals from these aquifers is exceeding the recharge rate, which can lead to contamination of drinking water sources and other consequences.

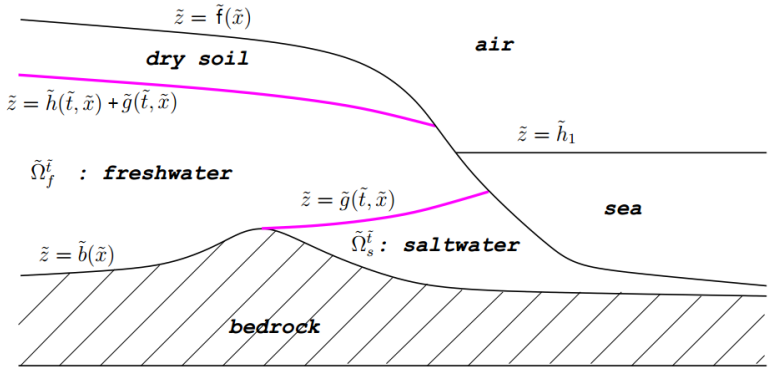


Figure 1.1 – Unconfined aquifer

1.3 Seawater intrusion in unconfined aquifer

We consider the space coordinates $(\tilde{x},\tilde{z}) \in \mathbb{R}^N \times \mathbb{R}$ where $\tilde{x}$ represents the horizonal coordinate, $\tilde{z}$ the vertical coordinate and $N = 1,2$. We assume that the interface between the saltwater and the bedrock is given as $\{\tilde{z} = \tilde{b}(\tilde{x})\}$, the surface of the soil as $\{\tilde{z} = \tilde{f}(\tilde{x})\}$, the interface between the saltwater and the freshwater, which are assumed to be unmiscible, can be written as

$$\Gamma_{\tilde{g}}^{\tilde{t}} = \left\{(\tilde{x},\tilde{z}) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde{z} = \tilde{g}(\tilde{t},\tilde{x})\right\},$$

and the interface between the freshwater and the dry soil can be written as

$$\Gamma_{\tilde{h}}^{\tilde{t}} = \left\{(\tilde{x},\tilde{z}) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde{z} = \tilde{h}(\tilde{t},\tilde{x}) + \tilde{g}(\tilde{t},\tilde{x})\right\}.$$

In addition we have that

$$\tilde{b} \leq \tilde{g} \leq \tilde{h} + \tilde{g} \leq \tilde{f} \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

and the fonctions $\tilde{b}$, $\tilde{g}$, $\tilde{h}$, $\tilde{f}$ are smooth enough. Moreover, we define the open sets

$$\tilde{\Omega}_f^{\tilde{t}} = \left\{(\tilde{x},\tilde{z}) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde{g}(\tilde{t},\tilde{x}) < \tilde{z} < \tilde{h}(\tilde{t},\tilde{x}) + \tilde{g}(\tilde{t},\tilde{x})\right\},$$

$$\tilde{\Omega}_s^{\tilde{t}} = \left\{(\tilde{x},\tilde{z}) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde{b}(\tilde{x}) < \tilde{z} < \tilde{g}(\tilde{t},\tilde{x})\right\},$$

$$\tilde{\Omega}^{\tilde{t}} = \left\{(\tilde{x},\tilde{z}) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde{z} < \tilde{f}(\tilde{x})\right\},$$

to be the set of freshwater, the set of saltwater in the porous medium and the set of the porous medium respectively.

The PDEs:
In what follows, we will index the parameters defined in Section 1.1 by α, where we set $\alpha = f$ in the case of a freshwater, $\alpha = s$ in the case of a saltwater. The density of the fluid α is given by

$$\tilde{\rho}_\alpha(\tilde{t},\tilde{x},\tilde{z}) = \begin{cases} \rho_\alpha^0 & \text{if} \quad (\tilde{x},\tilde{z}) \in \tilde{\Omega}_\alpha^{\tilde{t}} \\ 0 & \text{otherwise} \end{cases}$$

where ρ_α^0 is the mass density of the fluid α which is supposed to be a constant and the specific weight as

$$\gamma_\alpha = \rho_\alpha^0 g^0$$

where g^0 is the gravity constant. In what follows we suppose that $\theta = 1$ for simplicity.
We suppose that the flow velocity vector $\tilde{v}_\alpha$ satisfies:

$$\begin{cases} \widetilde{\mathrm{div}}_{\tilde{x}}\ \tilde{v}_\alpha^{\tilde{x}} + \partial_{\tilde{z}} \tilde{v}_\alpha^{\tilde{z}} = 0 \quad \text{on} \quad \tilde{\Omega}_\alpha^{\tilde{t}} \\ \tilde{v}_\alpha = -\tilde{\kappa}_\alpha(\tilde{x},\tilde{z})\ \widetilde{\nabla}(\tilde{p} + \gamma_\alpha \tilde{z}) \quad \text{on} \quad \tilde{\Omega}_\alpha^{\tilde{t}} \end{cases} \Bigg| \quad \text{for} \quad \alpha = f, s$$

$$\tilde{v}_f^{\tilde{x}}(\tilde{h}_{\tilde{x}} + \tilde{g}_{\tilde{x}}) - \tilde{v}_f^{\tilde{z}} + (\tilde{h}_{\tilde{t}} + \tilde{g}_{\tilde{t}}) = 0 \quad \text{on} \quad \left\{(\tilde{x},\tilde{z}) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde{z} = \tilde{h}(\tilde{t},\tilde{x}) + \tilde{g}(\tilde{t},\tilde{x}) < \tilde{f}(\tilde{x})\right\}$$

$$\tilde{v}_f^{\tilde{x}} \tilde{g}_{\tilde{x}} - \tilde{v}_f^{\tilde{z}} + \tilde{g}_{\tilde{t}} = 0 \quad \text{on} \quad \left\{(\tilde{x},\tilde{z}) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde{z} = \tilde{g}(\tilde{t},\tilde{x}) < \tilde{f}(\tilde{x})\right\}$$

$$\tilde{v}_s^{\tilde{x}} \tilde{g}_{\tilde{x}} - \tilde{v}_s^{\tilde{z}} + \tilde{g}_{\tilde{t}} = 0 \quad \text{on} \quad \left\{(\tilde{x},\tilde{z}) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde{z} = \tilde{g}(\tilde{t},\tilde{x}) < \tilde{f}(\tilde{x})\right\}$$

$$\tilde{v}_s^{\tilde{x}} \tilde{b}_{\tilde{x}} - \tilde{v}_s^{\tilde{z}} = 0 \quad \text{on} \quad \left\{(\tilde{x},\tilde{z}) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde{z} = \tilde{b}(\tilde{x}) < \tilde{g}(\tilde{t},\tilde{x})\right\}$$

$$\tilde{p} \quad \text{is continuous on} \quad \Gamma_{\tilde{g}}^{\tilde{t}}$$

$$\tilde{v}_f \cdot \tilde{n} \geq 0 \quad \text{across} \quad \partial\tilde{\Omega}^{\tilde{t}} \cap \partial\tilde{\Omega}_{\tilde{f}}^{\tilde{t}}$$

$$\tilde{g} \leq \tilde{h}_1 \quad \text{and} \quad \tilde{b} < \tilde{h} + \tilde{g} \quad \text{everywhere,} \tag{1.2}$$

with the following boundary condition

$$\begin{cases} \tilde{p}(\tilde{t},\tilde{x},\tilde{z}) = \begin{cases} \gamma_s(\tilde{h}_1 - \tilde{z}) & \text{if} \quad \tilde{z} = \tilde{h}(\tilde{t},\tilde{x}) + \tilde{g}(\tilde{t},\tilde{x}) = \tilde{f}(\tilde{x}) < \tilde{h}_1 \\ 0 & \text{otherwise} \end{cases} \\ \tilde{h}(\tilde{t},\tilde{x}) + \tilde{g}(\tilde{t},\tilde{x}) = \tilde{f}(\tilde{x}) \qquad \text{if} \quad \tilde{f}(\tilde{x}) \leq \tilde{h}_1. \end{cases} \tag{1.3}$$

Explication of system (1.2):

— The first equation is a consequence of the mass conservation law together with the fact that $\theta = 1$ and the density ρ_α is constant.

— The second equation is a consequence of Darcy's law with $\tilde{\kappa}_\alpha := \dfrac{1}{\theta}\dfrac{k_\alpha}{\mu_\alpha}$.

— The third, fourth, fifth and sixth equations imply that the interface $\tilde{\Gamma}_h^{\tilde{t}}$ moves by the freshwater velocity and that the interface $\tilde{\Gamma}_g^{\tilde{t}}$ moves by the freshwater and the saltwater velocities and that the two fluids move tangentially to $\{\tilde{z} = \tilde{b}(\tilde{x})\}$.

— The seventh line implies that we have some steady state. In fact, when we will be able to consider an interface between freshwater and saltwater, it means we suppose that we are in a state of equilibrium. At that moment, the pressure will be equal on either sides of this interface which imlpies the continuity.

— The eighth condition concerns the exterior normal unit vector where the unit vector $\tilde{n}$ is in the same direction as $\begin{pmatrix} -\widetilde{\nabla}\tilde{f}(\tilde{x}) \\ 1 \end{pmatrix}$ and it means that the freshwater can only

go out of the soil (supposing that there is no sources). Note that on the interface $\left\{(\tilde{x},\tilde{z})\in \mathbb{R}^N\times\mathbb{R},\quad \tilde{z}=\tilde{b}(\tilde{x})\right\}$ we have that $\tilde{v}_f\cdot\tilde{n}=0$ since there is no freshwater coming out from the impermeable rock.

— The final condition means that the saltwater in the soil is below the sea level which is a natural condition.

System (1.3) can be obtained naturally if we suppose that the atmosoheric pressure is zero and when $\tilde{\Gamma}_h^{\tilde{t}}$ has no contact with the sea, then its pressure is also zero.

New rescaling:

We suppose that there exists a small parameter $\varepsilon>0$ such that we have

$$\begin{cases} \tilde{x}=x\\ \tilde{z}=\varepsilon z\\ \tilde{t}=t/\varepsilon\\ \tilde{f}(\tilde{x})=\varepsilon f(x)\\ \tilde{b}(\tilde{x})=\varepsilon b(x)\\ \tilde{h}_1=\varepsilon h_1\\ \tilde{\kappa}_\alpha(\tilde{x},\tilde{z})=\kappa_\alpha(x,z)=\begin{pmatrix}\kappa_\alpha^{xx}(x,z) & \kappa_\alpha^{xz}(x,z)\\ \kappa_\alpha^{zx}(x,z) & \kappa_\alpha^{zz}(x,z)\end{pmatrix}\end{cases} \tag{1.4}$$

The parameter ε can be seen as the aspect ratio between the vertical dimension and the horizontal dimension.

Therefore, we consider the following rescaling

$$\begin{cases} \tilde{h}(\tilde{t},\tilde{x})=\varepsilon h^\varepsilon(t,x)\\ \tilde{g}(\tilde{t},\tilde{x})=\varepsilon g^\varepsilon(t,x)\\ \tilde{p}(\tilde{t},\tilde{x})=\varepsilon \bar{p}^\varepsilon(t,x)\\ \tilde{v}_\alpha^{\tilde{x}}(\tilde{t},\tilde{x})=\varepsilon v_\alpha^{x,\varepsilon}(t,x)\\ \tilde{v}_\alpha^{\tilde{z}}(\tilde{t},\tilde{x})=\varepsilon^2 v_\alpha^{z,\varepsilon}(t,x).\end{cases} \tag{1.5}$$

The vertical velocity is smaller than the horizontal velocity which is the Dupuit-Forchheimer assumption [73, 44].

We suppose that we have these formal asymptotics:

$$\begin{cases} h^\varepsilon & = h+\varepsilon\bar{h}_1+\varepsilon^2\bar{h}_2+\varepsilon^3\bar{h}_3+\cdots\\ g^\varepsilon & = g+\varepsilon g_1+\varepsilon^2 g_2+\cdots\\ \bar{p}^\varepsilon & = \bar{p}+\varepsilon\bar{p}_1^\varepsilon+\varepsilon^2\bar{p}_2^\varepsilon+\cdots\\ v_\alpha^{x,\varepsilon} & = v_\alpha^x+\varepsilon v_{\alpha,1}^x+\varepsilon^2 v_{\alpha,2}^x+\cdots\\ v_\alpha^{z,\varepsilon} & = v_\alpha^z+\varepsilon v_{\alpha,1}^z+\varepsilon^2 v_{\alpha,2}^z+\cdots\end{cases}$$

Set

$$\Omega_f^{t,\varepsilon}=\left\{(x,z)\in\mathbb{R}^N\times\mathbb{R},\quad g^\varepsilon(t,x)<z<h^\varepsilon(t,x)+g^\varepsilon(t,x)\right\},$$

$$\Omega_s^{t,\varepsilon}=\left\{(x,z)\in\mathbb{R}^N\times\mathbb{R},\quad b(x)<z<g^\varepsilon(t,x)\right\},$$

$$\Omega_f^{t}=\left\{(x,z)\in\mathbb{R}^N\times\mathbb{R},\quad g(t.x)<z<h(t,x)+g(t,x)\right\},$$

and

$$\Omega_s^{t}=\left\{(x,z)\in\mathbb{R}^N\times\mathbb{R},\quad b(x)<z<g(t,x)\right\}.$$

Using the new rescaling and substituting the formal asymptotics in (1.2) we get

$$\begin{cases} \left.\begin{array}{l} \mathrm{div}_x\ v_\alpha^x + \partial_z v_\alpha^z = 0 \quad \text{on} \quad \Omega_\alpha^t \\ v_\alpha^x = \bar\kappa_\alpha^{xx}(x,z)\nabla_x \bar p \quad \text{on} \quad \Omega_\alpha^t \\ \partial_z(\bar p + \gamma_\alpha z) = 0 \quad \text{on} \quad \Omega_\alpha^t \end{array}\right| \quad \text{for} \quad \alpha = f, s \\ v_f^x(h_x + g_x) - v_f^z + h_t + g_t = 0 \quad \text{across} \quad \big\{(x,z) \in \mathbb{R}^N \times \mathbb{R}, \quad z = h(t,x) + g(t,x) < f(x)\big\} \\ v_f^x g_x - v_f^z + g_t = 0 \quad \text{across} \quad \big\{(x,z) \in \mathbb{R}^N \times \mathbb{R}, \quad z = g(t,x) < f(x)\big\} \\ v_s^x g_x - v_s^z + g_t = 0 \quad \text{across} \quad \big\{(x,z) \in \mathbb{R}^N \times \mathbb{R}, \quad z = g(t,x) < f(x)\big\} \\ v_s^x b_x - v_s^z = 0 \qquad \text{sur} \quad \big\{(x,z) \in \mathbb{R}^N \times \mathbb{R}, \quad z = b(x) < g(t,x)\big\} \\ \bar p \quad \text{is continuous on} \quad \Gamma_g^t \\ -v_f^x \cdot \nabla_x f + v^z \geq 0 \quad \text{on} \quad \{z = h(t,x) + g(t,x) = f(x) > g(t,x)\} \\ g \leq h_1 \text{ and } b < h + g \quad \text{everywhere,} \end{cases} \tag{1.6}$$

where we have used the second and third equations of (1.2) and the new rescaling (1.5) to get that

$$\left\{\begin{array}{ll} \partial_z(\bar p^\varepsilon + \gamma_\alpha z) & = O(\varepsilon) = -\varepsilon(\kappa_\alpha^{zz}(x,z))^{-1}\left\{\varepsilon v_\alpha^{z,\varepsilon} + \kappa_\alpha^{zx}(x,z)\nabla_x \bar p^\varepsilon\right\} \\ v_\alpha^{x,\varepsilon} + \bar\kappa_\alpha^{xx}(x,z)\nabla_x \bar p_\varepsilon & = O(\varepsilon) = \varepsilon\kappa_\alpha^{xz}(x,z)(\kappa_\alpha^{zz}(x,z))^{-1} v_\alpha^{z,\varepsilon} \end{array}\right| \quad \text{in} \quad \Omega_\alpha^{t,\varepsilon}, \tag{1.7}$$

with

$$\bar\kappa_\alpha^{xx}(x,z) := \kappa_\alpha^{xx}(x,z) - (\kappa_\alpha^{zz}(x,z))^{-1}\kappa_\alpha^{xz}(x,z)\kappa_\alpha^{zx}(x,z). \tag{1.8}$$

For the sake of simplicity we assume that

$$\gamma_s \int_0^z \bar k_\alpha^{xx}(x,\bar z)\, d\bar z = z \cdot Id. \tag{1.9}$$

We obtain for the leading order term

$$\left\{\begin{array}{l} \partial_z(\bar p + \gamma_\alpha z) = 0 \\ \\ v_\alpha^x = -\bar\kappa_\alpha^{xx}(x,z)\nabla_x \bar p \end{array}\right| \quad \text{dans} \quad \Omega_\alpha^t \tag{1.10}$$

which implies the second and third term in (1.6). Now, integrating in z the third equation of (1.6) over $[z, h+g]$ and using the fact that $\alpha = s$ on $[z, g]$ and $\alpha = f$ on $[g, h+g]$ we get

$$\bar p(t,x,z) = \begin{cases} \gamma_s p_0(t,x) + \gamma_f(h(t,x) + g(t,x) - z) & \text{for } g(t,x) \leq z \leq h(t,x) + g(t,x) \\ \\ \gamma_s p_0(t,x) + \gamma_f h(t,x) + \gamma_s(g(t,x) - z) & \text{for } b(x) < z < g(t,x), \end{cases} \tag{1.11}$$

with

$$p_0(t,x) := \begin{cases} h_1 - f(x) & \text{if } h(t,x)+g(t,x) = f(x) < h_1 \\ \\ 0 & \text{otherwise.} \end{cases} \tag{1.12}$$

Let

$$\epsilon_0 = \frac{\gamma_s - \gamma_f}{\gamma_s} \in (0,1) \quad \text{which give us that} \quad \gamma_f \gamma_s^{-1} = 1 - (\frac{\gamma_s - \gamma_f}{\gamma_s}) = 1 - \epsilon_0.$$

We can write

$$\gamma_s^{-1} \nabla_x \bar{p}(t,\cdot) = \begin{cases} \nabla_x \left(p_0 + (1-\epsilon_0)(h+g)\right) & \text{on} \quad \Omega_f^t \\ \\ \nabla_x \left(p_0 + (1-\epsilon_0)h + g\right) & \text{on} \quad \Omega_s^t. \end{cases} \tag{1.13}$$

Now, integrating the first equation of (1.6) on $[g, h+g]$ and using the second equation of (1.10) with 1.9 and (1.13) we obtain

$$h_t = \mathrm{div}_x \ \left(h \nabla_x ((1-\epsilon_0)(h+g))\right) \quad \text{on} \quad \{h+g<f\}.$$

Moreover, the ninth line of (1.6) implies that the forth line of (1.6) still holds on $\{h+g=f>g\}$ with the condition that $h_t + g_t \geq 0$ on $\{h+g=f>g\}$. Which yields to

$$-g_t \leq \mathrm{div}_x \ \left(h \nabla_x ((1-\epsilon_0)(h+g))\right) \quad \text{on} \quad \{h+g=f>g\}.$$

Similarly, integrating the first line of (1.6) on $[b,g]$ we obtain that

$$g_t = \mathrm{div}_x \ \left((g-b) \nabla_x ((1-\epsilon_0)h+g)\right) \quad \text{on} \quad \{g<f\}.$$

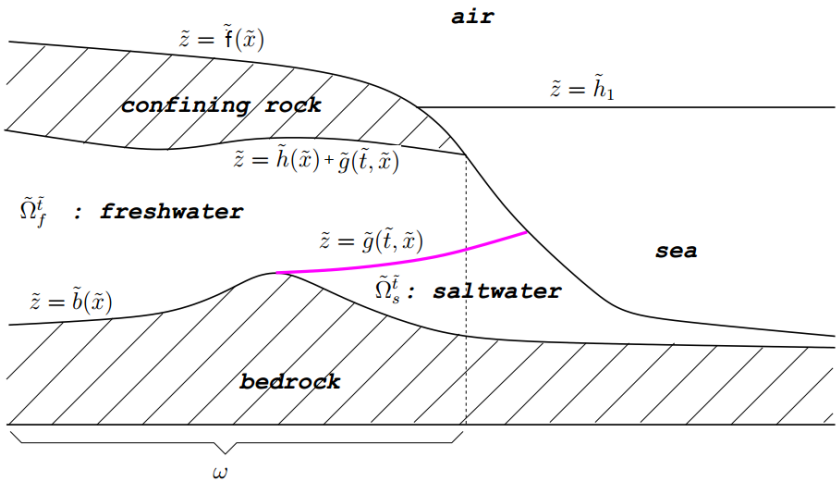
Now, using the fact that when $h+g=g=f<h_1$ we have $p_0 + (1-\varepsilon_0)h + \varepsilon_0 g = h_1$ we get

$$g_t \leq \mathrm{div}_x \ \left((g-b) \nabla_x ((1-\epsilon_0)h+g)\right) \quad \text{on} \quad \{h+g=g=f<h_1\}.$$

Therefore, h and g satisfy the following system

$$\begin{cases} b \leq g \leq h+g \leq f & \text{on} \quad [0,+\infty) \times \mathbb{R}^N \\ \\ h_t = \mathrm{div}_x \ \left(h \nabla_x (p_0 + (1-\epsilon_0)(h+g))\right) & \text{on} \quad \{h+g<f\} \\ \\ g_t = \mathrm{div}_x \ \left((g-b) \nabla_x (p_0 + (1-\epsilon_0)h+g)\right) & \text{on} \quad \{g<f\} \\ \\ -g_t \leq \mathrm{div}_x \ \left(h \nabla_x (p_0 + (1-\epsilon_0)(h+g))\right) & \text{on} \quad \{h+g=f>g\} \\ \\ g_t \leq \mathrm{div}_x \ \left((g-b) \nabla_x (p_0 + (1-\epsilon_0)h+g)\right) & \text{on} \quad \{h+g=g=f<h_1\}. \end{cases} \tag{1.14}$$

We remind that we can omit the pressure p_0 when $f > h_1$.



1.4 Seawater intrusion in confined aquifer

In the confined case, see the figure above, we have that $\Gamma_{\tilde{h}}^{\tilde{t}}$ is time independent interface, namely $h_t + g_t = 0$. The procedure is exactly the same only the boundary condition (1.3) is replaced by

$$\tilde{p}(\tilde{t}, \tilde{x}, \tilde{z}) = \gamma_s \max(0, \tilde{h}_1 - \tilde{z}) > 0 \quad \text{for } \tilde{z} = \tilde{f}(\tilde{x}) \quad \text{and } \tilde{x} \in \mathbb{R}^N \backslash \omega,$$

where

$$\omega = \left\{ \tilde{x} \in \mathbb{R}^N, (\tilde{h} + \tilde{g})(\tilde{x}) < \tilde{f}(\tilde{x}) \right\}.$$

Also the normalized pressure $p_0(x)$ is replaced by

$$p(t, x) = \begin{cases} \max(0, h_1 - (h + g)(x)) > 0 \quad \text{for } x \in \mathbb{R}^N \backslash \omega, \\ \\ \text{unknown} \quad \text{for } x \in \omega. \end{cases}$$

In the confined case, system (1.14) is still satisfied, where p in particular solves

$$\begin{cases} 0 = \operatorname{div}_x \left((h + g - b) \nabla_x (p + (1 - \epsilon_0)(h + g)) + \epsilon_0 (g - b) \nabla_x g \right) & \text{on} \quad \omega \\ \\ p(t, x) = p_1(x) := \max(0, h_1 - f(x)) > 0 & \text{across} \quad \partial\omega. \end{cases} \tag{1.15}$$

Chapter 2

Existence result for degenerate cross-diffusion system with application to seawater intrusion

Abstrat. This chapter is the subject of a paper written in collaboration with S. Issa, R. Monneau and M. Jazar, submitted for publication in "ESAIM: Control, Optimisation and Calculus of Variations". In this paper, we study degenerate parabolic system, which is strongly coupled. We prove general existence result, but the uniqueness remains an open question. Our proof of existence is based on a crucial entropy estimate which both control the gradient of the solution and the non-negativity of the solution. Our system is of porous medium type and our method applies to models in seawater intrusion.

2.1 Introduction

For the sake of simplicity, we will work on the torus $\Omega := \mathbb{T}^N = (\mathbb{R}/\mathbb{Z})^N$, with $N \geq 1$. Let $\Omega_T := (0,T) \times \Omega$ with $T > 0$. Let an integer $m \geq 1$. Our purpose is to study a class of degenerate strongly coupled parabolic system of the form

$$\partial_t u^i = \operatorname{div}\left(u^i \sum_{j=1}^{m} A_{ij} \nabla u^j \right) \qquad \text{in } \Omega_T, \quad \text{for } i = 1, \dots, m. \tag{2.1}$$

with the initial condition

$$u^i(0,x) = u_0^i(x) \geq 0 \qquad \text{a.e. in } \Omega, \quad \text{for } i = 1, \dots, m. \tag{2.2}$$

In the core of the paper we will assume that $A = (A_{ij})_{1 \leq i,j \leq m}$ is a real $m \times m$ matrix (not necessarily symmetric) that satisfies the following positivity condition: we assume that there exists $\delta_0 > 0$, such that we have

$$\xi^T A \, \xi \geq \delta_0 |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^m. \tag{2.3}$$

This condition can be weaken: see Subsection 2.4.1. Problem (2.1) appears naturally in the modeling of seawater intrusion (see Subsection 2.1.2).

2.1.1 Main results

To introduce our main result, we need to define the nonnegative entropy function Ψ:

$$\Psi(a) - \frac{1}{e} = \begin{cases} a \ln a & \text{for} \quad a > 0, \\ 0 & \text{for} \quad a = 0, \\ +\infty & \text{for} \quad a < 0, \end{cases} \tag{2.4}$$

which is minimal for $a = \dfrac{1}{e}$.

Theorem 2.1.1. ***(Existence for system (2.1))***
Assume that A satisfies (2.3). For $i = 1, \dots, m$, let $u_0^i \geq 0$ in Ω satisfying

$$\sum_{i=1}^{m} \int_\Omega \Psi(u_0^i) < +\infty, \tag{2.5}$$

where Ψ is given in (2.4). Then there exists a function $u = (u^i)_{1 \leq i \leq m} \in (L^2(0,T; H^1(\Omega)) \cap C([0,T); (W^{1,\infty}(\Omega))'))^m$ solution in the sense of distributions of (2.1),(2.2), with $u^i \geq 0$ a.e. in Ω_T, for $i = 1, \dots, m$. Moreover this solution satisfies the following entropy estimate for a.e. $t_1, t_2 \in (0,T)$, with $u^i(t_2) = u^i(t_2, \cdot)$:

$$\sum_{i=1}^{m} \int_\Omega \Psi(u^i(t_2)) + \delta_0 \sum_{i=1}^{m} \int_{t_1}^{t_2} \int_\Omega |\nabla u^i|^2 \leq \sum_{i=1}^{m} \int_\Omega \Psi(u_0^i), \tag{2.6}$$

where Ψ is given in (2.4).

Here $\|A\|$ is the matrix norm defined as

$$\|A\| = \sup_{|\xi|=1} |A\xi| \,. \tag{2.7}$$

Notice that the entropy estimate (2.6) guarantees that $\nabla u^i \in L^2(0,T;L^2(\Omega))$, and therefore allows us to define the product $u^i \sum_{i=1}^{m} A_{ij} \nabla u^j$ in (2.1). When our proofs were obtained, we realized that a similar entropy estimate has been obtained in [30], [17] and [24].

Remark 2.1.2. ***(Decreasing energy)***
If A is a symmetric matrix then a solution u of system (2.1) satisfies

$$\frac{d}{dt}\left(\sum_{i=1}^{m}\sum_{j=1}^{m}\int_\Omega \frac{1}{2} A_{ij} u^i u^j\right) = -\sum_{i=1}^{m}\int_\Omega u^i \left|\sum_{j=1}^{m} A_{ij}\nabla u^j\right|^2 .$$

2.1.2 Application to seawater intrusion

In this subsection, we describe briefly a model of seawater intrusion, which is particular case of our system (2.1).
An aquifer is an underground layer of a porous and permeable rock through which water can move. On the one hand coastal aquifers contain freshwater and on the other hand saltwater from the sea can enter in the ground and replace the freshwater. We refer to [9] for a general overview on seawater intrusion models.
Now let $\nu = 1 - \varepsilon_0 \in (0,1)$ where

$$\varepsilon_0 = \frac{\gamma_s - \gamma_f}{\gamma_s}$$

with γ_s and γ_f are the specific weight of the saltwater and freshwater respectively.

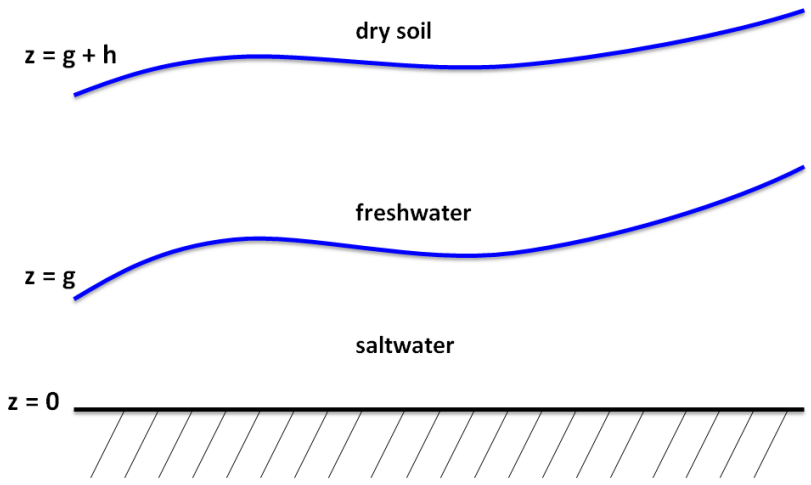


Figure 2.1 – Seawater intrusion in coastal aquifer

We assume that in the porous medium, the interface between the saltwater and the bedrock is given as $\{z = 0\}$, the interface between the saltwater and the freshwater, which are

assumed to be unmiscible, can be written as $\{z = g(t,x)\}$, and the interface between the freshwater and the dry soil can be written as $\{z = h(t,x) + g(t,x)\}$. Then the evolutions of h and g are given by a coupled nonlinear parabolic system (we refer to see [43]) of the form

$$\begin{cases} \partial_t h &= \operatorname{div}\{h\nabla(\nu(h+g))\} & \text{in } \Omega_T, \\ \partial_t g &= \operatorname{div}\{g\nabla(\nu h+g)\} & \text{in } \Omega_T, \end{cases} \tag{2.8}$$

This is a particular case of (2.1), where the 2×2 matrix

$$A = \begin{pmatrix} \nu & \nu \\ \nu & 1 \end{pmatrix} \tag{2.9}$$

satisfies (2.3).

2.1.3 Brief review of the litterature

The cross-diffusion systems, in particular the strongly coupled ones (for which the equations are coupled in the highest derivatives terms), are widely presented in different domains such as biology, chemistry, ecology, fluid mechanics and others. They are difficult to treat. Many of the standard results cannot be applied for such problems, such as the maximum principle. Hereafter, we cite several models where our method applies for most of them (see Section 2.4 for more generalizations on our problem).

In [66], Shigesada, Kawasaki and Teramoto proposed a two-species SKT model in one-dimensional space which arises in population dynamics. It can be written in a generalized form with m-species as

$$\partial_t u^i - \Delta\left[\left(\beta_i + \sum_{j=1}^m \alpha_{ij}u^j\right)u^i\right] = \left(a_i - \sum_{j=1}^m b_{ij}u^j\right)u^i, \quad \text{in } \Omega\times(0,T), \tag{2.10}$$

where u^i, for $i = 1,\dots,m$, denotes the population density of the i-th species and β_i, α_{ij}, a_i, b_{ij} are nonnegative constants. The existence of a global solution for such problem in arbitrary space dimension is studied in [74], where the quadratic form of the diffusion matrix is supposed positive definite. On the other hand, the two-species case was frequently studied, see for instance [52, 45, 76, 35, 67] for dimensions 1, 2, and [17, 62, 63, 18] for arbitrary dimension and appropriate conditions.

In [49], Lepoutre, Pierre and Rolland studied a relaxed model, without a term source of the form

$$\begin{cases} \partial_t u_i = \Delta\left[a_i(\tilde{u})u_i\right], \quad \tilde{u} = (\tilde{u}_i)_{1\le i\le I}, & \text{for } i = 1,\cdots,I, \\ \tilde{u}_i - \delta_i\Delta\tilde{u}_i = u_i \text{ with } \delta_i > 0, & \text{for } i = 1,\cdots,I, \end{cases}$$

in any dimension and for general nonlinearities a_i, which are only assumed to be continuous and bounded from below. They show the existence of a weak solution. Moreover, if the functions a_i are locally Lipschitz continuous then it is shown that this solution has more regularity and then is unique.

Another example of such problems is the electochemistry model studied by Choi, Huan and Lui in [21] where they consider the general form

$$\partial_t u^i = \sum_{\ell=1}^n\sum_{j=1}^m \frac{\partial}{\partial x_\ell}\left(a_\ell^{ij}(u)\frac{\partial u^j}{\partial x_\ell}\right), \quad u = (u^i)_{1\le i\le m} \quad \text{for} \quad i = 1,\dots,m, \tag{2.11}$$

and prove the existence of a weak solution of (2.11) under assumptions on the matrices $A_l(u) = (a_l^{ij}(u))_{1\leq i,j\leq m}$: it is continuous in u, its components are uniformly bounded with respect to u and its symmetric part is definite positive. Their strategy of proof seeks to use Galerkin method to prove the existence of solutions to the linearized system and then to apply Schauder fixed-point theorem. Then they apply the results obtained to an electrochemistry model.

A fourth example of cross-diffusion models is the chemotaxis model introduced in [50]. The global existence for classical solutions of this model is studied by Hillen and Painter in [38] where they considered

$$\begin{cases} \partial_t u &= \nabla\cdot(\nabla u - \chi(u,v)\nabla v), & t>0, x\in\Omega \\ \partial_t v &= \mu\Delta v + g(u,v), & t>0, x\in\Omega, \end{cases}$$

on a C^3- differentiable compact Riemannian manifold without boundary, where the function u describes the particle density, v is the density of the external signal, the chemotactic cross-diffusion χ is assumed to be bounded, and the function g describes production and degradation of the external stimulus. Another kind of chemotaxis model (the angiogenesis system) has been suggested and studied in [24]:

$$\begin{cases} \partial_t u &= \kappa\Delta u - \nabla\cdot(u\,\chi(v)\nabla v), & t>0, x\in\Omega \\ \partial_t v &= -v^m u, & t>0, x\in\Omega, \end{cases}$$

where $m\geq 1$ and κ is a constant.

Moreover, Alt and Luckhaus prove the existence in finite time of a solution for the following elliptic-parabolic problem

$$\partial_t b^i(u) - \operatorname{div}(a^i(b(u),\nabla u)) = f^i(b(u)), \quad \text{in } \Omega\times(0,T), \tag{2.12}$$

where $\Omega\subset R^N$ is open, bounded, and connected with Lipschitz boundary, b is monotone and continuous gradient and a is continuous and elliptic with some growth condition. This problem can be seen as a standard parabolic equation when $b(z)=z$.

Another problem is the Muskat Problem for Thin Fluid Layers of the form

$$\begin{cases} \partial_t f &= (1+R)\partial_x(f\partial_x f) + R\partial_x(f\partial_x g), \\ \partial_t g &= R_\mu\partial_x(g\partial_x f) + R_\mu\partial_x(g\partial_x g). \end{cases}$$

It models, [31], the motion of two fluids with different densities and viscosities in a porous meduim in one dimension, where f and g are the thickness of the two fluids and R, $R_\mu>0$ depending on the densities and the viscosities of the fluids. The authors in [31] studied the classical solutions of such problem. Moreover, weak solutions are studied in [30] by establishing two energy functionals and in [47] by using a gradient flow approach. Also the existence of a weak solutions for a strongly coupled fourth order parabolic system are established in [53, 48, 32].

2.1.4 Strategy of the proof

In (2.1), the elliptic part of the equation does not have a Lax-Milgram structure. Otherwise, our existence result is mainly based on the entropy estimate (2.6). It is difficult to get this entropy estimate directly (we do not have enough regularity to do it), so we proceed by approximations.

Approximation 1:
We discretize in time system (2.1), with a time step $\Delta t = T/K$, where $K \in \mathbb{N}^*$. Then for a given $u^n = (u^{i,n})_{1\leq i\leq m} \in (H^1(\Omega))^m$, we consider the implicit scheme which is an elliptic system:

$$\frac{u^{i,n+1} - u^{i,n}}{\Delta t} = \operatorname{div}\left\{ u^{i,n+1} \sum_{j=1}^{m} A_{ij} \nabla u^{j,n+1} \right\}. \tag{2.13}$$

Approximation 2:
We regularize the right-hand term of (2.13). To do that, we take $\eta > 0$ and $0 < \varepsilon < 1 < \ell$, and we choose the following regularization

$$\frac{u^{i,n+1} - u^{i,n}}{\Delta t} = \operatorname{div}\left\{ T^{\varepsilon,\ell}(u^{i,n+1}) \sum_{j=1}^{m} A_{ij} \nabla \rho_\eta \star \rho_\eta \star u^{j,n+1} \right\}, \tag{2.14}$$

where $T^{\varepsilon,\ell}$ is truncation operator defined as

$$T^{\varepsilon,\ell}(a) := \begin{cases} \varepsilon & \text{if } a \leq \varepsilon, \\ a & \text{if } \varepsilon \leq a \leq \ell, \\ \ell & \text{if } a \geq \ell, \end{cases} \tag{2.15}$$

and the mollifier $\rho_\eta(x) = \eta^{-N}\rho\left(x/\eta\right)$ with $\rho \in C_c^\infty(\mathbb{R}^N)$, $\rho \geq 0$, $\int_{\mathbb{R}^N} \rho = 1$ and $\rho(-x) = \rho(x)$. Now, with the convolution by ρ_η in (2.14), the term $\nabla\rho_\eta \star \rho_\eta \star u^{j,n+1}$ behaves like $u^{j,n+1}$. Note that, considering the $\mathbb{Z}^N$- periodic extension on $\mathbb{R}^N$ of $u^{j,n+1}$, the convolution $\rho_\eta \star u^{j,n+1}$ is possible over $\mathbb{R}^N$.
Approximation 3:
Let $\delta > 0$. We will add a second order term like $\delta\Delta u^i$ to equation (2.14) in order to obtain an elliptic one. More specifically, we consider $\operatorname{div}\left(\delta T^{\varepsilon,\ell}(u^i)\nabla u^i\right)$ instead of $\delta\Delta u^i$, to keep an entropy estimate.
Then we freeze the coefficients $u^{i,n+1}$ on the right-hand side to make a linear structure (these coefficients are now called $\delta T^{\varepsilon,\ell}(v^{i,n+1})$), we obtain the following modified system:

$$\frac{u^{i,n+1} - u^{i,n}}{\Delta t} = \operatorname{div}\left\{ T^{\epsilon,\ell}(v^{i,n+1}) \left(\sum_{j=1}^{m} A_{ij} \nabla \rho_\eta \star \rho_\eta \star u^{j,n+1} + \delta \nabla u^{i,n+1} \right) \right\}. \tag{2.16}$$

We will look for fixed points solutions $v^{i,n+1} = u^{i,n+1}$ of this modified system. Finally, we will recover the expected result dropping one after one all the approximations.

2.1.5 Organization of the paper

In Section 2.2, we recall some useful tools. In Section 2.3, we study system (2.1). By discretizing our problem on time, in Subsection 2.3.1, we obtain an elliptic problem. We use the Lax-Milgram theorem to show the existence of a unique solution to the linear problem (2.16). We demonstrate, in Subsection 2.3.2, the existence of a solution of the nonlinear problem, using the Schaefer's fixed point theorem.
Then we pass to the limit in the following order: $(\Delta t, \varepsilon) \to (0, 0)$ in Subsection 2.3.3, $(\ell, \eta) \to (\infty, 0)$ in Subsection 2.3.4 and $\delta \to 0$ in Subsection 2.3.5. Generalizations (including more general matrices A or tensors) will be presented in Section 2.4. We end with an Appendix showing some technical results in Section 2.5.

2.2 Preliminary tools

Theorem 2.2.1. (Schaefer's fixed point theorem) *[33, Theorem 4 page 504]*
Let X be a real Banach space. Suppose that

$$\Phi : X \to X$$

is a continuous and compact mapping. Assume further that the set

$$\{u \in X, \quad u = \lambda\Phi(u) \quad \text{for some} \quad \lambda \in [0,1]\}$$

is bounded. Then Φ has a fixed point.

Proposition 2.2.2. (Aubin's lemma) *[68]*
For any $T > 0$, and $\Omega = \mathbb{T}^N$, let E denote the space

$$E := \left\{ g \in L^2((0,T); H^1(\Omega)) \text{ and } \partial_t g \in L^2((0,T); H^{-1}(\Omega)) \right\},$$

endowed with the Hilbert norm

$$\|\omega\|_E = \left(\|\omega\|^2_{L^2(0,T;H^1(\Omega))} + \|\partial_t \omega\|^2_{L^2(0,T;H^{-1}(\Omega))} \right)^{\frac{1}{2}}.$$

The embedding

$$E \hookrightarrow L^2((0,T); L^2(\Omega)) \quad \text{is compact.}$$

On the other hand, it follows from [51, Proposition 2.1 and Theorem 3.1, Chapter 1] that the embedding

$$E \hookrightarrow C([0,T]; L^2(\Omega)) \quad \text{is continuous.}$$

Lemma 2.2.3. ***(Simon's Lemma)*** *[68]*
Let X, B and Y three Banach spaces, where $X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding. If $(g^n)_n$ is a sequence such that

$$\|g^n\|_{L^q(0,T;B)} + \|g^n\|_{L^1(0,T;X)} + \|\partial_t g^n\|_{L^1(0,T;Y)} \leq C,$$

where $1 < q \leq \infty$, and C is a constant independent of n, then $(g^n)_n$ is relatively compact in $L^p(0,T;B)$ for all $1 \leq p < q$.

Now we will present the variant of the original result of Simon's lemma [68, Corollary 6, page 87]. First of all, let us define the norm $\|.\|_{\mathrm{Var}([a,b);Y)}$ where Y is a Banach space with the norm $\|.\|_Y$.
For a function $g : [a,b) \to Y$, we set

$$\|g\|_{\mathrm{Var}([a,b);Y)} = \sup \sum_j \|g(a_{j+1}) - g(a_j)\|_Y \tag{2.17}$$

over all possible finite partitions:

$$a \leq a_0 < \cdots < a_k < b$$

Theorem 2.2.4. ***(Variant of Simon's Lemma)***
Let X, B and Y three Banach spaces, where $X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding. Let $(g^n)_n$ be a sequence such that

$$\|g^n\|_{L^1(0,T;X)} + \|g^n\|_{L^q(0,T;B)} + \|g^n\|_{Var([0,T);Y)} \leq C, \tag{2.18}$$

where $1 < q < \infty$, and C is a constant independent of n. Then $(g^n)_n$ is relatively compact in $L^p(0,T;B)$ for all $1 \leq p < q$.

Proof. **Step 1: Regularization of the sequence**
Let $\bar{\rho} \in C_c^\infty(\mathbb{R})$ with $\bar{\rho} \geq 0$, $\int_{\mathbb{R}} \bar{\rho} = 1$ and supp $\bar{\rho} \subset (-1,1)$. For $\varepsilon > 0$, we set

$$\bar{\rho}_\varepsilon(x) = \varepsilon^{-1}\bar{\rho}(\varepsilon^{-1}x).$$

We extend $g^n = g^n(t)$ by zero outside the time interval $[0,T)$. Because $q < +\infty$, we see that for each n, we choose some $0 < \varepsilon_n \to 0$ as $n \to +\infty$ such that

$$\|\bar{g}^n - g^n\|_{L^q(0,T;B)} \to 0 \quad \text{as} \quad n \to +\infty, \quad \text{with} \quad \bar{g}^n = \bar{\rho}_{\varepsilon_n} \star g^n \tag{2.19}$$

For any $\delta > 0$ small enough, we also have for n large enough (such that $\varepsilon_n < \delta$):

$$\|\bar{g}^n\|_{L^1(\delta,T-\delta;X)} \leq \|g^n\|_{L^1(0,T;X)} \leq C$$

and

$$\|\partial_t \bar{g}^n\|_{L^1(\delta,T-\delta;Y)} \leq \|g^n\|_{\mathrm{Var}([0,T);Y)} \leq C \tag{2.20}$$

Step 2: Checking (2.20)
By (2.18) there exists a sequence of step functions f_η which approximates uniformly g^n on $[0,T)$ as $\eta \to 0$, with moreover satisfies

$$\|f_\eta\|_{\mathrm{Var}([0,T);Y)} \to \|g^n\|_{\mathrm{Var}([0,T);Y)}.$$

Therefore we get easily (for $\varepsilon_n < \delta$)

$$\|\partial_t(\bar{\rho}_{\varepsilon_n} \star f_\eta)\|_{L^1(\delta,T-\delta;Y)} \leq \|f_\eta\|_{\mathrm{Var}([0,T);Y)}$$

which implies (2.20), when we pass to the limit as η goes to zero.
Step 3: Conclusion
We can then apply Corollary 6 in [68] to deduce that $\bar{g}^n$ is relatively compact in $L^p(0,T;B)$ for all $1 \leq p < q$. Because of (2.19), we deduce that this is also the case for the sequence $(g^n)_n$, which ends the proof of the Theorem. □

Besides the previous statement, several compactness results have been developed recently for piecewise constant functions of time resulting from a time discretization, see [20, 29] and [7, Proposition 3.3.1].

2.3 Existence for system (2.1)

Our goal is to prove Theorem 2.1.1 in order to get the existence of a solution for system (2.1).

2.3.1 Existence for the linear elliptic problem (2.16)

In this subsection we prove the existence, via Lax-Milgram theorem, of the unique solution for the linear elliptic system (2.16).
Let us recall our linear elliptic system. Assume that A is any $m \times m$ real matrix. Let $v^{n+1} = (v^{i,n+1})_{1\leq i\leq m} \in (L^2(\Omega))^m$ and $u^n = (u^{i,n})_{1\leq i\leq m} \in (H^1(\Omega))^m$. Then for all Δt, ε, ℓ, η, $\delta > 0$, with $\varepsilon < 1 < \ell$ and $\Delta t < \tau$ where τ is given in (2.22), we look for the solution $u^{n+1} = (u^{i,n+1})_{1\leq i\leq m}$ of the following system:

$$\left\{\begin{array}{rcl} \dfrac{u^{i,n+1}-u^{i,n}}{\Delta t} & = & \operatorname{div}\left\{J^i_{\epsilon,\ell,\eta,\delta}(v^{n+1},u^{n+1})\right\} \quad \text{in} \quad \mathcal{D}'(\Omega), \\ J^i_{\epsilon,\ell,\eta,\delta}(v^{n+1},u^{n+1}) & = & T^{\epsilon,\ell}(v^{i,n+1})\left\{\sum\limits_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^{j,n+1} + \delta\nabla u^{i,n+1}\right\}, \end{array}\right. \tag{2.21}$$

where $T^{\varepsilon,\ell}$ is given in (2.15).

Proposition 2.3.1. ***(Existence for system (2.21))***
Assume that A is any $m \times m$ real matrix. Let Δt, ε, ℓ, η, $\delta > 0$, with $\varepsilon < 1 < \ell$, such that

$$\Delta t < \frac{\delta\varepsilon\eta^2}{{C_0}^2\ell^2\left\|A\right\|^2} := \tau, \tag{2.22}$$

where

$$C_0 = \|\nabla\rho\|_{L^1(\mathbb{R}^N)}. \tag{2.23}$$

Then for $n \in \mathbb{N}$, for a given $v^{n+1} = (v^{i,n+1})_{1\leq i\leq m} \in (L^2(\Omega))^m$ and $u^n = (u^{i,n})_{1\leq i\leq m} \in (H^1(\Omega))^m$, there exists a unique function $u^{n+1} = (u^{i,n+1})_{1\leq i\leq m} \in (H^1(\Omega))^m$ solution of system (2.21). Moreover, this solution u^{n+1} satisfies the following estimate

$$\left(1-\frac{\Delta t}{\tau}\right)\left\|u^{n+1}\right\|^2_{(L^2(\Omega))^m} + \Delta t\varepsilon\delta\left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m} \leq \left\|u^n\right\|^2_{(L^2(\Omega))^m}, \tag{2.24}$$

where τ is given in (2.22).

Proof. The proof is done in four steps using Lax-Milgram theorem.
First of all, let us define for all $u^{n+1} = (u^{i,n+1})_{1\leq i\leq m}$ and $\varphi = (\varphi^i)_{1\leq i\leq m} \in (H^1(\Omega))^m$, the following bilinear form:

$$\begin{array}{rcl} a(u^{n+1},\varphi) & = & \displaystyle\sum_{i=1}^m\int_\Omega u^{i,n+1}\varphi^i + \Delta t\sum_{i,j=1}^m\int_\Omega T^{\varepsilon,\ell}(v^{i,n+1})A_{ij}\left(\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1}\right)\cdot\nabla\varphi^i \\ & & \displaystyle +\Delta t\delta\sum_{i=1}^m\int_\Omega T^{\varepsilon,\ell}(v^{i,n+1})\nabla u^{i,n+1}\cdot\nabla\varphi^i, \end{array}$$

which can be also rewritten as

$$\begin{array}{rcl} a(u^{n+1},\varphi) & = & \left\langle u^{n+1},\varphi\right\rangle_{(L^2(\Omega))^m} + \Delta t\left\langle T^{\varepsilon,\ell}(v^{n+1})\nabla\varphi, A\nabla\rho_\eta\star\rho_\eta\star u^{n+1}\right\rangle_{(L^2(\Omega))^m} \\ & & +\Delta t\delta\left\langle T^{\varepsilon,\ell}(v^{n+1})\nabla\varphi,\nabla u^{n+1}\right\rangle_{(L^2(\Omega))^m}, \end{array}$$

where $\langle\cdot,\cdot\rangle_{(L^2(\Omega))^m}$ denotes the scalar product on $(L^2(\Omega))^m$ and $T^{\varepsilon,\ell}(v^{n+1})\nabla\varphi = \left(T^{\varepsilon,\ell}(v^{i,n+1})\nabla\varphi^i\right)_{1\leq i\leq m}$. Also we define the following linear form:

$$L(\varphi) = \sum_{i=1}^{m}\int_\Omega u^{i,n}\varphi^i = \langle u^n,\varphi\rangle_{(L^2(\Omega))^m}.$$

Step 1: Continuity of a
For every $n\in\mathbb{N}$, u^{n+1} and $\varphi\in(H^1(\Omega))^m$, we have

$$\begin{aligned}|a(u^{n+1},\varphi)| &\leq \|u^{n+1}\|_{(L^2(\Omega)^m}\|\varphi\|_{(L^2(\Omega))^m} + \Delta t\ell\|A\|\|\nabla\rho_\eta\star\rho_\eta\star u^{n+1}\|_{(L^2(\Omega))^m}\|\nabla\varphi\|_{(L^2(\Omega))^m}\\ &\quad +\Delta t\delta\ell\|\nabla u^{n+1}\|_{(L^2(\Omega))^m}\|\nabla\varphi\|_{(L^2(\Omega))^m}\\ &\leq 3\max(1,\Delta t\ell\|A\|,\Delta t\delta\ell)\|u^{n+1}\|_{(H^1(\Omega))^m}\|\varphi\|_{(H^1(\Omega))^m}.\end{aligned}$$

where $\|A\|$ is given in (2.7) and we have used the fact that

$$\left\|\nabla\rho_\eta\star\rho_\eta\star u^{n+1}\right\|_{(L^2(\Omega))^m} \leq \left\|\nabla u^{n+1}\right\|_{(L^2(\Omega))^m}, \tag{2.25}$$

and

$$\varepsilon\leq T^{\varepsilon,\ell}(a)\leq\ell, \quad \text{for all } a\in\mathbb{R}. \tag{2.26}$$

Step 2: Coercivity of a
For all $\varphi\in(H^1(\Omega))^m$, we have that $a(\varphi,\varphi) = a_0(\varphi,\varphi)+a_1(\varphi,\varphi)$, where

$$a_0(\varphi,\varphi) = \|\varphi\|^2_{(L^2(\Omega))^m} + \Delta t\delta\left\langle T^{\varepsilon,\ell}(\varphi)\nabla\varphi,\nabla\varphi\right\rangle_{(L^2(\Omega))^m}$$

and

$$a_1(\varphi,\varphi) = \Delta t\left\langle T^{\varepsilon,\ell}(\varphi)\nabla\varphi, A\nabla\rho_\eta\star\rho_\eta\star\varphi\right\rangle_{(L^2(\Omega))^m}.$$

On the one hand, we already have the coercivity of a_0:

$$a_0(\varphi,\varphi) \geq \|\varphi\|^2_{(L^2(\Omega))^m} + \Delta t\delta\varepsilon\|\nabla\varphi\|^2_{(L^2(\Omega))^m}.$$

On the other hand, we have

$$\begin{aligned}|a_1(\varphi,\varphi)| &\leq \Delta t\ell\,\|A\|\,\|\nabla\rho_\eta\star\rho_\eta\star\varphi\|_{(L^2(\Omega))^m}\,\|\nabla\varphi\|_{(L^2(\Omega))^m}\\ &\leq \Delta t\ell\,\|A\|\left(\frac{1}{2\alpha}\|\nabla\rho_\eta\star\rho_\eta\star\varphi\|^2_{(L^2(\Omega))^m} + \frac{\alpha}{2}\|\nabla\varphi\|^2_{(L^2(\Omega))^m}\right)\\ &\leq \frac{\Delta t\ell^2\,\|A\|^2\,C_0^2}{2\delta\varepsilon\eta^2}\|\varphi\|^2_{(L^2(\Omega))^m} + \frac{\Delta t\varepsilon\delta}{2}\|\nabla\varphi\|^2_{(L^2(\Omega))^m},\end{aligned}$$

where in the second line we have used Young's inequality, and chosen $\alpha = \dfrac{\delta\varepsilon}{\|A\|\,\ell}$ in the third line, with C_0 is given in (2.23) and $\|A\|$ is given in (2.7). So we get that

$$a(\varphi,\varphi) \geq \left(1-\frac{\Delta t}{2\tau}\right)\|\varphi\|^2_{(L^2(\Omega))^m} + \frac{\Delta t\varepsilon\delta}{2}\|\nabla\varphi\|^2_{(L^2(\Omega))^m} \tag{2.27}$$

is coercive, since $\Delta t<\tau$ where τ is given in (2.22).

Step 3: Existence by Lax-Milgram

It is clear that L is linear and continuous on $(H^1(\Omega))^m$. Then by Step 1, Step 2 and Lax-Milgram theorem there exists a unique solution, u^{n+1}, of system (2.21).

Step 4: Proof of estimate (2.24)
Using (2.27) and the fact that $a(u^{n+1}, u^{n+1}) = L(u^{n+1})$ we get

$$\begin{aligned}\left(1-\frac{\Delta t}{2\tau}\right)\left\|u^{n+1}\right\|^2_{(L^2(\Omega))^m}+\frac{\Delta t\varepsilon\delta}{2}\left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m} &\leq \left\langle u^{i,n}, u^{i,n+1}\right\rangle_{(L^2(\Omega))^m}\\ &\leq \frac{1}{2}\left\|u^n\right\|^2_{(L^2(\Omega))^m}+\frac{1}{2}\left\|u^{n+1}\right\|^2_{(L^2(\Omega))^m},\end{aligned}$$

which gives us the estimate (2.24). □

2.3.2 Existence for the nonlinear time-discrete problem

In this subsection we prove the existence, using Schaefer's fixed point theorem, of a solution for the nonlinear time discrete-system (2.30) given below. Moreover, we also show that this solution satisfies a suitable entropy estimate.
First, to present our result we need to choose a function $\Psi_{\varepsilon,\ell}$ which is continuous, convex and satisfies that $\Psi''_{\varepsilon,\ell}(x) = \dfrac{1}{T^{\varepsilon,\ell}(x)}$, where $T^{\varepsilon,\ell}$ is given in (2.15). So let

$$\Psi_{\varepsilon,\ell}(a)-\frac{1}{e} = \begin{cases} \frac{a^2}{2\varepsilon}+a\ln\varepsilon-\frac{\varepsilon}{2} & \text{if } a\leq\varepsilon,\\ a\ln a & \text{if } \varepsilon<a\leq\ell,\\ \frac{a^2}{2\ell}+a\ln\ell-\frac{\ell}{2} & \text{if } a>\ell.\end{cases} \tag{2.28}$$

Let us introduce our nonlinear time discrete system: Assume that A satisfies (2.3). Let $u^0 = (u^{i,0})_{1\leq i\leq m} := u_0 = (u_0^i)_{1\leq i\leq m}$ that satisfies

$$C_1 := \sum_{i=1}^m \int_\Omega \Psi_{\varepsilon,\ell}(u_0^i) < +\infty, \tag{2.29}$$

such that $u_0^i \geq 0$ in Ω for $i = 1, \ldots, m$. Then for all Δt, ε, ℓ, η, $\delta > 0$, with $\varepsilon < 1 < \ell$ and $\Delta t < \tau$ where τ is given in (2.22), for $n \in \mathbb{N}$, we look for a solution $u^{n+1} = (u^{i,n+1})_{1\leq i\leq m}$ of the following system:

$$\begin{cases} \dfrac{u^{i,n+1}-u^{i,n}}{\Delta t} = \operatorname{div}\left\{J^i_{\varepsilon,\ell,\eta,\delta}(u^{n+1},u^{n+1})\right\} & \text{in } \mathcal{D}'(\Omega), \text{ for } n\geq 0\\ u^{i,0}(x) = u_0^i(x) & \text{in } \Omega,\end{cases} \tag{2.30}$$

where $J^i_{\varepsilon,\ell,\eta,\delta}$ is given in system (2.21), and $T^{\varepsilon,\ell}$ is given in (2.15).

Proposition 2.3.2. ***(Existence for system (2.30))***
Assume that A satisfies (2.3). Let $u_0 = (u_0^i)_{1\leq i\leq m}$ that satisfies (2.29), such that $u_0^i \geq 0$ a.e. in Ω for $i = 1, \ldots, m$. Then for all Δt, ε, ℓ, η, $\delta > 0$, with $\varepsilon < 1 < \ell$ and $\Delta t < \tau$ where τ is given in (2.22), there exists a sequence of functions $u^{n+1} = (u^{i,n+1})_{1\leq i\leq m} \in \left(H^1(\Omega)\right)^m$

for $n \in \mathbb{N}$, solution of system (2.30), that satisfies the following entropy estimate:

$$\sum_{i=1}^{m}\int_{\Omega}\Psi_{\varepsilon,\ell}(u^{i,n+1})+\delta\Delta t\sum_{i=1}^{m}\sum_{k=0}^{n}\int_{\Omega}|\nabla u^{i,k+1}|^2+\delta_0\Delta t\sum_{i=1}^{m}\sum_{k=0}^{n}\int_{\Omega}|\nabla\rho_\eta\star u^{i,k+1}|^2\leq\sum_{i=1}^{m}\int_{\Omega}\Psi_{\varepsilon,\ell}(u_0^i), \tag{2.31}$$

where $\Psi_{\varepsilon,\ell}$ is given in (2.28).

Proof. Our proof is based on the Schaefer's fixed point theorem. So we need to define, for a given $w := u^n = (u^{i,n})_{1\leq i\leq m} \in (L^2(\Omega))^m$ and $v := v^{n+1} = (v^{i,n+1})_{1\leq i\leq m} \in (L^2(\Omega))^m$, the map Φ as:

$$\begin{array}{rccc}\Phi & : & (L^2(\Omega))^m & \to & (L^2(\Omega))^m\\ & & v & \mapsto & u\end{array}$$

where $u := u^{n+1} = (u^{i,n+1})_{1\leq i\leq m} = \Phi(v^{n+1}) \in (H^1(\Omega))^m$ is the unique solution of system (2.21), given by Proposition 2.3.1.

Step 1: Continuity of Φ
Let us consider the sequence v_k such that

$$\begin{cases} v_k \in (L^2(\Omega))^m,\\ \\ v_k \longrightarrow v \quad \text{in} \quad (L^2(\Omega))^m. \end{cases} \tag{2.32}$$

We want to prove that the sequence $u_k = \Phi(v_k) \longrightarrow u = \Phi(v)$ to get the continuity of Φ. From the estimate (2.24), we deduce that u_k is bounded in $(H^1(\Omega))^m$. Therefore, up to a subsequence, we have

$$\begin{cases} u_k \rightharpoonup u \quad \text{weakly in} \ \ (H^1(\Omega))^m,\\ \text{and}\\ u_k \to u \quad \text{strongly in} \ \ (L^2(\Omega))^m, \end{cases}$$

where the strong convergence arises because Ω is compact. Thus, by the definition of the truncation operator $T^{\varepsilon,\ell}$, we can see that $T^{\varepsilon,\ell}$ is Lipschitz continuous together with (2.32) yield to

$$T^{\varepsilon,\ell}(v_k^i) \longrightarrow T^{\varepsilon,\ell}(v^i) \quad \text{in } L^2(\Omega), \qquad \text{for } i = 1,\dots,m.$$

Now we have

$$\frac{u_k^i - w^i}{\Delta t} = \operatorname{div}\left\{J^i_{\varepsilon,\ell,\eta,\delta}(v_k,u_k)\right\} \quad \text{ in } \mathcal{D}'(\Omega). \tag{2.33}$$

This system also holds in $H^{-1}(\Omega)$, because $J^i_{\varepsilon,\ell,\eta,\delta}(v_k,u_k) \in L^2(\Omega)$. Hence by multiplying this system by a test function in $(H^1(\Omega))^m$ and integrating over Ω for the bracket $\langle\cdot,\cdot\rangle_{H^{-1}(\Omega)\times H^1(\Omega)}$, we can pass directly to the limit in (2.33) as k tends to ∞, and we get

$$\frac{u^i - w^i}{\Delta t} = \operatorname{div}\{J^i_{\varepsilon,\ell,\eta,\delta}(v,u)\} \quad \text{ in } \mathcal{D}'(\Omega). \tag{2.34}$$

where we used in particular the weak L^2 - strong L^2 convergence in the product $T^{\varepsilon,\ell}(v_k)\nabla u_k$. Then $u = (u^i)_{1\leq i\leq m} = \Phi(v)$ is a solution of system (2.21). Finally, by uniqueness of the solutions of (2.21), we deduce that the limit u does not depend on the choice of the subsequence, and then that the full sequence converges:

$$u_k \to u \quad \text{strongly in} \quad (L^2(\Omega))^m, \quad \text{with} \quad u = \Phi(v).$$

Step 2: Compactness of Φ
By the definition of Φ we can see that for a bounded sequence $(v_k)_k$ in $(L^2(\Omega))^m$, $\Phi(v_k) = u_k$ converges strongly in $(L^2(\Omega))^m$ up to a subsequence, which implies the compactness of Φ.

Step 3: A priori bounds on the solutions of $v = \lambda\Phi(v)$
Let us consider a solution v of

$$v = \lambda\Phi(v) \quad \text{for some} \quad \lambda \in [0,1].$$

By (2.24) we see that there exists a constant $C_2 = C_2(\Delta t, \varepsilon, ...)$ such that for any given $w \in (L^2(\Omega))^m$, we have $\|\Phi(v)\|_{(H^1(\Omega))^m} \leq C_2 \|w\|_{(L^2(\Omega))^m}$. Hence $v = \lambda\Phi(v)$ is bounded.

Step 4: Existence of a solution
Now, we can apply Schaefer's fixed point Theorem (Theorem 2.2.1), to deduce that Φ has a fixed point u^{n+1} on $(L^2(\Omega))^m$. This implies the existence of a solution u^{n+1} of system (2.30).

Step 5: Proof of estimate (2.31)
We have,

$$\begin{aligned}
&\sum_{i=1}^{m} \int_\Omega \frac{\Psi_{\varepsilon,\ell}(u^{i,n+1}) - \Psi_{\varepsilon,\ell}(u^{i,n})}{\Delta t} \\
\leq\ & \sum_{i=1}^{m} \int_\Omega \left(\frac{u^{i,n+1} - u^{i,n}}{\Delta t} \right) \Psi'_{\varepsilon,\ell}(u^{i,n+1}) \\
=\ & \sum_{i=1}^{m} \left\langle \frac{u^{i,n+1} - u^{i,n}}{\Delta t}, \Psi'_{\varepsilon,\ell}(u^{i,n+1}) \right\rangle_{H^{-1}(\Omega)\times H^1(\Omega)} \\
=\ & -\sum_{i=1}^{m} \left\langle \delta T^{\varepsilon,\ell}(u^{i,n+1})\nabla u^{i,n+1} + T^{\varepsilon,\ell}(u^{i,n+1}) \sum_{j=1}^{m} A_{ij} \nabla\rho_\eta \star \rho_\eta \star u^{j,n+1}, \Psi''_{\varepsilon,\ell}(u^{i,n+1})\nabla u^{i,n+1} \right\rangle_{L^2(\Omega)} \\
=\ & -\sum_{i=1}^{m} \left\{ \delta \int_\Omega |\nabla u^{i,n+1}|^2 + \int_\Omega \sum_{j=1}^{m} \nabla\rho_\eta \star u^{i,n+1} A_{ij} \nabla\rho_\eta \star u^{j,n+1} \right\} \\
\leq\ & -\sum_{i=1}^{m} \delta \int_\Omega |\nabla u^{i,n+1}|^2 - \delta_0 \sum_{i=1}^{m} \int_\Omega |\nabla\rho_\eta \star u^{i,n+1}|^2,
\end{aligned}$$

where we have used, in the second line, the convexity inequality on $\Psi_{\varepsilon,\ell}$. In the third line, we used the fact that $\dfrac{u^{i,n+1} - u^{i,n}}{\Delta t} \in H^{-1}(\Omega)$ and $\nabla\Psi'_{\varepsilon,\ell}(u^{i,n+1}) = \Psi''_{\varepsilon,\ell}(u^{i,n+1})\nabla u^{i,n+1} \in L^2(\Omega)$ coming from the fact that $\Psi'_{\varepsilon,\ell} \in C^1(\mathbb{R})$, [14, Proposition IX.5, page 155], $\Psi''_{\varepsilon,\ell}(u^{i,n+1}) \in L^\infty(\Omega)$ and $\nabla u^{i,n+1} \in L^2(\Omega)$ for all $i = 1, \cdots, m$. Thus, in the fourth line we use that $u^{i,n+1}$ is a solution for system (2.30) where we have applied an integration by parts. In the fifth line, we used the transposition of the convolution (see for instance [14, Proposition IV.16, page 67]), and the fact that $\check{\rho}_\eta(x) = \rho_\eta(-x) = \rho_\eta(x)$. Finally, in the last line we use that A satisfies (2.3). Then by a straightforward recurrence we get estimate (2.31). This ends the proof of Proposition 2.3.2. $\square$

2.3.3 Passage to the limit as $(\Delta t, \varepsilon) \to (0,0)$

In this subsection we pass to the limit as $(\Delta t, \varepsilon) \to (0,0)$ in system (2.30) to get the existence of a solution for the continuous approximate system (2.37) given below.
First, let us define the function $\Psi_{0,\ell}$ as

$$\Psi_{0,\ell}(a) - \frac{1}{e} := \begin{cases} +\infty & \text{if } a < 0, \\ 0 & \text{if } a = 0, \\ a \ln a & \text{if } 0 < a \leq \ell, \\ \frac{a^2}{2\ell} + a \ln \ell - \frac{\ell}{2} & \text{if } a > \ell. \end{cases} \tag{2.35}$$

Now let us introduce our continuous approximate system. Assume that A satisfies (2.3). Let $u_0 = (u_0^i)_{1 \leq i \leq m}$ satisfying

$$C_3 := \sum_{i=1}^{m} \int_\Omega \Psi_{0,\ell}(u_0^i) < +\infty, \tag{2.36}$$

which implies that $u_0^i \geq 0$ a.e. in Ω for $i = 1, \ldots, m$. Then for all ℓ, η, $\delta > 0$, with $1 < \ell < +\infty$, we look for a solution $u = (u^i)_{1 \leq i \leq m}$ of the following system:

$$\begin{cases} \partial_t u^i &= div\left\{ J^i_{0,\ell,\eta,\delta}(u) \right\} & \text{in } \mathcal{D}'(\Omega_T), \\ J^i_{0,\ell,\eta,\delta}(u) &= T^{0,\ell}(u^i) \left\{ \sum_{j=1}^{m} A_{ij} \nabla \rho_\eta \star \rho_\eta \star u^j + \delta \nabla u^i \right\}, & \\ u^i(0,x) &= u_0^i(x) & \text{in } \Omega. \end{cases} \tag{2.37}$$

where $T^{0,\ell}$ is given in (2.15) for $\varepsilon = 0$, and we recall here $\Omega_T := (0,T) \times \Omega$.

Proposition 2.3.3. ***(Existence for system (2.37))***
Assume that A satisfies (2.3). Let $u_0 = (u_0^i)_{1 \leq i \leq m}$ satisfying (2.36). Then for all ℓ, η, $\delta > 0$ with $1 < \ell < +\infty$ there exists a function $u = (u^i)_{1 \leq i \leq m} \in (L^2(0,T; H^1(\Omega)) \cap C([0,T); L^2(\Omega)))^m$, with $u^i \geq 0$ a.e. in Ω_T, solution of system (2.37) that satisfies the following entropy estimate for a.e. $t_1, t_2 \in (0,T)$ with $u^i(t_1) = u^i(t_1, \cdot)$

$$\int_\Omega \sum_{i=1}^{m} \Psi_{0,\ell}(u^i(t_2)) + \delta \int_{t_1}^{t_2} \int_\Omega \sum_{i=1}^{m} \left|\nabla u^i\right|^2 + \delta_0 \int_{t_1}^{t_2} \int_\Omega \sum_{i=1}^{m} \left|\nabla \rho_\eta \star u^i\right|^2 \leq \int_\Omega \sum_{i=1}^{m} \Psi_{0,\ell}(u_0^i). \tag{2.38}$$

Proof. Our proof is based on the variant of Simon's Lemma (Theorem 2.2.4). Recall that $\Delta t = \dfrac{T}{K}$ where $K \in \mathbb{N}^*$ and $T > 0$ is given. We denote by C a generic constant independent of Δt and ε. For all $n \in \{0, \ldots, K-1\}$ and $i = 1, \ldots, m$, set $t_n = n\Delta t$ and let the piecewise constant in time function:

$$U^{i,\Delta t}(t,x) := u^{i,n+1}(x), \quad \text{for } t \in (t_n, t_{n+1}], \tag{2.39}$$

with $U^{i,\Delta t}(0,x) := u_0^i(x)$ satisfying (2.29).

Step 1: Upper bound on $\left\|\mathbf{U}^{\Delta t}\right\|_{(L^2(0,T;H^1(\Omega)))^m}$
We will prove that $U^{\Delta t}=(U^{i,\Delta t})_{1\leq i\leq m}$ satisfies

$$\int_0^T \|\nabla U^{\Delta t}(t)\|^2_{(L^2(\Omega))^m} \quad \leq \quad C.$$

For all $n\in\{0,\dots,K-1\}$ and $i=1,\dots,m$ we have

$$\nabla U^{i,\Delta t}(t,x)=\nabla u^{i,n+1}(x), \quad \text{for } t\in(t_n,t_{n+1}].$$

Then

$$\int_{t_n}^{t_{n+1}} \|\nabla U^{i,\Delta t}(t)\|^2_{L^2(\Omega)} \quad = \quad \Delta t\|\nabla u^{i,n+1}\|^2_{L^2(\Omega)}.$$

Hence

$$\begin{aligned}\int_0^T \|\nabla U^{\Delta t}(t)\|^2_{(L^2(\Omega))^m} &= \Delta t\sum_{k=0}^{K-1}\|\nabla u^{k+1}\|^2_{(L^2(\Omega)^m}\\ &\leq \frac{C_1}{\delta},\end{aligned}$$

where we have used the entropy estimate (2.31) with C_1 is given in (2.29). Hence, using Poincaré-Wirtinger's inequality we can get similarly an upper bound on $\displaystyle\int_0^T \left\|U^{i,\Delta t}\right\|^2_{(L^2(\Omega))^m}$ independently of Δt (using the fact that $\displaystyle\int_\Omega u^{i,n+1}=\int_\Omega u^{i,n}=\int_\Omega u^{i,0}$ by equation (2.30)) .

Step 2: Upper bound on $\left\|\mathbf{U}^{\Delta t}\right\|_{(\mathrm{Var}([0,T);H^{-1}(\Omega)))^m}$
We will prove that

$$\left\|U^{\Delta t}\right\|_{(\mathrm{Var}([0,T);H^{-1}(\Omega)))^m}\leq C.$$

We have for $i=1,\dots,m$

$$\begin{aligned}\left\|U^{i,\Delta t}\right\|_{\mathrm{Var}([0,T);H^{-1}(\Omega))} &= \sum_{n=0}^{K-1}\left\|U^{i,\Delta t}(t_{n+1})-U^{i,\Delta t}(t_n)\right\|_{H^{-1}(\Omega)}\\ &= \sum_{n=0}^{K-1}\left\|u^{i,n+1}-u^{i,n}\right\|_{H^{-1}(\Omega)}\\ &= \Delta t\sum_{n=0}^{K-1}\left\|\frac{u^{i,n+1}-u^{i,n}}{\Delta t}\right\|_{H^{-1}(\Omega)}\\ &\leq \Delta t\sum_{n=0}^{K-1}\left\|T^{\varepsilon,\ell}(u^{i,n+1})\left(\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1}+\delta\nabla u^{i,n+1}\right)\right\|_{L^2(\Omega)}\\ &\leq \ell\Delta t\sum_{n=0}^{K-1}\left\{\|A\|_\infty\sum_{j=1}^m\left\|\nabla\rho_\eta\star u^{j,n+1}\right\|_{L^2(\Omega)}+\delta\left\|\nabla u^{i,n+1}\right\|_{L^2(\Omega)}\right\}\\ &\leq C,\end{aligned}$$

where

$$\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{m} |A_{ij}|\,, \tag{2.40}$$

and we have used in the last inequality the entropy estimate (2.31), and the fact that

$$\Delta t \sum_{n=0}^{K-1} \left\|\nabla u^{i,n+1}\right\|_{L^2(\Omega)} \le \sqrt{T}\left(\Delta t \sum_{n=0}^{K-1} \left\|\nabla u^{i,n+1}\right\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

Step 3: $\mathbf{U^{i,\Delta t} \in L^p(0,T,L^2(\Omega))}$ with $\mathbf{p>2}$
The estimate (2.31) gives us that $U^{i,\Delta t} \in L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega))$ for $i = 1,\dots,m$. Using Sobolev injections we get $H^1(\Omega) \hookrightarrow L^{2+\alpha(N)}(\Omega)$, with $\alpha(N)>0$, and then $U^{i,\Delta t} \in L^2(0,T;L^{2+\alpha(N)}(\Omega))$. Hence by interpolation, we find that $U^{i,\Delta t} \in L^p(0,T;L^2(\Omega))$ with $\left(\frac{1}{p},\frac{1}{2}\right) = (1-\theta)\left(\frac{1}{\infty},\frac{1}{2}\right) + \theta\left(\frac{1}{2},\frac{1}{2+\alpha(N)}\right)$ and $\theta\in(0,1)$, i.e. for

$$p = \frac{4+4\alpha(N)}{2+\alpha(N)} > 2. \tag{2.41}$$

Step 4: Passage to the limit as $(\Delta \mathbf{t},\varepsilon)\to(0,0)$
By Steps 1,2 and 3 we have

$$\left\|U^{i,\Delta t}\right\|_{L^p(0,T;L^2(\Omega))} + \left\|U^{i,\Delta t}\right\|_{L^2(0,T;H^1(\Omega))} + \left\|U^{i,\Delta t}\right\|_{\mathrm{Var}([0,T);H^{-1}(\Omega))} \le C.$$

Then by noticing that $H^1(\Omega) \overset{compact}{\hookrightarrow} L^2(\Omega) \overset{continous}{\hookrightarrow} H^{-1}(\Omega)$, and applying the variant of Simon's Lemma (Theorem 2.2.4), we deduce that $(U^{i,\Delta t})_{\Delta t}$ is relatively compact in $L^2(0,T;L^2(\Omega))$, and there exists a function $U=(U^i)_{1\le i\le m} \in (L^2(0,T;H^1(\Omega)))^m$ such that, as $(\Delta t,\varepsilon)\to(0,0)$, we have (up to a subsequence)

$$U^{i,\Delta t} \to U^i \text{ strongly in } L^2(0,T;L^2(\Omega)).$$

By Step 1, we have $\nabla U^{i,\Delta t} \rightharpoonup \nabla U^i$ weakly in $L^2(0,T;L^2(\Omega))$. Now system (2.30) can be written as

$$\frac{U^{i,\Delta t}(t+\Delta t)-U^{i,\Delta t}(t)}{\Delta t} = \mathrm{div}\left\{J^i_{\varepsilon,\ell,\eta,\delta}(U^{i,\Delta t}(t+\Delta t),U^{i,\Delta t}(t+\Delta t))\right\} \quad \text{in } \mathcal{D}'(\Omega_T). \tag{2.42}$$

Multiplying this system by a test function in $\mathcal{D}(\Omega_T)$ and integrating over Ω_T, we can pass directly to the limit as $(\Delta t,\varepsilon)\to(0,0)$ in (2.42) to get

$$\partial_t U^i = \mathrm{div}\left(T^{0,\ell}(U^i)\left(\sum_{j=1}^{m} A_{ij}\nabla\rho_\eta \star \rho_\eta \star U^j + \delta\nabla U^i\right)\right) \quad \text{in } \mathcal{D}'(\Omega_T),$$

where we used the weak L^2 - strong L^2 convergence in the products such $T^{\varepsilon,\ell}(U^{i,\Delta t})\nabla U^{i,\Delta t}$ to get the existence of a solution of system (2.37).

Step 5: Recovering the initial condition
Let $\bar\rho \in C_c^\infty(\mathbb{R})$ with $\bar\rho\ge 0$, $\int_{\mathbb{R}}\bar\rho = 1$ and supp $\bar\rho \subset (-\frac{1}{2},\frac{1}{2})$. We set

$$\bar\rho_{\Delta t}(t) = \Delta t^{-1}\bar\rho(\Delta t^{-1}t), \text{ with } \bar\rho(t) = \bar\rho(-t).$$

Then we have

$$\begin{aligned}
\left\|\partial_t U^{\Delta t} \star \bar\rho_{\Delta t}\right\|^2_{(L^2(0,T;H^{-1}(\Omega)))^m} &= \sum_{i=1}^m \int_0^T \left\| \sum_{n=0}^{K-1} (u^{i,n+1}-u^{i,n})\delta_{t_{n+1}} \star \bar\rho_{\Delta t} \right\|^2_{H^{-1}(\Omega)} \\
&= \sum_{i=1}^m \sum_{n=0}^{K-1} \int_0^T (\Delta t \bar\rho_{\Delta t}(t-t_{n+1}))^2 \left\| \frac{u^{i,n+1}-u^{i,n}}{\Delta t} \right\|^2_{H^{-1}(\Omega)} \\
&= C_4 \Delta t \sum_{i=1}^m \sum_{n=0}^{K-1} \left\| \frac{u^{i,n+1}-u^{i,n}}{\Delta t} \right\|^2_{H^{-1}(\Omega)} \\
&\le C_4 \Delta t \sum_{n=0}^{K-1} \sum_{i=1}^m \int_\Omega \left| T^{\varepsilon,\ell}(u^{i,n+1}) \left(\sum_{j=1}^m A_{ij} \nabla \rho_\eta \star \rho_\eta \star u^{j,n+1} + \delta \nabla u^{i,n+1} \right) \right|^2 \\
&\le 2\, C_4 \ell^2 \Delta t \sum_{n=0}^{K-1} \sum_{i=1}^m \int_\Omega \left\{ \left(\sum_{j=1}^m A_{ij} \nabla \rho_\eta \star \rho_\eta \star u^{j,n+1} \right)^2 + \delta^2 \left|\nabla u^{i,n+1}\right|^2 \right\} \\
&\le 2\, C_4 \ell^2 \Delta t \sum_{n=0}^{K-1} \int_\Omega \left\{ \|A\|^2 \left\|\nabla \rho_\eta \star \rho_\eta \star u^{n+1}\right\|^2_{(L^2(\Omega))^m} + \delta^2 \left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m} \right\} \\
&\le 2\, C_4 \ell^2 \Delta t \sum_{n=0}^{K-1} \left\{ \|A\|^2 \|\nabla \rho_\eta \star u^{n+1}\|^2_{(L^2(\Omega))^m} + \delta^2 \left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m} \right\} \\
&\le 2\, C_4 \ell^2 C_1 \left(\frac{\|A\|^2}{\delta_0} + \delta \right) \le 2\, C_4 \ell^2 C_3 \left(\frac{\|A\|^2}{\delta_0} + \delta \right),
\end{aligned}$$

where $\delta_{t_{n+1}}$ is Dirac mass in $t = t_{n+1}$, C_1 as in (2.29), C_3 as in (2.36), $C_4 := \int_0^T \bar\rho(t)\,dt$, and we have used in the last line the entropy estimate (2.31). Clearly, $\bar\rho_{\Delta t} \star U_t^{i,\Delta t} \rightharpoonup U_t^i$ weakly in $L^2(0,T;H^{-1}(\Omega))$ as $(\Delta t,\varepsilon) \to 0$. Similarly we have that $\bar\rho_{\Delta t} \star U^{i,\Delta t} \to U^i$ strongly in $L^2(0,T;L^2(\Omega))$ since $U^{i,\Delta t} \to U^i$ in $L^2(0,T;L^2(\Omega))$. Then we deduce that $U^i \in \left\{ g \in L^2(0,T;H^1(\Omega)); g_t \in L^2(0,T;H^{-1}(\Omega)) \right\}$. And now $U^i(0,x)$ has sense, by Proposition 2.2.2, and we have that $U^i(0,x) = u_0^i(x)$ by Proposition 2.5.1.

Step 6: Proof of estimate (2.38)
By Step 4, there exists a function $U^i \in L^2(0,T;H^1(\Omega))$ such that the following holds true as $(\Delta t,\varepsilon) \to (0,0)$

$$\left\{ \begin{array}{lcl} U^{i,\Delta t} & \to & U^i \\ \nabla U^{i,\Delta t} & \rightharpoonup & \nabla U^i \\ \nabla \rho_\eta \star U^{i,\Delta t} & \to & \nabla \rho_\eta \star U^i \end{array} \right| \quad \text{in } \ L^2(0,T;L^2(\Omega)).$$

Now using the fact that the norm L^2 is weakly lower semicontinuous, with a sequence of integers n_2 (depending on Δt) such that $t_{n_2+1} \to t_2 \in (0,T)$ and

$$U^{i,\Delta t}(t_2) = U^{i,\Delta t}(t_{n_2+1}) = u^{n_2+1},$$

we get for $t_1 < t_2$

$$\int_{t_1}^{t_2} \int_\Omega \left|\nabla U^i\right|^2 \le \int_0^{t_2} \int_\Omega \left|\nabla U^i\right|^2 \le \liminf_{(\Delta t,\varepsilon)\to(0,0)} \int_0^{t_{n_2+1}} \int_\Omega \left|\nabla U^{i,\Delta t}\right|^2 = \liminf_{(\Delta t,\varepsilon)\to(0,0)} \Delta t \sum_{k=0}^{n_2} \int_\Omega |\nabla u^{i,k+1}|^2, \tag{2.43}$$

and

$$\int_{t_1}^{t_2}\int_\Omega \left|\nabla\rho_\eta \star U^i\right|^2 \le \int_0^{t_2}\int_\Omega \left|\nabla\rho_\eta \star U^i\right|^2 \le \liminf_{(\Delta t,\varepsilon)\to(0,0)} \Delta t \sum_{k=0}^{n_2}\int_\Omega |\nabla\rho_\eta \star u^{i,k+1}|^2. \quad (2.44)$$

Moreover, since we have $U^{i,\Delta t} \to U^i$ in $L^2(0,T;L^2(\Omega))$, we get that for a.e. $t \in (0,T)$ (up to a subsequence) $U^{i,\Delta t}(t,\cdot) \to U^i(t,\cdot)$ in $L^2(\Omega)$. For such t we have (up to a subsequence) $U^{i,\Delta t}(t,\cdot) \to U^i(t,\cdot)$ for a.e. in Ω. Moreover, by applying Lemma 2.5.2 we get that for a.e. $t \in (0,T)$

$$\Psi_{0,\ell}(U^i(t)) \le \liminf_{(\Delta t,\varepsilon)\to(0,0)} \Psi_{\varepsilon,\ell}(U^{i,\Delta t}(t)). \quad (2.45)$$

Integrating over Ω then applying Fatou's Lemma we get for a.e. $t_1 < t_2$

$$\sum_{i=1}^m \int_\Omega \Psi_{0,\ell}(U^i(t_2)) \le \int_\Omega \liminf_{(\Delta t,\varepsilon)\to(0,0)} \sum_{i=1}^m \Psi_{\varepsilon,\ell}(U^{i,\Delta t}(t_2)) \le \liminf_{(\Delta t,\varepsilon)\to(0,0)} \sum_{i=1}^m \int_\Omega \Psi_{\varepsilon,\ell}(u^{i,n_2+1}). \quad (2.46)$$

(2.43),(2.44) and (2.46) with the entropy estimate (2.31) give us that for a.e. $t_1 < t_2 \in (0,T)$

$$\begin{aligned}
&\sum_{i=1}^m \int_\Omega \Psi_{0,\ell}(U^i(t_2)) + \delta \sum_{i=1}^m \int_{t_1}^{t_2}\int_\Omega \left|\nabla U^i\right|^2 + \delta_0 \sum_{i=1}^m \int_{t_1}^{t_2}\int_\Omega \left|\nabla\rho_\eta \star U^i\right|^2 \\
\le\ & \liminf_{(\Delta t,\varepsilon)\to(0,0)} \sum_{i=1}^m \int_\Omega \Psi_{\varepsilon,\ell}(u^{i,n_2+1}) + \liminf_{(\Delta t,\varepsilon)\to(0,0)} \delta\Delta t \sum_{i=1}^m \sum_{k=0}^{n_2} \int_\Omega |\nabla u^{i,k+1}|^2 \\
&+ \liminf_{(\Delta t,\varepsilon)\to(0,0)} \delta_0 \Delta t \sum_{i=1}^m \sum_{k=0}^{n_2} \int_\Omega |\nabla\rho_\eta \star u^{i,k+1}|^2 \\
\le\ & \sum_{i=1}^m \int_\Omega \Psi_{\varepsilon,\ell}(u_0^i) \le \sum_{i=1}^m \int_\Omega \Psi_{0,\ell}(u_0^i),
\end{aligned}$$

which is estimate (2.38).

Step 7: Non-negativity of $\mathbf{U^i}$
Let $\Omega_i^\varepsilon(t) := \left\{x \in \Omega : U^{i,\Delta t}(t,x) \le \varepsilon\right\}$ for all $t \in [0,T]$ and $i = 1, \cdots, m$. By estimate (2.31), there exists a positive constant C independent of ε and Δt such that for all $i = 1, \ldots, m$ we have

$$\begin{aligned}
C &\ge \int_\Omega \Psi_{\varepsilon,\ell}(U^{i,\Delta t}) \\
&\ge \int_{\Omega_i^\varepsilon(t)} \Psi_{\varepsilon,\ell}(U^{i,\Delta t}) \\
&= \int_{\Omega_i^\varepsilon(t)} \frac{1}{e} + \frac{(U^{i,\Delta t})^2}{2\varepsilon} + U^{i,\Delta t}\ln\varepsilon - \frac{\varepsilon}{2} \\
&\ge \int_{\Omega_i^\varepsilon(t)} \frac{1}{e} + \frac{(U^{i,\Delta t})^2}{2\varepsilon} + \varepsilon\ln\varepsilon - \frac{1}{2} \\
&\ge \int_{\Omega_i^\varepsilon(t)} \frac{(U^{i,\Delta t})^2}{2\varepsilon} - \frac{1}{2},
\end{aligned}$$

i.e.

$$\int_{\Omega_i^\varepsilon(t)} \frac{(U^{i,\Delta t})^2}{2\varepsilon} \leq C + \frac{1}{2}. \tag{2.47}$$

Now by passing to the limit as $(\Delta t, \varepsilon) \to (0,0)$ in (2.47) we deduce that $\int_{\Omega_i^-(t)} \left|U^i\right|^2 = 0$, where $\Omega_i^-(t) := \left\{x \in \Omega : U^i(t,x) \leq 0\right\}$ for all $t \in [0,T]$ and $i = 1, \cdots, m$, which gives us that $(U^i)^- = 0$ in $L^2(\Omega)$, where $(U^i)^- = \min(0, U^i)$. □

Remark 2.3.4. *(**Another method following [51])***
Note that it would be also possible to use a theorem in Lions-Magenes [51, Chap. 3, Theorem 4.1, page 257]. This would prove in particular the existence of a unique solution for the following system:

$$\left\{\begin{array}{rcll} \partial_t u^i & = & \operatorname{div}\left\{J^i_{\epsilon,\ell,\eta,\delta}(v,u)\right\} & \text{in} \quad \mathcal{D}'(\Omega_T), \\ J^i_{\epsilon,\ell,\eta,\delta}(v,u) & = & T^{\epsilon,\ell}(v^i)\left\{\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^j + \delta\nabla u^i\right\}, & \\ u^i(0,x) & = & u_0^i(x) & \text{in} \quad \Omega, \end{array}\right. \tag{2.48}$$

where $T^{\varepsilon,\ell}$ is given in (2.15).
It would then be possible to find a fixed point solution $v = u$ of (2.48) to recover a solution of (2.37). We would have to justify again the entropy inequality (2.38).

2.3.4 Passage to the limit as $(\ell, \eta) \to (\infty, 0)$

In this subsection we pass to the limit as $(\ell, \eta) \to (\infty, 0)$ in system (2.37) to get the existence of a solution for system (2.49) given below (system independent of ℓ and η).
Let us introduce the system independant of ℓ and η. Asume that A satisfies (2.3). Let $u_0 = (u_0^i)_{1\leq i\leq m}$ satisfying (2.5). Then for all $\delta > 0$ we look for a solution $u = (u^i)_{1\leq i\leq m}$ of the following system:

$$\left\{\begin{array}{rcll} \partial_t u^i & = & div\left\{u^i \sum_{j=1}^m A_{ij}\nabla u^j + \delta u^i \nabla u^i\right\} & \text{in} \quad \mathcal{D}'(\Omega_T), \\ u^i(0,x) & = & u_0^i(x) & \text{a.e. in} \quad \Omega. \end{array}\right. \tag{2.49}$$

Proposition 2.3.5. *(**Existence for system (2.49)**)*
Assume that A satisfies (2.3). Let $u_0 = (u_0^i)_{1\leq i\leq m}$ satisfying (2.5). Then for all $\delta > 0$ there exists a function $u = (u^i)_{1\leq i\leq m} \in (L^2(0,T;H^1(\Omega)) \cap C([0,T);(W^{1,\infty}(\Omega))'))^m$, with $u^i \geq 0$ a.e. on Ω_T, solution of system (2.49), that satisfies the following entropy estimate for a.e. $t_1, t_2 \in (0,T)$ with $u^i(t_2) = u^i(t_2,.)$:

$$\int_\Omega \sum_{i=1}^m \Psi(u^i(t_2)) + \delta \int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^m \left|\nabla u^i\right|^2 + \delta_0 \int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^m \left|\nabla u^i\right|^2 \leq \int_\Omega \sum_{i=1}^m \Psi(u_0^i), \tag{2.50}$$

with Ψ is given in (2.4).

Proof. Let C be a generic constant independent of ℓ and η, and $u^\ell := (u^{i,\ell})_{1\leq i\leq m}$ a solution of system (2.37), where we drop the indices η and δ to keep light notations. The proof is

accomplished by passing to the limit as $(\ell,\eta)\to(\infty,0)$ in (2.37) and using Simon's lemma (Lemma 2.2.3), in order to get the existence result.

Step 1: Upper bound on $\partial_t \mathbf{u}^{\mathbf{i},\ell}$

As in Step 3 of the proof of Proposition 2.3.3, estimate (2.38) gives us that $u^{i,\ell}\in L^p(0,T,L^2(\Omega))$ with $p>2$ is given in (2.41). Let $q=\dfrac{2p}{p+2}>1$. It remains to prove that for $i=1,\dots,m$, $\left\|\partial_t u^{i,\ell}\right\|_{L^q(0,T;(W^{1,\infty})'(\Omega))}<C$. We have

$$
\begin{aligned}
&\|\partial_t u^{i,\ell}\|_{L^q(0,T;(W^{1,\infty}(\Omega))')}=\left(\int_0^T\left\|\partial_t u^{i,\ell}\right\|^q_{(W^{1,\infty}(\Omega))'}\right)^{\frac{1}{q}}\\
=&\left(\int_0^T\left\|div\left\{T^{0,\ell}(u^{i,\ell})\left(\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,\ell}+\delta\nabla u^{i,\ell}\right)\right\}\right\|^q_{(W^{1,\infty}(\Omega))'}\right)^{\frac{1}{q}}\\
\le&\left(\int_0^T\left(\int_\Omega\left|T^{0,\ell}(u^{i,\ell})\left(\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,\ell}+\delta\nabla u^{i,\ell}\right)\right|\right)^q\right)^{\frac{1}{q}}\\
\le&\left(\int_0^T\left(\int_\Omega\left|u^{i,\ell}\right|\left(\left|\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,\ell}\right|+\delta\left|\nabla u^{i,\ell}\right|\right)\right)^q\right)^{\frac{1}{q}}\\
\le&\left\|u^{i,\ell}\right\|_{L^p(0,T;L^2(\Omega))}\left\|\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,\ell}\right\|_{L^2(0,T;L^2(\Omega))}+\delta\left\|u^{i,\ell}\right\|_{L^p(0,T;L^2(\Omega))}\left\|\nabla u^{i,\ell}\right\|_{L^2(0,T;L^2(\Omega))}\\
\le&\left\|u^{i,\ell}\right\|_{L^p(0,T;L^2(\Omega))}\left(\|A\|_\infty\sum_{j=1}^m\left\|\nabla\rho_\eta\star u^{j,\ell}\right\|_{L^2(0,T;L^2(\Omega))}+\delta\left\|\nabla u^{i,\ell}\right\|_{L^2(0,T;L^2(\Omega))}\right)\le C,
\end{aligned}
$$

where we have used in the fifth line Holder's inequality (since we have $\dfrac{1}{q}=\dfrac{1}{p}+\dfrac{1}{2}$) and in the last line the entropy estimate (2.38).

Step 2: Passage to the limit as $(\ell,\eta)\to(\infty,0)$
In view of Step 1 of this proof and (2.38) we have that

$$
\left\|u^{i,\ell}\right\|_{L^p(0,T;L^2(\Omega))}+\left\|u^{i,\ell}\right\|_{L^2(0,T;H^1(\Omega))}+\left\|\partial_t u^{i,\ell}\right\|_{L^q(0,T;(W^{1,\infty}(\Omega))')}\le C,
$$

where $p>2$ is given in (2.41) and $q=\dfrac{2p}{p+2}>1$. Then by noticing that $H^1(\Omega)\overset{compact}{\hookrightarrow}L^2(\Omega)\overset{continous}{\hookrightarrow}(W^{1,\infty}(\Omega))'$, and applying Simon's Lemma (Lemma 2.3), we deduce that $(u^{i,\ell})_\ell$ is relatively compact in $L^2(0,T;L^2(\Omega))$, and there exists a function $u^i\in L^2(0,T;H^1(\Omega))$ such that, as $(\ell,\eta)\to(\infty,0)$, we have (up to a subsequence)

$$
u^{i,\ell}\to u^i \text{ strongly in } L^2(0,T;L^2(\Omega)).
$$

In addition, since $u^{i,\ell} \to u^i$ a.e., u^i is nonnegative a.e. hence $T^{0,\ell}(u^{i,\ell}) \to u^i$ strongly in $L^2(0,T;L^2(\Omega))$. Multiplying system (2.37) by a test function in $\mathcal{D}(\Omega_T)$ and integrating over Ω_T we can pass directly to the limit as $(\ell,\eta) \to (\infty,0)$, and we get

$$\partial_t u^i = \operatorname{div}\left\{ u^i \sum_{j=1}^{m} A_{ij} \nabla u^j + \delta u^i \nabla u^i \right\} \quad \text{in } \mathcal{D}'(\Omega_T).$$

where we used in particular the weak L^2 - strong L^2 convergence in the products such $T^{0,\ell}(u^{i,\ell})\nabla u^{i,\ell}$. Therefore, $u = (u^i)_{1\le i\le m}$ is a solution of system (2.49).

Step 3: Recovering the initial condition
Using Step 1 of this proof with the fact that $W^{1,1}(0,T;(W^{1,\infty}(\Omega))') \hookrightarrow C([0,T);(W^{1,\infty}(\Omega))')$ then $u^i(0,x)$ makes sense and $u^i(0,x) = u_0^i(x)$ for all $i = 1,\dots,m$, by Proposition 2.5.1.

Step 5: Proof of the estimate (2.50)
The proof is similar to Step 6 of the proof of Proposition 2.3.3. □

2.3.5 Passage to the limit as $\delta \to 0$

Proof. Let C be a generic constant independent of δ and $u^\delta := (u^{i,\delta})_{1\le i\le m}$ a solution of system (2.49). We follow the lines of proof of Proposition 2.3.5.
An upper bound on $u_t^{i,\delta}$ and estimate (2.50) allow us to apply Simon's Lemma (Lemma 2.3), then $(u^{i,\delta})_\delta$ is relatively compact in $L^2(0,T;L^2(\Omega))$, and there exists a function $u^i \in L^2(0,T;H^1(\Omega))$ such that, as $\delta \to 0$, we have (up to a subsequence)

$$u^{i,\delta} \to u^i \text{ strongly in } L^2(0,T;L^2(\Omega)),$$

and

$$\partial_t u^i = \operatorname{div}\left\{ u^i \sum_{j=1}^{m} A_{ij} \nabla u^j \right\} \quad \text{in } \mathcal{D}'(\Omega_T).$$

Similarly to Step 4 of the proof of Proposition 2.3.5 the initial condition is recoverd. Also estimate (2.6) can be easily obtained. □

Remark 2.3.6. ***(Passage to the limit as*** $(\ell,\eta,\delta) \to (\infty,0,0)$***)***
It is possible to pass to the limit in system (2.37) as $(\ell,\eta,\delta) \to (\infty,0,0)$ *at the same time: By using the entropy estimate (2.38) and applying Simon's Lemma on the sequence* $\rho_\eta \star u^{i,\ell}$ *instead of* $u^{i,\ell}$*. Moreover, to get the entropy estimate (2.6) it is sufficient to use the fact that* $\int_\Omega \Psi_{0,\ell}(\rho_\eta \star u^{i,\ell}) \le \int_\Omega \rho_\eta \star \Psi_{0,\ell}(u^{i,\ell})$*.*

2.4 Generalizations

2.4.1 Generalization on the matrix A

Assumption (2.3) can be weaken. Indead, we can assume that $A = (A_{ij})_{1\le i,j\le m}$ is a real $m \times m$ matrix that satisfies a positivity condition, in the sense that there exist two positive definite diagonal $m \times m$ matrices L and R and $\delta_0 > 0$, such that we have

$$\zeta^T LAR\zeta \ge \delta_0 |\zeta|^2, \quad \text{for all} \quad \zeta \in \mathbb{R}^m. \tag{2.51}$$

Remark 2.4.1. *(**Comments on the positivity condition (2.51)**)*
The assumption of positivity condition (2.51), generalize our problem for A not necessarily having a symmetric part positive definite. Here is an example of such a matrix, whose symmetric part is not definite positive, but the symmetric part of $L\,A\,R$ is definite positive for some suitable positive diagonal matrices L and R.
We consider

$$A = \begin{pmatrix} 1 & -a \\ 2a & 1 \end{pmatrix} \text{ with } |a| > 2.$$

Indeed,

$$A^{sym} = \frac{A^T + A}{2} = \begin{pmatrix} 1 & \frac{a}{2} \\ \frac{a}{2} & 1 \end{pmatrix},$$

satisfying $\det(A^{sym}) = 1 - \dfrac{a^2}{4} < 0$. *And let*

$$L = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

On the other hand,

$$B = L.A.R = \begin{pmatrix} 2 & -2a \\ 2a & 1 \end{pmatrix},$$

satisfies that

$$B^{sym} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

is definite positive.

Proposition 2.4.2. *(**The case where** $L = I_2$)*
Let A be a matrix that satisfies the positivity condition (2.51) with $L = I_2$. Then $\bar{u}$ is a solution for system (2.1) with the matrix $\bar{A} = A\,R$ (instead of A) if and only if $u^i = R_{ii}\,\bar{u}^i$ is a solution for system (2.1) with the matrix A.

Proposition 2.4.3. *(**The case where** $R = I_2$)*
Let $u^{n+1} = (u^{i,n+1})_{1\le i\le m}$ be a solution of system (2.30) with a matrix A satisfying the positivity condition (2.51) with $R = I_2$ and L a positive diagonal matrix. Then u^{n+1} satisfies the following entropy estimate

$$\begin{aligned} \sum_{i=1}^{m} \int_\Omega L_{ii} \Psi_{\varepsilon,\ell}(u^{i,n+1}) \;&+\; \delta\Delta t \min_{1\le i\le m}\{L_{ii}\} \sum_{i=1}^{m}\sum_{k=0}^{n} \int_\Omega |\nabla u^{i,k+1}|^2 \\ &+\; \delta_0 \Delta t \sum_{i=1}^{m}\sum_{k=0}^{n} \int_\Omega |\nabla \rho_\eta \star u^{i,k+1}|^2 \le \sum_{i=1}^{m} \int_\Omega L_{ii} \Psi_{\varepsilon,\ell}(u_0^i) \end{aligned}$$

Proof. Similarly to Step 5 of the proof of Proposition 2.3.2 we have

$$\begin{aligned}
\sum_{i=1}^{m}\int_\Omega L_{ii}\left(\frac{\Psi_{\varepsilon,\ell}(u^{i,n+1})-\Psi_{\varepsilon,\ell}(u^{i,n})}{\Delta t}\right) &\leq -\int_\Omega \sum_{i=1}^{m}\sum_{j=1}^{m} L_{ii}A_{ij}(\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1})\cdot\nabla u^{i,n+1}\\
&\quad -\delta\sum_{i=1}^{m}\int_\Omega L_{ii}|\nabla u^{i,n+1}|^2\\
&\leq \int_\Omega \sum_{i=1}^{m}\sum_{j=1}^{m}(\nabla\rho_\eta\star u^{j,n+1})L_{ii}A_{ij}(\nabla\rho_\eta\star u^{i,n+1})\\
&\quad -\delta\sum_{i=1}^{m}\int_\Omega L_{ii}|\nabla u^{i,n+1}|^2\\
&\leq -\delta_0\int_\Omega\sum_{i=1}^{m}|\nabla\rho_\eta\star u^{i,n+1}|^2\\
&\quad -\delta\min_{1\leq i\leq m}\{L_{ii}\}\sum_{i=1}^{m}\int_\Omega|\nabla u^{i,n+1}|^2,
\end{aligned}$$

where we have used, in the last line, the fact that the matrix A satisfies (2.51) with $R=I_2$. Then by a straightforward recurrence we get (2.52). □

Corollary 2.4.4. *Theorem 2.1.1 still hold true if we replace condition (2.3) by condition (2.51).*

2.4.2 Generalisation on the problem

The tensor case

Our study can be applied on a generalized systems of the form

$$\partial_t u^i=\sum_{j=1}^{m}\sum_{k=1}^{N}\sum_{l=1}^{N}\frac{\partial}{\partial x_k}\left(f_i(u^i)A_{ijkl}\frac{\partial u^j}{\partial x_l}\right)\qquad\text{for } i=1,\dots,m, \tag{2.52}$$

where f_i satisfies

$$\begin{cases}
f_i\in C(\mathbb{R}),\\
0\leq f_i(a)\leq C(1+|a|) & \text{for } a\in\mathbb{R} \text{ and } C>0,\\
0<f_i(a) & \text{for } a\in(0,a_0] \quad \text{with } a_0>0,\\
\displaystyle\int_{a_0}^{A}\frac{1}{f_i(a)}\,da<+\infty & \text{for all } A\geq a_0.
\end{cases}$$

An example for such f_i is

$$f_i(a)=\max\left(0,\min\left(a,\sqrt{|a-1|}\right)\right).$$

Moreover, $A = (A_{ijkl})_{i,j,k,l}$ is a tensor of order 4 that satisfies the following positivity condition: there exists $\delta_0 > 0$ such that

$$\sum_{i,j,k,l} A_{ijkl}\,\eta^i\,\eta^j\,\zeta_k\,\zeta_l \;\geq \delta_0|\eta|^2|\zeta|^2 \quad \text{for all } \eta \in \mathbb{R}^m, \zeta \in \mathbb{R}^N. \tag{2.53}$$

The entropy function Ψ_i is chosen such that Ψ_i is nonnegative, lower semi-continuous, convex and satisfies that $\Psi_i''(a) = \dfrac{1}{f_i(a)}$ for $i = 1, \ldots, m$. Our solution satisfies the following entropy estimate for a.e. $t > 0$

$$\sum_{i=1}^{m}\int_\Omega \Psi_i(u^i(t)) + \delta_0\sum_{i=1}^{m}\int_0^t\int_\Omega |\nabla u^i|^2 \leq \sum_{i=1}^{m}\int_\Omega \Psi_i(u_0^i). \tag{2.54}$$

To get this entropy we can apply the same strategy announced in Subsection 2.1.4 where $f_i(u^i)$ will be replaced by $T^{\varepsilon,\ell}(f_i(v^i))$ with $T^{\varepsilon,\ell}$ given in (2.15) and we use the fact that

$$\begin{aligned}
\int_\Omega \sum_{i,j,k,l} \frac{\partial u^i}{\partial x_k} A_{ijkl} \frac{\partial u^j}{\partial x_l} &= \sum_{n\in\mathbb{Z}^N}\sum_{i,j,k,l} \overline{\widehat{\left(\frac{\partial u^i}{\partial x_k}\right)}}(n) A_{ijkl} \widehat{\left(\frac{\partial u^j}{\partial x_l}\right)}(n) \\
&= \sum_{n\in\mathbb{Z}^N}\sum_{i,j,k,l} n_k\,\widehat{u^i}(n)\,A_{ijkl}\,n_l\,\widehat{u^j}(n) \\
&\geq \delta_0 \sum_{n\in\mathbb{Z}^N} |n|^2\,|\widehat{u}|^2 = \delta_0\,\|\nabla u\|^2_{(L^2(\Omega))^m}.
\end{aligned}$$

The variables coefficients case

Here the coefficients $A_{ij}(x,u)$ may depend continuously of (x,u). Then we have to take $\rho_\eta \star (A_{ij}(x,u)(\nabla\rho_\eta \star u^j))$ instead of $A_{ij}\nabla(\rho_\eta \star \rho_\eta \star u^j)$ in the approximate problem. We can consider a problem

$$\partial_t u^i = \operatorname{div}\left(u^i \sum_{j=1}^{m} A_{ij}(x,u)\nabla u\right) + g^i(x,u),$$

where the source terms are continuous with respect to the variable u and there exists a positive constant c such that

$$-c\,|u| \leq g^i(x,u) \leq c(1+|u|).$$

Laplace-type equations

Moreover, our method applies to models of the form

$$\partial_t u^i = \Delta(a_i(u)u^i) \quad \text{with } u = (u^i)_{1\leq i\leq m}, \tag{2.55}$$

under these assumptions:

$$\begin{cases}
a_i(u) \geq 0 \quad \text{if} \quad u^j \geq 0 \quad \text{for } j = 1, \ldots, m, \\
a_i \quad \text{is at most linear}, \\
a_i \in C^1(\mathbb{R}), \\
Sym\left(\left(\dfrac{\partial a_i}{\partial u_j}\right)_{i,j}\right) \geq \delta_0 I \quad \text{with} \quad \delta_0 > 0, \\
\dfrac{\partial a_i}{\partial u_j} \quad \text{are bounded from below for all } i,j = 1, \cdots, m
\end{cases} \tag{2.56}$$

where Sym denotes the symmetric part of a matrix. We can consider a particular case of (2.55) where $a_i(u) = \sum_{j=1}^{m} A_{ij}u^j$. Then problem (2.55) can be written as

$$\partial_t u^i = \operatorname{div}\left\{ u^i \sum_{j=1}^{m} A_{ij}\nabla u^j + \left(\sum_{j=1}^{m} A_{ij}u_j\right)\nabla u^i \right\}, \tag{2.57}$$

which can be also solved under these assumptions:

$$\begin{cases} A_{ij} \geq 0 \quad \text{for} \quad i,j = 1,\dots,m, \\ Sym(A) \geq \delta_0 I. \end{cases}$$

2.5 Appendix: Technical results

In this section we will present some technical results that are used in our proofs.

Proposition 2.5.1. ***(Recovering the initial condition)*** *Let Y be a Banach space with the norm $\|.\|_Y$. Consider a sequence $(g_m)_m \in C(0,T;Y)$ such that $\partial_t(g_m)$ is uniformly bounded in $L^q(0,T;Y)$ with $1 < q \leq \infty$, and $(g_m)_{|t=0} \to g_0$ in Y. Then there exists $g \in C(0,T;Y)$ such that $g_m \to g$ in $C(0,T;Y)$ and*

$$g_{|t=0} = g_0 \qquad \text{in } Y.$$

Proof. We have that for all $s < t \in (0,T)$

$$\begin{aligned} \|g_m(t) - g_m(s)\|_Y &= \left\| \int_s^t \partial_\tau(g_m)(\tau) \right\|_Y \\ &\leq \int_s^t \|\partial_\tau(g_m)(\tau)\|_Y \, ds \\ &\leq (t-s)^{\frac{q-1}{q}} \, \|\partial_\tau(g_m)(\tau)\|_{L^q(0,T;Y)} \leq (t-s)^{\frac{q-1}{q}} C, \end{aligned} \tag{2.58}$$

where we have used in the second line Holder's inequality, and the fact that $(g_m)_\tau$ is uniformly bounded in $L^q(0,T;Y)$. Since (2.58) implies the equicontinuity of $(g_m)_m$, by Arzelà-Ascoli theorem, there exists $g \in C(0,T;Y)$ such that $g_m \to g$ in $C(0,T;Y)$. Moreover, Taking $s = 0$ in (2.58) we get

$$\|g_m(t) - g_m(0)\|_Y \leq t^{\frac{q-1}{q}} C. \tag{2.59}$$

By passing to the limit in m in (2.59), we deduce that

$$\|g(t) - g_0\|_Y \leq t^{\frac{q-1}{q}} C$$

Particularly, for $t = 0$, we have

$$\|g(0) - g_0\|_Y = 0.$$

This implies the result. □

Lemma 2.5.2. ***(Convergence result)***
Let $(a_\varepsilon)_\varepsilon$ a real sequence such that $a_\varepsilon \to a_0$ as $\varepsilon \to 0$. Then we have

$$\Psi_{0,\ell}(a_0) \leq \liminf_{\varepsilon \to 0} \Psi_{\varepsilon,\ell}(a_\varepsilon),$$

where $\Psi_{\varepsilon,\ell}$ and $\Psi_{0,\ell}$ are given in (2.28) and (2.35) respectively.

Proof. Consider the case where $a_0 = 0$.
We suppose that the sequence $(a_\varepsilon)_\varepsilon \in (-\infty; \frac{1}{e}]$. Let $(b_\varepsilon)_\varepsilon \in (-\infty; \frac{1}{e}]$ a sequence that decreases to 0 as $\varepsilon \to 0$ with $b_\varepsilon > a_\varepsilon$. Since $\Psi_{\varepsilon,\ell}$ is decreasing on $(-\infty; \frac{1}{e}]$ we have $\Psi_{\varepsilon,\ell}(a_\varepsilon) \geq \Psi_{\varepsilon,\ell}(b_\varepsilon)$. Moreover, using the fact that $\Psi_{\varepsilon,\ell}(b_\varepsilon) \to 0 = \Psi_{0,\ell}(0)$ we get the result.
Otherwise, when $(a_\varepsilon)_\varepsilon \in (\frac{1}{e}; +\infty)$ the proof is the same as above but with taking $b_\varepsilon < a_\varepsilon$ since $\Psi_{\varepsilon,\ell}$ is nondecreasing in $(\frac{1}{e}; +\infty)$.
For the other cases, $a_0 < 0$ and $a_0 > 0$, the result is easily obtained. □

Chapter 3

Existence result for degenerate cross-diffusion system with constraint arising from a seawater intrusion

Abstract: This chapter is the subject of a paper written in collaboration with R. Monneau and M. Jazar, submitted for publication in "Mathematical Models and Methods in Applied Sciences". We consider a degenerate strongly-coupled nonlinear parabolic system with constraint, which arises from seawater intrusion model in confined aquifers. The existence of a nonnegative solution is obtained after establishing a suitable entropy estimate.

3.1 Introduction

3.1.1 Physical motivation and previous work

Seawater intrusion is one of the major concerns commonly found in coastal aquifers. It is the movement of seawater into the freshwater aquifers. In the modelling of such phenomenon, Jazar and Monneau proposed in [43] two reduced models in confined and unconfined aquifers, where the freshwater and the saltwater are assumed to be immiscible and one of the dimension is negligible with respect to the tow others. In this paper, we are concerned with the confined case (see Figure 3.1): we consider $\{z = 0\}$ the interface between the saltwater and the bedrock, $\{z = g(t, x)\}$ the interface between the saltwater and the freshwater and $\{z = h(t, x) + g(t, x) = f(x)\}$ the interface between the freshwater and the impermeable layer, where $f(x)$ is a given function.

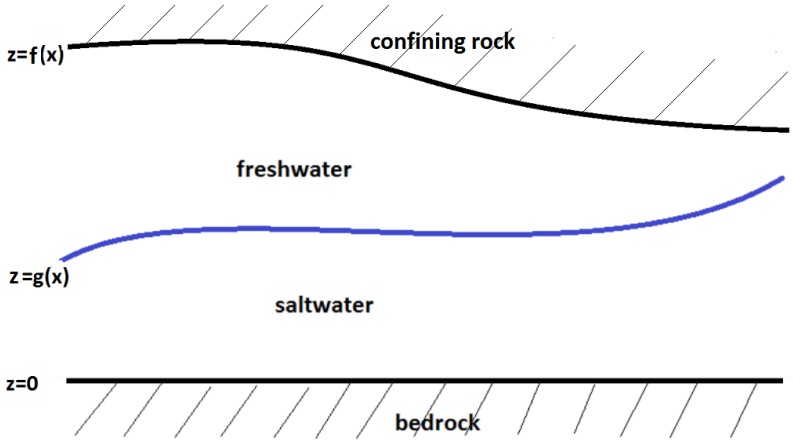


Figure 3.1 – Seawater intrusion in confined aquifer

Then the confined model reads

$$\begin{cases} \partial_t h &= \operatorname{div}\{h\nabla(p+\nu(h+g))\} & \text{in } [0,\infty)\times\mathbb{R}^N, \\ \partial_t g &= \operatorname{div}\{g\nabla(p+\nu h+g)\} & \text{in } [0,\infty)\times\mathbb{R}^N, \\ h+g &= f(x) & \text{in } [0,\infty)\times\mathbb{R}^N, \end{cases} \tag{3.1}$$

where $N = 2, 3$, p is the pressure on the top confining rock and $\nu = 1 - \varepsilon_0 \in (0, 1)$ with

$$\varepsilon_0 = \frac{\gamma_s - \gamma_f}{\gamma_s}$$

and γ_s and γ_f are the specific weight of the saltwater and freshwater respectively.

In this paper, we show existence result for a more generalized model of the form

$$\begin{cases} \partial_t u^i = \operatorname{div}\left\{u^i\nabla\left(p+\sum_{j=1}^m A_{ij}u^j\right)\right\} & \text{in } \Omega_T, \quad \text{for } i=1,\dots,m, \\ \sum_{i=1}^m u^i(t,x) = f(x) & \text{in } \Omega_T, \\ u^i(t,x) \geq 0 & \text{for a.e. } (t,x) \text{ in } \Omega_T, \end{cases} \tag{3.2}$$

where $\Omega_T := (0,T)\times\Omega$ with $T>0$ and $\Omega := \mathbb{T}^N = (\mathbb{R}/\mathbb{Z})^N$, with $N\geq 1$ and the initial condition

$$u^i(0,x) = u_0^i(x) \geq 0 \qquad \text{a.e. in } \Omega, \quad \text{for } i=1,\dots,m, \tag{3.3}$$

and

$$p(0,x) = p_0(x) \qquad \text{in } \Omega. \tag{3.4}$$

Here, p appears as a Lagrange multiplier of the constraint on $u=(u^i)_{1\leq i\leq m}$, given by the second line of (3.2).

The existence of a solution for (3.2) without the constraint and with $p=0$ is studied by many researchers with different settings, assumptions and particular cases [75, 74, 53, 47, 18, 17, 22]. The most relevant among them to the present paper is [3]. Let us mention that a different kind of seawater intrusion model in confined aquifers, which consists in a coupled system of an elliptic and a degenerate parabolic equation, has been studied in [61, 71, 23].

3.1.2 Main result

To introduce our main result, some definitions and assumptions are given.

The space $\mathbf{H}^1(\Omega)/\mathbb{R}$:
We define $H^1(\Omega)/\mathbb{R}$ as the space of functions of $H^1(\Omega)$, up to addition of constants. A natural norm is

$$\|p\|_{(H^1(\Omega)/\mathbb{R})} = \inf_{c\in\mathbb{R}} \|p-c\|_{H^1(\Omega)} = \left\| p - \frac{1}{|\Omega|}\int_\Omega p \right\|_{H^1(\Omega)}. \tag{3.5}$$

The function Ψ:
We define the nonnegative function Ψ as

$$\Psi(a) - \frac{1}{e} = \begin{cases} a\ln a & \text{for} \quad a>0, \\ 0 & \text{for} \quad a=0, \\ +\infty & \text{for} \quad a<0, \end{cases} \tag{3.6}$$

which is minimal for $a=\dfrac{1}{e}$.

The positivity condition:
The real $m\times m$ matrix $A=(A_{ij})_{1\leq i,j\leq m}$ is not necessarily symmetric and satisfies the following positivity condition: there exists $\delta_0>0$, such that we have

$$\xi^T A\,\xi \geq \delta_0|\xi|^2, \quad \text{for all } \xi\in\mathbb{R}^m. \tag{3.7}$$

This condition, as in [3], can be weaken: there exist two positive definite diagonal $m\times m$ matrices L and R and $\delta_0>0$, such that we have

$$\zeta^T LAR\,\zeta \geq \delta_0|\zeta|^2, \quad \text{for all} \quad \zeta\in\mathbb{R}^m. \tag{3.8}$$

In the core of this paper we will assume (3.7) for the sake of simplicity.

Now we state our main result.

Theorem 3.1.1. ***(Existence for (3.2))***
Assume that A satisfies (3.7) and that there exist $0<\ell_1<\ell_2<\infty$ such that

$$\ell_1 \leq f(x) \leq \ell_2, \quad \textit{for all } x\in\Omega \textit{ and } f\in H^1(\Omega). \tag{3.9}$$

For $i = 1, \dots, m$, let $u_0^i \geq 0$ in Ω satisfying

$$\sum_{i=1}^{m} \int_\Omega \Psi(u_0^i) < +\infty, \tag{3.10}$$

where Ψ is given in (3.6). Then there exists a function $u = (u^i)_{1\leq i\leq m} \in L^\infty(\Omega_T) \cap (L^2(0,T;H^1(\Omega)) \cap C([0,T);(W^{1,\infty}(\Omega))'))^m$, and a function $p \in L^2(0,T;H^1(\Omega)/\mathbb{R})$ such that (u,p) is a solution in the sense of distributions of (3.2)-(3.4), with $u^i \geq 0$ a.e. in Ω_T, for $i = 1, \dots, m$. Moreover, u satisfies the following entropy estimate for a.e. $t_1, t_2 \in (0,T)$, with $u^i(t_2) = u^i(t_2, \cdot)$

$$\sum_{i=1}^{m} \int_\Omega \Psi(u^i(t_2)) + \frac{\delta_0}{2} \sum_{i=1}^{m} \left\|\nabla u^i\right\|^2_{L^2(t_1,t_2;L^2(\Omega))} \leq \sum_{i=1}^{m} \int_\Omega \Psi(u_0^i) + C_0 \left\|\nabla f\right\|^2_{L^2(\Omega)}, \tag{3.11}$$

where $C_0 = \dfrac{m\,\ell_2^2 \left\|A\right\|^2}{2\,\ell_1^2 \delta_0}$ and p satisfies

$$\left\|\nabla p\right\|^2_{L^2(\Omega)} \leq \frac{\ell_2^2 \left\|A\right\|^2 m}{\ell_1} \left\|\nabla u\right\|^2_{(L^2(\Omega))^m}. \tag{3.12}$$

Together (3.11) and (3.12) yield to

$$\left\|\nabla p\right\|^2_{L^2(t_1,t_2;L^2(\Omega))} \leq C(\delta_0, \ell_1, \ell_2, m, \left\|A\right\|, \left\|f\right\|_{H^1(\Omega)}).$$

Here $\|A\|$ is the matrix norm defined as

$$\left\|A\right\| = \sup_{|\xi|=1} \left|A\xi\right|. \tag{3.13}$$

Notice that (3.11) and (3.12) allow us to define the products $u^i \sum_{i=1}^{m} A_{ij} \nabla u^j$ and $u^i \nabla p$ in (3.2).

Remark 3.1.2. *(Decreasing energy)*
If A is a symmetric matrix then a solution (u,p) of system (3.2) satisfies

$$\frac{d}{dt}\left(\sum_{i=1}^{m}\sum_{j=1}^{m} \int_\Omega \frac{1}{2} A_{ij} u^i u^j\right) = -\int_\Omega \left(\sum_{i=1}^{m} u^i \left|q^i\right|^2 - \left|\sum_{i=1}^{m} u^i q^i\right|^2\right) \leq 0,$$

where $q^i = \sum_{j=1}^{m} A_{ij} \nabla u^j$.

3.1.3 Encountered difficulties and strategy of the proof

This system is difficult to handle: we don't have a maximum principle, nor an entropy estimate and it is a strongly coupled degenerate system.
To this end, we proceed by approximations. The approximate system is then non degenerate, linear elliptic. We then pass to the limit in a way to preserve the obtained entropy.
We discretize in time our problem, we regularize the right hand side of (3.2), we add a term

like $\delta\Delta u$ and finally we truncate and freeze the coefficients. Therefore, an elliptic linear approximate system is obtained of the form:

$$\begin{cases} \dfrac{u^{i,n+1}-u^{i,n}}{\Delta t} &= \operatorname{div}\left\{F^i_{\varepsilon,\eta,\delta}(v^{n+1},u^{n+1},p^{n+1})\right\} \quad \text{in} \quad \mathcal{D}'(\Omega), \\ \displaystyle\sum_{j=1}^{m} u^{i,n+1}(x) &= f(x) \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega, \end{cases} \tag{3.14}$$

where

$$F^i_{\varepsilon,\eta,\delta}(v^{n+1},u^{n+1},p^{n+1}) = T^{\varepsilon,\ell_2}(v^{i,n+1})\left(\nabla p^{n+1} + \sum_{j=1}^{m} A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^{j,n+1} + \delta\nabla u^{i,n+1}\right), \tag{3.15}$$

$\Delta t = T/K$ is the time step with $K \in \mathbb{N}^*$, η, $\delta > 0$, $0 < \varepsilon < 1$, T^{ε,ℓ_2} is the truncation operator defined as

$$T^{\varepsilon,\ell_2}(a) := \begin{cases} \varepsilon & \text{if } a \le \varepsilon, \\ a & \text{if } \varepsilon \le a \le \ell_2, \\ \ell_2 & \text{if } a \ge \ell_2, \end{cases} \tag{3.16}$$

and the mollifier $\rho_\eta(x) = \eta^{-N}\rho(x/\eta)$ with $\rho \in C_c^\infty(\mathbb{R}^N)$, $\rho \ge 0$, $\int_{\mathbb{R}^N}\rho = 1$, $\int_{\mathbb{R}^N}\nabla\rho = 1$ and $\rho(-x) = \rho(x)$. Note that we consider the $\mathbb{Z}^N$- periodic extension on $\mathbb{R}^N$ of $u^{j,n+1}$. We will look for fixed points solutions $v^{i,n+1} = u^{i,n+1}$ of (3.14). Finally, we will recover the existence result by passing to the limit in all parameters.

The organization of the paper is as follow: In Section 3.2 we present the proof of Theorem 3.1.1. We prove the existence via Lax-Milgram theorem of a solution for (3.14) in Subsection 3.2.1. Then, in Subsection 3.2.2 we apply a fixed point theorem to get the existence of a solution for the nonlinear problem and we establish an entropy estimate. In Subsection 3.2.3, we pass to the limit as $(\Delta t, \varepsilon) \to (0,0)$ and in Subsection 3.2.4 we end the proof by passing to the limit as $\eta \to 0$ then $\delta \to 0$.

3.2 Proof of Main result

3.2.1 Existence for the linear elliptic problem (3.14)

Proposition 3.2.1. ***(Existence for system (3.14))***
Assume that A is any $m \times m$ real matrix. Let Δt, ε, η, $\delta > 0$, with $0 < \varepsilon < 1$ such that

$$\Delta t < \frac{\delta\varepsilon\eta^2}{2\ell_2^2\|A\|^2} := \tau. \tag{3.17}$$

Then for $n \in \mathbb{N}$, for a given $v^{n+1} = (v^{i,n+1})_{1\le i\le m} \in (L^2(\Omega))^m$ and $u^n = (u^{i,n})_{1\le i\le m} \in (H^1(\Omega))^m$, there exists a unique function $(u^{n+1},p^{n+1}) = ((u^{i,n+1})_{1\le i\le m},p^{n+1}) \in (H^1(\Omega))^m \times \left(H^1(\Omega)/\mathbb{R}\right)$ solution of system (3.14), in the sense of distributions. Moreover, this solution (u^{n+1},p^{n+1}) satisfies the following estimate

$$\begin{aligned} 2\left(\delta - \frac{\delta\Delta t}{\tau}\right)\left\|u^{n+1}\right\|^2_{(L^2(\Omega))^m} &+ \varepsilon\delta^2\Delta t\left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m} + m\varepsilon\Delta t\left\|\nabla p\right\|^2_{L^2(\Omega)} \\ &\le 4\delta\left\|u^n\right\|^2_{(L^2(\Omega))^m} + C(\ell_2,m,\varepsilon,\tau,\delta,\|A\|)\left\|f\right\|^2_{H^1(\Omega)} \end{aligned} \tag{3.18}$$

Proof of Proposition 3.2.1.
In order to prove the existence of a solution (u^{n+1}, p^{n+1}) of (3.14), it is sufficient to prove the existence of a solution $(\bar{u}^{n+1}, p^{n+1})$ of the following system

$$\left\{\begin{array}{rcll} \dfrac{\left(\bar{u}^{i,n+1}+\dfrac{f}{m}\right)-u^{i,n}}{\Delta t} & = & \mathrm{div}\left\{F^i_{\varepsilon,\eta,\delta}(v^{n+1},\left(\bar{u}^{n+1}+\dfrac{\vec{f}}{m}\right),p^{n+1})\right\} & \text{in} \quad \mathcal{D}'(\Omega), \\ \sum\limits_{j=1}^{m}\bar{u}^{i,n+1}(x) & = & 0 & \text{in } \Omega, \end{array}\right. \tag{3.19}$$

where $F^i_{\varepsilon,\eta,\delta}$ is given in (3.15) and $\vec{f}=(f,\cdots,f)$. System (3.19) can be written as

$$\left\{\begin{array}{rcll} \dfrac{\bar{u}^{i,n+1}-u^{i,n}}{\Delta t} & = & \mathrm{div}\left\{F^i_{\varepsilon,\eta,\delta}(v^{n+1},\bar{u}^{n+1},p^{n+1})\right\}-\dfrac{1}{m\Delta t}f & \\ & & +\dfrac{1}{m}\mathrm{div}\left\{T^{\varepsilon,\ell_2}(v^{i,n+1})\left(\sum\limits_{j=1}^{m}A_{ij}\nabla\rho_\eta\star\rho_\eta\star f+\delta\nabla f\right)\right\} & \text{in} \quad \mathcal{D}'(\Omega), \\ \sum\limits_{j=1}^{m}\bar{u}^{i,n+1}(x) & = & 0 & \text{in } \Omega. \end{array}\right. \tag{3.20}$$

To this end, let us define for all $(\bar{u}^{n+1},p^{n+1})=((\bar{u}^{i,n+1})_{1\le i\le m},p^{n+1})$ and $(\varphi,q)=((\varphi^i)_{1\le i\le m},q)\in (H^1(\Omega))^m\times\left(H^1(\Omega)/\mathbb{R}\right)$ such that $\sum_{i=1}^m\varphi^i=\sum_{i=1}^m\bar{u}^{i,n+1}=0$ in Ω, the following bilinear continuous form, obtained after multiplying the first line of system (3.20) by $\delta\varphi^i+q$, integrating over Ω then summing over i:

$$\begin{aligned} b\left((\bar{u}^{n+1},p^{n+1}),(\varphi,q)\right) \;:=\; & \delta\sum_{i=1}^{m}\int_\Omega\bar{u}^{i,n+1}\varphi^i \\ & +\Delta t\sum_{i=1}^{m}\int_\Omega T^{\varepsilon,\ell_2}(v^{i,n+1})\nabla p^{n+1}(\delta\nabla\varphi^i+\nabla q) \\ & +\Delta t\sum_{i=1}^{m}\int_\Omega T^{\varepsilon,\ell_2}(v^{i,n+1})\sum_{j=1}^{m}A_{ij}\nabla\rho_\eta\star\rho_\eta\star\bar{u}^{j,n+1}(\delta\nabla\varphi^i+\nabla q) \\ & +\delta\Delta t\sum_{i=1}^{m}\int_\Omega T^{\varepsilon,\ell_2}(v^{i,n+1})\nabla\bar{u}^{i,n+1}(\delta\nabla\varphi^i+\nabla q), \end{aligned}$$

and the following linear continuous form:

$$\begin{aligned} K(\varphi,q) \;:=\; & \delta\sum_{i=1}^{m}\int_\Omega u^{i,n}\varphi^i-\frac{\delta}{m}\sum_{i=1}^{m}\int_\Omega f\varphi^i \\ & -\frac{\Delta t}{m}\sum_{i=1}^{m}\int_\Omega T^{\varepsilon,\ell_2}(v^{i,n+1})\sum_{j=1}^{m}A_{ij}\nabla\rho_\eta\star\rho_\eta\star f(\delta\nabla\varphi^i+\nabla q) \\ & -\frac{\delta\Delta t}{m}\sum_{i=1}^{m}\int_\Omega T^{\varepsilon,\ell_2}(v^{i,n+1})\nabla f(\delta\nabla\varphi^i+\nabla q) \end{aligned} \tag{3.21}$$

Note that we used the fact that $\sum_{i=1}^{m}\int_\Omega u^{i,n}q = \int_\Omega fq$.

Step 1: Existence by Lax-Milgram
It remains to prove the coercivity of b to get the existence, by Lax-Milgram theorem, of a unique solution for system (3.20).
For all $(\varphi,q)=((\varphi^i)_{1\le i\le m},q)\in (H^1(\Omega))^m\times\left(H^1(\Omega)/\mathbb{R}\right)$ with $\sum_{i=1}^m\varphi^i=0$, we have that

$$\begin{aligned} b((\varphi,q),(\varphi,q)) &= \delta\sum_{i=1}^m\int_\Omega|\varphi^i|^2+\Delta t\sum_{i,j=1}^m\int_\Omega T^{\varepsilon,\ell_2}(v^{i,n+1})(\delta\nabla\varphi^i+\nabla q)^2\\ &\quad+\Delta t\sum_{i=1}^m\int_\Omega T^{\varepsilon,\ell_2}(v^{i,n+1})\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star\varphi^j\nabla q\\ &\quad+\delta\Delta t\sum_{i=1}^m\int_\Omega T^{\varepsilon,\ell_2}(v^{i,n+1})\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star\varphi^j\nabla\varphi^i\\ &= b_0((\varphi,q),(\varphi,q))+b_1((\varphi,q),(\varphi,q)), \end{aligned}$$

where

$$b_0((\varphi,q),(\varphi,q))=\delta\sum_{i=1}^m\int_\Omega|\varphi^i|^2+\Delta t\sum_{i=1}^m\int_\Omega T^{\varepsilon,\ell_2}(v^{i,n+1})(\delta\nabla\varphi^i+\nabla q)^2,$$

and

$$\begin{aligned} b_1((\varphi,q),(\varphi,q)) &= \Delta t\sum_{i=1}^m\int_\Omega T^{\varepsilon,\ell_2}(v^{i,n+1})\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star\varphi^j\nabla q\\ &\quad+\delta\Delta t\sum_{i=1}^m\int_\Omega T^{\varepsilon,\ell_2}(v^{i,n+1})\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star\varphi^j\nabla\varphi^i. \end{aligned}$$

On the one hand, we already have the coercivity of b_0:

$$b_0((\varphi,q),(\varphi,q)) \ge \delta\|\varphi\|^2_{(L^2(\Omega))^m}+m\varepsilon\Delta t\|\nabla q\|^2_{L^2(\Omega)}+\varepsilon\delta^2\Delta t\|\nabla\varphi\|^2_{(L^2(\Omega))^m}.$$

On the other hand, we have

$$\begin{aligned} |b_1(\varphi,\varphi)| &\le \Delta t\ell_2\|A\|\sqrt{m}\|\nabla\rho_\eta\star\rho_\eta\star\varphi\|_{(L^2(\Omega))^m}\|\nabla q\|_{L^2(\Omega)}\\ &\quad+\delta\Delta t\ell_2\|A\|\|\nabla\rho_\eta\star\rho_\eta\star\varphi\|_{(L^2(\Omega))^m}\|\nabla\varphi\|_{(L^2(\Omega))^m}\\ &\le \Delta t\ell_2\|A\|\sqrt{m}\left(\frac{1}{2d}\|\nabla\rho_\eta\star\rho_\eta\star\varphi\|^2_{(L^2(\Omega))^m}+\frac{d}{2}\|\nabla q\|^2_{L^2(\Omega)}\right)\\ &\quad+\delta\Delta t\ell_2\|A\|\left(\frac{1}{2c}\|\nabla\rho_\eta\star\rho_\eta\star\varphi\|^2_{(L^2(\Omega))^m}+\frac{c}{2}\|\nabla\varphi\|^2_{(L^2(\Omega))^m}\right)\\ &\le \frac{\varepsilon\delta^2\Delta t}{2}\|\nabla\varphi\|^2_{(L^2(\Omega))^m}+\frac{m\varepsilon\Delta t}{2}\|\nabla q\|^2_{L^2(\Omega)}+\frac{\Delta t\ell_2^2\|A\|^2}{\varepsilon\eta^2}\|\varphi\|^2_{(L^2(\Omega))^m}, \end{aligned}$$

where in the second line we have used Young's inequality, and chosen $c=\dfrac{\varepsilon\delta}{\ell_2\|A\|}$ and $d=\dfrac{\varepsilon\sqrt{m}}{\ell_2\|A\|}$ in the third line, with $\|A\|$ is given in (3.13).

So we get that

$$b((\varphi,q),(\varphi,q)) \geq \left(\delta - \frac{\delta\Delta t}{2\tau}\right)\|\varphi\|^2_{(L^2(\Omega))^m} + \frac{\varepsilon\delta^2\Delta t}{2}\|\nabla\varphi\|^2_{(L^2(\Omega))^m} + \frac{m\varepsilon\Delta t}{2}\|\nabla q\|^2_{L^2(\Omega)}, \quad (3.22)$$

is coercive, since $\Delta t < \tau$.

Step 2: Proof of estimate (3.18)
Using (3.22) and the fact that $b((\bar{u}^{n+1},p),(\bar{u}^{n+1},p)) = K((\bar{u}^{n+1},p))$ together with Young's inequality we get

$$\begin{aligned}
&\left(\delta - \frac{\delta\Delta t}{2\tau}\right)\left\|\bar{u}^{n+1}\right\|^2_{(L^2(\Omega))^m} + \frac{\varepsilon\delta^2\Delta t}{2}\left\|\nabla\bar{u}^{n+1}\right\|^2_{(L^2(\Omega))^m} + \frac{m\varepsilon\Delta t}{2}\left\|\nabla p^{n+1}\right\|^2_{L^2(\Omega)}\\
\leq\ & \delta\|u^n\|^2_{(L^2(\Omega))^m} + \frac{\delta}{2}\left\|\bar{u}^{n+1}\right\|^2_{(L^2(\Omega))^m} + \frac{\varepsilon\delta^2\Delta t}{4}\left\|\nabla\bar{u}^{n+1}\right\|^2_{(L^2(\Omega))^m} + \frac{m\varepsilon\Delta t}{4}\left\|\nabla p^{n+1}\right\|^2_{L^2(\Omega)}\\
&+C(\ell_2,m,\varepsilon,\tau,\delta,\|A\|)\|f\|^2_{H^1(\Omega)}.
\end{aligned}$$

Now replacing $\bar{u}^{i,n+1}$ by $u^{i,n+1} - \frac{1}{m}f$ we get estimate (3.18). □

3.2.2 Existence for the nonlinear time-discrete problem

In this subsection we prove the existence, using Schaefer's fixed point theorem, of a solution for the nonlinear time discrete confined system (3.25) given below.
For this purpose, let the function

$$\Psi_{\varepsilon,\ell_2}(a) - \frac{1}{e} = \begin{cases} \frac{a^2}{2\varepsilon} + a\ln\varepsilon - \frac{\varepsilon}{2} & \text{if } a \leq \varepsilon,\\ a\ln a & \text{if } \varepsilon < a \leq \ell_2,\\ \frac{a^2}{2\ell_2} + a\ln\ell_2 - \frac{\ell_2}{2} & \text{if } a \geq \ell_2, \end{cases} \quad (3.23)$$

which is continuous, convex and satisfies that $\Psi''_{\varepsilon,\ell_2}(x) = \dfrac{1}{T^{\varepsilon,\ell_2}(x)}$, where T^{ε,ℓ_2} is given in (3.16). Let us introduce our nonlinear time discrete system: Assume that A satisfies (3.7). Let $u^0 = (u^{i,0})_{1\leq i\leq m} := u_0 = (u_0^i)_{1\leq i\leq m}$ that satisfies

$$C_1 := \sum_{i=1}^m \int_\Omega \Psi_{\varepsilon,\ell_2}(u_0^i) < +\infty, \quad (3.24)$$

such that $u_0^i \geq 0$ in Ω for $i = 1,\ldots,m$. Then for all Δt, ε, η, $\delta > 0$, with $0 < \varepsilon < 1$ and $\Delta t < \tau$, for $n \in \mathbb{N}$, we look for a solution $(u^{n+1},p^{n+1}) = ((u^{i,n+1})_{1\leq i\leq m},p^{n+1})$ of the following nonlinear system:

$$\begin{cases} \dfrac{u^{i,n+1}-u^{i,n}}{\Delta t} = \operatorname{div}\left\{F^i_{\varepsilon,\eta,\delta}((u^{n+1},u^{n+1}),p^{n+1})\right\} & \text{in } \mathcal{D}'(\Omega),\\ \displaystyle\sum_{i=1}^m u^{i,n+1}(x) = f(x) & \text{in } \Omega, \end{cases} \quad (3.25)$$

where f satisfies (3.9).

Proposition 3.2.2. ***(Existence for system (3.25))***
There exists a sequence of functions $u^{n+1} = (u^{i,n+1})_{1\leq i\leq m} \in \left(H^1(\Omega)\right)^m$ *and a function* $p^{n+1} \in H^1(\Omega)/\mathbb{R}$ *such that* (u^{n+1}, p^{n+1}) *is a solution of system (3.25), that satisfies the entropy estimate*

$$\sum_{i=1}^{m}\int_\Omega \Psi_{\varepsilon,\ell_2}(u^{i,n+1}) \quad + \quad \frac{\delta}{2}\Delta t\sum_{i=1}^{m}\sum_{k=0}^{n}\left\|\nabla u^{i,k+1}\right\|^2_{L^2(\Omega)} + \frac{\delta_0}{2}\Delta t\sum_{i=1}^{m}\sum_{k=0}^{n}\left\|\nabla \rho_\eta \star u^{i,k+1}\right\|^2_{L^2(\Omega)}$$
$$\leq C_\delta \left\|\nabla f\right\|^2_{L^2(\Omega)} + \sum_{i=1}^{m}\int_\Omega \Psi_{\varepsilon,\ell_2}(u_0^i), \tag{3.26}$$

with

$$C_\delta = \frac{m\,\ell_2^2}{\ell_1}\left(\frac{\|A\|^2}{\delta_0} + \delta\right). \tag{3.27}$$

Moreover, we have

$$\sum_{k=0}^{n}\left\|\nabla p^{k+1}\right\|^2_{L^2(\Omega)} \leq 2\frac{m\,\ell_2^2\,\|A\|^2}{\ell_1}\sum_{k=0}^{n}\left\|\nabla \rho_\eta \star u^{k+1}\right\|^2_{(L^2(\Omega))^m} + 2\frac{\delta^2\ell_2^2 m}{\ell_1}\sum_{k=0}^{n}\left\|\nabla u^{k+1}\right\|^2_{(L^2(\Omega))^m}. \tag{3.28}$$

More precisely, together (3.28) and (3.26) imply that

$$\Delta t\sum_{k=0}^{n}\left\|\nabla p^{k+1}\right\|^2_{L^2(\Omega)} \leq C(\delta, \delta_0, \ell_1, \ell_2, \|A\|, \|f\|_{H^1(\Omega)}). \tag{3.29}$$

Proof of Proposition 3.2.2

Step 1: Existence of a solution for (3.25)
We define, for a given $w := u^n = (u^{i,n})_{1\leq i\leq m} \in (L^2(\Omega))^m$ and $v := v^{n+1} = (v^{i,n+1})_{1\leq i\leq m} \in (L^2(\Omega))^m$, the map θ as:

$$\begin{array}{rccc} \theta & : & (L^2(\Omega))^m & \to & (L^2(\Omega))^m \\ & & v & \mapsto & u \end{array}$$

where $u := u^{n+1} = (u^{i,n+1})_{1\leq i\leq m} = \theta(v^{n+1}) \in (H^1(\Omega))^m$ is such that (u, p^{n+1}) is the unique solution of system (3.14), given by Proposition 3.2.1. Moreover, we can prove that θ is continuous using the fact that $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact and that we have (3.18). Also, θ is a compact mapping and the set $\{u \in X, \quad u = \lambda\Phi(u) \quad \text{for some} \quad \lambda \in [0,1]\}$ is bounded. Then θ has a fixed point u^{n+1} on $(L^2(\Omega))^m$ by the Schaefer's fixed point theorem, Theorem 2.2.1. This implies the existence of a solution (u^{n+1}, p^{n+1}) for system (3.25).

Step 2: Proof of estimate (3.26)
Since $\Psi_{\varepsilon,\ell_2}$ is convex we have

$$\begin{aligned}
&\sum_{i=1}^{m}\int_\Omega \frac{\Psi_{\varepsilon,\ell_2}(u^{i,n+1})-\Psi_{\varepsilon,\ell_2}(u^{i,n})}{\Delta t} \leq \sum_{i=1}^{m}\int_\Omega \left(\frac{u^{i,n+1}-u^{i,n}}{\Delta t}\right)\Psi'_{\varepsilon,\ell_2}(u^{i,n+1})\\
=\ & -\sum_{i=1}^{m}\int_\Omega \Big(T^{\varepsilon,\ell_2}(u^{i,n+1})\nabla p^{n+1}+\delta T^{\varepsilon,\ell_2}(u^{i,n+1})\nabla u^{i,n+1}\\
&+T^{\varepsilon,\ell_2}(u^{i,n+1})\sum_{j=1}^{m}A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1}\Big)\Psi''_{\varepsilon,\ell_2}(u^{i,n+1})\nabla u^{i,n+1}\\
=\ & -\sum_{i=1}^{m}\left\{\delta\int_\Omega|\nabla u^{i,n+1}|^2+\int_\Omega\sum_{j=1}^{m}\nabla\rho_\eta\star u^{i,n+1}A_{ij}\nabla\rho_\eta\star u^{j,n+1}\right\}-\int_\Omega\nabla p^{n+1}\cdot\nabla f\\
\leq\ & -\sum_{i=1}^{m}\delta\int_\Omega|\nabla u^{i,n+1}|^2-\delta_0\sum_{i=1}^{m}\int_\Omega|\nabla\rho_\eta\star u^{i,n+1}|^2+\|\nabla f\|_{L^2(\Omega)}\left\|\nabla p^{n+1}\right\|_{L^2(\Omega)}. \quad (3.30)
\end{aligned}$$

In order to estimate the last term in (3.30) we take the sum on i of system (3.25) then we multiply by p^{n+1} and we integrate by parts,

$$\begin{aligned}
0=\sum_{i=1}^{m}\int_\Omega T^{\varepsilon,\ell_2}(u^{i,n+1})\left|\nabla p^{n+1}\right|^2 &+ \sum_{i=1}^{m}\int_\Omega T^{\varepsilon,\ell_2}(u^{i,n+1})\sum_{j=1}^{m}A_{ij}\left(\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1}\right)\cdot\nabla p^{n+1}\\
&+ \delta\sum_{i=1}^{m}\int_\Omega T^{\varepsilon,\ell_2}(u^{i,n+1})\nabla u^{i,n+1}\cdot\nabla p^{n+1}.
\end{aligned}$$

Therefore, using the fact that $\sum_{i=1}^{m}T^{\varepsilon,\ell_2}(u^{i,n+1})\geq\ell_1$ we get

$$\begin{aligned}
\ell_1\left\|\nabla p^{n+1}\right\|^2_{L^2(\Omega)} &\leq \ell_2\|A\|\sqrt{m}\left\|\nabla\rho_\eta\star\rho_\eta\star u^{n+1}\right\|_{(L^2(\Omega))^m}\left\|\nabla p^{n+1}\right\|_{L^2(\Omega)}\\
&\quad+\ell_2\,\delta\sqrt{m}\left\|\nabla u^{n+1}\right\|_{(L^2(\Omega))^m}\left\|\nabla p^{n+1}\right\|_{L^2(\Omega)}\\
&\leq \ell_2\|A\|\sqrt{m}\left(\frac{1}{2r}\left\|\nabla\rho_\eta\star u^{n+1}\right\|^2_{(L^2(\Omega))^m}+\frac{r}{2}\left\|\nabla p^{n+1}\right\|^2_{L^2(\Omega)}\right)\\
&\quad+\ell_2\,\delta\sqrt{m}\left(\frac{1}{2s}\left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m}+\frac{s}{2}\left\|\nabla p^{n+1}\right\|^2_{L^2(\Omega)}\right)\\
&\leq \frac{\ell_1}{2}\left\|\nabla p^{n+1}\right\|^2_{L^2(\Omega)}+\frac{\ell_2^2\|A\|^2m}{\ell_1}\left\|\nabla\rho_\eta\star u^{n+1}\right\|^2_{(L^2(\Omega))^m}\\
&\quad+\frac{\delta^2\ell_2^2m}{\ell_1}\left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m},
\end{aligned}$$

where $r=\dfrac{\ell_1}{2\ell_2\|A\|\sqrt{m}}$ and $s=\dfrac{\ell_1}{2\,\delta\,\ell_2\sqrt{m}}$, which yeilds to

$$\left\|\nabla p^{n+1}\right\|^2_{L^2(\Omega)}\leq\frac{2m\,\ell_2^2\|A\|^2}{\ell_1}\left\|\nabla\rho_\eta\star u^{n+1}\right\|^2_{(L^2(\Omega))^m}+\frac{2\delta^2\ell_2^2m}{\ell_1}\left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m}, \quad (3.31)$$

after a reccurence this yields to (3.28).

Now, using (3.31) then applying Young's inequality to the last term in (3.25) we get

$$\begin{aligned}
\|\nabla f\|_{L^2(\Omega)} \left\|\nabla p^{n+1}\right\|_{L^2(\Omega)} &\leq \frac{\sqrt{2m}\,\ell_2\,\|A\|}{\sqrt{\ell_1}} \|\nabla f\|_{L^2(\Omega)} \left\|\nabla \rho_\eta \star u^{n+1}\right\|_{(L^2(\Omega))^m} \\
&\quad + \frac{\delta \ell_2 \sqrt{2m}}{\sqrt{\ell_1}} \|\nabla f\|_{L^2(\Omega)} \left\|\nabla u^{n+1}\right\|_{(L^2(\Omega))^m} \\
&\leq \frac{\sqrt{2m}\,\ell_2\,\|A\|}{\sqrt{\ell_1}} \left(\frac{1}{2d} \|\nabla f\|^2_{L^2(\Omega)} + \frac{d}{2} \left\|\nabla \rho_\eta \star u^{n+1}\right\|^2_{(L^2(\Omega))^m} \right) \\
&\quad + \frac{\delta \ell_2 \sqrt{2m}}{\sqrt{\ell_1}} \left(\frac{1}{2c} \|\nabla f\|^2_{L^2(\Omega)} + \frac{c}{2} \left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m} \right) \\
&\leq \frac{\delta_0}{2} \left\|\nabla \rho_\eta \star u^{n+1}\right\|^2_{(L^2(\Omega))^m} + \frac{\delta}{2} \left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m} \\
&\quad + \frac{m\,\ell_2^2}{\ell_1} \left(\frac{\|A\|^2}{\delta_0} + \delta \right) \|\nabla f\|^2_{L^2(\Omega)} \qquad (3.32)
\end{aligned}$$

where $d = \dfrac{\delta_0 \sqrt{\ell_1}}{\sqrt{2m}\ell_2 \|A\|}$ and $c = \dfrac{\sqrt{\ell_1}}{\ell_2 \sqrt{2m}}$. Substituting (3.32) in (3.30) together with a reccurence yeild to (3.26). □

3.2.3 Passage to the limit as $(\Delta t, \varepsilon) \to (0, 0)$

Let us define the function Ψ_{0,ℓ_2} as

$$\Psi_{0,\ell_2}(a) - \frac{1}{e} := \begin{cases} +\infty & \text{if } a < 0, \\ 0 & \text{if } a = 0, \\ a \ln a & \text{if } 0 < a \leq \ell_2, \\ \frac{a^2}{2\ell_2} + a \ln \ell_2 - \frac{\ell_2}{2} & \text{if } a \geq \ell_2. \end{cases} \qquad (3.33)$$

Now let us introduce our continuous approximate system. Assume that A satisfies (3.7). Let $u_0 = (u_0^i)_{1 \leq i \leq m}$ satisfying

$$C_2 := \sum_{i=1}^{m} \int_\Omega \Psi_{0,\ell_2}(u_0^i) < +\infty, \qquad (3.34)$$

which implies that $u_0^i \geq 0$ a.e. in Ω for $i = 1, \ldots, m$. Then for all η, $\delta > 0$ we look for a solution $(u, p) = ((u^i)_{1 \leq i \leq m}, p)$ of the following system:

$$\begin{cases} \partial_t u^i &= \operatorname{div}\left\{ u^i \nabla \left(p + \sum_{j=1}^{m} A_{ij} \nabla \rho_\eta \star \rho_\eta \star u^j + \delta \nabla u^i \right) \right\} & \text{in} \quad \mathcal{D}'(\Omega_T), \\ \sum_{i=1}^{m} u^i(t,x) &= f(x) & \text{in} \quad \Omega_T, \\ u^i(t,x) &\geq 0 & \text{for a.e. } (t,x) \text{ in } \Omega_T. \end{cases} \qquad (3.35)$$

Proposition 3.2.3. ***(Existence for system (3.35))***
Assume that A satisfies (3.7). Let $u_0 = (u_0^i)_{1\le i\le m}$ satisfying (3.34). Then for all η, $\delta > 0$ there exists a function $u = (u^i)_{1\le i\le m} \in L^\infty(\Omega_T)\cap(L^2(0,T;H^1(\Omega))\cap C([0,T);L^2(\Omega)))^m$ and a function $p \in L^2(0,T;H^1(\Omega)/\mathbb{R})$ such that (u,p) is a solution in the sense of distributions of (3.35), with $u^i \ge 0$ a.e. in Ω_T, for $i = 1,\dots,m$. Moreover, u satisfies the following entropy estimate for a.e. $t_1, t_2 \in (0,T)$ with $u^i(t_2) = u^i(t_2,\cdot)$

$$\int_\Omega \sum_{i=1}^m \Psi_{0,\ell}(u^i(t_2)) + \frac{\delta}{2}\int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^m \left|\nabla u^i\right|^2 + \frac{\delta_0}{2}\int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^m \left|\nabla \rho_\eta \star u^i\right|^2 \le C_\delta \left\|\nabla f\right\|^2 + \int_\Omega \sum_{i=1}^m \Psi_{0,\ell}(u_0^i), \tag{3.36}$$

where C_δ is given in (3.27) and p satisfies

$$\left\|\nabla p\right\|^2_{L^2(\Omega)} \le \frac{\ell_2^2 m(\left\|A\right\| + \delta)^2}{\ell_1}\left\|\nabla u\right\|^2_{(L^2(\Omega))^m}. \tag{3.37}$$

Together (3.36) and (3.37) yield to

$$\left\|\nabla p\right\|_{L^2(t_1,t_2;L^2(\Omega))} \le C(\delta,\ell_1,\ell_2,\left\|A\right\|,m,\left\|f\right\|_{H^1(\Omega)}). \tag{3.38}$$

Proof of Proposition 3.2.3.

Step 1: Passage to the limit when $(\Delta t,\varepsilon)\to(0,0)$
For all $n \in \{0,\dots,K-1\}$ set $t_n = n\Delta t$ and let the piecewise constant in time functions:

$$u^{\Delta t}(t,x) = (u^{i,\Delta t}(t,x))_{1\le i\le m} := (u^{i,n+1}(x))_{1\le i\le m}, \quad \text{for } t \in (t_n,t_{n+1}]. \tag{3.39}$$

$$p^{\Delta t}(t,x) := p^{n+1}(x), \quad \text{for } t \in (t_n,t_{n+1}]. \tag{3.40}$$

Using (3.26) we obtain that $u^{i,\Delta t} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ for $i = 1,\cdots,m$. Together with (3.26) we can find a constant C independent on ε and Δt such that

$$\left\|u^{i,\Delta t}\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|u^{i,\Delta t}\right\|_{L^2(0,T;H^1(\Omega))} + \left\|u^{i,\Delta t}\right\|_{\mathrm{Var}([0,T);H^{-1}(\Omega))} \le C,$$

where the boundedness on $\left\|u^{i,\Delta t}\right\|_{\mathrm{Var}([0,T);H^{-1}(\Omega))} := \sum_{n=0}^{K-1}\left\|u^{i,\Delta t}(t_{n+1}) - u^{i,\Delta t}(t_n)\right\|_{H^{-1}(\Omega)}$ can be obtained by substituting $u^{i,\Delta t}(t_{n+1}) - u^{i,\Delta t}(t_n)$ by the right hand side of the first line of (3.25) multiplied by Δt and then using (3.26) and (3.29). Therefore, Lemma 2.2.4 implies that there exists a function $u = (u^i)_{1\le i\le m} \in \left(L^2(0,T;H^1(\Omega))\right)^m$ such that $u^{\Delta t}\to u$ strongly in $L^2(0,T;L^2(\Omega))$ as $(\Delta t,\varepsilon)\to(0,0)$. Moreover, (3.26) together with (3.29) and Poincarré-Wirtinger's inequality imply that there exists a function $p \in L^2(0,T;H^1(\Omega)/\mathbb{R})$ such that $p^{\Delta t}\to p$ weakly in $(L^2(0,T;H^1(\Omega)/\mathbb{R}))^m$ as $(\Delta t,\varepsilon)\to(0,0)$. Passing to the limit as $(\Delta t,\varepsilon)\to(0,0)$ in (3.25), by using in particular the weak L^2 - strong L^2 convergence in the products, we obtain that (u,p) is a solution of the following system

$$\begin{cases} \partial_t u^i & = & \mathrm{div}\left\{F^i_{0,\eta,\delta}(u,p)\right\} & \text{in } \mathcal{D}'(\Omega_T), \\ \displaystyle\sum_{i=1}^m u^i(t,x) & = & f(x) & \text{in } \Omega_T, \end{cases} \tag{3.41}$$

where

$$F^i_{0,\eta,\delta}(u,p) = T^{0,\ell_2}(u^i)\left(\nabla p + \sum_{j=1}^{m} A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^j + \delta\nabla u^i\right).$$

Step 2: Non-negativity of u

Let $\Omega^\varepsilon_i(t) := \left\{x \in \Omega : u^{i,\Delta t}(t,x) \le \varepsilon\right\}$ for all $t \in [0,T]$ and $i = 1, \cdots, m$. By estimate (3.26), there exists a positive constant C independent of ε and Δt such that for all $i = 1, \ldots, m$ we have

$$\begin{aligned} C &\ge \int_{\Omega^\varepsilon_i(t)} \Psi_{\varepsilon,\ell_2}(u^{i,\Delta t}) \\ &= \int_{\Omega^\varepsilon_i(t)} \frac{1}{e} + \frac{(u^{i,\Delta t})^2}{2\varepsilon} + u^{i,\Delta t}\ln\varepsilon - \frac{\varepsilon}{2} \\ &\ge \int_{\Omega^\varepsilon_i(t)} \frac{1}{e} + \frac{(u^{i,\Delta t})^2}{2\varepsilon} + \varepsilon\ln\varepsilon - \frac{1}{2} \\ &\ge \int_{\Omega^\varepsilon_i(t)} \frac{(u^{i,\Delta t})^2}{2\varepsilon} - \frac{1}{2}, \end{aligned}$$

i.e.

$$\int_{\Omega^\varepsilon_i(t)} \frac{(u^{i,\Delta t})^2}{2\varepsilon} \le C + \frac{1}{2}. \tag{3.42}$$

Now by passing to the limit as $(\Delta t, \varepsilon) \to (0,0)$ in (3.42) we deduce that $\int_{\Omega^-_i(t)} \left|u^i\right|^2 = 0$, where $\Omega^-_i(t) := \left\{x \in \Omega : u^i(t,x) \le 0\right\}$, which gives us that $(u^i)^- = 0$ in $L^2(\Omega)$, where $(u^i)^- = \min(0, u^i)$.

Note that the non-negativity of u^i, i.e. $0 \le u^i \le \ell_2$, implies that $T^{0,\ell_2}(u^i) = u^i$ for $i = 1, \cdots m$.

Step 3: Recovering the initial condition

Let $\bar\rho \in C^\infty_c(\mathbb{R})$ with $\bar\rho \ge 0$, $\int_{\mathbb{R}} \bar\rho = 1$ and supp $\bar\rho \subset (-\frac{1}{2}, \frac{1}{2})$. We set

$$\bar\rho_{\Delta t}(t) = \Delta t^{-1}\bar\rho(\Delta t^{-1}t), \text{ with } \bar\rho(t) = \bar\rho(-t).$$

Then we have

$$\begin{aligned} &\left\|\partial_t u^{\Delta t} \star \bar\rho_{\Delta t}\right\|^2_{(L^2(0,T;H^{-1}(\Omega)))^m} = \sum_{i=1}^{m}\int_0^T \left\|\sum_{n=0}^{K-1}(u^{i,n+1} - u^{i,n})\delta_{t_{n+1}} \star \bar\rho_{\Delta t}\right\|^2_{H^{-1}(\Omega)} \\ =\ & \sum_{i=1}^{m}\sum_{n=0}^{K-1}\int_0^T (\Delta t\bar\rho_{\Delta t}(t - t_{n+1}))^2 \left\|\frac{u^{i,n+1} - u^{i,n}}{\Delta t}\right\|^2_{H^{-1}(\Omega)} \\ =\ & C_3\Delta t\sum_{i=1}^{m}\sum_{n=0}^{K-1}\left\|\frac{u^{i,n+1} - u^{i,n}}{\Delta t}\right\|^2_{H^{-1}(\Omega)} \\ \le\ & C_3\Delta t\sum_{n=0}^{K-1}\sum_{i=1}^{m}\int_\Omega \left|T^{\varepsilon,\ell_2}(u^{i,n+1})\left(\sum_{j=1}^{m} A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^{j,n+1} + \delta\nabla u^{i,n+1} + \nabla p^{n+1}\right)\right|^2 \\ \le\ & 2\ C_3\ell_2^2\Delta t\sum_{n=0}^{K-1}\left\{\|A\|^2\|\nabla\rho_\eta \star u^{n+1}\|^2_{(L^2(\Omega))^m} + \delta^2\left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m} + \sqrt{m}\left\|\nabla p^{n+1}\right\|^2_{L^2(\Omega)}\right\}, \end{aligned}$$

where $\delta_{t_{n+1}}$ is Dirac mass in $t = t_{n+1}$, $C_3 := \int_0^T \bar{\rho}(t)\,dt$. Now, using (3.26) and (3.28) we obtain the uniform boundedness. Clearly, $\bar{\rho}_{\Delta t} \star \partial_t u^{i,\Delta t} \to \partial_t u^i$ weakly in $L^2(0,T;H^{-1}(\Omega))$ as $(\Delta t, \varepsilon) \to (0,0)$. Similarly we have that $\bar{\rho}_{\Delta t} \star u^{i,\Delta t} \to u^i$ strongly in $L^2(0,T;L^2(\Omega))$ since $u^{i,\Delta t} \to u^i$ in $L^2(0,T;L^2(\Omega))$. Then we deduce that $u^i \in \big\{g \in L^2(0,T;H^1(\Omega)); \partial_t g \in L^2(0,T;H^{-1}(\Omega))\big\}$. And now $u^i(0,x)$ has sense, by Proposition 2.2.2, and we have that $u^i(0,x) = u_0^i(x)$ by Proposition 2.5.1.

Step 4: Estimates (3.36) and (3.37)
On the one hand, taking the lim inf as $(\varepsilon, \Delta t) \to (0,0)$ in (3.26) and using the fact that the norm L^2 is lower semicontinuous we obtain that u satisfies the following entropy estimate for a.e. $t_1, t_2 \in (0,T)$

$$\begin{aligned}\int_\Omega \sum_{i=1}^m \Psi_{0,\ell_2}(u^i(t_2)) \quad &+ \quad \frac{\delta}{2}\sum_{i=1}^m \left\|\nabla u^i\right\|^2_{L^2(t_1,t_2;L^2(\Omega))} + \frac{\delta_0}{2}\sum_{i=1}^m \left\|\nabla \rho_\eta \star u^i\right\|^2_{L^2(t_1,t_2;L^2(\Omega))} \\ &\leq C_\delta \left\|\nabla f\right\|^2_{L^2(\Omega)} + \int_\Omega \sum_{i=1}^m \Psi_{0,\ell_2}(u_0^i). \end{aligned} \tag{3.43}$$

On the other hand, taking the sum over i of (3.35), multiplying by p and integrating over Ω with an integration by parts we get

$$\begin{aligned}\ell_1 \left\|\nabla p\right\|^2_{L^2(\Omega)} &\leq \ell_2 \left\|A\right\| \sqrt{m} \left\|\nabla \rho_\eta \star \rho_\eta \star u\right\|_{(L^2(\Omega))^m} \left\|\nabla p\right\|_{L^2(\Omega)} \\ &+ \ell_2 \delta \sqrt{m} \left\|\nabla u\right\|_{(L^2(\Omega))^m} \left\|\nabla p\right\|_{L^2(\Omega)} \\ &\leq \ell_2 \sqrt{m}\,(\delta + \left\|A\right\|) \left\|\nabla u\right\|_{(L^2(\Omega))^m} \left\|\nabla p\right\|_{L^2(\Omega)} \\ &\leq \ell_2 \sqrt{m}\,(\delta + \left\|A\right\|) \left(\frac{1}{2d}\left\|\nabla u\right\|^2_{(L^2(\Omega))^m} + \frac{d}{2}\left\|\nabla p\right\|^2_{L^2(\Omega)}\right) \\ &\leq \frac{\ell_1}{2}\left\|\nabla p\right\|^2_{L^2(\Omega)} + \frac{\ell_2^2 m(\left\|A\right\| + \delta)^2}{2\ell_1}\left\|\nabla u\right\|^2_{(L^2(\Omega))^m}, \end{aligned}$$

where we have used in the fourth line the Young's inequality, chosed in the fifth line $d = \dfrac{\ell_1}{\ell_2\sqrt{m}(\left\|A\right\| + \delta)}$, which yeild to (3.37). □

3.2.4 Passage to the limit as $\eta \to 0$ and $\delta \to 0$

Proof of Theorem 3.1.1

Step 1: Passage to the limit as $\eta \to 0$
Let $u^\eta := (u^{i,\eta})_{1 \leq i \leq m}$ be a solution of (3.35). This solution satisfies the fact that $0 \leq u^{i,\eta} \leq \ell_2$, for all $i = 1, \cdots, m$. Therefore, we have that $\partial_t u^\eta$ is uniformly bounded in

$L^2(0,T;(W^{1,\infty}(\Omega))')$. Indeed, we have

$$
\begin{aligned}
&\|\partial_t u^{i,\eta}\|_{L^2(0,T;(W^{1,\infty}(\Omega))')} = \left(\int_0^T \left\|\partial_t u^{i,\eta}\right\|^2_{(W^{1,\infty}(\Omega))'}\right)^{\frac{1}{2}} \\
\leq\ & \left(\int_0^T \left(\int_\Omega \left| u^{i,\eta}\left(\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^{j,\eta} + \delta\nabla u^{i,\eta} + \nabla p^\eta\right)\right|\right)^2\right)^{\frac{1}{2}} \\
\leq\ & \left(\int_0^T \left(\int_\Omega \left|u^{i,\eta}\right| \left(\left|\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^{j,\eta}\right| + \delta\left|\nabla u^{i,\eta}\right| + \left|\nabla p^\eta\right|\right)\right)^2\right)^{\frac{1}{2}} \\
\leq\ & \left\|u^{i,\eta}\right\|_{L^\infty(0,T;L^2(\Omega))} \left\|\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^{j,\eta}\right\|_{L^2(0,T;L^2(\Omega))} \\
&+\delta\left\|u^{i,\eta}\right\|_{L^\infty(0,T;L^2(\Omega))}\left\|\nabla u^{i,\eta}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|u^{i,\eta}\right\|_{L^\infty(0,T;L^2(\Omega))}\left\|\nabla p^\eta\right\|_{L^2(0,T;L^2(\Omega))} \\
\leq\ & \left\|u^{i,\eta}\right\|_{L^\infty(0,T;L^2(\Omega))}\left(\|A\|_\infty \sum_{j=1}^m \left\|\nabla\rho_\eta \star u^{j,\eta}\right\|_{L^2(0,T;L^2(\Omega))}\right. \\
&\left.+\ \delta\left\|\nabla u^{i,\eta}\right\|_{L^2(0,T;L^2(\Omega))} + \|\nabla p^\eta\|_{L^2(0,T;L^2(\Omega))}\right),
\end{aligned}
$$

where

$$
\|A\|_\infty = \max_{1\leq i\leq m}\sum_{j=1}^m |A_{ij}|, \tag{3.44}
$$

and we have used in the third inequality Holder's inequality (since we have $\frac{1}{2} = \frac{1}{\infty} + \frac{1}{2}$), together with (3.36), (3.37) which imply the uniform boundedness. Thus, we obtain

$$
\left\|u^{i,\eta}\right\|_{L^\infty(\Omega_T)} + \left\|u^{i,\eta}\right\|_{L^2(0,T;H^1(\Omega))} + \left\|\partial_t u^{i,\eta}\right\|_{L^2(0,T;(W^{1,\infty}(\Omega))')} \leq C, \tag{3.45}
$$

where C is independent of η. Together with Lemma 2.2.3 imply that we can find a function $u = (u^i)_{1\leq i\leq m} \in \left(L^2(0,T;H^1(\Omega))\right)^m$ and a function $p \in L^2(0,T;H^1(\Omega)/\mathbb{R})$ such that, as $\eta \to 0$, $u^\eta \to u$ strongly in $\left(L^2(0,T;L^2(\Omega))\right)^m$, $\nabla u^\eta \to \nabla u$ weakly in $\left(L^2(0,T;L^2(\Omega))\right)^m$ and $p^\eta \to p$ weakly in $L^2(0,T;H^1(\Omega)/\mathbb{R})$.

Moreover, since the strong convergence in $L^2(\Omega_T)$ implies the convergence almost everywhere in Ω_T, we get that $u^i \geq 0$ a.e. in Ω_T. Therefore, passing to the limit as $\eta \to 0$ in (3.41) we get that (u,p) is a solution of the following system

$$
\left\{\begin{array}{rcll}
\partial_t u^i & = & \operatorname{div}\left(u^i\left(\nabla p + \displaystyle\sum_{j=1}^m A_{ij}\nabla u^j + \delta\nabla u^i\right)\right) & \text{in } \mathcal{D}'(\Omega_T), \\
\displaystyle\sum_{i=1}^m u^i(t,x) & = & f(x) & \text{in } \Omega_T \\
u^i(x,0) & = & u_0^i(x) & \text{in } \Omega,
\end{array}\right. \tag{3.46}
$$

where the initial condition is recovered by Proposition 2.5.1, since $W^{1,1}(0,T;(W^{1,\infty}(\Omega))') \hookrightarrow C([0,T);(W^{1,\infty}(\Omega))')$.

Moreover, u satisfies for a.e. $t_1, t_2 \in (0,T)$

$$\int_\Omega \sum_{i=1}^m \Psi(u^i(t_2)) \quad + \quad \frac{\delta}{2}\sum_{i=1}^m \left\|\nabla u^i\right\|^2_{L^2(t_1,t_2;L^2(\Omega))} + \frac{\delta_0}{2}\sum_{i=1}^m \left\|\nabla u^i\right\|^2_{L^2(t_1,t_2;L^2(\Omega))}$$
$$\leq C_\delta \left\|\nabla f\right\|_{L^2(\Omega)} + \int_\Omega \sum_{i=1}^m \Psi(u_0^i). \tag{3.47}$$

and p satisfies

$$\|\nabla p\|^2_{L^2(\Omega)} \leq \frac{\ell_2^2 m(\|A\|+\delta)^2}{\ell_1} \|\nabla u\|^2_{(L^2(\Omega))^m} \leq C(\delta, \delta_0, \ell_1, \ell_2, \|A\|, \|f\|_{H^1(\Omega)}). \tag{3.48}$$

Step 2: Passage to the limit as $\delta \to 0$
Similarly, by (3.47), (3.48), Lemma 2.2.3 and Proposition 2.5.1 we can pass to the limit as $\delta \to 0$ in system (3.46) to get the existence result, announced in Theorem 3.1.1, of a solution for system (3.2) that satisfies (3.11) and (3.12). □

Chapter 4

Useful tools in geometric measure theory

Abstract: In this chapter we introduce the notion of varifolds, surfaces that may have singularities. Moreover, we associate to these surfaces a generalized mean curvature vector and a generalized velocity vector which coincide with the usual definitions in the case of a smooth surface. Also tools in analysis and geometric measure theory will be presented.

4.1 Introduction

Geometric measure theory is the study of geometric properties of sets through measure theory. It allows to extend tools from differential geometry to a much larger class of surfaces that are not necessarily smooth. Rectifiable sets, which are sets with the least possible regularity required to admit approximate tangent spaces, the area formula, which generalizes the concept of change of variables in integration, the coarea formula, which generalizes and adapts Fubini's Theorem to geometric measure theory and Radon measures are central in this theory [69].
The organization of this chapter is as follows: In Section 4.2 we present some preliminaries from measure theory such as Hausdorff measure, Radon measure, Radon-Nikodym derivative and approximate tangent space. In Section 4.3 we introduce the space of functions of bounded variation, as well as the area and co-area formulas. Section 4.4 is devoted to the notion of measure-function pairs and their weak and strong convergence. We introduce the notion of varifolds in Section 4.5 where we also define rectifiable and integral varifolds. Moreover we associate a generalized mean curvature vector and a generalized velocity to these varifolds. Finally, we present an overview of the link between the Allen-Cahn equation and the mean curvature flow, and in particular the Brakke flow.

Let $k \in \mathbb{N}$ and $n \in \mathbb{N}$ be such that $k \leq n$.

4.2 Preliminaries from measure theory

Definition 4.2.1. *(Hausdorff measure)*
The k-dimensional Hausdorff measure on $\mathbb{R}^n$ is defined by

$$\mathcal{H}^k(A) = \lim_{\delta \to 0} \mathcal{H}^k_\delta(A) \quad \text{for } A \subset \mathbb{R}^n,$$

where for each $\delta > 0$

$$\mathcal{H}^k_\delta(A) = \inf \left\{ \sum_{j=1}^{\infty} w_k \left(\frac{\operatorname{diam} C_j}{2} \right)^k \right\},$$

where w_k is the volume of the unit ball in $\mathbb{R}^k$ and the infimum is taken over all countable collections $C_1, C_2, \cdots$ of subsets of $\mathbb{R}^n$ such that $\operatorname{diam} C_j < \delta$ and $A \subset \bigcup_{j=1}^{\infty} C_j$.

Theorem 4.2.2. *(The case where $k = n$)*
We have that

$$\mathcal{H}^n(A) = \mathcal{L}^n(A), \quad \text{for every } A \subset \mathbb{R}^n$$

where $\mathcal{L}^n$ is the n-dimensional lebesgue measure on $\mathbb{R}^n$.

Definition 4.2.3. *(Radon measures)*
On a topological space Ω we say that a measure μ is a Radon measure if:
1) μ is a borel-regular measure in the sense that each Borel set is μ-measurable and for each subset $A \subset \Omega$ there is a Borel set $B \supset A$ such that $\mu(B) = \mu(A)$,
2) $\mu(K) < +\infty$ for all compact subset $K \subset \Omega$ (which enables us to integrate continuous functions with compact support).

Remark 4.2.4. *(Identification with linear functionals)*
We can identify the Radon measures on Ω with the non-negative linear functionals on $C_c(\Omega, \mathbb{R})$, so that we use the notation

$$\mu(\phi) = \int_\Omega \phi \, d\mu \quad \text{for all} \quad \phi \in C_c(\Omega, \mathbb{R}).$$

Also note that we have for every open $U \subset \Omega$ that

$$\mu(U) = \sup \left\{ \int_\Omega \phi \, d\mu : \phi \in C_c(\Omega, \mathbb{R}), |\phi| \leq 1, \text{support } \phi \subset U \right\}.$$

Theorem 4.2.5. *(Compactness theorem)*
Let $\{\mu_i\}$ be a sequence of Radon measures on Ω such that $\sup_i \mu_i(U) < +\infty$ for all $U \subset \Omega$ with $\bar{U}$ compact. Then there exist a Radon measure μ and a subsequence $\{\mu_{i'}\}$ such that $\mu_{i'} \to \mu$ in the sense that

$$\lim_{i' \to \infty} \mu_{i'}(\phi) = \mu(\phi) \quad \text{for each } \phi \in C_c(\Omega, \mathbb{R}).$$

Definition 4.2.6. *(Density function)*
The k-dimensional density of $A \subset \mathbb{R}^n$ at $a \in \mathbb{R}^n$ is defined as

$$\theta^k(A, a) = \lim_{r \to 0} \frac{\mathcal{H}^k(A \cap B_r(a))}{w_k r^k},$$

where $B_r(a)$ is the closed ball of radius r and center a. Similarly, the k-dimensional density of a measure μ on $\mathbb{R}^n$ at $a \in \mathbb{R}^n$ is given by

$$\theta^k(\mu, A, a) = \lim_{r \to 0} \frac{\mu(A \cap B_r(a))}{w_k r^k} \quad \text{for } A \subset \mathbb{R}^n.$$

Theorem 4.2.7. *(Radon-Nikodym derivative)*
Let μ_1 and μ_2 be two Radon measures on $\Omega \subset \mathbb{R}^n$. Then

$$\frac{d\mu_2}{d\mu_1}(x) := \lim_{\rho \to 0} \frac{\mu_2(B_\rho(x))}{\mu_1(B_\rho(x))}$$

exists μ_1-almost everywhere and is μ_1-measurable. Furthermore for any $A \subset \Omega$ we have

$$\mu_2(A) = \int_A \frac{d\mu_2}{d\mu_1} \, d\mu_1 + \mu_2^*(A),$$

where

$$\mu_2^* = \mu_2 \lfloor z$$

where z is a set of μ_1-measure zero (z is independent of A) and we denote $\mu_2 \lfloor z(B) = \mu_2(z \cap B)$ for all $B \subset \Omega$. μ_2^* is called the singular part of μ_2 with respect to μ_1. Also we can check that in the case where μ_2 is absolutely continuous with respect to μ_1 (i.e. in the sense that all the sets of μ_1 measure-zero also have μ_2 measure-zero and we note $\mu_2 << \mu_1$), then $\mu_2^* = 0$ and

$$\mu_2(A) = \int_A \frac{d\mu_2}{d\mu_1} \, d\mu_1.$$

Definition 4.2.8. *(Approximate tangent space)*
Let M be a $\mathcal{H}^k$-measurable subset of $\mathbb{R}^n$, θ a positive locally $\mathcal{H}^k$-integrable function on M and $\eta_{x,\lambda} : \mathbb{R}^n \mapsto \mathbb{R}^n$ a function defined as $\eta_{x,\lambda}(y) = \lambda^{-1}(y - x)$ with $y \in \mathbb{R}^n$ and $\lambda > 0$. If we have for all $\phi \in C_c(\mathbb{R}^n)$

$$\lim_{\lambda \downarrow 0} \int_{\eta_{x,\lambda}(M)} \phi(y)\theta(x + \lambda y)\, d\mathcal{H}^k(y) = \theta(x) \int_P \phi(y)\, d\mathcal{H}^k(y)$$

with P is a k-dimensional subspace of $\mathbb{R}^n$, then P is the approximate tangent space for M at x with respect to θ. We denote P as $\tan_x M$.

Definition 4.2.9. *(Approximate tangent space for a Radon measure)*
Suppose that μ is a Radon measure on $\mathbb{R}^n$, and for $x \in \mathbb{R}^n$, $\lambda > 0$, let $\mu_{x,\lambda}$ be the measure given by $\mu_{x,\lambda}(A) = \lambda^{-k}\mu(x + \lambda A)$. Suppose that for $\mu-$a.e. x there exist $\theta(x) \in (0, \infty)$ and a $k-$dimensional subspace $P \subset \mathbb{R}^n$ such that for all $\phi \in C_c(\mathbb{R}^n)$

$$\lim_{\lambda \to 0} \int \phi(y)\, d\mu_{x,\lambda}(y) = \theta(x) \int_P \phi(y)\, d\mathcal{H}^k(y).$$

Then P is called the approximate tangent space for μ at x, and $\theta(x)$ is called the multiplicity function.

In the next section we refer the reader to [36] for more details.

4.3 Preliminaries from analysis

Definition 4.3.1. *(BV functions)*
Let $\Omega \subset \mathbb{R}^n$ and $f \in L^1(\Omega)$. The function f is said to have bounded variation in Ω if $\int_\Omega |Df| < \infty$, where

$$\int_\Omega |Df| = \sup \left\{ \int_\Omega f \mathrm{divg}\, \mathrm{dx}; \mathrm{g} \in \mathrm{C}_0^1(\Omega; \mathbb{R}^\mathrm{n}), |\mathrm{g(x)}| \leq 1 \text{ for all } \mathrm{x} \in \Omega \right\}. \tag{4.1}$$

We define $BV(\Omega)$ as the space of all functions in $L^1(\Omega)$ with bounded variation.

Definition 4.3.2. *(The perimeter)*
Let E be a Borel set and Ω an open set in $\mathbb{R}^n$. The perimeter of E in Ω is defined as

$$P_\Omega(E) = \int_\Omega |D\chi_E|\,, \tag{4.2}$$

where χ_E is the characteristic function of E.

Theorem 4.3.3. *(Besicovitch covering theorem)*
Suppose that $\mathcal{B}$ is a collection of closed balls in $\mathbb{R}^n$, let A be the set of centers, and suppose that the set of all radii of balls in $\mathcal{B}$ is a bounded set. Then there are sub-collections $\mathcal{B}_1, \cdots, \mathcal{B}_N \subset \mathcal{B}$ with $N = N(n)$ such that each $\mathcal{B}_j$ is a pairwise disjoint (or empty) collection, and such that $\cup_{j=1}^N \mathcal{B}_j$ still covers A, namely that is $A \subset \cup_{j=1}^N (\cup_{B \in \mathcal{B}_j} B)$.

Theorem 4.3.4. *(Area formula)*
Let $f : \mathbb{R}^k \longmapsto \mathbb{R}^n$ be a Lipschitz function. Then, for any $\mathcal{L}^k-$measurable set $E \subset \mathbb{R}^k$, the function $\mathcal{H}^0(E \cap f^{-1}(y))$ is $\mathcal{H}^k-$ measurable in $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} \mathcal{H}^0(E \cap f^{-1}(y))\, d\mathcal{H}^k(y) = \int_E J_k\, f(x)\, dx,$$

where J_k denotes the $k-$dimensional Jacobian.

Theorem 4.3.5. *(Co-area formula)*
Let $f : M \longmapsto \mathbb{R}^N$ be a C^1 map with $N < k$ and M is a k-dimensional C^1 submanifold of $\mathbb{R}^n$. Then, if g is a non-negative $\mathcal{H}^k$-measurable function on M

$$\int_M (J_N f) g\, d\mathcal{H}^k = \int_{\mathbb{R}^N} \int_{f^{-1}(y)} g\, d\mathcal{H}^{k-N}\, d\mathcal{L}^N(y). \tag{4.3}$$

In particular if $f \in BV(\Omega)$ define $F_t = \{x \in \Omega; f(x) > t\}$. Then

$$\int_\Omega |Df| = \int_{-\infty}^{+\infty} \int_\Omega |D\chi_{F_t}|\, dx\, dt. \tag{4.4}$$

In the next section we introduce the notion of measure-function pair, see [39] for more details.

4.4 Measure-function pairs

Definition 4.4.1. *(Measure-function pair)*
Let $\Omega \subset \mathbb{R}^n$ be an open subset. Let μ be a positive Radon measure on Ω. Suppose that $f : \Omega \longmapsto \mathbb{R}^m$ is well defined $\mu-$almost everywhere, and that $f \in L^1(\mu, \mathbb{R}^m)$. Then we say that (μ, f) is a measure-function pair over Ω.

Definition 4.4.2. *(Convergence of measure-function pairs)*
Suppose $\{(\mu_i, f_i)\}_i$ and (μ, f) are measure-function pairs over Ω with values in $\mathbb{R}^m$. Suppose that $\lim_{i\to\infty} \mu_i = \mu$, in the sense of Radon-measures on Ω. Then we say (μ_i, f_i) converges to (μ, f) in the weak sense in Ω and write that

$$(\mu_i, f_i) \to (\mu, f) \quad \textit{weakly},$$

if

$$\lim_{i\to\infty} \int f_i \cdot \eta\, d\mu_i = \int f \cdot \eta\, d\mu, \tag{4.5}$$

for all $\eta \in C_c(\Omega, \mathbb{R}^m)$.

Theorem 4.4.3. *Let $F : \mathbb{R}^m \longmapsto [0, +\infty)$ be a continuous, convex function with super-linear growth at infinity, that is:*

$$\lim_{|y|\to\infty} \frac{F(y)}{|y|} = +\infty.$$

Suppose that $\{(\mu_i, f_i)\}_i$ are measure-function pairs over $\Omega \subset \mathbb{R}^n$ with values in $\mathbb{R}^m$. Suppose that μ is a Radon-measure on E and $\mu_i \to \mu$ as $i \to \infty$. Then the following properties are true:

(1) if

$$\sup_i \int F(f_i)\, d\mu_i < +\infty, \tag{4.6}$$

then some subsequence of $\{(\mu_i, f_i)\}_i$ *converges weakly to some* (μ, f) *for some* f *in the sense of (4.5).*

(2) if (4.6) holds and $(\mu_i, f_i) \to (\mu, f)$ *weakly then*

$$\liminf_{i\to\infty} \int F(f_i)\, d\mu_i \geq \int F(f)\, d\mu.$$

(3) if F *is strictly convex and for all* i *we have*

$$\int F(f_i)\, d\mu_i < +\infty, \tag{4.7}$$

then the following are equivalent:
(i) $(\mu_i, f_i) \to (\mu, f)$ *strongly.*
(ii) $(\mu_i, f_i) \to (\mu, f)$ *weakly and*

$$\int F(f_i)\, d\mu_i \to \int F(f)\, d\mu. \tag{4.8}$$

For instance, we can choose $F(y) = y^2$.

Proposition 4.4.4. *Let* $p,\ q \in (1, \infty)$ *such that* $\dfrac{1}{p} + \dfrac{1}{q} = 1$. *Suppose that* $\{\mu_i\}_i$ *and* μ *are Radon measures on* Ω *and that* $f_i \in L^p(\mu_i; \mathbb{R}^m)$, $f \in L^p(\mu; \mathbb{R}^m)$, $g_i \in L^q(\mu_i; \mathbb{R}^m)$ *and* $g \in L^q(\mu; \mathbb{R}^m)$. *Suppose further that* $(\mu_i, f_i) \to (\mu, f)$ *strongly with* $\|f_i\|_{L^p(\mu_i)}$ *is uniformly bounded and* $(\mu_i, g_i) \to (\mu, g)$ *weakly with* $\|g_i\|_{L^q(\mu_i)}$ *is uniformly bounded. Then*

$$(\mu_i, f_i \cdot g_i) \to (\mu, f \cdot g) \quad \textit{weakly}$$

with $\|f_i \cdot g_i\|_{L^1(\mu_i)}$ *is uniformly bounded.*

Next, we let $k < n$ and we introduce the notion of varifolds, see [69].

4.5 Varifolds

First of all let the Grassmanian $\mathbf{G}(n, k)$ be the space of k-dimensional linear subspaces of $\mathbb{R}^n$. For $S \in \mathbf{G}(n, k)$ we identify S with the orthogonal projection of $\mathbb{R}^n$ onto S. In the case where $k = n - 1$ and for $S \in \mathbf{G}(n, n-1)$ we have the relation $S = I - \nu \otimes \nu$ where ν is the unit orthogonal vector to S, I is the identity matrix and for vectors a and b in $\mathbb{R}^n$ we define the matrix $a \otimes b := (a_i b_j)_{1 \leq i,j \leq n}$.
We are interested in surfaces that may have singularities, that is why we will not restrict ourselves to manifolds. We will consider instead a family of "generalized surfaces" known as k-varifolds which is a class of Radon measures (so that it has nice compactness properties) defined over all the linear k-subspaces (tangential information) at each point (spatial information).

Definition 4.5.1. *(A general k-varifold)*
Let $\Omega \subset \mathbb{R}^n$ be open and $\boldsymbol{G}_k(\Omega) := \Omega \times \boldsymbol{G}(n,k)$. A general k- varifold in Ω denoted V is a Radon measure on $\boldsymbol{G}_k(\Omega)$;

$$V(\phi) := \int_{\boldsymbol{G}_k(\Omega)} \phi(x,S)\, dV(x,S) \quad \forall \phi \in C_c(\boldsymbol{G}_k(\Omega)).$$

We denote the set of all general k-varifolds in Ω by $\boldsymbol{V}_k(\Omega)$. We denote by $\|V\|$ the weight measure of V

$$\|V\|(\phi) := \int_{\boldsymbol{G}_k(\Omega)} \phi(x)\, dV(x,S) \quad \forall \phi \in C_c(\Omega), \tag{4.9}$$

more precisely it is the projection of V to the measures on $\mathbb{R}^n$.

Theorem 4.5.2. *(Countably k-rectifiable sets)*
A set $M \subset \mathbb{R}^n$ is countably k-rectifiable, if and only if there exists a positive locally $\mathcal{H}^k$-integrable function θ on M with respect to which the approximate tangent space $Tan_x M$ exists for $\mathcal{H}^k$-a.e. $x \in M$.

Definition 4.5.3. *(A rectifiable k-varifold)*
Let M be a countably k-rectifiable set and θ a locally $\mathcal{H}^k$ integrable function defined on M which is positive $\mathcal{H}^k$ a.e. in M. We say $V = V(M,\theta) \in \boldsymbol{V}_k(\Omega)$ is rectifiable if $V = \theta \mathcal{H}^k_{\lfloor M} \otimes \delta_{Tan_x M}$ i.e.

$$V(\phi) = \int_{G_k(\Omega)} \phi(x,S)\, dV(x,S) = \int_M \phi(x, Tan_x M)\theta(x)\, d\mathcal{H}^k(x) \quad \forall \phi \in C_c(\boldsymbol{G}_k(\Omega)),$$

where $Tan_x M$ is the approximate tangent space of M at x which exists $\mathcal{H}^k$ a.e. on M and which is represented by its orthogonal projection matrix onto $Tan_x M$. The function θ is called the multiplicity of V. We denote the set of all rectifiable k-varifolds in Ω by $\boldsymbol{RV}_k(\Omega)$.

Remark 4.5.4. *(Rectifiable Radon measure)*
A Radon measure μ on Ω is rectifiable if there exists $V \in \boldsymbol{RV}_k(\Omega)$ such that $\|V\| = \mu$.

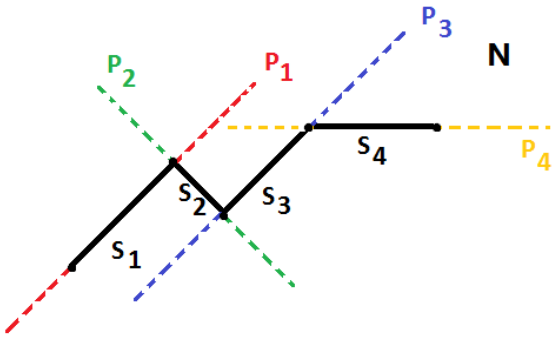
Definition 4.5.5. *(Integral k-varifolds)*
Let $V \in \boldsymbol{RV}_k(\Omega)$ be as in the previous definition. If in addition $\theta \in \mathbb{N}$ for $\mathcal{H}^k$ a.e. we say that V is integral. We denote the set of all rectifiable k-varifolds in Ω by $\boldsymbol{IV}_k(\Omega)$.

Examples:

Example 1: A 1-varifold associated to a line $D \subset \mathbb{R}^3$ is $V_D = \mathcal{H}^1_{\lfloor D} \otimes \delta_D$, ($\delta_D$ is the Dirac measure on $\mathbf{G}(3,1)$) which is a Radon measure on $\mathbb{R}^3 \times \mathbf{G}(3,1)$, in the sense that

$$\begin{aligned} V_D(\phi) &= \int_{\mathbb{R}^3 \times \mathbf{G}(3,1)} \phi(x,S)\, dV_D(x,S) \\ &= \int_{\mathbb{R}^3 \times \mathbf{G}(3,1)} \phi(x,S)\, d(\mathcal{H}^1_{\lfloor D} \otimes \delta_D)(x,S) \\ &= \int_{\mathbb{R}^3 \times \mathbf{G}(3,1)} \phi(x,S)\, d\mathcal{H}^1_{\lfloor D}(x)\, d\delta_D(S) \\ &= \int_D \phi(x,D)\, d\mathcal{H}^1(x), \end{aligned}$$

for all $\phi \in C_c(\mathbb{R}^3 \times \mathbf{G}(3,1))$.



Example 2: In the figure above, the curve $N \subset \mathbb{R}^2$ is consisting of 4 line segments S_1, S_2, S_3, S_4 of directions P_1, P_2, P_3, P_4. At each corner point we cannot define a tangent plane to the curve; however several planes are going through these points. So we associate a Radon measure to each such plane rather then defining a single tangent plane at each point (which is the case for differentiable manifolds). Furthermore, a 1-varifold associated to N can be the Radon measure $V_N = \sum_{i=1}^4 \mathcal{H}^1_{\lfloor S_i} \otimes \delta_{P_i}$ on $\mathbb{R}^2 \times \mathbf{G}(2,1)$, similarly in the sense that

$$V_N(\phi) = \sum_{i=1}^{4} \int_{S_i} \phi(x, P_i)\, d\mathcal{H}^1(x)$$

Example 3: We associate to a k-submanifold $M \subset \mathbb{R}^n$ the k-varifold $V_M = \mathcal{H}^k_{\lfloor M} \otimes \delta_{Tan_x M}$, i.e. $V_M(\phi) = \int_M \phi(x, Tan_x M)\, d\mathcal{H}^k(x)$. Note that we have $\|V_M\| = \mathcal{H}^k \lfloor_M$.

In what follows, we will associate to a varifold some geometric quantities which coincide with those of a smooth surface in the case where our varifold is the one considered in *Example 3*, namely $V = V_M$.
To begin with, we introduce the notion of a first variation. It is a function which assigns to any compactly supported C^1 vector field g on $\mathbb{R}^n$ the initial rate of change of the area of M under a smooth deformation of $\mathbb{R}^n$ with initial velocity g. Roughly, the first variation is due to the mean curvature of M.
The formula involves the tangential gradient and the tangential divergence of g on M which are defined by

$$\nabla_M g_i(x) := \sum_{j=1}^{k} (D_{\tau_j} g_i(x)) \cdot \tau_j = Tan_x M \cdot \nabla g_i, \tag{4.10}$$

which is the projection of ∇g_i onto $Tan_x M$ and

$$\mathrm{div}_M g(x) := \sum_{i=1}^{k} (D_{\tau_i} g(x)) \cdot \tau_i = Tan_x M \cdot \nabla g, \tag{4.11}$$

where $\{\tau_1, \tau_2, \cdots, \tau_k\}$ is an orthonormal basis for $Tan_x M$ and D_{τ_i} is the directional derivative with respect to τ_i.

Definition 4.5.6. *(The first variation of a C^1 submanifold)*
Suppose M is a $k-$dimensional compact C^1 submanifold of $\mathbb{R}^n$. Let $\{\phi_t\}_{0 \le t \le 1}$ be a $1-$parameter family of diffeomorphisms $M \to \mathbb{R}^n$ such that $\phi_0 = Id$ and $\left.\frac{\partial \phi_t}{\partial t}\right|_{t=0} = g(x)$.

Then we have

$$\frac{d}{dt}\mathcal{H}^k(\phi_t(M))\Big|_{t=0} = \int_M div_M g(x)\, d\mathcal{H}^k(x). \tag{4.12}$$

Definition 4.5.7. *(The first variation of a varifold)*
For $V \in \boldsymbol{V}_k(\Omega)$ let δV be the first variation of V so we have

$$\delta V(g) := \int_{G_k(\Omega)} \nabla g(x) \cdot S \, dV(x,S) \quad \forall g \in C^1_c(\Omega;\mathbb{R}^n). \tag{4.13}$$

Remark 4.5.8. *Note that in the case where we have $V = V_M$, see Example 3 above, the first variation of V_M with respect to $g \in C^1_c(\Omega;\mathbb{R}^n)$,*

$$\begin{aligned} \delta V_M(g) &:= \int_{G_k(\Omega)} \nabla g(x) \cdot S \, dV_M(x,S) \\ &= \int_{G_k(\Omega)} \nabla g(x) \cdot S \, d\mathcal{H}^k_{\lfloor M} \otimes \delta_{Tan_x M}(x,S) \\ &= \int_M \nabla g \cdot Tan_x M \, d\mathcal{H}^k(x), \end{aligned} \tag{4.14}$$

wich coincides with (4.12).

Furthermore, we define the total variation $\|\delta V\|$ to be

$$\|\delta V\|\, G = \sup\left\{\delta V(g); g \in C^1_c(\mathbb{R}^n;\mathbb{R}^n), spt\, g \subset G \text{ and } |g| \le 1\right\}, \quad \forall \text{ open } G \subset \mathbb{R}^n.$$

Now, consider the case where $k = n-1$ and let $\nu(x)$ the normal to M at $x \in M$. For all $i = 1, \cdots, n-1$ let $\gamma_i : (-1,1) \longrightarrow \mathbb{R}^n$ be a C^2 curve such that $\gamma_i(0) = x$, $\gamma_i((-1,1)) \subset M$ and $\gamma_i'(0) = \tau_i$. The normal curvature of M in the direction τ_i is expressed as

$$k_i := \gamma_i''(0) \cdot \nu.$$

This can be also seen as the second fondamental form of M at x, see for instance [69]. The mean curvature vector is then

$$h = \sum_{i=1}^{n-1} k_i \nu = \sum_{i=1}^{n-1} (\nu \cdot \gamma_i''(0))\nu. \tag{4.15}$$

Note that $\nu(\gamma_i(t)) \cdot \gamma_i'(t) = 0$ for all $t \in (-1,1)$ and $i = 1, \cdots, n-1$. After differentiating this relation with respect to t we get

$$\gamma_i''(t) \cdot \nu(\gamma_i(t)) + \gamma_i'(t) \frac{d}{dt}\nu(\gamma_i(t)) = 0.$$

Setting $t = 0$

$$\gamma_i''(0) \cdot \nu(x) = -\tau_i \cdot \frac{d}{dt}\nu(\gamma_i(t))|_{t=0} = -\tau_i \cdot D_{\tau_i}\nu.$$

Summing over i, multiplying by $\nu(x)$ and using (4.15) and (4.11) we obtain that

$$h = -\left(\mathrm{div}_M \nu\right)\nu.$$

In the case where $k < n-1$ we have that

$$h = -\sum_{\alpha=1}^{n-k} (\mathrm{div}_M \nu_\alpha)\, \nu_\alpha,$$

where $\nu_1, \cdots, \nu_{n-k}$ are vector fields satisfying $\nu_\alpha \cdot \nu_\beta = \delta_{\alpha\beta}$ and $\nu_\alpha(x) \in (Tan_x M)^\perp$ for all $\alpha, \beta \in \{1, \cdots, n-k\}$.

Definition 4.5.9. *(Mean curvature vector)*
Suppose M is an $k-$dimensional C^2 submanifold of $\mathbb{R}^n$ without boundary. Then the mean curvature vector h of M is given by

$$\int_M div_M g(x)\, d\mathcal{H}^k(x) = -\int_M g(x) \cdot h(x)\, d\mathcal{H}^k(x) \quad \forall g \in C_c^1(\Omega; \mathbb{R}^n). \tag{4.16}$$

Definition 4.5.10. *(Generalized mean curvature vector)*
Let $V \in \boldsymbol{V}_k(\Omega)$. If the total variation $\|\delta V\|$ of δV is locally bounded and absolutely continuous with respect to $\|V\|$, then by the Radon-Nikodym theorem, there exists a $\|V\|$ measurable vector field $h_V(.)$ such that

$$\delta V(g) = -\int_\Omega g(x) \cdot h_V(x)\, d\,\|V\|\,(x) \quad \forall g \in C_c^1(\Omega; \mathbb{R}^n). \tag{4.17}$$

The vector field $h_V(.)$ is called the generalized mean curvature vector of V.

Remark 4.5.11. *In the case where $V = V_M$ and using the fact that $\|V_M\| = \mathcal{H}^k \lfloor_M$ we obtain that (4.16) is equivalent to (4.17).*

Theorem 4.5.12. *(Allard's rectifiability theorem [4, page 450 5.5 (1)])*
Suppose $V \in \boldsymbol{V}_k(\Omega)$ and $\|\delta V\|$ is a Radon measure on Ω. Then

$$V \Big\lfloor_{\left\{x \in \Omega : \limsup_{r \downarrow 0} \frac{\|V_t\|(B_r(x))}{w_k r^k} > 0\right\} \times G(n,k)} \in \boldsymbol{RV}_k(\Omega).$$

Proposition 4.5.13. *(Normal velocity for smooth hypersurface)*
A smoooth family of hypersurfaces $\{M_t\}_{t \geq 0}$ has a normal velocity v if and only if

$$\frac{d}{dt} \int_{M_t} \phi d\mathcal{H}^{n-1} = \int_{M_t} (\nabla \phi \cdot v - \phi h \cdot v + \partial_t \phi)\, d\mathcal{H}^{n-1} \tag{4.18}$$

holds for all $\phi \in C_c^1(\Omega \times [0, \infty); \mathbb{R}^+)$ and $t \geq 0$, where h is the mean curvature vector of M_t.

Proof.
We will present here a formal proof of (4.18) in the case $n = 2$ and we refer the reader to see [58].
Fix for the instance the time t. We denote $\nu(x,t)$ the normal vector at the current point (x,t), $h(x,t) = \kappa\nu(x,t)$ the mean curvature vector at this point and $v(x,t) = c(x,t)\nu(x,t)$ the normal velocity at the same point. Let γ be a parametrization of M_t (which is a curve now). We parametrize $M_{t+\ell}$ then by $\gamma_{t+\ell}$ for ℓ small enough

$$\gamma_{t+\ell}(s) = \gamma(s) + r(\ell, s)\nu(\gamma(s)), \tag{4.19}$$

where r is such that $r(0,s)=0$, $\partial_\ell r(0,s)=c(\gamma(s),t)$. In what follows we will denote $\nu(s)$ instead of $\nu(\gamma(s),t)$. We have then

$$\begin{aligned}\int_{M_{t+\ell}}\phi(t+\ell,y)\,dx &= \int\phi(t+\ell,\gamma_{t+\ell}(s))\left|\frac{d}{ds}\gamma_{t+\ell}(s)\right|\,ds\\ &= \int\phi(t+\ell,\gamma(s)+r(\ell,s)\nu(s))\left|\frac{d}{ds}\gamma(s)+r_s(\ell,s)\nu(s)+r(\ell,s)\nu_s\right|\,ds\\ &= \int\phi(t+\ell,\gamma(s)+r(\ell,s)\nu(s))\left|\frac{d}{ds}\gamma(s)+r_s(\ell,s)\nu(s)-r(\ell,s)\kappa\frac{d}{ds}\gamma(s)\right|\,ds,\end{aligned}$$

where we have used in the last inequality the fact that $\frac{d}{ds}\nu(s)=-\kappa\frac{d}{ds}\gamma(s)$. Furthermore, we obtain

$$\begin{aligned}\frac{d}{d\ell}\left(\int_{M_{t+\ell}}\phi(t+\ell,x)\,dx\right)_{|\,\ell=0} &= \int(\partial_t\phi+r_\ell(0,s)\nu(s)\nabla\phi-\phi r_\ell(0,s)\kappa)\left|\frac{d}{ds}\gamma(s)\right|\,ds\\ &= \int_{M_t}\partial_t\phi+v\nabla\phi-c\kappa\phi\,dx.\end{aligned}$$

Which implies the desired result. □

Proposition 4.5.13 motivates the definition of generalized velocity for integral varifolds.

Definition 4.5.14. *(L^2-flow introduced in [59])*
Let $(V_t)_{t\in(0,T)}$ be an arbitrary family of integral varifolds such that $V:=\mathcal{L}^1\otimes V_t$ defines a Radon measure on $\Omega\times[0,T]$ and such that V_t has a generalized mean curvature $h_V(t,.)\in L^2(\|V_t\|,\mathbb{R}^n)$ for almost all $t\in(0,T)$.
If there exists a positive constant C and a vector field $v\in L^2(\|V\|,\mathbb{R}^n)$ such that

$$v(x,t)\perp T_x\|V_t\| \quad \text{for } \mu\text{-almost all } (x,t)\in\Omega\times[0,T]:=Q_T, \tag{4.20}$$

and

$$\left|\int_0^T\int_\Omega(\partial_t\eta+\nabla\eta\cdot v)\,d\|V_t\|\,dt\right|\le C\|\eta\|_{C^0(Q_T)}, \tag{4.21}$$

for all $\eta\in C_c^1(Q_T)$, then we call the evolution $(V_t)_{t\in(0,T)}$ an L^2-flow. A function $v\in L^2(\|V\|,\mathbb{R}^n)$ satisfying (4.20) and (4.21) is called a generalized velocity vector.

Remark 4.5.15. *This definition is based on (4.18): integrating (4.18) in time and using Holder's inequality we obtain (4.21).*

Definition 4.5.16. *(Push forward of varifolds)*
If N is a smooth Riemannian manifold and $F:M\longmapsto N$ is smooth, then F induces in a natural way a strongly continuous mapping $F_\#$ of the k-dimensional varifolds in M into the k-dimensional varifolds in N which has the property that $F_\#$ applied to the varifold associated with a k-dimensional submanifold of M is the k-dimensional varifold in N associated to the image of the submanifold under F.
In the case of general k-varifold V we have that for all Borel subsets B of $\boldsymbol{G}_k(N)$

$$F_\#V(B)=\int_{\{(x,S):(F(x),DF(x)(S))\in B\}}|J_kDF(x)\circ S|\,dV(x,S), \tag{4.22}$$

where J_k denotes the k-dimensional Jacobian.

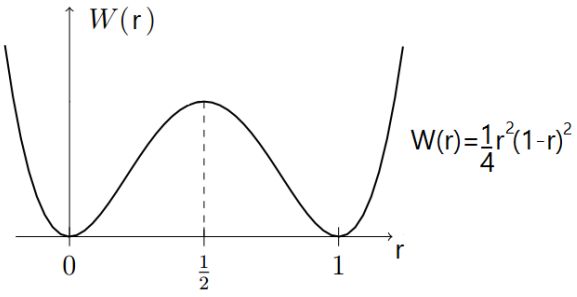
4.6 Phase field and mean curvature flow

We say that a family of surfaces evolves by mean curvature flow if the normal velocity at each point of the surface is equal to the mean curvature vector of the surface at this point. This kind of geometric flow has been frequently studied with different approaches.

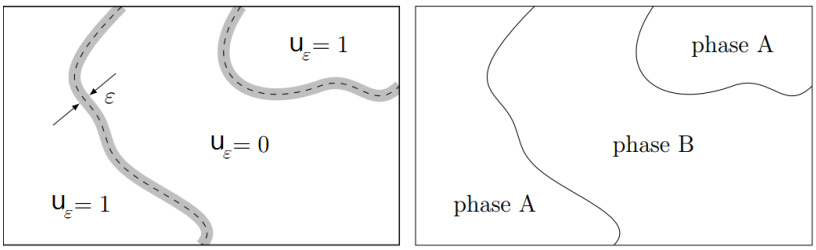
The phase field approach is based on an implicit description of the moving interface. It involves the Allen-Cahn equation which has been introduced in order to study phase separation problems [5]. Its link with the mean curvature flow have been studied several years ago. Given a small parameter $\varepsilon > 0$, the Allen-Cahn equation is written as

$$\partial_t u^\varepsilon = \Delta u^\varepsilon - \frac{1}{\varepsilon^2} W'(u^\varepsilon), \tag{4.23}$$

where W is an equal double well potential.



With the choice of W above, this equation will force to 0 the values of u^ε that are less than $1/2$, and to 1 the values of u^ε that are greater than $1/2$. As ε decreases the space domain Ω is gradually divided into two regions,where u^ε is close to 0 or 1. One expects that these regions are separated by diffuse transition zones, also called interfaces. The thickness of these transition zones will depend on the parameter ε. As $\varepsilon \to 0$, it is expected that the interfaces become sharp, as shown in the figure below. It has been proved that these sharp



interfaces move according to mean curvature flow [5, 19, 26, 28, 64]. One can interpret the Allen-Cahn equation as the gradient flow of the functional

$$E^\varepsilon(u^\varepsilon) := \int_\Omega \left(\frac{\varepsilon \, |\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx. \tag{4.24}$$

Let us also mention the result of Modica and Mortola [57] who proved that this functional Γ- converges to the perimeter function; let $0 < m < |\Omega|$ and $X := \left\{ u \in H^1(\Omega) : \int_\Omega u = m \right\}$.

We can extend the functional $E^\varepsilon(u)$ so that

$$E^\varepsilon(u) = \begin{cases} \int_\Omega \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \right) dx & \text{if } u \in X, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.25}$$

Then $E^\varepsilon(u)$ Γ- converges to $E^0(u)$ where

$$E^0(u) = \begin{cases} \sigma \int_\Omega |Du| & \text{if } u \in BV(\Omega) \text{ and } W(u) = 0 \text{ a.e. } x \in \Omega, \\ +\infty & \text{otherwise,} \end{cases} \tag{4.26}$$

with

$$\sigma = \int_0^1 \sqrt{2W(s)}\, ds. \tag{4.27}$$

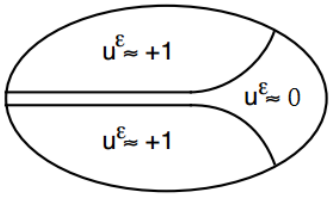
Another approach is the Brakke flow. It is the study of mean curvature flow in the context of non-smooth objects called varifolds [13], defined in Section 4.5. Brakke flows permit to define the mean curvature flow beyond the time of occurence of the first singularity, since they require very little smoothness. A Brakke flow is a family of rectifiable Radon measures V_t such that we have

$$\frac{d}{dt} \int \phi \, d \|V_t\| \leq \int (\nabla \phi \cdot h - |h|^2 \phi) \, d \|V_t\|,$$

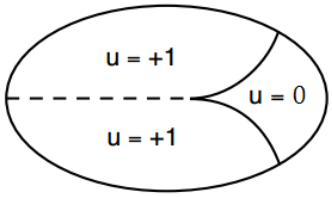
for all ϕ compactly supported test functions where h is the generalized mean curvature vector of the varifold V_t. Brakke flows have helped to study the size and the nature of the singular set [13].
Many studies deal with the convergence of the Allen-Cahn equation to a mean curvature flow in the sense of Brakke [41, 55, 65]. It consists of combining the phase field approach with Brakke flows. In order to avoid problems due to the cancellation of boundaries, one not only considers the evolution of phase boundaries but also the evolution of the Radon measures

$$d\mu_t^\varepsilon := \left(\frac{\varepsilon |\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx.$$

These measures coincide with the surface measures associated with the phase boundaries and they may be supported by additional hidden boundaries or may carry a higher multiplicity.



Phase separation for $\varepsilon \approx 0$



Phase separation as $\varepsilon \to 0$
showing **"hidden interface"**

Chapter 5

Singular limit in the sense of varifolds of an advection-reaction-diffusion equation arising from a chemotaxis-growth model

Abstract: This chapter is a joint work with D. Hilhorst. We study the singular limit of an advection-reaction-diffusion equation arising from a chemotaxis growth model. We associate to a family of diffuse interfaces a family of varifolds and we prove that the limit varifold is rectifiable and will move by a perturbed mean curvature flow.

5.1 Introduction

We consider an advection-reaction-diffusion equation which arises from a chemotaxis growth model proposed by Mimura and Tsujikawa [54] of the form

$$(P^\varepsilon) \quad \begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon - \nabla \cdot (u^\varepsilon \nabla \chi(q_\varepsilon)) + \dfrac{1}{\varepsilon^2} f(u^\varepsilon, \varepsilon\alpha) & \text{in } Q_T := \Omega \times (0,T), \\ u^\varepsilon(x,0) = u_0^\varepsilon(x) & \text{in } \Omega, \end{cases}$$

where $\Omega := \mathbb{T}^n$ is the n-torus, $0 < \varepsilon\alpha < \dfrac{1}{2}$, $f(r,\alpha) = r(1-r)(r-\frac{1}{2}+\alpha)$, χ is a smooth function, u_ε represents the population density and q_ε is the density of the chemotactic substance. The population is subject to three effects: diffusion, growth induced by the nonlinear term and a tendency of migrating towards higher gradients of chemotactic substance induced by the advection term. We are interested in studying the behavior of the family of smooth solutions $\{u^\varepsilon\}_{0<\varepsilon<1}$ of (P_ε) as $\varepsilon \to 0$ and we expect that the limit solution u will be a step function taking the values 0 or 1 almost everywhere on the domain, creating a sharp interface between these two regions. This sharp interface, which we will denote by Γ_t, obeys the law of motion, which is expressed as a perturbed motion by mean curvature:

$$v = h + (\nabla \chi(q) \cdot \nu)\,\nu + \frac{1}{\theta}\sqrt{2}\alpha\nu, \tag{5.1}$$

where v is the normal velocity to the interface, h is the mean curvature vector of the interface, θ is the multiplicity function which is equal to 1 in the case of a smooth interface and ν is the normal vector to the interface. We will suppose that q_ε is smooth and that there exist $C_q > 0$ and $q \in L^p(0,T;W^{2,p}(\Omega))$, with $1 \le p < \infty$ such that

$$\sup_{t\in[0,T]} (\|q_\varepsilon(.,t)\|_{L^\infty(\Omega)} + \|\nabla q_\varepsilon(.,t)\|_{L^\infty(\Omega)} + \|\Delta q_\varepsilon(.,t)\|_{L^\infty(\Omega)} + \|\nabla \Delta q_\varepsilon(.,t)\|_{L^\infty(\Omega)}) \le C_q, \tag{5.2}$$

for all $0 < \varepsilon < 1$ and

$$q_\varepsilon \to q \text{ in } L^2(Q_T) \text{ as } \varepsilon \to 0.$$

Problem (P^ε) can be written in the form:

$$\partial_t u^\varepsilon = \Delta u^\varepsilon - \frac{1}{\varepsilon^2}(W'(u^\varepsilon) - \varepsilon g_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)) \tag{5.3}$$

where $W(r) := \frac{1}{4}r^2(1-r)^2$ is an equal double well potential having two local minima at 0 and 1 and one local maximum at $\frac{1}{2}$. Note that $W'(r) := r(r-1)(r-\frac{1}{2})$ and that we can write the perturbation term as

$$\frac{1}{\varepsilon} g_\varepsilon := -\nabla\chi(q_\varepsilon)\cdot\nabla u^\varepsilon - u^\varepsilon \Delta\chi(q_\varepsilon) + \frac{\alpha}{\varepsilon} u^\varepsilon(1-u^\varepsilon) \tag{5.4}$$

We remark that one can find $\gamma \in (0, \frac{1}{2})$ and $k > 0$ such that

$$W''(r) \ge k, \quad \text{for all } \gamma \le \left| r - \frac{1}{2} \right| \le \frac{1}{2}. \tag{5.5}$$

The convergence result described above has been proved by Henry, Hilhorst and Schätzle [37] in the sense of viscosity solutions; they introduced a new unknown function w and they

considered the interface as a level-set of w. In addition, Mugnai and Röger [60] have proved this convergence result, in the sense of varifolds in a slightly more general context in the case of the space dimensions 2 and 3. Let us also mention the recent work of Takasao and Tonegawa [70] in arbitrary space dimension where they took

$$\frac{1}{\varepsilon} g_\varepsilon = -\vec{b} \cdot \nabla u_\varepsilon$$

with $\vec{b} \in L^q_{loc}([0,\infty); (W^{1,p}(\Omega))^n)$ and proved the convergence in the sens of varifolds. Note that the singular limit of a perturbed Allen-Cahn equation also in the case of chemotaxis problems, has been proved by [10] and [1]; they prove the convergence to a classical solution on a short time interval. In this chapter we will extend these results to the case that g_ε is given by (5.4) in arbitrary space dimension $n \geq 2$.
We define the energy functional

$$E^\varepsilon(u^\varepsilon) := \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx, \tag{5.6}$$

and we will associate to u^ε a Radon measure μ^ε_t on Ω

$$d\mu^\varepsilon_t := \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon(t,.)|^2 + \frac{W(u^\varepsilon(t,.))}{\varepsilon} \right) dx, \tag{5.7}$$

$$\text{i.e. } \mu^\varepsilon_t(\phi) = \int_\Omega \phi(x) d\mu^\varepsilon_t = \int_\Omega \phi(x) \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon(t,.)|^2 + \frac{W(u^\varepsilon(t,.))}{\varepsilon} \right) dx, \text{ for all } \phi \in C_c(\Omega), \tag{5.8}$$

which will behave more or less like a surface measure and a Radon measure μ^ε on Q_T as

$$d\mu^\varepsilon := \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx\, dt. \tag{5.9}$$

Let the discrepancy measure

$$d\xi^\varepsilon_t := \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon(t,.)|^2 - \frac{W(u^\varepsilon(t,.))}{\varepsilon} \right) dx. \tag{5.10}$$

Moreover, we define the varifold V^ε_t on $\Omega \times \mathbf{G}(n, n-1)$ as

$$V^\varepsilon_t(\eta) := \int_{\Omega \cap \{\nabla u^\varepsilon \neq 0\}} \eta \left(x, I - \frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|} \otimes \frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|} \right) d\mu^\varepsilon_t, \tag{5.11}$$

for all $\eta \in C_c(\Omega \times \mathbf{G}(n, n-1))$, where I is the identity matrix and for a vector a in $\mathbb{R}^n$ we define the matrix

$$a \otimes a := (a_i a_j)_{1 \leq i,j \leq n}.$$

Also note that we identify $S \in \mathbf{G}(n, n-1)$ with the corresponding orthogonal projection matrix of $\mathbb{R}^n$ onto S and we denote $S := I - \nu \otimes \nu$, with ν the unit normal vector to S. We define the scalar product of two $n \times n$ matrices A and B by

$$A \cdot B = \sum_{i=1}^n \sum_{j=1}^n A_{ij} B_{ij}.$$

Since we have that $\|V_t^\varepsilon\| = \mu_t^\varepsilon$, see (4.9) for the definition of the weight measure, also this varifold behaves like a surface measure but the test function takes an additional variable in $\mathbf{G}(n, n-1)$ which behaves as an approximate tangent space.
In addition, we set for all $t \in [0,T]$

$$D^\varepsilon(t) := \max\left\{1, \mu_t^\varepsilon(\Omega), \sup_{B_r(x)\subset\Omega} \frac{\mu_t^\varepsilon(B_r(x))}{w_{n-1}r^{n-1}}\right\}, \tag{5.12}$$

where w_{n-1} denote the volume of the unit $n-1$ dimensional ball, $B_r(x)$ is the n-dimensional ball of radius r and center x and for every open $B \subset \Omega$,

$$\mu_t^\varepsilon(B) = \sup\left\{\int_\Omega \phi(x)\, d\mu_t^\varepsilon(x),\ \phi \in C_c(B;\mathbb{R}),\ |\phi| \le 1\right\} = \int_B d\mu_t^\varepsilon.$$

Also, we suppose that the following hypotheses, which will denote by (H_0), are satisfied:

$$u_0^\varepsilon \in C^2(\bar\Omega), \tag{5.13}$$

$$\sup_\varepsilon E^\varepsilon(u_0^\varepsilon) \le E_0, \tag{5.14}$$

$$0 \le u_0^\varepsilon \le 1, \quad \text{on } \Omega, \text{ for all } \varepsilon > 0, \tag{5.15}$$

$$\varepsilon^i \left\|\nabla^i u_0^\varepsilon\right\|_{L^\infty(\Omega)} \le c_0, \quad \text{for } i = \{1,2,3\}, \tag{5.16}$$

there exists $0 < \beta < \frac{1}{2}$ such that

$$\sup_\Omega \left(\frac{\varepsilon\,|\nabla u_0^\varepsilon|^2}{2} - \frac{W(u_0^\varepsilon)}{\varepsilon}\right) \le \frac{1}{\varepsilon^\beta}, \tag{5.17}$$

there exists $0 < D_0 < \infty$ such that

$$D^\varepsilon(0) \le D_0. \tag{5.18}$$

Theorem 5.1.1. *(Main result)*
Let $n \ge 2$ and $\{u^\varepsilon\}_{0<\varepsilon<1}$ a family of smooth solutions of (P_ε) where $u^\varepsilon(x,0) = u_0^\varepsilon(x)$ satisfies the hypotheses (H_0) for all $0 < \varepsilon < 1$. Then we have, up to a subsequence denoted again by ε, that:

(i) there exists a function $u \in BV(Q_T;\{0,1\}) \cap L^\infty(0,T;BV(\Omega))$ such that

$$u^\varepsilon \to u \quad \text{strongly in } L^1(Q_T) \text{ as } \varepsilon \to 0. \tag{5.19}$$

(ii) there exists a Radon measure μ_t such that

$$\mu_t^\varepsilon \to \mu_t \quad \text{in the sense of Radon measures, for all } t \in [0,T] \text{ as } \varepsilon \to 0,$$

where μ_t^ε is defined in (5.9).

(iii) there exists a varifold V_t such that

$$V_t^\varepsilon \to V_t \quad \text{in the sense of varifolds, for all } t \in [0,T] \text{ as } \varepsilon \to 0,$$

where V_t^ε is defined in (5.11). Moreover $(V_t)_{t\in(0,T)}$ is an L^2-flow in the sense of Definition 4.5.14.

(iv) the law of motion of V_t is given by

$$v = h_{V_t} + g, \quad \mu - almost\ everywhere$$

where v is the generalized normal velocity of V_t, h_{V_t} is the generalized mean curvature vector of V_t and g satisfies

$$\int_{Q_T} \eta \cdot g \, d\mu := \lim_{\varepsilon\to 0} \int_{Q_T} -\eta \cdot \nabla u^\varepsilon g_\varepsilon \, dx\, dt \quad for\ all\ \eta \in C_c^1(Q_T;\mathbb{R}^n),$$

where g_ε is given in (5.4). In addition, on $\partial\{u=1\}$ we have

$$g = (\nabla\chi(q)\cdot\nu)\nu + \frac{1}{\theta}\sqrt{2}\alpha\nu,$$

where θ is the multiplicity function and ν is the inner normal vector to $\partial\{u=1\}$.

In a forthcoming work, we will try to extend our convergence result to the case of a more general function g_ε.

The remainder of the chapter is organized as follows: In Section 5.2 we formally derive the interface equation (5.1). In Section 5.3 we establish some uniform estimates such as the uniform boundedness of u^ε and of the energy functional $E^\varepsilon(u^\varepsilon)$. In Section 5.4 we prove the convergence of u^ε to a phase indicator function, namely we prove [Theorem 5.1.1, (i)]. Section 5.5 is devoted to the convergence of the diffuse surface area measure μ_t^ε, namely [Theorem 5.1.1, (ii)]. In Section 5.6 we establish a monotonicity formula which is similar to the one proposed by Ilmanen in [41] and we obtain an upper density ratio bound by using techniques from [70]. In Section 5.7 we prove a clearing out lemma [41, 55, 70] which leads to the vanishing of the discrepancy function as $\varepsilon \to 0$. The rectifiability almost everywhere of the limit varifold is obtained in Section 5.8 by the Allard's rectifiability theorem [4] where we also deduce from the Radon-Nikodym theorem the existence of a generalized mean curvature vector for the limit varifold. In Section 5.9 we prove that the multiplicity function is almost everywhere integer-valued modulo division by σ, where σ is given in (4.27) [72, 70, 40]. In Section 5.10 we prove that the limit varifold is an L^2-flow with generalized velocity v. We end the proof of Theorem 5.1.1 in Section 5.11 where we demonstrate that the interface moves according to the perturbed mean curvature flow (5.1) in arbitrary space dimension generalizing the result obtained by Mugnai and Röger [60, Section 5.3] in the case of space dimensions 2 and 3.

Note that in the proofs we will fix some constants $0 < \varepsilon_k < 1$ for $k \in \mathbb{N}$. These constants will satisfy that $\varepsilon_{k+1} < \varepsilon_k$.

5.2 Formal derivation of the interface motion equation

In this section, for the sake of completeness, we recall below a formal analysis performed by [1] (see also [2]), which allows to derive the interface equation (5.1).
Recall that the diffuse interface at time t is $\Gamma_t^\varepsilon := \left\{x \in \Omega : u^\varepsilon(x,t) = \frac{1}{2}\right\}$ with $\Gamma^\varepsilon :=$

$\cup_{0\le t\le T}\Gamma_t^\varepsilon$. The limit interface at time t will be denoted by Γ_t with $\Gamma := \cup_{0\le t\le T}\Gamma_t$. Denote also the signed distance function to Γ as

$$d(x,t) = \begin{cases} \operatorname{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t^+ \\ -\operatorname{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t^-, \end{cases}$$

where Ω_t^+ and Ω_t^- are the regions outside and inside the interface Γ_t respectively.
Outer expansion: assume that, away from the interface Γ, the solution u^ε of (P^ε) has the expansions

$$u^\varepsilon(x,t) = \begin{cases} 1 + \varepsilon u_1^+(x,t) + \varepsilon^2 u_2^+(x,t) + \cdots & \text{in } Q_T^+ := \cup_{0\le t\le T}\Omega^+ \times \{t\} \\ 0 + \varepsilon u_1^-(x,t) + \varepsilon^2 u_2^-(x,t) + \cdots & \text{in } Q_T^- := \cup_{0\le t\le T}\Omega^- \times \{t\}\,. \end{cases} \tag{5.20}$$

Inner expansion: assume that near the interface Γ

$$u^\varepsilon(x,t) = U_0(x,t,\xi) + \varepsilon U_1(x,t,\xi) + \varepsilon^2 U_2(x,t,\xi) + \cdots, \tag{5.21}$$

where $U_j(x,t,z)$, $j = 0,1,2,\cdots$, are defined for $x \in \Omega$, $t \ge 0$, $z \in \mathbb{R}$ and $\xi := d(x,t)/\varepsilon$. The stretched space variable ξ gives exactly the right spatial scaling to describe the rapid transition between the regions $\{u^\varepsilon \approx 0\}$ and $\{u^\varepsilon \approx 1\}$.
Normalization condition: we normalize U_0 in such a way that

$$U_0(x,t,0) = \frac{1}{2}.$$

Matching conditions: to make the inner and outer expansions consistent, we require that

$$\begin{aligned} U_0(x,t,+\infty) &= 1, \qquad & U_k(x,t,+\infty) &= u_k^+(x,t), \\ U_0(x,t,-\infty) &= 0, \qquad & U_k(x,t,-\infty) &= u_k^-(x,t), \end{aligned} \tag{5.22}$$

for all $k \ge 1$. The normalization condition and the matching condition for $k = 0$ will determine U_0 uniquely, which will then determine U_1.

In what follows we will substitute the inner expansion (5.21) into (P^ε) and we will collect the ε^{-2} and ε^{-1} terms. To that purpose, we compute the necessary terms to obtain

$$\begin{aligned} \partial_t u^\varepsilon &= \partial_t U_0 + \frac{\partial_t d}{\varepsilon}\partial_z U_0 + \varepsilon \partial_t U_1 + \partial_t d\,\partial_z U_1 + \cdots \\ \nabla u^\varepsilon &= \nabla U_0 + \frac{\nabla d}{\varepsilon}\partial_z U_0 + \varepsilon \nabla U_1 + \nabla d\,\partial_z U_1 + \cdots \\ \Delta u^\varepsilon &= \Delta U_0 + 2\frac{\nabla d}{\varepsilon}\cdot\nabla\partial_z U_0 + \partial_z U_0 \frac{\Delta d}{\varepsilon} + \partial_{zz} U_0 \frac{|\nabla d|^2}{\varepsilon^2} + \varepsilon \Delta U_1 \\ &\qquad + 2\nabla d\cdot\nabla\partial_z U_1 + \partial_z U_1 \Delta d + \partial_{zz} U_1 \frac{|\nabla d|^2}{\varepsilon} + \cdots \\ F(u^\varepsilon) &= F(U_0) + \varepsilon F'(U_0) U_1 + O(\varepsilon^2), \end{aligned}$$

where $F(u^\varepsilon) := -W'(u^\varepsilon)$ and the functions U_i $(i = 0,1)$, as well as their derivatives, are taken at the point $(x,t,d(x,t)/\varepsilon)$. Substituting in (P^ε) and collecting the ε^{-2} terms yield to

$$\partial_{zz} U_0 + f(U_0) \quad = 0.$$

In view of the normalization and matching conditions, we can now assert that $U_0(x,t,z) = U_0(z)$, where $U_0(z)$ is the unique solution of the stationary problem

$$\begin{cases} U_0'' + f(U_0) = 0 \\ U_0(-\infty) = 0, \quad U_0(0) = \frac{1}{2}, \quad U_0(+\infty) = 1. \end{cases} \tag{5.23}$$

This solution represents the first order approximation of the profile of a transition layer around the interface observed in the stretched coordinates.
Next we collect the ε^{-1} terms. Recalling that $\nabla\partial_z U_0 = 0$ and that $|\nabla d| = 1$ near Γ_t, we get

$$\partial_{zz}U_1 + f'(U_0)U_1 = U_0'(\partial_t d - \Delta d + \nabla\chi(q)\cdot\nabla d) - \alpha U_0(1-U_0). \tag{5.24}$$

Now we introduce the solvability condition in the next lemma.

Lemma 5.2.1 (Solvability condition)**.** *Let $A(z)$ be a bounded function on $-\infty < z < \infty$. Then the problem*

$$\begin{cases} \psi_{zz} + f'(U_0(z))\psi = A(z), & z \in \mathbb{R} \\ \psi(0) = 0, \quad \psi \in L^\infty(\mathbb{R}), \end{cases} \tag{5.25}$$

has a solution if and only if

$$\int_{\mathbb{R}} A(z)U_0'(z)dz = 0. \tag{5.26}$$

Moreover the solution, if it exists, is unique and satisfies

$$|\psi(z)| \le C\|A\|_{L^\infty} \qquad \text{for } z \in \mathbb{R}, \tag{5.27}$$

for some constant $C > 0$.

In view of Lemma 5.2.1, the solvability condition for (5.24) is given by

$$\int_{\mathbb{R}} \Big({U_0'}^2(z)(\partial_t d - \Delta d + \nabla\chi(q)\cdot\nabla d)(x,t) - \alpha\, U_0' U_0(1-U_0) \Big)\, dz = 0,$$

for all $(x,t) \in Q_T$. Hence we get

$$\partial_t d - \Delta d + \nabla\chi(q)\cdot\nabla d = \frac{\alpha \int_{\mathbb{R}} U_0' U_0(1-U_0)\, dz}{\int_{\mathbb{R}} {U_0'}^2(z)\, dz} = \frac{\alpha \int_0^1 s(1-s)\, ds}{\int_{\mathbb{R}} {U_0'}^2(z)\, dz}.$$

Moreover, multiplying equation (5.23) by U_0' and integrating over $(-\infty, z)$, we obtain

$$\begin{aligned} 0 &= \int_{-\infty}^{z} \big(U_0'' U_0' + f(U_0)U_0'\big)(s)ds \\ &= \frac{1}{2}{U_0'}^2(z) - W(U_0(z)), \end{aligned}$$

where we have also used the fact that $U_0(-\infty) = 0$ and $U_0'(-\infty) = 0$. This implies that

$$U_0'(z) = \sqrt{2}\, W(U_0(z))^{1/2},$$

and therefore

$$\begin{aligned} \int_{\mathbb{R}} {U_0'}^2(z)dz &= \sqrt{2}\int_{\mathbb{R}} U_0'(z)\, W(U_0(z))^{1/2} dz \\ &= \sqrt{2}\int_0^1 W(s)^{1/2} ds = \frac{\sqrt{2}}{2}\int_0^1 s(1-s). \end{aligned}$$

It then follows that

$$\partial_t d = \Delta d + \nabla\chi(q)\cdot\nabla d + \alpha\sqrt{2}. \tag{5.28}$$

We are now ready to derive the equation of interface motion. Since ∇d coincides with the outward normal unit vector to the hypersurface Γ_t, we have $\partial_t d(x,t) = -v_n$, where v_n is the normal velocity of the interface Γ_t. It is also known that the mean curvature κ of the interface is equal to $\Delta d/(N-1)$. Thus the equation of interface motion is given by:

$$v_n = -(N-1)\kappa + \nabla\chi(q)\cdot\nu + \alpha\sqrt{2} \quad \text{on } \Gamma_t. \tag{5.29}$$

Note that this formulation is equivalent to (5.1) in the case of a smooth interface with $v_n = v\cdot\nu$, $\theta = 1$ and $T < T_{max}$, where T_{max} is the maximal time before singularities occur.

5.3 A priori estimates

Proposition 5.3.1. *(Boundedness on u^ε)*
There exists $\varepsilon_1 = \varepsilon_1(\alpha, C_q)$ such that for all $0 < \varepsilon < \varepsilon_1$ we have that

$$0 \le u^\varepsilon \le 1 + \varepsilon \quad \text{in } Q_T. \tag{5.30}$$

Proof.
Let $M = 1+\varepsilon$. Suppose that we have $\max_{Q_T} u^\varepsilon > M$. Since u^ε is continuous in space and time and since we have (5.15) then there exists $(x_0,t_0)\in Q_T$ such that $u^\varepsilon(x_0,t_0) = M$. In addition,

$$\max_{\bar{Q}_T} u^\varepsilon(x,t) = \max_{t\in[0,T]}\left\{\max_{\bar{\Omega}} u^\varepsilon(x,t)\right\} > M, \tag{5.31}$$

and

$$\max_{\bar{\Omega}} u^\varepsilon(x,0) \le 1 < M. \tag{5.32}$$

Then there exists $t_1 \in (0,T]$ such that

$$\max_{\bar{\Omega}} u^\varepsilon(x,t_1) = M. \tag{5.33}$$

We choose $t_0 = t_1$ to be the smallest time such that (5.33) is satisfied; then we have $\max_{\bar{\Omega}\times[0,t_0]} u^\varepsilon(x,t) = M = u^\varepsilon(x_0,t_0)$. The maximum principle implies

$$\begin{aligned} 0 \le \partial_t u^\varepsilon(x_0,t_0) - \Delta u^\varepsilon(x_0,t_0) &= -M\Delta\chi(q_\varepsilon) + \frac{1}{\varepsilon^2}M(1-M)(M-\frac{1}{2}+\varepsilon\alpha) \\ &= -(1+\varepsilon)\Delta\chi(q_\varepsilon) - \frac{1}{\varepsilon^2}(1+\varepsilon)(\varepsilon)(\frac{1}{2}+\varepsilon+\varepsilon\alpha). \end{aligned} \tag{5.34}$$

In view of (5.2) for ε small enough depending on α and C_q, the right-hand side becomes negative which implies a contradiction and we have that $u^\varepsilon \le 1+\varepsilon$ on Q_T.
Now since 0 is a subsolution, by the comparison principle $u^\varepsilon \ge 0$ on Q_T. □

Proposition 5.3.2. *(Bounded measure)*
For all $t\in[0,T]$ there exists $0 < c_1 := c_1(E_0,\alpha,C_q,T)$ such that

$$E^\varepsilon(u^\varepsilon) = \mu_t^\varepsilon(\Omega) \le c_1. \tag{5.35}$$

Proof.

$$\begin{aligned}
\frac{d}{dt}E^\varepsilon(u^\varepsilon) &= \int_\Omega \varepsilon\nabla u^\varepsilon\nabla\partial_t u^\varepsilon + \frac{W'(u^\varepsilon)}{\varepsilon}\,\partial_t u^\varepsilon \\
&= -\int_\Omega \varepsilon\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)\partial_t u^\varepsilon \\
&= -\int_\Omega \varepsilon\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)^2 - \int_\Omega \varepsilon\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)(-\nabla\chi(q_\varepsilon)\cdot\nabla u^\varepsilon - u^\varepsilon\Delta\chi(q_\varepsilon)) \\
&\quad -\varepsilon\int_\Omega \left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)\frac{\alpha}{\varepsilon}u^\varepsilon(1-u^\varepsilon), \\
&\le -\int_\Omega \varepsilon\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)^2 + \int_\Omega \frac{\varepsilon}{2}\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)^2 + \int_\Omega \varepsilon\left|\nabla\chi(q_\varepsilon)\right|^2\left|\nabla u^\varepsilon\right|^2 \\
&\quad + \int_\Omega \varepsilon\left|u^\varepsilon\right|^2(\Delta\chi(q_\varepsilon))^2 + \int_\Omega \frac{\varepsilon}{2}\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)^2 + \int_\Omega \frac{\alpha^2}{2\varepsilon}(u^\varepsilon)^2(1-u^\varepsilon)^2 \\
&\le \int_\Omega \varepsilon\left|\nabla\chi(q_\varepsilon)\right|^2\left|\nabla u^\varepsilon\right|^2 + \varepsilon\left|u^\varepsilon\right|^2(\Delta\chi(q_\varepsilon))^2 + \frac{2\alpha^2}{\varepsilon}W(u^\varepsilon) \\
&\le \left(2\sup_\Omega\left|\nabla\chi(q_\varepsilon)\right|^2 + 2\alpha^2\right)E^\varepsilon(u^\varepsilon) + \varepsilon C_2(C_q) \\
&\le C_1(\alpha, C_q)E^\varepsilon(u^\varepsilon) + C_2(C_q) \qquad (5.36)
\end{aligned}$$

Therefore, Gronwall's inequality implies

$$E^\varepsilon(u^\varepsilon) \le (E_0 + \frac{C_2}{C_1})e^{C_1 T}.$$

□

Proposition 5.3.3. *(L^2-bound on the perturbation term)*
Let $g_\varepsilon(t,x,u^\varepsilon,\nabla u^\varepsilon)$ be defined as in (5.4). Then, there exists $0 < c_2 := c_2(c_1, C_q, \alpha, T)$ such that

$$\sup_{0<\varepsilon<1}\frac{1}{\varepsilon}\int_{Q_T}(g_\varepsilon)^2\,dx \le c_2. \qquad (5.37)$$

Proof.

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{Q_T}(g_\varepsilon)^2\,dx &\le 3\int_{Q_T}\left(\varepsilon\left|\nabla\chi(q_\varepsilon)\right|^2\left|\nabla u^\varepsilon\right|^2 + \varepsilon\left|u^\varepsilon\right|^2\left|\Delta\chi(q_\varepsilon)\right|^2 + \frac{1}{\varepsilon}\alpha^2(u^\varepsilon)^2(1-u^\varepsilon)^2\right) \\
&\le 6\left(\sup_{Q_T}\left|\nabla\chi(q_\varepsilon)\right|^2 + 2\alpha^2\right)\int_0^T E^\varepsilon(u^\varepsilon) + \varepsilon C(C_q, T). \qquad (5.38)
\end{aligned}$$

which, by (5.35), implies the result. □

We will denote

$$f_{q,\alpha} := \sup_{Q_T}\left|\nabla\chi(q_\varepsilon)\right|^2 + 2\alpha^2. \qquad (5.39)$$

Remark 5.3.4. *We remark that (5.37) is exactly the hypothesis made by Mugnai and Röger in a more general context in [60].*

Lemma 5.3.5. *Let u^ε be a solution of (P^ε) and let*

$$w_\varepsilon := -\varepsilon\Delta u^\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon);$$

then we have

$$E^\varepsilon\left[u^\varepsilon(.,t)\right] + \frac{\varepsilon}{2}\int_0^t\int_\Omega\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)^2 \leq E_0 + \frac{c_2}{2}, \tag{5.40}$$

for all $t > 0$,

$$\beta_\varepsilon := \int_0^T\int_\Omega(\varepsilon(\partial_t u^\varepsilon)^2 + \frac{1}{\varepsilon}w_\varepsilon^2)\,dx\,dt \leq c_2 + 2E_0, \tag{5.41}$$

and

$$\zeta_\varepsilon := \int_0^T\int_\Omega\left(\sqrt{\varepsilon}\partial_t u^\varepsilon + \frac{1}{\sqrt{\varepsilon}}w_\varepsilon\right)^2 dx\,dt \leq c_2. \tag{5.42}$$

Proof.
We have that

$$\begin{aligned}
\frac{d}{dt}E^\varepsilon(u^\varepsilon) &= \int_\Omega \varepsilon\nabla u^\varepsilon\nabla\partial_t u^\varepsilon + \frac{W'(u^\varepsilon)}{\varepsilon}\,\partial_t u^\varepsilon \\
&= -\int_\Omega \varepsilon\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)\partial_t u^\varepsilon \\
&= -\int_\Omega \varepsilon\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)^2 - \int_\Omega \varepsilon\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)\frac{g_\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)}{\varepsilon} \\
&\leq -\int_\Omega \frac{\varepsilon}{2}\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)^2 + \int_\Omega \frac{(g_\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon))^2}{2\varepsilon},
\end{aligned}$$

by integrating over $[0,t]$ we can get for any $t > 0$ that

$$E^\varepsilon\left[u^\varepsilon(.,t)\right] + \frac{\varepsilon}{2}\int_0^t\int_\Omega\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)^2 \leq E^\varepsilon\left[u_0^\varepsilon\right] + \int_0^t\int_\Omega\frac{(g_\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon))^2}{2\varepsilon} \leq E_0 + \frac{c_2}{2}, \tag{5.43}$$

which yields (5.40). Moreover, we have that

$$\begin{aligned}
\frac{1}{\varepsilon}\int_0^T\int_\Omega g_\varepsilon^2 &= \frac{1}{\varepsilon}\int_0^T\int_\Omega(\varepsilon\partial_t u^\varepsilon + w_\varepsilon)^2 = \zeta_\varepsilon \\
&= \int_0^T\int_\Omega(\varepsilon(\partial_t u^\varepsilon)^2 + \frac{1}{\varepsilon}w_\varepsilon^2) + 2\int_0^T\frac{d}{dt}E^\varepsilon(u^\varepsilon) \\
&= \int_0^T\int_\Omega(\varepsilon\partial_t u^\varepsilon)^2 + \frac{1}{\varepsilon}w_\varepsilon^2) + 2E^\varepsilon(u^\varepsilon(T,.)) - 2E^\varepsilon(u^\varepsilon(0,.)) \leq c_2,
\end{aligned}$$

which gives us that

$$\int_0^T\int_\Omega(\varepsilon(u_t^\varepsilon)^2 + \frac{1}{\varepsilon}w_\varepsilon^2) \leq c_2 + 2E_0.$$

□

5.4 Convergence of the phase function u^ε

We refer to Modica [56] for the results and the proofs presented in this section.

Theorem 5.4.1. *There exists a phase indicator function $u \in BV(Q_T;\{0,1\}) \cap L^\infty(0,T;BV(\Omega))$ such that up to a subsequence*

$$u^\varepsilon \to u \quad \text{in } L^1(Q_T) \text{ as } \varepsilon \to 0. \tag{5.44}$$

This result is a consequence of the three propositions below.

Proposition 5.4.2. *(Convergence in L^1)*
There exists a function $u \in L^1(Q_T)$ such that, up to a subsequence, $u^\varepsilon \to u$ in $L^1(Q_T)$ as $\varepsilon \to 0$.

Proof.
Let

$$\Phi(r) = \int_0^r \sqrt{W(s)/2}\, ds,$$

and define

$$v^\varepsilon := \Phi(u^\varepsilon)$$

for each ε.
Step 1. $\{v_\varepsilon\}_\varepsilon$ is bounded in $L^1(Q_T)$.
Since Φ is increasing, it follows that $0 \le \Phi(u^\varepsilon) \le \Phi(2) = \int_0^2 \sqrt{W(s)/2}\, ds$ so that $\Phi(u^\varepsilon)$ is bounded in $L^\infty(Q_T)$ and therefore in $L^1(Q_T)$.
Step 2. $\{v^\varepsilon\}_\varepsilon$ converges strongly in $L^1(Q_T)$.
We have that $\nabla v^\varepsilon(x) = \Phi'(u^\varepsilon(x))\nabla u^\varepsilon(x)$, then we have

$$\begin{aligned}
\int_\Omega |\nabla v^\varepsilon|\, dx &= \int_\Omega \sqrt{W(u^\varepsilon)}\, \frac{|\nabla u^\varepsilon|}{\sqrt{2}}\, dx \\
&\le \frac{1}{2}\int_\Omega \left(\frac{\varepsilon\, |\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx \\
&\le \frac{c_1}{2},
\end{aligned} \tag{5.45}$$

where we have used Young's inequality and (5.35) . Similarly, $\partial_t v^\varepsilon = \Phi'(u^\varepsilon)\partial_t u^\varepsilon$

$$\begin{aligned}
\int_{Q_T} |\partial_t v^\varepsilon|\, dx &= \int_{Q_T} \sqrt{W(u^\varepsilon)}\, \frac{|\partial_t u^\varepsilon|}{\sqrt{2}}\, dx \\
&\le \frac{1}{2}\int_{Q_T} \left(\frac{\varepsilon(\partial_t u^\varepsilon)^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx \\
&\le C(c_1, c_2, T, E_0),
\end{aligned} \tag{5.46}$$

where we have used (5.35) and (5.41). Hence, we deduce that $\{v^\varepsilon\}_\varepsilon$ is uniformly bounded in $BV(Q_T)$; therefore there exists a function $v \in BV(Q_T)$ and a subsequence of $\{v^\varepsilon\}_\varepsilon$ which we denote again by $\{v^\varepsilon\}_\varepsilon$ such that $v^\varepsilon \to v$ in $L^1(Q_T)$ as $\varepsilon \to 0$.
Step 3. $\{u^\varepsilon\}_\varepsilon$ converges strongly in $L^1(Q_T)$.
Let Φ^{-1} be the inverse function of Φ and define $u(x) := \Phi^{-1}(v(x))$. Since $v^\varepsilon \to v$ strongly in $L^1(Q_T)$ by Step 2 of this proof, it follows that $v^\varepsilon \to v$ a.e. in Q_T along a subsequence

which we will denote again by $\{v^\varepsilon\}$. Hence, since Φ^{-1} is continuous, $\Phi^{-1}(v^\varepsilon) \to \Phi^{-1}(v) = u$ a.e in Q_T. Finally, using the uniform boundeness of $u^\varepsilon = \Phi^{-1}(v^\varepsilon)$ and Lebesgue dominated convergence theorem we deduce that $u^\varepsilon \to u$ in $L^1(Q_T)$ up to a subsequence.

Proposition 5.4.3. *(Convergence of u^ε to a step function)*
$u = 0$ or $u = 1$ for almost all $(x,t) \in Q_T$.

Proof.
We have that

$$\begin{aligned}\int_{Q_T} W(u^\varepsilon) &= \varepsilon \int_0^T E^\varepsilon(u^\varepsilon) - \frac{\varepsilon^2}{2}\int_{Q_T} |\nabla u^\varepsilon|^2 \\ &\leq \varepsilon \int_0^T E^\varepsilon(u^\varepsilon) \leq c_1 T\varepsilon.\end{aligned}$$

Therefore, by Fatou's lemma and by using the fact that $u^\varepsilon \to u$ in $L^1(Q_T)$ we get

$$\int_{Q_T} W(u) \leq \liminf_{\varepsilon\to 0} \int_{Q_T} W(u^\varepsilon) \leq 0,$$

which implies that $W(u) = 0$ a.e. in Q_T so that $u = 0$ or $u = 1$ a.e. in Q_T. □

In the next Proposition we say that $u \in W^{1,1}(Q_T)$ in the sense that $u \in L^1(Q_T)$, $\partial_t u \in L^1(Q_T)$ and $\nabla u \in L^1(Q_T)$. We will also denote by D the weak derivative as in (4.1); in the case of $\phi \in W^{1,1}(\Omega)$ we have $\int_\Omega |D\phi| = \int_\Omega |\nabla\phi|$.

Proposition 5.4.4. *(BV bound on the limit)*
Denote D' the full derivative in time and space in $\mathbb{R}^n \times \mathbb{R}$ so that $\int_{Q_T} |D'u| := \int_{Q_T} \left|\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial t}\right)^T\right|$ in the case where $u \in W^{1,1}(Q_T)$. We have that

$$\int_{Q_T} \left|D'u\right| \leq c_3, \tag{5.47}$$

and

$$\max_{0\leq t\leq T} \int_\Omega |Du|\,(t,.) \leq c_4, \tag{5.48}$$

with $0 < c_3 := c_3(c_1, c_2, T, E_0)$ and $0 < c_4 := c_4(c_1)$.

Proof.
Recall that $v^\varepsilon := \Phi(u^\varepsilon)$ with $\Phi(r) = \int_0^r \sqrt{W(s)/2}\,ds$. By Step 2 of Proposition 5.4.2 $v^\varepsilon(t,.) \to v(t,.)$ in $L^1(\Omega)$ for almost every $t \in [0,T]$. Using the lower semicontinuity property and a computation similar to (5.45)

$$\int_\Omega |Dv| \leq \liminf_{\varepsilon\to 0} \int_\Omega |Dv^\varepsilon| \leq \frac{c_1}{2}. \tag{5.49}$$

By Proposition 5.4.3 and the coarea formula (see Theorem 4.3.5) we have that for every

$t \in [0,T]$

$$\begin{aligned}\int_\Omega |Dv| &= \int_{-\infty}^{+\infty} P_\Omega\left(\{x\in\Omega; v(x,t) > s\}\right) ds \\ &= \int_{-\infty}^{0} P_\Omega(\Omega) + \int_0^{\Phi(1)} P_\Omega(\{x\in\Omega; u(x,t)=1\}) \\ &= \Phi(1) P_\Omega(\{x\in\Omega; u(x,t)=1\}) \\ &= \Phi(1)\int_\Omega |Du|, \end{aligned} \tag{5.50}$$

where we have used the fact that

$$P_\Omega(\Omega) = \int_\Omega |D\chi_\Omega| = 0,$$

and Definition 4.3.2. Therefore, (5.50) give us

$$\int_\Omega |Du| = \frac{1}{\Phi(1)}\int_\Omega |Dv|. \tag{5.51}$$

Substituting in (5.49) we obtain (5.48).
The fact that, $v^\varepsilon \to v$ strongly in $L^1(Q_T)$ with (5.45) and (5.46) yield

$$\int_{Q_T} \left|D'v\right| \le \liminf_{\varepsilon\to 0}\int_{Q_T}\left|D'v^\varepsilon\right| \le C(c_1,c_2,T,E_0). \tag{5.52}$$

Similarily, the coarea formula gives

$$\begin{aligned}\int_{Q_T}\left|D'v\right| &= \int_{-\infty}^{+\infty} P_{Q_T}\left(\{(x,t)\in Q_T; v(x,t) > s\}\right) ds \\ &= \int_{-\infty}^{0} P_{Q_T}(Q_T) + \int_0^{\Phi(1)} P_{Q_T}(\{(x,t)\in Q_T : u(x,t)=1\}) \\ &= \Phi(1) P_{Q_T}(\{(x,t)\in Q_T : u(x,t)=1\}) \\ &= \Phi(1)\int_{Q_T}\left|D'u\right|. \end{aligned} \tag{5.53}$$

We deduce also in view of (5.52) that

$$\int_{Q_T}\left|D'u\right| = \frac{1}{\Phi(1)}\int_{Q_T}\left|D'v\right| \le C(c_1,c_2,T,E_0). \tag{5.54}$$

□

5.5 Convergence of diffuse surface area measures

In the next Lemma we want to prove that the time derivative of the measure μ_t^ε given in (5.7) is controlled.

Lemma 5.5.1. *There exists a constant $0 < c_5 := c_5(T, E_0, c_1, c_2)$ such that we have for all $\phi \in C^1(\Omega)$*

$$\int_0^T \left|\frac{d}{dt}\mu_t^\varepsilon(\phi)\right| := \int_0^T \left|\frac{d}{dt}\int_\Omega \phi\, d\mu_t^\varepsilon\right| \le c_5 \left\|\phi\right\|_{C^1(\Omega)}.$$

Proof.
For such ϕ we have that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \phi\, d\mu_t^\varepsilon &= \frac{d}{dt}\int_\Omega \phi\left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 + \frac{W(u^\varepsilon)}{\varepsilon}\right)dx \\ &= \int_\Omega \varepsilon\phi\nabla u^\varepsilon\nabla\partial_t u^\varepsilon + \phi\frac{W'(u^\varepsilon)}{\varepsilon}\partial_t u^\varepsilon \\ &= -\varepsilon\int_\Omega \operatorname{div}(\phi\nabla u^\varepsilon)\partial_t u^\varepsilon + \varepsilon\int_\Omega \phi\left(\partial_t u^\varepsilon\Delta u^\varepsilon - (\partial_t u^\varepsilon)^2 + \frac{g_\varepsilon\partial_t u^\varepsilon}{\varepsilon}\right) \\ &= -\varepsilon\int_\Omega \nabla\phi\nabla u^\varepsilon\partial_t u^\varepsilon - \varepsilon\int_\Omega \phi(\partial_t u^\varepsilon)^2 + \int_\Omega \phi g_\varepsilon\partial_t u^\varepsilon.\end{aligned}$$

Therefore, using Young's inequality

$$\int_0^T\left|\frac{d}{dt}\mu_t^\varepsilon(\phi)\right| \le \varepsilon\int_0^T\int_\Omega |\nabla\phi|\,|\nabla u^\varepsilon|\,|\partial_t u^\varepsilon| + \frac{1}{2\varepsilon}\int_0^T\int_\Omega g_\varepsilon^2\,|\phi| + \frac{3\varepsilon}{2}\int_\Omega |\phi|\,(\partial_t u^\varepsilon)^2. \tag{5.55}$$

Also, Young's inequality together with (5.35) and (5.41) imply

$$\varepsilon\int_0^T\int_\Omega |\nabla\phi|\,|\nabla u^\varepsilon|\,|\partial_t u^\varepsilon| \le \int_0^T\int_\Omega |\nabla\phi|\left(\varepsilon\frac{(\partial_t u^\varepsilon)^2}{2} + \varepsilon\frac{|\nabla u^\varepsilon|^2}{2}\right) \le C(E_0, c_1, c_2)\,\|\phi\|_{C^1(\Omega)}. \tag{5.56}$$

We obtain the expected result by combining (5.37), (5.41), (5.55) and (5.56). □

Proposition 5.5.2. *(Convergence of the measures μ_t^ε)*
There exist a family of Radon measures $\{\mu_t\}_{t\ge 0}$ and a subsequence which we will denote again by ε such that for all $t \in [0,T]$

$$\mu_t^\varepsilon \to \mu_t \text{ in the sense of Radon measures on } \Omega \text{ as } \varepsilon \to 0,$$

$$i.e.\ \int_\Omega \phi(x)\,d\mu_t^\varepsilon(x) \to \int_\Omega \phi(x)\,d\mu_t(x) \text{ as } \varepsilon\to 0 \text{ for all } \phi \in C_c(\Omega).$$

In addition, the function

$$t \longmapsto \mu_t(\phi) \quad \text{for all } \phi \in C^1(\bar\Omega),$$

is of bounded variation in $(0,T)$.
Moreover, we have for all $t \in [0,T]$

$$\mu_t \ge \sigma\,|Du(.,t)| \qquad \text{as Radon measures on } \Omega, \tag{5.57}$$

with σ is given in (4.27).

Proof. Let $\{\phi_k\}_{k=1}^\infty \subset C^1(\bar\Omega)$ be a dense subset in $C(\bar\Omega)$. From Lemma 5.5.1 we can see that the function $\mu_t^\varepsilon(\phi_k)$ is uniformly bounded in $BV(0,T)$. Then, using a diagonal argument we can find a function $m_k(t) \in BV(0,T)$ such that for all $k \in \mathbb{N}$, up to a subsequence

$$\mu_t^\varepsilon(\phi_k) \to m_k(t) \text{ strongly in } L^1(0,T) \text{ as } \varepsilon \to 0. \tag{5.58}$$

Therefore, (5.58) is valid for almost every $t \in (0,T)$. Denote by S_k the subset of $[0,T]$ where m_k has no discontinuities. Since m_k is of bounded variation, S_k is at most countable and

$$\mu_t^\varepsilon(\phi_k) \to m_k(t) \quad \text{for every } t \in [0,T]\backslash S_k. \tag{5.59}$$

Since S_k is a countable subset of $[0,T]$ we can use again a diagonal argument with (5.35) and the compactness property of Radon measures (see Theorem 4.2.5) to get that (5.59) is valid, up to a subsequence, for every $t \in [0,T]$ and that there exists a Radon measure μ_t on Ω such that $\lim_{\varepsilon\to 0} \mu_t^\varepsilon(\phi_k) = m_k(t) = \mu_t(\phi_k)$ for every $t \in [0,T]$. Therefore, since $\{\phi_k\}_{k=1}^\infty$ is dense in $C(\bar{\Omega})$ we get that $\mu_t^\varepsilon \to \mu^t$ as Radon measures for all $t \in [0,T]$.
Finally, to obtain (5.57) we combine (5.50), (5.49) and (5.45). Therefore, we get

$$\sigma \int_\Omega |Du| \, dx = 2 \int_\Omega |Dv| \, dx \leq 2 \liminf_{\varepsilon\to 0} \int_\Omega |Dv^\varepsilon| \, dx \leq \liminf_{\varepsilon\to 0} \mu_t^\varepsilon(\Omega) = \mu_t(\Omega).$$

□

5.6 Upper density ratio bound

In what follows we assume that there exist $0 < T_1 \leq T$ depending on ε and $D_1 \geq 2D_0$ such that

$$D^\varepsilon(t) \leq D_1 \quad \text{for all } t \in [0,T_1]. \tag{5.60}$$

The continuity in time of u^ε implies the continuity of $D^\varepsilon(t)$ together with the fact that $D_1 > D_0$ imply the existence of such T_1. In the end of this section we prove that $T_1 = T$ and we fix D_1.

Define ρ as the backward heat kernel

$$\rho_{(y,s)}(x,t) := \frac{1}{(4\pi(s-t))^{\frac{n-1}{2}}} e^{-\frac{|x-y|^2}{4(s-t)}}, \quad \text{for } x, y \in \mathbb{R}^n \text{ and } t < s.$$

Note that the backward heat kernel is scaled with respect to time as in $\mathbb{R}^{n-1}$ and not $\mathbb{R}^n$. Simple computations yield to

$$\nabla_x \rho = \frac{(x-y)}{2(s-t)}\rho,$$

$$\nabla_x^2 \rho = \left(\frac{(x-y)\otimes(x-y)}{4(s-t)^2} - \frac{I}{2(s-t)} \right) \rho,$$

$$\Delta_x \rho = \left(\frac{-n}{2(s-t)} + \frac{|x-y|^2}{4(s-t)^2} \right) \rho,$$

$$\partial_t \rho = \left(\frac{n-1}{2(s-t)} - \frac{|x-y|^2}{4(s-t)^2} \right) \rho.$$

Therefore, we have

$$\partial_t \rho + \Delta_x \rho = -\frac{\rho}{2(s-t)}, \tag{5.61}$$

and

$$\frac{(a\cdot\nabla_x\rho)^2}{\rho} + \big((I - a\otimes a)\cdot\nabla_x^2\rho\big) + \partial_t\rho = 0, \tag{5.62}$$

for any unit vector a. Let

$$\mathcal{L}^\varepsilon u^\varepsilon := \Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2} = \partial_t u^\varepsilon - \frac{1}{\varepsilon} g_\varepsilon.$$

Lemma 5.6.1. *For all $\phi \in C^2(\Omega)$ we have*

$$\varepsilon \int_\Omega \mathcal{L}^\varepsilon u^\varepsilon \nabla\phi \cdot \nabla u^\varepsilon \, dx = \int_\Omega \Delta\phi \, d\mu_t^\varepsilon - \varepsilon \int_\Omega \left(\nabla u^\varepsilon \otimes \nabla u^\varepsilon \cdot \nabla^2 \phi \right) \, dx. \tag{5.63}$$

Proof.
For such ϕ we obtain

$$\begin{aligned}
\int_\Omega \Delta\phi \, d\mu_t^\varepsilon &= \int_\Omega \Delta\phi \left(\frac{\varepsilon |\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx \\
&= -\frac{\varepsilon}{2} \int_\Omega \nabla\phi \cdot \nabla(|\nabla u^\varepsilon|^2) - \frac{1}{\varepsilon} \int_\Omega \nabla\phi \cdot \nabla u^\varepsilon W'(u^\varepsilon) \\
&= -\varepsilon \int_\Omega \sum_{i=1}^n \sum_{j=1}^n \partial_{x_j}\phi \, \partial_{x_i} u^\varepsilon \, \partial^2_{x_i x_j} u^\varepsilon - \frac{1}{\varepsilon} \int_\Omega \nabla\phi \cdot \nabla u^\varepsilon W'(u^\varepsilon) \\
&= \varepsilon \int_\Omega \sum_{i=1}^n \sum_{j=1}^n \partial_{x_j}\phi \, \partial_{x_j} u^\varepsilon \, \partial^2_{x_i x_i} u^\varepsilon + \varepsilon \int_\Omega \sum_{i=1}^n \sum_{j=1}^n \partial_{x_i}\partial_{x_j}\phi \, \partial_{x_i} u^\varepsilon \, \partial_{x_j} u^\varepsilon \\
&\quad -\frac{1}{\varepsilon} \int_\Omega \nabla\phi \cdot \nabla u^\varepsilon W'(u^\varepsilon) \\
&= \varepsilon \int_\Omega \nabla\phi \cdot \nabla u^\varepsilon \Delta u^\varepsilon + \varepsilon \int_\Omega \nabla^2\phi \cdot \nabla u^\varepsilon \otimes \nabla u^\varepsilon - \frac{1}{\varepsilon} \int_\Omega \nabla\phi \cdot \nabla u^\varepsilon W'(u^\varepsilon).
\end{aligned}$$

□

Lemma 5.6.2. *[41]*
Let

$$\rho_y^r(x) := \frac{1}{(\sqrt{2\pi} r)^{n-1}} \exp\left(-\frac{|x-y|^2}{2r^2} \right).$$

Then, $\rho_{(y,s)} = \rho_y^r$ when $r^2 = 2(s-t)$. Let μ be a Radon measure on $\mathbb{R}^n$ satisfying for some $D > 0$

$$\frac{\mu(B_R(x))}{w_{n-1} R^{n-1}} \le D \tag{5.64}$$

for $R > 0$ and $x \in \mathbb{R}^n$. Then we obtain the following:
(1) *For $r > 0$ and for $x \in \mathbb{R}^n$,*

$$\int_{\mathbb{R}^n} \rho_x^r(y) \, d\mu(y) \le D.$$

(2) *For r, $R > 0$ and for $x \in \mathbb{R}^n$,*

$$\int_{\mathbb{R}^n \setminus B_R(x)} \rho_x^r(y) \, d\mu(y) \le 2^{n-1} e^{-\frac{3R^2}{8r^2}} D.$$

(3) *For $\delta > 0$ there is $\gamma_1 > 0$ depending only on n and δ such that for x, $x_0 \in \mathbb{R}^n$ and $r > 0$ satisfying $|x - x_0| < \gamma_1 r$ we have*

$$\int_{\mathbb{R}^n} \rho_{x_0}^r(y) \, d\mu(y) \le (1+\delta) \int_{\mathbb{R}^n} \rho_x^r(y) \, d\mu(y) + \delta D.$$

(4) *For $\delta > 0$ there is $\gamma_2 > 0$ depending only on n and δ such that for $x \in \mathbb{R}^n$ and r, $R > 0$ satisfying $1 \le \frac{R}{r} \le 1 + \gamma_2$ we have*

$$\int_{\mathbb{R}^n} \rho_x^R(y) \, d\mu(y) \le (1+\delta) \int_{\mathbb{R}^n} \rho_x^r(y) \, d\mu(y) + \delta D.$$

Lemma 5.6.3. *(Huisken/Ilmanen monotonicity formula)*
There exists a constant $0 < c_6 := c_6(n)$ *such that*

$$\frac{d}{dt}\int_\Omega \tilde\rho\, d\mu_t^\varepsilon(x) \le \frac{1}{2(s-t)}\int_\Omega \tilde\rho\, d\xi_t^\varepsilon(x) + \frac{1}{2}\int_\Omega \frac{(g_\varepsilon)^2}{\varepsilon}\tilde\rho\, dx + c_6\, \mu_t^\varepsilon(B_{\frac12}(y)), \tag{5.65}$$

where

$$\tilde\rho_{(y,s)}(x,t) := \rho_{(y,s)}(x,t)\eta(x-y) = \frac{1}{(4\pi(s-t))^{\frac{n-1}{2}}} e^{-\frac{|x-y|^2}{4(s-t)}}\eta(x-y), \tag{5.66}$$

for $t < s$ *and* $x, y \in \Omega$ *with* η *a fixed radially symmetric cut-off function*

$$\eta(x) \in C_c^\infty(B_{\frac12}) \text{ with } \eta = 1 \text{ on } B_{\frac14}, 0 \le \eta \le 1. \tag{5.67}$$

One can check that

$$\int_\Omega \tilde\rho_{(y,s)}(x,t)\, dx \le \sqrt{4\pi(s-t)}. \tag{5.68}$$

Proof.
Let

$$\nu^\varepsilon = \frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}.$$

Also in view of (5.8)

$$\begin{aligned}
\frac{\partial}{\partial t}\int_\Omega \tilde\rho\, d\mu_t^\varepsilon &= \frac{\partial}{\partial t}\int_\Omega \tilde\rho\left(\frac{\varepsilon|\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon}\right) dx \\
&= \int_\Omega \partial_t\tilde\rho\, d\mu_t^\varepsilon + \varepsilon\int_\Omega \tilde\rho\nabla u^\varepsilon\cdot\nabla\partial_t u^\varepsilon + \frac{1}{\varepsilon}\int_\Omega \tilde\rho\, W'(u^\varepsilon)\partial_t u^\varepsilon \\
&= \int_\Omega \partial_t\tilde\rho\, d\mu_t^\varepsilon - \varepsilon\int_\Omega \tilde\rho\Delta u^\varepsilon\partial_t u^\varepsilon - \varepsilon\int_\Omega \nabla_x\tilde\rho\nabla u^\varepsilon\partial_t u^\varepsilon + \frac{1}{\varepsilon}\int_\Omega \tilde\rho W'(u^\varepsilon)\partial_t u^\varepsilon \\
&= \int_\Omega \partial_t\tilde\rho\, d\mu_t^\varepsilon - \varepsilon\int_\Omega \tilde\rho\Delta u^\varepsilon\left(\mathcal{L}^\varepsilon u^\varepsilon + \frac{1}{\varepsilon}g_\varepsilon\right) - \varepsilon\int_\Omega \nabla_x\tilde\rho\nabla u^\varepsilon\left(\mathcal{L}^\varepsilon u^\varepsilon + \frac{1}{\varepsilon}g_\varepsilon\right) \\
&\quad +\varepsilon\int_\Omega \tilde\rho\,(\Delta u^\varepsilon - \mathcal{L}^\varepsilon u^\varepsilon)\left(\mathcal{L}^\varepsilon u^\varepsilon + \frac{1}{\varepsilon}g_\varepsilon\right) \\
&= \int_\Omega \partial_t\tilde\rho\, d\mu_t^\varepsilon - \varepsilon\int_\Omega \nabla_x\tilde\rho\nabla u^\varepsilon\mathcal{L}^\varepsilon u^\varepsilon - \int_\Omega \nabla_x\tilde\rho\nabla u^\varepsilon g_\varepsilon - \varepsilon\int_\Omega \tilde\rho(\mathcal{L}^\varepsilon u^\varepsilon)^2 - \int \tilde\rho g_\varepsilon\mathcal{L}^\varepsilon u^\varepsilon \\
&= \int_\Omega \partial_t\tilde\rho\, d\mu_t^\varepsilon - \varepsilon\int_\Omega \tilde\rho\left(\mathcal{L}^\varepsilon u^\varepsilon + \frac{\nabla_x\tilde\rho\nabla u^\varepsilon}{\tilde\rho}\right)^2 + \varepsilon\int_\Omega \mathcal{L}^\varepsilon u^\varepsilon\nabla_x\tilde\rho\nabla u^\varepsilon + \varepsilon\int_\Omega \frac{(\nabla_x\tilde\rho\nabla u^\varepsilon)^2}{\tilde\rho} \\
&\quad -\int_\Omega g_\varepsilon\tilde\rho\left(\mathcal{L}^\varepsilon u^\varepsilon + \frac{\nabla_x\tilde\rho\nabla u^\varepsilon}{\tilde\rho}\right) \\
&\le \int_\Omega \left(\frac{(\nu^\varepsilon\cdot\nabla_x\tilde\rho)^2}{\tilde\rho} + \left((I-\nu^\varepsilon\otimes\nu^\varepsilon)\cdot\nabla_x^2\tilde\rho\right) + \partial_t\tilde\rho\right)\varepsilon|\nabla u^\varepsilon|^2\, dx \\
&\quad -\int_\Omega (\partial_t\tilde\rho + \Delta_x\tilde\rho)\, d\xi_t^\varepsilon + \frac{1}{2\varepsilon}\int_\Omega (g_\varepsilon)^2\tilde\rho \\
&\le \int_\Omega \frac{\tilde\rho}{2(s-t)}\, d\xi_t^\varepsilon + \frac{1}{2\varepsilon}\int_\Omega (g_\varepsilon)^2\tilde\rho + c_6\,\mu_t^\varepsilon(B_{\frac12}(y)),
\end{aligned}$$

where we have used the fact that $\mu_t^\varepsilon = \varepsilon|\nabla u^\varepsilon|^2 - \xi_t^\varepsilon$ together with (5.63), (5.61) and (5.62) with $a = \nu^\varepsilon = \dfrac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}$ and that ρ is uniformly bounded on $\{\nabla\eta \neq 0\}$ since $\{\nabla\eta \neq 0\} \subset$

$B_{\frac{1}{2}}\backslash B_{\frac{1}{4}}$ and we can always find a positive constant $c = c(n)$ such that $\ell^{\frac{n-1}{2}} e^{-\ell} \leq c$ for any $\ell > 0$; in particular for $\ell = \dfrac{1}{16(s-t)}$. □

Next, we recall a further property stated by [70, (4.24)].

Theorem 5.6.4. *If we have $\partial_t u - \Delta u = f$ on $B_2 \times [0,2]$ then we have for $j = 0$ or $j = 1$ and for $r \in (1,\infty)$ that*

$$\begin{aligned} \|\partial_t u\|_{L^r(B_1\times[j,2])} + \left\|\nabla^2 u\right\|_{L^r(B_1\times[j,2])} &\leq c(n,r)\Big(\|f\|_{L^r(B_2\times[0,2])} + \|\nabla u\|_{L^r(B_2\times[0,2])} \\ &\quad + \|u\|_{L^r(B_2\times[0,2])} + (1-j)\,\|u(.,0)\|_{W^{2,r}(B_2)}\Big). \end{aligned} \tag{5.69}$$

See [46, page 342, Theorem 9.1] for details.

In the next lemma we say that a function $f \in W_r^{2,1}(Q_T)$ if it is such that $f \in L^r(Q_T)$, $\partial_t f \in L^r(Q_T)$, $\nabla f \in L^r(Q_T)$ and $\nabla^2 f \in L^r(Q_T)$.
Also we say that a function $f \in C^{\frac{1}{2},\frac{1}{4}}(Q_T)$ if it is such that

$$\sup_{Q_T} |f| + \sup_{x,y\in\Omega,\ 0\leq t_1<t_2\leq T} \frac{|f(x,t_1) - f(y,t_2)|}{\max\left\{|x-y|^{\frac{1}{2}}, |t_1-t_2|^{\frac{1}{4}}\right\}} < \infty.$$

Lemma 5.6.5. *(Gradient bound)*
There exists a constant $0 < c_7 := c_7(n, W, \alpha, C_q, c_0)$ such that

$$\sup_{\Omega\times[0,T]} \varepsilon\,|\nabla u^\varepsilon| + \sup_{x,y\in\Omega,\ 0\leq t_1<t_2\leq T} \varepsilon^{\frac{3}{2}} \frac{|\nabla u^\varepsilon(x,t_1) - \nabla u^\varepsilon(y,t_2)|}{\max\left\{|x-y|^{\frac{1}{2}}, |t_1-t_2|^{\frac{1}{4}}\right\}} \leq c_7. \tag{5.70}$$

Proof.
We consider an arbitrary domain $B_{5\varepsilon}(x_0) \times [t_0, t_0 + 2\varepsilon^2] \subset \Omega \times [0,T]$. The rescaled problem becomes

$$\partial_t \tilde{u}^\varepsilon = \Delta \tilde{u}^\varepsilon - W'(\tilde{u}^\varepsilon) - \nabla\chi(\tilde{q}_\varepsilon)\cdot\nabla\tilde{u}^\varepsilon - \tilde{u}^\varepsilon \Delta\chi(\tilde{q}_\varepsilon) + \varepsilon\alpha\tilde{u}^\varepsilon(1-\tilde{u}^\varepsilon), \text{ on } B_5\times[0,2] \tag{5.71}$$

where $\tilde{u}^\varepsilon(x,t) := u^\varepsilon(\varepsilon x + x_0, \varepsilon^2 t + t_0)$ and $\tilde{q}_\varepsilon := q_\varepsilon(\varepsilon x + x_0, \varepsilon^2 t + t_0)$ for $(x,t) \in B_5 \times [0,2]$. Let $\phi \in C_c^1(B_5)$ be a cut-off function such that $\phi = 1$ on B_4 and $0 \leq \phi \leq 1$ on B_5. Multiplying (5.71) by $\tilde{u}^\varepsilon\phi^2$ and integrating over B_5 we get

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{1}{2}\int_{B_5} |\tilde{u}^\varepsilon|^2\,\phi^2\right) &= -\int_{B_5} |\nabla\tilde{u}^\varepsilon|^2\,\phi^2 - 2\int_{B_5} \tilde{u}^\varepsilon\phi\nabla\tilde{u}^\varepsilon\cdot\nabla\phi - \int_{B_5} W'(\tilde{u}^\varepsilon)\tilde{u}^\varepsilon\phi^2 \\ &\quad - \int_{B_5} \nabla\chi(\tilde{q}_\varepsilon)\cdot\nabla\tilde{u}^\varepsilon\tilde{u}^\varepsilon\phi^2 - \int_{B_5} \phi^2(\tilde{u}^\varepsilon)^2\Delta\chi(\tilde{q}_\varepsilon) + \varepsilon\alpha\int_{B_5}(\tilde{u}^\varepsilon)^2(1-\tilde{u}^\varepsilon)\phi^2. \end{aligned}$$

Integrating in time over $[0,2]$ and using Young's inequality we get

$$\begin{aligned} \frac{1}{2}\int_{B_5} |\tilde{u}^\varepsilon(.,2)|^2\,\phi^2 + \int_0^2\int_{B_5} |\nabla\tilde{u}^\varepsilon|^2\,\phi^2 &\leq \frac{1}{2}\int_{B_5} |\tilde{u}^\varepsilon(.,0)|^2\,\phi^2 + \frac{1}{4}\int_0^2\int_{B_5} |\nabla\tilde{u}^\varepsilon|^2\,\phi^2 \\ &\quad + 4\int_0^2\int_{B_5} (\tilde{u}^\varepsilon)^2\,|\nabla\phi|^2 + \frac{1}{4}\int_0^2\int_{B_5} |\nabla\tilde{u}^\varepsilon|^2\,\phi^2 \\ &\quad + \int_0^2\int_{B_5} |\nabla\chi(\tilde{q}_\varepsilon)|^2\,(\tilde{u}^\varepsilon)^2\phi^2 + C(W,\alpha,C_q,n). \end{aligned}$$

Therefore, we have

$$\int_0^2 \int_{B_4} |\nabla \tilde{u}^\varepsilon|^2 \, dx\, dt = \int_0^2 \int_{B_4} |\nabla \tilde{u}^\varepsilon|^2 \, \phi^2 \, dx\, dt \leq \int_0^2 \int_{B_5} |\nabla \tilde{u}^\varepsilon|^2 \, \phi^2 \, dx\, dt \leq C(W, \alpha, C_q, n). \tag{5.72}$$

Using Lemma 5.6.4 for $r = 2$ we get

$$\int_0^2 \int_{B_3} \left(|\partial_t \tilde{u}^\varepsilon|^2 + \left|\nabla^2 \tilde{u}^\varepsilon\right|^2 \right) dx\, dt \leq C(n, W, c_0, C_q, \alpha). \tag{5.73}$$

Now differentiating (5.71) with respect to x_i we obtain

$$\begin{aligned} \partial_t(\tilde{u}^\varepsilon_{x_i}) &= \Delta \tilde{u}^\varepsilon_{x_i} - W''(\tilde{u}^\varepsilon)\tilde{u}^\varepsilon_{x_i} - \nabla\chi(\tilde{q}_\varepsilon)\cdot\nabla\tilde{u}^\varepsilon_{x_i} - (\nabla\chi(\tilde{q}_\varepsilon))_{x_i}\cdot\nabla\tilde{u}^\varepsilon - \tilde{u}^\varepsilon_{x_i}\Delta\chi(\tilde{q}_\varepsilon) \\ &\quad -\tilde{u}^\varepsilon(\Delta\chi(\tilde{q}_\varepsilon))_{x_i} + \varepsilon\alpha\tilde{u}^\varepsilon_{x_i}(1-\tilde{u}^\varepsilon) - \varepsilon\alpha\tilde{u}^\varepsilon\tilde{u}^\varepsilon_{x_i}. \end{aligned} \tag{5.74}$$

Using (5.72), (5.73) and (5.16), the estimate (5.69) in Lemma 5.6.4 applied to (5.74) implies

$$\int_0^2 \int_{B_3} \left(|\nabla\partial_t \tilde{u}^\varepsilon|^2 + \left|\nabla^3 \tilde{u}^\varepsilon\right|^2 \right) dx\, dt \leq C(n, W, \alpha, C_q, c_0). \tag{5.75}$$

Hence we deduce that

$$\|\nabla \tilde{u}^\varepsilon\|_{W_2^{2,1}(B_3\times(0,2))} \leq C(W, n, \alpha, C_q, c_0).$$

Using Sobolev inequality we get

$$\|\nabla \tilde{u}^\varepsilon\|_{L^{\frac{2(n+1)}{n-1}}(B_3\times(0,2))} \leq C(W, n, \alpha, C_q, c_0).$$

We argue similarly with $r = \frac{2(n+1)}{n-1}$ in (5.69) and we repeat this argument until r is large enough so that $W_r^{2,1}(B_s \times (0,2)) \subset C^{\lambda,\frac{\lambda}{2}}(\bar{B}_s \times [0,2])$ for some ball of radius $s < 3$, with $\lambda = 2 - \dfrac{n+2}{r}$ if $r > \dfrac{n+2}{2}$ and $r \neq N+2$, see also [15]. We choose $\lambda = \frac{1}{2}$. Since the domain was arbitrary we get the desired estimate by rescaling back. □

Lemma 5.6.6. *(Upper bound on the discrepancy)*
There exists $0 < \varepsilon_2 := \varepsilon_2(n, c_7, C_q, \alpha)$ such that for all $0 < \varepsilon < \varepsilon_2$ we have

$$\sup_{x\in\Omega, t\in[0,T]} \xi^\varepsilon \leq 2\varepsilon^{-1/2}, \tag{5.76}$$

where $\xi^\varepsilon := \dfrac{\varepsilon\,|\nabla u^\varepsilon|^2}{2} - \dfrac{W(u^\varepsilon)}{\varepsilon}$.

Proof.
Let the arbitrary domain $B_3(x_0) \times [0,T] \subset Q_T$. By rescaling our domain as in the proof of Lemma 5.6.5, the rescaled problem can be written as

$$\partial_t \tilde{u}^\varepsilon = \Delta\tilde{u}^\varepsilon - W'(\tilde{u}^\varepsilon) - \nabla\chi(\tilde{q}_\varepsilon)\cdot\nabla\tilde{u}^\varepsilon - \tilde{u}^\varepsilon\Delta\chi(\tilde{q}_\varepsilon) + \varepsilon\alpha\tilde{u}^\varepsilon(1-\tilde{u}^\varepsilon), \text{ on } B_{3\varepsilon^{-1}}\times[0,\varepsilon^{-2}T] \tag{5.77}$$

Define the function

$$\xi := \frac{|\nabla\tilde{u}^\varepsilon|^2}{2} - W(\tilde{u}^\varepsilon) - G_\varepsilon(\tilde{u}^\varepsilon), \tag{5.78}$$

where

$$G_\varepsilon(r) := \varepsilon^{1/2}\left(1 - \frac{1}{8}(r-\frac{1}{2})^2\right).$$

Notice that we have

$$0 < G_\varepsilon(\tilde{u}^\varepsilon) < \varepsilon^{1/2},\ G'_\varepsilon W' \geq 0 \text{ and } G''_\varepsilon = -\frac{\varepsilon^{1/2}}{4}. \tag{5.79}$$

Now taking the gradient of (5.77) then multiplying by $\nabla\tilde{u}^\varepsilon$ we get

$$\begin{aligned}
\nabla\tilde{u}^\varepsilon \cdot \nabla\partial_t\tilde{u}^\varepsilon &= \nabla\tilde{u}^\varepsilon \cdot \nabla\Delta\tilde{u}^\varepsilon - W''\left|\nabla\tilde{u}^\varepsilon\right|^2 - \nabla\chi(\tilde{q}_\varepsilon)\otimes\nabla\tilde{u}^\varepsilon\cdot\nabla^2\tilde{u}^\varepsilon - \nabla^2\chi(\tilde{q}_\varepsilon)\cdot\nabla\tilde{u}^\varepsilon\otimes\nabla\tilde{u}^\varepsilon \\
&\quad - \left|\nabla\tilde{u}^\varepsilon\right|^2\Delta\chi(\tilde{q}_\varepsilon) - \tilde{u}^\varepsilon\nabla\tilde{u}^\varepsilon\cdot\nabla\Delta\chi(\tilde{q}_\varepsilon) + \varepsilon\alpha\left|\nabla\tilde{u}^\varepsilon\right|^2(1-\tilde{u}^\varepsilon) - \varepsilon\alpha\tilde{u}^\varepsilon\left|\nabla\tilde{u}^\varepsilon\right|^2 \\
&\leq \nabla\tilde{u}^\varepsilon \cdot \nabla\Delta\tilde{u}^\varepsilon - W''\left|\nabla\tilde{u}^\varepsilon\right|^2 - \nabla\chi(\tilde{q}_\varepsilon)\otimes\nabla\tilde{u}^\varepsilon\cdot\nabla^2\tilde{u}^\varepsilon - \nabla^2\chi(\tilde{q}_\varepsilon)\cdot\nabla\tilde{u}^\varepsilon\otimes\nabla\tilde{u}^\varepsilon \\
&\quad + \varepsilon\alpha\left|\nabla\tilde{u}^\varepsilon\right|^2(1-\tilde{u}^\varepsilon) + \varepsilon^2 C(c_7, C_q) + \varepsilon^3 C(c_7, C_q)
\end{aligned} \tag{5.80}$$

A simple computation gives us

$$\partial_t\xi = \nabla\tilde{u}^\varepsilon\cdot\nabla\partial_t\tilde{u}^\varepsilon - \partial_t\tilde{u}^\varepsilon(W' + G'_\varepsilon), \tag{5.81}$$

$$\nabla\chi(\tilde{q}_\varepsilon)\cdot\nabla\xi = (\nabla\chi(\tilde{q}_\varepsilon)\otimes\nabla\tilde{u}^\varepsilon)\cdot\nabla^2\tilde{u}^\varepsilon - (W' + G'_\varepsilon)\nabla\chi(\tilde{q}_\varepsilon)\cdot\nabla\tilde{u}^\varepsilon$$

$$\Delta\xi = \nabla\tilde{u}^\varepsilon\cdot\nabla(\Delta\tilde{u}^\varepsilon) + \left|\nabla^2\tilde{u}^\varepsilon\right|^2 - (W'' + G''_\varepsilon)\left|\nabla\tilde{u}^\varepsilon\right|^2 - (W' + G'_\varepsilon)\Delta\tilde{u}^\varepsilon.$$

Then we obtain after substituting (5.80) in (5.81) that

$$\begin{aligned}
\partial_t\xi + \nabla\chi(\tilde{q}_\varepsilon)\cdot\nabla\xi - \Delta\xi &\leq G''_\varepsilon\left|\nabla\tilde{u}^\varepsilon\right|^2 + W'(W' + G'_\varepsilon) - \left|\nabla^2\tilde{u}^\varepsilon\right|^2 - \nabla^2\chi(\tilde{q}_\varepsilon)\cdot\nabla\tilde{u}^\varepsilon\otimes\nabla\tilde{u}^\varepsilon \\
&\quad + \varepsilon\alpha\left|\nabla\tilde{u}^\varepsilon\right|^2(1-\tilde{u}^\varepsilon) + \left(\varepsilon^2 C(C_q) + \varepsilon C(\alpha)\right)\left(W' + G'_\varepsilon\right) \\
&\quad + \varepsilon^2 C(c_7, C_q) + \varepsilon^3 C(c_7, C_q).
\end{aligned} \tag{5.82}$$

Furthermore differentiating (5.78) with respect to x_i, taking the square, suming over i and applying Cauchy-Schwarz inequality we get

$$\begin{aligned}
\sum_{i=1}^n\left(\sum_{j=1}^n \partial_{x_j}\tilde{u}^\varepsilon\,\partial_{x_j x_i}\tilde{u}^\varepsilon\right)^2 &= \sum_{i=1}^n\left(\partial_{x_i}\xi + (W' + G'_\varepsilon)\partial_{x_i}\tilde{u}^\varepsilon\right)^2 \\
&= |\nabla\xi|^2 + 2(W' + G'_\varepsilon)\nabla\xi\cdot\nabla\tilde{u}^\varepsilon + (W' + G'_\varepsilon)^2\left|\nabla\tilde{u}^\varepsilon\right|^2 \\
&\leq \left|\nabla\tilde{u}^\varepsilon\right|^2\left|\nabla^2\tilde{u}^\varepsilon\right|^2.
\end{aligned} \tag{5.83}$$

Dividing (5.83) by $|\nabla\tilde{u}^\varepsilon|^2$ on $\{|\nabla\tilde{u}^\varepsilon| > 0\}$ and substituting the result in (5.82), we obtain

$$\begin{aligned}
\partial_t\xi + \nabla\chi(\tilde{q}_\varepsilon)\cdot\nabla\xi - \Delta\xi &\leq -(G'_\varepsilon)^2 - W'G' - \frac{2(W' + G'_\varepsilon)}{\left|\nabla\tilde{u}^\varepsilon\right|^2}\nabla\xi\cdot\nabla\tilde{u}^\varepsilon + G''_\varepsilon\left|\nabla\tilde{u}^\varepsilon\right| \\
&\quad + \varepsilon\alpha\left|\nabla\tilde{u}^\varepsilon\right|^2(1-\tilde{u}^\varepsilon) + \varepsilon C(\alpha)\left(W' + G'_\varepsilon\right) + O(\varepsilon^2)
\end{aligned} \tag{5.84}$$

Let $N := \sup_{B_{3\varepsilon^{-1}}\times[0,\varepsilon^{-2}T]}\left(\frac{|\nabla\tilde{u}^\varepsilon|^2}{2} - W(\tilde{u}^\varepsilon)\right)$ which is bounded by (5.70) (here we suppose also that it is positive) and $\phi \in C^\infty(B_{3\varepsilon^{-1}})$ be defined by

$$\phi(x,t) = \begin{cases} N & \text{on } B_{3\varepsilon^{-1}}\backslash B_{2\varepsilon^{-1}} \\ 0 & \text{on } B_{\varepsilon^{-1}}, \end{cases} \tag{5.85}$$

with

$$0 \le \phi \le N,\ |\nabla\phi| \le 2\varepsilon N,\ |\Delta\phi| \le 2n\varepsilon^2 N. \tag{5.86}$$

Let

$$\zeta := \xi - \phi. \tag{5.87}$$

Suppose for a contradiction that

$$\sup_{B_{\varepsilon^{-1}}\times[0,\varepsilon^{-2}T]} \xi \ge \varepsilon^{1/2}. \tag{5.88}$$

Due to the choice of ϕ and N we have that

$$\begin{cases} \zeta \le 0 & \text{on } (B_{3\varepsilon^{-1}}\backslash B_{2\varepsilon^{-1}})\times[0,\varepsilon^{-2}T], \\ \zeta \le \varepsilon^{1-\beta} & \text{on } B_{3\varepsilon^{-1}}\times\{0\} \text{ by (5.17)}, \\ \zeta \ge \varepsilon^{1/2} & \text{on } B_{\varepsilon^{-1}}\times[0,\varepsilon^{-2}T], \end{cases}$$

therefore there exists some interior maximum point of ζ where

$$\partial_t\zeta \ge 0,\ \nabla\zeta = 0,\ \Delta\zeta \le 0, \text{ and } \zeta \ge \varepsilon^{1/2}. \tag{5.89}$$

Combining (5.84), (5.87) and (5.89) we obtain

$$\begin{aligned} 0 \ \le\ & \Delta\phi - \nabla\chi(\tilde{q}_\varepsilon)\cdot\nabla\phi - (G'_\varepsilon)^2 - W'G'_\varepsilon - \frac{2(W'+G'_\varepsilon)}{|\nabla\tilde{u}^\varepsilon|^2}\nabla\phi\cdot\nabla\tilde{u}^\varepsilon + G''_\varepsilon|\nabla\tilde{u}^\varepsilon|^2 \\ & +\varepsilon\alpha|\nabla\tilde{u}^\varepsilon|^2(1-\tilde{u}^\varepsilon) + \varepsilon C(\alpha)\left(W'+G'_\varepsilon\right) + O(\varepsilon^2). \end{aligned} \tag{5.90}$$

Now using (5.79), (5.86) and the fact that $|\nabla\tilde{u}^\varepsilon|^2 \ge 2\xi \ge 2\zeta \ge 2\varepsilon^{1/2}$ we get

$$\begin{aligned} 0 \ \le\ & 2n\varepsilon^2 N + 2\varepsilon^2 NC(C_q) - (G'_\varepsilon)^2 - W'G'_\varepsilon + 2\sqrt{2}\left(\left|W'\right| + \left|G'_\varepsilon\right|\right)\varepsilon^{3/4}N - \frac{\varepsilon^{1/2}}{4}|\nabla\tilde{u}^\varepsilon|^2 \\ & +\varepsilon\alpha|\nabla\tilde{u}^\varepsilon|^2|1-\tilde{u}^\varepsilon| + \varepsilon C(\alpha)\left(\left|W'\right| + \left|G'_\varepsilon\right|\right) + O(\varepsilon^2). \end{aligned} \tag{5.91}$$

If $\tilde{u}^\varepsilon \in [\frac{1}{2}-\gamma, \frac{1}{2}+\gamma]$ we use the fact that $|\nabla\tilde{u}^\varepsilon|^2 = 2(\xi + W(\tilde{u}^\varepsilon) + G_\varepsilon(\tilde{u}^\varepsilon)) \ge W(\tilde{u}^\varepsilon) \ge W(\frac{1}{2}-\gamma) > 0$ which implies that

$$G''_\varepsilon(\tilde{u}^\varepsilon)|\nabla\tilde{u}^\varepsilon|^2 \le -\frac{\varepsilon^{1/2}}{4}\min_{r\in[\frac{1}{2}-\gamma,\frac{1}{2}+\gamma]} W(r)$$

and together with (5.91) yields a contradiction for sufficiently small ε depending on n, α and C_q.
If $\left|\tilde{u}^\varepsilon - \frac{1}{2}\right| > \gamma$ then we have

$$(G'_\varepsilon(\tilde{u^\varepsilon}))^2 \ge \frac{\varepsilon\,\gamma^2}{16} \quad \text{and} \quad W'(\tilde{u^\varepsilon})G'(\tilde{u^\varepsilon}) \ge \frac{\varepsilon^{1/2}\gamma}{4}\left|W'\right|.$$

In the case where $\tilde{u}^\varepsilon \notin \{0,1\}$ we have also a negative term of order $\varepsilon^{1/2}$ which similarly yeilds to a contradiction. Now consider the case where $\tilde{u}^\varepsilon = 0$; then the estimate (5.91) can

be written as

$$\begin{aligned} 0 &\leq 2n\varepsilon^2 N + 2\varepsilon^2 NC(C_q) - \varepsilon\frac{\gamma^2}{16} + 2\sqrt{2}\frac{\varepsilon^{1/2}}{8} \times \varepsilon^{3/4}N - \frac{\varepsilon^{1/2}}{4}\left|\nabla\tilde{u}^\varepsilon\right|^2 \\ &\quad +\varepsilon\alpha\left|\nabla\tilde{u}^\varepsilon\right|^2 + C(\alpha)\varepsilon \times \frac{\varepsilon^{1/2}}{8} + O(\varepsilon^2) \\ &\leq 2n\varepsilon^2 N + 2\varepsilon^2 NC(C_q) + C(\alpha)\frac{\varepsilon^{3/2}}{8} + 2\sqrt{2}\frac{\varepsilon^{5/4}}{8}N \\ &\quad -\varepsilon\frac{\gamma^2}{16} - \frac{\varepsilon^{1/2}}{8}\left|\nabla\tilde{u}^\varepsilon\right|^2 + O(\varepsilon^2), \end{aligned} \tag{5.92}$$

where we have used the fact that for sufficiently small ε depending on α we have

$$-\frac{\varepsilon^{1/2}}{4}\left|\nabla\tilde{u}^\varepsilon\right|^2 + \varepsilon\alpha\left|\nabla\tilde{u}^\varepsilon\right|^2 \leq -\frac{\varepsilon^{1/2}}{8}\left|\nabla\tilde{u}^\varepsilon\right|^2.$$

Since the right hand side of the inequality (5.92) is strictly negative for ε small enough we have reached a contradiction. A similar contradiction can be obtained in the case that $\tilde{u}^\varepsilon = 1$.
Since $\phi = 0$ on $B_{\varepsilon^{-1}}$, $G_\varepsilon < \varepsilon^{1/2}$ and the domain is arbitrary we get the result (5.76) by rescaling back. □

Lemma 5.6.7. *Let s, R, r be positive such that $0 \leq s - \left(\frac{R}{r}\right)^2 \leq T$ and $R \in (0, \frac{1}{2})$. Set $\tilde{s} = s - \left(\frac{R}{r}\right)^2$. Then there exists $c_8 := c_8(n) \geq 1$ such that, for any $y \in \Omega$, we have*

$$\int_\Omega \tilde{\rho}_{(y,s)}(x,\tilde{s})\, d\mu^\varepsilon_{\tilde{s}}(x) \leq \left(\frac{r}{\sqrt{4\pi}R}\right)^{n-1}\left\{\mu^\varepsilon_{\tilde{s}}(B_R(y)) + \mu^\varepsilon_{\tilde{s}}(B_{\frac{1}{2}}(y))e^{-r^2/16R}\right\} + c_8 D(\tilde{s})e^{-r^2/8}. \tag{5.93}$$

See [70, Lemma 4.3] for the proof.

Lemma 5.6.8. *Let γ be defined as in (5.5). Assume that $\frac{1}{2} - \gamma \leq u^\varepsilon(y,s) \leq \frac{1}{2} + \gamma$ with $s \in (0, T_1]$ and $y \in \Omega$. Then, for any $t \in [0, T_1]$ with $\max\left\{0, s - 2\varepsilon^{3/2}\right\} \leq t \leq s$ there exist $c_9 := c_9(c_8, D_1, c_7, n) > 1$, $0 < c_{10} := c_{10}(c_8, D_1, c_7, n) < 1$ and $0 < \varepsilon_3 := \varepsilon_3(c_1, c_6, C_q, \alpha, T)$ such that for all $0 < \varepsilon < \varepsilon_3$*

$$c_{10} \leq \frac{1}{R^{n-1}}\mu^\varepsilon_t(B_R(y)), \tag{5.94}$$

with $R = c_9(s + \varepsilon^2 - t)^{1/2}$.

Proof.
We have that

$$\begin{aligned} \int_\Omega \tilde{\rho}_{(y,s+\varepsilon^2)}(x,s)\, d\mu^\varepsilon_s(x) &= \int_\Omega \frac{1}{(4\pi\varepsilon^2)^{\frac{n-1}{2}}} e^{-\frac{|x-y|^2}{4\varepsilon^2}} \eta(x-y)\left(\frac{\varepsilon\left|\nabla u^\varepsilon\right|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon}\right) dx \\ &= \int_{\varepsilon^{-1}\Omega} \frac{e^{-\frac{|\tilde{x}|^2}{4}}}{(4\pi)^{\frac{n-1}{2}}} \eta(\varepsilon\tilde{x})\left(\frac{|\nabla\tilde{u}|^2}{2} + W(\tilde{u})\right) d\tilde{x}, \end{aligned} \tag{5.95}$$

where we have made the change of variable $\varepsilon\tilde{x} = x - y$ with $\tilde{u}(\tilde{x}, s) = u(\varepsilon\tilde{x} + y, s)$. Furthermore, the fact that $\frac{1}{2} - \gamma \le \tilde{u}(0, s) \le \frac{1}{2} + \gamma$ implies that for every $r_0 > 0$

$$\left|\tilde{u}^\varepsilon(\tilde{x}) - \frac{1}{2}\right| \le \left|\tilde{u}^\varepsilon(0) - \frac{1}{2}\right| + \sup_{x\in\Omega} |\nabla\tilde{u}^\varepsilon|\,|\tilde{x}| \le \gamma + c_7 r_0,$$

for all $\tilde{x} \in B_{r_0}(0)$. Therefore, taking r_0 small enough we can always find a constant $0 < c(c_7) < W(1/2) < 1$ such that $W(\tilde{u}^\varepsilon) > c(c_7)$ on $B_{r_0}(0)$. We obtain

$$\int_\Omega \tilde{\rho}_{(y,s+\varepsilon^2)}(x,s)\,d\mu_s^\varepsilon(x) \ge \int_{B_{r_0}(0)} C(n)W(\tilde{u})\,d\tilde{x} \ge C(n,c_7)w_n = 6\left(C(n,c_7)\,\frac{w_n}{6}\right) \ge 5\,c_{11}, \tag{5.96}$$

with $0 < c_{11} = c_{11}(n, c_7) < 1$. In addition, the monotonicity formula (5.65), (5.68), (5.76), and a computation similar to (5.38) give us

$$\frac{d}{d\lambda}\int_\Omega \tilde{\rho}_{(y,s+\varepsilon^2)}(x,\lambda)\,d\mu_\lambda^\varepsilon(x) \le 3f_{q,\alpha}\int_\Omega \tilde{\rho}_{(y,s+\varepsilon^2)}(x,\lambda)\,d\mu_\lambda^\varepsilon(x) + \frac{2\sqrt{\pi}\varepsilon^{-1/2}}{\sqrt{s-\lambda}} + C(c_1,c_6,C_q,T) \tag{5.97}$$

where $f_{q,\alpha}$ is given in (5.39). Multiplying (5.97) by $e^{\varepsilon^{-1}(s-\lambda)}$, integrating over $[t, s]$ and using the fact that

$$\frac{d}{d\lambda}\left(e^{\varepsilon^{-1}(s-\lambda)}\int_\Omega \tilde{\rho}\,d\mu_\lambda^\varepsilon(x)\right) = -\varepsilon^{-1}e^{\varepsilon^{-1}(s-\lambda)}\int_\Omega \tilde{\rho}\,d\mu_\lambda^\varepsilon(x) + e^{\varepsilon^{-1}(s-\lambda)}\frac{d}{d\lambda}\int_\Omega \tilde{\rho}\,d\mu_\lambda^\varepsilon(x)$$

we get

$$\begin{aligned} e^{\varepsilon^{-1}(s-\lambda)}\int_\Omega \tilde{\rho}\,d\mu_\lambda^\varepsilon(x)\Big|_{\lambda=t}^{s} \;\le\; & \left(3f_{q,\alpha} - \frac{1}{\varepsilon}\right)\int_t^s e^{\varepsilon^{-1}(s-\lambda)}\int_\Omega \tilde{\rho}\,d\mu_\lambda^\varepsilon(x)\,d\lambda + \varepsilon^{1/4}e^{2\varepsilon^{1/2}}4\sqrt{2\pi} \\ & + 2C(c_1,c_6,C_q,T)\varepsilon^{3/2}e^{2\varepsilon^{1/2}}. \end{aligned} \tag{5.98}$$

For sufficiently small ε depending on C_q, α, c_1, c_6 and T the right hand side can be made negative we choose also ε so that $e^{2\varepsilon^{1/2}} \le 5/2$ together with (5.96) yield to

$$\int_\Omega \tilde{\rho}\,d\mu_t^\varepsilon(x) \ge \frac{1}{e^{2\varepsilon^{1/2}}}\int_\Omega \tilde{\rho}\,d\mu_s^\varepsilon(x) \ge 2c_{11}. \tag{5.99}$$

Applying Lemma 5.6.7, with $\tilde{s}$ replaced by t, s replaced by $s+\varepsilon^2$, r replaced by $\sqrt{8\ln(2c_8D_1c_{11}^{-1})}$ (so that $c_8D_1e^{-\frac{r^2}{8}} = \frac{c_{11}}{2}$) and R replaced by $r(s+\varepsilon^2-t)^{1/2}$, we obtain

$$\int_\Omega \tilde{\rho}_{(y,s+\varepsilon^2)}(x,t)\,d\mu_t^\varepsilon(x) \le \left(\frac{r}{\sqrt{4\pi}R}\right)^{n-1}\left(\mu_t^\varepsilon(B_R(y)) + c_1e^{-r^2/16R^2}\right) + c_8D_1e^{-r^2/8}, \tag{5.100}$$

note that in view of $s - t \le 2\varepsilon^{3/2}$ we can have $R \le 1/2$ for sufficiently small ε depending on D_1, c_{11}, c_8. Therefore, using $r/R = 1/(s+\varepsilon^2-t)^{1/2} \ge 1/(\varepsilon^2+2\varepsilon^{3/2})^{1/2} \ge 1/(\sqrt{3}\varepsilon^{3/4})$ together with (5.99) we get for ε small enough

$$\left(\frac{r}{\sqrt{4\pi}R}\right)^{n-1}\mu_t^\varepsilon(B_R(y)) \ge c_{11}, \tag{5.101}$$

which implies the result with $c_9 = r$ and $c_{10} = r^{1-n}(\sqrt{4\pi})^{n-1}c_{11}$. □

In the next Lemma we suppose that $T_1 > 2\varepsilon^{3/2}$. This is not necessarily true in general and we will discuss in the proof of Lemma [5.6.10 (i)] the case that $T_1 \le 2\varepsilon^{3/2}$.

Lemma 5.6.9. *For any $r \in (\varepsilon^{3/4}, \frac{1}{2})$ and $t \in [2\varepsilon^{3/2}, T_1]$, there exist $0 < \varepsilon_4 := \varepsilon_4(k, C_q, \alpha)$ and $0 < c_{12} := c_{12}(n, k, D_1, c_9, c_{10})$ such that for all $0 < \varepsilon < \varepsilon_4$,*

$$\int_{B_r(y)} \left(\frac{\varepsilon\,|\nabla u^\varepsilon|^2}{2} - \frac{W(u^\varepsilon)}{\varepsilon} \right)_+ (x,t)\,dx \le c_{12}\varepsilon^{1/4} r^{n-1}. \tag{5.102}$$

Proof.
Let $y \in \Omega$, $r \in (\varepsilon^{3/4}, \frac{1}{2})$, $t_0 \in [2\varepsilon^{3/2}, T_1]$. Define

$$\tilde{A} := \left\{ x \in B_{2r}(y) :\ \text{for } t_0 - \varepsilon^{3/2} \le t \le t_0,\ \frac{1}{2} - \gamma \le u^\varepsilon(x,t) \le \frac{1}{2} + \gamma \right\}, \tag{5.103}$$

and

$$A := \left\{ x \in B_{2r + 2c_9\varepsilon^{3/4}}(y) : \operatorname{dist}(\tilde{A}, x) < 2c_9\varepsilon^{3/4} \right\}. \tag{5.104}$$

Claim 1: $\int_{A \cap B_r(y)} \left(\frac{\varepsilon|\nabla u^\varepsilon|^2}{2} - \frac{W(u^\varepsilon)}{\varepsilon} \right)_+ (x,t_0) \le c(n, D_1, c_6, c_7)\varepsilon^{1/4} r^{n-1}$.
Let $\mathcal{F} = \left\{ \bar{B}_{2c_9\varepsilon^{3/4}}(x) : x \in \tilde{A} \right\}$. $\mathcal{F}$ is a fine covering of A i.e. $A \subset \cup_{B \in \mathcal{F}} B$. Indeed, let $z \in A$. Then,

$$z \in B_{2r + 2c_9\varepsilon^{3/4}}(y) \quad \text{and} \quad \operatorname{dist}(\tilde{A}, z) = \inf_{w \in \tilde{A}} |w - z| < 2c_9\varepsilon^{3/4}.$$

Thus, there exists $x' \in \tilde{A}$ such that

$$\left| x' - z \right| < 2c_9\varepsilon^{3/4}.$$

Therefore, $z \in \bar{B}_{2c_9\varepsilon^{3/4}}(x')$.

Applying the Besicovitch covering theorem, Theorem 4.3.3, on $\mathcal{F}$ we get that there exists a set of pairwise disjoint balls $\left\{ B_{2c_9\varepsilon^{3/4}}(x_i) \right\}_{i=1}^N$ with $N(n)$ such that

$$A \subset \cup_{i=1}^N \bar{B}_{10c_9\varepsilon^{3/4}}(x_i), \qquad \text{with } x_i \in \tilde{A},\ \forall i = 1, \cdots, N. \tag{5.105}$$

For each x_i there exists t_i such that

$$t_0 - \varepsilon^{3/2} \le t_i \le t_0 \quad \text{and} \quad \frac{1}{2} - \gamma \le u^\varepsilon(x_i, t_i) \le \frac{1}{2} + \gamma.$$

Applying Lemma 5.6.8 with s replaced by t_i, y replaced by x_i, t replaced by $t_0 - 2\varepsilon^{3/2}$ and R replaced by $R_i = c_9(t_i + \varepsilon^2 - t_0 + 2\varepsilon^{3/2})^{1/2}$ we obtain

$$c_{10} R_i^{n-1} \le \mu^\varepsilon_{t_0 - 2\varepsilon^{3/2}}(B_{R_i}(x_i)) \quad \text{for } i = 1, \cdots, N. \tag{5.106}$$

Now, using the fact that

$$c_9(\varepsilon^{3/2} + \varepsilon^2)^{1/2} \le R_i \le c_9(2\varepsilon^{3/2} + \varepsilon^2)^{1/2} \le 2c_9\varepsilon^{3/4}, \tag{5.107}$$

we obtain using (5.106) and (5.107)

$$\begin{aligned} \mu^\varepsilon_{t_0 - 2\varepsilon^{3/2}}(B_{2c_9\varepsilon^{3/4}}(x_i)) &\ge \mu^\varepsilon_{t_0 - 2\varepsilon^{3/2}}(B_{R_i}(x_i)) \\ &\ge c_{10} R_i^{n-1} \\ &\ge c_{10} c_9^{n-1} (\varepsilon^{3/2} + \varepsilon^2)^{\frac{n-1}{2}} \\ &\ge c_{10} c_9^{n-1} \varepsilon^{\frac{3}{4}(n-1)}. \end{aligned} \tag{5.108}$$

Also we have that $B_{2c_9\varepsilon^{3/4}}(x_i) \subset B_{2r+2c_9\varepsilon^{3/4}}(y)$ for all $i = 1, \cdots, N$. Indeed, let $z \in B_{2c_9\varepsilon^{3/4}}(x_i)$. Then, $|z - x_i| \leq 2c_9\varepsilon^{3/4}$. Moreover, $|z - y| \leq |z - x_i| + |x_i - y| \leq 2c_9\varepsilon^{3/4} + 2r$, which implies that $z \in B_{2r+2c_9\varepsilon^{3/4}}(y)$. For this reason together with (5.108) and the fact that $\left\{B_{2c_9\varepsilon^{3/4}}(x_i)\right\}_{i=1}^N$ is a set of disjoint balls, we obtain

$$\begin{aligned} Nc_{10}c_9^{n-1}\varepsilon^{\frac{3}{4}(n-1)} &\leq \sum_{i=1}^N \mu^\varepsilon_{t_0-2\varepsilon^{3/2}}(B_{2c_9\varepsilon^{3/4}}(x_i)) \\ &= \mu^\varepsilon_{t_0-2\varepsilon^{3/2}}(\cup_{i=1}^N B_{2c_9\varepsilon^{3/4}}(x_i)) \\ &\leq \mu^\varepsilon_{t_0-2\varepsilon^{3/2}}(B_{2r+2c_9\varepsilon^{3/4}}(y)) \end{aligned} \tag{5.109}$$

Combining (5.105), (5.108) with (5.60) and the fact that $r > \varepsilon^{3/4}$

$$\begin{aligned} \mathcal{L}^n(A) &\leq N w_n (10c_9\varepsilon^{3/4})^n \\ &\leq \frac{10^n c_9 w_n}{c_{10}}\, \varepsilon^{3/4} \mu^\varepsilon_{t_0-2\varepsilon^{3/2}}(B_{2r+2c_9\varepsilon^{3/4}}(y)) \\ &\leq \frac{10^n c_9 w_n w_{n-1} D_1}{c_{10}} \varepsilon^{3/4}(2r + 2c_9\varepsilon^{3/4})^{n-1} \\ &\leq \frac{10^n c_9 w_n w_{n-1} D_1}{c_{10}} \varepsilon^{3/4}(2 + 2c_9)^{n-1} r^{n-1}. \end{aligned} \tag{5.110}$$

Now, (5.110) with (5.76) imply

$$\int_{A\cap B_r(y)} \left(\frac{\varepsilon\left|\nabla u^\varepsilon\right|^2}{2} - \frac{W(u^\varepsilon)}{\varepsilon}\right)_+ (x, t_0) \leq \mathcal{L}^n(A) 2\varepsilon^{-1/2} \leq c(n, D_1, c_9, c_{10})\varepsilon^{1/4} r^{n-1}. \tag{5.111}$$

Claim 2: $\int_{B_r(y)\setminus A} \left(\frac{\varepsilon|\nabla u^\varepsilon|^2}{2} - \frac{W(u^\varepsilon)}{\varepsilon}\right)_+ (x, t_0) \leq c(n, D_1, k)\varepsilon^{1/2} r^{n-1}$.
Computing the gradient of our problem (P^ε) we get

$$\begin{aligned} \partial_t(\nabla u^\varepsilon) &= \Delta(\nabla u^\varepsilon) - \frac{W''(u^\varepsilon)}{\varepsilon^2}\nabla u^\varepsilon - \nabla^2\chi(q_\varepsilon)\cdot\nabla u^\varepsilon - \nabla^2 u^\varepsilon \cdot \nabla\chi(q_\varepsilon) \\ &\quad -\nabla u^\varepsilon \Delta\chi(q_\varepsilon) - u^\varepsilon \nabla\Delta\chi(q_\varepsilon) + \frac{\alpha}{\varepsilon}\nabla u^\varepsilon - \frac{2\alpha}{\varepsilon} u^\varepsilon \nabla u^\varepsilon. \end{aligned} \tag{5.112}$$

Define $\phi \in Lip(B_{2r}(y))$ as

$$\phi(x) := \begin{cases} 1 & \text{if } x \in B_r(y)\setminus A, \\ 0 & \text{if } dist(x, B_r(y)\setminus A) \geq \varepsilon^{3/4}, \end{cases}$$

such that

$$|\nabla\phi| \leq 2\varepsilon^{-3/4} \quad \text{and} \quad 0 \leq \phi \leq 1. \tag{5.113}$$

Testing $\phi^2\nabla u^\varepsilon$ to (5.112) with an integration by parts we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega \frac{1}{2}\left|\nabla u^\varepsilon\right|^2 \phi^2\, dx &= -\int_\Omega \left|\nabla^2 u^\varepsilon\right|^2 \phi^2\, dx - 2\int_\Omega \phi\nabla\phi \otimes \nabla u^\varepsilon \cdot \nabla^2 u^\varepsilon - \frac{1}{\varepsilon^2}\int_\Omega W''(u^\varepsilon)\left|\nabla u^\varepsilon\right|^2 \phi^2\, dx \\ &\quad - \int_\Omega \phi^2 \nabla^2\chi(q_\varepsilon)\cdot\nabla u^\varepsilon \otimes \nabla u^\varepsilon - \int_\Omega \phi^2 \nabla^2 u^\varepsilon \cdot \nabla\chi(q_\varepsilon) \otimes \nabla u^\varepsilon \\ &\quad - \int_\Omega \phi^2 \left|\nabla u^\varepsilon\right|^2 \Delta\chi(q_\varepsilon) - \int_\Omega \phi^2 u^\varepsilon \nabla u^\varepsilon \cdot \nabla\Delta\chi(q_\varepsilon) \\ &\quad + \int_\Omega \frac{\alpha}{\varepsilon}\left|\nabla u^\varepsilon\right|^2 \phi^2 - \frac{2\alpha}{\varepsilon}\int_\Omega u^\varepsilon \left|\nabla u^\varepsilon\right|^2 \phi^2. \end{aligned}$$

Furthermore, we have that $spt\phi \cap \tilde{A} = \emptyset$. Indeed, $spt\phi \subset \left\{x : dist(x, B_r(y)\backslash A) < \varepsilon^{3/4}\right\}$. Also by the definition of A, $dist(\tilde{A}, B_r(y)\backslash A) = \inf_{x\in B_r(y)\backslash A} dist(\tilde{A}, x) > 2c_9\varepsilon^{3/4} > \varepsilon^{3/4}$, which imply that $spt\phi \cap \tilde{A} = \emptyset$. Therefore, by the definition of $\tilde{A}$ we get

$$u^\varepsilon(x,s) > \frac{1}{2} + \gamma \quad \text{or} \quad u^\varepsilon(x,s) < \frac{1}{2} - \gamma \quad \text{for } x \in spt\phi,\ s \in [t_0 - \varepsilon^{3/2}, t_0].$$

Then, Young's inequality together with (5.5) imply

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{1}{2}\left|\nabla u^\varepsilon\right|^2\phi^2\,dx \;\leq\; & -\int_\Omega \left|\nabla^2 u^\varepsilon\right|^2\phi^2\,dx + \frac{1}{4}\int_\Omega \left|\nabla^2 u^\varepsilon\right|^2\phi^2 + 4\int_\Omega |\nabla\phi|^2\left|\nabla u^\varepsilon\right|^2 - \frac{k}{\varepsilon^2}\int_\Omega \left|\nabla u^\varepsilon\right|^2\phi^2\,dx \\
& + \int_\Omega \left|\nabla^2\chi(q_\varepsilon)\right|\left|\nabla u^\varepsilon\right|^2\phi^2 + \frac{1}{4}\int_\Omega \left|\nabla^2 u^\varepsilon\right|^2\phi^2 + \int_\Omega |\nabla\chi(q_\varepsilon)|^2\left|\nabla u^\varepsilon\right|^2\phi^2 \\
& + \int_\Omega |\Delta\chi(q_\varepsilon)|\left|\nabla u^\varepsilon\right|^2\phi^2 + \frac{1}{2}\int_\Omega \left|\nabla u^\varepsilon\right|^2\phi^2 + \frac{1}{2}\int_\Omega (u^\varepsilon)^2\left|\nabla\Delta\chi(q_\varepsilon)\right|^2 \\
& + \int_\Omega \frac{\alpha}{\varepsilon}\left|\nabla u^\varepsilon\right|^2\phi^2.
\end{aligned}$$

Hence, for sufficiently small ε depending on C_q and α we can have

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\left|\nabla u^\varepsilon\right|^2\phi^2\,dx \leq \frac{-k}{2\varepsilon^2}\int_\Omega \left|\nabla u^\varepsilon\right|^2\phi^2 + 16\,\varepsilon^{-3/2}\int_{spt\phi}\left|\nabla u^\varepsilon\right|^2. \tag{5.114}$$

Since we have that

$$\begin{aligned}
\frac{d}{dt}\left(e^{-\frac{k}{\varepsilon^2}(t_0-t)}\int_\Omega \frac{1}{2}\left|\nabla u^\varepsilon\right|^2\phi^2\,dx\right) \;=\; & e^{-\frac{k}{\varepsilon^2}(t_0-t)}\frac{d}{dt}\int_\Omega \frac{1}{2}\left|\nabla u^\varepsilon\right|^2\phi^2\,dx \\
& + \frac{k}{\varepsilon^2}e^{-\frac{k}{\varepsilon^2}(t_0-t)}\int_\Omega \frac{1}{2}\left|\nabla u^\varepsilon\right|^2\phi^2\,dx,
\end{aligned}$$

and multiplying (5.114) by $e^{-\frac{k}{\varepsilon^2}(t_0-t)}$ and integrating over $[t_0 - \varepsilon^{3/2}, t_0]$ we obtain

$$\begin{aligned}
\int_\Omega \frac{1}{2}\left|\nabla u^\varepsilon(t_0,.)\right|^2\phi^2(x)\,dx \;\leq\; & e^{-\frac{k}{\varepsilon^{1/2}}}\int_\Omega \frac{1}{2}\left|\nabla u^\varepsilon(t_0-\varepsilon^{3/2},.)\right|^2\phi^2(x)\,dx \\
& + \int_{t_0-\varepsilon^{3/2}}^{t_0} e^{-\frac{k}{\varepsilon^2}(t_0-t)}\frac{16}{\varepsilon^{3/2}}\int_{spt\phi}\left|\nabla u^\varepsilon\right|^2\,dx\,dt.
\end{aligned} \tag{5.115}$$

Let

$$M := \sup_{t\in[t_0-\varepsilon^{3/2},t_0]}\int_{spt\phi}\frac{1}{2}\left|\nabla u^\varepsilon\right|^2(x,t)\,dx.$$

Therefore, we have

$$\int_\Omega \frac{1}{2}\left|\nabla u^\varepsilon(t_0,.)\right|^2\phi^2(x)\,dx \leq \left(e^{-\frac{k}{\varepsilon^{1/2}}} + \frac{32}{k}\varepsilon^{1/2}\right)M. \tag{5.116}$$

Noticing that $spt\phi \subset B_{2r}(y)$ we get

$$\varepsilon M \leq w_{n-1}D_1(2r)^{n-1}.$$

Then,

$$\int_{B_r(y)\backslash A}\frac{\varepsilon}{2}\left|\nabla u^\varepsilon\right|^2(x,t_0)\,dx \leq \int_\Omega \frac{\varepsilon}{2}\left|\nabla u^\varepsilon\right|^2\phi^2(x,t_0)\,dx \leq 33k^{-1}\varepsilon^{1/2}D_1w_{n-1}(2r)^{n-1}, \tag{5.117}$$

for sufficiently small ε depending on k, where we have used the fact that

$$e^{-k\varepsilon^{-1/2}} \leq \frac{1}{k\varepsilon^{-1/2}},$$

for sufficiently small ε depending on k and that $\phi = 1$ on $B_r(y)\backslash A$. Hence, *Claim* 1 and *Claim* 2 imply the result (5.102). □

Lemma 5.6.10. *For $t \in [0, T_1]$ with $t < s$ and $0 < \varepsilon < \varepsilon_4$, there exists $0 < c_{13} := c_{13}(c_{12}, n, T)$ such that we have*

$$\int_0^t \left(\frac{1}{2(s-\lambda)} \int_\Omega \left(\frac{\varepsilon |\nabla u^\varepsilon|^2}{2} - \frac{W(u^\varepsilon)}{\varepsilon} \right)_+ \tilde{\rho}_{(y,s)}(x,\lambda)\, dx \right) d\lambda \leq c_{13} \varepsilon^{\frac{1}{4}} |\log \varepsilon| . \tag{5.118}$$

Proof.
Case i: Let $t \leq 2\varepsilon^{\frac{3}{2}}$. For all $s > t$, we have in view of (5.68) and (5.76)

$$\begin{aligned}
\int_0^t \left(\frac{1}{2(s-\lambda)} \int_\Omega \left(\frac{\varepsilon |\nabla u^\varepsilon|^2}{2} - \frac{W(u^\varepsilon)}{\varepsilon} \right)_+ \tilde{\rho}_{(y,s)}(x,\lambda)\, dx \right) d\lambda &\leq \varepsilon^{-\frac{1}{2}} \sqrt{4\pi} \int_0^t \frac{1}{\sqrt{s-\lambda}}\, d\lambda \\
&\leq \varepsilon^{-\frac{1}{2}} \sqrt{4\pi} (\sqrt{s} - \sqrt{s-t}) \\
&\leq \varepsilon^{-\frac{1}{2}} \sqrt{4\pi} \sqrt{t} \\
&\leq 2\sqrt{2\pi} \varepsilon^{\frac{1}{4}}.
\end{aligned} \tag{5.119}$$

Case ii: Let $t > 2\varepsilon^{\frac{3}{2}}$ with $s - t \geq 2\varepsilon^{\frac{3}{2}}$.
Then, we get by using (5.76)

$$\begin{aligned}
\int_{2\varepsilon^{\frac{3}{2}}}^t \frac{1}{2(s-\lambda)} \int_{B_{\varepsilon^{\frac{3}{4}}}(y)} (\xi^\varepsilon)_+ \tilde{\rho}_{(y,s)}(x,\lambda)\, dx\, d\lambda &\leq \int_{2\varepsilon^{\frac{3}{2}}}^t \frac{(2\varepsilon^{-\frac{1}{2}})}{2(s-\lambda)} \int_{B_{\varepsilon^{\frac{3}{4}}}(y)} \frac{1}{(4\pi(s-\lambda))^{\frac{n-1}{2}}} e^{-\frac{|x-y|^2}{4(s-\lambda)}}\, dx\, d\lambda \\
&\leq \int_{2\varepsilon^{\frac{3}{2}}}^t \frac{\varepsilon^{-\frac{1}{2}}}{(4\pi)^{\frac{n-1}{2}} (s-\lambda)^{\frac{n+1}{2}}} \mathcal{L}_n(B_{\varepsilon^{\frac{3}{4}}})\, d\lambda \\
&\leq \frac{\varepsilon^{-\frac{1}{2}} \varepsilon^{\frac{3}{4}n} w_n}{(4\pi)^{\frac{n-1}{2}}} \int_{2\varepsilon^{\frac{3}{2}}}^t \frac{1}{(s-\lambda)^{\frac{n+1}{2}}}\, d\lambda \\
&= \frac{\varepsilon^{-\frac{1}{2}} \varepsilon^{\frac{3}{4}n} w_n}{(4\pi)^{\frac{n-1}{2}}} \frac{2}{1-n} \left[\left(s - 2\varepsilon^{\frac{3}{2}} \right)^{\frac{1-n}{2}} - (s-t)^{\frac{1-n}{2}} \right] \\
&\leq \frac{\varepsilon^{-\frac{1}{2}} \varepsilon^{\frac{3}{4}n} w_n}{(4\pi)^{\frac{n-1}{2}}} \frac{2}{(n-1)(s-t)^{\frac{n-1}{2}}} \\
&\leq \frac{2w_n}{(n-1)(8\pi)^{\frac{n-1}{2}}} \varepsilon^{\frac{1}{4}}.
\end{aligned} \tag{5.120}$$

Now, estimating outside the ball $B_{\varepsilon^{\frac{3}{4}}}(y)$

$$\begin{aligned}
&\int_{2\varepsilon^{\frac{3}{2}}}^{t} \frac{1}{2(s-\lambda)} \int_{\Omega\setminus B_{\varepsilon^{\frac{3}{4}}}(y)} (\xi^{\varepsilon})_+ \,\tilde{\rho}_{(y,s)}(x,\lambda)\,dx\,d\lambda \\
\leq\ & \int_{2\varepsilon^{\frac{3}{2}}}^{t} \frac{1}{2(4\pi)^{\frac{n-1}{2}}(s-\lambda)^{\frac{n+1}{2}}} \int_{B_{\frac{1}{2}}(y)\setminus B_{\varepsilon^{\frac{3}{4}}}(y)} e^{-\frac{|x-y|^2}{4(s-\lambda)}} (\xi^{\varepsilon})_+ \,dx\,d\lambda \\
\leq\ & \int_{2\varepsilon^{\frac{3}{2}}}^{t} \frac{1}{2(4\pi)^{\frac{n-1}{2}}(s-\lambda)^{\frac{n+1}{2}}} \int_0^1 \left(\int_{B_{\frac{1}{2}}(y)\setminus B_{\varepsilon^{\frac{3}{4}}}(y)\cap\left\{x:\, e^{-\frac{|x-y|^2}{4(s-\lambda)}}\geq \ell\right\}} (\xi^{\varepsilon})_+ \,dx \right) d\ell\,d\lambda \\
\leq\ & \int_{2\varepsilon^{\frac{3}{2}}}^{t} \frac{1}{2(4\pi)^{\frac{n-1}{2}}(s-\lambda)^{\frac{n+1}{2}}} \int_0^{e^{-\frac{1}{16(s-\lambda)}}} \int_{B_{\frac{1}{2}}(y)\setminus B_{\varepsilon^{\frac{3}{4}}}(y)} (\xi^{\varepsilon})_+ \,dx\,d\ell\,d\lambda \\
&+ \int_{2\varepsilon^{\frac{3}{2}}}^{t} \frac{d\lambda}{2(4\pi)^{\frac{n-1}{2}}(s-\lambda)^{\frac{n+1}{2}}} \int_{e^{-\frac{1}{16(s-\lambda)}}}^{e^{-\frac{\varepsilon^{\frac{3}{2}}}{4(s-\lambda)}}} \int_{B_{2\sqrt{(s-\lambda)\ln \ell^{-1}}}(y)} (\xi^{\varepsilon})_+ \,dx\,d\ell\,d\lambda \\
\leq\ & c(c_{12},n)\varepsilon^{\frac{1}{4}} \int_{2\varepsilon^{\frac{3}{2}}}^{t} \frac{e^{\frac{-1}{16(s-\lambda)}} + (s-\lambda)^{\frac{n-1}{2}}}{(s-\lambda)^{\frac{n+1}{2}}}\,d\lambda \\
\leq\ & c(c_{12},n)\varepsilon^{\frac{1}{4}} \left(\int_{2\varepsilon^{\frac{3}{2}}}^{t} e^{\frac{-1}{32(s-\lambda)}}\,d\lambda + \int_{2\varepsilon^{\frac{3}{2}}}^{t} \frac{1}{s-\lambda}\,d\lambda \right) \\
\leq\ & c(c_{12},n,T)\varepsilon^{\frac{1}{4}}(1+|\ln\varepsilon|),
\end{aligned} \tag{5.121}$$

where we have used (5.102) and the fact that

$$\int_{e^{-\frac{1}{16(s-\lambda)}}}^{e^{-\frac{\varepsilon^{3/2}}{4(s-\lambda)}}} (\ln \ell^{-1})^{\frac{n-1}{2}}\,d\ell = \int_{\frac{\varepsilon^{3/2}}{4(s-\lambda)}}^{\frac{1}{16(s-\lambda)}} p^{\frac{n-1}{2}}\, e^{-p}\,dp \leq c(n) e^{-\frac{\varepsilon^{3/2}}{8(s-\lambda)}} \leq c(n).$$

Case iii: Let $t > 2\varepsilon^{\frac{3}{2}}$ with $s - t \leq 2\varepsilon^{\frac{3}{2}}$.
In this case we will get

$$\begin{aligned}
&\int_0^t \left(\frac{1}{2(s-\lambda)} \int_\Omega (\xi^{\varepsilon})_+ \,\tilde{\rho}_{(y,s)}(x,\lambda)\,dx \right) d\lambda \\
=\ & \int_0^{s-2\varepsilon^{\frac{3}{2}}} \left(\frac{1}{2(s-\lambda)} \int_\Omega (\xi^{\varepsilon})_+ \,\tilde{\rho}_{(y,s)}(x,\lambda)\,dx \right) d\lambda + \int_{s-2\varepsilon^{\frac{3}{2}}}^{t} \left(\frac{1}{2(s-\lambda)} \int_\Omega (\xi^{\varepsilon})_+ \,\tilde{\rho}_{(y,s)}(x,\lambda)\,dx \right) d\lambda \\
:=\ & I_1 + I_2.
\end{aligned} \tag{5.122}$$

If $s - 2\varepsilon^{\frac{3}{2}} \leq \varepsilon^{\frac{3}{2}}$ the first term I_1 can be estimated as the first case. If $s - 2\varepsilon^{\frac{3}{2}} \geq \varepsilon^{\frac{3}{2}}$ then it can be estimated as the second case where we take $t = 2\varepsilon^{\frac{3}{2}}$. Now, to estimate the second term I_2 we use (5.76) to get

$$\begin{aligned}
I_2 &\leq \varepsilon^{-\frac{1}{2}}\sqrt{4\pi} \int_{s-2\varepsilon^{\frac{3}{2}}}^{t} \frac{1}{\sqrt{s-\lambda}}\,d\lambda \\
&\leq \varepsilon^{-\frac{1}{2}}\sqrt{4\pi}\sqrt{2\varepsilon^{\frac{3}{2}}} \\
&\leq 2\sqrt{2\pi}\varepsilon^{\frac{1}{4}}.
\end{aligned} \tag{5.123}$$

Combining the three cases we obtain the desired result. □

Remark 5.6.11. *(monotonicity formula)*
We now have that for $0 < \varepsilon < \varepsilon_4$, $y \in \Omega$, $t_0 < t_1 < s$ *with* $t_0, t_1 \in [0, T]$ *we have*

$$\int_\Omega \tilde{\rho}\, d\mu^\varepsilon_{t_1} + \int_{t_0}^{t_1} \frac{1}{2(s-t)} \int_\Omega |\xi^\varepsilon|\, \tilde{\rho}\, dx\, dt \;\leq\; \int_\Omega \tilde{\rho}\, d\mu^\varepsilon_{t_0} + c_{13}\varepsilon^{\frac{1}{4}} |\ln \varepsilon| + 3(t_1 - t_0) f_{q,\alpha} D_1 + \varepsilon(t_1 - t_0) C(Cq) + c_6(t_1 - t_0) D_1, \quad (5.124)$$

where $f_{q,\alpha}$ *is given in (5.39).*

Proposition 5.6.12. *Let* $c_{14} := c_{14}(n) > 1$, $c_{15} := c_{15}(n, c_6)$ *and* $\varepsilon_5 := \varepsilon_5(c_6, c_{13}, C_q) > 0$. *Suppose that for any* $0 \leq t_0 \leq t_1 \leq T_1$ *with* $t_1 - t_0 \leq 1$ *we have* $D^\varepsilon(t_1) = c_{14} D^\varepsilon(t_0)$ *et* $\sup_{t \in (t_0, t_1)} D^\varepsilon(t) \leq c_{14} D(t_0)$. *Then, for all* $0 < \varepsilon < \varepsilon_5$ *we have*

$$(t_1 - t_0) f_{q,\alpha} \geq c_{15}.$$

Proof.
In the case where $\sqrt{s - t_0} \geq \frac{1}{2}$ we have that

$$\int_\Omega \tilde{\rho}_{(y,s)}(x, t_0)\, d\mu^\varepsilon_{t_0} \leq \int_{B_{\frac{1}{2}(y)}} \frac{1}{(4\pi(s - t_0))^{\frac{n-1}{2}}}\, d\mu^\varepsilon_{t_0} \leq \frac{1}{(\frac{1}{2})^{n-1}} \mu^\varepsilon_{t_0}(B_{\frac{1}{2}}(y)) \leq c(n) D(t_0).$$

Otherwise, in the case where $\sqrt{s - t_0} < \frac{1}{2}$ we apply Lemma 5.6.7 with $r = 1$ and $R^2 = s - t_0$, together with the fact that we can always find a constant $c(n) > 0$ such that $\frac{1}{R^{n-1}} e^{-\frac{1}{R}} \leq c(n)$, we get

$$\int_\Omega \tilde{\rho}_{(y,s)}(x, t_0)\, d\mu^\varepsilon_{t_0} \leq c(n) D(t_0). \quad (5.125)$$

Therefore, (5.125) is true for any $s \geq t_0$ and for $c^* := c(n)$.

Now, let

$$1 < c_{14}(n) := \max\left\{ \frac{2(4^{n-1})}{w_{n-1}}, \frac{(1 + c_6 + c^*)(4\pi)^{\frac{n-1}{2}}}{w_{n-1} e^{-\frac{1}{4}}} \right\}.$$

By the definition of $D^\varepsilon(t)$ we have three possible cases.
Case 1: $D^\varepsilon(t_1) = \mu^\varepsilon_{t_1}(\Omega)$.
By a computations similar to that in (5.43) and (5.38) we have

$$\begin{aligned} \mu^\varepsilon_{t_1}(\Omega) &\leq \mu^\varepsilon_{t_0}(\Omega) + \frac{1}{2\varepsilon} \int_{t_0}^{t_1} \int_\Omega (g_\varepsilon)^2 \\ &\leq \mu^\varepsilon_{t_0}(\Omega) + 3(t_1 - t_0) f_{q,\alpha} \sup_{t \in (t_1, t_0)} D^\varepsilon(t) + \varepsilon(t_1 - t_0) C(C_q), \end{aligned} \quad (5.126)$$

where $f_{q,\alpha}$ is given in (5.39). Using the fact that $D^\varepsilon(t_1) = c_{14} D^\varepsilon(t_0)$, $\sup_{t \in (t_0, t_1)} D^\varepsilon(t) \leq c_{14} D^\varepsilon(t_0)$, $\frac{4^{n-1}}{w_{n-1}} \geq 1$ together with (5.126)

$$\begin{aligned} c_{14} D^\varepsilon(t_0) = D^\varepsilon(t_1) &\leq \frac{4^{n-1}}{w_{n-1}} \mu^\varepsilon_{t_1}(\Omega) \\ &\leq \frac{4^{n-1}}{w_{n-1}} \left(D^\varepsilon(t_0) + 3(t_1 - t_0) f_{q,\alpha} c_{14} D^\varepsilon(t_0) + \varepsilon(t_1 - t_0) C(C_q) \right). \end{aligned}$$

Then for sufficiently small ε depending on C_q we can choose $\dfrac{\varepsilon(t_1-t_0)C(C_q)}{D^\varepsilon(t_0)} \le \dfrac{1}{2}$, which makes sense since we have by definition that $D^\varepsilon(t) \ge 1$ for ever $t \in [0,T]$. Together with $\frac{4^{n-1}}{w_{n-1}} \le \frac{c_{14}}{2}$ we get that

$$(t_1-t_0)f_{q,\alpha} \ge \frac{1}{6c_{14}}. \tag{5.127}$$

Case 2: There exists $B_r(y) \subset \Omega$ such that $D^\varepsilon(t_1) = \dfrac{1}{w_{n-1}r^{n-1}}\,\mu^\varepsilon_{t_1}(B_r(y))$ and $r \ge \frac{1}{4}$.
In this case we have

$$\frac{w_{n-1}}{4^{n-1}} D^\varepsilon(t_1) \le w_{n-1}r^{n-1}D^\varepsilon(t_1) = \mu^\varepsilon_{t_1}(B_r(y)) \le \mu^\varepsilon_{t_1}(\Omega).$$

Therefore, arguing similarly as in the case 1 we get (5.127).

Case 3: There exists $B_r(y) \subset \Omega$ such that $D^\varepsilon(t_1) = \dfrac{1}{w_{n-1}r^{n-1}}\mu^\varepsilon_{t_1}(B_r(y))$ and $r < \frac{1}{4}$.
The monotonicity formula (5.65) together with (5.118) yeild to

$$\begin{aligned}\int_\Omega \tilde\rho_{(y,s)}(x,t_1)\,d\mu^\varepsilon_{t_1} \quad \le \quad & \int_\Omega \tilde\rho_{(y,s)}(x,t_0)\,d\mu^\varepsilon_{t_0} + c_{13}\varepsilon^{\frac14}\,|\ln\varepsilon| \\ & +3\,c_{14}(t_1-t_0)f_{q,\alpha}D^\varepsilon(t_0) + \varepsilon(t_1-t_0)C(C_q) \\ & +c_6(t_1-t_0)\left(D^\varepsilon(t_0) + 3c_{14}(t_1-t_0)f_{q,\alpha}D^\varepsilon(t_0) + \varepsilon(t_1-t_0)C(C_q)\right).\end{aligned} \tag{5.128}$$

For $r \le \frac14$ with $r^2 = s - t_1$

$$\begin{aligned}\int_\Omega \tilde\rho_{(y,s)}(x,t_1)\,d\mu^\varepsilon_{t_1} \ge \int_{B_r(y)} \rho_{(y,s)}(x,t_1)\,d\mu^\varepsilon_{t_1} \quad &\ge \quad \frac{e^{-\frac14}}{(4\pi)^{\frac{n-1}{2}}r^{n-1}}\mu^\varepsilon_{t_1}(B_r(y)) \\ &= \quad \frac{D^\varepsilon(t_1)w_{n-1}e^{-\frac14}}{(4\pi)^{\frac{n-1}{2}}} \\ &\ge \quad (1+c^*+c_6)D^\varepsilon(t_0).\end{aligned} \tag{5.129}$$

Choosing ε small enough depending on c_{13}, c_6 and C_q we can choose

$$\varepsilon C(C_q)(1+c_6) + c_{13}\varepsilon^{\frac14}\,|\ln\varepsilon| < \frac{1}{2}. \tag{5.130}$$

Using (5.128), (5.129) and (5.130) we get

$$(t_1-t_0)f_{q,\alpha} \ge \frac{1}{6c_{14}(1+c_6)}. \tag{5.131}$$

□

Theorem 5.6.13. *($T_1 = T$)*
Let $0 < T_b \le 1$ that satisfies $T_b\, f_{q,\alpha} \le c_{15}$. Then we have

$$D^\varepsilon(t) \le D_0\, c_{14}^{[\frac{t}{T_b}]+1} := D_1 \quad \textit{for all } t \in [0,T]. \tag{5.132}$$

Proof.
Suppose that there exists $t \in [0,T]$ such that (5.132) fails. Then there must have $T_1 \in (0,T)$ such that (5.132) is true for all $t \in (0,T_1)$ and $D^\varepsilon(T_1) = D_0 c_{14}^{[\frac{T_1}{T_b}]+1}$. We fix

$$D_1 := D_0\, c_{14}^{[\frac{T_1}{T_b}]+1},$$

then we have $D^\varepsilon(t) \le D_1$ for all $t \in [0,T_1]$.

Suppose that we have $T_1 \le T_b$. In this case we will have $D^\varepsilon(T_1) = c_{14}D_0$ and $\sup_{t\in(0,T_1)} D^\varepsilon(t) \le c_{14}D_0$ and we can apply Proposition 5.6.12 with $t_0 = 0$ and $t_1 = T_1$ to get that $T_1 f_{q,\alpha} \ge c_{15}$ which contradicts the fact that $T_1 < T_b$ and the assumption that $T_b f_{q,\alpha} < c_{15}$. Therefore, $T_1 \ge T_b$.

Suppose that we have $T_b \le T_1 < 2T_b$. In this case we will have $D^\varepsilon(T_1) = c_{14}^2 D_0$,

$$D^\varepsilon(t) \le c_{14}D_0 \quad \text{for all } t \in [0,T_b)$$

and

$$D^\varepsilon(t) \le c_{14}^2 D_0 \quad \text{for all } t \in [T_b,T_1].$$

Therefore, there exists $t_0 \in [T_b,T_1)$ such that $D^\varepsilon(t_0) = c_{14}D_0$. Applying Proposition 5.6.12 with $t_1 = T_1$ we also get a contradiction. By repeating the same argument we get a contradiction. □

5.7 Vanishing of the discrepancy

Lemma 5.7.1. *For any $(x',t') \in sptμ$ there exist a sequence $\{(x_i,t_i)\}_{i=1}^\infty$ and a subsequence denoted again by ε such that $t_i > 0$, $x_i \in \Omega$, $(x_i,t_i) \to (x',t')$ as $i \to \infty$ and $\frac{1}{2} - \gamma < u^\varepsilon(x_i,t_i) < \frac{1}{2} + \gamma$ for all $i \in \mathbb{N}$, where γ is given in (5.5).*

Proof.
Assume for a contradiction that there exists $0 < r_0 < \sqrt{t'}$ and a neighborhood $(B_{r_0}(x') \cap \Omega) \times (t'-r_0^2, t'+r_0^2)$ of (x',t') such that for small ε we have for all $(x,t) \in (B_{r_0}(x') \cap \Omega) \times (t'-r_0^2, t'+r_0^2)$ we have

$$u^\varepsilon > \frac{1}{2} + \gamma \quad \text{or} \quad u^\varepsilon < \frac{1}{2} - \gamma. \tag{5.133}$$

Computing the gradient of our problem (P^ε) we get

$$\begin{aligned} \partial_t(\nabla u^\varepsilon) &= \Delta(\nabla u^\varepsilon) - \frac{W''(u^\varepsilon)}{\varepsilon^2}\nabla u^\varepsilon - \nabla^2\chi(q_\varepsilon)\cdot\nabla u^\varepsilon - \nabla^2 u^\varepsilon \cdot \nabla\chi(q_\varepsilon) \\ &\quad -\nabla u^\varepsilon \Delta\chi(q_\varepsilon) - u^\varepsilon \nabla\Delta\chi(q_\varepsilon) + \frac{\alpha}{\varepsilon}\nabla u^\varepsilon - 2\frac{\alpha}{\varepsilon}u^\varepsilon \nabla u^\varepsilon. \end{aligned} \tag{5.134}$$

Let $\phi \in C_c^2(B_{r_0}(x'))$ such that

$$|\nabla\phi| \le \frac{2}{r_0}, \quad \phi|_{B_{\frac{r_0}{3}}} \equiv 1, \quad 0 \le \phi \le 1 \text{ on } B_{r_0}(x'). \tag{5.135}$$

Testing $\phi^2\nabla u^\varepsilon$ to (5.134) with an integration by parts we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{1}{2}|\nabla u^\varepsilon|^2\phi^2\,dx &= -\int_\Omega \left|\nabla^2 u^\varepsilon\right|^2\phi^2\,dx - 2\int_\Omega \phi\nabla\phi\otimes\nabla u^\varepsilon\cdot\nabla^2 u^\varepsilon - \frac{1}{\varepsilon^2}\int_\Omega W''(u^\varepsilon)|\nabla u^\varepsilon|^2\phi^2\,dx\\ &\quad -\int_\Omega \phi^2\nabla^2\chi(q_\varepsilon)\cdot\nabla u^\varepsilon\otimes\nabla u^\varepsilon - \int_\Omega \phi^2\nabla^2 u^\varepsilon\cdot\nabla\chi(q_\varepsilon)\otimes\nabla u^\varepsilon\\ &\quad -\int_\Omega \phi^2|\nabla u^\varepsilon|^2\Delta\chi(q_\varepsilon) - \int_\Omega \phi^2 u^\varepsilon\nabla u^\varepsilon\cdot\nabla\Delta\chi(q_\varepsilon)\\ &\quad +\int_\Omega \frac{\alpha}{\varepsilon}|\nabla u^\varepsilon|^2\phi^2 - \frac{2\alpha}{\varepsilon}\int_\Omega u^\varepsilon|\nabla u^\varepsilon|^2\phi^2.\end{aligned}$$

Since $u^\varepsilon > \frac{1}{2}+\gamma$ or $u^\varepsilon < \frac{1}{2}-\gamma$ we have that $W''(u^\varepsilon) > k$. Furthermore, Young's inequality implies

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{1}{2}|\nabla u^\varepsilon|^2\phi^2\,dx &\le -\int_\Omega \left|\nabla^2 u^\varepsilon\right|^2\phi^2\,dx + \frac{1}{4}\int_\Omega \left|\nabla^2 u^\varepsilon\right|^2\phi^2 + 4\int_\Omega |\nabla\phi|^2|\nabla u^\varepsilon|^2 - \frac{k}{\varepsilon^2}\int_\Omega |\nabla u^\varepsilon|^2\phi^2\,dx\\ &\quad +\int_\Omega \left|\nabla^2\chi(q_\varepsilon)\right||\nabla u^\varepsilon|^2\phi^2 + \frac{1}{4}\int_\Omega \left|\nabla^2 u^\varepsilon\right|^2\phi^2 + \int_\Omega |\nabla\chi(q_\varepsilon)|^2|\nabla u^\varepsilon|^2\phi^2\\ &\quad +\int_\Omega |\Delta\chi(q_\varepsilon)||\nabla u^\varepsilon|^2\phi^2 + \frac{1}{2}\int_\Omega |\nabla u^\varepsilon|^2\phi^2 + \frac{1}{2}\int_\Omega (u^\varepsilon)^2|\nabla\Delta\chi(q_\varepsilon)|^2\\ &\quad +\int_\Omega \frac{\alpha}{\varepsilon}|\nabla u^\varepsilon|^2\phi^2.\end{aligned}$$

which implies

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|\nabla u^\varepsilon|^2\phi^2\,dx \le \frac{-k}{2\varepsilon^2}\int_\Omega |\nabla u^\varepsilon|^2\phi^2 + \frac{16}{r_0^2}\int_{spt\phi}|\nabla u^\varepsilon|^2, \tag{5.136}$$

for sufficiently small ε depending on C_q, α. Since we have that

$$\begin{aligned}\frac{d}{dt}\left(e^{\frac{k}{\varepsilon^2}(t-t'+r_0^2)}\int_\Omega \frac{1}{2}|\nabla u^\varepsilon|^2\phi^2\,dx\right) &= e^{\frac{k}{\varepsilon^2}(t-t'+r_0^2)}\frac{d}{dt}\int_\Omega \frac{1}{2}|\nabla u^\varepsilon|^2\phi^2\,dx\\ &\quad +\frac{k}{\varepsilon^2}e^{\frac{k}{\varepsilon^2}(t-t'+r_0^2)}\int_\Omega \frac{1}{2}|\nabla u^\varepsilon|^2\phi^2\,dx,\end{aligned}$$

multiplying (5.136) by $e^{\frac{k}{\varepsilon^2}(t-t'+r_0^2)}$ and integrating over $[t'-r_0^2,\lambda]$ then multiplying by $e^{-\frac{k}{\varepsilon^2}(\lambda-t'+r_0^2)}$ give us

$$\begin{aligned}\int_\Omega \frac{1}{2}|\nabla u^\varepsilon|^2\phi^2(x,\lambda) &\le e^{-\frac{k}{\varepsilon^2}(\lambda-t'+r_0^2)}\int_\Omega \frac{1}{2}|\nabla u^\varepsilon|^2\phi^2(x,t'-r_0^2) &(5.137)\\ &\quad +\int_{t'-r_0^2}^{\lambda} e^{\frac{k}{\varepsilon^2}(t-t'+r_0^2)}\frac{16}{r_0^2}\int_{spt\phi}|\nabla u^\varepsilon|^2. &(5.138)\end{aligned}$$

Let

$$M := \sup_{\lambda\in[t'-r_0^2,t'+r_0^2]}\int_{spt\phi}\frac{1}{2}|\nabla u^\varepsilon|^2(x,\lambda)\,dx.$$

Therefore, we have

$$\int_\Omega \frac{1}{2}|\nabla u^\varepsilon|^2\phi^2(x,\lambda)\,dx \le \left(e^{-\frac{k}{\varepsilon^2}(\lambda-t'+r_0^2)} + 32\frac{\varepsilon^2}{kr_0^2}\right)M. \tag{5.139}$$

Since εM is uniformly bounded we deduce that

$$\lim_{\varepsilon\to 0} \sup_{\lambda\in[t'-\frac{r_0^2}{2},t'+r_0^2]} \int_\Omega \frac{\varepsilon}{2} \left|\nabla u^\varepsilon\right|^2 \phi^2(x,\lambda)\, dx = 0. \tag{5.140}$$

Next, since our u^ε solution is continuous we assume that $\frac{1}{2}+\gamma \le u^\varepsilon \le 1+\varepsilon$ on $B_{r_0}(x') \times (t'-r_0^2, t'+r_0^2)$. Testing $(u^\varepsilon-1)\phi^2$ to (P^ε) we get

$$\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega (u^\varepsilon-1)^2\phi^2 = \varepsilon\int_\Omega \Delta u^\varepsilon (u^\varepsilon-1)\phi^2 - \frac{1}{\varepsilon}\int_\Omega W'(u^\varepsilon)(u^\varepsilon-1)\phi^2 + \int_\Omega g_\varepsilon (u^\varepsilon-1)\phi^2,$$

where

$$\varepsilon\int_\Omega \Delta u^\varepsilon (u^\varepsilon-1)\phi^2 = -\varepsilon\int_\Omega |\nabla u|^2\phi^2 - 2\varepsilon\int_\Omega \phi\nabla\phi\cdot\nabla u^\varepsilon (u^\varepsilon-1) \le \varepsilon\int_\Omega |\nabla\phi|^2 (u^\varepsilon-1)^2$$

Integrating over $(t'-r_0^2, t'+r_0^2)$ we get

$$\frac{\varepsilon}{2}\int_\Omega (u^\varepsilon-1)^2\phi^2\, dx\Big|_{t=t'+r_0^2} - \frac{\varepsilon}{2}\int_\Omega (u^\varepsilon-1)^2\phi^2\, dx\Big|_{t=t'-r_0^2} \le$$
$$\varepsilon\int_{t'-r_0^2}^{t'+r_0^2} dt\int_\Omega |\nabla\phi|^2 (u^\varepsilon-1)^2 - \frac{1}{\varepsilon}\int_{t'-r_0^2}^{t'+r_0^2} dt\int_\Omega W'(u^\varepsilon)(u^\varepsilon-1)\phi^2\, dx + \int_{t'-r_0^2}^{t'+r_0^2} dt\int_\Omega g_\varepsilon (u^\varepsilon-1)\phi^2.$$

Hence,

$$\int_{t'-r_0^2}^{t'+r_0^2} dt\int_\Omega g_\varepsilon (u^\varepsilon-1)\phi^2 \le \|g_\varepsilon\|_{L^2(\Omega\times(t'-r_0^2,t'+r_0^2))} \|(u^\varepsilon-1)\|_{L^2(\Omega\times(t'-r_0^2,t'+r_0^2))} \le \sqrt{\varepsilon}\, c(c_2,\gamma).$$

Using the fact that

$$W'(s)(s-1) \ge k(s-1)^2 \ge c(W)W(s)$$

for some constant $c(W)>0$ if $\frac{1}{2}+\gamma \le s \le 1+\varepsilon$. Therefore, we get

$$\int_{t'-r_0^2}^{t'+r_0^2} dt\int_\Omega \frac{1}{\varepsilon} W(u^\varepsilon)\phi^2\, dx \quad \le \quad \varepsilon^{1/2} c(W,\gamma,c_2).$$

$$\frac{1}{\varepsilon}\int_{t'-r_0^2}^{t'+r_0^2} dt\int_\Omega W(u^\varepsilon)\phi^2\, dx \to 0 \text{ when } \varepsilon\to 0. \tag{5.141}$$

Thus (5.140) and (5.141) led us to

$$\int_{t'-\frac{r_0^2}{2}}^{t'+r_0^2} \left(\int_\Omega \phi^2\, d\mu_t^\varepsilon\right) dt \to 0 \text{ when } \varepsilon\to 0.$$

Then,

$$\int_{t'-\frac{r_0^2}{2}}^{t'+r_0^2} \left(\int_\Omega \phi^2\, d\mu_t\right) dt \to 0 \text{ when } \varepsilon\to 0.$$

This shows that $(x',t') \notin spt\mu$ and we get a contradition. □

Lemma 5.7.2. *There exist* $0 < \delta_0 := \delta_0(n, c_7)$, $0 < \eta_0 := \eta_0(C_q, D_1, \alpha)$, $0 < \gamma_0 := \gamma_0(\gamma, c_7)$ *such that the following holds: If*

$$\int_\Omega \eta(x-y)\rho_{(y,s)}(x,t)\,d\mu_s(y) < \delta_0, \tag{5.142}$$

for some $0 \le t < s < \frac{T}{2}$ *with* $s-t \le \eta_0$ *and* $x \in \Omega$, *then* $(x',t') \notin spt\mu$ *for all* $x' \in B_{\gamma_0 r}(x) \cap \Omega$, *where* $t' = 2s - t$ *and* $r = \sqrt{2(s-t)}$.

Proof.
Assume for a contradiction that $(x',t') \in spt\mu$. Then by Lemma 5.7.1 there exists a sequence $\{(x_i,t_i)\}_{i=1}^\infty$ and a sequence denoted for simplicity ε such that $(x_i,t_i) \to (x',t')$ as $i \to \infty$ and $\left|u^\varepsilon(x_i,t_i) - \frac{1}{2}\right| < \gamma$ for all $i \in \mathbb{N}$.
Let $r_i := \gamma_0\varepsilon$, where $\gamma_0 > 0$ will be chosen later, and $T_i := t_i + r_i^2$.
We have that for $y \in B_{r_i}(x_i)$,

$$\left|u^\varepsilon(y,t_i) - \frac{1}{2}\right| \le \gamma_0 \sup_{x\in\Omega} \varepsilon\,|\nabla u^\varepsilon(x,t_i)| + \left|u^\varepsilon(x_i,t_i) - \frac{1}{2}\right| \le c_7\gamma_0 + \gamma. \tag{5.143}$$

For this reason we can always find a positif constant c such that for sufficiently small $\gamma_0 > 0$ depending on γ, c_7 and for $y \in B_{r_i}(x_i)$ we have $W(u^\varepsilon(y,t_i)) \ge c$.
Furthermore, using the fact that $\mu_t^\varepsilon(\Omega) \ge \displaystyle\int_\Omega \frac{W(u^\varepsilon(x,t))}{\varepsilon}\,dx$ for all $t \ge 0$ and $0 < \varepsilon < 1$ we can always find a constant δ_0, depending on n and c_7, satisfying

$$\begin{aligned}
&\int_{B_{r_i}(x_i)} \eta(y-x_i)\rho_{(x_i,T_i)}(y,t_i)\,d\mu_{t_i}^\varepsilon(y) \ge \\
&\frac{1}{(4\pi r_i^2)^{\frac{n-1}{2}}} \int_{B_{r_i}(x_i)} \eta(y-x_i)\exp\left(-\frac{|y-x_i|^2}{4r_i^2}\right)\frac{W(u^\varepsilon(y,t_i))}{\varepsilon}\,dy \ge \\
&\frac{c}{(4\pi\gamma_0^2)^{\frac{n-1}{2}}\varepsilon^n} \int_{B_{r_i}(x_i)} \exp\left(-\frac{|y-x_i|^2}{4r_i^2}\right)dy \ge \delta_0.
\end{aligned} \tag{5.144}$$

In addition, we choose $t' - s = s - t \le \eta_0$ small enough depending on C_q, D_1, α so that

$$3(t_i - s)f_{q,\alpha}D_1 + \varepsilon(t_i - s)C(Cq) + c_6(t_i - s)D_1 \le \frac{\delta_0}{2}.$$

Substituting in (5.124) we get

$$\int_{B_{r_i}(x_i)} \eta(y-x_i)\rho_{(x_i,T_i)}(y,t_i)\,d\mu_{t_i}^\varepsilon(y) \le \int_{B_{r_i}(x_i)} \eta(y-x_i)\rho_{(x_i,T_i)}(y,s)\,d\mu_s^\varepsilon(y) + c_{13}\varepsilon^{\frac{1}{4}}\,|\ln\varepsilon| + \frac{\delta_0}{2}.$$

Passing to the limit when $i \to \infty$

$$\frac{\delta_0}{2} \le \int_\Omega \eta(y-x')\rho_{(x',t')}(y,s)\,d\mu_s(y). \tag{5.145}$$

Assuming (5.142) and using Lemma 5.6.2 (3) for any $\delta > 0$ we have

$$\begin{aligned}
\int_\Omega \eta(y-x')\rho_{(x',t')}(y,s)\,d\mu_s(y) &= \int_\Omega \eta(y-x')\rho_{x'}^r(y)\,d\mu_s(y) \\
&\le (1+\delta)\int_\Omega \eta(y-x)\rho_x^r(y)\,d\mu_s(y) + \delta D_0 \\
&= (1+\delta)\int_\Omega \eta(y-x)\rho_{(y,s)}(x,t)\,d\mu_s(y) + \delta D_0 \\
&\le \delta_0(1+\delta) + \delta D_0.
\end{aligned}$$

Since $\delta > 0$ is arbitrary we can choose it such that $\delta_0(1+\delta) + \delta D_0 \le \frac{\delta_0}{4}$. Then we get a contradiction

$$\frac{\delta_0}{2} \le \int_\Omega \eta(y-x')\rho_{(x',t')}(y,s)\, d\mu_s(y) \le \frac{\delta_0}{4}.$$

Hence $(x',t') \notin spt\mu$. □

Lemma 5.7.3. *(forward density lower bounds)*
For $T > 0$, let δ_0 the constant introduced in Lemma 5.7.2. Then we have $\mu(Z^-(T)) = 0$, where

$$Z^-(T) := \left\{ (x,t) \in spt\mu, \quad 0 \le t \le \frac{T}{2} : \limsup_{s \searrow t} \int_\Omega \eta(y-x)\rho_{(y,s)}(x,t)\, d\mu_s(y) < \delta_0 \right\}.$$

Proof.
For $0 < \tau < \frac{r_0^2}{2}$ define

$$Z^\tau := \left\{ (x,t) \in spt\mu, \quad 0 \le t \le \frac{T}{2} : \int_\Omega \eta(x-y)\rho_{(y,s)}(x,t)\, d\mu_s(y) < \delta_0, \text{for } t < s < t+\tau \right\}.$$

If we take a sequence $\tau_m > 0$ with $\lim_{m\to\infty} \tau_m = 0$, then $Z^- \subset \cup_{m=1}^\infty Z^{\tau_m}$. Hence, we only need to show $\mu(Z^\tau) = 0$.
Let $(x,t) \in Z^\tau$ be fixed and we define

$$P(x,t) := \left\{ (x',t') : \gamma_0^{-2} \left|x'-x\right|^2 < \left|t'-t\right| < \tau \right\}.$$

We claim that $P(x,t) \cap Z^\tau = \emptyset$. Indeed, suppose for a contradiction that $(x',t') \in P(x,t) \cap Z^\tau$. Assume $t' > t$ and put $s = \frac{1}{2}(t+t')$. Then $t < s \le t+\tau$, $|x-x'| < \gamma_0\sqrt{|t'-t|} = \gamma_0\sqrt{2(s-t)}$ and

$$\int_\Omega \eta(y-x)\rho_{(y,s)}(x,t)\, d\mu_s(y) < \delta_0.$$

Hence by Lemma 5.7.2, $(x',t') \notin spt\mu$, which contradicts $(x',t') \in Z^\tau$. If $t' < t$, by the similar argument, we obtain $(x',t') \notin spt\mu$ which is a contradiction. This proves $P(x,t) \cap Z^\tau = \emptyset$.
For a fixed $(x_0,t_0) \in \Omega \times (\tau, \frac{T}{2})$, define

$$Z^{\tau,x_0,t_0} := Z^\tau \cap \left(B_1(x_0) \times (t_0-\tau, t_0+\tau)\right).$$

Then Z^τ is a countable union of Z^{τ,x_m,t_m} with (x_m,t_m) spaced appropriately. Hence we only need to show that $\mu(Z^{\tau,x_0,t_0}) = 0$. Denote Z^{τ,x_0,t_0} by Z'. By the Besicovitch covering theorem, for $0 < \rho \le 1$, we may find a covering of $\pi_\Omega(Z') := \{x \in \Omega : (x,t) \in Z'\}$ by a collection of balls $\{B_{r_i}(x_i)\}_{i=1}^\infty$ contained in $B_1(x_0)$ with intersection controlled by $c(n)$, where $(x_i,t_i) \in Z'$, $r_i \le \rho$, so that

$$\sum_{i=1}^\infty w_n r_i^n \le c(n)\mathcal{L}^n\left(B_1(x_0)\right). \tag{5.146}$$

For such covering, we find

$$Z' \subset \cup_{i=1}^\infty B_{r_i}(x_i) \times \left(t_i - r_i^2\gamma_0^{-2}, t_i + r_i^2\gamma_0^{-2}\right). \tag{5.147}$$

Indeed, if $(x,t) \in Z'$, then $x \in B_{r_i}(x_i)$ for some $i \in \mathbb{N}$. Since $P(x_i,t_i) \cap Z^\tau = \emptyset$, we have

$$|t-t_i| \leq |x-x_i|^2 \gamma_0^{-2} < r_i^2 \gamma_0^{-2}.$$

Therefore we obtain by (5.147) and (5.146)

$$\begin{aligned} \mu(Z') &\leq \sum_{i=1}^{\infty} \mu(B_{r_i}(x_i) \times (t_i - r_i^2\gamma_0^{-2}, t_i + r_i^2\gamma_0^{-2})) \leq \sum_{i=1}^{\infty} 2\gamma_0^{-2} D_1 r_i^{n+1} w_{n-1} \\ &\leq 2c(n)\rho\gamma_0^{-2} D_1 w_n^{-1} \mathcal{L}^n\left(B_1(x_0)\right). \end{aligned}$$

Letting $\rho \to 0$ we get $\mu(Z') = 0$. This concludes the proof. □

Proposition 5.7.4. *(Vanishing of the discrepancy)*
There exists a subsequence denoted again by ε and a Radon measure $|\xi|$ such that

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \phi \, d\,|\xi_t^\varepsilon| \, dt = \int_0^T \int_\Omega \phi \, d\,|\xi|$$

for all $0 < T < \infty$ and $\phi \in C_c(\Omega \times [0,\infty))$. Furthermore, we have

$$|\xi| = 0 \quad on \quad \Omega \times (0,\infty). \tag{5.148}$$

Proof.
Since we have that $|\xi_t^\varepsilon|(\Omega) \leq \mu_t^\varepsilon(\Omega) \leq c_1$ the compactness theorem for Radon measures implies the existence of such $|\xi|$.
Once we prove that

$$\limsup_{s \searrow t} \int_\Omega \tilde{\rho} \, d\mu_s(y) = 0 \tag{5.149}$$

for $|\xi|$-almost all (x,t) we get the result: (5.149) implies that $|\xi|(\Omega \times (0,T) \backslash Z^-(T)) = 0$. In addition, since we have that $|\xi| \leq \mu$ by Lemma 5.7.3 we get that $|\xi|(\Omega \times (0,T)) = 0$.
The monotonicity formula implies

$$\begin{aligned} \int_\Omega \tilde{\rho} \, d\mu_s^\varepsilon + \int_0^s \frac{1}{2(s-t)} \int_\Omega |\xi^\varepsilon| \, \tilde{\rho} \, dx \, dt &\leq \int_\Omega \tilde{\rho} \, d\mu_0^\varepsilon + c_{13}\varepsilon^{\frac{1}{4}} |\ln \varepsilon| + 3s f_{q,\alpha} D_1 \\ &\quad + \varepsilon s C(Cq) + c_6 s D_1, \end{aligned} \tag{5.150}$$

which yields to

$$\int_0^s \frac{1}{2(s-t)} \int_\Omega |\xi^\varepsilon| \, \tilde{\rho} \, dx \, dt < +\infty.$$

Then passing to the limit $\varepsilon \to 0$

$$\int_0^s \int_\Omega \frac{1}{2(s-t)} \tilde{\rho} \, |d\xi| < +\infty. \tag{5.151}$$

Now integrating over $(y,s) \in \Omega \times (0,T)$ with respect to $d\mu_s \, ds$ we get

$$\int_0^T ds \int_\Omega d\mu_s \int_0^s \int_\Omega \frac{1}{2(s-t)} \tilde{\rho} \, |d\xi| < +\infty. \tag{5.152}$$

Which gives us by Fubini theorem that

$$\int_0^T \int_\Omega \left(\int_t^T \int_\Omega \frac{1}{2(s-t)} \tilde{\rho} \, d\mu_s(y) \, ds \right) |d\xi|(x,t) < +\infty. \tag{5.153}$$

Therefore,

$$\int_t^T \int_\Omega \frac{1}{2(s-t)} \tilde{\rho}\, d\mu_s(y)\, ds < +\infty, \tag{5.154}$$

for $|\xi|$-almost all $(x,t) \in \Omega \times (0,T)$.
For $t < s$ define $\sigma := \log(s-t)$ and

$$h(s) := \int_\Omega \tilde{\rho}\, d\mu_s(y).$$

Therefore, (5.154) can be written as

$$\int_{-\infty}^{\log(T-t)} h(t+e^\sigma)\, d\sigma < +\infty. \tag{5.155}$$

This implies that h is small enough and we can find a decreasing sequence $\{\sigma_i\}_{i=1}^\infty$ such that for all i we have

$$\sigma_i \to -\infty,\ \sigma_i - \sigma_{i+1} < \theta \text{ and } h(t+e^{\sigma_i}) < \theta. \tag{5.156}$$

Using (5.124) with $\varepsilon \to 0$, $t_0 = t + e^{\sigma_i}$, $t_1 = t + e^\sigma$ and the fact that $\rho_{(y,t+e^\sigma)}(x,t) = \rho_{(x,t+2e^\sigma)}(y, t+e^\sigma)$ we obtain

$$\begin{aligned} h(t+e^\sigma) &= \int_\Omega \eta(y-x)\rho_{(y,t+e^\sigma)}(x,t)\, d\mu_{t+e^\sigma}(y) \\ &= \int_\Omega \eta(y-x)\rho_{(x,t+2e^\sigma)}(y,t+e^\sigma)\, d\mu_{t+e^\sigma}(y) \\ &\le \int_\Omega \eta(y-x)\rho_{(x,t+2e^\sigma)}(y,t+e^{\sigma_i})\, d\mu_{t+e^{\sigma_i}}(y) + c(\theta) \\ &= \int_\Omega \eta(x-y)\rho_x^{R_i}(y)\, d\mu_{t+e^{\sigma_i}}(y) + c(\theta), \end{aligned} \tag{5.157}$$

where $c(\theta) \to 0$ as $\theta \to 0$ and $R_i^2 = 2(2e^\sigma - e^{\sigma_i})$.
In addition,

$$h(t+e^{\sigma_i}) = \int_\Omega \eta(y-x)\rho_{(y,t+e^{\sigma_i})}(x,t)\, d\mu_{t+e^{\sigma_i}}(y) = \int_\Omega \eta(x-y)\rho_x^{r_i}(y)\, d\mu_{t+e^{\sigma_i}}(y) \le \theta, \tag{5.158}$$

where $r_i^2 = 2e^{\sigma_i}$. Note that we have $R_i \ge r_i$ and $R_i^2/r_i^2 < 2e^\theta - 1 \approx 1$ since $0 \le \sigma - \sigma_i < \theta$ and θ can be made small enough. Furthermore, Lemma 5.6.2 (4) implies for any $\delta > 0$

$$\int_\Omega \eta(x-y)\rho_x^{R_i}(y)\, d\mu_{t+e^{\sigma_i}}(y) \le (1+\delta) \int_\Omega \eta(x-y)\rho_x^{r_i}(y)\, d\mu_{t+e^{\sigma_i}}(y) + \delta D_0. \tag{5.159}$$

By (5.157), (5.158) and (5.159) we obtain

$$h(t+e^\sigma) \le (1+\delta)\theta + \delta D_0 + c(\theta), \tag{5.160}$$

which implies (5.149) since θ and δ are arbitrary. □

5.8 Rectifiability of the limit varifold

We will denote by μ a measure on $\Omega \times [0,T]$ defined as

$$d\mu := \lim_{\varepsilon \to 0} d\mu_t^\varepsilon \, dt = d\mu_t \, dt,$$

and

$$(\operatorname{spt} \mu)_t := \{x \in \Omega; (x,t) \in \operatorname{spt} \mu\}.$$

Lemma 5.8.1. *(The support of μ)*
For all $t \in (0,T]$ we have that

$$spt\, \mu_t \subset (spt\, \mu)_t. \tag{5.161}$$

Proof.
First of all, corresponding to a $\phi \in C_c^1(\Omega; \mathbb{R}^+)$ there exists a constant $0 < \zeta_0 := \zeta_0(E_0, c_1, c_2, C_q, \alpha, \|\phi\|_{C^1(\Omega)})$ such that

$$\mu_t^\varepsilon(\phi) - \zeta_0 \left(\int_0^t \sup_\Omega |\nabla \chi(q_\varepsilon(.,s))|^2 \, ds + t \right) \quad \text{is decreasing on } [0,T]. \tag{5.162}$$

Indeed, by a similar computation as in (5.55), (5.56) and (5.38) we get that

$$\begin{aligned} \frac{d}{dt}\mu_t^\varepsilon(\phi) &\leq \int_\Omega |\nabla \phi| \left(\varepsilon \frac{(\partial_t u^\varepsilon)^2}{2} + \varepsilon \frac{|\nabla u^\varepsilon|^2}{2} \right) + \frac{1}{2\varepsilon} \int_\Omega (g_\varepsilon)^2 \phi \\ &\leq c(E_0, c_1, c_2, C_q, \|\phi\|_{C^1(\Omega)}) + 3(\sup_\Omega |\nabla \chi(q_\varepsilon)|^2 + 2\alpha^2) \|\phi\|_{C^1(\Omega)} \mu_t^\varepsilon(\Omega) \\ &\leq \zeta_0 (\sup_\Omega |\nabla \chi(q_\varepsilon)|^2) + \zeta_0. \end{aligned} \tag{5.163}$$

Now, fix $t_0 \in (0,T]$. Suppose for a contradiction that we have $x \in \operatorname{spt} \mu_{t_0}$ and $(x,t_0) \notin \operatorname{spt} \mu$. This implies that there exists $r > 0$ such that we have $\mu\left(B_r(x) \times (t_0 - r^2, t_0 + r^2)\right) = 0$. For this purpose, let $\phi \in C_c^1(B_r(x); \mathbb{R}^+)$ with $\phi = 1$ on $B_{\frac{r}{2}}(x)$. Since $x \in \operatorname{spt} \mu_{t_0}$ we get that $\mu_{t_0}(\phi) > 0$. Let $h > 0$ small enough using (5.162) with $t_0 - h < t_0$ we get

$$\mu_{t_0}(\phi) \leq \mu_{t_0 - h}(\phi) + \zeta_0 \int_{t_0 - h}^{t_0} \sup_\Omega |\nabla \chi(q_\varepsilon(.,s))|^2 \, ds + \zeta_0 h.$$

which can be written as

$$\mu_{t_0 - h}(\phi) \geq \mu_{t_0}(\phi) - o(1) \geq \frac{\mu_{t_0}(\phi)}{2} > 0,$$

where $o(1) \to 0$ as $h \to 0$, which implies that $(x, t_0) \in spt\mu$. Therefore, we get a contradiction. □

Corollary 5.8.2. *Let $U \subset \Omega$ an open subset. There exists $0 < c_{16} := c_{16}(n, D_1, \delta_0)$ such that for every $t \in (0,T]$ we have*

$$\mathcal{H}^{n-1}\left((spt\, \mu)_t \cap U\right) \leq c_{16} \liminf_{r \to 0} \mu_{t - r^2}(U). \tag{5.164}$$

Proof.
Let $K \subset U$ be a compact subset and set $X_t := (\operatorname{spt}\mu)_t \cap K$. Similarly to (5.145) we have that for $(x,t) \in X_t$ and for r small enough

$$\delta_0 \leq \int_\Omega \tilde{\rho}_{(x,t)}(y, t-r^2)\, d\mu_{t-r^2}(y). \tag{5.165}$$

Now, let $0 < L < \frac{1}{2r}$ we obtain using (5.132) that

$$\begin{aligned}
&(4\pi r^2)^{\frac{n-1}{2}} \int_{\Omega\setminus B_{rL}(x)} \tilde{\rho}_{(x,t)}(y, t-r^2)\, d\mu_{t-r^2}(y) \leq \int_{B_{\frac{1}{2}}\setminus B_{rL}} e^{-\frac{|x-y|^2}{4r^2}}\, d\mu_{t-r^2}(y) \\
\leq\ & \int_0^1 \int_{B_{\frac{1}{2}}\setminus B_{rL}\cap\left\{x:\, e^{-\frac{|x-y|^2}{4r^2}} \geq \ell\right\}} d\mu_{t-r^2}(y)\, d\ell \\
\leq\ & \int_0^{e^{-\frac{1}{16r^2}}} \mu(B_{\frac{1}{2}}\setminus B_{rL})\, d\ell + \int_{e^{-\frac{1}{16r^2}}}^{e^{-\frac{L^2}{4}}} \mu(B_{2r\sqrt{\ln \ell^{-1}}})\, d\ell \\
\leq\ & D_1 w_{n-1}\left(\frac{1}{2^{n-1}} e^{-\frac{1}{16r^2}} + (2r)^{n-1} \int_{\frac{L^2}{4}}^{\frac{1}{16r^2}} s^{\frac{n-1}{2}} e^{-s}\, ds\right) \\
\leq\ & D_1 w_{n-1} r^{n-1} 2^{n-1} \int_{\frac{L^2}{4}}^{\infty} s^{\frac{n-1}{2}} e^{-s}\, ds.
\end{aligned}$$

For sufficiently large L depending on n, D_1 and δ_0 we have

$$\int_{\Omega\setminus B_{rL}(x)} \tilde{\rho}_{(x,t)}(y, t-r^2)\, d\mu_{t-r^2}(y) \leq \frac{\delta_0}{2}. \tag{5.166}$$

Together (5.165) and (5.166) imply

$$\frac{\delta_0}{2} \leq \int_{B_{rL}(x)} \tilde{\rho}_{(x,t)}(y, t-r^2)\, d\mu_{t-r^2}(y) \leq \frac{1}{(4\pi r^2)^{\frac{n-1}{2}}} \mu_{t-r^2}(B_{rL}(x)). \tag{5.167}$$

Let $\mathcal{B} = \left\{\bar{B}_{rL}(x) \subset U; x \in X_t\right\}$ be a covering of X_t. By the Besicovitch covering theorem, there existe a constannt N depending only on n and a finite subcollection $\mathcal{B}_1, \cdots, \mathcal{B}_{N(n)}$ such that

$$X_t \subset \cup_{i=1}^{N(n)} \bigcup \mathcal{B}_i. \tag{5.168}$$

Thus, we obtain using (5.167)

$$\begin{aligned}
\mathcal{H}^{n-1}_{2rL}(X_t) &\leq \sum_{i=1}^{N(n)} \sum_{\bar{B}_{rL}(x)\in\mathcal{B}_i} w_{n-1}(rL)^{n-1} && (5.169)\\
&\leq \sum_{i=1}^{N(n)} \frac{2w_{n-1}L^{n-1}}{(4\pi)^{\frac{n-1}{2}}\delta_0} \sum_{\bar{B}_{rL}(x)\in\mathcal{B}_i} \mu_{t-r^2}(B_{rL}(x)) && (5.170)\\
&\leq \sum_{i=1}^{N(n)} \frac{2w_{n-1}L^{n-1}}{(4\pi)^{\frac{n-1}{2}}\delta_0} \mu_{t-r^2}(U) && (5.171)\\
&= \frac{2w_{n-1}L^{n-1}N(n)}{(4\pi)^{\frac{n-1}{2}}\delta_0} \mu_{t-r^2}(U). && (5.172)
\end{aligned}$$

Finally we get the desired result by letting $r \to 0$.

Proposition 5.8.3. *(First variation)*
For all $\eta \in C_c^1(\Omega;\mathbb{R}^n)$ we have

$$\begin{aligned}\delta V_t^\varepsilon(\eta) &= \int_\Omega (\eta\cdot\nabla u^\varepsilon)(\varepsilon\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon})\,dx + \int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} \nabla\eta\cdot(\nu^\varepsilon\otimes\nu^\varepsilon)d\xi_t^\varepsilon \\ &\quad - \int_{\Omega\cap\{|\nabla u^\varepsilon|=0\}} \nabla\eta\cdot I\frac{W(u^\varepsilon)}{\varepsilon}\,dx, \end{aligned} \tag{5.173}$$

and for a.e. t

$$\delta V_t(\eta) = \lim_{\varepsilon\to 0}\delta V_t^\varepsilon(\eta) = \lim_{\varepsilon\to 0}\int_\Omega (\eta\cdot\nabla u^\varepsilon)(\varepsilon\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon})\,dx. \tag{5.174}$$

Moreover, the total variation $\|\delta V_t\|$ is a Radon measure.

Proof.
Let $\eta\in C_c^1(\Omega;\mathbb{R}^n)$. An integration by parts yields to

$$\begin{aligned}\int_\Omega (\eta\cdot\nabla u^\varepsilon)\Delta u^\varepsilon &= -\int_\Omega \nabla(\eta\cdot\nabla u^\varepsilon)\nabla u^\varepsilon \\ &= -\int_\Omega \nabla\eta\cdot\nabla u^\varepsilon\otimes\nabla u^\varepsilon - \int_\Omega \eta\cdot\nabla(\frac{|\nabla u^\varepsilon|^2}{2}) \\ &= -\int_\Omega \nabla\eta\cdot\nabla u^\varepsilon\otimes\nabla u^\varepsilon + \int_\Omega \mathrm{div}(\eta)\frac{|\nabla u^\varepsilon|^2}{2}, \end{aligned} \tag{5.175}$$

note that $\mathrm{div}(\eta) = \nabla\eta\cdot I$. Moreover,

$$\int_{\Omega\cap\{\nabla u^\varepsilon\neq 0\}} \nabla\eta\cdot I\frac{W(u^\varepsilon)}{\varepsilon}\,dx = -\int_{\Omega\cap\{\nabla u^\varepsilon=0\}} \nabla\eta\cdot I\frac{W(u^\varepsilon)}{\varepsilon}\,dx - \int_\Omega(\eta\cdot\nabla u^\varepsilon)\frac{W'(u^\varepsilon)}{\varepsilon}\,dx. \tag{5.176}$$

The first variation of V_t^ε with respect to η is

$$\begin{aligned}\delta V_t^\varepsilon(\eta) &= \int_{\Omega\times\mathbf{G}(n,n-1)} \nabla\eta(x)\cdot S\,dV_t^\varepsilon(x,S) \\ &= \int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} \nabla\eta(x)\cdot(I-\nu^\varepsilon\otimes\nu^\varepsilon)\,d\mu_t^\varepsilon(x) \\ &= \int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} \nabla\eta(x)\cdot I\left(\varepsilon\frac{|\nabla u^\varepsilon|^2}{2}+\frac{1}{\varepsilon}W(u^\varepsilon)\right)dx \\ &\quad - \int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} \nabla\eta(x)\cdot\nu^\varepsilon\otimes\nu^\varepsilon\left(\varepsilon\frac{|\nabla u^\varepsilon|^2}{2}+\frac{1}{\varepsilon}W(u^\varepsilon)\right)dx \\ &= \int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} \varepsilon\nabla\eta\cdot\nabla u^\varepsilon\otimes\nabla u^\varepsilon + \int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} \varepsilon(\eta\cdot\nabla u^\varepsilon)\Delta u^\varepsilon - \frac{1}{\varepsilon}\int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} (\eta.\nabla u^\varepsilon)W'(u^\varepsilon) \\ &\quad - \int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} \nabla\eta\cdot\nu^\varepsilon\otimes\nu^\varepsilon\left(\varepsilon\frac{|\nabla u^\varepsilon|^2}{2}+\frac{1}{\varepsilon}W(u^\varepsilon)\right)dx \end{aligned}$$

$$
\begin{aligned}
= & \int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} \varepsilon(\eta\cdot\nabla u^\varepsilon)\Delta u^\varepsilon - \frac{1}{\varepsilon}\int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} (\eta.\nabla u^\varepsilon)W'(u^\varepsilon) \\
& + \int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} \varepsilon\nabla\eta\cdot\left(\frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}\otimes\frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}\right)|\nabla u^\varepsilon|^2 \\
& - \int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} \nabla\eta\cdot\left(\frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}\otimes\frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}\right)\frac{\varepsilon\,|\nabla u^\varepsilon|^2}{2} - \int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} \nabla\eta\cdot\nu^\varepsilon\otimes\nu^\varepsilon\left(\frac{1}{\varepsilon}W(u^\varepsilon)\right) \\
= & \int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} \varepsilon(\eta\cdot\nabla u^\varepsilon)\Delta u^\varepsilon - \frac{1}{\varepsilon}\int_{\Omega}(\eta.\nabla u^\varepsilon)W'(u^\varepsilon) - \frac{1}{\varepsilon}\int_{\{\nabla u^\varepsilon=0\}}\nabla\eta\cdot I\,W(u^\varepsilon) \\
& + \int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} \nabla\eta\cdot(\nu^\varepsilon\otimes\nu^\varepsilon)\left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 - \frac{1}{\varepsilon}W(u^\varepsilon)\right),
\end{aligned}
$$

which yields to (5.173). Proposition 5.7.4 together with the dominated convergence theorem imply

$$
\lim_{\varepsilon\to 0}\int_\Omega |\xi_t^\varepsilon(.,t)|\,dx = 0 \quad \text{for a.e. } t\geq 0. \tag{5.177}
$$

For such t the last two terms in (5.173) vanish as $\varepsilon\to 0$ since also we have $\mu_t^\varepsilon = \varepsilon\,|\nabla u^\varepsilon|^2 - \xi_t^\varepsilon$. Moreover, by (5.40) and Fatou's lemma we have

$$
\liminf_{\varepsilon\to 0}\int_\Omega \varepsilon\left(\Delta u^\varepsilon(.,t) - \frac{W'(u^\varepsilon(.,t))}{\varepsilon^2}\right)^2 dx < \infty \quad \text{for a.e. } t\geq 0. \tag{5.178}
$$

Choosing t that satisfies (5.177) and (5.178) we get for $\eta\in C_c^1(\Omega;\mathbb{R}^n)$ such that $\sup_\Omega|\eta|\leq 1$

$$
\begin{aligned}
\delta V_t(\eta) &= \lim_{\varepsilon\to 0}\delta V_t^\varepsilon(\eta) = \lim_{\varepsilon\to 0}\int_\Omega(\eta\cdot\nabla u^\varepsilon)\left(\varepsilon\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon}\right)dx \\
&\leq \liminf_{\varepsilon\to 0}\left(\int_\Omega \varepsilon\,|\eta\nabla u^\varepsilon|^2\right)^{1/2}\left(\int_\Omega\varepsilon\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)^2\right)^{1/2} \\
&\leq \liminf_{\varepsilon\to 0}\left(\int_\Omega \varepsilon\,|\nabla u^\varepsilon|^2\right)^{1/2}\left(\int_\Omega\varepsilon\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)^2\right)^{1/2} < \infty,
\end{aligned} \tag{5.179}
$$

which implies that the total variation $\|\delta V_t\|$ is a Radon measure. □

Proposition 5.8.4. *(The rectifiability of the limit varifold)*
Any convergent subsequence $\{V_t^\varepsilon\}_{0<\varepsilon<1}$ satisfying (5.177) and (5.178) converges to the unique varifold V_t associated with μ_t. Moreover, $\|V_t\| = \mu_t$ is rectifiable for a.e. $t>0$, namely

$$
V_t(\eta) = \int_{\Omega\cap\{|\nabla u|\neq 0\}}\eta(x,P(x,t))\,d\mu_t, \quad \textit{for all } \eta\in C_c(\Omega\times\boldsymbol{G}(n,n-1)), \tag{5.180}
$$

with $P := I-\nu\otimes\nu$ denotes the projection onto the tangential plane $Tan_x\mu_t$ and ν denotes the normal to P.

Proof.
Since $\|V_t^\varepsilon\| = \mu_t^\varepsilon$ together with (5.35) and the compactness theorem for Radon measures there exists a subsequence denoted again $\{V_t^\varepsilon\}_\varepsilon$ which converges to a varifold V_t for all $t\geq 0$. Moreover, since we have $\lim_{\varepsilon\to 0}\|V_t^\varepsilon\| = \|V_t\|$ and by using Proposition 5.5.2 we get

$\|V_t\| = \mu_t$.
The total variation $\|\delta V_t\|$ is a Radon measure for t satisfying (5.177) and (5.178). Furthermore, using the fact that

$$\mu_t\left(\left\{x \in \text{support}\mu_t : \limsup_{r\downarrow 0} \frac{\mu_t(B_r(x))}{w_{n-1}r^{n-1}} \leq s\right\}\right) \leq 2^{n-1}s\mathcal{H}^{n-1}(\text{support } \mu_t)$$

for any $s > 0$ (see [69, page 11 3.2(2)] for more details), and that $\mathcal{H}^{n-1}(\text{support } \mu_t) < \infty$ by Lemma 5.8.1 and Corollary 5.8.2 we obtain

$$\mu_t\left(\left\{x \in \text{support}\mu_t : \lim_{r\downarrow 0} r^{1-n}\mu_t(B_r(x)) = 0\right\}\right) = 0.$$

Then V_t can be written as

$$V_t = V_t \left\lfloor_{\left\{x\in\Omega:\limsup_{r\downarrow 0} r^{1-n}\|V_t\|(B_r(x))>0\right\}} \times G(n, n-1)\right. ,$$

which is rectifiable for t satisfying (5.177) and (5.178) by the Allard's rectifiability theorem (Theorem 4.5.12). V_t is determined uniquely by $\|V_t\| = \mu_t$ so μ_t is rectifiable. Finally, any converging subsequence satisfying (5.178) and (5.177) has the same limit V_t. □

Proposition 5.8.5. *(Generalized mean curvature vector)*
The limit varifold V_t has a generalized mean curvature vector h_{V_t} for a.e. $t > 0$ and it satisfies

$$\int_\Omega \phi\,|h_{V_t}|^2\,d\,\|V_t\| \leq \liminf_{\varepsilon\to 0}\int_\Omega \varepsilon\phi\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)^2 dx < \infty, \tag{5.181}$$

for all $\phi \in C_c(\Omega;\mathbb{R}^+)$.

Proof.
By the same calculation in (5.179) we have for $\eta \in C_c^1(\Omega;\mathbb{R}^n)$

$$|\delta V_t(\eta)| \leq \left(\int_\Omega |\eta|^2\,d\mu_t\right)^{1/2} \liminf_{\varepsilon\to 0}\left(\varepsilon\left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)^2\right)^{1/2}. \tag{5.182}$$

This implies that $\|\delta V_t\| << \|V_t\| = \mu_t$. Thus by the Radon-Nikodym theorem there exists a $\|V_t\|$ measurable vector field h_{V_t} such that

$$\delta V_t(\eta) = -\int_\Omega \eta\cdot h_{V_t}\,d\,\|V_t\|\,. \tag{5.183}$$

Which can also be written as

$$\lim_{\varepsilon\to 0}\int_\Omega (\eta\cdot\nabla u^\varepsilon)\left(\varepsilon\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon}\right)dx = -\int_\Omega \eta\cdot h_{V_t}\,d\,\|V_t\|\,. \tag{5.184}$$

Now, let $T : C_c^1(\Omega;\mathbb{R}^n) \longmapsto \mathbb{R}$ defined as $T(\eta) = \int_\Omega \eta\cdot h_{V_t}\,d\mu_t$ for all $\eta \in C_c^1(\Omega;\mathbb{R}^n)$. Using the fact that μ_t is rectifiable we have that $C_c^1(\Omega;\mathbb{R}^n) \hookrightarrow L^2(\mu_t;\mathbb{R}^n)$ is dense and therefore there exists $\tilde{T} : L^2(\mu_t;\mathbb{R}^n) \longmapsto \mathbb{R}$ such that $\tilde{T}(\eta) = T(\eta)$ for all $\eta \in C_c^1(\Omega;\mathbb{R}^n)$. By (5.182) and (5.183) we get that $\tilde{T}$ is linear and bounded then by the Riesz representation theorem there exists a function $\tilde{h} \in L^2(\mu_t;\mathbb{R}^n)$ such that $\tilde{T}(\eta) = \int_\Omega \eta\cdot\tilde{h}\,d\mu$ for all $\eta \in L^2(\mu_t;\mathbb{R}^n)$.

Which implies that $h_{V_t} = \tilde{h}$ almost everywhere and $h_{V_t} \in L^2(\mu_t; \mathbb{R}^n)$.
Since η in (5.183) is arbitrary take η the approximation of h_{V_t} and replace it in (5.182) to get

$$\int_\Omega |h(V_t)|^2 \, d\|V_t\| \leq \liminf_{\varepsilon\to 0} \int_\Omega \varepsilon \left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)^2 dx < \infty,$$

which is (5.181) with $\phi = 1$. Now let $\phi \in C_c(\Omega; \mathbb{R}^+)$ and $\{\phi_k\}_k \in C_c^1(\Omega; \mathbb{R}^+)$ an approximation of ϕ. Doing the same calculation but with $\phi_k\eta$ instead of η then passing to the limit as $k \to \infty$ we obtain

$$\left|\int_\Omega \phi\eta \cdot h_{V_t} \, d\mu_t\right| \leq \left(\int_\Omega \phi \, |\eta|^2 \, d\mu_t\right)^{1/2} \liminf_{\varepsilon\to 0} \left(\varepsilon\phi \left(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}\right)^2\right)^{1/2}. \tag{5.185}$$

Again using an approximation $\eta_k \in C_c^1(\Omega; \mathbb{R}^n)$ of h_{V_t} and passing to the limit as $k \to \infty$ we get (5.181). □

5.9 Integrality of the limit varifold

We first present two propositions introduced in [70]. Proposition 5.9.1 states that if some quantities are controlled, then we have a lower bound on a measure in terms of a sum of densities of vertically aligned points; in the proof the authors decompose the domain horizontally so that each separated domain contains approximately one sheet of diffused interface. The original idea comes from [4] and it has first been used in the context of the diffused interface problem in [40] in the elliptic case and in [72] in the parabolic case where a time derivative term is added in the estimates. We omit the proof and we refer to [70] for more details.
Proposition 5.9.2 shows that the energy behaves more or less like a one-dimensional simple ODE solution if certain quantities are controlled, see [40, 72, 70].

Proposition 5.9.1. *(Densities of vertically aligned points[70, Subsection 7.1].)*
Let $0 < R, E_0 < \infty$, $0 < s < 1$ and $N \in \mathbb{N}$. Suppose that there exists $0 < \varrho < 1$ with the following properties:
(1) *$\boldsymbol{Y}$ is a finite subset of $\mathbb{R}^n$ that has no more than $N+1$ elements, and $\boldsymbol{Y} \subset \{(0, \cdots, 0, x_n) : x_n \in \mathbb{R}\}$. For some $0 < a < R$ we have $|y - z| > 3a$ for y, $z \in \boldsymbol{Y}$ with $y \neq z$.*
(2) *$diam\, \boldsymbol{Y} < \varrho R$.*
(3) *We have $\Psi \in C^2(\{y \in \mathbb{R}^n : dist(y, \boldsymbol{Y}) < R\})$.*
(4) *For all $x \in \boldsymbol{Y}$ and $a \leq r \leq R$,*

$$\int_{B_r(x)} |\xi_\varepsilon(\Psi)| + \left(1 - (\nu_n)^2\right) \varepsilon \, |\nabla\Psi|^2 + \varepsilon \, |\nabla\Psi| \left|\Delta\Psi - \frac{W'(\Psi)}{\varepsilon^2}\right| dy \leq \varrho r^{n-1}, \tag{5.186}$$

where

$$\xi_\varepsilon(\Psi) := \frac{\varepsilon \, |\nabla\Psi|^2}{\varepsilon} - \frac{W(\Psi)}{\varepsilon}, \tag{5.187}$$

and $\nu = (\nu_1, \cdots, \nu_n) = \frac{\nabla\Psi}{|\nabla\Psi|}$.
(5) *For all $x \in \boldsymbol{Y}$,*

$$\int_a^R \frac{d\tau}{\tau^n} \int_{B_\tau(x)} (\xi_\varepsilon)_+ \, dy \leq \varrho. \tag{5.188}$$

(6) *For all* $x \in \boldsymbol{Y}$ *and* $a \le r \le R$,

$$\int_{B_r(x)} \varepsilon \left|\nabla \Psi\right|^2 \, dy \le E_0 r^{n-1}. \tag{5.189}$$

Then we have

$$\sum_{x \in \boldsymbol{Y}} \frac{1}{a^{n-1}} \int_{B_a(x)} e_\varepsilon(\Psi) \le s + \frac{1+s}{R^{n-1}} \int_{\{x: dist(\boldsymbol{Y},x)<R\}} e_\varepsilon(\Psi), \tag{5.190}$$

where

$$e_\varepsilon(\Psi) := \frac{\varepsilon \left|\nabla \Psi\right|^2}{\varepsilon} + \frac{W(\Psi)}{\varepsilon}.$$

Proposition 5.9.2. *(The* ε*-scale estimate [70, Proposition 7.2].)*
Given $0 < s,\ b,\ \beta < 1$, *and* $1 < c < \infty$, *there exist* $0 < \varrho,\ \varepsilon_* < 1$ *and* $1 < L(n, W) < \infty$ *with the following property:*
Assume $0 < \varepsilon < \varepsilon_*$, $\Phi \in C^2(B_{4\varepsilon L})$ *and*

$$\sup_{B_{4\varepsilon L}} \varepsilon \left|\nabla \Phi\right| \le c, \quad \sup_{x,y \in B_{4\varepsilon L}} \varepsilon^{\frac{3}{2}} \frac{\left|\nabla \Phi(x) - \nabla \Phi(y)\right|}{|x-y|^{\frac{1}{2}}} \le c, \quad b < \Phi(0) < 1 - b,$$

$$\int_{B_{4\varepsilon L}(x)} \left(\left|\xi_\varepsilon(\Phi)\right| + \left(1 - (\nu_n)^2\right) \varepsilon \left|\nabla \Phi\right|^2 \right) dx \le \varrho (4\varepsilon L)^{n-1}$$

and

$$\sup_{B_{4\varepsilon L}} (\xi_\varepsilon)_+ \le \varepsilon^{-\beta}.$$

Then for $J := B_{3\varepsilon L} \cap \{(0, \cdots, 0, x_n)\}$ *we have*

$$\inf_{x \in J} \partial_{x_n} \Phi(x) > 0 \quad or \quad \sup_{x \in J} \partial_{x_n} \Phi(x) < 0, \quad and \quad [b, 1-b] \subset \Phi(J).$$

We also have

$$\left| \sigma - \frac{1}{w_{n-1} (L\varepsilon)^{n-1}} \int_{B_{\varepsilon L}} e_\varepsilon \right| \le s.$$

In what follows we will replace $\Omega \times (0,T)$ by $B_3(0) \times (0,2)$. Let $t \in [0,2]$ and define

$$Z_\gamma := \left\{ x \in B_3(0) : \left| u^\varepsilon(x,t) - \frac{1}{2} \right| \le \gamma \right\}.$$

Lemma 5.9.3. *(Relation between the value of* u^ε *and the distance to the interface.)*
There exist $0 < c_{17} := c_{17}(n,k)$ *and* $0 < \varepsilon_6 := \varepsilon_6(k, c_7, \alpha, C_q)$ *such that if* $(x_0, t_0) \in B_1(0) \times (1,2)$ *with*

$$u^\varepsilon(x_0,t_0) < 1 - \varepsilon^\delta \ or \ u^\varepsilon(x_0,t_0) > \varepsilon^\delta, \tag{5.191}$$

where $0 < \delta < 1$ *satisfies*

$$\frac{1}{c_{17} \left|\ln \varepsilon\right|} < \delta < \frac{1}{c_{17} \varepsilon \left|\ln \varepsilon\right|}, \tag{5.192}$$

then, we have that for $\varepsilon < \varepsilon_6$

$$dist(x_0, Z_\gamma) \le c_{17} \delta \varepsilon \left| ln \varepsilon \right|. \tag{5.193}$$

Proof.
Define $\tilde{r} := c_{17}\delta\,|\ln\varepsilon|$. Consider the domain $B_{\varepsilon\tilde{r}}(x_0)\times(t_0-\varepsilon^2\tilde{r}^2,t_0)\subset B_3(0)\times(0,2)$ i.e. since $\varepsilon\tilde{r}<1$ and $(x_0,t_0)\in B_1(0)\times(1,2)$. We rescale the domain so that

$$\partial_t\tilde{u}^\varepsilon=\Delta\tilde{u}^\varepsilon-W'(\tilde{u}^\varepsilon)-\nabla\chi(\tilde{q}_\varepsilon)\cdot\nabla\tilde{u}^\varepsilon-\tilde{u}^\varepsilon\Delta\chi(\tilde{q}_\varepsilon)+\varepsilon\alpha\tilde{u}^\varepsilon(1-\tilde{u}^\varepsilon)\text{ on }B_{\tilde{r}}\times[-\tilde{r}^2,0],\tag{5.194}$$

where $\tilde{u}^\varepsilon(x,t):=u^\varepsilon(\varepsilon x+x_0,\varepsilon^2t+t_0)$ and $\tilde{q}_\varepsilon:=q_\varepsilon(\varepsilon x+x_0,\varepsilon^2t+t_0)$ for $(x,t)\in B_{\tilde{r}}\times[-\tilde{r}^2,0]$. Next, for the comparison principle, we define the following problem

$$\begin{cases}\Delta\tilde{\Psi}=\frac{k}{4}\tilde{\Psi} & \text{on }\mathbb{R}^n,\\ \tilde{\Psi}(0)=1.\end{cases}\tag{5.195}$$

We can check that

$$\tilde{\Psi}(x)=\frac{1}{|\mathbb{S}^{n-1}|}\int_{\{|y|=1\}}e^{\sqrt{\frac{k}{4}}y\cdot x}\,dy$$

is a solution for (5.195) which can be also written as

$$\tilde{\Psi}(x)=\frac{2\pi^{\frac{n-1}{2}}}{|\mathbb{S}^{n-1}|\,\Gamma\left(\frac{n-1}{2}\right)}\int_0^\pi e^{\sqrt{\frac{k}{4}}|x|\cos\phi_1}\sin^{n-2}\phi_1\,d\phi_1=\frac{n-1}{2\pi}\int_0^\pi e^{\sqrt{\frac{k}{4}}|x|\cos\phi_1}\sin^{n-2}\phi_1\,d\phi_1,$$

where ϕ_1 is the angle between x and y so that $\frac{x}{|x|}\cdot y=cos\phi_1$. Thus, $\tilde{\Psi}$ can be seen as a radial solution which growth exponentially as $|x|\to\infty$ and takes its minimum at 0. Hence, $\tilde{\Psi}\geq 1$ on $\mathbb{R}^n$. Then, we define a comparison function $\Psi(x,t):=\tilde{\Psi}\exp(\frac{-kt}{4})$. We can prove that there exists c_{17} depending on k and n such that Ψ satisfies the following

$$\begin{cases}\partial_t\Psi=\Delta\Psi-\frac{k}{2}\Psi & \text{on }\mathbb{R}^n\times(-\infty,0),\\ \Psi(x,t)\geq\exp(\frac{|x|+|t|}{c_{17}}) & \text{on }\mathbb{R}^n\times(-\infty,0)\backslash B_1^{n+1}(0,0),\\ \Psi(0,0)=1,\end{cases}\tag{5.196}$$

By the definition of $\tilde{r}$ we have $1-\varepsilon^\delta e^{\tilde{r}/c_{17}}=1-\varepsilon^\delta e^{\ln\varepsilon^{-\delta}}=0$. Now, suppose for a contradiction that

$$\tilde{u}^\varepsilon(0,0)<1-\varepsilon^\delta\quad\text{and}\quad\inf_{B_{\tilde{r}}(0)\times(-\tilde{r}^2,0)}\tilde{u}^\varepsilon(x,t)>\frac{1}{2}+\gamma.\tag{5.197}$$

Next, define $\phi_\varepsilon:=1-\varepsilon^\delta\Psi$ which will satisfies according to (5.196)

$$\partial_t\phi_\varepsilon=-\varepsilon^\delta\partial_t\Psi=-\varepsilon^\delta\Delta\Psi+\varepsilon^\delta\frac{k}{2}\Psi=\Delta\phi_\varepsilon+\frac{k}{2}(1-\phi_\varepsilon),\tag{5.198}$$

$$\phi_\varepsilon(x,t)=1-\varepsilon^\delta\Psi\leq 1-\varepsilon^\delta e^{\frac{\tilde{r}}{c_{17}}}=0<\frac{1}{2}+\gamma\leq\tilde{u^\varepsilon},\text{ on }\partial_0\left(B_{\tilde{r}}(0)\times(-\tilde{r}^2,0)\right),\tag{5.199}$$

where $\partial_0\left(B_{\tilde{r}}(0)\times(-\tilde{r}^2,0)\right)$ represents the parabolic boundary and we also have

$$\phi_\varepsilon(0,0)=1-\varepsilon^\delta\Psi(0,0)=1-\varepsilon^\delta>\tilde{u}^\varepsilon(0,0).\tag{5.200}$$

Thus, ϕ_ε satisfies

$$\begin{cases}\partial_t\phi_\varepsilon=\Delta\phi_\varepsilon+\frac{k}{2}(1-\phi_\varepsilon) & \text{on }B_{\tilde{r}}(0)\times(-\tilde{r}^2,0)\\ \phi_\varepsilon(x,t)-\tilde{u}^\varepsilon(x,t)<0 & \text{on }\partial_0\left(B_{\tilde{r}}(0)\times(-\tilde{r}^2,0)\right)\\ \phi_\varepsilon(0,0)-\tilde{u}^\varepsilon(0,0)>0,\end{cases}\tag{5.201}$$

Therefore, $\phi_\varepsilon - \tilde{u}^\varepsilon$ has a parabolic interior positive maximum point where at this point we have

$$\begin{aligned} 0 &\leq \partial_t(\phi_\varepsilon - \tilde{u}^\varepsilon) - \Delta(\phi_\varepsilon - \tilde{u}^\varepsilon) \\ &= \frac{k}{2}(1-\phi_\varepsilon) + W'(\tilde{u}^\varepsilon) + \nabla\chi(\tilde{q}_\varepsilon)\cdot\nabla\tilde{u}^\varepsilon + \tilde{u}^\varepsilon\Delta\chi(\tilde{q}_\varepsilon) - \varepsilon\alpha\tilde{u}^\varepsilon(1-\tilde{u}^\varepsilon) \\ &\leq -\frac{k}{2}(1-\phi_\varepsilon) + \varepsilon c(C_q, c_7) + \varepsilon^2 c(C_q) + \varepsilon c(\alpha) \\ &\leq -\frac{k}{2}\varepsilon^\delta + \varepsilon c(C_q, c_7, \alpha), \end{aligned} \tag{5.202}$$

where we have used the fact that W' is increasing on $[\frac{1}{2}+\gamma, 1+\varepsilon]$, since here we have $W'' > k > 0$, and that $\phi_\varepsilon > \tilde{u}^\varepsilon$ at the maximum point to obtain

$$W'(\tilde{u}^\varepsilon) < W'(\phi_\varepsilon) = W'(\phi_\varepsilon) - W'(1) \leq k(\phi_\varepsilon - 1).$$

Thus, since $0 < \delta < 1$, (5.202) implies a contradiction for sufficiently small ε depending on α, C_q, c_7 and k.
Similarily, we can prove that

$$\sup_{B_{\varepsilon\tilde{r}(x_0)}\times(t_0-\varepsilon^2\tilde{r}^2,t_0)} u^\varepsilon(x,t) > \frac{1}{2} - \gamma, \text{ whenever } u^\varepsilon(x_0,t_0) > \varepsilon^\delta,$$

which implies our result. □

Lemma 5.9.4. *(The volume of the r-neighborhood of the interface.)*
For $t_0 \in (1,2)$, and $0 < r < \frac{1}{2}$ define

$$Z_{r,t_0} := \left\{ x \in B_1(0) : \inf_{B_r(x)\times(t_0-r^2,t_0)} \left| u^\varepsilon - \frac{1}{2} \right| < \gamma \right\}.$$

There exist $0 < c_{18} := c_{18}(n, c_7, W, D_1)$ and $0 < \varepsilon_7$ such that for $0 < \varepsilon < \varepsilon_7$ we have

$$\mathcal{L}^n\left(Z_{r,t_0}\right) \leq c_{18} r. \tag{5.203}$$

Proof.
Let $x_0 \in Z_{r,t_0}$. For $(x_1,t_1) \in B_r(x_0)\times(t_0-r^2,t_0)$ with $\frac{1}{2}-\gamma < u^\varepsilon(x_1,t_1) < \frac{1}{2}+\gamma$, we have by direct computation that

$$\int_\Omega \tilde{\rho}_{(x_1,t_1+\varepsilon^2)}(x,t_1)\, d\mu^\varepsilon_{t_1}(x) \geq \int_{B_{\varepsilon^{-1}}(0)} \frac{e^{-\frac{|\tilde{x}|^2}{4}}}{(4\pi)^{\frac{n-1}{2}}}\, \eta(\varepsilon\tilde{x}) W(\tilde{u})\, d\tilde{x} \geq 3\eta_1 \tag{5.204}$$

for some constant $\eta_1(n, W, c_7)$, where we have made the change of variable $\varepsilon\tilde{x} = x - x_1$ with $\tilde{u}(\tilde{x}, t_1) = u(\varepsilon\tilde{x} + x_1, t_1)$. We also used the fact that $\frac{1}{2} - \gamma \leq \tilde{u}(0,t_1) \leq \frac{1}{2} + \gamma$. Using (5.124) with t_0 replaced by $t_0 - 2r^2$, y replaced by x_1 and s replaced by $t_1 + \varepsilon^2$, we get for a suitable ε and r

$$\int_\Omega \tilde{\rho}_{(x_1,t_1+\varepsilon^2)}(x,t_1)\, d\mu^\varepsilon_{t_1}(x) \leq \int_\Omega \tilde{\rho}_{(x_1,t_1+\varepsilon^2)}(x,t_0-2r^2)\, d\mu^\varepsilon_{t_0-2r^2}(x) + \eta_1. \tag{5.205}$$

Therefore,

$$2\eta_1 \le \int_\Omega \tilde{\rho}_{(x_1,t_1+\varepsilon^2)}(x, t_0 - 2r^2)\, d\mu^\varepsilon_{t_0-2r^2}(x). \tag{5.206}$$

Using (5.132) we can choose a suitably large $c_{19}(D_1)$ such that

$$\int_{\Omega\setminus B_{c_{19}r}(x_1)} \tilde{\rho}_{(x_1,t_1+\varepsilon^2)}(x, t_0 - 2r^2)\, d\mu^\varepsilon_{t_0-2r^2}(x) \le \eta_1. \tag{5.207}$$

Finally, we obtain

$$\eta_1 \le \int_{B_{c_{19}r}(x_1)} \tilde{\rho}_{(x_1,t_1+\varepsilon^2)}(x, t_0 - 2r^2)\, d\mu^\varepsilon_{t_0-2r^2}(x). \tag{5.208}$$

Note that we have

$$\tilde{\rho}_{(x_1,t_1+\varepsilon^2)(x,t_0-2r^2)} \le \frac{1}{(4\pi(t_1+\varepsilon^2-t_0+2r^2))^{\frac{n-1}{2}}} \le \frac{1}{r^{n-1}}.$$

Together with the fact that $B_{c_{19}(x_1)} \subset B_{(c_{19}+1)r}(x_0)$ imply that

$$\mu^\varepsilon_{t_0-2r^2}(B_{c_{19}r}(x_0)) \ge \eta_1 r^{n-1}. \tag{5.209}$$

Next, let $\mathbf{B} = \{B_{c_{19}r}(x) : x \in Z_{r,t_0}\}$ which is a covering of Z_{r,t_0} by closed balls centered at $x \in Z_{r,t_0}$. By the Besicovitch covering theorem, there exist a finite sub-collection $\mathbf{B'}$ of N balls such that

$$Z_{r,t_0} \subset \bigcup_{\mathbf{B'}} B_{c_{19}r}(x).$$

Thus (5.209) implies

$$N\eta_1 r^{n-1} \le \sum_{\mathbf{B'}} \mu^\varepsilon_{t_0-2r^2}(B_{c_{19}r}(x)) \le \mu^\varepsilon_{t_0-2r^2}(B_3(0)) \le 3^{n-1} w_{n-1} D_1.$$

Then,

$$\mathcal{L}^n(Z_{r,t_0}) \le N(c_{19}r)^n w_n \le \frac{3^{n-1} D_1 (c_{19})^n (w_n)^2}{\eta_1} r,$$

which shows the statment of the Lemma. □

Proposition 5.9.5. *(Small amount of energy on $\{u^\varepsilon \le b$ or $u^\varepsilon \ge 1-b\}$.)*
Let $0 < s < 1$ be given. There exist $0 < b < 1$ and $0 < \varepsilon_8 := \varepsilon_8(W, c_{17}, c_{18})$ such that we have

$$\int_{B_1(0)\times\{t\}\cap\{u^\varepsilon \le b \text{ or } u^\varepsilon \ge 1-b\}} \frac{W(u)}{\varepsilon} \le s, \tag{5.210}$$

for all $t \in (1,2)$ and $\varepsilon \le \varepsilon_8$.

Proof.
Let $t_0 \in (1,2)$ and $1 \le J \in \mathbb{N}$ will be fixed later. Define for $j = 1, \dots, J$,

$$A_j := \left\{ x \in B_3(0) :\ \varepsilon^{\frac{1}{2j}} \le u^\varepsilon(x,t_0) \le \varepsilon^{\frac{1}{2j+1}} \quad \text{or} \quad 1-\varepsilon^{\frac{1}{2j+1}} \le u^\varepsilon(x,t_0) \le 1-\varepsilon^{\frac{1}{2j}} \right\}. \tag{5.211}$$

In order to apply Lemma 5.9.3 with $\delta = \frac{1}{2^j}$ we choose J and ε such that

$$c_{17} |\ln \varepsilon| \leq \frac{1}{\varepsilon} \quad \text{and} \quad c_{17} \frac{1}{2^J} |\ln \varepsilon| \geq 1, \tag{5.212}$$

and we obtain

$$\inf_{B_{\tilde{r}}(x)\times(t_0-\tilde{r}^2,t_0)} \left| u^\varepsilon - \frac{1}{2} \right| < \gamma$$

where $\tilde{r} = c_{17} \frac{1}{2^j} |\ln \varepsilon|$ which implies that

$$A_j \subset Z_{\varepsilon\tilde{r},t_0}$$

again applying Lemma 5.9.4 we obtain for all $j = 1, \ldots, J$

$$\mathcal{L}^n(A_j) \leq \frac{c_{18}c_{17}}{2^j} \varepsilon |\ln \varepsilon| .$$

Now, let b satisfying $\varepsilon^{\frac{1}{2J+1}} \in (b, \sqrt{b}]$ and $\frac{1}{2} - b > \gamma$ so that

$$Y := \left\{ x \in B_1(0) : \ \varepsilon^{\frac{1}{2}} \leq u^\varepsilon(x, t_0) \leq b \quad \text{or} \quad 1 - b \leq u^\varepsilon(x, t_0) \leq 1 - \varepsilon^{\frac{1}{2}} \right\} \subset \cup_{j=1}^J A_j. \tag{5.213}$$

On A_j, more precisly on $\left\{ x : 1 - \varepsilon^{\frac{1}{2^{j+1}}} \leq u^\varepsilon(x, t_0) \leq 1 - \varepsilon^{\frac{1}{2^j}} \right\}$ we apply Taylor-Lagrange inequality to get

$$\frac{W(u^\varepsilon)}{\varepsilon} \leq \left(\max_{[-1,1]} \left| W'' \right| \right) \frac{(1 - u^\varepsilon)^2}{2\varepsilon} \leq \frac{\left(\max_{[-1,1]} |W''| \right)}{2\varepsilon} \left(\varepsilon^{\frac{1}{2^{j+1}}} \right)^2 \leq c(W) \varepsilon^{\frac{1}{2^j} - 1}.$$

Otherwise, on $\left\{ x : \varepsilon^{\frac{1}{2^j}} \leq u^\varepsilon(x, t_0) \leq \varepsilon^{\frac{1}{2^{j+1}}} \right\}$ by using also the fact that W is symmetric with respect to $\frac{1}{2}$, again Taylor-Lagrange inequality implies

$$\frac{W(u^\varepsilon)}{\varepsilon} = \frac{W(1 - u^\varepsilon)}{\varepsilon} \leq \left(\max_{[-1,1]} \left| W'' \right| \right) \frac{u^2}{2\varepsilon} \leq \frac{\left(\max_{[-1,1]} |W''| \right)}{2\varepsilon} \left(\varepsilon^{\frac{1}{2^{j+1}}} \right)^2 \leq c(W) \varepsilon^{\frac{1}{2^j} - 1}.$$

Therefore,

$$\begin{aligned} \int_Y \frac{W(u^\varepsilon)}{\varepsilon} &\leq \sum_{j=1}^J \int_{A_j} \frac{W(u^\varepsilon)}{\varepsilon} \leq c(W) c_{17} c_{18} |\ln \varepsilon| \sum_{j=1}^J 2^{-j} \varepsilon^{2^{-j}} \\ &\leq c(W, c_{18}, c_{17}) |\ln \varepsilon| \int_1^{J+1} 2^{-t} \varepsilon^{2^{-t}} \, dt = \frac{c(W, c_{18}, c_{17})}{\ln 2} (\varepsilon^{\frac{1}{2^{J+1}}} - \varepsilon^{\frac{1}{2}}) \\ &\leq c(W, c_{18}, c_{17}) \sqrt{b}. \end{aligned} \tag{5.214}$$

We can choose b so that (5.214) is smaller than $\frac{s}{2}$. Similarly, we can apply Lemma 5.9.3 with $\delta = \frac{2}{3}$ and Lemma 5.9.4 with $r = c_{17}\varepsilon |\ln \varepsilon|$, to get that

$$\int_{\left\{ \frac{1}{2} - \sqrt{\varepsilon} \leq \left| u^\varepsilon - \frac{1}{2} \right| \leq \frac{1}{2} - \varepsilon^{\frac{2}{3}} \right\}} \frac{W(u^\varepsilon)}{\varepsilon} \leq c(W, c_{17}, c_{18}) \varepsilon |\ln \varepsilon| , \tag{5.215}$$

which can be made smaller than $\frac{s}{4}$. Finally, using $0 \leq u^\varepsilon \leq 1 + \varepsilon$

$$\int_{\left\{ \frac{1}{2} - \varepsilon^{\frac{2}{3}} \leq \left| u^\varepsilon - \frac{1}{2} \right| \right\}} \frac{W(u^\varepsilon)}{\varepsilon} \leq c(W) (\varepsilon^{\frac{2}{3}})^2 \frac{1}{\varepsilon} \leq c(W) \varepsilon^{\frac{1}{3}} \leq \frac{s}{4}, \tag{5.216}$$

which is also true for sufficiently small ε. (5.214), (5.215) and (5.216) give the result. □

Proposition 5.9.6. *(Integrality of the limit varifold)*
The multiplicity function defined as

$$\theta(x) := \lim_{r\to 0} \frac{\mu_t(B_r(x))}{w_{n-1}r^{n-1}},$$

satisfies that for almost every $t > 0$

$$\sigma^{-1}\theta(x,t) \in \mathbb{N} \text{ for } \mu_t\text{-almost every } x \text{ in } \Omega,$$

where $\sigma := \int_0^1 \sqrt{2W(s)}\, ds$.

Proof.
In what follows we let the sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ in a way that $\varepsilon_i \to 0$ as $i \to \infty$. In what follows we fix $t \in [0,T]$ such that Proposition 5.8.4 holds true. For such t we have by Young's inequality

$$c_h(t) := \sup_{\varepsilon_i} \int_\Omega \varepsilon_i \left|\Delta u^{\varepsilon_i} - \frac{W'(u^{\varepsilon_i})}{\varepsilon_i^2}\right| |\nabla u^{\varepsilon_i}|\, dx < \infty.$$

Define for any $m \in \mathbb{N}$ the sets

$$A_{i,m} := \left\{ x \in \Omega : \int_{B_r(x)} \varepsilon_i \left|\Delta u^{\varepsilon_i} - \frac{W'(u^{\varepsilon_i})}{\varepsilon_i^2}\right| |\nabla u^{\varepsilon_i}| \le m\, \mu_t^{\varepsilon_i}(B_r(x)), \text{ for all } 0 \le r \le \frac{1}{2} \right\}, \tag{5.217}$$

$$A_m := \{x \in \Omega : \text{ there exist } x_i \in A_{i,m} \text{ for infinitely many } i \text{ with } x_i \to x\}, \tag{5.218}$$

and

$$A := \cup_{m=1}^{\infty} A_m. \tag{5.219}$$

Claim 1: $\mu_t(\Omega\backslash A) = 0$.
Suppose that there exists a compact set $K \subset \Omega\backslash A$ such that $\mu_t(K) \ge \frac{1}{2}\mu_t(\Omega\backslash A)$. Which implies by using (5.219) that $K \subset \Omega\backslash A_m$ for every $m \in \mathbb{N}$. Moreover we can always find an open set O_m such that $K \subset O_m$ and $O_m \cap A_{i,m} = \emptyset$ for large i. Let $\phi_m \in C_c(O_m; \mathbb{R}^+)$ such that $\phi_m = 1$ on K and $\le \phi_m \le 1$. Therefore, by using the Besicovitch covering theorem to $\Omega\backslash A_{i,m}$, there exists $c(n)$ such that for j large enough

$$\mu_t(K) \le \int_\Omega \phi_m\, d\mu_t = \lim_{i\to\infty} \int_{\Omega\backslash A_{j,m}} \phi_m\, d\mu_t^{\varepsilon} \le \frac{c(n)c_h(t)}{m}, \tag{5.220}$$

Now by taking m large enough we get Claim 1.

Thus, for μ_t a.e. x there exists $m \in \mathbb{N}$ such that $x \in A_m$. Also the rectifiability of μ_t implies that μ_t a.e. x has an approximate tangent space. We fix any such x, after a suitable translation and rotation we may consider $x = 0$ with an approximate tangent space $P := \{x_n = 0\}$. Now, let $\Phi_{r_i}(x) = \frac{x}{r_i}$ where r_i is a sequence such that $\lim_{i\to\infty} r_i = 0$. Then the push-forward of the varifold V_t, see Definition 4.5.16, is

$$\begin{aligned}
(\Phi_{r_i})_{\#}V_t(\phi) &= \int_{\Phi_{r_i}(M)} \phi(x, Tan_x \Phi_{r_i}(M))\theta(\Phi_{r_i}^{-1}x)\, d\mathcal{H}^{n-1}(x) \\
&= \int_{\frac{M}{r_i}} \phi(x, Tan_x \frac{M}{r_i})\theta(r_i x)\, d\mathcal{H}^{n-1}(x) \\
&\to \theta(0)\int_P \phi(x,P)\, d\mathcal{H}^{n-1}(x), \quad \text{as } i \to \infty,
\end{aligned} \tag{5.221}$$

and so $\lim_{i\to\infty}(\Phi_{r_i})_{\#}V_t = \theta\, V_P$, where $V_P = \mathcal{H}^{n-1}\lfloor_P \otimes \delta_P$ is the unit density rectifiable varifold derived from P. Since we have supposed that there exists $m \in \mathbb{N}$ such that $0 \in A_m$, by the definition of A_m we get that there exists a sequence $x_i \in A_{i,m}$ such that $x_i \to_{i\to\infty} 0$. Hence, up to a subsequence we assume that

$$\lim_{i\to\infty}(\Phi_{r_i})_{\#}V_t^{\varepsilon_i} = \theta\, V_P, \tag{5.222}$$

$$\lim_{i\to\infty}\frac{x_i}{r_i} = 0, \tag{5.223}$$

and

$$\tilde{\varepsilon_i} := \frac{\varepsilon_i}{r_i} \quad \text{with } \lim_{i\to\infty}\tilde{\varepsilon_i} = 0. \tag{5.224}$$

Next, we rescale our coordinates as follow

$$\tilde{x} := \frac{x}{r_i},$$

and we define

$$\tilde{u}^{\tilde{\varepsilon_i}}(\tilde{x}) := u^{\varepsilon_i}(r_i\tilde{x}).$$

Similarly, $\tilde{\xi}^{\tilde{\varepsilon_i}}$ is defined according to $\tilde{\varepsilon_i}$ and $\tilde{u}^{\varepsilon_i}$. By (5.148) we obtain that

$$\lim_{i\to\infty}\int_{B_3}\left|\tilde{\xi}^{\tilde{\varepsilon_i}}\right| d\tilde{x} = \lim_{i\to\infty}\frac{1}{r_i^{n-1}}\left|\xi^{\varepsilon_i}(B_{3r_i})\right| = 0. \tag{5.225}$$

For $y \in B_2$, $0 < r < 2$, i large enough so $rr_i < \frac{1}{2}$ and $0 < \varepsilon_i < \varepsilon_5$ we have that

$$\int_0^r\int_{B_{\tilde{\tau}}(y)}\frac{1}{\tilde{\tau}^n}(\tilde{\xi}^{\tilde{\varepsilon_i}})_+\, d\tilde{x}\, d\tilde{\tau} = \int_0^{rr_i}\int_{B_\tau(r_iy)}\frac{1}{\tau^n}(\xi^{\varepsilon_i})_+\, dx\, d\tau \le 2w_n\varepsilon_i^{\frac{1}{4}} + c_{12}\varepsilon_i^{\frac{1}{4}}\left|\ln\varepsilon_i\right|, \tag{5.226}$$

where we had used (5.76) in the case where $\tau \in (0, \varepsilon_i^{\frac{3}{4}})$ and (5.102) in the case where $\tau \in (\varepsilon_i^{\frac{3}{4}}, rr_i)$. Hence, (5.226) goes to 0 as $i \to \infty$.
Moreover, since $x_i \in A_{i,m}$ together with (5.223) and (5.132)

$$\begin{aligned}\tilde{\varepsilon_i}\int_{B_3}\left|\Delta\tilde{u}^{\tilde{\varepsilon_i}} - \frac{W'(\tilde{u}^{\tilde{\varepsilon_i}})}{\tilde{\varepsilon_i}^2}\right|\left|\nabla\tilde{u}^{\tilde{\varepsilon_i}}\right| d\tilde{x} &= \frac{\varepsilon_i}{r_i^{n-2}}\int_{B_{3r_i}}\left|\Delta u^{\varepsilon_i} - \frac{W'(u^{\varepsilon_i})}{\varepsilon_i^2}\right|\left|\nabla u^{\varepsilon_i}\right| dx \\ &\le \frac{m}{r_i^{n-2}}\mu_t^{\varepsilon_i}(B_{4r_i}(x_i)) \le m4^{n-1}w_{n-1}D_1r_i\end{aligned} \tag{5.227}$$

which also vanishes as $i \to \infty$.

Now, let $\nu = (\nu_1, \cdots, \nu_n) = \dfrac{\nabla\tilde{u}^{\tilde{\varepsilon_i}}}{|\nabla\tilde{u}^{\tilde{\varepsilon_i}}|}$, and $\tilde{V}_t^{\tilde{\varepsilon_i}}$ to be the varifold defined in (5.11) but according to $\tilde{u}^{\tilde{\varepsilon_i}}$. Note that $\tilde{V}_t^{\tilde{\varepsilon_i}} = (\Phi_{r_i})_{\#}V_t^{\varepsilon_i}$.

Claim 2: $\displaystyle\int_{B_3}(1-(\nu_n)^2)\tilde{\varepsilon_i}\left|\nabla\tilde{u}_{\tilde{\varepsilon_i}}\right|^2 d\tilde{x} \to 0$ as $i \to \infty$.
Define the continuous function $\psi : \pm\nu \in \mathbb{S}^{n-1}/\{\pm 1\} \longmapsto 1-\nu_n^2$. For all $\phi \in C_c(\mathbb{R}^n)$

$$\begin{aligned}\tilde{V}_t^{\tilde{\varepsilon_i}}(\phi\psi) = \int\phi(\tilde{x})(1-(\nu_n)^2)\, d\left\|\tilde{V}_t^{\tilde{\varepsilon_i}}\right\|(\tilde{x}) &\to \theta V_P(\phi\psi) \\ &= \theta\int_P\phi(\tilde{x})\psi((0,\cdots,\pm 1))\, d\mathcal{H}^{n-1}(\tilde{x}) \\ &= 0,\end{aligned} \tag{5.228}$$

which implies our claim.

Now, fix a positive integer N such that we have

$$\theta \in [(N-1)\sigma, N\sigma). \tag{5.229}$$

For the sake of simplicity we will drop the tilde in what follows.

Let $s > 0$ arbitrary. Using (5.226) with Proposition 5.9.5 we can find $0 < b < 1$ such that

$$\int_{B_3 \cap \{u^\varepsilon \le b \text{ or } u^\varepsilon \ge 1-b\}} \left(\frac{\varepsilon_i \, |\nabla u^{\varepsilon_i}|^2}{2} + \frac{W(u^{\varepsilon_i})}{\varepsilon_i} \right) \le s \tag{5.230}$$

for $0 < \varepsilon_i < \varepsilon_8$. We now choose ϱ and L in Proposition 5.9.2 corresponding to s, b, and c_7. In what follows we choose $R = 2$ and $a = L\varepsilon_i$ in Proposition 5.9.1. Define

$$\begin{aligned} G_i \;:=\; & B_2 \cap \{b \le u^{\varepsilon_i} \le 1-b\} \cap \Bigg\{ x : \int_{B_r(x)} \varepsilon_i \left| \Delta u^{\varepsilon_i} - \frac{W'(u^{\varepsilon_i})}{\varepsilon_i^2} \right| |\nabla u^{\varepsilon_i}| + |\xi^{\varepsilon_i}| \\ & + (1 - (\nu_n)^2) \varepsilon_i \, |\nabla u^{\varepsilon_i}|^2 \le \varrho \, \mu_t^{\varepsilon_i}(B_r(x)) \text{ for } \varepsilon_i L \le r \le 1 \Bigg\}. \end{aligned} \tag{5.231}$$

Claim 3: $\mu_t^{\varepsilon_i}(B_2 \cap \{b \le u^{\varepsilon_i} \le 1-b\} \setminus G_i) \to 0$.
The Besicovitch covering theorem implies that there exists $N(n)$ such that

$$\begin{aligned} \mu_t^{\varepsilon_i}(B_2 \cap \{b \le u^{\varepsilon_i} \le 1-b\} \setminus G_i) \;\le\; & \frac{N(n)}{\varrho} \int_{B_3} \varepsilon_i \left| \Delta u^{\varepsilon_i} - \frac{W'(u^{\varepsilon_i})}{\varepsilon_i^2} \right| |\nabla u^{\varepsilon_i}| + |\xi^{\varepsilon_i}| \\ & + (1 - (\nu_n)^2) \varepsilon_i \, |\nabla u^{\varepsilon_i}|^2 \, dx, \end{aligned} \tag{5.232}$$

which goes to 0 as $i \to \infty$.

Claim 4: For $x \in G_i$ and $L\varepsilon_i \le r \le 1$ we have $\mu_t^{\varepsilon_i}(B_r(x)) \ge (\sigma - 2s) w_{n-1} r^{n-1}$ for sufficiently large i.
Define

$$h_i := \Delta u^{\varepsilon_i} - \frac{W'(u^{\varepsilon_i})}{\varepsilon_i}. \tag{5.233}$$

Multiplying (5.233) by $y \cdot \nabla u^{\varepsilon_i} \xi_1$, where $\xi_1(y)$ is a smooth approximation of the characteristic function χ_{B_r}, and integrating by parts then letting $\xi_1 \to \chi_{B_r}$ we get

$$\frac{d}{d\tau} \left(\frac{1}{\tau^{n-1}} \int_{B_\tau} e_\varepsilon \right) + \frac{1}{\tau^n} \int_{B_\tau} \xi^\varepsilon + \varepsilon h_i (y \cdot \nabla u^\varepsilon)) - \frac{\varepsilon}{\tau^{n+1}} \int_{\partial B_\tau} (y \cdot \nabla u^\varepsilon)^2 = 0.$$

Integrating over $(L\varepsilon_i, r)$ and using the fact that $x_i \in G_i$ we obtain

$$\begin{aligned} \frac{1}{\tau^{n-1}} \mu_t^{\varepsilon_i}(B_\tau(x)) \Big|_{\tau = L\varepsilon_i}^{r} \;&\ge\; w_{n-1} D_1 (r - \varepsilon_i L) - \int_{L\varepsilon_i}^{r} \varrho \frac{\mu_t^{\varepsilon_i}(B_\tau(x))}{\tau^{n-1}} \, d\tau & (5.234) \\ &\ge\; o(1) - w_{n-1} D_1 \varrho, & (5.235) \end{aligned}$$

then using the fact that *Claim 4* is satisfied for $r = \varepsilon_i L$, with $2s$ replaced by s, according to Proposition 5.9.2 and choosing $D_1 \rho < s$ we get the result.

Claim 5: $\#(P^{-1}(x) \cap G_i \cap \{u^{\varepsilon_i} = \ell\}) \le N - 1$ for every $x \in P \cap B_1$ and $b \le \ell \le 1 - b$.

Assume that *Claim 5* is not valid and let $\mathbf{Y}$ be the set of an N elements. Choose $R = 1$, $\phi = u^{\varepsilon_i}$, $a = L\varepsilon_i$ we get by Proposition 5.9.1

$$\sum_{y\in Y} \frac{1}{(L\varepsilon_i)^{n-1}} \mu_t^{\varepsilon_i}(B_{L\varepsilon_i}(y)) \leq s + (1+s)\mu_t^{\varepsilon_i}(\{z : \operatorname{dist}(Y,z) < 1\}), \tag{5.236}$$

Also we have that

$$\lim_{i\to\infty} \mu_t^{\varepsilon_i}(\{z : \operatorname{dist}(Y,z) < 1\}) = \theta w_{n-1}.$$

Claim 4 yields to

$$N(\sigma - 2s)w_{n-1} \leq s + (1+s)\theta w_{n-1} \tag{5.237}$$

which is a contradiction for sufficiently small s depending on σ, θ and n.

Claim 6: $\theta = (N-1)\sigma$. We have that, as $i \to \infty$

$$P_{\#}\hat{V}_t^{\varepsilon_i}(\phi) = \int_{\{x_n \leq 1\}} \phi(P(x), P)\, |\Lambda_{n-1} P \circ (I - \nu\otimes\nu)|\, d\mu_t^{\varepsilon_i} \to \theta V_P, \tag{5.238}$$

where $\hat{V}_t^{\varepsilon_i} := V_t^{\varepsilon_i} \lfloor_{\{x_n\leq 1\}\times G(n,n-1)}$ and the $n-1$-dimensional Jacobian

$$|J_{n-1} P \circ (I - \nu\otimes\nu)| = |\nu_n|$$

Let B_1 the $n-1$ dimensional ball of radius 1 and centered at the origin. Noticing that $P \cap B_1 = B_1$ we get

$$\begin{aligned}
w_{n-1}\theta &= \theta\mathcal{H}^{n-1}(B_1) = \|\theta V_P\|(B_1) = \lim_{i\to\infty} \left\| P_{\#}\hat{V}_t^{\varepsilon_i} \right\|(B_1) = \lim_{i\to\infty} \int_{B_1} |\nu_n|\, d\mu_t^{\varepsilon_i} \\
&\leq \lim_{i\to\infty} \left(\int_{B_1\cap\{b\leq u^{\varepsilon_i}\leq 1-b\}\cap G_i} |\nu_n|\, d\mu_t^{\varepsilon_i} + \int_{B_1\cap\{b\leq u^{\varepsilon_i}\leq 1-b\}\setminus G_i} |\nu_n|\, d\mu_t^{\varepsilon_i} \right. \\
&\quad \left. + \int_{B_1\cap\{u^{\varepsilon_i} > 1-b \text{ or } u^{\varepsilon_i} < b\}} |\nu_n|\, d\mu_t^{\varepsilon_i} \right) \\
&\leq \lim_{i\to\infty} \int_{B_1\cap\{b\leq u^{\varepsilon_i}\leq 1-b\}\cap G_i} |\nu_n|\, |\nabla u^{\varepsilon}|\, \sqrt{2W(u^{\varepsilon_i})}\, dx + 2s \\
&\leq \lim_{i\to\infty} \left(\int_b^{1-b} d\tau \int_{\{u^{\varepsilon_i}=\tau\}\cap B_1\cap G_i} |\nu_n|\, \sqrt{2W(\tau)}\, d\mathcal{H}^{n-1} \right) + 2s \\
&\leq \lim_{i\to\infty} \int_b^{1-b} \sqrt{2W(\tau)} \int_{\{x_n=0\}} \mathcal{H}^0\left(\{u^{\varepsilon_i} = \tau\}\cap G_i \cap P^{-1}(x)\cap B_1\right) d\mathcal{H}^{n-1}(x)\, d\tau + 2s \\
&\leq \sigma(N-1)w_{n-1} + 2s.
\end{aligned} \tag{5.239}$$

where we have used Claim 3, Claim 5, (5.230) the co-area formula, the area formula and the fact that

$$\begin{aligned}
\frac{\varepsilon_i}{2}|\nabla u^{\varepsilon_i}|^2 + \frac{W(u^{\varepsilon_i})}{\varepsilon_i} - |\nabla u^{\varepsilon_i}|\sqrt{2W(u^{\varepsilon_i})} &= \left(\frac{\sqrt{\varepsilon_i}}{\sqrt{2}}|\nabla u^{\varepsilon_i}| - \sqrt{\frac{W(u^{\varepsilon_i})}{\varepsilon_i}} \right)^2 \\
&\leq \left| \frac{\varepsilon_i |\nabla u^{\varepsilon_i}|^2}{2} - \frac{W(u^{\varepsilon_i})}{\varepsilon_i} \right| = |\xi_t^{\varepsilon_i}|.
\end{aligned} \tag{5.240}$$

Since $s > 0$ is arbitrary *Claim 6* is proved which ends the proof. □

5.10 Existence of generalized velocity

We denote

$$d\tilde{\mu}_t^\varepsilon := \varepsilon \left|\nabla u^\varepsilon(.,t)\right|^2 \, dx,$$

$$d\tilde{\mu}^\varepsilon := \varepsilon \left|\nabla u^\varepsilon\right|^2 \, dx\, dt,$$

and for all $\eta \in C_c(\Omega; \mathbf{G}(n, n-1))$

$$\tilde{V}_t^\varepsilon(\eta) := \int \eta(x, P^\varepsilon(x,t)) d\tilde{\mu}_t^\varepsilon.$$

Lemma 5.10.1. *(Convergence of the approximate velocities)*
Define for each $0 < \varepsilon < 1$ the approximate velocity $v^\varepsilon :\to \mathbb{R}^n$ by

$$v^\varepsilon := \begin{cases} -\dfrac{\partial_t u^\varepsilon}{|\nabla u^\varepsilon|} \dfrac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|} & \text{if } |\nabla u^\varepsilon| \neq 0, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{5.241}$$

Then there exists a function $v \in L^2(\mu, \mathbb{R}^n)$ such that

$$(\tilde{\mu}^\varepsilon, v^\varepsilon) \to (\mu, v) \quad \text{weakly as measure-function pairs} \tag{5.242}$$

and such that we have

$$\int_{Q_T} |v|^2 \, d\mu \leq \liminf_{\varepsilon \to 0} \int_{Q_T} \varepsilon (\partial_t u^\varepsilon)^2 \, dx\, dt. \tag{5.243}$$

Proof.
Noticing that $\tilde{\mu}^\varepsilon = \mu^\varepsilon + \xi^\varepsilon$ and since we have $|\xi^\varepsilon| \to 0$ and $\mu^\varepsilon \to \mu$ as Radon measures on Q_T, we get that

$$\tilde{\mu}^\varepsilon \to \mu \quad \text{as Radon measure on } Q_T. \tag{5.244}$$

Moreover, we have that $(\tilde{\mu}^\varepsilon, v^\varepsilon)$ is a function-measure pair on Q_T with

$$\int_{Q_T} |v^\varepsilon|^2 \, d\tilde{\mu}^\varepsilon = \int_{|\nabla u^\varepsilon| \neq 0} \frac{(\partial_t u^\varepsilon)^2}{|\nabla u^\varepsilon|^2} \varepsilon \left|\nabla u^\varepsilon\right|^2 \, dx\, dt \leq \int_{Q_T} \varepsilon (\partial_t u^\varepsilon)^2 \, dx\, dt \leq c_2 + 2E_0. \tag{5.245}$$

Then there exists a function $v \in L^2(\mu, \mathbb{R}^n)$ such that we have (5.242), which is equivalent to

$$\lim_{\varepsilon \to 0} -\varepsilon \int_{Q_T} \eta \cdot \nabla u^\varepsilon \partial_t u^\varepsilon \, dx\, dt = \int_{Q_T} \eta \cdot v \, d\mu_t, \tag{5.246}$$

for all $\eta \in C_c(Q_T; \mathbb{R}^n)$. □

Lemma 5.10.2. *(Normal velocity)*
For μ-almost all $(x,t) \in Q_T$ we have

$$v(x,t) \perp T_x \mu_t. \tag{5.247}$$

Proof.
Define for each ε the function

$$\hat{v}_\varepsilon(x,t,S) := v_\varepsilon(x,t) \quad \text{for all } (x,t,S) \in Q_T \times \mathbf{G}(n,n-1). \tag{5.248}$$

Therefore, we get

$$\begin{aligned}\int_{Q_T\times\mathbf{G}(n,n-1)} \hat{v}_\varepsilon^2(t,x,S)\, d\tilde{V}^\varepsilon(t,x,S) &= \int_{Q_T} \hat{v}_\varepsilon^2(t,x,P_\varepsilon)\, d\tilde{\mu}^\varepsilon(x,t) \\ &= \int_{Q_T} v_\varepsilon^2(x,t)\, d\tilde{\mu}^\varepsilon(x,t) \le C(\lambda,E_0), \end{aligned} \tag{5.249}$$

which implies that there exists a function $\hat{v} \in L^2(V;\mathbb{R}^n)$ such that

$$\left(\tilde{V}^\varepsilon, \hat{v_\varepsilon}\right) \to (V,\hat{v})$$

weakly as measure-function pairs. Moreover, let the function

$$h \in C_c(\mathbb{R}^{n\times n}) \quad \text{such that } h(S)=1 \quad \text{for all projections } S.$$

Therefore, using (5.180) we obtain for all $\eta \in C_c(Q_T;\mathbb{R}^n)$

$$\begin{aligned}\int_{Q_T} \eta(x,t)\cdot\hat{v}(t,x,P(x,t))\, d\mu(x,t) &= \int_{Q_T} \eta(x,t)h(P)\cdot\hat{v}(t,x,P(x,t))\, d\mu(x,t) \\ &= \int_{Q_T\times\mathbf{G}(n,n-1)} \eta(x,t)h(S)\cdot\hat{v}(x,t,S)\, dV(x,t,S) \\ &= \lim_{\varepsilon\to0}\int_{Q_T\times\mathbf{G}(n,n-1)} \eta(x,t)h(S)\cdot\hat{v}_\varepsilon(t,x,S)\, dV^\varepsilon(x,t,S) \\ &= \lim_{\varepsilon\to0}\int_{Q_T} \eta(x,t)h(P)\cdot\hat{v}_\varepsilon(x,t,P)\, d\mu^\varepsilon(x,t) \\ &= \lim_{\varepsilon\to0}\int_{Q_T} \eta(x,t)\cdot v_\varepsilon(x,t)\, d\mu^\varepsilon(x,t) \\ &= \int_{Q_T} \eta(x,t)\cdot v(x,t)\, d\mu(x,t), \end{aligned}$$

which implies that

$$\hat{v}(t,x,P(x,t)) = v(x,t) \quad \text{for}\mu\text{-almost all } (x,t)\in Q_T.$$

Also, we have for all $\eta \in C_c(Q_T;\mathbb{R}^n)$ that

$$\begin{aligned}\int_{Q_T} \eta(x,t)\cdot P(x,t)v(x,t)\, d\mu(x,t) &= \int_{Q_T} \eta(x,t)h(P)\cdot P(x,t)\hat{v}(t,x,P)\, d\mu(x,t) \\ &= \int_{Q_T\times\mathbf{G}(n,n-1)} \eta(x,t)h(S)\cdot S\hat{v}(t,x,S)\, dV(t,x,S) \\ &= \lim_{\varepsilon\to0}\int_{Q_T\times\mathbf{G}(n,n-1)} \eta(x,t)h(S)\cdot S\hat{v}_\varepsilon(t,x,S)\, dV^\varepsilon(t,x,S) \\ &= \lim_{\varepsilon\to0}\int_{Q_T} \eta(x,t)h(P^\varepsilon)\cdot P^\varepsilon v_\varepsilon(x,t)\, d\tilde{\mu}^\varepsilon(x,t) \\ &= 0. \end{aligned}$$

This gives us that

$$P(x,t)v(x,t) = 0 \quad \text{for} \mu\text{-almost all } (x,t) \in Q_T,$$

which implies the result. □

Theorem 5.10.3. *(L^2-flow)*
We have that $v \in L^2(\mu;\mathbb{R}^n)$ is the generalized velocity of the family $(V_t)_{t\in(0,T)}$ in the sense of definition (4.5.14).

Proof.
We have that for all $\eta \in C_c^1(Q_T)$

$$\begin{aligned}
\int_{Q_T} \eta \left(\sqrt{\varepsilon}\partial_t u^\varepsilon + \frac{1}{\sqrt{\varepsilon}} w_\varepsilon\right)^2 &= \int_{Q_T} \eta \left(\varepsilon(\partial_t u^\varepsilon)^2 + \frac{1}{\varepsilon} w_\varepsilon^2\right) \qquad (5.250)\\
&\quad +2\int_{Q_T} \eta\, \partial_t u^\varepsilon \left(-\varepsilon\Delta u^\varepsilon + \frac{1}{\varepsilon} W'(u^\varepsilon)\right)\\
&= \int_{Q_T} \eta \left(\varepsilon(\partial_t u^\varepsilon)^2 + \frac{1}{\varepsilon} w_\varepsilon^2\right) dx\, dt + 2\varepsilon \int_{Q_T} \nabla\eta\, \partial_t u^\varepsilon \nabla u_\varepsilon\\
&\quad +2\varepsilon \int_{Q_T} \eta \nabla \partial_t u_\varepsilon \nabla u^\varepsilon + \frac{2}{\varepsilon}\int_{Q_T} \eta\, (\partial_t W(u_\varepsilon))\\
&= \int_{Q_T} \eta \left(\varepsilon(\partial_t u^\varepsilon)^2 + \frac{1}{\varepsilon} w_\varepsilon^2\right) dx\, dt - 2\int_{Q_T} \partial_t\eta\, d\mu^\varepsilon\\
&\quad +2\int_{Q_T} \varepsilon \nabla\eta \cdot \partial_t u_\varepsilon \nabla u_\varepsilon\, dx\, dt. \qquad (5.251)
\end{aligned}$$

Also, we have by Lemma 5.10.1

$$\lim_{\varepsilon\to 0} \int_{Q_T} \varepsilon \nabla\eta \cdot \partial_t u_\varepsilon \nabla u_\varepsilon\, dx\, dt = -\lim_{\varepsilon\to 0} \int_{Q_T} \varepsilon \nabla\eta \cdot v_\varepsilon\, |\nabla u_\varepsilon\,|^2\, dx\, dt = -\int_{Q_T} \nabla\eta \cdot v\, d\mu. \quad (5.252)$$

Therefore, we obtain by using (5.41), (5.42) together with (5.252)

$$\left|\int_{Q_T} (\partial_t\eta + \nabla\eta \cdot v)\, d\mu\right| \le c(c_2, E_0)\, \|\eta\|_{C(Q_T)}\,. \qquad (5.253)$$

□

5.11 Convergence to a perturbed mean curvature flow

Theorem 5.11.1. *(Convergence to perturbed mean curvature flow)*
There exists a function $g \in L^2(\mu;\mathbb{R}^n)$ such that up to a subsequence we have for all $\eta \in C_c(Q_T;\mathbb{R}^n)$

$$\int_{Q_T} \eta \cdot v\, d\mu = \int_{Q_T} \eta \cdot (h_{V_t} + g)\, d\mu, \qquad (5.254)$$

where

$$\int_{Q_T} \eta \cdot g\, d\mu := \lim_{\varepsilon\to 0} \int_{Q_T} -\eta \cdot \nabla u^\varepsilon g_\varepsilon\, dx\, dt. \qquad (5.255)$$

Proof.
Let

$$\vec{g_\varepsilon} := -\frac{g_\varepsilon}{\varepsilon\left|\nabla u^\varepsilon\right|}\nu_\varepsilon. \tag{5.256}$$

Then we get that

$$\int_{Q_T} |\vec{g_\varepsilon}|^2 \, d\tilde{\mu}^\varepsilon(x,t) = \int_{Q_T} \frac{(g_\varepsilon)^2}{\varepsilon^2\left|\nabla u^\varepsilon\right|^2}\varepsilon\left|\nabla u^\varepsilon\right|^2 \, dx\, dt = \frac{1}{\varepsilon}\int_{Q_T}(g_\varepsilon)^2\, dx\, dt \le c_2. \tag{5.257}$$

Therefore, there exists a function $g \in L^2(\mu;\mathbb{R}^n)$ such that we have for all $\eta \in C_c(Q_T;\mathbb{R}^n)$

$$\lim_{\varepsilon\to 0}\int_{Q_T}\eta\cdot\vec{g_\varepsilon}\, d\tilde{\mu}^\varepsilon = \int_{Q_T}\eta\cdot g\, d\mu, \tag{5.258}$$

which is equivalent to

$$\lim_{\varepsilon\to 0}\int_{Q_T} -\eta\cdot\nabla u^\varepsilon g_\varepsilon\, d\tilde{\mu}^\varepsilon = \int_{Q_T}\eta\cdot g\, d\mu. \tag{5.259}$$

Now multiplying our problem (P^ε) with $-\eta\cdot\nabla u^\varepsilon$ and integrating over Q_T we get

$$\begin{aligned} -\varepsilon\int_{Q_T}\eta\cdot\partial_t u^\varepsilon\nabla u^\varepsilon\, dx\, dt &= -\varepsilon\int_{Q_T}\eta\cdot\nabla u^\varepsilon\Delta u^\varepsilon + \varepsilon\int_{Q_T}\eta\cdot\nabla u^\varepsilon W'(u^\varepsilon) - \int_{Q_T} g_\varepsilon\eta\cdot\nabla u^\varepsilon\, dx\, dt \\ &= \int_{Q_T}\eta\cdot\nabla u^\varepsilon w_\varepsilon - \int_{Q_T} g_\varepsilon\eta\cdot\nabla u^\varepsilon\, dx\, dt. \end{aligned} \tag{5.260}$$

Now passing to the limit as $\varepsilon\to 0$ together with (5.246), (5.184) and (5.255) we get

$$\int_{Q_T}\eta\cdot v\, d\mu = \int_{Q_T}\eta\cdot(h_{V_t}+g)\, d\mu \tag{5.261}$$

□

Proposition 5.11.2. *We have that*

$$g = (\nabla\chi(q)\cdot\nu)\nu + \frac{1}{\theta}\sqrt{2}\alpha\nu \quad \mathcal{H}^{n-1}\ a.e.\ on\ \partial^*\{u=1\},$$

where ∂^* denotes the reduced boundary, g is defined in (5.255), $\nu = -\dfrac{\nabla u}{|\nabla u|}$ is the inner normal to $\partial^*\{u=1\}$ and θ is the multiplicity function.

Proof.
According to (5.255) we need to compute for all $\eta\in C_c(Q_T;\mathbb{R}^n)$

$$\begin{aligned} \int_{Q_T} -\eta\cdot\nabla u^\varepsilon g_\varepsilon\, dx\, dt &= \varepsilon\int_{Q_T}\eta\cdot\nabla u^\varepsilon\otimes\nabla u^\varepsilon\cdot\nabla\chi(q_\varepsilon) + \varepsilon\int_{Q_T}\eta\cdot\nabla u^\varepsilon u^\varepsilon\Delta\chi(q_\varepsilon) \\ &\quad -\alpha\int_{Q_T}\eta\cdot\nabla u^\varepsilon u^\varepsilon(1-u^\varepsilon) \\ &:= I_1 + I_2 + I_3, \end{aligned} \tag{5.262}$$

where I_1, I_2 and I_3 are the first, the second and the third term on the right-hand-side respectively. We can also write

$$I_1 = \int_{Q_T} -\eta \cdot (P_\varepsilon - I)\nabla\chi(q_\varepsilon)\, d\tilde{\mu}^\varepsilon,$$

where $P_\varepsilon := I - \nu^\varepsilon \otimes \nu^\varepsilon$. Moreover, we have

$$\int_{Q_T} |\nabla\chi(q_\varepsilon)|^2 \, d\tilde{\mu}^\varepsilon \leq 2 \sup_{Q_T}(|\nabla\chi(q_\varepsilon)|^2)\mu^\varepsilon(Q_T) \leq c(C_q, c_1, T).$$

Thus, using the fact that $\tilde{\mu}^\varepsilon \to \mu$ in the sense of Radon measures we deduce that there exists a function $b \in L^2(\mu; \mathbb{R}^n)$ such that

$$(-\nabla\chi(q_\varepsilon), \tilde{\mu^\varepsilon}) \to (b, \mu) \quad \text{weakly as measure-function pairs.} \tag{5.263}$$

Now we will prove that $(P^\varepsilon, \tilde{\mu}^\varepsilon) \to (P, \mu)$ strongly as measure-function pairs. To this end we first prove the uniform boundedness of P^ε in $L^2(\tilde{\mu}^\varepsilon; \mathbb{R}^{n\times n})$,

$$\begin{aligned}
\int_{Q_T} |P^\varepsilon|^2 \, d\tilde{\mu}^\varepsilon &= \int_{Q_T} \left(\sum_{i=1}^n \sum_{j=1}^n \left(\delta_{ij} - \frac{\partial_{x_i} u^\varepsilon}{|\nabla u^\varepsilon|} \frac{\partial_{x_j} u^\varepsilon}{|\nabla u^\varepsilon|} \right)^2 \right) \varepsilon \, |\nabla u^\varepsilon|^2 \, dx\, dt \\
&= \varepsilon \int_{Q_T} \left(n\,|\nabla u^\varepsilon|^2 + |\nabla u^\varepsilon|^2 - 2\,|\nabla u^\varepsilon|^2 \right) dx\, dt \\
&= \varepsilon(n-1) \int_{Q_T} |\nabla u^\varepsilon|^2 \leq c(n, c_1, T),
\end{aligned} \tag{5.264}$$

which implies that there exists $Y \in L^2(\mu; \mathbb{R}^{n\times n})$ such that for all $\eta \in C_c(Q_T; \mathbb{R}^{n\times n})$.

$$\lim_{\varepsilon\to 0} \int_{Q_T} \eta \cdot P^\varepsilon d\tilde{\mu}^\varepsilon = \int_{Q_T} \eta \cdot Y d\mu. \tag{5.265}$$

Now we prove that

$$Y(x,t) = P(x,t) \quad \text{for} - \mu \text{ almost all} (x,t) \in Q_T.$$

Let $h \in C_c(\mathbb{R}^{n\times n})$ such that $h(S) = 1$ for all projections S. Using (5.180) we have for all $\eta \in C_c(Q_T; \mathbb{R}^{n\times n})$

$$\begin{aligned}
\int_{Q_T} \eta(x,t) \cdot Y(x,t)\, d\mu &= \lim_{\varepsilon\to 0} \int_{Q_T} h(P^\varepsilon)\eta(x,t) \cdot P^\varepsilon(x,t) d\tilde{\mu}^\varepsilon && (5.266)\\
&= \lim_{\varepsilon\to 0} \int_{Q_T \times \mathbf{G}(n,n-1)} h(S)\eta(x,t) \cdot S \, d\tilde{V}^\varepsilon(x,t,S) && (5.267)\\
&= \int_{Q_T \times \mathbf{G}(n,n-1)} h(S)\eta(x,t) \cdot S \, dV(x,t,S) && (5.268)\\
&= \int_{Q_T} h(P)\eta(x,t) \cdot P(x,t)\, d\mu && (5.269)\\
&= \int_{Q_T} \eta(x,t) \cdot P(x,t)\, d\mu. && (5.270)
\end{aligned}$$

which implies that

$$Y(x,t) = P(x,t) \quad \text{for} - \mu \text{ almost all} (x,t) \in Q_T.$$

In addition, let $\phi \in C_c(Q_T \times \mathbb{R}^{n\times n})$ such that $\phi(x,t,S) = |S|^2$ for all projections. We obtain that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{Q_T} |P^\varepsilon|^2 \, d\tilde{\mu}^\varepsilon &= \lim_{\varepsilon\to 0}\int_{Q_T\times \mathbf{G}(n,n-1)} \phi(x,t,S)\, d\tilde{V}^\varepsilon(x,t,S) \\ &= \int_{Q_T\times \mathbf{G}(n,n-1)} \phi(x,t,S)\, dV(x,t,S) \\ &= \int_{Q_T} |P|^2 \, d\mu. \end{aligned} \tag{5.271}$$

Therefore $(P_\varepsilon, \tilde{\mu}^\varepsilon) \to (P,\mu)$ strongly as measure-function pairs. We then obtain by Proposition 4.4.4 that

$$\int_{Q_T} -\eta\cdot(P_\varepsilon - Id)\nabla\chi(q_\varepsilon)\, d\tilde{\mu}^\varepsilon = \int_{Q_T} \eta\cdot(P-Id)\, b\, d\mu.$$

Now, by (5.2) we have q_ε is uniformly bounded in $L^\infty(0,T;W^{2,\infty}(\Omega))$. Using the fact that $W^{2,\infty}(\Omega) \hookrightarrow C^1(\bar{\Omega})$ compactly then as $\varepsilon \to 0$ we have for all $t \in (0,T)$

$$q_\varepsilon(t,.) \to q(t,.) \quad \text{strongly in } C^1(\bar{\Omega}) \quad \text{as } \varepsilon \to 0,$$

which implies that

$$\nabla\chi(q_\varepsilon(t,.)) \to \nabla\chi(q(t,.)) \quad \text{uniformly on } \Omega \text{ as } \varepsilon \to 0.$$

Also, we have that in view of (5.263)

$$\lim_{\varepsilon\to 0}\int_\Omega -\nabla\chi(q_\varepsilon(t,.))\cdot\eta(t,.) d\tilde{\mu}^\varepsilon_t = \int_\Omega b(t,.)\cdot\eta(t,.)\, d\mu_t,$$

which implies that

$$b = -\nabla\chi(q) \quad \text{on the support of } \mu.$$

Now, estimating the second term I_2 by using Holder's inequality we get that

$$|I_2| \le \sqrt{\varepsilon}\, c(n, c_1, C_q),$$

so that I_2 vanishes as $\varepsilon \to 0$.
Concerning the term I_3, we let

$$\vec{f_\varepsilon} := \frac{\alpha u^\varepsilon(1-u^\varepsilon)}{\varepsilon\,|\nabla u^\varepsilon|}\nu_\varepsilon \quad \text{on } \{\nabla u^\varepsilon \neq 0\},$$

where $\nu_\varepsilon = \dfrac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}$. Then we have

$$\begin{aligned}\int_{Q_T}\left|\vec{f_\varepsilon}\right|^2 d\tilde{\mu}^\varepsilon &= \int_{Q_T} \frac{\alpha^2 (u^\varepsilon)^2(1-u^\varepsilon)^2}{\varepsilon^2\,|\nabla u^\varepsilon|^2}\varepsilon\,|\nabla u^\varepsilon|^2\, dx\, dt \\ &= \int_{Q_T} \alpha^2\frac{(u^\varepsilon)^2(1-u^\varepsilon)^2}{\varepsilon}\, dx\, dt = \int_{Q_T} 4\alpha^2\frac{W(u^\varepsilon)}{\varepsilon}\, dx\, dt \le c(c_1,T,\alpha).\end{aligned}$$

Thus, $\vec{f_\varepsilon}$ is uniformly bounded in $L^2(\tilde{\mu}^\varepsilon;\mathbb{R}^n)$ which implies the existence of $f \in L^2(\mu;\mathbb{R}^n)$ such that we have as $\varepsilon \to 0$

$$(\vec{f_\varepsilon}, \tilde{\mu}^\varepsilon) \to (f, \mu) \quad \text{weakly as measure-function pairs.}$$

Letting $K(r) := \int_0^r \alpha s(1-s)\,ds$ we can write

$$\int_{Q_T} \vec{f_\varepsilon}\, d\tilde{\mu}^\varepsilon = \int_{Q_T} \alpha\, \eta \cdot \nabla u^\varepsilon u^\varepsilon (1-u^\varepsilon)\, dx\, dt = \int_{Q_T} \eta \cdot \nabla K(u^\varepsilon)\, dx\, dt.$$

Moreover, $K(u^\varepsilon) \to K(u)$ in $L^1(Q_T)$, $K(u^\varepsilon(t,.))$ is uniformly bounded in $BV(\Omega)$ (see the proofs of Proposition 5.4.2 and Proposition 5.4.4) and

$$\nabla K(u^\varepsilon(t,.)) \to K(1)\nabla u(t,.) \quad \text{as Radon measures on } \Omega, \quad \text{see (5.50).}$$

Using the Lebesgue dominated convergence theorem we obtain that for all $\eta \in C_c(Q_T;\mathbb{R}^n)$

$$\int_{Q_T} f \cdot \eta\, d\mu = K(1) \int_{Q_T} \eta \cdot \nabla u\, dx\, dt = \alpha\sqrt{2}\sigma \int_{Q_T} \eta \cdot \nabla u\, dx\, dt,$$

where σ is given in (4.27). Now using the fact that $d\mu = \theta\sigma\,|\nabla u|\; dx\, dt$ on $\partial^*\{u=1\}$ we deduce the result. □

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