



# Maxwell field on the Reissner-Nordström-De Sitter manifold: decay and conformal scattering

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Mokdad Mokdad. Maxwell field on the Reissner-Nordström-De Sitter manifold: decay and conformal scattering. Mathematical Physics [math-ph]. Université de Bretagne occidentale - Brest, 2016. English. NNT : 2016BRES0060 . tel-01502657

**HAL Id: tel-01502657**

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UNIVERSITE  
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LOIRE

**THÈSE / UNIVERSITÉ DE BRETAGNE OCCIDENTALE**

*sous le sceau de l'Université Bretagne Loire*

pour obtenir le titre de

**DOCTEUR DE L'UNIVERSITÉ DE BRETAGNE OCCIDENTALE**

*Mention : Mathématiques*

**École Doctorale SICMA**

présentée par

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Préparée à l'Université de Bretagne Occidentale,  
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de Bretagne Atlantique

# Champs de Maxwell en Espace-temps de Reissner- Nordstrøm-De Sitter : Décroissance et Scattering Conforme

**Thèse soutenue le 30 septembre 2016**

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# MAXWELL FIELD ON THE REISSNER-NORDSTRØM-DE SITTER MANIFOLD: DECAY AND CONFORMAL SCATTERING

Ph.D Thesis  
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September 30, 2016



### *Acknowledgements*

*Firstly, I would like to thank my family for their great support and encouragement without which I would have not made it through. To the persons close to my heart, to the dear ones, to my friends, and to everyone who was next to me during the preparation of my Ph.D, to all of them, I say thank you, thank you very much from the bottom of my heart!*

*Special thanks for the referees, Professor Emmanuel Humbert and Professor Juan Valiente-Kroon, for reading the manuscript. I would like to express my gratitude to the members of the examination committee: MME Michèle Benyounes, M. Dietrich Häfner, M. Johannes Huisman, M. Emmanuel Humbert, M. Jérémie Joudioux, M. Jean-Philippe Nicolas, ,and M. Juan Valiente-Kroon.*

*Finally but most gratefully, I thank my advisor Professor Jean-Philippe Nicolas for supervising this thesis. He has been a wise teacher whose guidance was indispensable, a great man of wonderful spirit, and foremost and most importantly, a real friend whom I am so lucky to have. Thank you!*

*Mokdad Mokdad*



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# Résumé / Abstract

## Résumé:

Nous étudions les champs de Maxwell à l'extérieur de trous noirs de Reissner-Nordstrom-de Sitter. Nous commençons par étudier la géométrie de ces espaces-temps: nous donnons une condition sous laquelle la métrique admet trois horizons puis dans ce cadre nous construisons l'extension analytique maximale d'un trou noir de Reissner-Nordstrom-de Sitter. Nous donnons ensuite une description générale des champs de Maxwell en espace-temps courbe, de leur décomposition en composantes spinorielle ainsi que de leur énergie. La première étude analytique établit la décroissance ponctuelle de champs de Maxwell à l'extérieur d'un trou noir de Reissner-Nordstrom-de Sitter ainsi que la décroissance uniforme de l'énergie sur un hyperboloïde qui s'éloigne dans le futur. Ce chapitre utilise des méthodes de champs de vecteurs (estimations d'énergie géométriques) dans l'esprit des travaux de Pieter Blue. Enfin nous construisons une théorie du scattering conforme pour les champs de Maxwell à l'extérieur du trou noir. Ceci consiste en la résolution du problème de Goursat pour les champs de Maxwell à la frontière isotrope de l'extérieur du trou noir, constituée des horizons du trou noir et horizons cosmologiques futurs et passés. Les estimations de décroissance uniforme de l'énergie sont cruciales dans cette partie.

## Abstract:

We study Maxwell fields on the exterior of Reissner-Nordstrom-de Sitter black holes. We start by studying the geometry of these spacetimes: we give the condition under which the metric admits three horizons and in this case we construct the maximal analytic extension of the Reissner-Nordstrom-de Sitter black hole. We then give a general description of Maxwell fields on curved spacetimes, their decomposition into spin components, and their energies. The first result establishes the pointwise decay of the Maxwell field in the exterior of a Reissner-Nordstrom-de Sitter black hole, as well as the uniform decay of the energy flux across a hyperboloid that recedes in the future. This chapter uses the vector fields methods (geometric energy estimates) in the spirit of the work of Pieter Blue. Finally, we construct a conformal scattering theory for Maxwell fields in the exterior of the black hole. This amounts to solving the Goursat problem for Maxwell fields on the null boundary of the exterior region, consisting of the future and past black hole and cosmological horizons. The uniform decay estimates of the energy are crucial to the construction of the conformal scattering theory.





# INTRODUCTION

The year 2015 marked the 100th anniversary of Albert Einstein’s presentation of the complete **Theory of General Relativity** to the Prussian Academy. A hundred years have passed and Einstein’s general theory of relativity is still the most accurate description of gravity that we ever had. According to this theory, gravity is the manifestation of the curvature of spacetime, a Lorentzian 4-manifold consisting of all the events in “space” and “time”, where these two concepts merge into one. The field equations that govern the laws of gravity relate the presence of energy and momentum to the curvature of a Lorentzian metric which is a solution of the equations. The tensorial form of the equations is,

$$\mathbf{G}_{ab} + \Lambda \mathbf{g}_{ab} = \frac{8\pi G}{c^4} \mathbf{T}_{ab}$$

The unknown in the equations is the Lorentzian metric  $\mathbf{g}_{ab}$  which is a non-degenerate symmetric (0,2)-tensor of signature  $(+, -, -, -)$ <sup>1</sup>. The Einstein tensor  $\mathbf{G}_{ab}$  is

$$\mathbf{G}_{ab} = \mathbf{R}_{ab} - \frac{1}{2} \mathbf{g}_{ab} \mathbf{R} ,$$

where  $\mathbf{R}_{ab}$  is the Ricci curvature of the metric  $\mathbf{g}_{ab}$  and  $\mathbf{R}$  is the scalar curvature of the metric. These curvature quantities are given by the Riemann curvature tensor  $\mathbf{R}_{abcd}$  which itself is locally given in terms of the Christoffel symbols of the metric:

$$\Gamma^c_{ab} = \frac{1}{2} \mathbf{g}^{cd} (\partial_a \mathbf{g}_{bd} + \partial_b \mathbf{g}_{ad} - \partial_d \mathbf{g}_{ab}) ,$$

and

$$\mathbf{R}^a_{bcd} = \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \Gamma^a_{ce} \Gamma^e_{db} - \Gamma^a_{de} \Gamma^e_{cb} .$$

The scalar curvature is the trace of the Ricci curvature which in turn is given by the trace of the Riemann curvature tensor:

$$\mathbf{R}_{ab} = \mathbf{R}^c_{acb} \quad \text{and} \quad \mathbf{R} = \mathbf{R}^a_a .$$

$\mathbf{T}_{ab}$  is the *energy-momentum tensor*<sup>2</sup> determined by the matter, energy, and momentum, present in the spacetime. The rest are constants:  $\Lambda$  is the cosmological constant,  $G$  is the gravitational constant of Newton, and finally  $c$  is the speed of light in vacuum.

---

<sup>1</sup>Or  $(-, +, +, +)$ , the difference is a matter of taste in most situations. In this work, we shall carry on with the convention in the text above.

<sup>2</sup>Also called stress-energy-momentum tensor or *stress-energy tensor*.

# BLACK HOLES

The curvature terms in Einstein's field equations contain first and second order partial derivatives of the metric and they are a highly nonlinear system of partial differential equations, which makes them very hard to solve in general. However, several families of exact solutions are known. The trivial solution in the vacuum case, i.e. when the energy-momentum tensor vanishes, with a zero cosmological constant is the simplest Lorentzian metric on  $\mathbb{R}^4$ ,

$$\mathbf{g} = dt^2 - dx^2 - dy^2 - dz^2 ,$$

which is known as the Minkowski metric. Minkowski spacetime is flat, meaning that the Riemann curvature tensor vanishes identically. The second best-known solution is the Schwarzschild metric. This solution of the Einstein vacuum equations with zero cosmological constant, describes an empty spacetime outside a non-rotating and uncharged spherical body of mass  $M$  and radius  $R = 2MGc^{-2}$  by a metric  $\mathbf{g}$  whose spherical coordinate expression is

$$\mathbf{g} = \left(1 - \frac{R}{r}\right) dt^2 - \left(1 - \frac{R}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin(\theta)^2 d\varphi^2) .$$

The metric describes the spacetime region with  $r > R$ , called the exterior Schwarzschild solution, nevertheless, one could, mathematically at least, assume that the space inside the region where the body is supposed to be, is another empty region of spacetime given by the same metric expression but for  $0 < r < R$ , which is called the interior Schwarzschild solution. When viewed with these coordinates, they appear as two completely separate solutions with no physical connection between them, separated by an apparent singularity at  $r = R$ . However, viewed as a Lorentzian manifold, the singularity at  $r = R$  is a mere coordinate singularity due to this particular choice of coordinates. In fact, the Kruskal-Szekeres coordinates extends the original Schwarzschild spacetime and cover the entirety of  $r > 0$ . The Kruskal-Szekeres extension is the maximal analytic extension of the Schwarzschild spacetime, and it describes a theoretical eternal black/white hole. It shows that the hypersurface at  $r = R$  is not singular but rather a regular null hypersurface that acts like a barrier which can be crossed only in one direction and is therefore an event horizon, hence the name black/white hole. (Figure 1)

The singularity at  $r = 0$  is different. This is a genuine physical or geometrical singularity since the scalar curvature

$$\mathbf{R} = \frac{12R^2}{r^6}$$

clearly blows up, and since this is a scalar quantity, it means no coordinate transformation could resolve the singularity at  $r = 0$ .

One form of energy that can induce gravity is light, i.e. electromagnetic radiation. A source-free Maxwell field on spacetime is a 2-form  $F$  satisfying Maxwell's equations:

$$dF = 0 \quad ; \quad d \star F = 0$$

where  $d$  is the exterior differentiation and  $\star$  is the Hodge star operator. The Maxwell system describes the phenomena of electromagnetism. The presence of a Maxwell field

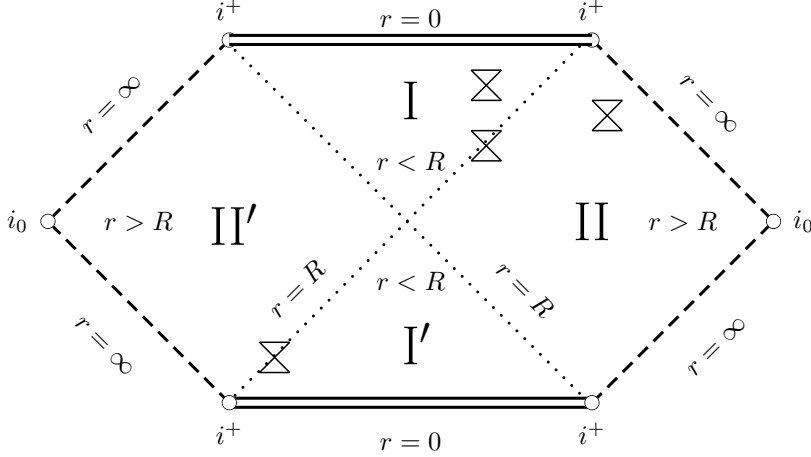


Figure 1: *Penrose-Carter conformal diagram of Kruskal-Szekeres spacetime, the maximal extension of a Schwarzschild black hole (I and II).*

curves spacetime around it. Its effect is given by the electromagnetic energy-momentum tensor

$$\mathbf{T}_{ab} = \frac{1}{4} \mathbf{g}_{ab} F^{cd} F_{cd} - F_{ac} F_b{}^c .$$

The Einstein-Maxwell equations are Einstein's field equations with  $\mathbf{T}_{ab}$  given by the electromagnetic energy-momentum tensor of a Maxwell field. A known family of exact solutions to the Einstein-Maxwell equations with no cosmological constant generalizes the Schwarzschild solution by describing the spacetime outside a rotating charged black hole. This family of solutions is known as the Kerr-Newman black holes, and each member of the family is uniquely described by three real parameters of the body: a mass  $M$ , a charge  $Q$ , and an angular momentum  $a$ . The Schwarzschild solution corresponds to the case when  $a$  and  $Q$  are both zero. If only  $a = 0$ , the solution is called Reissner-Nordström black hole and it describes a non-rotating but charged black hole. When  $Q = 0$  but  $a \neq 0$ , the solution represents a rotating black hole, also known as a Kerr black hole. When a cosmological constant is present, a solution of the Einstein-Maxwell equations will have what can be called a de Sitter ( $\Lambda > 0$ ) or anti-de Sitter ( $\Lambda < 0$ ) “aspect”. The simplest solution to Einstein's vacuum equations with a positive (negative) cosmological constant is the (anti) de Sitter spacetime. The de Sitter spacetime is the analogue in Minkowski spacetime, of a sphere in ordinary Euclidean space. It is maximally symmetric, has constant positive scalar curvature, and is simply connected. It can be visualized as hyperboloid in a 5-dimensional flat Lorentzian manifold. Spherically symmetric asymptotically de Sitter spacetimes that are solutions to the Einstein-Maxwell equations have metrics similar to the Schwarzschild metric. For example, in spherical coordinates, the metric of such a spacetime typically is of the form:

$$\mathbf{g} = f(r) dt^2 - \frac{1}{f(r)} dr^2 - r^2 (d\theta^2 + \sin(\theta)^2 d\varphi^2) ,$$

and

$$f(r) = 1 - \frac{2MG}{c^2 r} + \frac{Q^2}{r^2} - \Lambda r^2. \quad (\text{I})$$

With all of  $M$ ,  $Q$ , and  $\Lambda$  equal to zero, i.e.  $f(r) = 1$ , we get Minkowski spacetime. When all but  $\Lambda$  are equal to zero, we get the de Sitter spacetime. When all but  $M$  are zero, this is Schwarzschild's black hole. and when only  $Q$  is zero we have the Schwarzschild-de Sitter spacetime. Alternatively, if only  $\Lambda$  equals zero, the spacetime is a Reissner-Nordström black hole. Finally, if all three parameters are non zero, we get the *Reissner-Nordström-de Sitter black hole*, which in this work, is the spacetime we are interested in.

*In the rest of the manuscript, the metric is presented in units where both  $G$  and  $c$  are 1. Furthermore, we assume  $M, \Lambda > 0$  and  $Q \neq 0$ , and there is no rotation ( $a = 0$ ).*

Therefore, the Reissner-Nordström-de Sitter solution we work with is a solution of the Einstein-Maxwell field equations in the presence of a positive cosmological constant  $\Lambda$ , that models a non-rotating spherically symmetric charged black hole with mass  $M$  and a charge  $Q$ , in a de Sitter background. The de Sitter background means that there is a cosmological horizon beyond which lies a dynamic region that stretches to infinity, while the Reissner-Nordström nature entails that near the singularity, depending on the relation between the mass and the charge, one has a succession of static and dynamic regions separated by horizons. In our case, we work with three horizons corresponding to  $r_1, r_2$ , and  $r_3$  the three positive zeros of  $f$ , where there is a static region in the interior of the black hole nearest to the singularity, another static region in the exterior which we call  $\mathcal{N} = \mathbb{R}_t \times ]r_2, r_3[_r \times \mathcal{S}_\omega^2$ , and an interior dynamic region separating the two static regions, and finally a dynamic region near infinity. A detailed discussion of this black hole spacetime is done in chapter 1. More on exact solutions and on Einstein's general theory of relativity can be found in classical books such as [28, 74, 142].

*«An small index of used symbols and terminologies is situated at the end of this report.»*

## RESULTS OBTAINED IN THE THESIS

Our work covers essentially different but closely related topics. The main contributions are:

1. **Photon Sphere:** *Finding the necessary and sufficient conditions on the parameters of the Reissner-Nordström-de Sitter metric to have three horizons, and locating the photon sphere.*
2. **Decay:** *Proving pointwise decay in time and uniform decay of the energy flux across achronal hypersurfaces for Maxwell fields on  $\mathcal{N}$ .*
3. **Conformal Scattering:** *Solving the Goursat Problem and constructing a conformal scattering theory for the Maxwell fields on  $\tilde{\mathcal{N}}$ .*

Before presenting our results in precise statements, let us heuristically discuss how these topics relate to each other, which is an important aspect when dealing with the problems addressed by these results.

**The Experiments.** Consider the following classical experiments. Imagine a pulse of light emitted at some time somewhere in  $\mathcal{N}$  the exterior static region, and propagating outwards with a spherical wave front that grows with time. Suppose that an observer in this region, whose line world is time-like, is trying to study the propagating light wave. A natural question such an observer may want to address is whether the measured energy of this pulse of light is conserved for all times, and more subtly, how does the energy confined to a specific part of space changes with time. These are not trivial questions particularly because of the *trapping effect* produced by the extreme gravity of the black hole: There is a place outside the black hole, in  $]r_2, r_3[_r \times \mathcal{S}_\omega^2$  as we show later, where light can orbit the black hole; it swipes a cylindrical hypersurface called the **photon sphere**. Thus a priori, the *local energy* may not decay. The answer depends on the observer of course, yet, a negative answer to the question of **decay** could mean that the spacetime is unstable. The stability problem is a central issue in general relativity. Such considerations about decay are also linked to whether information about the pulse and its history can always be retrieved from its remnants or from its trace on the boundary of the observable space far in the future, i.e. on  $\partial\mathcal{N}$ . In other words, we would like to know if the pulse can be completely characterized along with its entire evolution by observing its asymptotic profile in the distant future. The question is of course valid if we reverse the time orientation replacing the above experiments by similar ones about the past.

**Decoupled System.** In this work, the “pulses” we deal with are test Maxwell fields on a fixed Reissner-Nordström-de Sitter background. That is to say, given a fixed Maxwell field, a solution of Maxwell’s equations, we use its energy-momentum tensor in Einstein’s field equations with a positive cosmological constant. The solution of these field equations is then a Reissner-Nordström-de Sitter black hole spacetime, it is in this sense a solution of the coupled Einstein-Maxwell equations. We fix this solution of the coupled system, and on this fixed spacetime we study solutions to Maxwell’s equations without taking into account the backreaction on the metric. This is a decoupled system.

**Spin Reduction.** The spin components of a Maxwell field<sup>1</sup>, satisfy a coupled system of wave equations, in which the equation for the middle component is decoupled. Since the trapping term contributing to the growth of the conformal energy of the Maxwell field, is multiplied by the absolute value of the middle component, an analysis of the wave-like equation allows us to obtain uniform bounds on the conformal charge of the middle component, and on the trapping error term, and eventually obtain the desired decay for Maxwell fields.

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<sup>1</sup>These are the three complex scalar components of the field on a spin-frame, they characterize the field completely (see (49))

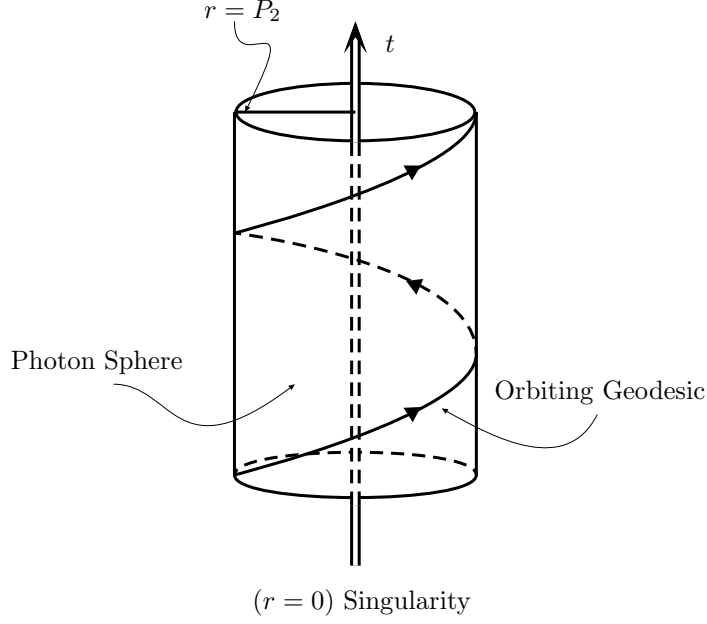


Figure 2: A null geodesic orbiting the black hole at the photon sphere which is a space-like hypersurface.

**Photon Sphere and Decay.** The photon sphere is a space-like hypersurface foliated by purely rotational null geodesics that orbit the black hole at fixed distance (see section 1.1.2 and figure (2)). Since singularities of a Maxwell field travel along null geodesics, the photon sphere is a priori an obstacle for decay of the local energy. This is also true for highly localized high energy wave packets which may orbit the black hole from outside and remain in the neighbourhood of the photon sphere for arbitrary lengths of time without crossing either horizon. The trapping effect manifests itself as an error term also in the conformal charge conservation. In order to overcome this a priori obstruction and show that there is indeed a decay in the local energy in any compact region of  $]r_2, r_3[_r \times \mathcal{S}_\omega^2$  in spite of this trapping of geodesics, we need to use a smooth radial vector field that is pointing away from the photon sphere where the maximum of the effective potential is, so the weight of the vector field needs to vanish and change sign at the photon sphere. We therefore need to locate the photon sphere in order to define our radial vector field adequately.

**Uniform Decay and Conformal Scattering.** If we want to construct a complete scattering theory, i.e. a theory in which the scattering operator is an isomorphism and thus have a complete characterization of Maxwell fields by their asymptotic profiles (their traces on the horizons), we need to make sure that no information is lost at time-like infinities  $i^\pm$ . This is guaranteed if no part of the energy is concentrated at  $i^\pm$ , which is equivalent to saying that the energy flux across an achronal hypersurface decays to zero as the hypersurface approaches  $i^+$  or  $i^-$ . This is the uniform decay we need and prove.

**Generic Spherically Symmetric Spacetimes.** *The decay and the scattering results are true for a wide class of spherically symmetric spacetimes specified in section 3.2.3. The methods we use rely on no special properties that are unique to the Reissner-Nordström-de Sitter black hole. The proofs we present apply without any major change to a generic spherically symmetric spacetime of the type referred to above. In particular, we make no use of the particular form of the horizon function  $f$  defined above in (I), but rather we use some of its general properties.*

## STATEMENT OF RESULTS

We now give the full statements of the results. The theorems are titled and numbered as they appear in the chapters.

### • Three Positive Zeros and One Photon Sphere

**Proposition 1.** *The horizon function  $f$  has exactly three positive distinct zeros if and only if*

$$Q \neq 0 \quad \text{and} \quad 0 < \Lambda < \frac{1}{12Q^2} \quad \text{and} \quad M_1 < M < M_2 ,$$

*where  $M_1$  and  $M_2$  are two positive numbers depending on  $M, Q$  and  $\Lambda$ , and given in (4). In this case, there is exactly one photon sphere and it is located in  $\mathcal{N}$ .*

In what follows,  $\mathbb{O}$  is the set of vector fields that generate rotations and  $\mathbb{T}$  is  $\mathbb{O} \cup \{T\}$ ,  $r_*$  is the Regge-Wheeler coordinate, and  $F$  is a Maxwell field whose spin components in a given frame are  $\Phi_1$ ,  $\Phi_0$ , and  $\Phi_{-1}$ . The energy  $E_T[F](S)$  and the conformal energy  $E_K[F](S)$  of  $F$  are the integrals over a hypersurface  $S$  of the energy-momentum tensor of the field contracted respectively against the time-like vector field  $T = \partial_t$ , and against the conformal<sup>1</sup> vector field  $K = (t^2 + r_*^2)\partial_t + 2tr_*\partial_{r_*}$ . We assume that  $F$  is non-stationary, i.e. has no  $l = 0$  mode in the spin-weighted spherical harmonics. We also assume that on  $\{t = 0\}$ ,  $F$  and its required Lie derivatives have finite energies and conformal energies. We note that the results are stated for  $t \geq 0$  but similar results hold for  $t \leq 0$ .

### • Uniform Decay

**Theorem 34.** *Let  $t_0 \geq 0$  be a real parameter. Let  $S$  be any achronal future oriented smooth hypersurface, such that its union with  $\Sigma_0 = \{0\} \times \mathbb{R} \times \mathcal{S}^2$  is the boundary of an open submanifold of  $\mathcal{N}$ , and such that on  $S$ ,  $t \geq |r_*| + t_0$ . Then there is a constant  $C > 0$  independent of  $t_0$ ,  $F$  and  $S$ , such that*

$$E_T[F](S) \leq t_0^{-2}C \left( \sum_{k=0}^1 E_K[\mathcal{L}_{\mathbb{O}}^k F](0) + \sum_{k=0}^5 E_T[\mathcal{L}_{\mathbb{O}}^k F](0) \right) .$$

---

<sup>1</sup>This vector field is not conformally Killing on our spacetime. The name is taken from the Minkowski case where it is indeed conformally Killing.



- **Pointwise Decay**

**Theorem 37.** *Let  $[r_{*1}, r_{*2}]$  be a compact interval of  $r_*$ . Then there is a constant  $C_{(r_{*1}, r_{*2})} > 0$  such that for all  $t \geq 0$ ,  $r_* \in [r_{*1}, r_{*2}]$ ,  $\omega \in \mathcal{S}^2$ ,*

$$|\Phi_1| + |\Phi_0| + |\Phi_{-1}| \leq C_{(r_{*1}, r_{*2})} t^{-1} \left( \sum_{k=0}^4 E_K[\mathcal{L}_{\mathbb{T}}^k F](0) + \sum_{k=0}^8 E_T[\mathcal{L}_{\mathbb{T}}^k F](0) \right).$$

In what follows,  $\mathcal{H}^+$  denotes the space of finite energy on the future null boundary of  $\bar{\mathcal{N}}$  consisting of the two future horizons  $\mathcal{H}_2^+$  and  $\mathcal{H}_3^+$  and the two bifurcation spheres.  $\mathcal{H}^-$  is the past analogue of  $\mathcal{H}^+$ .

- **Goursat Problem and Scattering Operator**

**Theorem 40.** *For  $(\phi_+, \phi_-) \in \mathcal{H}^+$  there is a unique finite energy Maxwell field  $F$  defined on  $\bar{\mathcal{N}}$ , such that*

$$(\Phi_1|_{\mathcal{H}_2^+}, \Phi_{-1}|_{\mathcal{H}_3^+}) = (\phi_+, \phi_-). \quad (\text{II})$$

*Furthermore, a similar statement is true for  $\mathcal{H}^-$ , and the scattering operator*

$$\mathbb{S} : \mathcal{H}^- \longrightarrow \mathcal{H}^+$$

*is an isometry which takes data from one space to the other through their common Maxwell field solution which satisfies (II).*

Now that we have stated our results and discussed the connections between them, we shall give a separate overview of each of the two main topics: decay and conformal scattering. More elaborate versions of the following two sections can be found in the introductions to chapters 3 and 4.

## DECAY

Our motivations for studying decay are twofold: On the one hand, we use part of the decay results (uniform decay) obtained here to construct a complete conformal scattering theory later in chapter 4, on the other hand, the subject of energy bounds and decay using Morawetz estimates in general relativity has gained attention in the last decades largely due to its fundamental role in the analysis of the nonlinear stability of the spacetimes of general relativity. We shall discuss the first later.

## Motivation by Stability Problems

A basic problem in the general theory of relativity is the question of stability of Minkowski spacetime, that is, whether any asymptotically flat initial data set which is sufficiently close to the trivial one gives rise to a global (i.e. geodesically complete) solution of the vacuum Einstein equations that remains globally close to Minkowski spacetime. The local existence of solutions of the initial value problem was proven by Y. Choquet-Bruhat [29] in 1952. In 1983 partial results were obtained by H. Friedrich [69] using conformal methods, and in the early 1990's, the global nonlinear stability of Minkowski spacetime was established in the important work of D. Christodoulou and S. Klainerman [34]<sup>1</sup>. The main tool they used for the energy estimates is the vector field method developed by Klainerman which generalizes the multiplier method in the works of C.S. Morawetz. They first obtain precise decay estimates [32] for the Bianchi equations (spin-2 zero rest-mass fields) on Minkowski which model linearized gravity on Minkowski spacetime. Then they prove that the same decay rates are still valid for the full Einstein equations. These works and methods motivated many research projects in the years afterwards on the topics of decay, stability, and initial-value problems in general relativity. Currently, many groups are concentrating on the stability of Kerr black holes<sup>2</sup>. This is relevant in the contexts of some main problems of the theory, such as the weak cosmic censorship conjecture. Proving the Kerr black-hole stability is a major step towards solving these problems. The multiplier method, the vector field method, and its generalizations, are being employed to obtain the required uniformly bounded energies and to prove Morawetz estimates for solutions of the wave equation on black-hole spacetimes, motivated by the fact that proving boundedness and decay in time for solutions to the scalar wave equation on the asymptotically flat exterior of the Kerr spacetime is an important model problem for the full black-hole stability problem. However, there are some fundamental difficulties in the Kerr case, mostly because of the lack of symmetries, the trapping effect ranging over a radial interval, and there is no positive conserved quantity since Kerr black holes do not admit global time-like Killing vector fields.

## An Overview of Decay

Many of the decay results in the literature are for solutions of wave equations, but this has important consequences for many other systems such as Maxwell's equations. We present a quick overview on the history of decay estimates summarizing some methods used to obtain them and how they evolved to become more adaptable to different geometries.

### Basic Notions

Consider the strong Huygens principle:

**Theorem** (Huygens Principle). *If the initial data,  $(u(0, x), \partial_t u(0, x))$  with  $x \in \mathbb{R}^3$ , for the*

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<sup>1</sup>See also [31, 33] for a summary of the proof. A revisit of the proof can be found in [93]

<sup>2</sup>Works addressing the question of the stability of the Schwarzschild manifold can be found in [39, 56, 59, 81, 88, 132].

wave equation

$$\partial_t^2 u - \partial_{x_1}^2 u - \partial_{x_2}^2 u - \partial_{x_3}^2 u = 0 .$$

are supported in the ball  $B(0, R)$ , then the associated solution  $u$  satisfies

$$u(t, x) = 0 \quad \text{for all } |t| > |x| + R$$

When an equation satisfies the strong Huygens principle, fields generated by compactly supported data decay infinitely fast and, at each point, become identically zero after a finite amount of time. Despite this *pointwise decay*, there is a quantity determined by the solution which is conserved for all times due to the time translation symmetry of the system. This can be seen by multiplying the wave equation with the time derivative of the solution, called the *multiplier*, and rearranging,

$$\partial_t u \left( \partial_t^2 u - \sum_{i=1}^3 \partial_{x_i}^2 u \right) = \frac{1}{2} \partial_t \left( (\partial_t u)^2 + \sum_{i=1}^3 (\partial_{x_i} u)^2 \right) - \sum_{i=1}^3 \partial_{x_i} (\partial_t u \partial_{x_i} u),$$

if we now integrate the right hand side over a spacetime slab  $[t_1, t_2] \times \mathbb{R}^3$  and using the fact that  $u$  is a solution for the equation and that it has a compact support in space for all  $t$ , we arrive at the following identity:

$$\int_{\{t_1\} \times \mathbb{R}^3} \left( (\partial_t u)^2 + \sum_{i=1}^3 (\partial_{x_i} u)^2 \right) dx = \int_{\{t_2\} \times \mathbb{R}^3} \left( (\partial_t u)^2 + \sum_{i=1}^3 (\partial_{x_i} u)^2 \right) dx .$$

This quantity is called the (total) *energy*  $E[u](t)$  and it is conserved:  $E[u](t) = E[u](0)$ . However, the *local energy*

$$E[u](D, t) = \int_{\{t\} \times D} \left( (\partial_t u)^2 + \sum_{i=1}^3 (\partial_{x_i} u)^2 \right) dx$$

in any bounded region of space, for example  $D = \{x \in \mathbb{R}^3; |x| \leq R\}$ , is clearly not conserved and becomes zero after the wave leaves the region  $D$ .

The strong Huygens principle for the wave equation on flat spacetime is only valid in odd space dimensions starting at three. More general wave equations with a potential or on curved spacetimes satisfy a weak Huygens principle which says essentially that the local energy decays. One may then ask at what rate the local energy decays for, say, smooth compactly supported data. For example, in two space dimensions the rate of pointwise decay of solutions to the above wave equation can be exactly  $t^{-1}$ , and thus the local energy decays as  $t^{-2}$ .

## The Multiplier Method

The method of multipliers originated from the so-called Friedrichs' ABC method that dates back to K.O. Friedrichs in the 1950's. The method of multipliers was used in the 1960's and

1970's to prove uniform decay results for the homogeneous linear wave equation ( $\square u = 0$ ) outside obstacles. C.S. Morawetz was the first to succeed in proving local energy decay for star-shaped obstacles with Dirichlet boundary condition using this method in 1961 [109]. In this work, the effects of scaling and the spread into space on the solution for the wave equation and its local energy is captured using the scaling multiplier

$$Su = t\partial_t u + r\partial_r u + u$$

and the following local energy decay is established

$$E[u](R, t) \leq \frac{C}{t} E[u](0) ,$$

where  $R$  is a region bounded between the obstacle and an outside sphere, and  $C > 0$  depends on the obstacle and the support of the initial data. This estimate then gives a pointwise decay of rate  $t^{-\frac{1}{2}}$ . A year later, Morawetz used the multiplier

$$K = \xi^2 \partial_\xi u + \eta^2 \partial_\eta u + (\xi + \eta)u = (t^2 + r^2)\partial_t u + 2tr\partial_r u + 2tu \quad ; \quad \xi = t - r , \eta = t + r ,$$

in her work [110] to improve on the results of [109] and get faster decay rates of  $t^{-1}$  for the pointwise decay and  $t^{-2}$  for the local energy. Morawetz was motivated by the fact that for large times the disturbance is expected to be radiating outwards, and there will be little dependence on the angles, so,  $ru$  will approach a solution of  $\partial_r^2 w - \partial_t^2 w = 0$  for which an appropriate multiplier is  $Nw = p_1 \partial_r w + p_2 \partial_t w$ . The multiplier  $K$  is in fact related to a “time” translation: If we apply the Kelvin transformation on the coordinates  $(t, r, \theta, \varphi)$  given by,

$$\hat{t} = \frac{t}{r^2 - t^2} , \quad \hat{r} = \frac{r}{r^2 - t^2}$$

leaving the angular variables unchanged, we can see that

$$\partial_{\hat{t}} = 2tr\partial_r + (r^2 + t^2)\partial_t .$$

This transformation is conformal and takes the cone  $r^2 = t^2$  at the origin to a cone at infinity and vice versa. It is not a surprise then that this vector field is appropriate for studying the asymptotic behaviour of the solution. Moreover, in 1968 [113] Morawetz uses a radial multiplier of the form

$$\zeta(r)(\partial_r u + r^{-1}u)$$

where  $-\zeta(r)$  is a bump function around the origin, to obtain uniform integrated local energy estimates for the non-linear Klein-Gordon equation  $\square u + mu + P(u) = 0$ ,

$$\int_0^T E[u](\Omega, t) dt \leq K E[u](0) ,$$

where  $\Omega$  is a finite region in space and  $K$  a positive constant depending only on  $\Omega$  (its volume).

These multipliers and their corresponding vector fields have all found many important applications, most notably in General Relativity. For more on Morawetz's work we refer to [107, 111].

## Vector Field Method

The vector field method is a flexible tool generalizing the multiplier method by making use of well adapted vector fields, related to symmetries or approximate symmetries of the equations, to derive decay estimates and thus to control the long time behaviour of solutions. The basics of this method has two aspects: The vector fields are used to define generalized energy norms, and, if they commute with the equations then one can derive identities for the energy norms obtained. In the mid 1980's, S. Klainerman introduced the notion of generalized energy norms defined from the conformal group, whose elements have useful commuting properties among themselves and with the D'Alembertian, to obtain energy estimates and prove decay for solutions of the wave equation on  $\mathbb{R}^{n+1}$  [89, 91, 92]. These works of Klainerman were in essence a combination of the local energy decay estimates of C.S. Morawetz [110] and the conformal method of Y. Choquet-Bruhat and D. Christodoulou [30]. If  $\mathbb{A}$  is a set of vector fields and  $s \in \mathbb{N}$  we define the following norm of a function  $u$  on  $\mathbb{R}^{n+1}$  by

$$\|u(t)\|_{\mathbb{A},s,p} = \left( \sum_{k=0}^s \sum_{X_{i_j} \in \mathbb{A}} \int_{\mathbb{R}^n} |(X_{i_1} \dots X_{i_k} u)(t, x)|^p dx \right)^{\frac{1}{p}}.$$

Klainerman uses such norms for different subsets of the conformal group in place of  $\mathbb{A}$  to get what he calls global Sobolev inequalities (which nowadays are known as Klainerman-Sobolev inequalities) of the form

$$|u(t, x)| \leq h(t, |x|) \|u(t)\|_{\mathbb{A},s,p}$$

for functions  $u$  with

$$\|u\|_{\mathbb{A},s,p}^{\#} = \sup_{t \geq 0} \|u(t)\|_{\mathbb{A},s,p} < \infty.$$

In the same papers he also gets decay estimates of the form

$$|u(t, x)| \leq d(t) \|u\|_{\mathbb{A},s,p}^{\#}.$$

Many results concerning the long-time and global existence were subsequently obtained using the methods of Klainerman: [9, 73, 82, 90, 92] and of course, the important work of D. Christodoulou and S. Klainerman on the stability of Minkowski spacetime [32, 34]

The usefulness of the vector field method is best seen, although not exclusively<sup>1</sup>, in view of Noether's theorem in the case of general field equations derived from a quadratic action in the context of a Lagrangian theory. Let  $\phi$  be a general field on a globally hyperbolic, time-orientable, spacetime  $(M, g)$  and assume there is a scalar Lagrangian  $L$  which depends on the field and its derivatives (and possibly position in spacetime), used to define an action  $S$  as the integral of  $L$  on  $M$ . The field equations are the Euler-Lagrange equations:

$$\frac{\delta L}{\delta \phi} - \nabla^a \frac{\delta L}{\delta \nabla^a \phi} = 0.$$

One can define from the field and the Lagrangian a symmetric 2-tensor  $\mathbf{T}$  called the *energy-momentum tensor* depending on the field and its derivatives, which by the Euler-Lagrange

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<sup>1</sup>See for example [5].

equations turns out to be divergence free. The energy associated with a vector field  $X$  and evaluated on a hypersurface  $\Sigma$  is,

$$E_X(\Sigma) = \int_{\Sigma} \mathbf{T}_{ab} X^b d\sigma^a ,$$

where  $\alpha_a d\sigma^a$  is the 3-form  $\star\alpha$  given by the Hodge star operator for any 1-form  $\alpha$ . If  $\Sigma$  is space-like and if the energy-momentum tensor satisfies the dominant energy condition, the energy will be positive definite if  $X$  is time-like. By Stokes' theorem, we have the following law

$$E_X(\Sigma_{t_2}) - E_X(\Sigma_{t_1}) = \int_{\Omega_{t_1, t_2}} \nabla^a (\mathbf{T}_{ab} X^b) d\text{Vol}_g = \int_{\Omega_{t_1, t_2}} {}^{(X)}\pi_{ab} \mathbf{T}^{ab} d\text{Vol}_g ,$$

where  ${}^{(X)}\pi_{ab} = \mathcal{L}_X g_{ab} = 2\nabla_{(a} X_{b)}$  is called the deformation tensor or Killing form of  $X$ . This law is called the deformation law. A vector field  $X$  is a conformal Killing vector field if the deformation tensor of  $X$  is proportional to the metric by a scalar factor  $\lambda$ , and  $X$  is a Killing vector field when  $\lambda = 0$ . We see then that when  $X$  is Killing the deformation law entails that the energy is conserved. In general, energy estimates are obtained by controlling the deformation term  ${}^{(X)}\pi_{ab} \mathbf{T}^{ab}$ , and in that case one says that one has an (almost) conservation law. A symmetry operator for an equation, in the simplest case, is defined to be a differential operator that commutes with the equations. When  $Y$  is Killing, the operator  $\mathcal{L}_Y$  generated by Lie differentiation with respect to  $Y$  is a symmetry operator for the wave equation and for Maxwell's equations among others. This means that when  $\mathbb{Y}$  is a set of Killing vector fields, one has identities for the energies defined using these vector fields but also for all Lie derivatives of the solutions with respect to these vector fields, at all orders. This adds on the control of the energies and allows better estimates and rates of decay. We note that these energy identities are of course nothing but deformation laws for Klainerman's generalized energies. D. Christodoulou and S. Klainerman use arguments similar to the one above with the symmetry generators  $T, S$ , and  $K$  in [32] to obtain uniform bounds on the generalized energies and then, by means of Klainerman's global Sobolev inequalities, obtain the decay estimates for Maxwell and spin-2 equations. The latter are formally identical to the Bianchi identities and thus relevant to the understanding of the Einstein field equations. In fact, the methods they developed in [32] in the study of the spin-2 equations in Minkowski spacetime prepared for the subsequent study of the nonlinear stability of the Minkowski metric, as mentioned at the beginning of this introduction.

## Some Recent Works

The literature centred around decay estimates in general relativity is vast, so we refer to some recent works where additional references can be found. In particular, Blue's paper [22] about the decay of Maxwell fields in Schwarzschild in 2008 is central to our work, in fact, we show that the methods used in [22] can be applied to the case of RNdS black holes. In their paper of 1999 on a nonlinear Schrödinger equation [94], I. Laba and A. Soffer introduced a Morawetz vector field on the Schwarzschild spacetime. They also introduced a modified radial Morawetz multiplier, known as the Soffer-Morawetz multiplier, based on the work

of C.S. Morawetz, J.V. Ralston, and W.A. Strauss [115]. Through the 2000's, these tools were used on Schwarzschild's spacetime with further adaptations in the works of P. Blue, A. Soffer, and J. Sterbenz [20, 21, 22, 23, 24, 26] and in this present work, to help control the trapping terms. Similar problems were independently studied by M. Dafermos and I. Rodnianski [41, 42]. M. Dafermos and I. Rodnianski in 2011 proved uniform boundedness for the wave equation on slow Kerr backgrounds [45]. Also, D. Tataru and M. Tohaneanu obtained local decay for energy on Kerr and other asymptotically flat stationary spacetimes [137, 138]. A paper by J.-F. Bony and D. Häfner in 2008 [27] addresses the decay of the local energy for the wave equation on the de Sitter-Schwarzschild metric. Several decay estimates with rates were obtained in the early 2010's: J. Luk [100]; M. Tohaneanu [140]; M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman [43, 44, 47]. There is also a paper in 2013 by L. Andersson P. Blue, and J.-P. Nicolas on wave equations with trapping and complex potential [6]. Two recent papers in 2015 were published by L. Andersson and P. Blue: [4] proving uniform energy bounds for Maxwell fields on Schwarzschild's spacetime, and [5] in which they generalize the vector field method to take the hidden symmetries of Kerr spacetime into account and obtain an integrated Morawetz estimate and uniform bounds for a model energy for the wave equation. M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman's work on scattering for the wave equation on the Kerr metric [46] contains decay results and the uniform energy equivalence needed for conformal scattering (see J.-P. Nicolas [120]). There is a more recent paper by L. Andersson, T. Bäckdahl, and P. Blue [3] in 2016 proving a new integrated local energy decay estimate for Maxwell fields outside a Schwarzschild black hole using a new superenergy tensor. There have been works on Price's law (see [127, 128]), as [105] in 2012 by J. Metcalfe, D. Tataru, and M. Tohaneanu. And using different techniques (see [62]) F. Finster, N. Kamran, J. Smoller, and S.-T. Yau obtain decay estimates for Dirac and the wave equation on Kerr and Kerr-Newman black holes [58, 60, 61, 63, 64].

## Maxwell Fields

The behaviour of Maxwell fields is well-known in flat spacetime, at any point in space the effect of a signal dies off but the total energy carried by the signal is preserved, carried off in fact to infinity, as seen, for example, in the works of C.S. Morawetz in 1974 [114], and D. Christodoulou and S. Klainerman [32] in 1990 with rates of  $t^{-5/2}$  obtained using the full conformal group. In Schwarzschild, a rate of  $t^{-3}$  was obtained in regions bounded away from the horizon and null infinity, by R.H. Price in 1972 [127], and later by R.H. Price and L.M. Burko in 2004 [129]. Only time and spherical symmetries are available in the Schwarzschild case, so the vector field method produces slower rates of  $t^{-1}$ , as in P. Blue [22], however, the conformal energy associated to the conformal Morawetz vector field can be used to control all the components of the field. We prove here that this is also the case for generic spherically symmetric static black holes by working out the details on RNdS black holes; the results can be extended to more general situations including cosmological black holes. In 2015 J Sterbenz, D Tataru [134] obtained local energy decays for Maxwell fields on a general asymptotically flat spherically symmetric spacetime, thus they do not cover cases with positive cosmological constant. Works in situations similar to ours were done by

A. Vasy and P. Hintz [78, 79, 80] in 2015, using methods from microlocal analysis and it seems that their work needs a positive cosmological constant (maybe with the exception of flat spacetime), whereas the vector field method which we use applies equally well with or without a (positive) cosmological constant under the conditions of section 3.2.3.

Before discussing the method we use, it is worth mentioning that there is a resemblance between Maxwell’s equations and the spin-2 equations. The symmetries of a spin 2 field extend the antisymmetry of a Maxwell field, and the two systems of equations have similarities. If the vacuum Einstein equations are satisfied, then the Ricci curvature vanishes, and the Weyl curvature satisfies the spin-2 field equations. In Minkowski spacetime, the spin-2 field equations models the linearization of Einstein’s equations about the Minkowski solution. This is the motivation for studying the spin-2 field in [34]. However, this is not true for the linearization about other solutions. Nevertheless, we expect that an analysis using the vector field method and Morawetz estimates will apply to the linearized gravity system.

The arguments used in our work follow the same philosophy as in the works [22, 25, 26, 42] using the vector field method. The major obstacle is the trapping effect:

**Trapping Effect.** The conformal energy is not conserved because of trapping. It can be seen as the main “error” which is generated by the divergence of the conformal energy density. This effect can be overcome by introducing a radial vector field which points away from the photon sphere. This is a modified Morawetz radial multiplier of the form  $A\partial_{r_*}$ , where  $A$  is a continuously differentiable function of  $r_*$  that changes sign at the photon sphere, marked at  $r_* = 0$ .

The work can be divided into three main steps. In *the first step*, the conformal energy, defined by the conformal Morawetz vector field, of a Maxwell field is not conserved but can be controlled by the conformal charge<sup>1</sup> of the middle (or spin-weight zero) component of the field which satisfies a wave-like equation decoupled from the other components.

**Wave Analysis.** The conformal charge of the solutions to the wave-like equation is not conserved either. *The second step* is to control the error term using a radial Soffer-Morawetz multiplier which allows us to obtain a uniform bound on the conformal charge of the wave. Because this wave-like equation is actually simpler than the covariant wave equation, the usual analysis on the local energy of the wave equation is replaced by an analysis of an energy localized inside the light cone, and no decomposition on the spherical harmonics is required. Through some Hardy estimates, the trapping term is controlled by the energy generated by the radial multiplier and the integral of the energy localized inside the light cone. Since the trapping term controls the growth of the conformal charge, and since the energy is conserved, this gives a linear bound on the conformal charge. The linear bound can be improved to a uniform one. A uniform bound on the trapping term is also obtained.

*The third step* is to use the conformal energy to control norms of the Maxwell field. The

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<sup>1</sup>This is the conformal energy of the solution to the wave equation, but to avoid confusion with the conformal energy of the Maxwell field, we call it a conformal charge.



generalized energy and conformal energy of the Maxwell field generated by the rotation group are conserved and control the energy and the conformal charge of the middle spin component which in turn control the trapping term by the uniform bound. Thus, the conformal energy of the Maxwell field is controlled by the generalized energy and conformal energy of the initial data through a uniform bound. Since the integral of the trapping term has been controlled in the entire  $r_*$ -range, we have a uniform bound on the energy flux through any achronal hypersurface. This can be improved to a uniform decay rate of  $t^{-2}$ . The integrand in the conformal energy behaves like  $t^2$  times the Maxwell field components squared. Since the conformal energy is bounded, the field components decay in  $L^2_{\text{loc}}$  like  $t^{-1}$ . Sobolev estimates can be used to convert  $L^2_{\text{loc}}$  decay for derivatives into  $L^\infty_{\text{loc}}$  decay. For this, we need decay on the spatial derivatives of the Maxwell field. To control the radial derivatives, we use the structure of the Maxwell equations.

## CONFORMAL SCATTERING

In the classic experiment of scattering one has a field propagating in a medium with an obstacle ; an incoming plane wave hits the obstacle and scatters away from it as a superposition of outgoing plane waves. *Scattering theory* is a way of summarizing this evolution by constructing the scattering operator, a map that associates the future and past asymptotic behaviours of the solution and whose existence and invertibility comes from the one-to-one correspondence between the field and its asymptotic behaviours. Radar systems make use of this correspondence to gain information on the obstacle from the asymptotic behaviour of the scattered wave. This reconstruction is the aim of *inverse scattering*.

### Brief History of Analytic Scattering

Scattering theory proved to be a useful tool in the framework of general relativity to study the asymptotic influence of the geometry of spacetime on fields. Although in this current work we do not use an analytic approach to scattering, we very briefly touch on the history of the subject because this is part of the origin of conformal scattering and it helps to understand what new features the conformal approach bring to the domain. In 1980 S. Chandrasekhar [28] used the stationary approach to study quasi-normal modes of black hole spacetimes. Around the same time, M. Reed and B. Simon published their classic book “*Scattering Theory*” [131]. Then starting in 1985 time-dependent scattering of classical and quantum fields on the exterior of a Schwarzschild black hole were first studied by J. Dimock and B. Kay in [53, 54, 55, 56]. And in the 1990’s, A. Bachelot produced an important series of papers starting with scattering theories for classical fields and culminating in the first rigorous proof of the Hawking effect [7, 8, 10, 11, 12, 13]. J.-P. Nicolas in 1995 developed a scattering theory for classical massless Dirac fields and obtained partial results on the asymptotic profiles for a non linear Klein-Gordon equation on spherical black-hole metrics [117, 118]. W.M. Jin in 1998 constructed wave operators in the massive case [85], and F. Melnyk in 2003 obtained a complete scattering for massive charged Dirac fields and a

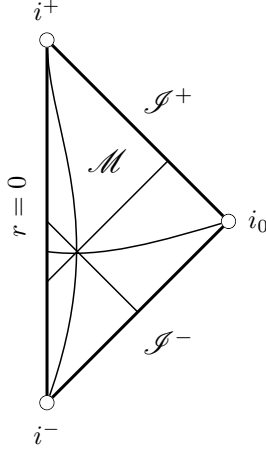


Figure 3: *Penrose diagram of  $\hat{\mathcal{M}}$  the conformal compactification of Minkowski spacetime  $\mathcal{M}$  with time-like, space-like, and null curves.*

proof of the Hawking effect for quantum massive spin-1/2 fields [103, 104]. In 1999 I. Laba and A. Soffer [94] obtained complete scattering for the non-linear Schrödinger equation on Schwarzschild manifolds. Then in 1992 S. De De Bièvre, P. Hislop and I.M. Sigal [52] constructed a scattering theory for the wave equation on asymptotically hyperbolic manifolds by means of a Mourre estimate. A complete scattering theory for the wave equation in the asymptotically flat case was subsequently obtained by D. Häfner in 2001 using the Mourre theory [75]. Time-dependent scattering theories on Kerr black holes for massive Dirac fields were obtained by D. Häfner and J.-P. Nicolas in 2003-2004 [76, 77]. In 2005 T. Daudé produced scattering theories for Dirac fields in various spacetimes [48, 49, 50, 51]. In 2014 M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman developed a scattering theory for the wave equation on Kerr black holes [46].

## Conformal Scattering

In the present work, we construct a *Conformal Scattering* theory. Conformal scattering is a geometrical approach to time-dependent scattering based on Penrose *conformal compactification*: a rescaling of the metric and the fields using conformal factors. The *scattering operator* is the fundamental object in the theory. Here, the asymptotics of the solution are given as restrictions of the conformally rescaled solution on past and future null infinities and are called *radiation fields*. With suitable energy estimates, which is a crucial step in the theory, the scattering data completely characterizes the solution. This is a *Goursat problem*, where data is given at null infinity instead of some space-like hypersurface as in the non-characteristic case. The resolution of the Goursat problem is the core of conformal scattering theory.

## The Main Ingredients

We describe the essential steps of the general strategy of conformal scattering. We shall assume for simplicity the existence of a global time-like, possibly causal at the conformal boundary, Killing vector field  $\tau$  (see [101]).

**Conformal compactification.** A *globally hyperbolic spacetime*  $(\mathcal{M}, g)$ , with a suitable asymptotic structure, such as asymptotic flatness, is rescaled and replaced by an “un-physical” Lorentzian manifold with boundary  $(\hat{\mathcal{M}}, \hat{g})$ , the *conformal compactification* of  $\mathcal{M}$ , with  $\partial\hat{\mathcal{M}} = \mathcal{I}$  that represents points at infinity of  $(\mathcal{M}, g)$ , and  $\text{int}\hat{\mathcal{M}} = \mathcal{M}$ . The new metric is *conformally related* to the original metric by

$$\hat{g} = \Omega^2 g ,$$

for an appropriate choice of a smooth non-negative boundary function  $\Omega$  defined on  $\hat{\mathcal{M}}$ , called the *conformal factor*, such that it is positive on  $\mathcal{M}$  and becomes zero on  $\mathcal{I}$ , the asymptotic regions where  $g$  becomes infinite, and  $d\Omega|_{\mathcal{I}} \neq 0$  (figure 3). What is important is to define things in a way such that the new metric has some differentiability<sup>1</sup> at the boundary hypersurface  $\mathcal{I}$ . Now, the asymptotics of  $\mathcal{M}$  can be studied using local techniques on  $\hat{\mathcal{M}}$ , without resorting to complicated limiting arguments when studying, for example, the radiation fields of a physical field on the original spacetime. A *conformally invariant equation* is an equation defined on  $\mathcal{M}$  for  $g$  such that whenever  $\Phi$  is a solution to the equation, then for some  $s \in \mathbb{R}$ , the *rescaled field*<sup>2</sup>  $\hat{\Phi} := \Omega^s \Phi$  is a solution to the same equation but defined on  $\hat{\mathcal{M}}$  for the rescaled metric  $\hat{g}$ . Examples of conformally invariant equations are the conformal wave equation, Dirac’s and Maxwell’s equations. Working with this class of equations ensures that we can study the rescaled field on the rescaled spacetime and gain information on the behaviour of the physical field in the physical spacetime. In the cases of black holes, part or all of the conformal boundary will be the horizon or horizons. Horizons are finite null hypersurfaces for the physical metric and when the whole conformal boundary is made of horizons, conformal rescalings are not required ; even in such a case we talk about conformal scattering because we use the same approach based on the resolution of a Goursat problem at the null boundary. Note that such cases are more amenable to extending the method to non-conformally invariant equations since there is no conformal rescaling involved. See [122, 123, 124, 125, 126] for more on conformal compactification.

**Cauchy problem: Defining the trace operators.** The scattering operator is defined using two operators called the past and the future *trace operators*  $T^\pm$ . The future (past) trace operator associates to data at  $t = 0$  data at  $t = +\infty$ , i.e. on the future part  $\mathcal{I}^+$  of  $\mathcal{I}$  ( $t = -\infty$ ,  $\mathcal{I}^-$ ). In general, one defines a normed energy space  $\mathcal{H}$  on a Cauchy hypersurface of the compactified spacetime  $\hat{\mathcal{M}}$  and normed energy spaces  $\mathcal{H}^\pm$  on the

<sup>1</sup>We note that not all spacetimes admit a conformal compactification with the needed regularity of the rescaled metric at the boundary.

<sup>2</sup>See [125] for the precise definition.

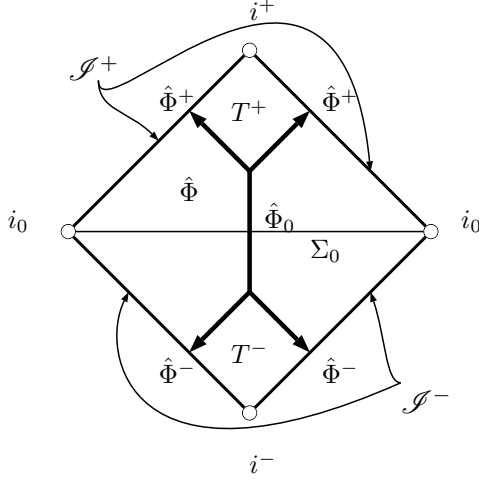


Figure 4: *The trace operators  $T^\pm$ .*

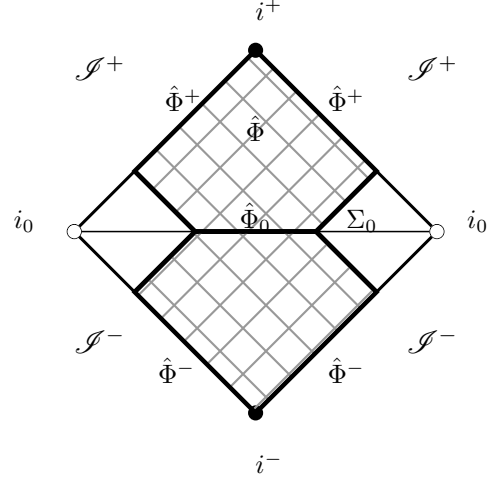


Figure 5: *The case of regular  $i^\pm$ .*

boundary parts  $\mathcal{I}^\pm$ . The energy norms are defined by contracting the time-like or causal vector field  $\tau$  with the stress-energy tensor  $\mathbf{T}$  of the equation. The future (past) trace operator then takes finite energy data in  $\mathcal{H}$  defined on the Cauchy hypersurface to finite energy data in  $\mathcal{H}^\pm$  on  $\mathcal{I}^\pm$ . The general construction of the future trace operator involves solving the Cauchy problem from a Cauchy hypersurface, then the image of the initial data by the future trace operator is the trace (restriction) of the solution, or part of the solution, to the future boundary  $\mathcal{I}^+$ . The past trace operator is defined similarly (figure 4).

Let us for the sake of this general overview assume that the equations studied are linear. This entails that the trace operators themselves are linear operators, yet, this is not an absolute necessity for the construction of a conformal scattering theory, see [86] for example.

**Energy estimates: The trace operators are one-to-one and have closed ranges.** We first construct the trace operators for a dense subset of the finite energy space  $\mathcal{H}$ , such as smooth compactly supported functions. Then we prove uniform energy estimates both ways between the initial Cauchy data in the dense subset and their images under the trace operator. This entails that the trace operator extends uniquely as a bounded one-to-one linear map from  $\mathcal{H}$  to  $\mathcal{H}^+$  (or  $\mathcal{H}^-$ ), with closed range. With the right conditions, we can work entirely on the rescaled spacetime, which has the important advantage that all the hypersurfaces involved are regular finite hypersurfaces, in particular  $\mathcal{I}^\pm$ . By Stokes' theorem we obtain the required energy identities (figure 5). In the case of black hole spacetimes, time-like infinities are singular. This constitutes an important difficulty and we need a sufficient decay of the solutions to the equations so that we can rule out the accumulation of energy at time-like infinities. As shown in figure 6, Stokes' theorem implies the following energy identity,

$$\mathcal{E}_{\tau, \Sigma_0} = \mathcal{E}_{\tau, \mathcal{I}_s^+} + \mathcal{E}_{\tau, S_s}.$$

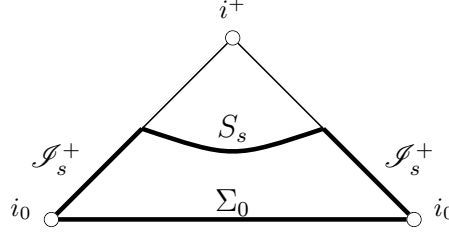


Figure 6: *The closed hypersurfaces of the compactified spacetime.*

If

$$\lim_{s \rightarrow +\infty} \mathcal{E}_{\tau, S_s} = 0 ,$$

then the conservation law follows. This is the required decay. Clearly, the same can be done in the past direction. Obtaining the desired decay is usually a separate problem that has its own subject. This is partly why we proved the decay results of chapter 3.

**Goursat problem: The trace operators are onto.** The third and last step in defining the scattering operator is to prove that the trace operators we defined are surjective, this comes down to solving the Goursat problem on a null hypersurface for data in dense subsets of the finite energy spaces  $\mathcal{H}^\pm$ . One can solve the characteristic Cauchy problem using uniform energy estimates, weak convergence and compactness methods [83, 101]. In some cases, some “reversible” modifications to the setting is needed before applying those methods or the results they produce. This is what we do in section 4.2, following the construction done in [121], we are then able to apply the results of [83] directly.

**Scattering operator** With the Goursat problem solved, the trace operators  $T^\pm$  become isometries between the boundary energy spaces  $\mathcal{H}^\pm$  on  $\mathcal{I}^\pm$  and the initial energy space  $\mathcal{H}$  on  $\Sigma_0$ . We can then define the scattering operator  $S : \mathcal{H}^- \rightarrow \mathcal{H}^+$  by  $S = T^+ \circ (T^-)^{-1}$  and it is an isometry.

## History

The techniques of conformal compactification were introduced by R. Penrose around 1964 [122, 123, 124, 125, 126]. In the same period of early 1960’s F.G. Friedlander introduced his notion of *radiation fields* [65, 66, 67]. Penrose in [122] explicitly states that scattering is a motivation for introducing the conformal compactification technique: “The technique affords a covariant approach to the definition of radiation fields in general relativity.” Meanwhile, P.D. Lax and S.R. Phillips developed their theory of scattering [98] in 1967, based on a translation representative of the solution which is reinterpreted as an asymptotic profile of the field along outgoing null geodesics, analogous to Friedlander’s radiation field. Fifteen years after Penrose discussed radiations fields in the conformal setting, Friedlander saw the connection between the Lax-Phillips theory of scattering and his notion of radiation fields, and in 1980 the first actual conformal scattering theory appeared in his founding paper

[68]. The principle of the construction was first to reinterpret the scattering theory as the well-posedness of the Goursat problem for the rescaled equation at null infinity, then to solve this Goursat problem. Friedlander as well as J.C. Baez, I.E. Segal and Zhou Z.F. who pushed his ideas further in 1989-1990 [14, 15, 16, 17, 18] worked exclusively on static backgrounds. In 1990 [17], L. Hörmander solved the Goursat problem for a wave equation on generic null hypersurfaces in a spatially compact spacetime [83]. With this, and knowing that constructing conformal scattering theories amounts to solving a Goursat problem on a compactified spacetime, the road to non-stationary spacetimes was clear, yet, no one pushed it in this direction until 2004 when L. Mason and J.P. Nicolas picked up Friedlander's ideas and applied them to fields on generically non-stationary asymptotically simple spacetimes [101, 102]. J. Joudioux in 2012 [87] constructed a conformal scattering theory for a non-linear wave equation on non-stationary backgrounds. And in 2013 J.P. Nicolas produced a paper [121] on a conformal scattering theory for the wave equation on Schwarzschild black holes. In these recent works, [87, 101, 121], the resolution of the Goursat problem is based on methods following the work of Hörmander [83]. In the present work, we directly apply [83] to show that the Goursat problem for Maxwell fields on Reissner-Nordström-de Sitter black holes is well-posed.

## Remarks

The ultimate purpose of conformal scattering is to use conformal methods to construct scattering theories, not to reinterpret existing scattering theories in conformal terms. The idea of replacing spectral analysis by conformal geometry is the door to the extension of scattering theories to general non-stationary situations, which may be inaccessible to spectral methods. Note that in [68, 101, 121], the reinterpretation is done in addition to the conformal construction, giving more insight on questions such as the required decay for a conformal scattering theory, or whether a conformal scattering theory and a scattering theory defined in terms of wave operators are equivalent or not.

It is worth mentioning that the conformal scattering we construct here is done without conformal compactification! This is because the scattering data is taken on the horizons. Nevertheless, the results we obtain can be applied to any spherically symmetric spacetime satisfying the conditions stated in section 3.2.3 of chapter 3 with a conformal compactification when needed, the rest goes through essentially without modification.

## CONTENTS OF THE CHAPTERS

This thesis is divided into four chapters. The first two are introductory chapters about the Reissner-Nordström-de Sitter Black Holes and Maxwell Fields. The last two chapters are the core of the thesis and contain the important results<sup>1</sup>: decay and conformal scattering.

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<sup>1</sup>In addition to the result about the Photon sphere in chapter 1.

## Chapter 1: Reissner-Nordstrøm-de Sitter Black Holes

This chapter is devoted to the study of the Reissner-Nordstrøm-de Sitter black holes and some of their properties that will lay the groundwork for obtaining decay and conformal scattering of Maxwell fields.

**Section 1.1:** We start the section by presenting the necessary and sufficient conditions (5) on the parameters  $M, Q$ , and  $\Lambda$  of the RNdS metric so that it has three horizons. We then verify our claim regarding these conditions along with the fact that there is a photon sphere only at one value of  $r > 0$  and it is located in the exterior static region. This is Proposition 1, and to our knowledge, this is not present in the literature.

**Section 1.2:** This section is a detailed construction of the maximal analytic extension of the RNdS manifold in the case of three horizons, and a discussion about some of its aspects.

## Chapter 2: Maxwell Fields on the Exterior Static Region

In this chapter we introduce Maxwell's equations and fields on the Reissner-Nordstrøm-de Sitter manifold, and we prove some preliminary facts about the Maxwell system which will be used in the subsequential chapters.

**Section 2.1:** After introducing the equations and some notations, we rewrite them first in tetrad formalism using a null tetrad. Then we reformulate the equations as an evolution system of three equations and a spatial constraint equation. We finally decompose them on spin-weighted spherical harmonics.

**Section 2.2:** We define the energy of a Maxwell field and study the Cauchy problem for Maxwell's equations using two approaches: symmetric hyperbolic systems and wave equations.

**Section 2.3:** Here, we show that the pure charge solutions, i.e. which only have the  $l = 0$ -mode in the spin-weighted spherical harmonics, are the only time-periodic solutions with finite energy, which is what we call stationary solutions. These solutions of Maxwell's equations are excluded since they do not decay.

**Section 2.4:** In this section we discuss Maxwell potentials and the Lagrangian formulation of electromagnetism using potentials. We also use the potential formulation of Maxwell's equations to show that smooth compactly supported data are dense in the constraint subspace of the finite energy space.

## Chapter 3: Decay

The aim of chapter 3 is to prove decay results for the Maxwell field on the Reissner-Nordström-de Sitter Black Hole. Both pointwise and uniform decay results are consequences of the bounds on the conformal energy obtained from the wave analysis on the middle spin component of the field where we followed the work in [22]. The details in the sections of the chapter are as follows:

**Section 3.1:** This section is devoted to the analysis of the wave equation satisfied by the middle spin component of the field. We show that the energy for the wave equation is conserved and derive estimates for the conformal charge. Following [22], we use these estimates, a Morawetz estimate involving a radial multiplier and Hardy-type estimates to obtain a uniform bound on the conformal charge. We note that this is where the exclusion of stationary solutions become necessary so that we can control the  $L^2$ -norm of the wave solution by the norm of its angular derivatives: the uniform bound we get controls the integral of the trapping term multiplied by the angular derivative of the middle spin component.

**Section 3.2:** The second section of the chapter is the decay results. We introduce some norms on the Maxwell 2-form and discuss the energies of the field. We get an almost conservation law, describing the effect of trapping on the conformal energy defined by the Morawetz vector field  $K$ , where the significance of the photon sphere is again seen. We then relate the wave energy of the middle component and that of the full field. Using these results a uniform bound on the conformal energy is obtained. At this point we state and prove the decay results in section 3.2.2. Finally, in section 3.2.3 we specify under what conditions this work and these decay results can be extended to other spherically symmetric spacetimes.

## Chapter 4: Conformal Scattering

In this part of the manuscript we address the topic of conformal scattering on the exterior region of RNdS black holes, and construct a scattering operator establishing the correspondence between null data on past horizons and null data on future horizons of region III. It is divided into two sections, we present an overview of their contents.

**Section 4.1:** In this section we construct the trace operators and show that they are injective and norm preserving after establishing conservation laws. We start the section by expressing the Maxwell field in null tetrad formalisms adapted to the geometry of our spacetime. We also briefly discuss the Newman-Penrose formalism in a general normalized null tetrad. We next define the energy spaces on the horizons associated to the smoothly extended vector field  $T$  given by  $\partial_t$  on  $\mathcal{N}$ . By the decay results of Theorem 34 on achronal hypersurfaces that we obtain in chapter 3, we get a conservation law between the Cauchy hypersurface and the horizons. Afterwards, we solve the Cauchy problem which allows us to define each trace operator as a partial isometry, due to energy conservation.



**Section 4.2:** Showing that the trace operators are invertible, i.e. isometries between the full spaces of finite energy, requires solving the Goursat problem on the horizons which we do in this section. It suffices to solve the Goursat problem for the dense subset of smooth compactly supported data. The idea goes as follows. We first show that the triplet  $\hat{\Phi}$  of the spin components of the field in suitable tetrads satisfies a system of coupled wave equations,

$$\hat{W}\hat{\Phi} = 0 . \tag{III}$$

This allows us to transform the problem from Maxwell's equations to wave equations. Following [121], and thanks to the fact that the Goursat data is supported away from time-like infinity  $i^+$ , we can adapt the setting to the framework of Hörmander's results in [83]. Although [83] does not deal with coupled systems, our wave equations (III) are coupled in a special way that allows us to apply Hörmander's results directly. This guarantees the well-posedness of the Goursat problem for our system of wave equations. The difficulty now is to reinterpret the solution of the wave equations as a Maxwell field. The main idea of the proof is that if Maxwell's compacted equations, i.e. the equations satisfied by the spin components in the tetrad formalism, were expressed as  $E_i = 0$  for  $i = 1 \dots 4$ , then the  $E_i$  themselves satisfy a system of wave equations if the spin components satisfy (III). Therefore we are able to apply Hörmander's results repeatedly to prove that  $E_i = 0$ . The way the equations are coupled in these systems when expressed using null tetrads near each horizon, provides the grounds for the desired results. The key point is to use finite propagation speed and the constraint equations on the horizons to prove the initial conditions on the horizons, and thus  $E_i = 0$  on the whole spacetime by the well-posedness of the Goursat problem for the wave equation [83].

## Appendices

At the end of the report, we have two appendices. Some geometric facts and tools that we use throughout this report are presented in appendix A, notably, a variant of the divergence theorem which is adequate for Lorentzian geometry. Appendix B is about spinor notations and facts relating the concepts used here to the ones usually found in the literature.

# Chapter 1

## Reissner-Nordstrøm-de Sitter Black Holes

### 1.0 Introduction

This chapter is devoted to the study of the Reissner-Nordstrøm-de Sitter black holes and some of their properties that will lay the groundwork for obtaining decay and conformal scattering of Maxwell fields.

One of the spherically symmetric solutions of Einstein-Maxwell Field equations in the presence of a positive cosmological constant is the Reissner-Nordstrøm-de Sitter solution (which we sometimes abbreviate as *RNdS*). It models a non-rotating spherically symmetric charged black hole with mass and a charge, in a de Sitter background. The de Sitter background means that there is a cosmological horizon beyond which lies a dynamic region that stretches to infinity, while the Reissner-Nordstrøm nature entails that near the singularity, depending on the relation between the mass and the charge, one has a succession of static and dynamic regions separated by horizons. Some discussions on Reissner-Nordstrøm-de Sitter black holes can be found in [95, 96] and the references within.

The Reissner-Nordstrøm-de Sitter metric is given in spherical coordinates by

$$g_{\mathcal{M}} = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2d\omega^2, \quad (1)$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \Lambda r^2, \quad (2)$$

and  $d\omega^2$  is the Euclidean metric on the 2-Sphere,  $\mathcal{S}^2$ , which in spherical coordinates is,

$$d\omega^2 = d\theta^2 + \sin(\theta)^2 d\varphi^2,$$

and  $g_{\mathcal{M}}$  is defined on  $\mathcal{M} = \mathbb{R}_t \times ]0, +\infty[_r \times \mathcal{S}_{\theta, \varphi}^2$ . Here  $M$  is the mass of the black hole,  $Q$  is its charge, and  $\Lambda$  is the cosmological constant. We assume that  $Q$  is real and non zero, and  $M$  and  $\Lambda$  are positive.

The metric in these coordinates appear to have singularities at  $r = 0$  and at the zeros of  $f$ . Only the singularity at  $r = 0$  is a real geometric singularity at which the curvature blows up. The apparent singularities at the zeros of  $f$  are artificial and due to this particular choice of coordinates. The regions of spacetime where  $f$  vanishes are essential features of the geometry of the black hole, they are called event horizons or *horizons* for short, and  $f$  is called the *horizon function*. If  $f$  has three positive zeros and one negative, then the zeros in the positive range corresponds in an increasing order respectively to the *Cauchy horizon* or *inner horizon*, the *horizon of the black hole* or the *outer horizon*, and the *cosmological horizon*. In this case, as we shall see,  $f$  changes sign at each horizon and one has static and dynamic regions separated by these horizons.

In this work, we are interested in the decay in time of Maxwell fields in the static region between the horizon of the black hole and the cosmological horizon, which we refer to as the *exterior static region*. This part of spacetime contains a *photon sphere*, i.e. null geodesics orbiting the black hole at fixed  $r$ . This is a priori an obstacle for the decay but we shall see that the field decays in spite of the existence of a photon sphere. We are also interested in constructing a conformal scattering theory on the exterior static region. For this, we need to have access to the boundary of the region corresponding to infinite  $t$ -values. This boundary is part of the maximal analytic extension of the spacetime which we construct in this chapter.

This chapter has two main sections:

**Section 1.1:** We start the section by presenting the necessary and sufficient conditions (5) on the parameters  $M, Q$ , and  $\Lambda$  of the RNdS metric so that it has three horizons. We then verify our claim regarding these conditions along with the fact that there is a photon sphere only at one value of  $r > 0$  and it is located in the exterior static region. This is Proposition 1, and up to our knowledge, this is not in the literature.

**Section 1.2:** This section is a detailed construction of the maximal analytic extension of the the RNdS manifold in the case of three horizons (5) at  $0 < r_1 < r_2 < r_3$ . We start by exploring some properties of the black hole in the RNdS coordinates  $(t, r, \omega)$ . We then discuss the Regge-Wheeler  $r_*$  coordinate and use it to obtain coordinate expressions of the radial null geodesics. We define next the Eddington-Finkelstein advanced and retarded coordinates and extensions, showing that the event horizons are not singular but in fact are regular null hypersurfaces for the extended metric. The place where horizons of the same  $r$ -value “meet” is asymptotic to all of the Eddington-Finkelstein charts, these are the bifurcation spheres. To cover these spheres we need the Kruskal-Szekeres extensions. Each of these new extensions now cover all the horizons at  $r = r_i$  and the bifurcation sphere where they intersect. Finally, we use the Kruskal-Szekeres charts to cover the manifold of the maximal analytic extension. We discuss its causal structure, and some properties of timelike singularity at  $r = 0$ .

## 1.1 Photon Sphere

In this section we study the horizon function  $f$  given in (2) of the RNdS metric (1). We put,

$$R = \frac{1}{\sqrt{6\Lambda}} \quad ; \quad \Delta = 1 - 12Q^2\Lambda \quad ; \quad m_1 = R\sqrt{1 - \sqrt{\Delta}} \quad ; \quad m_2 = R\sqrt{1 + \sqrt{\Delta}} \quad (3)$$

$$M_1 = m_1 - 2\Lambda m_1^3 \quad ; \quad M_2 = m_2 - 2\Lambda m_2^3 . \quad (4)$$

and we consider the following conditions,

$$Q \neq 0 \quad \text{and} \quad 0 < \Lambda < \frac{1}{12Q^2} \quad \text{and} \quad M_1 < M < M_2 . \quad (5)$$

The main result of this section is:

**Proposition 1** (Three Positive Zeros and One Photon Sphere). *The function  $f$  has exactly three positive distinct zeros if and only if (5) holds. In this case, there is exactly one photon sphere in the static exterior region of the black hole defined by the portion between the largest two zeros.*

The proof is divided into parts. First, we study the conditions on  $M, Q$ , and  $\Lambda$  for  $f$  to have three positive zeros, and then we show that there is only one photon sphere.

### 1.1.1 The Zeros of the Horizon Function

The zeros of the function  $f$  are the roots of the polynomial

$$r^2 f(r) = P(r) = -\Lambda r^4 + r^2 - 2Mr + Q^2 . \quad (6)$$

Let us show that  $P$  has exactly three positive and one negative real roots if and only if (5) holds. We will prove this in two lemmata.

**Lemma 2.** *The polynomial  $P$  has three positive roots if and only if*

$$P'(R) > 0 \quad \text{and} \quad P(s_1) < 0 \quad \text{and} \quad P(s_2) > 0 , \quad (7)$$

*where  $0 < s_1 < s_2$  are the two positive roots of  $P'$ .*

*Proof.* The expressions of  $P'$  and  $P''$  are

$$P'(r) = -4\Lambda r^3 + 2r - 2M , \quad P''(r) = -12\Lambda r^2 + 2 ,$$

and so  $P''(R) = 0$ . Because  $R$  is the only positive root of  $P''(r)$  and  $P''(0) = 2$ ,  $P'$  is increasing on  $[0, R]$  and decreasing on  $[R, +\infty[$  with a local maximum at  $R$ . If  $P'(R)$  is non positive, and since  $P'(0) = -2M < 0$ , then  $P'$  is everywhere non positive on  $[0, +\infty[$ . Thus,  $P$  is decreasing on  $[0, +\infty[$ , and has only one root there as it decreases from  $P(0) = Q^2 > 0$

to  $-\infty$ . Therefore, a necessary condition for  $P$  to have three positive roots is that  $P'(R)$  be positive. Clearly,

$$P'(R) > 0 \quad \Leftrightarrow \quad M < \frac{2}{3}R. \quad (8)$$

As  $P'(0) < 0$ , and  $\lim_{r \rightarrow \pm\infty} P'(r) = \mp\infty$ , then having a positive local maximum at  $R$  implies that  $P'$  has exactly two roots  $0 < s_1 < R < s_2$  on the positive axis, and one on the negative axis. Also  $P'$  changes sign after passing through each of its roots  $s_1$  and  $s_2$ , which means that  $P(s_1)$  and  $P(s_2)$  are respectively the local minimum and the local maximum of  $P$  over the interval  $[0, +\infty[$ . We can conclude the following:

- If  $P(s_1) > 0$ , then  $P$  has one positive root  $x$ , with  $s_2 < x$ .
- If  $P(s_1) = 0$ , then  $P$  has two positive roots  $s_1$  and  $x$ , with  $s_1 < s_2 < x$ .
- If  $P(s_1) < 0$ , then :
  - If  $P(s_2) < 0$ , then  $P$  has one positive root  $x$ , with  $x < s_1$ .
  - If  $P(s_2) = 0$ , then  $P$  has two positive roots  $x$  and  $s_2$ , with  $0 < x < s_1 < s_2$ .
  - If  $P(s_2) > 0$ , then  $P$  has three positive roots  $r_1, r_2$ , and  $r_3$ , with  $0 < r_1 < s_1 < r_2 < s_2 < r_3$ .

This concludes the proof. □

Instead of finding  $s_1$  and  $s_2$  explicitly, we will, using the next lemma, transform the conditions in (7) to those in (5) directly.

**Lemma 3.** *If  $P'(R) > 0$ , with  $s_1$  and  $s_2$  the positive roots of  $P'$ , then*

$$P(s_1) < 0 \quad \text{if and only if} \quad P'(m_1) < 0; \quad (9)$$

$$P(s_2) > 0 \quad \text{if and only if} \quad P'(m_2) > 0, \quad (10)$$

where  $m_1$  and  $m_2$  are defined in (3).

*Proof.* We first note that

$$\begin{aligned} P(r) &= -\Lambda r^4 + r^2 - 2Mr + Q^2 \\ &= rP'(r) + T(r) \end{aligned}$$

where  $T$  is the polynomial

$$T(r) = 3\Lambda r^4 - r^2 + Q^2.$$

So,  $P(s_1) = T(s_1)$  and  $P(s_2) = T(s_2)$ . Therefore if we study the sign of  $T$  we shall know the sign of  $P(s_1)$  and  $P(s_2)$ . Let  $\bar{T}(r^2) = T(r)$ , i.e.

$$\bar{T}(r) = 3\Lambda r^2 - r + Q^2,$$

which has discriminant  $\Delta = 1 - 12\Lambda Q^2$ . We investigate the different cases.

- If  $\Delta < 0$  then  $\bar{T}$  has no real roots and is always positive, and hence so is  $T$ . In particular, this means that  $T(s_1)$  and  $T(s_2)$  are both positive, which is not the desired case.
- If  $\Delta = 0$  then  $R$  is a double root for  $\bar{T}$  and it is non negative. It follows that  $T$  is also non negative, and the conditions of (7) cannot be satisfied.
- Finally, if  $\Delta > 0$  which is  $\Lambda < \frac{1}{12Q^2}$ , then  $\bar{T}$  has two positive roots  $m_1^2$ ,  $m_2^2$  and hence  $\pm m_1$ ,  $\pm m_2$  are the roots of  $T$ , and  $T$  is positive on  $[0, m_1[$ , negative on  $]m_1, m_2[$ , and positive on  $]m_2, +\infty[$ .

Thus, noting that  $s_1$  and  $m_1$  are strictly less than  $R$ , and  $s_2$  and  $m_2$  are strictly greater than  $R$  when  $\Lambda < \frac{1}{12Q^2}$  and assuming (8), we see that

$$P(s_1) = T(s_1) < 0 \quad \text{if and only if} \quad m_1 < s_1 \quad (11)$$

$$P(s_2) = T(s_2) > 0 \quad \text{if and only if} \quad m_2 < s_2. \quad (12)$$

Now the key point which makes the right hand sides of (11) and (12) more useful is that  $P'$  is strictly monotonic on each side of  $R$ , and that  $P'(s_1) = P'(s_2) = 0$ . By applying  $P'$  to (11) and (12), one gets (9) and (10). □

Summarizing, recalling  $M_1$  and  $M_2$  from (4) and noting that  $P'(m_1) = -4\Lambda R^3(1 - \sqrt{\Delta})$ , we see that  $P$  has three positive roots if and only if

1.  $Q \neq 0$  and
2.  $0 < \Lambda < \frac{1}{12Q^2}$  and
3.  $P'(R) > 0$  i.e.  $M < \frac{2}{3}R$  and
4.  $P'(m_1) < 0$  i.e.  $M_1 < M$  and
5.  $P'(m_2) > 0$  i.e.  $M < M_2$ .

It remains to check the consistency of all of this and reduce it to (5). In fact, the only thing we need to show is that

$$0 < M_1 < M_2 < \frac{2}{3}R$$

whenever  $0 < \Lambda < \frac{1}{12Q^2}$  i.e.  $0 < \Delta < 1$ , and  $Q \neq 0$ . Consider the polynomial  $A(x) = x - 2\Lambda x^3$ . We have

$$\lim_{x \rightarrow \pm\infty} A(x) = \mp\infty$$

when  $\Lambda > 0$ , and the roots of  $A$  are zero and  $\pm a$  where

$$a = \frac{1}{\sqrt{2\Lambda}}.$$

Also,  $A$  is positive on  $]0, a[$  with  $R$  its local positive maximum on  $x \geq 0$  and  $A(R) = \frac{2}{3}R$ . Moreover,

$$\begin{aligned} 0 < m_1 < R < a, \\ 0 < m_2 < R\sqrt{2} < a, \end{aligned}$$

and since  $A(m_i) = M_i$ , it follows that

$$0 < M_1, M_2 < \frac{2}{3}R.$$

Finally, to see that  $M_1 < M_2$  we note that

$$M_2 - M_1 = (m_2 - m_1)(1 - 2\Lambda(m_2^2 + m_1m_2 + m_1^2)) = (m_2 - m_1) \left(1 - \frac{2 - \sqrt{1 - \Delta}}{3}\right) > 0.$$

### 1.1.2 Photon Sphere

Henceforth and unless otherwise specified, we will always assume that the conditions in (5) hold. We will denote the three positive zeros of  $f$  by  $0 < r_1 < r_2 < r_3$ . The hypersurfaces  $r = r_i$  for  $i = 1, 2, 3$  are respectively the *inner horizon*, the *outer horizon*, and the *cosmological horizon*. The *exterior static region* in which we are interested is  $\mathcal{N} = \mathbb{R}_t \times ]r_2, r_3[ \times \mathcal{S}_\omega^2$  where  $\omega = (\theta, \varphi)$ .

Let us now precise what we mean by a photon sphere and continue the proof of Proposition 1, namely, that there is only one photon sphere and it is situated in  $\mathcal{N}$ . We recall the definition of the Christoffel symbols of the Levi-Civita connection determined by the RNdS metric:

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

In the coordinates  $(t, r, \theta, \varphi) = (x^0, x^1, x^2, x^3)$ , the non zero Christoffel symbols are:

$$\begin{aligned} \Gamma_{01}^0 = -\Gamma_{11}^1 = \frac{f'}{2f} \quad ; \quad \Gamma_{00}^1 = \frac{ff'}{2} \quad ; \quad \Gamma_{22}^1 = -rf \quad ; \quad \Gamma_{33}^1 = -rf \sin(\theta)^2 \\ \Gamma_{12}^2 = \Gamma_{13}^3 = \frac{1}{r} \quad ; \quad \Gamma_{33}^2 = -\cos(\theta) \sin(\theta) \quad ; \quad \Gamma_{23}^3 = \cot(\theta). \end{aligned} \quad (13)$$

If we take a non zero purely rotational vector field along the angle  $\varphi$  it will be of the form

$$\mathcal{V} = \alpha \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial \varphi},$$

and the condition for it to be null is  $g(\mathcal{V}, \mathcal{V}) = 0$ , which is

$$f(r) = \frac{\beta^2 r^2 \sin(\theta)^2}{\alpha^2}. \quad (14)$$

Therefore, a photon sphere could only exist in the regions where  $f \geq 0$ , but we work in the static region outside the black hole, that is  $r_3 > r > r_2$ , thus  $f > 0$ . Also, condition (14) implies that

$$\mathcal{V} = \alpha \left( \frac{\partial}{\partial t} \pm \frac{\sqrt{f}}{r \sin(\theta)} \frac{\partial}{\partial \varphi} \right), \quad (15)$$

and so it is enough to examine the case where  $\alpha = 1$ .

Given  $\mathcal{V}$  in this form, i.e.

$$\mathcal{V} = \left( 1, 0, 0, \pm \frac{\sqrt{f}}{r \sin(\theta)} \right),$$

we calculate

$$\begin{aligned} \nabla_{\mathcal{V}} \mathcal{V} &= \mathcal{V}^a (\partial_a \mathcal{V}^c + \mathcal{V}^b \Gamma_{ab}^c) \partial_c \\ &= \Gamma_{00}^c \partial_c + (\mathcal{V}^3)^2 \Gamma_{33}^c \partial_c \\ &= \Gamma_{00}^1 \partial_r + (\mathcal{V}^3)^2 (\Gamma_{33}^1 \partial_r + \Gamma_{33}^2 \partial_\theta) \\ &= f \left( \frac{f'}{2} - \frac{f}{r} \right) \partial_r - \frac{\cot(\theta) f}{r^2} \partial_\theta. \end{aligned}$$

Thus, and since we have  $f > 0$ , we see that

$$\begin{aligned} \nabla_{\mathcal{V}} \mathcal{V} = 0 &\Leftrightarrow f \left( \frac{f'}{2} - \frac{f}{r} \right) = 0 \quad \text{and} \quad \frac{\cot(\theta) f}{r^2} = 0 \\ &\Leftrightarrow r f' - 2f = 0 \quad \text{and} \quad \theta = \frac{\pi}{2} \end{aligned} \quad (16)$$

One then can see that if we assume from the beginning that  $\theta = \frac{\pi}{2}$ , the integral curves of  $\mathcal{V}$  at the zeros of  $r f'(r) - 2f(r)$  are geodesics. Hence, by the spherical symmetry, we get a full “sphere” of null geodesics outside the black hole, this is referred to as the *photon sphere* around the black hole.

As we can see

$$r f'(r) - 2f(r) = \frac{6M}{r} - \frac{4Q^2}{r^2} - 2,$$

thus by studying the polynomial

$$S(r) = -r^2 + 3Mr - 2Q^2$$

we can determine the zeros. The discriminant of  $S$  is

$$\Delta_S = 9M^2 - 8Q^2 = (3M - 2\sqrt{2}Q)(3M + 2\sqrt{2}Q)$$

which is positive if

$$M > \frac{2\sqrt{2}|Q|}{3}.$$



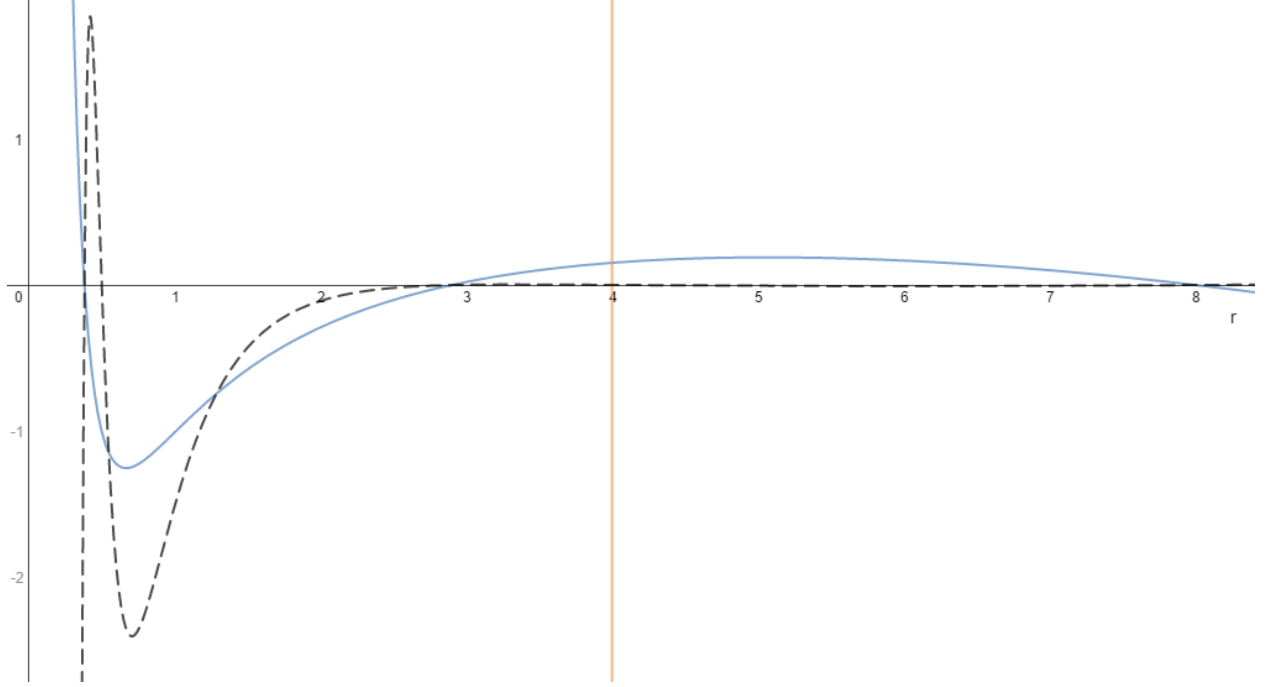


Figure 1.1:  $Q = 1$ ,  $M = 1.5$ ,  $\Lambda = 0.01$ . The function  $f$  is the continuous curve and the coefficient of radial acceleration  $f(2^{-1}f' - r^{-1}f)$  is the dotted curve. The vertical line ( $r=4$ ) is the photon sphere  $r = P_2$ .

The two roots, if they exist, have the expressions

$$P_1 = \frac{3M - \sqrt{\Delta_S}}{2} \quad \text{and} \quad P_2 = \frac{3M + \sqrt{\Delta_S}}{2}. \quad (17)$$

We can directly show that the last inequality holds when (5) is satisfied, however, by studying the sign of  $rf' - 2f$  near the zeros of  $f$ , not only can one show that it has two zeros but also one can know their positions relative to the horizons, which is the important thing. This is Proposition 1 and the argument is in its proof which we will present now.

*Continuation of the Proof of Proposition 1.* We showed that  $f$  has three positive zeros  $r_1, r_2$ , and  $r_3$  if (5) holds. Note that  $f$  and  $P$ , given in (6), are both smooth and have the same sign over  $]0, +\infty[$ , and we know the sign of  $P$  everywhere. In a small interval around  $r_1$ ,  $f$  is decreasing since it is positive to the left of  $r_1$  and negative to its right, thus  $f' < 0$  over this interval. Shrinking the interval if necessary, it follows that in the acceleration of the vector field  $\mathcal{V}$  (see (16))

$$\nabla_{\mathcal{V}}\mathcal{V} = f \left( \frac{f'}{2} - \frac{f}{r} \right) \partial_r$$

the factor  $f(2^{-1}f' - r^{-1}f)$  is negative to the left of  $r_1$  and positive to its right. Using exactly the same logic, the last statement holds true for  $r_2$  and  $r_3$  also. (Figure 1.1)

Since the acceleration vector field is continuous, it must vanish in order to change sign. And since its zeros are  $\{r_1, r_2, r_3, P_1, P_2\}$  (see (17)), then by the above argument the zeros

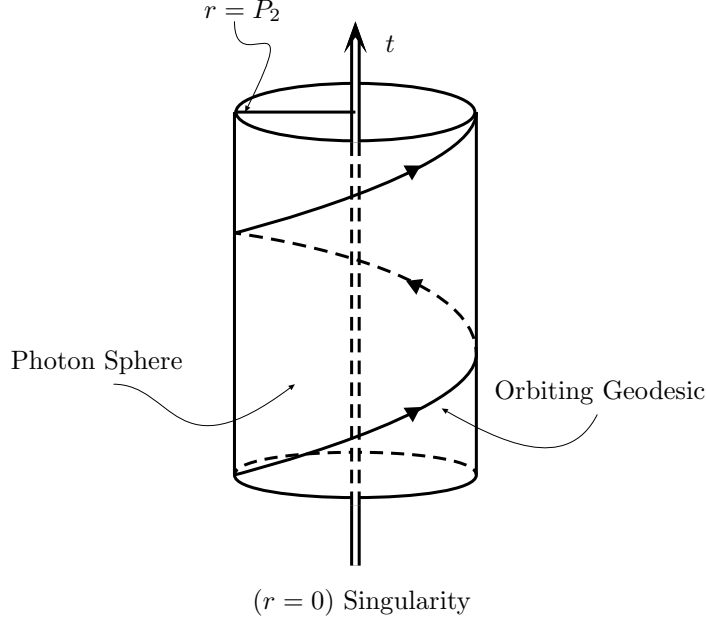


Figure 1.2: A null geodesic orbiting the black hole at the photon sphere  $\{r = P_2\}$  which is a spacelike hypersurface.

are necessarily ordered as follows:  $r_1 < P_1 < r_2 < P_2 < r_3$ , which is what we wanted to prove.  $\square$

Note that  $\{r = P_1\}$  is *not* a photon sphere since  $f$  is negative on  $]r_1, r_2[$  and so the rotational vector  $\mathcal{V}$  is necessarily spacelike. This means that there are no orbits inside the black hole horizon, which is consistent with the fact that this region is dynamic. We also note that in spite the covering of the photon sphere by null geodesics it is not a null hypersurface, as a matter of fact,  $\partial_r$  is a normal to the photon sphere hypersurface, and therefore it is spacelike. (Figure 1.2)

## 1.2 Maximal Analytic Extension

In the RNdS coordinates  $(t, r, \omega) \in \mathcal{M} = \mathbb{R}_t \times ]0, +\infty[_r \times \mathcal{S}_\omega^2$  the metric

$$g = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2d\omega^2, \quad (18)$$

appears to be singular at  $r = r_i$  where the factor  $f^{-1}(r)$  blows up. So, the metric  $g$  in these coordinates is actually defined on  $\mathcal{M}$  with these hypersurfaces removed. The removal of these hypersurfaces disconnects the spacetime and divides it into four open regions  $U_i$ ,  $i$  from 1 to 4. Usually, a spacetime is defined to be a connected smooth Lorentzian 4-manifold, so, we consider each of these open regions separately, and we write  $g_i$  for  $g|_{U_i}$  when necessary. To understand the meaning of these coordinate singularities, we shall extend each of these

regions by analytic extensions covering the horizons. We say that a connected *analytic Lorentzian 4-manifold*<sup>1</sup>  $(\tilde{U}, \tilde{g})$  is an *analytic extension* of  $(U_i, g_i)$  if the latter is isometrically embedded in the first and its image is a proper subset. Since we only consider analytic extensions, we shall refer to them simply as extensions, and an *inextendible spacetime* will be a spacetime which has no extension.

Intuitively, the places to extend are near the removed hypersurfaces, and just as in other spacetimes, like Schwarzschild, there is a known way of doing it by simple changes of coordinates on each region which when extended to their maximal natural domains of definition produce extensions of the whole spacetime  $\mathcal{M}$  including the hypersurfaces  $\{r = r_i\}$ . These are the Eddington-Finkelstein coordinates and extensions, and they are covered with single coordinate charts<sup>2</sup>. However, as shall be seen, each of these extensions separately is not maximal, that is, has an extension itself. From the possible time orientations on  $\mathcal{M}$ , we shall see that the different Eddington-Finkelstein extensions are in some sense complementary, and can be done all at the same time to give a bigger extension. This extension consists of an infinite number of different overlapping Eddington-Finkelstein coordinate charts, yet, there will be in the maximal extension isolated spheres of radii  $r_i$ s, called the *bifurcation spheres* or *crossing spheres*, which are not covered by Eddington-Finkelstein coordinate chart. To cover these, we need to introduce new coordinate charts on  $\mathcal{M}$  and extend them, they are called the Kruskal-Szekeres coordinates. It turns out that the maximal extension can be completely covered using three families of Kruskal-Szekeres coordinate domains.

Moreover, the maximal analytic extension of RNdS satisfies a rather stronger inextendibility property. Besides being inextendible in the sense we described above, it is also *locally inextendible*: It has no open non-empty subset whose closure is non-compact and can be embedded in an analytic manifold with a relatively compact image. With this taken into account, there is a unique maximal connected analytic extension of RNdS which is locally inextendible, as long as we do not make identifications that change the topology.

The two dimensional diagrams presented in this section are two dimensional cross-sections of the spacetime at fixed generic angular direction  $\omega_0 = (\theta_0, \varphi_0)$ , or equivalently, they are quotients of the spacetime by the action of the rotation group.

### 1.2.1 Reissner-Nordström-de Sitter Coordinates

We start by reviewing some properties of the RNdS coordinates  $(t, r, \omega)$ . Consider the following open subsets of  $\mathcal{M}$ , which we also refer to by I, II, III, and IV, respectively:

$$\begin{aligned} U_1 &= \mathbb{R}_t \times ]0, r_1[ \times \mathcal{S}_{\theta, \varphi}^2 ; \\ U_2 &= \mathbb{R}_t \times ]r_1, r_2[ \times \mathcal{S}_{\theta, \varphi}^2 ; \\ U_3 &= \mathbb{R}_t \times ]r_2, r_3[ \times \mathcal{S}_{\theta, \varphi}^2 ; \\ U_4 &= \mathbb{R}_t \times ]r_3, +\infty[ \times \mathcal{S}_{\theta, \varphi}^2 , \end{aligned}$$

---

<sup>1</sup>An analytic  $n$ -manifold is a topological space with an atlas whose charts are analytically related, i.e. the transition maps between the charts are analytic functions of  $\mathbb{R}^n$ . Also, if it is Lorentzian, we require the metric defined on it to be analytic.

<sup>2</sup>Except for the fact that  $S^2$  needs multiple charts to fully cover it.

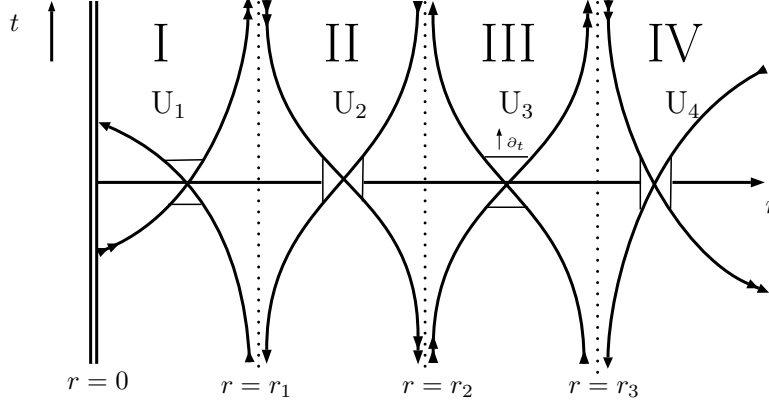


Figure 1.3: The open sets  $U_i$ 's of  $\mathcal{M}$ , also referred to as regions I, II, III, and IV. The radial null geodesics shown are integral curves of the null vector fields  $Y^\pm = f^{-1}\partial_t \pm \partial_r$ , with the light cones shown where they meet, and the arrows on the geodesics show the increasing direction of their affine parameters.

and let  $I_i$  be the corresponding interval of  $r$  such that

$$U_i = \mathbb{R}_t \times I_i \times \mathcal{S}_{\theta, \varphi}^2. \quad (19)$$

We orient  $\mathcal{M}$  by requiring  $(\partial_t, \partial_r, \partial_\theta, \partial_\varphi)$  to be a positively oriented frame. On the other hand, because  $\mathcal{M}$  is not connected as a Lorentzian manifold when we remove the hypersurfaces at  $r = r_i$ , there is no canonical way of defining a continuous time-orientation on it a priori. For example, while  $\partial_t$  is timelike in I and III, it is spacelike in II and IV where  $\partial_r$  is timelike. When we do decay in chapter 3 we choose  $\partial_t$  to be future-oriented in III, but this says nothing about the time-orientation of  $\partial_r$  or any other causal vector in regions II and IV. In other words,  $\mathcal{M}$  admits more than two time-orientations. In effect, each connected component has exactly two time-orientations,  $\pm\partial_t$  for I and III, and  $\pm\partial_r$  for II and IV. This amounts to a total of sixteen different configurations for time-orienting  $\mathcal{M}$ . We shall see that each configuration is isometrically embedded, via a time-orientation preserving embedding<sup>1</sup>, in a connected part of the maximal extension. When we want to distinguish between different time-orientations, we shall designate  $(U_1, +\partial_t)$  by I, and  $(U_1, -\partial_t)$  by I', and the same for  $(U_3)$ . The time-orientation on the other regions is indicated similarly, with II and IV for  $+\partial_r$ , and II' and IV' for  $-\partial_r$ .

We note that  $\mathcal{M}$  admits no global timelike Killing vector field. Only regions  $U_1$  and  $U_2$  admit a timelike Killing vector field, and hence are *stationary*, and in fact, since this vector field is  $\partial_t$  which is orthogonal to the foliation by the spacelike hypersurfaces  $\{t = cst\}$ , regions I and III are *static*. Regions II and IV are *dynamic* (not stationary) which implies, in particular, that no observer or light can “hover” or orbit at a fixed distance from the singularity at  $r = 0$ .

<sup>1</sup>We say that an isometric embedding preserves time-orientation if it maps future oriented causal vectors to future oriented ones.

Since  $(\mathcal{M}, g)$  is a spherically symmetric spacetime, radial null geodesics are particularly important. First, consider a radial null geodesic  $\gamma$  of  $\mathcal{M}$ , that is, satisfying the geodesic equations

$$\ddot{x}^k + \dot{x}^i \dot{x}^j \Gamma_{ij}^k = 0 ,$$

which reduce to

$$\begin{aligned} \ddot{t} &= -\dot{t} \dot{r} \frac{f'}{2f} , \\ \ddot{r} &= 0 , \end{aligned}$$

and  $\gamma(s) = (t(s), r(s), \omega_0)$  for some fixed  $\omega_0$ , with  $\dot{\gamma}(s) = \dot{t}(s)\partial_t|_{\gamma(s)} + \dot{r}(s)\partial_r|_{\gamma(s)}$  a null vector, i.e.

$$\dot{t}^2 = \frac{\dot{r}^2}{f^2} .$$

Then,  $r = c_1 s + c_2$  for some constants  $c_1$  and  $c_2$ , so  $r$  is an affine parameter, and we have  $\dot{t} = \pm c_1 f^{-1}$ . Therefore,  $\gamma$  is an integral curve of a vector field of the form  $c(f^{-1}\partial_t \pm \partial_r)$  for some non zero constant  $c$ , and hence it is sufficient to study the integral curves of the vector fields<sup>1</sup>  $Y^\mp = f^{-1}\partial_t \pm \partial_r$  that generates the others (figure 1.3).

Two particular features of the spacetime can be seen from the radial null geodesics and the directions of the light cones in regions I and IV. The singularity at  $r = 0$  has a timelike nature and is more of a “place” in space, which can therefore be avoided, to the eye of an observer in this region. The end-points of the null geodesics at  $r = \infty$ , denoted by  $\mathcal{I}$ , can be understood as a smooth spacelike boundary by means of a conformal rescaling<sup>2</sup>. The timelike singularity is of course due to the charge of the black hole or the Reissner-Nordström aspect of the spacetime, and at the other end, the spacelike null infinity is nothing but a manifestation of the cosmological constant, i.e. the de Sitter background of the spacetime.

### 1.2.2 Regge-Wheeler Charts

If  $\gamma^-$  is an integral curve of  $Y^- = f^{-1}\partial_t + \partial_r$ , then  $r$  is an affine parameter of  $\gamma^-$ , and

$$\frac{d(t \circ \gamma^-)}{dr}(r) = \frac{1}{f(r)} .$$

Thus,  $t(\gamma^-(r))$  is, up to an integration constant, nothing but the Regge-Wheeler coordinate function  $r_*(r)$  which we will presently define, and  $\gamma^-(r) = (r_*(r) + C, r, \omega_0)$  for some constant  $C$ . Similarly, an integral curve of  $Y^+ = f^{-1}\partial_t - \partial_r$  is of the form  $\gamma^+(s) = (C - r_*(-s), -s, \omega_0)$  defined for  $s < 0$ . If we choose  $\partial_t$  to be future-oriented on  $U_3$ , the null vector fields  $Y^\mp$  will be

<sup>1</sup>The choice of the sign in the names of the vector fields  $Y^- = f^{-1}\partial_t + \partial_r$  and  $Y^+ = f^{-1}\partial_t - \partial_r$  may seem strange or unpleasant at first, but we feel it is more natural this way since an integral curve of  $Y^-$  is a curve of constant  $u_-$  where  $u_- = t - r_*$  is the retarded time coordinate of the Eddington-Finkelstein Extension, and that of  $Y^+$  is given by a constant  $u_+ = t + r_*$ . See section 1.2.3

<sup>2</sup> $\mathcal{I}$  is perhaps better described as an “instant of time” in the infinite future or past of an observer in region IV.

future-oriented on  $U_3$ . Thus,  $\gamma^+$  is by definition an incoming null geodesic since it is future-directed and  $r$  is a decreasing function of its affine parameter  $s$ , while  $\gamma^-$  is an outgoing null geodesic for similar reasons.

The *Regge-Wheeler radial coordinate* function (also known as the *tortoise coordinate*), is defined by

$$\frac{dr}{dr_*} = f(r) \quad \text{and} \quad r_* = 0 \text{ when } r = P_2, \quad (20)$$

where  $P_2$  is the localization of the photon sphere outside the black hole given by (17)<sup>1</sup>. To get the explicit expression of  $r_*$  in terms of  $r$ , let the four zeros of  $f$  be  $r_i$  with  $r_0 < 0 < r_1 < r_2 < r_3$ , and let us write  $f$  as

$$f(r) = -\frac{\Lambda}{r^2} \prod_{i=0}^3 (r - r_i).$$

We integrate,

$$r_*(r) = \int_{P_2}^r \frac{1}{f(s)} ds = \int_{P_2}^r -\frac{s^2}{\Lambda} \prod_{i=0}^3 \frac{1}{s - r_i} ds = \sum_{i=0}^3 a_i \ln |r - r_i| + a, \quad (21)$$

where

$$a_i = -\frac{r_i^2}{\Lambda} \prod_{j \neq i} \frac{1}{(r_i - r_j)} \quad ; \quad a = -\sum_{i=0}^3 a_i \ln |P_2 - r_i|.$$

We note that  $a_0, a_2 > 0$  and  $a_1, a_3 < 0$ ,  $f'(r_i) = \frac{1}{a_i}$ , and  $dr = f dr_*$  on  $I_i$  define in (19). Since  $f$  has a constant sign on each interval  $I_i$ , each  $r_{*i}(r) := r_{*|I_i}(r)$  is a monotonic function on  $I_i$ , and in fact, analytic. Thus, on each  $U_i$ , we define the Regge-Wheeler coordinates  $(t, r_{*i}, \omega) \in U_i^* = \mathbb{R} \times I_i^* \times \mathcal{S}^2$ , where  $I_i^* = r_*(I_i)$ . The metric in these coordinates is

$$g = f(r)(dt^2 - dr_{*i}^2) - r^2 d\omega^2,$$

where  $r = r(r_{*i})$  is the inverse function of  $r_{*i}(r)$ .

We shall usually drop the  $i$  in  $r_{*i}$  (and in other coordinates later) for clarity. The ordered basis  $(\partial_t, \partial_{r_*}, \partial_\theta, \partial_\varphi)$  is positively oriented on  $U_1^*$  and  $U_3^*$ , and negatively oriented on the other two domains. To determine the intervals  $I_i^*$  we calculate the limits of  $r_*(r)$  at the singularity  $r = 0$ , at the horizons  $r = r_i$ , and at infinity  $r = +\infty$ .

First,

$$\lim_{r \rightarrow 0} r_*(r) = r_*(0) = b \in \mathbb{R},$$

and from the signs of the coefficients  $a_i$ s, we have the two sided limits:

$$\begin{aligned} \lim_{r \rightarrow r_1} r_*(r) &= +\infty, \\ \lim_{r \rightarrow r_2} r_*(r) &= -\infty, \\ \lim_{r \rightarrow r_3} r_*(r) &= +\infty, \end{aligned} \quad (22)$$

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<sup>1</sup>This choice of the origin for  $r_*$  is justified when we deal with the trapping term in Lemma 31.

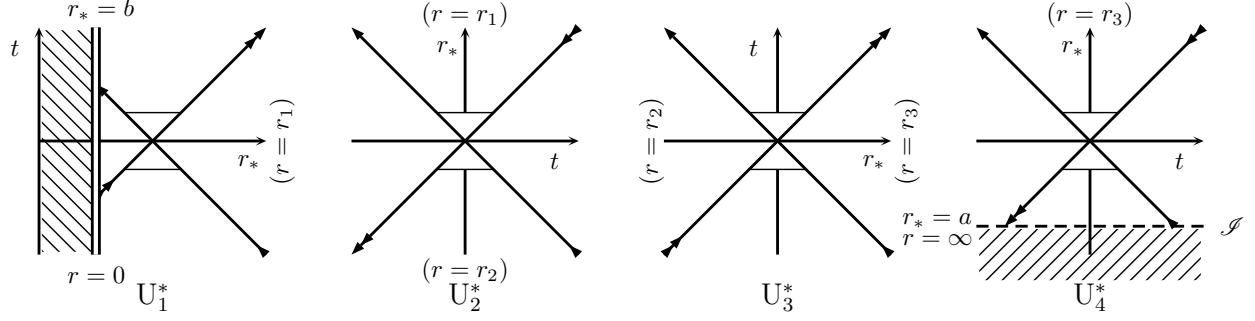


Figure 1.4: *Oriented Regge-wheeler charts defined on  $U_i$ . The null geodesics are integral curves of  $Y^\mp = f^{-1}(\partial_t \pm \partial_{r_*})$  (lines at  $\pm 45^\circ$ ). The hypersurfaces  $r = r_i$  (indicated in parenthesis) are off the chart since they are limits of  $r_*$ .*

and,

$$\lim_{r \rightarrow +\infty} r_*(r) = a ,$$

since as  $r$  tends to  $+\infty$  we have

$$r_*(r) - a \sim \sum_{i=0}^3 a_i \ln(r) = \ln(r) \sum_{i=0}^3 a_i ,$$

but for any four distinct non zero numbers  $x_1, x_2, x_3, x_4$  we have

$$\sum_{j=1}^4 x_j^2 \prod_{k \neq j} \frac{1}{(x_j - x_k)} = 0 ,$$

and so,  $\sum_{i=0}^3 a_i = 0$ . Therefore,  $I_1^* = ]b, +\infty[$ ,  $I_2^* = I_3^* = \mathbb{R}$ , and  $I_4^* = ]-\infty, a[$ .

In  $(t, r_*, \omega)$ -coordinates,  $Y^\mp = f^{-1}(\partial_t \pm \partial_{r_*})$  and we readily see that the images of their integral curves  $\gamma^\mp$  are straight lines at  $45^\circ$  (figure 1.4). The intuitive meaning of a straight line in a spacetime diagram is the worldline of a particle moving with a constant speed in the given coordinate reference frame, this is the case here too. If we consider  $r_*$  as the radial coordinate, then the coordinate speeds of  $\gamma^\mp$  are indeed constants and equal to 1, i.e.

$$\frac{(\dot{\gamma}^\mp)^1}{(\dot{\gamma}^\mp)^0} = 1.$$

Compared to their coordinate speeds in the  $(t, r, \omega)$ -coordinates which is  $f$ , the effect of slowing down near the hypersurfaces  $\{r = r_i\}$  in the  $(t, r, \omega)$ -coordinates is not apparent anymore in the  $(t, r_*, \omega)$ -coordinates but at the cost of pushing the hypersurfaces  $\{r = r_i\}$  to infinite  $r_*$ -values. This is why  $r_*$  is sometimes called the “*tortoise coordinate*”. Finally, the boundary hypersurface  $r_* = a$  is conformally spacelike, and so the spacelike nature of  $\mathcal{S}$  is more revealed in the Regge-Wheeler coordinates.

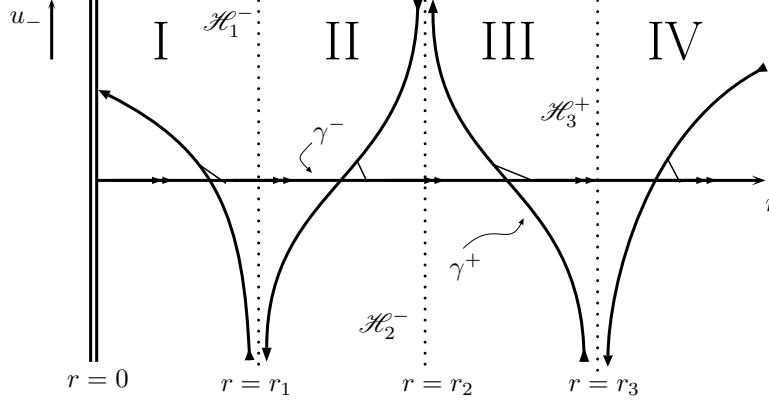


Figure 1.5:  $\mathcal{M}_F^-$  and the integral curves of  $Y^\mp$ .

### 1.2.3 Eddington-Finkelstein Extensions

Let us temporarily fix a time-orientation on  $\mathcal{M}$ . Let  $\partial_t$  be future oriented on  $U_1$  and  $U_3$ , so,  $Y^\mp = f^{-1}\partial_t \pm \partial_r$  are future-oriented there, and we choose the orientation given by  $\partial_r$  on  $U_2$  and  $U_4$ , then  $Y^-$  is future-oriented while  $Y^+$  is past-oriented in regions  $U_2$  and  $U_4$ . Since we defined incoming and outgoing geodesics to be future-directed,  $\gamma^-$  will be an outgoing radial null geodesic on  $\mathcal{M}$ , while there are no incoming radial null geodesics in the dynamic regions for this particular time-orientation. In the  $(t, r, \omega)$ -coordinates, the coordinate expression of  $\gamma^-$  has discontinuities at  $r = r_i$  since its  $t$ -coordinate blows up because  $r_*$  does, but this could be a mere bad choice of coordinate. To check this, we use the coordinates given by the flows of  $Y^-$ , i.e. using the geodesics  $\gamma^-$  themselves as coordinate lines: For each point  $p = (t_p, r_p, \omega_0)$  of the spacetime in the plane  $\{\omega = \omega_0\}$ , there is a unique  $C_p \in \mathbb{R}$  such that  $\gamma_p^-(r) = (r_*(r) + C_p, r, \omega_0)$  passes through  $p$ , and  $p$  can be given the new coordinates  $(C_p, r_p, \omega_0)$ , with  $t_p = r_*(r_p) + C_p$ . We thus define a new coordinate  $u_- := t - r_*$ , this is Eddington-Finkelstein retarded null coordinate<sup>1</sup>. The Eddington-Finkelstein retarded coordinate chart on  $U_i$  is

$$(u_{-i}, r, \omega) \in \mathbb{R}_{u_-} \times I_{ir} \times \mathcal{S}_\omega^2,$$

with  $u_{-i} = t - r_{*i}$ , defined is called the . In this chart the metric is:

$$g = f(r)du_{-i}^2 + 2du_{-i}dr - r^2d\omega^2, \quad (23)$$

This expression of the metric is analytic for all values  $(u_-, r, \omega) \in \mathbb{R} \times ]0, +\infty[ \times \mathcal{S}^2$ , including  $r = r_i$ . The Lorentzian manifold  $\mathcal{M}^- = \mathbb{R}_{u_-} \times ]0, +\infty[ \times \mathcal{S}_\omega^2$  with the metric (23) is called the *retarded Eddington-Finkelstein extension* of the RNdS manifold. Taking the orientation of  $\mathcal{M}$ ,  $(\partial_{u_-}, \partial_r, \partial_\theta, \partial_\varphi)$  is positively oriented on  $\mathcal{M}^-$ , and when  $\partial_r$  is chosen to be future-oriented<sup>2</sup>, we denote  $\mathcal{M}^-$  by  $\mathcal{M}_F^-$  and call it the *future retarded extension*(figure 1.5).

<sup>1</sup>A null, time, or space coordinate is one whose level surfaces are null, spacelike, or timelike hypersurfaces respectively

<sup>2</sup>This is *not* the coordinate vector field  $\partial_r$  of the chart  $(t, r, \omega)$ . If we denote the Eddington-Finkelstein retarded coordinates by  $(u_-, r_-(=r), \omega)$  then  $\partial_{r_-} = f^{-1}\partial_t + \partial_r = Y^-$ .



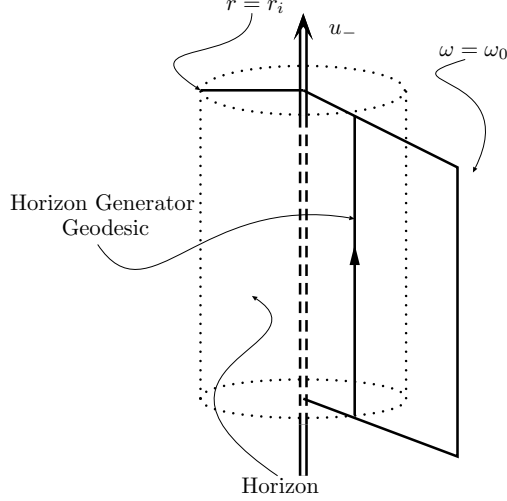


Figure 1.6: *Integral curves of  $e^{\frac{1}{2}f'(r_i)u_-}\partial_{u_-}$  at  $r = r_i$  are null geodesics that generate the horizon  $\{r = r_i\}$ .*

On  $\mathcal{M}_F^-$ ,  $Y^-$  is a smooth null vector field and it is equal to  $\partial_r$  (in the retarded coordinates). Its integral curves  $\gamma^-$  are outgoing radial null geodesics<sup>1</sup>, and they are just straight lines of constant  $u_-$  and  $\omega$ :  $\gamma^-(r) = (C, r, \omega_0)$ . These geodesics are maximal and go through the hypersurfaces  $\{r = r_i\}$  without any peculiar behaviour showing that the picture given by the RNdS coordinates is not complete and no real geometric singularities are present at  $r = r_i$ , and the only real singularity is at  $r = 0$  where the curvature becomes infinite. Since all future-directed causal curves in regions II and IV are outgoing, the hypersurfaces at  $r = r_i$ , the zeros of  $f$ , act like one way barriers which can be crossed only from the inside ( $r < r_i$ ), and all events happening beyond them ( $r > r_i$ ) are (for observers on the other side where  $r < r_i$ ) hidden behind the horizons ( $r = r_i$ ). We therefore call these hypersurfaces at  $r = r_i$  *event horizons*, and hence  $f$  is called the *horizon function* [28].

For an observer in III, light coming from the singularity and passing through the first two event horizons of the black hole is travelling forward in time and hence is from the past, therefore the observer will consider the singularity to be in the past as well as the *past inner horizon*  $\mathcal{H}_1^- = \mathbb{R}_{u_-} \times \{r = r_1\} \times \mathcal{S}_\omega^2$ , and the *past outer horizon*  $\mathcal{H}_2^- = \mathbb{R}_{u_-} \times \{r = r_2\} \times \mathcal{S}_\omega^2$ , which are now regular null hypersurfaces. Similarly, the observer can only send but never receive any signal from the last horizon and  $\mathcal{I}$ . In this extension, we denote  $\mathcal{I}$  by  $\mathcal{I}^+$  since it lies in the future of the observer, and so does the *future cosmological horizon*  $\mathcal{H}_3^+ = \mathbb{R}_{u_-} \times \{r = r_3\} \times \mathcal{S}_\omega^2$  which is a regular null hypersurface for the metric (23). The null horizons are generated by null geodesics each lying in a fixed angular plane (figure 1.6). This means that at the horizon some “photons hover” in place at  $r = r_i$  and  $\omega = \omega_0$ .

Although the outgoing geodesics  $\gamma^-$  are inextendible in the extension  $\mathcal{M}_F^-$ , the incoming radial null geodesics (in the static regions I and III)  $\gamma^+(s) = (C - 2r_*(-s), -s + c, \omega_0)$  are not. For this reason,  $\mathcal{M}_F^-$  is also called the *outgoing Eddington-Finkelstein extension*.

<sup>1</sup>Other outgoing geodesics are  $-\gamma^+$  the integral curves of  $-Y^+$  in the dynamic regions  $U_2$  and  $U_4$  where there are only outgoing geodesics.

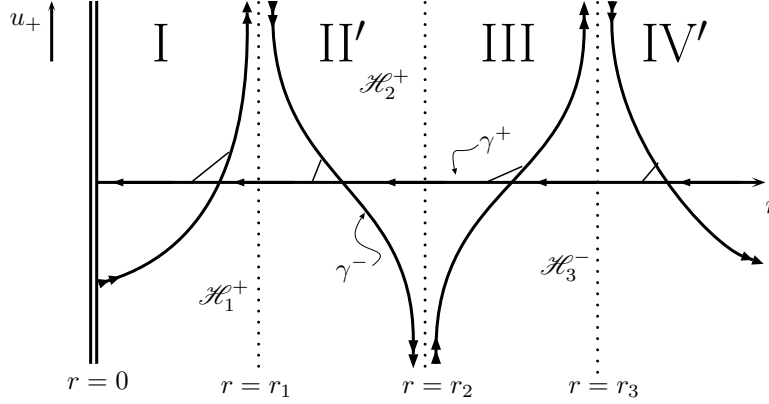


Figure 1.7:  $\mathcal{M}_F^+$  and the integral curves of  $Y^\mp$ .

Nonetheless, if we reverse the time-orientation, outgoing and incoming will be reversed, and the integral curves of  $-Y^-$  will be the incoming geodesics crossing the horizons. We refer to  $\mathcal{M}^-$  with this time-orientation as  $\mathcal{M}_P^-$  the *past retarded extension*. Of course, this is not the only extension in which the incoming geodesics are inextendible, had we chosen the opposite time-orientation on  $U_2$  and  $U_4$  so that  $\mathcal{M}$  is time-oriented by  $\partial_t$  and  $-\partial_r$  of the  $(t, r, \omega)$ -chart, the same procedure with  $Y^+$  and  $Y^-$  exchanging places would have lead instead to the *advanced Eddington-Finkelstein null coordinate*  $u_+ = t + r_*$  and to a new extension  $\mathcal{M}^+$  covered by a single chart  $(u_+, r, \omega) \in \mathbb{R}_{u_+} \times ]0, +\infty[ \times \mathcal{S}_\omega^2 = \mathcal{M}^+$  endowed with the analytic metric

$$g = f(r)du_+^2 - 2du_+dr - r^2d\omega^2, \quad (24)$$

where  $(\partial_{u_+}, \partial_r, \partial_\theta, \partial_\varphi)$  is positively oriented and  $-\partial_r$  is future-oriented. This is the *future advanced Eddington-Finkelstein extension*  $\mathcal{M}_F^+$  (figure 1.7). Similarly, with  $\partial_r$  future-oriented we get the *past advanced Eddington-Finkelstein extension*  $\mathcal{M}_P^+$ . The picture given by  $\mathcal{M}_F^+$  and the one given by  $\mathcal{M}_P^-$  are alike but not quiet the same as we shall presently see. In both, the singularity and the horizons at  $r = r_1$  and  $r = r_2$  are in the future of region III, while the horizon at  $r = r_3$  is in the past of the region where also past null infinity  $\mathcal{I}^-$  is. In  $\mathcal{M}_F^+$ , we have the *future inner horizon*  $\mathcal{H}_1^+ = \mathbb{R}_{u_+} \times \{r = r_1\} \times \mathcal{S}_\omega^2$ , the *future outer horizon*  $\mathcal{H}_2^+ = \mathbb{R}_{u_+} \times \{r = r_2\} \times \mathcal{S}_\omega^2$ , and the *past cosmological horizon*  $\mathcal{H}_3^- = \mathbb{R}_{u_+} \times \{r = r_3\} \times \mathcal{S}_\omega^2$ . For the past extensions,  $\mathcal{M}_P^\pm$ , we sometimes denote the horizon by a minus sign when we want to specify:  $-\mathcal{H}_i^\pm$ .

With these four extensions in hand we can see what the different regions of  $\mathcal{M}$  represent. Although, and as seen from the geodesics  $\gamma^\pm$ , none of the extensions is locally inextendible and of course non is geodesically complete<sup>1</sup>, yet, when combined they give us an almost full picture: For *almost* any radial null geodesic in  $\mathcal{M}$  there is an Eddington-Finkelstein extension for which the given geodesic is future-directed and maximal i.e. extending from the singularity to  $\mathcal{I}$ . We say almost because the null generators of the horizons are not

<sup>1</sup>In fact, even with the four combined we still do not have a geodesically complete spacetime because of the singularity at  $r = 0$  beyond which the metric cannot be smoothly or even continuously extended. So we do not expect the maximal extension to be geodesically complete.

maximal in any Eddington-Finkelstein extension, even when combined.

From the convention of labelling the regions with primed and unprimed roman numbers to indicate time-orientation, we can easily follow the parts of different Eddington-Finkelstein extensions which are isometric in an orientation preserving manner: Different parts that carry the same label describe exactly the same geometry, the same orientation, and the same time-orientation, so, the difference is merely a change of coordinates. Each labelled region will be covered by exactly two of these extensions, however, this is not the case for the horizons. For example,  $\mathcal{M}_F^\pm$  both agree on III, but in  $\mathcal{M}_F^-$  the null geodesic  $\gamma^-$  intersects the past outer horizon  $\mathcal{H}_2^-$  and the future cosmological horizon  $\mathcal{H}_3^+$ , i.e. the hypersurfaces  $r = r_2$  and  $r = r_3$ , whereas in  $\mathcal{M}_F^+$ ,  $\gamma^-$  never touches the hypersurfaces  $r = r_2$  and  $r = r_3$ , the future outer horizon  $\mathcal{H}_2^+$  and the past cosmological horizon  $\mathcal{H}_3^-$ . In fact,  $\mathcal{H}_2^+$  and  $\mathcal{H}_3^-$  are asymptotic to  $\mathcal{M}_F^-$  where  $u_- = \pm\infty$ . Thus, the horizons can not be identified with each other, this is why we distinguish between future and past horizons.

### 1.2.4 Kruskal-Szekeres Extensions

With the Eddington-Finkelstein extensions we *almost* have the full picture since even if we extend using the four extensions at the same time, we still do not get a locally inextendible manifold. To see why it is the case, let us examine what do we mean by doing all Eddington-Finkelstein extensions at the same time.

Each of the previous extensions is done basically by a change of coordinates on a region  $U_i$ , then noticing that the metric in the new coordinates is analytic on a domain bigger than the original domain of the new chart, and then  $\mathcal{M}$  is isometrically embedded in this bigger domain. We follow a similar strategy here, however we are not going to cover the new extension, or even  $\mathcal{M}$  for this matter, by a single coordinate chart, we need three, which is related to having three horizons.

We start by defining on  $U_i$  the *double null coordinates*  $u_{-i} = t - r_{*i}$  and  $u_{+i} = t + r_{*i}$ . We have  $(u_{-i}, u_{+i}, \omega) \in \hat{U}_i$  with:

$$\begin{aligned}\hat{U}_1 &= \{(u_{-1}, u_{+1}) \in \mathbb{R}^2; u_{+1} - u_{-1} > 2b\} \times \mathcal{S}_\omega^2; \\ \hat{U}_2 &= \mathbb{R}_{u_{-2}} \times \mathbb{R}_{u_{+2}} \times \mathcal{S}_\omega^2; \\ \hat{U}_3 &= \mathbb{R}_{u_{-3}} \times \mathbb{R}_{u_{+3}} \times \mathcal{S}_\omega^2; \\ \hat{U}_4 &= \{(u_{-4}, u_{+4}) \in \mathbb{R}^2; u_{+4} - u_{-4} < 2a\} \times \mathcal{S}_\omega^2.\end{aligned}$$

The frame  $(\partial_{u_{-i}}, \partial_{u_{+i}}, \partial_\theta, \partial_\varphi)$  is positively oriented on  $\hat{U}_1$  and  $\hat{U}_3$ , and negatively oriented on  $\hat{U}_2$  and  $\hat{U}_4$ . The metric in these coordinates is,

$$g = f(r)du_{-i}du_{+i} - r^2d\omega^2, \quad (25)$$

where  $r$  is implicitly given by  $u_{+i} - u_{-i} = 2r_{*i}(r)$ .

To put these charts in the context of the Eddington-Finkelstein extensions, we need to choose orientations on the  $\hat{U}_i$ s. We can then use the radial null geodesics and the extensions  $\mathcal{M}_F^\pm$  and  $\mathcal{M}_P^\pm$  to determine the asymptotics of the oriented double null coordinates charts.

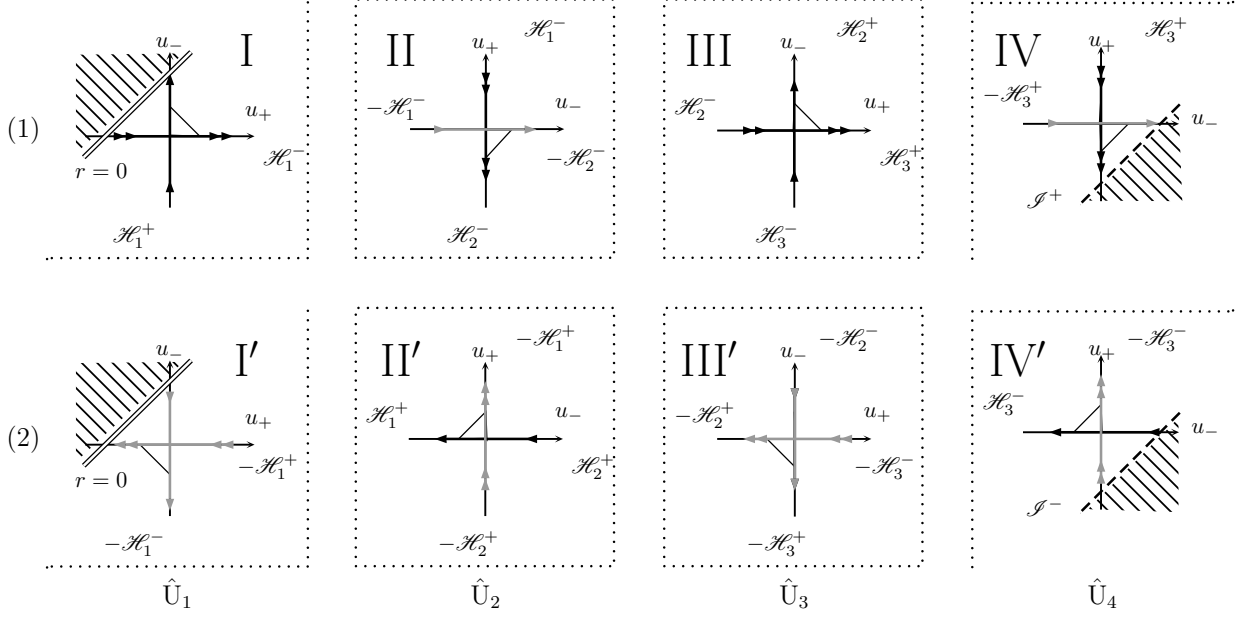


Figure 1.8: Oriented double null coordinates on  $\hat{U}_i$ . In (1), we have time-orientation given by  $\partial_t$  and  $\partial_r$ , while in (2) by  $-\partial_t$  and  $-\partial_r$ . Incoming and outgoing radial null geodesics are integral curves of  $Y^\mp = 2f^{-1}\partial_{u_\pm}$  and of  $-Y^\mp$  (shown in gray). The horizons  $\pm\mathcal{H}_i^\pm$  (dotted lines) are asymptotic to the charts.

This is figure 1.8. The next step is to “glue” the charts along their common (asymptotic) horizons, and since we want to understand these charts as oriented coordinate systems on Eddington-Finkelstein extensions, there is only one way of putting them together, which is shown in figure 1.9.

It is clear that we have left out where the four null hypersurfaces  $\pm\mathcal{H}_i^\pm$  (for the same  $i$ ) meet, this is a sphere  $\mathcal{S}_i$  called the *bifurcation sphere* since the hypersurface  $r = r_i$  bifurcates into four horizons. To see that the missing spheres are regular and the metric can be analytically extended on them, we need to define new coordinates on  $\hat{U}_i$  for which we can identify (glue) the corresponding horizons as regular hypersurfaces and not just asymptotically. On the one hand, to bring the horizons back to finite coordinate values, one choice is exponential functions of the null coordinates  $u_-$  and  $u_+$  with the correct weights, this is the Kruskal-Szekeres choice of coordinates. On the other hand, the metric  $g_{\Omega_0}$ , defined on the two dimensional space  $\{\omega = \omega_0\}$  and locally given by  $f(r)du_-du_+$ , is locally conformally flat as we can see from the double null coordinates expression. The only coordinates transformation that leaves it in such a form, is if one of the new coordinates is function of  $u_-$  only, and the other is function of  $u_+$  only. The simplest of such transformations would be

$$U_{+i} = \beta_+ e^{\alpha_+ u_{+i}} \quad \text{and} \quad U_{-i} = \beta_- e^{\alpha_- u_{-i}},$$

with non zero constant weights  $\alpha_\pm$  and  $\beta_\pm$  (indexed by  $i$  but we drop it for now). We have,

$$dU_{+i} = \alpha_+ U_{+i} du_{+i} \quad ; \quad dU_{-i} = \alpha_- U_{-i} du_{-i},$$

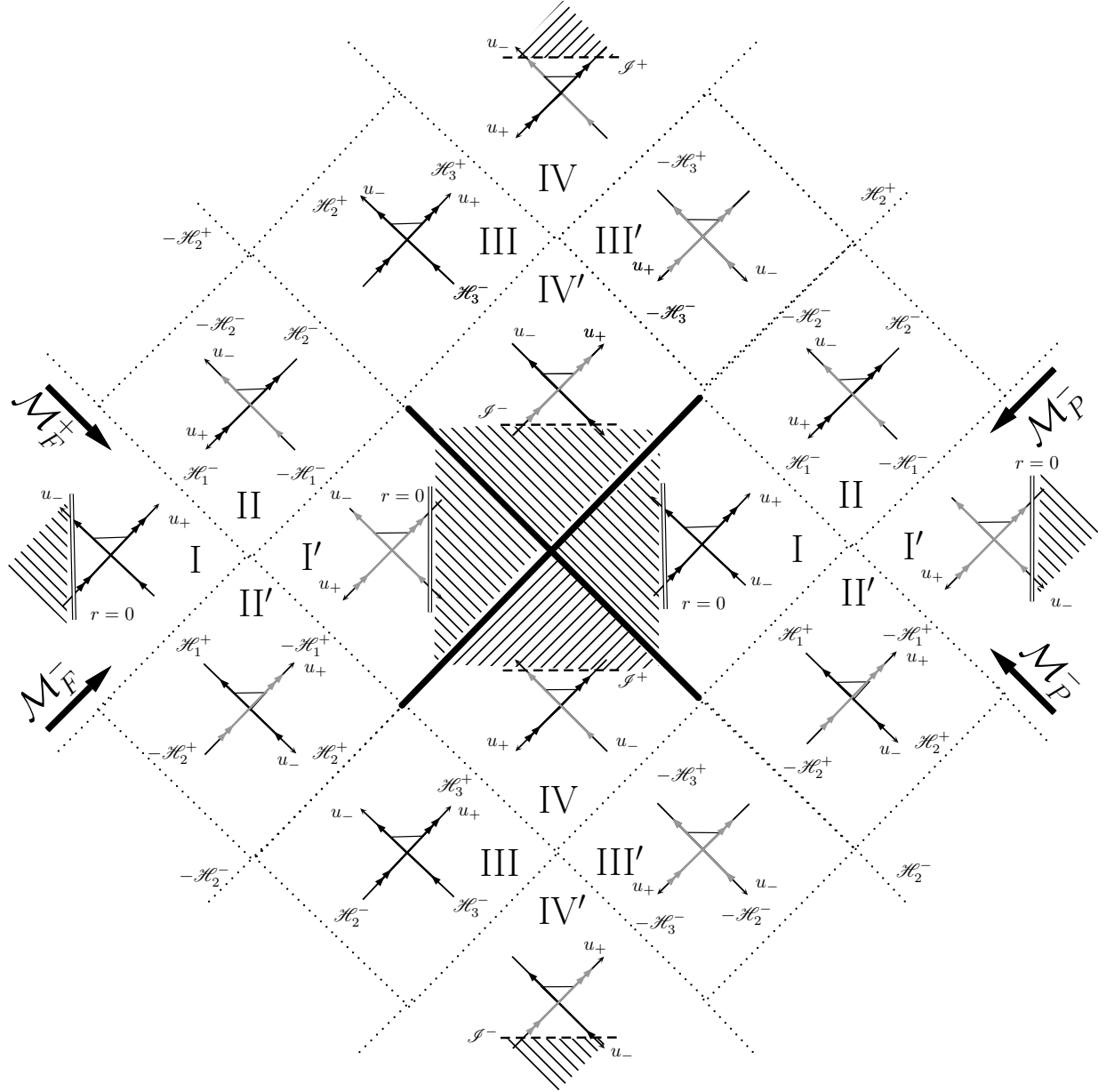


Figure 1.9: *The gluing of double null coordinates along their asymptotic horizons, covering the different Eddington-Finkelstein extensions, with the pattern repeating infinitely. See figure 1.8 for the legend.*

so

$$dU_{+i}dU_{-i} = \alpha_+\alpha_-U_{+i}U_{-i}du_{+i}du_{-i}.$$

Thus, the metric would be

$$g = \frac{f(r)}{\alpha_+\alpha_-U_{+i}U_{-i}}dU_{+i}dU_{-i} - r^2d\omega^2,$$

and we need to express  $r$  in terms of  $U_{+i}$  and  $U_{-i}$ . In fact,

$$U_{+i}U_{-i} = \beta_+\beta_-e^{(\alpha_+-\alpha_-)r_{*i}}e^{(\alpha_++\alpha_-)t}, \quad (26)$$

and in order to define  $r$  as a function of  $(U_{-i}, U_{+i})$  using this relation, we must eliminate the  $t$  variable from the relation, so, we take  $\alpha_+ = -\alpha_- =: \alpha_i$ . This simplifies the above expression,

$$U_{+i}U_{-i} = \beta_+\beta_-e^{2\alpha_i r_{*i}} = \beta_+\beta_-A_i \prod_{j=0}^3 |r - r_j|^{2\alpha_i a_j} =: h_i(r), \quad (27)$$

where  $A_i = e^{2\alpha_i a}$ .  $h_i$  is a bijective function of  $r$  defined on  $I_i$ , since so is  $r_{*i}$ . Thus,  $r(U_{-i}, U_{+i}) = h_i^{-1}(U_{+i}U_{-i})$  is a one-to-one function from  $h_i(I_i)$  onto  $I_i$ , and actually analytic since,

$$\frac{dh_i}{dr}(r) = \beta_+\beta_- \frac{2\alpha_i}{f(r)} e^{2\alpha_i r_{*i}(r)} \neq 0.$$

It follows that

$$g = \frac{f(r)}{-\alpha_i^2 U_{+i}U_{-i}}dU_{+i}dU_{-i} - r^2d\omega^2, \quad (28)$$

is also analytic on the domain

$$U'_i = \{(U_{-i}, U_{+i}) \in \mathbb{R}^2; \beta_+U_{+i} > 0; \beta_-U_{-i} > 0; U_{+i}U_{-i} \in h_i(I_i)\} \times \mathcal{S}_\omega^2.$$

If we want  $(\partial_{U_{-i}}, \partial_{U_{+i}}, \partial_\theta, \partial_\varphi)$  to be positively oriented everywhere, then we are bound to  $\beta_+\beta_- < 0$  on  $U_1$  and  $U_3$ , and  $\beta_+\beta_- > 0$  on  $U_2$  and  $U_4$ . This means that  $h_1, h_3$  are negative and  $h_2, h_4$  are positive. There is no serious restriction in assuming that  $|\beta_\pm| = 1$ , since it is their sign which is interesting to us. Accordingly, we have

$$h_i(r) = (-1)^i e^{2\alpha_i r_{*i}} = (-1)^i A_i \prod_{j=0}^3 |r - r_j|^{2\alpha_i a_j}.$$

As before, we have defined a new coordinate system, we now try to extend its domain of definition, keeping in mind that we wish to assign finite double null coordinates for the horizons. We see from the expression of  $h_i$  that for a good choice of  $\alpha_i$   $h_i$ , and hence the domain of the chart, can be extended analytically to an interval containing a horizon at  $r = r_j$  where  $r_j$  is a boundary point of  $I_i$  different than zero. Thus, the choice of  $\alpha_i$  is self suggesting as  $1/a_j$  for some  $j$ . However, if we use  $1/a_j$ , we shall run into trouble when extending the metric since  $U_{+i}U_{-i} = h_i(r)$  will be zero at  $r = r_j$  and  $h_i$  will contain one more power of  $(r - r_j)$  than  $f$ , so the metric will blow up, we thus take  $\alpha_i = \frac{1}{2a_j}$ .

Therefore, for  $i = 1, 2, 3$ , let  $\alpha_i = \alpha_{i+1} = \frac{1}{2a_i}$ , then  $A_i = A_{i+1}$  and the function

$$H_i(r) = (-1)^i A_i (r_i - r) \prod_{j \neq i, j=0}^3 |r - r_j|^{\frac{a_j}{a_i}} = \begin{cases} h_i(r) & r \in I_i \\ 0 & r = r_i \\ h_{i+1}(r) & r \in I_{i+1} \end{cases},$$

is continuous on  $I_i \cup \{r_i\} \cup I_{i+1}$ . Moreover, since  $f$  has the opposite sign of  $a_i$  over  $I_i$ , and the same sign over  $I_{i+1}$ ,  $H_i$  is monotonic on its domain:

$$\begin{aligned} \frac{dH_i}{dr}|_{I_i} &= \frac{(-1)^i}{a_i f} e^{\frac{1}{a_i} r_{*i}(r)} \\ \frac{dH_i}{dr}(r_i) &= (-1)^{i+1} A_i \prod_{j \neq i, j=0}^3 |r_i - r_j|^{\frac{a_j}{a_i}} \\ \frac{dH_i}{dr}|_{I_{i+1}} &= \frac{(-1)^{i+1}}{a_i f} e^{\frac{1}{a_i} r_{*i+1}(r)}, \end{aligned}$$

so,  $H_1$  and  $H_3$  are increasing, while  $H_2$  is decreasing. Thus,  $H_i$  is an analytic bijection from  $I_i \cup \{r_i\} \cup I_{i+1}$  onto its image, and its inverse is also analytic. To find the domain of the inverse function we take the limits. From the limits of  $r_*$  in (22) we have:

$$-\infty < \lim_{r \rightarrow 0} H_1(r) = H_1(0) = -e^{\frac{b}{a_1}} := B < 0,$$

we also have

$$\lim_{r \rightarrow r_2} H_1(r) = +\infty.$$

Thus,  $H_1 : ]0, r_2[ \rightarrow ]B, +\infty[$ . Similarly,

$$\begin{aligned} \lim_{r \rightarrow r_1} H_2(r) &= +\infty, \\ \lim_{r \rightarrow r_3} H_2(r) &= -\infty, \end{aligned}$$

so,  $H_2 : ]r_1, r_3[ \rightarrow ]-\infty, +\infty[$ . Also,

$$\begin{aligned} \lim_{r \rightarrow r_2} H_3(r) &= -\infty, \\ \lim_{r \rightarrow +\infty} H_3(r) &= e^{\frac{a}{a_3}} =: A > 0, \end{aligned}$$

and  $H_3 : ]r_2, \infty[ \rightarrow ]-\infty, A[$ . Using the  $H_i$ s and the formal expression (28), we can define three Lorentzian manifolds  $(\mathcal{K}_i, g_{\mathcal{K}_i})$  for  $i = 1, 2, 3$ , called the Kruskal-Szekeres extensions, as follows:

$$\mathcal{K}_i = \{(U_{-i}^*, U_{+i}^*) \in \mathbb{R}^2; U_{+i}^* U_{-i}^* \in H_i(I_i \cup \{r_i\} \cup I_{i+1})\} \times \mathcal{S}_\omega^2,$$

and

$$g_{\mathcal{K}_i} = \frac{-4a_i^2 f(r)}{H_i(r)} dU_{+i}^* dU_{-i}^* - r^2 d\omega^2, \quad (29)$$

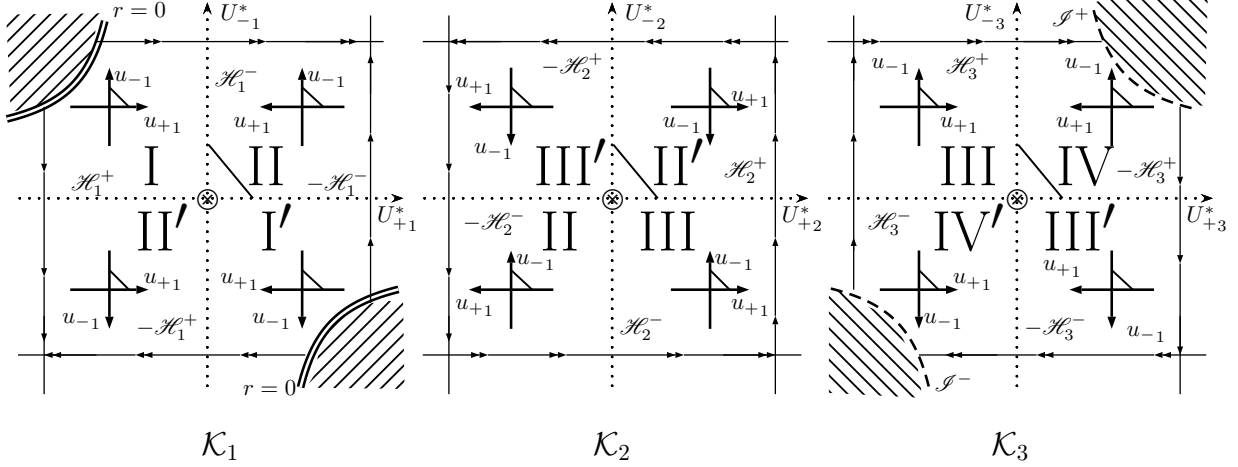


Figure 1.10: The Kruskal-Szekeres extensions  $\mathcal{K}_i$ s with the radial null geodesics  $\gamma^\pm$ .

where  $r(U_{-i}^*, U_{+i}^*) = H_i^{-1}(U_{+i}^* U_{-i}^*)$ . Note that  $g_{\mathcal{K}_i}$  is indeed analytic since the factor  $(r - r_i)$  in  $H_i(r)$  in the denominator is cancelled out by the same factor coming from  $f(r)$ , and thus the metric is regular on  $r = r_i = \{U_{+i}^* U_{-i}^* = 0\}$ , in particular, it is regular on the bifurcation sphere  $(U_{-i}^*, U_{+i}^*) = (0, 0)$ .

To see these manifolds as local extensions of the Eddington-Finkelstein manifolds and of  $\mathcal{M}$ , let us embed the  $U_i$ s in them via the transformation

$$U_{+i}^* = \beta_{+j} e^{\frac{1}{2a_i} u_{+j}} \quad \text{and} \quad U_{-i}^* = \beta_{-j} e^{-\frac{1}{2a_i} u_{-i}} ,$$

for  $j = i, i + 1$ , where  $u_{\pm i} = t \pm r_{*i}$ . If we want the transformation to be orientation preserving with  $U_i$  oriented by  $(\partial_t, \partial_r, \partial_\theta, \partial_\varphi)$  being positively oriented, then as we mentioned above, we must have  $\beta_{+i}\beta_{-i}$  positive for  $i = 2, 4$  and negative for  $i = 1, 3$ . Then, from the definition of  $r(U_{-i}^*, U_{+i}^*)$  and  $H_i(r)$ , we see that two “diagonally opposite quadrants” of  $\mathcal{K}_i$  are each isometric to  $U_i$ , and the other two “quadrants” to  $U_{i+1}$ , and the horizons at  $r = r_i$  corresponds to the “axis” of  $\mathcal{K}_i$ , of course each of these parts of  $\mathcal{K}_i$  is a product with  $\mathcal{S}^2$ . We note also that since  $H_i(r)$  and  $f(r)$  have opposite signs,  $\partial_{U_{-i}^*} + \partial_{U_{+i}^*}$  is timelike on  $\mathcal{K}_i$ . The choice of this vector being future or past oriented is equivalent to fixing the sign of each  $\beta_{\pm j}$ . These choices can be decided alternatively and equivalently by following the geodesics of  $Y^\pm$  guided by figure 1.9, where  $Y^\pm$  are now given in the Kruskal-Szekeres coordinates by

$$Y^\pm = \frac{1}{a_i f(r)} U_{\mp i}^* \partial_{U_{\mp i}^*} .$$

We note that since

$$Y^- = \frac{1}{a_i f(r)} U_{+i}^* \partial_{U_{+i}^*} = \frac{H_i(r)}{a_i f(r) U_{-i}^*} \partial_{U_{+i}^*} ,$$

$Y^-$  is actually defined and smooth on  $\mathcal{K}_i \setminus \{U_{+i}^* = 0\}$ . Similarly for  $Y^+$ . The geodesics along the horizons are given by  $Y_{\mathcal{H}_i}^\mp = \pm \frac{1}{2a_i} \partial_{U_{\pm i}^*}$  on  $U_{\mp i}^* = 0$ . Figure 1.10 summarizes all of this



when  $\partial_{U_{-i}^*} + \partial_{U_{+i}^*}$  is future-oriented. We remark that using this coordinates change we can recover  $t$  as a function of  $(U_{-i}^*, U_{+i}^*)$  through

$$\frac{U_{+i}^*}{U_{-i}^*} = \beta_{+j}\beta_{-j}e^{\frac{t}{2a_i}}. \quad (30)$$

### 1.2.5 The Maximal Extension

The Kruskal-Szekeres extensions can be used as an atlas for the maximal extension. The maximal analytic extension of  $\mathcal{M}$  is a Lorentzian manifold  $\mathcal{M}^*$  covered by an atlas  $\mathfrak{A}^*$  consisting of coordinate charts given by the  $\mathcal{K}_i$ s, and is endowed with the metric  $g^*$  given locally as  $g_{\mathcal{K}_i}$  (or simply  $g$ ). The manifold  $\mathcal{M}^*$  which we shall consider is in fact a Penrose conformal diagram (cross-product the two sphere), in other words, it is (conformal) compactification of the Kruskal-Szekeres extensions after which we identify the corresponding regions. So, equipped with the usual topology, let

$$\mathcal{M}^* = \left( \mathbb{R}^2 \setminus \left( \bigcup_{k,l \in \mathbb{Z}} S_{k,l} \right) \right) \times \mathcal{S}^2,$$

where  $S_{k,l}$  is the square block

$$S_{k,l} = \left\{ (x, y) \in \mathbb{R}^2; \frac{\pi}{2} \leq x\sqrt{2} - 2k\pi \leq 3\frac{\pi}{2}; -\frac{\pi}{2} \leq y\sqrt{2} - 2l\pi \leq \frac{\pi}{2} \right\},$$

and let the atlas be

$$\mathfrak{A}^* = \{(A_{k,l}, \phi_{k,l}), (B_{k,l}, \chi_{k,l}), (C_{k,l}, \psi_{k,l}) ; k, l \in \mathbb{Z}\},$$

with the charts defined as follows: Let  $n = l - k$  and  $m = l + k$ , and set

$$X = \frac{1}{\sqrt{2}}(y + x) \quad ; \quad Y = \frac{1}{\sqrt{2}}(y - x),$$

the opens<sup>1</sup>  $A_{k,l}$ ,  $B_{k,l}$ , and  $C_{k,l}$  are

$$\begin{aligned} A_{k,l} &= \left\{ (x, y) \in \mathbb{R}^2; \tan(X) \tan(Y) > -1 ; -\frac{\pi}{2} < X - m\pi < \frac{\pi}{2} ; -\frac{\pi}{2} < Y - n\pi < \frac{\pi}{2} \right\} \times \mathcal{S}^2 \\ B_{k,l} &= \left\{ (x, y) \in \mathbb{R}^2; -\frac{\pi}{2} < X - \left(m + \frac{1}{2}\right)\pi < \frac{\pi}{2} ; -\frac{\pi}{2} < Y - \left(n + \frac{1}{2}\right)\pi < \frac{\pi}{2} \right\} \times \mathcal{S}^2 \\ C_{k,l} &= \left\{ (x, y) \in \mathbb{R}^2; \tan(X) \tan(Y) < 1 ; \right. \\ &\quad \left. -\frac{\pi}{2} < X - (m + 1)\pi < \frac{\pi}{2} ; -\frac{\pi}{2} < Y - n\pi < \frac{\pi}{2} \right\} \times \mathcal{S}^2, \end{aligned}$$

---

<sup>1</sup>Here, we ignore the fact that the 2-sphere needs multiple charts to cover it.

and the chart bijections are

$$\begin{aligned}\phi_{k,l} : A_{k,l} &\longrightarrow \mathcal{K}_1 ; \\ \chi_{k,l} : B_{k,l} &\longrightarrow \mathcal{K}_2 ; \\ \psi_{k,l} : C_{k,l} &\longrightarrow \mathcal{K}_3 ,\end{aligned}$$

given by

$$\begin{aligned}\phi_{k,l}(x, y, \omega) &= (U_{-1}, U_{+1}, \omega) = (\sqrt{-B} \tan(Y), \sqrt{-B} \tan(X), \omega) ; \\ \chi_{k,l}(x, y, \omega) &= (U_{-2}, U_{+2}, \omega) = \left( \tan \left( Y - \frac{\pi}{2} \right), \tan \left( X - \frac{\pi}{2} \right), \omega \right) ; \\ \psi_{k,l}(x, y, \omega) &= (U_{-3}, U_{+3}, \omega) = (\sqrt{A} \tan(Y), \sqrt{A} \tan(X), \omega) .\end{aligned}$$

Finally, the metric on  $\mathcal{M}^*$  is  $g^*$  whose coordinate expression on each chart domain is given by (29). This extension is the maximal analytic extension of RNdS manifold. It is maximal in the sense that it is locally inextensible. It is also unique if the topology is not changed.

The structure of  $\mathcal{M}^*$  is shown in figure 1.11. First, we note that the metric is analytic and well behaved at all points of  $\mathcal{M}^*$ , including the horizons which are now given by  $U_{+i}U_{-i} = 0$  and the bifurcation spheres  $(U_{+i}, U_{-i}) = (0, 0)$ . The RNdS radius  $r$  is a scalar field on  $\mathcal{M}^*$ , but the same can not be said regarding the time parameter  $t$ , which is given in the different regions through (30) as shown in the diagram, and is not defined on the horizons where it becomes infinite.

$\mathcal{M}^*$  contains infinitely many isometric copies of the original spacetime  $\mathcal{M}$ . Each consists of four regions numbered from one to four in roman, possibly primed or mixed. There are sixteen (infinite) families of these copies, each family corresponds to one of the sixteen different ways of time-orienting  $\mathcal{M}$ . Four of the families correspond to the Eddington-Finkelstein extensions of  $\mathcal{M}$ . Examples of the others along with these four are shown in figure 1.12.

The causal structure of  $\mathcal{M}^*$  can also be seen from figure 1.9. Upon choosing a time orientation on  $\mathcal{M}^*$ , say  $\partial_{U_{-i}^*} + \partial_{U_{+i}^*}$ , then all future directed timelike causal curves in region IV end at  $\mathcal{I}^+$ , and all past directed causal curves end at  $\mathcal{I}^-$ . Unlike Minkowski, Schwarzschild, Reissner-Nordström, or Kerr spacetimes, in RNdS, null infinity or  $\mathcal{I}$  is not a null “hypersurface”, instead it is spacelike due to the De Sitter nature of our spacetime. Using the conformal factor  $\sqrt{|f^{-1}|}$  one can define the metric on this hypersurface, and see that it is indeed spacelike for the conformal metric, but the conformal metric will not be analytic or even smooth on  $\mathcal{M}^*$ . In coordinates,  $\mathcal{I}$  is given by  $U_{+3}U_{-3} = A$  which also corresponds to  $r = \infty$ , and its spacelike nature produces a behaviour near infinity similar to that of a spacelike singularity. Near  $\mathcal{I}^+$ , future-directed causal curves are bound to “go to infinity” once they enter region IV. Of course, unlike the spacelike singularity in Schwarzschild, no observer or light ray can reach infinity in a finite amount of an affine parameter of these null and timelike geodesics, so no geodesic incompleteness is caused by the dynamics of region IV.

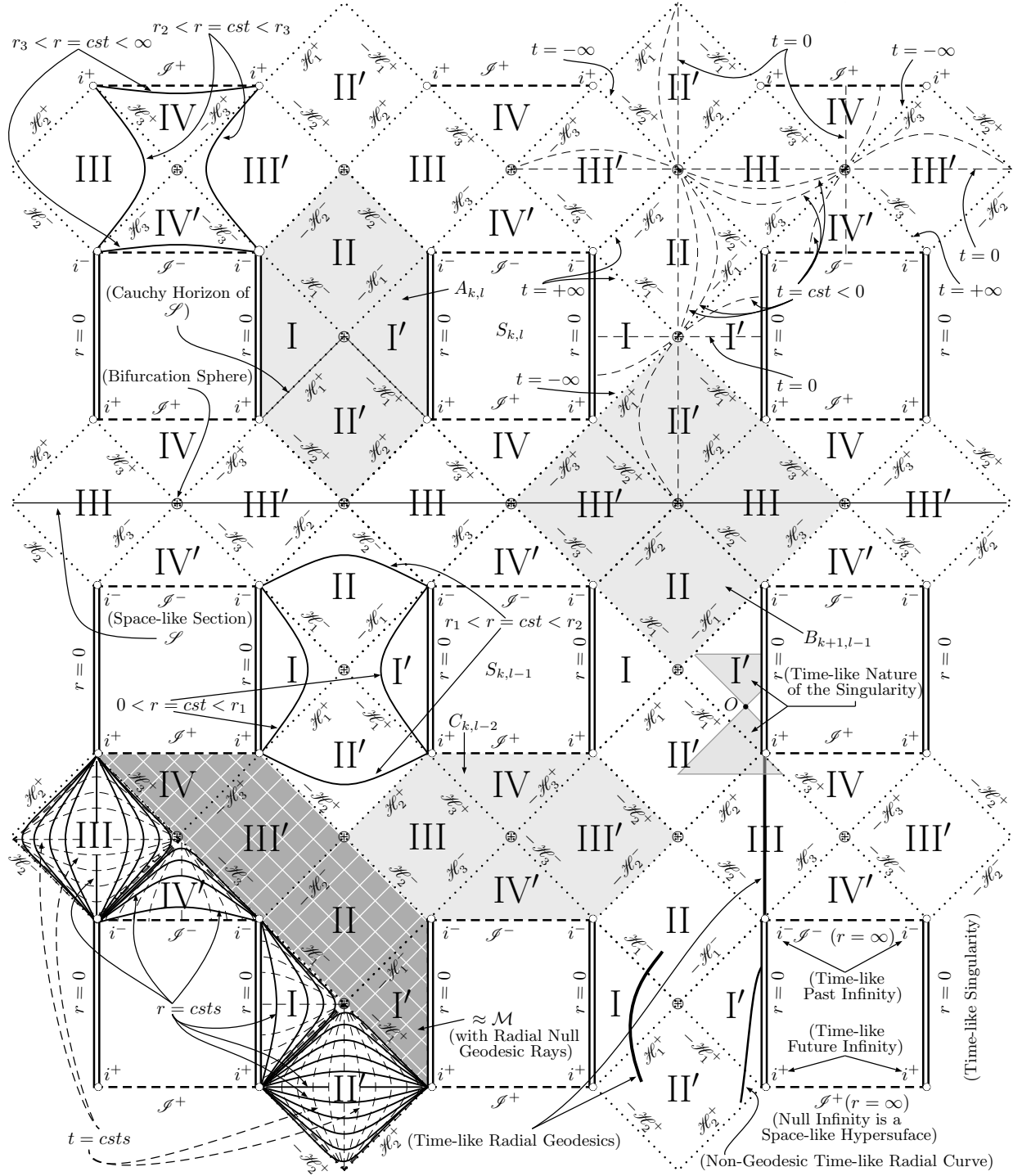


Figure 1.11: The Structure of  $\mathcal{M}^*$ .

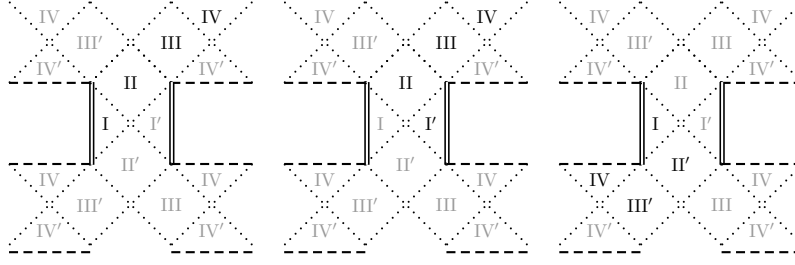


Figure 1.12: *Examples of different time-orientations on  $\mathcal{M}$  as connected subsets of  $\mathcal{M}^*$ .*

The geodesic incompleteness comes from the singularity at  $r = 0$ : Radial null geodesics hit the singularity in a finite amount of their affine parameter, so,  $\mathcal{M}^*$  is geodesically incomplete. However, null and timelike curves can avoid hitting the singularity and go from region II' to region II passing through the “wormhole”. This indicates that the singularity is timelike. Despite geodesic incompleteness, the spacetime is timelike geodesically complete as the singularity is repulsive, due to the Reissner-Nordström nature of our spacetime, To see why, consider for simplicity radial timelike geodesics: A radial curve  $\gamma(\tau) = (t(\tau), r(\tau), \omega_0)$ , for some constant angular coordinates, is geodesic if

$$\ddot{t} = -\dot{t}\dot{r}\frac{f'}{f}; \quad (31)$$

$$\ddot{r} = -\frac{f'}{2} \left( f\dot{t}^2 + \frac{\dot{r}^2}{f} \right), \quad (32)$$

assuming  $f \neq 0$  which is the case near  $r = 0$ , and where dot denote differentiating with respect to  $\tau$ . In addition, we have  $g(\dot{\gamma}, \dot{\gamma}) = \text{constant} = E > 0$  i.e.

$$f\dot{t}^2 - \frac{\dot{r}^2}{f} = E, \quad (33)$$

So, (32) becomes

$$-2\ddot{r} = f'E.$$

If we multiple both sides of this equality by  $\dot{r}$  then integrate in  $\tau$  we obtain

$$\dot{r}^2 + fE = C,$$

where  $C$  is the integration constant. We see from (33) that  $C = f^2\dot{t}^2$  and hence  $C \geq 0$ . Thus,

$$\dot{r}^2 = C - fE, \quad (34)$$

but  $f > 0$  on  $0 < r < r_1$  and in fact  $f \rightarrow +\infty$  as  $r \rightarrow 0$ , which puts a constraint on  $r$  preventing it from reaching zero. This means that there must be a turning point in the curve  $\gamma$  after which  $r$  starts to increase again. So, even objects in free-fall directly i.e. radially towards the singularity get ejected to the other region II. Therefore, no timelike geodesic can hit the singularity. The timelike nature of the singularity also means that there are

points in the spacetime whose both future and past null cones meet the singularity inside the same region I. Another consequence of this nature is the absence of a (global) Cauchy hypersurface, as there are inextendible timelike curves of arbitrarily small length which start and end at the singularity. For instance, the spacelike hypersurface  $\mathcal{S}$  in figure 1.11 is a Cauchy hypersurface for regions covered by the domains  $B_{k,l-1}$  and  $C_{k,l-1}$  for all  $k \in \mathbb{Z}$ . Yet, there are future-directed and past-directed inextendible timelike curves of  $\mathcal{M}^*$  which do not intersect  $\mathcal{S}$ . Such curves hit the singularity inside region I and never cross the horizons at  $r = r_1$  towards  $\mathcal{S}$ , therefore, data in regions I do not depend on data at  $\mathcal{S}$ . The hypersurfaces  $-\mathcal{H}_1^- \cup \mathcal{H}_1^-$  and  $-\mathcal{H}_1^+ \cup \mathcal{H}_1^+$  bounding regions II and II' in  $B_{k,l-1}$  (for all  $k$ ) are said to be Cauchy horizons for the spacelike section  $\mathcal{S}$  (see [74]).

Each point of the diagram shown in figure 1.11 is a 2-sphere of  $\mathcal{M}^*$ , or of  $\mathcal{S}^\pm$  which are conformal spacelike 3-hypersurfaces. The segments labelled by  $r = 0$  where spherical coordinates are singular, are 1-dimensional lines of singular points (of the metric) representing the center of the black hole at different times. We note that the singularity does not touchy null infinity in reality. The corners of the removed squares  $S_{k,l}$ s, labelled by  $i^+$  and  $i^-$  and called future and past timelike infinities respectively, are distinct from the segments of  $r = 0$  because there are plenty of inextendible timelike curves that do not hit the singularities. For example, the timelike curves of constant  $r$  in regions III ( $r_2 < r < r_3$ ) of the form  $\gamma(\tau) = (\tau, C, \omega_0)$  for  $\tau \in \mathbb{R}$  in the  $(t, r, \omega)$ -coordinates never approach the singularity, and one of them is a geodesic, namely when  $f'(C) = 0$ . These future-directed timelike curves originate at  $i^-$  and finish at  $i^+$ . We also note that there are extendible timelike curves that have no end points in the closure of  $\mathcal{M}^*$  such as those given locally by  $t = 0$  in regions II and II'.

# Chapter 2

## Maxwell Fields on the Exterior Static Region

### 2.0 Introduction

In this chapter we introduce *Maxwell's equations* and fields on the Reissner-Nordstrøm-de Sitter manifold, and we prove some preliminary facts about the Maxwell's system which will be used in the subsequential chapters.

One of the most celebrated set of equations in classical physics is the Maxwell's system that describes the phenomena of electromagnetism. In vacuum the famous equations are:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}\end{aligned}$$

Here,  $\mathbf{E}$  is the electric vector field and  $\mathbf{B}$  is the magnetic vector field, and  $c$  is the speed of light in vacuum. The divergence operator is  $\nabla \cdot$  and the curl operator is  $\nabla \times$ .

In the notation of differential forms, the Maxwell field is a 2-form  $F$  on the spacetime, and Maxwell's equations are:

$$dF = 0 \quad ; \quad d \star F = 0$$

where  $d$  is the exterior differentiation and  $\star$  is the Hodge star operator. By projecting  $F$  onto a spacetime frame one retrieves the electric and magnetic fields in that frame.

In this work, we deal with test Maxwell fields on a fixed Reissner-Nordstrøm-de Sitter background. That is to say, given a fixed Maxwell field that determines the energy-momentum tensor in Einstein's equations with a cosmological constant, a solution of these

coupled Einstein-Maxwell equations is then a Reissner-Nordström-de Sitter black hole spacetime which we fix. On this fixed spacetime we study solutions to Maxwell equations without taking into account the effect of their presence on the energy-momentum tensor of Einstein's equations. This is described as a *decoupled system*.

In this chapter, we want to study the Maxwell field on the *static exterior* of the Reissner-Nordström-de Sitter black hole, which we denote by

$$\mathcal{N} = \mathbb{R}_t \times ]r_2, r_3[_r \times \mathcal{S}_\omega^2 = \mathbb{R}_t \times \mathbb{R}_{r_*} \times \mathcal{S}_\omega^2. \quad (35)$$

This is done as follows:

**Section 2.1:** After introducing the equations, we rewrite them first in tetrad formalism using a null tetrad. From that we reformulate the equations as an evolution system of three equations and a spatial constraint equation. We then reduce the dimensionality of the problem to 2-dimensions by decomposing on spin-weighted spherical harmonics.

**Section 2.2:** We define the energy of a Maxwell field and study the Cauchy problem of Maxwell's equations using two approaches: first by directly showing the well-posedness of the evolution equations as a symmetric hyperbolic system which preserves the constraint; the second approach is to show that the Cauchy problem determined by the evolution system for smooth, compactly supported, and constrained data is equivalent to a Cauchy problem for a system of wave equations.

**Section 2.3:** Here, we show that the pure charge solutions, i.e. which have a  $l = 0$ -mode in the spin-weighted spherical harmonics, are the only time-periodic solutions with finite energy, which is what we call stationary solutions. These solutions of Maxwell's equations are excluded since they do not decay.

**Section 2.4:** In this section we discuss Maxwell potentials and the Lagrangian formulation of electromagnetism using potentials. We also use the potential formulation of Maxwell's equations to show that smooth compactly supported data are dense in the constraint subspace of the finite energy space.

**Notation.** We shall use, as much as possible, the following notation conventions: We denote spin components by a capital greek letter (usually  $\Phi_i$ ) and bold the letter to indicate the vector defined by the components (like  $\mathbf{\Phi} = (\Phi_1, \Phi_0, \Phi_{-1})$ ). Similarly we shall denote data defined on the initial hypersurface by a small greek letter (as  $\phi_i$ ) and bold it to say that it is a vector ( $\boldsymbol{\phi}$ ), moreover, we use the same letter capitalized for the associated solution ( $\mathbf{\Phi}$ ). Likewise for harmonic coefficients (usually by the letter  $\Psi_{mn}^l$ ).

## 2.1 The Equations

We recall from chapter 1 that, for  $r > 0$ , we have defined the Regge-Wheeler coordinate function  $r_*$  by

$$r_*(r) = \sum_{i=0}^3 a_i \ln |r - r_i| + a \quad ; \quad a_i = -\frac{r_i^2}{\Lambda} \prod_{j \neq i} \frac{1}{(r_i - r_j)} \quad ; \quad a = -\sum_{i=0}^3 a_i \ln |P_2 - r_i|$$

where  $\{r = P_2\}$  is the photon sphere hypersurface.

We now introduce the chart  $(t, r_*, \theta, \varphi)$  over  $\mathcal{N}$ , where  $r_* \in \mathbb{R}_{r_*}$  is By (20), or equivalently

$$\frac{dr_*}{dr} = \frac{1}{f(r)} > 0,$$

we see that  $r_*$  is a strictly increasing continuous function of  $r$  (thus a bijection) over the interval  $]r_2, r_3[$ , and ranges from  $-\infty$  to  $+\infty$ . We also have  $\partial_{r_*} = f \partial_r$  and  $dr = f dr_*$ . The RNdS metric in these coordinates is:

$$g_{\mathcal{N}} = f(r)(dt^2 - dr_*^2) - r^2 d\omega^2. \quad (36)$$

It will be useful for us in calculations to have the Christoffel symbols in the coordinates  $(t, r_*, \theta, \varphi) = (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ . The non zero symbols are:

$$\begin{aligned} \tilde{\Gamma}_{01}^0 = \tilde{\Gamma}_{00}^1 = \tilde{\Gamma}_{11}^1 &= \frac{f'}{2} \quad ; \quad \tilde{\Gamma}_{22}^1 = -r \quad ; \quad \tilde{\Gamma}_{33}^1 = -r \sin(\theta)^2 \\ \tilde{\Gamma}_{13}^3 = \tilde{\Gamma}_{12}^2 &= \frac{f}{r} \quad ; \quad \tilde{\Gamma}_{33}^2 = -\cos(\theta) \sin(\theta) \quad ; \quad \tilde{\Gamma}_{23}^3 = \cot(\theta). \end{aligned} \quad (37)$$

Let  $F$  be a 2-form on the RNdS manifold  $\mathcal{N}$ . As we saw, the source free *Maxwell's equations* can be written as

$$\delta F = 0, \quad (38)$$

$$dF = 0, \quad (39)$$

where  $\delta = \star d \star$ , and  $\star$  is the Hodge star operator, or in abstract index notation,

$$\nabla^a F_{ab} = 0, \quad (40)$$

$$\nabla_{[a} F_{bc]} = 0, \quad (41)$$

and in coordinate form these translate to the following two sets of equations,

$$g^{ac} (\partial_c F_{ab} - F_{db} \Gamma_{ca}^d - F_{ad} \Gamma_{cb}^d) = 0 \quad \forall b, \quad (42)$$

$$\partial_c F_{ab} + \partial_b F_{ca} + \partial_a F_{bc} = 0 \quad \forall a, b, c. \quad (43)$$



If taken in the coordinates  $(t, r_*, \theta, \varphi) = (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ , (42) becomes respectively for  $b = 0, \dots, 3$ :

$$\partial_1 F_{10} + V \partial_2 F_{20} + V \sin(\theta)^{-2} \partial_3 F_{30} + 2rV F_{10} + V F_{20} \cot(\theta) + f' F_{01} = 0, \quad (44)$$

$$\partial_0 F_{10} + V \partial_2 F_{21} + V \sin(\theta)^{-2} \partial_3 F_{31} + V F_{21} \cot(\theta) = 0, \quad (45)$$

$$\partial_1 F_{12} + \partial_0 F_{20} + V \sin(\theta)^{-2} \partial_3 F_{32} = 0, \quad (46)$$

$$\partial_1 F_{13} + \partial_0 F_{30} + V \partial_2 F_{23} + V F_{32} \cot(\theta) = 0. \quad (47)$$

where  $V = fr^{-2}$ .

As much as equations (38)-(41) are elegant and simple they are not the most convenient form for us to use in all arguments and calculations, and evidently neither are their expressions in coordinates. Thence, we need to look at the equations from different angles and in doing so explore the appropriate tools that will prove useful in diverse occasions. These are the tetrad formalism, the evolution system and the constraint, and the spin-weighted spherical harmonics.

We shall assume for the moment that we only deal with smooth solutions so that regularity is not an issue for now. We shall worry about it when we study the existence of solutions of the Cauchy problem in section 2.2.

### 2.1.1 Tetrad Formalism

Instead of working with the components of the Maxwell field in a coordinate basis, it is more convenient to use the components of the field in a general basis of the tangent space which might not be the canonical basis given by the coordinates. At each point, one defines a set of four vectors, called the tetrad, that forms a basis for the tangent space at that point. One can then reformulate the field equations using this tetrad. In general relativity, it is natural to project on a null tetrad, which consists of two real null vectors and two conjugate null complex vectors usually defined as  $X \pm iY$  for  $X$  and  $Y$  two spacelike real vectors. Here, we use a null tetrad  $\mathcal{N}$  given by two null real vectors and a two conjugate null complex vector tangent to the 2-Sphere  $\mathcal{S}^2$ :

$$\begin{aligned} L &= \partial_t + \partial_{r_*} \\ N &= \partial_t - \partial_{r_*} \\ M &= \partial_\theta + \frac{i}{\sin(\theta)} \partial_\varphi \\ \bar{M} &= \partial_\theta - \frac{i}{\sin(\theta)} \partial_\varphi \end{aligned} \quad (48)$$

We shall call this tetrad the “*stationary tetrad*”. Using this tetrad, we can represent the Maxwell field by three complex scalar functions called the *spin components*<sup>1</sup> of the Maxwell

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<sup>1</sup>For the name “spin”, see appendix B.

field associated to the given tetrad, and defined by:

$$\begin{aligned}\Phi_1 &= F(L, M) \\ \Phi_0 &= \frac{1}{2} (V^{-1} F(L, N) + F(\bar{M}, M)) \\ \Phi_{-1} &= F(N, \bar{M})\end{aligned}\tag{49}$$

The conventional definition of spin components of an anti-symmetric tensor is slightly different. One normally defines it without the extra factor of  $V^{-1}$  in the middle component  $\Phi_0$ . However the way we project the field on the tetrad is more convenient for studying decay in the region  $\mathcal{N}$ . Our conventions are the same as P. Blue's [22]. Also the usual way to label the components is different, conventionally, they are indexed by 0, 1, and 2 (see footnote 1 page 67). For more on these notations see Appendix B.

Note that the tetrad we use, unlike those in the *Newman-Penrose formalism*, are not normalized: A *normalized tetrad* is such that the inner product of the two null real vectors of the tetrad with each other equals 1, and the product of the null complex vector with its conjugate is  $-1$ , while all other products are zero. The formalism we use is a form of Geroch-Held-Penrose formalism (GHP), which does not require normalization. The form of Maxwell's equations in this formalism is usually referred to as Maxwell's *compact equations* [125].

In this framework Maxwell's equations translate as follows.

**Lemma 4** (Maxwell Compacted Equations).  *$F$  satisfies Maxwell's equations (38) and (39) if and only if its spin components  $(\Phi_1, \Phi_0, \Phi_{-1})$  in the stationary tetrad satisfy the compacted equations*

$$N\Phi_1 = VM\Phi_0,\tag{50}$$

$$L\Phi_0 = \bar{M}_1\Phi_1,\tag{51}$$

$$N\Phi_0 = -M_1\Phi_{-1},\tag{52}$$

$$L\Phi_{-1} = -V\bar{M}\Phi_0.\tag{53}$$

where  $M_1 = M + \cot(\theta)$  and  $\bar{M}_1$  is its conjugate.

*Proof.* First, let us express  $\Phi_{-1}, \Phi_0, \Phi_1$  in terms of the components of  $F$  in the coordinates  $(t, r_*, \theta, \varphi) = (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ . We have,

$$\begin{aligned}F(L, M) &= F_{02} + \frac{i}{\sin(\theta)} F_{03} + F_{12} + \frac{i}{\sin(\theta)} F_{13}, \\ F(L, N) &= 2F_{10}, \\ F(\bar{M}, M) &= \frac{2i}{\sin(\theta)} F_{23}, \\ F(N, \bar{M}) &= F_{02} + \frac{i}{\sin(\theta)} F_{30} + F_{21} + \frac{i}{\sin(\theta)} F_{13},\end{aligned}$$

thus,

$$\begin{aligned}
\Phi_1 &= F_{02} + \frac{i}{\sin(\theta)} F_{03} + F_{12} + \frac{i}{\sin(\theta)} F_{13}, \\
\Phi_0 &= V^{-1} F_{10} + \frac{i}{\sin(\theta)} F_{23}, \\
\Phi_{-1} &= F_{02} + \frac{i}{\sin(\theta)} F_{30} + F_{21} + \frac{i}{\sin(\theta)} F_{13}.
\end{aligned} \tag{54}$$

Therefore,

$$\begin{aligned}
N\Phi_1 &= (\partial_t - \partial_{r_*}) \left( F_{02} + \frac{i}{\sin(\theta)} F_{03} + F_{12} + \frac{i}{\sin(\theta)} F_{13} \right) \\
&= \partial_0 F_{02} + \frac{i}{\sin(\theta)} \partial_0 F_{03} + \partial_0 F_{12} + \frac{i}{\sin(\theta)} \partial_0 F_{13} \\
&\quad + \partial_1 F_{20} + \frac{i}{\sin(\theta)} \partial_1 F_{30} + \partial_1 F_{21} + \frac{i}{\sin(\theta)} \partial_1 F_{31},
\end{aligned} \tag{55}$$

and

$$\begin{aligned}
VM\Phi_0 &= V \left( \partial_\theta + \frac{i}{\sin(\theta)} \partial_\varphi \right) \left( V^{-1} F_{10} + \frac{i}{\sin(\theta)} F_{23} \right) \\
&= \partial_2 F_{10} + \frac{i}{\sin(\theta)} \partial_3 F_{10} + \frac{iV}{\sin(\theta)} F_{32} \cot(\theta) \\
&\quad + \frac{iV}{\sin(\theta)} \partial_2 F_{23} + \frac{V}{\sin(\theta)^2} \partial_3 F_{32}.
\end{aligned} \tag{56}$$

From (43) we have,

$$\begin{aligned}
\partial_2 F_{10} &= \partial_0 F_{12} + \partial_1 F_{20}, \\
\partial_3 F_{10} &= \partial_0 F_{13} + \partial_1 F_{30},
\end{aligned}$$

so using these two equations, (46), and (47) to compare the terms of (55) and (56), we see that (50) holds.

We also have,

$$\begin{aligned}
L\Phi_0 &= (\partial_t + \partial_{r_*}) \left( V^{-1} F_{10} + \frac{i}{\sin(\theta)} F_{23} \right) \\
&= V^{-1} \partial_0 F_{10} + \frac{i}{\sin(\theta)} \partial_0 F_{23} + \frac{i}{\sin(\theta)} \partial_1 F_{23} \\
&\quad + V^{-1} \partial_1 F_{10} + 2r F_{10} - V^{-1} f' F_{10},
\end{aligned} \tag{57}$$

and

$$\begin{aligned}
\bar{M}_1 \Phi_1 &= \left( \partial_\theta - \frac{i}{\sin(\theta)} \partial_\varphi + \cot(\theta) \right) \left( F_{02} + \frac{i}{\sin(\theta)} F_{03} + F_{12} + \frac{i}{\sin(\theta)} F_{13} \right) \\
&= \partial_2 F_{02} + \partial_2 F_{12} + \frac{i}{\sin(\theta)} \partial_2 F_{03} + \frac{i}{\sin(\theta)} \partial_2 F_{13} \\
&\quad + \frac{i}{\sin(\theta)} \partial_3 F_{20} + \frac{1}{\sin(\theta)^2} \partial_3 F_{03} + \frac{i}{\sin(\theta)} \partial_3 F_{21} \\
&\quad + \frac{1}{\sin(\theta)^2} \partial_3 F_{13} + \cot(\theta) F_{02} + \cot(\theta) F_{12}.
\end{aligned} \tag{58}$$

Using,

$$\begin{aligned}
\partial_0 F_{23} &= \partial_3 F_{20} + \partial_2 F_{03} \\
\partial_1 F_{23} &= \partial_3 F_{21} + \partial_2 F_{13},
\end{aligned}$$

which follows from (43), in addition to (44) and (45), one can see that the right hand sides of (57) and (58) are equal, showing that (51) holds.

Next we will prove (52). For this we calculate,

$$\begin{aligned}
N \Phi_0 &= V^{-1} \partial_0 F_{10} + \frac{i}{\sin(\theta)} \partial_0 F_{23} + \frac{i}{\sin(\theta)} \partial_1 F_{32} \\
&\quad + V^{-1} \partial_1 F_{01} + 2r F_{01} + V^{-1} f' F_{10}.
\end{aligned} \tag{59}$$

$$\begin{aligned}
-M_1 \Phi_{-1} &= \left( \partial_\theta + \frac{i}{\sin(\theta)} \partial_\varphi - \cot(\theta) \right) \left( F_{02} + \frac{i}{\sin(\theta)} F_{30} + F_{21} + \frac{i}{\sin(\theta)} F_{13} \right) \\
&= \partial_2 F_{20} + \frac{i}{\sin(\theta)} \partial_2 F_{03} + \partial_2 F_{12} + \frac{i}{\sin(\theta)} \partial_2 F_{31} \\
&\quad + \frac{i}{\sin(\theta)} \partial_3 F_{20} + \frac{1}{\sin(\theta)^2} \partial_3 F_{30} + \frac{i}{\sin(\theta)} \partial_3 F_{12} \\
&\quad + \frac{1}{\sin(\theta)^2} \partial_3 F_{13} + \cot(\theta) F_{20} + \cot(\theta) F_{12}.
\end{aligned} \tag{60}$$

From (43) we get

$$\begin{aligned}
\partial_0 F_{23} &= \partial_3 F_{20} + \partial_2 F_{03} \\
\partial_1 F_{32} &= \partial_2 F_{31} + \partial_3 F_{12},
\end{aligned}$$

By these, (44), and (45), the required equality of (60) and (59) follows.

Finally,

$$\begin{aligned}
L \Phi_{-1} &= (\partial_t + \partial_{r_*}) \left( F_{02} + \frac{i}{\sin(\theta)} F_{30} + F_{21} + \frac{i}{\sin(\theta)} F_{13} \right) \\
&= \partial_0 F_{02} + \frac{i}{\sin(\theta)} \partial_0 F_{30} + \partial_0 F_{21} + \frac{i}{\sin(\theta)} \partial_0 F_{13} \\
&\quad + \partial_1 F_{02} + \frac{i}{\sin(\theta)} \partial_1 F_{30} + \partial_1 F_{21} + \frac{i}{\sin(\theta)} \partial_1 F_{13},
\end{aligned} \tag{61}$$

and

$$\begin{aligned}
-V\bar{M}\Phi_0 &= -V\left(\partial_\theta - \frac{i}{\sin(\theta)}\partial_\varphi\right)\left(V^{-1}F_{10} + \frac{i}{\sin(\theta)}F_{23}\right) \\
&= \partial_2 F_{01} + \frac{iV}{\sin(\theta)}\cot(\theta)F_{23} + \frac{iV}{\sin(\theta)}\partial_2 F_{32} \\
&\quad + \frac{iV}{\sin(\theta)}\partial_2 F_{32} + \frac{i}{\sin(\theta)}\partial_3 F_{10} + \frac{V}{\sin(\theta)^2}\partial_3 F_{32}.
\end{aligned} \tag{62}$$

Again (43) gives

$$\begin{aligned}
\partial_2 F_{01} &= \partial_0 F_{21} + \partial_1 F_{02} \\
\partial_3 F_{10} &= \partial_1 F_{30} + \partial_0 F_{13}
\end{aligned}$$

which combined with (46) and (47) shows that (61) and (62) are equal, and (53) is proved.  $\square$

### 2.1.2 3+1 Decomposition: Evolution and Constraint

The compacted equations in Lemma 4 can be split into an evolution system and a constraint equation,

$$\begin{aligned}
\partial_t \Phi_1 &= \partial_{r_*} \Phi_1 + VM\Phi_0, \\
\partial_t \Phi_0 &= \bar{M}_1 \Phi_1 - \partial_{r_*} \Phi_0, \\
\partial_t \Phi_0 &= -M_1 \Phi_{-1} + \partial_{r_*} \Phi_0, \\
\partial_t \Phi_{-1} &= -\partial_{r_*} \Phi_{-1} - V\bar{M}\Phi_0.
\end{aligned}$$

Adding the two middle equations we get three evolution equations governing the components,

$$\partial_t \Phi_1 = \partial_{r_*} \Phi_1 + VM\Phi_0, \tag{63}$$

$$\partial_t \Phi_0 = \frac{1}{2}(\bar{M}_1 \Phi_1 - M_1 \Phi_{-1}), \tag{64}$$

$$\partial_t \Phi_{-1} = -\partial_{r_*} \Phi_{-1} - V\bar{M}\Phi_0, \tag{65}$$

and by subtracting instead, we get a constraint equation which involves spatial derivatives only,

$$\partial_{r_*} \Phi_0 = \frac{1}{2}(\bar{M}_1 \Phi_1 + M_1 \Phi_{-1}). \tag{66}$$

For the evolution system we define the Maxwell Hamiltonian  $H$  by:

$$H = -i \begin{pmatrix} \partial_{r_*} & VM & 0 \\ \frac{1}{2}\bar{M}_1 & 0 & -\frac{1}{2}M_1 \\ 0 & -V\bar{M} & -\partial_{r_*} \end{pmatrix}, \tag{67}$$

and so, the system can be expressed as:

$$\partial_t \Phi = iH\Phi, \quad \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_0 \\ \Phi_{-1} \end{pmatrix}.$$

We define the *evolution operator*  $E = \partial_t - iH$  and the *constraint operator*  $D$  as

$$D = \begin{pmatrix} \frac{1}{2}\bar{M}_1 & -\partial_{r_*} & \frac{1}{2}M_1 \end{pmatrix}.$$

Therefore, Maxwell's system is equivalent to

$$D\Phi = 0 \quad \text{and} \quad E\Phi = 0.$$

We shall discuss the properties of the evolution system in more details when we solve the Cauchy problem.

### 2.1.3 Spin-Weighted Spherical Harmonics

To further simplify Maxwell's equations we use a basis of *spin-weighted spherical harmonics* of  $L^2(\mathcal{S}^2)$  to decompose the functions  $(\Phi_1, \Phi_0, \Phi_{-1})(t, r_*)$  on  $\mathcal{S}^2$  and subsequently reduces the equations to 2-dimensions. These will be helpful in showing that the Cauchy problem is well-posed, and for excluding the non-decaying solutions. These spherical functions form an orthonormal basis, denoted by  $\{W_{mn}^l(\theta, \varphi); l, m, n \in \mathbb{Z}; l \geq 0, -l \leq m, n \leq l\}$ , satisfying:

$$W_{mn}^l(\theta, \varphi) = e^{-in\varphi} u_{mn}^l(\theta) \in L^2(\mathcal{S}^2), \quad (68)$$

$$\int_0^\pi |u_{mn}^l(\theta)|^2 \sin(\theta) d\theta = \frac{1}{2\pi}, \quad (69)$$

$$\frac{d^2 u_{mn}^l}{d\theta^2} + \cot(\theta) \frac{du_{mn}^l}{d\theta} + \left( l(l+1) - \frac{n^2 - 2mn \cos(\theta) + m^2}{\sin(\theta)^2} \right) u_{mn}^l = 0.$$

In addition, the functions  $u_{mn}^l$  satisfy:

$$\frac{du_{mn}^l}{d\theta} - \frac{m - n \cos(\theta)}{\sin(\theta)} u_{mn}^l = -i \sqrt{(l+n)(l-n+1)} u_{m,n-1}^l \quad (70)$$

$$\frac{du_{mn}^l}{d\theta} + \frac{m - n \cos(\theta)}{\sin(\theta)} u_{mn}^l = -i \sqrt{(l+n+1)(l-n)} u_{m,n+1}^l \quad (71)$$

$$\frac{du_{mn}^l}{d\theta} - \frac{n - m \cos(\theta)}{\sin(\theta)} u_{mn}^l = -i \sqrt{(l+m)(l-m+1)} u_{m-1,n}^l \quad (72)$$

$$\frac{du_{mn}^l}{d\theta} + \frac{n - m \cos(\theta)}{\sin(\theta)} u_{mn}^l = -i \sqrt{(l+m+1)(l-m)} u_{m+1,n}^l \quad (73)$$

For more on the spin-weighted spherical harmonics the reader may refer to [70, 125, 126], and [7, 119] for usage in contexts similar to ours.

Assuming that the function  $\Phi_m(t, r_*)$  of spin-weight<sup>1</sup>  $m$  is in  $L^2(\mathcal{S}^2)$ , then the angular variables can be separated from the temporal and the radial variables, since the function can

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<sup>1</sup>The conformal weight and the spin weight are respectively related to the way the component change when we rescale the complex vector of the tetrad by a complex constant and the conjugate vector by the conjugate constant, and when rescaling the first null vector of the tetrad by a real constant and the second by the inverse constant. More precisely, the components transform as powers of the real rescaling constant, the power being the index of the component.

be expanded on the basis  $\{W_{mn}^l\}$  as follows:

$$\Phi_m(t, r_*, \theta, \varphi) = \sum_{l=|m|}^{+\infty} \sum_{n=-l}^l \Psi_{mn}^l(t, r_*) W_{mn}^l(\theta, \varphi). \quad (74)$$

with

$$\Psi_{mn}^l(t, r_*) = \int_{S^2} \Phi_m(t, r_*, \theta, \varphi) \bar{W}_{mn}^l(\theta, \varphi) d^2\omega. \quad (75)$$

From the way  $W_{mn}^l$  are constructed, it is conventional to assume that if  $m > l$  or  $m < -l$  then  $W_{mn}^l = 0$ . The same holds for  $n$ . Thus, the above summation can be taken over  $l = 0, \dots, +\infty$  and still means exactly the same thing. spin-weighted spherical harmonics we can reduce the dimensionality of Maxwell's equations, this is the next lemma.

**Lemma 5.** *The system*

$$\begin{cases} D\Phi = 0 \\ E\Phi = 0 \end{cases}$$

*reduces to: For all  $l > 0$ ,  $-l \leq n \leq l$ , and  $\alpha = i\sqrt{l(l+1)}$ ,*

$$i\partial_t \Psi_1 = \partial_{r_*} \Psi_1 - \alpha V \Psi_0, \quad (76)$$

$$i\partial_t \Psi_0 = \frac{\alpha}{2} (\Psi_{-1} - \Psi_1), \quad (77)$$

$$i\partial_t \Psi_{-1} = -\partial_{r_*} \Psi_{-1} + \alpha V \Psi_0, \quad (78)$$

$$\partial_{r_*} \Psi_0 = -\frac{\alpha}{2} (\Psi_{-1} + \Psi_1), \quad (79)$$

where we have, for simplicity, dropped the indices  $l$  and  $n$  and put  $\Psi_m = \Psi_{mn}^l$ .

In addition, for  $l = 0$ ,  $i\partial_t \Psi_{00}^0 = 0$  and  $\partial_{r_*} \Psi_{00}^0 = 0$ , i.e.  $\Psi_{00}^0$  is a constant.

*Proof.* To get the desired equations we need to calculate

$$M\Phi_0 \quad ; \quad \bar{M}\Phi_0 \quad ; \quad \bar{M}_1\Phi_1 \quad ; \quad M_1\Phi_{-1}.$$

It is enough to carry out these calculations on  $W_{mn}^l$  in place of  $\Phi_m$ . First, from (68) we have

$$\begin{aligned} \partial_\theta W_{mn}^l &= e^{-in\varphi} \frac{du_{mn}^l}{d\theta}, \\ \partial_\varphi W_{mn}^l &= -ine^{-in\varphi} u_{mn}^l. \end{aligned}$$

It follows that

$$\begin{aligned} MW_{mn}^l &= e^{-in\varphi} \left( \frac{du_{mn}^l}{d\theta} + \frac{n}{\sin(\theta)} u_{mn}^l \right), \\ \bar{M}W_{mn}^l &= e^{-in\varphi} \left( \frac{du_{mn}^l}{d\theta} - \frac{n}{\sin(\theta)} u_{mn}^l \right), \end{aligned}$$

and so

$$\begin{aligned} M_1 W_{mn}^l &= e^{-in\varphi} \left( \frac{du_{mn}^l}{d\theta} + \frac{n + \cos(\theta)}{\sin(\theta)} u_{mn}^l \right), \\ \bar{M}_1 W_{mn}^l &= e^{-in\varphi} \left( \frac{du_{mn}^l}{d\theta} - \frac{n - \cos(\theta)}{\sin(\theta)} u_{mn}^l \right). \end{aligned}$$

Consequently, using (72) and (73) we get

$$MW_{0n}^l = -\alpha W_{1n}^l \quad (80)$$

$$\bar{M}W_{0n}^l = -\alpha W_{-1n}^l \quad (81)$$

$$M_1 W_{-1n}^l = -\alpha W_{0n}^l \quad (82)$$

$$\bar{M}_1 W_{1n}^l = -\alpha W_{0n}^l \quad (83)$$

Now we expand  $E\Phi = 0$  and  $D\Phi = 0$  as in (74). (63) becomes

$$\begin{aligned} i\partial_t \sum_{l=1}^{+\infty} \sum_{n=-l}^l \Psi_{1n}^l W_{1n}^l &= \partial_{r_*} \left( \sum_{l=1}^{+\infty} \sum_{n=-l}^l \Psi_{1n}^l W_{1n}^l \right) + VM \left( \sum_{l=0}^{+\infty} \sum_{n=-l}^l \Psi_{0n}^l W_{0n}^l \right), \\ \sum_{l=1}^{+\infty} \sum_{n=-l}^l (i\partial_t \Psi_{1n}^l) W_{1n}^l &= \sum_{l=1}^{+\infty} \sum_{n=-l}^l (\partial_{r_*} \Psi_{1n}^l) W_{1n}^l + \sum_{l=0}^{+\infty} \sum_{n=-l}^l (V \Psi_{0n}^l) MW_{0n}^l, \end{aligned}$$

and by the above calculations this is

$$\sum_{l=1}^{+\infty} \sum_{n=-l}^l (i\partial_t \Psi_{1n}^l) W_{1n}^l = \sum_{l=1}^{+\infty} \sum_{n=-l}^l (\partial_{r_*} \Psi_{1n}^l) W_{1n}^l + \sum_{l=1}^{+\infty} \sum_{n=-l}^l (-\alpha V \Psi_{0n}^l) W_{1n}^l,$$

i.e.

$$\sum_{l=1}^{+\infty} \sum_{n=-l}^l (i\partial_t \Psi_{1n}^l - \partial_{r_*} \Psi_{1n}^l + \alpha V \Psi_{0n}^l) W_{1n}^l = 0,$$

and since  $W_{1n}^l$  are linearly independent, (76) follows. (78) is obtained by the same calculations changing only some signs.

Next, we treat (64) in a similar manner. Expanding, we have,

$$i\partial_t \sum_{l=0}^{+\infty} \sum_{n=-l}^l \Psi_{0n}^l W_{0n}^l = \frac{1}{2} \left( \sum_{l=1}^{+\infty} \sum_{n=-l}^l (\Psi_{1n}^l \bar{M}_1 W_{1n}^l - \Psi_{-1n}^l M_1 W_{-1n}^l) \right).$$

Using what we calculated above, we have

$$i\partial_t \sum_{l=0}^{+\infty} \sum_{n=-l}^l \Psi_{0n}^l W_{0n}^l = \left( \sum_{l=1}^{+\infty} \sum_{n=-l}^l \alpha (\Psi_{-1n}^l - \Psi_{1n}^l) W_{0n}^l \right).$$

Hence,

$$i\partial_t \Psi_{00}^0 W_{00}^0 + \sum_{l=1}^{+\infty} \sum_{n=-l}^l \left( i\partial_t \Psi_{0n}^l - \frac{\alpha}{2} (\Psi_{-1n}^l - \Psi_{1n}^l) \right) W_{0n}^l = 0,$$



and we get (77) with  $i\partial_t\Psi_{00}^0 = 0$ . The last equation, (79), can be obtained using these calculations with the required obvious changes.  $\square$

It follows from the above results that the operator  $H$ , and hence  $E$ , when applied to  $\Phi$  decompose into the operators

$$H_n^l = -i \begin{pmatrix} \partial_{r_*} & -\alpha V & 0 \\ -\frac{1}{2}\alpha & 0 & \frac{1}{2}\alpha \\ 0 & \alpha V & -\partial_{r_*} \end{pmatrix} \quad \text{acting on} \quad \Psi_n^l = \begin{pmatrix} \Psi_{1n}^l \\ \Psi_{0n}^l \\ \Psi_{-1n}^l \end{pmatrix}, \quad (84)$$

where  $\Phi$  and  $\Psi_n^l$  are related by (74). We then set  $E_n^l = \partial_t - iH_n^l$ . Likewise for  $D$ , we have

$$D_n^l = \begin{pmatrix} -\frac{1}{2}\alpha & -\partial_{r_*} & -\frac{1}{2}\alpha \end{pmatrix}. \quad (85)$$

Therefore, we can restate Lemma 5 simply as:

$$\begin{cases} D\Phi = 0 \\ E\Phi = 0 \end{cases} \iff \begin{cases} D_n^l \Psi_n^l = 0 \\ E_n^l \Psi_n^l = 0 \end{cases} \quad \forall l, n. \quad (86)$$

**Remark 6.** Note that  $D_n^l H_n^l = 0$ . In effect, we have  $DH = 0$  which is immediate once we note that the second order operators  $M_1\bar{M}$  and  $\bar{M}_1M$  are one and the same. The fact that  $DH = 0$  is what ensures the compatibility of the constraint with the evolution system. As a matter of fact, what is more important, is that  $M_1\bar{M} = \bar{M}_1M$  and that they are the Laplacian on the sphere written in spherical coordinates, which is essential and, unsurprisingly, plays a central role throughout this work, from the Cauchy problem to the energy estimates and the decay, as well as for Scattering and the Goursat problem, as we shall see later.

## 2.2 Cauchy Problem

The initial-value problem for Maxwell's equations can be solved either as first order evolution equations with a constraint, or, as second order wave equations. In this section we solve the Cauchy problem on  $\mathcal{N}$  by two methods: First, Using the *symmetric hyperbolicity* of the first order evolution system and the preservation of the constraint under the evolution. Second, using three (complex) coupled wave equations on the spin components. It is worth mentioning that these are not the only first and second order approaches that can be employed to solve the Cauchy problem.

The well-posedness of the Cauchy problem is closely related to the concept of globally hyperbolic spacetimes. A space-time is said to be *globally hyperbolic* if it admits a *global Cauchy hypersurface*, that is a spacelike hypersurface which every inextendible causal (non-spacelike) curve intersects exactly once<sup>1</sup>. In reality, such a space-time is diffeomorphic to

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<sup>1</sup>This is not the original definition of globally hyperbolic, that was due to Leray [99]. For the definition given in the text, see [74]

a product  $\mathbb{R} \times \Sigma$  where  $\Sigma$  is a 3-dimensional manifold and its copies  $\{t\} \times \Sigma$  are Cauchy surfaces [19, 72]. Clearly,  $\mathcal{N} = \mathbb{R}_t \times \mathbb{R}_{r_*} \times \mathcal{S}_\omega^2$  with the RNdS metric is a globally hyperbolic Lorentzian manifold, foliated by the Cauchy hypersurfaces  $\Sigma_t = \{t\} \times \Sigma$  and  $\Sigma = \mathbb{R}_{r_*} \times \mathcal{S}_\omega^2$  on which we define our function spaces.

Define on  $\Sigma$  the space of complex-valued square integrable functions,

$$L^2(\Sigma) = \{\phi : \Sigma \longrightarrow \mathbb{C} ; \|\phi\|_{L^2(\Sigma)}^2 := \int_{\Sigma} |\phi|^2 dr_* d^2\omega < \infty\},$$

where  $d^2\omega = \sin(\theta)d\theta d\varphi$  is the Lebesgue measure on the 2-sphere. We also need the weighted  $L^2$ -space

$$L_V^2(\Sigma) = \{\phi : \Sigma \longrightarrow \mathbb{C} ; \|\phi\|_{L_V^2(\Sigma)}^2 := \int_{\Sigma} 2V|\phi|^2 dr_* d^2\omega < \infty\},$$

which is a Hilbert space since it is isomorphic<sup>1</sup> to  $L^2(\Sigma)$ .

The space of finite energy data on  $\Sigma$  is defined as

$$\mathcal{H} = \{\phi : \Sigma \longrightarrow \mathbb{C}^3 ; \|\phi\|_{\mathcal{H}}^2 = \langle \phi, \phi \rangle_{\mathcal{H}} < \infty\} \quad (87)$$

where

$$\langle \phi, \chi \rangle_{\mathcal{H}} = \frac{1}{4} \int_{\Sigma} \phi_1 \bar{\chi}_1 + 2V\phi_0 \bar{\chi}_0 + \phi_{-1} \bar{\chi}_{-1} dr_* d^2\omega, \quad (88)$$

and so

$$\|\phi\|_{\mathcal{H}}^2 = \frac{1}{4} \int_{\Sigma} |\phi_1|^2 + 2V|\phi_0|^2 + |\phi_{-1}|^2 dr_* d^2\omega. \quad (89)$$

We shall also denote (89) by  $E_T[F](t)$ . This norm is the natural energy associated with Maxwell's equations and it can be defined geometrically (see (209) and (228) of section 3.2.1). We note that  $\mathcal{H}$  is a Hilbert space as  $\mathcal{H} = L^2(\Sigma) \times L_V^2(\Sigma) \times L^2(\Sigma)$  and

$$\|\phi\|_{\mathcal{H}}^2 = \frac{1}{4} \left( \|\phi_1\|_{L^2(\Sigma)}^2 + \|\phi_0\|_{L_V^2(\Sigma)}^2 + \|\phi_{-1}\|_{L^2(\Sigma)}^2 \right).$$

Despite the fact that the restriction to  $\Sigma_t$  of a finite energy solution of Maxwell's equations is in  $\mathcal{H}$  as we shall see later on, it is not true that any data in  $\mathcal{H}$  gives rise to a Maxwell field. This is because the constraint operator  $D$  is tangent to  $\Sigma$ . If we pick data in  $\mathcal{H}$  that do not satisfy the constraint equation, there would not be a solution associated to this data. Taking this into consideration, we cannot hope that the solution to the Cauchy problem for the evolution equations  $E\Phi = 0$  with general data in  $\mathcal{H}$  to be a Maxwell field. We need to choose data that satisfy the constraint, in the sense of distributions, that is data in the *constraint subspace*

$$\mathcal{U} = \{\phi \in \mathcal{H} ; D\phi = 0\}. \quad (\text{See Remark 18}) \quad (90)$$

$\mathcal{U}$  is a Hilbert space as the next lemma shows.

**Lemma 7.** *The subspace  $\mathcal{U}$  of  $\mathcal{H}$  is a Hilbert space when endowed with  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ .*

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<sup>1</sup>Via the unitary transformation  $\phi \in L^2(\Sigma) \longrightarrow (2V)^{-\frac{1}{2}}\phi \in L_V^2(\Sigma)$ .

*Proof.* Since  $\mathcal{H}$  is a complete space, we only need to show that  $\mathcal{U}$  is a closed subspace. Let  $(\phi^p)_p$  be a sequence in  $\mathcal{U}$  that converges in  $\mathcal{H}$  to  $\phi$ , which is to say that it converges component wise in  $L^2$ -norms. Since  $\mathcal{H}$  is a  $L^2$ -space, it is embedded in the distributions space, and hence  $(\phi^p)_p$  converges in the sense of distributions to  $\phi$ , and so  $(\partial_i \phi^p)_p$  converges to  $\partial_i \phi$ , from which we must have that the sequence  $(D\phi^p)_p$  converges to  $D\phi$ . Therefore  $D\phi = 0$  and  $\phi \in \mathcal{U}$ .  $\square$

We now solve the Cauchy problem.

**Proposition 8** (Cauchy Problem). *Let  $\phi \in \mathcal{U}$ , then the initial value problem*

$$\begin{cases} E\Phi = 0 \\ \Phi|_{\Sigma} = \phi \end{cases} \quad (91)$$

*has a unique solution  $\Phi \in \mathcal{C}(\mathbb{R}, \mathcal{U})$ . Moreover,*

$$\|\Phi(t)\|_{\mathcal{H}} = \|\phi\|_{\mathcal{H}} \quad \forall t \in \mathbb{R}. \quad (92)$$

We present two methods for proving this result. But first, we have this simple but useful lemma.

**Lemma 9.** *If  $\Phi$  is a solution of (91), then  $D\Phi = 0$ .*

*Proof.* From Remark 6,  $DH = 0$ , so we have

$$\begin{cases} \partial_t D\Phi = D\partial_t \Phi = iDH\Phi = 0 \\ (D\Phi)|_{\Sigma} = D\Phi|_{\Sigma} = D\phi = 0 \end{cases} \quad (93)$$

since  $\phi \in \mathcal{U}$ . Thus  $D\Phi = 0$  globally, since  $D\Phi$  solves the above simple initial-value problem.  $\square$

Consequently, solutions to (91) are automatically solutions of Maxwell's equations. With this observation, we see that notwithstanding that Maxwell (compacted) equations are overdetermined (4 equations and 3 unknowns), the constraint is compatible with the evolution Cauchy problem and propagates with the solution. In the conclusion, this shows that the Cauchy problem for Maxwell's equations is well-posed and equivalent to (91), when the data is chosen in  $\mathcal{U}$ .

### 2.2.1 Cauchy Problem - Method 1: Symmetric Hyperbolic Evolution Equations

We use Leray theory for hyperbolic equations, in particular, a version presented in [139], to show that (91) admits a unique solution for data in  $\mathcal{H}$ . We first prove the result for smooth compactly supported data using the separability of the equations. Then, from smooth compactly supported data we extend to  $\mathcal{H}$  by density. Finally, thanks to Lemma 9 we restrict

back to data in  $\mathcal{U}$ . It is appropriate in this occasion to mention that alternatively but in the same spirit, one can use spectral analysis and Stone's theorem to get the same result [131, 133].

Consider the Cauchy problem (91) with smooth compactly supported data. From Lemma 5 and the paragraph after it, follows that (91) implies

$$\begin{cases} E_n^l \Psi_n^l &= 0 \\ \Psi_n^l(0, r_*) &= \psi_n^l(r_*) \end{cases} \quad (94)$$

with  $\psi_n^l$  the spin-weighted spherical harmonic coefficients of  $\phi$ , furthermore,  $\psi_n^l \in (\mathcal{C}_0^\infty(\mathbb{R}))^3$ . By (74) and the fact that  $\{W_{mn}^l\}$  is an orthonormal family, we have, for  $\phi, \mu \in \mathcal{H}$  with harmonic coefficients  $\psi_n^l, \nu_n^l$  respectively,

$$\langle \phi, \mu \rangle_{\mathcal{H}} = \sum_{l,n} \frac{1}{4} \int_{\mathbb{R}} \psi_{1n}^l \bar{\nu}_{1n}^l + 2V \psi_{0n}^l \bar{\nu}_{0n}^l + \psi_{-1n}^l \bar{\nu}_{-1n}^l dr_* .$$

And hence if we set,

$$\langle \psi_n^l, \nu_n^l \rangle_{\mathcal{H}_{r_*}} := \frac{1}{4} \int_{\mathbb{R}} \psi_{1n}^l \bar{\nu}_{1n}^l + 2V \psi_{0n}^l \bar{\nu}_{0n}^l + \psi_{-1n}^l \bar{\nu}_{-1n}^l dr_* ,$$

then  $\psi_n^l, \nu_n^l \in \mathcal{H}_{r_*}$  the completion of  $\mathcal{C}_0^\infty(\mathbb{R}; \mathbb{C}^3)$  with respect to the norm

$$\|\psi\|_{\mathcal{H}_{r_*}}^2 = \frac{1}{4} \int_{\mathbb{R}} |\psi_1|^2 + 2V |\psi_0|^2 + |\psi_{-1}|^2 dr_* ,$$

and we have

$$\langle \phi, \mu \rangle_{\mathcal{H}} = \sum_{l,n} \langle \psi_n^l, \nu_n^l \rangle_{\mathcal{H}_{r_*}} . \quad (95)$$

Using the following classical result, the next lemma shows that (94) is well-posed for data in  $(\mathcal{C}_0^\infty(\mathbb{R}))^3$ .

We first define the *symbol classes*  $S_{1,0}^m$  as given in [139]. If  $B = B(x, D)$  is a differential operator on  $\mathbb{R}_x^n$  given by

$$B(x, D) = \sum_{|\beta| \leq k} a_\beta(x) D^\beta$$

with  $D^\beta = D_1^{\beta_1} \dots D_n^{\beta_n}$ ,  $D_j = -i\partial_{x^j}$ , then the symbol of the operator  $B$  is the function  $B(x, \xi)$  on  $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$  given by,

$$B(x, \xi) = \sum_{|\beta| \leq k} a_\beta(x) \xi^\beta .$$

We say, for an operator  $B(x, D)$  with  $x \in \mathbb{R}^n$ , that  $B(x, \xi) \in S_{1,0}^m$  if

$$\|D_x^\beta D_\xi^\gamma B(x, \xi)\| < C_{\gamma\beta} (1 + \|\xi\|^2)^{\frac{1}{2}(m-|\gamma|)} \quad (96)$$

for all  $\gamma, \beta \in \mathbb{N}^n$ . The next theorem can be found in [139].

**Theorem 10** (Symmetric Hyperbolic Evolution Systems). *Consider the first-order system with  $u : \mathbb{R}_t \mathbb{R}_x^n \rightarrow \mathbb{R}^m$ :*

$$\begin{cases} \partial_t u = A(t, x, D_x)u + w(t, x) \\ u(0) = v \end{cases} \quad (97)$$

*such that  $A(t, x, \xi) \in S_{1,0}^1$  with smooth dependence on  $t$ , i.e.*

$$\|D_t^j D_x^\beta D_\xi^\gamma A(x, \xi)\| < C_{j\gamma\beta}(1 + \|\xi\|^2)^{\frac{1}{2}(m-|\gamma|)} . \quad (98)$$

*Here  $A(t, x, \xi)$  is a  $m \times m$  matrix-valued function. The system is said satisfy the symmetric hyperbolicity condition if:*

$$A(t, x, \xi)^* + A(t, x, \xi) \in S_{1,0}^1. \quad (99)$$

*If in addition to symmetric hyperbolicity, we have ,*

$$v \in H^s(\mathbb{R}^n) \text{ and } w \in \mathcal{C}(\mathbb{R}, H^s(\mathbb{R}^n)) , \quad s \in \mathbb{R}$$

*then there is a unique solution  $u$  to (97), satisfying*

$$u \in \mathcal{C}(\mathbb{R}, H^s(\mathbb{R}^n)) \cap \mathcal{C}^1(\mathbb{R}, H^{s-1}(\mathbb{R}^n)) \quad (100)$$

We now show that (94) is well-posed for data in  $(\mathcal{C}_0^\infty(\mathbb{R}))^3$ .

**Lemma 11.** *Let  $l \geq 0$  and  $\psi_n^l \in (\mathcal{C}_0^\infty(\mathbb{R}))^3$ , then the Cauchy problem given by (94) has a unique solution  $\Psi_n^l \in \mathcal{C}^\infty(\mathbb{R}, (\mathcal{C}_0^\infty(\mathbb{R}))^3)$ , and*

$$\|\Psi_n^l(t)\|_{\mathcal{H}_{r_*}} = \|\psi_n^l\|_{\mathcal{H}_{r_*}} \quad \forall t \in \mathbb{R}. \quad (101)$$

*Proof.* Let us drop the  $l$  and  $n$  indices for the proof . By Theorem 10 we only to show that the operator  $A = iH_n^l$  is in the right symbol class and satisfies the hypothesis of symmetric hyperbolicity, that is, we need to show that  $A$  is a first order skew-symmetric operator. More precisely,

$$A(r_*, \xi) \in S_{1,0}^1, \quad (102)$$

and the symmetric hyperbolicity condition:

$$A(r_*, \xi)^* + A(r_*, \xi) \in S_{1,0}^0. \quad (103)$$

where  $A(r_*, \xi)^*$  is the adjoint matrix of  $A(r_*, \xi)$  defined with respect to the energy inner product  $\langle \cdot, \cdot \rangle_{\mathcal{H}_{r_*}}$ . Here,

$$A(r_*, \xi) = \begin{pmatrix} i\xi & -\alpha V & 0 \\ -\frac{1}{2}\alpha & 0 & \frac{1}{2}\alpha \\ 0 & \alpha V & -i\xi \end{pmatrix}$$

is a matrix-valued function, so we choose the max norm on matrices to show (102). The inequality obviously holds for  $\beta, \gamma \geq 1$ , since its left hand side is just zero. If  $\beta = 0$  and  $\gamma \geq 1$ , then we have

$$\|D_\xi^\gamma A(r_*, \xi)\|_\infty = |D_\xi^\gamma \xi| = \begin{cases} 1 & \gamma = 1 , \\ 0 & \gamma > 1 . \end{cases}$$

Thus,

$$\|D_\xi^\gamma A(r_*, \xi)\|_\infty < C(1 + |\xi|^2)^{\frac{1}{2}(1-|\gamma|)}.$$

On the other hand, if  $\gamma = 0$ , we have

$$\|D_{r_*}^\beta A(r_*, \xi)\|_\infty = |\alpha| \|\partial_{r_*}^\beta V\| = |\alpha| \|(f\partial_r)^\beta V\|.$$

But,  $f$  and  $V$  are smooth functions of  $r$ , and hence they and all their derivatives are bounded on  $[r_2, r_3]$ , so,

$$\|D_{r_*}^\beta A(r_*, \xi)\|_\infty < C'_\beta |\alpha| < C_\beta (1 + |\xi|^2)^{\frac{1}{2}},$$

and (102) is proved. Next, a straightforward calculation shows that for  $\psi_1, \psi_2 \in (\mathcal{C}_0^\infty(\mathbb{R}))^3$

$$\langle A(r_*, \xi) \psi_1, \psi_2 \rangle_{\mathcal{H}_{r_*}} = \langle \psi_1, -A(r_*, \xi) \psi_2 \rangle_{\mathcal{H}_{r_*}}.$$

Therefore  $A(r_*, \xi)^* = -A(r_*, \xi)$  and the symmetric hyperbolicity condition is trivially satisfied. Therefore, there exist a unique solution  $\Psi$  for (94) satisfying (100). However, since our data are in  $(\mathcal{C}_0^\infty(\mathbb{R}))^3$ , the Sobolev spaces can be replaced by  $(\mathcal{C}_0^\infty(\mathbb{R}))^3$  where the compact support is a consequence of finite speed propagation. To get smoothness in time, we iteratively apply the evolution equations in 101 to get differentiability in time from that in space variables. Therefore,  $\Psi \in \mathcal{C}^\infty(\mathbb{R}, (\mathcal{C}_0^\infty(\mathbb{R}))^3)$ .

Finally, if we set  $\|\Psi\|_{\mathcal{H}_{r_*}}^2(t) := \|\Psi(t)\|_{\mathcal{H}_{r_*}}^2$ , and differentiate,

$$\begin{aligned} \partial_t \|\Psi\|_{\mathcal{H}_{r_*}}^2(t) &= \partial_t \langle \Psi, \Psi \rangle_{\mathcal{H}_{r_*}}(t) \\ &= \langle \partial_t \Psi, \Psi \rangle_{\mathcal{H}_{r_*}}(t) + \langle \Psi, \partial_t \Psi \rangle_{\mathcal{H}_{r_*}}(t) \\ &= \langle A \Psi, \Psi \rangle_{\mathcal{H}_{r_*}}(t) + \langle \Psi, A \Psi \rangle_{\mathcal{H}_{r_*}}(t) \\ &= \langle A \Psi(t), \Psi(t) \rangle_{\mathcal{H}_{r_*}} + \langle \Psi(t), A \Psi(t) \rangle_{\mathcal{H}_{r_*}} \\ &= \langle \Psi(t), -A \Psi(t) \rangle_{\mathcal{H}_{r_*}} + \langle \Psi(t), A \Psi(t) \rangle_{\mathcal{H}_{r_*}} = 0, \end{aligned}$$

so,  $\|\Psi\|_{\mathcal{H}_{r_*}}^2$  is constant in time and equals to  $\|\psi\|_{\mathcal{H}_{r_*}}^2$ .  $\square$

As a consequence of the previous lemma, the linear operator

$$\begin{array}{ccc} U_t : (\mathcal{C}_0^\infty(\mathbb{R}))^3 & \longrightarrow & (\mathcal{C}_0^\infty(\mathbb{R}))^3 \\ \psi & \longrightarrow & \Psi(t) \end{array}$$

which to data in  $(\mathcal{C}_0^\infty(\mathbb{R}))^3$  associates the corresponding unique solution of (94) at time  $t$ , is a unitary operator.

We now reconstruct the solution to (91) for smooth compactly supported data from solutions to (94). Similar to Fourier expansion, in spherical harmonic expansion, the regularity (and convergence) of the series is closely related to the decay of the coefficients as  $l \rightarrow \infty$ . As a matter of fact, the series converges to a smooth function if and only if the coefficients decay faster than any power of  $l$ .

Let  $\phi \in (\mathcal{C}_0^\infty(\Sigma))^3$  and write its spherical harmonic expansion as

$$\phi = \sum_{l,n} \psi_n^l \odot \mathbf{W}_n^l ; \quad \text{with } \mathbf{W}_n^l = \begin{pmatrix} W_{1n}^l \\ W_{0n}^l \\ W_{-1n}^l \end{pmatrix}$$

where  $\odot$  denote the Hadamard matrix product<sup>1</sup> defined as the component wise product:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \odot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ y_1 y_2 \\ z_1 z_2 \end{pmatrix}.$$

**Lemma 12.** *If  $\phi \in \mathcal{C}_0^\infty(\Sigma))^3$  then (91) admits a unique solution  $\Phi \in \mathcal{C}^\infty(\mathbb{R}, (\mathcal{C}_0^\infty(\Sigma))^3)$ , and energy is conserved:*

$$\|\Phi(t)\|_{\mathcal{H}} = \|\phi\|_{\mathcal{H}} \quad \forall t \in \mathbb{R}. \quad (104)$$

*Proof.* If  $\Phi \in \mathcal{C}^\infty(\mathbb{R}, (\mathcal{C}_0^\infty(\Sigma))^3)$  is a solution of (91), then (104) follows from (95) and (101), which guarantees uniqueness.

As  $\phi = \sum_{l,n} \psi_n^l \odot \mathbf{W}_n^l$ , then  $\psi_n^l \in \mathcal{C}_0^\infty(\mathbb{R})^3$  for each  $l, n$  ( $l > 0$  since  $\phi$  has compact support). Let  $\Psi(t) = U_t \psi$  be the solution to (94).

Define the operator

$$P = \begin{pmatrix} M\bar{M}_1 & 0 & 0 \\ 0 & M_1\bar{M} & 0 \\ 0 & 0 & \bar{M}M_1 \end{pmatrix},$$

then from (80)-(83) we see that  $P\mathbf{W}_n^l = \alpha^2 \mathbf{W}_n^l$ . It follows that

$$\int_{S^2} (P^k \phi) \odot \bar{\mathbf{W}}_n^l d^2\omega = \alpha^{2k} \psi_n^l.$$

Thus, interpreting absolute values and inequalities component wise, by Cauchy-Schwartz we have

$$\begin{aligned} |\psi_n^l| &\leq |\alpha|^{-2k} \int_{S^2} |(P^k \phi) \odot \bar{\mathbf{W}}_n^l| d^2\omega \\ &\leq |\alpha|^{-2k} \left( \int_{S^2} |P^k \phi|^2 d^2\omega \odot \int_{S^2} |\mathbf{W}_n^l|^2 d^2\omega \right) \\ &= |\alpha|^{-2k} \int_{S^2} |P^k \phi|^2 d^2\omega, \end{aligned}$$

for all  $k \in \mathbb{N}^*$ , i.e.  $\psi_n^l$  decays faster than any power of  $l$ . Since  $U_t$  is continuous,  $\Psi_n^l(t)$  also has the same decay for all  $t$ . Thereby,  $\Phi = \sum_{l,n} \Psi_n^l \odot \mathbf{W}_n^l$  is a well defined solution of (91) in  $\mathcal{C}(\mathbb{R}, (\mathcal{C}^\infty(\Sigma))^3)$ . In fact,  $\Phi \in \mathcal{C}(\mathbb{R}, (\mathcal{C}_0^\infty(\Sigma))^3)$  by finite propagation speed as  $\phi$  is compactly supported.

To get regularity in time, let  $k$  be a natural number, then  $(iH)^k \phi \in \mathcal{C}_0^\infty(\Sigma))^3$ , in consequence,  $(iH_n^l)^k \psi_n^l$  decays faster than  $l^{-p}$  for any  $p \in \mathbb{N}$ , and so does  $U_t(iH_n^l)^k \psi_n^l$  for all

---

<sup>1</sup>As easily seen, this product is commutative, associative, and distributive over addition. The Hadamard product has many different notations including the one we use. One can also find the notation  $\circ$  (see [130] for example), but we avoid using  $\circ$  as it usually denotes composition of functions.

$t$ . Accordingly,  $\sum_{l,n} (U_t (iH_n^l)^k \psi_n^l) \odot \mathbf{W}_n^l$  is well defined. Now, since  $(iH_n^l)^k \Psi_n^l$  is again a solution of (94) with  $(iH_n^l)^k \psi_n^l$  as data, the operators  $U_t$  and  $iH$  commute. Therefore,

$$\begin{aligned} \sum_{l,n} (U_t (iH_n^l)^k \psi_n^l) \odot \mathbf{W}_n^l &= \sum_{l,n} ((iH_n^l)^k U_t \psi_n^l) \odot \mathbf{W}_n^l = \sum_{l,n} ((iH_n^l)^k \Psi_n^l) \odot \mathbf{W}_n^l \\ &= (iH)^k \sum_{l,n} \Psi_n^l \odot \mathbf{W}_n^l = (iH)^k \Phi = \partial_t^k \Phi, \end{aligned}$$

concluding the proof.  $\square$

On these grounds,  $U_t : \phi \mapsto \Phi(t)$  is a norm preserving operator on  $(\mathcal{C}_0^\infty(\Sigma))^3$  which to data on  $\Sigma$  associates the solution, at time  $t$ , of the corresponding Cauchy problem (91). By the density of  $\mathcal{C}_0^\infty(\Sigma)$  in  $L^2(\Sigma)$ ,  $(\mathcal{C}_0^\infty(\Sigma))^3$  is dense in  $\mathcal{H}$ . Thus,  $U_t$  extends in a unique manner to a unitary operator on  $\mathcal{H}$ , which associates  $\Phi(t) := U_t \phi$  to data  $\phi \in \mathcal{H}$ . Then  $\Phi \in \mathcal{C}(\mathbb{R}, \mathcal{H})$  is a solution of (91) with data in  $\mathcal{H}$ . From Lemma 9, we can see that  $U_t$  restricts to a unitary operator on  $\mathcal{U}$ , in other words,  $\Phi(t) = U_t \phi \in \mathcal{C}(\mathbb{R}, \mathcal{U})$  if  $\phi \in \mathcal{U}$ . We have proved Proposition 8.

## 2.2.2 Cauchy Problem - Method 2: Wave equations

In this short paragraph, we first show that solutions to Maxwell's compacted equations given in Lemma 4 satisfy a system of coupled wave equations. Then we show that the Cauchy problem defined in Proposition 8, and the Cauchy problem for the wave equations derived here, are equivalent when we restrict to smooth compactly supported data.

To see that the spin components of the Maxwell field satisfy a system of coupled wave equations, we only need to do a simple calculation using Lemma 4. Applying  $L$  to (50) then using (51) we get a second order equation on  $\Phi_1$ . Also, by applying  $L$  to (51) and using (53) we get a second order equation on  $\Phi_0$ . Similarly, if we apply  $N$  to (53) then by (52) we get a second order equation on  $\Phi_{-1}$ . These equations are:

$$LN\Phi_1 - VM\bar{M}_1\Phi_1 - \dot{V}M\Phi_0 = 0, \quad (105)$$

$$LN\Phi_0 - VM_1\bar{M}\Phi_0 = 0 \quad (106)$$

$$LN\Phi_{-1} - V\bar{M}M_1\Phi_{-1} - \dot{V}\bar{M}\Phi_0 = 0, \quad (107)$$

where  $\dot{V} = \partial_{r_*} V = f \partial_r V$ . Define the operator

Note the symmetry between the first and the last equations, as a matter of fact, the symmetry between  $\Phi_1$  and  $\Phi_{-1}$ . This is due to the unconventional way of projecting the tensor field on the tetrad  $\{L, N, M, \bar{M}\}$ .

Let us see that  $W$  is indeed a modified d'Alembertian. On the one hand we have the geometric d'Alembertian,

$$\square u = \square_g u = \nabla^\alpha \nabla_\alpha u = g^{\mathbf{ab}} (\partial_{\mathbf{a}} \partial_{\mathbf{b}} u - \Gamma_{\mathbf{ab}}^{\mathbf{c}} \partial_{\mathbf{c}} u)$$



which is,

$$\square u = \frac{1}{f} (\partial_t^2 u - \partial_{r_*}^2 u) - \frac{2}{r} \partial_{r_*} u - \frac{1}{r^2} \Delta_{S^2} u ,$$

with the spherical Laplacian being

$$\Delta_{S^2} = \partial_\theta^2 + \frac{1}{\sin(\theta)^2} \partial_\varphi^2 + \cot(\theta) \partial_\theta .$$

On the other hand,

$$LN = NL = \partial_t^2 - \partial_{r_*}^2 \quad \text{and} \quad M_1 \bar{M} = \partial_\theta^2 + \frac{1}{\sin(\theta)^2} \partial_\varphi^2 + \cot(\theta) \partial_\theta ,$$

As  $M_1 \bar{M}$  is real, it must be equal to its conjugate:  $M_1 \bar{M} = \bar{M}_1 M = \Delta_{S^2}$ . Thus,

$$W_{22} := LN - VM_1 \bar{M} = f \square + 2rV \partial_{r_*} ,$$

and since  $f > 0$ ,  $W_{22}u = 0$  if and only if  $\square u + 2r^{-1} \partial_{r_*} u = 0$ . The other two equations are also easily seen as wave equations of the form  $\square + Q$  where  $Q$  is a smooth first order linear operator, since<sup>1</sup>  $M \bar{M}_1 = \bar{M}_1 M - [\bar{M}, M] + M(\cot(\theta))$ . It is important to note that the coupling happens only in the lower order terms, that is to say, the box operator appears only in the diagonal. And since  $W_{22}\Phi_0 = 0$  has no coupling terms, one can solve for  $\Phi_0$  as a solution of a source free wave equation, and then view the other two equations as wave equations with a source given by angular derivatives of  $\Phi_0$ .

Throughout the proof of the next lemma, we use the classical result that for wave equations in one scalar unknown with source, i.e. of the form  $\square u + Qu = h$  where  $Q$  is a first order differential operator and  $h$  a source term, the Cauchy problem

$$\begin{cases} \square u + Qu = h \\ u|_\Sigma = u_1 \\ (\partial_t u)|_\Sigma = u_2 \end{cases}$$

is well-posed in the set of smooth compactly supported data ([83, 99]).

**Lemma 13.** *Let  $\phi \in (C_0^\infty(\Sigma))^3 \cap \mathcal{U}$  and consider the following Cauchy problems,*

$$\begin{cases} E\Phi = 0 \\ \Phi|_\Sigma = \phi \end{cases} \tag{108}$$

$$\begin{cases} W\Phi = 0 \\ \Phi|_\Sigma = \phi \\ (\partial_t \Phi)|_\Sigma = iH\phi \end{cases} \tag{109}$$

*Then  $\Phi$  is a solution of (108) if and only if it is a solution of (109).*

---

<sup>1</sup>The singularity at  $\theta = 0$  is merely a coordinate singularity of the spherical coordinates.

*Proof.* Assume  $\Phi$  is a solution of (108). Since  $\phi \in \mathcal{U}$  then  $D\phi = 0$ , then by Lemma 9,  $D\Phi = 0$ . So,  $E\Phi = 0$  and  $D\Phi = 0$ , which in turn implies that  $W\Phi = 0$ . The initial conditions of (109) are automatically satisfied and hence  $\Phi$  is a solution of (109).

Assume now that  $\Phi$  solves (109).  $W$  can be decomposed in terms of  $E$  and  $D$ . In effect,  $W = E^*E + D^*D$  with  $E^* = \partial_t + iH$  and

$$D^* = \begin{pmatrix} -VM \\ \partial_{r_*} \\ -V\bar{M} \end{pmatrix}.$$

One way to see this is by directly calculating  $H^2$  and observing that  $\partial_t^2 + H^2 = W - D^*D$ , i.e.  $W = \partial_t^2 - (iH)^2 + D^*D$ . Note also that  $W_{22} = \partial_t^2 + DD^*$ . And so, since  $DH = 0$ , we have

$$DW = D\partial_t^2 + DH^2 + DD^*D = (\partial_t^2 - DD^*)D = W_{22}D.$$

Thus, since  $W\Phi = 0$ , we have,  $W_{22}D\Phi = 0$ .  $D$  only involves derivatives tangent to  $\Sigma$ , so,  $(D\Phi)|_\Sigma = D(\Phi|_\Sigma) = D\phi = 0$  from (109). In addition, and also from the initial conditions,  $(\partial_t D\Phi)|_\Sigma = D((\partial_t \Phi)|_\Sigma) = DH\phi = 0$ . Therefore,  $D\Phi$  solves the Cauchy problem

$$\begin{cases} W_{22}D\Phi = 0 \\ (D\Phi)|_\Sigma = 0 \\ (\partial_t D\Phi)|_\Sigma = 0, \end{cases}$$

so  $D\Phi$  must be zero by uniqueness of solution. It follows right away from  $W\Phi = 0$  that  $E^*E\Phi = 0$ .

Finally, consider

$$WE\Phi = E^*EE\Phi - D^*DE\Phi = EE^*E\Phi - \partial_t D^*D\Phi = 0.$$

Now since  $E^*E\Phi = 0$  then  $(E^*E\Phi)|_\Sigma = 0$  i.e.  $(\partial_t^2 \Phi)|_\Sigma = ((iH)^2 \Phi)|_\Sigma = (iH)^2 \phi$ . Thus

$$(\partial_t(E\Phi))|_\Sigma = (\partial_t^2 \Phi - iH\partial_t \Phi)|_\Sigma = (\partial_t^2 \Phi)|_\Sigma - iH((\partial_t \Phi)|_\Sigma) = (\partial_t^2 \Phi)|_\Sigma - iH(iH\phi) = 0.$$

One last step is required. We have  $(\partial_t \Phi)|_\Sigma = iH\phi$  which is nothing but  $(E\Phi)|_\Sigma = 0$ . We just showed that

$$\begin{cases} WE\Phi = 0 \\ (E\Phi)|_\Sigma = 0 \\ (\partial_t E\Phi)|_\Sigma = 0, \end{cases}$$

hence  $E\Phi = 0$ .

□

This lemma shows that Maxwell's Cauchy problem is well-posed for smooth compactly supported data, and by finite propagation speed the solution is compactly supported for all times.

## 2.3 Stationary Solutions

Evidently not all solutions of Maxwell's equations decay in time, take for example the case where  $\Phi$  is a non zero constant vector, then it satisfies (4) and clearly does not decay as it does not change with time. Even solutions starting in  $\mathcal{U}$ , i.e. having finite energy, may not decay in time: Consider the constant vector  $\Phi = (\Phi_1 = 0, \Phi_0 = C \neq 0, \Phi_{-1} = 0)$ , it has finite energy, yet it does not decay. Since Maxwell's equations are linear, the last example shows that solutions (even with finite energy) having charge do not decay, where by the charge of a Maxwell field we mean the constant  $\Psi_{00}^0$  (See Lemma 5), i.e. solutions having non zero  $l = 0$  part. So, we need to exclude such solutions in order to prove decay.

More generally, *time-periodic solutions*, also called *stationary solutions*, do not decay, these are solutions of the form  $\phi(r_*, \omega)e^{i\lambda t}$  ( $\lambda$  is real), so they are solutions to  $\partial_t \Phi = i\lambda \Phi$ . Requiring them to be solutions of  $E\Phi = 0$  means that they are eigenfunctions of the Hamiltonian  $H$ . Therefore, solutions that are eigenfunctions of  $H$  do not decay and must be excluded.

So we look for solutions of Maxwell's equations  $E\Phi = 0$  and  $D\Phi = 0$ , with finite energy such that

$$H\Phi = \lambda\Phi \quad , \quad \lambda \in \mathbb{R}, \quad (110)$$

i.e.,

$$i\lambda\Phi_1 = VM\Phi_0 + \partial_{r_*}\Phi_1, \quad (111)$$

$$i\lambda\Phi_0 = \frac{1}{2}(\bar{M}_1\Phi_1 - M_1\Phi_{-1}), \quad (112)$$

$$i\lambda\Phi_{-1} = -\partial_{r_*}\Phi_{-1} - V\bar{M}\Phi_0. \quad (113)$$

The finite energy assumption  $\|\Phi(t)\|_{\mathcal{H}} < \infty$  for all  $t$ , is equivalent to:

$$\Phi_1, \frac{\sqrt{f}}{r}\Phi_0, \Phi_{-1} \in L^2(\Sigma). \quad (114)$$

Adding the requirement of finite energy, and as the next proposition says, the only admissible time-periodic solutions are exactly the pure charge solutions  $\Phi = (0 \ C \ 0)$ .

**Proposition 14** (Stationary Solutions). *If  $\Phi = (\Phi_1, \Phi_0, \Phi_{-1})$  is a finite energy stationary solution of Maxwell's equations, then*

$$\Phi = \begin{pmatrix} 0 \\ C \\ 0 \end{pmatrix} \quad \text{where } C \text{ is a complex constant.} \quad (115)$$

We postpone the proof of this proposition to the end of this section. First we restate the above as conditions on the spin-weighted spherical harmonic coefficients. The proof is exactly as in Lemma 5.

**Lemma 15.** *If  $\Phi$  is a stationary solution and  $\Psi_n^l$  are its harmonic coefficients, then the components of each  $\Psi_n^l$  satisfy an ordinary differential system: For  $l > 0$ ,*

$$i\lambda\Psi_1 = \partial_{r_*}\Psi_1 - \alpha V\Psi_0, \quad (116)$$

$$i\lambda\Psi_0 = \frac{\alpha}{2}(\Psi_{-1} - \Psi_1), \quad (117)$$

$$i\lambda\Psi_{-1} = -\partial_{r_*}\Psi_{-1} + \alpha V\Psi_0, \quad (118)$$

$$\partial_{r_*}\Psi_0 = -\frac{\alpha}{2}(\Psi_{-1} + \Psi_1), \quad (119)$$

where we have, for simplicity, dropped the indices  $l$  and  $n$  and put  $\Psi_m = \Psi_{mn}^l$ .

For  $l = 0$ ,  $i\lambda\Psi_{00}^0 = 0$  and  $\partial_{r_*}\Psi_{00}^0 = 0$ .

Similar to the full components, the coefficients  $\Psi_n^l$  are solutions of coupled wave-like equations. This is stated in the following corollary.

**Corollary 16.** *Let  $l > 0$ . Keeping the same notations as in the lemma above, the coefficients  $\Psi_n^l$  of  $\Phi$ , satisfy:*

$$\partial_{r_*}^2\Psi_0 + \lambda^2\Psi_0 = -\alpha^2 V\Psi_0, \quad (120)$$

$$\partial_{r_*}^2\Psi_1 + \lambda^2\Psi_1 = -\alpha^2 V\Psi_1 + \alpha\dot{V}\Psi_0, \quad (121)$$

$$\partial_{r_*}^2\Psi_{-1} + \lambda^2\Psi_{-1} = -\alpha^2 V\Psi_{-1} + \alpha\dot{V}\Psi_0, \quad (122)$$

$$\Psi_{\pm 1} \in L^2(\mathbb{R}_{r_*}), \quad (123)$$

$$\frac{\sqrt{f}}{r}\Psi_0 \in L^2(\mathbb{R}_{r_*}). \quad (124)$$

where, as before,  $\dot{V} = \partial_{r_*}V$

*Proof.* We indicate differentiation with respect to the variable  $r_*$  by a dot,

$$\dot{\Psi}_m = \partial_{r_*}\Psi_m.$$

First, differentiating (119), then using the other three equations of Lemma 15, we get

$$\begin{aligned} \ddot{\Psi}_0 &= -\frac{\alpha}{2}(\dot{\Psi}_{-1} + \dot{\Psi}_1) \\ &= -\frac{\alpha}{2}(i\lambda\Psi_1 + \alpha V\Psi_0 - i\lambda\Psi_{-1} + \alpha V\Psi_0) \\ &= \frac{i\lambda\alpha}{2}(\Psi_{-1} - \Psi_1) - \alpha^2 V\Psi_0 \\ &= -\lambda^2\Psi_0 - \alpha^2 V\Psi_0. \end{aligned}$$

For the second equation, we do a similar computation,

$$\begin{aligned} i\lambda\dot{\Psi}_1 &= \ddot{\Psi}_1 - \alpha\dot{V}\Psi_0 - \alpha V\dot{\Psi}_0 \\ -\lambda^2\Psi_1 + i\lambda\alpha V\Psi_0 &= \ddot{\Psi}_1 - \alpha\dot{V}\Psi_0 - \alpha V\left(-\frac{\alpha}{2}(\Psi_{-1} + \Psi_1)\right) \\ -\lambda^2\Psi_1 + \alpha V\left(\frac{\alpha}{2}(\Psi_{-1} - \Psi_1)\right) &= \ddot{\Psi}_1 - \alpha\dot{V}\Psi_0 - \alpha V\left(-\frac{\alpha}{2}(\Psi_{-1} + \Psi_1)\right) \\ -\lambda^2\Psi_1 - \alpha^2 V\Psi_1 &= \ddot{\Psi}_1 - \alpha\dot{V}\Psi_0. \end{aligned}$$

The third equation is obtained in the same manner.

The last two conditions of the corollary are direct consequences of (114) and the normalization of  $W_{mn}^l$  (see (69)).  $\square$

Before proving Proposition 14, we need the following technical lemma<sup>1</sup>.

**Lemma 17.** *Let prime denote differentiation with respect to  $t$ . Consider a differential equation of the form given by (120), i.e.*

$$u''(t) + \lambda^2 u(t) = v(t)u(t) \quad (125)$$

where  $\lambda \in \mathbb{R}$ ,  $v$  is continuous on  $\mathbb{R}$ , and  $v, t^2 v \in L^1(\mathbb{R})$ . Also consider

$$z''(t) + \lambda^2 z(t) = 0. \quad (126)$$

Let  $z_1$  and  $z_2$  be two linearly independent solutions of (126), then for every  $a, b \in \mathbb{C}$  there exists a solution  $u$  of (125) such that

$$u(t) = \alpha(t)z_1(t) + \beta(t)z_2(t),$$

with

$$\lim_{t \rightarrow +\infty} \alpha(t) = a \quad \text{and} \quad \lim_{t \rightarrow +\infty} \beta(t) = b.$$

Moreover, the general solution of (125) can be written in the form

$$u = C_1 u_+ + C_2 u_-, \quad (127)$$

for  $C_1, C_2$  constants, and

$$\begin{aligned} u_+ &= \alpha_+(t)z_1(t) + \beta_+(t)z_2(t), \\ u_- &= \alpha_-(t)z_1(t) + \beta_-(t)z_2(t), \end{aligned}$$

with

$$\begin{aligned} \lim_{t \rightarrow +\infty} \alpha_+(t) &= 1 & ; & & \lim_{t \rightarrow +\infty} \beta_+(t) &= 0 \\ \lim_{t \rightarrow +\infty} \alpha_-(t) &= 0 & ; & & \lim_{t \rightarrow +\infty} \beta_-(t) &= 1. \end{aligned}$$

*Proof.* Let  $u$  be a solution of (125), and let  $A$  and  $B$  be two functions such that

$$Az_1 + Bz_2 = u, \quad (128)$$

$$A'z_1 + B'z_2 = 0, \quad (129)$$

---

<sup>1</sup>The proof of this lemma is due to [141], we present it here for the sake of completeness and because we shall use parts of it in other arguments.

then, differentiating the first equation twice and substituting the derivative of the second equation, we get,

$$\begin{aligned} u'' &= A'z_1' + Az_1'' + B'z_2' + Bz_2'' \\ &= A'z_1' + B'z_2' - \lambda^2(Az_1 + Bz_2), \\ vu - \lambda^2u &= A'z_1' + B'z_2' - \lambda^2u, \end{aligned}$$

thus we have the first order differential system

$$\begin{aligned} A'z_1 + B'z_2 &= 0 \\ A'z_1' + B'z_2' &= v(Az_1 + Bz_2). \end{aligned}$$

Solving for  $A'$  and  $B'$  we find

$$A' = \frac{vz_2}{W}(Az_1 + Bz_2) \quad (130)$$

$$B' = \frac{-vz_1}{W}(Az_1 + Bz_2), \quad (131)$$

where  $W = z_1'z_2 - z_2'z_1$ . Note that  $W' = z_1''z_2 - z_2''z_1 = \lambda^2z_2z_1 - \lambda^2z_1z_2 = 0$ , so  $W$  is a constant, and the fact that it is not equal to zero follows from  $z_1$  and  $z_2$  being linearly independent solutions of (126). Therefore,

$$A(t) = A(0) + \frac{1}{W} \int_0^t v(s)z_2(s)(A(s)z_1(s) + B(s)z_2(s))ds, \quad (132)$$

$$B(t) = B(0) - \frac{1}{W} \int_0^t v(s)z_1(s)(A(s)z_1(s) + B(s)z_2(s))ds. \quad (133)$$

Thus, we have

$$\begin{aligned} |A(t)| + |B(t)| &\leq |A(0)| + |B(0)| + \\ &\quad \frac{1}{|W|} \int_0^t |v|(|A||z_1| + |B||z_2|)(|z_1| + |z_2|)ds. \end{aligned} \quad (134)$$

If  $\lambda \neq 0$ , we set  $z_1(t) = e^{i\lambda t}$  and  $z_2(t) = e^{-i\lambda t}$ , and so  $|z_1| = |z_2| = 1$ . Hence, the above inequality reduces to

$$|A(t)| + |B(t)| \leq |A(0)| + |B(0)| + \frac{2}{|W|} \int_0^t |v|(|A| + |B|)ds.$$

Now using Gronwall's inequality it follows that

$$|A(t)| + |B(t)| \leq (|A(0)| + |B(0)|)e^{\frac{2}{|W|} \int_0^t |v|ds},$$

and since  $v \in L^1(\mathbb{R})$ ,  $A$  and  $B$  are bounded on  $\mathbb{R}^+$ .

If  $\lambda = 0$ , set  $z_1(t) = 1$  and  $z_2(t) = t$ . Then (134) becomes

$$|A(t)| + |B(t)| \leq |A(0)| + |B(0)| + \frac{1}{|W|} \int_0^t |v|(|A| + |B|s)(1+s)ds.$$

Since  $A$  and  $B$  are continuous, they are bounded on  $0 \leq t \leq 1$ . For  $t \geq 1$ , we have

$$\begin{aligned} \int_0^t |v|(|A| + |B|s)(1+s)ds &= \int_0^1 |v|(|A| + |B|s)(1+s)ds + \int_1^t |v|(|A| + |B|s)(1+s)ds \\ &\leq 2 \int_0^1 |v|(|A| + |B|)ds + 2 \int_1^t |v|s^2(|A| + |B|)ds. \end{aligned}$$

Again, by continuity, the integral over  $[0, 1]$  is just a finite constant. So,

$$|A(t)| + |B(t)| \leq |A(0)| + |B(0)| + C + \frac{2}{|W|} \int_1^t |v|s^2(|A| + |B|)ds,$$

and by Gronwall's inequality

$$|A(t)| + |B(t)| \leq (|A(0)| + |B(0)| + C)e^{\frac{2}{|W|} \int_1^t |v|s^2ds},$$

which implies that  $A$  and  $B$  are bounded, because  $s^2v \in L^1(\mathbb{R})$ . Therefore,  $A$  and  $B$  are bounded functions over  $\mathbb{R}^+$  for all  $\lambda \in \mathbb{R}$ . It follows that for  $i = 1, 2$ ,

$$\left| \int_0^\infty v z_i (A z_1 + B z_2) ds \right| \leq \begin{cases} C \int_0^\infty |v| ds < \infty & ; \lambda \neq 0 \\ C_1 + C_2 \int_1^\infty |v|s^2 ds < \infty & ; \lambda = 0 \end{cases}.$$

From this and the two integral equations (132) and (133), we can see that  $\lim_{t \rightarrow \infty} A(t)$  and  $\lim_{t \rightarrow \infty} B(t)$  exist and are finite.

Since all the coefficient functions of the system formed by (130) and (131) are continuous, the initial value problem of this system has a unique solution. So, let  $(A_i(t), B_i(t))$  for  $i = 1, 2$ , be two solutions with

$$A_1(0) = 1 \quad ; \quad B_1(0) = 0 \quad ; \quad A_2(0) = 0 \quad ; \quad B_2(0) = 1,$$

and denote

$$\lim_{t \rightarrow \infty} A_i(t) = a_i \quad ; \quad \lim_{t \rightarrow \infty} B_i(t) = b_i.$$

Note that  $A_1 B_2 - A_2 B_1$  is a constant since

$$(A_1 B_2 - A_2 B_1)' = A_1' B_2 - A_2' B_1 + A_1 B_2' - A_2 B_1',$$

and by (130) and (131)

$$\begin{aligned} A_1' B_2 &= \frac{v z_2}{W} (A_1 B_2 z_1 + B_2 B_1 z_2), \\ A_2 B_1' &= -\frac{v z_1}{W} (A_1 A_2 z_1 + A_2 B_1 z_2), \\ A_2' B_1 &= \frac{v z_2}{W} (A_2 B_1 z_1 + B_2 B_1 z_2), \\ A_1 B_2' &= -\frac{v z_1}{W} (A_1 A_2 z_1 + A_1 B_2 z_2), \end{aligned}$$

so,

$$\begin{aligned}
(A_1B_2 - A_2B_1)' &= \frac{vz_2}{W}(A_1B_2z_1 + B_2B_1z_2 - A_2B_1z_1 - B_2B_1z_2) \\
&\quad + \frac{vz_1}{W}(A_1A_2z_1 + A_2B_1z_2 - A_1A_2z_1 - A_1B_2z_2) \\
&= \frac{vz_1z_2}{W}(A_1B_2 - A_2B_1 + A_2B_1 - A_1B_2) = 0.
\end{aligned}$$

In fact  $A_1B_2 - A_2B_1 = 1$  since  $A_1(0)B_2(0) - A_2(0)B_1(0) = 1$ , and so,  $a_1b_2 - a_2b_1 = 1$ .

Next, given  $a, b \in \mathbb{C}$ , put

$$\begin{aligned}
\alpha(t) &= (b_2a - a_2b)A_1(t) + (ba_1 - ab_1)A_2(t), \\
\beta(t) &= (b_2a - a_2b)B_1(t) + (ba_1 - ab_1)B_2(t).
\end{aligned}$$

We can see that

$$\lim_{t \rightarrow +\infty} \alpha(t) = a \quad \text{and} \quad \lim_{t \rightarrow +\infty} \beta(t) = b.$$

In order to check that

$$u(t) = \alpha(t)z_1(t) + \beta(t)z_2(t),$$

is indeed a solution of (125), just note that  $A_iz_1 + B_iz_2$  are by construction solutions of the same linear equation (125). This concludes the first part of the proof.

For the second part, we need to show that the two solutions  $u_+$  and  $u_-$  defined in the lemma are linearly independent. For this, it is enough to show that the Wronskian of these two solutions does not vanish, i.e.

$$u_+(t)u'_-(t) - u'_+(t)u_-(t) \neq 0, \quad \forall t.$$

First, we have,

$$\begin{aligned}
\alpha_+(t) &= b_2A_1(t) - b_1A_2(t), \\
\beta_+(t) &= b_2B_1(t) - b_1B_2(t), \\
\alpha_-(t) &= -a_2A_1(t) + a_1A_2(t), \\
\beta_-(t) &= -a_2B_1(t) + a_1B_2(t).
\end{aligned}$$

Now,

$$\begin{aligned}
u_+u'_- &= u_+(\alpha'_-z_1 + \beta'_-z_2 + \alpha_-z'_1 + \beta_-z'_2) \\
&= u_+(-a_2A'_1z_1 + a_1A'_2z_1 - a_2B'_1z_2 + a_1B'_2z_2 + \alpha_-z'_1 + \beta_-z'_2),
\end{aligned}$$

but by (129) we have  $A'_iz_1 + B'_iz_2 = 0$ , and so

$$\begin{aligned}
u_+u'_- &= u_+(\alpha_-z'_1 + \beta_-z'_2) \\
&= (\alpha_+z_1 + \beta_+z_2)(\alpha_-z'_1 + \beta_-z'_2).
\end{aligned}$$

Similarly, we have

$$u_-u'_+ = (\alpha_-z_1 + \beta_-z_2)(\alpha_+z'_1 + \beta_+z'_2).$$



Thus

$$\begin{aligned}
u_+ u'_- - u_- u'_+ &= \alpha_+ \beta_- (z_1 z'_2 - z_2 z'_1) + \alpha_- \beta_+ (z'_1 z_2 - z_1 z'_2) \\
&= (z_1 z'_2 - z_2 z'_1) (\alpha_+ \beta_- - \alpha_- \beta_+) \\
&= -W (\alpha_+ \beta_- - \alpha_- \beta_+),
\end{aligned}$$

but

$$\begin{aligned}
\alpha_+ \beta_- - \alpha_- \beta_+ &= (b_2 A_1 - b_1 A_2) (-a_2 B_1 + a_1 B_2) - (-a_2 A_1 + a_1 A_2) (b_2 B_1 - b_1 B_2) \\
&= (A_1 B_2 - A_2 B_1) (a_1 b_2 - a_2 b_1) = 1.
\end{aligned}$$

Therefore

$$u_+ u'_- - u_- u'_+ = -W,$$

which is a constant, and from the beginning of the proof we see that if  $\lambda \neq 0$  then  $W = 2i\lambda$ , and if  $\lambda = 0$  then  $W = -1$ , and in both cases it is a non zero constant. Hence, our solutions  $u_+$  and  $u_-$  are linearly independent, and the lemma is proved.  $\square$

With these results, we are now ready to prove Proposition 14.

*Proof of Proposition 14.* First we treat the case where  $l = 0$ . By the last sentence of Lemma 15,  $\Psi_{00}^0$  is independent of  $r_*$ , and if  $\lambda \neq 0$  then it is zero. If  $\lambda = 0$  then using (110), we are left with a constant. The non zero constant case corresponds to a non zero solution of the Newman-Penrose equations that has a finite energy.

Now we consider the case  $l > 0$ . Let  $u$  be a solution of (125), i.e. of the form

$$u = C_1 u_+ + C_2 u_-,$$

with  $u_{\pm}$  having the properties mentioned in the Lemma 17. Assume first that  $\lambda \neq 0$ . Since

$$|u_{\pm}(r_*) e^{\mp i \lambda r_*} \xrightarrow{r_* \rightarrow +\infty} 1$$

$u_{\pm} \notin L^2(\mathbb{R})$ . As shown in the proof of Lemma 17,

$$u'_{\pm} = \alpha_{\pm} z'_1 + \beta_{\pm} z'_2,$$

and similarly  $u'_{\pm} \notin L^2(\mathbb{R})$ . This means that, for  $\lambda \neq 0$ , if  $u$  or  $u' \in L^2(\mathbb{R})$ , then  $u$  is not a solution of (125) unless it is zero.

Since  $\Psi_0$  satisfies equation (125) with  $v = l(l+1)V$ , and since  $V$  and  $V r_*^2$  are continuous functions of  $r_*$ , and are in  $L^1(\mathbb{R}_{r_*})$ , then by Lemma 17,  $\Psi_0$  is given by (127). Now assume that (123) is satisfied, then by (79) we have  $\partial_{r_*} \Psi_0 \in L^2(\mathbb{R}_{r_*})$ , thus  $\Psi_0$  must be zero by the above argument. This in turn when combined with (121) and (122) implies that  $\Psi_{\pm 1}$  also satisfy (125) with  $v = l(l+1)V$ . It follows that  $\Psi_{\pm 1}$  are both zero.

Now assume  $\lambda = 0$  ( $l > 0$ ). Let  $z_1(t) = 1$  and  $z_2(t) = t$  be the two linearly independent solutions of  $z''(t) = 0$ . Consider  $u$  a solution of (125) with  $\lambda = 0$ . From the proof of Lemma 17 we can see that

$$u = A + Bt \quad ; \quad A' + B't = 0,$$

for some functions  $A$  and  $B$ , from which it follows that  $u' = B$ . Since we know that  $B$  has a limit at infinity, then if  $u' \in L^2(\mathbb{R})$ , we must have  $\lim_{t \rightarrow +\infty} B(t) = 0$ . Also  $u$  can be written as a linear combination of the solutions  $u_{\pm}$ ,

$$u = C_1 u_+ + C_2 u_-.$$

But  $u_+ = \alpha_+(t) + \beta_+(t)t$ , and

$$\begin{aligned}\alpha_+(t) &= b_2 A_1(t) - b_1 A_2(t), \\ \beta_+(t) &= b_2 B_1(t) - b_1 B_2(t),\end{aligned}$$

with

$$b_i = \lim_{t \rightarrow +\infty} B_i(t),$$

and assuming  $u' \in L^2(\mathbb{R})$  implies that  $b_i = 0$  and so  $u_+ = 0$ . Thus,  $u = C_2 u_-$  and

$$u' = C_2 u'_- = C_2 \beta_- ,$$

but  $\lim_{t \rightarrow +\infty} \beta_- = 1$ , so  $C_2 = 0$ . Therefore  $u = 0$ .

Again,  $\Psi_0$  satisfies (125), and by (79) and (123) we have  $\partial_{r_*} \Psi_0 \in L^2(\mathbb{R}_{r_*})$ . Hence  $\Psi_0 = 0$ . In the proposition we assume that  $\Phi$  is a solution of Maxwell's equations, and thereby (76) and (78) hold. And since  $\lambda = 0$  and  $\Psi_0 = 0$ ,  $\Psi_{\pm}$  are constant functions in  $L^2(\mathbb{R}_{r_*})$ , i.e.  $\Psi_{\pm} = 0$ .

Accordingly, if  $\Phi$  satisfies the statement of the proposition then, noting that  $W_{00}^0$  is just a constant, which is easy to see from the definition, we have

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_0 \\ \Phi_{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ CW_{00}^0 \\ 0 \end{pmatrix},$$

which is what we aimed to prove. □

The solutions we will consider from now on are finite energy solutions with no stationary part, and by Proposition 14, these are the finite energy solutions in the orthogonal complement of the  $l = 0$  subspace, that is solutions of the form:

$$\Phi_{\pm 1}(t, r_*, \theta, \varphi) = \sum_{l=1}^{+\infty} \sum_{n=-l}^l \Psi_{\pm 1n}^l(t, r_*) W_{\pm 1n}^l(\theta, \varphi) \quad \Psi_{\pm 1n}^l \in L^2(\mathbb{R}_{r_*}), \quad (135)$$

$$\Phi_0(t, r_*, \theta, \varphi) = \sum_{l=1}^{+\infty} \sum_{n=-l}^l \Psi_{0n}^l(t, r_*) W_{0n}^l(\theta, \varphi) \quad \frac{\sqrt{f}}{r} \Psi_{0n}^l \in L^2(\mathbb{R}_{r_*}). \quad (136)$$

**Remark 18.** From now on,  $\mathcal{U}$  will denote the space of finite energy Maxwell data of the above form, i.e. without stationary parts.

## 2.4 Maxwell Potential

We briefly discuss the existence of a potential to a Maxwell solution, and the formulation of the Cauchy problem in terms of the potential.

A *potential*  $A$  of a Maxwell tensor field  $F$  is a 1-form such that  $F = dA$ . If we treat electromagnetism as a theory of potential 1-forms instead of 2-forms<sup>1</sup>, then Maxwell's equations can be derived as the field equations from the principle of stationary action with a scalar Lagrangian. The action is defined as the integral of the Lagrangian scalar. One get the field equations, also known as Euler-Lagrange equations, which are satisfied by the field when the field is a critical point of the action, hence the name of the principle.

Let  $A$  is a 1-form, and let  $F$  be the 2-form  $F = 2dA$ , i.e.

$$F_{ab} = \nabla_a A_b - \nabla_b A_a \quad (137)$$

The Lagrangian of electromagnetism is

$$\mathcal{L} = \frac{1}{2} F_{ab} F^{ab} .$$

The Euler-Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta A_b} - \nabla^a \frac{\delta \mathcal{L}}{\delta \nabla^a A_b} = 0 .$$

are then

$$\nabla^a F_{ab} = 0 .$$

Of course the other equation,  $dF = 0$ , follows from the identity  $d^2 = 0$ . For the Lagrangian  $\mathcal{L}$  there is a symmetric 2-tensor that is divergence-free whenever the Euler-Lagrange equations are satisfied, it is called the energy-momentum tensor, and is given by

$$\mathbf{T}_{ab} = \frac{1}{2} \left( \nabla_a A_c \frac{\delta \mathcal{L}}{\delta \nabla^b A_c} - g_{ab} \mathcal{L} \right) .$$

The electromagnetic energy-momentum tensor, can be written in terms of  $F$  as:

$$\mathbf{T}_{ab} = \frac{1}{4} g_{ab} F^{cd} F_{cd} - F_{ac} F_b{}^c .$$

This tensor is divergence-free if Maxwell equations are satisfied, even when it is not derived from a Lagrangian (see Lemma 29).

Clearly, the potential is not unique because of the gauge freedom coming from the fact that  $d^2 = 0$ . For example, if  $F = dA$  then  $F = dA'$  for all  $A' = A + dh$  with  $h$  a scalar function. While not all Maxwell 2-forms are exact, i.e. are the differential of 1-forms, the Poincare lemma says that locally all closed  $p$ -forms are exact. Since a Maxwell 2-form  $F$  is

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<sup>1</sup>In real life, electromagnetic potentials can be measured only with a gauge uncertainty, which is exactly measuring the 2-form. Potentials are therefore not considered as physical quantities.

closed by definition, then locally one can always find a potential that determines  $F$ , and any two potentials for  $F$  differs, at least locally, by exactly a differential since their difference is closed.

However, the global existence of a Maxwell potential is related to the topology of the spacetime. For example, on any contractible spacetime, that is, a spacetime that can be continuously retracted to a point, Poincaré lemma holds globally, and hence all Maxwell solutions are exact and have global potentials, moreover, the gauge freedom is reduced to a choice of a function defined globally.

A particularly useful tool to determine whether on a given spacetime global potentials exist or not, is to check the de Rham cohomology groups. Since every exact form is closed, then the quotient of the set of closed  $p$ -forms by exact  $p$ -forms is a well defined group, called the  $p$ -th de Rham cohomology group and denote by  $H_{dR}^p(\mathcal{E})$ , where  $\mathcal{E}$  is the concerned spacetime or manifold in general. Since  $H_{dR}^p(\mathcal{E})$  is the set of equivalence classes of closed forms that differs by an exact form, if  $H_{dR}^p(\mathcal{E})$  is trivial, i.e. is the zero group, then all closed  $p$ -forms are exact.

For Maxwell fields, we are interested in the first and second de Rham cohomology groups. This is because when the first group is trivial then closed 1-forms are exact, and so the gauge freedom in the Maxwell potential is then a choice of a globally defined function. This is the case, for example, in simply connected spacetimes. The lemma below shows that the only obstruction for finding a global potential is part of the charge, namely, the “magnetic charge”: The charge  $\Psi_{00}^0$  is a complex constant, the real part is called the *electric charge* of the Maxwell field, and the imaginary part is called the *magnetic charge*.

**Lemma 19.** *If  $F$  is a Maxwell field on  $\mathcal{N}$  with spin components  $\Phi$  whose spin-weighted harmonic coefficients are  $\Psi_n^l$ , then  $F$  has a global potential if and only if the imaginary part of  $\Psi_{00}^0$  vanish.*

*Proof.* Our spacetime is  $\mathcal{N} = R^2 \times \mathcal{S}^2$ , the second de Rham cohomology group is non-trivial,  $H_{dR}^2(\mathcal{N}) \simeq \mathbb{R}$ . Consider the orientation form  $\omega = \sin(\theta)d\theta \wedge d\varphi$  on  $\mathcal{S}^2$  as a 2-form on  $\mathcal{N}$ . Clearly,  $d\omega = 0$ , so  $[\omega] \in H_{dR}^2(\mathcal{N})$ , and since  $\int_{\mathcal{S}^2} \omega = \text{Vol}_{\mathcal{S}^2}$  the volume of  $\mathcal{S}^2$ , it is non zero, and so  $\omega$  is not exact, by Stokes. Furthermore,  $\omega$  is a solution for Maxwell’s equations:  $d\omega = 0$  and  $d \star \omega = 0$  as  $\star \omega = V dt \wedge dr$ .

Let  $\omega_k = k\omega$  for  $k \in \mathbb{R}$ . Since for  $k \neq k'$ ,  $\omega_k - \omega_{k'}$  is a non zero multiple of  $\omega$ ,  $\omega_k - \omega_{k'}$  is not exact. So for any  $[\sigma] \in H_{dR}^2(\mathcal{N})$ ,  $\exists k \in \mathbb{R}$  such that  $\omega_k \in [\sigma]$ . Thus,  $\sigma - \omega_k = dA$  for some 1-form  $A$  globally defined. Therefore, if  $\sigma$  is a solution to Maxwell’s equations then so is  $dA$ . And so, any Maxwell solution can be written as  $\omega_k + dA$ .

It follows that there are some  $k$  and  $A$  such that  $F = \omega_k + dA$ . We recall from (54) that  $\Phi_0 = V^{-1}F_{10} + \frac{i}{\sin(\theta)}F_{23}$  where  $F_{23}$  is the  $d\theta \wedge d\varphi$  component. So  $F_{23} = k \sin(\theta) + d_{\mathcal{S}^2}(A^\varphi d\varphi - A^\theta d\theta)$ . From (75),

$$\Psi_{00}^0 = \int_{\mathcal{S}^2} \Phi_0 \bar{W}_{00}^0 d^2\omega ,$$

now,  $W_{00}^0$  can be taken to be  $(4\pi)^{-\frac{1}{2}}$ , so

$$\begin{aligned}\Psi_{00}^0 &= \frac{1}{2\sqrt{\pi}} \int_{S^2} V^{-1} F_{10} + \frac{i}{\sin(\theta)} F_{23} d^2\omega \\ &= \frac{1}{2V\sqrt{\pi}} \int_{S^2} F_{10} d^2\omega + \int_{S^2} \frac{i}{2\sqrt{\pi} \sin(\theta)} (k \sin(\theta) + d_{S^2}(A^\varphi d\varphi - A^\theta d\theta)) d^2\omega ,\end{aligned}$$

but the integral of the differential term is zero by the Stokes. Thus,

$$\Psi_{00}^0 = \frac{1}{2V\sqrt{\pi}} \int_{S^2} F_{10} d^2\omega + \frac{ik}{2\sqrt{\pi}} \text{Vol}_{S^2},$$

and hence,  $F$  is a differential if and only if  $\Psi_{00}^0$  is a real constant<sup>1</sup>.  $\square$

We excluded charged solutions since they do not decay. In particular, this excludes solutions with non zero magnetic charge. Thus, solutions that are of interest to us are defined globally by a potential. Furthermore, the first order de Rham cohomology group of  $\mathcal{N}$  vanishes. Thus we can fix the gauge by choosing a globally defined function.

A usual gauge (partial) fixing in general relativity is the Lorenz<sup>2</sup> gauge. The Lorenz gauge is a restriction on the choice of the potential by the requiring it to be divergence-free. That is, if we have  $F = dA$ , then the Lorenz gauge is

$$\nabla^\alpha A_\alpha = 0. \quad (138)$$

The advantage of this constraint on  $A$ , particularly in general relativity, is that it is Lorentz invariant. However, there is some freedom left in the choice of  $A$ :  $A$  is defined up to a harmonic function. If  $h$  is a smooth function such that  $\square h := \nabla^\alpha \nabla_\alpha h = 0$  then we still have for  $A' = A + dh$ ,  $F = dA'$  and  $\nabla^\alpha A'_\alpha = 0$ .

If  $A$  is a potential for a Maxwell field, and if it satisfies the Lorenz gauge condition, then Maxwell's equations entails the  $A$  satisfies the following second order equation:

$$\square A_a + R_{ca} A^c = 0. \quad (139)$$

where  $\square = \nabla^\alpha \nabla_\alpha$  and  $R_{ca}$  is Ricci curvature tensor. This can be seen from (40) by applying  $\nabla^a$  to (137) and using the Lorenz gauge condition to replace the  $\nabla^a \nabla_b A_a$  term by the curvature term. Although equations (139) are a hyperbolic system, they are valid under the constraint Lorenz gauge constraints, it is therefore not clear that one can approach finite energy data for (139) by smooth compactly supported data still satisfying the Lorenz gauge condition. This is in fact possible thanks to the a fact that a sufficient condition for  $A_a$  to be a potential for a Maxwell field and to satisfy the Lorenz gauge condition is that  $A$  satisfies the Dirac equation, which is hyperbolic. We shall say more about this in appendix B, since we use spinors to discuss it. For now, this means that:

**Lemma 20.** *(Density) Smooth compactly supported Maxwell data that satisfy the constraint equation are dense in  $\mathcal{U}$ , the constrained space of finite energy.*

*Proof.* See (327) in appendix B.  $\square$

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<sup>1</sup> $\Psi_{00}^0$  is a constant by Lemma 5.

<sup>2</sup>This is Ludvig Lorenz and not Hendrik Antoon Lorentz.

# Chapter 3

## Decay

### 3.0 Introduction

In this chapter, we discuss the topic of decay, in particular, the decay of Maxwell fields in the exterior static region of RNdS black holes. Our motivations are twofold: On the one hand, we use part of the decay results (uniform decay) obtained here to construct a complete conformal scattering theory later in chapter 4, on the other hand, the subject of energy bounds and decay using Morawetz estimates in general relativity has gained attention in the last decades largely due to its fundamental role in the analysis of the nonlinear stability of spacetimes. The first motivations is discussed in chapter 4, so let us in short address the second.

#### 3.0.1 Motivation by Stability Problems

Global stability problems in the general theory of relativity require specific information about the asymptotic behaviour of the solutions to Einstein's equations, and often a control provided by precise decay estimates for test fields on the background spacetime is crucial to access these informations.

#### Minkowski Stability

A basic problem in general relativity is the question of stability of Minkowski spacetime, that is, whether any asymptotically flat initial data set which is sufficiently close to the trivial one gives rise to a global (i.e. geodesically complete) solution of the Einstein vacuum equations that remains globally close to Minkowski spacetime. The local existence of solutions of the initial value problem was proven by Y. Choquet-Bruhat [29] in 1952. In 1983 partial results were obtained by H. Friedrich [69] using conformal methods, and in the early 1990's, the global nonlinear stability of Minkowski spacetime was established in the important work of D. Christodoulou and S. Klainerman [34]<sup>1</sup>. The main tool they used for the energy estimates

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<sup>1</sup>See also [31, 33] for a summary of the proof. A revisit of the proof can be found in [93]

is the vector field method developed by Klainerman which generalizes the multiplier method in the works of C.S. Morawetz. They first obtain precise decay estimates [32] for the Bianchi equations (spin-2 zero rest-mass fields) on Minkowski which model linearized gravity on Minkowski spacetime. Then they prove that the same decay rates are still valid for the full Einstein equations. The perturbed spacetimes they construct has global features resembling those of Minkowski spacetime: a foliation by maximal spacelike slices given by the level hypersurfaces of a time function; an optical function whose level hypersurfaces describe the structure of future null infinity; a family of almost Killing and conformal Killing vector fields related to the time and optical functions. The symmetries and almost symmetries they use to get conserved and almost conserved quantities and to define the basic energy norms, are generated by these almost Killing and conformally Killing vector fields. The vector fields are the generator of time translation  $T$ , the generators of the Lorentz group (generators of rotations and boosts)  $\Omega_{ij}$  for  $i, j = 0, \dots, 3$ , the generator of scaling (dilation)  $S$ , and the generator of inverted time translation (conformal Morawetz vector field)  $K$ . In fact, the Lie derivatives along these vector fields are used to define the basic quantities, which give better control in the estimates. This work was the first step towards the proof of the stability of Kerr spacetime, which is a crucial question for the understanding of the large time evolution of black holes.

## Kerr Stability

Currently, many groups are concentrating on the stability of Kerr black holes<sup>1</sup>. Asymptotically flat vacuum initial data for the evolution problem in general relativity are expected to give rise to spacetimes that can be decomposed into regions each of which approaches a Kerr black hole. The Kerr black hole spacetime is expected to be the unique, stationary, asymptotically flat, vacuum spacetime containing a nondegenerate Killing horizon<sup>2</sup> [1]. This is relevant in the context of some main problems of the theory, such as the weak cosmic censorship conjecture. Proving the Kerr black-hole stability is a major step towards solving these problems. The multiplier method, the vector field method, and its generalizations, are being employed to obtain the required uniformly bounded energies and to prove Morawetz estimates for solutions of the wave equation on black-hole spacetimes, motivated by the fact that proving boundedness and decay in time for solutions to the scalar wave equation on the asymptotically flat exterior of the Kerr spacetime is an important model problem for the full black-hole stability problem. However, there are some fundamental difficulties in the Kerr case, mostly because of the lack of symmetries, the trapping effect ranging over a radial interval, and there is no positive conserved quantity since Kerr black holes do not admit global timelike Killing vector fields.

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<sup>1</sup>Works addressing the question of the stability of the Schwarzschild manifold can be found in [39, 56, 59, 81, 88, 132].

<sup>2</sup>A Killing horizon is a null hypersurface defined by the vanishing of the norm of a Killing vector field.

### 3.0.2 An Overview of Decay

Many of the decay results in the literature are for solutions of wave equations. Let alone that waves themselves are interesting as physical and mathematical objects, the reason behind the extensive study is that these prototype equations are important model problems but also appear as a fundamental part of the structure in many problems where their analysis is essential. For example, they model the propagation of several systems including Schrödinger, gravity, and of course Maxwell's equations that we are studying in the present work, in which as we shall see, a wave analysis on the middle spin component of the field plays a central role in obtaining the decay results. We present a quick overview on the history of decay estimates summarizing some methods used to obtain them and how they evolved to become more adaptable to different geometries.

#### Basic Notions

Consider the simple scalar wave equation on  $\mathbb{R}^{1+3}$ ,

$$\partial_t^2 u - \partial_{x_1}^2 u - \partial_{x_2}^2 u - \partial_{x_3}^2 u = 0 .$$

The explicit formula for solutions to the wave equation in one space dimension was due to D'Alembert. In three space dimensions, the wave equation admits radial solutions of the form

$$u = \frac{h(|x| - t)}{|x|}$$

where  $x \in \mathbb{R}^3$ ,  $|x| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$ , and  $h$  is any twice differentiable function on  $\mathbb{R}$ . Let  $u$  be such a solution and say  $h$  is a smooth function which is compactly supported in  $]0, +\infty[$ . This solution radiates away from the origin at speed 1 as  $t$  increases, and for some  $R > 0$ , it identically vanishes<sup>1</sup> for  $t > |x| + R$  (figure 3.1). Such a solution models a disturbance starting in a bounded region which then spreads outward and reaches every point in space, but for each point and after a finite amount of time, there is no disturbance left at all. In fact, this is true for all the solutions of the above wave equation on  $\mathbb{R}^{1+3}$  which start in confined regions, perhaps this is not seen as directly as in the above simple case, but it follows from Kirchhoff's formula (19th century) which can be proved by the method of spherical means. An equation having this property is said to satisfy the strong Huygens principle:

**Theorem** (Huygens Principle). *If the initial data,  $(u(0, x), \partial_t u(0, x))$  with  $x \in \mathbb{R}^3$ , for the above wave equation are supported in the ball  $B(0, R)$ , then the associated solution  $u$  satisfies,*

$$u(t, x) = 0 \quad \text{for all } |t| > |x| + R$$

For equations satisfying the strong Huygens principle, if we start from compactly supported initial data, the field decays infinitely fast in time because at each point in space it vanishes identically after a certain time. Despite this *pointwise decay*, there is a quantity determined by the solution which is conserved for all times due to the time translation

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<sup>1</sup>Actually these particular solutions, i.e.  $h$  having such support, vanish for  $t \geq |x|$ .



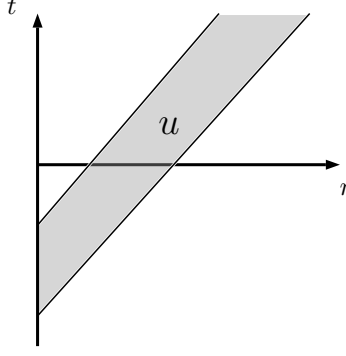


Figure 3.1: The support of  $u$  is contained in this diagonal strip and therefore  $u$  propagates exactly at speed 1. Here  $r = |x|$ .

symmetry of the system. This can be seen by multiplying the wave equation with the time derivative of the solution, called the *multiplier*, and rearranging,

$$\partial_t u \left( \partial_t^2 u - \sum_{i=1}^3 \partial_{x_i}^2 u \right) = \frac{1}{2} \partial_t \left( (\partial_t u)^2 + \sum_{i=1}^3 (\partial_{x_i} u)^2 \right) - \sum_{i=1}^3 \partial_{x_i} (\partial_t u \partial_{x_i} u),$$

if we now integrate the right hand side over a spacetime slab  $[t_1, t_2] \times \mathbb{R}^3$  and using the fact that  $u$  is a solution for the equation and that it has a compact support in space for all  $t$ , we arrive at the following identity:

$$\int_{\{t_1\} \times \mathbb{R}^3} \left( (\partial_t u)^2 + \sum_{i=1}^3 (\partial_{x_i} u)^2 \right) dx = \int_{\{t_2\} \times \mathbb{R}^3} \left( (\partial_t u)^2 + \sum_{i=1}^3 (\partial_{x_i} u)^2 \right) dx .$$

This quantity is called the (total) *energy*  $E[u](t)$  and it is conserved:  $E[u](t) = E[u](0)$ . However, the *local energy*

$$E[u](D, t) = \int_{\{t\} \times D} \left( (\partial_t u)^2 + \sum_{i=1}^3 (\partial_{x_i} u)^2 \right) dx$$

in any bounded region of space  $D = \{x \in \mathbb{R}^3; |x| \leq R\}$  is clearly not conserved and becomes zero after the wave leaves the region  $D$ .

The strong Huygens principle for the wave equation on flat spacetime is only valid in odd space dimensions starting at three. More general wave equations with a potential or on curved spacetimes satisfy a weak Huygens principle which says essentially that the local energy decays. One may then ask at what rate the local energy decays for, say, smooth compactly supported data. By a rate of decay in time for the local energy we mean a function  $d(t)$  that tends to zero when  $t$  tends to infinity and such that,

$$E[u](D, t) \leq d(t) E[u](0) .$$

For example, in two space dimensions the rate of pointwise decay of solutions to the above wave equation can be exactly  $t^{-1}$ , and thus the local energy decays as  $t^{-2}$ . The question is then when to expect that a solution should tend to zero with time pointwise and at which rate. The main obstruction to decay is the existence of finite energy stationary solutions, i.e. of the form  $e^{i\lambda t}v(x)$ . Provided we avoid such solutions, vector field and multiplier methods can be applied to obtain decay rates in fairly general situations.

## The Multiplier Method

The method of multipliers originated from the so-called Friedrichs' ABC method that dates back to K.O. Friedrichs in the 1950's. The method was used first to obtain decay and energy estimates in non-relativistic situations of geometrical optics, possibly outside of an obstacle whose geometry is known, and then was later applied to relativistic theories. The idea of this method is to multiply the equation with a factor  $Mu$ , where  $M$  is a linear first-order differential operator, defined as

$$Mu = Au + B \cdot \nabla u + C\partial_t u$$

and then to express the product as a divergence or at least an identity of the form

$$\text{divergence term} + \text{remaining terms} = 0$$

which is then integrated over a domain in  $\mathbb{R}^{1+n}$ ; the required estimates are derived by controlling the remainder. The multiplier method generalizes the ABC method: Suppose  $L$  is a differential operator of order  $k$  and consider the expression  $MuLu$ , where  $M$  is a differential operator of order  $k-1$ . With the right mix of derivatives, one hopes that  $MuLu$  can be written as  $\text{div}(Qu) + Ru$ , where  $Qu$  and  $Ru$  are quadratic expressions in the derivatives up to order  $k-1$ . The method of multipliers was used in the 1960's and 1970's to prove uniform decay results for the homogeneous linear wave equation ( $\square u = 0$ ) outside obstacles. C.S. Morawetz was the first to succeed in proving local energy decay for star-shaped obstacles with Dirichlet boundary condition using this method in 1961 [109]. In this work, the effects of scaling and the spread into space on the solution for the wave equation and its local energy is captured using the scaling multiplier

$$Su = t\partial_t u + r\partial_r u + u$$

and the following local energy decay is established

$$E[u](R, t) \leq \frac{C}{t} E[u](0),$$

where  $R$  is a region bounded between the obstacle and an outside sphere, and  $C > 0$  depends on the obstacle and the support of the initial data. This estimate then gives a pointwise decay of rate  $t^{-\frac{1}{2}}$ . A year later, Morawetz used the multiplier

$$K = \xi^2 \partial_\xi u + \eta^2 \partial_\eta u + (\xi + \eta)u = (t^2 + r^2)\partial_t u + 2tr\partial_r u + 2tu \quad ; \quad \xi = t - r, \eta = t + r,$$

in her work [110] to improve on the results of [109] and get faster decay rates of  $t^{-1}$  for the pointwise decay and  $t^{-2}$  for the local energy. She was motivated by the fact that for large times the disturbance is expected to be radiating outwards, and there will be little dependence on the angles, so,  $ru$  will approach a solution of  $\partial_r^2 w - \partial_t^2 w = 0$  for which an appropriate multiplier is  $Nw = p_1 \partial_r w + p_2 \partial_t w$ . The multiplier  $K$  is in fact related to a “time” translation: If we apply the Kelvin transformation on the coordinates  $(t, r, \theta, \varphi)$  given by,

$$\hat{t} = \frac{t}{r^2 - t^2}, \quad \hat{r} = \frac{r}{r^2 - t^2}$$

and leaving the angular variables the same, we can see that

$$\partial_{\hat{t}} = 2tr\partial_r + (r^2 + t^2)\partial_t.$$

This transformation is conformal and takes the cone  $r^2 = t^2$  at the origin to a cone at infinity and vice versa. It is not a surprise then that this vector field is appropriate for studying the asymptotic behaviour of the solution. Moreover, in 1968 [113] Morawetz used a radial multiplier of the form

$$\zeta(r)(\partial_r u + r^{-1}u)$$

where  $-\zeta(r)$  is a bump function around the origin, to obtain uniform integrated local energy estimates for the non-linear Klein-Gordon equation  $\square u + mu + P(u) = 0$ ,

$$\int_0^T E[u](\Omega, t) dt \leq K E[u](0),$$

where  $\Omega$  is a finite region in space and  $K$  a positive constant depending only on  $\Omega$  (its volume). She then uses this estimate to prove that the local energy decays but without giving a rate. Also in the same paper, it is proven that the  $L^2$ -norm of the solution decays. Before this work, also in 1968, a similar but complex radial multiplier was used by C.S. Morawetz and D. Ludwig [108] on a wave operator.

These multipliers and their corresponding vector fields have all found many important applications, most notably in General Relativity<sup>1</sup>. We mention an interesting work of Morawetz and W.A. Strauss [116] on decay and scattering for a nonlinear relativistic wave equation using these methods. Morawetz also established decay properties for Maxwell fields in [107]. For more on her work, we also refer to [111].

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<sup>1</sup>Of course, the results of Morawetz in other fields were as important, especially in the field of geometrical optics, and have been built upon and improved: Better decay rates have been achieved, as in odd dimensions  $n \geq 3$ , Huygens principle has been shown to imply an exponential rate of decay whenever there is some sort of decay by P.D. Lax, C.S. Morawetz, and R.S. Phillips in 1963 [97], and then by Morawetz [112] in 1966. Moreover, the class of obstacles under consideration has been enlarged using the method of multipliers after generalizing the multipliers to suit the geometry of the obstacle. Other wider generalizations later followed, W.A. Strauss [136] proved uniform local energy decay for the homogeneous linear wave equation using the Straussian vector fields. Then these Straussian vector fields were generalized by C.S. Morawetz, J.V. Ralston, and W.A. Strauss [115], by constructing a pseudo-differential operator  $P(x, D)$  (coming from a function  $p(x, \xi)$  called the “escape function”), and finally setting  $Pu$  as a multiplier.

## Vector Field Method

The vector field method is a flexible tool generalizing the multiplier method by making use of well adapted vector fields, related to symmetries or approximate symmetries of the equations, to derive decay estimates and thus to control the long time behaviour of solutions. The basics of this method has two aspects: The vector fields are used to define generalized energy norms, and, if they commute with the equations then one can derive identities for the energy norms considered. In the mid 1980's, S. Klainerman introduced the notion of generalized energy norms defined from the conformal group, which is generated by the vector fields  $T, S, K$ , and the  $\Omega_{ij}$ 's and whose elements have useful commuting properties among themselves and with the D'Alembertian, to obtain energy estimates and prove decay for solutions of the wave equation on  $\mathbb{R}^{n+1}$  [89, 91, 92]. These works of Klainerman were in essence a combination of the local energy decay estimates of C.S. Morawetz [110] and the conformal method of Y. Choquet-Bruhat and D. Christodoulou [30]. If  $\mathbb{A}$  is a set of vector fields and  $s \in \mathbb{N}$  we define the following norm of a function  $u$  on  $\mathbb{R}^{n+1}$  by

$$\|u(t)\|_{\mathbb{A},s,p} = \left( \sum_{k=0}^s \sum_{X_{i_j} \in \mathbb{A}} \int_{\mathbb{R}^n} |(X_{i_1} \dots X_{i_k} u)(t, x)|^p dx \right)^{\frac{1}{p}}.$$

Klainerman uses such norms for different subsets of the conformal group in place of  $\mathbb{A}$  to get what he calls global Sobolev inequalities (which are now known as Klainerman-Sobolev inequalities) of the form

$$|u(t, x)| \leq h(t, |x|) \|u(t)\|_{\mathbb{A},s,p}$$

for functions  $u$  with

$$\|u\|_{\mathbb{A},s,p}^{\#} = \sup_{t \geq 0} \|u(t)\|_{\mathbb{A},s,p} < \infty.$$

In the same papers he also gets decay estimates of the form

$$|u(t, x)| \leq d(t) \|u\|_{\mathbb{A},s,p}^{\#}.$$

Many results concerning the long-time and global existence were subsequently obtained using the methods of Klainerman. Klainerman himself used the results we hinted at above to prove long-time existence for a family of nonlinear wave equations [92], and using the same methods, he obtained existence and decay results (of rates  $t^{-\frac{5}{2}}$ ) for nonlinear Klein-Gordon equations on Minkowski spacetime [90]. Other works in the domain include L. Hörmander [82] in 1987 on nonlinear hyperbolic equations; A. Bachelot [9] in 1988 on Dirac-Klein-Gordon systems; J. Ginibre, A. Soffer, and G. Velo [73] in 1992 for the critical non-linear wave equation; and of course, the important work of D. Christodoulou and Klainerman on the stability of Minkowski spacetime [34].

Another important work of D. Christodoulou and S. Klainerman in 1990 is their paper [32] which studies the asymptotics of linear field equations in Minkowski spacetime. This paper was in fact the preparatory foundation for the proof of the nonlinear stability of Minkowski spacetime and in it, the vector field method took its standard current form, which was used

in many works on decay estimates that came later, and initiated, along with the proof of the stability of Minkowski spacetime, the project of proving Kerr stability. In this paper they derive uniform decay estimates for solutions to linear field equations in Minkowski spacetime which give precise information on the asymptotic behaviour of the solutions. It is based on geometric considerations of energy and generalized energy estimates. Their method relies on Klainerman's systematic use of the invariance properties of the field equations with respect to the conformal group of the Minkowski spacetime<sup>1</sup>, and was then extended to nonlinear cases, in particular to Einstein's vacuum equations [34].

The usefulness of the vector field method is best seen, although not exclusively<sup>2</sup>, in view of Noether's theorem in the case of general field equations derived from a quadratic action in the context of a Lagrangian theory. Let  $\phi$  be a general field on a general spacetime  $(M, g)$  and assume there is a scalar Lagrangian  $L$  which depends on the field and its derivatives (and possibly position in spacetime), used to define an action  $S$  as the integral of  $L$  on  $M$ . The field equations governing the behaviour of the field are derived by the "principle of least action", that is to say that  $\phi$  satisfies the field equations if it is a minimizer (or a critical point) of the action. These field equations are then a simple relation between the variation of the Lagrangian with respect to the field and its variation with respect to the field's derivatives, and are called the Euler-Lagrange equations:

$$\frac{\delta L}{\delta \phi} - \nabla^a \frac{\delta L}{\delta \nabla^a \phi} = 0 .$$

One can then define from the field and the Lagrangian a symmetric 2-tensor  $\mathbf{T}$  called the *energy-momentum tensor* (or stress-energy tensor) depending on the field and its derivatives (usually quadratically), which by the Euler-Lagrange equations turns out to be divergence-free<sup>3</sup>(see [74, 142] for example)<sup>4</sup>. The energy associated with a vector field  $X$  and evaluated on a hypersurface  $\Sigma$  is,

$$E_X(\Sigma) = \int_{\Sigma} \mathbf{T}_{ab} X^b d\sigma^a ,$$

where  $\alpha_a d\sigma^a$  is the 3-form  $\star\alpha$  given by the Hodge star operator for any 1-form  $\alpha$ .  $E_X(\Sigma)$  is sometimes referred to as the geometric energy. If  $\Sigma$  is spacelike and  $M$  is time-orientable we choose the normal on  $\Sigma$  to be future-oriented<sup>5</sup>, since if the energy-momentum tensor satisfies the *dominant energy condition*:

$$\mathbf{T}(V, W) \geq 0 , \quad \text{whenever } V \text{ and } W \text{ are future oriented, causal vectors.}$$

the above expression of the energy will be positive definite if  $X$  is timelike and future-oriented. When the spacetime is globally hyperbolic<sup>6</sup> or foliated by hypersurfaces  $\Sigma_t$  of constant time,

<sup>1</sup>Thus differing from the previous methods of analysing the fundamental solution.

<sup>2</sup>See for example [5].

<sup>3</sup>Although the natural way of obtaining an energy-momentum tensor is by means of a Lagrangian, one can as well directly consider 2-tensors with the desired properties and which might not be derived from a Lagrangian.

<sup>4</sup>For electromagnetic fields represented by 2-forms on the manifold, we actually vary the (local) potential and not the field (the 2-form) itself. See section 2.4.

<sup>5</sup>Depending on the sign conventions.

<sup>6</sup>Admitting a global Cauchy hypersurface. See [74].

then by Stokes' theorem, or more precisely by the divergence theorem, we have the following law: If  $\Omega_{t_1, t_2}$  is the region enclosed between  $\Sigma_{t_1}$  and  $\Sigma_{t_2}$ , then by the properties of  $\mathbf{T}$  we have,

$$E_X(\Sigma_{t_2}) - E_X(\Sigma_{t_1}) = \int_{\Omega_{t_1, t_2}} \nabla^a (\mathbf{T}_{ab} X^b) d\text{Vol}_g = \int_{\Omega_{t_1, t_2}} {}^{(X)}\pi_{ab} \mathbf{T}^{ab} d\text{Vol}_g ,$$

where  ${}^{(X)}\pi_{ab} = \mathcal{L}_X g_{ab} = 2\nabla_{(a} X_{b)}$  is called the deformation tensor of  $X$ . This law is called the deformation law. A vector field  $X$  is a conformal Killing vector field if the deformation tensor of  $X$  is proportional to the metric by a scalar factor  $\lambda$ , and  $X$  is a Killing vector field when  $\lambda = 0$ . We see then that when  $X$  is Killing the deformation law entails that the energy is conserved. The same happens when  $X$  is conformal Killing *and* the energy-momentum tensor is trace-free. In general, energy estimates are obtained by controlling the deformation term  ${}^{(X)}\pi_{ab} \mathbf{T}^{ab}$ , and in that case one says that one has an (almost) conservation law. A symmetry operator for an equation is defined to be a differential operator that takes solutions to solutions; in simple cases, the symmetry operator commutes with the equations. When  $Y$  is Killing, the Lie differentiation  $\mathcal{L}_Y$  with respect to  $Y$  is a symmetry operator for the wave equation and for Maxwell's equations among others. This means that when  $\mathbb{Y}$  is a set of Killing vector fields, one has identities for the energies defined using these vector fields but also for all Lie derivatives of the solutions with respect to these vector fields, at all orders. This adds on the control of the energies and allows better estimates and rates of decay.

In Minkowski spacetime, the conformal group is generated by conformal Killing vector fields, but only time and time inverted translation (or time acceleration as called in [32]) generators are timelike and thus can be associated with a positive definite energy. D. Christodoulou and S. Klainerman use arguments similar to the one above with the symmetry generators  $T, S$ , and  $K$  in [32] to obtain uniform bounds on the generalized energies and then, by means of Klainerman's global Sobolev inequalities, obtain the decay estimates for Maxwell and spin-2 equations. The latter are formally identical to the Bianchi identities for the Riemann curvature tensor and thus relevant to the understanding of the Einstein field equations. In fact, the methods they developed in [32] in the study of the spin-2 equations in Minkowski spacetime prepared for the subsequent study of the nonlinear stability of the Minkowski metric, as mentioned at the beginning of this introduction.

## Some Recent Works

The literature centred around decay estimates in general relativity is vast, so we refer to some recent works where additional references can be found. In particular, Blue's paper [22] about the decay of Maxwell fields in Schwarzschild in 2008 is central to our work, in fact, we show that the methods used in [22] can be applied to the case of RNdS black holes. Furthermore, this work shows that already existing methods of vector fields and Morawetz estimates can be applied to generic spherically symmetric black holes including the case of positive cosmological constant, with no real modifications (see section 3.2.3). In their paper of 1999 on nonlinear Schrödinger equation [94], I. Laba and A. Soffer introduced a Morawetz vector field on the Schwarzschild spacetime. They also introduce a modified

radial Morawetz multiplier, known as Soffer-Morawetz multiplier, based on the work of C.S. Morawetz, J.V. Ralston, and W.A. Strauss [115] (also see footnote 1). Through the 2000's, these tools were used on Schwarzschild's spacetime with further adaptations in the works of P. Blue, A. Soffer, and J. Sterbenz [20, 21, 22, 23, 24, 26] and in this present work, to help control the trapping terms. The radial Morawetz vector field that made these estimates possible is centred about the orbiting null geodesics. In 2000, in a paper on Maxwell fields on Schwarzschild's spacetime [84], W. Inglese and F. Nicolò give specific asymptotic estimates for different components of the field. A variant of the problem considered by P. Blue and J. Sterbenz in 2006 [26], about the uniform decay of local energy for wave equations, was independently studied by M. Dafermos and I. Rodnianski [42] in 2009 with a stronger estimate obtained near the event horizon (see also [41] by the same authors). M. Dafermos and I. Rodnianski in 2008 proved decay results for the wave equation on Schwarzschild-de Sitter spacetimes [40]. The same authors proved uniform boundedness for the wave equation on slow Kerr backgrounds in 2011 [45]. In the same year, and using the same methods, D. Tataru and M. Tohaneanu obtained local decay for energy also on Kerr [138], and later in 2013 D. Tataru extended the results to asymptotically flat stationary spacetimes [137]. A paper by J.-F. Bony and D. Häfner in 2008 [27] addresses the decay and non-decay of the local energy for the wave equation on the de Sitter-Schwarzschild metric. Several decay estimates with rates were obtained in the early 2010's: J. Luk in 2012 [100]; M. Tohaneanu [140]; M. Dafermos and I. Rodnianski in 2010 [43, 44] and in 2014 with Y. Shlapentokh-Rothman [47]. There is also a paper in 2013 by L. Andersson P. Blue, and J.-P. Nicolas on wave equations with trapping and complex potential that appear in the Maxwell and linearized Einstein systems on the exterior of a rotating black hole [6]. Two recent papers in 2015 were published by L. Andersson and P. Blue: [4] proving uniform energy bounds for Maxwell fields on Schwarzschild, and [5] in which they generalize the vector field method to take the hidden symmetries of Kerr spacetime into account (also see [2] for second order symmetry operators) and obtain an integrated Morawetz estimate and uniform bounds for a model energy for the wave equation. M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman's work on scattering for the wave equation on Kerr [46] contains decay results and the uniform energy equivalence needed for conformal scattering (see J.-P. Nicolas [120]). There is a more recent paper by L. Andersson, T. Bäckdahl, and P. Blue [3] in 2016 proving a new integrated local energy decay estimate for Maxwell fields outside a Schwarzschild black hole using a new superenergy tensor  $\mathbf{H}_{ab}$  defined in terms of the Maxwell field and its first derivatives. There have been works on Price's law (see [127, 128]), such as [105] in 2012 by J. Metcalfe, D. Tataru, and M. Tohaneanu. Finally, using different techniques (an integral representation of the propagators, see [62]) F. Finster, N. Kamran, J. Smoller, and S.-T. Yau obtain decay estimates for: Dirac on the Kerr-Newman spacetime [58] in 2002, and [61] in 2003; for the wave equation on Kerr [63, 64] in 2006 (corrected in 2008), and by F. Finster and J. Smoller [60] in 2008 also for the wave equation on Kerr.

## Maxwell Fields

Interest in the asymptotic behaviour of solutions to Maxwell's equations goes back at least to the 1970's [107], yet most of the literature is on scalar wave equations. It turns out

that some features of the Maxwell field can be captured in the behaviour of its components which are governed by wave-like equations, and results on the latter can be applied to study Maxwell systems. The behaviour of Maxwell fields is well-known in flat spacetime, at any point in space the effect of a signal dies off but the total energy carried by the signal is preserved, carried off in fact to infinity, as seen, for example, in the works of C.S. Morawetz in 1974 [114], and D. Christodoulou and S. Klainerman [32] in 1990 with rates of  $t^{-5/2}$  obtained using the full conformal group. In Schwarzschild, a rate of  $t^{-3}$  was obtained in regions bounded away from the horizon and null infinity, by R.H. Price in 1972 [127], and later by R.H. Price and L.M. Burko in 2004 [129]. Only time and spherical symmetries are available in this case, so the vector field method produces a slower rate of  $t^{-1}$ , as in P. Blue [22], however, the conformal energy associated to the conformal Morawetz vector field can be used to control all the components of the field, and no spherical harmonic decomposition is required. We prove here that this is also the case for generic spherically symmetric static black holes by working out the details on RNdS black holes; the results can be extended to more general situations including cosmological black holes. In 2015 J Sterbenz and D Tataru [134] obtained local energy decay for Maxwell fields on a general spherically symmetric spacetime but which is required to be asymptotically flat, thus they do not cover cases with positive cosmological constant. Also in 2015, J. Metcalfe, D. Tataru, and M. Tohaneanu studied the pointwise decay properties of solutions to the Maxwell system on a class of non-stationary asymptotically flat backgrounds [106]. Decay of waves and non-scalar fields (including Maxwell) on cosmological backgrounds with a de Sitter character were recently treated in the works of A. Vasy and P. Hintz [78, 79, 80] in 2015, using methods from microlocal analysis and it seems that their work needs positive cosmological constant (maybe with the exception of flat spacetime), whereas the vector field method which we use applies equally well with or without a (positive) cosmological constant under the conditions of section 3.2.3.

Before discussing the method we use, it is worth mentioning that there is a resemblance between Maxwell's equations and the spin-2 equations. A spin-2 field can be seen as a covariant 4-tensor with the following symmetries:

$$\begin{aligned} W_{abcd} &= -W_{bacd} , \\ W_{abcd} &= -W_{abdc} , \\ W_{[abc]d} &= 0 , \\ W_{abc}{}^a &= 0 , \end{aligned}$$

satisfying the equations,

$$\begin{aligned} \nabla^a W_{abcd} &= 0 , \\ \nabla_{[e} W_{ab]cd} &= 0 . \end{aligned}$$

The symmetries of a spin 2 field extend the antisymmetry of a Maxwell field, and the two systems of equations have similarities. If the Einstein vacuum equations are satisfied, then



the Ricci curvature vanishes, and the Weyl curvature satisfies the spin-2 field equations. In Minkowski spacetime, the spin-2 field equations models the linearization of Einstein's equations about the Minkowski solution. If one introduces a perturbed metric on Minkowski spacetime and treats the Weyl tensor as a tensor field on the original space-time, then, using the flatness of the background and the vanishing of the Christoffel symbols in cartesian coordinates, the difference between the covariant derivative of the Weyl tensor with respect to the perturbed metric and the original metric will be second order in the perturbation. Thus, ignoring second and higher order terms, the perturbed Weyl tensor satisfies the spin 2 field equations on the original metric. In this sense, the spin 2 field equations are the linearization of the Einstein vacuum equation about Minkowski spacetime. This is the motivation for studying the spin-2 field in [34]. However, this is not true for the linearization about other solutions. When linearizing around a curved space-time, the Christoffel symbols do not vanish, and the linearized Einstein equations do not reduce to the spin 2 field equations. Nevertheless, we expect that an analysis using the vector field method and Morawetz estimates will apply to the linearized gravity system. The linearized gravity equations are more complicated than the spin-2 field equations because there are terms involving the perturbed Christoffel symbols contracted against the unperturbed and non-vanishing Weyl tensor.

The arguments used in our work follow the same philosophy as in the works [22, 25, 26, 42] using the vector field method. The major obstacle is the trapping effect:

**Tapping Effect.** The conformal vector field

$$K = (t^2 + r_*^2)\partial_t + 2tr_*\partial_{r_*}$$

where  $r_*$  is the Regge-Wheeler coordinate, is timelike away from the  $t = \pm r_*$  hypersurfaces where it is null, and is used to introduce a positive definite quantity, a conformal energy. It is not conserved because of trapping. The presence of null geodesics at the photon sphere manifests itself through the trapping terms which are positive around the photon sphere. They appear as a contribution governing the growth of the conformal energy. It can be seen as the main “error” which is generated by the divergence of the conformal energy density. This effect can be overcome by introducing a radial vector field which points away from the photon sphere. This is a modified Morawetz radial multiplier of the form  $A\partial_{r_*}$ , where  $A$  is a continuously differentiable function of  $r_*$  that changes sign at the photon sphere, marked at  $r_* = 0$ .

The work can be divided into three main steps. In *the first step*, the conformal energy, defined by the conformal Morawetz vector field, of a Maxwell field is not conserved but can be controlled by the conformal charge<sup>1</sup> of the middle (or spin-weight zero) component of the field which satisfies a wave-like equation decoupled from the other components. This reduces the problem from spin-1 to spin-0, this is the so-called “spin-reduction”.

**Wave Analysis.** The conformal charge of the solutions to the wave-like equation is not conserved either. *The second step* is to control the error term using a radial Soffer-Morawetz

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<sup>1</sup>This is the conformal energy of the solution to the wave equation, but to avoid confusion with the conformal energy we call it a conformal charge.

multiplier which allows us to obtain a uniform bound on the conformal charge of the wave. Because this wave-like equation is actually simpler than the covariant wave equation, the usual analysis on the local energy of the wave equation is replaced by an analysis of an energy localized inside the light cone, and no decomposition on the spherical harmonics is required. Through some Hardy estimates, the trapping term is controlled by the energy generated by the radial multiplier and the integral of the energy localized inside the light cone. Since the trapping term controls the growth of the conformal charge, and since the energy (generated by time translation) is conserved, this gives a linear bound on the conformal charge. Using the Cauchy-Schwarz estimate and an integration by parts, the linear bound is improved to a uniform one. This also gives a uniform bound on the trapping term.

*The third step* is to use the conformal energy to control norms of the Maxwell field. The generalized energy and conformal energy of the Maxwell field generated by the rotation group are conserved and control the energy and the conformal charge of the middle spin component which in turn control the trapping term by the uniform bound. Thus, the conformal energy of the Maxwell field is controlled by the generalized energy and conformal energy of the initial data through a uniform bound. Since the integral of the trapping term has been controlled in the entire  $r_*$ -range, we have a uniform bound on the energy flux through any achronal hypersurface. This can be improved to a uniform decay rate of  $t^{-2}$ . The integrand in the conformal energy behaves like  $t^2$  times the Maxwell field components squared. Since the conformal energy is bounded, the field components decay in  $L^2_{\text{loc}}$  like  $t^{-1}$ . Control on radial derivatives is the main thing that we need to improve this into pointwise decay. Sobolev estimates can be used to convert  $L^2_{\text{loc}}$  decay for derivatives into  $L^\infty_{\text{loc}}$  decay. For this, we need decay on the spatial derivatives of the Maxwell field. From spherical symmetry, the Lie derivative of the Maxwell field along angular Killing vectors also satisfies the Maxwell equations and has the same type of local  $L^2$  decay as the field. Since differentiating in the radial direction does not generate a symmetry, the Lie derivative in that direction will not solve the Maxwell equations. To control the radial derivatives, we use the structure of the Maxwell equations. Using the time translation symmetry, we can control Lie time derivatives in  $L^2_{\text{loc}}$ . In a fixed, compact range of  $r_*$ -values, the covariant derivatives of the coordinate basis vectors are linear combinations of coordinate basis vectors with bounded smooth coefficients. We are working in  $L^2$  where we already control all the components. Thus, we control the difference between components of the covariant derivative in a direction and the covariant derivative of the components of the Maxwell tensor.

### 3.0.3 Summary of Sections

The aim of chapter 3 is to prove decay results for the Maxwell field on the Reissner-Nordström-de Sitter Black Hole. We prove two types of decay: The first is a decay of the energy of the Maxwell field on achronal hypersurfaces in the static region as the hypersurfaces approach timelike infinity, with quadratic decay rate. The second decay result is a pointwise decay in time with a rate of  $t^{-1}$ , also in the static region of the spacetime. Both results are consequences of the bounds on the conformal energy obtained from the wave analysis on the middle spin component of the field where we follow the work in [22]. The

above general outline of the work is detailed in the sections of the chapter as follows:

**Section 3.1:** This section is devoted to the analysis of the wave equation satisfied by the middle spin component of the field. We show that the energy for the wave equation is conserved and derive estimates for the conformal charge. Following [22], we use these estimates and a Morawetz estimate using a radial multiplier to obtain a uniform bound on the conformal charge where the Hardy estimates are needed. We note that this is where the exclusion of stationary solutions becomes necessary so that we can control the  $L^2$ -norm of the wave solution by the norm of its angular derivatives: the uniform bound we get controls the integral of the trapping term multiplied by the angular derivative of the wave solution.

**Section 3.2:** The second section of the chapter is the decay results. We introduce some norms on the Maxwell 2-form and discuss the energies of the field. The stress-energy tensor is used to define the energies on a hypersurface, we then write them for the Cauchy hypersurface  $\{t = 0\}$  in terms of the spin components. We get an almost conservation law, describing quantitatively the influence of the trapping effect on the conformal energy defined by the Morawetz vector field  $K$ , where the significance of the photon sphere is manifested. We then relate the wave energy of the middle component and that of the full field, bounding the former by the energy and the conformal energy of derivatives of the latter. Using these results a uniform bound on the conformal energy is obtained. At this point we state and prove the decay results in section 3.2.2. Finally, in section 3.2.3 we specify under what conditions this work and these decay results can be extended to other spacetimes, which include a wide range of spherically symmetric spacetimes.

## 3.1 Energy Estimates

In this section we derive several estimates that will help us prove decay. As we said in the introduction of this chapter, we need to obtain a uniform bound on the middle component, which satisfies a wave-like equation, in order to control the energies the Maxwell field. According, we shall first analyse the wave-like equation by prove different estimates on the energy and the conformal charge of the solutions. We then use Morawetz estimates to control the conformal charge and obtain the uniform bound.

### 3.1.1 The Wave-Like Equation

Recall that  $\Phi_0$  satisfies the following wave equation:

$$\partial_t^2 \Phi_0 = \partial_{r_*}^2 \Phi_0 + V \Delta_{S^2} \Phi_0. \quad (140)$$

We use  $\Delta$  as another symbol for the operator  $\Delta_{S^2}$ . Similarly,  $\nabla$  will designate the Levi-Civita connection on the sphere  $\nabla_{g_{S^2}}$ .

Since the box notation

$$\square u = \square_g u = \nabla^\alpha \nabla_\alpha u = g^{\mathbf{ab}}(\partial_{\mathbf{a}} \partial_{\mathbf{b}} u - \Gamma_{\mathbf{ab}}^{\mathbf{c}} \partial_{\mathbf{c}} u) \quad (141)$$

is reserved for the geometric wave equation on the RNdS manifold, we will use the  $\tilde{\square}$  symbol for our wave equation so that

$$\tilde{\square} u = \partial_t^2 u - \partial_{r_*}^2 u - V \Delta u \quad (142)$$

and the equation  $\tilde{\square} u = 0$  is nothing but (140).

We also use the dot notation to designate the scalar product on 1-forms on  $\mathcal{S}^2$  as well as for the divergence of a 1-form on  $\mathcal{S}^2$ . So we denote, for  $\alpha$  and  $\beta$ , 1-forms on  $\mathcal{S}^2$ ,

$$\alpha \cdot \beta := \alpha^a \beta_a = g_{\mathcal{S}^2}^{-1}(\alpha, \beta) \quad ; \quad |\alpha|^2 = g_{\mathcal{S}^2}^{-1}(\alpha, \alpha) \quad ; \quad \nabla \cdot \alpha := \nabla^a \alpha_a \quad (143)$$

We can readily see that for a smooth function  $u$ ,

$$\nabla \cdot (u\alpha) = \nabla u \cdot \alpha + u \nabla \cdot \alpha := g_{\mathcal{S}^2}^{-1}(\nabla u, \alpha) + u \nabla \cdot \alpha. \quad (144)$$

and

$$\nabla \cdot \nabla u = \nabla^a \nabla_a u = \Delta u. \quad (145)$$

Finally, throughout this section,  $C$  will designate a constant that may change from a line to another and which depends *neither* on  $t, r_*, \omega$ , *nor* on the solution  $u$ .

## Energy and Conformal Charge

For solutions of  $\tilde{\square} u = 0$ , there are two important quantities, one of which is conserved and the other is controlled, and are related to the time-translation vector field  $T = \partial_t$  and Morawetz vector field  $K = (t^2 + r_*^2)\partial_t + 2tr_*\partial_{r_*}u$ . These are the associated energy and the conformal charge, and they are given by the integral of their respective densities:

$$e = (\partial_t u)^2 + (\partial_{r_*} u)^2 + V |\nabla u|^2, \quad (146)$$

$$e_{\mathcal{C}} = \frac{1}{2}(t^2 + r_*^2)e + 2tr_*\partial_t u \partial_{r_*} u + e. \quad (147)$$

$$E[u](t) = \frac{1}{2} \int_{\Sigma_t} e dr_* d^2 w, \quad (148)$$

$$E_{\mathcal{C}}[u](t) = \frac{1}{2} \int_{\Sigma_t} e_{\mathcal{C}} dr_* d^2 w, \quad (149)$$

where  $\Sigma_t = \{t\} \times \mathbb{R} \times \mathcal{S}^2$ .

These densities are positive quantities noting that the conformal charge density  $e_{\mathcal{C}}$  can be written as the sum of squares,

$$e_c = \frac{1}{4}[(t + r_*)(\partial_t + \partial_{r_*})u]^2 + \frac{1}{4}[(t - r_*)(\partial_t - \partial_{r_*})u]^2 + \frac{1}{2}(t^2 + r_*^2)V|\nabla u|^2 + e. \quad (150)$$

The conservation laws can be obtained using a multiplier method, which is what we do in Lemma 22 and its proof, nonetheless, we obtain the conservation law first using a geometrical approach in which we use the Lagrangian method on the wave equation  $\square u = 0$ , thus relating  $T$  and  $E[u](t)$  is a geometric way.

The natural energy associated to the wave equation  $\square u = 0$  is generated by a stress-energy tensor contracted with the Killing vector field  $T = \partial_t$  which describes a static observer at infinity. The stress-energy tensor for the wave equation  $\square \phi = 0$  in the abstract index notation is given by

$$\mathbf{S}_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi.$$

For solutions of the wave equation, the stress-energy is divergence-free. More generally,

$$\nabla^a \mathbf{S}_{ab} = \nabla_b \phi \square \phi. \quad (151)$$

**Lemma 21.** *The energy of solutions to  $\tilde{\square} u = 0$  is conserved, i.e.*

$$E[u](t) = E[u](0) \quad \forall t \in \mathbb{R}.$$

*Proof.* First we show that

$$\tilde{\square} u = r f \square \left( \frac{u}{r} \right) - \frac{f f'}{r} u. \quad (152)$$

Calculating directly from (141), we have

$$\begin{aligned} \square u &= \frac{1}{f} (\partial_t^2 u - \partial_{r_*}^2 u) - \frac{2}{r} \partial_{r_*} u - \frac{1}{r^2} \left( \partial_\theta^2 u + \frac{1}{\sin(\theta)^2} \partial_\varphi^2 u + \cot(\theta) \partial_\theta u \right) \\ &= \frac{1}{f} (\partial_t^2 u - \partial_{r_*}^2 u) - \frac{2}{r} \partial_{r_*} u - \frac{1}{r^2} \Delta u. \end{aligned}$$

The only terms that still need to be computed are those with partial derivative with respect to  $r_*$ ,

$$\partial_{r_*} \left( \frac{u}{r} \right) = \frac{1}{r} \partial_{r_*} u - \frac{f}{r^2} u, \quad (153)$$

$$\partial_{r_*}^2 \left( \frac{u}{r} \right) = \frac{1}{r} \partial_{r_*}^2 u - 2 \frac{f}{r^2} \partial_{r_*} u + u \frac{f}{r^3} (2f - r f'). \quad (154)$$

Putting these terms in the above expression of  $\square u$  and using (142) we get (152).

The natural energy associated to the wave equation is:

$$\tilde{E}[\phi](t) = \int_{\Sigma_t} \mathbf{S}_{ab} \hat{T}^a T^b d\sigma \quad (155)$$

where  $d\sigma$  is the induced measure defined on  $\Sigma_t$ . If  $\phi$  is a solution of the wave equation then the energy defined above is conserved. In general, for any  $\phi$

$$\tilde{E}[\phi](t) - \tilde{E}[\phi](0) = \int_{[0,t] \times \mathbb{R} \times \mathcal{S}^2} (\nabla^a \mathbf{S}_{ab}) T^b d^4x . \quad (156)$$

where  $T = \partial_t$ ,  $\hat{T} = f^{-1}T$ , and  $d^4x$  is the 4-volume measure on the RNdS manifold. To see why (156) is true, we start with the difference and apply the divergence theorem (see 318 in Appendix A), noting that  $\hat{T}^a$  is the unit normal vector field on  $\Sigma_t$ ,

$$\begin{aligned} \tilde{E}[\phi](t) - \tilde{E}[\phi](0) &= \int_{\Sigma_t} \mathbf{S}_{ab} T^b \hat{T}^a d\sigma - \int_{\Sigma_0} \mathbf{S}_{ab} T^b \hat{T}^a d\sigma \\ &= \int_{[0,t] \times \mathbb{R} \times \mathcal{S}^2} \nabla^a (\mathbf{S}_{ab} T^b) d^4x \\ &= \int_{[0,t] \times \mathbb{R} \times \mathcal{S}^2} [(\nabla^a \mathbf{S}_{ab}) T^b + (\nabla^a T^b) \mathbf{S}_{ab}] d^4x , \end{aligned}$$

but  $\mathbf{S}_{ab}$  is symmetric, so

$$(\nabla^a T^b) \mathbf{S}_{ab} = \frac{1}{2} (\nabla^a T^b + \nabla^b T^a) \mathbf{S}_{ab} ,$$

and  $T$  is Killing i.e.

$$0 = \mathcal{L}_T g^{ab} = \nabla^a T^b + \nabla^b T^a .$$

On the other hand,

$$d\sigma = \hat{T} \lrcorner d^4x = \left( f^{-\frac{1}{2}} \partial_t \right) \lrcorner (f r^2 dt \wedge dr_* \wedge d^2\omega) = f^{\frac{1}{2}} r^2 dr_* \wedge d^2\omega ,$$

where  $d^2\omega = \sin(\theta) d\theta \wedge d\varphi$  is the Euclidean area element on  $\mathcal{S}^2$ . And so,

$$\tilde{E}[\phi](t) = \int_{\Sigma_t} \mathbf{S}_{00} r^2 dr_* d^2\omega$$

with

$$\mathbf{S}_{00} = \frac{1}{2} \left( (\partial_t \phi)^2 + (\partial_{r_*} \phi)^2 + \frac{f}{r^2} |\nabla \phi|^2 \right) ,$$

$$\nabla \phi = \partial_\theta \phi d\theta + \partial_\varphi \phi d\varphi , \quad \text{and} \quad |\nabla \phi|^2 = \nabla \phi \cdot \nabla \phi = (\partial_\theta \phi)^2 + \sin(\theta)^{-2} (\partial_\varphi \phi)^2 .$$

We can write

$$\tilde{E}[\phi](t) = \frac{1}{2} \int_{\Sigma_t} \left( (\partial_t(r\phi))^2 + (r \partial_{r_*} \phi)^2 + \frac{f}{r^2} |\nabla(r\phi)|^2 \right) dr_* d^2\omega ,$$

and since the set of smooth compactly supported functions in the variable  $r_*$  is dense in the finite energy space, we can apply a double integration by parts on the middle term of the energy, and making use of the previous calculations in (153) and (154) we get

$$\begin{aligned}
\int_{\Sigma_t} (r \partial_{r_*} \phi)^2 dr_* d^2\omega &= - \int_{\Sigma_t} \phi \partial_{r_*} (r^2 \partial_{r_*} \phi) dr_* d^2\omega \\
&= - \int_{\Sigma_t} r \phi \left( 2f \partial_{r_*} \left( \frac{1}{r} r \phi \right) + r \partial_{r_*}^2 \left( \frac{1}{r} r \phi \right) \right) dr_* d^2\omega \\
&= - \int_{\Sigma_t} r \phi \left( -2 \frac{f^2}{r^2} r \phi - r \left( \frac{f f'}{r^2} r \phi - 2 \frac{f^2}{r^3} r \phi \right) + \partial_{r_*}^2 (r \phi) \right) dr_* d^2\omega \\
&= - \int_{\Sigma_t} r \phi \partial_{r_*}^2 (r \phi) dr_* d^2\omega + \int_{\Sigma_t} (r \phi)^2 \frac{f f'}{r} dr_* d^2\omega \\
&= \int_{\Sigma_t} (\partial_{r_*} (r \phi))^2 dr_* d^2\omega + \int_{\Sigma_t} (r \phi)^2 \frac{f f'}{r} dr_* d^2\omega .
\end{aligned}$$

Therefore, if we set  $u = r \phi$  then,

$$\tilde{E}[\phi](t) = E[u](t) + \frac{1}{2} \int_{\Sigma_t} \frac{f f'}{r} u^2 dr_* d^2\omega . \quad (157)$$

By (152),  $u$  is a solution of (140) if and only if  $\phi$  satisfies

$$\square \phi = \frac{f'}{r} \phi ,$$

It follows from (151) and (156) that

$$\begin{aligned}
\tilde{E}[\phi](t) - \tilde{E}[\phi](0) &= \int_{[0,t] \times \mathbb{R} \times \mathcal{S}^2} (\nabla^a \mathbf{S}_{ab}) T^b d^4x \\
&= \int_{[0,t] \times \mathbb{R} \times \mathcal{S}^2} \square \phi (\nabla_b \phi) T^b d^4x \\
&= \int_{[0,t] \times \mathbb{R} \times \mathcal{S}^2} (f' r^{-1} \phi) (\partial_t \phi) f r^2 dt dr_* d^2\omega \\
&= \int_{\mathbb{R} \times \mathcal{S}^2} r f f' \int_{[0,t]} \phi \partial_t \phi dt \quad dr_* d^2\omega \\
&= \frac{1}{2} \int_{\mathbb{R} \times \mathcal{S}^2} r f f' (\phi(t)^2 - \phi(0)^2) dr_* d^2\omega \\
&= \frac{1}{2} \int_{\Sigma_t} r f f' (\phi^2) dr_* d^2\omega - \frac{1}{2} \int_{\Sigma_0} r f f' (\phi^2) dr_* d^2\omega \\
&= \frac{1}{2} \int_{\Sigma_t} \frac{f f'}{r} u^2 dr_* d^2\omega - \frac{1}{2} \int_{\Sigma_0} \frac{f f'}{r} u^2 dr_* d^2\omega .
\end{aligned}$$

Evaluating the left hand side using (157) we obtain the conservation law.  $\square$

In order to control the conformal charge  $E_C[u]$  we use the *Morawetz multiplier*  $K(u)$ : developing  $K(u)\tilde{\square}u = 0$  and then integrating by parts. In fact, the conformal charge is more associated with  $T + K$  rather than just  $K$ . As we shall see, near the end of the proof of the next lemma, we use the fact  $E[u]$  is conserved to obtain control on the error term of the conformal charge, which is essentially the same as using the multiplier  $(T + K)(u)$  in place of  $K(u)$ .

**Lemma 22.** *If  $\tilde{\square}u = 0$ , then there is a non-negative compactly supported smooth function of  $r_*$ ,  $\chi_{trap}$ , such that for all  $t_1, t_2 > 0$ ,*

$$E_C[u](t_2) - E_C[u](t_1) \leq \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} t \chi_{trap} |\nabla u|^2 dt dr_* d^2\omega . \quad (158)$$

*Proof.* We develop  $K(u)\tilde{\square}u = 0$ . First, we have

$$\partial_t u \tilde{\square}u = \frac{1}{2} \partial_t e - \partial_{r_*} (\partial_t u \partial_{r_*} u) - \nabla \cdot (V \partial_t u \nabla u) , \quad (159)$$

since,

$$\begin{aligned} \partial_t (|\nabla u|^2) &= 2 (\nabla u \cdot \nabla (\partial_t u)) ; \\ \frac{1}{2} \partial_t e &= \partial_t u \partial_t^2 u + \partial_{r_*} u \partial_{tr_*}^2 u + V (\nabla u \cdot \nabla (\partial_t u)) ; \\ \partial_{r_*} (\partial_t u \partial_{r_*} u) &= \partial_{r_*} u \partial_{tr_*}^2 u + \partial_t u \partial_{r_*}^2 u ; \\ \nabla \cdot (V \partial_t u \nabla u) &= V (\nabla u \cdot \nabla (\partial_t u)) + V \partial_t u \nabla \cdot \nabla u . \end{aligned}$$

Similarly,

$$\partial_{r_*} u \tilde{\square}u = -\frac{1}{2} \partial_{r_*} e + \partial_t (\partial_{r_*} u \partial_t u) - \nabla \cdot (V \partial_{r_*} u \nabla u) + \frac{1}{2} \partial_{r_*} (V) |\nabla u|^2 + V \partial_{r_*} (|\nabla u|^2) .$$

Next we integrate  $K(u)\tilde{\square}u = 0$  over the domain  $\Omega = [t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2$  and use integration by parts. For simplicity, let,

$$dx = dt dr_* d^2\omega , \quad \text{and} \quad d\varsigma = dr_* d^2\omega .$$

We divide the integral into two parts,

$$\int_{\Omega} K(u) \tilde{\square}u dx = \int_{\Omega} (t^2 + r_*^2) \partial_t u \tilde{\square}u dx + \int_{\Omega} 2tr_* \partial_{r_*} u \tilde{\square}u dx .$$

For the first term of the first part, by integration by parts in the  $t$  variable, we have:

$$\frac{1}{2} \int_{\Omega} (t^2 + r_*^2) \partial_t e dx = \frac{1}{2} \left( \int_{\Sigma_{t_2}} (t^2 + r_*^2) e d\varsigma - \int_{\Sigma_{t_1}} (t^2 + r_*^2) e d\varsigma \right) - \int_{\Omega} t e dx .$$



We do a similar integration by parts for the second term but this time in the  $r_*$  variable noting that  $u$  is taken to be compactly supported in  $r_*$ .

$$\int_{\Omega} (t^2 + r_*^2) \partial_{r_*} (\partial_t u \partial_{r_*} u) dx = -2 \int_{\Omega} r_* \partial_t u \partial_{r_*} u dx .$$

The last term in the first part of the integral is zero by the divergence theorem on  $\mathcal{S}^2$ ,

$$\int_{\Omega} (t^2 + r_*^2) \nabla \cdot (V \partial_t u \nabla u) dx = 0 .$$

Next, and using the same technique, we have:

$$\begin{aligned} \frac{1}{2} \int_{\Omega} 2tr_* \partial_{r_*} e dx &= - \int_{\Omega} t e dx ; \\ 2 \int_{\Omega} tr_* \partial_t (\partial_{r_*} u \partial_t u) dx &= 2 \left( \int_{\Sigma_{t_2}} tr_* \partial_{r_*} u \partial_t u d\varsigma - \int_{\Sigma_{t_1}} tr_* \partial_{r_*} u \partial_t u d\varsigma \right) - 2 \int_{\Omega} r_* \partial_{r_*} u \partial_t u dx ; \\ 2 \int_{\Omega} tr_* \nabla \cdot (V \partial_t u \nabla u) dx &= 0 . \end{aligned}$$

Finally, an integration by parts on the last term yields,

$$\begin{aligned} \int_{\Omega} (tr_* \partial_{r_*} (V) |\nabla u|^2 + 2tr_* V \partial_{r_*} (|\nabla u|^2)) dx &= - \int_{\Omega} (tr_* \partial_{r_*} (V) |\nabla u|^2 + 2tV |\nabla u|^2) dx \\ &= - \int_{\Omega} t (r_* \partial_{r_*} V + 2V) |\nabla u|^2 dx . \end{aligned}$$

Putting everything together, we get:

$$\begin{aligned} \int_{\Omega} K(u) \tilde{\square} u &= 0 \\ \iff \frac{1}{2} \int_{\Sigma_{t_2}} \left( \frac{1}{2} (t^2 + r_*^2) e + 2tr_* \partial_t u \partial_{r_*} u \right) d\varsigma &- \frac{1}{2} \int_{\Sigma_{t_1}} \left( \frac{1}{2} (t^2 + r_*^2) e + 2tr_* \partial_t u \partial_{r_*} u \right) d\varsigma \\ &= \int_{\Omega} t (r_* \partial_{r_*} V + 2V) |\nabla u|^2 dx . \end{aligned}$$

By conservation of the energy  $E[u](t)$ ,

$$\int_{\Sigma_{t_2}} e d\varsigma - \int_{\Sigma_{t_1}} e d\varsigma = 0 ,$$

thus,

$$E_C[u](t_2) - E_C[u](t_1) = \int_{\Omega} t (r_* \partial_{r_*} V + 2V) |\nabla u|^2 dx = \int_{\Omega} 2tV \mathcal{T} |\nabla u|^2 dx .$$

where

$$\mathcal{T} = \frac{r_*}{2V} \partial_{r_*} V = 1 + r_* \left( \frac{f'}{2} - \frac{f}{r} \right) + 1, \quad (160)$$

is called the *trapping term*. From the sign argument in the proof of Proposition 1 page 38, we can see that as  $r_*$  goes to minus infinity,  $r$  approaches  $r_2$  the middle zero of  $f$ , and  $f'(r_2) > 0$ . Also, we see that as  $r_*$  goes to plus infinity,  $r$  approaches  $r_3$  the largest zero of  $f$ , and  $f'(r_3) < 0$ . This means that the limits of  $\mathcal{T}$  are negative at both infinities, and so,  $\mathcal{T}$  is positive only on a compact interval of  $r_*$ . Now since  $V > 0$ , there exists some non-negative compactly supported function  $\chi_{trap}$  of  $r_*$  which dominates  $2V\mathcal{T}$ . This proves (158).  $\square$

As we mentioned before, Lemma 21 can also be proved using a multiplier method. From (159) we see directly that

$$\int_{[0,t] \times \mathbb{R} \times \mathcal{S}^2} \partial_t u \tilde{\square} u dt dr_* d^2\omega = E[u](t) - E[u](0)$$

which is the conservation law since the integrals of the last two terms of the right hand side of (159) are zero by the divergence theorem.

### 3.1.2 Morawetz Estimate

In to obtain a uniform bound on the conformal charge, we use a weighted radial Soffer-Morawetz multiplier that points away from the photon sphere at  $r_* = 0$  so its weight changes sign there. The error terms coming from the multiplier method can be controlled by energy localized inside the light cone,

$$E_\ell[u](t) = \int_{\{t\} \times \{|r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} e dr_* d^2\omega. \quad (161)$$

Using this multiplier and suitable Hardy estimates, we control the error term of the conformal charge by the energy localized inside the light cone. This in turn is controlled by the conformal charge multiplied by  $t^2$ , the is because, inside the light cone, the conformal charge density controlled the energy density times a factor of  $t^2$ . This factor of  $t$  compensate for compensate for the factor in the linear bound on the conformal charge, allowing us to obtain the uniform bound we need.

For the rest of this section,  $u$  will be a smooth function of the form (136). Also, as before, let

$$dx = dt dr_* d^2\omega, \quad \text{and} \quad d\zeta = dr_* d^2\omega.$$

We say that two functions  $v$  and  $w$  are equivalent over a set  $A$  and write  $v \sim w$ , if there exists a positive constant  $C > 0$  such that for all  $x \in A$ ,

$$\frac{1}{C} v(x) \leq w(x) \leq C v(x).$$

## Useful Estimates and Identities

We will need a couple of estimates for the solutions of (140). Since we exclude the stationary solutions with finite energy of the Maxwell field equations, we can then benefit from the following estimates.

**Lemma 23.** *If  $u$  is of the form (136), then*

$$\int_{S^2} u^2 d^2\omega \leq \frac{1}{2} \int_{S^2} |\nabla u|^2 d^2\omega . \quad (162)$$

*Proof.* Recall that

$$M_1 \bar{M} = \Delta .$$

From (80)-(83) the proof of Lemma 5,

$$M_1 \bar{M} W_{0n}^l = -l(l+1) W_{0n}^l .$$

Applying an integration by parts on the sphere and using (136), noting that  $l(l+1) \geq 2$  for  $l \geq 1$ , we have:

$$\begin{aligned} \int_{S^2} |\nabla u|^2 d^2\omega &= \int_{S^2} \nabla u \cdot \nabla u d^2\omega \\ &= - \int_{S^2} u \Delta u d^2\omega \\ &= - \int_{S^2} u \Delta \left( \sum_{l=1}^{+\infty} \sum_{n=-l}^l \Psi_{0n}^l(t, r_*) W_{0n}^l(\theta, \varphi) \right) d^2\omega \\ &= - \int_{S^2} u \sum_{l=1}^{+\infty} \sum_{n=-l}^l \Psi_{0n}^l(t, r_*) \Delta W_{0n}^l(\theta, \varphi) d^2\omega \\ &= - \int_{S^2} u \sum_{l=1}^{+\infty} \sum_{n=-l}^l \Psi_{0n}^l(t, r_*) (-l(l+1)) W_{0n}^l(\theta, \varphi) d^2\omega \\ &\geq 2 \int_{S^2} u^2 d^2\omega . \end{aligned}$$

□

We now establish some Hardy-like estimates.

**Lemma 24.** *Let  $t \geq 1$ ,  $0 < \sigma$ , and let  $\xi$  be a non-negative function of  $r_*$  which is positive in an open non-empty subinterval of  $|r_*| \leq \frac{1}{2}$ , and  $u$  be a smooth compactly supported function. Then,*

$$\int_{\{t\} \times \{|r_*| \leq \frac{1}{2}\} \times S^2} \frac{u^2}{(1 + (r_*)^2)^{\sigma+1}} d\varsigma \leq C \int_{\{t\} \times \{|r_*| \leq \frac{1}{2}\} \times S^2} \left( \frac{(\partial_{r_*} u)^2}{(1 + (r_*)^2)^\sigma} + \xi u^2 \right) d\varsigma . \quad (163)$$

Moreover, if  $u$  is given by (136) then,

$$\int_{\{t\} \times \{|r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} \frac{u^2}{1+r_*^2} d\varsigma \leq CE_\ell[u](t), \quad (164)$$

*Proof.* For simplicity, we denote  $u(t, s = r_*, \omega)$  by  $u(s)$  since  $t$  is given and fixed, and  $\omega$  is irrelevant for this calculation. Let  $\alpha \geq 0$  for now. For  $s_1 > 0$  we have,

$$\begin{aligned} \frac{u(s_1)^2}{(1+s_1)^{\alpha+1}} - u(0)^2 &= \int_0^{s_1} \partial_s \left( \frac{u^2}{(1+s)^{\alpha+1}} \right) ds \\ &= \int_0^{s_1} \left( \frac{2u\partial_s u}{(1+s)^{\alpha+1}} - \frac{(\alpha+1)u^2}{(1+s)^{\alpha+2}} \right) ds. \end{aligned}$$

But for  $s \geq 0$ ,

$$\frac{1}{(1+s)^\alpha} \left( \frac{u}{1+s} \sqrt{\frac{(\alpha+1)}{2}} - \partial_s u \sqrt{\frac{2}{(\alpha+1)}} \right)^2 \geq 0,$$

hence,

$$\left( \frac{\alpha+1}{2} \right) \left( \frac{u^2}{(1+s)^{\alpha+2}} \right) + \left( \frac{2}{\alpha+1} \right) \left( \frac{(\partial_s u)^2}{(1+s)^\alpha} \right) \geq \frac{2u\partial_s u}{(1+s)^{\alpha+1}}.$$

Thus,

$$\frac{u(s_1)^2}{(1+s_1)^{\alpha+1}} - u(0)^2 \leq -\frac{\alpha+1}{2} \int_0^{s_1} \frac{u^2}{(1+s)^{\alpha+2}} ds + \frac{2}{\alpha+1} \int_0^{s_1} \frac{(\partial_s u)^2}{(1+s)^\alpha} ds,$$

and since

$$\frac{u(s_1)^2}{(1+s_1)^{\alpha+1}} \geq 0,$$

we have,

$$\int_0^{s_1} \frac{u^2}{(1+s)^{\alpha+2}} ds \leq \frac{4}{(\alpha+1)^2} \int_0^{s_1} \frac{(\partial_s u)^2}{(1+s)^\alpha} ds + \frac{2}{\alpha+1} u(0)^2.$$

Since the Max norm and the Euclidean norm are equivalent over  $\mathbb{R}^2$ , there are some  $a, b > 0$  such that for all  $s \in \mathbb{R}$ ,

$$a(1+|s|) \leq \sqrt{1+s^2} \leq b(1+|s|),$$

and so,

$$\frac{1}{C(\beta)} (1+|s|)^\beta \leq (1+s^2)^{\beta/2} \leq C(\beta) (1+|s|)^\beta \quad \beta \geq 0, \quad (165)$$

implying that,

$$\int_0^{s_1} \frac{u^2}{(1+s^2)^{1+\alpha/2}} ds \leq C \int_0^{s_1} \frac{(\partial_s u)^2}{(1+s^2)^{\alpha/2}} ds + Cu(0)^2. \quad (166)$$

The same estimate holds true for the function  $v$  defined by  $v(s) = u(-s)$ . Thus, for all  $s_1, s_2 > 0$  and all  $u$  smooth,

$$\int_{-s_2}^{s_1} \frac{u^2}{(1+s^2)^{1+\alpha/2}} ds \leq C \int_{-s_2}^{s_1} \frac{(\partial_s u)^2}{(1+s^2)^{\alpha/2}} ds + Cu(0)^2.$$

In particular, this holds over  $[-s_1 - s_0, s_1 - s_0]$  with  $|s_0| < s_1$ , and for the function  $v(s) = u(s + s_0)$ , i.e.

$$\int_{-s_1}^{s_1} \frac{u^2}{(1+(s-s_0)^2)^{1+\alpha/2}} ds \leq C \int_{-s_1}^{s_1} \frac{(\partial_s u)^2}{(1+(s-s_0)^2)^{\alpha/2}} ds + Cu(s_0)^2. \quad (167)$$

Since the function

$$\left( \frac{1+(s-s_0)}{1+s^2} \right)^\beta$$

is positive and tends to 1 at both infinities, we have the following equivalence: For all  $s, s_0 \in \mathbb{R}$  and  $|s_0| < a$  with  $a > 0$  and  $\beta \geq 0$ , there exists a constant  $\tilde{C}(a, \beta) > 0$ , depending on  $a$  and  $\beta$  only, such that,

$$\frac{1}{\tilde{C}(a, \beta)} (1+s^2)^\beta \leq (1+(s-s_0)^2)^\beta \leq \tilde{C}(a, \beta) (1+s^2)^\beta.$$

Using this equivalence, (167) becomes: For  $|s_0| < a \leq s_1 \in \mathbb{R}$  and  $\alpha \geq 0$ , there exists a constant  $C(a, \alpha) > 0$  such that,

$$\int_{-s_1}^{s_1} \frac{u^2}{(1+s^2)^{1+\alpha/2}} ds \leq C(a, \alpha) \int_{-s_1}^{s_1} \frac{(\partial_s u)^2}{(1+s^2)^{\alpha/2}} ds + C(a, \alpha) u(s_0)^2. \quad (168)$$

Set  $s_1 = \frac{1}{2}t$ ,  $a = \frac{1}{2}$ , and  $\sigma = \frac{1}{2}\alpha > 0$ , let  $\xi$  be any non-negative function of  $r_*$  which is positive on  $[c, d] \subset ]-a, a[$  with  $c < d$ . We integrate over  $\mathcal{S}^2$  and  $s_0 \in [c, d]$  where  $\xi$  is bounded below away from zero. Since  $[c, d] \subset ]-a, a[$ , we can extend the integration domain in the  $s_0$  variable to  $]-a, a[$ . Thus, there is some  $C > 0$ , which depends on  $\xi$  and  $\sigma$  only, such that (163) holds.

Similarly, in (168), let  $\alpha = 0$ ,  $a = \frac{3}{4}$ , and  $s_1 = \frac{3}{4}t$  ( $t \geq 1$ ). Integrating over  $s_0 \in [-\frac{3}{4}, \frac{3}{4}]$  and then over the 2-sphere  $\mathcal{S}^2$ , we get

$$\int_{\{t\} \times \{|r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} \frac{u^2}{1+r_*^2} d\varsigma \leq C \int_{\{t\} \times \{|r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} (\partial_{r_*} u)^2 d\varsigma + C \int_{\{t\} \times \{|r_*| \leq \frac{3}{4}\} \times \mathcal{S}^2} u^2 d\varsigma.$$

Since the continuous function  $fr^{-2}$  is positive for all  $r_* \in ]-\infty, +\infty[$ , then there is some  $\delta > 0$  such that  $\frac{1}{\delta} fr^{-2} > 1$  for  $|r_*| \leq \frac{3}{4}$ . If  $u$  is of the form (136), then by Lemma 23 we have,

$$\int_{\{t\} \times \{|r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} \frac{u^2}{1+r_*^2} d\varsigma \leq C \int_{\{t\} \times \{|r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} (\partial_{r_*} u)^2 d\varsigma + C \int_{\{t\} \times \{|r_*| \leq \frac{3}{4}\} \times \mathcal{S}^2} \frac{f}{r^2} |\nabla u|^2 d\varsigma,$$

from which (164) follows. □

Finally, we derive some identities on  $\mathcal{S}^2$ . Let  $\Theta_1$ ,  $\Theta_2$ , and  $\Theta_3$  be the generators of rotation around the  $x$ ,  $y$ , and  $z$  axes in  $\mathbb{R}^3$ :

$$\begin{aligned}\Theta_1 &= z\partial_y - y\partial_z, \\ \Theta_2 &= z\partial_x - x\partial_z, \\ \Theta_3 &= x\partial_y - y\partial_x.\end{aligned}$$

**Lemma 25.** *Let  $u$  be a smooth function on  $\mathcal{S}^2$  then,*

$$|\nabla u|^2 = \sum_{i=1}^3 (\Theta_i u)^2, \quad (169)$$

$$\Delta u = \sum_{i=1}^3 \Theta_i^2 u, \quad (170)$$

$$\int_{\mathcal{S}^2} |\Delta u|^2 d^2\omega = \int_{\mathcal{S}^2} \sum_{i=1}^3 |\nabla \Theta_i u|^2 d^2\omega. \quad (171)$$

*Proof.* We prove the first two identities in spherical coordinates  $(\theta, \varphi)$ . On  $\mathcal{S}^2$  we have,

$$\begin{aligned}x &= \sin(\theta) \cos(\varphi), \\ y &= \sin(\theta) \sin(\varphi), \\ z &= \cos(\theta),\end{aligned}$$

so,

$$\begin{aligned}\Theta_1 &= \sin(\varphi)\partial_\theta + \cot(\theta)\cos(\varphi)\partial_\varphi, \\ \Theta_2 &= \cos(\varphi)\partial_\theta - \cot(\theta)\sin(\varphi)\partial_\varphi, \\ \Theta_3 &= \partial_\varphi.\end{aligned}$$

Thus, noting that

$$\begin{aligned}|\nabla u|^2 &= (\partial_\theta u)^2 + \frac{1}{\sin(\theta)^2} (\partial_\varphi u)^2, \\ \Delta u &= \partial_\theta^2 u + \frac{1}{\sin(\theta)^2} \partial_\varphi^2 u + \cot(\theta) \partial_\theta u,\end{aligned}$$

a straightforward calculation gives (169) and (170). To show (171) we use the following properties of the commutators of the generators of rotation which are a direct calculation,

$$\begin{aligned}[\Theta_1, \Theta_2] &= \Theta_3, \\ [\Theta_2, \Theta_3] &= \Theta_1, \\ [\Theta_3, \Theta_1] &= \Theta_2,\end{aligned}$$

or using the Levi-Civita symbol,

$$\Theta_i \Theta_j - \Theta_j \Theta_i = [\Theta_i, \Theta_j] = \varepsilon_{ij}^k \Theta_k = \sum_{k=1}^3 \varepsilon_{ijk} \Theta_k, \quad (172)$$

where the Levi-Civita symbol is

$$\varepsilon_{ij}^k = \varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3); \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3); \\ 0 & \text{otherwise} \end{cases} \quad (173)$$

We also need the skew-symmetry of the rotation generators:

$$\int_{S^2} v \Theta_i u d^2\omega = - \int_{S^2} u \Theta_i v d^2\omega . \quad (174)$$

To prove this, it is enough to show that it holds for  $\partial_\varphi$ , since any other  $\Theta_i$  can be changed into  $\partial_\varphi$  via a permutation of the variables  $(x, y, z)$ . Indeed, let  $u$  be a smooth function on  $S^2$ ,

$$\int_{S^2} \Theta_i u d^2\omega = \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} (\partial_\varphi u) \sin(\theta) d\theta d\varphi = \int_{\theta=0}^{\pi} (u(2\pi) - u(0)) \sin(\theta) d\theta = 0 ,$$

since  $u(2\pi) = u(0)$  as  $\varphi = 0$  and  $\varphi = 2\pi$  correspond to the same points on the sphere. Applying this to the product  $vu$  in place of  $u$  we get (174). Now we prove (171). We have,

$$\begin{aligned} \int_{S^2} |\nabla u|^2 d^2\omega &= \int_{S^2} \left( \sum_{i=1}^3 \Theta_i^2 u \right) \left( \sum_{j=1}^3 \Theta_j^2 u \right) d^2\omega \quad \text{by (170)} \\ &= - \sum_{i,j=1}^3 \int_{S^2} \Theta_i u \Theta_i \Theta_j^2 u d^2\omega \quad \text{by (174)} \\ &= - \sum_{i,j=1}^3 \int_{S^2} \Theta_i u (\varepsilon_{ij}^k \Theta_k + \Theta_j \Theta_i) \Theta_j u d^2\omega \quad \text{by (172)} \\ &= - \sum_{i,j=1}^3 \int_{S^2} (\varepsilon_{ij}^k \Theta_i u \Theta_k \Theta_j u + \Theta_i u \Theta_j \Theta_i \Theta_j u) d^2\omega \\ &= - \sum_{i,j=1}^3 \int_{S^2} \varepsilon_{ij}^k \Theta_i u \Theta_k \Theta_j u d^2\omega + \sum_{i,j=1}^3 \int_{S^2} \Theta_j \Theta_i u \Theta_i \Theta_j u d^2\omega \\ &= - \sum_{i,j=1}^3 \int_{S^2} \varepsilon_{ij}^k \Theta_i u \Theta_k \Theta_j u d^2\omega + \sum_{i,j=1}^3 \int_{S^2} (\Theta_i \Theta_j u)^2 d^2\omega \\ &\quad - \sum_{i,j=1}^3 \int_{S^2} \varepsilon_{ij}^k \Theta_k u \Theta_i \Theta_j u d^2\omega \end{aligned}$$

By the antisymmetric nature of the Levi-Civita symbol, the first and the last integrals cancel out. So, we are left with

$$\int_{S^2} |\nabla u|^2 d^2\omega = \sum_{i,j=1}^3 \int_{S^2} (\Theta_i \Theta_j u)^2 d^2\omega .$$

By (169) the right hand side is nothing but

$$\int_{\mathcal{S}^2} \sum_{i=1}^3 |\nabla \Theta_i u|^2 d^2 \omega .$$

□

It is an easy calculation to show that the  $\Theta_i$ 's are Killing vector fields of the full metric (and of the Euclidean metric on  $\mathcal{S}^2$ ) and thus satisfy the Killing equation,

$$\nabla_\alpha (\Theta_i)_\beta + \nabla_\beta (\Theta_i)_\alpha = \partial_\alpha (\Theta_i)_\beta + \partial_\beta (\Theta_i)_\alpha - 2\tilde{\Gamma}_{\alpha\beta}^\gamma (\Theta_i)_\gamma = 0 .$$

Consequently, we can make use of the following important property of Killing vector fields.

**Lemma 26.** *Let  $X$  be any smooth vector field on a Lorentzian manifold endowed with the Levi-Civita connection. In abstract index notation, let the deformation tensor of  $X$  be*

$$^{(X)}\pi_{ab} = \nabla_a X_b + \nabla_b X_a .$$

*Let  $u$  be a smooth function then,*

$$\nabla^a \nabla_a (X^b \nabla_b u) = (\nabla^a {}^{(X)}\pi_{ab}) \nabla^b u + \frac{1}{2} (\nabla_b {}^{(X)}\pi_a{}^b) \nabla^a u + {}^{(X)}\pi_{ab} \nabla^a \nabla^b u + X^b \nabla_b \nabla^a \nabla_a u .$$

*Thus, if  $X$  is Killing, i.e. satisfies the Killing equation:*

$$^{(X)}\pi_{ab} = 0 ,$$

*then  $X$  commutes with the wave operator,*

$$\nabla^a \nabla_a (X^b \nabla_b u) = \square(X(u)) = X(\square u) = X^b \nabla_b \nabla^a \nabla_a u . \quad (175)$$

*Proof.* The proof is a direct computation and then using the symmetries of the curvature tensor of the metric. We have,

$$\begin{aligned} \nabla^a \nabla_a (X^b \nabla_b u) &= \nabla^a ((\nabla_a X^b) \nabla_b u + X^b \nabla_a \nabla_b u) \\ &= (\nabla^a \nabla_a X^b) \nabla_b u + 2(\nabla^a X^b)(\nabla_a \nabla_b u) + X^b \nabla^a \nabla_a \nabla_b u . \end{aligned}$$

Since  $\nabla$  is torsion-free then,

$$\nabla_a \nabla_b u = \nabla_b \nabla_a u ,$$

and so,

$$2(\nabla^a X^b)(\nabla_a \nabla_b u) = {}^{(X)}\pi_{ab} \nabla^a \nabla^b u .$$

The connection being torsion-free also implies also that for a vector field  $Y$ ,

$$R_{abc}{}^d Y^c = (\nabla_a \nabla_b - \nabla_b \nabla_a) Y^d ,$$



where  $R_{abc}{}^d$  is the curvature tensor. Thus,

$$X^b \nabla^a \nabla_a \nabla_b u = X^b \nabla_a \nabla_b \nabla^a u = R_{abc}{}^a X^b \nabla^c u + X^b \nabla_b \nabla^a \nabla_a u .$$

Adding and subtracting terms we also have,

$$\begin{aligned} (\nabla^a \nabla_a X^b) \nabla_b u &= (\nabla^a (X) \pi_{ab}) \nabla^b u - (\nabla_a \nabla_b X^a) \nabla^b u \\ &= (\nabla^a (X) \pi_{ab}) \nabla^b u - R_{abc}{}^a X^c \nabla^b u + \frac{1}{2} (\nabla_b (X) \pi_a{}^a) \nabla^b u . \end{aligned}$$

Everything is there except that we have additional curvature terms. Due to the compatibility of the connection with the metric,

$$R_{abcd} = -R_{bacd} = R_{badc} = R_{dcba} .$$

and the terms involving the curvature tensor cancel out.  $\square$

Therefore, as  $\Theta_i$ 's are Killing, they commute with the wave operator. Since they also commute with  $r$ , we have

$$\tilde{\square}(\Theta_i(u)) = \Theta_i(\tilde{\square}u) , \quad (176)$$

where  $\tilde{\square}$  was defined in (142). This implies that  $\Theta_i u$  is a solution of  $\tilde{\square}v = 0$  when  $u$  is. Observing that the  $\Theta_i$ 's commute with  $\Delta$  since they are also Killing on the Sphere, under the assumptions of Lemma 23 we have,

$$\int_{S^2} (\Theta_i u)^2 d^2\omega \leq \frac{1}{2} \int_{S^2} |\nabla \Theta_i u|^2 d^2\omega , \quad (177)$$

which can be directly seen from the proof of the Lemma by commuting with  $\Delta$  after the first integration by parts. Summing over  $i$  then using (169) and (171) adds the following inequality to Lemma 23,

$$\int_{S^2} u^2 d^2\omega \leq \frac{1}{2} \int_{S^2} |\nabla u|^2 d^2\omega \leq \frac{1}{4} \int_{S^2} |\Delta u|^2 d^2\omega . \quad (178)$$

## Uniform Bound on the Conformal Charge

We use a radial multiplier:

$$\gamma_{(u)} = g \partial_{r_*} u + \frac{1}{2} (\partial_{r_*} g) u ,$$

where  $u$  is a smooth solution of (140) and  $g$  in this paragraph denotes a function of the  $t$  and  $r_*$  variables only. We start with the following lemma.

**Lemma 27.** *Let  $\gamma_{(u)}$  be as above. Set*

$$E_\gamma[u](t) = \int_{\Sigma_t} \gamma_{(u)} \partial_t u d\varsigma ,$$

then,

$$2E_\gamma[u](t_2) - 2E_\gamma[u](t_1) = - \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} (2(\partial_{r_*} g)(\partial_{r_*} u)^2 - \frac{1}{2}u^2(\partial_{r_*}^3 g) - g|\nabla u|^2 \partial_{r_*} V - 2(\partial_t u)(\partial_t g)(\partial_{r_*} u) - u(\partial_t u) \partial_{tr_*}^2 g) dx . \quad (179)$$

*Proof.* We use dot for time derivative and prime for derivative with respect to the radial variable  $r_*$ . Since

$$2E_\gamma[u](t_2) - 2E_\gamma[u](t_1) = \int_{[t_1, t_2]} 2\dot{E}_\gamma[u](t) dt = \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} \partial_t(2\dot{u}\gamma_{(u)}) dx .$$

Using (140) we calculate,

$$\partial_t(2\dot{u}\gamma_{(u)}) = 2\gamma_{(u)}u'' + 2\frac{f}{r^2}\gamma_{(u)}\Delta u + 2\dot{u}\dot{\gamma}_{(u)} .$$

But

$$\begin{aligned} \gamma_{(u)}\Delta u &= \nabla \cdot (\gamma_{(u)}\nabla u) - \nabla \gamma_{(u)} \cdot \nabla u , \\ 2\gamma_{(u)}u'' &= \partial_{r_*}(gu'' + uu'g') - 2u''g' - uu'g'' , \\ 2\dot{u}\dot{\gamma}_{(u)} &= \partial_{r_*}(g\dot{u}^2) + u\dot{u}g' + 2\dot{u}gu' , \end{aligned}$$

and

$$\begin{aligned} \nabla \gamma_{(u)} \cdot \nabla u &= \frac{1}{2}\partial_{r_*}(g|\nabla u|^2) , \\ uu'g'' &= \frac{1}{2}\partial_{r_*}(u^2g'') - \frac{1}{2}u^2g''' . \end{aligned}$$

Thus

$$\begin{aligned} \partial_t(2\gamma_{(u)}\dot{u}) &= \partial_{r_*} \left( u''g + uu'g' - \frac{f}{r^2}g|\nabla u|^2 - \frac{1}{2}u^2g'' + g\dot{u}^2 \right) + 2\nabla \cdot \left( \frac{f}{r^2}\gamma_{(u)}\nabla u \right) \\ &\quad + \frac{1}{2}u^2g''' + u\dot{u}g' + 2\dot{u}gu' - \partial_{r_*} \left( \frac{f}{r^2} \right) g|\nabla u|^2 - 2g'u'^2 . \end{aligned}$$

Integrating over  $[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2$  we get (179) since the integrals of the first two terms are zero.  $\square$

We are now ready to establish the uniform bound estimates of this section.

**Proposition 28** (Uniform Bound). *Let  $u$  be a smooth solution of (140) of the form (136), and  $\chi$  any compactly supported smooth function. There exists a constant  $C > 0$  such that for all  $t \geq 0$  we have,*

$$E_C[u](t) \leq C(E_C[u](0) + E[\Delta^2 u](0)) , \quad (180)$$

$$\int_{[0, +\infty[ \times \mathbb{R} \times \mathcal{S}^2} t\chi|\nabla u|^2 dt dr_* d^2\omega \leq C(E_C[u](0) + E[\Delta^2 u](0)) . \quad (181)$$

*Proof.* Recall that

$$V = \frac{f}{r^2}.$$

Let  $t \geq 1$ . Following [22], we set

$$\begin{aligned} h(r_*) &= \int_0^{r_*} \frac{1}{(1 + (\epsilon y)^2)^\sigma} dy, \\ g &= t\mu h, \end{aligned}$$

where,  $\sigma \in [1, 2]$ ,  $\epsilon > 0$ , and  $\mu = \tilde{\mu}(\frac{r_*}{t})$  with  $\tilde{\mu}$  a smooth function with compactly supported in  $] -\frac{3}{4}, \frac{3}{4}[$  which is identically 1 on  $[-\frac{1}{2}, \frac{1}{2}]$ . Note that  $h$  is bounded.

The idea now is bound  $E_\gamma[u]$  and the error terms in Lemma 27 by the local energy  $E_\ell[u]$ . The bounds on the error terms are uniform, while the bound on  $E_\gamma[u]$  is linear. Using the Hardy estimates and these bounds we get a linear bound on the error term of the conformal charge and thus on the conformal charge itself. This linear bound can be improved to a uniform one using the fact that inside the light cone, the conformal charge density bounds the energy density times  $t^2$ . As before, dot and prime indicate differentiation with respect to  $t$  and  $r_*$  respectively.

We calculate the terms in (179) to get,

$$2E_\gamma[u](t_2) - 2E_\gamma[u](t_1) = - \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} (2t\mu h' u'^2 - t\mu V' h |\nabla u|^2) dx \quad (182)$$

$$+ \frac{1}{2} \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} t\mu h''' u^2 dx \quad (183)$$

$$- \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} 2th\mu' u'^2 dx \quad (184)$$

$$+ \frac{1}{2} \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} tu^2(3\mu'h'' + 3\mu''h' + \mu'''h) dx \quad (185)$$

$$+ \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} (2\dot{u}\dot{g}u' + u\dot{u}\dot{g}') dx. \quad (186)$$

All these integrals are in fact over the domain  $[t_1, t_2] \times \{|r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2$  because of  $\mu$  and its derivatives.

We start by bounding (184)-(186) by the integral of local energy  $E_\ell$ . To do so, we use (165):

Since  $\tilde{\mu}$  is constant on  $[-\frac{1}{2}, \frac{1}{2}]$ , its derivatives (denoted  $\frac{d^n \tilde{\mu}}{d\tau^n}$ ) are supported away from

zero, namely in  $] -\frac{3}{4}, -\frac{1}{2}] \cup [\frac{1}{2}, \frac{3}{4}[$ . For  $\sigma \in [1, 2]$  we have,

$$\begin{aligned} |h| &\leq C, \\ |h'| &= \frac{1}{(1 + (\epsilon r_*)^2)^\sigma} \sim \frac{1}{(1 + r_*^2)^\sigma} \leq \frac{1}{1 + r_*^2} \leq C, \\ |h'| &= \frac{1}{(1 + (\epsilon r_*)^2)^\sigma} \sim \frac{1}{(1 + |r_*|)^{2\sigma}} \leq \frac{1}{1 + |r_*|} \leq C, \end{aligned}$$

and for  $t \geq 1$ ,

$$\partial_{r_*}^n \mu = \frac{1}{t^n} \frac{d^n \tilde{\mu}}{d\tau^n} \left( \frac{r_*}{t} \right) \leq \frac{C}{t^n} \leq C,$$

and if in addition  $|\frac{r_*}{t}| \in [\frac{1}{2}, \frac{3}{4}[$  then,

$$\begin{aligned} \frac{t}{|r_*|} &\sim \frac{t}{1 + |r_*|}, \\ 1 < \frac{4}{3} &\leq \frac{t}{|r_*|} \leq 2, \\ |h''| &= \frac{2\epsilon^2 \sigma |r_*|}{(1 + (\epsilon r_*)^2)^{\sigma+1}} \sim \frac{1}{(1 + r_*^2)^{\sigma+\frac{1}{2}}} \leq \frac{1}{1 + r_*^2} \leq C. \end{aligned}$$

Therefore, by (164),

$$\begin{aligned} \left| \int_{\{t\} \times \mathbb{R} \times \mathcal{S}^2} 2th\mu' u'^2 d\varsigma \right| &\leq C \int_{\{t\} \times \{\frac{1}{2}t \leq |r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} u'^2 d\varsigma \leq CE_\ell[u](t), \\ \left| \frac{1}{2} \int_{\{t\} \times \mathbb{R} \times \mathcal{S}^2} 3tu^2 \mu' h'' d\varsigma \right| &\leq C \int_{\{t\} \times \{\frac{1}{2}t \leq |r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} 3tu^2 \left( \frac{1}{t} \right) \left( \frac{1}{1 + r_*^2} \right) d\varsigma \leq CE_\ell[u](t) \\ \left| \frac{1}{2} \int_{\{t\} \times \mathbb{R} \times \mathcal{S}^2} 3tu^2 \mu'' h' d\varsigma \right| &\leq C \int_{\{t\} \times \{\frac{1}{2}t \leq |r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} 3tu^2 \left( \frac{1}{t^2} \right) \left( \frac{1}{1 + r_*^2} \right) d\varsigma \leq CE_\ell[u](t) \\ \left| \frac{1}{2} \int_{\{t\} \times \mathbb{R} \times \mathcal{S}^2} tu^2 \mu''' h d\varsigma \right| &\leq C \int_{\{t\} \times \{\frac{1}{2}t \leq |r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} tu^2 \left( \frac{1}{t^3} \right) \frac{t^2}{|r_*|^2} d\varsigma \quad \left( \frac{t^2}{|r_*|^2} > 1 \right) \\ &\leq C \int_{\{t\} \times \{\frac{1}{2}t \leq |r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} tu^2 \left( \frac{1}{t^3} \right) \frac{t^2}{1 + r_*^2} d\varsigma \leq CE_\ell[u](t). \end{aligned}$$

This treats (184) and (185). For (186) we do the same.

$$\begin{aligned}
\dot{g} &= h \left( \mu - \frac{1}{t} \frac{d\tilde{\mu}}{d\tau} \left( \frac{r_*}{t} \right) \right) \leq C, \\
\dot{g}' &= h \left( \frac{1}{t} \frac{d\tilde{\mu}}{d\tau} \left( \frac{r_*}{t} \right) - \frac{1}{t^2} \frac{d^2\tilde{\mu}}{d\tau^2} \left( \frac{r_*}{t} \right) \right) + h' \left( \mu - \frac{1}{t} \frac{d\tilde{\mu}}{d\tau} \left( \frac{r_*}{t} \right) \right) \\
&\leq C \left( \frac{1}{t} \right) \frac{t}{1+|r_*|} + C \frac{1}{1+|r_*|} \leq C \frac{1}{1+|r_*|}, \quad \text{for } \left| \frac{r_*}{t} \right| \in \left[ \frac{1}{2}, \frac{3}{4} \right].
\end{aligned}$$

Now using this and (165) and the inequality  $(a-b)^2 \geq 0$ , we have:

$$\begin{aligned}
\left| \int_{\{t\} \times \mathbb{R} \times \mathcal{S}^2} (2\dot{u}g'u' + u\dot{u}g') d\varsigma \right| &\leq C \int_{\{t\} \times \{|r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} |\dot{u}u'| d\varsigma + C \int_{\{t\} \times \{\frac{1}{2}t \leq |r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} |\dot{u}| \frac{|u|}{1+|r_*|} d\varsigma \\
&\leq C \int_{\{t\} \times \{|r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} (|\dot{u}|^2 + |u'|^2) d\varsigma \\
&\quad + C \int_{\{t\} \times \{|r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} \left( |\dot{u}|^2 + \frac{u^2}{(1+|r_*|)^2} \right) d\varsigma \\
&\leq CE_\ell[u](t).
\end{aligned}$$

The same calculation gives,

$$|E_\gamma[u](t)| \leq CtE_\ell[u](t). \quad (187)$$

Recapitulating, we have shown that for  $t \geq 1$ ,

$$\begin{aligned}
2E_\gamma[u](t_2) - 2E_\gamma[u](t_1) &\leq - \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} (2t\mu h' u'^2 - t\mu V' h |\nabla u|^2) dx + \frac{1}{2} \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} t\mu h''' u^2 dx \\
&\quad + C \int_{t_1}^{t_2} E_\ell[u](t) dt.
\end{aligned}$$

We still need to treat the first two terms on the right hand side. Since  $h' > 0$ ,  $h$  is increasing and has the sign of  $r_*$ . By the argument used in the proof of Proposition 1,

$$V' = \frac{f}{r^3} (rf' - 2f)$$

has the opposite sign of  $r_*$ . So,  $hV' \leq 0$  with the equality holding only at  $r_* = 0$  where both functions vanish. Thus, the first term is negative. We need to control the second. We have,

$$|h'''| = \frac{2\epsilon^2\sigma(2\sigma+1)(\epsilon r_*)^2 - 1}{(1+(\epsilon r_*)^2)^{\sigma+2}} \leq \frac{2\epsilon^2\sigma((2\sigma+1)(\epsilon r_*)^2 + 2\sigma + 1)}{(1+(\epsilon r_*)^2)^{\sigma+2}} = \frac{2\epsilon^2\sigma(2\sigma+1)}{(1+(\epsilon r_*)^2)^{\sigma+1}},$$

so,

$$\frac{1}{2} \int_{\{t\} \times \mathbb{R} \times \mathcal{S}^2} t \mu h''' u^2 d\varsigma \leq \epsilon^2 \sigma (2\sigma + 1) \int_{\{t\} \times \{|r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} t \mu \frac{u^2}{(1 + (\epsilon r_*)^2)^{\sigma+1}} d\varsigma, \quad (188)$$

We divide the integral into two parts. One which we bound by the local energy and one on which we apply (163) and eventually get absorbed by the right hand side of (182). Since  $t \geq 1$  and  $\sigma \geq 1$ ,

$$\begin{aligned} \epsilon^2 \sigma (2\sigma + 1) \int_{\{t\} \times \{\frac{1}{2}t \leq |r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} \mu \frac{t}{(1 + (\epsilon r_*)^2)} \frac{u^2}{(1 + (\epsilon r_*)^2)^\sigma} d\varsigma &\leq C \int_{\{t\} \times \{\frac{1}{2}t \leq |r_*| \leq \frac{3}{4}t\} \times \mathcal{S}^2} \frac{t^2}{r_*^2} \frac{u^2}{(1 + (r_*)^2)} d\varsigma \\ &\leq C E_\ell[u](t), \end{aligned}$$

using (164).

For the other part we need to keep track of the constants, in particular those involved in (163).  $-V'h$  is non-negative and vanishes only at zero, and so does

$$-V' \int_0^{r_*} \frac{1}{(1 + y^2)^\sigma} dy \leq -V'h.$$

So, using Lemma 23 and (163),

$$\int_{\{t\} \times \{|r_*| \leq \frac{1}{2}t\} \times \mathcal{S}^2} \frac{u^2}{(1 + (\epsilon r_*)^2)^{\sigma+1}} d\varsigma \leq C \int_{\{t\} \times \{|r_*| \leq \frac{1}{2}t\} \times \mathcal{S}^2} \left( \frac{(\partial_{r_*} u)^2}{(1 + (\epsilon r_*)^2)^\sigma} \right. \quad (189)$$

$$\begin{aligned} &\quad \left. -V' \left( \int_0^{r_*} \frac{1}{(1 + y^2)^\sigma} dy \right) u^2 \right) d\varsigma \\ &\leq C \int_{\{t\} \times \{|r_*| \leq \frac{1}{2}t\} \times \mathcal{S}^2} (2h'u'^2 - V'h|\nabla u|^2) d\varsigma, \quad (190) \end{aligned}$$

Now,  $\mu \equiv 1$  over  $\{|r_*| \leq \frac{1}{2}t\}$  so,

$$\frac{1}{2} \int_{\{t\} \times \{|r_*| \leq \frac{1}{2}t\} \times \mathcal{S}^2} t \mu h''' u^2 d\varsigma \leq \epsilon^2 \sigma (2\sigma + 1) C \int_{\{t\} \times \{|r_*| \leq \frac{1}{2}t\} \times \mathcal{S}^2} (2t \mu h' u'^2 - t \mu V'h |\nabla u|^2) d\varsigma.$$

We choose  $\epsilon^2 < \frac{1}{2\sigma(2\sigma+1)C}$ , so

$$2E_\gamma[u](t_2) - 2E_\gamma[u](t_1) \leq -\frac{1}{2} \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} (2t \mu h' u'^2 - t \mu V'h |\nabla u|^2) dx + C \int_{t_1}^{t_2} E_\ell[u](t) dt,$$

i.e.

$$\frac{1}{2} \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} (2t \mu h' u'^2 - t \mu V'h |\nabla u|^2) dx \leq -2E_\gamma[u](t_2) + 2E_\gamma[u](t_1) + C \int_{t_1}^{t_2} E_\ell[u](t) dt.$$

Using (176) we see that this estimate holds equally for  $\Theta_i u$ . Summing over  $i$  the estimate for  $\Theta_i u$ , and using (169) and (171) we get,

$$\frac{1}{2} \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} t \mu (2h' |\nabla u'|^2 - V' h (\Delta u)^2) dx \leq -2E_\gamma[\nabla u](t_2) + 2E_\gamma[\nabla u](t_1) + C \int_{t_1}^{t_2} E_\ell[\nabla u](t) dt . \quad (191)$$

where the energy terms denote,

$$E_\gamma[\nabla u](t) := \sum_{i=1}^3 E_\gamma[\Theta_i u](t) \quad ; \quad E_\ell[\nabla u](t) := \sum_{i=1}^3 E_\ell[\Theta_i u](t) ,$$

motivated by (169) and (171). Now, applying the estimate (163) for  $\Theta_i u$  then using (177), we get an estimate for  $\Theta_i u$  analogous to (190). Again, summing over  $i$  and using (169) and (171), still keeping  $t \geq 1$ , we have,

$$\int_{\{t\} \times \{|r_*| \leq \frac{1}{2}t\} \times \mathcal{S}^2} t \frac{|\nabla u|^2}{(1 + (\epsilon r_*)^2)^{\sigma+1}} d\varsigma \leq C \int_{\{t\} \times \mathbb{R} \times \mathcal{S}^2} t \mu (2h' |\nabla u'|^2 - V' h (\Delta u)^2) d\varsigma ,$$

Let  $a \geq 1$  such that  $\text{supp}(\chi_{\text{trap}}) \subseteq [-a/2, a/2]$  (see (158)). Then for some constant  $C > 0$  and all  $r_* \in [-a/2, a/2]$

$$\chi_{\text{trap}} \leq C \frac{1}{(1 + (\epsilon r_*)^2)^{\sigma+1}} ,$$

and hence for  $t \geq a$  ,

$$\int_{\{t\} \times \mathbb{R} \times \mathcal{S}^2} t \chi_{\text{trap}} |\nabla u|^2 d\varsigma = \int_{\{t\} \times \{|r_*| \leq \frac{1}{2}t\} \times \mathcal{S}^2} t \chi_{\text{trap}} |\nabla u|^2 d\varsigma \quad (192)$$

$$\leq C \int_{\{t\} \times \{|r_*| \leq \frac{1}{2}t\} \times \mathcal{S}^2} t \frac{|\nabla u|^2}{(1 + (\epsilon r_*)^2)^{\sigma+1}} d\varsigma . \quad (193)$$

Integrating from  $a$  to  $t' \geq a$ , then using (191), we get

$$\int_{[a, t'] \times \mathbb{R} \times \mathcal{S}^2} t \chi_{\text{trap}} |\nabla u|^2 dx \leq C \left| -2E_\gamma[\nabla u](t) \Big|_a^{t'} \right| + C \int_a^{t'} E_\ell[\nabla u](t) dt . \quad (194)$$

Now using (158), we then have

$$E_C[u](t') \leq E_C[u](a) + C \left| -2E_\gamma[\nabla u](t) \Big|_a^{t'} \right| + C \int_a^{t'} E_\ell[\nabla u](t) dt .$$

By (187),

$$E_C[u](t') \leq E_C[u](a) + C \sup_{t \in [0, t']} (t E_\ell[\nabla u](t)) + C \int_0^{t'} E_\ell[\nabla u](t) dt .$$

Since  $V > 0$  and  $\chi_{trap}$  has a compact support then  $\chi_{trap} \leq CV$ . For  $0 \leq t_2 \leq 1$ , we have,

$$\int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} t \chi_{trap} |\nabla u|^2 dx \leq C \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} V |\nabla u|^2 dx \leq C \int_{t_1}^{t_2} E[u](t) dt \leq C \int_{t_1}^{t_2} E_C[u](t) dt . \quad (195)$$

And for  $t_1 \geq 1$ ,

$$\int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} t \chi_{trap} |\nabla u|^2 dx \leq C \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} t^2 V |\nabla u|^2 dx \leq C \int_{t_1}^{t_2} E_C[u](t) dt . \quad (196)$$

Therefore, for all  $t_1, t_2 \geq 0$ , by (158) we have

$$E_C[u](t_2) - E_C[u](t_1) \leq \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} t \chi_{trap} |\nabla u|^2 dx \leq C \int_{t_1}^{t_2} E_C[u](t) dt . \quad (197)$$

By Gronwall's inequality,

$$E_C[u](t_2) \leq E_C[u](t_1) e^{c(t_2 - t_1)} , \quad (198)$$

then taking  $t_1 = 0$  and  $t_2 = a$  we get an exponential bound on the conformal energy,

$$E_C[u](a) \leq C E_C[u](0) ,$$

with  $C$  depending on  $a$  but not on  $u$ . Thus, for  $t' \geq a$ ,

$$E_C[u](t') \leq C \left( E_C[u](0) + \sup_{t \in [0, t']} (t E_\ell[\nabla u](t)) + \int_0^{t'} E_\ell[\nabla u](t) dt \right) . \quad (199)$$

Since  $E_\ell[\nabla u](t) \leq C E[\nabla u](t) = C E[\nabla u](0)$  by Lemma 21, this gives a linear bound on the conformal charge,

$$E_C[u](t) \leq C(1+t)(E_C[u](0) + E[\nabla u](0)) , \quad t \geq a . \quad (200)$$

Next, we derive an estimate for the local energy. Let  $\mathcal{S}_t = \{t\} \times [-\frac{3}{4}t, \frac{3}{4}t] \times \mathcal{S}^2$ , and recall (145). Integrating by parts then using Cauchy-Schwartz inequality, then a double integration by parts followed by another Cauchy-Schwartz inequality, we have:

$$\begin{aligned} E_\ell[\nabla u](t) &= \int_{\mathcal{S}_t} (|\nabla \dot{u}|^2 + |\nabla u'|^2 + V(\nabla \cdot \nabla u)^2) d\varsigma \\ &= \int_{\mathcal{S}_t} ((-\Delta \dot{u})\dot{u} + (-\Delta u')u' + V(-\nabla \Delta u) \cdot \nabla u) d\varsigma \\ &\leq \left( \int_{\mathcal{S}_t} (\Delta \dot{u})^2 d\varsigma \right)^{\frac{1}{2}} \left( \int_{\mathcal{S}_t} \dot{u}^2 d\varsigma \right)^{\frac{1}{2}} + \left( \int_{\mathcal{S}_t} (\Delta u')^2 d\varsigma \right)^{\frac{1}{2}} \left( \int_{\mathcal{S}_t} u'^2 d\varsigma \right)^{\frac{1}{2}} \\ &\quad + \left( \int_{\mathcal{S}_t} V |\nabla \Delta u|^2 d\varsigma \right)^{\frac{1}{2}} \left( \int_{\mathcal{S}_t} V |\nabla u|^2 d\varsigma \right)^{\frac{1}{2}} \\ &\leq \left( \left( \int_{\mathcal{S}_t} (\Delta^2 \dot{u})^2 d\varsigma \right)^{\frac{1}{2}} \left( \int_{\mathcal{S}_t} \dot{u}^2 d\varsigma \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \int_{\mathcal{S}_t} \dot{u}^2 d\varsigma \right)^{\frac{1}{2}} \end{aligned}$$



$$\begin{aligned}
& + \left( \left( \int_{\mathcal{S}_t} (\Delta^2 u')^2 d\varsigma \right)^{\frac{1}{2}} \left( \int_{\mathcal{S}_t} u'^2 d\varsigma \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \int_{\mathcal{S}_t} u'^2 d\varsigma \right)^{\frac{1}{2}} \\
& + \left( \left( \int_{\mathcal{S}_t} V |\nabla \Delta^2 u|^2 d\varsigma \right)^{\frac{1}{2}} \left( \int_{\mathcal{S}_t} V |\nabla u|^2 d\varsigma \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \int_{\mathcal{S}_t} V |\nabla u|^2 d\varsigma \right)^{\frac{1}{2}} \\
& = \left( \int_{\mathcal{S}_t} (\Delta^2 \dot{u})^2 d\varsigma \right)^{\frac{1}{4}} \left( \int_{\mathcal{S}_t} \dot{u}^2 d\varsigma \right)^{\frac{3}{4}} + \left( \int_{\mathcal{S}_t} (\Delta^2 u')^2 d\varsigma \right)^{\frac{1}{4}} \left( \int_{\mathcal{S}_t} u'^2 d\varsigma \right)^{\frac{3}{4}} \\
& + \left( \int_{\mathcal{S}_t} V |\nabla \Delta^2 u|^2 d\varsigma \right)^{\frac{1}{4}} \left( \int_{\mathcal{S}_t} V |\nabla u|^2 d\varsigma \right)^{\frac{3}{4}} .
\end{aligned}$$

Using Hölder's inequality,

$$\sum_k \left| x_k^{\frac{1}{4}} y_k^{\frac{3}{4}} \right| \leq \left| \sum_k x_k \right|^{\frac{1}{4}} \left| \sum_k y_k \right|^{\frac{3}{4}} ,$$

on the last line of the above estimate, we arrive at,

$$E_\ell[\nabla u](t) \leq (E_\ell[\Delta^2 u](t))^{\frac{1}{4}} (E_\ell[u](t))^{\frac{3}{4}} .$$

But  $E_\ell[\Delta^2 u](t) \leq CE[\Delta^2 u](t)$ , and for  $|r_*| < \frac{3}{4}t$ ,

$$\frac{t - r_*}{t} > \frac{1}{4} \quad ; \quad \frac{t + r_*}{t} > \frac{1}{4} ,$$

so using (150) we have,

$$\begin{aligned}
\frac{e_C}{t^2} & \geq \frac{1}{64} ((\dot{u} + u')^2 + (\dot{u} - u')^2) + \frac{1}{2} \left(1 + \frac{r_*^2}{t^2}\right) |\nabla u|^2 V + \frac{e}{t^2} \\
& = \frac{1}{32} (\dot{u}^2 + u'^2) + \frac{1}{32} V |\nabla u|^2 + \left(\frac{15}{32} + \frac{r_*^2}{2t^2}\right) |\nabla u|^2 V + \frac{e}{t^2} \\
& \geq \frac{e}{32} .
\end{aligned}$$

Thus,

$$E_\ell[u](t) \leq C \frac{E_C[u](t)}{t^2} .$$

Therefore,

$$E_\ell[\nabla u](t) \leq C \left( E[\Delta^2 u](t) \right)^{\frac{1}{4}} \left( \frac{E_C[u](t)}{t^2} \right)^{\frac{3}{4}} . \tag{201}$$

Substituting for  $E_\ell[\nabla u](t)$  in (199) using (201) we have, for  $t' \geq a$

$$\begin{aligned}
E_C[u](t') & \leq C \left( E_C[u](0) + \sup_{t \in [a, t']} \left( t \left( E[\Delta^2 u](t) \right)^{\frac{1}{4}} \left( \frac{E_C[u](t)}{t^2} \right)^{\frac{3}{4}} \right) \right. \\
& \quad \left. + \int_a^{t'} \left( E[\Delta^2 u](t) \right)^{\frac{1}{4}} \left( \frac{E_C[u](t)}{t^2} \right)^{\frac{3}{4}} dt \right) ,
\end{aligned}$$

by Lemma 21 and the linear bound (200),

$$E_C[u](t') \leq C \left( E_C[u](0) + \left( E[\Delta^2 u](0) \right)^{\frac{1}{4}} \left( E_C[u](0) + E[\nabla u](0) \right)^{\frac{3}{4}} \sup_{t \in [a, t']} \left( t \left( \frac{1+t}{t^2} \right)^{\frac{3}{4}} \right) \right. \\ \left. + \left( E[\Delta^2 u](0) \right)^{\frac{1}{4}} \left( E_C[u](0) + E[\nabla u](0) \right)^{\frac{3}{4}} \int_a^{t'} \left( \frac{1+t}{t^2} \right)^{\frac{3}{4}} dt \right),$$

but for  $t \geq a \geq 1$ ,

$$\left( \frac{1+t}{t^2} \right)^{\frac{3}{4}} \leq C t^{-\frac{3}{4}},$$

so,

$$E_C[u](t) \leq C \left( E_C[u](0) + t^{\frac{1}{4}} \left( E[\Delta^2 u](0) \right)^{\frac{1}{4}} \left( E_C[u](0) + E[\nabla u](0) \right)^{\frac{3}{4}} \right).$$

From (178) one has,

$$E[u](t) \leq E[\nabla u](t) \leq E[\Delta u](t),$$

but as it is easily seen that  $\Delta u$  is again a solution of (140) of the form (136), then

$$E[\Delta u](t) \leq E[\nabla \Delta u](t) \leq E[\Delta^2 u](t).$$

Thus, replacing  $E[\nabla u](0)$  by  $E[\Delta^2 u](0)$  in the above estimate and upon adding positive terms we have for  $t \geq a$ ,

$$E_C[u](t) \leq C(1 + t^{\frac{1}{4}}) \left( E_C[u](0) + E[\Delta^2 u](0) \right). \quad (202)$$

Doing the same again but using this estimate gives the uniform bound, that is, substituting for  $E_\ell[\nabla u](t)$  by (201) in (199) then using (202) this time yields,

$$E_C[u](t) \leq C \left( E_C[u](0) + E[\Delta^2 u](0) \right),$$

for  $t \geq a$ . One can use the exponential bound to get the estimate over  $0 \leq t \leq a$ . So, using (198):

$$E_C[u](t) \leq E_C[u](0)e^{Ct} \leq CE_C[u](0).$$

This proves (180), and in doing so, as a matter of fact, we have essentially proved (181) also. If  $\chi$  is any function of  $r_*$  with a compact support, and with  $b \geq 1$  depending on  $\chi$  and playing the role of  $a$ , then, similar to the case of  $\chi_{trap}$ , one has the corresponding version of (196). From that, and after applying (187), we obtain for  $t' \geq b \geq 1$  that

$$\int_{[b, t'] \times \mathbb{R} \times \mathcal{S}^2} t \chi_{trap} |\nabla u|^2 dx \leq C \left( \sup_{t \in [0, t']} (t E_\ell[\nabla u](t)) + \int_0^{t'} E_\ell[\nabla u](t) dt \right). \quad (203)$$

Then the fact that  $E_\ell[\nabla u](t) \leq CE[\nabla u](t) = CE[\nabla u](0)$  gives a linear bound on the integral over the interval  $[b, t']$ . As before, using (201) we substitute for  $E_\ell[\nabla u](t)$  in (203), after which we use the linear bound to obtain a  $t^{1/4}$  bound. Then repeat the same thing again using the  $t^{1/4}$  bound we get the uniform bound over  $[b, t']$ . Finally, an inequality similar to the second part of (197) holds true for  $\chi$  over  $[t_1, t_2] = [0, b]$ . From it and from the uniform bound (or any bound) on the conformal energy, the uniform bound follows.  $\square$

## 3.2 Decay of the Maxwell Field

We start this section by introducing some useful notations. Consider the following sets of smooth vector fields:

$$\begin{aligned}\mathbb{O} &= \{\Theta_1, \Theta_2, \Theta_3\} , \\ \mathbb{T} &= \{T = \partial_t\} \cup \mathbb{O} , \\ \mathbb{X} &= \{R = \partial_{r_*}\} \cup \mathbb{T} .\end{aligned}\tag{204}$$

We also use the hatted version of a letter to indicate the corresponding normalized vector field.

We define the following pointwise norms on smooth tensors fields: Let  $F$  be a  $(0, m)$ -tensor field, and  $A$  a set of vector fields. We set

$$|F|_A^2 = \sum_{Y_1, \dots, Y_m \in A} |F(Y_1, \dots, Y_m)|^2 \tag{205}$$

and it is a norm when  $A$  is a spanning set<sup>1</sup>. If in addition  $F$  is smooth,  $n$  is a non-negative integer, and  $B$  is set of vector fields, we can define the higher order quantity:

$$|F|_{A,n,B}^2 = \sum_{k=0}^n \sum_{X_1, \dots, X_k \in B} \sum_{Y_1, \dots, Y_m \in A} |(\mathcal{L}_{X_k} \dots \mathcal{L}_{X_1} F)(Y_1, \dots, Y_m)|^2 , \tag{206}$$

where  $\mathcal{L}_X$  is the Lie derivative with respect to the vector field  $X$ . When  $A$  is a spanning set,  $|F|_{A,n,B}^2$  is a norm.

When working with inequalities and  $A$  is a spanning set, we sometimes use the same notation for the uniformly equivalent norm defined by dropping the squares in (206).

Another norm for the Maxwell field is the spin norm defined by  $|\Phi_1| + |\Phi_0| + |\Phi_{-1}|$  or the equivalent norm  $(|\Phi_1|^2 + |\Phi_0|^2 + |\Phi_{-1}|^2)^{1/2}$  (See (49)). It is easy to see that this norm is uniformly equivalent to the norm  $|F|_{\mathbb{X}}$  by noting that the frames  $\{\partial_\theta, \frac{1}{\sin(\theta)}\partial_\varphi\}$  and  $\mathbb{O}$  can be expressed as linear combinations of each other with bounded smooth coefficient functions. For example, the vector field

$$X = \frac{1}{\sin(\theta)}\partial_\varphi = \frac{1}{\sin(\theta)}\Theta_3$$

can be written as

$$X = \frac{\rho(\theta, \varphi)}{\sin(\theta)}\Theta_3 + \frac{(1 - \rho(\theta, \varphi))}{\cos(\theta)}(\cos(\varphi)\Theta_1 - \sin(\varphi)\Theta_2) , \tag{207}$$

with  $0 \leq \rho \leq 1$  a smooth function on the sphere compactly supported away from the pole  $(0, 0, 1)$  i.e.  $\theta = 0$ .

---

<sup>1</sup>A spanning set  $A$  of a vector space is a subset of the space with the property that every vector in the space can be written as a linear combination of vectors in  $A$  only

An essential property of Maxwell's equations is that if  $F$  is a solution and  $X$  is a Killing vector field then  $\mathcal{L}_X F$ , the Lie derivative of  $F$  with respect to  $X$ , is again a solution of the equations. To see why, we start by (39). Using Cartan's identity for Lie derivatives on differential forms we immediately get

$$d(\mathcal{L}_X F) = d(i_X dF + di_X F) = di_X dF = \mathcal{L}_X(dF) = 0 ,$$

in other words, the commutator  $[\mathcal{L}_X, d] = 0$  on forms, and this is true for any vector field  $X$ , not necessarily Killing. For (38), we can use the expression of the Hodge star in the abstract index formalism:

$$(\star\alpha)_{a_{k+1}\dots a_n} = \frac{1}{k!} \alpha^{a_1\dots a_k} \omega_{a_1\dots a_n} .$$

where  $\alpha$  is a differential  $k$ -form and  $\omega$  is the  $n$ -volume form given by the metric. When  $X$  is Killing,  $[\mathcal{L}_X, \star] = 0$  since the divergence of a Killing vector field vanishes by (320). To see that this is true, we note that

$$(\mathcal{L}_X \star \alpha)_{a_{k+1}\dots a_n} = \frac{1}{k!} \mathcal{L}_X (\alpha^{a_1\dots a_k} \omega_{a_1\dots a_n}) = \frac{1}{k!} ((\mathcal{L}_X \alpha)^{a_1\dots a_k} \omega_{a_1\dots a_n} + \alpha^{a_1\dots a_k} (\mathcal{L}_X \omega)_{a_1\dots a_n}) .$$

The last term is zero because of (312), and the term before it is  $\star(\mathcal{L}_X \alpha)$ , which means that the two operators commute. Hence  $\mathcal{L}_X F$  is a solution of (38) if  $X$  is Killing.

As before, we assume that our Maxwell field  $F$  is a non-stationary solution with finite energy, i.e. satisfying (135) and (136).

### 3.2.1 Energies of the Maxwell Field

Motivated by the Lagrangian theory, we consider an energy-momentum tensor  $\mathbf{T}_{ab}$  that is a  $(0, 2)$ -symmetric tensor i.e.  $\mathbf{T}_{ab} = \mathbf{T}_{(ab)}$ , and that is divergence-free i.e.  $\nabla^a \mathbf{T}_{ab} = 0$ . Let  $X$  be a vector field and  ${}^{(X)}\pi_{ab} = \nabla_a X_b + \nabla_b X_a$  be its deformation tensor. If  $\mathcal{U}$  is an open submanifold of  $\mathcal{N}$  with a piecewise  $\mathcal{C}^1$ -boundary  $\partial\mathcal{U}$ , then by the divergence theorem and the properties of  $\mathbf{T}$ , we have for  $\eta$  a normal vector to  $\partial\mathcal{U}$  and  $\tau$  a transverse one such that  $\eta^a \tau_a = 1$ :

$$\begin{aligned} \int_{\partial\mathcal{U}} \mathbf{T}_{ab} X^b \eta^a i_\tau d^4x &= \int_{\mathcal{U}} \nabla^a (\mathbf{T}_{ab} X^b) d^4x \\ &= \int_{\mathcal{U}} (X^b \nabla^a \mathbf{T}_{ab} + \mathbf{T}_{ab} \nabla^a X^b) d^4x \\ &= \frac{1}{2} \int_{\mathcal{U}} {}^{(X)}\pi_{ab} \mathbf{T}^{ab} d^4x . \end{aligned} \tag{208}$$

This is exactly what we did in the proof of Lemma 21, and we saw that it is particularly interesting when  $X$  is Killing and thus its deformation tensor vanishes. Motivated by this, we define the energy of a general 2-form  $F$  which the energy-momentum tensor depends on (aside from the metric), on an oriented smooth hypersurface  $S$  to be:

$$E_X[F](S) = \int_S (X \lrcorner \mathbf{T})^\sharp \lrcorner d^4x . \tag{209}$$

Of course, it is understood that we are not integrating the 3-form but its restriction on  $S$ , which is the pull back of the form by the inclusion map. We can choose  $\eta_S$  and  $\tau_S$  to be respectively vector fields normal and transverse to  $S$ , such that their scalar product is one (see (314)), then by (317), we have

$$E_X[F](S) = \int_S \mathbf{T}_{ab} X^b \eta_S^a i_{\tau_S} d^4x . \quad (210)$$

The definition is independent of the choice of  $\eta_S$  and  $\tau_S$ . As we can see, this will lead to a conservation law when  $X$  is Killing. There is of course a more physical and a more “natural” motivation for this quantity to be called energy. In fact, if  $S$  is a spacelike hypersurface and  $X = \partial_t$ , then (210) is indeed the energy measured at an instant of time by an observer whose frame of reference is defined by the integral curves of the vector field  $X$ . For our purpose, we shall mainly consider spacelike or null slices defined as level-hypersurfaces of a smooth function.

From section 2.4, we know that if the Maxwell field is given as an exterior derivative of some 1-form, one can then define a Lagrangian, and by varying the 1-form, the Euler-Lagrange equations are going to be Maxwell’s equations. Using this Lagrangian, it is possible to define an energy-momentum tensor which by the Euler-Lagrange equations is divergence-free. However, since in general not all Maxwell fields admit a global potential, we shall take the same energy-momentum tensor and show by direct calculations that it is divergence-free using Maxwell field equations. Let  $F$  be a Maxwell field and consider the  $(0, 2)$ -symmetric tensor

$$\mathbf{T}_{ab} = \frac{1}{4} g_{ab} F^{cd} F_{cd} - F_{ac} F_b{}^c . \quad (211)$$

In the following lemma we summarize some of the properties of this tensor that are most important to us in this work.

**Lemma 29.** *Let  $\mathbf{T}$  be defined as in (211), and recall the definition of the vectors  $L$  and  $N$  from (48). We have:*

$$\mathbf{T}_{ab} = \mathbf{T}_{(ab)} , g^{ab} \mathbf{T}_{ab} = 0 , \quad (212)$$

$$\nabla^a \mathbf{T}_{ab} = 0 , \quad (213)$$

$$\mathbf{T}_{ab} L^a L^b = r^{-2} |\Phi_1|^2 , \quad (214)$$

$$\mathbf{T}_{ab} L^a N^b = f r^{-4} |\Phi_0|^2 , \quad (215)$$

$$\mathbf{T}_{ab} N^a N^b = r^{-2} |\Phi_{-1}|^2 . \quad (216)$$

*Proof.* The trace-freeness is immediate:

$$g^{ab} \mathbf{T}_{ab} = \mathbf{T}^a{}_a = \frac{4}{4} F^{cd} F_{cd} - F_{ac} F^{ac} = 0 .$$

For the divergence, from (40) we have,

$$\begin{aligned} \nabla^a \mathbf{T}_{ab} &= \frac{1}{2} g_{ab} F_{cd} \nabla^a F^{cd} - F_b^c \nabla^a F_{ac} - F_{ac} \nabla^a F_b^c \\ &= \frac{1}{2} F^{ac} \nabla_b F_{ac} - F^{ac} \nabla_a F_{bc} . \end{aligned}$$

And from (41),

$$\nabla_b F_{ac} = \nabla_c F_{ab} + \nabla_a F_{bc} ,$$

which entails

$$\nabla^a \mathbf{T}_{ab} = \frac{1}{2} F^{ac} \nabla_c F_{ab} - \frac{1}{2} F^{ac} \nabla_a F_{bc} = 0$$

upon swapping the indices  $a$  and  $c$  in the first term, then using the fact that  $F$  is antisymmetric.

For the last three properties (214)-(216), we have, from (54),

$$|\Phi_1|^2 = (F_{02} + F_{12})^2 + \frac{1}{\sin(\theta)^2} (F_{03} + F_{13})^2 , \quad (217)$$

$$|\Phi_0|^2 = \frac{r^4}{f^2} (F_{01})^2 + \frac{1}{\sin(\theta)^2} (F_{23})^2 , \quad (218)$$

$$|\Phi_{-1}|^2 = (F_{02} - F_{12})^2 + \frac{1}{\sin(\theta)^2} (F_{03} - F_{13})^2 . \quad (219)$$

Since both  $L$  and  $N$  are null, then for  $X \in \{L, N\}$

$$\mathbf{T}_{ab} X^a X^b = -g^{cd} F_{ac} F_{bd} X^a X^b ,$$

which in a local coordinate basis expands to

$$\mathbf{T}(X, X) = -\frac{1}{f} (F_{\mathbf{a}0} X^{\mathbf{a}})^2 + \frac{1}{f} (F_{\mathbf{a}1} X^{\mathbf{a}})^2 + \frac{1}{r^2} (F_{\mathbf{a}2} X^{\mathbf{a}})^2 + \frac{1}{r^2 \sin(\theta)^2} (F_{\mathbf{a}3} X^{\mathbf{a}})^2 ,$$

since  $F_{00} = F_{11} = 0$ , the first two terms cancel out when we evaluate  $L$  and  $N$  with their coordinate expressions. This proves (214) and (216). For the middle equation, we need to calculate  $\mathbf{T}(L, N)$ . We have

$$\mathbf{T}(L, N) = \mathbf{T}_{00} - \mathbf{T}_{11} ,$$

but,

$$\mathbf{T}_{00} = \frac{1}{4} f F_{\mathbf{c}\mathbf{d}} F^{\mathbf{c}\mathbf{d}} - F_{0\mathbf{c}} F_0^{\mathbf{c}} = \frac{1}{4} f F_{\mathbf{c}\mathbf{d}} F^{\mathbf{c}\mathbf{d}} - f F_{0\mathbf{c}} F^{0\mathbf{c}} ,$$

similarly,

$$\mathbf{T}_{11} = -\frac{1}{4} f F_{\mathbf{c}\mathbf{d}} F^{\mathbf{c}\mathbf{d}} + f F_{1\mathbf{c}} F^{1\mathbf{c}} ,$$

thus,

$$\begin{aligned}
\mathbf{T}_{00} - \mathbf{T}_{11} &= f \left( \frac{1}{2} F_{\mathbf{cd}} F^{\mathbf{cd}} - F_{0\mathbf{c}} F^{0\mathbf{c}} - F_{1\mathbf{c}} F^{1\mathbf{c}} \right) \\
&= f \left( \sum_{0 \leq c < d \leq 3} F_{\mathbf{cd}} F^{\mathbf{cd}} - \sum_c F_{0\mathbf{c}} F^{0\mathbf{c}} - \sum_c F_{1\mathbf{c}} F^{1\mathbf{c}} \right) \\
&= f \left( \sum_{1 \leq c < d \leq 3} F_{\mathbf{cd}} F^{\mathbf{cd}} - \sum_c F_{1\mathbf{c}} F^{1\mathbf{c}} \right) \\
&= f \left( \sum_{2 \leq c < d \leq 3} F_{\mathbf{cd}} F^{\mathbf{cd}} - F_{10} F^{10} \right) \\
&= f (F_{23} F^{23} - F_{10} F^{10}) = \frac{1}{f} (F_{01})^2 + \frac{f}{r^4 \sin(\theta)^2} (F_{23})^2,
\end{aligned}$$

which proves (215).  $\square$

Identities (214), (215), and (216) have two important consequences. First, let  $V$  be a future oriented causal vector with no angular components, i.e. of the form

$$V = \alpha \partial_t + \beta \partial_{r_*} \quad \text{with} \quad \alpha \geq |\beta|. \quad (220)$$

Since any such vector can be expressed as a linear combination of the vectors  $L$  and  $N$  with non-negative coefficients, more precisely,

$$V = \left( \frac{\alpha + \beta}{2} \right) L + \left( \frac{\alpha - \beta}{2} \right) N,$$

the next corollary is immediate.

**Corollary 30** (Dominant Energy Condition). *Let  $\mathbf{T}$  be the stress-energy tensor of (211), and let  $V$  and  $W$  be two future (past) oriented causal vectors with no angular components then,*

$$\mathbf{T}(V, W) \geq 0. \quad (221)$$

The energy-momentum tensor of the Maxwell field satisfies a stronger positivity condition. Actually, the corollary holds true for all future oriented causal vectors, even with non zero angular components. This is called the dominant energy condition. The proof of the full dominant energy condition is much easier to see using spinor notations. This is (326) which is discuss in the appendix B.

The second consequence of these identities is related to the definition of the energy of the Maxwell field. We can expand the left hand side of (214)-(216) to get

$$r^{-2} |\Phi_1|^2 = \mathbf{T}_{00} + 2\mathbf{T}_{01} + \mathbf{T}_{11}, \quad (222)$$

$$f r^{-4} |\Phi_0|^2 = \mathbf{T}_{00} - \mathbf{T}_{11}, \quad (223)$$

$$r^{-2} |\Phi_{-1}|^2 = \mathbf{T}_{00} - 2\mathbf{T}_{01} + \mathbf{T}_{11}, \quad (224)$$

and from this, we can compute the components of the stress-energy tensor in terms of the spin components,

$$\mathbf{T}_{00} = \frac{1}{4r^2} \left( |\Phi_1|^2 + \frac{2f}{r^2} |\Phi_0|^2 + |\Phi_{-1}|^2 \right), \quad (225)$$

$$\mathbf{T}_{01} = \frac{1}{4r^2} (|\Phi_1|^2 - |\Phi_{-1}|^2), \quad (226)$$

$$\mathbf{T}_{11} = \frac{1}{4r^2} \left( |\Phi_1|^2 - \frac{2f}{r^2} |\Phi_0|^2 + |\Phi_{-1}|^2 \right). \quad (227)$$

We can see that the energy of the Maxwell field defined in (89) of section 2.2 and the one given by (210) are equal. If  $S = \Sigma_t = \{t\} \times \mathbb{R} \times \mathcal{S}^2$ , then its unit normal is  $\hat{T} = f^{-\frac{1}{2}} \partial_t$ , and taking the transverse vector to be  $\hat{T}$  also, we see that

$$\begin{aligned} E_T[F](\Sigma_t) &= \int_{\Sigma_t} \mathbf{T}_{ab} T^b \hat{T}^a i_{\hat{T}} d^4x \\ &= \int_{\Sigma_t} \mathbf{T}_{00} f^{-\frac{1}{2}} f^{\frac{1}{2}} r^2 dr_* d^2\omega \\ &= \frac{1}{4} \int_{\Sigma_t} |\Phi_1|^2 + \frac{2f}{r^2} |\Phi_0|^2 + |\Phi_{-1}|^2 dr_* d^2\omega = E_T[F](t). \end{aligned} \quad (228)$$

This gives, in addition to a good motivation for considering (89), from (208) in the introductory argument of this section, that it is a conserved quantity as the vector field  $T$  is Killing, i.e.

$$E_T[F](t) = E_T[F](0). \quad (229)$$

And since the Lie derivative of a Maxwell field  $F$  with respect to a Killing vector field  $X$  is again a Maxwell field, its energy is conserved as well. So, for  $X_1, \dots, X_k$  Killing,

$$E_T[\mathcal{L}_{X_1} \dots \mathcal{L}_{X_k} F](t) = E_T[\mathcal{L}_{X_1} \dots \mathcal{L}_{X_k} F](0). \quad (230)$$

To get decay for the Maxwell field we will need to control another energy on  $\Sigma_t$  defined by the vector field

$$K = (t^2 + r_*^2) \partial_t + 2tr_* \partial_{r_*}. \quad (231)$$

Using the advanced and retarded coordinates  $u_+ = t + r_*$  and  $u_- = t - r_*$  it becomes

$$K = \frac{1}{2} (u_+^2 L + u_-^2 N). \quad (232)$$

To compute  $E_K[F](t) := E_K[F](\Sigma_t)$ , which we shall call the conformal energy, we have to compute  $\mathbf{T}_{0a} K^a$ . From the above form of  $K$ , (225), and (226) we have

$$\begin{aligned} \mathbf{T}_{0a} K^a &= \frac{1}{2} (u_+^2 \mathbf{T}_{0a} L^a + u_-^2 \mathbf{T}_{0a} N^a) \\ &= \frac{1}{2} (u_+^2 (\mathbf{T}_{00} + \mathbf{T}_{01}) + u_-^2 (\mathbf{T}_{00} - \mathbf{T}_{01})) \\ &= \frac{1}{8r^2} \left( u_+^2 \left( 2|\Phi_1|^2 + \frac{2f}{r^2} |\Phi_0|^2 \right) + u_-^2 \left( 2|\Phi_{-1}|^2 + \frac{2f}{r^2} |\Phi_0|^2 \right) \right) \\ &= \frac{1}{4r^2} \left( u_+^2 |\Phi_1|^2 + (u_+^2 + u_-^2) \frac{f}{r^2} |\Phi_0|^2 + u_-^2 |\Phi_{-1}|^2 \right). \end{aligned}$$



Therefore, the conformal energy is

$$\begin{aligned} E_K[F](t) &= \int_{\Sigma_t} \mathbf{T}_{ab} K^b \hat{T}^a i_{\hat{T}} d^4x \\ &= \frac{1}{4} \int_{\Sigma_t} u_+^2 |\Phi_1|^2 + (u_+^2 + u_-^2) \frac{f}{r^2} |\Phi_0|^2 + u_-^2 |\Phi_{-1}|^2 dr_* d^2\omega . \end{aligned} \quad (233)$$

The next result is rather important and central to our work, it is sometimes referred to as the Trapping Effect. From it and its proof, a great part of the importance of the photon sphere, the Regge-Wheeler coordinate, and the wave analysis we did in section 3.1.1 is revealed.

**Lemma 31** (Trapping Lemma). *There is a non-negative compactly supported function  $\chi_{trap}$  of  $r_*$ , depending only on the geometry of the spacetime, such that for  $F$  a solution to Maxwell's equations, we have*

$$E_K[F](t_2) - E_K[F](t_1) \leq \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} t \chi_{trap} |\Phi_0|^2 dt dr_* d^2\omega . \quad (234)$$

*Proof.* This is a matter of applying the divergence theorem (208). For this we need to calculate the deformation tensor of  $K$ . Since  $T^b = f dt$  and  $R^b = -f dr_*$ , then from (231),

$$K_b = (t^2 + r_*^2) T_b + 2tr_* R_b ,$$

so,

$$\begin{aligned} \nabla_a K_b &= T_b \nabla_a (t^2 + r_*^2) + (t^2 + r_*^2) \nabla_a T_b + R_b \nabla_a (2tr_*) + 2tr_* \nabla_a R_b \\ &= T_b (2tf^{-1} T_a - 2r_* f^{-1} R_a) + (t^2 + r_*^2) \nabla_a T_b + R_b (2r_* f^{-1} T_a - 2tf^{-1} R_a) + 2tr_* \nabla_a R_b \\ &= 2f^{-1} t (T_a T_b - R_a R_b) + 2tr_* \nabla_a R_b + 2f^{-1} r_* (T_a R_b - T_b R_a) + (t^2 + r_*^2) \nabla_a T_b . \end{aligned}$$

Symmetrizing, the third term vanishes since it is skew, and the last term also vanishes as  $T$  is Killing. Thus, the only term that we need to calculate is the covariant derivative of  $R_a$ . By (37) we have,

$$\begin{aligned} \nabla R^b &= \left( \partial_a R_b - \tilde{\Gamma}_{ab}^c R_c \right) dx^a \otimes dx^b \\ &= \partial_{r_*} R_1 dr_*^2 - \sum_a \tilde{\Gamma}_{aa}^1 R_1 (dx^a)^2 \\ &= \frac{ff'}{2} dt^2 - \frac{ff'}{2} dr_*^2 + \frac{f}{r} d\omega^2 \\ &= \frac{f}{r} g_{\mathcal{M}} + f \left( \frac{f'}{2} - \frac{f}{r} \right) (dt^2 - dr_*^2) . \end{aligned}$$

Now, since this is a symmetric tensor,  $\nabla_a R_b = \nabla_{(a} R_{b)}$ . Therefore,

$$\begin{aligned} {}^K \pi_{ab} &= 2\nabla_{(a} K_{b)} = 4f^{-1} t (T_a T_b - R_a R_b) + 4tr_* \nabla_a R_b \\ &= 4f^{-1} t (T_a T_b - R_a R_b) + 4tr_* f r^{-1} g_{\mathcal{M}} + 4tr_* f^{-1} \left( \frac{f'}{2} - \frac{f}{r} \right) (T_a T_b - R_a R_b) \\ &= 4tr_* f r^{-1} g_{ab} + 4f^{-1} t \left( 1 + r_* \left( \frac{f'}{2} - \frac{f}{r} \right) \right) (T_a T_b - R_a R_b) . \end{aligned}$$

Recall the trapping term defined in (160),

$$\mathcal{T} = 1 + r_* \left( \frac{f'}{2} - \frac{f}{r} \right) .$$

Contracting the deformation tensor of  $K$  with the energy-momentum tensor of the Maxwell field  $\mathbf{T}$ , and using (212) and (223) we obtain

$${}^K\pi_{ab}\mathbf{T}^{ab} = 4tr^{-4}\mathcal{T}|\Phi_0|^2 .$$

In virtue of the definition of the conformal energy given by (233), and through (208) we arrive at

$$E_K[F](t_2) - E_K[F](t_1) = \int_{[t_1, t_2] \times \mathbb{R} \times \mathcal{S}^2} 2t \frac{f}{r^2} \mathcal{T} |\Phi_0|^2 dt dr_* d^2\omega .$$

Finally, our choice of  $\chi_{trap}$  at the end of the proof of Lemma 22 ensures that the statement of the lemma holds true.  $\square$

The trapping lemma shows that the conformal energy of the Maxwell field is controlled by the integral of the middle component. This is why we did the wave analysis in section 3.1.1. As the middle component satisfies the wave-like equation (140), then the bounds we obtained in Proposition 28 can be used to establish bounds on the conformal energy. For this purpose, we need to find a relation between the energy of the middle term and the full Maxwell tensor. Two 1-forms are particularly important for the discussion. We set

$$\alpha = i_L F , \quad \text{and} \quad \underline{\alpha} = i_N F . \quad (235)$$

We see that

$$\Phi_1 = \alpha(M) = \alpha_\theta + \frac{i}{\sin(\theta)} \alpha_\varphi ,$$

and

$$\Phi_{-1} = \underline{\alpha}(\bar{M}) = \underline{\alpha}_\theta - \frac{i}{\sin(\theta)} \underline{\alpha}_\varphi .$$

Using this, a simple calculation shows that

$$\begin{aligned} (\bar{M} + \cot(\theta))\Phi_1 &= \partial_\theta \alpha_\theta + \frac{1}{\sin(\theta)^2} \partial_\varphi \alpha_\varphi + \cot(\theta) \alpha_\theta + \frac{i}{\sin(\theta)} (\partial_\theta \alpha_\varphi - \partial_\varphi \alpha_\theta) , \\ (M + \cot(\theta))\Phi_{-1} &= \partial_\theta \underline{\alpha}_\theta + \frac{1}{\sin(\theta)^2} \partial_\varphi \underline{\alpha}_\varphi + \cot(\theta) \underline{\alpha}_\theta - \frac{i}{\sin(\theta)} (\partial_\theta \underline{\alpha}_\varphi - \partial_\varphi \underline{\alpha}_\theta) . \end{aligned}$$

Since we are going to work only with operations defined on the sphere  $\mathcal{S}^2$  when dealing with these 1-forms, then the  $t$  and  $r_*$  variables play no role in our calculations. So one might replace these forms with their projections on the sphere of radius  $r_*$ . The  $r_*$  will only be a rescaling factor, moreover, from (235) and then (217) and (219),

$$\begin{aligned} (\alpha_\theta)^2 + \frac{1}{\sin(\theta)^2} (\alpha_\varphi)^2 &= (F_{02} + F_{12})^2 + \frac{1}{\sin(\theta)^2} (F_{03} + F_{13})^2 = |\Phi_1|^2 , \\ (\underline{\alpha}_\theta)^2 + \frac{1}{\sin(\theta)^2} (\underline{\alpha}_\varphi)^2 &= (F_{02} - F_{12})^2 + \frac{1}{\sin(\theta)^2} (F_{03} - F_{13})^2 = |\Phi_{-1}|^2 , \end{aligned}$$

and this motivates us to consider them as covector fields on the sphere  $\mathcal{S}^2$ . Therefore we set

$$\alpha = \alpha_\theta d\theta + \alpha_\varphi d\varphi, \quad \underline{\alpha} = \underline{\alpha}_\theta d\theta + \underline{\alpha}_\varphi d\varphi,$$

$$|\alpha|^2 := g_{\mathcal{S}^2}(\alpha, \alpha), \quad |\underline{\alpha}|^2 := g_{\mathcal{S}^2}(\underline{\alpha}, \underline{\alpha}),$$

and so

$$|\alpha|^2 = |\Phi_1|^2, \quad |\underline{\alpha}|^2 = |\Phi_{-1}|^2. \quad (236)$$

Taking this interpretation of the 1-forms, we now see that

$$(\bar{M} + \cot(\theta))\Phi_1 = \nabla^a \alpha_a + i\epsilon^{ab} \nabla_a \alpha_b, \quad (237)$$

$$(M + \cot(\theta))\Phi_{-1} = \nabla^a \underline{\alpha}_a - i\epsilon^{ab} \nabla_a \underline{\alpha}_b, \quad (238)$$

where  $\epsilon^{ab}$  is the Levi-Civita tensor on  $\mathcal{S}^2$  defined by

$$\epsilon^{\mathbf{ab}} = \frac{1}{\sqrt{|g_{\mathcal{S}^2}|}} \varepsilon^{\mathbf{ab}} = \frac{1}{\sin(\theta)} \varepsilon^{\mathbf{ab}}$$

and  $\varepsilon^{ab}$  is the Levi-Civita symbol given in (173).

With these notations, we can now relate the energies of  $\Phi_0$  and  $F$ .

**Lemma 32.** *Let  $F$  be a Maxwell field and let  $(\Phi_1, \Phi_0, \Phi_{-1})$  be its spin components and consider the energy and the conformal charge defined in (148) and (149). Then*

$$E[\Phi_0](t) \leq C \sum_{i=1}^3 E_T[\mathcal{L}_{\Theta_i} F](t), \quad (239)$$

$$E[\Delta_{\mathcal{S}^2}^2 \Phi_0](t) \leq C \sum_{X_1, \dots, X_5 \in \mathbb{O}} E_T[\mathcal{L}_{X_1} \dots \mathcal{L}_{X_5} F](t), \quad (240)$$

$$E_C[\Phi_0](t) \leq C \sum_{i=1}^3 (E_K[\mathcal{L}_{\Theta_i} F](t) + E_T[\mathcal{L}_{\Theta_i} F](t)). \quad (241)$$

*Proof.* Let  $\Sigma_t = \{t\} \times \mathbb{R} \times \mathcal{S}^2$ . By (236) we have

$$E_T[F](t) = \frac{1}{4} \int_{\Sigma_t} |\alpha|^2 + 2V|\Phi_0|^2 + |\underline{\alpha}|^2 dr_* d^2\omega.$$

Thus, to find  $E_T[\mathcal{L}_{\Theta_i} F](t)$  we need to calculate the middle spin component of  $\mathcal{L}_{\Theta_i} F$  and the corresponding 1-forms. For a 2-form  $\beta$  and two vectors  $X$  and  $Y$  such that  $[X, Y] = 0$ , we have for all vectors  $Z$

$$\begin{aligned} (\mathcal{L}_Y i_X \beta)(Z) &= \mathcal{L}_Y((i_X \beta)(Z)) - (i_X \beta)(\mathcal{L}_Y Z) \\ &= \mathcal{L}_Y(\beta(X, Z)) - \beta(X, \mathcal{L}_Y Z) \\ &= (\mathcal{L}_Y \beta)(X, Z) + \beta(\mathcal{L}_Y X, Z) + \beta(X, \mathcal{L}_Y Z) - \beta(X, \mathcal{L}_Y Z) \\ &= (i_X \mathcal{L}_Y \beta)(Z). \end{aligned}$$

Thus,

$$i_L(\mathcal{L}_{\Theta_i}F) = \mathcal{L}_{\Theta_i}(i_LF) = \mathcal{L}_{\Theta_i}\alpha,$$

and the same for  $\underline{\alpha}$ . This treats the forms, for the middle component we need the commutators of  $M$  and  $\bar{M}$  with the  $\Theta_i$ 's. Direct calculation gives,

$$\begin{aligned} [\Theta_1, M] &= \frac{-i \cos(\varphi)}{\sin(\theta)} M \quad ; \quad [\Theta_1, \bar{M}] = \frac{i \cos(\varphi)}{\sin(\theta)} \bar{M} \\ [\Theta_2, M] &= \frac{i \sin(\varphi)}{\sin(\theta)} M \quad ; \quad [\Theta_2, \bar{M}] = \frac{-i \sin(\varphi)}{\sin(\theta)} \bar{M} \\ [\Theta_3, M] &= 0 \quad ; \quad [\Theta_3, \bar{M}] = 0 . \end{aligned}$$

Hence, the middle component of  $\mathcal{L}_{\Theta_i}F$  is

$$\begin{aligned} \frac{1}{2} (V^{-1}(\mathcal{L}_{\Theta_i}F)(L, N) + (\mathcal{L}_{\Theta_i}F)(\bar{M}, M)) &= \frac{1}{2} \Theta_i (V^{-1}F(L, N) + F(\bar{M}, M)) \\ &\quad - \frac{1}{2} (F(\mathcal{L}_{\Theta_i}\bar{M}, M) + F(\bar{M}, \mathcal{L}_{\Theta_i}M)) \\ &= \frac{1}{2} \Theta_i (V^{-1}F(L, N) + F(\bar{M}, M)) \\ &= \Theta_i(\Phi_0) . \end{aligned}$$

Therefore, and using (169),

$$\sum_{i=1}^3 E_T[\mathcal{L}_{\Theta_i}F](t) = \frac{1}{4} \int_{\Sigma_t} \sum_{i=1}^3 (|\mathcal{L}_{\Theta_i}\alpha|^2 + |\mathcal{L}_{\Theta_i}\underline{\alpha}|^2) + 2V|\nabla\Phi_0|^2 dr_* d^2\omega . \quad (242)$$

Next we want to substitute for the  $t$  and  $r_*$  derivatives of  $\Phi_0$  in  $E[\Phi_0](t)$  in terms of derivatives of the 1-forms  $\alpha$  and  $\underline{\alpha}$ . From (51) and (52) we have

$$|L\Phi_0|^2 + |N\Phi_0|^2 = |\bar{M}\Phi_1 + \cot(\theta)\Phi_1|^2 + |M\Phi_{-1} + \cot(\theta)\Phi_{-1}|^2 .$$

On the one hand, by the parallelogram rule,

$$|L\Phi_0|^2 + |N\Phi_0|^2 = 2|\partial_t\Phi_0|^2 + 2|\partial_{r_*}\Phi_0|^2 ,$$

and on the other hand, by (237) and (238),

$$|(\bar{M}1 + \cot(\theta))\Phi_1|^2 + |(M + \cot(\theta))\Phi_{-1}|^2 = (\nabla^a\alpha_a)^2 + (\epsilon^{ab}\nabla_a\alpha_b)^2 + (\nabla^a\underline{\alpha}_a)^2 + (\epsilon^{ab}\nabla_a\underline{\alpha}_b)^2 .$$

Thus,

$$E[\Phi_0](t) = \frac{1}{4} \int_{\Sigma_t} (\nabla^a\alpha_a)^2 + (\epsilon^{ab}\nabla_a\alpha_b)^2 + (\nabla^a\underline{\alpha}_a)^2 + (\epsilon^{ab}\nabla_a\underline{\alpha}_b)^2 + 2V|\nabla\Phi_0|^2 dr_* d^2\omega .$$

So, (239) follows if we show that

$$(\nabla^a \alpha_a)^2 + (\epsilon^{ab} \nabla_a \alpha_b)^2 \leq C \sum_{i=1}^3 |\mathcal{L}_{\Theta_i} \alpha|^2, \quad (243)$$

$$(\nabla^a \underline{\alpha}_a)^2 + (\epsilon^{ab} \nabla_a \underline{\alpha}_b)^2 \leq C \sum_{i=1}^3 |\mathcal{L}_{\Theta_i} \underline{\alpha}|^2.$$

We will workout the  $\alpha$ 's case only since the case of  $\underline{\alpha}$  is completely analogous. We start by showing that

$$\sum_{i=1}^3 |\mathcal{L}_{\Theta_i} \alpha|^2 = \nabla_a \alpha_b \nabla^a \alpha^b + |\alpha|^2. \quad (244)$$

We have

$$\begin{aligned} \sum_{i=1}^3 |\mathcal{L}_{\Theta_i} \alpha|^2 &= \sum_{i=1}^3 (\Theta_i^b \nabla_b \alpha_a + \alpha_b \nabla_a \Theta_i^b) (\Theta_i^c \nabla_c \alpha^a + \alpha_c \nabla^a \Theta_i^c) \\ &= \nabla_b \alpha_a \nabla^a \alpha^b + 2 \alpha_c \nabla_b \alpha_a \sum_{i=1}^3 \Theta_i^b \nabla^a \Theta_i^c + \alpha_b \alpha_c \sum_{i=1}^3 \nabla_a \Theta_i^b \nabla^a \Theta_i^c. \end{aligned}$$

A quite lengthy but very straight forward computation<sup>1</sup> shows that

$$\sum_{i=1}^3 \Theta_i^b \Theta_i^c = g_{S^2}^{bc} \quad ; \quad \sum_{i=1}^3 \Theta_i^b \nabla^a \Theta_i^c = 0 \quad ; \quad \sum_{i=1}^3 \nabla_a \Theta_i^b \nabla^a \Theta_i^c = g_{S^2}^{bc},$$

and (244) follows immediately.

Furthermore, if we denote the components of  $\nabla \alpha$  in the  $(\omega_1 = \theta, \omega_2 = \varphi)$  coordinates by

$$\alpha_{\mathbf{a};\mathbf{b}} := \partial_{\mathbf{b}} \alpha_{\mathbf{a}} + \mathcal{S}^2 \Gamma_{\mathbf{ab}}^c \alpha_c,$$

then

$$\begin{aligned} (\nabla^a \alpha_a)^2 &= (g_{S^2}^{11} \alpha_{1;1} + g_{S^2}^{22} \alpha_{2;2})^2, \\ (\epsilon^{ab} \nabla_a \alpha_b)^2 &= g_{S^2}^{22} (\alpha_{1;2} - \alpha_{2;1})^2. \end{aligned}$$

And so, using the identities  $2(a^2 + b^2) \geq (a \pm b)^2$ , then the fact that  $g_{S^2}^{11} = 1$ , and finally (244), we have

$$\begin{aligned} (\nabla^a \alpha_a)^2 + (\epsilon^{ab} \nabla_a \alpha_b)^2 &\leq 2 \left( (g_{S^2}^{11} \alpha_{1;1})^2 + (g_{S^2}^{22} \alpha_{2;2})^2 + g_{S^2}^{22} (\alpha_{1;2}^2 + \alpha_{2;1}^2) \right) \\ &= 2 \nabla_a \alpha_b \nabla^a \alpha^b \\ &\leq 2 \sum_{i=1}^3 |\mathcal{L}_{\Theta_i} \alpha|^2, \end{aligned}$$

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<sup>1</sup>It is worth mentioning that in addition to checking these identities by hand, we had also carried out these calculations using Sage which proves to be a great symbolic computational program for differential geometry, in both calculus and algebra.

which proves (239).

Similar to what we did in the beginning of the proof to obtain (242), one has

$$\begin{aligned} \sum_{X_1, \dots, X_5 \in \mathbb{O}} E_T[\mathcal{L}_{X_1} \dots \mathcal{L}_{X_5} F](t) &= \frac{1}{4} \sum_{X_1, \dots, X_5 \in \mathbb{O}} \int_{\Sigma_t} 2V |\mathcal{L}_{X_1} \dots \mathcal{L}_{X_5} \Phi_0|^2 \\ &\quad + |\mathcal{L}_{X_1} \dots \mathcal{L}_{X_5} \alpha|^2 + |\mathcal{L}_{X_1} \dots \mathcal{L}_{X_5} \underline{\alpha}|^2 dr_* d^2\omega . \end{aligned}$$

By (170), we see that

$$\mathbb{A}^2 \Phi_0 = \sum_{i,j=1}^3 \Theta_i^2 \Theta_j^2 \Phi_0 .$$

if we set

$${}^{ij} \Phi_0 = \Theta_i^2 \Theta_j^2 \Phi_0 ,$$

then  ${}^{ij} \Phi_0$  is the middle component of  $\mathcal{L}_{\Theta_i}^2 \mathcal{L}_{\Theta_j}^2 F$ . Using the identity  $n(a_1^2 + \dots + a_n^2) \leq (a_1 + \dots + a_n)^2$  we have

$$E \left[ \mathbb{A}^2 \Phi_0 \right] (t) = E \left[ \sum_{i,j=1}^3 {}^{ij} \Phi_0 \right] (t) \leq C \sum_{i,j=1}^3 E \left[ {}^{ij} \Phi_0 \right] (t) ,$$

then from (239) applied to  ${}^{ij} \Phi_0$  we get

$$E \left[ {}^{ij} \Phi_0 \right] (t) \leq C \sum_{k=1}^3 E_T[\mathcal{L}_{\Theta_k} \mathcal{L}_{\Theta_i}^2 \mathcal{L}_{\Theta_j}^2 F](t) .$$

Summing over  $i$  and  $j$  and noting that

$$\sum_{i,j,k=1}^3 E_T[\mathcal{L}_{\Theta_k} \mathcal{L}_{\Theta_i}^2 \mathcal{L}_{\Theta_j}^2 F](t) \leq \sum_{X_1, \dots, X_5 \in \mathbb{O}} E_T[\mathcal{L}_{X_1} \dots \mathcal{L}_{X_5} F](t) ,$$

we obtain (240).

The final inequality of the lemma is obtain using the same techniques. Indeed, the conformal energy of  $\mathcal{L}_{\Theta_i} F$  can be written as

$$E_K[\mathcal{L}_{\Theta_i} F](t) = \frac{1}{4} \int_{\Sigma_t} u_+^2 |\mathcal{L}_{\Theta_i} \alpha|^2 + u_-^2 |\mathcal{L}_{\Theta_i} \underline{\alpha}|^2 + (u_+^2 + u_-^2) V |\mathcal{L}_{\Theta_i} \Phi_0|^2 dr_* d^2\omega ,$$

and on the other side, using (150) the conformal charge of  $\Phi_0$  is

$$E_C[\Phi_0](t) = \frac{1}{4} \int_{\Sigma_t} \frac{1}{2} (u_+^2 |L\Phi_0|^2 + u_-^2 |N\Phi_0|^2) + (u_+^2 + u_-^2) V |\nabla \Phi_0|^2 dr_* d^2\omega + E[\Phi_0](t) ,$$

but since as we saw,

$$\begin{aligned} |L\Phi_0|^2 &= |(\bar{M}1 + \cot(\theta))\Phi_1|^2 = (\nabla^a \alpha_a)^2 + (\epsilon^{ab} \nabla_a \alpha_b)^2 \\ |N\Phi_0|^2 &= |(M + \cot(\theta))\Phi_{-1}|^2 = (\nabla^a \underline{\alpha}_a)^2 + (\epsilon^{ab} \nabla_a \underline{\alpha}_b)^2 , \end{aligned}$$

thus applying (243) and (239), we prove (241) and the lemma.  $\square$

Those relations, along with all the bounds that we established, will allow us to complete the list of energy estimates required to prove the decay results. We shall use the following compact notation for summations of energies. For  $Y = T$  or  $K$  and  $\mathbb{A}$  a set of (Killing) vector fields,

$$E_Y[\mathcal{L}_{\mathbb{A}}^k F](t) := \sum_{X_1, \dots, X_k \in \mathbb{A}} E_Y[\mathcal{L}_{X_1} \dots \mathcal{L}_{X_k} F](t) . \quad (245)$$

**Proposition 33** (Uniform bound on Conformal Energy). *Let  $F$  be a Maxwell field and  $n$  a non-negative integer, then*

$$\int_{[0, +\infty[ \times \mathbb{R} \times \mathcal{S}^2} t \chi_{trap} |\Phi_0|^2 dt dr_* d^2 \omega \leq C \left( E_K[\mathcal{L}_0 F](0) + \sum_{k=0}^5 E_T[\mathcal{L}_0^k F](0) \right) , \quad (246)$$

$$E_K[\mathcal{L}_{\mathbb{T}}^n F](t) \leq C \left( \sum_{k=0}^{n+1} E_K[\mathcal{L}_{\mathbb{T}}^k F](0) + \sum_{k=0}^{n+5} E_T[\mathcal{L}_{\mathbb{T}}^k F](0) \right) . \quad (247)$$

*Proof.* Since we assume that  $\Phi_0$  is of the form (136), then using (162)

$$\int_{[0, +\infty[ \times \mathbb{R} \times \mathcal{S}^2} t \chi_{trap} |\Phi_0|^2 dt dr_* d^2 \omega \leq \int_{[0, +\infty[ \times \mathbb{R} \times \mathcal{S}^2} t \chi_{trap} |\nabla \Phi_0|^2 dt dr_* d^2 \omega ,$$

which in turn by (181) yields

$$\int_{[0, +\infty[ \times \mathbb{R} \times \mathcal{S}^2} t \chi_{trap} |\Phi_0|^2 dt dr_* d^2 \omega \leq C (E_c[\Phi_0](0) + E[\Delta^2 \Phi_0](0)) ,$$

and (246) follows from (240) and (241).

Applying the trapping lemma for  $\mathcal{L}_{X_1} \dots \mathcal{L}_{X_n} F$  over  $[0, t]$ , then summing over  $X_i \in \mathbb{T}$ , (234) becomes

$$E_K[\mathcal{L}_{\mathbb{T}}^n F](t) - E_K[\mathcal{L}_{\mathbb{T}}^n F](0) \leq \sum_{X_1 \dots X_n \in \mathbb{T}_{[0, t] \times \mathbb{R} \times \mathcal{S}^2}} \int_{[0, t] \times \mathbb{R} \times \mathcal{S}^2} t \chi_{trap} |\mathcal{L}_{X_1} \dots \mathcal{L}_{X_n} \Phi_0|^2 dt dr_* d^2 \omega ,$$

which upon using (246) to bound the right hand side gives (247).  $\square$

### 3.2.2 Decay Results

#### Uniform Decay

From the uniform bound on the integral of the trapping term, (246), the conformal energy is controlled on any achronal future oriented smooth hypersurface (i.e. a smooth hypersurface with future oriented casual normal) such that its union with  $\Sigma_t = \{t\} \times \mathbb{R} \times \mathcal{S}^2$  is the boundary of an open submanifold of  $\mathcal{N}$ . To see why, let  $S$  be such a hypersurface and

$\mathcal{U}$  the corresponding open subset of  $[0, +\infty[ \times \mathbb{R}_{r_*} \times \mathcal{S}_{\theta, \varphi}^2$ , then by (208) and the last step in the proof of Lemma (31), the trapping lemma, we have

$$\int_{\partial \mathcal{U}} \mathbf{T}_{ab} K^b \eta^a i_\tau d^4x = \frac{1}{2} \int_{\mathcal{U}} {}^{(K)}\pi_{ab} \mathbf{T}^{ab} d^4x \leq \int_{[0, +\infty[ \times \mathbb{R} \times \mathcal{S}^2} t \chi_{trap} |\Phi_0|^2 dt dr_* d^2\omega .$$

So, by (246), we have a uniform bound for all such hypersurfaces  $S$ ,

$$E_K[F](S) = \int_S \mathbf{T}_{ab} K^b \eta^a i_\tau d^4x \leq C \left( \sum_{k=0}^1 E_K[\mathcal{L}_0^k F](0) + \sum_{k=0}^5 E_T[\mathcal{L}_0^k F](0) \right) . \quad (248)$$

**Theorem 34** (Uniform Decay). *Let  $t_0 \geq 0$  be a real parameter. Let  $F$  with spin components  $(\Phi_1, \Phi_0, \Phi_{-1})$ , be a non-stationary finite energy solution of Maxwell's equations (38) and (39), that is, satisfying (135) and (136). Let  $S$  be any achronal future oriented smooth hypersurface, such that its union with  $\Sigma_0 = \{0\} \times \mathbb{R} \times \mathcal{S}^2$  is the boundary of an open submanifold of  $\mathcal{N}$ , and such that on  $S$ ,  $t \geq |r_*| + t_0$ . If  $F$  and its first five Lie derivatives with respect to  $\mathbb{O}$  in (204) have finite energies and conformal energies on  $\Sigma_0$ , then the energy of the Maxwell field on  $S$ , defined by (210) for  $X = T = \partial_t$ , decays like  $t_0^{-2}$ .*

*In fact, there is a constant  $C > 0$  independent of  $t_0$ ,  $F$ ,  $(t, r_*, \omega)$ , and  $S$ , such that*

$$E_T[F](S) \leq t_0^{-2} C \left( \sum_{k=0}^1 E_K[\mathcal{L}_0^k F](0) + \sum_{k=0}^5 E_T[\mathcal{L}_0^k F](0) \right) . \quad (249)$$

*Proof.* If we show that

$$E_T[F](S) \leq t_0^{-2} E_K[F](S) \quad (250)$$

then (248) gives the desired decay. To prove (250), it is sufficient to show that

$$\mathbf{T}_{ab} \eta_S^a (K^b - t_0^2 T^b) \geq 0 ,$$

which by the dominant energy condition (Corollary 30) and the remark after it, reduces to showing that  $K^b - t_0^2 T^b$  is a future oriented causal vector field, that is, proving that

$$t^2 + r_*^2 - t_0^2 \geq |2tr_*| \quad \text{i.e.} \quad (t - |r_*|)^2 \geq t_0^2 ,$$

but this is true on  $S$  by the hypothesis of the theorem.  $\square$

Another way of looking at this results is by considering a collection of hypersurfaces  $\{S_{t_0}; t_0 \geq 0\}$  indexed by  $t_0$ , and the theorem says that the energy flux across these surfaces decays as  $t_0$  goes to infinity. If one takes the collection of parabolas  $\{S_{t_0} : t = \sqrt{1 + r_*^2} + t_0; t_0 \geq 0\}$ , then we see that the energy decays as the surfaces approach the timelike infinity. This kind of decay is particularly useful in the construction of scattering theories on spacetimes.



## Pointwise Decay

We shall divide the proof of the pointwise decay into lemmata. Let  $F$  be a non-stationary finite energy solution of Maxwell's equations, and recall the norms defined in the begin of this section, We start by the following estimate.

**Lemma 35.** *Let  $[r_{*1}, r_{*2}]$  be a compact interval of  $r_*$ . Then there is a constant  $C_{(r_{*1}, r_{*2})} > 0$  such that for all  $t \geq 0$ ,*

$$\int_{\{t\} \times (r_{*1}, r_{*2}) \times \mathcal{S}^2} t^2 |F|_{\mathbb{X}}^2 dr_*^2 d\omega^2 \leq C E_K[F](t) \quad (251)$$

*Proof.* Let  $a = 2 \max\{|r_{*1}|, |r_{*2}|\}$ . If  $t \geq a$ , then for all  $r_* \in [r_{*1}, r_{*2}]$  we have  $t \geq 2|r_*|$  and so

$$\begin{aligned} t - r_* &\geq \frac{1}{2}t \Rightarrow (t - r_*)^2 \geq \frac{1}{4}t^2, \\ t + r_* &\geq \frac{1}{2}t \Rightarrow (t + r_*)^2 \geq \frac{1}{4}t^2. \end{aligned}$$

Thus, for  $t \geq a$ , from the equivalence of norms discussed in the beginning of this section,

$$\begin{aligned} E_K[F](t) &= \frac{1}{4} \int_{\Sigma_t} (t + r_*)^2 |\Phi_1|^2 + (t^2 + r_*^2) V |\Phi_0|^2 + (t - r_*)^2 |\Phi_{-1}|^2 dr_* d^2\omega \\ &\geq C_{(r_{*1}, r_{*2})} \int_{\{t\} \times (r_{*1}, r_{*2}) \times \mathcal{S}^2} t^2 (|\Phi_1|^2 + |\Phi_0|^2 + |\Phi_{-1}|^2) dr_* d^2\omega \\ &\geq C_{(r_{*1}, r_{*2})} \int_{\{t\} \times (r_{*1}, r_{*2}) \times \mathcal{S}^2} t^2 |F|_{\mathbb{X}}^2 dr_* d^2\omega. \end{aligned}$$

If for some  $t' \in [0, a]$  we have  $E_K[F](t') = 0$ , then since its integrant is a continuous function, it vanishes over  $(r_{*1}, r_{*2}) \times \mathcal{S}^2$ , and so,  $|\Phi_1(t')| = |\Phi_0(t')| = |\Phi_{-1}(t')| = 0$ . By the uniqueness of the solution to the Cauchy problem and the linearity of Maxwell's equations,  $F$  must be identically zero for all  $t$ , and the inequality trivially holds. Thus, for  $F \neq 0$ ,  $E_K[F](t) \neq 0$  for all  $t$ , and so the function

$$(E_K[F](t))^{-1} \int_{\{t\} \times (r_{*1}, r_{*2}) \times \mathcal{S}^2} t^2 |F|_{\mathbb{X}}^2 dr_* d^2\omega$$

is continuous and hence bounded over the interval  $[0, a]$ , which is what we want to prove.  $\square$

We will also need some Sobolev estimates which are direct consequences of classical results.

**Lemma 36.** 1. Let  $u$  be a smooth function on  $\bar{U}$  with  $U = (a, b) \subset \mathbb{R}$ . Let  $H^1(U)$  be a Sobolev space over  $U$ . Then,

$$\sup_{\bar{U}} |u| \leq C \|u\|_{H^1(U)} = C \left( \int_{(a,b)} |u|^2 + |u'|^2 ds \right)^{\frac{1}{2}}. \quad (252)$$

2. Let  $u$  be a smooth function on the Sphere  $\mathcal{S}^2$ . Then,

$$\sup_{\mathcal{S}^2} |u| \leq C \left( \int_{\mathcal{S}^2} |u|^2 + |\nabla u|^2 + |\Delta u|^2 d^2\omega \right)^{\frac{1}{2}}. \quad (253)$$

*Proof.* Since  $u$  is smooth, the first estimate is immediate from the classical Morrey's local inequality (see [57] for example). For the second estimate, we know from the Sobolev estimate on the Sphere that  $H^2(\mathcal{S}^2) \subset C^0(\mathcal{S}^2)$  with

$$\sup_{\mathcal{S}^2} |u| \leq C \left( \int_{\mathcal{S}^2} |u|^2 + |\nabla^2 u|^2 d^2\omega \right)^{\frac{1}{2}}.$$

where  $|\nabla^2 u|^2 = |g_{\mathcal{S}^2}^{ij} g_{\mathcal{S}^2}^{kl} (\nabla^2 u)_{il} (\nabla^2 u)_{jk}|$ . By the divergence theorem,

$$\int_{\mathcal{S}^2} \Delta(|\nabla u|^2) d^2\omega = 0.$$

Then using Bochner-Weitzenbock-Lichnerowicz formula,

$$\frac{1}{2} \Delta(|\nabla u|^2) = |\nabla^2 u|^2 + (\nabla u) \cdot (\nabla \Delta u) + \text{Ric}_{\mathcal{S}^2}(\nabla u, \nabla u),$$

and applying an integration by parts then using the fact that  $\text{Ric}_{\mathcal{S}^2} \geq C g_{\mathcal{S}^2}$  for some  $C > 0$ , we get (253).  $\square$

We now state the pointwise decay and complete the proof.

**Theorem 37** (Pointwise Decay). *Let  $[r_{*1}, r_{*2}]$  be a compact interval of  $r_*$ . Then there is a constant  $C_{(r_{*1}, r_{*2})} > 0$  such that if  $F$  with spin components  $(\Phi_1, \Phi_0, \Phi_{-1})$ , is a non-stationary finite energy solution of Maxwell's equations (38) and (39), that is, satisfying (135) and (136), then for all  $t \geq 0$ ,  $r_* \in [r_{*1}, r_{*2}]$ ,  $\omega = (\theta, \varphi) \in \mathcal{S}^2$ ,*

$$|\Phi_1| + |\Phi_0| + |\Phi_{-1}| \leq C_{(r_{*1}, r_{*2})} t^{-1} \left( \sum_{k=0}^4 E_K[\mathcal{L}_{\mathbb{T}}^k F](0) + \sum_{k=0}^8 E_T[\mathcal{L}_{\mathbb{T}}^k F](0) \right). \quad (254)$$

*Proof.* In this proof, the constant  $C$  may depend on the interval  $[r_{*1}, r_{*2}]$ . Since the covariant derivative and the Lie derivative of a smooth vector field with respect to another smooth vector field is again a smooth vector field, then it is a linear combination of the coordinate vector fields  $\partial_t, \partial_{r_*}, \partial_\theta$ , and  $\partial_\varphi$  with smooth coefficient functions. Thus, if we have  $X$  and

$Y$  in  $\mathbb{X}$  from (204), then  $\mathcal{L}_X Y$  and  $\nabla_X Y$  will be independent of  $t$ , and over  $[r_{*1}, r_{*2}] \times \mathcal{S}^2$  their coefficient functions in the the frame  $(\partial_t, \partial_{r_*}, \partial_\theta, \partial_\varphi)$  will be bounded by a constant  $C$  depending only on  $[r_{*1}, r_{*2}] \times \mathcal{S}^2$ . By the paragraph just before (207), we see that this is also true in the frame  $\mathbb{X}$ .

To prove the decay, we need the control

$$|\mathcal{L}_R \mathcal{L}_{T_1} \mathcal{L}_{T_2}(F(X_1, X_2))| \leq C|F|_{\mathbb{X}, 3, \mathbb{T}}, \quad (255)$$

$$|\mathcal{L}_R \mathcal{L}_{T_1}(F(X_1, X_2))| \leq C|F|_{\mathbb{X}, 2, \mathbb{T}}, \quad (256)$$

$$|\mathcal{L}_R(F(X_1, X_2))| \leq C|F|_{\mathbb{X}, 1, \mathbb{T}}, \quad (257)$$

for all  $T_1, T_2 \in \mathbb{T}$  and  $X_1, X_2 \in \mathbb{X}$ .

Let  $X_1, X_2, X_3 \in \mathbb{X}$ , then

$$X_1(F(X_2, X_3)) = (\mathcal{L}_{X_1} F)(X_2, X_3) + F(\mathcal{L}_{X_1} X_2, X_3) + F(X_2, \mathcal{L}_{X_1} X_3), \quad (258)$$

and so,

$$|X_1(F(X_2, X_3))| \leq C(|(\mathcal{L}_{X_1} F)(X_2, X_3)| + |F|_{\mathbb{X}}) \leq C|F|_{\mathbb{X}, 1, \mathbb{X}}. \quad (259)$$

If we apply  $X_4 \in \mathbb{X}$  to (258), then expand  $X_4(\mathcal{L}_{X_1} F)(X_2, X_3)$  as in (258) with  $\mathcal{L}_{X_1} F$  replacing  $F$ , then by (259) we get

$$|X_4 X_1(F(X_2, X_3))| \leq C|F|_{\mathbb{X}, 2, \mathbb{X}}.$$

Repeating this process  $n$  times, we have the iterated estimate

$$|\mathcal{L}_{X_1} \dots \mathcal{L}_{X_k}(F(X, Y))| \leq C|F|_{\mathbb{X}, k, \mathbb{X}}.$$

In particular, if we choose  $k$  vectors  $X_i \in \mathbb{T}$  and  $X, Y \in \mathbb{X}$ , we obtain

$$|\mathcal{L}_{X_1} \dots \mathcal{L}_{X_k}(F(X, Y))| \leq C|F|_{\mathbb{X}, k, \mathbb{T}}. \quad (260)$$

Similar to (258), we also have for  $X_1, X_2, X_3 \in \mathbb{X}$ ,

$$X_1(F(X_2, X_3)) = (\nabla_{X_1} F)(X_2, X_3) + F(\nabla_{X_1} X_2, X_3) + F(X_2, \nabla_{X_1} X_3), \quad (261)$$

If we take  $T_1, T_2 \in \mathbb{T}$  then by applying  $\mathcal{L}_{T_1} \mathcal{L}_{T_2}$  to (261) with  $T_3 \in \mathbb{T}$  instead of  $X_1 \in \mathbb{X}$ , keeping  $X_2, X_3 \in \mathbb{X}$ , we get

$$\begin{aligned} \mathcal{L}_{T_1} \mathcal{L}_{T_2}(T_3(F(X_2, X_3))) &= \mathcal{L}_{T_1} \mathcal{L}_{T_2}((\nabla_{T_3} F)(X_2, X_3)) \\ &\quad + \mathcal{L}_{T_1} \mathcal{L}_{T_2}(F(\nabla_{T_3} X_2, X_3)) + \mathcal{L}_{T_1} \mathcal{L}_{T_2}(F(X_2, \nabla_{T_3} X_3)). \end{aligned}$$

By (260), the left hand side and the last two terms of the right hand side are controlled by  $|F|_{\mathbb{X}, 3, \mathbb{T}}$  and  $|F|_{\mathbb{X}, 2, \mathbb{T}}$  respectively, and hence

$$|\mathcal{L}_{T_1} \mathcal{L}_{T_2}((\nabla_{T_3} F)(X_2, X_3))| \leq C|F|_{\mathbb{X}, 3, \mathbb{T}}. \quad (262)$$

Similarly one has

$$\begin{aligned} |\mathcal{L}_{T_1}((\nabla_{T_3} F)(X_2, X_3))| &\leq C|F|_{\mathbb{X}, 2, \mathbb{T}}. \\ |(\nabla_{T_3} F)(X_2, X_3)| &\leq C|F|_{\mathbb{X}, 1, \mathbb{T}}. \end{aligned} \quad (263)$$

Let  $T_1, \dots, T_4 \in \mathbb{T}$ , then starting from (261) again, we apply two Lie derivatives and obtain

$$\begin{aligned} \mathcal{L}_{T_1} \mathcal{L}_{T_2}(R(F(T_3, T_4))) &= \mathcal{L}_{T_1} \mathcal{L}_{T_2}((\nabla_R F)(T_3, T_4)) \\ &\quad + \mathcal{L}_{T_1} \mathcal{L}_{T_2}(F(\nabla_R T_3, T_4)) + \mathcal{L}_{T_1} \mathcal{L}_{T_2}(F(T_3, \nabla_R T_4)) . \end{aligned}$$

From (41),

$$(\nabla_R F)(T_3, T_4) = -(\nabla_{T_3} F)(R, T_4) - (\nabla_{T_4} F)(T_3, R) .$$

and so we have,

$$\begin{aligned} |\mathcal{L}_{T_1} \mathcal{L}_{T_2}(R(F(T_3, T_4)))| &\leq |\mathcal{L}_{T_1} \mathcal{L}_{T_2}((\nabla_{T_3} F)(R, T_4))| + |\mathcal{L}_{T_1} \mathcal{L}_{T_2}((\nabla_{T_4} F)(T_3, R))| + \\ &\quad + |\mathcal{L}_{T_1} \mathcal{L}_{T_2}(F(\nabla_R T_3, T_4))| + |\mathcal{L}_{T_1} \mathcal{L}_{T_2}(F(T_3, \nabla_R T_4))| , \end{aligned}$$

and all the terms on the right hand side are controlled by (262) and (260). Thus,

$$|\mathcal{L}_{T_1} \mathcal{L}_{T_2}(R(F(T_3, T_4)))| \leq C|F|_{\mathbb{X}, 3, \mathbb{T}} .$$

Similarly, we have,

$$\begin{aligned} |\mathcal{L}_{T_1}(R(F(T_3, T_4)))| &\leq C|F|_{\mathbb{X}, 2, \mathbb{T}} , \\ |R(F(T_3, T_4))| &\leq C|F|_{\mathbb{X}, 1, \mathbb{T}} . \end{aligned}$$

Next, choose an a local orthonormal frame  $\{rV, rW\}$  on  $U \subset \mathcal{S}^2$ , so that

$$\{f^{-\frac{1}{2}}\partial_t, f^{-\frac{1}{2}}\partial_{r_*}, V, W\}$$

is a local orthonormal frame on the spacetime. In this frame, for  $X \in \mathbb{X}$ , (40) is

$$f^{-1}((\nabla_{\partial_t} F)(\partial_t, X) - (\nabla_R F)(R, X)) - (\nabla_V F)(V, X) - (\nabla_W F)(W, X) = 0 . \quad (264)$$

Shrinking  $U$  if necessary,  $V$  and  $W$  can be locally written as linear combinations of vectors in  $\mathbb{O}$  with smooth coefficients, and since  $\mathcal{S}^2$  is locally compact, the coefficients and all their derivatives are bounded in some neighbourhood around each point. By covering  $\mathcal{S}^2$  with such  $U$ 's, and as  $\mathcal{S}^2$  is compact, a finite subcover of all these neighbourhoods, means that we have a uniform bound for all these coefficients defined on the subcover. Applying  $\mathcal{L}_{T_1} \mathcal{L}_{T_2}$ , for  $T_1, T_2 \in \mathbb{T}$ , to (264) with  $V$  and  $W$  replaced by there expressions in terms of  $\Theta_i$ 's with bounded coefficients, then using (262) and (263), we get the control

$$|\mathcal{L}_{T_1} \mathcal{L}_{T_2}((\nabla_R F)(R, X))| \leq C|F|_{\mathbb{X}, 3, \mathbb{T}} .$$

This control, by virtue of (261) gives the required control

$$|\mathcal{L}_{T_1} \mathcal{L}_{T_2}(R(F(R, X)))| \leq C|F|_{\mathbb{X}, 3, \mathbb{T}} .$$

Similarly,

$$\begin{aligned} |\mathcal{L}_{T_1}(R(F(R, X)))| &\leq C|F|_{\mathbb{X}, 2, \mathbb{T}} , \\ |R(F(R, X))| &\leq C|F|_{\mathbb{X}, 1, \mathbb{T}} . \end{aligned}$$

This covers (255)-(257).

Let  $X_1, X_2 \in \mathbb{X}$ , from (253),

$$|F(X_1, X_2)|^2 \leq C \int_{\mathcal{S}^2} |F(X_1, X_2)|^2 + |\nabla(F(X_1, X_2))|^2 + |\Delta(F(X_1, X_2))|^2 d^2\omega .$$

Then using (169) and (170),

$$\begin{aligned} |F(X_1, X_2)|^2 &\leq C \int_{\mathcal{S}^2} |F(X_1, X_2)|^2 + \sum_{i=1}^3 (|\Theta_i(F(X_1, X_2))|^2 + |\Theta_i^2(F(X_1, X_2))|^2) d^2\omega \\ &\leq C \int_{\mathcal{S}^2} |F(X_1, X_2)|^2 + \sum_{T_1, T_2 \in \mathbb{T}} (|\mathcal{L}_{T_1}(F(X_1, X_2))|^2 + |\mathcal{L}_{T_1}\mathcal{L}_{T_2}(F(X_1, X_2))|^2) d^2\omega . \end{aligned}$$

By (252),

$$\begin{aligned} \int_{\{t\} \times \{r_*\} \times \mathcal{S}^2} |\mathcal{L}_{T_1}\mathcal{L}_{T_2}(F(X_1, X_2))|^2 d^2\omega &\leq C \int_{\{t\} \times (r_{*1}, r_{*2}) \times \mathcal{S}^2} |\mathcal{L}_{T_1}\mathcal{L}_{T_2}(F(X_1, X_2))|^2 \\ &\quad + |\mathcal{L}_{T_1}\mathcal{L}_{T_2}(R(F(X_1, X_2)))|^2 dr_* d^2\omega , \\ \int_{\{t\} \times \{r_*\} \times \mathcal{S}^2} |\mathcal{L}_{T_1}(F(X_1, X_2))|^2 d^2\omega &\leq C \int_{\{t\} \times (r_{*1}, r_{*2}) \times \mathcal{S}^2} |\mathcal{L}_{T_1}(F(X_1, X_2))|^2 \\ &\quad + |\mathcal{L}_{T_1}(R(F(X_1, X_2)))|^2 dr_* d^2\omega , \\ \int_{\{t\} \times \{r_*\} \times \mathcal{S}^2} |(F(X_1, X_2))|^2 d^2\omega &\leq C \int_{\{t\} \times (r_{*1}, r_{*2}) \times \mathcal{S}^2} |(F(X_1, X_2))|^2 + |(R(F(X_1, X_2)))|^2 dr_* d^2\omega . \end{aligned}$$

Summing over all  $X_1, X_2 \in \mathbb{X}$ , then using (255), (256), (257), and (260), we have

$$|F|_{\mathbb{X}}^2 \leq C \int_{\{t\} \times (r_{*1}, r_{*2}) \times \mathcal{S}^2} |F|_{\mathbb{X}, 3, \mathbb{T}}^2 dr_* d^2\omega .$$

Finally, since for  $T_1, \dots, T_k \in \mathbb{T}$ ,  $\mathcal{L}_{T_1} \dots \mathcal{L}_{T_k} F$  is again a Maxwell field, then applying (251) for  $\mathcal{L}_{T_1} \dots \mathcal{L}_{T_k} F$ , we have

$$\int_{\{t\} \times (r_{*1}, r_{*2}) \times \mathcal{S}^2} t^2 |\mathcal{L}_{T_1} \dots \mathcal{L}_{T_k} F|_{\mathbb{X}}^2 dr_* d^2\omega \leq C E_K[\mathcal{L}_{T_1} \dots \mathcal{L}_{T_k} F](t) ,$$

Taking summation over all  $T_i \in \mathbb{T}$  and over  $k$  from 0 to 3, we obtain,

$$\int_{\{t\} \times (r_{*1}, r_{*2}) \times \mathcal{S}^2} t^2 |F|_{\mathbb{X}, 3, \mathbb{T}}^2 \leq C \sum_{k=0}^3 E_K[\mathcal{L}_{\mathbb{T}}^k F](t) ,$$

and therefore,

$$|F|_{\mathbb{X}}^2 \leq C t^{-2} E_K[\mathcal{L}_{\mathbb{T}}^k F](t) ,$$

and the decay follows from (247). □

### 3.2.3 Generic Spherically Symmetric Black Hole Spacetimes

We conclude these results by a remark on generic spherically symmetric black hole spacetimes, and specify what conditions on the metric are needed for our results to hold on a more general spacetime than RNdS.

Let the manifold  $\mathcal{M} = \mathbb{R}_t \times ]0, +\infty[_r \times \mathcal{S}_{\theta, \varphi}^2$  be equipped with the metric defined in (1), with  $f$  a smooth function on  $]0, +\infty[_r$ . Consider the following conditions on  $f$ :

1. There are three real numbers  $r_i$ ,  $0 \leq r_1 < r_2 < r_3 \leq +\infty$ , and these are the only possible zeros of  $f$ .
2. If  $0 < r_i < +\infty$  then  $f(r_i) = 0$  and  $f'(r_i) \neq 0$ .
3.  $f(r) < 0$  for  $r \in ]r_1, r_2[$  and  $f(r) > 0$  for  $r \in ]r_2, r_3[$
4. If  $r_3$  is infinite then  $f(r) = 1 - Cr^{-1} + O(r^{-2})$  for  $C > 0$ , and for  $k = 1, 2, 3$ ,  $\partial_r^k f(r) = O(r^{-k-1})$ , as  $r \rightarrow +\infty$ .

All these properties are satisfied by usual spherical black holes, like Schwarzschild, Reissner-Nordström, Yasskin, asymptotically flat, or De Sitter, etc ...

If  $f$  indeed satisfies these conditions, then all of the above decay results hold equally true for such a generic spherically symmetric spacetimes. Our arguments make no use of the particular form of  $f$  in RNdS, but rather these properties of  $f$ , and so the required arguments for a general spacetime are the same.



# Chapter 4

## Conformal Scattering

### 4.0 Introduction

In the classic experiment of scattering one has a field propagating in a medium with an obstacle ; an incoming plane wave hits the obstacle and scatters away from it as a superposition of outgoing plane waves. *Scattering theory* is a way of summarizing this evolution, which may involve complicated intermediate interactions of the field, described as the solution to an evolution equation, by constructing the map that, to the asymptotic behaviour of the solution in the distant past (incoming wave), associates its asymptotic behaviour in the distant future (outgoing wave). This can be done provided the asymptotic behaviour characterizes the solution completely. Radar systems make use of this characterization of the solution by its asymptotic profile to gain information about the medium and the obstacles it contains. This reconstruction is the aim of *inverse scattering*.

#### 4.0.1 Brief History of Analytic Scattering

Scattering theory proved to be a useful tool in the framework of general relativity to study the asymptotic influence of the geometry of spacetime on fields. Although in this current work we do not use an analytic approach to scattering, we very briefly touch on the history of the subject because this is part of the origin of conformal scattering and it helps to understand what new features the conformal approach bring to the domain. Scattering theory in black holes spacetimes played an essential role in the rigorous description of phenomena like superradiance, the Hawking effect, and quasi-normal modes (resonances of black holes which are related to gravitational waves). In 1980 S. Chandrasekhar [28] used the stationary approach, resorting to a Fourier transformation in time, to study quasi-linear modes of black hole spacetimes such as Schwarzschild, Reissner-Nordström, and Kerr. Chandrasekhar's work systematically used the Newman-Penrose formalism to develop stationary scattering theories described in terms of the scattering matrix of transmission and reflection coefficients. And around the same time, M. Reed and B. Simon published “*Scattering Theory*” the third volume of their classic series [131]. Then time-dependent scattering (based on the comparison of dynamics) of classical and quantum fields on the exterior of a Schwarzschild black hole



were first studied by J. Dimock in 1985 [53] and by J. Dimock and B. Kay in 1986 and 1987 [54, 55, 56]. And in the 1990's, A. Bachelot produced an important series of papers starting with scattering theories for classical fields, Maxwell in 1990 and 1991 [7, 10], Klein-Gordon in 1994 [11] and on the Hawking effect for a spherical gravitational collapse in 1997 [8], 1999 [12] and 2000 [13]. J.-P. Nicolas in 1995 developed a scattering theory for classical massless Dirac fields [117], and a work on a non linear Klein-Gordon equation on the Schwarzschild metric (and other similar geometries) with partial scattering results obtained by conformal methods in 1995[118]. W.M. Jin in 1998 contributed to the subject with a construction of wave operators in the massive case [85], and F. Melnyk in 2003 obtained a complete scattering for massive charged Dirac fields [103] and the Hawking effect for charged, massive spin-1/2 fields [104]. In 1999 I. Laba and A. Soffer [94] obtained complete scattering for the nonlinear Schrödinger equation on Schwarzschild manifolds. Then people started using commutator methods and Mourre theory. This led to scattering theories on the Kerr metric. One paper appeared in 1992 due to S. De Bièvre, P. Hislop and I.M. Sigal [52] on scattering theory for the wave equation on non-compact manifolds by means of a Mourre estimate. A complete scattering theory for the wave equation, on stationary, asymptotically flat space-times, was subsequently obtained by D. Häfner in 2001 using the Mourre theory [75]. Time-dependent scattering theories on Kerr black holes were obtained by D. Häfner in 2003 [76] and in 2004 by D. Häfner and J.-P. Nicolas for massless Dirac fields using a Mourre estimate [77]. In 2005 T. Daudé produced scattering theories for Dirac fields in various spacetimes [48, 49, 50], and in 2010 he published results on time-dependent scattering for charged Dirac [51], before moving to several works on inverse scattering in general relativity. In 2014 M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman developed scattering theory for the scalar wave equation on Kerr exterior backgrounds in the subextremal case [46].

## 4.0.2 Conformal Scattering

In the present work, we construct a *Conformal Scattering* theory. Conformal scattering is a geometrical approach to time-dependent scattering based on Penrose *conformal compactification*: a rescaling of the metric and the fields using conformal factors. This enables the definition of a *scattering operator*, the fundamental object in the theory. This operator associates to the asymptotic behaviour of the solution in the distant past, its asymptotic behaviour in the distant future. Here, the asymptotics of the solution are the scattering data and are given as restrictions of the conformally rescaled solution on past and future null infinities and are called *radiation fields*. With suitable energy estimates, which is a crucial step in the theory, the scattering data completely characterizes the solution. This can be viewed as an initial-value problem which is a characteristic Cauchy problem, also called a *Goursat problem*, where data is given at null infinity instead of some spacelike hypersurface as in the non-characteristic case. The resolution of the Goursat problem is the core of conformal scattering theory.

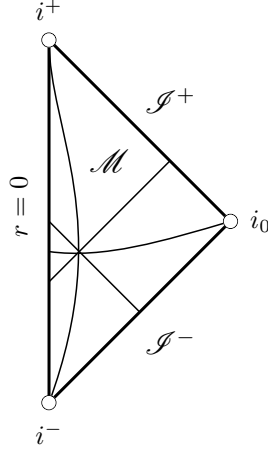


Figure 4.1: Penrose diagram of  $\hat{\mathcal{M}}$  the conformal compactification of Minkowski spacetime  $\mathcal{M}$  with timelike, spacelike, and null curves.

## The Main Ingredients

We describe the essential steps of the general strategy of conformal scattering.

**Conformal compactification.** In the words of R. Penrose, conformal compactification is a technique to “make infinity finite”. A *globally hyperbolic spacetime*<sup>1</sup>  $(\mathcal{M}, g)$ , with suitable asymptotic structure, such as asymptotic flatness, is rescaled and replaced by an “unphysical” Lorentzian manifold with boundary  $(\hat{\mathcal{M}}, \hat{g})$ , the *conformal compactification* of  $\mathcal{M}$ , with  $\partial\hat{\mathcal{M}} = \mathcal{I}$  that represents points at infinity of  $(\mathcal{M}, g)$ , and  $\text{int}\hat{\mathcal{M}} = \mathcal{M}$ . The new metric is *conformally related* to the original metric by

$$\hat{g} = \Omega^2 g ,$$

for an appropriate choice of a smooth non-negative boundary function  $\Omega$  defined on  $\hat{\mathcal{M}}$ , called the *conformal factor*, such that it is positive on  $\mathcal{M}$  and becomes zero on  $\mathcal{I}$ , the asymptotic regions where  $g$  becomes infinite, and  $d\Omega|_{\mathcal{I}} \neq 0$  (figure 4.1). What is important is to define things in a way such that the new metric has some differentiability on the boundary hypersurface  $\mathcal{I}$ . Now, the asymptotics of  $\mathcal{M}$  can be studied using local techniques on  $\hat{\mathcal{M}}$ , without resorting to complicated limiting arguments when studying, for example, the radiation fields of a physical field on the original spacetime. A *conformally invariant equation* is an equation defined on  $\mathcal{M}$  for  $g$  such that whenever  $\Phi$  is a solution to the equation, then for some  $s \in \mathbb{R}$ , the *rescaled field*<sup>2</sup>  $\hat{\Phi} := \Omega^s \Phi$  is a solution to the same equation but defined on  $\hat{\mathcal{M}}$  for the rescaled metric  $\hat{g}$ . Examples of conformally invariant equations are the conformal wave equations, Dirac equation, and Maxwell’s equations. Working with this class of

<sup>1</sup>A spacetime that admits a spacelike hypersurface that intersect every inextendible causal curve exactly once.

<sup>2</sup>See [125] for the precise definition.

equations that admit such rather explicit transformation law under conformal rescaling ensures that we can study the equation on the rescaled spacetime and gain information on its behaviour in the physical spacetime. Conformal scattering theories have been obtained on generic non-stationary spacetimes [86, 101], but let us here assume the existence of a global Killing timelike (causal) vector field  $\tau$  for simplicity. As the just cited works illustrate, this symmetry assumption can be relaxed to more general situations such as *asymptotically simple spacetimes* defined in [35, 36, 37, 38]. We note that not all spacetimes admit a conformal compactification with the needed regularity of the rescaled metric at the boundary, this is in fact related to the decay of the Weyl curvature at infinity. When the required compactification exists, different parts of the boundary will correspond to different ways of going to infinity (along spacelike, timelike, or null curves). Also, in the cases of black holes, part or all of the conformal boundary will be the horizon or horizons. Horizons are finite null hypersurfaces for the physical metric and when the whole conformal boundary is made of horizons, conformal rescalings are not required ; even in such a case we talk about conformal scattering because we use the same approach based on the resolution of a Goursat problem at the null boundary. Note that such cases are more amenable to extending the method to non-conformally invariant equations since there is no conformal rescaling involved. For more details on the topic of conformal rescaling and compactification we refer to [122, 123, 124, 125, 126].

**Cauchy problem: Defining the trace operators.** The scattering operator is defined using two operators called the past and the future *trace operators*  $T^\pm$ . The past trace operator associates to data at some finite instant of time ( $t = 0$ ), data in the infinite past ( $t = -\infty$ ). The future trace operator associates to data at that finite instant, data in the infinite future ( $t = +\infty$ ). In general, one defines a normed energy space  $\mathcal{H}$  on a Cauchy hypersurface of the compactified spacetime  $\hat{\mathcal{M}}$  and a normed energy spaces  $\mathcal{H}^\pm$  on the boundary parts  $\mathcal{J}^\pm$ . The energy norms are defined by contracting the timelike or causal vector field  $\tau$  with the stress-energy tensor  $\mathbf{T}$  of the studied equations in order to define the energy current  $J_a = \tau^b \mathbf{T}_{ab}$ , and the norm is then the energy flux across the considered hypersurface:

$$\mathcal{E}_{\tau, \Sigma_0} = \int_{\Sigma_0} \tau^a \mathbf{T}_{ab} d\sigma^b \quad \text{and} \quad \mathcal{E}_{\tau, \mathcal{J}^\pm} = \int_{\mathcal{J}^\pm} \tau^a \mathbf{T}_{ab} d\hat{\sigma}^b .$$

The future (past) trace operator then takes finite energy data in  $\mathcal{H}$  defined on the Cauchy hypersurface to finite energy data in  $\mathcal{H}^+$  ( $\mathcal{H}^-$ ) on the future (past) part of the boundary  $\mathcal{J}^+$  ( $\mathcal{J}^-$ ). The general construction of the future operator goes as follows: For a given finite energy data  $\hat{\Phi}_0$  on the spacelike Cauchy hypersurface  $\Sigma_0$  we solve the Cauchy problem on  $\hat{\mathcal{M}}$  to get a solution  $\hat{\Phi}$  of our equation. The future radiation field or the image of  $\hat{\Phi}_0$  by the future trace operator is then the trace (a restriction) of the solution  $\hat{\Phi}$  to the future boundary  $\mathcal{J}^+$ , i.e.  $T^+(\hat{\Phi}_0) = \hat{\Phi}|_{\mathcal{J}^+}$ . The past trace operator is defined similarly (figure 4.2). Of course, not all constructions follow this exact steps. Depending on the asymptotic structure of the spacetime and the equations we are studying, some intermediate steps may be required, and the definition of the

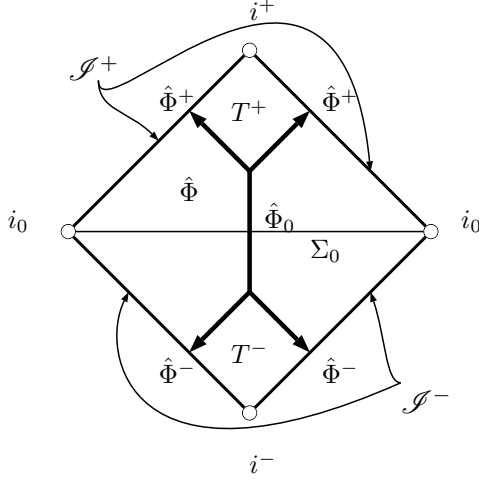


Figure 4.2: The trace operators  $T^\pm$ .

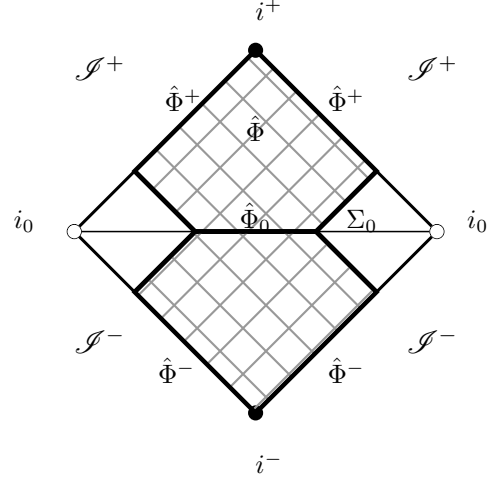


Figure 4.3: The case of regular  $i^\pm$ .

trace operator may differ slightly. For example, while the above scheme generally describes the situation of the wave equation on Minkowski spacetime, additional steps are needed for different spacetimes depending on the nature of the timelike and spacelike infinities,  $i^\pm$  and  $i_0$  (see [121]). On the other hand, the trace operators for Maxwell's equations do not associate to the initial Cauchy data the full restriction of the field but rather a part of it, this is because of the constraint equations that should be satisfied by the solution to the evolution problem. This is the case we treat in this work. For other situations we refer for example to [87, 101, 121].

Let us for the sake of this general overview assume that the studied equations are linear, this entails that the trace operators themselves are linear operators, yet, this is not an absolute necessity for the construction of a conformal scattering theory, see [86] for example.

### Energy estimates: The trace operators are one-to-one and have closed ranges.

The next step is to show that the trace operators are injective or one-to-one. In fact, the above construction of the trace operators is usually done first for a dense subset of the finite energy space  $\mathcal{H}$  on  $\Sigma_0$  such as smooth compactly supported functions. If one proves uniform energy estimates both ways between the initial Cauchy data in the dense subset and their images under the trace operator, then the operator extends to the whole of  $\mathcal{H}$  as a one-to-one map with a closed range. In some cases, one can prove exact energy identities, and the trace operators preserve the energy norms in this case, they are partial isometries. Ways of getting the uniform estimates depend on the structure of the spacetime at infinity and the properties of the stress-energy tensor. If the stress-energy tensor of the original unrescaled equations is divergence-free i.e. conserved, and conformally invariant, as for the Maxwell's equations, then working with the rescaled quantities  $\hat{\Phi}$  and  $\hat{g}$  has the important advantage of seeing all the involved hypersurfaces as regular hypersurfaces at finite distances, in particular  $\mathcal{I}^\pm$ . If we are on Minkowski spacetime, a simple application of Stokes' theorem, or more

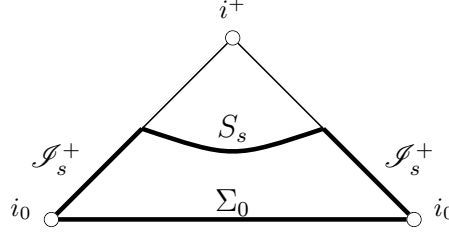


Figure 4.4: *The closed hypersurfaces of the compactified spacetime.*

precisely the divergence theorem, yields the required energy identities:

$$\mathcal{E}_{\tau, \Sigma_0} = \mathcal{E}_{\tau, \mathcal{I}^\pm}.$$

Even if the rescaled metric is singular at  $i_0$ , as long as the initial data is supported away from  $i_0$ , finite propagation speed guarantees that the solution does not see the singular spacelike infinity since it is zero in a neighbourhood of it, and the above technique can be applied without essential modification thanks to the density of compactly supported functions in the energy space (figure 4.3). In the case of black hole spacetimes, timelike infinities are singular. This constitutes an important difficulty and finite propagation speed will not help us here since the singularity lies in the future of any initial data no matter how small its compact support maybe. What we need is sufficient decay results of the solutions to the equations so that we can rule out the accumulation of energy at timelike infinities. In such situations the estimates can be obtained as follows. Consider an achronal hypersurface  $S_s$  ( $s > 0$ ) for the rescaled metric that forms a regular closed hypersurface with the future boundary  $\mathcal{I}^+$  and  $\Sigma_0$ <sup>1</sup> as shown in figure 4.4. The divergence theorem now implies that

$$\mathcal{E}_{\tau, \Sigma_0} = \mathcal{E}_{\tau, \mathcal{I}_s^+} + \mathcal{E}_{\tau, S_s}.$$

Assume that  $S_s$  accumulates on  $i^+$  as  $s \rightarrow +\infty$ . Here is where the decay is needed, namely to show that

$$\lim_{s \rightarrow +\infty} \mathcal{E}_{\tau, S_s} = 0,$$

and the conservation law follows:

$$\mathcal{E}_{\tau, \Sigma_0} = \mathcal{E}_{\tau, \mathcal{I}^+}.$$

Clearly, the same can be done in the past direction. Obtaining the desired decay is usually a separate problem that has its own subject. This is partly why we proved the decay results of chapter 3. In a different setting, such as the wave equation on the Schwarzschild metric, the energy estimates are not as direct since the stress-energy tensor is not conformally invariant, and hence the stress-energy tensor of the rescaled equation is not conserved. However, it happens that one can recover the conservation

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<sup>1</sup>Except possibly for  $i_0$  when it is singular, but the compact support keeps us from running into trouble there.

law for the wave equation on Schwarzschild black hole spacetimes since the error term is a divergence [121], here too a decay result [41] is needed to ensure no information is lost at the singular  $i^\pm$ . The current decay results use techniques that require too precise local information for a scattering theory. It is however not clear what are the minimal decay assumptions needed for conformal scattering.

**Goursat problem: The trace operators are onto.** The third and last step in defining the scattering operator is to prove that the trace operators we defined are surjective, this comes down to solving the Goursat problem on a null hypersurface for data in dense subsets of the finite energy spaces  $\mathcal{H}^\pm$ , usually smooth and compactly supported functions. This means that we need to find for a given smooth compactly supported Goursat data, say  $\hat{\Phi}^+$  on  $\mathcal{I}^+$ , a  $\hat{\Phi}_0 \in \mathcal{H}$  such that  $T^+(\hat{\Phi}_0) = \hat{\Phi}^+$ , or taking into account the well-posedness of the Cauchy problem, we must be able to find a finite energy solution to the equations that has  $\hat{\Phi}^+$  as its trace on  $\mathcal{I}^+$ . One way of solving the characteristic Cauchy problem is to approach the null conformal boundary by spacelike hypersurfaces. Goursat data are projected as part of the Cauchy data on the spacelike slices by means of congruences of null geodesics in the neighbourhood of  $\mathcal{I}$ . The solution to the Goursat problem is then obtained using uniform energy estimates, weak convergence, and compactness methods [83, 101]. In some case, some “reversible” modifications to the setting is needed before applying the methods just mentioned or the results they produce. For example, one can still apply the results of [83] where spatial compactness is needed, to spacetimes that are not spatially compact by a cut-extend construction that transports the problem into a framework suitable for [83], as in section 4.2 following the construction done in [121] where the situation is more subtle due to the singularity at  $i_0$ .

**Scattering operator** With the Goursat problem solved, the trace operators  $T^\pm$  become isometries between the boundary energy spaces  $\mathcal{H}^\pm$  on  $\mathcal{I}^\pm$  and the initial energy space  $\mathcal{H}$  on  $\Sigma_0$ . We can then define the scattering operator  $S : \mathcal{H}^- \rightarrow \mathcal{H}^+$  by  $S = T^+ \circ (T^-)^{-1}$  and it is an isometry. The construction of the scattering operator relies on a choice of Cauchy hypersurface used to construct the trace operators  $T^\pm$ , however, the scattering operator maps the past radiation fields to the future radiation fields independently of the choice of the intermediate spacelike hypersurface and the theory is in fact truly covariant as Penrose hinted in [122].

## History

The introduction of “points at infinity” in a consistent way where these points constitute a hypersurface boundary  $\mathcal{I}$  to a manifold whose interior is conformally identical with the original space-time, was first done by R. Penrose around 1964 [122, 123, 124] and presented in his classic book with W. Rindler [125, 126] in the 1980’s. This idea was first motivated by the fact that massless free-field equations are conformally invariant if interpreted in a suitable way, so their behaviour at “infinity” can be studied at this hypersurface. In the same period of early 1960’s F.G. Friedlander introduced his notion of *radiation fields* [65, 66,

[67]: In spherical coordinates, a radiation field of a solution  $u(t, r, \omega)$  to the wave equation is a function  $v(t, \omega)$  on  $\mathbb{R} \times \mathcal{S}^2$  given by the limit

$$v(t, \omega) = \lim_{r \rightarrow +\infty} ru(t + r, r, \omega) .$$

Penrose in [122] explicitly states that scattering is a motivation for introducing the conformal compactification technique: “The technique affords a covariant approach to the definition of radiation fields in general relativity.” Meanwhile, P.D. Lax and S.R. Phillips developed their theory of scattering [98] in 1967. The Lax-Phillips scattering theory for the wave equation on flat spacetime is based on a translation representative of the solution which is reinterpreted as an asymptotic profile of the field along outgoing null geodesics, analogous to Friedlander’s radiation field. Fifteen years after Penrose discussed radiations fields in the conformal setting, Friedlander saw the connection between Lax-Phillips theory of scattering and his notion of radiation fields, and in 1980 the first actual conformal scattering theory appeared in his founding paper [68]. The paper treated the case of the conformal wave equation in a static asymptotically flat spacetime with a fast enough decay at infinity to ensure a smooth conformal compactification including at spacelike and timelike infinities. The principle of the construction was first to reinterpret the scattering theory as the well-posedness of the Goursat problem for the rescaled equation at null infinity, then to solve this Goursat problem. Friedlander as well as J.C. Baez, I.E. Segal and Zhou Z.F. who pushed his ideas further in 1989-1990 [14, 15, 16, 17, 18] worked exclusively on static backgrounds. Right after [17], L. Hörmander solved the Goursat problem for a wave equation on generic null hypersurfaces in a spatially compact spacetime [83]. With this, and knowing that constructing conformal scattering theories amounts to solving a Goursat problem on a compactified spacetime, the road to non-stationary spacetimes was clear, yet, no one pushed it in this direction until 2004 when L. Mason and J.P. Nicolas picked up Friedlander’s ideas and applied them to scalar waves<sup>1</sup>, Dirac, and Maxwell fields on generically non-stationary asymptotically simple spacetimes [101]. J. Joudioux in 2012 [87] constructed a conformal scattering theory for a non-linear wave equation on non-stationary backgrounds. And in 2013 J.P. Nicolas produced a paper [121] on a conformal scattering theory for the wave equation on Schwarzschild black holes. In these recent works, [87, 101, 121] and the current work, the resolution of the Goursat problem is based on methods following the work of Hörmander [83] which deal with the Goursat problem using energy estimates for the wave equation, weak convergence, and compactness. The data in [83] is given on a general weakly spacelike Lipschitz hypersurface (including null), then the problem is solved by changing the equation using a parameter in front of the Laplacian in the wave equation to slow down the propagation speed so that the given weakly spacelike Lipschitz hypersurface becomes spacelike for the modified equation<sup>2</sup>. While in [101] the energy estimates of [83] are used, the authors, instead of slowing down the propagation speed, approach null infinity by spacelike hypersurfaces without changing the equation. Here in our work we directly apply [83] to show that the Goursat problem for Maxwell fields on Reissner-Nordström-de Sitter black holes is well-posed.

<sup>1</sup>The result on waves was completed in another paper in 2009 [102] by the same authors

<sup>2</sup>The resolution of the Goursat problem on a Lipschitz spacelike hypersurface is done by approximation with smooth spacelike hypersurfaces then using the well-posedness of the Cauchy problem on them.

The ultimate purpose of conformal scattering is to use conformal methods to construct scattering theories, not to reinterpret existing scattering theories in conformal terms. The idea of replacing spectral analysis by conformal geometry is the door to the extension of scattering theories to general non-stationary situations, which may be inaccessible to spectral methods. Note that in [68, 101, 121], the reinterpretation is done in addition to the conformal construction, giving more insight on questions such as the required decay for a conformal scattering theory, or whether a conformal scattering theory and a scattering theory defined in terms of wave operators are equivalent or not: Some spectral scattering theories cannot be reinterpreted as conformal scattering, but when the spacetime has the right asymptotic structure and the equation considered is conformally invariant, the question is valid. For the time being, the methods used require these two conditions, however it is interesting to know whether and how they can be extended to more general situations of conformally non-invariant equations which include the massive cases.

### 4.0.3 Work Done

In this part of the manuscript we address the topic of conformal scattering on the exterior region of RNdS black holes, and construct a scattering operator establishing the correspondence between null data on past horizons and null data on future horizons of region III. It is divided into two sections, we present an overview of their contents.

**Section 4.1:** In this section we construct the trace operators and show that they are injective and norm preserving after establishing conservation laws. We start the section by expressing the Maxwell field in null tetrad formalisms adapted to the geometry of our spacetime. Namely, the outgoing and incoming tetrad, which are parallelly transported respectively along outgoing and incoming radial null geodesics (see section 4.1). Each tetrad is useful to access the relevant components of the Maxwell field on different parts of the future and past total horizons  $\mathcal{H}^\pm := \mathcal{H}_2^\pm \cup \mathcal{H}_3^\pm$ . However, we do not project the field on these tetrads in the conventional way: Similar to what we did in chapter 2, the real part of the middle component is rescaled using the horizon function  $f$  of the RNdS metric. This simplifies the expressions of the compacted equations and the calculations. We also briefly discuss the Newman-Penrose formalism in a general normalized null tetrad, which can be found in details in the literature [28]. This will be used in solving the Cauchy problem on the closure of the exterior region. We next define the energy spaces on the horizons associated to the smoothly extended vector field  $T$  given by  $\partial_t$  on  $\mathcal{N}$ . The expression of the energy norm on each horizon in terms of the spin components is the integral of the restriction of the first or last component to the horizon. Accordingly, the energy spaces on  $\mathcal{H}^\pm$  will be simple  $L^2$ -spaces<sup>1</sup>:  $L^2(\mathcal{H}_2^\pm) \times L^2(\mathcal{H}_3^\pm)$ . Because the vector field  $T$  is Killing and causal (but not everywhere timelike) on  $\bar{\mathcal{N}}$ , then for a Maxwell field that is smooth and compactly supported at each instant  $t$ , and for an achronal hypersurface  $S^+(s)$  that intersects both horizons  $\mathcal{H}_2^+$

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<sup>1</sup>This is known in the literature [101]: On each horizon we have two constraint equations, this means that we do not need the restriction of the full field on the horizon to retrieve the complete information it carries, and accordingly the function space is “smaller” than what one might expect at first.



and  $\mathcal{H}_3^+$ , the divergence theorem gives an energy identity between the energy on the initial Cauchy hypersurface  $\Sigma_0$  on one side and the sum of the energies on  $S^+(s)$ ,  $\mathcal{H}_2^+(s)$ , and  $\mathcal{H}_3^+(s)$  on the other side, where  $\mathcal{H}_i^+(s)$  is the part of the horizon  $\mathcal{H}_i$  in the past of  $S^+(s)$ . By the decay results of Theorem 34 on achronal hypersurfaces that we obtained in chapter 3, the energy on  $S^+(s)$  tends to zero when the hypersurface gets far away from  $\Sigma_0$  and closer to future timelike infinity  $i^+$ , while at the same time,  $\mathcal{H}_2^+(s)$  and  $\mathcal{H}_3^+(s)$  cover more and more of their horizons. The energy identity then reduces to a conservation law between  $\Sigma_0$  and the future horizons, thereby, the energy of the smooth compactly supported Cauchy data is equal to the energy of the Goursat data. Next, we solve the Cauchy problem for smooth compactly supported data on  $\Sigma_0$  by formulating Maxwell's equations as an evolution system of three equations on the spin components of the field, and one spatial constraint equation which is preserved by the evolution equations. This is done by extending the spacetime to a globally hyperbolic cylindrical spacetime foliated by Cauchy hypersurfaces that are the level sets of a time function on the cylinder, and then using Theorem 10 on topologically trivial open subsets that cover the cylinder. The well-posedness of the Cauchy problem allows us to define each trace operator as an isometry, due to energy conservation, from the space of smooth compactly supported and constrained data to its image in the finite energy space defined on the horizons, by taking the data at  $t = 0$  and assigning to it the trace, or restriction, of the first and third spin components involved in the energies of the corresponding Maxwell solution on each horizon. By density, the trace operator then extends to the whole space of finite energy constrained Cauchy data as a partial isometry or an isometry into its range. Of course, the same is true for the past trace operator.

**Section 4.2:** Showing that the trace operators are invertible, i.e. isometries between the full spaces of finite energy, requires solving the Goursat problem on the horizons which we do in this section. Here too, it suffices to solve the Goursat problem for the dense subset of smooth compactly supported data. The idea goes as follows. We first show that the triplet  $\hat{\Phi}$  of the spin components of the field in the outgoing and incoming tetrads satisfies a system of coupled wave equations,

$$\hat{W}\hat{\Phi} = 0 . \quad (*)$$

This allows us to transform the problem from Maxwell's equations to wave equations. The two constraint equations on a horizon allow us to determine the rest of the spin components of the field on the horizon from one of them, the one appearing in the expression of the energy on that horizon which is the Goursat data for Maxwell's equations. Thus, we can define the Goursat data for the wave equations from the Goursat data for the Maxwell's equations. Following [121], and thanks to the fact that the Goursat data is supported away from timelike infinity  $i^+$ , we can cut out a neighbourhood of  $i^+$  and extend the remainder of the future of  $\Sigma_0$  to a globally hyperbolic cylinder and extend the hypersurface  $\Sigma_0$  with its boundary bifurcation spheres to  $\mathcal{S}^3$ , and the future horizons as the graph of a Lipschitz continuous function on  $\mathcal{S}^3$ . We can with this simple construction adapt the setting to the framework of Hörmander's results in [83] that prove the well-posedness of the Goursat problem for a general wave equation. Although [83] does not deal with coupled systems, our wave equations (\*) are coupled in a special way: The middle component of the field satisfies a decoupled source-free

wave equation, while it is coupled with each of the other two components separately and in the source term only. Thus, we apply Hörmander results for the decoupled equation of the middle component and then for the other two equations with the terms depending on the middle component being now a fixed known source. This guarantees the existence of the solution to our system of wave equations. The difficulty now is to reinterpret the solution of the wave equations as a Maxwell field. The main idea of the proof is to use the fact that one can go back and forth from Maxwell's equations (with perturbations) to wave equations by successive applications of the Maxwell operator: If Maxwell's compacted equations, i.e. the equations satisfied by the spin components in the tetrad formalism, were expressed as  $E_i = 0$  for  $i = 1 \dots 4$ , then the  $E_i$  themselves satisfy a system of wave equations if the spin components satisfy (\*). Therefore we are able to apply Hörmander's results repeatedly to prove that  $E_i = 0$ . The way the equations are coupled in these systems when expressed using the outgoing (incoming) null tetrad provides the ground to get the desired results. The key point is to use finite propagation speed and to note that solutions to the wave equations (\*) vanish along with all their derivatives in a neighbourhood of  $i^+$ . This along with the constraint equations on the horizons can be used to prove the initial conditions  $E_i|_{\mathcal{H}^+} = 0$ , and thus  $E_i = 0$  on the whole spacetime by the well-posedness of the Goursat problem for the wave equation [83].

It is worth mentioning that the conformal scattering we construct here is done without conformal compactification! This is because scattering data is taken on the horizons which are regular null hypersurfaces for the original metric on the maximal extension of RNdS black hole. Nevertheless, the results we obtain can be applied to any spherically symmetric spacetime satisfying the conditions stated in section 3.2.3 of chapter 3 with a conformal compactification when needed, the rest goes through essentially without modifications since Maxwell's equations are conformally invariant and in fact the rescaled Maxwell field tensor is equal to the unrescaled one, and the stress-energy tensor is also conformally invariant.

## 4.1 Trace Operators

The first step in defining the scattering operator is to define the trace operators. In this section we express Maxwell's equations in different null tetrads adapted to the geometry of the spacetime. We define the energy function spaces on the horizons, and we obtain an energy identity up to  $i^+$ . We then solve the Cauchy problem on the closure and thereby define the trace operators.

### 4.1.1 Maxwell Field on the Closure of the Static Exterior Region

We need to study Maxwell fields up to the horizons, i.e. on

$$\bar{\mathcal{N}} \tag{265}$$

the closure of  $\mathcal{N}$  in  $\mathcal{M}^*$ . The boundary of  $\mathcal{N}$  consists of the future and past outer horizons  $\mathcal{H}_2^\pm$ , the future and past cosmological horizons  $\mathcal{H}_3^\pm$ , and the two bifurcation spheres  $\mathcal{S}_2$  and

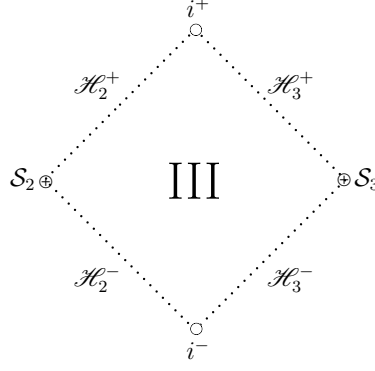


Figure 4.5:  $\bar{\mathcal{N}}$ : the closure of  $\mathcal{N}$  in  $\mathcal{M}^*$ .

$\mathcal{S}_3$ , in addition we have the two singular timelike infinities  $i^\pm$  which are not part of  $\bar{\mathcal{N}}$  (figure 4.5).

Let  $F$  be a Maxwell field on  $\bar{\mathcal{N}}$ . To write down its spin components we need to define a tetrad at each point of  $\bar{\mathcal{N}}$ . Simply extending the old tetrad  $\{L, N, M, \bar{M}\}$  to the boundary wont work, particularly because one of the null vectors  $L$  and  $N$ , which are given by  $\partial_t \pm f \partial_r$  in  $(t, r, \omega)$ -coordinates, will always vanish on two of the horizons. For example, in  $(u_-, r, \omega)$ -coordinates,  $L = f \partial_r$  and thus it vanishes on  $\mathcal{H}_3^+$  and  $\mathcal{H}_2^-$ , which means that the tetrad  $\{L, N, M, \bar{M}\}$  is singular there and does not form a basis of the tangent space. The same thing happens to  $N$  on  $\mathcal{H}_2^+$  and  $\mathcal{H}_3^-$  as can be seen in the  $(u_+, r, \omega)$ -coordinates. However, if we rescale  $L$  by the factor  $f^{-1}$ , the tetrad  $\{\hat{L} = f^{-1}L = \partial_r, N = 2\partial_{u_-} - f\partial_r, M, \bar{M}\}$  becomes a regular basis on  $\mathcal{M}_F^-$ , and in particular, on  $\mathcal{H}_3^+$  and  $\mathcal{H}_2^-$ . We define the spin components of  $F$  in this tetrad as:

$$\begin{aligned}\hat{\Phi}_1 &= F(\hat{L}, M) \\ \Phi_0 &= \frac{1}{2} \left( \hat{V}^{-1} F(\hat{L}, N) + F(\bar{M}, M) \right) \\ \Phi_{-1} &= F(N, \bar{M})\end{aligned}\tag{266}$$

where  $\hat{V} = f^{-1}V = r^{-2}$ . Since the integral curves of  $\hat{L}$  are the outgoing radial null geodesics, we refer to this tetrad as the *outgoing tetrad*, and the spin components as the *outgoing components*. Compared to (49), the components in the stationary tetrad  $\{L, N, M, \bar{M}\}$ , we have  $\hat{\Phi}_1 = f^{-1}\Phi_1$  while the other two components stay the same, hence we denote them by the same letters. The *incoming tetrad*  $\{L, \hat{N} = f^{-1}N, M, \bar{M}\}$  and the associated components are defined similarly. Neither of these two new tetrads defines a frame on the entire spacetime  $\bar{\mathcal{N}}$ , however, we can use the two tetrads with a partition of unity subordinate to the open sets  $\mathcal{M}_F^-$  and  $\mathcal{M}_F^+$  to define a tetrad that extends to all horizons. Using the relation  $\hat{\Phi}_1 = f^{-1}\Phi_1$  with the other components being the same, it is readily found that Maxwell compacted

equations (4) in the outgoing tetrad take the following form:

$$N\hat{\Phi}_1 = \hat{V}M\Phi_0 + f'\hat{\Phi}_1, \quad (267)$$

$$\hat{L}\Phi_0 = \bar{M}_1\hat{\Phi}_1, \quad (268)$$

$$N\Phi_0 = -M_1\Phi_{-1}, \quad (269)$$

$$\hat{L}\Phi_{-1} = -\hat{V}\bar{M}\Phi_0. \quad (270)$$

More generally, by Stiefel [135, 143],  $\bar{\mathcal{N}}$  is *parallelizable*<sup>1</sup>, so one can always define a smooth normalized global tetrad on it [71]. Let  $\{l, n, m, \bar{m}\}$  be such a tetrad. The normalization condition is:  $l^a n_a = 1$  and  $m^a \bar{m}_a = -1$ , while all other products are zero, so here we are working in the *Newman-Penrose formalism*, and one can write Maxwell's equations in this formalism in a way similar to what we did for the above tetrads. To use the notations of Chandrasekhar [28], let us rename the tetrad as:

$$e_{(1)} = l ; e_{(2)} = n ; e_{(3)} = m ; e_{(4)} = \bar{m} ,$$

and let the components of  $F$  and  $g$  in this tetrad be

$$F_{(a)(b)} = F(e_{(a)}, e_{(b)}) = F_{ij}e_{(a)}^i e_{(b)}^j \quad ; \quad g_{(a)(b)} = g(e_{(a)}, e_{(b)}) = g_{ij}e_{(a)}^i e_{(b)}^j ,$$

where  $i, j, k \dots$  indicate tensor coordinate components. and let  $g^{(a)(b)}$  be the inverse of  $g_{(a)(b)}$ . From the normalization condition we have

$$g_{(a)(b)} = g^{(a)(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \quad (271)$$

and we define,

$$e_{(a)i} = g_{ij}e_{(a)}^j \quad ; \quad e_i^{(a)} = g^{(a)(b)}e_{(b)i}.$$

Clearly, we have,

$$g^{(a)(b)} = g^{ij}e_i^{(a)}e_j^{(b)} ; e_i^{(a)}e_{(b)}^i = \delta_{(b)}^{(a)} ; e_i^{(a)}e_{(a)}^j = \delta_i^j ; g_{ij} = e_i^{(a)}e_{(a)j} .$$

The equations we shall present are found in [28]. The *intrinsic derivative* of  $F$  is defined to be

$$F_{(a)(b)|(c)} := e_{(a)}^i e_{(b)}^j F_{ij;k} e_{(c)}^k ,$$

with “;” denoting covariant differentiation:  $F_{ij;k} = (\nabla_{\partial_k} F)(\partial_i, \partial_j)$ . A direct calculation shows that Maxwell's equations (42) and (43) can be written respectively as:

$$g^{(a)(c)}F_{(a)(b)|(c)} = 0 \quad ; \quad F_{[(a)(b)|(c)]} = 0 . \quad (272)$$

---

<sup>1</sup>A manifold is said to be parallelizable if it admit a continuous global frame, i.e. a set of continuous vector fields that is a basis for the tangent space at each point of the manifold.

If we define the spin components of  $F$  in the tetrad  $(e_{(a)})$  as:

$$\begin{aligned}\phi_1 &= F(l, m) = F_{(1)(3)} \\ \phi_0 &= \frac{1}{2} (F(l, n) + F(\bar{m}, m)) = \frac{1}{2} (F_{(1)(2)} + F_{(4)(3)}) \\ \phi_{-1} &= F(n, \bar{m}) = F_{(2)(4)} ,\end{aligned}\tag{273}$$

then the compacted Maxwell's equations follow from the above equations:

$$\begin{aligned}\phi_{0|(1)} - \phi_{1|(4)} &= 0 ; & \phi_{0|(3)} - \phi_{1|(2)} &= 0 ; \\ \phi_{0|(2)} + \phi_{-1|(3)} &= 0 ; & \phi_{0|(4)} + \phi_{-1|(1)} &= 0 .\end{aligned}$$

To write these equations in terms of directional derivatives of the  $\phi_i$ 's with respect to the tetrad  $(e_{(a)})$ , we need the *Ricci rotation-coefficients* of the tetrad  $(e_{(a)})$  which are defined by

$$\gamma_{(c)(a)(b)} = e_{(c)}^k e_{(a)k;i} e_{(b)}^i ,$$

and denoted by special symbols:

$$\begin{aligned}\kappa &= \gamma_{311} ; & \rho &= \gamma_{314} ; & \varepsilon &= \frac{1}{2}(\gamma_{341} + \gamma_{211}) ; \\ \sigma &= \gamma_{313} ; & \mu &= \gamma_{243} ; & \gamma &= \frac{1}{2}(\gamma_{212} + \gamma_{342}) ; \\ \lambda &= \gamma_{244} ; & \tau &= \gamma_{312} ; & \alpha &= \frac{1}{2}(\gamma_{214} + \gamma_{344}) ; \\ \nu &= \gamma_{242} ; & \pi &= \gamma_{241} ; & \beta &= \frac{1}{2}(\gamma_{213} + \gamma_{343}) .\end{aligned}$$

Accordingly, the four compacted equations expand as:

$$\mathbf{l} := l\phi_0 - \bar{m}\phi_1 - (\pi - 2\alpha)\phi_1 - 2\rho\phi_0 - \kappa\phi_{-1} = 0 ,\tag{274}$$

$$\mathbf{n} := n\phi_0 + m\phi_{-1} - (\tau - 2\beta)\phi_{-1} + 2\mu\phi_0 - \nu\phi_1 = 0 ,\tag{275}$$

$$\mathbf{m} := m\phi_0 - n\phi_1 - (\mu - 2\gamma)\phi_1 - 2\tau\phi_0 + \sigma\phi_{-1} = 0 ,\tag{276}$$

$$\bar{\mathbf{m}} := \bar{m}\phi_0 + l\phi_{-1} - (\rho - 2\varepsilon)\phi_{-1} + 2\pi\phi_0 - \lambda\phi_1 = 0 .\tag{277}$$

Finally, the energy-momentum tensor of a Maxwell field given in (211) is replaced by

$$\mathbf{T}_{(a)(b)} = \frac{1}{4} g_{(a)(b)} F^{(c)(d)} F_{(c)(d)} - F_{(a)(c)} F_{(b)}^{(c)} .\tag{278}$$

If follows that,

$$\mathbf{T}_{(1)(1)} = -F_{(1)(c)} F_{(1)}^{(c)} = 2F_{(1)(3)} F_{(1)(4)} = 2F_{(1)(3)} \overline{F_{(1)(3)}} = 2|\phi_1|^2 .$$

Similarly, we get:

$$\begin{aligned}\mathbf{T}_{(1)(1)} &= 2|\phi_1|^2 ; & \mathbf{T}_{(1)(3)} &= 2\phi_1 \bar{\phi}_0 ; \\ \mathbf{T}_{(3)(4)} = \mathbf{T}_{(1)(2)} &= 2|\phi_0|^2 ; & \mathbf{T}_{(2)(3)} &= -2\phi_0 \bar{\phi}_{-1} ; \\ \mathbf{T}_{(2)(2)} &= 2|\phi_{-1}|^2 ; & \mathbf{T}_{(3)(3)} &= -2\phi_1 \bar{\phi}_{-1} .\end{aligned}\tag{279}$$

### 4.1.2 Function Space and Energy Identity

Assume that  $F$  is a smooth Maxwell field defined on  $\bar{\mathcal{N}}$ . The energy flux of the Maxwell field across an oriented hypersurface of  $\bar{\mathcal{N}}$  is defined to be the quantity (210) with respect to the smooth vector field  $T$  which is given by  $\partial_t$  in the RNdS coordinate, and by  $\partial_{u_{\pm}}$  on  $\mathcal{M}^{\pm}$ , and vanishes on the bifurcation spheres.

For any Cauchy hypersurface of constant  $t$  the expression of the energy flux across it is given by (228). Across the horizons  $\mathcal{H}_3^+$  and  $\mathcal{H}_2^-$  the energy flux can be expressed using the outgoing tetrad defined above and the outgoing spin components in (266). Precisely, in the retarded coordinates  $(u_-, r, \omega)$ ,  $N = 2\partial_{u_-} + f\partial_r$  is normal to these two horizons and is equal to  $2T = 2\partial_{u_-}$  on these null hypersurfaces, in addition,  $\hat{L} = \partial_r$  is transverse to them and  $g(\partial_{u_-}, \partial_r) = 1$ . So, we take  $\eta_{\mathcal{H}_3^+} = \frac{1}{2}N$  and  $\tau_{\mathcal{H}_3^+} = \hat{L}$ , and we have

$$E_T[F](\mathcal{H}_3^+) = \frac{1}{4} \int_{\mathcal{H}_3^+} \mathbf{T}_{ab} N^a N^b i_{\hat{L}} d^4x ,$$

and from (216),

$$E_T[F](\mathcal{H}_3^+) = -\frac{1}{4} \int_{\mathcal{H}_3^+} |\Phi_{-1}|^2 du_- \wedge d^2\omega ,$$

where we have chosen to orient  $\mathcal{H}_3^+$  by  $\partial_r$  so that  $i_{\hat{L}} d^4x$  is a positively oriented volume form on it, and the above quantity is thus positive. In other words,  $(\partial_{u_-}, \partial_\theta, \partial_\varphi)$  is a negatively oriented frame on the horizon and so is the chart  $(u_-, \omega)$ , hence,

$$E_T[F](\mathcal{H}_3^+) = \frac{1}{4} \int_{\mathbb{R}_{u_-} \times S^2} |\Phi_{-1}|^2 du_- d^2\omega . \quad (280)$$

The expression of  $E_T[F](\mathcal{H}_2^-)$  is exactly the same. As for the other two horizons  $\mathcal{H}_2^+$  and  $\mathcal{H}_3^-$  which are covered by the advanced coordinates  $(u_+, r, \omega)$ , we orient them by  $\hat{N} = -\partial_r$  and use the incoming tetrad and the spin components analogously to (266), to have,

$$E_T[F](\mathcal{H}_2^+) = \frac{1}{4} \int_{\mathcal{H}_2^+} \mathbf{T}_{ab} L^a L^b i_{\hat{N}} d^4x = \frac{1}{4} \int_{\mathbb{R}_{u_+} \times S^2} |\Phi_1|^2 du_+ d^2\omega , \quad (281)$$

and  $E_T[F](\mathcal{H}_3^-)$  has the same expression.

This gives us the definition of finite energy on the horizons  $\mathcal{H}_i^{\pm}$ . In addition, compared to the expression of the energy flux (228) on a spacelike slice of constant  $t$ , we can almost see the conservation law up to the horizons: If we take “limits” as  $t$  goes to  $\pm\infty$ , the surface  $\Sigma_t$  approaches  $\mathcal{H}_2^{\pm} \cup \mathcal{H}_3^{\pm}$  respectively, and since  $f = 0$  on the horizons, we formally have

$$\lim_{t \rightarrow \pm\infty} E_T[F](\Sigma_t) = E_T[F](\mathcal{H}_2^{\pm}) + E_T[F](\mathcal{H}_3^{\pm}),$$

but because of the energy conservation in (229), one expects the following conservation law

$$E_T[F](\Sigma_0) = E_T[F](\mathcal{H}_2^{\pm}) + E_T[F](\mathcal{H}_3^{\pm}). \quad (282)$$

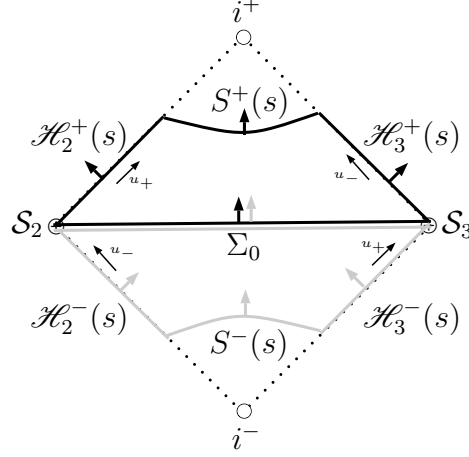


Figure 4.6: The hypersurfaces forming two closed hypersurfaces (black and gray). The thick arrows indicate the orientation of the surface, while the thin arrows indicate the direction of increasing coordinate  $u_\pm$  from  $-\infty$  to  $+\infty$ .

Thus, we define the energy spaces on the horizons  $\mathcal{H}_i^\pm$  to be the completions of  $\mathcal{C}_0^\infty(\mathcal{H}_i^\pm)$  with respect to the norms

$$\|\phi\|_{\mathcal{H}_2^\pm}^2 = \pm \frac{1}{2} \int_{\mathcal{H}_2^\pm} |\phi|^2 du_\pm \wedge d^2\omega, \quad ; \quad \|\phi\|_{\mathcal{H}_3^\pm}^2 = \mp \frac{1}{2} \int_{\mathcal{H}_3^\pm} |\phi|^2 du_\mp \wedge d^2\omega, \quad (283)$$

and on the future and past *total horizons*  $\mathcal{H}^\pm := \mathcal{H}_2^\pm \cup \mathcal{H}_3^\pm$  we define  $\mathcal{H}^\pm$  to be the completions of  $\mathcal{C}_0^\infty(\mathcal{H}_2^\pm) \times \mathcal{C}_0^\infty(\mathcal{H}_3^\pm)$  with respect to the addition norm

$$\|(\phi_\pm, \phi_\mp)\|_{\mathcal{H}^\pm}^2 = \frac{1}{2} \|\phi_\pm\|_{\mathcal{H}_2^\pm}^2 + \frac{1}{2} \|\phi_\mp\|_{\mathcal{H}_3^\pm}^2, \quad (284)$$

We now use the decay results Theorem 34 obtained in chapter 3 to show that these norms are conserved for smooth compactly supported data. Consider the hypersurfaces

$$S^\pm(s) = \{(t, r_*, \omega) \in \mathbb{R} \times \mathbb{R} \times \mathcal{S}^2 ; t = \pm \sqrt{1 + r_*^2} + s ; \pm s \geq 0\}. \quad (285)$$

$S^+(s)$  actually intersects  $\mathcal{H}^+$  in two spheres, one in each of the horizons  $\mathcal{H}_2^+$  and  $\mathcal{H}_3^+$ , namely at

$$\{s\}_{u_+} \times \{r_2\}_r \times \mathcal{S}^2 \quad \text{and} \quad \{s\}_{u_-} \times \{r_3\}_r \times \mathcal{S}^2$$

respectively (figure 4.6). Therefore if we set

$$\begin{aligned} \mathcal{H}_2^+(s) &= ]-\infty, s[_{u_+} \times \{r_2\}_r \times \mathcal{S}^2, \\ \mathcal{H}_3^+(s) &= ]-\infty, s[_{u_-} \times \{r_3\}_r \times \mathcal{S}^2, \end{aligned}$$

then these hypersurfaces along with  $\Sigma_0$  and  $S^+(s)$ , in addition to the bifurcation spheres  $\mathcal{S}_2$  and  $\mathcal{S}_3$ , form a closed hypersurface in  $\mathcal{N}$ . The same goes for  $S^-(s)$ , as shown in figure 4.6.

Thus, if  $F$  is a smooth solution of Maxwell's equations which is compactly supported for each  $t$ , then since  $T$  is Killing on  $\bar{\mathcal{N}}$ , we have by (208),

$$E_T[F](\Sigma_0) = E_T[F](\mathcal{H}_2^+(s)) + E_T[F](\mathcal{H}_3^+(s)) + E_T[F](S^+(s)).$$

$E_T[F](\mathcal{H}_2^+(s))$  and  $E_T[F](\mathcal{H}_3^+(s))$  are two positive increasing functions of  $s$ , and from the positiveness of  $E_T[F](S^+(s))$ , their sum is bounded from above by  $E_T[F](\Sigma_0)$ . Thus they have limits when  $s$  tends to  $+\infty$ , and these limits are  $E_T[F](\mathcal{H}_2^+)$  and  $E_T[F](\mathcal{H}_3^+)$ . Thanks to Theorem 34,

$$\lim_{s \rightarrow +\infty} E_T[F](S^+(s)) = 0,$$

and the conservation law (282) is proved. The same holds true with past horizons and  $S^-(s)$ .

### 4.1.3 Cauchy Problem up to the Horizons and Trace Operators

We show that the future (past) trace operator, which to data on  $\Sigma_0$  associates unique data on the future (past) total horizon, is well-defined by showing that the Maxwell Cauchy problem on  $\Sigma_0$  is well-posed up to the horizons.

Although the vector field  $\partial_t$  extends smoothly to  $\bar{\mathcal{N}}$ , it is null on the horizons to which it is tangent and normal, and since the horizons are null hypersurfaces, they are not adequate Cauchy hypersurfaces. We need to use an adapted 3+1 decomposition of the geometry by defining a new global time function  $\tau$  on  $\bar{\mathcal{N}}$  whose gradient is smooth, future-oriented, and timelike, and whose level surfaces are smooth Cauchy hypersurfaces that foliate  $\bar{\mathcal{N}}$ . We extend the spacetime to a globally hyperbolic spacetime cylinder ( $\mathcal{C} = \mathbb{R} \times \mathcal{S}^3, \tilde{g}$ ), and we extend the leaves of the new foliation to  $\mathcal{C}$  as copies of  $\mathcal{S}^3$ , thus our Cauchy hypersurfaces are compact.

**Proposition 38.** *Let  $F_0 : \Sigma_0 \rightarrow \Omega^2(\bar{\mathcal{N}})$  be a smooth 2-form on  $\Sigma_0$  with compact support satisfying the constraint (66). Then, there is a unique smooth 2-form  $F$  defined on  $\bar{\mathcal{N}}$  which is a solution to Maxwell's equations (38) and (39) such that  $F|_{\Sigma_0} = F_0$ .*

*Moreover, by density, the Cauchy problem on  $\bar{\mathcal{N}}$  is well-posed in  $\mathcal{U}$ , the constrained space of finite energy on  $\Sigma_0$  define in (90).*

*Proof.* To extend the metric  $g$  to  $\mathcal{C}$  in a smooth manner, we need first to cut out the singularities at  $i^\pm$ . Let  $P_\pm \in \bar{\mathcal{N}}$  be two points in the neighbourhoods of  $i^\pm$  respectively. We remove from  $\bar{\mathcal{N}}$  the two subsets  $I^\pm(P_\pm)$ , the future of  $P_+$  and the past of  $P_-$  respectively<sup>1</sup>, and we call the remaining space  $\bar{\mathcal{N}}_P = \bar{\mathcal{N}} \setminus \cup I^\pm(P_\pm)$ . Since the metric  $g$  is smooth up to the boundary of  $\bar{\mathcal{N}}_P$ ,  $(\bar{\mathcal{N}}_P, g)$  can be extended as a cylindrical globally hyperbolic spacetime  $(\mathcal{C}, \tilde{g})$  foliated by Cauchy hypersurface  $\tilde{\Sigma}_\tau$  which are all diffeomorphic to  $\mathcal{S}^3$ . In fact,  $\bar{\mathcal{N}}_P$  is compact, so, we can embed it in  $[-a, a]_\tau \times \mathcal{S}^3$  (figure 4.7).  $\Sigma_0$  is extended to  $\tilde{\Sigma}_0 \simeq \mathcal{S}^3$ , and the initial data  $F_0$  is extended by zero to the rest of  $\tilde{\Sigma}_0$ , and finally, we consider Maxwell's equations as defined on  $\mathcal{C}$ .

<sup>1</sup>The future of a point  $P$  of the spacetime is the set of all points in the spacetime that are distinct from  $P$  and can be reached from  $P$  by a future-directed timelike curve. The past of a point is defined analogously.



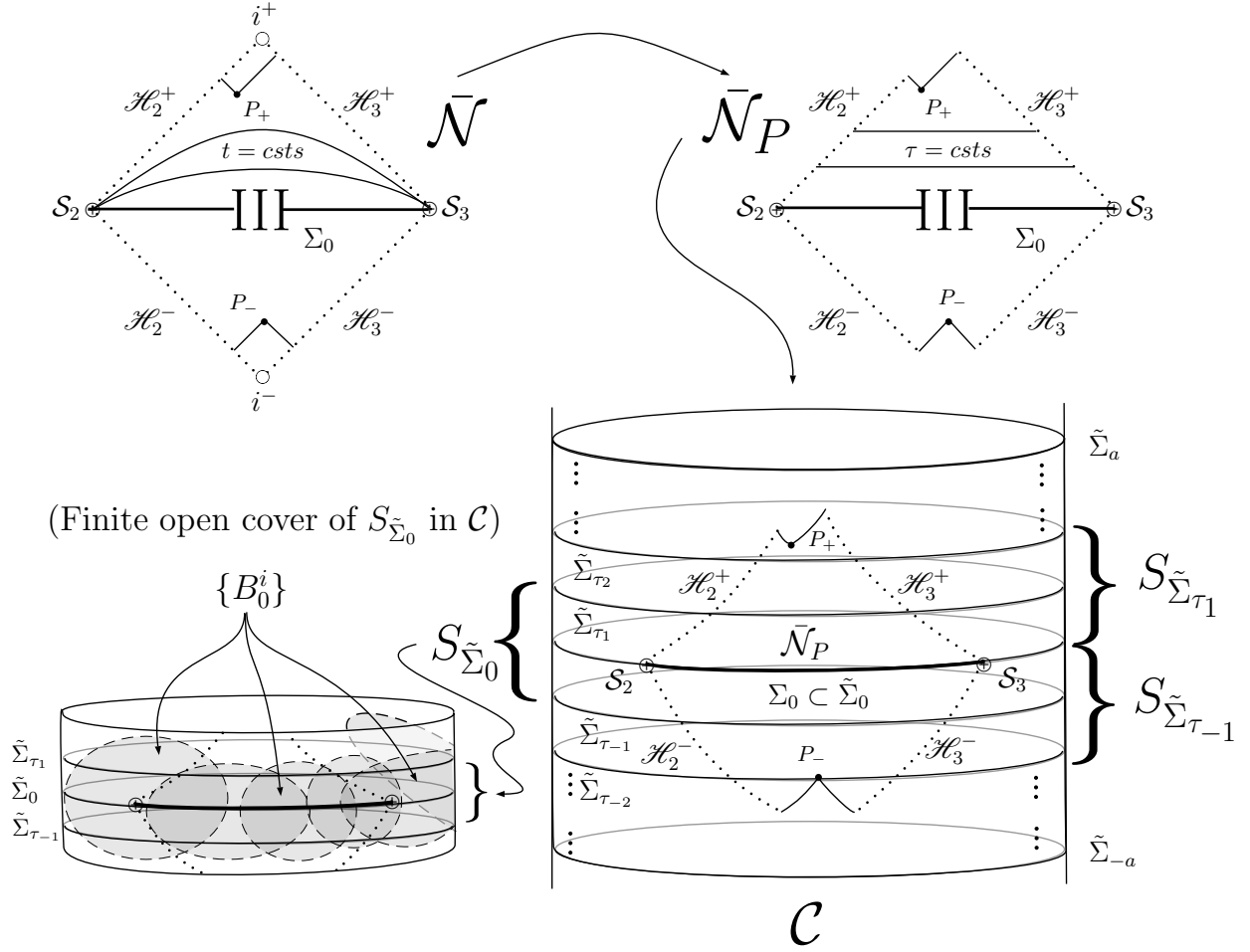


Figure 4.7: *The setting of the Cauchy problem on  $\bar{\mathcal{N}}$ .*

As in Proposition 8, we apply Theorem 10 (see [139]), however, since we are not working in  $\mathbb{R}^n$ , we need to restrict to topologically trivial open subsets of  $\mathcal{C}$ , and construct the solution step by step to cover  $\tilde{\mathcal{N}}_P$ . In effect, let  $\{B_0^i\}$  be a finite open covering of  $\tilde{\Sigma}_0$  consisting of relatively compact, geodesically complete, open subsets of  $\mathcal{C}$ . Since  $\{B_0^i\}$  is a finite collection of open sets, there exist  $\epsilon > 0$  such that  $S_{\tilde{\Sigma}_0} = [-\epsilon, \epsilon]_\tau \times \mathcal{S}^3 \subset \cup_i B_0^i$ . For every  $\tau \in [-a, a]$ , let  $\{B_\tau^i\}$  and  $S_{\tilde{\Sigma}_\tau}$  be similar to  $\{B_0^i\}$  and the stripe  $S_{\tilde{\Sigma}_0}$  but on  $\tilde{\Sigma}_\tau$ . By compactness, there is a natural number  $p$  such that the finite family  $\{S_{\tilde{\Sigma}_{\tau_j}} ; -p \leq j \leq p, j \in \mathbb{N}\}$ , shown in figure 4.7, covers  $[-a, a]_\tau \times \mathcal{S}^3$ , with  $\{B_j^i\}$  the corresponding cover of  $S_{\tilde{\Sigma}_{\tau_j}}$ . If we now show that, in each  $B_0^i$ , there is a unique solution of the Maxwell Cauchy problem with data  $F_0|_{\tilde{\Sigma}_0 \cup B_0^i}$ , then by uniqueness we have a solution  $\tilde{F}$  of the Cauchy problem on  $\tilde{\Sigma}_0$  defined on  $S_{\tilde{\Sigma}_0}$ . We now, similarly, solve the Cauchy problems on  $\tilde{\Sigma}_{\tau_{\pm 1}}$  with data given by  $\tilde{F}|_{\tilde{\Sigma}_{\tau_{\pm 1}}}$ . We then have a solution defined up to  $\tilde{\Sigma}_{\tau_{\pm 2}}$ . Continuing doing so, we find the solution  $\tilde{F}$  defined on  $[-a, a]_\tau \times \mathcal{S}^3$  to the Cauchy problem with the (extended) data  $F_0$  on  $\tilde{\Sigma}_0$ . We simply set  $F = \tilde{F}|_{\tilde{\mathcal{N}}_P}$ , but the points  $P_\pm$  can be chosen in arbitrarily small neighbourhoods of  $i^\pm$ , which entails by uniqueness that  $F$  is defined and smooth on  $\tilde{\mathcal{N}}$ .

It remains to show that the Cauchy problem with data  $F_0|_{K_i}$ , where  $K_i = \tilde{\Sigma}_0 \cup B_0^i$ , has a unique smooth solution in  $B_0^i$ . We perform a 3+1 decomposition of the metric which allows us to formulate Maxwell's system as three evolution equations with one spatial constraint equation. Let

$$\mathcal{T}^a = N \nabla^a \tau,$$

be the future timelike unit normal to  $K_i$ , where the lapse function  $N$  is given by

$$\frac{1}{N^2} = \|\nabla \tau\|^2 = \tilde{g}_{ab} \nabla^a \tau \nabla^b \tau.$$

The product structure on  $B_0^i$  is fixed by identifying points on the leaves of the new foliation along the integral lines of  $\mathcal{T}^a$ . If we subtract  $\tilde{g}$  from  $\mathcal{T}_a \mathcal{T}_b$  we get a Riemannian metric  $h_{ab}$  which is the restriction of  $-\tilde{g}$  on the tangent space of  $K_i$  since  $\mathcal{T}$  is normal to it. Thus the metric can be written as:

$$\tilde{g}_{ab} = \mathcal{T}_a \mathcal{T}_b - h_{ab}, \quad \text{or,} \quad \tilde{g} = N^2 d\tau^2 - h,$$

and since  $\mathcal{T}^a \mathcal{T}_a = 1$ ,

$$\mathcal{T}^a h_{ab} = g_{ab} \mathcal{T}^a - \mathcal{T}^a \mathcal{T}_a \mathcal{T}_b = 0.$$

Define on  $B_0^i$  a smooth normalized null tetrad  $\{l, n, m, \bar{m}\}$  such that  $l + n = \sqrt{2} \mathcal{T}$ . Maxwell's equations in the tetrad formalism are given by equations (274)-(277), which are represented as  $\mathbf{l} = 0$ ;  $\mathbf{n} = 0$ ;  $\mathbf{m} = 0$ ;  $\bar{\mathbf{m}} = 0$ . We also define the 1-form

$$\mathbf{I}_a = \mathbf{l} n_a - \mathbf{n} l_a - \mathbf{m} \bar{m}_a - \bar{\mathbf{m}} m_a,$$

so that  $\mathbf{I}_a = 0$  is nothing but Maxwell's equations in this tetrad since  $l, n, m, \bar{m}$  are linearly independent. The spacelike constraint is

$$\mathcal{T}^a \mathbf{I}_a = 0 \quad (C) \quad (286)$$

as  $\sqrt{2}\mathcal{T}^a\mathbf{I}_a = \mathbf{l} - \mathbf{n}$  involves derivatives tangent to  $\tilde{\Sigma}_0$  only, since, for  $\boldsymbol{\phi} = (\phi_1, \phi_0, \phi_{-1})$ ,  $\sqrt{2}\mathcal{T}^a\mathbf{I}_a$  is

$$D\boldsymbol{\phi} := (l - n)\phi_0 - \bar{m}\phi_1 - m\phi_{-1} - (\pi - 2\alpha)\phi_1 - 2\rho\phi_0 - \kappa\phi_{-1} + (\tau - 2\beta)\phi_{-1} - 2\mu\phi_0 + \nu\phi_1,$$

The rest is the evolution equations:

$$\mathbf{E}_a = \mathbf{I}_a - \mathcal{T}^b\mathbf{I}_b\mathcal{T}_a = 0 \quad (E). \quad (287)$$

To see that (E) is indeed an evolution system, we note that from the space/time decomposition and the product structure we chose, the vector field

$$\partial_\tau = N\mathcal{T},$$

is defined independently of the choice of coordinates on  $\tilde{\Sigma}_0$ . Now since  $\mathcal{T}^a\mathbf{E}_a = 0$ , (E) is equivalent to

$$\mathbf{l} + \mathbf{n} = 0 \quad \mathbf{m} = 0 \quad \bar{\mathbf{m}} = 0,$$

which in turn is

$$\begin{aligned} \partial_\tau\phi_1 &= \frac{2}{N\sqrt{2}}m\phi_0 + \frac{1}{N}\mathcal{X}\phi_1 + \mathbf{\Gamma}_1\boldsymbol{\phi}, \\ \partial_\tau\phi_0 &= \frac{1}{N\sqrt{2}}(\bar{m}\phi_1 - m\phi_{-1}) + \mathbf{\Gamma}_2\boldsymbol{\phi}, \\ \partial_\tau\phi_{-1} &= \frac{-2}{N\sqrt{2}}\bar{m}\phi_0 - \frac{1}{N}\mathcal{X}\phi_{-1} + \mathbf{\Gamma}_3\boldsymbol{\phi}, \end{aligned}$$

where  $\sqrt{2}\mathcal{X} = l - n$  and  $\mathbf{\Gamma}_i\boldsymbol{\phi}$  are the zero order terms. Therefore, the above system can be written as

$$\partial_\tau\boldsymbol{\phi} = \mathbf{H}\boldsymbol{\phi},$$

where  $\mathbf{H} = \mathbf{L} + \mathbf{\Gamma}$ ,  $\mathbf{L}$  being a matrix-valued first order operator and  $\mathbf{\Gamma}$  a matrix-valued function of zero order terms, given by

$$\mathbf{L} = \frac{1}{N\sqrt{2}} \begin{pmatrix} \sqrt{2}\mathcal{X} & 2m & 0 \\ \bar{m} & 0 & -m \\ 0 & -2\bar{m} & -\sqrt{2}\mathcal{X} \end{pmatrix} \quad \text{and} \quad \mathbf{\Gamma} = \begin{pmatrix} \mathbf{\Gamma}_1 \\ \mathbf{\Gamma}_2 \\ \mathbf{\Gamma}_3 \end{pmatrix}.$$

Because we are working on a relatively compact domain, and  $\mathbf{H}$  is a smooth first order operator, the above system is a hyperbolic system, more precisely,  $\mathbf{H}$  satisfies (98). Moreover, the Maxwell energy flux across  $K_i$  is

$$E_{\mathcal{T}}[F](K_i) = \int_{K_i} \mathbf{T}_{ab}\mathcal{T}^a\mathcal{T}^b i_{\mathcal{T}} d\text{Vol}_{\tilde{g}},$$

which by (279) is

$$E_{\mathcal{T}}[F](K_i) = \int_{K_i} |\phi_1|^2 + 2|\phi_0|^2 + |\phi_{-1}|^2 d\text{Vol}_{\tilde{\Sigma}_0},$$

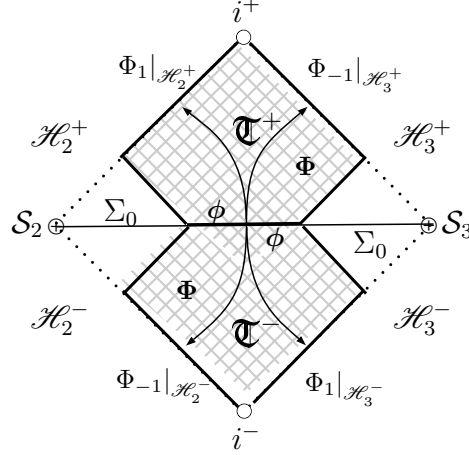


Figure 4.8: *The Trace operators defined for smooth compactly supported Maxwell Cauchy data.*

and  $d\text{Vol}_{\tilde{\Sigma}_0}$  is the volume form  $i_{\mathcal{T}}d\text{Vol}_{\tilde{g}}$  on  $\tilde{\Sigma}_0$ . This integral defines a norm induced by an inner product on data  $\phi \in (C^\infty(K_i))^3$ . We have  $\mathbf{L}(\tau, \tilde{x}, \xi)^* = -\mathbf{L}(\tau, \tilde{x}, \xi)$  for this inner product, from which follows that  $\mathbf{H}(\tau, \tilde{x}, \xi)^* + \mathbf{H}(\tau, \tilde{x}, \xi)$  is of zero order terms only, and hence satisfies the symmetric hyperbolicity condition (99). Thence by Theorem 10, (E) has a unique smooth solution  $\tilde{\phi}$  defined on  $B_0^i$  such that  $\tilde{\phi}|_{K_i} = \phi_0$  which corresponds to the initial data  $F_0|_{K_i}$ .

Finally, if we show that  $D\tilde{\phi} = 0$ , then  $\tilde{F}$  given by  $\tilde{\phi}$  will be a solution of Maxwell's equations on  $B_0^i$ .

Since the initial data  $\phi$  satisfies the constraint  $D\phi = 0$  then by the evolution system, this constraint is preserved. This in fact follows from the identity<sup>1</sup>  $\nabla^a I_a = 0$  which is easiest to see in spinor notations. This is proved in Appendix B (see (325)). Thus, with  $\nabla^a I_a = 0$  and  $\mathbf{E}_a = I_a - T^b I_b T_a = 0$ , we have,

$$0 = \nabla^a I_a = \nabla^a (T^b I_b T_a) = T_a \nabla^a (T^b I_b) + T^b I_b \nabla^a T_a ,$$

which is a simple first order linear differential equation on the function  $u = T^a I_a$ , of the form

$$\partial_\tau u + vu = 0 ,$$

with the initial condition  $u|_{\tau=0} = 0$  satisfied. And hence The proposition is proved.  $\square$

## Trace Operators

We now define the trace operators. Since we only showed that the energy is conserved for smooth fields with compact supports for each  $t$ , we first define the future and past trace operators by

$$\mathfrak{T}^\pm : (C^\infty(\Sigma))^3 \cap \mathcal{U} \longrightarrow \mathcal{H}^\pm ,$$

<sup>1</sup>This is related to the simple fact that for any anti-symmetric (0,2)-tensor  $F_{ab}$ , i.e. a 2-form,  $\nabla^a \nabla^b F_{ab} = 0$  due to the symmetries of the curvature tensor of  $\nabla$  whenever it is the Levi-Civita connection.

and as follows: Let  $F_0$  be the 2-form on  $\Sigma_0$  whose spin components in the stationary tetrad  $\{L, N, M, \bar{M}\}$  are the initial Cauchy data  $\phi = (\phi_1, \phi_0, \phi_{-1}) \in (\mathcal{C}_0^\infty(\Sigma))^3 \cap \mathcal{U}$ , and let  $\tilde{F}$  be the unique solution to the Cauchy problem defined in Proposition 38 on  $\mathcal{N}$  with  $\tilde{F}|_{\Sigma_0} = F_0$ , and whose spin components in the tetrad  $\{L, N, M, \bar{M}\}$  are  $\Phi = (\Phi_1, \Phi_0, \Phi_{-1})$ , then

$$\mathfrak{T}^\pm(\phi) = (\Phi_{\pm 1}|_{\mathcal{H}_2^\pm}, \Phi_{\mp 1}|_{\mathcal{H}_3^\pm}),$$

as figure 4.8 illustrates, and by (282)

$$\|\phi\|_{\mathcal{H}} = \|\mathfrak{T}^\pm(\phi)\|_{\mathcal{H}^\pm}. \quad (288)$$

By the density of  $(\mathcal{C}_0^\infty(\Sigma))^3 \cap \mathcal{U}$  in  $\mathcal{U}$ ,  $\mathfrak{T}^\pm$  extend to bounded operators that is,

$$\mathfrak{T}^\pm : \mathcal{U} \longrightarrow \mathcal{H}^\pm,$$

with closed range, and still satisfy (288).

## 4.2 Scattering operator and Goursat Problem

The main result of this section is that the trace operators defined above are invertible and hence isometries, allowing us to introduce the scattering operator. Since an isometry is a surjective norm preserving linear map between Hilbert spaces, all that is left is to show that the trace operators are surjective.

More precisely, let  $(\phi_\pm, \phi_\mp) \in \mathcal{H}^\pm$ , we wish to show that there exist some Cauchy data  $\phi \in \mathcal{U}$  such that  $\mathfrak{T}^\pm(\phi) = (\phi_\pm, \phi_\mp)$ , and since the trace operators are injective, the Cauchy data  $\phi$  is unique if it exists. By the well-posedness of the Cauchy problem, this will mean that there is a unique finite energy solution  $\Phi$  such that  $\Phi|_{\Sigma_0} = \phi$ , and by the definition of  $\mathfrak{T}^\pm$  we have

$$\mathfrak{T}^\pm(\Phi|_{\Sigma_0}) = (\Phi_{\pm 1}|_{\mathcal{H}_2^\pm}, \Phi_{\mp 1}|_{\mathcal{H}_3^\pm}),$$

and hence

$$(\Phi_{\pm 1}|_{\mathcal{H}_2^\pm}, \Phi_{\mp 1}|_{\mathcal{H}_3^\pm}) = (\phi_\pm, \phi_\mp). \quad (289)$$

Therefore, what we want to do is to solve the characteristic Cauchy problem, also known as the Goursat problem, on the total horizons  $\mathcal{H}^\pm$ . We do so by showing that the ranges of the trace operators contain dense subsets of the Hilbert spaces  $\mathcal{H}^\pm$ , and since norm preserving linear maps take complete normed spaces to complete ones, this means that the ranges are equal to  $\mathcal{H}^\pm$ . Thus, by density, it is enough to consider *Goursat data* in  $\mathcal{C}_0^\infty(\mathcal{H}_2^\pm) \times \mathcal{C}_0^\infty(\mathcal{H}_3^\pm)$ . As the future and the past cases are analogous, we only work out the case of the future trace operator. To further simplify the problem, we take advantage of the linearity of Maxwell's equations and assume that the non-trivial part of the initial (Goursat) data is only on one horizon, i.e. we treat smooth compactly supported data of the form, say,  $(0, \phi_-) \in \mathcal{H}^+$  with  $\phi_- \in \mathcal{C}_0^\infty(\mathcal{H}_3^+)$ , which represent the trace of an outgoing Maxwell solution. The case of  $(\phi_+, 0) \in \mathcal{H}^+$  is completely analogous.

## Goursat Problem

To solve the Goursat problem we use the results of L. Hörmander [83] by first converting the initial-value problem from Maxwell's equations to wave equations, then following J.-P. Nicolas [121] in his approach of putting the problem in a framework for which Hörmander's results apply. The idea is then to reinterpret the solution of the wave equations obtained by using the results of [83] as a Maxwell field.

As we shall restrict our attention to the future cosmological horizon, let us consider the outgoing tetrad and the corresponding spin components of the Maxwell field in details. If  $F$  is a Maxwell field, then just as the spin components  $\Phi$  satisfy coupled wave equations, the outgoing components  $\hat{\Phi}$  do too.

**Lemma 39.** *Let  $\hat{\Phi} = (\hat{\Phi}_1, \Phi_0, \Phi_{-1})$  be the outgoing spin components of a smooth Maxwell field defined on  $\mathcal{M}_F^-$ , then  $\hat{\Phi}$  satisfies the wave equation*

$$\hat{W}\hat{\Phi} = \begin{pmatrix} \hat{W}_{11} & -\hat{V}'M & 0 \\ 0 & \hat{W}_{00} & 0 \\ 0 & -V'\bar{M} & \hat{W}_{0-1} \end{pmatrix} \begin{pmatrix} \hat{\Phi}_1 \\ \Phi_0 \\ \Phi_{-1} \end{pmatrix} = 0, \quad (290)$$

where differentiation with respect to  $r$  is indicated by a prime,  $N_1 = N - f'$ , and the diagonal entries are<sup>1</sup>

$$\hat{W}_{11} := \hat{L}N_1 - \hat{V}M\bar{M}_1, \quad \hat{W}_{00} := \hat{L}N - \hat{V}M_1\bar{M}, \quad \hat{W}_{0-1} := \hat{L}N - \hat{V}\bar{M}M_1. \quad (291)$$

*Proof.* We denote the left hand side of Maxwell's equations (267)-(270) as:

$$N_1\hat{\Phi}_1 - \hat{V}M\Phi_0 =: E_1; \quad (292)$$

$$\hat{L}\Phi_0 - \bar{M}_1\hat{\Phi}_1 =: E_2; \quad (293)$$

$$N\Phi_0 + M_1\Phi_{-1} =: E_3; \quad (294)$$

$$\hat{L}\Phi_{-1} + \hat{V}\bar{M}\Phi_0 =: E_4. \quad (295)$$

As  $[N, \hat{L}] = f'\hat{L}$ , i.e.  $N_1\hat{L} = \hat{L}N$ , and  $M_1\bar{M} = \bar{M}_1M$ , we have:

$$\hat{L}E_1 + \hat{V}ME_2 = \hat{W}_{11}\hat{\Phi}_1 - \hat{V}'M\Phi_0; \quad (296)$$

$$N_1E_2 + \bar{M}_1E_1 = \hat{W}_{00}\Phi_0; \quad (297)$$

$$\hat{L}E_3 - M_1E_4 = \hat{W}_{00}\Phi_0; \quad (298)$$

$$N_1E_4 - \hat{V}\bar{M}E_3 = \hat{W}_{0-1}\Phi_{-1} - V'\bar{M}\Phi_0. \quad (299)$$

---

<sup>1</sup>The indices of  $\hat{W}_{ij}$  indicate their expressions:  $\hat{W}_{ij} = \hat{L}I(N) - \hat{V}J(M, M_1)$  with

$$i = \begin{cases} 0 & \text{if } I(N) = N; \\ 1 & \text{if } I(N) = N_1, \end{cases} \quad j = \begin{cases} 1 & \text{if } J(M, M_1) = M\bar{M}_1; \\ 0 & \text{if } J(M, M_1) = M_1\bar{M} = \bar{M}_1M; \\ -1 & \text{if } J(M, M_1) = \bar{M}M_1. \end{cases}$$

Finally, to see that  $\hat{W}$  is indeed a modified d'Alembertian just note that

$$\hat{L}N - \hat{V}M_1\bar{M} = \square + r\hat{V}(f\hat{L} - N).$$

□

We now look at Maxwell's equations on  $\mathcal{H}_3^+$ , in particular the first and the third,  $E_1 = 0$  and  $E_2 = 0$ . Since  $N$  is tangent to the horizon, equations (267) and (269) are tangent to it, i.e. contain only tangential derivatives:

$$N_1|_{\mathcal{H}_3^+}\hat{\Phi}_1|_{\mathcal{H}_3^+} - \hat{V}(r_3)M\Phi_0|_{\mathcal{H}_3^+} = 0, \quad (300)$$

$$N|_{\mathcal{H}_3^+}\Phi_0|_{\mathcal{H}_3^+} + M_1\Phi_{-1}|_{\mathcal{H}_3^+} = 0. \quad (301)$$

These are the constraints on the horizon. Thus, they must be satisfied by the restriction of the field's spin components. It follows that if the Goursat data  $\phi_- \in \mathcal{C}_0^\infty(\mathcal{H}_3^+)$  is to be viewed as part of a Maxwell field, namely  $\Phi_{-1}|_{\mathcal{H}_3^+}$ , then the other two components of the field are determined uniquely on the horizon by  $\phi_-$  through the above constraints and the requirement that they vanish in a neighbourhood of  $i^+$ , this is because (300) and (301) force them to vanish identically from  $i^+$  to the support of the Goursat data  $\phi_-$ . We choose them to be zero near  $i^+$  since as we shall presently see, this allows us to apply Hörmander's result. Therefore, for  $\phi_- \in \mathcal{C}_0^\infty(\mathcal{H}_3^+)$  we define  $\phi_0, \hat{\phi}_+ \in \mathcal{C}^\infty(\mathcal{H}_3^+)$  consecutively by the constraints initial-value problems in  $\mathcal{H}_3^+$ :

$$(C_1) \begin{cases} 2\partial_{u_-}\phi_0 = M_1\phi_- \\ \phi_0|_{\mathcal{S}_p} = 0 \end{cases} ; \quad (C_2) \begin{cases} (2\partial_{u_-} - f'(r_3))\hat{\phi}_+ = \hat{V}(r_3)M\phi_0 \\ \hat{\phi}_+|_{\mathcal{S}_p} = 0 \end{cases} \quad (302)$$

where  $\mathcal{S}_p$  is any sphere of  $\mathcal{H}_3^+$  in the future of the support of  $\phi_-$ . The supports of  $\phi_0$  and  $\hat{\phi}_+$  may touch the bifurcation sphere  $\mathcal{S}_3$ , but this is no problem since  $\mathcal{S}_3$  is a finite smooth sphere in  $\bar{\mathcal{N}}$  where the Cauchy hypersurface meets the future cosmological horizon, and no real scattering happens at  $\mathcal{S}_3$ . We refer to the triplet

$$\hat{\phi} = (\hat{\phi}_+, \phi_0, \phi_-)$$

also as the Goursat data, since it will be the *Goursat data* for the wave equations (290).

In [83], the author consider Lorentzian manifolds with a time function whose level hypersurfaces are compact and spacelike. The work is actually done for product manifolds of the form  $\mathbb{R} \times X$  where  $X$  is smooth compact manifold without boundary on which a time dependent Riemannian metric is defined, and the Laplace-Beltrami operator is defined with respect to a fixed Riemannian density. The paper studies the well-posedness of the Cauchy problem set on weakly spacelike hypersurfaces that are the graphs of Lipschitz functions over  $X$ , for wave equations of the form:

$$\square u + Qu = h, \quad (303)$$

where  $\square$  is a the modified d'Alembertian while  $Q$  is a first order operator of essentially bounded measurable coefficients, and  $h$  a source. In fact, Hörmander's results are valid

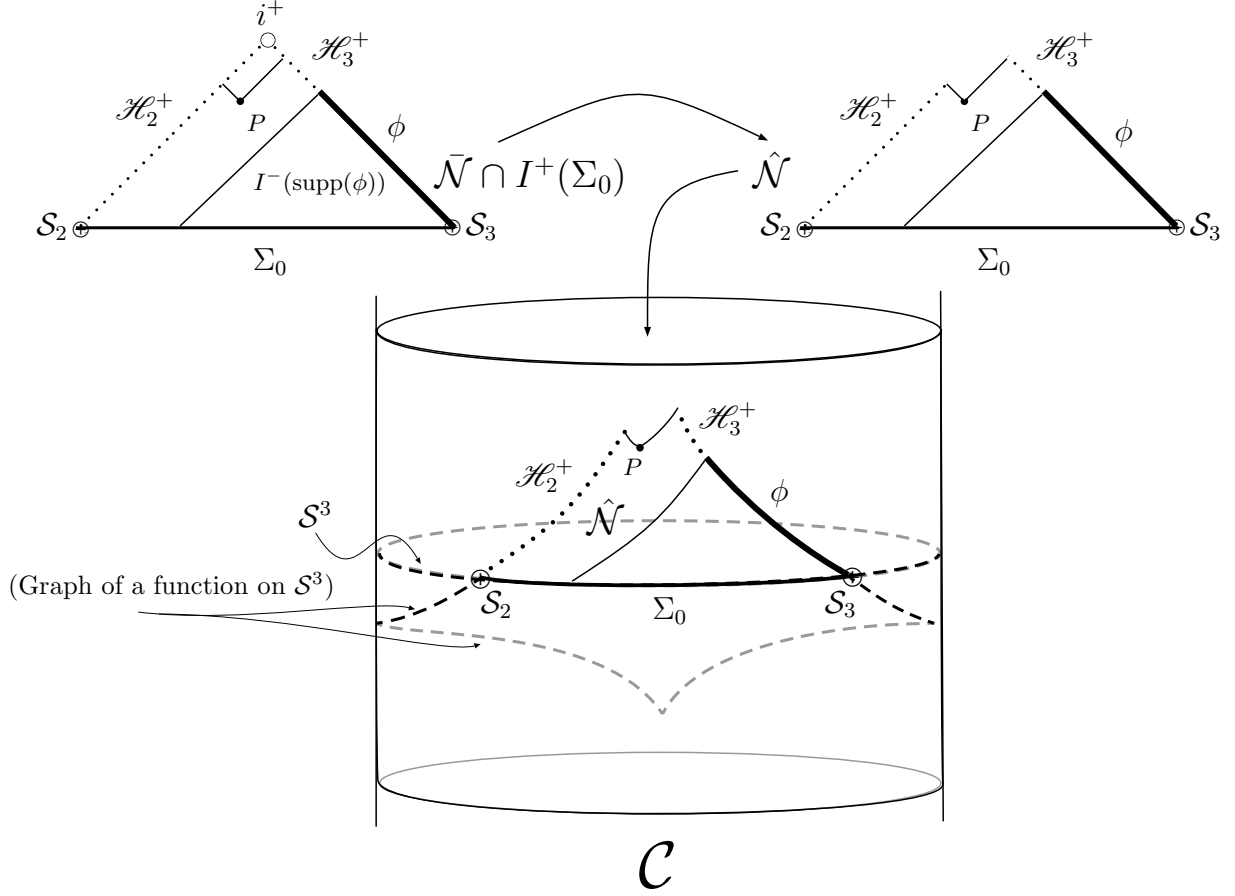


Figure 4.9: *The construction done to understand the Goursat problem for data  $\phi$  supported away from  $i^+$  in a framework suited to Hörmander's result.*

for globally hyperbolic and spatially compact spacetimes, since the product structure can be recovered by global hyperbolicity, while any non-degenerate change in the metric or the volume density, entails in the d'Alembertian a change that can be absorbed into the first order operator  $Q$ .

In what comes next, we need the well-posedness of the Goursat problem on the future total horizon for different wave equations (see Lemma 39) that are of the form (303). And although our spacetime is not spatially compact (without boundary), as long as the Goursat data is smooth and supported away from  $i^+$ , then its compact support in  $\mathcal{H}_3^+ \cup \mathcal{S}_3$  enables us to transform the problem into a framework suitable for Hörmander's results. Following the work of J.-P. Nicolas [121], this is done through a construction similar to the one we did for the Cauchy problem in the previous section.

We pick any point  $P$  whose future does not intersect the support of the Goursat data on the horizon, that is, a point in the future of the past of the data. We then remove the future of this point, and set

$$\hat{\mathcal{N}} = (\bar{\mathcal{N}} \setminus I^+(p)) \cap I^+(\Sigma_0),$$



where  $I^+(\Sigma_0)$  is the future of the Cauchy hypersurface  $\Sigma_0$  in  $\bar{\mathcal{N}}$ . We now extend  $\hat{\mathcal{N}}$  as a globally hyperbolic cylindrical spacetime ( $\mathcal{C} = \mathbb{R} \times \mathcal{S}^3, \tilde{g}$ ). We extend  $\Sigma_0$  as  $\mathcal{S}^3$  and the remaining part of the future total horizon as the graph of a Lipschitz function over  $\mathcal{S}^3$ , and the data by zero on the rest of the extended hypersurface. Then [83] guarantees the existence of a unique smooth solution on  $\mathcal{C}$  to the wave equation we consider. We take the restriction of the solution to  $\hat{\mathcal{N}}$ . Finite propagation speed then ensures that the solution is zero in the future of the past of the Goursat data (figure 4.9).

Moreover, despite the fact that (290) is a coupled system of three scalar wave equations, the coupling happens only on lower order terms, meaning that  $\square$  is in the diagonal only. Thus, the work in [83], where a single scalar wave equation (not a system) with scalar source is treated, can be applied to our case, when put in the above framework, with only a slight modification<sup>1</sup>. However in truth, the results in [83] can be applied to (290) directly and without any modification at all since (290) can be considered as three separate single scalar wave equations, two of which have a source, and one is source-free. This is because the middle component,  $\Phi_0$ , satisfies the decoupled source-free wave equation  $\hat{W}_{00}\Phi_0 = 0$ , and the coupling is only between the middle component and each of the other components separately, and hence the terms depending on  $\Phi_0$  in the other two equations can simply be viewed as source terms after solving  $\hat{W}_{00}\Phi_0 = 0$ .

**Theorem 40** (Goursat Problem). *For  $\phi_- \in \mathcal{C}_0^\infty(\mathcal{H}_3^+)$  there is a unique smooth, finite energy, Maxwell field  $F$  defined on  $\bar{\mathcal{N}}$ , with  $\hat{\Phi} = (\Phi_1, \Phi_0, \Phi_{-1})$  its spin components in the stationary tetrad, such that*

$$(\Phi_1|_{\mathcal{H}_2^+}, \Phi_{-1}|_{\mathcal{H}_3^+}) = (0, \phi_-).$$

*Proof.* Finite energy is immediate from the law of conservation of energy (282), and thus uniqueness follows directly from the injectivity of the future trace operator and the well-posedness of the Cauchy problem on  $\bar{\mathcal{N}}$ .

Let  $\phi_0$  and  $\hat{\phi}_+$  be given by  $\phi_-$  and (302), so that

$$N_1|_{\mathcal{H}_3^+}\hat{\phi}_+ - \hat{V}(r_3)M\phi_0 = 0, \quad (a)$$

$$N|_{\mathcal{H}_3^+}\phi_0 + M_1\phi_- = 0, \quad (b)$$

and set  $\hat{\phi} = (\hat{\phi}_+, \phi_0, \phi_-)$ . We now extend  $\hat{\phi}$  by zero to  $\mathcal{H}_2^+$ . The reason we do so, is because Hörmander's results apply to Goursat data defined on a generalized Cauchy hypersurface<sup>2</sup>, so we consider our data to be defined on the future total horizon  $\mathcal{H}^+$ . By [83] there is a unique smooth solution  $\hat{\Phi} = (\hat{\Phi}_1, \Phi_0, \Phi_{-1})$  to the Goursat problem

$$\begin{cases} \hat{W}\hat{\Phi} = 0 \\ \hat{\Phi}|_{\mathcal{H}^+} = \hat{\phi} \end{cases} \quad (304)$$

defined on  $\bar{\mathcal{N}} \cap (\Sigma_0 \cup I^+(\Sigma_0))$ , and by finite propagation speed and local uniqueness,  $\hat{\Phi}$  is

<sup>1</sup>The general operator  $Q$  of first and lower order terms in the equation considered in [83] is controlled by a priori estimates giving exponential bounds in [83]. If the lower order term is a matrix instead of a simple scalar potential, it can be controlled in the same manner, and the proof goes through unchanged.

<sup>2</sup>Weakly spacelike hypersurface such that every inextendible timelike curve intersect it only once.

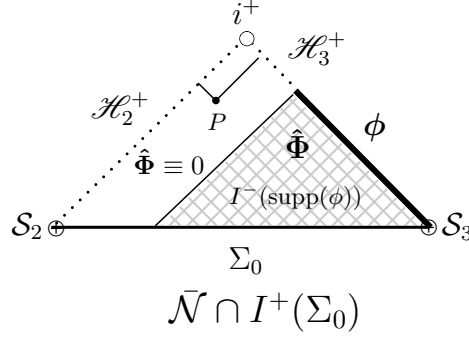


Figure 4.10: *The solution of the wave equations and its support.*

zero on  $I^+(I^-(\text{supp}\phi))$ , of course except for the part of the horizon where the support lies (figure 4.10).

We now reinterpret  $\hat{\Phi}$  as the spin components<sup>1</sup> of a solution to the Goursat problem on  $\mathcal{H}^+$  for Maxwell's equations with data  $(0, \phi_-)$ . Let  $\hat{W}\hat{\Phi} = (\hat{\Omega}_1, \Omega_0, \Omega_{-1})$ , then using (296)-(299) only, we have

$$N_1\hat{\Omega}_1 - \hat{V}M\Omega_0 = \hat{W}_{01}E_1 + f\hat{V}'ME_2 ; \quad (305)$$

$$\hat{L}\Omega_0 - \hat{V}\bar{M}_1\hat{\Omega}_1 = \hat{W}_{10}E_2 ; \quad (306)$$

$$N_1\Omega_0 + M_1\Omega_{-1} = \hat{W}_{00}E_3 ; \quad (307)$$

$$\hat{L}\Omega_{-1} + \hat{V}\bar{M}\Omega_0 = \hat{W}_{1-1}E_4 - \hat{V}'ME_3 . \quad (308)$$

where

$$\hat{W}_{01} = \hat{L}N - \hat{V}M\bar{M}_1 ; \quad (309)$$

$$\hat{W}_{10} = \hat{L}N_1 - \hat{V}M_1\bar{M} ; \quad (310)$$

$$\hat{W}_{1-1} = \hat{L}N_1 - \hat{V}\bar{M}_1M . \quad (311)$$

Since (304) holds, then on the one hand, we see that the  $E_i$ 's are solutions of coupled wave equations, and on the other hand, the constraints (a) and (b) implies that  $E_1|_{\mathcal{H}^+} = 0$  and  $E_3|_{\mathcal{H}^+} = 0$ . It follows that  $E_3$  is a solution of the Goursat problem

$$\begin{cases} \hat{W}_{00}E_3 = 0 \\ E_3|_{\mathcal{H}^+} = 0 \end{cases}$$

and hence  $E_3 = 0$ . This has an immediate effect on  $E_4$  by (299), i.e.  $N_1E_4 = 0$ , and in particular, we now have  $N_1|_{\mathcal{H}^+}E_4|_{\mathcal{H}^+} = 0$ . But since  $\hat{\Phi}$  is zero in a neighbourhood of  $i^+$  which intersects the horizon, all the derivatives of its components vanish as well, among

<sup>1</sup>Although the outgoing tetrad is singular on  $\mathcal{H}_2^+$ ,  $\hat{\Phi}$  vanishes on a neighbourhood of  $\mathcal{H}_2^+$ , so we can take our generalized Cauchy hypersurface to be  $\mathcal{H}_3^+ \cup S'$ , where  $S'$  is a null hypersurface in  $I^+(I^-(\text{supp}\phi))$  as in figure 4.11.

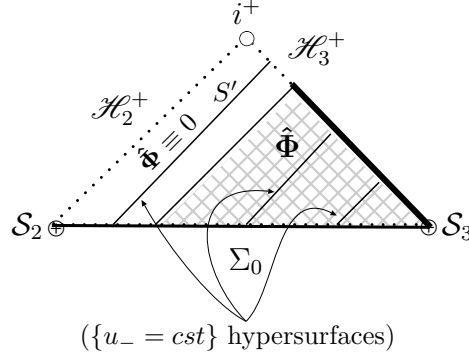


Figure 4.11: *The foliation by the hypersurfaces  $\{u_- = cst\}$  for the equation  $N_1 E_4 = 0$ .*

which are  $\hat{L}\Phi_{-1}$  and  $\bar{M}\Phi_0$ . (295) then means that  $E_4|_{\mathcal{H}^+} = 0$ , and therefore  $E_4$  solves the Goursat problem

$$\begin{cases} \hat{W}_{1-1} E_4 = 0 \\ E_4|_{\mathcal{H}^+} = 0 \end{cases}$$

and so  $E_4 = 0$ . Note that we could have alternatively used equation  $N_1 E_4 = 0$  directly to show that  $E_4 = 0$ : Because of the form of  $\text{supp}(\hat{\Phi})$ ,  $\hat{L}\Phi_{-1}$  and  $\bar{M}\Phi_0$  are zero on a hypersurface  $S'$  of constant  $u_-$  lying in the future of  $I^-(\text{supp}\phi)$ , and so  $E_4|_{S'} = 0$  (figure 4.11). Now for the simple transport equation  $N_1 E_4 = 0$ , the initial-value problem

$$\begin{cases} N_1 E_4 = 0 \\ E_4|_{S'} = 0 \end{cases}$$

is well-posed and has a unique solution, thus,  $E_4 = 0$ .

Because only  $E_1$  is tangential to the horizon while  $E_2$  is the one satisfying a source-free wave equation among the two, we need to use both at the same time. The fact that  $E_1|_{\mathcal{H}^+} = 0$  implies that  $N_1|_{\mathcal{H}^+} E_2|_{\mathcal{H}^+} = 0$  by (297) and the fact that  $N$  is tangent to the horizon. Now by the above argument of zero derivatives near  $i^+$ ,  $E_2|_{\mathcal{H}^+}$  itself is zero on some sphere at the horizon, say  $\mathcal{S}_p$ . Therefore  $E_2|_{\mathcal{H}^+}$  in turn solves

$$\begin{cases} N_1|_{\mathcal{H}^+} E_2|_{\mathcal{H}^+} = 0 \\ (E_2|_{\mathcal{H}^+})|_{\mathcal{S}_p} = 0 \end{cases}$$

which is a well-posed initial-value problem on the 2-surface  $\mathcal{S}_p$  in the horizon. Thus,  $E_2|_{\mathcal{H}^+} = 0$ , and so,

$$\begin{cases} \hat{W}_{10} E_2 = 0 \\ E_2|_{\mathcal{H}^+} = 0 \end{cases}$$

i.e.  $E_2 = 0$ . For  $E_1$ , we now have two options, both follow from what we have so far. Either we consider  $E_1$  as the solution of the Goursat problem

$$\begin{cases} \hat{W}_{01} E_1 = 0 \\ E_1|_{\mathcal{H}^+} = 0 \end{cases}$$

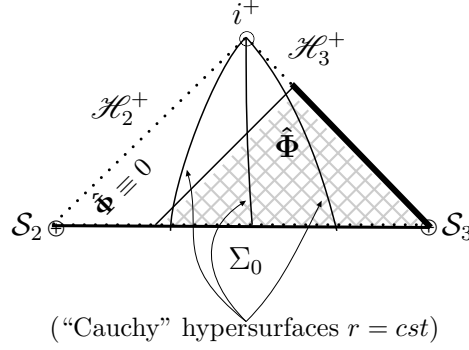


Figure 4.12: *The foliation by the hypersurfaces  $\{r = cst\}$  for the equation  $\hat{L}E_1 = 0$ .*

where the initial condition is given by the constraint (a), or, we use (296) as a simple initial-value problem

$$\begin{cases} \hat{L}E_1 = 0 \\ E_1|_{\mathcal{H}_3^+} = 0 \end{cases}$$

and since  $\mathcal{H}_3^+$  is a hypersurface of constant  $r$ , the problem is well-posed (figure 4.12). Both methods entails that  $E_1 = 0$ .

Therefore,  $\hat{\Phi}$  are the outgoing components of a Maxwell field  $F$ . The well-posedness of the Cauchy problem on  $\Sigma_0$  for Maxwell's equations ensures the global definition of  $F$  on  $\bar{\mathcal{N}}$  as a smooth solution. The only thing left to prove is that  $F$  has zero trace on the future outer horizon, which, since  $F$  is smooth up to the horizons, follows from the relation  $f\hat{\Phi}_1 = \Phi_1$  and the fact that  $\hat{\Phi}_1$  vanishes on a neighbourhood of the future outer horizon  $\mathcal{H}_2^+$ .  $\square$

## Scattering Operator

Theorem 40 shows that the trace operators  $\mathfrak{T}^\pm$  have inverses and are in effect isometries from  $\mathcal{U}$  to  $\mathcal{H}^\pm$ .

The scattering operator is the map  $\mathfrak{S} : \mathcal{H}^- \longrightarrow \mathcal{H}^+$  defined as follows:

$$\mathfrak{S} = \mathfrak{T}^+ \circ (\mathfrak{T}^-)^{-1}$$



# Appendix A

## Geometric Tools

One important tool that we use frequently is the divergence theorem. We present a version of this theorem which we think is better suited for Lorentzian manifolds than the usual one found in textbooks on Riemannian geometry. We start by defining the divergence of a smooth vector field. Let  $\mathcal{U}$  be an oriented smooth  $n$ -manifold, with  $\omega$  a positively oriented volume form, i.e. determining the orientation on  $\mathcal{U}$ , and let  $X$  be a smooth vector field on it. The divergence of  $X$  is defined to be the unique function  $\text{div}X$  such that,

$$\mathcal{L}_X\omega = (\text{div}X)\omega . \quad (312)$$

If the orientation on  $\mathcal{U}$  is given by a pseudo-Riemannian metric  $g$ , i.e.  $\omega = \text{d}V_g$ , then the above definition of  $\text{div}X$  coincides with the more familiar one, which is locally defined as:

$$\frac{1}{\sqrt{|g|}}\partial_i \left( \sqrt{|g|}X^i \right) , \quad (313)$$

where  $|g|$  is the absolute value of the determinant of the metric  $g$ . In fact, if  $\omega = \sqrt{|g|}\text{d}x^1 \wedge \cdots \wedge \text{d}x^n$  then, denoting  $\frac{\partial}{\partial x^i}$  by  $\partial_i$ , we have,

$$\begin{aligned} (\mathcal{L}_X\omega)(\partial_1, \dots, \partial_n) &= X \left( \sqrt{|g|} \cdot 1 \right) - \sqrt{|g|}\omega(\mathcal{L}_X\partial_1, \partial_2, \dots, \partial_n) - \cdots - \sqrt{|g|}\omega(\partial_1, \partial_2, \dots, \mathcal{L}_X\partial_n) \\ &= X \left( \sqrt{|g|} \right) - \sqrt{|g|}\omega(-\partial_1 X^1 \partial_1, \partial_2, \dots, \partial_n) - \\ &\quad \cdots - \sqrt{|g|}\omega(\partial_1, \partial_2, \dots, -\partial_n X^n \partial_n) \\ &= X^i \partial_i \left( \sqrt{|g|} \right) + \sqrt{|g|}\partial_i X^i \\ &= \frac{1}{\sqrt{|g|}}\partial_i \left( \sqrt{|g|}X^i \right) \omega(\partial_1, \dots, \partial_n) . \end{aligned}$$

Equivalently, one can define the divergence of a vector field by another local expression:  $X^i_{;i} := \partial_i X^i + X^k \Gamma^i_{ik}$ , which is of course equal to the former one. Indeed, since both local expressions are independent of the choice of coordinates (for the  $X^i_{;i}$  expression, this is a direct calculation using the transformation identities), it suffices to show that they are equal

in one coordinate system. In particular, we can choose a normal coordinate chart centred at a point, then on the one hand we have  $|g| = 1$  at this point and hence, from (313),  $\text{div} X = \partial_i X^i$ . On the other hand, in normal coordinates the Christoffel symbols  $\Gamma_{jk}^i$  are zero at the center of the coordinate chart, and so  $X^i_{;i} = \partial_i X^i$ . Therefore,  $\text{div} X = X^i_{;i}$  in any chart. Thus, in abstract index notation the divergence of a vector field is  $\nabla_a X^a$ .

If  $\mathcal{U}$  is a smooth oriented manifold with boundary, then its boundary always admits a boundary defining function, i.e. a smooth function  $f : \mathcal{U} \rightarrow [0, +\infty)$  with  $f^{-1}(0) = \partial\mathcal{U}$  and  $df_p \neq 0$  for all  $p \in \partial\mathcal{U}$ . This means that the boundary is always orientable: one can consider a Riemannian metric on  $\mathcal{U}$ , then the normalized gradient of  $f$  is a unit normal vector field to the boundary which is nowhere tangent to it, and hence defines an orientation. If  $\mathcal{U}$  is endowed with a pseudo-Riemannian metric  $g$ , then the gradient of  $f$  with respect to this metric is again a normal vector field along the boundary but this time it could be tangent to the boundary at some of its points where it is null, and hence unnormalizable. And since the metric is non-degenerate, the normal bundle is of rank one<sup>1</sup> and generated by the gradient of  $f$ , so there might not be a normal that can be used to define an orientation on the boundary nor can it be directly used in the divergence theorem since it may not be of unit length. One way around this is to use a vector field whose scalar product with a normal is 1, such vector field is automatically transverse (nowhere tangent) to the boundary. It is always possible to find such a transverse vector field. Indeed, if  $N$  is any smooth nowhere vanishing vector field along the boundary, or along any immersed hypersurface  $\mathcal{V}$  for this matter, then around each point  $p$  of the submanifold there is a local coordinate chart of  $\mathcal{U}$  such that for some  $i$ ,  $g_p(N_p, \partial_{i|p}) > 0$  since the metric is non-degenerate. By smoothness of  $g$ ,  $N$  (if extended locally to a smooth vector field on  $\mathcal{U}$ ), and  $L_p = \partial_i$  around  $p$ , there is some neighbourhood  $U_p$  of  $p$  in  $\mathcal{U}$  such that  $g(N, L_p) > 0$  on  $U_p \cap \mathcal{V}$ . Let  $\{\Psi_p\}$  be a partition of unity subordinate to the open collection<sup>2</sup>  $\{U_p\}$  and define

$$\tilde{L} = \sum_{p \in \mathcal{V}} \Psi_p L_p ,$$

then  $\tilde{L}$  is a smooth vector field on  $\mathcal{V}$  when restricted to it, and  $g(N, \tilde{L})_p = \alpha(p) > 0$  for all  $p \in \mathcal{V}$ . If we set

$$L = \frac{1}{\alpha} \tilde{L} \tag{314}$$

we get the desired vector field.

**Lemma 41** (Divergence Theorem). *Let  $\mathcal{U}$  be an oriented smooth  $n$ -manifold with boundary (possibly empty), with  $\omega$  a positively oriented volume form, i.e. determining the orientation on  $\mathcal{U}$ , and the boundary  $\partial\mathcal{U}$  is outward oriented (Stokes' orientation), and let  $X$  be a smooth vector field on  $\mathcal{U}$ . If  $\mathcal{U}$  is compact or  $X$  is compactly supported then,*

---

<sup>1</sup>Let  $N_1$  and  $N_2$  be two normals to a hypersurface  $S$  of a manifold  $M^n$  equipped with a non-degenerate metric  $g$ . If  $\{E_i\}$  is a basis of the tangent space of  $M$  with  $\{E_i; i = 2 \dots n\}$  a basis of that of  $S$ , then necessarily  $g(N_i, E_1) = \alpha_i \neq 0$  for  $i = 1, 2$ , and for any vector  $Y$  on  $M$ ,  $g(N_1 - \frac{\alpha_1}{\alpha_2} N_2, Y) = 0$ . Thus  $N_1 = a N_2$  with  $a \neq 0$ .

<sup>2</sup>We assume that the supports of partition of unity is a locally finite family, i.e. every point belongs to only finitely many supports of the family.

$$\int_{\partial\mathcal{U}} i_X \omega = \int_{\mathcal{U}} \mathcal{L}_X \omega = \int_{\mathcal{U}} \operatorname{div} X \omega , \quad (315)$$

Moreover, if the orientation on  $\mathcal{U}$  is given by a pseudo-Riemannian metric  $g$ , i.e.  $\omega = \operatorname{d}V_g$ , then (315) can be reformulated as:

$$\int_{\partial\mathcal{U}} N(X) i_L \operatorname{d}V_g = \int_{\mathcal{U}} \mathcal{L}_X \operatorname{d}V_g = \int_{\mathcal{U}} \operatorname{div} X \operatorname{d}V_g , \quad (316)$$

where  $N$  is a conormal field to  $\partial\mathcal{U}$ , i.e.  $N^\sharp$  is a normal vector field, and  $L$  is a vector field transverse to  $\partial\mathcal{U}$ , such that  $N(L) = 1$ .

*Proof.* (315) is a direct consequence of Stokes' theorem and Cartan's identity. In fact, since  $i_X \omega$  is a  $(n-1)$ -form on  $\mathcal{U}$ , then by Stokes' theorem,

$$\int_{\partial\mathcal{U}} i_X \omega = \int_{\mathcal{U}} \operatorname{d}i_X \omega ,$$

and by Cartan's identity,

$$\mathcal{L}_X \omega = \operatorname{d}i_X \omega + i_X \operatorname{d}\omega = \operatorname{d}i_X \omega .$$

To prove (316) we only need to show that

$$i_X \operatorname{d}V_g = N(X) i_L \operatorname{d}V_g . \quad (317)$$

Indeed, since  $i_L N = N(L) = 1$  and by the property of the interior product  $i$ , we have,

$$i_L(N \wedge i_X \operatorname{d}V_g) = i_L N \wedge i_X \operatorname{d}V_g - N \wedge i_L i_X \operatorname{d}V_g ,$$

so,

$$i_X \operatorname{d}V_g = i_L(N \wedge i_X \operatorname{d}V_g) + N \wedge i_L i_X \operatorname{d}V_g ,$$

again, and since  $N \wedge \operatorname{d}V_g = 0$ ,

$$0 = i_X(N \wedge \operatorname{d}V_g) = i_X N \wedge \operatorname{d}V_g - N \wedge i_X \operatorname{d}V_g ,$$

i.e.

$$N \wedge i_X \operatorname{d}V_g = i_X N \wedge \operatorname{d}V_g .$$

Thus,

$$i_X \operatorname{d}V_g = N(X) i_L \operatorname{d}V_g + N \wedge i_L i_X \operatorname{d}V_g ,$$

but,  $N \wedge i_L i_X \operatorname{d}V_g = 0$  on  $\partial\mathcal{U}$  because if  $X_1, \dots, X_{n-1}$  are vector fields tangent to  $\partial\mathcal{U}$  then,

$$(N \wedge i_L i_X \operatorname{d}V_g)(X_1, \dots, X_{n-1}) = C \sum_{\sigma \in \mathfrak{S}} \operatorname{sgn}(\sigma) N(X_{\sigma(1)}) (i_L i_X \operatorname{d}V_g)(X_{\sigma(2)}, \dots, X_{\sigma(n-1)}) ,$$

where  $\mathfrak{S}$  is some subset of the permutation group, and this wedge product is zero since  $N(X_i) = 0$  as  $N$  is conormal to  $\partial\mathcal{U}$ .  $\square$



Note that if the normal vector field can be normalized, which is always the case if the metric is Riemannian, and is only true if the hypersurface is timelike in the Lorentzian case, one can then choose the transverse vector to be the normal itself and thus recovering the well known form of this theorem:

$$\int_{\partial\mathcal{U}} N(X) dV_{\tilde{g}} = \int_{\mathcal{U}} \text{div} X dV_g, \quad \text{or,} \quad \int_{\partial\mathcal{U}} N_a X^a dV_{\tilde{g}} = \int_{\mathcal{U}} \nabla_a X^a dV_g \quad (318)$$

$\tilde{g}$  being the induced metric on  $\partial\mathcal{U}$ , and  $dV_{\tilde{g}} = i_{N^\sharp} dV_g$ .

Killing vector fields have the nice property of vanishing divergence. A vector field  $X$  is said to be Killing if the metric is conserved along the flow of  $X$ , i.e.  $\mathcal{L}_X g = 0$ . Since the Levi-Civita connection is torsion-free and compatible with the metric, we have, for any vectors  $Y$  and  $Z$ ,

$$\begin{aligned} \mathcal{L}_X g_{ab} Y^a Z^b &= X(g_{ab} Y^a Z^b) - g_{ab} (\mathcal{L}_X Y)^a Z^b - g_{ab} Y^a (\mathcal{L}_X Z)^b \\ &= g_{ab} X^c (\nabla_c Y^a) Z^b + g_{ab} Y^a X^c (\nabla_c Z^b) - g_{ab} (X^c (\nabla_c Y^a) - Y^c (\nabla_c X^a)) Z^b \\ &\quad - g_{ab} (X^c (\nabla_c Z^b) - Z^c (\nabla_c X^b)) Y^a \\ &= g_{ab} Y^c (\nabla_c X^a) Z^b + g_{ab} Y^a Z^c (\nabla_c X^b) \\ &= (\nabla_c X_b) Y^c Z^b + (\nabla_c X_a) Y^a Z^c = (\nabla_a X_b + \nabla_b X_a) Y^a Z^b. \end{aligned}$$

Thus,  $\mathcal{L}_X g_{ab} = \nabla_a X_b + \nabla_b X_a = 2\nabla_{(a} X_{b)}$ , and so for Killing fields,

$$\nabla_a X_b - \nabla_b X_a = 0, \quad (319)$$

consequently,

$$0 = g_{ab} (\nabla_a X_b - \nabla_b X_a) = 2\nabla_a X^a, \quad (320)$$

hence  $\text{div} X = 0$ . Equation (319) is called the Killing equation, and the  $(0, 2)$ -tensor involved is sometimes called the deformation tensor or Killing tensor, denoted

$${}^X \pi_{ab} = 2\nabla_{(a} X_{b)}. \quad (321)$$

Sometimes it is useful to see more directly the dependence of the integral on the hypersurface as we did in (316) for a boundary hypersurface using a normal and a transverse vector field. The existence of these vector fields is a consequence of the fact that  $S$  is an oriented hypersurface of a pseudo-Riemannian oriented Manifold as the following lemma guarantees.

**Lemma 42.** *If  $S$  is a smooth orientable hypersurface of a smooth orientable  $n$ -manifold  $M$ , then  $S$  admits a nowhere vanishing smooth 1-form  $\alpha$  defined on a neighbourhood of  $S$  in  $M$  with the property that  $\alpha_p \equiv 0$  on  $T_p S$ , for all  $p \in S$ . Such a 1-form is unique up to a multiplication by a smooth function that does not vanish.*

*Proof.* Let  $p \in S$ . We first show that if  $\beta$  and  $\beta'$  are any two non zero covectors of  $M$  at  $p$  that are identically zero on  $T_p S$  then  $\beta = a\beta'$  for  $a \neq 0$ . Consider a basis  $\{E_i; i = 2, \dots, n\}$  of  $T_p S$  and complete it to a basis  $\{E_i\}$  of  $T_p M$ , and denote by  $\{E_i^*\}$  the dual basis. Let  $\beta$

be non zero covector of  $M$  at  $p$  such that  $\beta = 0$  on  $T_p S$  ( $E_1^*$  is such a covector), then for  $Y \in T_p M$ ,

$$\beta(Y) = Y^i \beta(E_i) = Y^1 \beta(E_1) = \beta(E_1) E_1^*(Y),$$

so  $\beta = \beta(E_1) E_1^*$  with  $\beta(E_1) \neq 0$  since  $\beta$  is not the zero covector.

Next, fix some orientation on  $S$  and  $M$ . Since an immersed hypersurface is locally embedded, and to avoid unnecessary complication, we assume that  $S$  is an embedded connected hypersurface of  $M$  whom in turn we assume to be connected. So, we can find  $(U, x^1, \dots, x^n)$  an oriented slice chart for  $S$  in  $M$  around  $p$ , i.e. an oriented chart of  $M$  such that  $x^n = 0$  on  $U \cap S$  and  $p \in U$ . Then  $dx^n(X) = 0, \forall X \in T_p S$ , moreover,  $(U \cap S, x^1, \dots, x^{n-1})$  is a smooth chart of  $S$ . If  $(V, y^1, \dots, y^n)$  is another similar chart around  $p$ , then  $dy^n = a dx^n$  with  $a(q) \neq 0 \forall q \in V \cap U$ . Let  $\{(U_\gamma, x_\gamma)\}$  be a covering of  $S$  by such slice charts, and let  $\{\Psi_\gamma\}$  be a partition of unity subordinate to this cover. Set  $\alpha_\gamma = dx_\gamma^n$  if  $(\tilde{U}, \tilde{x}_\gamma) = (U_\gamma \cap S, x_\gamma^1, \dots, x_\gamma^{n-1})$  is a positively oriented chart of  $S$ , and  $\alpha_\gamma = -dx_\gamma^n$  if the chart is negatively oriented. Define

$$\alpha = \sum_{\gamma} \Psi_\gamma \alpha_\gamma,$$

then  $\alpha$  is a smooth 1-form defined on a neighbourhood of  $S$  in  $M$  and  $\alpha_p$  is identically zero on  $T_p S$  for all  $p \in S$ . To see that  $\alpha$  is nowhere vanishing, take  $p \in S$ , then  $p \in \text{supp} \Psi_\gamma$  for finitely many  $\gamma$ 's only. Label the finite collection by  $i = 0, \dots, m$ , so that

$$\alpha_p = \sum_{i=0}^m \Psi_i(p) \alpha_{ip}.$$

For every  $i$ , we have  $dx_i^n = a_i dx_0^n$  with  $a_i \neq 0$  (non vanishing smooth function), and hence the Jacobian determinant  $\mathbf{J}_{0i}$  of the transition map from  $(U_i, x_i)$  to  $(U_0, x_0)$  is:

$$\mathbf{J}_{0i} = \begin{vmatrix} \frac{\partial x_i^1}{\partial x_0^1} & \cdots & \frac{\partial x_i^1}{\partial x_0^{n-1}} & \frac{\partial x_i^1}{\partial x_0^n} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial x_i^{n-1}}{\partial x_0^1} & \cdots & \frac{\partial x_i^{n-1}}{\partial x_0^{n-1}} & \frac{\partial x_i^{n-1}}{\partial x_0^n} \\ 0 & \cdots & 0 & a_i \end{vmatrix} = a_i \begin{vmatrix} \frac{\partial x_i^1}{\partial x_0^1} & \cdots & \frac{\partial x_i^1}{\partial x_0^{n-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_i^{n-1}}{\partial x_0^1} & \cdots & \frac{\partial x_i^{n-1}}{\partial x_0^{n-1}} \end{vmatrix} = a_i \mathbf{J}_{0i}^S,$$

where  $\mathbf{J}_{0i}^S$  is the Jacobian determinant of the transition map from  $(\tilde{U}_i, \tilde{x}_i)$  to  $(\tilde{U}_0, \tilde{x}_0)$ . We chose the collection  $\{(U_\gamma, x_\gamma)\}$  to be oriented charts of  $M$ , thus  $\mathbf{J}_{0i} > 0$  for all  $i$ . Therefore, if the charts  $(\tilde{U}_i, \tilde{x}_i)$  and  $(\tilde{U}_0, \tilde{x}_0)$  determine the same orientation on  $S$ , then on the one hand we have  $\alpha_i = a_i \alpha_0$ , and on the other hand  $\mathbf{J}_{0i}^S > 0$  and hence  $a_i > 0$ . Similarly, if they give opposite orientations on  $S$ , then  $\alpha_i = -a_i \alpha_0$ , and  $\mathbf{J}_{0i}^S < 0$  so  $a_i < 0$ . Whence, it is always the case that  $\alpha_i = b_i \alpha_0$  for  $b_i > 0$ . Thus,

$$\alpha_p = \sum_{i=0}^m \Psi_i(p) \alpha_{ip} = \left( \sum_{i=0}^m \Psi_i(p) b_i(p) \right) \alpha_{0p}.$$

which is clearly non zero from the definition of  $\alpha_\gamma$ .  $\square$

If  $M$  is equipped with a non-degenerate metric  $g$ , then  $N^a = g^{ab} \alpha_b$  is a smooth normal to  $S$  and any two normals are collinear as shown in footnote 1.



# Appendix B

## Spinors

In this appendix we provide a minimum amount of notions about spinors that is needed to treat some topics in this work, and to complete ideas we have postponed until we discuss spinors. Here we give a general “definition” for what a spinor is. For a full account on spinors we refer to the classic of R. Penrose and W. Rindler, “**Spinors and space-time**” Volumes I [125] and II [126]. First let us recall some notation conventions in the abstract index formalism as described by Penrose and Rindler in [125].

### Abstract Index Formalism

The *abstract indices* are denoted by light face latin letters, small  $(a, b, c, \dots)$  for tensor indices; and capital  $(A, B, C, \dots)$  for spinor indices. They do not any reference to bases or coordinate systems and all expressions or development involving them are intrinsic.

The concrete indices define the components in reference to a coordinate basis. They are denoted by bold face latin letters, small  $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots)$  for tensor indices and capital  $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots)$  for spinor indices.

The concrete tensor indices take their values in  $\{0, 1, 2, 3\}$  whereas the concrete spinor indices take their values in  $\{0, 1, \}$ . Moreover, the Einstein summation convention applies for an index appearing twice, once up, once down, in the same term. For abstract indices this means contraction, whereas for concrete indices it means a sum over all the values.

### Spinor Structure

A *spinor* at a point  $x$  of a space-time  $\mathcal{M}$  is an ordered pair of complex numbers associated with an orthonormal basis of the tangent space  $T_x\mathcal{M}$ , which transforms in a specific way under a continuous change of basis. The most unusual aspect of this transformation law is that a spinor changes sign when the basis completes a rotation of  $2\pi$  about a fixed axis and thereby returns to its original configuration. Therefore, a spinor has two possible values

in a given orthonormal basis, and cannot be physically measurable. However, every real null vector can be expressed as the tensor product of a spinor and its complex conjugate. In this sense, a spinor may be viewed as a square root of a null vector. For a Lorentzian 4-dimensional non-compact manifold  $\mathcal{M}$  to admit spinor structure it is sufficient that  $\mathcal{M}$  is parallelizable (see [71]). Our manifolds  $\mathcal{N}$  and  $\mathcal{N}^1$  are then parallelizable (see (35) and (265)).

Let us describe the spinor algebra, using the abstract index formalism. We denote by  $\mathbb{S}^A$  the spin bundle over  $\mathcal{M}$  and  $\mathbb{S}^{A'}$  its complex conjugate structure. The dual bundles are respectively denoted by  $\mathbb{S}_A$  and  $\mathbb{S}_{A'}$ . The complexified tangent bundle is recovered as the tensor product of  $\mathbb{S}^A$  and  $\mathbb{S}^{A'}$ , i.e.

$$\begin{aligned} T^a \mathcal{M} \otimes \mathbb{C} &= \mathbb{S}^A \otimes \mathbb{S}^{A'} , \\ T_a \mathcal{M} \otimes \mathbb{C} &= \mathbb{S}_A \otimes \mathbb{S}_{A'} . \end{aligned}$$

An abstract tensor index  $a$  is thus understood as an unprimed spinor index  $A$  and a primed spinor index  $A'$  put together:  $a = AA'$ .

The spin bundle  $\mathbb{S}^A$  is equipped with a canonical symplectic form  $\varepsilon_{AB}$ . It is used to raise and lower spinor indices, but due to its skew-symmetry  $\varepsilon_{AB} = -\varepsilon_{BA}$ , the order is important:

$$\varepsilon^{AB} \kappa_B = \kappa^A , \quad \kappa^A \varepsilon_{AB} = \kappa_B$$

Similarly,  $\mathbb{S}^{A'}$  is equipped with  $\overline{\varepsilon_{AB}} = \bar{\varepsilon}_{A'B'}$ , which is simply denoted by  $\varepsilon_{A'B'}$ . These symplectic forms are compatible with Lorentzian metric as follows:

$$g_{ab} = g_{AA'BB'} = \varepsilon_{AB} \varepsilon_{A'B'} .$$

## Electromagnetic Spinor

Any 2-antisymmetric real tensor  $F_{ab}$ , and hence the Maxwell tensor in particular, can be expressed in terms of a symmetric 2-spinor  $\phi_{AB} = \phi_{A'B'}$ :

$$F_{ab} = F_{AA'BB'} = \phi_{AB} \varepsilon_{A'B'} + \bar{\phi}_{A'B'} \varepsilon_{AB} . \quad (322)$$

If  $F$  is a Maxwell field satisfying the source-free equations (40) and (41), then the corresponding spinor is called the electromagnetic spinor, and Maxwell's equations on the spinor are:

$$\begin{aligned} \nabla^{A'B} \phi_{AB} &= \nabla^{AB'} \bar{\phi}_{A'B'} \\ \nabla^{A'B} \phi_{AB} + \nabla^{AB'} \bar{\phi}_{A'B'} &= 0 , \end{aligned}$$

and therefore Maxwell's equations reduces to

$$\nabla^{A'B} \phi_{AB} = 0 , \text{ and } \phi_{AB} = \phi_{(AB)} . \quad (323)$$

---

<sup>1</sup>Note the  $\mathcal{N}$  is not a compact manifold, the timelike singularities are limiting points but not part of it

Note that for a general symmetric spinor  $\phi_{AB}$ , not necessarily satisfying (323), we have

$$\nabla^{BB'} \nabla^{A'}_{B'} \phi_{AB} = 0 . \quad (324)$$

This identity follows from the symmetries of the curvature tensor. A proof is found [125] page 322 equation (5.1.54).

## Spin-Frame and Newman-Penrose Formalism

Recall from sections 2.1.1 and 4.1.1 the definition of a Newman-Penrose tetrad, and let  $\{l, n, m, \bar{m}\}$  be Newman-Penrose tetrad. Any such tetrad is related to the spinor structure of the spacetime, and we can associate to the tetrad a unitary spin-frame  $(o^A, \iota^A)$ , defined uniquely up to an overall sign factor by the following requirements,

$$l^a = o^A o^{A'} ; \quad n^a = \iota^A \iota^{A'} ; \quad m = o^A \iota^{A'} ; \quad \bar{m} = \iota^A o^{A'} .$$

The spin-frame is said to be unitary if  $o_A \iota^A = 1$ . The *spin components* of a symmetric spinor  $\phi_{AB}$  in the spin-frame  $(o^A, \iota^A)$  are:

$$\begin{aligned} \phi_{00} &:= o^A o^B \phi_{AB} , \\ \phi_{01} &:= o^A \iota^B \phi_{AB} , \\ \phi_{11} &:= \iota^A \iota^B \phi_{AB} . \end{aligned}$$

To see that the scalars  $\Phi_1, \Phi_0$ , and  $\Phi_{-1}$  defined in (49) are indeed the spin components of a Maxwell field, consider the spin-frame  $o^A$  and  $\iota^A$  given by

$$\begin{aligned} L &= o^A o^{A'} , \\ N &= \iota^A \iota^{A'} , \\ M &= o^A \iota^{A'} , \\ \bar{M} &= \iota^A o^{A'} , \end{aligned}$$

and so we have,

$$o_A \iota^A = \sqrt{2} f^{\frac{1}{2}} \quad o_A o^A = \iota_A \iota^A = 0 .$$

We note that since the tetrad  $\{L, N, M, \bar{M}\}$  is not normalized, the associated spin frame will not be unitary. A simple calculation using (49) shows that

$$\begin{aligned} \phi_{00} &:= o^A o^B \phi_{AB} = \Phi_1 , \\ \phi_{01} &:= o^A \iota^B \phi_{AB} = \Phi_0 , \\ \phi_{11} &:= \iota^A \iota^B \phi_{AB} = -\Phi_{-1} . \end{aligned}$$

In a given spin-frame, Maxwell's equations (323) can be projected on a null tetrad as in Newman-Penrose formalism, similar to what we did in Lemma 4 and (273). here too, there

is a Newman-Penrose formalism for the spinorial equations:

$$\begin{aligned} L^b \nabla^A_{B'} \phi_{AB} &= o^A o^{A'} \nabla^A_{B'} \phi_{AB} = 0 , \\ N^b \nabla^A_{B'} \phi_{AB} &= \iota^A \iota^{A'} \nabla^A_{B'} \phi_{AB} = 0 , \\ M^b \nabla^A_{B'} \phi_{AB} &= o^A \iota^{A'} \nabla^A_{B'} \phi_{AB} = 0 , \\ \bar{M}^b \nabla^A_{B'} \phi_{AB} &= \iota^A o^{A'} \nabla^A_{B'} \phi_{AB} = 0 , \end{aligned}$$

In fact these equations are exactly equations (274) - (277). That is,  $I_a = \nabla^B_{A'} \phi_{AB}$ , and by (324)

$$\nabla^a I_a = 0 \quad (325)$$

## Energy-Momentum Tensor and Dominant Energy Condition

The energy-momentum tensor for a Maxwell field has a very simple form when written in spinor notation:

$$\mathbf{T}_{ab} = \phi_{AB} \phi_{A'B'} .$$

If  $l^a$  and  $n^a$  are any two null vectors, then there are  $\mu^A$  and  $\nu^A$  two spinors such that,

$$l^a = \mu^A \mu^{A'} \quad \text{and} \quad n^a = \nu^A \nu^{A'}$$

If  $l$  and  $n$  are future-oriented null vectors, we then have

$$T_{ab} l^a n^b = |\phi_{AB} \mu^A \nu^B|^2 \geq 0 . \quad (326)$$

Since any future oriented causal vector can be written linear combination of two null vectors, this inequality applies to any two future-oriented causal vectors. This proves the dominant energy condition.

## Maxwell Potential

As we saw in section 2.4, a 1-form  $A_a$  satisfying the Lorenz fixation, is a global potential for a Maxwell field if it satisfies the hyperbolic equation (139). Before giving the proof, we need some last notions from spinors: duality of a bivector. A bivector  $F_{ab}$  is an anti-symmetric  $F_{ab} = -F_{ba}$ , possibly complex, tensor of type (0,2), or in spinor form,  $F_{AA'BB'} = -F_{BB'AA'}$ . The dual of a bivector  $F_{ab}$  is

$${}^*F_{ABA'B'} := iF_{ABB'A'} .$$

We also define anti-self-dual and self-dual parts of  $F_{ab}$  respectively by

$${}^-F_{ab} := \frac{1}{2}(F_{ab} + i{}^*F_{ab}) ; {}^+F_{ab} := \frac{1}{2}(F_{ab} - i{}^*F_{ab}) .$$

Consequently, every bivector is (uniquely) the sum of a self-dual and anti-self-dual bivectors. Now we show that  $A_a$  satisfies both conditions, i.e. Lorenz condition and Maxwell's equations if

$$\nabla^{A'}_A A_{A'B} = 0 . \quad (327)$$

To prove that this is indeed the case, we take the skew-part (i.e. the trace in  $(A, B)$ ) of (327), this gives the Lorenz gauge condition (138). The symmetric part gives the self-dual part of the exterior derivative of  $A_a$ . The anti-self-dual part of the derivative is then automatically an anti-self-dual Maxwell field.

This proof is due to A.R. Gover and J.P. Nicolas in their paper “*Conformal scattering of Maxwell potentials*” which is in preparation.





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# Champs de Maxwell en Espace-temps de Reissner-Nordstrøm-De Sitter : Décroissance et Scattering Conforme

## Résumé

Nous étudions les champs de Maxwell à l'extérieur de trous noirs de Reissner-Nordstrom-de Sitter . Nous commençons par étudier la géométrie de ces espaces-temps : nous donnons une condition sous laquelle la métrique admet trois horizons puis dans ce cadre nous construisons l'extension analytique maximale d'un trou noir de Reissner-Nordstrom-de Sitter. Nous donnons ensuite une description générale des champs de Maxwell en espace-temps courbe, de leur décomposition en composantes spinorielle ainsi que de leur énergie. La première étude analytique établit la décroissance ponctuelle de champs de Maxwell à l'extérieur d'un trou noir de Reissner-Nordstrom-de Sitter ainsi que la décroissance uniforme de l'énergie sur un hyperboloïde qui s'éloigne dans le futur. Ce chapitre utilise des méthodes de champs de vecteurs (estimations d'énergie géométriques) dans l'esprit des travaux de Pieter Blue. Enfin nous construisons une théorie du scattering conforme pour les champs de Maxwell à l'extérieur du trou noir. Ceci consiste en la résolution du problème de Goursat pour les champs de Maxwell à la frontière isotrope de l'extérieur du trou noir, constituée des horizons du trou noir et horizons cosmologiques futurs et passés. Les estimations de décroissance uniforme de l'énergie sont cruciales dans cette partie.

**Mots clés :** Mathématiques, Géométrie, Physique mathématique, Relativité générale, Géométrie différentielle, Géométrie Conforme, Géométrie Lorentzienne, Analyse, EDP, Décroissance des champs, Trou Noir, Théorie de Scattering Conforme.

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## Maxwell Field on the Reissner-Nordstrøm-de Sitter Manifold : Decay and Conformal Scattering

### Abstract :

We study Maxwell fields on the exterior of Reissner-Nordstrom-de Sitter black holes. We start by studying the geometry of these spacetimes: we give the condition under which the metric admits three horizons and in this case we construct the maximal analytic extension of the Reissner-Nordstrom-de Sitter black hole. We then give a general description of Maxwell fields on curves spacetimes, their decomposition into spin components, and their energies. The first result establishes the pointwise decay of the Maxwell field in the exterior of a Reissner-Nordstrom-de Sitter black hole, as well as the uniform decay of the energy flux across a hyperboloid that recedes in the future. This chapter uses the vector fields methods (geometric energy estimates) in the spirit of the work of Pieter Blue. Finally, we construct a conformal scattering theory for Maxwell fields in the exterior of the black hole. This amounts to solving the Goursat problem for Maxwell fields on the null boundary of the exterior region, consisting of the future and past black hole and cosmological horizons. The uniform decay estimates of the energy are crucial to the construction of the conformal scattering theory.

**Keywords :** Mathematics, Geometry, Mathematical Physics, General Relativity, Differential Geometry, Conformal Geometry, Lorentzian Geometry, Analysis, PDE, Decay of fields, Black Holes, Conformal Scattering Theory.