Mathematical study of complex fluids in anisotropic geometries

Andrei Ichim

N° d'Ordre : D.U. 2760

UNIVERSITÉ BLAISE PASCAL

U.F.R. Sciences et Technologies

ÉCOLE DOCTORALE DES SCIENCES FONDAMENTALES

N° 892

THÈSE

présentée pour obtenir le grade de

DOCTEUR D'UNIVERSITÉ

Spécialité : **Mathématiques Appliquées**

Par :

Andrei Ichim

TITRE DE LA THÈSE :

Étude mathématique du comportement de fluides complexes dans des géométries anisotropes

Soutenue le 5 Décembre 2016

devant le jury composé de :

Rapporteurs :	Didier Bresch	-	Université de Savoie
	Geneviève Raugel	-	Université Paris-Sud
Examinateurs :	Anne-Laure Dalibard	-	Université Pierre et Marie Curie
	Julien Olivier	-	Institut de Mathématiques de Marseille
Directeurs :	Laurent Chupin	-	Université Blaise Pascal
	Sébastien Martin	-	Université Paris Descartes

Remerciements

Je tiens d'abord à exprimer mes sincères remerciements à *Laurent Chupin* pour m'avoir offert la possibilité de réaliser cette thèse. Ce fut un véritable plaisir de travailler avec un mathématicien si talentueux. Ses conseils avisés et son soutien permanent ont contribué décisivement à la finalisation de ce travail. Je remercie également *Sébastien Martin* pour la gentillesse et la patience sans borne dont il a fait preuve. Nos échanges scientifiques durant mes visites à Paris ont été toujours fructueuses et enrichissantes.

Je voudrais remercier les rapporteurs *Didier Bresch* et *Geneviève Raugel* pour leur lecture attentive de ma thèse, et pour leurs suggestions et commentaires qui m'ont aidé à améliorer ce manuscrit. Merci également à *Anne-Laure Dalibard* et *Julien Olivier* pour avoir participé au jury.

Pendant la durée de ma thèse, j'ai bénéficié de très bonnes conditions au Laboratoire de Mathématiques à Clermont-Ferrand. J'en remercie ses membres, en particulier ceux de l'équipe EDPAN ainsi que les personnels des secrétariats, de la bibliothèque et de l'imprimerie, pour avoir répondu promptement à toutes mes demandes.

Enfin, je voudrais remercier tous mes amis et ma famille en Roumanie. Leur soutien permanent m'a permis de surmonter bien des difficultés pendant ces années.

Résumé

Cette thèse est consacrée à l'étude mathématique des écoulements complexes dans des tubes minces. Les difficultés ne sont pas seulement liées à la rhéologie complexe, mais aussi aux conditions au bord sur la pression en entrée et en sortie (qui sont moins habituelles, mais réalistes du point de vue physique).

Dans une première partie, des écoulements quasi-newtoniens stationnaires sont étudiés. D'abord, on utilise la petitesse du domaine pour montrer l'existence de la solution. Ensuite, on écrit un développement asymptotique de cette solution et on calcule formellement ses coefficients. Finalement, on justifie rigoureusement la validité de ce dévelopement en démontrant des estimations d'erreur.

Dans une deuxième partie, on considère des écoulemets de fluides visco-élastiques décrits par la loi d'Oldroyd en régime stationnaire. Le modèle que nous avons choisi contient un terme diffusif en contrainte, dont l'ordre de grandeur est lié à la petitesse du domaine. Similairement à la première partie, un dévelopement asymptotique est completement justifié du point de vue mathématique. Dans le cas particulier de domaines axisymétriques une solution numérique est cherchée afin de la comparer à la solution obtenue via la technique asymptotique.

Dans une dernière partie, on étudie les équations de Navier-Stokes non stationnaires. Un résultat d'existence des solutions fortes pour des données petites est démontré. Malheureusement, la méthode directe ne nous a pas permis pas d'avoir suffisamment de contrôle par rapport à la petitesse du domaine. Pour obtenir le résultat désiré, on utilise l'approche à la Kato, basé sur la théorie de C_0 semi-groupes.

Mots clefs : fluides non-newtoniens, tubes minces, analyse asymptotique, conditions au bord en pression, existence globale.

Abstract

This thesis is devoted to the mathematical analysis of complex flows in thin pipes. The difficulties stem not only from the complex rheology, but also from the boundary conditions used involving the pressure (which are rather atypical, but realistic from a physical point of view).

In the first part, we study stationary, quasi-newtonian flows. The existence of a solution is shown using the smallness of the domain as a key ingredient. Furthermore, an asymptotic expansion of this solution is sought and its coefficients are formally computed. Lastly, the validity of this expansion is rigorously justified by proving error estimates.

In the second part, we consider visco-elastic flows represented by Oldroyd's law in stationary regime. The model which we have chosen contains a diffusive stress term, whose order of magnitude is related to the smallness of the domain. Similarly to the first part, a complete asymptotic expansion in mathematically justified. For the special case of axisymmetric domains a numerical solution is sought in order to compare it against the one obtained via the asymptotic technique.

In the last part we study the non stationary Navier-Stokes equations. An existence result of strong solutions for small initial data is proven. Unfortunately, the direct method – based on energy estimates – doesn't give us an optimal control of the smallness constant with respect to the size of the domain. To obtain the desired result, we employ the method of C_0 semigroups of linear operators.

Keywords: non-newtonian fluids, thin pipes, asymptotic analysis, pressure boundary condition, global existence.

Contents

Chapter 1

Introduction

1.1 Sur les écoulements non-newtoniens dans des domaines anisotropes

1.1.1 Problématique générale

L'écoulement des fluides est un des phénomènes les plus importants étudié par les chercheurs grâce à ses nombreuses applications pratiques - les circulations atmosphérique et océanique, le mouvement à l'intérieur des étoiles, les applications dans l'aéronautique ou l'étude de l'écoulement du sang en sont des exemples. Au cours de cette thèse, on va se concentrer sur les écoulements complexes (non-newtoniens) dans des petits tubes – le sang qui s'écoule dans les artères et veines est un exemple de ce type.
La loi qui décrit l'écoulement d'un fluide newtonien dans un tube cylindrique a été découverte par Poiseuille (1840), et s'écrit de la manière

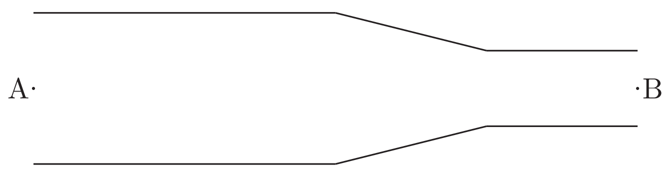
$$Q = \frac{\pi R^4}{8\eta L}\Delta p, \tag{1.1}$$

où

- Q est le débit,
- η est la viscosité du fluide,
- R est le rayon du tube,
- L est la longueur du tube,
- Δp est le gradient de la pression (la différence de pression aux extrémités de la canalisation).

En fait, le résultat est encore plus précis : le champ de la vitesse est parallèle à l'axe du tube et a un profil parabolique, la vitesse maximale étant la vitesse au

centre du tube.
La formule de Poiseuille est applicable dans les domaines qui ont une section verticale constante. Par contre, si on pense aux vaisseaux sanguins ou, plus généralement, aux réseaux tubulaires c'est clair qu'il peut y avoir des constrictions (ou des élargissements) qui vont modifier la géométrie. Considérons le cas suivant:



Il est naturel de se demander quelle est la relation entre la pression en A et la pression en B, ou s'il existe une relation explicite entre la pression et la vitesse. Dans le cas d'un fluide parfait (sans viscosité) la réponse est donnée par la loi de Bernoulli (1738), qui exprime le fait que :

$$\rho\frac{u^2}{2} + p \equiv constant. \tag{1.2}$$

où ρ est la densité et u la vitesse du fluide (supposée constante). Les résultats (1.1)–(1.2) sont obtenus en utilisant des modèles simplifiés (en négligeant certains termes). En général, les équations qui modélisent les écoulements sont très compliquées et on ne peut pas espérer les résoudre explicitement. Dans cette thèse, nous allons chercher des modèles dégénérés (réduits) qui soient des approximations des modèles initiaux dans certaines géométries. Bien entendu les modèles obtenus ne seront pas aussi simples que ceux donnés par (1.1)–(1.2)! Soulignons les trois aspects importants sur lesquels on va se concentrer au cours de notre travail :

- *le caractère complexe (non-newtonien)* des fluides considérés (on aura une description plus détaillée dans la section suivante).
- *la géométrie anisotrope* - nous considérons des écoulements dans des tubes irréguliers fins (pour lesquels le diamètre est très inférieur à la longueur).
- *les conditions au bord en pression* que nous allons utiliser sont moins usuelles. Du point de vue mathématique, on préfère travailler avec des conditions au bord en vitesse, car cela facilite l'obtention des résultats théoriques. Par contre, dans certaines situations il est plus convenable (et plus naturel) d'utiliser les conditions en pression.

1.1.2 Aspects de modélisation

Les équations qui décrivent l'écoulement de fluides sont de deux types :

- les équations qui représentent des lois physiques de conservation valables pour tous les milieux continus (la conservation de la masse et la conservation de la quantité de mouvement). Elles s'écrivent de la façon suivante:

$$div\, u = 0,$$
$$\rho(\partial_t u + (u \cdot \nabla)u) = f + div\, \sigma,$$

  où ρ est la densité et u le champ de vitesses. Le terme f correspond aux forces (externes) en volume qui agissent sur le fluide (e.g. gravité, forces électromagnétiques, forces de Coriolis). La contrainte σ est une mesure de forces (internes) de surface qui sont générées par les interactions moléculaires. Le tenseur de contraintes σ se décompose de façon suivante :

$$\sigma = -pI + \tau,$$

  où p est la pression hydrostatique et τ est le déviateur de contraintes.

- les lois constitutives qui décrivent le comportement spécifique d'une certaine classe de fluides.

Un fluide newtonien est un milieu continu pour lequel le déviateur τ est une fonction linéaire du tenseur des taux de déformation Du (la partie symétrique du gradient de la vitesse). Cette condition s'écrit sous la forme suivante:

$$\tau = 2\eta Du,$$

η étant la viscosité du fluide. Donc, le modèle newtonien exprime le fait qu'un seul paramètre est suffisant pour décrire toutes les propriétés – mais, comme déjà annoncé, ce n'est pas toujours vrai. Effectivement, il existe beaucoup de fluides qu'on trouve dans la vie quotidienne qui ont un fort caractère non-newtonien (e.g. les polymères, la peinture, la mousse à raser, le dentifrice, le sang et les autres fluides biologiques). Cela s'explique par le fait que, au niveau microscopique, ils sont un mélange d'un solvant et des micro-structures qui flottent dans ce solvant. Notamment, le sang est une suspension de vésicules déformables (les globules rouges) dans un solvant (le plasma). Pour donner une explication qualitative, considérons l'écoulement de cisaillement, c'est-à-dire l'écoulement entre deux surfaces parallèles dont l'une est fixe et l'autre se déplace dans une direction parallèle aux surfaces à une vitesse constante U. Le taux de cisaillement $\dot{\gamma}$ est défini comme $\frac{U}{h}$, où h est la la distance entre les deux parois. La contrainte de cisaillement τ_c est le rapport entre la force de cisaillement et la surface de contact. Selon la loi newtonienne, la relation entre τ_c et $\dot{\gamma}$ devrait s'exprimer de la manière suivante:

$$\frac{\tau_c}{\dot{\gamma}} = \eta \equiv \text{constant}.$$

Par contre, les expériences viscométriques faites pour une large plage de taux de cisaillements ont montré que la viscosité n'est pas constante, mais elle dépend de $\dot{\gamma}$. Les fluides *quasi-newtoniens* sont des fluides dont τ dépend d'une manière explicite de Du, mais cette relation n'est plus linéaire comme c'est le cas pour les fluides newtoniens. Mathématiquement, on écrit:

$$\tau = 2\eta(|Du|)Du.$$

En particulier, si la fonction η est décroissante, on appelle ce type de fluides *rhéofluidifiants* - le sang est un exemple remarquable de fluide rhéofluidifiant [5], voir aussi la Figure 1.1.

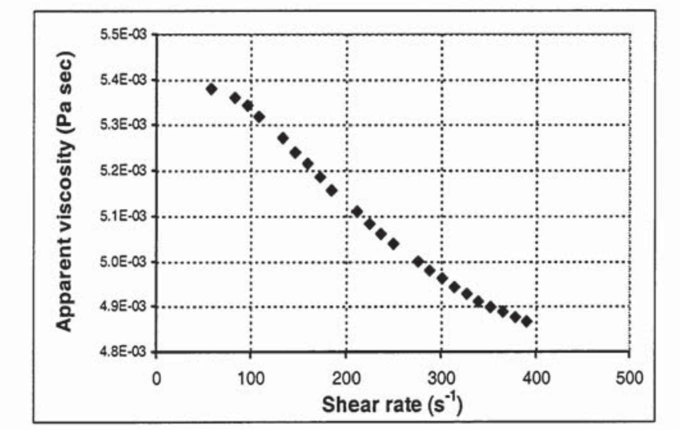


Figure 1.1: Viscosité du sang en fonction de vitesse de cisaillement (d'après [9])

Cette propriété est attribuée à la présence des globules rouges qui modifient leur dynamique pour des différentes valeurs du taux de cisaillement (pour une description détaillée, voir [15]). De nombreuses lois ont été proposées pour décrire le comportement des fluides quasi-newtoniens, selon la fonction η. Un modèle très populaire est la loi de puissance, introduite par W. Ostwald (1925):

$$\eta(|Du|) = \eta_s |Du|^{n-1},$$

où $\eta_s, n > 0$. La popularité de ce modèle est due au fait qu'il est applicable pour une très large classe d'écoulements. Néanmoins, cette loi peut devenir irréaliste dans le regime rhéofluidifiant $n < 1$ (celui qui nous intéresse plutôt), car la viscosité devient non bornée lorsque $|Du| \to 0$, ce qui, évidemment, n'est pas vrai. Une loi qui n'a plus ce désavantage a été proposé par P. Carreau (1968), et s'écrit sous la forme suivante:

$$\eta(|Du|) = \eta_\infty + (\eta_0 - \eta_\infty)(1 + \lambda |Du|^2)^{\frac{r-1}{2}},$$

où $\eta_0 \geq \eta_\infty > 0, r \geq 0$. Bien que les lois présentées ci-dessus décrivent d'une manière satisfaisante le caractère rhéofluidifiant de fluides, elles ne sont pas suffisantes pour prendre en compte des aspects encore plus complexes des écoulements. Mentionnons quelques uns de ces phénomènes (des descriptions plus détaillées sont disponibles dans [5]):

- l'effet Weissenberg – un fluide entraîné par une tige tournante monte le long de la tige si la vitesse de rotation est suffisamment grande.
- le gonflement en sortie de filière.
- la détente élastique – un fluide versé d'un récipient peut être coupé et il remonte dans le récipient.

Les premiers deux effets sont dus à l'anisotropie de contraintes normales – par contre, dans un fluide newtonien (ou bien quasi-newtonien) soumis à un écoulement de cisaillement les contraintes normales sont égales entre elles. La dernière expérience montre que le fluide a un caractère élastique bien prononcé (ou bien il a une mémoire des déformations précédentes) – c'est pour cela qu'on appelle ce type de fluides *visco-élastiques*. Le caractère visco-élastique du sang a été observé, pour la première fois dans [44] – pour une analyse plus récente et détaillée voir [15]. Maxwell (1867) a fait une première tentative pour caractériser les propriétés élastiques d'un fluide – il a supposé que le matériau est une association en série d'un ressort (de constante d'élongation G) et d'une masse (de viscosité η) – voir la figure 1.2. La contrainte τ est imposée à l'ensemble.

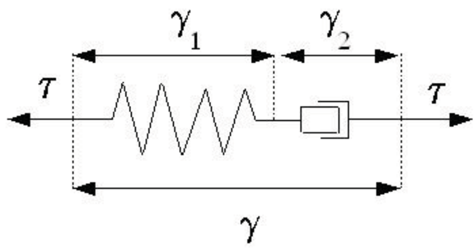


Figure 1.2: Schéma du modèle de Maxwell

On peut donc écrire

$$\tau = G\gamma_1 = \eta \frac{\partial \gamma_2}{\partial t},$$

où γ_1 et γ_2 sont les déformations subies par le ressort et la masse respectivement. On obtient immédiatement la *forme différentielle* du modèle de Maxwell:

$$\lambda \frac{\partial \tau}{\partial t} + \tau = \eta \frac{\partial \gamma}{\partial t} \tag{1.3}$$

où $\gamma = \gamma_1 + \gamma_2$ et $\lambda = \frac{\eta}{G}$. Bien sûr γ représente la déformation totale subie par le système. Le paramètre λ – appelé le temps de relaxation – sert à expliquer le comportement non-usuel de certains matériaux qui ont à la fois un aspect visqueux et élastique. Si on le sollicite rapidement, sur un intervalle de temps expérimental $t_{\mathrm{exp}} \ll \lambda$, la réponse est élastique ($\tau \sim G\gamma$). Par contre, si $\lambda \ll t_{\mathrm{exp}}$ la réponse est

visqueuse ($\tau \sim \eta\dot{\gamma}$).
En intégrant la relation (1.3) on obtient la *forme intégrale* du modèle:

$$\tau(t) = \int_{-\infty}^{t} Ge^{\frac{t-s}{\lambda}}\dot{\gamma}(s)ds. \tag{1.4}$$

Cette dernière formule exprime le fait que la contrainte dépend de l'histoire du taux de déformations avec un poids (la fonction de relaxation $s \mapsto Ge^{\frac{-s}{\lambda}}$). Un modèle un peu plus compliqué a été proposé par Jeffrey (1929), qui a supposé que le fluide est un couplage (en série) entre une masse (de viscosité η_1) et une association en parallèle d'un ressort (de constante d'élongation G) et d'une masse (de viscosité η_2). L'équation différentielle associée au modèle de Jeffrey est la suivante:

$$\lambda_1\frac{\partial\tau}{\partial t} + \tau = \eta_1\Big(\frac{\partial\gamma}{\partial t} + \lambda_2\frac{\partial^2\gamma}{\partial t^2}\Big), \tag{1.5}$$

où γ et τ ont la même signification que dans le modèle de Maxwell. Par ailleurs, $\lambda_1 = \frac{\eta_1+\eta_2}{G}$ est le temps de relaxation et $\lambda_2 = \frac{\eta_2}{G}$ est le temps de retard.
Bien entendu, les modèles 1D de Maxwell et Jeffrey sont très simplistes – il existe des modèles multidimensionnels (différentiels et intégraux) beaucoup plus compliqués (voir [5] et [1]). Une géneralisation 3D du modèle de Jeffrey s'écrit sous la forme suivante (voir [5])

$$\lambda_1\frac{D}{Dt}\tau + \tau = 2\eta\Big(Du + \lambda_2\frac{D}{Dt}Du\Big),$$

où $\frac{D}{Dt}$ est une dérivée en temps objective, c'est-à-dire invariante par toutes transformations euclidiennes. Le coefficient λ_2 mesure le rapport entre la partie élastique de la partie visqueuse du fluide : il est compris entre 0 et λ_1. Le cas $\lambda_1 = \lambda_2 = 0$ correspond à un fluide newtonien, alors que le cas $\lambda_2 = 0, \lambda_1 \neq 0$ correspond au cas d'un fluide purement élastique. Ensuite, on décompose le tenseur τ dans la partie newtonienne $\tau_s = 2\eta\frac{\lambda_2}{\lambda_1}Du$ et la partie purement élastique τ_e. En introduisant le paramètre de retard $r = 1 - \frac{\lambda_2}{\lambda_1}$ et en notant $\sigma := \tau_e$ on déduit

$$\lambda_1\frac{D}{Dt}\sigma + \sigma = 2\eta r Du,$$

Nous allons nous intéresser à une de ces dérivées objectives, utilisée dans le modèle d'Oldroyd. Plus concrètement,

$$\frac{D\sigma}{Dt} = \frac{\partial\sigma}{\partial t} + (u\cdot\nabla)\sigma + \big(\sigma\cdot Wu - Wu\cdot\sigma\big) + a(\sigma\cdot Du + Du\cdot\sigma)$$

où Wu est la partie anti-symétrique du gradient de la vitesse (le tenseur de vorticité), et a est un paramètre dans $[-1, 1]$.

1.1.3 État de l'art : Résultats dans des domaines quelconques

Les équations qui interviennent dans les modèles présentés ci-dessus sont complexes. L'étude mathématique de ces équations (avec des conditions initiales et au bord) a fait l'objet de nombreuses publications scientifiques au cours des 80 dernières années. Notons que, en dépit de tous ces travaux, le cas newtonien pose lui-même des questions difficiles et non résolues : pour les équations de Navier-Stokes en 3D, on ne dispose d'aucun résultat d'existence et d'unicité de solutions fortes! Mentionnons rapidement les principaux résultats mathématiques liés aux problèmes que nous allons considérer.

- Les premiers résultats mathématiques concernant le problème de Navier-Stokes sont dus à Leray (1934), qui a démontré pour la première fois l'existence de solutions faibles. Depuis lors, plusieurs mathématiciens ont étudié l'unicité et la régularité de solutions de Leray, notamment J. Serrin [41], O. Ladyshenkaya [26], J. L. Lions [27] et R. Temam [42]. L'idée essentielle pour montrer l'existence de solutions consiste à utiliser les estimations d'énergie. Une autre approche est d'employer la théorie des semi-groupes d'opérateurs introduite par Kato et Fujita [13], [24] et utilisée après par Y. Giga et T. Miyakawa [16] dans un cadre plus général.

- L'étude mathématique des modèles quasi-newtoniens a été initiée par O. Ladyshenkaya [25], et continuée par de nombreux auteurs, notamment J.L. Lions [27]. Un premier résultat de régularité a été obtenu par J. Málek et al. [31].

- Par ailleurs, les premières contributions théoriques concernant les modèles viscoélastiques sont dues à M. Renardy [39], qui a démontré un résultat d'existence dans le cas stationnaire. Dans le cas instationnaire, un premier résultat d'existence locale en temps ou à données petites a été obtenu par J.C. Saut et C. Guillopé [19]. En utilisant des techniques fines d'analyse P. Lions et N. Masmoudi [28] ont obtenu un résultat d'existence (de solutions faibles) global en temps (dans le cas où le paramètre a introduit dans la section précédente est 0).

- En ce qui concerne les conditions au bord en pression il existe très peu de résultats mathématiques. L'étude mathématique des équations de Navier-Stokes avec des conditions au bord en pression a été initiée au début des années '90 par Conca et al. [10] dans le cas stationnaire. D'autre part, dans le cas non-stationnaire un résultat d'existence d'une solution locale faible en 2D a été démontre par S. Marušić [32].

1.1.4 État de l'art : Résultats dans des domaines minces

Du point de vue mathématique, travailler dans des domaines minces permet d'obtenir des meilleurs résultats. Il est bien connu que les équations de Navier-Stokes en

3D ont une solution globale forte si les données initiales sont petites. G. Raugel et G. Sell ([38],[37]) ont montré que, dans un domaine du type $\omega \times (0, \varepsilon)$ (où $\omega \subset \mathbb{R}^2$ est ouvert) si les données initiales sont dans des "large sets" B_ε (c'est-à-dire que $\mu(B_\varepsilon) \to \infty$ lorsque $\varepsilon \to 0$), les équations de Navier-Stokes (avec de conditions au bord périodiques pour la vitesse) ont une solution globale forte en 3D. Leurs travaux ont été généralisés par R. Temam et M. Ziane ([43],[35]) dans le cas de diverses conditions au bord pour la vitesse. Un résultat pour une classe plus générale de domaines a été obtenu par J. Avrin [2](dans le cas des conditions au bord Dirichlet). En ce qui concerne les conditions au bord en pression , un résultat similaire a été obtenu par W. Jäger et A. Mikelić [22].
D'autre part, travailler dans des domaines minces augmente considérablement le coût des calculs numériques. C'est pour cela que, dans de tels domaines, on prefère rechercher un modèle plus simple qui soit une approximation du modèle initial. En particulier, nous étudions le cas des domaines anisotropes où l'épaisseur est très inférieure à la longueur - on appelle ces domaines canaux (ou tubes) fins. Notamment, l'écoulement sanguin est un exemple de ce type. Effectivement, le diamètre des vaisseaux sanguins dans le corps humain varie entre 15 mm pour les grandes artères et 8μm pour les capillaires – bien sûr, la longueur de ces vaisseaux est beaucoup plus grande que leur diamètre.
Il y a une vaste littérature consacrée aux écoulements de fluides en film mince; nous n'allons présenter que les résultats liés à notre travail :

- De nombreux problèmes sont liés à la lubrification (c'est-à-dire l'écoulement de cisaillement dans un espace mince entre deux surfaces). Dans le cas des fluides quasi-newtoniens, le résultat le plus général a été obtenu par Sac-Epée et Taous [40], qui ont montré la convergence du modèle vers une équation limite. En ce qui concerne les fluides viscoélastiques, la justification rigoureuse de passage à la limite vers un modèle limite a été faite par Bayada et al. dans [3].

- Par contre, dans le cas des écoulements dans des petits tubes, les résultats existants concernent plutôt les fluides newtoniens. Une analyse asymptotique sur les fluides newtoniens incompressibles se trouve dans [17], alors que le cas compressible a été traité dans [34]. E. Marušić-Paloka s'est aussi intéressé aux effets de la courbure sur les écoulements dans des petits tubes [33]. Les écoulements dans des réseaux tubulaires ont été étudiés par G. Panasenko et al. [6], [8].

1.1.5 Technique générale

La technique générale que nous employons dans deux des trois chapitres de cette thèse est la suivante :

- Soit $\varepsilon \ll 1$ un petit paramètre qui mesure habituellement le rapport entre le diamètre et la longueur du tube.

- Montrer l'existence et l'unicité de la solution (notéappelée générique U_ε en général) au problème initial.
- Écrire formellement $U_\varepsilon = U_0 + \varepsilon U_1 + \ldots$
- Décrire l'algorithme pour trouver (ou, au moins, montrer l'existence de) tous les U_n.
- Justifier rigoureusement que l'ansatz ci-dessus est valide, en prouvant un résultat de convergence.
- Faire une étude numérique.

1.2 Description des résultats obtenus

1.2.1 Chapitre 2

Dans ce chapitre on s'intéresse aux écoulements quasi-newtoniens dans des tubes minces. Plus précisement, on va étudier le comportement asymptotique de ce type des fluides et justifier les résultats d'une manière rigoureuse.
Par rapport aux travaux mentionnés, soulignons les point forts des nôtres :

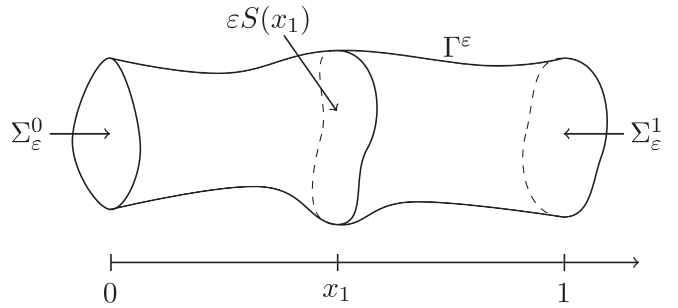
- On travaille avec des conditions au bord moins usuelles (mais réalistes du point de vue physique), ce qui rend l'analyse mathématique du modèle plus difficile.
- La fonction de viscosité est plus générale que celle utilisée dans les travaux précédents.

On voudrait insister sur l'importance des conditions au bord (en pression) car, dans certaines situations réelles, c'est plus naturel (et plus convenable) d'imposer la pression sur une partie du bord – par exemple, c'est le cas des vaisseaux sanguins ou des systèmes hydrauliques.
Au cours de cette thèse, on va supposer que le domaine dans lequel les fluides s'écoulent est un petit tube "irrégulier", dont la géométrie est décrite par

$$\Omega_\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 \in (0,1),\ (x_2, x_3) \in \varepsilon S(x_1)\}, \qquad (1.6)$$

où $\varepsilon > 0$ est un petit paramètre. Sans limiter la généralité, nous avons supposé que la longueur du tube est 1, car c'est le rapport entre le diamètre et la longueur du tube qui nous intéresse plutôt (c'est-à-dire qu'on peut avoir des canaux soit très longs, soit très fins).
Les équations qui décrivent l'écoulement stationnaire des fluides quasi-newtoniens s'écrivent sous la forme suivante :

$$\begin{cases} (u \cdot \nabla)u + \nabla p = 2div\,(\eta(|D(u)|^2)Du), \\ div\, u = 0. \end{cases} \qquad (1.7)$$



Évidemment, la frontière du domaine se compose de trois parties :

$$\Gamma_\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 \in (0,1),\ (x_2, x_3) \in \varepsilon\partial S(x_1)\},$$
$$\Sigma_0^\varepsilon = \varepsilon S(0),\ \Sigma_1^\varepsilon = \varepsilon S(1).$$

Bien sûr, il faut ajouter les conditions aux bords, qui s'écrivent

$$\begin{cases} u = 0 & \text{sur} \quad \Gamma_\varepsilon, \\ u \times n = 0 & \text{sur} \quad \Sigma_0^\varepsilon \cup \Sigma_1^\varepsilon, \\ p = p_i & \text{sur} \quad \Sigma_i^\varepsilon. \end{cases} \tag{1.8}$$

Les résultats obtenus sont divisés en trois parties:

Étape 1 Nous montrons que le problème (1.7)-(1.8) a au moins une solution faible *pour ε assez petit.* Comme déjà mentionné, les premières investigations sur le caractère bien-posé des équations Navier-Stokes avec ces conditions au bord ont été faites dans [10]. Les auteurs ont montré que, si on impose les conditions au bord (1.8) le problème de *Stokes* est bien posé. Le problème de Navier-Stokes, par contre, est plus difficile à cause du terme inertiel $u \cdot \nabla u$. Une manière de contourner cette difficulté est d'imposer plutôt la pression dynamique $p + \frac{|u|^2}{2}$ au lieu de la pression. Néanmoins, nous n'allons pas utiliser cette alternative car ces conditions au bord ne sont pas "compatibles" (dans un sens à préciser) avec la technique asymptotique que nous allons décrire à l'étape suivante. On est donc obligé d'utiliser la petitesse de ε, ainsi que des inégalités "optimales" (dans le sens qu'on prend en compte la dépendance des différentes constantes par rapport à ε) pour arriver à contrôler le terme inertiel. Par ailleurs, les techniques (de compacité et monotonie) employées pour montrer l'existence de la solution sont classiques, connues depuis les années '60 [27]. Finalement, il faut démontrer que la pression récupérée via la technique de De Rahm vérifie la dernière condition dans (1.8). Même si la démonstration est assez élémentaire elle n'est pas immédiate et, à ma connaissance, n'existait pas dans la littérature.

Étape 2 En faisant le changement de variables $\widetilde{x_2} = \frac{x_2}{\varepsilon}, \widetilde{x_3} = \frac{x_3}{\varepsilon}$ on se ramène au domaine $\Omega = \{(x_1, \widetilde{x_2}, \widetilde{x_3}) \in \mathbb{R}^3 | \; x_1 \in (0,1), (\widetilde{x_2}, \widetilde{x_3}) \in S(x_1)\}$. Les estimations a priori montrées à l'étape précédente nous suggèrent d'introduire les développements asymptotiques suivants

$$\begin{cases} u_1 = \varepsilon^2 u_1^0 + \varepsilon^4 u_1^1 + \cdots + \varepsilon^{2n+2} u_1^n + \ldots, \\ u_k = \varepsilon^3 u_k^0 + \varepsilon^5 u_k^1 + \cdots + \varepsilon^{2n+3} u_k^n + \ldots, \qquad \forall\, k \in \{2,3\} \\ p = p^0 + \varepsilon^2 p^1 + \cdots + \varepsilon^{2n} p^n + \ldots, \end{cases}$$

où $u = (u_1, u_2, u_3)$. Cette parité des puissances s'explique par le fait que, à cause de la géometrie particulière, le vrai paramètre asymptotique est ε^2; de plus, le décalage entre u_1 et u_k est dû à la condition d'incompressibilité.
En utilisant le développement Taylor de η, et, en injectant tous ces développements dans le système on obtient, à l'ordre principal

$$\begin{cases} -\eta^0 \Delta_k u_1^0 + \partial_{x_1} p^0 = 0 & \text{sur } S(x_1), \\ u_1^0 = 0 & \text{sur } \partial S(x_1), \\ \partial_2 p^0 = \partial_3 p^0 = 0, \\ \partial_{x_1} \int_{S(x_1)} u_1^0 = 0, \; p|_{x_1=0} = p_0, \, p|_{x_1=1} = p_1, \end{cases} \tag{1.9}$$

où $\Delta_k = \partial_{x_2}^2 + \partial_{x_3}^2$, $\eta^0 = \eta(0)$. Observons que ces systèmes admettent une solution explicite, qui s'écrit

$$p^0(x_1) = p_0 + \frac{p_1 - p_0}{\int_0^1 \frac{dx_1}{\beta(x_1)}} \int_0^{x_1} \frac{dy_1}{\beta(y_1)},$$
$$u_1^0(x_1, x_k) = \frac{p_1 - p_0}{\int_0^1 \frac{dx_1}{\beta(x_1)}} \cdot \frac{W(x_1, x_k)}{\beta(x_1)},$$

où W est la solution du problème Dirichlet

$$\begin{cases} \Delta_k W = \frac{1}{\eta^0} & \text{sur } S(x_1), \\ W = 0 & \text{sur } \partial S(x_1) \end{cases}$$

$\eta^0 = \eta(0)$ et $\beta(x_1) = \int_{S(x_1)} W$.
Ensuite, on décrit soigneusement l'algorithme pour trouver de manière récurrente tous les u^n et p^n.

Étape 3 Finalement, on montre que les développements *formels* introduits à l'étape précédente sont valides. Plus précisement, on revient d'abord aux variables microscopiques (en faisant le changement inverse de variables $x_2 = \varepsilon \widetilde{x_2}, x_3 = \varepsilon \widetilde{x_3}$) et on considère les approximations U^n et P^n d'ordre n de u et p.

$$\begin{cases} U_1^n = \varepsilon^2 u_1^0 + \varepsilon^4 u_1^1 + \cdots + \varepsilon^{2n+2} u_1^n, \\ U_k^n = \varepsilon^3 u_k^0 + \varepsilon^5 u_k^1 + \cdots + \varepsilon^{2n+3} u_k^n, \qquad \forall\, k \in \{2,3\} \\ P^n = p^0 + \varepsilon^2 p^1 + \cdots + \varepsilon^{2n} p^n. \end{cases}$$

C'est ici que le choix des conditions au bord devient pertinent - on voit que U^n et P^n satisfont (1.8). Si on avait choisi d'autre conditions au bord cela n'aurait pas été vrai, car il n'y aurait pas eu moyen de s'assurer que U^n et P^n vérifient les conditions sur Σ_i^ε. Et, évidemment ce fait est essentiel pour démontrer le résultat principal, qui nous dit :

$$\|u - U^n\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^{2n+3},$$
$$\|p - P^n\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{2n+2},$$

pour une constante C indépendante de ε.

1.2.2 Chapitre 3

Dans ce chapitre, on étudie le comportement asymptotique des fluides visco-élastiques décrits par la loi d'Oldroyd dans des petits tubes. Pour les domaines axisymétriques une solution numérique est cherchée afin de la comparer à la solution obtenue via la technique asymptotique.
Nous allons travailler avec le modèle d'Oldroyd précédemment introduit avec un terme additif diffusif en contrainte $D_\varepsilon \Delta\sigma$ (ce modèle a été proposé par A.W. El-Kareh et L.G. Leal [12]). Effectivement, les expériences ont montré qu'il existe un tel terme – il est toutefois négligé habituellement étant trop petit par rapport aux autres termes de l'équation. En tenant compte de ce fait, nous allons supposer que $D_\varepsilon = \varepsilon^2$ - même s'il n'y a aucune justification physique de ce choix, nous affirmons qu'il est nécessaire pour obtenir un raccord asymptotique non-trivial.
Les équations adimensionnalisées s'écrivent :

$$\begin{cases} -(1-r)\Delta u + \mathrm{Re}(u\cdot\nabla)\cdot u + \nabla p = div\,\sigma + f_\varepsilon, \\ \mathrm{We}\big((u\cdot\nabla)\sigma + g(\sigma,\nabla u)\big) + \sigma - \varepsilon^2\Delta\sigma = 2rDu, \\ div\,u = 0, \end{cases} \tag{1.10}$$

où $\mathrm{Re} > 0$ est le nombre de Reynolds, $\mathrm{We} > 0$ est le nombre de Weissenberg, et

$$g(\sigma,\nabla u) = \big(\sigma\cdot Wu - Wu\cdot\sigma\big) + a(\sigma\cdot Du + Du\cdot\sigma).$$

Au niveau des conditions au bord, on considère:

$$\begin{cases} u = 0 \quad \text{sur} \quad \Gamma_\varepsilon, \\ \sigma = 0 \quad \text{sur} \quad \Gamma_\varepsilon, \\ u|_{\Sigma_0^\varepsilon} = u|_{\Sigma_1^\varepsilon}\,,\ \sigma|_{\Sigma_0^\varepsilon} = \sigma|_{\Sigma_1^\varepsilon}\,, \\ \partial_1 u|_{\Sigma_0^\varepsilon} = \partial_1 u|_{\Sigma_1^\varepsilon}\,,\ \partial_1\sigma|_{\Sigma_0^\varepsilon} = \partial_1\sigma|_{\Sigma_1^\varepsilon}\,, \\ p|_{\Sigma_0^\varepsilon} = p_0,\ p|_{\Sigma_1^\varepsilon} = p_1. \end{cases} \tag{1.11}$$

Dans la première partie nous montrons l'existence d'une solution faible (u,σ,p) en utilisant des méthodes classiques de compacité. La deuxième partie est similaire à celle du chapitre précédent – on écrit les développements formels de u, σ et p et on obtient, à l'ordre principal :

$$\begin{cases} -(1-r)\Delta_k u_1^0 + \partial_1 p_0 = f_1 + \partial_2 \sigma_{12}^0 + \partial_3 \sigma_{13}^0, \\ \sigma_{12}^0 - \Delta_k \sigma_{12}^0 = r\partial_2 u_1^0, \\ \sigma_{13}^0 - \Delta_k \sigma_{13}^0 = r\partial_3 u_1^0, \\ \partial_2 p_0 = \partial_3 p_0 = 0, \\ \partial_1 \int_{S(x_1)} u_1^0 = 0,\ p|_{x_1=0} = p_0, p|_{x_1=1} = p_1. \end{cases} \tag{1.12}$$

C'est ici qu'on voit la pertinence du coefficient du terme diffusif $\varepsilon^2\Delta\sigma$ - si on avait choisi une autre puissance de ε, on aurait obtenu un découplage de u et σ et donc on se serait ramené essentiellement au même modèle que dans le cas newtonien. Notons que ces systèmes n'ont pas une solution explicite (contrairement au cas quasi-newtonien), même si les géométries de $S(x_1)$ sont très simples. Par ailleurs, on détermine, par récurrence, tous les termes dans les développements asymptotiques de u, σ et p.
La troisième partie est dédiée à la démonstration des estimations d'erreur. Effectivement, on montre :

$$\begin{aligned} &\|u - U^n\|_{H^1(\Omega_\varepsilon)} \le C\varepsilon^{2n+3}, \\ &\|\sigma - S^n\|_{H^1(\Omega_\varepsilon)} \le C\varepsilon^{2n+2}, \\ &\|p - P^n\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^{2n+2}, \end{aligned}$$

où U^n, S^n et P^n sont les approximations d'ordre n de u, σ et p, et C est indépendante de ε.
Dans la dernière partie nous considérons le cas d'un domaine axisymétrique, c'est-à-dire un domaine 3D obtenu par la rotation d'un domaine 2D autour d'un axe. Nous montrons d'abord d'une manière rigoureuse que la solution dans un tel domaine est aussi axisymétrique (dans un sens bien précisé). Ensuite, nous présentons un algorithme numérique afin de trouver cette solution (cela devient possible grâce à la réduction de la dimension) et finalement nous la comparons à la solution de (1.12) afin de valider notre résultat.

1.2.3 Chapitre 4

Dans ce dernier chapitre, on s'intéresse à l'analyse mathématique du problème évolutif de Navier-Stokes, décrit par les équations:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) \cdot u + \nabla p = f, \\ div\, u = 0, \\ u(0) = u^0, \end{cases} \tag{1.13}$$

avec les conditions au bord (1.8). On voit que la seule différence entre le modèle ci-dessus et le modèle classique avec des conditions Dirichlet au bord est le fait qu'on

impose la pression p au lieu de la première composante de la vitesse u_1 sur Σ_i^ε. On verra que cette petite modification complique beaucoup l'analyse mathématique car on n'est plus capable d'employer les mêmes techniques pour montrer l'existence de la solution.
Les résultats obtenus sont divisés en deux parties :

- Dans un premier temps, nous travaillons dans le domaine renormalisé Ω – d'abord, nous montrons un résultat de régularité pour l'opérateur de Stokes $Au = -\Delta u + \nabla p$ associé aux conditions au bord (1.8). Effectivement, nous démontrons que, si $Au \in L^2(\Omega)$, alors $u \in H^2(\Omega)$ et, de plus

$$\|u\|_{H^2(\Omega)} \leq C(\|Au\|_{L^2(\Omega)} + p_0 + p_1).$$

  Bien évidemment, en faisant le changement

$$p \mapsto p + (p_0 - p_1)x_1 - p_0,$$

  il s'ensuit qu'on peut considérer $p_0 = p_1 = 0$. Dans ce cas, l'opérateur A est linéaire et, de plus

$$\|u\|_{H^2(\Omega)} \leq C\|Au\|_{L^2(\Omega)}.$$

  Par ailleurs, en utilisant le résultat ci-dessus et des estimations d'énergie, nous montrons l'existence d'une unique solution forte

$$u \in L^2(0,T; H^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega))$$

  pour des données initiales petites dans le sens où :

$$\|f\|^2_{L^2(0,T;L^2(\Omega))} + \|u^0\|^2_{H^1(\Omega)} \leq K, \tag{1.14}$$

  pour un $K = K(\Omega)$.

- Dans la deuxième partie nous revenons au domaine petit Ω_ε – dans ce cas il s'ensuit que la constante K dans (1.14) dépend de ε. Malheureusement, en utilisant la méthode ci-dessus, nous n'avons pas réussi à déterminer la dépendence de K par rapport à ε. Par conséquent, nous avons axé notre attention sur la théorie des semi-groupes d'opérateurs linéaires. Effectivement, si on définit

$$\mathcal{H}_\varepsilon = \{u \in (C^\infty(\Omega_\varepsilon))^3 \,|\, div\, u = 0,\ u = 0 \text{ on } \Gamma_\varepsilon,\ u \times n = 0 \text{ on } \Sigma_0^\varepsilon \cup \Sigma_1^\varepsilon\}$$

  et $H_\varepsilon = \overline{\mathcal{H}_\varepsilon}^{L^2}$, alors l'opérateur de Stokes

$$A_\varepsilon : D(A_\varepsilon) \subset H_\varepsilon \to H_\varepsilon, \quad A_\varepsilon = -P(\Delta),$$

  (où P est la projection sur H_ε) est le générateur d'un semi-groupe C_0 analytique dans H_ε, qu'on appelle e^{-tA_ε}. Par ailleurs, on réécrit le système (1.13) comme:

$$u(t) = e^{-tA_\varepsilon}u^0 + \int_0^t A_\varepsilon^{1/4} e^{-(t-s)A_\varepsilon} Bu(s)ds + \int_0^t e^{-(t-s)A_\varepsilon} Pf(s)ds, \tag{1.15}$$

où $Bu(t) = -PA_\varepsilon^{-1/4}(u(t) \cdot \nabla)u(t)$. Le résultat obtenu est le suivant : pour chaque $\delta > 0$, $\varepsilon < 1$ et $u^0 \in D(A^{1/2})$, $f \in L^\infty(0,\infty; D(A^{1/2}))$ tels que

$$\|A^{1/2}u^0\| \leq K_1\varepsilon^{\delta-1/2} \quad \text{et} \quad \|f\|_{L^\infty L^2(\Omega_\varepsilon)} \leq K_2\varepsilon^{\delta-3/2},$$

il existe une unique solution au problème (1.15) dans l'espace

$$Y = \{u \in L^\infty(0,\infty; D(A^{1/2})) \mid \sup_{t\geq 0} \|A^{1/2}u(t)\| \leq K_3\varepsilon^{\delta-1/2}\},$$

où K_1, K_2, K_3 sont des constantes absolues (indépendentes de ε et δ). On voit toute de suite la pertinence de ce résultat : pour ε très petit, les domaines admissibles qui assurent l'existence d'une solution unique deviennent *très larges*.

Chapter 2

Asymptotic behaviour of a class of incompressible, quasi-Newtonian fluids in thin pipes

2.1 Geometry and Notations

The notations introduced in this chapter will be valid throughout this thesis, apart from the second section in the last chapter where additional notations will be introduced. We consider a small parameter $\varepsilon > 0$ (throughout this material we shall always have $\varepsilon \leq 1$) and we suppose that the fluids flow through a simply connected, thin domain Ω_ε of the form

$$\Omega_\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 \in (0,1),\ (x_2, x_3) \in \varepsilon S(x_1)\},$$

where $S(x_1)$ are sufficiently regular domains – we also suppose that the application $x_1 \mapsto S(x_1)$ is smooth enough (C^2 regularity is sufficient). For the sake of simplicity we are going to assume $\Omega_\varepsilon \subset \Omega_\varepsilon^* = [0,1] \times [0,\varepsilon] \times [0,\varepsilon]$ – and the three-part boundary of Ω_ε is given by the following:

$$\begin{aligned}&\Gamma_\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 \in (0,1),\ (x_2, x_3) \in \varepsilon \partial S(x_1)\},\\ &\Sigma_0^\varepsilon = \varepsilon S(0),\ \Sigma_1^\varepsilon = \varepsilon S(1)\end{aligned}$$

(see below Figure 2.1). Moreover, we suppose that Σ_0^ε and Σ_1^ε are perpendicular to $e_1 = (1,0,0)$.

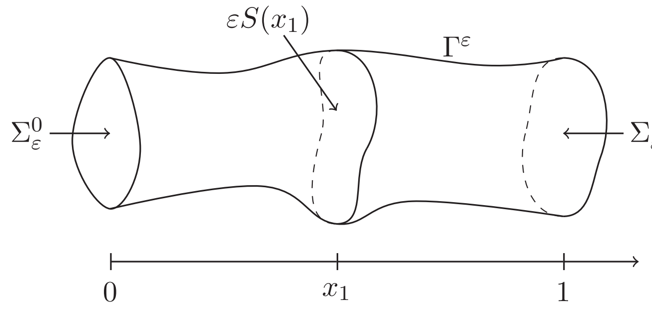


Figure 2.1: The thin domain

As always, n will denote the outward normal vector to $\partial\Omega_\varepsilon$. We make the conventions $\Omega_1 = \Omega$, $\Gamma_1 = \Gamma$, $\Sigma_i^1 = \Sigma_i$.
We use the classical Lebesgue spaces $L^p(\Omega_\varepsilon)$ $(1 \leq p \leq \infty)$, endowed with the usual norms $\|\cdot\|_{L^p}$, as well as the classical Sobolev spaces $H^s(\Omega_\varepsilon)$ $(s \in \mathbb{N})$ with their norms $\|\cdot\|_{H^s}$. For simplicity, we denote by $\|\cdot\|$ the L^2 norm since we use it quite frequently. Moreover, we denote by $H^{1/2}(\Omega_\varepsilon)$ the space of boundary traces of H^1 functions, and by $H^{-1/2}(\Omega_\varepsilon)$ its dual (see [18] for a detailed presentation).
Whenever X is a Banach space and $T > 0$, we shall denote $L^pX = L^p(0,T;X)$ for $1 \leq p \leq \infty$.
While the dot product "$\cdot$" is the standard notation in $\mathbb{R}^3$, for matrices we employ the matrix product

$$A : B = \sum_{i,j=1}^{3} a_{ij} b_{ij},$$

whenever $A = (a_{ij})$, $B = (b_{ij})$.
Furthermore, we make the following convention: if $v : \Omega_\varepsilon \to \mathbb{R}^3(\mathbb{R})$ then we will denote $\widetilde{v} : \Omega \to \mathbb{R}^3(\mathbb{R})$ the application defined by

$$\widetilde{v}(x_1, x_2, x_3) = v(x_1, \varepsilon x_2, \varepsilon x_3).$$

While we consistently use Einstein's summation convention, we shall only use k to denote summation (or repeated index) for $k \in \{2, 3\}$.
Finally, we denote by C various constants independent of ε.

2.2 Introduction of the problem

As already discussed, in this chapter we deal with incompressible quasi-newtonian fluids flowing through thin 3D pipes, with prescribed pressure at the ends. We shall suppose that the flow is stationary. The equations can be written as:

$$\begin{cases} (u \cdot \nabla)u + \nabla p = 2div\,(\eta(I(u))Du), \\ div\, u = 0, \end{cases} \tag{2.1}$$

with u the velocity, p the pressure and η the viscosity function depending on

$$I(u) = \sum_{i,j=1}^{3} (D_{ij}(u))^2,$$

where

$$D_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big).$$

While we shall work with a general η, we will later see how it applies to a realistic model (for instance, the case of Carreau fluids).
With respect to the boundary conditions, we impose

$$\begin{cases} u = 0 & \text{on} \quad \Gamma_\varepsilon, \\ u \times n = 0 & \text{on} \quad \Sigma_0^\varepsilon \cup \Sigma_1^\varepsilon, \\ p = p^i & \text{on} \quad \Sigma_i^\varepsilon. \end{cases} \tag{2.2}$$

with $p^0, p^1 > 0$ given constants. Throughout this thesis, we shall always denote the pressure drop

$$p_d = p^0 - p^1.$$

Note that the second condition is equivalent to $u_2 = u_3 = 0$ on $\Sigma_0^\varepsilon \cup \Sigma_1^\varepsilon$. Let us make the following elementary yet important observation: for every divergence free vector field u that satisfies (2.2) we can well define the flux of u to be the following quantity

$$\Phi_u := \int_{\varepsilon S(x_1)} u_1,$$

in the sense that the term on the right hand side is independent of x_1.

In the next section we prove the existence of the solution to problem (2.1)–(2.2), while in the final section we derive a complete asymptotic expansion and prove an error estimate in order to justify its validity.

2.3 The existence theorem

The proof of the existence of the solution is based on classical compactness and monotony arguments, and is very similar to the one in [29]. There are two main differences – the first one is that we use different boundary conditions, the importance of which shall become apparent when discussing the asymptotic expansion. On the other hand, there is the problem of controlling the inertial term. While the author of [29] uses a specific property of the viscosity function, we shall use the smallness of ε together with some sharp Poincaré and Sobolev type of inequalities for thin domains.
We say that (u, p) is a weak solution to problem (2.1)–(2.2) if it satisfies (2.1) in

the sense of distributions and (2.2) in the sense of traces.
We will suppose that $\eta : \mathbb{R}_+ \to \mathbb{R}$ is a function satisfying the following properties:

$$\eta \in C(\mathbb{R}_+), \tag{2.3}$$

$$0 < \eta_0 \le \eta(x) \le \eta_1 \qquad \forall\, x \in \mathbb{R}_+, \tag{2.4}$$

$$(\eta(x^2)x - \eta(y^2)y)(x-y) \ge 0 \qquad \forall\, x, y \in \mathbb{R}_+. \tag{2.5}$$

Let us note that (2.5) is equivalent to the fact that the mapping $x \mapsto \eta(x^2)x$ is increasing. In particular, this is true if η is differentiable and

$$\frac{d}{dx}\big(\eta(x^2)x\big) \ge 0, \quad \forall\, x \in \mathbb{R}_+. \tag{2.6}$$

In order to introduce the variational problem, let us consider the space

$$V = \{v \in H^1(\Omega_\varepsilon)^3 \,|\, div\, v = 0,\ v = 0 \ \text{ on } \ \Gamma_\varepsilon,\ v \times n = 0 \ \text{ on } \ \Sigma_0^\varepsilon \cup \Sigma_1^\varepsilon\}.$$

We will say that $u \in V$ is the solution of the variational problem if:

$$\int_{\Omega_\varepsilon} (u \cdot \nabla)u \cdot v + 2\int_{\Omega_\varepsilon} \eta(I(u))D_{ij}(u)D_{ij}(v) = p_d \Phi_v \qquad \forall\, v \in V. \tag{2.7}$$

It is a simple matter to verify that if (u, p) is a smooth solution to (2.1)–(2.2) then u is a solution of (2.7). Before passing to the statement and the proof of the converse, let us introduce two results that will prove useful:

Lemma 2.3.1. *If $f \in \big(L^2(\Omega_\varepsilon)\big)^3$ such that $div\, f \in L^{3/2}(\Omega_\varepsilon)$. Then $f \cdot n \in H^{-1/2}(\partial\Omega_\varepsilon)$ in the following sense*

$$\big\langle f \cdot n, \phi \big\rangle_{\partial\Omega_\varepsilon} = \int_{\Omega_\varepsilon} \phi\, div\, f + f \cdot \nabla\phi, \quad \forall\, \phi \in H^1(\Omega_\varepsilon)$$

(where $\big\langle \cdot, \cdot \big\rangle_{\partial\Omega_\varepsilon}$ denotes the pairing between $H^{-1/2}(\partial\Omega_\varepsilon)$ and $H^{1/2}(\partial\Omega_\varepsilon)$), meaning that the term on the right hand side is only dependent on the boundary trace value of ϕ and that it defines a linear continuous functional on $H^{1/2}(\partial\Omega_\varepsilon)$.

Proof. Using Exercise III.3.1 in [14] we can find a $g \in \big(W^{1,3/2}(\Omega_\varepsilon)\big)^3$ such that

$$div\, g = div\, f.$$

From Exercise II.4.3 in [14] we get that

$$\int_{\partial\Omega_\varepsilon} \phi g \cdot n = \int_{\Omega_\varepsilon} \phi\, div\, g + g \cdot \nabla\phi, \quad \forall\, \phi \in H^1(\Omega_\varepsilon).$$

By using the continuous inclusion of $H^{1/2}(\partial\Omega_\varepsilon)$ into $L^3(\partial\Omega_\varepsilon)$ it is immediate that the term on the left hand side defines a linear continuous functional on $H^{1/2}(\partial\Omega_\varepsilon)$.

Next, by standard Sobolev embeddings, we have $g \in \big(W^{1,3/2}(\Omega_\varepsilon)\big)^3 \subset \big(L^2(\Omega_\varepsilon)\big)^3$ and so $f - g \in \big(L^2(\Omega_\varepsilon)\big)^3$, while $div\,(f-g) = 0 \in \big(L^2(\Omega_\varepsilon)\big)^3$. Hence by using Theorem 2.5 in [18] we derive $(f-g)\cdot n \in H^{-1/2}(\partial\Omega_\varepsilon)$ and

$$\big\langle (f-g)\cdot n, \phi \big\rangle_{\partial\Omega_\varepsilon} = \int_{\Omega_\varepsilon} (f-g)\cdot\nabla\phi, \quad \forall\, \phi \in H^1(\Omega_\varepsilon).$$

Hence, for $h \in H^{1/2}(\partial\Omega_\varepsilon)$, then given $\phi_h \in H^1(\Omega_\varepsilon)$ such that $\phi_h = h$ on $\partial\Omega_\varepsilon$, the application

$$h \mapsto \int_{\Omega_\varepsilon} \phi_h\, div\, f + f\cdot\nabla\phi_h$$

is well defined and determines a linear continuous functional on $H^{1/2}(\partial\Omega_\varepsilon)$, being the sum of two applications enjoying this property. The proof is then completed. □

Lemma 2.3.2. *If $f \in L^2(\Omega_\varepsilon)$ and $a \in \big(H^{1/2}(\partial\Omega_\varepsilon)\big)^3$ such that*

$$\int_{\Omega_\varepsilon} f = \int_{\partial\Omega_\varepsilon} a\cdot n,$$

then there exists $v \in \big(H^1(\Omega_\varepsilon)\big)^3$ such that:

$$\begin{aligned}
&div\, v = f \quad on\ \Omega_\varepsilon,\\
&v = a \quad on\ \partial\Omega_\varepsilon,\\
&\|v\|_{H^1} \le C_\varepsilon(\|f\| + \|a\|_{H^{1/2}}).
\end{aligned}$$

Proof. See Exercise III.3.5 in [14]. □

Proposition 2.3.1. *Let u be a solution to* (2.7)*. Then there exists $p \in L^2(\Omega_\varepsilon)$ such that (u,p) satisfy* (2.1) *in the sense of distributions and* (2.2) *in the sense of traces.*

Proof. Let $\langle\cdot,\cdot\rangle$ be the pairing between V and its dual space V'. Since $\mathcal{V} = \{v \in \big(C_0^\infty(\Omega_\varepsilon)\big)^3,\ div\, v = 0\} \subset V$ it easily follows from (2.7) that

$$\langle -2div\,(\eta(I(u))Du) + (u\cdot\nabla)\cdot u - \overline{p_d},\ v\rangle = 0,$$

for all $v \in \mathcal{V}$, with the brackets understood in the distributional sense (clearly, $\overline{p_d} \in V'$ is given by $\langle\overline{p_d}, v\rangle = p_d\Phi_v$). A simple application of the de Rham theory enables us to retrieve the pressure $p \in L^2(\Omega_\varepsilon)$ such that (2.1) holds in the distributional sense. To conclude we need to verify the last boundary conditions in (2.2) (involving the pressure). At first glance it looks strange, since L^2 functions don't have boundary trace, but we shall see the exact sense in which these equalities are understood. Observe that

$$div\,\big(-2\eta(I(u))Du_i + pe_i\big) = -(u\cdot\nabla)u_i \in L^{3/2}(\Omega_\varepsilon), \quad \forall\, i \in \{1,2,3\},$$

and since $-2\eta(I(u))Du_i + u_i u + pe_i \in \big(L^2(\Omega_\varepsilon)\big)^3$ we can use Lemma 2.3.1 to get $\big(-2\eta(I(u))Du_i + pe_i\big)\cdot n \in H^{-1/2}(\partial\Omega_\varepsilon)$ and

$$\langle(-2\eta(I(u))Du_i+pe_i)\cdot n, \phi\rangle_{\partial\Omega_\varepsilon} = -\int_{\Omega_\varepsilon}\phi(u\cdot\nabla)u_i + \int_{\Omega_\varepsilon} -2\eta(I(u))D_{ij}u\partial_j\phi + p\partial_i\phi, \tag{2.8}$$

for all $\phi \in H^1(\Omega_\varepsilon)$ and all $i\in\{1,2,3\}$. Let $v\in V$ be arbitrary. By taking v_i in place of ϕ in (2.8) and summing for i, we derive, using (2.7) that

$$\langle -2\eta(I(u))\partial_1 u_1 + p, v_1\rangle_{\Sigma_0^\varepsilon} - \langle -2\eta(I(u))\partial_1 u_1 + p, v_1\rangle_{\Sigma_1^\varepsilon} = p^0\int_{\Sigma_0^\varepsilon} v_1 - p^1\int_{\Sigma_1^\varepsilon} v_1. \tag{2.9}$$

Next, we show that $\partial_1 u\cdot n = 0$ in $H^{-1/2}(\partial\Omega_\varepsilon)$. First, observe that $\partial_1 u \in L^2(\Omega_\varepsilon)$ and $div\,(\partial_1 u) = 0$, so that, by Lemma 2.3.1, $\partial_1 u\cdot n \in H^{-1/2}(\partial\Omega_\varepsilon)$ and

$$\langle \partial_1 u\cdot n, \phi\rangle_{\partial\Omega_\varepsilon} = \int_{\Omega_\varepsilon}\partial_1 u\cdot\nabla\phi, \quad \forall\,\phi\in H^1(\Omega_\varepsilon).$$

It remains to show that the right hand side is 0. We prove for $\phi \in C^2(\overline{\Omega_\varepsilon})$, and conclude by density argument. We have

$$\begin{aligned}\int_{\Omega_\varepsilon}\partial_1 u\cdot\nabla\phi &= -\int_{\Omega_\varepsilon} u\cdot\nabla\partial_1\phi + \int_{\partial\Omega_\varepsilon} u\cdot\nabla\phi n_1\\ &= \int_{\Omega_\varepsilon} div\, u\,\partial_1\phi - \int_{\partial\Omega_\varepsilon}\partial_1\phi u\cdot n + \int_{\partial\Omega_\varepsilon} u\cdot\nabla\phi n_1,\end{aligned}$$

but, since $u\in V$, and $n_2 = n_3 = 0$ on Σ_0^ε, Σ_1^ε the conclusion readily follows.
So it is clear $\partial_1 u_1 = 0$ on $\Sigma_0^\varepsilon, \Sigma_1^\varepsilon$, and we can rewrite (2.9) as

$$\langle p - p^0, v_1\rangle_{\Sigma_0^\varepsilon} = \langle p - p^1, v_1\rangle_{\Sigma_1^\varepsilon}.$$

Let $w_0 \in H^{1/2}(\Sigma_0^\varepsilon)$, $w_1\in H^{1/2}(\Sigma_1^\varepsilon)$ be such that $\int_{\Sigma_0^\varepsilon} w_0 = \int_{\Sigma_1^\varepsilon} w_1$. By using Lemma 2.3.2, we can find a $v\in V$ such that $v_1 = w_i$ on Σ_i^ε.
For all $h\in L^2(\Sigma_i^\varepsilon)$, call T_h^i the functional $u\mapsto \int_{\Sigma_i^\varepsilon} hu$ in $H^{-1/2}(\Sigma_i^\varepsilon)$; extend this notion for elements in $H^{-1/2}(\Sigma_i^\varepsilon)$. The above result can be translated to

$$T_1^0 w_0 = T_1^1 w_1 \implies T_{p-p^0}^0 w_0 = T_{p-p^1}^1 w_1.$$

It follows that $\ker T_1^0 \subset \ker T_{p-p^0}^0$, so that $\left(\ker T_{p-p^0}^0\right)^\perp \subset \left(\ker T_1^0\right)^\perp$. But $\dim\left(\ker T_1^0\right)^\perp = 1$, which leads to $\dim\left(\ker T_{p-p^0}^0\right)^\perp \in \{0,1\}$. Either way, $T_{p-p^0}^0 = kT_1^0 = T_k^0$ for some $k \in \mathbb{R}$. Similarly, $T_{p-p^1}^1 = T_k^1$ with the same k. Consequently,

$$\begin{cases} p = p^0 + k & \text{on} \quad \Sigma_0^\varepsilon, \\ p = p^1 + k & \text{on} \quad \Sigma_1^\varepsilon, \end{cases}$$

thus ending the proof. □

In virtue of a Poincaré inequality – see Lemma 2.3.5 below – it follows that $\|\nabla u\|$ is a norm on V equivalent to the H^1 norm – and from now on we will consider V endowed with this norm.

A simple calculation – integrating twice by parts – shows that if $u \in \mathcal{X} = \{v \in (C^2(\overline{\Omega_\varepsilon}))^3 \mid v = 0 \text{ on } \Gamma_\varepsilon,\, v \times n = 0 \text{ on } \Sigma_0^\varepsilon \cup \Sigma_1^\varepsilon\}$ then

$$\sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} \partial_{x_i} u_j \partial_{x_j} u_i = \int_{\Omega_\varepsilon} (div\, u)^2.$$

Since $\mathcal{X}$ is dense in

$$X = \{v \in (H^1(\Omega_\varepsilon))^3 \mid v = 0 \text{ on } \Gamma_\varepsilon,\, v \times n = 0 \text{ on } \Sigma_0^\varepsilon \cup \Sigma_1^\varepsilon\},$$

it follows that, for all $u \in V$,

$$\sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} \partial_{x_i} u_j \partial_{x_j} u_i = 0,$$

so we can write

$$\int_{\Omega_\varepsilon} (D_{ij}(u))^2 = \frac{1}{2} \int_{\Omega_\varepsilon} (\partial_{x_i} u_j)^2 = \frac{\|\nabla u\|^2}{2}.$$

Let us consider the mapping $A : V \to V'$ defined by

$$\langle A(u), v\rangle = 2 \int_{\Omega_\varepsilon} \eta(I(u)) D_{ij}(u) D_{ij}(v) \qquad \forall\, u, v \in V.$$

Lemma 2.3.3. *The operator A is continuous and monotone, i.e. satisfies*

$$\langle A(u) - A(u'),\, u - u'\rangle \geq 0, \qquad \forall\, u, u' \in V.$$

Proof. First we prove continuity. Let $u_n \to u$ in V. Obviously we can choose a subsequence u_m such that

$$I(u_m) \to I(u) \quad \text{a.e.} \tag{2.10}$$

We can write

$$|\langle A(u_m) - A(u), v\rangle| \le 2\int_{\Omega_\varepsilon} \big|\big(\eta(I(u_m)) - \eta(I(u))\big) D_{ij}(u) D_{ij}(v)\big|$$
$$+ 2\int_{\Omega_\varepsilon} \big|\eta(I(u_m))(D_{ij}(u_m) - D_{ij}(u)) D_{ij}(v)\big|.$$

We look at the two terms separately. We have

$$2\int_{\Omega_\varepsilon} \big|\big(\eta(I(u_m)) - \eta(I(u))\big) D_{ij}(u) D_{ij}(v)\big| \le \alpha_m \|\nabla v\|, \tag{2.11}$$

with

$$\alpha_m = 6\Big(\int_{\Omega_\varepsilon} \big(\eta(I(u_m)) - \eta(I(u))\big)^2 I(u)\Big)^{1/2} \to 0, \tag{2.12}$$

owing to the continuity and the boundedness of η, (2.10) and the use of Lebesgue's dominated convergence theorem. Secondly,

$$2\int_{\Omega_\varepsilon} \big|\eta(I(u_m))(D_{ij}(u_m) - D_{ij}(u)) D_{ij}(v)\big| \le \eta_1 \|\nabla(u_m - u)\| \|\nabla v\|. \tag{2.13}$$

Therefore from (2.11),(2.12) and (2.13) we derive

$$A(u_m) \to A(u) \quad \text{in } V'. \tag{2.14}$$

We have actually proven more. From any subsequence u_k of u_n we can choose a subsequence u_m for which (2.14) holds. Therefore we have $A(u_n) \to A(u) \quad \text{in } V'$, showing the continuity of A.

To prove the monotony of A, let us write

$$\langle A(u) - A(u'),\, u - u'\rangle = 2\int_{\Omega_\varepsilon} \Big[\eta(I(u))I(u) + \eta(I(u'))I(u')$$
$$- \big(\eta(I(u)) + \eta(I(u'))\big) D_{ij}(u) D_{ij}(u')\Big].$$

From the elementary Cauchy Schwarz inequality

$$D_{ij}(u) D_{ij}(u') \le \sqrt{I(u)I(u')}$$

we deduce

$$\langle A(u) - A(u'),\, u - u'\rangle \ge 2\int_{\Omega_\varepsilon} \Big(\eta(I(u))\sqrt{I(u)} - \eta(I(u'))\sqrt{I(u')}\Big)\Big(\sqrt{I(u)} - \sqrt{I(u')}\Big),$$

and the conclusion follows from (2.5). $\square$

Define now $L : V \to V'$ by

$$\langle L(u), v\rangle = \int_{\Omega_\varepsilon} (u\cdot\nabla)u\cdot v, \quad \forall\, v \in V.$$

Lemma 2.3.4. *If $u_n \to u$ weakly in V, then $L(u_n) \to L(u)$ weakly in V'.*

Proof. Taking into account the compact inclusion of V into $(L^4(\Omega_\varepsilon))^3$ we deduce that we can choose a subsequence u_k such that $u_k \to u$ strongly in $(L^4(\Omega_\varepsilon))^3$. Pick any $v \in V$. We can write

$$|\langle L(u_k) - L(u), v\rangle| \le \left|\int_{\Omega_\varepsilon} ((u_k - u)\cdot\nabla)u_k\cdot v\right| + \left|\int_{\Omega_\varepsilon} (u\cdot\nabla)(u_k - u)\cdot v\right|.$$

The second term is convergent to 0 since the mapping $z \mapsto \int_{\Omega_\varepsilon}(u\cdot\nabla)z\cdot v$ is linear continuous on V, while for the first one we can write

$$\left|\int_{\Omega_\varepsilon} ((u_k - u)\cdot\nabla)u_k\cdot v\right| \le \|\nabla u_k\|\|u_k - u\|_{L^4}\|v\|_{L^4},$$

and since u_k is bounded being weakly convergent, we obtain that $\langle L(u_k), v\rangle \to \langle L(u), v\rangle$. Reasoning as in Lemma 2.3.3, we get the desired result. $\square$

We continue with a technical result, that will prove very useful

Lemma 2.3.5. *For any $u \in H^1(\Omega_\varepsilon)$ such that $u = 0$ on Γ_ε in the sense of traces, the following inequalities hold:*

$$\|u\| \le \varepsilon\|\nabla u\|,$$
$$\|u\|_{L^4} \le c_0\varepsilon^{1/4}\|\nabla u\|,$$

Moreover, for all $u \in V$ we have

$$|\Phi_u| \le \varepsilon^2\|\nabla u\|.$$

Proof. The first is a classical Poincaré inequality in thin domains – see, for instance Proposition 2.1. in [43].

The proof for the second inequality is largely inspired from Ladyzhenskahya – see [26]. The sharper constant is obtained for our specific geometry - there is also a small adjustment to make up for the lack of Dirichlet boundary conditions on Σ_i^ε. Suppose $u \in \mathcal{X}_1$, where

$$\mathcal{X}_1 = \{u \in C^1(\Omega_\varepsilon), |\ \operatorname{supp} u \subset \Omega_\varepsilon \cup \Sigma_0^\varepsilon \cup \Sigma_1^\varepsilon\},$$

that is the functions which are 0 near Γ^ε. The conclusion will follow through a density argument. Evidently, we can extend the function u to be 0 on $\Omega_\varepsilon^* \setminus \Omega_\varepsilon$. We can write

$$\int_{\Omega_\varepsilon} u^4 \le \int_0^1 dx_1 \int_0^\varepsilon \max_{x_2} u^2 dx_3 \int_0^\varepsilon \max_{x_3} u^2 dx_2. \tag{2.15}$$

We have

$$u^2(x_1,x_2,x_3) = 2\int_0^{x_3} u(x_1,x_2,y_3)\partial_{x_3}u(x_1,x_2,y_3)dy_3 \le 2\int_0^{\varepsilon} |u\partial_{x_3}u|dx_3.$$

Hence, by Cauchy Schwarz we get

$$\max_{x_3} u^2 \le 2\Big(\int_0^{\varepsilon} u^2dx_3\Big)^{1/2}\Big(\int_0^{\varepsilon}(\partial_{x_3}u)^2dx_3\Big)^{1/2}.$$

By integrating with respect to x_2 and using Cauchy Schwarz again we obtain

$$\int_0^{\varepsilon}\max_{x_3} u^2dx_2 \le 2\Big(\int_{S_\varepsilon} u^2dx'\Big)^{1/2}\Big(\int_{S_\varepsilon}(\partial_{x_3}u)^2dx'\Big)^{1/2},$$

where $S_\varepsilon = [0,\varepsilon]\times[0,\varepsilon]$ and $dx' = dx_2dx_3$. Similarly, one can prove

$$\int_0^{\varepsilon}\max_{x_2} u^2dx_3 \le 2\Big(\int_{S_\varepsilon} u^2dx'\Big)^{1/2}\Big(\int_{S_\varepsilon}(\partial_{x_2}u)^2dx'\Big)^{1/2}, \tag{2.16}$$

and so by using (2.15) we get

$$\int_{\Omega_\varepsilon} u^4 \le 4\max_{x_1}\int_{S_\varepsilon} u^2dx'\Big(\int_{\Omega_\varepsilon}(\partial_{x_2}u)^2\Big)^{1/2}\Big(\int_{\Omega_\varepsilon}(\partial_{x_3}u)^2\Big)^{1/2}. \tag{2.17}$$

For all $z_1\in[0,1]$ we have

$$\begin{aligned} u^2(x_1,x_2,x_3) &= u^2(z_1,x_2,x_3) + 2\int_{z_1}^{x_1} u(y_1,x_2,x_3)\partial_{x_1}u(y_1,x_2,x_3)dy_1 \\ &\le u^2(z_1,x_2,x_3) + 2\int_0^1 |u\partial_{x_1}u|dx_1. \end{aligned}$$

By integrating with respect to z_1 it readily follows that

$$u^2 \le \int_0^1 u^2dx_1 + 2\Big(\int_0^1 u^2dx_1\Big)^{1/2}\Big(\int_0^1(\partial_{x_1}u)^2dx_1\Big)^{1/2},$$

and so we obtain

$$\max_{x_1}\int_{S_\varepsilon} u^2 \le \int_{\Omega_\varepsilon} u^2 + 2\Big(\int_{\Omega_\varepsilon} u^2\Big)^{1/2}\Big(\int_{\Omega_\varepsilon}(\partial_{x_1}u)^2\Big)^{1/2}. \tag{2.18}$$

From (2.17) and (2.18) we derive

$$\int_{\Omega_\varepsilon} u^4 \le 4(\|u\|^2 + 2\|u\|\|\partial_{x_1}u\|)\|\partial_{x_2}u\|\|\partial_{x_3}u\|,$$

and with some elementary algebraic manipulations (as well as the Poincaré inequality above) we get

$$\int_{\Omega_\varepsilon} u^4 \leq \varepsilon\left(\frac{8}{3\sqrt{3}} + 2\varepsilon\right)\|\nabla u\|^4,$$

and, since we have assumed that $\varepsilon \leq 1$ we get that

$$c_0 = \sqrt[4]{\frac{8}{3\sqrt{3}} + 2}.$$

The last inequality is immediate if we consider that

$$|\Phi_u| = \left|\int_0^1 \Phi_u\right| = \left|\int_{\Omega_\varepsilon} u\right| \leq \varepsilon\|u\| \leq \varepsilon^2\|\nabla u\|,$$

where we have used Cauchy Schwarz and the Poincaré inequality previously proved. □

Theorem 2.3.1. *Assume that η satisfies* (2.3)–(2.5). *Provided $\varepsilon > 0$ is small enough, there exists a solution u of the variational problem* (2.7) *satisfying*

$$\|\nabla u\| \leq \frac{2p_d\varepsilon^2}{\eta_0}. \tag{2.19}$$

Proof. Since V is a separable space, let $\{z_1, z_2, \dots\}$ be a dense basis in V; moreover let V_n be the finite dimensional subspace generated by $\{z_1, \dots, z_n\}$. We begin by showing that the following problem: Find $u \in V_n$ such that

$$\langle A(u) + L(u), v\rangle = p_d\Phi_v \qquad \forall\, v \in V_n$$

has at least one solution $u_n \in V_n$.
Let us consider the applications $P_n : V_n \to V_n$ defined by

$$(P_n(w), v) = \langle A(w) + L(w), v\rangle - p_d\Phi_v \qquad \forall\, w, v \in V_n.$$

We intend to use the following classical result, whose proof can be found in [27].

Lemma 2.3.6. *Let $P : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping such that, for some $\rho > 0$,*

$$(P(\xi), \xi) \geq 0 \quad \forall\xi \quad \text{such that} \quad |\xi| = \rho.$$

Then there exists ξ with $|\xi| \leq \rho$ such that $P(\xi) = 0$.

By Lemma 2.3.5 we get

$$\begin{aligned}
&|\langle L(w), w\rangle| \le \int_{\Omega_\varepsilon} |(w\cdot\nabla)w\cdot w| \le c_0^2\varepsilon^{1/2}\|\nabla w\|^3 \le c_0^2\|\nabla w\|^3,\\
&|p_d\Phi_v| \le p_d\varepsilon^2\|\nabla w\|.
\end{aligned}$$

Also we have

$$\langle A(w), w\rangle = 2\int_{\Omega_\varepsilon} \eta(I(w))I(w) \ge \eta_0\|\nabla w\|^2,$$

so we obtain

$$(P_n(w), w) \ge \eta_0\|\nabla w\|^2 - c_0^2\|\nabla w\|^3 - p_d\varepsilon^2\|\nabla w\|.$$

Hence, if $\varepsilon \le \frac{\eta_0}{2\sqrt{p_d}c_0}$, then for

$$\|\nabla w\| = \rho_\varepsilon = \frac{\eta_0 - \sqrt{\eta_0^2 - 4c_0^2\varepsilon^2}}{2c_0^2}$$

$(P_n(w), w) \ge 0$, and so we can apply Lemma 2.3.6 to find $u_n \in V_n$, $\|\nabla u_n\| \le \rho_\varepsilon$ with $P_n(u_n) = 0$. Clearly ρ_ε is independent of n so we can choose a subsequence still denoted by u_n such that

$$\begin{aligned}
u_n &\to u \quad \text{weakly in } V,\\
u_n &\to u \quad \text{strongly in } L^4.
\end{aligned}$$

Moreover we have

$$\|\nabla u\| \le \liminf \|\nabla u_n\| \le \rho_\varepsilon = \frac{2p_d\varepsilon^2}{\eta_0 + \sqrt{\eta_0^2 - 4c_0^2\varepsilon^2}} \le \frac{2p_d\varepsilon^2}{\eta_0}. \tag{2.20}$$

Clearly $A(u_n)$ is bounded in V', hence we can choose another subsequence such that

$$A(u_n) \to \chi \quad \text{weakly in } V'. \tag{2.21}$$

Let us now fix $n \ge 1$. Then for all $m \ge n$ we have

$$\langle A(u_m), z_n\rangle + \langle L(u_m), z_n\rangle = p_d\Phi_{z_n}.$$

By Lemma 2.3.4 and (2.21) we can pass to the limit as $m \to \infty$ to get

$$\langle \chi, z_n\rangle + \langle L(u), z_n\rangle = p_d\Phi_{z_n}.$$

Since this is true for all n and $\{z_1, z_2, \dots\}$ is dense in V, it easily follows

$$\langle \chi, v\rangle + \langle L(u), v\rangle = p_d\Phi_v \quad \forall\, v \in V. \tag{2.22}$$

To conclude, it remains to prove that $A(u) = \chi$. Since u_n is the solution of the approximate problem, we can write

$$\langle A(u_n), u_n \rangle + \langle L(u_n), u_n \rangle = p_d \Phi_{u_n}.$$

Using similar arguments as those in the proof of Lemma 2.3.4 we derive $\langle L(u_n), u_n \rangle \to \langle L(u), u \rangle$, and so

$$\lim_{n\to\infty} \langle A(u_n), u_n \rangle = p_d \Phi_u - \langle L(u), u \rangle. \tag{2.23}$$

By Lemma 2.3.3, we have

$$\langle A(u_n),\, u_n - v \rangle \geq \langle A(v),\, u_n - v \rangle, \quad \forall\, v \in V.$$

Using (2.23) we can pass to the limit in the above equation to get

$$-\langle L(u), u \rangle - p_d \Phi_u - \langle \chi, v \rangle \geq \langle A(v),\, u - v \rangle, \quad \forall\, v \in V.$$

By taking $v = u$ in (2.22) we get $\langle \chi, u \rangle = p_d \Phi_u$, so that

$$\langle \chi - A(v),\, u - v \rangle \geq 0, \quad \forall\, v \in V,$$

which can be further written as

$$t \langle \chi - A(u + tv), v \rangle \leq 0, \quad \forall\, t \in \mathbb{R},\, v \in V.$$

Using the continuity of A, we derive the desired result. □

2.4 Asymptotic expansion

By making the change of variables $x_k \mapsto \frac{x_k}{\varepsilon}$, we can rewrite the equations in the rescaled domain Ω as:

$$\begin{cases} \widetilde{u}_1 \partial_{x_1} \widetilde{u}_1 + \dfrac{\widetilde{u}_k \partial_{x_k} \widetilde{u}_1}{\varepsilon} + \partial_{x_1} \widetilde{p} = 2\partial_{x_1}(\widetilde{\eta} \partial_{x_1} \widetilde{u}_1) + \dfrac{\partial_{x_k}\big(\widetilde{\eta}(\partial_{x_1} \widetilde{u}_k + \frac{\partial_{x_k} \widetilde{u}_1}{\varepsilon})\big)}{\varepsilon}, \\ \widetilde{u}_1 \partial_{x_1} \widetilde{u}_2 + \dfrac{\widetilde{u}_k \partial_{x_k} \widetilde{u}_2}{\varepsilon} + \dfrac{\partial_{x_2} \widetilde{p}}{\varepsilon} = \partial_{x_1}\big(\widetilde{\eta}(\partial_{x_1} \widetilde{u}_2 + \dfrac{\partial_{x_2} \widetilde{u}_1}{\varepsilon})\big) + \dfrac{\partial_{x_k}(\widetilde{\eta}(\partial_{x_2} \widetilde{u}_k + \partial_{x_k} \widetilde{u}_2))}{\varepsilon^2}, \\ \widetilde{u}_1 \partial_{x_1} \widetilde{u}_3 + \dfrac{\widetilde{u}_k \partial_{x_k} \widetilde{u}_3}{\varepsilon} + \dfrac{\partial_{x_3} \widetilde{p}}{\varepsilon} = \partial_{x_1}\big(\widetilde{\eta}(\partial_{x_1} \widetilde{u}_3 + \dfrac{\partial_{x_3} \widetilde{u}_1}{\varepsilon})\big) + \dfrac{\partial_{x_k}(\widetilde{\eta}(\partial_{x_k} \widetilde{u}_3 + \partial_{x_3} \widetilde{u}_k))}{\varepsilon^2}, \\ \partial_{x_1} \widetilde{u}_1 + \dfrac{\partial_{x_k} \widetilde{u}_k}{\varepsilon} = 0, \end{cases}$$

with $\widetilde{\eta} = \eta(I(\widetilde{u}))$. An elementary calculation and Lemma 2.3.5 show that

$$\|\widetilde{u}\|_{L^2(\Omega)} = \frac{1}{\varepsilon} \|u\| \leq \|\nabla u\| \leq C\varepsilon^2.$$

The above estimate, as well as fair intuitions leads us to *formally* write

$$\begin{cases} \widetilde{u}_1 = \varepsilon^2 \widetilde{u}_1^0 + \varepsilon^4 \widetilde{u}_1^1 + \cdots + \varepsilon^{2n+2} \widetilde{u}_1^n + \dots, \\ \widetilde{u}_k = \varepsilon^3 \widetilde{u}_k^0 + \varepsilon^5 \widetilde{u}_k^1 + \cdots + \varepsilon^{2n+3} \widetilde{u}_k^n + \dots, \qquad \forall\, k \in \{2,3\} \\ \widetilde{p} = \widetilde{p}^0 + \varepsilon^2 \widetilde{p}^1 + \cdots + \varepsilon^{2n} \widetilde{p}^n + \dots, \end{cases}$$

where $\widetilde{u}_\alpha^m$, $\widetilde{p}^m$ are independent of ε for all $m \geq 0$, $\alpha \in \{1,2,3\}$. The boundary conditions are expected to become

$$\begin{cases} \widetilde{u}_\alpha^m = 0 & \text{on } \Gamma \quad \forall\, m \geq 0,\ \alpha \in \{1,2,3\}, \\ \widetilde{u}_k^m = 0 & \text{on } \Sigma_0 \cup \Sigma_1 \quad \forall\, m \geq 0,\ k \in \{2,3\}, \\ \widetilde{p}^0(i) = p^i & \text{on } \Sigma_i, \\ \widetilde{p}^m(i) = 0 & \text{on } \Sigma_i \quad \forall\, m \geq 1. \end{cases} \tag{2.24}$$

Writing (formally) the Taylor expansion of η in 0 and plugging in the above expansion, we arrive at

$$\widetilde{\eta} = \widetilde{A}_0 + \varepsilon^2 \widetilde{A}_1 + \cdots + \varepsilon^{2n} \widetilde{A}_n + \dots,$$

where $\widetilde{A}_n$ depends only on $\widetilde{u}^i$ for $i < n$. Note $\widetilde{A}_0 = \eta(0) := \eta^0$.
Plugging in the expansions and identifying the coefficients of ε^{-1} in the second equation, of ε^0 in the first equation and taking (2.24) into account we get $\partial_{x_2} \widetilde{p}^0 = \partial_{x_3} \widetilde{p}^0 = 0$ and

$$\begin{cases} -\eta^0 \Delta_k \widetilde{u}_1^0 + \partial_{x_1} \widetilde{p}^0 = 0 & \text{on } S(x_1), \\ \widetilde{u}_1^0 = 0 & \text{on } \partial S(x_1). \end{cases} \tag{P_1^0}$$

where Δ_k denotes the Laplacian in two variables – $\Delta_k = \partial_{x_2}^2 + \partial_{x_3}^2$. Observe that we can solve for $\widetilde{u}_1^0(x_1, x_k) = \partial_{x_1} \widetilde{p}^0(x_1) W(x_1, x_k)$, where W is the unique solution of the problem

$$\begin{cases} \Delta_k \widetilde{v} = \frac{1}{\eta^0} & \text{on } S(x_1), \\ \widetilde{v} = 0 & \text{on } \partial S(x_1), \end{cases}$$

To find $\widetilde{p}^0$ let us use (a priori) the following compatibility relation

$$\int_{S(x_1)} \partial_{x_1} \widetilde{u}_1^0 = 0. \tag{C_0}$$

If we define

$$\beta(x_1) = \int_{S(x_1)} W(x_1, x_k),$$

then

$$\widetilde{p}^0(x_1) = p^0 + \frac{p^1 - p^0}{\int_0^1 \frac{dx_1}{\beta(x_1)}} \int_0^{x_1} \frac{dy_1}{\beta(y_1)},$$

$$\widetilde{u}_1^0(x_1, x_k) = \frac{p^1 - p^0}{\int_0^1 \frac{dx_1}{\beta(x_1)}} \cdot \frac{W(x_1, x_k)}{\beta(x_1)}.$$

Due to the imposed regularity requirements we have $\widetilde{u}_1^0 \in C^2(\overline{\Omega})$ and $\widetilde{p}^0 \in C^1(\overline{\Omega})$. It is easy to check that $(\widetilde{u}_1^0, \widetilde{p}^0)$ satisfy (P_1^0) in the classical sense, and that $\widetilde{u}_1^0$ verifies (C_0) for all x_1. The key ingredient for that – and a property which we shall further use – is that whenever $\widetilde{w} \in C^1(\overline{\Omega})$ satisfies $\widetilde{w} = 0$ on Γ, in virtue of Reynolds' transport theorem

$$\int_{S(x_1)} \partial_{x_1} \widetilde{w} = \partial_{x_1} \int_{S(x_1)} \widetilde{w}, \qquad \forall\, x_1.$$

Moreover, we have $\widetilde{p}^0(i) = p^i$.
To determine all $\widetilde{u}_\alpha^m$ we proceed by complete induction. Let $j \geq 1$. Assume we have found (sufficiently) smooth $\widetilde{u}_1^i$ such that $\int_{S(x_1)} \partial_{x_1} \widetilde{u}_i^0 = 0$ for all $i < j$ and $\widetilde{u}_k^i$ for all $i < j-1$ (nothing if $j = 1$). We show how to determine $\widetilde{u}_1^j, \widetilde{u}_k^{j-1}, \widetilde{p}^j$ such that

$$\int_{S(x_1)} \partial_{x_1} \widetilde{u}_1^j = 0. \tag{C_j}$$

Again, plugging in the expansion and identifying in the second and third equation the coefficients of ε^{2j-1} and in the last one those of ε^{2j}, together with (2.24), we obtain

$$\begin{cases} -\eta^0 \partial_{x_k}^2 \widetilde{u}_2^{j-1} + \partial_{x_2} \widetilde{p}^j = \widetilde{f}_2^j & \text{on } S(x_1), \\ -\eta^0 \partial_{x_k}^2 \widetilde{u}_3^{j-1} + \partial_{x_3} \widetilde{p}^j = \widetilde{f}_3^j & \text{on } S(x_1), \\ \partial_{x_k} \widetilde{u}_k^{j-1} = -\partial_{x_1} \widetilde{u}_1^{j-1} & \text{on } S(x_1), \\ \widetilde{u}_2^{j-1} = \widetilde{u}_3^{j-1} = 0 & \text{on } \partial S(x_1), \end{cases} \tag{P_k^{j-1}}$$

with

$$\begin{aligned} \widetilde{f}_l^j = &\sum_{i+m=j-2} \partial_{x_1}(\widetilde{A}_i \partial_{x_1} \widetilde{u}_l^m) + \sum_{\substack{i+m=j-1\\ i\geq 1}} \big(\partial_{x_1}(\widetilde{A}_i \partial_{x_l} \widetilde{u}_1^m) + 2\partial_{x_l}(\widetilde{A}_i \partial_{x_l} \widetilde{u}_l^m)\big) \\ &+ \sum_{\substack{i+m=j-1\\ i\geq 1}} \partial_{x_l}(\widetilde{A}_i \partial_{x_2} \widetilde{u}_3^m + \partial_{x_3} \widetilde{u}_2^m) - \sum_{i+m=j-3} \big(\widetilde{u}_1^i \partial_{x_1} \widetilde{u}_l^m + \widetilde{u}_k^i \partial_{x_k} \widetilde{u}_l^m\big). \end{aligned}$$

for $l \in \{2, 3\}$. Observe that $\widetilde{f}_l^j$ are known, since they depend on terms $\widetilde{u}_1^i$ with $i < j$ and $\widetilde{u}_k^i$ with $i < j-1$ - moreover they are smooth by the induction hypothesis. In virtue of the compatibility relation

$$\int_{S(x_1)} \partial_{x_1} \widetilde{u}_{j-1}^0 = 0,$$

it follows (see [14]) that the Stokes system has a unique regular solution $(\widetilde{u}_k^{j-1}, \widetilde{p}^j)$ that satisfies (P_k^{j-1}) in the classical sense. Note that $\widetilde{p}^j$ is determined up to a constant (for every x_1). Hence we can write

$$\widetilde{p}^j(x_1, x_k) = \widetilde{q}^j(x_1, x_k) + \widetilde{r}^j(x_1),$$

where $\widetilde{q}^j$ is fixed and $\widetilde{r}^j$ is to be determined.
Finally, plugging the expansion in the first equation and identifying the coefficients of ε^{2j} and using once again (2.24) we get

$$\begin{cases} -\eta^0 \partial^2_{x_k} \widetilde{u}^j_1 = -\partial_{x_1} \widetilde{r}^j + \widetilde{f}^j_1 & \text{on } S(x_1), \\ \widetilde{u}^j_1 = 0 & \text{on } \partial S(x_1), \end{cases} \tag{P^j_1}$$

with

$$\widetilde{f}^j_1 = \sum_{i+m=j-1} \big(2\partial_{x_1}(\widetilde{A}_i \partial_{x_1} \widetilde{u}^m_1) + \partial_{x_k}(\widetilde{A}_i \partial_{x_1} \widetilde{u}^m_k)\big) + \sum_{\substack{i+m=j \\ i\geq 1}} \partial_{x_k}(\widetilde{A}_i \partial_{x_k} \widetilde{u}^m_1)$$
$$- \sum_{i+m=j-2} \big(\widetilde{u}^i_1 \partial_{x_1} \widetilde{u}^m_1 + \widetilde{u}^i_k \partial_{x_k} \widetilde{u}^m_1\big) - \partial_{x_1} \widetilde{q}^j.$$

We can separate into two problems

$$\begin{cases} -\Delta_k \widetilde{v} = \frac{1}{\eta^0} \widetilde{f}^j_1 & \text{on } S(x_1), \\ \widetilde{v} = 0 & \text{on } \partial S(x_1), \end{cases}$$

and

$$\begin{cases} -\Delta_k \widetilde{v} = -\frac{1}{\eta^0} \partial_{x_1} \widetilde{r}^j & \text{on } S(x_1), \\ \widetilde{v} = 0 & \text{on } \partial S(x_1). \end{cases}$$

The first problem is a homogeneous Dirichlet, and so it has a uniquely determined solution, say $\widetilde{u}^{j,1}_1$. Observe that the second problem has the solution $\widetilde{u}^{j,2}_1 = \partial_{x_1} \widetilde{r}^j W$, where W is as previously introduced. Note that if

$$\int_{S(x_1)} \widetilde{u}^{j,1}_1 = \gamma(x_1),$$

then we define

$$\widetilde{r}^j(x_1) = C_1 + \int_0^{x_1} \frac{C - \gamma(x)}{\beta(x)} dx,$$

where C, C_1 can be determined uniquely by the restrictions $\widetilde{r}^j(0) = \widetilde{r}^j(1) = 0$. Hence we have completely found

$$\widetilde{u}^j_1 = \widetilde{u}^{j,1}_1 + \widetilde{u}^{j,2}_1.$$

Again, it is fairly easy to see that $\widetilde{u}^j_1$ is regular, satisfies (P^j_1) in the classical sense and also the condition (C_j). This ends the induction step, and thereby we have determined $\widetilde{u}^n$, $\widetilde{p}^n$ for all $n \geq 0$.
Before we continue, we introduce some terminology. We call Ω perfect near the boundary – or (PNB) – if there exists $\delta > 0$ such that $S(x_1) = S(0)$ for all $x_1 \in (0, \delta)$ and $S(x_1) = S(1)$ for all $x_1 \in (1 - \delta, 1)$. Given Ω a (PNB) domain

we say that $\widetilde{u} \in C(\overline{\Omega})$ has the (PNB) property if $\widetilde{u}(x_1, x_k) = \widetilde{u}(0, x_k)$ for all $x_1 \in (0, \delta)$, $x_k \in S(0)$ and $\widetilde{u}(x_1, x_k) = \widetilde{u}(1, x_k)$ for all $x_1 \in (1-\delta, 1)$, $x_k \in S(1)$. Whenever these terms are 0 we say that $\widetilde{u}$ is 0 near the boundary. Obviously, if $\widetilde{u} \in C^m(\overline{\Omega})$ has the (PNB) property then all the derivatives up to the order m enjoy the same property; moreover $\partial_{x_1}^{\alpha_1} \partial_{x_k}^{\alpha_k} \widetilde{u} = 0$ whenever $\alpha_1 > 0$. Finally, it is trivial to see that sum/product of functions with the (PNB) property has it also, while if one terms is 0 near the boundary then so is the product.

Lemma 2.4.1. *Assume that Ω is (PNB). Then*

$$\widetilde{u}_k^j(0) = \widetilde{u}_k^j(1) \equiv 0 \quad \forall j \geq 0$$
$$\widetilde{p}^j(0) = \widetilde{p}^j(1) \equiv 0 \quad \forall j \geq 1.$$

Proof. We shall prove by induction the following: $\widetilde{u}_1^j, \widetilde{u}_k^j, \widetilde{q}^j$ have the (PNB) property, $\widetilde{u}_k^j(0) = \widetilde{u}_k^j(1) \equiv 0$ for all $j \geq 0$ and $\widetilde{p}^j(0) = \widetilde{p}^j(1) \equiv 0 \quad \forall j \geq 1$. Observe that, by definition, $\widetilde{u}_1^0$ has the (PNB) property. This implies $\partial_{x_1} \widetilde{u}_1^0 = 0$ near the boundary, so that the Stokes system (P_k^{j-1}) with $j = 1$ has only trivial solution near the boundary – which completes the first step of induction. Next, assume we have been able to prove the statement for all $i < j$. Looking at $\widetilde{f}_1^j$, note that it has the (PNB) property, being a combination of terms that have the (PNB) property – it follows easily that $\widetilde{u}_1^j$ has the (PNB) property. For $\widetilde{f}_k^{j+1}$ more can be told – not only does it have the (PNB) property, but, since it is a combination of the terms of the form $\widetilde{u}_k^i$ and $\partial_{x_1} \widetilde{u}_\alpha^i$ $(\alpha \in \{1,2,3\}, i < j)$ then it is 0 near the boundary – hence the system (P_k^{j-1}) (with $j+1$ in place of j) has only trivial solution near the boundary, thus completing the induction. $\square$

Let

$$u_1^n(x_1, x_k) = \sum_{j=0}^{n} \varepsilon^{2j+2} \widetilde{u}_1^j\Big(x_1, \frac{x_k}{\varepsilon}\Big),$$
$$u_l^n(x_1, x_k) = \sum_{j=0}^{n} \varepsilon^{2j+3} \widetilde{u}_l^j\Big(x_1, \frac{x_k}{\varepsilon}\Big), \quad l \in \{2,3\},$$
$$p^n(x_1, x_k) = \sum_{j=0}^{n} \varepsilon^{2j} \widetilde{p}^j\Big(x_1, \frac{x_k}{\varepsilon}\Big).$$

Observe that, if Ω is (PNB), owing to Lemma 2.4.1, $u^n \in V \cap C^2(\overline{\Omega_\varepsilon})$ and $p^n \in C^1(\overline{\Omega_\varepsilon})$ with $p^n(i) = p^i$ for all $n \geq 0$. We can now state the main result in this section:

Theorem 2.4.1. *Let $n \in \mathbb{N}$ be fixed. Suppose that Ω is (PNB), η satisfies* (2.3)-(2.5) *and, in addition,*

$$\exists\, c > 0 \text{ such that } |\eta(a) - \eta(b)| \leq c|\sqrt{a} - \sqrt{b}| \qquad \forall\, a, b \geq 0, \tag{2.25}$$
$$\eta \in C^{n+1}(\mathbb{R}_+). \tag{2.26}$$

Then the following estimates hold:

$$\|\nabla(u-u^n)\| \le C\varepsilon^{2n+4}, \tag{2.27}$$
$$\|p-p^n\| \le C\varepsilon^{2n+2}.$$

Proof. In order to simplify the calculations let us introduce $A(i), u_1(i), u_k(i) : \Omega_\varepsilon \to \mathbb{R}$ for $i \le n$ defined by

$$A(i)(x_1,x_k) = \varepsilon^{2i}\widetilde{A}_i\Big(x_1, \frac{x_k}{\varepsilon}\Big),$$
$$u_1(i)(x_1,x_k) = \varepsilon^{2i+2}\widetilde{u}_1^i\Big(x_1, \frac{x_k}{\varepsilon}\Big),$$
$$u_l(i)(x_1,x_k) = \varepsilon^{2i+3}\widetilde{u}_l^i\Big(x_1, \frac{x_k}{\varepsilon}\Big), \quad l \in \{2,3\}.$$

Multiplying the equation (P_1^j) by ε^{2j} and adding for $j \in \{0,\dots,n\}$ we obtain

$$\sum_{i+j\le n-1} \Big(2\partial_{x_1}\big(A(i)\partial_{x_1}u_1(j)\big) + \partial_{x_k}\big(A(i)\partial_{x_1}u_k(j)\big)\Big) + \sum_{i+j\le n} \partial_{x_k}(A(i)\partial_{x_k}u_1(j)) -$$
$$- \sum_{i+j\le n-2} \big(u_1(i)\partial_{x_1}u_1(j) + u_k(i)\partial_{x_k}u_1(j)\big) - \partial_{x_1}p^n = 0.$$

Using (2.26) we can write the Taylor development (up to the order n, with the Lagrange remainder) of η in 0, and so we derive – properly this time

$$\eta(I(u^n)) = B_0 + \varepsilon^2 B_1 + \cdots + \varepsilon^{2N} B_N$$

for some $N > n$ (actually, $N = (2n+3)(n+1)$). It is an easy matter to verify that $B_i = A_i$ for $i \le n$. If $(u^n \cdot \nabla)u^n = w^n$ and $2div\,(\eta(I(u^n))Du^n) = t^n$ then a simple calculation shows

$$w_1^n = \sum_{i,j=0}^{n} \big(u_1(i)\partial_{x_1}u_1(j) + u_k(i)\partial_{x_k}u_1(j)\big),$$

$$t_1^n = \sum_{i\ge 0, j\le n} \Big(2\partial_{x_1}\big(B(i)\partial_{x_1}u_1(j)\big) + \partial_{x_k}\big(B(i)\partial_{x_1}u_k(j)\big)\Big) + \sum_{i\ge 0, j\le n} \partial_{x_k}(B(i)\partial_{x_k}u_1(j)),$$

where

$$B(i)(x_1,x_k) = \varepsilon^{2i}\widetilde{B}_i\Big(x_1, \frac{x_k}{\varepsilon}\Big).$$

Therefore, we can write

$$w_1^n + \partial_{x_1}p^n = t_1^n + R_1^n,$$

where the rest is given by

$$R_1^n = \sum_{i+j\ge n-1} \big(u_1(i)\partial_{x_1}u_1(j) + u_k(i)\partial_{x_k}u_1(j)\big) - \sum_{\substack{i+j>n \\ j\le n}} \partial_{x_k}(B(i)\partial_{x_k}u_1(j))$$
$$- \sum_{\substack{i+j\ge n \\ j\le n}} \big(2\partial_{x_1}(B(i)\partial_{x_1}u_1(j) + \partial_{x_k}(B(i)\partial_{x_1}u_k(j))\big).$$

Observe that all the terms in the above development contain powers of ε of at least $2n+2$, and since we proved all $\widetilde{u}_\alpha^m$ are sufficiently regular (at least $C^2(\overline{\Omega_\varepsilon})$), we can write $\|R_1^n\|_{L^\infty(\Omega_\varepsilon)} \le C\varepsilon^{2n+2}$. In a completely similar manner it can be proved that for $k \in \{2,3\}$ we have

$$w_k^n + \partial_{x_k} p^{n+1} = t_k^n + R_k^n,$$

with $\|R_k^n\|_{L^\infty(\Omega_\varepsilon)} \le C\varepsilon^{2n+3}$. Combining the three equations we obtain

$$(u^n \cdot \nabla)u^n + \nabla p^{n+1} = 2div\,(\eta(I(u^n))Du^n) + R^n, \tag{2.28}$$

with the rest satisfying

$$\|R^n\|_{L^\infty(\Omega_\varepsilon)} \le C\varepsilon^{2n+2}. \tag{2.29}$$

Multiplying (2.28) by a test function $v \in V$ and integrating over Ω_ε we obtain

$$\int_{\Omega_\varepsilon} (u^n \cdot \nabla)u^n \cdot v + 2\int_{\Omega_\varepsilon} \eta(I(u^n))D_{ij}(u^n)D_{ij}(v) - p_d\Phi_v = \int_{\Omega_\varepsilon} R^n \cdot v, \qquad \forall\, v \in V. \tag{2.30}$$

By taking in (2.7) and (2.30) $v = u - v^n$ and subtracting the two so obtained eqalities we get

$$\begin{aligned} &2\int_{\Omega_\varepsilon} \big(\eta(I(u))D_{ij}(u) - \eta(I(u^n))D_{ij}(u^n)\big)D_{ij}(u-u^n) = \\ &\qquad = \int_{\Omega_\varepsilon} \big((u^n\cdot\nabla)u^n - (u\cdot\nabla)u\big)\cdot(u-u^n) - \int_{\Omega_\varepsilon} R^n\cdot(u-u^n). \end{aligned} \tag{2.31}$$

Note that

$$\begin{aligned} &\left|\int_{\Omega_\varepsilon} \big((u^n-u)\cdot\nabla u^n\big)\cdot(u-u^n)\right| \le C\varepsilon^{5/2}\|\nabla(u^n-u)\|^2, \\ &\left|\int_{\Omega_\varepsilon} \big(u\cdot\nabla(u^n-u)\big)\cdot(u-u^n)\right| \le C\varepsilon^{5/2}\|\nabla(u^n-u)\|^2, \\ &\left|\int_{\Omega_\varepsilon} R^n\cdot(u-u^n)\right| \le C\varepsilon^{2n+4}\|\nabla(u^n-u)\|, \end{aligned}$$

using the inequalities in Lemma 2.3.5, (2.29) and the elementary following inequality

$$\|\nabla u^n\| \le C\varepsilon^2,$$

so that if C_n is the term on right hand side in (2.31) then

$$|C_n| \le \frac{\eta_0}{3}\|\nabla(u-u^n)\|^2 + C\varepsilon^{2n+4}\|\nabla(u-u^n)\|, \tag{2.32}$$

provided ε is small enough. Again, if D_n is the term on the left hand side in (2.31) then

$$D_n = 2\int_{\Omega_\varepsilon} \eta(I(u))D_{ij}^2(u-u^n) - 2\int_{\Omega_\varepsilon} \big(\eta(I(u)) - \eta(I(u^n))\big)D_{ij}(u^n)D_{ij}(u-u^n). \tag{2.33}$$

Using (2.25) in (2.33) and taking into account the elementary $\|D_{ij}(u^n)\|_{L^\infty} \le C\varepsilon$, it follows – after using a Cauchy-Schwarz inequality that

$$\begin{aligned} D_n &\ge \eta_0\|\nabla(u-u^n)\|^2 - C\varepsilon\|\sqrt{I(u)} - \sqrt{I(u^n)}\|\|\nabla(u-u^n)\| \\ &\ge \eta_0\|\nabla(u-u^n)\|^2 - C\varepsilon\|\nabla(u-u^n)\|^2 \ge \frac{2\eta_0}{3}\|\nabla(u-u^n)\|^2 \end{aligned} \tag{2.34}$$

for small enough ε – to derive the second part of the above inequality we have used the trivial

$$|\sqrt{I(u)} - \sqrt{I(u^n)}| \le \sqrt{\sum_{1\le i,j\le 3}\big(D_{ij}(u-u^n)\big)^2}.$$

Combining (2.32) and (2.34) we get

$$\|\nabla(u-u^n)\| \le C\varepsilon^{2n+4}. \tag{2.35}$$

In order to obtain the pressure estimate, we are going to use the following result:

Lemma 2.4.2. *Let $f : \Omega_\varepsilon \to \mathbb{R}$ with $f \in L^2(\Omega_\varepsilon)$ and*

$$\int_{\Omega_\varepsilon} f = 0.$$

Then there exists $\phi \in \big(H_0^1(\Omega_\varepsilon)\big)^3$ such that

$$\begin{aligned} div\,\phi &= f \quad on \quad \Omega_\varepsilon, \\ \|\nabla\phi\| &\le C\varepsilon^{-1}\|f\|. \end{aligned}$$

Proof of Lemma 2.4.2. As usual, let $\tilde f : \Omega \to \mathbb{R}$ be given by $\tilde f(x_1, x_k) = f(x_1, \varepsilon x_k)$. Since $\int_\Omega \tilde f = 0$ we can use Lemma 2.3.2 to find a $\widetilde\psi : \Omega \to \mathbb{R}^3$ with $\widetilde\psi \in \big(H_0^1(\Omega)\big)^3$ and $div\,\widetilde\psi = \tilde f$. We can now define

$$\phi(x_1, x_k) = \Big(\widetilde\psi_1\big(x_1, \frac{x_k}{\varepsilon}\big),\ \varepsilon\widetilde\psi_2\big(x_1, \frac{x_k}{\varepsilon}\big),\ \varepsilon\widetilde\psi_3\big(x_1, \frac{x_k}{\varepsilon}\big)\Big).$$

Clearly $div\,\phi = f$ and $\phi = 0$ on $\partial\Omega_\varepsilon$. To conclude, let us observe that

$$\|\nabla\phi\| \le \|\nabla\widetilde\psi\| \qquad \text{and} \qquad \|f\| = \varepsilon\|\tilde f\|,$$

so that the conclusion is achieved, with the constant explicitly given by

$$C = \frac{\|\nabla\widetilde{\psi}\|}{\|\tilde{f}\|}.$$

$\square$

Returning to the proof of the theorem, let us note that, since p is defined up to a constant we can choose it such that

$$\int_{\Omega_\varepsilon} p - p^{n+1} = 0.$$

From Lemma 2.4.2, it follows that there exists $\phi_n \in \left(H_0^1(\Omega_\varepsilon)\right)^3$ such that:

$$\begin{aligned} &div\, \phi_n = p - p^{n+1} \quad \text{on} \quad \Omega_\varepsilon, \\ &\|\nabla\phi_n\| \le C\varepsilon^{-1}\|p - p^{n+1}\|. \end{aligned}$$

Consider the first equation in (2.1) – understood in the distributional sense – and (2.28). By subtracting the two equalities and then applying $\phi_n \in \left(H_0^1(\Omega_\varepsilon)\right)^3$ we obtain

$$\begin{aligned} \|p - p^{n+1}\|^2 + 2\int_{\Omega_\varepsilon} &\big(\eta(I(u))D_{ij}(u) - \eta(I(u^n))D_{ij}(u^n)\big)D_{ij}(\phi_n) = \\ &= \int_{\Omega_\varepsilon} \big((u^n\cdot\nabla)u^n - (u\cdot\nabla)u\big)\cdot\phi_n - \int_{\Omega_\varepsilon} R^n\cdot\phi_n. \end{aligned}$$

Proceeding in a very similar manner to the way we determined the velocity estimates – while also using these estimates – it is easy to establish

$$\|p - p^{n+1}\| \le C\varepsilon^{2n+3},$$

from which the pressure estimate follows. Together with (2.35), this completes the proof. $\square$

We conclude by making some observations.

- Consider the concrete case of a Carreau fluid, whose viscosity law can be expressed as

  $$\eta_C(x) = \eta_\infty + (\eta_0 - \eta_\infty)(1+\lambda x)^{\frac{r-1}{2}}, \quad \lambda > 0,\ 0 < \eta_\infty \le \eta_0.$$

  Note that condition (2.26) is verified for all $n \in \mathbb{N}$, while if $r \le 1$ we have

  $$\eta_\infty \le \eta(x) \le \eta_0 \qquad \forall\, x \ge 0,$$

and, if $\varphi(x) = \eta_C(x^2)$ then

$$|\varphi'(x)| = (1-r)\lambda(\eta_0 - \eta_\infty)x(1+\lambda x^2)^{\frac{r-3}{2}} \le \frac{(1-r)\sqrt{\lambda}(\eta_0 - \eta_\infty)}{2},$$

which easily implies that (2.4) and (2.25) are satisfied. Lastly, if $\psi(x) = x\eta_C(x^2)$ then

$$\psi'(x) > (\eta_0 - \eta_\infty)(1+\lambda x^2)^{\frac{r-3}{2}}(1 + r\lambda x^2),$$

from which follows that, if $r \ge 0$, (2.5) is also verified. Hence, provided $r \in [0,1]$, all the conditions are satisfied, so the results proved can be applied for these types of fluids.

- It is easy to see that, if we add (2.25) to the conditions verified by η in Theorem 2.3.1, we can prove that the solution is unique. The proof mimics the one for the velocity estimate in Theorem 2.4.1.

- The (PNB) condition on Ω is sufficient – whether is it necessary or not it is not clear. However, some geometric condition is definitely required to obtain the desired result. To see this, let us suppose that $S(x_1) = (1+x_1)S(0)$, with $S(0) = D\big((\frac{1}{2}, \frac{1}{2}); \frac{1}{4}\big)$ (the open disk centred at $(\frac{1}{2}, \frac{1}{2})$ with radius 1/4) – see below Figure 2.2.

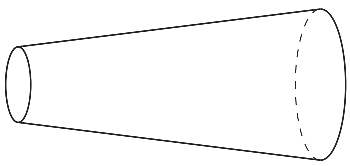


Figure 2.2: Domain without the (PNB) property.

We are going to show that, in spite of the simplicity of the geometry, the estimate (2.27) does not hold even for $n = 0$. Assume the contrary. An elementary calculation leads us to

$$\widetilde{u}_1^0(x_1, x_k) = a_1 \cdot \left[\frac{(x_2 - \frac{1}{2})^2 + (x_3 - \frac{1}{2})^2}{(1+x_1)^4} - \frac{1}{(1+x_1)^2}\right]$$

for some constant $a_1 \neq 0$ so that

$$\partial_{x_1}\widetilde{u}_1^0(0, x_k) = a_2\big((x_2 - \frac{1}{2})^2 + (x_3 - \frac{1}{2})^2\big) + a_3 \not\equiv 0,$$

(since $a_2, a_3 \neq 0$) and so the Stokes system (P_k^{j-1}), for $j = 1$ and $x_1 = 0$, has non-trivial solution. Assume, with no loss of generality, that $\widetilde{u}_2^0(0) \not\equiv 0$. It is immediate to see that

$$\|\nabla(\widetilde{u} - \widetilde{u}^0)\|_0 \leq \|\nabla(u - u^0)\| \leq C\varepsilon^3,$$

and so the trace inequality yields

$$\|\widetilde{u}_2 - \widetilde{u}_2^0\|_{L^2(S(0))} \leq C\varepsilon^3.$$

But since this holds for all $\varepsilon > 0$ and the member on left hand side and C are independent of ε, it follows that $\widetilde{u}_2^0(0) = \widetilde{u}_2(0) \equiv 0$, which is a contradiction.

Chapter 3

Stationary diffusive Oldroyd model in thin pipes

3.1 Introduction and Statement of the problem

In this chapter we study the stationary flow of an incompressible Oldroyd fluid with diffusive stress of order ε^2 in a thin tube with diameter to length ratio of order ε. Firstly, we prove the existence of a solution for our problem - its uniqueness will be shown later under additional hypothesis on ε. In terms of actually finding this solution, since the full 3D model is very difficult to deal with, we shall discuss two ways of reducing its complexity. The first method is the same as the one used in the previous chapter: it involves writing the asymptotic expansion of the rescaled solution with respect to ε, formally computing its coefficients and finally showing the validity of this expansion by proving a convergence result. The second possibility is to reduce the 3D problem to a 2D one in the particular case of an axisymmetric domain. Two numerical algorithms for solving this reduced problem are presented, and the corresponding solutions are compared to the one obtained via the first method.
The equations describing the model presented in the general introduction can be written as

$$\begin{cases} -(1-r)\Delta u + \mathrm{Re}(u\cdot\nabla)\cdot u + \nabla p = div\,\sigma + f_\varepsilon, \\ \mathrm{We}\big((u\cdot\nabla)\sigma + g_a(\sigma,\nabla u)\big) + \sigma - \varepsilon^2\Delta\sigma = 2rDu, \\ div\,u = 0, \end{cases} \tag{3.1}$$

where

$$g_a(\sigma,\nabla u) = \big(\sigma\cdot Wu - Wu\cdot\sigma\big) + a(\sigma\cdot Du + Du\cdot\sigma),$$

with

$$Wu = \frac{\nabla u - (\nabla u)^T}{2}, \qquad Du = \frac{\nabla u + (\nabla u)^T}{2},$$

and where the unknowns are the velocity u, the pressure p and the stress tensor σ. By f_ε we have denoted the exterior forces acting on the fluid. Here $\mathrm{Re} > 0$ is

the Reynolds number, We >0 the Weissenberg number, $r \in (0,1)$ the retardation parameter, and $a \in [-1,1]$.
We suppose that the domain Ω_ε has the (PNB) property introduced in the previous chapter.
As for the boundary conditions we consider

$$\begin{cases} u = 0 \quad \text{on} \quad \Gamma_\varepsilon, \\ \sigma = 0 \quad \text{on} \quad \Gamma_\varepsilon, \\ u_{|\Sigma_0^\varepsilon} = u_{|\Sigma_1^\varepsilon},\ \sigma_{|\Sigma_0^\varepsilon} = \sigma_{|\Sigma_1^\varepsilon}, \\ \partial_1 u_{|\Sigma_0^\varepsilon} = \partial_1 u_{|\Sigma_1^\varepsilon},\ \partial_1 \sigma_{|\Sigma_0^\varepsilon} = \partial_1 \sigma_{|\Sigma_1^\varepsilon} \\ p_{|\Sigma_0^\varepsilon} - p_{|\Sigma_1^\varepsilon} = p_d. \end{cases} \tag{3.2}$$

For the remainder of this chapter, we will make two assumptions:

1. The first assumption is about the parameter $a \in [-1,1]$ introduced in order to describe different objective (frame indifferent) time derivatives. For technical reason, we will assume
$$a = 0.$$
From a physical point of view, this choice corresponds to the Jaumann (or co-rotational) derivative. This restriction is used to obtain the existence of a stationary solution in the next section. Except for this theoretical section, it seems that the more general case $a \neq 0$ can be treated in the same way. In particular, the asymptotic model obtained in the case $\varepsilon \to 0$ is independent of the value of a.

2. The second assumption relates to the the force term f_ε. We assume that it satisfies one of the following conditions:

   (F1) There exist smooth $\widetilde{f} : \Omega_\varepsilon \to \mathbb{R}^3$ such that $f_\varepsilon(x_1, x_k) = \widetilde{f}(x_1, \frac{x_k}{\varepsilon})$.

   (F2) $\sup_\varepsilon \|f_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} < \infty$.

3.2 The existence problem

The proof of the existence of a weak solution is based on classical energy estimates and compactness arguments. We proceed very carefully to show that some boundary conditions are verified in a very weak sense. Define $H^1_{per}(\Omega_\varepsilon) = \{v \in H^1(\Omega_\varepsilon) |\, v_{|\Sigma_0^\varepsilon} = v_{|\Sigma_1^\varepsilon}\}$. Consider the following spaces

$$V = \{v \in \left(H^1_{per}(\Omega_\varepsilon)\right)^3 \,|\, div\, v = 0,\ v = 0 \quad \text{on} \quad \Gamma_\varepsilon\},$$
$$W = \{\tau \in \mathbb{M}_{3\times 3}\left(H^1_{per}(\Omega_\varepsilon)\right) \,|\, \tau_{ij} = \tau_{ji},\ \tau = 0 \quad \text{on} \quad \Gamma_\varepsilon\}.$$

The variational problem can be written as: Find $(u,\sigma) \in V \times W$ such that, for all $(v,\tau) \in V \times W$,

$$\begin{cases} (1-r)\int_{\Omega_\varepsilon} \nabla u : \nabla v + \mathrm{Re}\int_{\Omega_\varepsilon} (u\cdot\nabla)u\cdot v + \int_{\Omega_\varepsilon} \sigma : Dv = \int_{\Omega_\varepsilon} f_\varepsilon v + p_d \Phi_v, \\ \mathrm{We}\int_{\Omega_\varepsilon} \big((u\cdot\nabla)\sigma + g_0(\sigma,\nabla u)\big) : \tau + \int_{\Omega_\varepsilon} \sigma : \tau + \varepsilon^2 \int_{\Omega_\varepsilon} \nabla\sigma : \nabla\tau = 2r\int_{\Omega_\varepsilon} Du : \tau. \end{cases} \tag{3.3}$$

Evidently, every classical solution

$$(u,\sigma,p) \in \big(C^2(\Omega_\varepsilon)\cap C^1(\overline{\Omega_\varepsilon})\big) \times \big(C^2(\Omega_\varepsilon)\cap C^1(\overline{\Omega_\varepsilon})\big) \times \big(C^1(\Omega_\varepsilon)\cap C(\overline{\Omega_\varepsilon})\big)$$

that verifies (3.1) and (3.2) is a solution to (3.3). The goal is to prove that there exists a solution to the problem (3.3), which is also a weak solution to the problem (3.1)–(3.2).

Theorem 3.2.1. *For all $\varepsilon > 0$ the problem* (3.3) *admits at least one solution $(u,\sigma) \in V \times W$, which satisfies*

$$\|\nabla u\| \le C\varepsilon^2, \quad \|\nabla\sigma\| \le C\varepsilon. \tag{3.4}$$

Proof. We consider the Hilbert space $V \times W$ endowed with the scalar product:

$$\big\langle (u_1,\sigma_1),(u_2,\sigma_2)\big\rangle = (u_1,u_2)_V + \varepsilon^2(\sigma_1,\sigma_2)_W,$$

where

$$(u_1,u_2)_V = \int_{\Omega_\varepsilon} \nabla u_1 : \nabla u_2, \quad (\sigma_1,\sigma_2)_W = \int_{\Omega_\varepsilon} \nabla\sigma_1 : \nabla\sigma_2.$$

Let $\{v_1,\dots,v_n,\dots\}$ be a dense basis in V, $\{\tau_1,\dots,\tau_n,\dots\}$ be a dense basis in W, and let V_n, W_n be the subspaces generated by $\{v_1,\dots,v_n\}$ and $\{\tau_1,\dots,\tau_n\}$ respectively. We show that the approximate problem: find $(u_n,\sigma_n) \in V_n \times W_n$ such that (3.3) holds for all $(v,\tau) \in V_n \times W_n$ has a solution satisfying (3.4). Define the application $P_n : V_n \times W_n \to V_n \times W_n$ by

$$\begin{aligned} \big\langle P_n(u,\sigma),(v,\tau)\big\rangle = 2r\bigg((1-r)\int_{\Omega_\varepsilon} \nabla u : \nabla v + \mathrm{Re}\int_{\Omega_\varepsilon} (u\cdot\nabla)u\cdot v + \int_{\Omega_\varepsilon} \sigma : Dv \\ - \int_{\Omega_\varepsilon} f_\varepsilon v - p_d\Phi_v\bigg) + \bigg(\mathrm{We}\int_{\Omega_\varepsilon} \big((u\cdot\nabla)\sigma + g_0(\sigma,\nabla u)\big) : \tau + \int_{\Omega_\varepsilon} \sigma : \tau \\ + \varepsilon^2 \int_{\Omega_\varepsilon} \nabla\sigma : \nabla\tau - 2r\int_{\Omega_\varepsilon} Du : \tau\bigg). \end{aligned}$$

We have the following

$$\int_{\Omega_\varepsilon} (u\cdot\nabla)u\cdot u = \int_{\Omega_\varepsilon} (u\cdot\nabla)\sigma : \sigma = \int_{\Omega_\varepsilon} \big(\sigma\cdot W(u) - W(u)\cdot\sigma\big) : \sigma = 0,$$

for all $(u, \sigma) \in V_n \times W_n$. By using Lemma 2.3.5 we can write

$$\int_{\Omega_\varepsilon} \big|f_\varepsilon u\big| + p_d \Phi_u \leq 2r\varepsilon^2 (C_f + p_d) \|\nabla u\|,$$

where $C_f = \|\tilde{f}\|$ in the case (F1), while $C_f = \sup_\varepsilon \|f_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}$ in the case (F2). Hence if $\xi = (u, \sigma)$ (with $\|\xi\|^2 = \|\nabla u\|^2 + \varepsilon^2 \|\nabla \sigma\|^2$), we have

$$\big\langle P_n(\xi), \xi \big\rangle \geq 2r(1-r)\|\xi\|^2 - \varepsilon^2 (C_f + p_d)\|\xi\|.$$

So for

$$\|\xi\| = \frac{C_f + p_d}{1-r} \varepsilon^2,$$

we have $\big\langle P_n(\xi), \xi \big\rangle \geq 0$. Since P_n is clearly continuous, using Lemma 2.3.6 we derive the existence of $(u_n, \sigma_n) \in V_n \times W_n$ with $P_n(u_n, \sigma_n) = 0$, and, in addition, it follows easily

$$\|\nabla u_n\| \leq \frac{C_f + p_d}{1-r} \varepsilon^2, \qquad \|\nabla \sigma_n\| \leq \frac{C_f + p_d}{1-r} \varepsilon. \tag{3.5}$$

Since $P_n(u_n, \sigma_n)$ was constructed as a sum of two linear independent functionals, it follows that (u_n, σ_n) is a solution to the approximate problem above mentioned. Using (3.5) and the compact embedding of H^1 into L^4 we have, on a subsequence,

$$u_m \to u \quad \text{weakly in } V, \tag{3.6a}$$

$$\sigma_m \to \sigma \quad \text{weakly in } W, \tag{3.6b}$$

$$u_m \to u \quad \text{strongly in } L^4, \tag{3.6c}$$

$$\sigma_m \to \sigma \quad \text{strongly in } L^4. \tag{3.6d}$$

It is immediate that (u, σ) satisfies (3.4); we show next that (u, σ) is the desired solution. Fix $n \geq 1$, and $(v, \tau) \in V_n \times W_n$. Then for $m \geq n$ (in the above subsequence sense), we can write

$$\begin{cases} (1-r)\int_{\Omega_\varepsilon} \nabla u_m : \nabla v + \mathrm{Re} \int_{\Omega_\varepsilon} (u_m \cdot \nabla) u_m \cdot v + \int_{\Omega_\varepsilon} \sigma_m : Dv = \int_{\Omega_\varepsilon} f_\varepsilon v + p_d \Phi_v, \\ \mathrm{We} \int_{\Omega_\varepsilon} \big((u_m \cdot \nabla)\sigma_m + g_0(\sigma_m, \nabla u_m)\big) : \tau + \int_{\Omega_\varepsilon} \sigma_m : \tau + \varepsilon^2 \int_{\Omega_\varepsilon} \nabla \sigma_m : \nabla \tau = 2r \int_{\Omega_\varepsilon} D(u_m) : \tau. \end{cases}$$

To prove the convergence, let us write the following estimates

$$\begin{aligned} \left| \int_{\Omega_\varepsilon} (u_m \cdot \nabla) u_m \cdot v - \int_{\Omega_\varepsilon} (u \cdot \nabla) u \cdot v \right| &\leq \int_{\Omega_\varepsilon} \big|((u_m - u) \cdot \nabla) u_m \cdot v\big| \\ &+ \int_{\Omega_\varepsilon} \big|(u \cdot \nabla)(u_m - u) \cdot v\big| \\ &\leq \|u_m - u\|_{L^4} \|\nabla u_m\| \|v\|_{L^4} \\ &+ \int_{\Omega_\varepsilon} \big|(u \cdot \nabla)(u_m - u) \cdot v\big|, \end{aligned}$$

$$
\begin{aligned}
\left| \int_{\Omega_\varepsilon} (u_m \cdot \nabla)\sigma_m : \tau - \int_{\Omega_\varepsilon} (u \cdot \nabla)\sigma : \tau \right| &\le \int_{\Omega_\varepsilon} \big|((u_m - u) \cdot \nabla)\sigma_m : \tau\big| \\
&\quad + \int_{\Omega_\varepsilon} \big|(u \cdot \nabla)(\sigma_m - \sigma) : \tau\big| \\
&\le \|u_m - u\|_{L^4} \|\nabla \sigma_m\| \|\tau\|_{L^4} \\
&\quad + \int_{\Omega_\varepsilon} \big|(u \cdot \nabla)(\sigma_m - \sigma) : \tau\big|,
\end{aligned}
$$

$$
\begin{aligned}
\left| \int_{\Omega_\varepsilon} g_0(\sigma_m, \nabla u_m) : \tau - \int_{\Omega_\varepsilon} g_0(\sigma, \nabla u) : \tau \right| &\le \int_{\Omega_\varepsilon} \big|g_0(\sigma_m - \sigma, \nabla u_m) : \tau\big| \\
&\quad + \int_{\Omega_\varepsilon} \big|g_0(\sigma, \nabla(u_m - u)) : \tau\big| \\
&\le \|\sigma_m - \sigma\|_{L^4} \|\nabla u_m\| \|\tau\|_{L^4} \\
&\quad + \int_{\Omega_\varepsilon} \big|g_0(\sigma, \nabla(u_m - u)) : \tau\big|.
\end{aligned}
$$

Owing to (3.6a)–(3.6d) and the continuous embedding of L^4 into H^1, we easily derive that all the terms on the left hand side are convergent to 0. The remaining linear terms are trivial. Hence, in particular, (u, σ) satisfies (3.3) for all (v_n, τ_n), $n \ge 1$. We readily derive that (u, σ) is a solution to (3.3). □

Proposition 3.2.1. *There exists $p \in L^2(\Omega_\varepsilon)$ such that (u, σ, p) satisfies* (3.1) *in the sense of distributions and* (3.2) *in the sense of traces.*

Proof. The proof is very similar to that of Theorem 2.3.1. The only non trivial part is to show the boundary conditions involving $\partial_1 u$, $\partial_1 \sigma$ and p in (3.2) hold true in some sense.
As in the proof of Theorem 2.3.1 we can derive

$$
\langle -(1-r)\partial_1 u_1 + p, v_1 \rangle_{\Sigma_0^\varepsilon} - \langle -(1-r)\partial_1 u_1 + p, v_1 \rangle_{\Sigma_1^\varepsilon} = p_d \Phi_v. \tag{3.7}
$$

for all $v \in V$. We show next $\partial_1 u_1|_{\Sigma_0^\varepsilon} = \partial_1 u_1|_{\Sigma_1^\varepsilon}$. Note that, just like in Theorem 2.3.1 we get

$$
\langle \partial_1 u \cdot n, \phi \rangle_{\partial\Omega_\varepsilon} = \int_{\Omega_\varepsilon} \partial_1 u \cdot \nabla \phi, \quad \forall\, \phi \in H^1(\Omega_\varepsilon).
$$

Take any $\phi \in C^2(\overline{\Omega_\varepsilon})$ such that $\phi|_{\Sigma_0^\varepsilon} = \phi|_{\Sigma_1^\varepsilon}$. We have

$$
\begin{aligned}
\int_{\Omega_\varepsilon} \partial_1 u \cdot \nabla \phi &= -\int_{\Omega_\varepsilon} u \cdot \nabla \partial_1 \phi + \int_{\partial\Omega_\varepsilon} u \cdot \nabla \phi n_1 \\
&= \int_{\Omega_\varepsilon} div\, u\, \partial_1 \phi - \int_{\partial\Omega_\varepsilon} \partial_1 \phi u \cdot n + \int_{\partial\Omega_\varepsilon} u \cdot \nabla \phi n_1,
\end{aligned}
$$

but, since $u \in V$ and $n_2 = n_3 = 0$ on Σ_0^ε, Σ_1^ε we get

$$\langle \partial_1 u_1, \phi \rangle_0 - \langle \partial_1 u_1, \phi \rangle_1 = \int_{\partial\Omega_\varepsilon} \partial_2\phi u_2 + \partial_3\phi u_3 = 0,$$

using that $u \in V$ and the periodicity of ϕ; hence, by a density argument we obtain the desired result. The term involving $\partial_1\sigma$ is completely similar, while the recovery of the boundary condition for the pressure is done exactly as in the proof of Theorem 2.3.1. □

3.3 Asymptotic expansion

From now on, we are going to assume that f_ε satisfies condition (F1).
Consider the applications $\widetilde{u} : \Omega \to \mathbb{R}^3$, $\widetilde{\sigma} : \Omega \to \mathbb{M}_{3\times 3}(\mathbb{R})$, $\widetilde{p} : \Omega \to \mathbb{R}$ defined by

$$\begin{aligned}
\widetilde{u}(x_1, x_k) &= u(x_1, \varepsilon x_k),\\
\widetilde{\sigma}(x_1, x_k) &= \sigma(x_1, \varepsilon x_k),\\
\widetilde{p}(x_1, x_k) &= p(x_1, \varepsilon x_k).
\end{aligned}$$

By a priori estimates, we easily derive

$$\|\widetilde{u}\| \leq C\varepsilon^2, \quad \|\widetilde{\sigma}\| \leq C\varepsilon.$$

So let us *formally* write

$$\begin{aligned}
\widetilde{u}_1 &= \varepsilon^2 \widetilde{u}_1^0 + \cdots + \varepsilon^{2n+2}\widetilde{u}_1^n + \ldots\\
\widetilde{u}_k &= \varepsilon^3 \widetilde{u}_k^0 + \cdots + \varepsilon^{2n+3}\widetilde{u}_k^n + \ldots\\
\widetilde{p} &= \widetilde{p}^0 + \cdots + \varepsilon^{2n}\widetilde{p}^n + \ldots\\
\widetilde{\sigma}_{11} &= \varepsilon^2 \widetilde{\sigma}_{11}^0 + \cdots + \varepsilon^{2n+2}\widetilde{\sigma}_{11}^n + \ldots\\
\widetilde{\sigma}_{1k} &= \varepsilon \widetilde{\sigma}_{1k}^0 + \cdots + \varepsilon^{2n+1}\widetilde{\sigma}_{1k}^n + \ldots\\
\widetilde{\sigma}_{lm} &= \varepsilon^2 \widetilde{\sigma}_{lm}^0 + \cdots + \varepsilon^{2n+2}\widetilde{\sigma}_{lm}^n + \ldots
\end{aligned}$$

for $l, m \in \{2, 3\}$. Upon writing the equations in the rescaled domain Ω, plugging in the expansions above and identifying the appropriate powers of ε we get the following equations:

$$-(1-r)\partial_k^2 \widetilde{u}_1^j + \partial_1 \widetilde{p}_j = \partial_1 \widetilde{\sigma}_{11}^{j-1} + \partial_k \widetilde{\sigma}_{1k}^j + \widetilde{h}_1^j, \tag{V1}$$

$$-(1-r)\partial_k^2 \widetilde{u}_2^j + \partial_2 \widetilde{p}_{j+1} = \partial_1 \widetilde{\sigma}_{12}^j + \partial_k \widetilde{\sigma}_{2k}^j + \widetilde{h}_2^j, \tag{V2}$$

$$-(1-r)\partial_k^2 \widetilde{u}_3^j + \partial_3 \widetilde{p}_{j+1} = \partial_1 \widetilde{\sigma}_{13}^j + \partial_k \widetilde{\sigma}_{3k}^j + \widetilde{h}_3^j, \tag{V3}$$

$$
\begin{aligned}
\mathrm{We}\bigg(\sum_{i=0}^{j-1} \Big(\widetilde{u}_\alpha^i \partial_\alpha \widetilde{\sigma}_{11}^{j-i-1} - \partial_1 \widetilde{u}_k^i \widetilde{\sigma}_{1k}^{j-i-1} \Big) + \sum_{i=0}^{j} \partial_k \widetilde{u}_1^i \widetilde{\sigma}_{1k}^{j-i} \bigg) + \widetilde{\sigma}_{11}^j \\
- \partial_1^2 \widetilde{\sigma}_{11}^{j-1} - \partial_k^2 \widetilde{\sigma}_{11}^j = 2r \partial_1 \widetilde{u}_1^j,
\end{aligned} \tag{E1}
$$

$$
\begin{aligned}
\mathrm{We} \sum_{i=0}^{j-1} \widetilde{u}_\alpha^i \partial_\alpha \widetilde{\sigma}_{12}^{j-i-1} + \frac{\mathrm{We}}{2} \bigg(\sum_{i=0}^{j-2} \Big(\partial_1 \widetilde{u}_2^i \widetilde{\sigma}_{11}^{j-i-2} - \partial_1 \widetilde{u}_k^i \widetilde{\sigma}_{2k}^{j-i-2} \Big) \\
+ \sum_{i=0}^{j-1} \Big(- \partial_2 \widetilde{u}_1^i \widetilde{\sigma}_{11}^{j-i-1} + \big(\partial_3 \widetilde{u}_2^i - \partial_2 \widetilde{u}_3^i \big) \widetilde{\sigma}_{13}^{j-i-1} + \partial_k \widetilde{u}_1^i \widetilde{\sigma}_{2k}^{j-i-1} \Big) \bigg) \\
+ \widetilde{\sigma}_{12}^j - \partial_1^2 \widetilde{\sigma}_{12}^{j-1} - \partial_k^2 \widetilde{\sigma}_{12}^j = r(\partial_1 \widetilde{u}_2^{j-1} + \partial_2 \widetilde{u}_1^j),
\end{aligned} \tag{E2}
$$

$$
\begin{aligned}
\mathrm{We} \sum_{i=0}^{j-1} \widetilde{u}_\alpha^i \partial_\alpha \widetilde{\sigma}_{13}^{j-i-1} + \frac{\mathrm{We}}{2} \bigg(\sum_{i=0}^{j-2} \Big(\partial_1 \widetilde{u}_3^i \widetilde{\sigma}_{11}^{j-i-2} - \partial_1 \widetilde{u}_k^i \widetilde{\sigma}_{3k}^{j-i-2} \Big) \\
+ \sum_{i=0}^{j-1} \Big(- \partial_3 \widetilde{u}_1^i \widetilde{\sigma}_{11}^{j-i-1} + \big(\partial_2 \widetilde{u}_3^i - \partial_3 \widetilde{u}_2^i \big) \widetilde{\sigma}_{12}^{j-i-1} + \partial_k \widetilde{u}_1^i \widetilde{\sigma}_{3k}^{j-i-1} \Big) \bigg) \\
+ \widetilde{\sigma}_{13}^j - \partial_1^2 \widetilde{\sigma}_{13}^{j-1} - \partial_k^2 \widetilde{\sigma}_{13}^j = r(\partial_1 \widetilde{u}_3^{j-1} + \partial_3 \widetilde{u}_1^j),
\end{aligned} \tag{E3}
$$

$$
\begin{aligned}
\mathrm{We}\bigg(\sum_{i=0}^{j-1} \Big(\widetilde{u}_\alpha^i \partial_\alpha \widetilde{\sigma}_{22}^{j-i-1} + \partial_1 \widetilde{u}_2^i \widetilde{\sigma}_{12}^{j-i-1} + \partial_2 \widetilde{u}_2^i \widetilde{\sigma}_{22}^{j-i-1} + \big(\partial_3 \widetilde{u}_2^i - \partial_2 \widetilde{u}_3^i \big) \widetilde{\sigma}_{23}^{j-i-1} \Big) \\
- \sum_{i=0}^{j} \partial_2 \widetilde{u}_1^i \widetilde{\sigma}_{12}^{j-i} \bigg) + \widetilde{\sigma}_{22}^j - \partial_1^2 \widetilde{\sigma}_{22}^{j-1} - \partial_k^2 \widetilde{\sigma}_{22}^j = 2r \partial_2 \widetilde{u}_2^j,
\end{aligned} \tag{E4}
$$

$$
\begin{aligned}
\mathrm{We} \sum_{i=0}^{j-1} \widetilde{u}_\alpha^i \partial_\alpha \widetilde{\sigma}_{23}^{j-i-1} + \frac{\mathrm{We}}{2} \bigg(\sum_{i=0}^{j-1} \Big(\partial_1 \widetilde{u}_3^i \widetilde{\sigma}_{12}^{j-i-1} + \partial_1 \widetilde{u}_2^i \widetilde{\sigma}_{13}^{j-i-2} \\
+ \big(\partial_2 \widetilde{u}_3^i - \partial_3 \widetilde{u}_2^i \big) \big(\widetilde{\sigma}_{22}^{j-i-1} - \widetilde{\sigma}_{33}^{j-i-1} \big) \Big) - \sum_{i=0}^{j} \Big(\partial_3 \widetilde{u}_1^i \widetilde{\sigma}_{12}^{j-i} + \partial_2 \widetilde{u}_1^i \widetilde{\sigma}_{13}^{j-i} \Big) \bigg) \\
+ \widetilde{\sigma}_{23}^j - \partial_1^2 \widetilde{\sigma}_{23}^{j-1} - \partial_k^2 \widetilde{\sigma}_{23}^j = r(\partial_2 \widetilde{u}_3^j + \partial_3 \widetilde{u}_2^j)
\end{aligned} \tag{E5}
$$

$$
\begin{aligned}
\mathrm{We}\bigg(\sum_{i=0}^{j-1} \Big(\widetilde{u}_\alpha^i \partial_\alpha \widetilde{\sigma}_{33}^{j-i-1} + \partial_1 \widetilde{u}_3^i \widetilde{\sigma}_{13}^{j-i-1} + \partial_3 \widetilde{u}_3^i \widetilde{\sigma}_{33}^{j-i-1} + \big(\partial_2 \widetilde{u}_3^i - \partial_3 \widetilde{u}_2^i \big) \widetilde{\sigma}_{23}^{j-i-1} \Big) \\
- \sum_{i=0}^{j} \partial_3 \widetilde{u}_1^i \widetilde{\sigma}_{13}^{j-i} \bigg) + \widetilde{\sigma}_{33}^j - \partial_1^3 \widetilde{\sigma}_{33}^{j-1} - \partial_k^2 \widetilde{\sigma}_{33}^j = 2r \partial_3 \widetilde{u}_3^j,
\end{aligned} \tag{E6}
$$

$$\partial_1 \widetilde{u}_1^j + \partial_k \widetilde{u}_k^j = 0, \tag{D}$$

where

$$\widetilde{h}_1^j = (1-r)\partial_1^2 \widetilde{u}_1^{j-1} - \mathrm{Re}\sum_{i=0}^{j-2}\left(\widetilde{u}_1^i \partial_1 \widetilde{u}_1^{j-i-2} + \widetilde{u}_k^i \partial_k \widetilde{u}_1^{j-i-2}\right) + \widetilde{f}_1^j,$$

$$\widetilde{h}_l^j = (1-r)\partial_1^2 \widetilde{u}_l^{j-1} - \mathrm{Re}\sum_{i=0}^{j-3}\left(\widetilde{u}_1^i \partial_1 \widetilde{u}_l^{j-i-3} + \widetilde{u}_k^i \partial_k \widetilde{u}_l^{j-i-3}\right) + \widetilde{f}_l^j$$

for $l \in \{2,3\}$, with $\widetilde{f}_\alpha^j = \widetilde{f}_\alpha$ for $j = 0$ and $\widetilde{f}_\alpha^j = 0$ for $j > 0$; note that all $\widetilde{u}_\alpha^m$, $\widetilde{\sigma}_{\alpha\beta}^m$, $\widetilde{p}^m$ are independent of ε for all $m \geq 0$, $\alpha, \beta \in \{1,2,3\}$. The boundary conditions are expected to become

$$\begin{cases} \widetilde{u}_\alpha^m = 0 \quad \text{on } \Gamma \quad \forall\, m \geq 0,\ \alpha \in \{1,2,3\}, \\ \widetilde{\sigma}_{\alpha\beta}^m = 0 \quad \text{on } \Gamma \quad \forall\, m \geq 0,\ \alpha,\beta \in \{1,2,3\}, \\ \widetilde{u}^m|_{\Sigma_0} = \widetilde{u}^m|_{\Sigma_1},\ \widetilde{\sigma}^m|_{\Sigma_0} = \widetilde{\sigma}^m|_{\Sigma_1}, \\ \partial_{x_1}\widetilde{u}^m|_{\Sigma_0} = \partial_{x_1}\widetilde{u}^m|_{\Sigma_1},\ \partial_{x_1}\widetilde{\sigma}^m|_{\Sigma_0} = \partial_{x_1}\widetilde{\sigma}^m|_{\Sigma_1} \quad \forall\, m \geq 0, \\ \widetilde{p}^0(0) - \widetilde{p}^0(1) = p_d, \\ \widetilde{p}^m(i) = 0 \quad \text{on } \Sigma_i \quad \forall\, m \geq 1. \end{cases} \tag{3.8}$$

We have tacitly assumed that terms with negative upper coefficients are 0.
By taking $j = 0$ in (V1),(E2),(E3) and $j = -1$ in (V2),(V3) we derive

$$\begin{cases} -(1-r)\partial_k^2 \widetilde{u}_1^0 + \partial_1, \widetilde{p}_0 = \widetilde{f}_1 + \partial_2 \widetilde{\sigma}_{12}^0 + \partial_3 \widetilde{\sigma}_{13}^0, \\ \widetilde{\sigma}_{12}^0 - \partial_k^2 \widetilde{\sigma}_{12}^0 = r\partial_2 \widetilde{u}_1^0, \\ \widetilde{\sigma}_{13}^0 - \partial_k^2 \widetilde{\sigma}_{13}^0 = r\partial_3 \widetilde{u}_1^0, \\ \partial_2 \widetilde{p}_0 = \partial_3 \widetilde{p}_0 = 0. \end{cases} \tag{P_1^0}$$

We shall use the superposition principle to solve this problem. First, consider – for all x_1 – the problem

$$\begin{cases} -(1-r)\Delta_k \widetilde{v} = \widetilde{f}_1 & \text{on } S(x_1), \\ \widetilde{v} = 0 & \text{on } \partial S(x_1), \end{cases}$$

where $\Delta_k = \partial_2^2 + \partial_3^2$. Call the unique smooth solution of the Dirichlet problem $\widetilde{v}_1$, and let us denote

$$\int_{S(x_1)} \partial_1 \widetilde{v}_1 = \gamma(x_1).$$

Secondly, consider the problem

$$\begin{cases} -(1-r)\Delta_k \widetilde{v} + 1 = div\, \widetilde{\tau} & \text{on } S(x_1), \\ \widetilde{\tau} - \Delta_k \widetilde{\tau} = r\nabla_k \widetilde{v} & \text{on } S(x_1), \\ \widetilde{v} = 0,\ \widetilde{\tau} = 0 & \text{on } \partial S(x_1). \end{cases}$$

Note that the system has a unique solution. The existence part can be proved via a classical approximation technique, using the following energy estimate:

$$E(\widetilde{v},\widetilde{\tau}) = r(1-r)\|\nabla_k\widetilde{v}\|^2_{L^2(S(x_1))} + \|\widetilde{\tau}\|^2_{L^2(S(x_1))} + \|\nabla_k\widetilde{\tau}\|^2_{L^2(S(x_1))} = -r\int_{S(x_1)}\widetilde{v} \\ \leq r\|\nabla_k\widetilde{v}\|_{L^2(S(x_1))}.$$

To show the uniqueness, observe that, if $(\widetilde{v}^1,\widetilde{\tau}^1)$ and $(\widetilde{v}^2,\widetilde{\tau}^2)$ then

$$E(\widetilde{v}^1-\widetilde{v}^2,\widetilde{\tau}^1-\widetilde{\tau}^2) = 0,$$

and the conclusion easily follows. The regularity of the solution can be shown by alternating between the two elliptic problems (considered separately) - see [7]. Let us call this solution $(\widetilde{v}_2,\widetilde{\tau}_2,\widetilde{\tau}_3)$. Then it is clear that $(\widetilde{u}^0_1,\widetilde{\sigma}^0_{12},\widetilde{\sigma}^0_{13}) = (\widetilde{v}_1 + \partial_1\widetilde{p}_0\widetilde{v}_2, \partial_1\widetilde{p}_0\widetilde{\tau}_2, \partial_1\widetilde{p}_0\widetilde{\tau}_3)$ is a solution of (P^0_1) with $\widetilde{p}_0$ determined by imposing

$$\partial_1\left(\partial_1\widetilde{p}_0\int_{S(x_1)}\widetilde{v}_2\right) = -\gamma(x_1)$$

and $\widetilde{p}_0(0)-\widetilde{p}_0(1) = p_d$. Note that $\widetilde{p}_0$ is defined up to a constant, but $\widetilde{u}^0_1,\widetilde{\sigma}^0_{12},\widetilde{\sigma}^0_{13}$ are uniquely defined. Due to the imposed regularity requirements, we have $\widetilde{u}^0_1,\widetilde{\sigma}^0_{12},\widetilde{\sigma}^0_{13} \in C^2(\overline{\Omega})$ and $\widetilde{p}^0 \in C^1(\overline{\Omega})$. So we have shown that $(\widetilde{u}^0_1,\widetilde{p}^0,\widetilde{\sigma}^0_{12},\widetilde{\sigma}^0_{13})$ satisfy (P^0_1) in the classical sense; moreover, the following compatibility condition is verified:

$$\int_{S(x_1)}\partial_{x_1}\widetilde{u}^0_1 = 0. \qquad (C_0)$$

The geometrical assumption on the domain implies that $\widetilde{u}^0_1(x_1) = \widetilde{u}^0_1(1-x_1)$ and $\widetilde{\sigma}^0_{1k}(x_1) = \widetilde{\sigma}^0_{1k}(1-x_1)$ for $x_1 \in [0,\delta)$, so that $\widetilde{u}_1$, $\widetilde{\sigma}_{1k}$ have the (PNB) property as introduced in the first chapter. Clearly, $\partial_1\widetilde{u}^0_1(0) = \partial_1\widetilde{u}^0_1(1) = \partial_1\widetilde{\sigma}_{1k}(0) = \partial_1\widetilde{\sigma}_{1k}(1) = 0$, so that all conditions in (3.8) are satisfied thus far.
Next, proceed by induction. Let $j \geq 0$. Suppose we have determined $\widetilde{u}^i_1,\widetilde{\sigma}^i_{1k},\widetilde{p}_i$ with $\int_{S(x_1)}\partial_1\widetilde{u}^i_1 = 0$ for all $i \leq j$ and $\widetilde{\sigma}^i_{11},\widetilde{u}^i_k,\widetilde{\sigma}^i_{22},\widetilde{\sigma}^i_{23},\widetilde{\sigma}^i_{33}$ for all $i < j$ (nothing if $j = 0$) all smooth and with the (PNB) property. We show how to determine $\widetilde{\sigma}^j_{11},\widetilde{u}^j_k,\widetilde{\sigma}^j_{22},\widetilde{\sigma}^j_{23},\widetilde{\sigma}^j_{33}$ and $\widetilde{u}^{j+1}_1,\sigma^{j+1}_{1k},\widetilde{p}_{j+1}$ with $\int_{S(x_1)}\partial_1\widetilde{u}^{j+1}_1 = 0$ – all smooth and with the (PNB) property.
Using (E1) we derive

$$\widetilde{\sigma}^j_{11} - \partial^2_k\widetilde{\sigma}^j_{11} = \widetilde{f}^j_{11}$$

with known smooth $\widetilde{f}^j_{11}$ having the (PNB) property, which can be solved as an elliptic problem – with homogeneous Dirichlet boundary conditions – on every

$S(x_1)$. By considering (E4), (E5) and (E6), and also (V2), (V3) and (D) we obtain

$$\begin{cases} -(1-r)\partial_k^2 \widetilde{u}_2^j + \partial_2 \widetilde{p}_{j+1} = \partial_k \widetilde{\sigma}_{2k}^j + \widetilde{g}_2^j, \\ -(1-r)\partial_k^2 \widetilde{u}_3^j + \partial_3 \widetilde{p}_{j+1} = \partial_k \widetilde{\sigma}_{3k}^j + \widetilde{g}_3^j, \\ \widetilde{\sigma}_{22}^j - \partial_k^2 \widetilde{\sigma}_{22}^j = 2r\partial_2 \widetilde{u}_2^j + \widetilde{f}_{22}^j, \\ \widetilde{\sigma}_{23}^j - \partial_k^2 \widetilde{\sigma}_{23}^j = r(\partial_2 \widetilde{u}_3^j + \partial_3 \widetilde{u}_2^j) + \widetilde{f}_{23}^j, \\ \widetilde{\sigma}_{33}^j - \partial_k^2 \widetilde{\sigma}_{33}^j = 2r\partial_3 \widetilde{u}_3^j + \widetilde{f}_{33}^j, \\ \partial_k \widetilde{u}_k^j = -\partial_1 \widetilde{u}_1^j, \end{cases} \tag{P_k^j}$$

where all the $\widetilde{g}^j$'s and $\widetilde{f}^j$'s are known, smooth and have the the (PNB) property. The above is a coupled inhomogeneous Stokes and elliptic system in 2D, and, if we supplement Dirichlet boundary conditions we get

$$\begin{cases} -(1-r)\Delta_k \widetilde{v}^j + \nabla_k \widetilde{p}_{j+1} = div_k\, \widetilde{\tau}^j + \widetilde{g}^j & \text{on } S(x_1), \\ \widetilde{\tau}^j - \Delta_k \widetilde{\tau}^j = 2rD_k \widetilde{v}^j + \widetilde{F}^j & \text{on } S(x_1), \\ div_k\, \widetilde{v}^j = -\partial_1 \widetilde{u}_1^j & \text{on } S(x_1), \\ \widetilde{v}^j, \widetilde{\tau}^j = 0 & \text{on } \partial S(x_1). \end{cases}$$

Owing to the condition $\int_{S(x_1)} \partial_1 \widetilde{u}_1^j = 0$, use Lemma 2.3.2 to find $\widetilde{z}^j$ (not necessarily smooth) such that

$$\begin{cases} div_k\, \widetilde{z}^j = -\partial_1 \widetilde{u}_1^j & \text{on } S(x_1), \\ \widetilde{z}^j = 0 & \text{on } \partial S(x_1). \end{cases}$$

Evidently we can choose $\widetilde{z}^j$ to have the (PNB) property. Therefore, in order to prove the existence of a solution to problem (P_k^j) it suffices to prove the existence to the following problem

$$\begin{cases} -(1-r)\Delta_k \widetilde{w}^j + \nabla_k \widetilde{p}_{j+1} = div_k\, \widetilde{\tau}^j + \widetilde{g}^{j,1} & \text{on } S(x_1), \\ \widetilde{\tau}^j - \Delta_k \widetilde{\tau}^j = 2rD_k \widetilde{w}^j + \widetilde{F}^{j,1} & \text{on } S(x_1), \\ div_k\, \widetilde{w}^j = 0 & \text{on } S(x_1), \\ \widetilde{w}^j, \widetilde{\tau}^j = 0 & \text{on } \partial S(x_1). \end{cases}$$

The above system is a linearized version of our initial problem, and the existence of a unique solution is shown using the Lax-Milgram theorem. The regularity is obtained by alternating between the Stokes problem and the elliptic ones in (P_k^j). Moreover, since the source terms have the (PNB) property, this solution has it also. Note that $\widetilde{p}_{j+1}$ is defined up to a constant for all x_1 – say $\widetilde{r}_{j+1}(x_1)$. By taking $j+1$ instead of j in (V1),(E2),(E3) we derive

$$\begin{cases} -(1-r)\partial_k^2 \widetilde{u}_1^{j+1} + \partial_1 \widetilde{r}_{j+1} = \widetilde{g}_1^{j+1} + \partial_2 \widetilde{\sigma}_{12}^{j+1} + \partial_3 \widetilde{\sigma}_{13}^{j+1} \\ \widetilde{\sigma}_{12}^{j+1} - \partial_k^2 \widetilde{\sigma}_{12}^{j+1} = r\partial_2 \widetilde{u}_1^{j+1} + \widetilde{f}_{12}^{j+1} \\ \widetilde{\sigma}_{13}^{j+1} - \partial_k^2 \widetilde{\sigma}_{13}^{j+1} = r\partial_3 \widetilde{u}_1^{j+1} + \widetilde{f}_{13}^{j+1} \end{cases} \tag{P_1^{j+1}}$$

with known, smooth, periodic $\widetilde{g}_1^{j+1}, \widetilde{f}_{12}^{j+1}, \widetilde{f}_{13}^{j+1}$. This is handled in exactly the same manner as problem (P_1^0), with the sole difference that $\widetilde{r}_{j+1}$ is uniquely defined by the conditions $\widetilde{r}_{j+1}(0) = \widetilde{r}_{j+1}(1) = 0$. The proof is then complete.

3.4 Error estimates

Let

$$
\begin{aligned}
u_1^n(x_1, x_k) &= \sum_{j=0}^{n} \varepsilon^{2j+2} \widetilde{u}_1^j\Big(x_1, \frac{x_k}{\varepsilon}\Big),\\
u_l^n(x_1, x_k) &= \sum_{j=0}^{n} \varepsilon^{2j+3} \widetilde{u}_k^j\Big(x_1, \frac{x_k}{\varepsilon}\Big), \quad l \in \{2,3\},\\
p_n(x_1, x_k) &= \sum_{j=0}^{n} \varepsilon^{2j} \widetilde{p}_j\Big(x_1, \frac{x_k}{\varepsilon}\Big),\\
\sigma_{11}^n(x_1, x_k) &= \sum_{j=0}^{n} \varepsilon^{2j+2} \widetilde{\sigma}_{11}^j\Big(x_1, \frac{x_k}{\varepsilon}\Big),\\
\sigma_{1l}^n(x_1, x_k) &= \sum_{j=0}^{n} \varepsilon^{2j+1} \widetilde{\sigma}_{1l}^j\Big(x_1, \frac{x_k}{\varepsilon}\Big), \quad l \in \{2,3\},\\
\sigma_{lm}^n(x_1, x_k) &= \sum_{j=0}^{n} \varepsilon^{2j+2} \widetilde{\sigma}_{lm}^j\Big(x_1, \frac{x_k}{\varepsilon}\Big), \quad l, m \in \{2,3\}.
\end{aligned}
$$

Theorem 3.4.1. *For all $n \geq 0$, the following estimates hold true:*

$$
\begin{aligned}
\|\nabla(u - u^n)\| &\leq C\varepsilon^{2n+4},\\
\|\nabla(\sigma - \sigma^n)\| &\leq C\varepsilon^{2n+3},\\
\|p - p_n\| &\leq C\varepsilon^{2n+2}.
\end{aligned}
$$

Proof. From the previous section we get that $(u^n, \sigma^n) \in V \times W$. Also, (u^n, σ^n, p^n) are smooth and satisfy (3.2). After some calculations we get

$$
\begin{cases}
-(1-r)\Delta u^n + \mathrm{Re}(u^n \cdot \nabla) \cdot u^n + \nabla p_{n+1} = div\, \sigma^n + f_\varepsilon + R^n,\\
\mathrm{We}\big((u^n \cdot \nabla)\sigma^n + g_0(\sigma^n, \nabla u^n)\big) + \sigma^n - \varepsilon^2 \Delta \sigma^n = 2rDu^n + Q^n,\\
div\, u^n = 0,
\end{cases} \tag{3.9}
$$

where R^n, Q^n have complicated expressions that we would rather omit. Still, as in the proof of Theorem 2.4.1 we can prove

$$\|R^n\|_{L^\infty} \leq C\varepsilon^{2n+2},\ \|Q^n\|_{L^\infty} \leq C\varepsilon^{2n+3}. \tag{3.10}$$

It easily follows

$$\begin{cases}(1-r)\int_{\Omega_\varepsilon}\nabla u^n:\nabla v+\mathrm{Re}\int_{\Omega_\varepsilon}(u^n\cdot\nabla)u^n\cdot v+\int_{\Omega_\varepsilon}\sigma^n:Dv=\int_{\Omega_\varepsilon}f_\varepsilon v+p_d\Phi_v+\int_{\Omega_\varepsilon}R^n v,\\ \mathrm{We}\int_{\Omega_\varepsilon}\big((u^n\cdot\nabla)\sigma^n+g_0(\sigma^n,\nabla u^n)\big):\tau+\int_{\Omega_\varepsilon}\sigma^n:\tau+\varepsilon^2\int_{\Omega_\varepsilon}\nabla\sigma^n:\nabla\tau=2r\int_{\Omega_\varepsilon}Du^n:\tau+\int_{\Omega_\varepsilon}Q^n:\tau\end{cases} \tag{3.11}$$

By taking in (3.3) and (3.11) $v = u - u^n$ and $\tau = \sigma - \sigma^n$ and making some elementary manipulations, we obtain

$$\begin{aligned}
&2r(1-r)\|\nabla(u-u^n)\|^2+\|\sigma-\sigma^n\|^2+\varepsilon^2\|\nabla(\sigma-\sigma^n)\|^2\le 2r\left|\int_{\Omega_\varepsilon}R^n(u-u^n)\right|\\
&+2r\mathrm{Re}\left(\left|\int_{\Omega_\varepsilon}(u-u^n)\cdot\nabla u^n\cdot(u-u^n)\right|+\left|\int_{\Omega_\varepsilon}u\cdot\nabla(u-u^n)\cdot(u-u^n)\right|\right)\\
&+\mathrm{We}\left(\left|\int_{\Omega_\varepsilon}(u-u^n)\cdot\nabla\sigma^n:(\sigma-\sigma^n)\right|+\left|\int_{\Omega_\varepsilon}u\cdot\nabla(\sigma-\sigma^n):(\sigma-\sigma^n)\right|\right.\\
&\left.+\left|\int_{\Omega_\varepsilon}g_0(\nabla(u-u^n),\sigma^n):(\sigma-\sigma^n)\right|+\left|\int_{\Omega_\varepsilon}g_0(\nabla u,\sigma-\sigma^n):(\sigma-\sigma^n)\right|\right)\\
&+\left|\int_{\Omega_\varepsilon}Q^n(\sigma-\sigma^n)\right|.
\end{aligned}$$

By using Hölder's inequality, (3.10), Lemma 2.3.5, estimates in Theorem 3.2.1, as well as the following trivial

$$\|\sigma_n\|_{L^\infty}\le C\varepsilon,\quad \|\nabla\sigma_n\|_{L^\infty}\le C,$$

we derive the following estimates:

$$\left|\int_{\Omega_\varepsilon} R^n \cdot (u-u^n)\right| \le C\varepsilon^{2n+4}\|\nabla(u^n-u)\|,$$
$$\left|\int_{\Omega_\varepsilon} Q^n(\sigma-\sigma^n)\right| \le C\varepsilon^{2n+5}\|\nabla(\sigma-\sigma^n)\|,$$
$$\left|\int_{\Omega_\varepsilon} (u-u^n)\cdot\nabla u^n\cdot(u-u^n)\right| \le C\varepsilon^{3}\|\nabla(u-u^n)\|^2,$$
$$\left|\int_{\Omega_\varepsilon} u\cdot\nabla(u-u^n)\cdot(u-u^n)\right| \le C\varepsilon^{5/2}\|\nabla(u-u^n)\|^2,$$
$$\left|\int_{\Omega_\varepsilon} (u-u^n)\cdot\nabla\sigma^n:(\sigma-\sigma^n)\right| \le C\varepsilon^{2}\|\nabla(u^n-u)\|\|\nabla(\sigma-\sigma^n)\| \le C\varepsilon(\|\nabla(u^n-u)\|^2+\varepsilon^2\|\sigma-\sigma^n\|^2)$$
$$\left|\int_{\Omega_\varepsilon} u\cdot\nabla(\sigma-\sigma^n):(\sigma-\sigma^n)\right| \le C\varepsilon^{5/2}\|\nabla(\sigma-\sigma^n)\|^2,$$
$$\left|\int_{\Omega_\varepsilon} g_0(\nabla(u-u^n),\sigma^n):(\sigma-\sigma^n)\right| \le C\varepsilon\|\nabla(u^n-u)\|\|\sigma-\sigma^n\| \le C\varepsilon(\|\nabla(u^n-u)\|^2+\varepsilon^2\|\sigma-\sigma^n\|^2),$$
$$\left|\int_{\Omega_\varepsilon} g_0(\nabla u,\sigma-\sigma^n):(\sigma-\sigma^n)\right| \le C\varepsilon^{5/2}\|\nabla(\sigma-\sigma^n)\|^2.$$

Hence, for ε small, we can derive

$$\|\nabla(u-u^n)\|^2+\varepsilon^2\|\nabla(\sigma-\sigma^n)\|^2 \le C\varepsilon^{2n+4}(\|\nabla(u-u^n)\|+\|\nabla(\sigma-\sigma^n)\|),$$

from which follows easily

$$\begin{aligned}\|\nabla(u-u^n)\| &\le C\varepsilon^{2n+4},\\ \|\nabla(\sigma-\sigma^n)\| &\le C\varepsilon^{2n+3}.\end{aligned}$$

To derive a pressure estimate, use Lemma 2.4.2 to find $\phi_n \in \left(H_0^1(\Omega_\varepsilon)\right)^3$ such that:

$$\begin{aligned}&div\,\phi_n = p-p_{n+1} \quad \text{on} \quad \Omega_\varepsilon,\\ &\|\nabla\phi_n\| \le C\varepsilon^{-1}\|p-p_{n+1}\|.\end{aligned}$$

Consider the first equation in (3.1) – understood in distributional sense – and (3.9). By subtracting the two and then applying $\phi_n \in \left(H_0^1(\Omega_\varepsilon)\right)^3$ we obtain

$$\begin{aligned}\|p-p_{n+1}\|^2 + (1-r)\int_{\Omega_\varepsilon}&\nabla(u-u^n)\cdot\nabla\phi_n\\ &= \int_{\Omega_\varepsilon}\big((u^n\cdot\nabla)u^n-(u\cdot\nabla)u\big)\cdot\phi_n - \int_{\Omega_\varepsilon} R^n\cdot\phi_n.\end{aligned}$$

Proceeding in a very similar manner to the way we determined the velocity estimates – while also using these estimates – it is easy to establish

$$\|p - p_{n+1}\| \leq C\varepsilon^{2n+3},$$

from which the pressure estimate easily follows. □

Remark 1. *The result we have just proved shows, in particular, that for ε small enough, the solution to problem* (3.1)–(3.2) *is unique.*

3.5 Axisymmetric equation

In view of the last remark in the previous section, we shall suppose that ε is small enough so that the solution to our problem is uniquely defined.
Henceforth, suppose that the domain Ω_ε is axisymmetric, namely that is obtained by rotating around the x_1 axis the following 2D domain

$$D_\varepsilon = \{(x_1, x_3) \in \mathbb{R}^2 \,|\, x_1 \in (0,1),\, x_3 \in (0, \varepsilon h(x_1))\}.$$

for some smooth $h : [0,1] \to \mathbb{R}_+^*$, such that $h(x_1) = h(0) = h(1 - x_1)$ for all $x_1 \in (0, \delta)$.
A comprehensive study of various equations in axisymmetric domains can be found in [4]. The goal of this section is to show that the weak solution is "axisymmetric" in a sense that we shall properly define provided the source term enjoys the same property. In [30] a characterization of regular axisymmetric functions related to the Navier-Stokes equation is given. Here we show how to extend those results to the case of weak solutions. Although the result is very intuitive, the proof – while elementary – is quite technical.
We start by introducing cylindrical coordinates:

$$\begin{aligned} x_1 &= x, \\ x_2 &= z \sin\theta, \\ x_3 &= z \cos\theta, \end{aligned}$$

with $x \in [0,1]$, $z \in [0, \varepsilon h(x_1)]$, $\theta \in [0, 2\pi)$. We write

$$\begin{cases} u = u_x(x,z,\theta)e_x + u_z(x,z,\theta)e_z + u_\theta(x,z,\theta)e_\theta, \\ \sigma = \sigma_{xx}(x,z,\theta)e_x \otimes e_x + \cdots + \sigma_{\theta\theta}(x,z,\theta)e_\theta \otimes e_\theta, \end{cases}$$

with

$$e_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_z = \begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix}, \quad e_\theta = \begin{pmatrix} 0 \\ \cos\theta \\ -\sin\theta \end{pmatrix}.$$

Consider
$$\mathcal{X} = C^1(\Omega_\varepsilon) \cap C(\overline{\Omega_\varepsilon}) \cap H^1(\Omega_\varepsilon).$$
Note that if $u \in \mathcal{X}^3$ and $\sigma \in \mathbb{M}_{3\times 3}(\mathcal{X})$ then $u_x, u_z, u_\theta, \sigma_{xx}, \dots, \sigma_{\theta\theta} \in C^1(D_\varepsilon \times (0, 2\pi))$, and so we can define
$$\begin{aligned}
\mathcal{X}_s &= \{u \in \mathcal{X}^3 \,|\, \partial_\theta u_x = \partial_\theta u_z = \partial_\theta u_\theta = 0\},\\
\mathcal{Y}_s &= \{\sigma \in \mathbb{M}_{3\times 3}(\mathcal{X}) \,|\, \sigma_{ij} = \sigma_{ji}, \partial_\theta \sigma_{xx} = \cdots = \partial_\theta \sigma_{\theta\theta} = 0\}.
\end{aligned}$$

Lemma 3.5.1. *Given $u \in \mathcal{X}^3$ then $u_x, u_z, u_\theta \in C(\overline{D_\varepsilon \times (0, 2\pi)})$, and, for all $\theta \in (0, 2\pi]$ $u_x(\cdot, \cdot, \theta), u_z(\cdot, \cdot, \theta), u_\theta(\cdot, \cdot, \theta) \in C^1(D'_\varepsilon)$, where $D'_\varepsilon = D_\varepsilon \cup \{(x, 0) \,|\, x \in (0, 1)\}$. Moreover, for all $x \in (0, 1)$ and $\theta \in [0, \pi)$ we have*
$$u_z(x, 0, \theta) = -u_z(x, 0, \theta + \pi), \tag{3.12a}$$
$$u_\theta(x, 0, \theta) = -u_\theta(x, 0, \theta + \pi), \tag{3.12b}$$
$$\partial_z u_x(x, 0, \theta) = -\partial_z u_x(x, 0, \theta + \pi). \tag{3.12c}$$
In particular, if $u \in \mathcal{X}_s$ then $u_z(x, 0) = u_\theta(x, 0) = 0$ and $\partial_z u_x(x, 0) = 0$.

Proof. The proof is trivial and follows from the definition of u_x, u_z, u_θ. $\square$

Observe that if $u \in \mathcal{X}^3$ then
$$\int_{\partial\Omega_\varepsilon} u^2 = \int_{T_\varepsilon} (u_x^2 + u_z^2 + u_\theta^2) z,$$
where $T_\varepsilon = \partial(D_\varepsilon \times (0, 2\pi)) \setminus \{(x, 0, \theta) \,|\, x \in (0, 1), \theta \in (0, 2\pi)\}$. Also, if $u \in \mathcal{X}_s$, then
$$\|u\|_{H^1}^2 = 2\pi \int_{D_\varepsilon} \big(u_x^2 + u_z^2 + u_\theta^2 + (\partial_x u_x)^2 + (\partial_z u_x)^2 + (\partial_x u_z)^2 + (\partial_z u_z)^2 + (\partial_x u_\theta)^2 + (\partial_z u_\theta)^2\big) z + \frac{u_z^2}{z} + \frac{u_\theta^2}{z}.$$
This motivates the introduction of the following spaces
$$\begin{aligned}
H &= \{u \in L^2_z(D_\varepsilon) \,|\, \nabla u \in \big(L^2_z(D_\varepsilon)\big)^2\},\\
H' &= \{u \in L^2_{z^{-1}}(D_\varepsilon) \,|\, \nabla u \in \big(L^2_z(D_\varepsilon)\big)^2\},
\end{aligned}$$
where, if $w \geq 0$ on D_ε, then $L^2_w(D_\varepsilon)$ is the classical weighted Lebesgue space, endowed with the scalar product
$$(u, v)_{L^2_w} = \int_{D_\varepsilon} uvw.$$

The spaces H and H' become Hilbert spaces when endowed with the following scalar products:

$$(u,v)_H = 2\pi\big((u,v)_{L^2_z} + (\partial_x u, \partial_x v)_{L^2_z} + (\partial_z u, \partial_z v)_{L^2_z}\big),$$
$$(u,v)_{H'} = 2\pi\big((u,v)_{L^2_z} + (u,v)_{L^2_{z^{-1}}} + (\partial_x u, \partial_x v)_{L^2_z} + (\partial_z u, \partial_z v)_{L^2_z}\big).$$

The application $T : \big(\mathcal{X}_s, \|\cdot\|_{H^1}\big) \to H \times H' \times H'$ defined by $Tu = (u_x, u_z, u_\theta)$ is an isometry, and so if X_s is the completion of $\mathcal{X}_s$ in H^1 then T can be uniquely extended to an isometry – still denoted by $T : X_s \to H \times H' \times H'$.
As we could see in Lemma 3.5.1, if $u \in \mathcal{X}_s$ then $u_x, u_z, u_\theta \in \mathcal{C} = \{(v_x, v_z, v_\theta) \in C^1(D'_\varepsilon) \cap C(\overline{D_\varepsilon}) \,|\, \partial_z v_x(x,0) = 0, v_z(x,0) = v_\theta(x,0) = 0\}$. We can now show that this completely defines the elements in $\mathcal{X}_s$.

Lemma 3.5.2. *The operator T is bijective from $\mathcal{X}_s$ to $\mathcal{C}$.*

Proof. Given $(u_x, u_z, u_\theta) \in \mathcal{C}$, then we have

$$u_1(x_1,x_2,x_3) = u_x(x_1,z),$$
$$u_2(x_1,x_2,x_3) = u_z(x_1,z)\frac{x_2}{z} + u_\theta(x_1,z)\frac{x_3}{z},$$
$$u_3(x_1,x_2,x_3) = u_z(x_1,z)\frac{x_3}{z} - u_\theta(x_1,z)\frac{x_2}{z},$$

with $r = \sqrt{x_2^2 + x_3^2}$. The only non-trivial part is to show that $u, \nabla u$ are well-defined on $z = 0$. This is obvious for u_1 and $\partial_1 u_1$, while

$$\partial_k u_1(x_1,x_2,x_3) = \partial_z u_x(x_1,z)\frac{x_k}{z}$$

(for $k \in \{2,3\}$) is 0 on $z = 0$, as the product of a function which tends to 0 as $z \to 0$ and a bounded function. The same idea is used to show that $u_k, \partial_1 u_k$ are 0 on $z = 0$, for $k \in \{2,3\}$. Finally,

$$\partial_2 u_2(x_1,x_2,x_3) = \partial_z u_z(x_1,z)\frac{x_2^2}{z^2} + u_z(x_1,z)\frac{x_3^2}{z^3} + \partial_z u_\theta(x_1,z)\frac{x_2 x_3}{z^2} - u_\theta(x_1,z)\frac{x_2 x_3}{z^3}$$
$$= \partial_z u_z(x_1,z) + \Big(\frac{u_z(x_1,z)}{z} - \partial_z u_z(x_1,z)\Big)\frac{x_3^2}{z^2} + \frac{x_2 x_3}{z^2}\Big(\frac{-u_\theta(x_1,z)}{z} + \partial_z u_\theta(x_1,z)\Big).$$

Since $u_z(x_1,0) = u_\theta(x_1,0) = 0$, the second and the third term in the above equation are convergent to 0, while the first is convergent to $\partial_z u_z(x_1,0)$. The rest of the terms are treated exactly in the same manner. $\square$

Clearly, all the above considerations can be extended to σ, and we define Y_s to be the completion of $\mathcal{Y}_s$ with respect to the H^1 norm. Define

$$V_s = \{u \in X_s \cap (H^1_{per})^3 \,|\, div\, u = 0, u = 0 \text{ on } \Gamma_\varepsilon\},$$
$$W_s = \{\sigma \in Y_s \cap \mathbb{M}_{3\times3}(H^1_{per}) \,|\, \sigma = 0 \text{ on } \Gamma_\varepsilon\}.$$

Let us denote $L_1(u,\sigma), L_2(u,\sigma)$ the operators involved in the variational formulation (3.3). We have proven in the first part that there exists exactly one solution to the problem

$$\begin{cases} \langle L_1(u,\sigma), v\rangle = 0 & \text{for all } v \in V, \\ \langle L_2(u,\sigma), \tau\rangle = 0 & \text{for all } \tau \in W. \end{cases}$$

In a completely similar manner, it can be proven that there is $(\widehat{u},\widehat{\sigma}) \in V_s \times W_s$ such that

$$\begin{cases} \langle L_1(\widehat{u},\widehat{\sigma}), \widehat{v}\rangle = 0 & \text{for all } \widehat{v} \in V_s, \\ \langle L_2(\widehat{u},\widehat{\sigma}), \widehat{\tau}\rangle = 0 & \text{for all } \widehat{\tau} \in W_s. \end{cases}$$

We intend to prove that $(\widehat{u},\widehat{\sigma})$ is in fact a solution to (3.3). The key ingredient will be the following:

Lemma 3.5.3. *Assume that $f \in \mathcal{X}_s$. Then there exists bounded operators $\Phi : V \to V_s$ and $\Psi : W \to W_s$ with $\Phi v := v^s$ and $\Psi\tau := \tau^s$ such that, for all $(v,\tau) \in V \times W$,*

$$\langle L_1(\widehat{u},\widehat{\sigma}), v\rangle = \langle L_1(\widehat{u},\widehat{\sigma}), v^s\rangle\,, \tag{3.13a}$$

$$\langle L_2(\widehat{u},\widehat{\sigma}), \tau\rangle = \langle L_2(\widehat{u},\widehat{\sigma}), \tau^s\rangle \tag{3.13b}$$

for all $(v,\tau) \in V \times W$.

Note that this readily implies that $(\widehat{u},\widehat{\sigma})$ is solution to the variational problem (3.3).

Proof. We prove only for Φ, since the other one is similar. Let $v \in \mathcal{X}^3$. Then, by Lemma 3.5.1 $v_x, v_z, v_\theta \in C(\overline{D_\varepsilon \times (0,2\pi)})$, and, for all $\theta \in [0,2\pi)$, $v_x(\cdot,\cdot,\theta), v_z(\cdot,\cdot,\theta), v_\theta(\cdot,\cdot,\theta) \in C^1(D'_\varepsilon)$. Denote

$$v^s_x(x,z) = \frac{1}{2\pi}\int_0^{2\pi} v_x(x,z,\theta)d\theta,$$

$$v^s_z(x,z) = \frac{1}{2\pi}\int_0^{2\pi} v_z(x,z,\theta)d\theta,$$

$$v^s_\theta(x,z) = \frac{1}{2\pi}\int_0^{2\pi} v_\theta(x,z,\theta)d\theta.$$

Clearly $v^s_x, v^s_z, v^s_\theta \in C(\overline{D_\varepsilon}) \cap C^1(D'_\varepsilon)$, and, from (3.12a),(3.12b),(3.12c) it follows that $\partial_z v^s_x(x,0) = 0, v^s_z(x,0) = v^s_\theta(x,0) = 0$. Hence, from Lemma 3.5.2, we get that $v^s (= T^{-1}(v^s_x, v^s_z, v^s_\theta)) \in \mathcal{X}_s$. Moreover, it is elementary to see that

$$\|v^s\|_{H^1(\Omega_\varepsilon)} \le \|v\|_{H^1(\Omega_\varepsilon)}. \tag{3.14}$$

If we denote $v^{as} = v - v^s$, then we have the following

$$\int_0^{2\pi} v^{as}_x(x,z,\theta)d\theta = \int_0^{2\pi} v^{as}_z(x,z,\theta)d\theta = \int_0^{2\pi} v^{as}_\theta(x,z,\theta)d\theta = 0.$$

So if $(u, \sigma) \in \mathcal{X}_s \times \mathcal{Y}_s$, then by a change of variables it can be proved that

$$\langle L_1(u, \sigma), v^{as} \rangle = 0,$$

and so by a density argument it follows

$$\langle L_1(\widehat{u}, \widehat{\sigma}), v \rangle = \langle L_1(\widehat{u}, \widehat{\sigma}), v^s \rangle\,,$$

which implies that (3.13a) is true for all $v \in \mathcal{X}^3$. From (3.14) we get that Φ is continuous on $\mathcal{X}^3$, and so it can be uniquely extended to H^1, and (3.14) is still verified for all $v \in H^1$. To conclude we need to show that $\Phi(V) \subset V_s$, that is, for all $v \in V$, v^s is divergence free and satisfies the same boundary conditions as v. The continuity of Φ readily implies that $v^s = 0$ on Γ_ε, while using the following inequalities

$$\begin{aligned} &\|div\, v^s\|_{L^2(\Omega_\varepsilon)} \leq \|div\, v\|_{L^2(\Omega_\varepsilon)} && \forall\, v \in \mathcal{X}^3, \\ &\|v^s(0) - v^s(1)\|_{L^2(\Sigma_0^\varepsilon)} \leq \|v(0) - v(1)\|_{L^2(\Sigma_0^\varepsilon)} && \forall\, v \in \mathcal{X}^3, \end{aligned}$$

which are easily derived by changing of variables, we derive that $div\, v^s = 0$ and $v^s \in H^1_{per}$ through a density argument. $\square$

We will actually prove more. Note that

$$\begin{aligned} V_s &= V_s' \oplus V_s'', \\ W_s &= W_s' \oplus W_s'', \end{aligned}$$

where

$$\begin{aligned} V_s' &= \{v \in V_s \,|\, v_\theta = 0\}, \\ V_s'' &= \{v \in V_s \,|\, v_x = v_z = 0\}, \\ W_s' &= \{\sigma \in W_s \,|\, \sigma_{x\theta} = \sigma_{z\theta} = 0\}, \\ W_s'' &= \{\sigma \in W_s \,|\, \sigma_{xx} = \sigma_{xz} = \sigma_{zz} = \sigma_{\theta\theta} = 0\}. \end{aligned}$$

We intend to show that, provided $f \in \{g \in \mathcal{X}_s \,|\, g_\theta = 0\}$, the solution is in $V_s' \times W_s'$. We use the same trick as before. First, there exists $(\widetilde{u}, \widetilde{\sigma}) \in V_s' \times W_s'$ such that

$$\begin{aligned} \langle L_1(\widetilde{u}, \widetilde{\sigma}), \widetilde{v} \rangle &= 0, \\ \langle L_2(\widetilde{u}, \widetilde{\sigma}), \widetilde{\tau} \rangle &= 0, \end{aligned}$$

for all $(\widetilde{v}, \widetilde{\tau}) \in V_s' \times W_s'$. Pick any $(\widehat{v}, \widehat{\tau}) \in V_s \times W_s$. Then it can be expressed as $(\widetilde{v}, \widetilde{\tau}) + (\overline{v}, \overline{\tau})$ with $(\widetilde{v}, \widetilde{\tau}) \in V_s' \times W_s'$ and $(\overline{v}, \overline{\tau}) \in V_s'' \times W_s''$. To conclude, it rests to prove that

$$\begin{aligned} \langle L_1(\widetilde{u}, \widetilde{\sigma}), \overline{v} \rangle &= 0, \\ \langle L_2(\widetilde{u}, \widetilde{\sigma}), \overline{\tau} \rangle &= 0, \end{aligned}$$

which follows easily from changing the variables.
Finally, we'd like to prove that the recovered pressure is also "axisymmetric" in some sense. We look at $\partial_\theta p$ as a distribution in $D_\varepsilon \times (0, 2\pi)$. Let $q \in C_0^1(D_\varepsilon \times (0, 2\pi))$ be arbitrary but fixed. Then if $v = T^{-1}(0, 0, q)$, evidently $v \in C_0^1(\Omega_\varepsilon)$. The de-Rahm theory leads us to

$$\langle L_1(\widetilde{u}, \widetilde{\sigma}), v\rangle = \int_{\Omega_\varepsilon} p\, div\, v. \tag{3.15}$$

Taking into account the definition of v and the fact that $(\widetilde{u}, \widetilde{\sigma}) \in V_s' \times W_s'$, we get that

$$\langle L_1(\widetilde{u}, \widetilde{\sigma}), v\rangle = 0. \tag{3.16}$$

From (3.15) and (3.16) we derive

$$0 = \int_{\Omega_\varepsilon} p\, div\, v = \int_{D_\varepsilon \times (0,2\pi)} p \partial_\theta q,$$

so that $\partial_\theta p = 0$ in the distributional sense.
Let us summarize the results obtained in this section:

Proposition 3.5.1. *Suppose that $f \in \{g \in \mathcal{X}_s \,|\, g_\theta = 0\}$. Then the solution (u, p) to problem (3.1)–(3.2) belongs to the axisymmetric space $V_s' \times W_s'$. Moreover, the pressure satisfies $\partial_\theta p = 0$ in the distributional sense.*

In cylindrical coordinates, the variational problem writes: Find $(u, \sigma) \in T(V_s' \times W_s')$ such that

$$\begin{aligned}
&\int_{D_\varepsilon} (1-r)\big(z(\partial_x u_x \partial_x v_x + \partial_z u_x \partial_z v_x + \partial_x u_z \partial_x v_z + \partial_z u_z \partial_z v_z) + \frac{u_z v_z}{z}\big) \\
&+ z(\sigma_{xx}\partial_x v_x + \sigma_{xz}(\partial_x v_z + \partial_z v_x) + \sigma_{zz}\partial_z v_z) + \sigma_{\theta\theta} v_z \\
&+ z(u_x \partial_x + u_z \partial_z)(u_x v_x + u_z v_z) = \int_{D_\varepsilon} z f_1 v_x,
\end{aligned} \tag{AS1}$$

$$\begin{aligned}
&\int_{D_\varepsilon} z(u_x\partial_x + u_z\partial_z)(\sigma_{xx}\tau_{xx} + 2\sigma_{xz}\tau_{xz} + \sigma_{zz}\tau_{zz} + \sigma_{\theta\theta}\tau_{\theta\theta}) \\
&+ z(\partial_x u_z - \partial_z u_x)(\sigma_{xz}(\tau_{zz} - \tau_{xx}) - \tau_{xz}(\sigma_{zz} - \sigma_{xx})) \\
&+ z(\sigma_{xx}\tau_{xx} + 2\sigma_{xz}\tau_{xz} + \sigma_{zz}\tau_{zz} + \sigma_{\theta\theta}\tau_{\theta\theta}) \\
&+ \varepsilon^2\big(z(\partial_x\sigma_{xx}\partial_x\tau_{xx} + \partial_z\sigma_{xx}\partial_z\tau_{xx} + 2\partial_x\sigma_{xz}\partial_x\tau_{xz} + 2\partial_z\sigma_{xz}\partial_z\tau_{xz} + \partial_x\sigma_{zz}\partial_x\tau_{zz} \\
&+ \partial_z\sigma_{zz}\partial_z\tau_{zz} + \partial_x\sigma_{\theta\theta}\partial_x\tau_{\theta\theta} + \partial_z\sigma_{\theta\theta}\partial_z\tau_{\theta\theta}) + \frac{\sigma_{xz}\tau_{xz} + \sigma_{zz}\tau_{zz} + \sigma_{\theta\theta}\tau_{\theta\theta}}{z}\big) \\
&= \int_{D_\varepsilon} z(\tau_{xx}\partial_x u_x + \tau_{xz}(\partial_x u_z + \partial_z u_x) + \tau_{zz}\partial_z u_z) + \tau_{\theta\theta} u_z
\end{aligned} \tag{AS2}$$

for all $(v,\tau) \in T(V_s' \times W_s')$. The boundary conditions translate to

$$\begin{cases} u, \sigma = 0 & \text{on } \gamma_\varepsilon, \\ u_z = \sigma_{xz} = \sigma_{zz} = \sigma_{\theta\theta} = 0 & \text{on } C, \\ u, \sigma, p \text{ are } x_1 \text{ periodic}, \end{cases}$$

where we have written the lower and the upper part of the boundary of D_ε as

$$C = \{(x,0) \,|\, x \in [0,1]\}, \quad \gamma_\varepsilon = \{(x, \varepsilon h(x)) \,|\, x \in [0,1]\}.$$

3.6 Numerical results

Observe that equations (AS1), (AS2) can be written

$$\begin{cases} A_1(u,v) + N_1(u,u,v) + A_2(\sigma,v) = l(v), \\ N_2(u,\sigma,\tau) + N_3(u,\sigma,\tau) + A_3(\sigma,\tau) = A_4(u,\tau), \end{cases}$$

where

$$A_1(u,v) = \int_{D_\varepsilon} (1-r)\big(z(\partial_x u_x \partial_x v_x + \partial_z u_x \partial_z v_x + \partial_x u_z \partial_x v_z + \partial_z u_z \partial_z v_z) \\ + \frac{u_z v_z}{z}\big),$$

$$A_2(\sigma,v) = \int_{D_\varepsilon} z(\sigma_{xx}\partial_x v_x + \sigma_{xz}(\partial_x v_z + \partial_z v_x) + \sigma_{zz}\partial_z v_z) + \sigma_{\theta\theta} v_z$$

$$A_3(\sigma,\tau) = \int_{D_\varepsilon} z(\sigma_{xx}\tau_{xx} + 2\sigma_{xz}\tau_{xz} + \sigma_{zz}\tau_{zz} + \sigma_{\theta\theta}\tau_{\theta\theta}) \\ + \varepsilon^2\big(z(\partial_x\sigma_{xx}\partial_x\tau_{xx} + \partial_z\sigma_{xx}\partial_z\tau_{xx} + 2\partial_x\sigma_{xz}\partial_x\tau_{xz} + 2\partial_z\sigma_{xz}\partial_z\tau_{xz} + \partial_x\sigma_{zz}\partial_x\tau_{zz} \\ + \partial_z\sigma_{zz}\partial_z\tau_{zz} + \partial_x\sigma_{\theta\theta}\partial_x\tau_{\theta\theta} + \partial_z\sigma_{\theta\theta}\partial_z\tau_{\theta\theta}) + \frac{\sigma_{xz}\tau_{xz} + \sigma_{zz}\tau_{zz} + \sigma_{\theta\theta}\tau_{\theta\theta}}{z}\big),$$

$$A_4(u,\tau) = \int_{D_\varepsilon} z(\tau_{xx}\partial_x u_x + \tau_{xz}(\partial_x u_z + \partial_z u_x) + \tau_{zz}\partial_z u_z) + \tau_{\theta\theta} u_z,$$

$$N_1(u,w,v) = \int_{D_\varepsilon} z(u_x\partial_x + u_z\partial_z)(w_x v_x + w_z v_z),$$

$$N_2(u,\sigma,\tau) = \int_{D_\varepsilon} z(u_x\partial_x + u_z\partial_z)(\sigma_{xx}\tau_{xx} + 2\sigma_{xz}\tau_{xz} + \sigma_{zz}\tau_{zz} + \sigma_{\theta\theta}\tau_{\theta\theta}),$$

$$N_3(u,\sigma,\tau) = \int_{D_\varepsilon} z(\partial_x u_z - \partial_z u_x)(\sigma_{xz}(\tau_{zz} - \tau_{xx}) - \tau_{xz}(\sigma_{zz} - \sigma_{xx})),$$

$$l(v) = \int_{D_\varepsilon} z f_1 v_x.$$

Numerically, we will be interested in the following discrete problem: Find $(u_h, \sigma_h, p_h) \in V_h \times W_h \times Q_h$ such that

$$\begin{cases} A_1(u_h, v_h) + N_1(u_h, u_h, v_h) + A_2(\sigma_h, v_h) + \displaystyle\int_{D_\varepsilon} p_h div\, v_h = l(v_h), \\ N_2(u_h, \sigma_h, \tau_h) + N_3(u_h, \sigma_h, \tau_h) + A_3(\sigma_h, \tau_h) = A_4(u_h, \tau_h), \\ \displaystyle\int_{D_\varepsilon} q_h div\, u_h = 0, \end{cases} \tag{3.17}$$

for all $(v_h, \tau_h, q_h) \in V_h \times W_h \times Q_h$ where V_h, W_h, Q_h are appropriate finite-dimensional subspaces. The numerical analysis of the above problem is outside the scope of this paper - we refer to [11] for a detailed study of the axisymmetric Navier-Stokes system. Here, we shall present two algorithms for solving the problem (3.17) numerically by linearization. We drop the subscript h for convenience.

Method I. Start with $u^0 = 0, \sigma^0 = 0$. Define $u^{n+1}, \sigma^{n+1}, p^{n+1}$ by

$$\begin{cases} A_1(u^{n+1}, v) + N_1(u^n, u^{n+1}, v) + A_2(\sigma^n, v) + \displaystyle\int_{D_\varepsilon} p^{n+1} div\, v = l(v), \\ N_2(u^n, \sigma_h, \tau) + N_3(u^{n+1}, \sigma^{n+1}, \tau) + A_3(\sigma^{n+1}, \tau) = A_4(u^{n+1}, \tau), \\ \displaystyle\int_{D_\varepsilon} q div\, u^{n+1} = 0 \end{cases}$$

for all $(v, \tau, q) \in V_h \times W_h \times Q_h$.

Method II. Start with $u^0 = 0, \sigma^0 = 0$. Define $u^{n+1}, \sigma^{n+1}, p^{n+1}$ by

$$\begin{cases} A_1(u^{n+1}, v) + N_1(u^n, u^{n+1}, v) + A_2(\sigma^{n+1}, v) + \displaystyle\int_{D_\varepsilon} p^{n+1} div\, v = l(v), \\ N_2(u^n, \sigma_h, \tau) + N_3(u^n, \sigma^{n+1}, \tau) + A_3(\sigma^{n+1}, \tau) = A_4(u^n, \tau), \\ \displaystyle\int_{D_\varepsilon} q div\, u^{n+1} = 0 \end{cases}$$

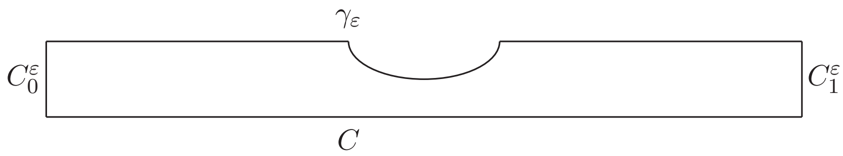
for all $(v, \tau, q) \in V_h \times W_h \times Q_h$.

Note the subtle but crucial difference. While the first method involves splitting the system and treating them separately, the second one solves the system as a whole. The first method is computationally faster but has the disadvantage of not converging for $r \geq 0.88$. Let us now consider a specific example on which we will base our numerical simulations. Assume that the domain is axisymmetric and that D_ε is completely described by the boundary (see the picture below)

$$\begin{aligned}
C_0^\varepsilon &= \{(0,y)\,|\,y \in (0,\varepsilon)\},\\
C_1^\varepsilon &= \{(1,y)\,|\,y \in (0,\varepsilon)\},\\
C &= \{(x,0)\,|\,x \in (0,1)\},\\
\gamma_\varepsilon &= \{(x,\varepsilon h(x))\,|\,x \in (0,1)\},
\end{aligned}$$

where

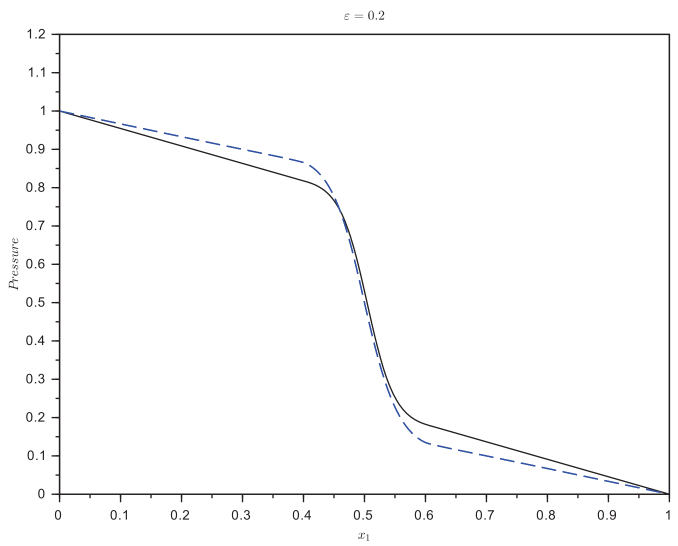
$$h(x) = \begin{cases} 1, & x \in (0,0.4),\\ 1 - \frac{1}{2}\sin(5\pi(x-0.4)), & x \in (0.4,0.6),\\ 1, & x \in (0.6,1). \end{cases}$$



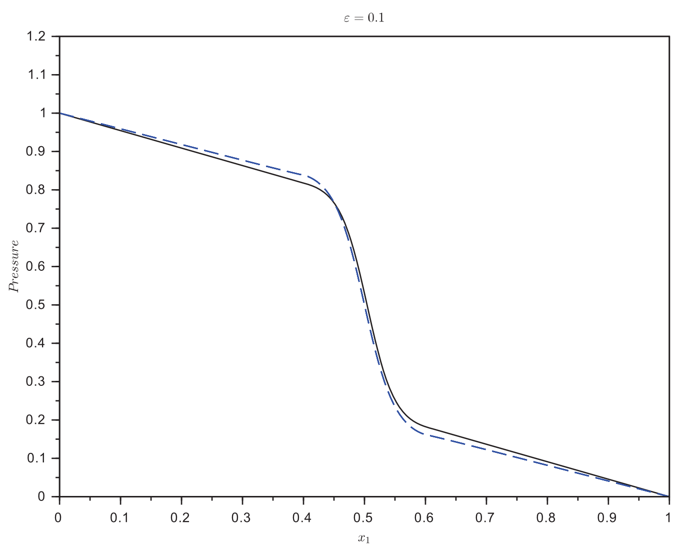
Further we present numerical results relating the pressures obtained via the methods presented thus far. When computing the numerical solution, no significant different was found between Method I and Method II (at the required precision related to the convergence process). In the next four graphs we compare p^ε (dashed) – the pressures calculated through Method I (or Method II since there is virtually no difference between them) to and p_a (black) the pressure computed through the asymptotic technique for $\varepsilon \in \{0.2, 0.1, 0.05, 0.01\}$. Observe that we are not plotting p^ε, but rather its average with respect to the radial direction, that is:

$$\overline{p}^\varepsilon(x_1) = \frac{1}{\varepsilon h(x_1)} \int_0^{\varepsilon h(x_1)} p^\varepsilon(x_1,z)dz.$$

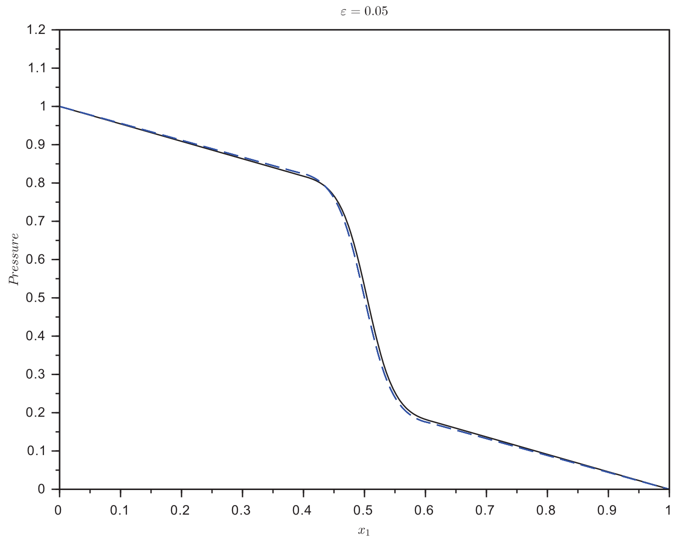
The simulations have ben done using the FreeFem++ solver – we have chosen to work with P2/P2/P1 finite elements (for the velocity, stress and pressure respectively). We have used everywhere $r = 0.5$, $p_d = 1$, $f_1 = 0$. Computations for other values of r and p_d have been performed without significant qualitative modification of the results.

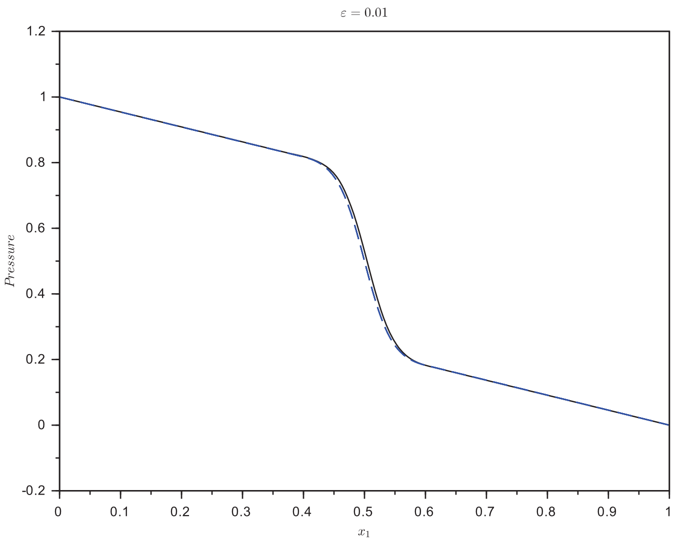


$\varepsilon = 0.2$
Pressure
x_1

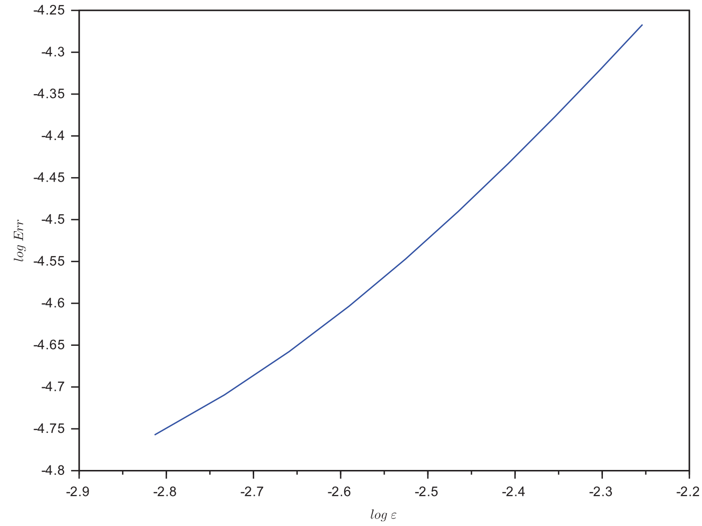


$\varepsilon = 0.1$
Pressure
x_1





Furthermore, we show the error $\|\overline{p}^{\varepsilon} - p_a\|_{L^2(0,1)}$ versus ε in a $\log - \log$ plot (for $\varepsilon \in [0.05, 0.1]$). The plot is close to a line of slope 1, which is consistent with the results shown in Section 3.



We conclude by considering the case of a non axis-symmetric domain, namely a cylinder with an upper constriction (see figure below). The domain is formally described by defining

$$S(x_1) = \begin{cases} D(0,1), & x \notin (0.4, 0.6), \\ D(0,1) \cap \{(x_2, x_3) \,|\, x_3 \leq 10|x_1 - 0.5|\}, & x \in [0.4, 0.6]. \end{cases}$$

where $D(0,1) = \{(x_2, x_3) \,|\, x_2^2 + x_3^2 \leq 1\}$. We compute numerically the asymptotic pressure (dashed line) in order to show the influence of the geometry on the flow - in black, the asymptotic pressure recovered in the axisymmetric case (where the constriction is double in size).

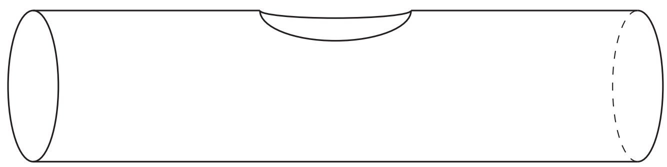
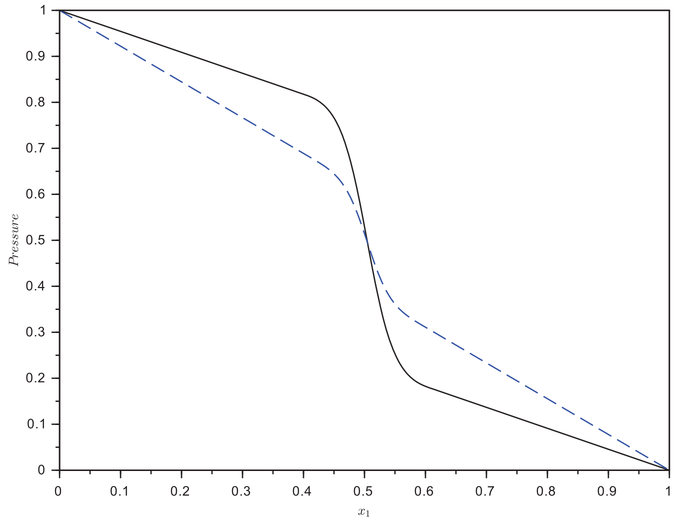


Figure 3.1: Non axis-symmetric domain



Pressure
x_1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Chapter 4

Existence of global strong solutions for the Navier Stokes equation with pressure boundary conditions

4.1 Introduction and problem statement

The results obtained in the first two chapters apply to *stationary* flows. In this chapter we shall study the *evolutionary* case, while using once more boundary conditions involving the pressure. We shall restrict ourselves to the study of the Navier-Stokes equations. The equations can be written as

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) \cdot u + \nabla p = f, \\ div\, u = 0, \\ u(0) = u^0, \end{cases} \tag{4.1}$$

with u the velocity and p the pressure being the unknowns, while f represents the exterior forces acting on the fluid and u^0 the initial velocity. With respect to the boundary conditions, we impose

$$\begin{cases} u = 0 & \text{on} \quad \Gamma, \\ u \times n = 0 & \text{on} \quad \Sigma_0 \cup \Sigma_1, \\ p = p^i & \text{on} \quad \Sigma_i. \end{cases} \tag{4.2}$$

where $p^0, p^1 > 0$ are given constants. This final chapter is divided as follows:

- In the first part we work in the renormalized domain Ω and use classical energy estimates in order to prove the existence of global strong solutions for small initial data. Beforehand, we discuss the properties of the Stokes operator related to our boundary conditions and prove a regularity result similar to the one for Dirichlet boundary conditions.

- In the second part we come back to the thin domain Ω_ε and show, using the theory of linear semigroups, the existence of global strong solutions *when the initial data belongs to "large sets"*.

4.2 The Stokes operator

For the remainder of this chapter, assume that:

$$\measuredangle(\Sigma_i, \Gamma) \geq \pi/2 \quad \forall\, i \in \{1, 2\} \tag{4.3}$$

(note that this is a weaker condition than the (PNB) one used in the previous chapters).
Let $f \in L^2(\Omega)$ and let us consider the stationary Stokes problem

$$\begin{cases} -\Delta u + \nabla p = f, \\ div\, u = 0, \end{cases} \tag{4.4}$$

with the boundary conditions (4.2). Call this system problem (S). The goal of this section is to show a regularity result for problem (S), similar to that for the standard case of Dirichlet boundary conditions. As in Chapter 2, it is easy to see that the problem (S) has a unique weak solution $(u, p) \in (H^1, L^2)$. The Stokes operator $A : V \to V'$ is defined as follows

$$\langle Au, v \rangle = \int_\Omega \nabla u : \nabla v - p_d \Phi_v, \quad \forall\, v \in V.$$

We intend to prove that $f \in L^2 \implies (u, p) \in H^2 \times H^1$ and

$$\|u\|_{H^2} + \|p\|_{H^1} \leq C(\|f\| + p^0 + p^1)$$

The regularity result follows closely the presentation in [14] – note that the proof is very long and technical. Since the study of this problem is well outside the scope of this thesis, we are going to sketch only the main ideas in showing how the proof should be modified to take into account the changes in the boundary conditions. We shall use the same notations as in [14] for the convenience of the reader. The proof is divided into 2 parts:

- Interior estimates (identical to the general case).
- Estimates near the boundary.

For the second part, cover $\partial\Omega \setminus (\partial\Sigma_0 \cup \partial\Sigma_1)$ with a finite number of open balls such that the balls covering Σ_0 and Σ_1 are distinct from those covering Γ (this can be done, owing to condition (4.3)). Near Γ, the same techniques used in [14] can be employed, since we have Dirichlet boundary conditions. It remains to prove the estimates near Σ_0 (for Σ_1, it is evidently the same thing). To achieve the proof we need only to show a result similar to the one in Theorem IV.5.1. in [14]. Without restricting the generality, we can assume that $0 \in \Sigma_0$. Before presenting the result we make some remarks on notation. We write $\Sigma = \{x \in \mathbb{R}^3 \,|\, x_1 = 0\}$ and $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 \,|\, x_1 \geq 0\}$. For $(0, x') \in \Sigma_0$ ($x' = (x_2, x_3) \in \Sigma$) and $r > 0$ we denote $B_r = \{x \,|\, x_1 > 0, \|x\| < r\}$. Given any function defined on $\mathbb{R}^3$ (or $\mathbb{R}^3_+$) we shall use the notation $f(0) = f(0, x')$. Moreover, by D^1u we denote the gradient of u, while D^2u represents the matrix of second derivatives of u. The homogeneous Sobolev spaces for $E = \mathbb{R}^3$ or $E = \mathbb{R}^3_+$ are defined as follows:

$$D^{i,2}(E) = \{u \in L^1_{loc}(E) \,|\, D^i u \in L^2(E)\}, \quad \forall\, i \in \{1, 2\}.$$

with their respective seminorms $|\cdot|_{D^{i,2}(E)}$. Finally, the space of traces of functions in $D^{1,2}(\mathbb{R}^3_+)$ is $D^{1/2,2}(\Sigma)$, which turns into a Banach space with respect to the norm $\|\cdot\|_{1/2,2}$. A detailed study of these spaces can be found in [14]. We can now pass to the statement and sketch of the proof of the main result.

Theorem 4.2.1. *Let $r > 0$ such that $B_r \subset \Omega$. Then for all $r' < r$, we have*

$$u \in H^2(B_{r'}), \quad p \in H^1(B_{r'}),$$

and, in addition,

$$\|u\|_{H^2(B_{r'})} + \|p\|_{H^1(B_{r'})} \leq C(\|f\|_{L^2(B_r)} + p^0 + \|u\|_{H^1(B_r)} + \|p\|_{L^2(B_r)}).$$

Sketch of the proof. Let us denote

$$K_{B_r} = \|f\|_{L^2(B_r)} + p^0 + \|u\|_{H^1(B_r)} + \|p\|_{L^2(B_r)}.$$

Consider $r' < r'' < r$. Take $\varphi \in C^\infty(\mathbb{R}^3_+)$ with

$$\begin{cases} \varphi = 1 & \text{on} \quad B_{r'}, \\ \varphi = 0 & \text{on} \quad \mathbb{R}^3_+ \setminus B_{r''}. \end{cases}$$

Define

$$\begin{cases} v = \varphi u,\ \pi = \varphi p & \text{on} \quad B_r, \\ v, \pi = 0 & \text{on} \quad \mathbb{R}^3_+ \setminus B_r. \end{cases}$$

Then we can write

$$\begin{cases} -\Delta v + \nabla \pi = F & \text{on } \mathbb{R}^3_+, \\ div\, v = g & \text{on } \mathbb{R}^3_+, \\ v_2 = v_3 = 0 & \text{on } \Sigma, \\ \pi = \pi^0 & \text{on } \Sigma, \end{cases} \tag{4.5}$$

where $F=\varphi f-2\nabla u\cdot\nabla\varphi-u\Delta\varphi+p\nabla\varphi,\ g=u\nabla\varphi,\ \pi^0=\varphi p^0$. It is clear that

$$\begin{aligned}&\|F\|_{L^2(\mathbb{R}^3_+)}\le C(\|f\|_{L^2(B_r)}+\|u\|_{H^1(B_r)}+\|p\|_{L^2(B_r)}),\\&\|g\|_{H^1(\mathbb{R}^3_+)}\le C\|u\|_{H^1(B_r)},\\&\|\pi^0\|_{H^1(\Sigma)}\le Cp^0.\end{aligned}\tag{4.6}$$

The main idea is to show that the problem (4.5) has a unique solution $(u,p)\in D^{2,2}(\mathbb{R}^3_+)\times D^{1,2}(\mathbb{R}^3_+)$. By taking the divergence in the first equation in (4.5), we get the following equation for π:

$$\begin{cases}\Delta\pi=div\,h & \text{on } \mathbb{R}^3_+,\\ \pi=\pi^0 & \text{on } \Sigma,\end{cases}\tag{4.7}$$

with $h=F+\nabla g\in L^2(\mathbb{R}^3_+)$. We can approximate F in L^2 norm by $F_n\in C^3_0(\mathbb{R}^3_+)$ and g in $D^{1,2}$ norm by $g_n\in C^4_0(\overline{\mathbb{R}^3_+})$ - see Theorem II.7.8 in [14]. Hence $h_n=F_n+\nabla g_n\to h$ in L^2. By classical reflection techniques, extend h_n to $C^3_0(\mathbb{R}^3)$ functions $\widetilde{h}_n$ such that

$$\|\widetilde{h}_n-\widetilde{h}_m\|_{L^2(\mathbb{R}^3)}\le C\|h_n-h_m\|_{L^2(\mathbb{R}^3_+)}.\tag{4.8}$$

Then $r_n=E*\widetilde{h}_n\in C^3(\mathbb{R}^3)$ satisfies $\Delta r_n=\widetilde{h}_n$, with E the fundamental solution to the Laplace equation $\left(E(x)=\frac{1}{4\pi|x|}\right)$, and the following estimate holds

$$\|D^2(r_n-r_m)\|_{L^2(\mathbb{R}^3)}\le C\|\widetilde{h}_n-\widetilde{h}_m\|_{L^2(\mathbb{R}^3)},\tag{4.9}$$

like in Exercise II.11.9 in [14]. Note that $\pi^1_n=div\,r_n\in C^2(\mathbb{R}^3)$ solves $\Delta\pi^1_n=div\,\widetilde{h}_n$, and, using (4.8) and (4.9) we obtain

$$\|D^1(\pi^1_n-\pi^1_m)\|_{L^2(\mathbb{R}^3)}\le C\|h_n-h_m\|_{L^2(\mathbb{R}^3_+)}.$$

Since $h_n\to h$ in $L^2(\mathbb{R}^3_+)$, it follows that π^1_n is Cauchy in $D^{1,2}(\mathbb{R}^3)$ so that $\pi^1_n\to\pi^1$ in $D^{1,2}(\mathbb{R}^3)$ and

$$\|D^1\pi^1\|_{L^2(\mathbb{R}^3)}\le C\|h\|_{L^2(\mathbb{R}^3_+)}.\tag{4.10}$$

From Theorem II.10.2 in [14], we have that $\pi^1(0)\in D^{1/2,2}(\Sigma)$ and, with (E2),

$$\|\pi^1(0)\|_{1/2,2}\le C\|D^1\pi^1\|_{L^2(\mathbb{R}^3)}\le C\|h\|_{L^2(\mathbb{R}^3_+)}.$$

Next we show that $\pi^1_n(0)\in L^2(\Sigma)$ for all n. Clearly, there exists $R_n>0$ such that supp $\widetilde{h}_n\subset B(0,R_n)$. Since $\pi^1_n(0)\in C(\Sigma)$ we need only to show that

$$\int_{|x'|>2R_n}|\pi^1_n(0)|^2dx'<\infty.\tag{4.11}$$

We have that

$$|\pi_n^1(x)| = |div\, E * \widetilde{h}_n| \leq C \left| \int_{|y|<R_n} \frac{|x_i - y_i|}{|x-y|^3} \widetilde{h}_n(y) dy \right|,$$

and so it is trivial to show that

$$\begin{aligned}
\int_{|x'|>2R_n} |\pi_n^1(0)|^2 &\leq C \int_{|x'|>2R_n} \left(\int_{|y|<R_n} \frac{1}{|(0,x') - y|^2} dy \right)^2 dx' \\
&\leq C \int_{|x'|>2R_n} \frac{1}{(|x'| - R_n)^4} dx' \\
&\leq C \int_{R_n}^{\infty} \frac{1}{r^4} dr < \infty,
\end{aligned}$$

thus proving (4.11). Following Exercise II.11.10 in [14], there exist $\pi_n^2 \in D^{1,2}(\mathbb{R}^3_+)$ such that

$$\begin{cases} \Delta \pi_n^2 = 0 & \text{on } \mathbb{R}^3_+, \\ \pi_n^2 = \pi^0 - \pi_n^1(0) & \text{on } \Sigma, \end{cases}$$

satisfying

$$\begin{cases} \|D^1\pi_n^2 - D^1\pi_m^2\|_{L^2(\mathbb{R}^3)} \leq C\|\pi_m^2(0) - \pi_n^1(0)\|_{1/2,2}, \\ \|D^1\pi_n^2\|_{L^2(\mathbb{R}^3)} \leq C\|\pi_0 - \pi_n^1(0)\|_{1/2,2} \leq C(\|h\|_{L^2(\mathbb{R}^3_+)} + p^0). \end{cases}$$

Hence π_n^2 is Cauchy in $D^{1,2}(\mathbb{R}^3_+)$, so that $\pi_n^2 \to \pi^2$ in $D^{1,2}(\mathbb{R}^3_+)$ and, moreover,

$$\|D^1\pi_n^2\|_{L^2(\mathbb{R}^3)} \leq C(\|h\|_{L^2(\mathbb{R}^3_+)} + p^0). \tag{4.12}$$

The uniqueness of the solution to the problem (4.7) is proved in [14] - see remark II.11.3. Hence it is easy to show that the function $\pi = \pi^1 + \pi^2$ is the desired solution. Taking into account (4.10) and (4.12), as well as the expression for h and estimates (4.6) we get

$$\|D^1\pi\|_{L^2(\mathbb{R}^3)} \leq CK_{B_r}. \tag{4.13}$$

Using the same complete arguments as above and the estimate (4.13) we can prove that $v_2, v_3 \in D^{2,q}(\mathbb{R}^3_+)$ and

$$\|D^2 v_{2,3}\|_{L^2(\mathbb{R}^3)} \leq CK_{B_r}.$$

The final step is to find v_1. We have

$$\begin{cases} -\Delta v_1 = F_1 - \partial_1 \pi, \\ \partial_1 v_1 = k, \end{cases} \tag{4.14}$$

where $k = g - \partial_2 v_2 - \partial_3 v_3 \in D^{1,p}(\mathbb{R}^3_+)$. Note that this is not, a-priori, a Neumann problem since the second condition has to hold in all half-space. However, observe that

$$\partial_1(F_1 - \partial_1 \pi) = -\Delta k,$$

and this compatibility condition ensures the existence of the solution. One can readily prove that the solution is actually unique in the space $D^{2,2}(\mathbb{R}^3_+)/M$, where

$$M = \{\Phi(x_2, x_3) \,|\, \Delta\Phi = 0\}.$$

To obtain estimates on the solution, reason the same as for π to find $v_1^1 \in D^{2,2}(\mathbb{R}^3)$ such that $-\Delta v_1^1 = F_1 - \partial_1 \pi$ in $\mathbb{R}^3$ and

$$\|D^2 v_1^1\|_{L^2(\mathbb{R}^3)} \leq C\|F_1 - \partial_1 \pi\|_{L^2(\mathbb{R}^3_+)} \leq C K_{B_r}.$$

Finally, consider the problem

$$\begin{cases} \Delta v_1^2 = 0 & \text{on } \mathbb{R}^3_+, \\ \partial_1 v_1^2(0) = k(0) - \partial_1 v_1^1(0) & \text{on } \Sigma. \end{cases} \tag{4.15}$$

Since $k, \partial_1 v_1^1 \in D^{1,2}(\mathbb{R}^3_+)$ we have $\phi = k(0) - \partial_1 v_1^1(0) \in D^{1/2,2}(\Sigma)$ and

$$\|\phi\|_{1/2,2} \leq C\|D^1(k - \partial_1 v_1^1)\|_{L^2(\mathbb{R}^3_+)} \leq C K_{B_r}.$$

It is well-known that a solution to the problem (4.15) is given by

$$v_1^2(x) = -\frac{1}{8\pi}\int_\Sigma \frac{\phi(y)}{|x-y|}dy,$$

and so

$$D_{x_i} v_1^2 = -\frac{1}{8\pi}\int_\Sigma \frac{\phi(y)(x_i - y_i)}{|x-y|^3}dy$$

for all $i \in \{1, 2, 3\}$. By using Theorem II.11.6. in [14], we get that $D_{x_i} v_1^2 \in D^{1,2}$ and

$$\|D^1(D_{x_i} v_1^2)\|_{L^2(\mathbb{R}^3_+)} \leq C\|\phi\|_{1/2,2} \leq C K_{B_r}.$$

Combining the obtained results, we have that the unique solution in the quotient space $D^{2,2}(\mathbb{R}^3_+)/M$ to the problem (4.14) satisfies

$$|v_1|_{D^{2,2}(\mathbb{R}^3_+)/M} \leq C K_{B_r},$$

which completes the proof. $\square$

4.3 Existence result via energy estimates

Observe that, without restricting the generality, we can suppose that $p^0 = p^1 = 0$. Indeed, by defining

$$q(x_1, x_k) = p(x_1, x_k) + p_d x_1 - p^0,$$

we have $q(0) = q(1) = 0$, and the first equation in (4.1) becomes

$$\partial_t u - \Delta u + (u \cdot \nabla) \cdot u + \nabla q = f + p_d e_1.$$

Since the constant extra term $p_d e_1$ is irrelevant in the mathematical analysis of the system, our reasoning is complete.
Then the operator A becomes

$$\langle Au, v \rangle = \int_\Omega \nabla u : \nabla v, \quad \forall\, v \in V.$$

Note that A is linear; moreover, as in the standard case, there exists a basis in V formed of eigenvectors of the operator A. Let $\{z_1, z_2, \dots\}$ be this basis and let $\{\lambda_1, \lambda_2, \dots\}$ be the corresponding eigenvalues. Obviously, we can choose $\{z_1, z_2, \dots\}$ to be orthonormal in L^2. The regularity result implies that there exists $C_1 > 0$ such that

$$\|u\|_{H^2} \leq C_1 \|Au\|, \quad \forall\, u \in H^2 \cap V.$$

Hence $u \mapsto \|Au\|$ is a norm on $H^2 \cap V$ equivalent to the H^2 norm. Moreover, from standard Sobolev embeddings, we have that there exists a constant C_2 such that

$$\|u\|_{L^\infty} \leq C_2 \|u\|_{H^2}, \quad \forall\, u \in H^2.$$

By combining this results we get that

$$\|u\|_{L^\infty} \leq C_0 \|Au\|, \quad \forall\, u \in H^2 \cap V.$$

where $C_0 = C_1 C_2$.
The goal of this section is to prove the following result:

Theorem 4.3.1. *Let $T > 0$ be arbitrary. Provided $f \in L^2L^2$, $u^0 \in V$ such that $\|f\|_{L^2L^2}$ and $\|u^0\|_V$ are small enough there exists exactly one $u \in L^2H^2 \cap L^\infty V$ with $u' \in L^2L^2$ such that*

$$\begin{aligned}
&\int_\Omega u' \cdot w + \nabla u : \nabla w + (u \cdot \nabla)u \cdot w = \int_\Omega f \cdot w \quad \forall w \in V, \textit{a.e. } t \in [0,T], \\
&u(0) = u^0.
\end{aligned} \tag{4.16}$$

Proof. Note first that the second equality in (4.16) has a sense since u is at least $C([0,T]; L^2)$. To prove the result, we discuss the linearized version of the problem, namely

$$\begin{aligned}
&\int_\Omega u' \cdot w + \nabla u : \nabla w + (v \cdot \nabla)u \cdot w = \int_\Omega f \cdot w \quad \forall w \in V, \text{a.e. } t \in [0,T], \\
&u(0) = u^0.
\end{aligned} \tag{4.17}$$

Let us first consider the following space:

$$X = \{u \in L^2H^2 \cap L^\infty V,\, u' \in L^2L^2\},$$

endowed with the following norm, which makes it a Banach space:

$$\|u\|_X^2 = \|u\|_{L^\infty V}^2 + \|Au\|_{L^2L^2}^2 + \|u'\|_{L^2L^2}^2.$$

Let $B_k(X) = \{u \in X,\, \|u\|_X \leq k\}$. We intend to show that, provided $\|v\|_X$ is small enough, the problem (4.17) has a solution $u \in X$. This allows us to define the mapping $T : X \to X$, $T(v) = u$. The idea is to prove that, if k is small, $T(B_k(X)) \subset B_k(X)$ and T is a contraction. From Banach's fixed point theorem, we obtain the desired solution to problem (4.16).
We start by looking for an approximate solution to the problem (4.17) of the form

$$u_n(t,x) = \sum_{i=1}^n g_{in}(t) z_i(x),$$

chosen such that

$$\begin{aligned} &g'_{in} + \sum_{j=1}^n g_{jn} \int_\Omega \nabla z_j : \nabla z_i + (v(t)\cdot\nabla) z_j \cdot z_i = \int_\Omega f \cdot z_i, \\ &g_{in}(0) = \int_\Omega u_n^0 \cdot z_i, \end{aligned} \tag{4.18}$$

for all $i \in \{1, 2, \dots, n\}$, where u_n^0 is the projection of u^0 on $V_n = \operatorname{span}\{z_1, z_2, \dots, z_n\}$. Clearly

$$\|u_n^0\|_V \leq \|u^0\|_V, \quad \forall\, n. \tag{4.19}$$

Since $v \in C([0,T]; L^2)$, the function $t \mapsto \int_\Omega (v(t)\cdot\nabla) z_j \cdot z_i$ is continuous. Hence, by using a classical result in the theory of ordinary differential equations, we get that the above system has a unique solution g_n defined on a maximal interval $[0, t_n)$. We show next that $t_n = T$, for all n. By multiplying in (4.18) by $\lambda_i g_{in}$ and summing for $i \in \{1, 2, \dots, n\}$ we get that

$$\frac{d}{dt}\left(\frac{\|u_n\|_V^2}{2}\right) + \|Au_n\|^2 + \int_\Omega (v(t)\cdot\nabla u_n)\cdot Au_n = \int_\Omega f \cdot Au_n,$$

and, using the Hölder inequality, we obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{\|u_n\|_V^2}{2}\right) + \|Au_n\|^2 &\leq \|v(t)\|_{L^\infty}\|u_n\|_V\|Au_n\| + \|f\|\|Au_n\| \\ &\leq \|v(t)\|_{L^\infty}\|u_n\|_V\|Au_n\| + \frac{\|f\|^2}{2} + \frac{\|Au_n\|^2}{2}. \end{aligned}$$

Take $s_n < t_n$. By integrating from 0 to $t \leq s_n$ in the above equation it follows that

$$\frac{\|u_n(t)\|_V^2}{2} + \frac{1}{2}\int_0^t \|Au_n\|^2 \leq \|v\|_{L^2L^\infty} \sup_{[0,s_n]} \|u_n\|_V \|Au_n\|_{L^2(0,s_n;L^2)} + \frac{\|f\|_{L^2L^2}^2}{2} + \frac{\|u_n^0\|_V^2}{2}$$
$$\leq \frac{\|v\|_{L^2L^\infty}}{2}\left(\sup_{[0,s_n]} \|u_n\|_V^2 + \int_0^{s_n} \|Au_n\|^2\right) + \frac{\|f\|_{L^2L^2}^2}{2} + \frac{\|u_n^0\|_V^2}{2}.$$

By taking the supremum in the above inequality and using (4.19) we get

$$(1 - C_0\|v\|_X)\left(\sup_{[0,s_n]} \|u_n\|_V^2 + \int_0^{s_n} \|Au_n\|^2\right) \leq \|f\|_{L^2L^2}^2 + \|u^0\|_V^2,$$

and so, if $\|v\|_X < \frac{1}{C_0}$ we get

$$\sup_{t\in[0,s_n]} \|u_n\|_V^2 \leq M,$$
$$\int_0^{s_n} \|Au_n\|^2 \leq M,$$

for some M independent of s_n and n. Since this is true for all $s_n < t_n$ – the supposed maximum domain of definition, it follows that g_n and hence u_n can be properly defined on $[0,T]$. Moreover, we have

$$(1 - C_0\|v\|_X)\left(\|u_n\|_{L^\infty V}^2 + \int_0^T \|Au_n\|^2\right) \leq \|f\|_{L^2L^2}^2 + \|u^0\|_V^2, \tag{4.20}$$

and so we have proven

$$u_n \text{ is bounded in } L^\infty V \cap L^2H^2. \tag{4.21}$$

By multiplying in (4.18) by g'_{in} and summing for $i \in \{1, 2, \dots, n\}$ we derive

$$\|u'_n\|^2 + \frac{d}{dt}\left(\frac{\|u_n\|_V^2}{2}\right) + \int_\Omega (v(t)\cdot\nabla)u_n \cdot u'_n = \int_\Omega f \cdot u'_n,$$

By using the same arguments as above it follows easily the following estimate

$$(1 - C_0\|v\|_X)\left(\int_0^T \|u'_n\|^2 + \|u_n\|_{L^\infty V}^2\right) \leq \|f\|_{L^2L^2}^2 + \|u^0\|_V^2 \tag{4.22}$$

and so, if $\|v\|_X < \frac{1}{C_0}$

$$u'_n \text{ is bounded in } L^2L^2. \tag{4.23}$$

From (4.21) and (4.23) we get that, on a subsequence

$$\begin{cases} u_n \to u \text{ weakly in } L^2H^2, \\ u_n' \to u' \text{ weakly in } L^2L^2, \\ u_n \to u \text{ weakly star in } L^\infty V. \end{cases}$$

Since there are no nonlinear terms, it is now trivial to pass to the limit (for instance weakly $L^2(0,T)$) to find that u is verifies the first equality in (4.17). The recovery of the initial condition in (4.17) is done as in the standard case – see for instance [42]. Let then $v \in B_k(X)$ with $k < \frac{1}{C_0}$. Then $u \in X$, and, from (4.20) and (4.22) we have

$$\|u\|_X^2 \leq \frac{\|f\|_{L^2L^2}^2 + \|u^0\|_V^2}{1 - kC_0}, \tag{4.24}$$

Obviously, for fixed $k > 0$ we can choose $\|f\|_{L^2L^2}^2 + \|u^0\|_V^2$ small enough so that $\|u\|_X \leq k$. It rests to prove that T is a contraction. Let then $v_1, v_2 \in X$ and u_1, u_2 the corresponding solutions. Then

$$\int_\Omega u_i' \cdot w + \eta A u_i \cdot w + (v_i \cdot \nabla) u_i \cdot w = \int_\Omega f \cdot w \quad \text{a.e. } t \in [0,T],\ i \in \{1,2\}.$$

Let us denote $\overline{v} = v_1 - v_2, \overline{u} = u_1 - u_2$. Note that we would like to take in the above equations $w = A\overline{u}$ and subtract them. However, we cannot legitimately do so since $A\overline{u}$ does not belong to V. To circumvent this obstruction we take as test function z_i and multiply by $\lambda_i \overline{g}_{in}$ - $\overline{g}_{in} = g_{in1} - g_{in2}$ where g_{in1}, g_{in2} are the functions appear in the Galerkin approximations of u_1, u_2. Hence we obtain

$$\int_\Omega u_i' \cdot A\overline{u}_n + Au_i \cdot A\overline{u}_n + (v_i \cdot \nabla) u_i \cdot A\overline{u}_n = \int_\Omega f \cdot A\overline{u}_n \quad \text{a.e. } t \in [0,T],\ i \in \{1,2\}.$$

Since clearly $A\overline{u}_n \to A\overline{u}$ weakly L^2L^2 we can pass to the weak $L^2(0,T)$ limit in the above equation to find

$$\int_\Omega u_i' \cdot A\overline{u} + Au_i \cdot A\overline{u} + (v_i \cdot \nabla) u_i \cdot A\overline{u} = \int_\Omega f \cdot A\overline{u} \quad \text{a.e. } t \in [0,T],\ i \in \{1,2\},$$

and so by subtracting the two equations we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{\|\overline{u}\|_V^2}{2}\right) + \|A\overline{u}\|^2 &= \int_\Omega (\overline{v} \cdot \nabla) u_1 \cdot A\overline{u} + \int_\Omega (v_2 \cdot \nabla)\overline{u} \cdot A\overline{u} \\ &\leq \|\overline{v}\|_{L^\infty} \|u_1\|_V \|A\overline{u}\| + \|v_2\|_{L^\infty} \|\overline{u}\|_V \|A\overline{u}\|. \end{aligned}$$

Integrating from 0 to T in the above inequality we find

$$\begin{aligned} \frac{\|\overline{u}\|_{L^\infty V}^2}{2} + \|A\overline{u}\|_{L^2L^2}^2 &\leq \|\overline{v}\|_{L^2L^\infty} \|u_1\|_{L^\infty V} \|A\overline{u}\|_{L^2L^2} + \|v_2\|_{L^2L^\infty} \|\overline{u}\|_{L^\infty V} \|A\overline{u}\|_{L^2L^2} \\ &\leq C_0 k \left(\frac{\|\overline{u}\|_X^2}{2} + \|\overline{u}\|_X \|\overline{v}\|_X\right). \end{aligned} \tag{4.25}$$

In a completely similar manner it can be proven that

$$\|\overline{u}'\|^2_{L^2L^2} + \frac{1}{2}\|\overline{u}\|^2_{L^\infty V} \le C_0 k \left(\frac{\|\overline{u}\|^2_X}{2} + \|\overline{u}\|_X \|\overline{v}\|_X \right). \tag{4.26}$$

By using (4.25) and (4.26) we then obtain

$$\|\overline{u}\|^2_X \le C_0 k \|\overline{u}\|^2_X + 2C_0 k \|\overline{u}\|_X \|\overline{v}\|_X,$$

and it follows that

$$\|\overline{u}\|_X \le \frac{2C_0 k}{1 - C_0 k} \|\overline{v}\|_X.$$

Hence, if k is small enough so that

$$k < \frac{1}{3C_0}, \tag{4.27}$$

the mapping T is a contraction from $B_k(X)$ to $B_k(X)$. The uniqueness of the solution is standard – see Theorem 3.4 in [42]. □

Let us conclude this section by making some final remarks:

- Note that, if we choose $k = \frac{1}{4C_0}$, then for any f and u^0 such that

  $$\|f\|^2_{L^2L^2} + \|u^0\|^2_V \le \frac{3}{64C_0^2},$$

  conditions (4.24) and (4.27) are verified, so the existence is ensured.

- Returning to the thin domain Ω_ε, we would like to see how the condition on the initial data relates to ε. However, this is not an easy matter, since we have found no way of controlling the dependence of C_1 with respect to ε (unlike C_2, see below). In [43] the independence of C_1 with respect to ε is proved in particular thin domains and with particular boundary conditions. Unfortunately, we have not been able to replicate that result for our problem.

- Finally, let us see how the constant C_2 behaves with respect to ε. Following the proof of Theorem 9.12 in [7] we find that, for $u \in V$, we have:

  $$\|u\|_{L^\infty} \le C\varepsilon^{1/2} \|\nabla u\|_{L^6}.$$

  Reasoning the same as in the proof of Lemma 2.3.5, we can prove that:

  $$\|\nabla u\|_{L^6} \le C\|u\|_{H^2}.$$

  Combining the above results, we derive $C_1 = C\varepsilon^{1/2}$.

4.4 The semigroup approach

Since the results derived in the previous section did not allow us to obtain a "good" dependence on the initial data with respect to ε in order for the problem to have a global strong solution, we shall present an alternative method that will enable us to obtain the desired result using the theory of semigroups of linear operators. The results in this section have been obtained, in a more general by J. Avrin [2] (in the case of Dirichlet boundary conditions). The proofs presented here are, however, significantly simpler. Consider the space

$$\mathcal{H}_\varepsilon = \{u \in (C^\infty(\Omega_\varepsilon))^3 \,|\, div\, u = 0,\ u = 0 \text{ on } \Gamma_\varepsilon,\ u \times n = 0 \text{ on } \Sigma_0^\varepsilon \cup \Sigma_1^\varepsilon\},$$

and let H_ε be the closure of $\mathcal{H}_\varepsilon$ in $(L^2(\Omega_\varepsilon))^3$. Clearly, H_ε is a Hilbert space with respect to the L^2 scalar product $(\cdot,\cdot)$ – the corresponding norm is, as always, denoted $\|\cdot\|$. As usual, the Stokes operator A_ε can be defined as

$$A_\varepsilon : D(A_\varepsilon) \subset H_\varepsilon \to H_\varepsilon, \quad A_\varepsilon = -P\Delta,$$

where P denotes the projection onto H_ε. The domain $D(A_\varepsilon)$ can be characterised using the regularity results proved in the second section as

$$D(A_\varepsilon) = \{u \in (H^2(\Omega_\varepsilon))^3 \,|\, div\, u = 0,\ u = 0 \text{ on } \Gamma_\varepsilon,\ u \times n = 0 \text{ on } \Sigma_0^\varepsilon \cup \Sigma_1^\varepsilon\}.$$

Obviously, we can write

$$(A_\varepsilon u, v) = \int_{\Omega_\varepsilon} \nabla u : \nabla v. \tag{4.28}$$

We begin with the first result:

Proposition 4.4.1. *The operator $-A_\varepsilon$ generates an analytical C_0-semigroup of contractions on H_ε.*

Proof. From (4.28) it follows that $(-A_\varepsilon u, u) \leq 0$, so $-A_\varepsilon$ is dissipative. Given $f \in H_\varepsilon$, it is immediate to find, by Lax-Milgram, a weak solution $u \in V$ to the problem

$$u + A_\varepsilon u = f,$$

which implies that $A_\varepsilon u \in (L^2(\Omega_\varepsilon))^3$, and so, by the regularity result proved in the second section of this chapter, $u \in (H^2(\Omega_\varepsilon))^3 \cap V = D(A_\varepsilon)$. From Theorem 4.3 in [36], it follows that $-A_\varepsilon$ generates a C_0-semigroup of contractions $S_\varepsilon(t)$ on H_ε. Using this and the fact that $-A_\varepsilon$ is symmetric (which follows from (4.28)), by Lemma 1.6.1 in [45], we get that $-A_\varepsilon$ is self-adjoint. Finally, from Corollary 7.1.1 in [45] we obtain that $S_\varepsilon(t)$ is analytical. $\square$

We shall denote by e^{-tA_ε} the semigroup $S_\varepsilon(t)$. The next Lemma collects useful results on the semigroup e^{-tA_ε}.

Lemma 4.4.1. *Let $0 < \alpha < 1$. Then:*

$$\|e^{-tA_\varepsilon}\|_{\mathcal{L}(H_\varepsilon)} \leq e^{-\frac{t}{\varepsilon^2}}, \tag{4.29}$$

$$\|A^\alpha e^{-tA_\varepsilon}\|_{\mathcal{L}(H_\varepsilon)} \leq c_0 t^{-\alpha}, \tag{4.30}$$

$$\int_0^t \|A^\alpha e^{-(t-s)A_\varepsilon}\|_{\mathcal{L}(H_\varepsilon)} ds \leq 2c_0 \varepsilon^{2-2\alpha}\Gamma(1-\alpha), \tag{4.31}$$

for all $t > 0$ with some c_0 independent of ε.

Proof. From Remark 7.2.1 in [45] we have that

$$\|e^{-tA_\varepsilon}\|_{\mathcal{L}(H_\varepsilon)} \leq e^{-\lambda t},$$

where

$$\lambda = \inf\{(A_\varepsilon u, u) \,|\, u \in D(A_\varepsilon),\ \|u\| = 1\}.$$

From (4.28) and the Poincaré inequality in Lemma 2.3.5 we derive that $\lambda \geq \frac{1}{\varepsilon^2}$, from which we get (4.29).
Following the proofs of Theorems 6.13 and 5.2 in [36], we get the existence of a c_0 independent of ε so that (4.30) holds true. Lastly, we can write

$$\int_0^t \|A^\alpha e^{-(t-s)A_\varepsilon}\|_{\mathcal{L}(H_\varepsilon)} ds \leq \int_0^t \|A^\alpha e^{-\frac{t-s}{2}A_\varepsilon}\|_{\mathcal{L}(H_\varepsilon)} \|e^{-\frac{t-s}{2}A_\varepsilon}\|_{\mathcal{L}(H_\varepsilon)} ds$$
$$\leq c_0 2^\alpha \int_0^\infty s^{-\alpha} e^{-\frac{s}{2\varepsilon^2}} ds = 2c_0\varepsilon^{2-2\alpha} \int_0^\infty s^{-\alpha} e^{-s} ds = 2c_0\varepsilon^{2-2\alpha}\Gamma(1-\alpha),$$

thus completing the proof. □

We are now ready to proceed to the statement and the proof of the main result. Henceforth we drop the subscript ε for simplicity. It should be noted that our work in this section has been heavily influenced by [13].
We can rewrite the system (3.1)–(3.2) in abstract form

$$\begin{cases} u'(t) = -Au(t) + Fu(t) + Pf(t), \quad t > 0, \\ u(0) = u^0, \end{cases} \tag{4.32}$$

where $Fu(t) = -P(u(t)\cdot\nabla)u(t)$. The idea is to prove the existence of a mild solution that will be regular enough ($C([0,\infty); D(A^{1/2})) \cap L^\infty(0,\infty; D(A^{1/2}))$). Firstly, we express (4.32) into integral form:

$$u(t) = e^{-tA}u^0 + \int_0^t A^{1/4} e^{-(t-s)A} Bu(s) ds + \int_0^t e^{-(t-s)A} Pf(s) ds, \tag{4.33}$$

where

$$Bu(t) = -A^{-1/4} P(u(t)\cdot\nabla)u(t). \tag{4.34}$$

We have chosen to write it in this manner since, for $u(t) \in D(A^{1/2})$ we have only $(u(t)\cdot\nabla)u(t) \in L^{3/2}$ – as shown in [16], the operator $A^{-1/4}$ has a unique continuous extension to $L^{3/2}$ and so the relation (4.34) has a sense. Moreover, following [13], we can prove useful results on B:

$$\|Bu\| \leq c_1\|A^{1/2}u\|^2, \tag{4.35}$$

$$\|Bu - Bv\| \leq c_1\|A^{1/2}(u-v)\|(\|A^{1/2}u\| + \|A^{1/2}v\|), \tag{4.36}$$

for all $u, v \in D(A^{1/2})$, with some c_1 independent of ε.
We are now ready to state and prove the main result:

Theorem 4.4.1. *Let $u^0 \in D(A^{1/2})$ and $f \in L^\infty L^2$ be given. Let also $0 < \alpha < \frac{1}{2}$. Then, for $\varepsilon < 1$ small enough, the problem (4.33) has a solution $u \in C([0,\infty); D(A^{1/2})) \cap L^\infty(0,\infty; D(A^{1/2}))$, which is unique in the space*

$$Y = \Big\{u \in L^\infty(0,\infty; D(A^{1/2})) \mid \sup_{t\geq 0}\|A^{1/2}u(t)\| \leq \frac{1}{2c\varepsilon^\alpha}\Big\},$$

where c is a constant that will be later specified.

Proof. Define, as in [13], the sequence

$$u_{n+1}(t) = u_0(t) + \int_0^t A^{1/4}e^{-(t-s)A}Bu_n(s)ds, \tag{4.37}$$

for all $n \geq 0$, where

$$u_0(t) = e^{-tA}u^0 + \int_0^t e^{-(t-s)A}Pf(s)ds.$$

Let us first prove that $u_n \in L^\infty(0,\infty; D(A^{1/2}))$ for all n and derive a uniform bound. Using (4.29) and (4.31) we get that

$$\|A^{1/2}u_0(t)\| \leq \|A^{1/2}u^0\| + 2c_0\varepsilon\Gamma(\frac{1}{2})\|f\|_{L^\infty L^2}$$

for all $t \geq 0$, showing that $u_0 \in L^\infty(0,\infty; D(A^{1/2}))$. Assume that, for a given $n \geq 0$, $u_n \in L^\infty(0,\infty; D(A^{1/2}))$ and let $k_n = \sup_{t\geq 0}\|A^{1/2}u_n(t)\|$. Taking (4.37) into account and using (4.29), (4.31) and (4.35) we obtain

$$\|A^{1/2}u_{n+1}(t)\| \leq k_0 + 2c_0c_1\varepsilon^{1/2}\Gamma(\frac{1}{4})k_n^2,$$

for all $t \geq 0$. Hence $u_{n+1} \in L^\infty(0,\infty; D(A^{1/2}))$ and, moreover,

$$k_{n+1} \leq k_0 + c\varepsilon^{1/2}k_n^2,$$

where $c = 2c_0c_1\Gamma(\frac{1}{4})$. Since we have supposed that $\varepsilon < 1$ and $\alpha < 1/2$, we can write

$$k_{n+1} \leq k_0 + c\varepsilon^\alpha k_n^2,$$

By induction, it can be easily seen that, provided

$$k_0 \leq \frac{1}{4c\varepsilon^\alpha} \tag{4.38}$$

we have

$$k_n \leq \frac{1}{2c\varepsilon^\alpha}, \quad \forall\, n \geq 1. \tag{4.39}$$

Obviously we can choose ε small enough so that (4.38) holds. From (4.37) we derive

$$A^{1/2}(u_{n+1}(t) - u_n(t)) = \int_0^t A^{3/4}e^{-(t-s)A}(Bu_n(s) - Bu_{n-1}(s))ds, \quad \forall\, n \geq 0, \tag{4.40}$$

where we make the convention $u_{-1}(t) \equiv 0$. Let

$$\beta_n = \sup_{t\geq 0} \|A^{1/2}(u_{n+1}(t) - u_n(t))\|.$$

Taking (4.40) into account and using (4.36),(4.31) and (4.39) we readily obtain

$$\beta_n \leq \varepsilon^{1/2-\alpha}\beta_{n-1}.$$

Since $\varepsilon < 1$, this immediately gives the uniform convergence of $A^{1/2}u_n$ to a limit in H say v. The Poincaré inequality implies that u_n is uniformly convergent to a limit in H that we call u. Since the operator $A^{1/2}$ is closed it follows that $u(t) \in D(A^{1/2})$ and $A^{1/2}u(t) = v(t)$ for all $t \geq 0$. In order to show that u is a solution to problem (4.33) we pass to the limit in equation (4.37) using the above results as well as (4.29), (4.31) and (4.35). The continuity of u in $D(A^{1/2})$ is immediate.
To prove uniqueness, let $u^1 \in Y$ be another solution to problem (4.33) and let $\overline{u} = u - u^1$. We have

$$A^{1/2}\overline{u}(t) = \int_0^t A^{3/4}e^{-(t-s)A}(Bu(s) - Bu^1(s))ds,$$

which leads to

$$\|A^{1/2}\overline{u}(t)\| \leq \varepsilon^{1/2-\alpha} \sup_{s\geq 0} \|A^{1/2}\overline{u}(s)\|$$

for all $t \geq 0$, and hence $\overline{u}(t) = 0$, which achieves the proof. □

Let us summarize the results in this section:
For all $\delta > 0$, provided that the initial data belong to the "large sets"

$$\|A^{1/2}u^0\| \leq K_1\varepsilon^{\delta-1/2} \quad \text{and} \quad \|f\|_{L^\infty L^2} \leq K_2\varepsilon^{\delta-3/2},$$

there exists a solution to the problem (4.33) which is unique in the "large space"

$$Y = \{u \in L^\infty(0,\infty; D(A^{1/2})) \mid \sup_{t\geq 0} \|A^{1/2}u(t)\| \leq K_3\varepsilon^{\delta-1/2}\},$$

where K_1, K_2, K_3 are independent of ε and δ.

Bibliography

[1] G. Astarita and G. Marrucci. *Principles of non-Newtonian fluid mechanics.* McGraw-Hill, 1974.

[2] Joel D. Avrin. Large-eigenvalue global existence and regularity results for the Navier-Stokes equation. *J. Differential Equations*, 127(2):365–390, 1996.

[3] G. Bayada, L. Chupin, and B. Grec. Viscoelastic fluids in thin domains: a mathematical proof. *Asymptot. Anal.*, 64(3-4):185–211, 2009.

[4] C. Bernardi, M. Dauge, Y. Maday, and M. Azaïez. *Spectral methods for axisymmetric domains.* Gauthier-Villars Paris, 1999.

[5] R. B. Bird, R. C. Armstrong, O. Hassager, and C. F. Curtiss. *Dynamics of polymeric liquids*, volume 1. Wiley New York, 1977.

[6] F. Blanc, O. Gipouloux, G. Panasenko, and A. M. Zine. Asymptotic analysis and partial asymptotic decomposition of domain for Stokes equation in tube structure. *Math. Models Methods Appl. Sci.*, 9(9):1351–1378, 1999.

[7] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[8] G. Cardone, R. Fares, and G. P. Panasenko. Asymptotic expansion of the solution of the steady Stokes equation with variable viscosity in a two-dimensional tube structure. *J. Math. Phys.*, 53(10):103702, 21, 2012.

[9] W. Chang, D. Tzebotich, L. P. Lee, and D. Liepmann. Blood flow in simple microchannels. In *Microtechnologies in Medicine and Biology, 1st Annual International, Conference On. 2000*, pages 311–315. IEEE, 2000.

[10] C. Conca, F. Murat, and O. Pironneau. The Stokes and Navier-Stokes equations with boundary conditions involving the pressure. *Japan. J. Math. (N.S.)*, 20(2):279–318, 1994.

[11] S. Deparis. *Numerical Analysis of Axisymmetric Flows and Methods for Fluid-Structure Interaction Arising in Blood Flow Simulation.* PhD thesis, École Polytechnique Fédérale de Lausanne, 2004.

[12] A. W. El-Kareh and L. G. Leal. Existence of solutions for all deborah numbers for a non-newtonian model modified to include diffusion. *J. Non-Newtonian Fluid Mech.*, 33(3):257–287, 1989.

[13] H. Fujita and T. Kato. On the Navier-Stokes initial value problem I. *Arch. Rational Mech. Anal.*, 16(4):269–315, 1964.

[14] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations*. Springer Monographs in Mathematics. Springer, New York, second edition, 2011. Steady-state problems.

[15] E. Ghiringhelli. *Viscoélasticité du sang et du caillot*. PhD thesis, Université de Grenoble, 2014.

[16] Y. Giga and T. Miyakawa. Solutions in L_r of the Navier-Stokes initial value problem. *Arch. Rational Mech. Anal.*, 89(3):267–281, 1985.

[17] O. Gipouloux and E. Marušić-Paloka. Asymptotic behaviour of the incompressible Newtonian flow through thin constricted fracture. In *Multiscale problems in science and technology (Dubrovnik, 2000)*, pages 189–202. Springer, Berlin, 2002.

[18] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.

[19] C. Guillopé and J.-C. Saut. Existence results for the flow of viscoelastic fluids with a differential constitutive law. *Nonlinear Anal.*, 15(9):849–869, 1990.

[20] C. Hu. Navier-Stokes equations in 3D thin domains with Navier friction boundary condition. *J. Differential Equations*, 236(1):133–163, 2007.

[21] D. Iftimie, G. Raugel, and G. R. Sell. Navier-Stokes equations in thin 3D domains with Navier boundary conditions. *Indiana Univ. Math. J.*, 56(3):1083–1156, 2007.

[22] W. Jäger and A. Mikelić. On the effective equations of a viscous incompressible fluid flow through a filter of finite thickness. *Comm. Pure Appl. Math.*, 51(9-10):1073–1121, 1998. Dedicated to the memory of Fritz John.

[23] W. Jäger and A. Mikelić. On the roughness-induced effective boundary conditions for an incompressible viscous flow. *J. Differential Equations*, 170(1):96–122, 2001.

[24] T. Kato and H. Fujita. On the nonstationary Navier-Stokes system. *Rend. Sem. Mat. Univ. Padova*, 32:243–260, 1962.

[25] O. A. Ladyženskaja. New equations for the description of the motions of viscous incompressible fluids, and global solvability for their boundary value problems. *Trudy Mat. Inst. Steklov.*, 102:85–104, 1967.

[26] O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow.* Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2. Gordon and Breach, Science Publishers, New York-London-Paris, 1969.

[27] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod; Gauthier-Villars, Paris, 1969.

[28] P. L. Lions and N. Masmoudi. Global solutions for some Oldroyd models of non-Newtonian flows. *Chinese Ann. Math. Ser. B*, 21(2):131–146, 2000.

[29] W. G. Litvinov. Models for laminar and turbulent flows of viscous and nonlinear viscous fluids. In *Recent developments in theoretical fluid mechanics (Paseky, 1992)*, volume 291 of *Pitman Res. Notes Math. Ser.*, pages 78–102. Longman Sci. Tech., Harlow, 1993.

[30] J.-G Liu and W.-C. Wang. Characterization and regularity for axisymmetric solenoidal vector fields with application to Navier-Stokes equation. *SIAM J. Math. Anal.*, 41(5):1825–1850, 2009.

[31] J. Málek, J. Necas, and M. Ruzicka. On weak solutions to a class of non-Newtonian incompressible fluids in bounded three-dimensional domains: the case $p \geq 2$. *Adv. Differential Equations*, 6(3):257–302, 2001.

[32] S. Marušić. On the Navier-Stokes system with pressure boundary condition. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 53(2):319–331, 2007.

[33] E. Marušić-Paloka. The effects of flexion and torsion on a fluid flow through a curved pipe. *Appl. Math. Optim.*, 44(3):245–272, 2001.

[34] E. Marušić-Paloka and M. Starčević. Asymptotic analysis of an isothermal gas flow through a long or thin pipe. *Math. Models Methods Appl. Sci.*, 19(4):631–649, 2009.

[35] I. Moise, R. Temam, and M. Ziane. Asymptotic analysis of the Navier-Stokes equations in thin domains. *Topol. Methods Nonlinear Anal.*, 10(2):249–282, 1997. Dedicated to Olga Ladyzhenskaya.

[36] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[37] G. Raugel and G. R. Sell. Navier-Stokes equations in thin 3D domains. III. Existence of a global attractor. In *Turbulence in fluid flows*, volume 55 of *IMA Vol. Math. Appl.*, pages 137–163. Springer, New York, 1993.

[38] G. Raugel and G. R. Sell. Navier-Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions. *J. Amer. Math. Soc.*, 6(3):503–568, 1993.

[39] M. Renardy. Existence of slow steady flows of viscoelastic fluids with differential constitutive equations. *Z. Angew. Math. Mech.*, 65(9):449–451, 1985.

[40] J.-M. Sac-Épée and K. Taous. On a wide class of nonlinear models for non-Newtonian fluids with mixed boundary conditions in thin domains. *Asymptot. Anal.*, 44(1-2):151–171, 2005.

[41] J. Serrin. The initial value problem for the Navier-Stokes equations. In *Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962)*, pages 69–98. Univ. of Wisconsin Press, Madison, Wis., 1963.

[42] R. Temam. *Navier-Stokes equations. Theory and numerical analysis*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Studies in Mathematics and its Applications, Vol. 2.

[43] R. Temam and M. Ziane. Navier-Stokes equations in three-dimensional thin domains with various boundary conditions. *Adv. Differential Equations*, 1(4):499–546, 1996.

[44] G. B Thurston. Viscoelasticity of human blood. *Biophys. J.*, 12(9):1205–1217, 1972.

[45] Ioan I. Vrabie. *C_0-semigroups and applications*, volume 191 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 2003.