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Nesrine Ben Khalifa

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ACADÉMIE D'AIX-MARSEILLE
UNIVERSITÉ D'AVIGNON ET DES PAYS DE VAUCLUSE

THESIS

A thesis submitted in partial fulfillment for the degree of Doctor of Philosophy

In Computer Science

Doctoral School 536 « Agrosiences et Sciences »
Laboratoire d'Informatique (EA 4128)

Evolutionary Games with Nonuniform Interactions and Delays

Presented by Nesrine BEN KHALIFA

Defended on December 16, 2016 before the jury members:

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ACADÉMIE D'AIX-MARSEILLE
UNIVERSITÉ D'AVIGNON ET DES PAYS DE VAUCLUSE

THÈSE

présentée à l'Université d'Avignon et des Pays de Vaucluse
pour obtenir le diplôme de DOCTORAT

SPÉCIALITÉ : Informatique

École Doctorale 536 « Agrosiences et Sciences »
Laboratoire d'Informatique (EA 4128)

Jeux Évolutionnaires avec des Interactions Non Uniformes et Délais

Présentée par Nesrine BEN KHALIFA

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Résumé

Mots clés : Théorie des jeux, théorie des jeux évolutionnaires, stratégie évolutivement stable, dynamique de réplication, délais, bifurcation de Hopf, Hawk-Dove.

La théorie des jeux évolutionnaires est un outil qui permet d'étudier l'évolution des stratégies dans une population composée d'un grand nombre d'agents qui interagissent d'une façon continue et aléatoire. Dans cette théorie, il y a deux concepts essentiels qui sont la stratégie évolutivement stable (ESS), et la dynamique de réplication. Une stratégie évolutivement stable est une stratégie, qui, si adoptée par toute la population, ne peut pas être envahie par une autre stratégie "mutante" utilisée par une petite fraction de la population [1]. Ce concept statique est un raffinement de l'équilibre de Nash, et il ne peut pas renseigner, par exemple, sur la durée du temps nécessaire pour que l'ESS élimine la stratégie mutante. La dynamique de réplication, originalement proposée par Taylor and Jonker [2], est un modèle dynamique qui permet de prédire l'évolution de la fraction de chaque stratégie dans la population en fonction du temps, en réponse aux gains des stratégies et l'état de la population.

Dans cette thèse, nous proposons dans une première partie une extension de la dynamique de réplication classique en y introduisant des délais hétérogènes et aléatoires. En effet, la plupart des phénomènes qui se produisent prennent un temps incertain avant d'avoir des résultats. Nous étudions l'effet de la distribution des délais sur la stabilité de l'ESS dans la dynamique de réplication et nous considérons les distributions uniforme, exponentielle, et Gamma (ou Erlang). Dans les cas des distributions uniforme et Gamma, nous trouvons la valeur critique de la moyenne à laquelle la stabilité de l'équilibre est perdue et des oscillations permanentes apparaissent. Dans le cas de la distribution exponentielle, nous montrons que la stabilité de l'équilibre ne peut être perdue, et ce pour toute valeur de la moyenne de la distribution. Par ailleurs, nous montrons que la distribution exponentielle peut affecter la stabilité de l'ESS quand une seule stratégie subit un délai aléatoire issu de cette distribution. Nous étudions également le cas où les délais sont discrets et nous trouvons une condition suffisante et indépendante des valeurs des délais pour la stabilité de l'équilibre. Dans tous les cas, nous montrons que les délais aléatoires sont moins risqués que les délais constants pour la stabilité de l'équilibre, vu que la valeur moyenne critique des délais aléatoires est toujours supérieure de celle des délais constants.

En outre, nous considérons comme paramètre de bifurcation la moyenne de la distribution des délais et nous étudions les propriétés de la solution périodique qui apparaît à la bifurcation de Hopf, et ce en utilisant une méthode de perturbation non linéaire. En effet, à la bifurcation de Hopf, une oscillation périodique stable apparaît dont l'amplitude est fonction de la moyenne de la distribution. Nous déterminons analytiquement l'amplitude de l'oscillation au voisinage de la bifurcation de Hopf en fonction du paramètre de bifurcation et de la matrice des jeux dans les cas des distributions de Dirac, uniforme, Gamma et discrète, et nous appuyons nos résultats avec des simulations numériques.

Dans une deuxième partie, nous considérons une population hétérogène composée de plusieurs communautés qui interagissent d'une manière non-uniforme. Pour chaque communauté, nous définissons les matrices des jeux et les probabilités d'interaction avec les autres communautés. Dans ce contexte, nous définissons trois ESS avec différents niveaux de stabilité contre les mutations: un ESS fort, un ESS faible et un ESS intermédiaire. Nous définissons un ESS fort comme suit: si toute la population adopte l'ESS, alors l'ESS ne peut pas être envahi par une petite fraction de mutants composée d'agents de toutes les communautés. Pour l'ESS faible, chaque communauté ne peut être envahie par une petite fraction de mutants de cette communauté (mutants locaux). Par contre, une population qui adopte l'ESS intermédiaire ne peut être envahie par une petite fraction de mutants en considérant une seule utilité qui est la somme des utilités de toutes les communautés. Dans le cas de deux communautés, nous montrons qu'aucun ESS fort ne peut exister, et nous trouvons les conditions d'existence des ESS faibles et intermédiaires en fonction des probabilités d'interaction et des matrices des jeux. Par ailleurs, nous étudions la dynamique de réplication et nous montrons que l'ESS intermédiaire est toujours asymptotiquement stable alors que la condition de la stabilité faible de l'ESS ne garantit pas la stabilité asymptotique.

En outre, motivés par l'omniprésence des délais dans la plupart des phénomènes réels, nous introduisons dans la dynamique de réplication deux types de délais: délais spatiaux et délais stratégiques. Les délais spatiaux apparaissent dans les interactions qui impliquent deux agents de deux communautés différentes (interactions hétérogènes). Nous montrons que ce type de délais n'affecte pas la stabilité de l'équilibre. Les délais stratégiques sont associés aux stratégies et apparaissent dans tous les types d'interactions. Dans ce cas, nous trouvons la valeur critique de ce délai à partir de laquelle la stabilité de l'équilibre est perdue en faveur d'oscillations permanentes.

Finalement, nous appliquons les résultats obtenus sur le jeu de Hawk-Dove, qui est une simple caractérisation des conflits sur une ressource rare, très étudiée dans la théorie des jeux évolutionnaires, et qui permet de prédire le niveau d'agressivité dans la population. Nous considérons une population composée de deux groupes d'agents avec deux niveaux d'agressivité asymétriques. En fonction des probabilités des interactions et des paramètres du jeu (coût de l'affrontement, valeur de la ressource), nous étudions l'existence des différents types de ESS ainsi que la dynamique de réplication avec délais.

Abstract

Keywords: Game theory, evolutionary game theory, evolutionarily stable strategy, replicator dynamics, time delay, Hopf bifurcation, Hawk-Dove.

In this dissertation, we study evolutionary game theory which is a mathematical tool used to model and predict the evolution of strategies in a population composed of a large number of players. In this theory, there are two basic concepts which are the evolutionarily stable strategy (ESS) and the replicator dynamics. The ESS is originally defined as follows [1]: if all the population adopts the ESS, then no alternative strategy used by a sufficiently small fraction of the population can invade the population. The ESS is a static concept and a refinement of a Nash equilibrium. It does not allow us, for example, to estimate the time required for the ESS to overcome the mutant strategy, neither to predict the asymptotic distribution of strategies in the population. The replicator dynamics, originally introduced in [2], is a model of evolution of strategies according to which the growth rate of a given strategy is proportional to how well this strategy performs relative to the average payoff in the population.

In the first part of this work, we propose an extended version of the replicator dynamics which takes into account heterogeneous random delays. Indeed, in many situations, the presence of uncertain delays is ubiquitous. We first consider continuous delays and we study the effect of the distribution of delays on the asymptotic stability of the mixed equilibrium in the replicator dynamics. In the case of uniform and Gamma delay distributions, we find the critical mean delay at which a Hopf bifurcation is created and the stability of the mixed equilibrium is lost. When the distribution of delays is exponential, we prove that the stability of the equilibrium cannot be affected by the delays. However, when only one strategy is delayed according to the exponential distribution, the asymptotic stability of the ESS can be lost. In all the cases, we show that the critical mean delay value is higher than that of constant delays, and thus random delays are less threatening than constant delays. In addition, we consider discrete delays and one of our results is that, when the instantaneous term is dominant, that is when the probability of zero delay is sufficiently high, the stability of the ESS cannot be lost.

Furthermore, by taking as a bifurcation parameter the mean delay distribution, we examine the properties of the bifurcating periodic solution created near the Hopf bifurcation using a nonlinear perturbation method. Indeed, near the Hopf bifurcation, a

stable periodic oscillation appears whose amplitude depends on the value of the bifurcation parameter. We give a closed-form expression of the amplitude of the periodic solution and we validate our results with numerical simulations.

In the second part, we consider an heterogeneous population composed of several communities which interact in a nonuniform manner. Each community has its own set of strategies, payoffs, and interaction probabilities. Indeed, individuals of a population have many inherent differences that favor the appearance of groups or clusters. In this scenario, we define three ESS with different levels of stability against mutations: strong, weak, and intermediate ESS, and we examine their connection to each other. A strong ESS is a strategy that, when adopted by all the population, cannot be invaded by a sufficiently small fraction of mutants composed of agents from all the communities. In contrast, a weak ESS is a strategy wherein each community resists invasion by a sufficiently small fraction of mutants in that community (local mutants). In the intermediate ESS, the population adopting the ESS cannot be invaded by a small fraction of mutants when we consider the total fitness of the population rather than the fitness of each community separately. In the case of two communities, we show that no strong ESS can exist, but under some conditions on the payoffs and the interaction probabilities, intermediate and weak ESS may exist. Further, we complete our static analysis by introducing the replicator dynamics in the case of two interacting communities. We prove that the intermediate ESS is asymptotically stable in the replicator dynamics whereas the weak stability condition does not ensure the asymptotic stability in the replicator dynamics.

Furthermore, we introduce two types of delays in the replicator dynamics: strategic delay and spatial delay. The strategic delay is associated to the strategies used by the players and can be defined as the time interval a strategy takes to have a consequence on the fitness of the interacting players. It is more realistic to consider that a strategy does not have an immediate effect but takes some time to produce results. As in a one population scenario, we prove that there exists a critical value of the strategic delay at which the stability of the mixed equilibrium is lost. The spatial delay depends on the types of interactions and appears in heterogeneous interactions only, that is interactions that involve players from different communities. Indeed, it comes from the latency induced by the individual types when they interact. Interestingly, we show that spatial delays do not affect the asymptotic stability of the equilibrium.

In the third part, we apply our results to the Hawk-Dove game which is a well studied model in the theory of evolutionary games. In the one stage Hawk-Dove game, two individuals compete for a scarce resource and have two possible strategies: an aggressive behavior and a peaceful one. We consider a population composed of two communities that have asymmetric fighting abilities and we analyze the existence of different types of ESS in function of the interaction probabilities and the parameters of the game (intra-community and inter-community fighting costs, resource value). We also examine the stability of the replicator dynamics in this context.

Introduction

Many complex systems exist where a large number of agents interact and make interdependent choices. In economics, for example, the decision makers have to make interdependent decisions and for each decision maker, the outcome of his action depends also on the actions of his opponents. Furthermore, agents may form clusters or groups, and exhibit special types of interactions such as the formation of coalitions or cooperations. In social networks, users can form groups or communities sharing some common features. The development of a framework that enables the analysis of a competitive context is therefore necessary, and evolutionary game theory provides an analytical framework that allows to study such complex systems. Evolutionary game theory has a wide range of applications in many areas such as social sciences [3, 4], computer science, where some examples of applications can be found in multiple access protocols [5, 6, 7, 8, 9, 10], multi-homing [11], resources competition in the Internet [12], and wireless networks [13, 14].

In this thesis, we study evolutionary games in an extended framework. The contribution of this work can be classified into three parts: (i) stability and Hopf bifurcations in the replicator dynamics with heterogeneous and random delays, (ii) evolutionary games in interacting communities, and (iii) application to the Hawk-Dove game.

Evolutionary game theory is a mathematical tool used to model and study the evolution of strategies in large populations. In this framework, agents of a large population are continually involved in random and pairwise interactions. A strategy is characterized by (i) the proportion of the players using it in the population, and (ii) its fitness which is a measure of its success and depends on the proportions of all other strategies in the population. In this theory, there are two main concepts which are the evolutionarily stable strategy (ESS) and the replicator dynamics. The ESS is a static concept and is a refinement of the Nash equilibrium. It is originally defined as follows: if all the population adopts the ESS, then no alternative strategy used by a sufficiently small fraction of the population can invade the population [1]. The replicator dynamics is a model that enables the prediction of the time evolution of strategy frequencies in the population.

Replicator Dynamics with Random Delays

In the first part of our work, we extend the classic replicator dynamics by introducing heterogeneous random delays. In the evolutionary game literature, most of the works on delayed evolutionary dynamics have considered only the case of a single deterministic delay [15, 16, 17, 18, 19]. However, in many real-world applications, we observe that time delays are heterogeneous, uncertain, and may arise under various forms. For example, in social networks, the users react to delayed information and the delays experienced by the users are not the same but rather heterogeneous. In economics, the investments do not have an immediate result, but take some time to have consequences, which is usually uncertain. Unlike Previous works on delayed replicator dynamics which considered a single fixed delay, we consider here random continuous delays and we study the effect of the delay distribution on the stability of the interior equilibrium. In addition, we study the replicator dynamics subject to several discrete delays. We believe that our work is the first attempt to examine the stability of the replicator dynamics with random delays.

Our major result is that random delays have less threats than constant delays. Indeed, the critical mean value of the random delay at which the stability is lost is higher than that of a constant delay. The method we used to study the asymptotic stability of the replicator dynamics with delays is based on the linearization of the replicator dynamic around the equilibrium and analyzing the stability of the linearized system. There is asymptotic stability of the linearized system if all the roots of the associated characteristic equation have negative real parts. Usually, the characteristic equation has an infinite number of complex roots, which makes the problem of analyzing the sign of the roots a challenging one. A transition from stability to instability occurs when a root passes through the imaginary axis and if, furthermore, the derivative of the root in function of the delay at the critical delay is positive, then the asymptotic stability is definitely lost and cannot be regained again, otherwise, there would appear a phenomenon of stability switches [20].

We considered different delay distributions in the replicator dynamics and we here summarize the major contribution in this part, presented in Chapters 2 and 3:

- Under the exponential distribution, we show that the mixed ESS is asymptotically stable for any value of the distribution parameter. However, when only one strategy is delayed with an exponential delay distribution, we prove that a Hopf bifurcation may occur depending on the game payoffs. At the Hopf bifurcation, the asymptotic stability is lost and the replicator dynamics exhibits a stable periodic oscillation in the proportions of strategies in the population.
- For the uniform and Gamma (or Erlang) distributions, we derive conditions for the asymptotic stability of the mixed ESS. We show that random delays require more flexible stability conditions than constant delays. In addition, we illustrate

the effects of the Gamma distribution parameter on the critical mean delay and the frequency of oscillations near the Hopf bifurcation.

- In case of random discrete delays, we determine the critical mean delay and the critical frequency of oscillations. Furthermore, we derive a delay-independent stability condition for the asymptotic stability of the mixed ESS. We also show that, when the instantaneous term is dominant, that is, when the probability of zero delay is sufficiently high, then the asymptotic stability follows. Moreover, we examine a special case of asymmetric delays, where only one strategy is delayed.

We validated our analysis through numerical simulations. Interestingly, our results show that the consequences of delays depend not only on the values of delays, but also on the distribution according to which the delays are drawn and the symmetry (or not) across the strategies.

Furthermore, in Chapter 4, we continue our study of random delays and we examine the properties of the periodic solution (or limit cycle) created at the Hopf bifurcation using Poincaré-Lindstedt perturbation method. This method enables us to determine whether the Hopf bifurcation is supercritical, in which case the bifurcating periodic solution is stable, or subcritical, in which case the bifurcating periodic solution is unstable, as well as the properties of the bifurcating solution. In our context, we prove the appearance of a stable periodic oscillation near the Hopf bifurcation (supercritical bifurcation) and we determine the amplitude of the periodic solution. Indeed, at the critical value of the bifurcation parameter, a stable periodic oscillation with a small amplitude appears, and as the bifurcation parameter moves away from this value, the amplitude of oscillation increases further. We determined analytically the growth rate of the amplitude in function of the bifurcation parameter in the cases of Dirac, uniform, Gamma, and discrete delay distributions. Our results are corroborated with numerical simulations. To the best of our knowledge, our work is the first attempt to investigate the Hopf bifurcation in the replicator dynamics with random delays.

Evolutionary Games in Interacting Communities

In the second part, we consider an heterogeneous population composed of communities which interact in a nonuniform manner. Indeed, we observe that agents in a population cannot be completely similar and there are usually cultural, religious, language, or other inherent differences among them. Instead of considering a well-mixed population as in the classical framework of evolutionary games, we explore games in which the population is composed of several communities. Each community has its own set of strategies, payoff matrix and resulting outcomes. Our work focuses on different types of individuals, such that any pairwise interaction does not lead to the same fitness, depending on the type of individuals that are competing, and not only the strategy used. In addition,

each community may interact with any other community with different probabilities. To the best of our knowledge, this is the first work that considers nonuniform interaction feature among communities. In this scenario of nonuniform interactions, we introduce new concepts of ESS, we examine their connections to each other, and we investigate the impact of the interaction probabilities on their existence. More precisely, we define the ESSs with different levels of stability as follows:

- **Strong ESS:** The stability condition guarantees that no alternative strategy can invade the population. This condition states that all communities have an incentive to remain at the ESS when a rare alternative strategy is used by mutants in all the communities that form the population.
- **Weak ESS:** The stability condition guarantees the stability, for each community, against a local fraction of mutants in a single community.
- **Intermediate ESS:** This equilibrium considers the global fitness of the whole population instead of a single community. It guarantees that all the population cannot earn a higher total payoff when deviating from the ESS.

As a remark, we note that an ESS can be fully mixed, in which case all strategies are present in all the communities; partially mixed where at least one community uses a mixed strategy and another uses a pure strategy; and fully pure where all communities use pure strategies. We show that any fully mixed Nash equilibrium is not a strong ESS wherein all communities using this equilibrium cannot be invaded by a small group from all communities with a mutant strategy. But under some assumptions on the payoff and interaction probabilities, this mixed equilibrium is an ESS when we consider the global fitness of the population rather than the fitness of each community separately.

In this framework, we aim to connect the analysis of the stability in a static concept and the steady state of the replicator dynamics. For the evolutionary dynamics, we introduce the replicator dynamics under nonuniform interactions in communities. We study the relationship between the steady state of the replicator dynamics and the ESSs with different levels of stability. In particular, we show that the fully mixed intermediate ESS is asymptotically stable in the replicator dynamics. In contrast, the condition of weak stability does not ensure the stability condition of the replicator dynamics.

Furthermore, we study the effect of time delays on the stability of the replicator dynamics in a population composed of communities. We distinguish and define two types of delays:

- **Strategic delay** which is the delay associated with the strategies of the players. We suppose that each action takes some fixed time interval to have a consequence on the fitness of the interacting players. As in one population scenario [21, 16], we show that for large strategic delays, the mixed intermediate ESS may become an unstable state for the replicator dynamics where the profile of the population fluctuates around the ESS.

-
- **Spatial delay** which is associated to the communities of the competing players and comes from the latency induced by the individual types when they interact. In fact, we can assume that in some situations, the delay of a pairwise interaction between individuals from the same community can be lower than the delay observed when individuals from different communities are interacting. It is more intuitive that delays are more likely to occur in mixed interactions than in homogeneous ones. For example, in a social network, individuals from the same community will share faster some content/information as there is some kind of confidence between them, whereas, a content coming from an individual from another community may yield to a careful behavior and then increases the outcome delay of the interaction. Thus, delays are more likely to happen in mixed interactions engaging agents from different communities than in homogeneous interactions that involve similar individuals. We prove that the replicator dynamics converges to the fully mixed intermediate ESS for any value of the spatial delay.

The two types of delays yield two different formulae of the expected utilities of strategies, and then different consequences on the asymptotic stability of the mixed ESS are intuitively expected.

Applications

In the third part, we apply our results to the Hawk-Dove game which is one of the most studied examples in evolutionary games. The Hawk-Dove game is a model for determining the degree of aggressiveness in the population. In the one stage Hawk-Dove game, two individuals compete for a rare resource and there are two possible actions or behaviors: an aggressive one and a peaceful one. Depending on the value of the resource and the fighting costs, this game can predict the proportion of aggressive agents in the population. Our new model of heterogeneous population enables us to study the evolution of aggressiveness within different species of animals having asymmetric fighting abilities and interacting in a nonuniform manner. Considering a population composed of two communities of hawks and doves that have an asymmetric level of aggressiveness, we study the existence of different types of ESS (strong, weak, intermediate, fully mixed, partially mixed, fully pure) in function of the interaction probabilities and the fighting costs (intra and inter-community fighting costs). In addition, we study this game on random graphs.

Organization of the Document

The remaining of this thesis is structured as follows:

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- In Chapter 1, we present the main concepts of evolutionary game theory. In addition, we present some major related works that cover delayed evolutionary dynamics and spatial evolutionary games. We also present some interesting applications of the theory of evolutionary games in networking, language, control and optimization problems.
 - In Chapter 2, we study the replicator dynamics with random continuous delays and we examine the asymptotic stability of the mixed equilibrium in a class of anti-coordination games. In the cases of uniform and Gamma distributions of delays, we find the critical value of the mean delay at which a Hopf bifurcation occurs and the asymptotic stability is lost. We illustrate our results with numerical examples.
 - In Chapter 3, we study the replicator dynamics subject to random discrete delays. We thoroughly examine the asymptotic stability of the mixed ESS in this scenario, and we derive a delay-independent stability condition.
 - In Chapter 4, we examine the properties of the periodic solution created at the Hopf bifurcation using Lindstedt perturbation method. We validate our results through numerical simulations.
 - In Chapter 5, we consider an heterogeneous population composed of different communities that are interacting in a nonuniform manner. In this context, we define new ESS with different levels of stability against mutations. In the case of two interacting communities, we thoroughly study the existence of different ESS in function of the interaction probabilities and the game payoffs.
 - In Chapter 6, we study the replicator dynamics in the context of two interacting communities. We examine the relationship between the ESS and the stationary points of the replicator dynamics. Furthermore, we examine the stability of the replicator dynamics subject to strategic and spatial delays.
 - In Chapter 7, we apply the theoretical results obtained in the two previous chapters to the Hawk-Dove game. We examine the existence of the different types of the ESS and the stability of the replicator dynamics in function of the interaction probabilities and the parameters of the game (intra and inter-community fighting costs, resource value).
 - In Chapter 8, we summarize the major contributions of the thesis and present some possible perspectives.

Chapter 1

Introduction to Evolutionary Game Theory

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The work presented in this dissertation is related to evolutionary game theory which is an analytical framework that enables the study of complex systems composed of a large number of agents. Therefore, we introduce in this chapter the main concepts of evolutionary game theory. We also give an overview of the major relevant related works to evolutionary game theory.

The present chapter is organized as follows:

- First, we present a general overview of game theory and then evolutionary game theory: the original framework and the main concepts of evolutionarily stable strategy and evolutionary dynamics;
- In Sections 1.3, 1.4, 1.5, and 1.6, we give an overview of deterministic, delayed, stochastic and spatial evolutionary dynamics, respectively;
- In Section 1.7, we present some applications of evolutionary game theory such as networking, control and optimization problems.
- In Section 1.8, we conclude the chapter.

1.1 Game Theory

Game Theory was developed in [22] as a mathematical approach to analyze situations in which agents (decision makers) make interdependent and conflicting decisions. Game theory has many applications in economics [23], social sciences, biology, and networking as well [24, 13]. It is based on the following assumptions: (i) rationality of the players: the players are considered to be rational and self-interested; (ii) the payoff of a player depends not only on his strategy but also on the strategies or actions used by the other players.

One of the main concepts in game theory is that of a Nash equilibrium. At the Nash equilibrium, no player can improve his payoff by unilaterally deviating from his strategy, that is, the strategy of each player is a best-response to the other players' strategies. However, one property of the Nash equilibrium is that it does not ensure the social efficiency. To illustrate this limitation, we consider the prisoner's dilemma game where each player can choose either to cooperate or defect. If both players cooperate, then each one gets a payoff R , if both defect then each one gets a payoff P , and if one cooperates and the other defects then the cooperator gets a payoff S and the defector gets a payoff T where $T > R > P > S$. The highest payoff is thus obtained by the defector when his opponent chooses to cooperate. At the Nash equilibrium, both players choose to defect; however, the social efficiency is obtained when both cooperate, but this profile is not a Nash equilibrium.

1.2 Evolutionary Game Theory

Evolutionary game theory originated in biology where it was introduced by [1] to model competitions among species. In the original framework, a large number of individuals with different phenotypes are continuously and randomly paired to meet and interact. At each interaction, the engaged individuals obtain a payoff which is interpreted as an increase in their Darwinian fitness. A phenotype corresponds to a pure strategy, and the goal is to predict the relative abundance of each strategy in the population. In two-strategy games, the outcome of a single interaction is described by the following matrix:

$$\mathbf{G} = \begin{array}{c} \begin{array}{cc} & A & B \\ A & a & b \\ B & c & d \end{array} \end{array}, \quad (1.1)$$

where a (b) is the payoff obtained by a player using strategy A when it interacts with an opponent using strategy A (B), c (d) is the payoff obtained by a player using strategy B when interacting with an opponent using strategy A (B).

A mixed strategy is a probability distribution over the pure strategies (represented as a row vector). If one individual chooses a mixed strategy p and interacts with an individual who plays mixed strategy q , the expected fitness J of the first individual is obtained through:

$$J(p, q) = p\mathbf{G}q'.$$

We denote by s the population state, that is the fraction of the population using strategy A , so $1 - s$ is the fraction of the population using strategy B . Let U_X denote the expected payoff of strategy X which depends on the state of population s . Then we have:

$$U_A = (1 \ 0)\mathbf{G}(s \ 1-s)' = as + b(1-s),$$

and

$$U_B = (0 \ 1)\mathbf{G}(s \ 1-s)' = cs + d(1-s).$$

The average payoff in the population is denoted by \bar{U}_s and is given by:

$$\bar{U}_s = sU_A + (1-s)U_B.$$

In n -strategy games, we can similarly derive the expected payoffs of strategies from the payoff matrix. The average payoff in the population is the sum of expected payoffs of each strategy weighted by its fraction in the population.

Evolutionarily stable strategy (ESS): An ESS is a strategy, such that when adopted by all the population, cannot be invaded by a small fraction of individuals using another strategy, called mutants. In a multi-population setting, however, various ESS definitions are proposed in the literature which differ in the level of stability. For example, an ESS definition given by Cressman [25], referred to as a weak ESS or *Cressman* ESS,

is a strategy where at least one sub-population sticks to the ESS when a small whole fraction of mutants is introduced in the population. An alternative ESS definition with a stronger stability condition is given by Taylor [26], in which the total payoff of the non-mutants in all sub-populations is higher than mutants' total payoff.

We consider an homogeneous population where the whole population adopts a mixed strategy q , and we suppose that a fraction ε of *mutants* deviates to mixed strategy p . Strategy q is an ESS if $\forall p \neq q$, there exists some $\varepsilon_p > 0$ such that $\forall \varepsilon \in (0, \varepsilon_p)$:

$$J(p, \varepsilon p + (1 - \varepsilon)q) < J(q, \varepsilon p + (1 - \varepsilon)q).$$

In other words, this strict inequality says that an ESS defeats any small mutations (relative to ε) of the population profile. In that sense, the equilibrium concept of ESS is said to be more robust than the Nash Equilibrium, because it is robust against the deviation of a fraction of players, and not only one. The following proposition allows to characterize an ESS through its stability properties.

Proposition 1 ([27]). *A mixed strategy $q \in \Delta(\mathcal{C})$ is an ESS if and only if it satisfies the following conditions:*

- *Nash Condition* : $J(p, q) \leq J(q, q) \quad \forall p$,
- *Stability Condition* : $J(p, q) = J(q, q) \Rightarrow J(p, p) < J(q, p) \quad \forall p \neq q$.

An ESS can be viewed from two angles. The first one is through the mixed strategy used by any individual into a global population. A second point of view, is to look at the population profile considering that each individual plays pure strategy. As we are interested in the dynamics of strategies inside a population, we prefer to consider the ESS as a population profile. In this case, it is commonly called an evolutionarily stable state but by abuse of definition concepts, it is also called ESS. In the case of pairwise interactions where the outcome is defined as a matrix game, the existence and uniqueness of ESS is well known and proved in [27]. Let $\delta_1 = b - d$, $\delta_2 = c - a$, and $\delta = \delta_1 + \delta_2$. If $\delta_1 > 0$ and $\delta_2 > 0$, or $\delta_1 < 0$ and $\delta_2 < 0$, then the game has a unique mixed Nash equilibrium given by: $s^* = \frac{\delta_1}{\delta_1 + \delta_2}$. If $\delta_1 > 0$ and $\delta_2 > 0$, then s^* is a unique mixed ESS.

Heterogeneous populations: In evolutionary game theory (EGT), the success of a given strategy depends on the frequency of all strategies represented in the population, and successful strategies spread over the population. Unfortunately, the theory developed in this field has mostly focused on the homogeneous population case in which a given individual may equally likely interact with any other member of the population. Then, the success of any individual depends on the frequency of all other strategies represented in the population. In many examples such as social networks, however, the population is composed of several communities or groups that can be seen as clusters [28, 29, 30]. Therefore, community strategies are influenced by interactions inside the

community and also with other communities. Besides, the interactions among individuals are inherently nonuniform, and individuals are more likely to meet and interact with some agents than others because of spatial barriers, language or cultural differences [31]. In biology, some animal species are strongly territorial [32], and territories vary in quality [33]. Hence, an animal may fight against animals from different species and the payoff depends on the species. For example, the probability that an animal being hurt or killed is higher if it meets a larger animal than smaller animal.

1.3 Deterministic Evolutionary Dynamics

1.3.1 Replicator Dynamics

The replicator dynamics is the most common evolutionary dynamics [2, 34, 27, 35, 36] and it investigates the relative frequencies of strategies in a population (the fraction of the population using a given strategy). In this dynamics, the growth rate of a strategy is proportional to the difference between the expected payoff of that strategy and the average payoff in the population. In biology, the replicator dynamics describes the time evolution of the frequency of phenotypes across the population. In social sciences, the replicator equation can be seen as an imitation process where each agent imitates a randomly chosen player with a probability proportional to the difference in payoffs between the strategies. Formally, the replicator dynamics is given by:

$$\frac{ds_i(t)}{dt} = s_i(t)(U_i(t) - \bar{U}(s_i(t))), \quad (1.2)$$

where s_i is the proportion of strategy i in the population, U_i is the expected payoff of strategy i , and $\bar{U} = \sum_i s_i U_i$ is the average fitness in the population. In n -strategy games, the replicator equation is an $(n - 1)$ dimensional system ordinary differential equation (ODE) since $\sum_i s_i = 1$. The $(n - 1)$ dimensional replicator equation is equivalent to the classical Lotka-Volterra equations of n species [27].

When there are two strategies, denoted by A and B , the replicator dynamics is given by:

$$\begin{aligned} \frac{ds(t)}{dt} &= s(t)[U_A(t) - \bar{U}(s(t))], \\ &= s(t)[1 - s(t)][U_A(t) - U_B(t)], \end{aligned}$$

where $s(t)$ and $1 - s(t)$ are the proportions of strategies A and B in the population at instant t , respectively. We note that the replicator dynamics belongs to a class of *noninnovative* dynamics, which means that a strategy that is not currently present in the population cannot reappear.

1.3.2 Relation between ESS and Evolutionary Dynamics

The major question posed in the EGT literature is related to the stability of a steady state which leads to a refinement of the Nash equilibrium. Much of work on evolution has studied the relationship between the steady state of the replicator dynamics and the ESS concept [2, 37, 38, 27, 39, 36]. Taylor and Jonker [2], established conditions under which one may infer the existence of a stable state under the replicator dynamics given an evolutionarily stable strategy. In [27] (pages 69-71), the authors proved that, in matrix games (i) a Nash equilibrium of the (one stage) game is a rest point of the replicator dynamics, (ii) if a strategy is evolutionarily stable then it is asymptotically stable in the replicator dynamics, (iii) if a strategy is evolutionarily stable and it is fully mixed, then it is globally asymptotically stable in the replicator dynamics. We should point out that any asymptotically stable equilibrium is not necessarily evolutionarily stable. Numerical examples with three strategy games which prove this claim are given in [39, 27].

However, the results mentioned above can fail to be true in multi-populations, asymmetric games [27], more than two strategies [40, 41], or spatial settings. In the last two decades, there have been several attempts to relax the assumption derived by Taylor and Jonker [2] in order to explore games in which agents only interact with their neighbors on a lattice [42, 43, 44] or on a random graph [45, 46, 47, 48, 49]. These modifications on the replicators dynamics lead to lose the connection between the stable equilibrium of the replicator and ESS. Indeed, under some payoffs, stable states have no corresponding analogue neither in the replicator dynamics nor in the analysis of ESS [42].

1.4 Delayed Evolutionary Dynamics

The majority of works in evolutionary dynamics have studied the replicator dynamics without taking into account the delay effects, assuming that the interactions have an immediate effect on the fitness of strategies. The expected payoff of a strategy is then considered as a function of the frequency of strategies in the population at the current moment. However, many examples in biology and economics show that the impact of actions is not immediate and their effects only appear after some time interval. Therefore, a more realistic model of the replicator dynamics would take into consideration some time delays. Another interpretation of time delays can be given as follows: the players continuously update their strategies according to their estimation of strategy payoffs. The players cannot have an immediate information about payoffs and the delay can correspond to the time transfer of the information (*social delay*) [18, 19, 50, 51]. Another biological interpretation of delays is given in [18] in which individuals born at time $t - \tau$ may take part in contests when they become mature at time t or equivalently they are born τ units of time after their parents played and received payoffs. In this bi-

ological model, it is shown that the stability of the replicator dynamics is unaffected for any value of τ . Note that the introduction of delays in the replicator equation yields a delayed differential equation (DDE), which is more challenging to analyze since it is an infinite dimensional system [52] (unlike an ODE which is a finite dimensional system). Usually, the characteristic equation associated with a DDE is transcendental rather than polynomial and admits an infinite number of complex roots.

1.4.1 Symmetric Delays

A previous work on delayed evolutionary dynamics is given in [16]. The authors investigated the effect of delays in two-strategy games which have a unique interior equilibrium, stable when there is no delay (anti-coordination games). The delay is assumed to be fixed and symmetric across the strategies i.e. all strategies are subject to the same delay value. The fitness of a given strategy at time t is written as a function of the proportion of each strategy at time $t - \tau$. More precisely, the fitness functions (or expected payoffs) of strategies A and B are given by:

$$U_A(t, \tau) = as(t - \tau) + b(1 - s(t - \tau)), \quad (1.3)$$

$$U_B(t, \tau) = cs(t - \tau) + d(1 - s(t - \tau)). \quad (1.4)$$

Using linear analysis methods, the authors demonstrated the existence of a critical value of delay τ_{cr} at which a Hopf bifurcation occurs, i.e. a change of stability of an equilibrium point due to change in parameters, and the stability of the interior equilibrium is lost. For the values of τ larger than τ_{cr} , the population state oscillates around the equilibrium. A similar result was proved by the authors in [18] in their social-type model.

In these previous works, the authors limited their analysis to two-strategy games subject to symmetric delays across the strategies. Besides, they established only a linear analysis and did not examine the stability of the Hopf bifurcation and the resulting limit cycle's amplitude and frequency (which requires a nonlinear analysis).

An analysis of the Hopf Bifurcation in two strategy games is given in [53]. The limit cycle is a periodic oscillation created at the Hopf bifurcation. Taking the delay τ as a bifurcation parameter, the authors examined the properties of the Hopf bifurcation (supercritical or subcritical, amplitude, and period) using a method based on the normal form theory and the center manifold theorem established in [54]. Another method for determining the stability of the Hopf bifurcation and the resulting limit cycle for a class of delay differential equations is given in [55]. The authors gave an explicit formula for determining the amplitude of the limit cycle in function of all the coefficients of the delay differential equation including nonlinear coefficients.

A similar nonlinear analysis is introduced in three strategy games (Rock-Paper-Scissors games) in [56]. The Rock-Paper-Scissors (RPS) game has three strategies R_1 ,

R_2 , and R_3 where R_1 is beaten by R_2 , which is beaten by R_3 , which is beaten by R_1 . The payoffs of this game are given by the following matrix [27, 35] (the entry ij specifies the payoff to strategy i when used against strategy j) :

$$\begin{pmatrix} 0 & -a_2 & b_3 \\ b_1 & 0 & -a_3 \\ -a_1 & b_2 & 0 \end{pmatrix},$$

where $a_i, b_i > 0$. As a remark, we note that RPS games are characterized by the existence of an interior Nash equilibrium which is asymptotically stable in the replicator dynamics but is not an evolutionarily stable strategy [27].

In [56], the authors analyzed the stability of three-strategy replicator dynamics and the existence of Hopf bifurcations. Since there are three strategies, the replicator equation reduces to a two dimensional system. Using nonlinear analysis methods, the authors determined the amplitude and the frequency of the bifurcating limit cycle.

1.4.2 Asymmetric Delays

In the references [21, 15], the authors considered fixed asymmetric delays across the different strategies, that is, each strategy has its own delay value, in a multiple access game context. They studied the replicator dynamics and derived new stability conditions in this context of asymmetric delays. The fitness functions of strategies A and B in this scenario can be derived as follows:

$$U_A(t, \tau_1) = as(t - \tau_1) + b(1 - s(t - \tau_1)), \quad (1.5)$$

$$U_B(t, \tau_2) = cs(t - \tau_2) + d(1 - s(t - \tau_2)), \quad (1.6)$$

where τ_1 and τ_2 are the delays associated to strategy A and B respectively, $s(t)$ and $1 - s(t)$ are the proportions of the population using strategies A and B respectively at time t . For instance, the authors derived the critical value of one delay under which the mixed ESS is asymptotically stable for any value of the second delay. Furthermore, in [21], the authors considered the Nyquist stability criterion to examine numerically the stability of the nonlinear replicator dynamics (without resorting to the linearization).

1.4.3 Interaction-dependent Delays

A recent work on delayed replicator dynamics [17] considered a new delayed fitness function where the delays depend on the type of interactions. The authors considered two models: in the first model called *full* delay model, all interactions are assumed to have a delay τ . The expected payoff of a given strategy is thus a function of the delayed frequency of strategies, and all the terms in the fitness are delayed. In the second

model, called *off-diagonal* delay model, only mixed interactions are delayed, that is interactions between individuals using different strategies. The fitness of a strategy is then composed of delayed (opposite strategy) terms and non delayed (similar strategy) terms. Then, the authors considered a stochastic combination of the two models: the *full* delay model with a probability γ and the *off-diagonal* delay model with probability $1 - \gamma$. The resulting model can be interpreted as follows: interactions between the same strategies (i.e. between opponents using the same strategies) may have a delay interval τ with probability γ or may have an immediate effect with probability $1 - \gamma$; whereas interactions between opposite strategies always induce a delay τ . Therefore, in two-strategy games, the fitness functions (or expected utilities) of strategies A and B are given by:

$$U_A(t, \tau) = a(\gamma s(t) + (1 - \gamma)s(t - \tau)) + b(1 - s(t - \tau)), \quad (1.7)$$

$$U_B(t, \tau) = cs(t - \tau) + d(\gamma(1 - s(t)) + (1 - \gamma)(1 - s(t - \tau))). \quad (1.8)$$

In this model, the authors established a complete analysis of the Hopf bifurcation using Poincaré-Lindstedt's method. In particular, the authors identified the region at which a Hopf bifurcation may exist, in function of the payoffs and the parameters of the model, and using the results in [55], they determined the amplitude of the limit cycle generated at the Hopf bifurcation.

In [57], the authors studied the impact of fixed delays in N-Person Stag Hunt game and showed that delays may lead to the disappearance of coexistence and a population composed of defectors only. We notice that some other interesting related works on delayed evolutionary games are proposed in [58, 59, 60, 61].

1.5 Stochastic Replicator Dynamics

A stochastic version of the replicator dynamics is proposed in [62] to model stochastic effects. The authors considered that the change in the fitness of individuals in a pairwise encounter is a random variable that is subject to random continuous perturbations. The continuous perturbations are not fully considered in the classic evolutionary dynamics and may have non negligible effects. The stochastic influences in natural selection may be due to random mutations, errors and immigration. Approximating the sources of perturbations by a Wiener process, the authors defined the time evolution of the relative frequency of strategy i as follows:

$$ds_i(t) = s_i(t)(U_i(t) - \bar{U}(t))dt + \sigma dW(t), \quad (1.9)$$

where $W(t)$ is a continuous, white noise process with mean zero and variance σ^2 . We observe that a term capturing the stochastic effects is added to the classic version of the replicator dynamics. Unlike the notion of an evolutionarily stable strategy which

is characterized by the resistance to some rare and isolated mutation, the authors introduced, furthermore, the notion of a stochastically stable equilibrium (SSE) which is defined by its robustness to continuous and simultaneous random perturbations. The SSE is clearly a refinement of the notion of an evolutionarily stable strategy. Formally, the SSE is the limiting distribution to which solutions of equation (1.9) converge as the noise vanishes (i.e. σ tends toward zero), and $t \rightarrow \infty$. This notion of SSE has also the advantage of addressing the issue of equilibrium selection: indeed, the authors showed that some ESSs may be stochastically stable and others not; and in some cases none of the ESSs is stochastically stable. The stochastically stable set minimizes some potential function. In a single population and two-strategy games, the authors gave an explicit formula to compute the stochastically stable set.

1.6 Spatial Evolutionary Dynamics

Continuous Space Setting

One of the limitations of standard evolutionary dynamics is that they do not take into account the spatial effects. Indeed, the notion of space or population structure is fully abstracted away in the original framework of evolutionary games, in which it is assumed that every player may interact with any other player in the population with equal probabilities and the outcome of an interaction depends solely on the strategies used. In the literature, there are some extents to the replicator and logit dynamics to include spatial effects. In continuous space settings, this has been traditionally done by introducing into the standard evolutionary dynamics a diffusion term that captures the continuous spatial variations, which yields a reaction-diffusion equation [63]. As an example, in n strategy games and one dimensional space, the time evolution of strategy i is given by:

$$\frac{\partial s_i(x,t)}{\partial t} = s_i(x,t) \left[U_i(x,t) - \bar{U}(x,t) \right] + D \frac{\partial^2 s_i(x,t)}{\partial x^2}, \quad (1.10)$$

where x stands for the spatial variable and D is a diffusion term that captures the diffusion rate of strategies over space [63]. The approach used to study such systems is based on partial differential equations. Spatial variations may have non-negligible effects as they may lead to the appearance of spatial patterns and they may address the problem of equilibrium selection. In [64, 65], the authors demonstrated the appearance of a traveling wave front solution that enables the selection of one ESS in a class of games which have two pure ESSs (coordination games). A similar result is proved in [63] in a density-dependent model of evolutionary dynamics. More recently, a different approach for including the spatial variations in the evolutionary dynamics is proposed in [66]. In this work, the authors introduced in the standard replicator dynamics a symmetric kernel that gives a positive weight to the payoff of interactions in function of the spatial proximity between the competing agents.

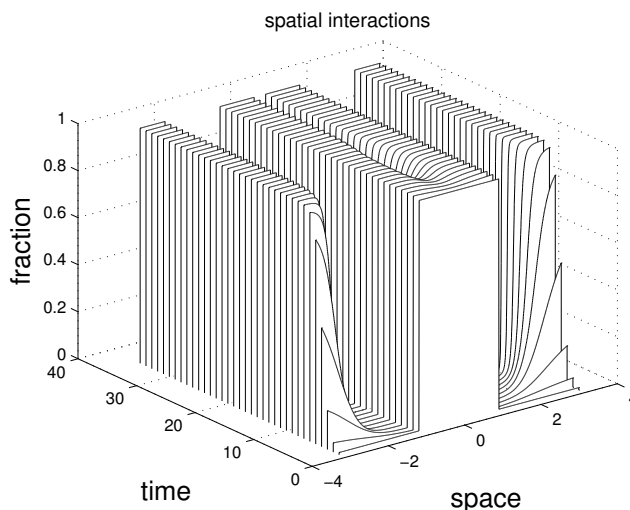


Figure 1.1: Appearance of a periodic spatial structure where $a = 1$, $b = 5$, $c = 3.3$, and $d = 3$ (anti-coordination games), Figure inspired from [66].

The traditional assumption of uniform interactions is thus relaxed by putting more weight into interactions between close neighbors whereas interactions between agents whose relative distance is large are weakened. Their approach yields an integro-differential equation with temporal and spatial variables that capture the effects of the spatial distance in the outcome of interactions. In this context, the authors demonstrated the existence of traveling front solutions in coordination games and the appearance of periodic spatial structures in anti-coordination games (see the illustration in Fig. 1.1). As a remark, we observe here that the nonuniform feature of the interactions among the agents may also be explained by the different levels of confidence or trust between each other which can be asymmetric.

Evolutionary Games On Static Graphs

Traditional assumptions in evolutionary games consisting of (i) an infinite population and (ii) a well mixed population where an agent equally likely interacts with any other opponent in the population, are also relaxed in several works that studied evolutionary dynamics on graphs or structured populations. These works include [42, 67, 47, 68, 69, 29, 30], to name a few. In graphs, the interactions are localized and an agent may interact with opponents in his local neighborhood only. In this framework, the players are represented by the vertices of the graph and the edges of the graph specifies who interacts with whom. A graph can be static when the edges are fixed, or dynamic in which case links between nodes can be broken and new others can be created [70, 68].

A pioneering work in [42] considered evolutionary dynamics on a regular graph, that is a graph where all nodes have the same fixed number of neighbors. Specifically, the

authors studied the Prisoner's Dilemma game considering as a spatial structure a lattice where each node has four neighbors. In this framework, the strategy update mechanism is as follows: at each iteration or time step, each node gets involved in pairwise encounters with all its neighbors (four neighbors with periodic boundary conditions) and its obtained payoffs in the encounters are summed up. Then, each node either switches to the strategy of the neighbor who has the maximum total payoff or it keeps its strategy if it has the highest score among its neighbors. The authors showed that, depending on the initial configuration and the parameters of the game, this deterministic spatial version of the Prisoner's Dilemma may generate a variety of spatial patterns, and even a chaotic spatial structure in which cooperators and defectors coexist in fluctuating proportions. More interestingly, when the initial configuration is symmetric, the authors gave an approximation of the asymptotic frequency of cooperators and proved its consistency with numerical simulations.

In [69], the authors studied update mechanisms for the evolutionary dynamics on cycles, which are regular graphs where each node has two neighbors. Specifically, they considered three different update rules which are Birth-Death (BD), Death-Birth (DB) and Imitation (IM) rules. A player's payoff is the sum of payoffs obtained in its pairwise interactions with its two immediate neighbors. An update rule specifies the mechanism according to which the strategies evolve in the graph. In the BD rule, an individual is selected proportional to its fitness for reproduction and its offspring replaces a randomly chosen neighbor. In the DB rule, a random individual is eliminated and replaced by a neighbor chosen proportionally to its fitness. In the IM rule, a random individual is chosen to revise its strategy and select a neighbor to imitate proportionally to fitness (it is more likely to imitate neighbors with higher fitness). The simple structure of the graph enables the exact computation of the probability for one strategy to dominate and replace the other strategy (fixation probability) in two strategy coordination games. For the Prisoner's Dilemma game, the authors showed that for DB or IM rules, cooperators can be favored over defectors while the BD rule always favors defectors. The authors considered also an interesting case in which a player can interact with any other player in the graph and not only players in his vicinity (global interactions) but can imitate only an adjacent player (local updating). In [47], the authors considered the update mechanisms described above in regular graphs with a fixed degree k .

Evolutionary Games On Dynamic Graphs

In [68], the authors studied coordination games (which admit two pure Nash equilibria) on dynamical graphs. In this work, the population is represented by a dynamic graph with asymmetric weights that represent the level of trust a node has with each neighbor. It is assumed that each node updates its level of trust with neighbors in function of payoffs it gets in encounters, and can even break the links with some vertices and create new neighbors. For the strategy update algorithm, the authors used a myopic best-response rule according to which a node switches to a strategy that maximizes its fitness obtained through weighted interactions with all its neighbors. A node has only

to know its strategy, the payoff matrix, its neighbors and their strategies. In the evolutionary dynamics, the authors considered that at each time step a randomly chosen node has the opportunity to revise its strategy according to some probability distribution and switches to the action that maximizes its payoff given the profile of its neighbors. Then, the active node can break the link with the neighbor that yields the lower payoff and create a new link with another node. Furthermore, the active node updates the weights with its neighbors. For coordination games, the authors showed that the dynamic graph evolves towards the polarization of the two strategies in two separate homogeneous clusters. Evolutionary games on dynamic graphs are also studied in [70].

1.7 Applications

1.7.1 Hawk-Dove Game

The Hawk-Dove game is a simple characterization of contests between two animals over a resource such as a food item, a territory or a mating partner [71]. For both animals, two kinds of behaviors are possible: an aggressive behavior that escalates and fights for the resource and a peaceful behavior that does never fight. The animal which gets the resource increases in biological fitness (or reproductive success). Depending on the value of the resource and the cost of fighting, this game predicts whether the aggressive behavior would dominate the population or whether hawks and doves would coexist at evolutionarily stability and at which proportions. The evolutionarily stable strategy is determined by analyzing a single interaction.

In evolutionary game literature, variants of the Hawk-Dove game exist. For example, in [72] a dynamic version of the Hawk-Dove game is proposed. In this version, it is assumed that each player (animal) has a state that corresponds to its level of energy reserves. A strategy of a player specifies which action to take as a function of its state. Assuming that an animal must minimize its probability of dying, the authors established a new ESS according to which an animal plays a hawk if its energy reserves are below some critical value, and plays dove otherwise. Furthermore, any single mutant that adopts other strategy than the ESS would get a strictly lower fitness. In [73], the author considered an heterogeneous population composed of two groups of hawks and doves that have different fighting abilities and which are linked via migration. Assuming that migration occurs at a much faster time scale than the game dynamics (Hawk-Dove game), the author studied the dynamics of the full population. In function of the migration rates, fighting costs, and resource value, new stationary solutions can appear. The classic hawk-dove game with asymmetric time delays associated with strategies (social delay) is studied in [21]. Other interesting extended versions of the Hawk-Dove game are proposed in [74, 75, 76, 77, 78, 79, 80].

A spatial version of the Hawk-Dove game is proposed in [81]. In this work, the

interactions are localized, that is, each agent interacts with neighboring agents only, and an agent changes his strategy if his neighbors are doing better than himself. The considered graph is a two-dimensional square lattice with periodic boundary conditions. The authors showed that the fraction of Hawks is lower in this spatial framework than in the classic evolutionary game theory. Furthermore, the authors proved that the spatial setting favors the success of other strategies, such as Retaliator and Bully, that do not prosper in the classic setting.

Originated in biology, the Hawk-Dove game lends itself well to various networking problems as well such as power control or medium access control. In [82, 83, 84], a semi-dynamic version of the hawk-dove game applied to power control is introduced. In this game, the aggressive behavior stands for transmitting at a high power level while the peaceful behavior is associated to transmitting at a low power level. Each mobile station (player) has a state that corresponds to its energy level. The action used by a player determines its immediate fitness and its future state. Moreover, it is assumed that a player can use only the same strategy during its lifetime. The goal of a player is to maximize its overall amount of data sent during its lifetime. The authors identified in this context a paradox in which the fraction of a population choosing the peaceful behavior at the evolutionarily stability decreases as the initial energy state of players increases. In [85], the authors applied the Hawk-Dove game to congestion control where the aggressive behavior corresponds to using a high-speed TCP version to be used over the Internet. Another application of the Hawk-Dove game in the medium access control is considered in [86]. Finally, an extension of the Hawk-Dove in an economic context is proposed in [87].

1.7.2 Control and Optimization Problems

The replicator equation has been used in control and optimization problems in large systems. For instance, in [88], the authors extended the classic framework of evolutionary games in which the interactions are assumed to occur through pairwise encounters by considering interactions that include a large and random number of players following a Poisson process. Therefore, a framework that integrates evolutionary games and Poisson games [89] is developed in this work. The authors considered an epidemic problem in a large population of heterogeneous players where each player has two possible actions: either to adopt or not a software protection. The utility of each action depends on the fraction of players choosing to invest and of those choosing not to invest, the protection cost, recovery cost, and the parameters of the Poisson process.

In this context, the authors demonstrated the existence of multiple Nash equilibria. Moreover, the authors derived the replicator equation that models the behavior of the population and predicts the proportions of the strategies in the long run. Interestingly, according to the folk theorem in evolutionary games according to which a rest point of

the replicator dynamics is a Nash equilibrium [90], the replicator equation determines the equilibria without having to explicitly compute them. Based on this result and considering as a global function to optimize the global revenue of the controller (who fixes the price of virus protection), the authors considered a two time scale dynamical system: the game dynamics that occurs at a fast time scale and which determines the asymptotic behavior of the population (fraction of the population using each strategy), and a gradient descent algorithm according to which the controller updates the price of the virus software in function of the profile of the population. The authors demonstrated that the coupled dynamics converge to an optimal price for the controller.

An evolutionary game theoretic approach for the congestion control is proposed in [91, 12, 92]. In [91], the authors used an evolutionary game theoretic approach to study a congestion control problem in multimedia traffic. In this work, the authors considered a framework where the applications (or equivalently the users) sharing a bottleneck link must choose the rate at which they receive data. Each user behaves selfishly and aims to maximize its quality of service (QoS) by choosing a specific action among a finite set of possible actions (data rates). The authors considered a two-layer model where the upper layer consists of an evolutionary game approach where the applications (or equivalently users) have to decide (repeatedly) about the rate at which they receive data; and a lower layer that captures the network performance and the QoS perceived by the users.

The authors studied both the Nash equilibria and the dynamic process according to which the users update their data rates. To model the dynamic strategy adjustment process, the authors used a version of the replicator equation coupled with a performance model. More specifically, the authors defined a Markov chain wherein the state is the number of users using every data rate. The transition from a given state to another state is proportional to the difference in the utilities perceived by a user in the two states (as in the two-strategy replicator dynamics). The authors proved the existence of a stationary distribution and examined the relationship between the Nash equilibria and the steady states of the dynamic model of strategy adjustment. We notice that in this work, all the players are engaged in an interaction (group interactions) unlike the classic framework where only two players get involved in an interaction (pairwise interactions). Moreover, through numerical examples, the authors showed that the steady state (Nash equilibrium) may not be Pareto optimal and examined the effects of the buffer capacity, link capacity, and the number of users on the system equilibrium.

1.7.3 Evolution of Language and Sociology

In [93], the authors proposed an evolutionary language game to study the evolution of vocabulary defined as a specific association between signals and objects. In this framework, the population is composed of individuals (animals) able to make a finite number

of sounds to design a finite number of objects. In the proposed model, both individuals involved in an interaction are considered as speakers and listeners. Using matrix notation, the authors defined a language as the probabilities for each object to be associated with a given sound for a speaker and also the probability for each sound to be associated with an object for a listener. The payoff of an interaction between two individuals is defined as the sum of the probabilities for each one to convey information to the other. A successful communication results in an increase in fitness (number of offsprings) for the speaker. The authors studied and compared three processes for language learning: (i) parental learning, in which children follow the language of their parents (ii) role model learning in which case children imitate successful individuals (which have the higher payoffs), (iii) random learning: where children imitate a randomly chosen individual. Using computer simulations, the authors showed that parental and random learning outperform random learning, because in the first two cases there is a higher average asymptotic payoff and faster convergence to a common language.

In [94], the authors used an evolutionary game theoretic approach to study the evolution of cultural traits in populations. The authors considered two types of cultural traits: adaptive traits which promote survival and reproduction, and maladaptive traits which reduce the fitness. Using adequate dynamics (differential equations), the authors modeled the evolution of traits in function of the rate of invention of new cultural traits, the rate of losing traits, the rate of transforming an adaptive trait into a maladaptive one.

1.8 Conclusion

In this introductory chapter, we presented the main concepts of evolutionary game theory and a literature overview of the major related works that cover delayed and spatial evolutionary dynamics. In particular, we detailed how different models of delayed fitness functions have been studied in the literature. We also examined thoroughly existing related works to evolutionary games on graphs (discrete, continuous, static, and dynamic graphs). In addition, we presented some interesting applications of this theory in the fields of networking (such as multiple access games and power control games), control and optimization, and biology (evolution of language and Hawk-Dove game).

In the next chapter, we propose to study the replicator dynamics with heterogeneous random delays in a single population.

Chapter 2

Continuous Random Delays in Replicator Dynamics

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Evolutionary game theory provides an analytical framework for studying and modeling the interactions among a large number of individuals. The individuals get involved in random pairwise interactions over and over and at each interaction, an individual gets a payoff that is a function of his strategy and others' strategies. The payoff of a strategy translates into a growth rate of the frequency of that strategy in the population. In evolutionary game dynamics, it is assumed that the growth rate of a strategy is a function of the current frequencies of strategies in the population; However, this assumption can fail to be true in many situations. Indeed, most of phenomena occurring in real-world complex systems take an uncertain time delay to have consequences. The objective of this chapter is to introduce in the replicator dynamics heterogeneous and random delays and to examine the effects of these delays on the convergence to the mixed ESS.

This chapter is structured as follows:

- In Section 2.1, we present motivations for our work and a brief literature overview;
- In Section 2.2, we examine the stability of the replicator dynamics with continuous delays. We consider uniform, exponential, and Gamma distributions;
- In Section 2.3, we study the replicator dynamics when only one strategy is delayed;
- Finally, in Section 2.4, we conclude the chapter.

2.1 Introduction

In the modeling of delays, it is usually assumed that the time delay is fixed [16, 19, 21], but in reality, the time delay is heterogeneous and random. Indeed, most of phenomena occurring in real-world complex systems, especially in the economics system, have not an immediate effect but appear after some random delay. For example, economical investments produce revenues only in the future and the time delay is uncertain. Time delays have been incorporated into biological models to represent resource regeneration times, maturation periods, feeding times, reaction times, etc. Hence, the presence of random time delays in the payoff of individuals leads to a delayed replicator equations, which may exhibit new evolutionary outcome. In examining social networks and engineering systems we often face random delays. In the scenario of social networks, users react to delayed information, and the delay is not the same for all users [51]. In large populations, each user may experience a variable delay that depends on the conditions in which the action is taken. Therefore, considering heterogeneous and random delays would be more realistic. To the best of our knowledge, this work is the first attempt to study the randomly delayed replicator dynamics. We aim to study thoroughly the impact of the delay distribution on the stability of the mixed ESS.

In evolutionary game literature, it is well known that time delays may cause the appearance of a Hopf bifurcation and oscillation in solutions of replicator dynamics. Many papers studied the stability of the Hopf bifurcation and bifurcation period solution [17, 56]. In [17], the authors analyzed the bifurcation and resulting limit cycle using Lindstedt's method when the fitness is fully delayed or when delay appears in interactions between opposite strategies only. A stability analysis in three-strategy replicator dynamics with delay has been studied in [56].

2.2 Replicator Dynamics with Continuous Random Delays

In this section, we introduce in the replicator dynamics continuous random delays. When a player uses a strategy at time t , he would receive his payoff after a random delay τ , it means at time $t + \tau$. Then, its expected utility is determined only at that instant, i.e. $U(t + \tau)$. Equivalently, the expected payoff of a strategy at the current time is a function of the state of the population at some previous random time [95]. If the delay is equal to τ , then the expected payoff of strategy A at time t is determined by:

$$U_A(t, \tau) = as(t - \tau) + b(1 - s(t - \tau)),$$

if $t \geq \tau$, it is 0 otherwise. Let $p(\tau)$ be the probability distribution of delays whose support is $[0, \infty[$. As we consider a large population, every player can experience a different positive delay. Thus, we consider the expected payoff of all the players choosing strategy A by averaging the payoffs of all individuals and then all possible delays as:

$$U_A(t) = \int_0^\infty p(\tau)U_A(t, \tau)d\tau.$$

The expected payoff of strategy A is then given by:

$$U_A(t) = a \int_0^\infty p(\tau)s(t - \tau)d\tau + b[1 - \int_0^\infty p(\tau)s(t - \tau)d\tau]. \quad (2.1)$$

Similarly, the expected payoff of strategy B is:

$$U_B(t) = c \int_0^\infty p(\tau)s(t - \tau)d\tau + d[1 - \int_0^\infty p(\tau)s(t - \tau)d\tau]. \quad (2.2)$$

Then, considering expected delay, we can write the replicator dynamics as follows:

$$\begin{aligned} \dot{s}(t) &= s(t)[1 - s(t)][U_A(t) - U_B(t)], \\ &= s(t)[1 - s(t)][-\delta \int_0^\infty p(\tau)s(t - \tau)d\tau + \delta_1], \end{aligned} \quad (2.3)$$

where the dot denotes the differentiation with respect to time. In order to examine the local stability of the ESS s^* , we can make a linearization of the replicator dynamics around this equilibrium and observe the stability of the linearized equation. We suppose there is a small perturbation around s^* defined by $x(t) = s(t) - s^*$. The replicator dynamics is then given by:

$$\dot{x}(t) = -\delta(x(t) + s^*)(1 - x(t) - s^*) \int_0^\infty p(\tau)x(t - \tau)d\tau. \quad (2.4)$$

Keeping only linear terms in x in the previous equation, we get the linearized equation:

$$\dot{x}(t) = -\delta s^*(1 - s^*) \int_0^\infty p(\tau)x(t - \tau)d\tau. \quad (2.5)$$

The characteristic equation corresponding to the delay-differential equation (2.5) is given by:

$$\lambda + \delta s^*(1 - s^*) \int_0^\infty p(\tau)e^{-\lambda\tau}d\tau = 0. \quad (2.6)$$

From the theory of delay-differential equations, the mixed equilibrium is asymptotically stable if and only if all roots of the characteristic equation (2.6) have negative real parts [96, 20, 97]. In the next subsections, we study the cases of uniform, exponential, and Gamma distributions.

2.2.1 Uniform Delay

In this case, $p(\tau) = \frac{1}{\tau_{max}}$ for $0 \leq \tau \leq \tau_{max}$, and 0 otherwise. The characteristic equation reduces to:

$$\lambda + \frac{D}{\tau_{max}} \int_0^{\tau_{max}} e^{-\lambda\tau}d\tau = 0, \quad (2.7)$$

where $D = \delta s^*(1 - s^*)$. Note that $\lambda = 0$ is not a root of (2.7). The result about the asymptotic stability of the mixed ESS is summarized in the following proposition.

Proposition 2. *The mixed ESS is asymptotically stable in the replicator dynamics with uniform distribution if and only if $\tau_{max} < \tau_{cr}$, where $\tau_{cr} = \frac{\pi^2}{2D}$.*

Proof. We suppose that $\tau_{max} < \tau_{cr}$. If λ is a real root of (2.7), then it is clear that λ cannot be positive. Let $\lambda = u + iv$ be a complex root of (2.7) with $u > 0$ and $v > 0$ (without loss of generality we assume that $v > 0$ since if $u + iv$ a solution of (2.7), then $u - iv$ is also a solution). We aim to find a contradiction and hence, we prove that no root of (2.7) with a positive real part can exist. We have:

$$v = \frac{D}{v\tau_{max}} \int_0^{\tau_{max}v} e^{-\alpha\frac{u}{v}} \sin(\alpha)d\alpha. \quad (2.8)$$

Let $\tau_{\max}v = 2k\pi + \gamma$ with $k \geq 0$ and $0 \leq \gamma < 2\pi$. From (2.8), we get:

$$v^2 = \frac{D}{\tau_{\max}} \int_0^{2k\pi+\gamma} e^{-\alpha \frac{u}{v}} \sin(\alpha) d\alpha \leq \frac{D(k+1)}{\tau_{\max}} \int_0^\pi \sin(\alpha) d\alpha = \frac{2(k+1)D}{\tau_{\max}}.$$

Thus, we have $v^2 \leq \frac{2(k+1)D}{\tau_{\max}}$. Since $\tau_{\max} < \tau_{cr}$, we get $v\tau_{\max} < (k+1)^{\frac{1}{2}}\pi$. But, $v\tau_{\max} = 2k\pi + \gamma$ with $0 \leq \gamma < 2\pi$, which finally yields $k = 0$ and $v\tau_{\max} < \pi$. On the other hand, we have $u = -\frac{D}{\tau_{\max}v} \int_0^{\tau_{\max}v} e^{-\alpha \frac{u}{v}} \cos(\alpha) d\alpha$. Let us study the sign of the right-hand side of this equation. If $\tau_{\max}v \leq \frac{\pi}{2}$, then we obtain $u < 0$, this is a contradiction. If $\tau_{\max}v > \frac{\pi}{2}$, then we have,

$$\begin{aligned} \int_0^{\tau_{\max}v} e^{-\alpha \frac{u}{v}} \cos(\alpha) d\alpha &= \int_0^{\frac{\pi}{2}} e^{-\alpha \frac{u}{v}} \cos(\alpha) d\alpha + \int_{\frac{\pi}{2}}^{\tau_{\max}v} e^{-\alpha \frac{u}{v}} \cos(\alpha) d\alpha \\ &\geq e^{-\frac{\pi u}{2v}} \int_0^{\frac{\pi}{2}} \cos(\alpha) d\alpha + e^{-\frac{\pi u}{2v}} \int_{\frac{\pi}{2}}^{\tau_{\max}v} \cos(\alpha) d\alpha > 0. \end{aligned}$$

We obtain $u < 0$, which is a contradiction with the initial assumption of $u > 0$. This proves the sufficient condition.

Furthermore, by implicit differentiation, we derive:

$$\mathcal{R}e \left[\frac{d\lambda(\tau_{\max})}{d\tau_{\max}} \right]_{\tau_{\max}=\tau_{cr}} = Dv^2 > 0.$$

Therefore, when $\tau_{\max} = \tau_{cr}$, the asymptotic stability is lost and cannot be regained. \square

We conclude that when $\tau_{\max} < \tau_{cr}$, the mixed ESS s^* is asymptotically stable, otherwise, the mixed ESS becomes unstable. In the case of a constant delay, the critical delay, which we denote by τ_{c0} , is given by $\frac{\pi}{2D}$ (see [16] for more details). Clearly, τ_{c0} is smaller than τ_{cr} . Therefore, this random feature of delay makes the possible values of delay for which the ESS is asymptotically stable larger. The mixed ESS is less sensitive to the delay when it is random.

We make numerical simulations of the trajectories of solutions of (2.3) to compare the impact of uniformly at random delays with constant delays. We fixed the game parameters to $a = 1$, $b = 5$, $c = 2$, $d = 3$, then $\tau_{cr} = 7.4$ time units. In Figure 2.1-left, we considered a constant delay of $\tau = 2.4$ time units, whereas in Figure 2.1-right, we considered uniformly at random delays with a mean equal to 2.4 time units ($\tau_{\max} = 4.8$ time units). In other words, we fixed the mean delay of the uniform distribution to the same value of the constant delay. In case of constant delays, the replicator dynamics exhibit oscillations around the ESS, and when the delays are random, the replicator dynamics converge to the ESS. This confirms that random delays are less threatening than constant delays.

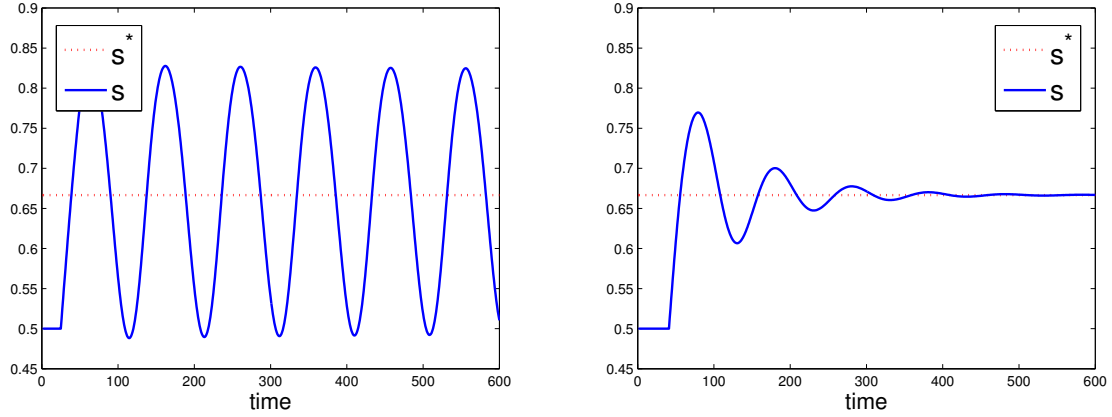


Figure 2.1: Numerical solutions of the replicator dynamics with constant delay (left) and uniformly at random delays (right), where $a = 1$, $b = 5$, $c = 2$, $d = 3$, $\delta = 3$, $\tau_{cr} = 7.4$ time units. Left, constant delay setting, $\tau = 2.4$. Right, random delay setting, $\tau_{\max} = 4.8$ time units.

2.2.2 Exponential Delay

We consider an exponential distribution of delays with parameter $\beta > 0$, $p(\tau) = \beta e^{-\beta\tau}$ whose support is $[0, \infty[$. Under this distribution, the characteristic equation (2.6) becomes:

$$\lambda + \beta D \int_0^{\infty} e^{-(\beta+\lambda)\tau} d\tau = 0. \quad (2.9)$$

We can establish the following proposition.

Proposition 3. *The mixed ESS s^* is asymptotically stable in the replicator dynamics for any value of β of the exponential distribution.*

Proof. We shall prove that all roots of (2.9) have negative real parts for any $\beta > 0$. Let α be a real solution of (2.9), it is clear that α cannot be positive. Let $\lambda = u + iv$ a complex solution of (2.9) with $u > 0$ and $v > 0$ (without loss of generality, we assume that $v > 0$, since if $u + iv$ a solution of the characteristic equation, $u - iv$ is also a solution). We aim to prove that no root with a positive real part can exist. Separating real and imaginary parts in (2.9), we get:

$$u + \beta D \int_0^{\infty} e^{-(\beta+u)\tau} \cos(v\tau) d\tau = 0, \quad (2.10)$$

$$v - \beta D \int_0^{\infty} e^{-(\beta+u)\tau} \sin(v\tau) d\tau = 0. \quad (2.11)$$

2.2. Replicator Dynamics with Continuous Random Delays

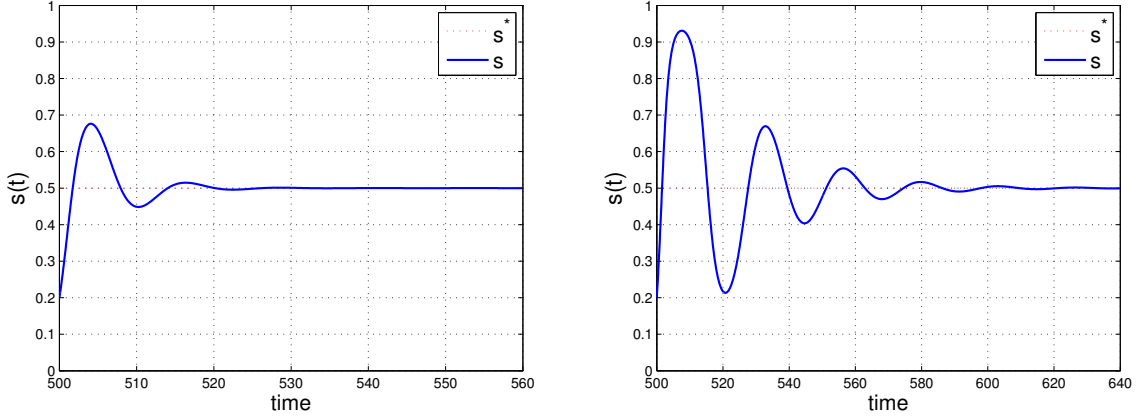


Figure 2.2: Convergence to the ESS for different values of β under the exponential distribution. Left, $\beta = 0.41$. Right, $\beta = 0.01$.

Furthermore, we have:

$$\begin{aligned} \int_0^{\infty} e^{-(\beta+u)\tau} \sin(\tau v) d\tau &= \frac{1}{v} \int_0^{\infty} e^{-(\beta+u)\frac{z}{v}} \sin(z) dz \\ &= \frac{I}{\beta + u}, \end{aligned}$$

with $I = \int_0^{\infty} e^{-(\beta+u)\frac{z}{v}} \cos(z) dz$. Taking into account of the equations above, we can write (2.10) and (2.11) as follows:

$$\begin{aligned} u + \beta D \frac{I}{v} &= 0, \\ v - \beta D \frac{I}{\beta + u} &= 0. \end{aligned}$$

The two previous equations yield $u = \frac{-\beta}{2} < 0$. This result proves that the real parts of the roots are always negative, which results in the asymptotic stability. \square

By virtue of Proposition 3, we conclude that the stability of the mixed ESS is unaffected under the exponential distribution for any value of β . This can be explained by the fact that small delays are much more likely to occur than larger delays. Thus, large delays cannot have a destabilizing effect. We depict in Fig. 2.2 solutions of the replicator dynamics for two different values of β . When $\beta = 0.41$, then the mean delay of the exponential distribution is 2.4 and we observe convergence to the ESS, unlike the case of a constant delay where we observed permanent oscillations in Fig. 2.1-Left. The value of β affects only the convergence rate.

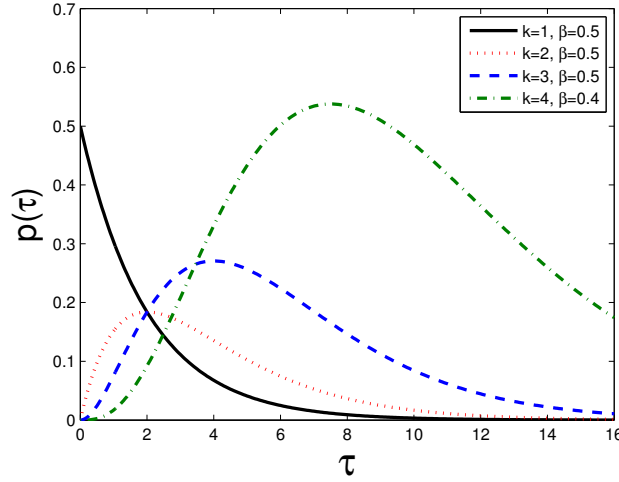


Figure 2.3: Gamma distribution of delays for different values of k and β .

2.2.3 Gamma/ Erlang Distribution

We consider an Erlang distribution (or Gamma distribution) of delays with support $[0, \infty[$ and parameters $k \geq 1$ and $\beta > 0$. The probability distribution in this case is given by $p(\tau; k, \beta) = \frac{\beta^k \tau^{k-1} e^{-\beta\tau}}{(k-1)!}$ and the mean of this delay distribution is $\frac{k}{\beta}$.

To better illustrate the effects of the parameters k and β on the delay distribution, we depict in Fig. 2.3 the Gamma distribution for some values of these two parameters. We clearly see that, as the value of the parameter k increases, more weight is put on the higher values of delays, that is larger values of delays are more probable and a user is expected to experience a larger value of delays. Thus, the critical mean delay is expected to decrease as the value of the parameter k grows. We propose to find a closed-form expression of the critical mean delay at which the stability of the mixed ESS is lost. The characteristic equation associated with this delay distribution is given by:

$$\lambda + D \int_0^\infty \frac{\beta^k}{(k-1)!} \tau^{k-1} e^{-(\beta+\lambda)\tau} d\tau = 0, \quad (2.12)$$

which can be written as:

$$\lambda + D \frac{\beta^k}{(k-1)! (\beta + \lambda)^k} \int_0^\infty z^{k-1} e^{-z} dz = 0.$$

Therefore, the characteristic equation reduces to:

$$\lambda + D \frac{\beta^k}{(\beta + \lambda)^k} = 0, \text{ or } \lambda(\beta + \lambda)^k + D\beta^k = 0.$$

The zero solution of the linearized equation is asymptotically stable if $\text{Re}(\lambda) < 0$, and unstable if $\text{Re}(\lambda) > 0$. A stability transition corresponds to the appearance of a pure imaginary solution of the characteristic equation ($\text{Re}(\lambda) = 0$) as β decreases. We aim to find the value of β at which a root passes through the imaginary axis. Substituting λ with iw_0 (with $w_0 > 0$) and β with β_c in the previous equation we get:

$$iw_0(\beta_c + iw_0)^k + D\beta_c^k = 0.$$

Equivalently, we have,

$$iw_0(\beta_c^2 + w_0^2)^{\frac{k}{2}} e^{ik\theta} + D\beta_c^k = 0,$$

with $\cos(\theta) = \frac{\beta_c}{(\beta_c^2 + w_0^2)^{\frac{1}{2}}}$ and $\sin(\theta) = \frac{w_0}{(\beta_c^2 + w_0^2)^{\frac{1}{2}}}$. Separating real and imaginary parts in the previous equation, we derive:

$$\begin{aligned} \cos(k\theta) &= 0 \\ D\beta_c^k - w_0(\beta_c^2 + w_0^2)^{\frac{k}{2}} \sin(k\theta) &= 0. \end{aligned}$$

which yields:

$$k\theta = \frac{\pi}{2} + 2n\pi, \quad n \in \mathbb{N},$$

$$D\beta_c^k = w_0(\beta_c^2 + w_0^2)^{\frac{k}{2}}.$$

Since $\cos(\theta) = \frac{\beta_c}{(\beta_c^2 + w_0^2)^{\frac{1}{2}}}$, $\cos(\frac{\pi}{2k} + \frac{2n\pi}{k}) = \frac{\beta_c}{(\beta_c^2 + w_0^2)^{\frac{1}{2}}}$ and $w_0^2 = \beta_c^2 \frac{\sin^2(\frac{\pi}{2k} + \frac{2n\pi}{k})}{\cos^2(\frac{\pi}{2k} + \frac{2n\pi}{k})}$. As β decreases from infinity, the first pure imaginary root appears when $n = 0$. From Eq. (2.13), we have:

$$D^2\beta_c^{2k} = v^2(\beta_c^2 + w_0^2)^k.$$

Substituting w_0^2 with $\beta_c^2 \frac{\sin^2(\frac{\pi}{2k})}{\cos^2(\frac{\pi}{2k})}$ in the equation above and taking the square root of both sides, we get:

$$\beta_c = D \frac{\cos^{k+1}(\frac{\pi}{2k})}{\sin(\frac{\pi}{2k})}. \quad (2.13)$$

The critical mean delay is therefore given by:

$$\frac{k}{\beta_c} = \frac{k}{D} \frac{\sin(\frac{\pi}{2k})}{\cos^{k+1}(\frac{\pi}{2k})}. \quad (2.14)$$

At this value of the mean delay, a pure imaginary root of the characteristic equation exists. We can establish the following sufficient condition of stability when the delays follow an Erlang distribution with parameter $k \geq 1$.

Proposition 4. *The mixed ESS s^* is asymptotically stable in the replicator dynamics if and only if $\beta > \beta_c$ with $\beta_c = D \frac{\cos^{k+1}(\frac{\pi}{2k})}{\sin(\frac{\pi}{2k})}$.*

Proof. Let $\beta > \beta_c$. Let $\lambda = u + iv$ be a solution of the characteristic equation with $u > 0$. We suppose without loss of generality that $v > 0$ (since if $u + iv$ is a solution of the characteristic equation, then $u - iv$ is also a solution). We aim to find a contradiction and to prove that no root with a positive real part can exist when $\beta > \beta_c$. Substituting λ with $u + iv$ and separating real and imaginary parts in the characteristic equation, we get:

$$\begin{aligned} u + [(\beta + u)^2 + v^2]^{-\frac{k}{2}} D \beta^k \cos(k\theta) &= 0, \\ v - [(\beta + u)^2 + v^2]^{-\frac{k}{2}} D \beta^k \sin(k\theta) &= 0, \end{aligned}$$

with $\cos(\theta) = \frac{\beta + u}{[(\beta + u)^2 + v^2]^{\frac{1}{2}}}$ and $\sin(\theta) = \frac{v}{[(\beta + u)^2 + v^2]^{\frac{1}{2}}}$. The system above can be written as follows:

$$\begin{aligned} u &= \frac{-D \beta^k \cos(k\theta)}{[(\beta + u)^2 + v^2]^{\frac{k}{2}}}, \\ v &= \frac{D \beta^k \sin(k\theta)}{[(\beta + u)^2 + v^2]^{\frac{k}{2}}}. \end{aligned}$$

From the equations above, we get:

$$\begin{aligned} \cos(k\theta) < 0 &\Rightarrow \frac{\pi}{2k} < \theta, \\ v^2 < D^2 \frac{\beta^{2k}}{[(\beta + u)^2 + v^2]^k} &< D^2 \frac{\beta^{2k}}{[\beta^2 + v^2]^k}, \end{aligned}$$

which yields $\cos(\theta) < \cos(\frac{\pi}{2k})$. In addition, we check that $\frac{\beta}{[\beta^2 + v^2]^{\frac{1}{2}}} < \frac{\beta + u}{[(\beta + u)^2 + v^2]^{\frac{1}{2}}}$; and then we have $\frac{\beta}{[\beta^2 + v^2]^{\frac{1}{2}}} < \cos(\theta) < \cos(\frac{\pi}{2k})$. From this inequality, we obtain:

$$\beta^2 \frac{\sin^2(\frac{\pi}{2k})}{\cos^2(\frac{\pi}{2k})} < v^2. \quad (2.15)$$

On the other hand, we have $v^2 < D^2 \frac{\beta^{2k}}{[\beta^2 + v^2]^k}$ and $\frac{\beta^{2k}}{[\beta^2 + v^2]^k} < \cos^{2k}(\frac{\pi}{2k})$, which yields:

$$v^2 < D^2 \cos^{2k}(\frac{\pi}{2k}). \quad (2.16)$$

Using (2.15) and (2.16), we get $\beta^2 < D^2 \frac{\cos^{2k+2}(\frac{\pi}{2k})}{\sin^2(\frac{\pi}{2k})}$, which is in contradiction with the initial assumption of $\beta > \beta_c$. Therefore, there is no root with a positive real part when $\beta > \beta_c$, and the local asymptotic stability follows.

Now, let us prove the necessary condition. By differentiating Eq. (2.12) with respect to β , we get :

$$\Re\left(\frac{d\lambda(\beta)}{d\beta}\right)_{\beta=\beta_c} < 0.$$

Therefore, as β decreases, the roots of characteristic equation cross the imaginary axis only from the left to the right and once the stability is lost, it cannot be regained again. The stability is persistently lost when $\beta = \beta_c$. \square

When $\beta = \beta_c$, the asymptotic stability of the mixed ESS is lost in favor of a periodic oscillation with a frequency given by:

$$w_0 = \beta_c \frac{\sin(\frac{\pi}{2k})}{\cos(\frac{\pi}{2k})}. \quad (2.17)$$

By substituting β_c with its value given in (2.13), we can write w_0 as follows:

$$w_0 = D \cos^k\left(\frac{\pi}{2k}\right). \quad (2.18)$$

The critical mean delay is $\frac{k}{\beta_c} = \frac{k \sin(\frac{\pi}{2k})}{D \cos^{k+1}(\frac{\pi}{2k})}$, which is larger than $\frac{\pi}{2D}$, the critical fixed delay. Thus, the ESS is less sensitive to a random delay than a fixed delay. When $k = 2$, $\beta_c = \frac{D}{2}$, and the critical mean delay is given by $\frac{4}{D}$. When $k = 1$, $\beta_c = 0$ and the ESS is asymptotically stable for any value of $\beta > 0$. This confirms our result on the exponential delay obtained in the previous section.

In this paragraph, we propose to illustrate the effect of the parameter k on the critical mean delay and critical frequency of oscillations. In Fig. 2.4-left, we display the critical mean delay $\frac{k}{\beta_c}$ given in (2.14) in function of k . We see that the critical mean delay decreases dramatically as the parameter k increases, that is the instability becomes more probable as k grows. This result could be intuitively expected while examining the shape of the Gamma distribution depicted in Fig. 2.3. In fact, as the parameter k grows, the higher values of delay get more probable, which affects the stability of the equilibrium. Furthermore, we display in Fig. 2.4-right, the critical frequency of oscillations given in (2.18), in function of the parameter k . We observe that, as k increases, the critical frequency becomes larger and the potential of instability emphasizes. For instance, when $k = 2$, $w_0 = 0.18$ and this value increases to 0.29 when $k = 5$.

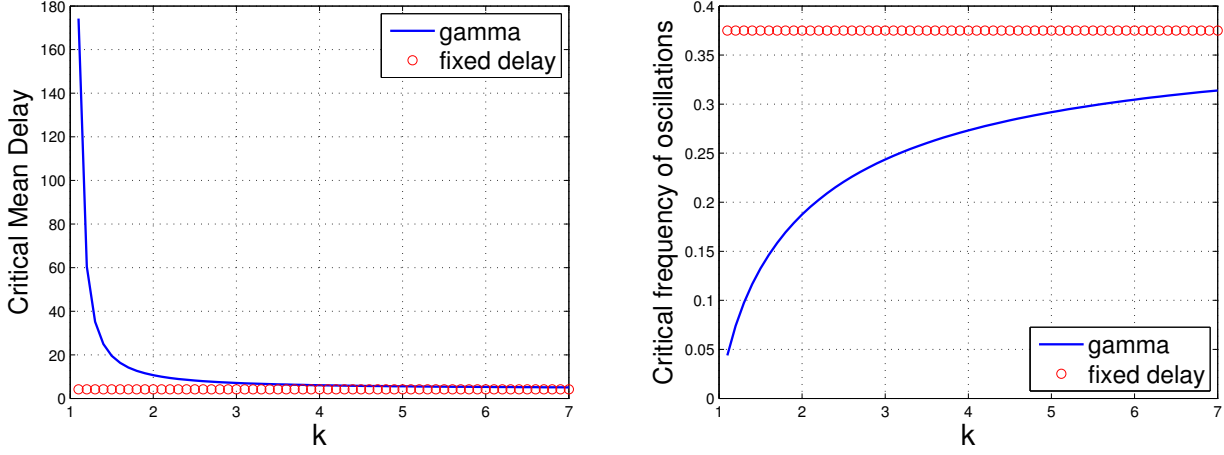


Figure 2.4: Left, the critical mean value of delays in function of the parameter k under the Gamma distribution. Right, the critical frequency of oscillations in function of the parameter k under the Gamma distribution, where $a = -0.5$, $b = 3$, $c = 0$, and $d = 1.5$.

2.3 Replicator Dynamics with Asymmetric Delays

In this section, we study the replicator dynamics with one delayed strategy. Indeed, there exist many scenarios in which this situation holds. In networking, in particular for the multiple access game [21], the action of transmitting may be delayed, while keeping silent would have no payoff and no delay. For the Hawk-Dove game, the aggressive behavior may have a delayed payoff due to the fighting time, whereas the peaceful type would not fight and no delay is incurred. The time delay may represent the time of fighting and it is random since it depends on whether the opponent is aggressive or not.

In the following, we suppose that only strategy A incurs a random delay and strategy B has no delay. In this case, the expected payoff of strategy A is given by:

$$U_A(t) = a \int_0^\infty p(\tau) s(t - \tau) d\tau + b \left[1 - \int_0^\infty p(\tau) s(t - \tau) d\tau \right].$$

The expected payoff of strategy B remains unchanged and is given by:

$$U_B(t) = cs(t) + d(1 - s(t)).$$

Following the same procedure in the previous sections, we write the linearized replicator dynamics as follows:

$$\dot{x}(t) = D \left[(a - b) \int_0^\infty p(\tau) x(t - \tau) d\tau + (d - c)x(t) \right], \quad (2.19)$$

with $D = s^*(1 - s^*)$. The characteristic equation corresponding to the equation above is given by:

$$\lambda - D(a - b) \int_0^\infty p(\tau) e^{-\lambda\tau} d\tau - D(d - c) = 0. \quad (2.20)$$

Exponential Delay

We consider an exponential distribution and we suppose that $p(\tau) = \beta e^{-\beta\tau}$ with $\beta > 0$. The characteristic equation can then be written as:

$$\lambda - D\beta(a-b)\frac{1}{\lambda + \beta} - D(d-c) = 0. \quad (2.21)$$

Let $\lambda^* = iw_0$ with $w_0 > 0$ be a pure imaginary root of the characteristic equation. Substituting λ with λ^* in (2.21) and separating the real and imaginary parts, we get the following system:

$$\begin{aligned} 1 + \frac{D\beta_c(a-b)}{\beta_c^2 + w_0^2} &= 0, \\ \frac{D\beta_c^2(a-b)}{\beta_c^2 + w_0^2} + D(d-c) &= 0, \end{aligned}$$

which yields,

$$\begin{aligned} -D\beta_c(a-b) &= \beta_c^2 + w_0^2, \\ \frac{D\beta_c^2(a-b)}{\beta_c^2 + w_0^2} &= -D(d-c). \end{aligned}$$

Finally, we get:

$$-D\beta_c(a-b) = \beta_c^2 + w_0^2, \quad (2.22)$$

$$\beta_c = D(d-c). \quad (2.23)$$

Therefore, we conclude that as β decreases, at the value of $\beta_c = D(d-c)$, a stability switch occurs. We can establish the following result:

Proposition 5. • *Let $b > a$, the interior equilibrium s^* is asymptotically stable if and only if $\beta > \beta_c$ where $\beta_c = D(d-c)$. Furthermore, the critical frequency of oscillations is given by:*

$$w_0 = \sqrt{\delta D\beta_c}.$$

- *Let $a > b$, the interior equilibrium is asymptotically stable for any $\beta > 0$.*

Proof. • Let $a - b < 0$. We shall prove that if $\beta > \beta_c$, then all roots of (2.21) have negative real parts, and if $\beta < \beta_c$, there exists a root with a positive real part. Let $\lambda = u + iv$ be a root of (2.21) with $v \geq 0$. Substituting λ with $u + iv$ in (2.21), we get:

$$u + iv - \frac{D\beta(a-b)}{u + \beta + iv} - D(d-c) = 0.$$

Separating real and imaginary parts in the equation above, we get:

$$1 + \frac{D\beta(a-b)}{(u+\beta)^2 + v^2} = 0,$$

$$u - \frac{D\beta(a-b)(u+\beta)}{(u+\beta)^2 + v^2} - D(d-c) = 0.$$

We obtain,

$$(\beta + u)^2 + v^2 = -\beta D(a-b),$$

$$2u = -\beta + D(d-c).$$

Therefore, $u < 0$ if $\beta > \beta_c$. And consequently, $u > 0$ if $\beta < \beta_c$.

• Let $a > b$, then all roots of (2.21) are real, and $c - d > 0$ (since $\delta > 0$). Let α be a real root of (2.21), we have:

$$\alpha^2 + \alpha[\beta - D(d-c)] + \beta D\delta = 0.$$

The discriminant of the equation above is given by:

$$\mathcal{X} = [\beta + D(d-c)]^2 + 4\beta D(a-b) > 0.$$

Since $\delta > 0$, and $\beta - D(d-c) > 0$, the roots of the characteristic equation are negative, and the local asymptotic stability follows. \square

Remarkably, we observe that the exponential delay may affect the stability of the mixed ESS. This scenario could not be observed when both strategies have the same delay which follows the exponential distribution. Indeed, in this case, the asymmetry property between the delays of the two strategies yields a new scenario and may increase the possibility of the instability. It is worth mentioning that, considering asymmetric delays across the strategies or only one delayed strategy rather than symmetric delays, yields a delay differential equation with more terms, which is more challenging to analyze. Therefore, we could find closed-form results in the case of exponential delay only.

In order to better illustrate our results, we make numerical simulations, and we display the numerical solutions of the replicator dynamics in the scenario where only one strategy is delayed and subject to exponential delay distribution. When the game parameters are fixed as follows: $a = -1.5$, $b = 3$, $c = 0$, $d = 1.5$, there exists a unique mixed ESS given by $s^* = 0.5$, the critical value of the parameter β predicted analytically in Proposition 5 is given by $\beta_c = 0.37$, and the critical mean delay is equal to 2.7 time units (since $b > a$, a Hopf bifurcation exists, see Proposition 5). In Fig. 2.5-left, we fixed the value of β to 0.6, and consequently the mean delay is 1.67 time units. In this case, we observe that the replicator dynamics converges to the mixed ESS and no oscillations appear. In contrast, in Fig. 2.5-right, we decreased the value of β to 0.2 and the mean delay in this case is given by 5 time units. As a consequence of this value of β , the replicator dynamics exhibits permanent oscillations around the mixed ESS. The numerical simulations are in coherence with our analytical results.

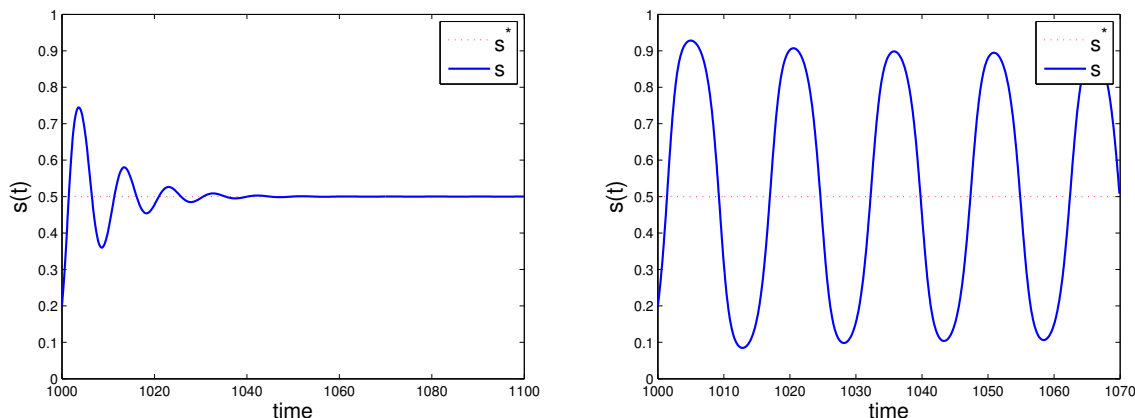


Figure 2.5: Numerical solutions of the replicator dynamics with one delayed strategy (exponential distribution), where $a = -0.5$, $b = 3$, $c = 0$, $d = 1.5$, $s^* = 0.5$. Left, $\beta = 0.6 > \beta_c = 0.37$. Right, $\beta = 0.2 < \beta_c$.

2.4 Conclusion

In this chapter, we studied the asymptotic stability of the mixed ESS in presence of heterogeneous and random continuous delays. Indeed, many scenarios exist where the delays are not the same for all the users and are variable in function of the conditions in which the actions are taken. For example, in social networks, users react to delayed information and each user may have its own delay. Considering a large population of players, we suppose that each player may experience a delay that can be seen as an independent and identically distributed (i.i.d.) random variable drawn from some probability distribution. By averaging over all the users in the population, we derived the expected utilities of strategies and then the replicator dynamics.

To investigate the stability of the replicator dynamics with random continuous delays, we resorted to a linearization of the replicator dynamics, analyzed the associated characteristic equation, and derived closed-form results. We proved that when the distribution of delays is uniform, a large mean value of the delays results in the instability of the mixed ESS and the appearance of permanent oscillations in the proportions of strategies in the population. We found analytically the critical value of the mean delay beyond which the stability is lost. Moreover, when the delays follow the Erlang (or Gamma) distribution, we found analytically the critical value of the mean delay at which the Hopf bifurcation occurs and the stability is persistently lost. We also determined the period of oscillations of the bifurcating periodic solution. Interestingly, under the exponential delay distribution, we proved that the mixed ESS remains asymptotically stable for any value of the mean delay. In all the cases, the critical mean delay is larger than the critical delay obtained when the delay is constant. We illustrated our results with numerical examples.

However, when there is only one delayed strategy, the results are not the same as those obtained in case of symmetric delays. For the exponential delay distribution, there may exist a Hopf bifurcation, a scenario which cannot be observed when both strategies are delayed. Indeed, in this case, the asymmetry property between the delays of the two strategies yields a new scenario and make the problem more challenging to analyze. Technically, considering asymmetric delays across the strategies or only one delayed strategy yields a delay differential equation with more terms, which makes the analysis more complex.

In the next chapter, we propose to examine the stability of the replicator dynamics in presence of random discrete delays.

Chapter 3

Discrete Random Delays in Replicator Dynamics

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In this chapter, we continue our study of the replicator dynamics in presence of random delays. We propose to examine thoroughly the stability of the mixed equilibrium considering discrete distributions of delays. In fact, time delays are ubiquitous and may arise under various forms. Taking account of discrete random delays is a novelty since most of works related to evolutionary dynamics have considered deterministic delays only. In order to derive analytical results in the context of stochastic delays, we used techniques inspired from the theory of delay differential equations [20], as well as results established in previous works. Our results are corroborated with numerical simulations.

This chapter is structured as follows:

- In Section 3.1, we examine the stability of the replicator dynamics with one, two, and several discrete random delays;
- In Section 3.2, we study the replicator dynamics when only one strategy is delayed;
- Finally, we conclude the chapter in Section 3.3.

3.1 Replicator Dynamics with Discrete Random Delays

In this section, we consider symmetric and random discrete delays. We start by the case of one delay, then we consider two delays, and finally we extend our study to the multiple-delay case.

3.1.1 Replicator Dynamics with One Delay

We suppose in this section that a strategy, either A or B , would take a delay τ with probability p or no delay with probability $1 - p$. When p is high (near one), then the interactions are more likely to be delayed, whereas when p is low (near zero), then most of interactions are not delayed. Therefore, we might expect that the critical delay decreases as the probability of a delayed strategy or interaction grows. We propose to investigate in this scenario the asymptotic stability of the mixed ESS. In this case, the replicator dynamics is given by:

$$\frac{ds(t)}{dt} = s(t)(1 - s(t))(-p\delta s(t - \tau) - (1 - p)\delta s(t) + b - d). \quad (3.1)$$

Let $x(t) = s(t) - s^*$. Substituting s with x in the equation (4.34), we get:

$$\begin{aligned} \frac{dx(t)}{dt} = & -(1 - p)\delta \gamma x(t) - p\delta \gamma x(t - \tau) - p\delta(1 - 2s^*)x(t)x(t - \tau) - (1 - p)\delta \times \\ & (1 - 2s^*)x(t)^2 + p\delta x(t - \tau)x(t)^2 + (1 - p)\delta x(t)^3, \end{aligned} \quad (3.2)$$

Keeping only linear terms in x in the previous replicator dynamics equation, we get the following linearized equation:

$$\frac{dx(t)}{dt} = -(1-p)\delta\gamma x(t) - p\delta\gamma x(t-\tau). \quad (3.3)$$

The associated characteristic equation is given by:

$$\lambda + p\delta\gamma \exp(-\lambda\tau) + (1-p)\delta\gamma = 0. \quad (3.4)$$

The characteristic equation allows us to derive results about the local asymptotic stability of the mixed ESS. We summarize the stability properties in this scenario in the following proposition.

Proposition 6. • If $p \leq 0.5$, then the mixed ESS is asymptotically stable in the replicator dynamics for any value of τ ,

- If $p > 0.5$, then a Hopf bifurcation exists when $\tau = \tau_{cr}$ where $\tau_{cr} = \frac{\arccos(-\frac{1-p}{p})}{\delta\gamma\sqrt{2p-1}}$, at which the asymptotic stability of the mixed ESS is lost.

Proof. • Let $\lambda = u + iv$, where $v > 0$ a root of (3.4). We suppose that $u > 0$ and we aim to prove that $p > 0.5$. Substituting λ by $u + iv$ in equation (3.4) and separating the real and imaginary parts, we derive:

$$u + (1-p)\delta\gamma = -p\delta\gamma e^{-u\tau} \cos(v\tau), \quad (3.5)$$

$$v = p\delta\gamma e^{-u\tau} \sin(v\tau). \quad (3.6)$$

which yields,

$$(u + (1-p)\delta\gamma)^2 + v^2 = p^2 \delta^2 \gamma^2 e^{-2u\tau}. \quad (3.7)$$

Since $u > 0$, we conclude the following inequalities,

$$(u + (1-p)\delta\gamma)^2 + v^2 \leq p^2 \delta^2 \gamma^2, \quad (3.8)$$

$$((1-p)\delta\gamma)^2 \leq (u + (1-p)\delta\gamma)^2 + v^2. \quad (3.9)$$

Finally, we obtain,

$$(1-p) < p, \quad (3.10)$$

and consequently,

$$p > 0.5. \quad (3.11)$$

Therefore, we conclude that $u < 0$ for any $p \leq 0.5$.

- Let $\lambda^* = iw_0$, where $w_0 > 0$, be a pure imaginary root of the characteristic equation. From (3.4), we get:

$$iw_0 + p\gamma\delta \exp(-iw_0\tau_{cr}) + (1-p)\gamma\delta = 0, \quad (3.12)$$

which yields,

$$\begin{aligned} \cos(w_0\tau_{cr}) &= -\frac{1-p}{p}, \\ \sin(w_0\tau_{cr}) &= \frac{w_0}{p\gamma\delta}. \end{aligned}$$

Therefore, we obtain:

$$\tau_{cr} = \frac{\text{acos}\left(-\frac{1-p}{p}\right)}{\delta\gamma\sqrt{2p-1}}, \quad (3.13)$$

$$w_0 = \delta\gamma\sqrt{2p-1}, \quad (3.14)$$

where 'acos' denotes the 0 to π branch of the inverse cosine function. Furthermore, we have,

$$\mathcal{R}e \frac{d\lambda(\tau)}{d\tau} \Big|_{\tau=\tau_{cr}} = \frac{w_0^2}{(1+(1-p)\delta\gamma\tau_{cr})^2 + \tau_{cr}^2 w_0^2} > 0,$$

which means that when τ is near τ_{cr} and $\tau > \tau_{cr}$, two roots gain positive parts as τ passes through τ_{cr} . Therefore, when $p \geq 0.5$, a Hopf bifurcation exists at τ_{cr} at which the asymptotic stability of the mixed equilibrium is lost.

□

As a remark, we observe that when $p = 1$, the critical delay is given by $\tau_{cr} = \pi/2\delta\gamma$ and this value coincides with the critical delay determined in the Dirac distribution case.

In Fig. 3.1, we plot the critical delay τ_{cr} and the frequency of oscillations at the Hopf bifurcation w_0 , in function of p , the probability of a delayed strategy. The range of p over which a Hopf bifurcation may exist is $]0.5, 1]$; when p is outside this range, the mixed equilibrium is asymptotically stable for any value of τ . We observe that as p increases, the critical delay decreases, and thus the potential of instability increases. For instance, when $p = 0.6$, the critical delay value is given by 20.5 time units, whereas this value decreases to 9.4 time units when $p = 0.8$. In addition, the frequency of oscillations at the Hopf bifurcation grows as p increases, which emphasizes the instability property. For example, when $p = 0.6$, $w_0 = 0.11$, while $w_0 = 0.19$ when $p = 0.8$.

Furthermore, it is interesting to compare the results derived in our scenario with those obtained in the classical case of a single delay. Therefore, we displayed in Fig.

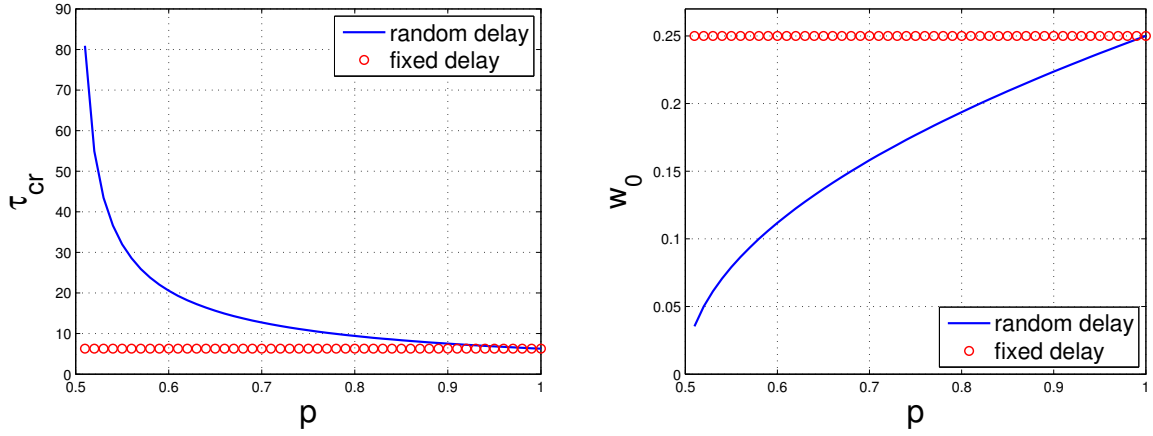


Figure 3.1: Left, the critical delay τ_{cr} in function of p . Right, the frequency of oscillations at the Hopf bifurcation w_0 , in function of p , where $a = -0.5$, $b = 1$, $c = 0$, and $d = 0.5$.

3.1 the critical delay (which we denote by τ_{c0} and the frequency of oscillation (which we denote by w_{c0}) in the case of a single delay. We observe that τ_{cr} (as defined in proposition 6) is always larger than τ_{c0} and they coincide only when $p = 1$. Similarly, w_0 is always smaller than w_{c0} and they coincide when $p = 1$, in which case the two scenarios are exactly the same.

3.1.2 Replicator Dynamics with Two Delays

In this section, we extend our analysis to cover the case of two distinct delays. We suppose that a strategy used by a player would take a delay τ_1 with probability p_1 , a delay τ_2 with probability p_2 , or no delay ($\tau_0 = 0$) with probability p_0 ($p_0 = 1 - p_1 - p_2$). Equivalently, this means that a player would receive a payoff after τ_1 time units with probability p_1 or τ_2 time units with probability p_2 , or immediately after using a strategy with probability p_0 . Using the same notation as in the previous chapter, we can write the expected payoff of strategy A as follows:

$$U_A(t) = a[p_0s(t) + p_1s(t - \tau_1) + p_2s(t - \tau_2)] + b[1 - p_0s(t) - p_1s(t - \tau_1) - p_2s(t - \tau_2)].$$

Similarly,

$$U_B(t) = c[p_0s(t) + p_1s(t - \tau_1) + p_2s(t - \tau_2)] + d[1 - p_0s(t) - p_1s(t - \tau_1) - p_2s(t - \tau_2)].$$

Indeed, we introduced in the utility the state or the profile of the population at some previous instants weighted by the probabilities of delays. The replicator dynamics is then given by:

$$\dot{s}(t) = s(t)(1 - s(t))[-\delta(p_0s(t) + p_1s(t - \tau_1) + p_2s(t - \tau_2)) + \delta_1].$$

Making a linearization of the equation above around s^* we get:

$$\dot{x}(t) \approx -\delta s^*(1-s^*)[p_0x(t) + p_1x(t-\tau_1) + p_2x(t-\tau_2)]. \quad (3.15)$$

Equation (3.15) was studied in [98, 99, 100, 101]. We use the following proposition in [100] to conclude about the stability of the mixed ESS in the replicator dynamics.

Proposition 7. [100] *Let $\dot{x}(t) = -a_0x(t) - a_1x(t-h_1) - a_2x(t-h_2)$, where $h_1 > 0$ and $h_2 > 0$. If one of the following conditions holds:*

- $0 < a_0, |a_1| + |a_2| < a_0,$
- $0 < a_0 + a_1 + a_2, |a_1|h_1 + |a_2|h_2 < \frac{a_0 + a_1 + a_2}{|a_0| + |a_1| + |a_2|},$
- $0 < a_0 + a_1, |a_1|h_1 < \frac{a_0 + a_1 - |a_2|}{|a_0| + |a_1| + |a_2|},$
- $0 < a_0 + a_2, |a_2|h_2 < \frac{a_0 + a_2 - |a_1|}{|a_0| + |a_1| + |a_2|}.$

Then the equation above is asymptotically (and exponentially) stable.

From this proposition, the next corollary follows by substituting the parameters a_i , $i = 0, 1, 2$ with those in the linearized replicator dynamics.

Corollary 1. *If one of the following conditions holds:*

- (i) $p_1 + p_2 < p_0,$
- (ii) $p_1\tau_1 + p_2\tau_2 < \frac{1}{\delta s^*(1-s^*)},$
- (iii) $p_1\tau_1 < \frac{p_0 + p_1 - p_2}{\delta s^*(1-s^*)},$
- (iv) $p_2\tau_2 < \frac{p_0 + p_2 - p_1}{\delta s^*(1-s^*)},$

then s^ is asymptotically stable in the replicator dynamics.*

The condition of stability (i) is independent of the values of delays, τ_1 and τ_2 and is equivalent to $p_0 > 0.5$. This means that, when the probability of the non delayed term is sufficiently high, the asymptotic stability of the ESS cannot be lost for any value of τ_1 and τ_2 . This result is coherent with that obtained when the delays follow an exponential distribution. Indeed, we proved that this distribution does not affect the stability of the ESS which may be explained by the shape of the exponential distribution. The condition (ii) gives an upper bound of the mean delay. The last two conditions give an upper bound of the values of one delay for which the stability of the interior equilibrium is unaffected for any value of the second delay.

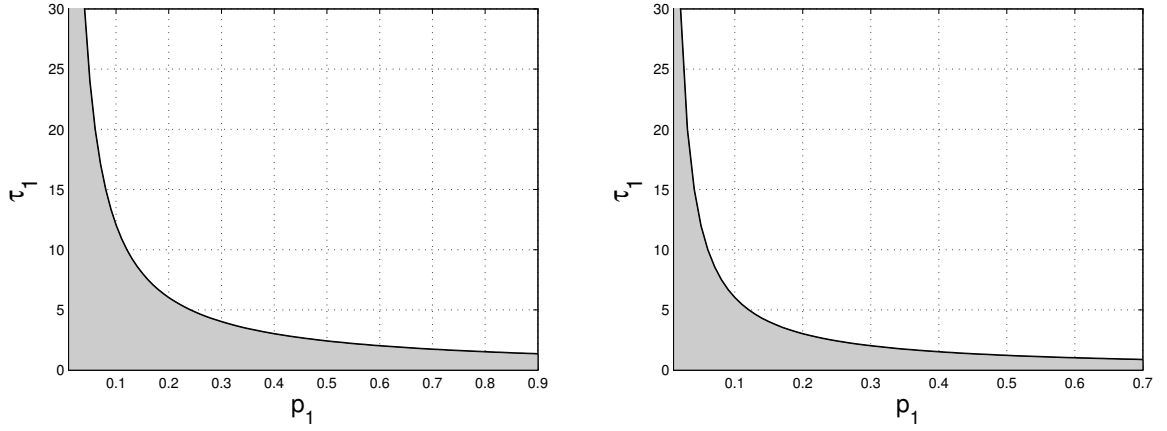


Figure 3.2: When τ_1 is in the shaded region, then the mixed equilibrium is asymptotically stable for any value of τ_2 , where $a = 1$, $b = 5$, $c = 2$, and $d = 3$. Left $p_2 = 0.1$. Right $p_2 = 0.3$.

In order to better understand these results, we illustrate in Fig. 3.2, the region of stability given in the condition (iii) in the corollary above. When τ_1 is in the shaded region, the asymptotic stability cannot be lost for any value of τ_2 . This result is interesting because it gives a sufficient stability condition on one parameter only τ_1 (or τ_2). Similarly to our previous observations, the critical delay becomes smaller as the probability of delays grows.

3.1.3 Replicator Dynamics with Several Delays

We consider that a strategy used by a player would take a delay τ_k with probability p_k , where $k = 0, \dots, q$, $\tau_0 = 0$ and $\sum_{k=0}^q p_k = 1$. We are interested in finding a delay-independent stability condition. Doing the same as in the previous sections, we can write the linearized replicator dynamics as follows:

$$\dot{x}(t) = -\delta s^*(1 - s^*) \left[p_0 x(t) + \sum_{k=1}^q p_k x(t - \tau_k) \right].$$

Following [102], we derive the following necessary and sufficient delay-independent stability condition for the asymptotic stability of the interior equilibrium (the proof in [102], Theorem 3.2).

Proposition 8. [102] Let $\dot{x}(t) = a_0 x(t) + \sum_{k=1}^q b_k x(t - h_k)$ where $a_0 < 0$. The zero solution is asymptotically stable independently of delays if and only if $-a_0 > \sum_{k=1}^q |b_k|$ or if

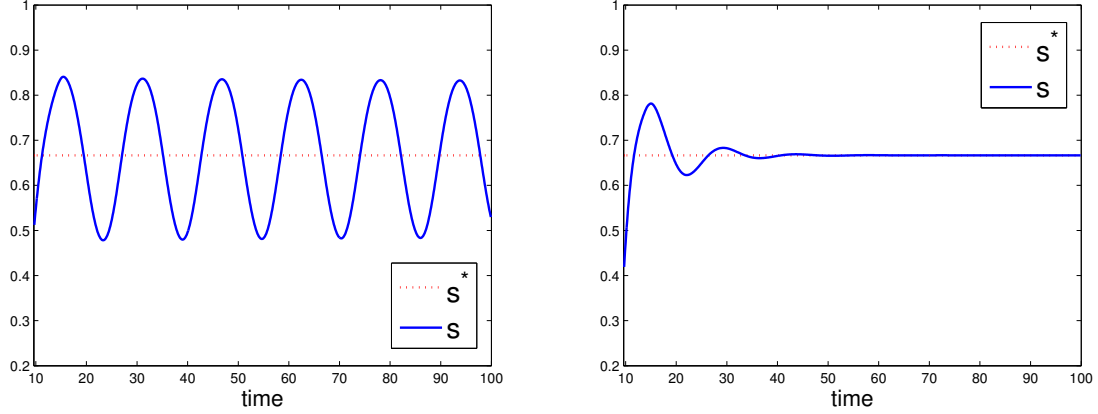


Figure 3.3: Numerical solutions of the replicator dynamics with discrete random delays, in both figures: $a = 1$, $b = 5$, $c = 2$, $d = 3$, $\tau_0 = 0$, $\tau_1 = 4.4$, $\tau_2 = 5.2$ time units. Left $p_0 = 0.3$, $p_1 = 0.2$ and $p_2 = 0.5$. Right $p_0 = 0.6$, $p_1 = 0.2$ and $p_2 = 0.2$.

$$-a_0 = \sum_{k=1}^q |b_k| \text{ but } a_0 + \sum_{k=1}^q b_k \neq 0.$$

The next corollary immediately follows.

Corollary 2. *The mixed ESS is asymptotically stable in the replicator dynamics for any τ_k , if and only if $p_0 \geq \frac{1}{2}$.*

In Figure 3.3, we depict the trajectories of solutions of the replicator equation for $\tau_1 = 4.4$ and $\tau_2 = 5.2$ time units. In the left subfigure, we fixed p_0 and p_1 to 0.3 and 0.2 respectively; whereas in the right subfigure, we fixed p_0 and p_1 to 0.6 and 0.2 respectively. The results observed here corroborate our previous results. Though the values of delays are unchanged, the persistent oscillations observed in the left subfigure disappear in the right subfigure because, in this case, the probability of $\tau = 0$ is sufficiently high, and the delay-independent stability condition in corollary 2 is satisfied. Thus, the values of delays do not have a destabilizing consequence. We are also interested in the case where there are several delays which are all positive (i.e. $p_0 = 0$). We aim to look whether the asymptotic stability can be preserved and for which conditions. We use the following result [103]:

Proposition 9. *Let $\dot{x}(t) = -\sum_{k=1}^q a_k x(t - \tau_k)$, with $\tau_k > 0$ for all k and $\sum_{k=1}^q a_k > 0$. If*

$$\sum_{k=1}^q \left(\sum_{i=k}^q a_i \right) |\tau_k - \tau_{k-1}| < \frac{\sum_{k=1}^q a_k}{\sum_{k=1}^q |a_k|},$$

where $\tau_0 = \frac{1}{e \sum_{k=1}^q a_k}$. Then, the equation above is asymptotically (and exponentially) stable.

We conclude the following condition:

Corollary 3. *If*

$$\sum_{k=1}^q \left[\sum_{i=k}^q p_i \right] |\tau_k - \tau_{k-1}| < \frac{1}{\delta s^* (1 - s^*)},$$

where $\tau_0 = \frac{1}{\delta s^* (1 - s^*)} e$. Then the interior equilibrium is asymptotically (and exponentially) stable.

3.2 Replicator Dynamics with Asymmetric Delays

In this section, we consider the case where only one strategy is delayed. We suppose that strategy A would have a delay τ with probability p or no delay with probability $1 - p$. The expected payoff to strategy A is then given by:

$$\begin{aligned} U_A(s) &= a((1-p)s(t) + ps(t-\tau)) + b\left((1-p)(1-s(t)) + p(1-s(t-\tau))\right), \\ &= (1-p)(a-b)s(t) + p(a-b)s(t-\tau) + b. \end{aligned}$$

The expected payoff to strategy B remains unchanged and is given by:

$$U_B(t) = cs(t) + d(1-s(t)).$$

The replicator dynamics is then given by:

$$\dot{s}(t) = s(t)(1-s(t)) \left(s(t)((1-p)(a-b) - c + d) + p(a-b)s(t-\tau) + b - d \right).$$

The linearized replicator dynamics around the interior stationary point is given by:

$$\dot{x}(t) = s^*(1-s^*) \left(((1-p)(a-b) - c + d)x(t) + p(a-b)x(t-\tau) \right). \quad (3.16)$$

Let $\gamma = s^*(1-s^*)$. We recall that $s^* = \frac{b-d}{b-d+c-a}$, $\gamma = \frac{(b-d)(c-a)}{(b-d+c-a)^2}$, and $b > d$, and $c > a$. (Recall that we consider anti-coordination games). Following the same procedure in the previous section, we derive the characteristic equation associated with equation (3.16):

$$\lambda - \gamma((1-p)(a-b) - c + d) - \gamma p(a-b)e^{-\lambda\tau} = 0.$$

Let $\tau = \tau_{cr}$ and $\lambda = iw_0$. Separating real and imaginary parts in the characteristic equation, we get the following system:

$$\begin{aligned} \cos(w_0\tau_{cr}) &= -\frac{p(a-b) - c + d}{p(a-b)}, \\ \sin(w_0\tau_{cr}) &= -\frac{w_0}{\gamma p(a-b)}. \end{aligned}$$

Squaring and summing up the two previous equations, we get:

$$w_0^2 = \gamma^2(b-d+c-a)\left((1-2p)(a-b) - (c-d)\right).$$

The critical frequency is real if and only if $((1-2p)(a-b) - (c-d)) > 0$. In this case, it is given by:

$$w_0 = (b-d)(c-a)\sqrt{\frac{(1-2p)(a-b) - c + d}{(b-d+c-a)^3}}. \quad (3.17)$$

From 3.17, the critical delay is given by:

$$\tau_{cr} = \frac{1}{\gamma\sqrt{(b-d+c-a)\left((1-2p)(a-b) - (c-d)\right)}} \operatorname{acos}\left(-\frac{(1-p)(a-b) - c + d}{p(a-b)}\right).$$

Or equivalently,

$$\tau_{cr} = \frac{1}{(b-d)(c-a)}\sqrt{\frac{(b-d+c-a)^3}{(1-2p)(a-b) - c + d}} \operatorname{acos}\left(-\frac{(1-p)(a-b) - c + d}{p(a-b)}\right).$$

When $p = 1$, we have the deterministic case in which the strategy A has a fixed delay of value τ . We have the following result on the stability of the mixed ESS.

Proposition 10. *The mixed ESS s^* is asymptotically stable in the replicator dynamics with one delayed strategy if $\tau < \tau_{cr}$ where $\tau_{cr} =$*

$$\frac{1}{(b-d)(c-a)}\sqrt{\frac{(b-d+c-a)^3}{(1-2p)(a-b) - c + d}} \operatorname{acos}\left(-\frac{(1-p)(a-b) - c + d}{p(a-b)}\right).$$

Proof. The characteristic equation is given by:

$$\lambda - \gamma((1-p)(a-b) - c + d) - p\gamma(a-b)e^{-\lambda\tau} = 0$$

Let $\lambda = u + iv$ a root of the characteristic equation with $v > 0$ (without loss of generality). We aim to prove that when $u > 0$, then $\tau > \tau_{cr}$. Separating real and imaginary parts in the characteristic equation, we get:

$$u - A - B\exp(-u\tau)\cos(v\tau) = 0, \quad (3.18)$$

$$v + B\exp(-u\tau)\sin(v\tau) = 0, \quad (3.19)$$

where $A = \gamma((1-p)(a-b) - c + d)$ and $B = p\gamma(a-b)$. From (3.18) and (3.19), we derive:

$$\begin{aligned} \cos^2(v\tau) &> \frac{A^2}{B^2}, \\ v^2 &\leq B^2 - A^2, \end{aligned}$$

which finally yields $\tau > \tau_{cr}$. Therefore, we conclude that when $\tau < \tau_{cr}$, the real parts of the roots of the characteristic equation are negative and the local asymptotic stability follows. \square

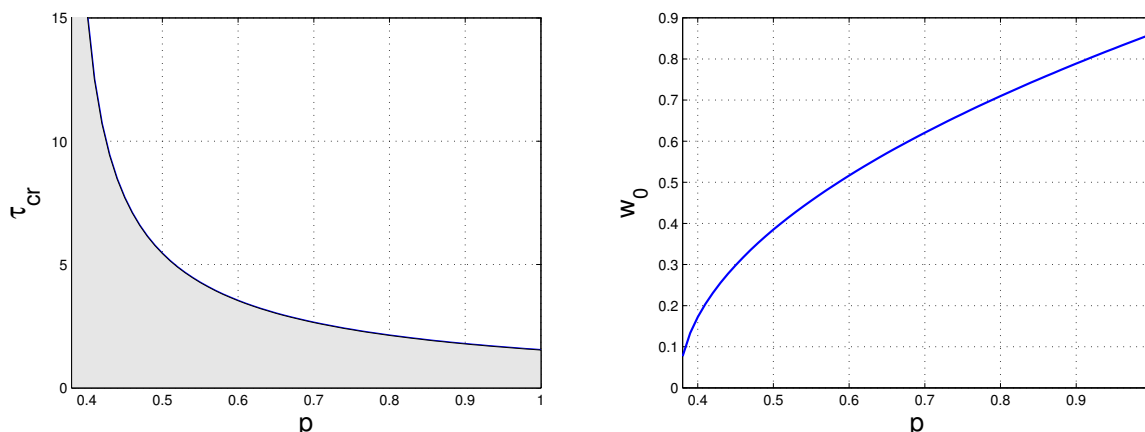


Figure 3.4: Left, the shaded region of stability bounded by the critical delay τ_{cr} . Right, the frequency of oscillations near the Hopf bifurcation, where $a = 1$, $b = 5$, $c = 2$, $d = 3$.

A first observation is that, even when only one strategy is delayed, the existence of a Hopf bifurcation and the loss of stability are possible. This proposition allows us to identify the region of stability of the mixed equilibrium in function of the probability p and the game parameters. In Fig. 3.4-left, we display the region of stability of the mixed ESS when only one strategy is delayed. This region is bounded by the critical delay derived in Proposition 10. We observe that the critical delay decreases significantly as p grows. Furthermore, we plot in Fig. 3.4-right the frequency of oscillations near the Hopf bifurcation. We see that w_0 increases in function of p , which means that the instability property gets more stressed as p grows. For instance, when $p = 0.5$, $w_0 = 0.38$, whereas w_0 increases to 0.7 when p equals 0.8.

3.3 Conclusions

In this chapter, we studied the stability of the interior equilibrium in the replicator dynamics in presence of several discrete delays. We introduced in the fitness functions delayed profiles of the population.

Considering one discrete and random delay, we derived the critical delay value at which the asymptotic stability of the ESS is lost. When there are two delays, we illustrated through a numerical example how the asymptotic stability of the mixed ESS can be changed by keeping the values of delays unchanged and varying only the probabilities of delays. In addition, we examined the case in which there are several discrete delays and we derived a necessary and sufficient delay-independent stability condition. We also studied the case where there is one delayed strategy and we derived the critical value of delay in function of the probabilities of delays.

In the next chapter, we propose to examine the stability and the properties of the bifurcating limit cycle created at the Hopf bifurcation in the replicator dynamics with random delays.

Chapter 4

Hopf Bifurcations in Replicator Dynamics with Random Delays

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In this chapter, we examine the behavior of the Hopf bifurcation in the two-strategy replicator dynamics subject to heterogeneous and random delays. In evolutionary games, we assume that a strategy would take an uncertain time interval to have a consequence on the fitness. For the sufficiently small values of the mean delay, the replicator dynamics converges to the mixed ESS. As the mean delay increases, a change in the stability of the equilibrium (Hopf bifurcation) may occur at which a stable periodic oscillation, the limit cycle, is created. We use the nonlinear Poincaré-Lindstedt's perturbation method to determine its amplitude. Our theoretical results are in a good agreement with results obtained through numerical simulations.

This chapter is structured as follows:

- In Section 4.1, we examine the stability of the Hopf bifurcation in the replicator dynamics, with Dirac, uniform, and Gamma distributions. We also consider a case of discrete delays;
- In Section 4.2, we make numerical simulations to make comparison with our theoretical results;
- Finally, we conclude the chapter in Section 4.3.

4.1 Hopf Bifurcations in the Replicator Dynamics

We consider a population composed of a large and finite number of players. Each player would experience a delay that can be seen as an i.i.d. random variable drawn from some probability distribution. Let $p(\tau)$ be the probability according to which the delays are drawn with $\tau \in [0, \infty[$ and $\int_0^\infty p(\tau)d\tau = 1$; the support of $p(\tau)$ may be finite. Taking the average over all the players in the population, we can write the utilities of the strategies as follows:

$$\begin{aligned} U_A(t) &= a \int_0^\infty p(\tau)s(t-\tau)d\tau + b(1 - \int_0^\infty p(\tau)s(t-\tau)d\tau), \\ U_B(t) &= c \int_0^\infty p(\tau)s(t-\tau)d\tau + d(1 - \int_0^\infty p(\tau)s(t-\tau)d\tau). \end{aligned}$$

Therefore, the replicator dynamics can be written as:

$$\frac{ds(t)}{dt} = s(t)(1-s(t))(-\delta \int_0^\infty p(\tau)s(t-\tau)d\tau + b - d). \quad (4.1)$$

Let $x(t) = s(t) - s^*$. Substituting s with x in the previous equation, we get:

$$\begin{aligned} \frac{dx(t)}{dt} &= -\delta\gamma \int_0^\infty p(\tau)x(t-\tau)d\tau - \delta(1-2s^*)x(t) \int_0^\infty p(\tau)x(t-\tau)d\tau + \\ &\quad \delta x^2(t) \int_0^\infty p(\tau)x(t-\tau)d\tau, \end{aligned} \quad (4.2)$$

which is of the form,

$$\frac{dx(t)}{dt} = A \int_0^\infty p(\tau)x(t-\tau)d\tau + Bx(t) \int_0^\infty p(\tau)x(t-\tau)d\tau + Cx^2(t) \int_0^\infty p(\tau)x(t-\tau)d\tau, \quad (4.3)$$

where $A = -\delta\gamma$, $B = -\delta(1 - 2s^*)$, $C = \delta$, and $\gamma = s^*(1 - s^*)$. Near the Hopf bifurcation, the solution of the replicator dynamics equation (4.2) can be approximated as:

$$x(t) = A_m \cos(w_0 t).$$

To examine the bifurcating solution, we define a small parameter ε and a new variable u as follows:

$$x(t) = \varepsilon u(t).$$

Furthermore, we stretch time by defining a new variable Ω as follows:

$$T = \Omega t.$$

The equivalent replicator dynamics equation can then be written as:

$$\Omega \frac{du(T)}{dT} = A \int_0^\infty p(\tau)u(T - \Omega\tau)d\tau + \varepsilon B u(T) \int_0^\infty p(\tau)u(T - \Omega\tau)d\tau + \varepsilon^2 C u^2(T) \times \int_0^\infty p(\tau)u(T - \Omega\tau)d\tau. \quad (4.4)$$

In addition, we make a series expansion of Ω as follows:

$$\Omega = w_0 + \varepsilon^2 k_2 + O(\varepsilon^3).$$

Indeed, we omit the $O(\varepsilon)$ term because it turns out to be a zero, (it will be multiplied by a secular term, therefore it should be removed). We also make the following series expansion:

$$u(T) = u_0(T) + \varepsilon u_1(T) + \varepsilon^2 u_2(T) + O(\varepsilon^3).$$

Finally, in equation (4.4), we expand out and collect terms of the same order in ε . By setting the secular terms which yield a resonance effect to zero, we get the amplitude of the limit cycle.

We aim to determine the properties of the Hopf bifurcation in the replicator dynamics subject to random delays. In the remaining of this chapter, we consider the following delay distributions: Dirac, uniform, and Gamma distributions. We also consider a case of stochastic and discrete delays.

4.1.1 Dirac Distribution

We suppose there is a single fixed delay of a value τ . The replicator dynamics in (4.1) reduces to:

$$\frac{ds(t)}{dt} = s(t)(1-s(t))(-\delta s(t-\tau) + b - d). \quad (4.5)$$

Thus, equation (4.3) reduces to:

$$\frac{dx(t)}{dt} = Ax(t-\tau) + Bx(t)x(t-\tau) + Cx(t-\tau)x^2(t). \quad (4.6)$$

A Hopf bifurcation occurs when a pair of conjugate pure imaginary roots of the characteristic equation crosses the imaginary axis, that is when:

$$\tau_{cr} = \frac{\pi}{2\delta\gamma}, \text{ and } w_0 = \delta\gamma. \quad (4.7)$$

These formulae will be used later in solving the DDE. By making a change of variable as mentioned in the previous section, we can write the replicator dynamics equation (4.4) as follows:

$$\Omega \frac{du(T)}{dT} = Au(T - \Omega T) + \varepsilon Bu(T - \Omega\tau) + \varepsilon^2 Cu^2(T)u(T - \Omega\tau). \quad (4.8)$$

In order to examine the behavior of the bifurcating periodic solution, we should take into account of all the terms in the replicator dynamics equation, including nonlinear terms. This is in opposite to the approach taken in the previous chapters, which consists in making a linearization of the replicator dynamics around the equilibrium point. The linearization approach allows us to determine the stability conditions of the equilibrium point but is not sufficient to analyze the behavior of the periodic solution created near the Hopf bifurcation. From the replicator dynamics equation (4.8), we can examine the behavior of the bifurcating periodic solution. The following proposition summarizes the properties of the bifurcating limit cycle.

Proposition 11. *Let $P = -20A^3$ and $Q = 5CA^2\tau_{cr} - 3B^2A\tau_{cr} - B^2$, the amplitude of the bifurcating limit cycle is given by:*

$$A_m = \sqrt{\frac{P}{Q}}\mu,$$

where $\mu = \tau - \tau_{cr}$. Furthermore, the Hopf bifurcation is supercritical.

Proof. Since we take as a bifurcation parameter τ , we make the following series expansion around τ_{cr} :

$$\begin{aligned} \tau &= \tau_{cr} + \varepsilon^2 \hat{\mu} + O(\varepsilon^3) \\ &= \tau_{cr} + \mu + O(\varepsilon^3). \end{aligned}$$

In addition, we have:

$$\Omega = w_0 + \varepsilon^2 k_2 + O(\varepsilon^3), \quad (4.9)$$

We expand $u(T - \Omega\tau)$ as follows :

$$\begin{aligned} u(T - \Omega\tau) &= u(T - (w_0 + \varepsilon^2 k_2 + \dots)(\tau_{cr} + \varepsilon^2 \hat{\mu})) \\ &= u(T - w_0 \tau_{cr} - \varepsilon^2 (k_2 \tau_{cr} + w_0 \hat{\mu}) + \dots) \\ &= u(T - w_0 \tau_{cr}) - \varepsilon^2 (k_2 \tau_{cr} + w_0 \hat{\mu}) u'(T - w_0 \tau_{cr}) + O(\varepsilon^3). \end{aligned} \quad (4.10)$$

Substituting the series expansions above and collecting terms of similar order in ε in equation (4.8), we get the following system of equations which we resolve recursively:

$$\bullet \quad w_0 \frac{du_0(T)}{dT} - Au_0(T - w_0 \tau_{cr}) = 0 \quad (4.11)$$

$$\bullet \quad w_0 \frac{du_1(T)}{dT} - Au_1(T - w_0 \tau_{cr}) = Bu_0(T)u_0(T - w_0 \tau_{cr}) \quad (4.12)$$

$$\begin{aligned} \bullet \quad w_0 \frac{du_2(T)}{dT} - Au_2(T - w_0 \tau_{cr}) &= -k_2 \frac{du_0(T)}{dT} - A(k_2 \tau_{cr} + w_0 \hat{\mu}) u_0'(T - w_0 \tau_{cr}) + \\ &Bu_1(T)u_0(T - w_0 \tau_{cr}) + Bu_0(T)u_1(T - w_0 \tau_{cr}) \\ &+ Cu_0(T - w_0 \tau_{cr})u_0^2(T). \end{aligned} \quad (4.13)$$

A solution of (4.11) is of the form:

$$u_0(T) = \hat{A}_m \cos(T). \quad (4.14)$$

We substitute (4.14) into (4.12) to get the following equation in u_1 :

$$w_0 \frac{du_1(T)}{dT} - Au_1(T - w_0 \tau_{cr}) = \frac{B\hat{A}_m^2}{2} \sin(2T).$$

Let $u_1(T) = m_1 \sin(2T) + m_2 \cos(2T)$ be a solution of the previous DDE. Solving the previous equation for u_1 yields:

$$m_1 = \hat{A}_m^2 \frac{B}{10A}, \quad \text{and} \quad m_2 = 2m_1.$$

We used the relation $w_0 = \delta\gamma = -A$. Finally, after using the relation $\cos(3x) = 4\cos^3(x) - 3\cos(x)$ and $\sin(3x) = 3\sin(x) - 4\sin^3(x)$, and setting the secular terms ($\cos(T)$ and $\sin(T)$ terms that yield a resonance effect) to zero in equation (4.13), we get the amplitude of the bifurcating periodic solution:

$$\hat{A}_m^2 = \frac{-20A^3 \hat{\mu}}{5CA^2 \tau_{cr} - 3B^2 A \tau_{cr} - B^2}.$$

Multiplying both sides by ε^2 in the equation above, we get:

$$A_m^2 = \frac{-20A^3 \mu}{5CA^2 \tau_{cr} - 3B^2 A \tau_{cr} - B^2}.$$

Since the Hopf bifurcation occurs when τ is near τ_{cr} and $\tau > \tau_{cr}$, then $\mu > 0$ and the bifurcation is supercritical. \square

Our result is in coherence with the results in [55]. The result above means that the amplitude of the bifurcating limit cycle is proportional to $\sqrt{\tau - \tau_{cr}}$.

4.1.2 Uniform Distribution

When the delays are i.i.d. random variables drawn from the uniform distribution, that is when:

$$p(\tau) = \frac{1}{\tau_{max}} \text{ for } \tau \in [0, \tau_{max}] \text{ and zero otherwise,} \quad (4.15)$$

the replicator dynamics can be written as:

$$\frac{ds(t)}{dt} = s(t)(1-s(t)) \left(-\delta \int_0^{\tau_{max}} \frac{1}{\tau_{max}} s(t-\tau) d\tau + b - d \right). \quad (4.16)$$

Let $x(t) = s(t) - s^*$. Substituting s with x in the previous equation, we get:

$$\begin{aligned} \frac{dx(t)}{dt} = & A \int_0^{\tau_{max}} \frac{1}{\tau_{max}} x(t-\tau) d\tau + Bx(t) \int_0^{\tau_{max}} \frac{1}{\tau_{max}} x(t-\tau) d\tau + \\ & Cx^2(t) \int_0^{\tau_{max}} \frac{1}{\tau_{max}} x(t-\tau) d\tau, \end{aligned} \quad (4.17)$$

where A , B , and C are defined in the previous subsection. In Chapter 2, we proved that, at the Hopf bifurcation, we have:

$$\tau_{cr} = \frac{\pi^2}{2D}, \text{ and } w_0 = \frac{\pi}{\tau_{cr}},$$

with $D = \gamma\delta$. As in the previous section, we define $x = \varepsilon u$ and $T = \Omega t$. We obtain the following equation,

$$\begin{aligned} \Omega \frac{du(T)}{dT} = & A \int_0^{\tau_{max}} \frac{1}{\tau_{max}} u(T - \Omega\tau) d\tau + \varepsilon B u(T) \int_0^{\tau_{max}} \frac{1}{\tau_{max}} u(T - \Omega\tau) d\tau + \\ & \varepsilon^2 C u^2(T) \int_0^{\tau_{max}} \frac{1}{\tau_{max}} u(T - \Omega\tau) d\tau. \end{aligned} \quad (4.18)$$

By expanding out the equation above, collecting terms of the same order in ε , and removing the secular terms, we obtain the amplitude of the limit cycle. The properties of the bifurcating limit cycle are brought out in the next proposition.

Proposition 12. *Let $P = 8A^2$ and $Q = \tau_{cr}(B^2 - 2CA)$. The amplitude of the bifurcating limit cycle is given by:*

$$A_m = \sqrt{\frac{P}{Q}} \mu,$$

where $\mu = \tau_{max} - \tau_{cr}$. Furthermore, the Hopf bifurcation is supercritical.

Proof. Since we take as a bifurcation parameter τ_{max} , we make the following series expansion:

$$\begin{aligned}\tau_{max} &= \tau_{cr} + \varepsilon^2 \hat{\mu} + O(\varepsilon^3) \\ &= \tau_{cr} + \mu + O(\varepsilon^3).\end{aligned}$$

Note that the $O(\varepsilon)$ term is omitted because it will be multiplied by a secular term, and then removed. We recall that

$$\begin{aligned}u(T - \Omega\tau) &= u(T - (w_0 + k_2\varepsilon^2 + \dots)\tau) \\ &= u(T - w_0\tau) - k_2\varepsilon^2\tau u'(T - w_0\tau) + O(\varepsilon^3).\end{aligned}$$

In addition, we have,

$$\begin{aligned}\int_0^{\tau_{max}} u(T - \Omega\tau) d\tau &= \int_0^{\tau_{cr}} u(T - \Omega\tau) d\tau + \int_{\tau_{cr}}^{\tau_{cr} + \varepsilon^2\mu} u(T - \Omega\tau) d\tau \\ &= \int_0^{\tau_{cr}} u_0(T - w_0\tau) d\tau + \varepsilon \int_0^{\tau_{cr}} u_1(T - w_0\tau) d\tau + \\ &\quad \varepsilon^2 \left(\int_0^{\tau_{cr}} (u_2(T - w_0\tau) - k_2\tau u_0'(T - w_0\tau)) d\tau + \mu u_0(T - w_0\tau) \right) \\ &\quad + O(\varepsilon^3).\end{aligned}$$

Taking account of the series expansions above, we collect terms of the same order in ε in (4.18), we get the following system of equations, which we resolve recursively:

$$\bullet \quad w_0\tau_{cr} \frac{du_0(T)}{dT} - A \int_0^{\tau_{cr}} u_0(T - w_0\tau) d\tau = 0 \quad (4.19)$$

$$\bullet \quad w_0\tau_{cr} \frac{du_1(T)}{dT} - A \int_0^{\tau_{cr}} u_1(T - w_0\tau) d\tau = Bu_0(T) \int_0^{\tau_{cr}} u_0(T - w_0\tau) d\tau \quad (4.20)$$

$$\begin{aligned}\bullet \quad w_0\tau_{cr} \frac{du_2(T)}{dT} + (k_2\tau_{cr} + w_0\hat{\mu}) \frac{du_0(T)}{dT} &= A \int_0^{\tau_{cr}} u_2(T - w_0\tau) d\tau + \left(A\hat{\mu} + \frac{Ak_2\tau_{cr}}{w_0} \right) \\ u_0(T - w_0\tau_{cr}) + \left(-\frac{Ak_2}{w_0} + Bu_1(T) + Cu_0^2(T) \right) &\int_0^{\tau_{cr}} u_0(T - w_0\tau) d\tau.\end{aligned} \quad (4.21)$$

A solution of (4.19) is of the form,

$$u_0(T) = \hat{A}_m \cos(T). \quad (4.22)$$

Then, we have:

$$\begin{aligned}\int_0^{\tau_{cr}} u_0(T - w_0\tau) d\tau &= \hat{A}_m \int_0^{\tau_{cr}} \cos(T - w_0\tau) d\tau \\ &= 2 \frac{\hat{A}_m}{w_0} \sin(T).\end{aligned}$$

Let $u_1(T) = m_1 \sin(2T) + m_2 \cos(2T)$ be a solution of (4.20). Then, we have,

$$\begin{aligned} \int_0^{\tau_{cr}} u_1(T - w_0 \tau) d\tau &= m_1 \int_0^{\tau_{cr}} \sin 2(T - w_0 \tau) d\tau + m_2 \int_0^{\tau_{cr}} \cos 2(T - w_0 \tau) d\tau \\ &= 0. \end{aligned}$$

The result above can be obtained by remembering that $\cos(w_0 \tau_{cr}) = -1$ and $\sin(w_0 \tau_{cr}) = 0$. Therefore, equation (4.20) can be reduced to:

$$w_0 \tau_{cr} \frac{du_1(T)}{dT} = \frac{\hat{A}_m^2 B}{w_0} \sin(2T).$$

Considering that $u_1(T) = m_1 \sin(2T) + m_2 \cos(2T)$, we get:

$$m_1 = 0, \quad \text{and} \quad m_2 = -\frac{B \hat{A}_m^2}{2w_0^2 \tau_{cr}}.$$

On the other hand, equation (4.21) can be written as:

$$\begin{aligned} w_0 \tau_{cr} \frac{du_2(T)}{dT} - A \int_0^{\tau_{cr}} u_2(T - w_0 \tau) d\tau &= -(k_2 \tau_{cr} + w_0 \hat{\mu}) \frac{du_0(T)}{dT} + \left(A \hat{\mu} + \frac{A k_2 \tau_{cr}}{w_0} \right) \times \\ u_0(T - w_0 \tau_{cr}) + \left(-\frac{A k_2}{w_0} + B u_1(T) + C u_0^2(T) \right) &\int_0^{\tau_{cr}} u_0(T - w_0 \tau) d\tau, \end{aligned} \quad (4.23)$$

which yields,

$$w_0 \tau_{cr} \frac{du_2(T)}{dT} - A \int_0^{\tau_{cr}} u_2(T - w_0 \tau) d\tau = \hat{A}_m F_1 \sin(T) + \hat{A}_m F_2 \cos(T) + F_3 \sin(3T) + F_4 \cos(3T),$$

where,

$$F_1 = (k_2 \tau_{cr} + \hat{\mu} w_0) - 2 \frac{A k_2}{w_0^2} - \frac{B m_2}{w_0} + \frac{C \hat{A}_m^2}{2w_0}, \quad \text{and} \quad F_2 = -A \hat{\mu} - A \frac{k_2 \tau_{cr}}{w_0}.$$

By removing secular terms that yield a resonant effect, that is by setting $F_1 = 0$ and $F_2 = 0$, we get:

$$\hat{A}_m^2 = \frac{8A^2 \hat{\mu}}{\tau_{cr} (B^2 - 2CA)}.$$

Multiplying both sides by ε^2 in the equation above, we get:

$$A_m^2 = \frac{8A^2 \mu}{\tau_{cr} (B^2 - 2CA)}.$$

Since $\mu > 0$, then the bifurcating limit cycle is stable and the Hopf bifurcation is supercritical. \square

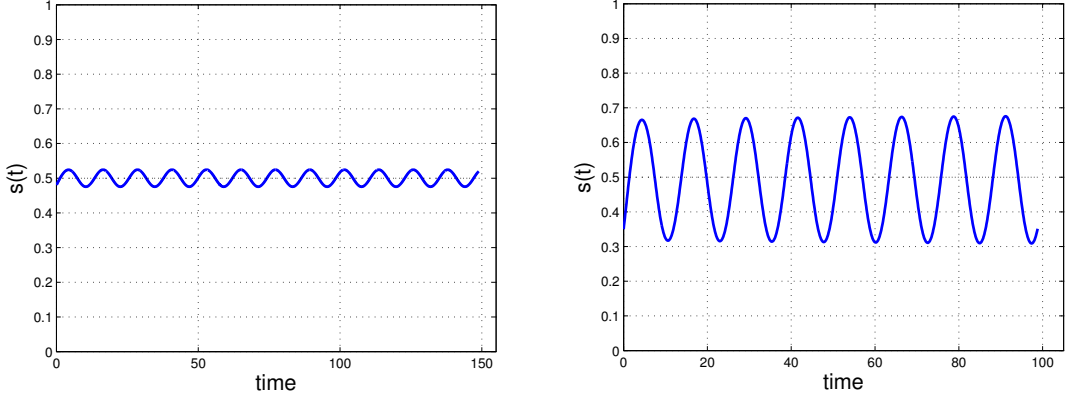


Figure 4.1: Stable limit cycle with the uniform distribution with $\tau_{cr} = 6.58$ time units. Left $\mu = 0.001$, $\tau_{max} = \tau_{cr} + \mu$. Right $\mu = 0.03$, $\tau_{max} = \tau_{cr} + \mu$, where $a = -0.5$, $b = 3$, $c = 0$, and $d = 1.5$.

The amplitude of the bifurcating periodic solution is proportional to $\sqrt{\tau_{max} - \tau_{cr}}$. When the value of τ_{max} is near and superior to τ_{cr} , the replicator dynamics exhibits a stable periodic oscillation in the proportions of the strategies in the population. We illustrate in Fig. 4.1, the stable limit cycle occurring near the Hopf bifurcation under the uniform distribution. In the left-subfigure, we fixed μ to 0.001 time units whereas in the right-subfigure, μ is fixed to 0.03 time units. We recall that $\mu = \tau_{max} - \tau_{cr}$. In the first case, we observe that the stable limit cycle has a very small amplitude, and by increasing τ_{max} , we see in the second case a limit cycle with an amplitude of approximately 0.18. The amplitude, indeed, increases significantly as τ_{max} moves away from τ_{cr} .

4.1.3 Gamma Distribution

We consider a Gamma distribution of delays with support $[0, \infty[$ and parameters $k \geq 1$ and $\beta > 0$. The probability distribution in this case is given by:

$$p(\tau; k, \beta) = \frac{\beta^k \tau^{k-1} e^{-\beta \tau}}{\Gamma(k)},$$

where $\Gamma(k) = (k-1)!$ (Gamma function). The mean of the Gamma distribution is $\frac{k}{\beta}$.

In Chapter 2, we proved that a Hopf bifurcation occurs when,

$$\beta_c = D \frac{\cos^{k+1}\left(\frac{\pi}{2k}\right)}{\sin\left(\frac{\pi}{2k}\right)}, \quad (4.24)$$

and the frequency of oscillations near the bifurcation point is given by:

$$w_0 = D \cos^k\left(\frac{\pi}{2k}\right). \quad (4.25)$$

As a remark, we observe that when $k = 1$, the Gamma distribution coincides with the exponential distribution, and $\beta_c = 0$. Therefore, there cannot exist a Hopf bifurcation in this case. In the following, we suppose that $k \geq 2$. Now, let us determine the properties of the limit cycle in the neighborhood of the bifurcation. We define I as follows:

$$I = \int_0^\infty \frac{\beta^k}{\Gamma(k)} \tau^{k-1} e^{-\beta\tau} u(T - \Omega\tau) d\tau. \quad (4.26)$$

The equation (4.4) can then be written as,

$$\Omega \frac{du(T)}{dT} = AI + \varepsilon Bu(T)I + \varepsilon^2 Cu^2(T)I, \quad (4.27)$$

where $\Omega = w_0 + k_2\varepsilon^2 + O(\varepsilon^3)$. The properties of the bifurcating limit cycles are given in the following proposition.

Proposition 13. *Let P and Q be as follows:*

$$P = (k+1) \frac{A}{\beta_c} \left(1 + \frac{w_0^2}{\beta_c^2}\right)^{-\frac{k}{2}} - \frac{k}{k+1} \left(1 + \frac{w_0^2}{\beta_c^2}\right)^{\frac{1}{2}} - \frac{w_0}{\beta_c},$$

and

$$Q = \frac{B\beta_c}{2(k+1)A} \left(1 + \frac{w_0^2}{\beta_c^2}\right)^{\frac{1}{2}} \left(F_1 \frac{w_0}{\beta_c} + F_2\right) - B \frac{w_0}{\beta_c} \left(1 + \frac{w_0^2}{\beta_c^2}\right)^{-\frac{k+1}{2}} \left(F_2 + \frac{F_1}{2} \left(\frac{w_0}{\beta_c} - 1\right)\right) + \frac{C}{4} \left(1 + \frac{w_0^2}{\beta_c^2}\right)^{-\frac{k}{2}}.$$

The amplitude of the bifurcating limit cycle is given by:

$$A_m = \sqrt{\frac{P}{Q}\mu}, \quad (4.28)$$

where $\mu = \beta - \beta_c$, F_1 , and F_2 are defined in (4.32) and (4.33), respectively. Furthermore, the Hopf bifurcation is supercritical.

Proof. Since we take as a bifurcation parameter β , we can make a series expansion of β as follows:

$$\beta = \beta_c + \varepsilon^2 \hat{\mu} + O(\varepsilon^3).$$

By remembering that $(\beta_c + \varepsilon^2 \hat{\mu})^k = \beta_c^k + k\beta_c^{k-1} \varepsilon^2 \hat{\mu} + O(\varepsilon^3)$, and by making a series

expansion in (4.26), we get,

$$\begin{aligned}
 I &= \int_0^\infty \frac{(\beta_c + \varepsilon^2 \hat{\mu})^k}{\Gamma(k)} \tau^{k-1} e^{-(\beta_c + \varepsilon^2 \hat{\mu})\tau} u(T - \Omega\tau) d\tau \\
 &= \int_0^\infty \frac{(\beta_c + \varepsilon^2 \hat{\mu})^k}{\Gamma(k)} \tau^{k-1} e^{-\beta_c \tau} (1 - \varepsilon^2 \hat{\mu} \tau) u(T - \Omega\tau) d\tau \\
 &= \frac{\beta_c^k}{\Gamma(k)} \int_0^\infty \tau^{k-1} e^{-\beta_c \tau} u_0(T - w_0\tau) d\tau + \varepsilon \frac{\beta_c^k}{\Gamma(k)} \int_0^\infty \tau^{k-1} e^{-\beta_c \tau} u_1(T - w_0\tau) d\tau + \frac{\varepsilon^2}{\Gamma(k)} \times \\
 &\quad \int_0^\infty \beta_c^k \tau^{k-1} e^{-\beta_c \tau} \left(u_2(T - w_0\tau) - k_2 \tau u_0'(T - w_0\tau) - \hat{\mu} \tau u_0(T - w_0\tau) \right) d\tau + \\
 &\quad \frac{\varepsilon^2 k \beta_c^{k-1} \hat{\mu}}{\Gamma(k)} \int_0^\infty \tau^{k-1} e^{-\beta_c \tau} u_0(T - w_0\tau) d\tau + O(\varepsilon^3).
 \end{aligned}$$

Then, taking account of the previous expansions, and collecting terms of similar order in ε , we get the following equations which we resolve recursively:

$$\bullet \quad w_0 \frac{du_0(T)}{dT} - A \frac{\beta_c^k}{\Gamma(k)} \int_0^\infty \tau^{k-1} e^{-\beta_c \tau} u_0(T - w_0\tau) d\tau = 0 \quad (4.29)$$

$$\bullet \quad w_0 \frac{du_1(T)}{dT} - A \frac{\beta_c^k}{\Gamma(k)} \int_0^\infty \tau^{k-1} e^{-\beta_c \tau} u_1(T - w_0\tau) d\tau = B u_0(T) \frac{\beta_c^k}{\Gamma(k)} \int_0^\infty \tau^{k-1} e^{-\beta_c \tau} \times \\
 u_0(T - w_0\tau) d\tau, \quad (4.30)$$

$$\bullet \quad w_0 \frac{du_2(T)}{dT} - A \frac{\beta_c^k}{\Gamma(k)} \int_0^\infty \tau^{k-1} e^{-\beta_c \tau} u_2(T - w_0\tau) d\tau = -k_2 \frac{du_0(T)}{dT} - A k_2 \frac{\beta_c^k}{\Gamma(k)} \\
 \int_0^\infty \tau^k e^{-\beta_c \tau} u_0'(T - w_0\tau) d\tau - \frac{A \beta_c^k \mu}{\Gamma(k)} \int_0^\infty \tau^k e^{-\beta_c \tau} u_0(T - w_0\tau) d\tau + \frac{B \beta_c^k}{\Gamma(k)} u_0(T) \times \\
 \int_0^\infty \tau^{k-1} e^{-\beta_c \tau} u_1(T - w_0\tau) d\tau + \frac{B \beta_c^k}{\Gamma(k)} u_1(T) \int_0^\infty \tau^{k-1} e^{-\beta_c \tau} u_0(T - w_0\tau) d\tau + \\
 \frac{C \beta_c^k}{\Gamma(k)} u_0^2(t) \int_0^\infty \tau^{k-1} e^{-\beta_c \tau} u_0(T - w_0\tau) d\tau + \frac{kA}{\Gamma(k)} \beta_c^{k-1} \hat{\mu} \int_0^\infty \tau^{k-1} e^{-\beta_c \tau} u_0(T - w_0\tau) d\tau. \quad (4.31)$$

Let $u_1 = m_1 \sin(2T) + m_2 \cos(2T)$ be a solution of (4.30). Solving (4.30) in u_1 , we get:

$$m_1 = F_1 \hat{A}_m^2, \quad \text{and} \quad m_2 = F_2 \hat{A}_m^2,$$

where,

$$F_1 = - \frac{\frac{AB}{2} \left(1 + \frac{w_0^2}{\beta_c^2}\right)^{-\frac{k}{2}} \left(1 + 4 \frac{w_0^2}{\beta_c^2}\right)^{-\frac{k}{2}} \cos(k\theta_1)}{4w_0^2 + A^2 \left(1 + 4 \frac{w_0^2}{\beta_c^2}\right)^{-k} + 4w_0 A \left(1 + 4 \frac{w_0^2}{\beta_c^2}\right)^{-\frac{k}{2}} \sin(k\theta_1)}, \quad (4.32)$$

and

$$F_2 = -\frac{\frac{B}{2}(1 + \frac{w_0^2}{\beta_c^2})^{-\frac{k}{2}}(2w_0 + A(1 + 4\frac{w_0^2}{\beta_c^2})^{-\frac{k}{2}}\sin(k\theta_1))}{4w_0^2 + A^2(1 + 4\frac{w_0^2}{\beta_c^2})^{-k} + 4w_0A(1 + 4\frac{w_0^2}{\beta_c^2})^{-\frac{k}{2}}\sin(k\theta_1)}, \quad (4.33)$$

$\theta_1 = \text{atan}(\frac{2w_0}{\beta_c})$, and 'atan' denotes the 0 to $\frac{\pi}{2}$ branch of the inverse tangente function. On the other hand, equation (4.31) can be written as:

$$w_0 \frac{du_2(T)}{dT} - A \frac{\beta_c^k}{\Gamma(k)} \int_0^\infty \tau^{k-1} e^{-\beta_c \tau} u_2(T - w_0 \tau) d\tau = G \sin(T) + K \cos(T) + L \sin(3T) + M \cos(3T),$$

where,

$$G = k_2 \hat{A}_m - A \hat{A}_m \beta_c^k (k+1) (\beta_c^2 + w_0^2)^{-\frac{k+2}{2}} (k_2 w_0 + \beta_c \hat{\mu}) + \frac{C \beta_c^k \hat{A}_m^3}{4} (\beta_c^2 + w_0^2)^{-\frac{k}{2}} + \hat{A}_m \beta_c^{k-1} (\beta_c^2 + w_0^2)^{-\frac{k+1}{2}} (-\frac{m_2}{2} B \beta_c w_0 + \frac{m_1}{2} B \beta_c^2 + k A \hat{\mu} w_0),$$

and

$$K = \beta_c^k (\beta_c^2 + w_0^2)^{-\frac{k+1}{2}} \hat{A}_m (-k_2 \beta_c A (k+1) (\beta_c^2 + w_0^2)^{-\frac{1}{2}} + A w_0 \hat{\mu} (k+1) (\beta_c^2 + w_0^2)^{-\frac{1}{2}} + \frac{m_2}{2} B \beta_c + \frac{m_1}{2} w_0 B + k A \hat{\mu}).$$

Finally, by setting $K = 0$ and $G = 0$, we obtain the amplitude given in equation (4.28). \square

As in the previous sections, the amplitude of the bifurcating limit cycle is proportional to $\sqrt{\beta_c - \beta}$. Note that the bifurcation occurs when β is near β_c and $\beta < \beta_c$, therefore the quotient P/Q should be negative. When β is near and below β_c , then the replicator dynamics exhibits a stable periodic oscillation in the proportion of strategies in the population.

4.1.4 Discrete Delays

We suppose in this section that a strategy, either A or B , would take a delay τ with probability p or no delay with probability $1 - p$. In this case, the replicator dynamics is given by:

$$\frac{ds(t)}{dt} = s(t)(1 - s(t))(-p\delta s(t - \tau) - (1 - p)\delta s(t) + b - d).$$

Let $x(t) = s(t) - s^*$. Substituting s with x in the equation (4.34), we get:

$$\begin{aligned} \frac{dx(t)}{dt} = & -(1-p)\delta\gamma x(t) - p\delta\gamma x(t-\tau) - p\delta(1-2s^*)x(t)x(t-\tau) - (1-p)\delta \times \\ & (1-2s^*)x^2(t) + p\delta x(t-\tau)x^2(t) + (1-p)\delta x^3(t), \end{aligned}$$

which is of the form:

$$\frac{dx(t)}{dt} = a_1x(t) + b_1x(t-\tau) + c_1x(t)x(t-\tau) + d_1x^2(t) + e_1x(t-\tau)x^2(t) + f_1x^3(t),$$

where $a_1 = -(1-p)\delta\gamma$, $b_1 = -p\delta\gamma$, $c_1 = -p\delta(1-2s^*)$, $d_1 = -(1-p)\delta(1-2s^*)$, $e_1 = p\delta$, $f_1 = (1-p)\delta$. In the previous chapter, we proved that a Hopf bifurcation occurs when $p > 0.5$ and $\tau_{cr} = \frac{\arccos(-\frac{1-p}{p})}{\delta\gamma\sqrt{2p-1}}$. The properties of the bifurcating limit cycle are brought out in the next proposition.

Proposition 14. *Let P and Q be defined as follows:*

$$P = 4b_1^3(b_1 - a_1)(a_1 + b_1)^2(-5b_1 + 4a_1),$$

and

$$\begin{aligned} Q = & 5e_1b_1^6\tau_{cr} + a_1e_1b_1^5\tau_{cr} - 15a_1f_1b_1^5\tau_{cr} - 3c_1^2b_1^2\tau_{cr} - 7c_1d_1b_1^5\tau_{cr} - 4d_1^2b_1^5\tau_{cr} + \\ & 6a_1^2e_1b_1^4\tau_{cr} - 3a_1^2f_1b_1^4\tau_{cr} + 7c_1^2a_1b_1^4\tau_{cr} + 19c_1d_1a_1b_1^4\tau_{cr} + 18d_1^2a_1b_1^4\tau_{cr} + \\ & 2a_1^3e_1b_1^3\tau_{cr} + 12a_1^3f_1b_1^3\tau_{cr} - 12c_1^2a_1^2b_1^3\tau_{cr} - 26c_1d_1a_1^2b_1^3\tau_{cr} - 8d_1^2a_1^2b_1^3\tau_{cr} - \\ & 8a_1^4e_1b_1^2\tau_{cr} + 8c_1^2a_1^3b_1^2\tau_{cr} + 8c_1d_1a_1^3b_1^2\tau_{cr} + 15f_1b_1^5 - 15a_1e_1b_1^4 + 3a_1f_1b_1^4 - \\ & c_1^2b_1^4 - 9c_1d_1b_1^4 - 18d_1^2b_1^4 - 3a_1^2e_1b_1^3 - 12a_1^2f_1b_1^3 + 11c_1^2a_1b_1^3 + 33c_1d_1a_1b_1^3 + \\ & 12d_1^2a_1b_1^3 + 12a_1^3e_1b_1^2 - 14c_1^2a_1^2b_1^2 - 18c_1d_1a_1^2b_1^2 + 4c_1^2a_1^3b_1. \end{aligned}$$

The amplitude of the bifurcating limit cycle is given by:

$$A_m = \sqrt{\frac{P}{Q}\mu},$$

where $\mu = \tau - \tau_{cr}$. Furthermore, the Hopf bifurcation is supercritical.

Proof. The proof follows by carrying out the same procedure as in the previous sections. \square

This proposition gives a closed-form expression of the amplitude of bifurcating periodic solution. Indeed, when τ is inferior to τ_{cr} , the mixed ESS is asymptotically stable. But when τ is near and superior to τ_{cr} , a stable periodic oscillation appears whose amplitude is proportional to $\sqrt{\tau - \tau_{cr}}$.

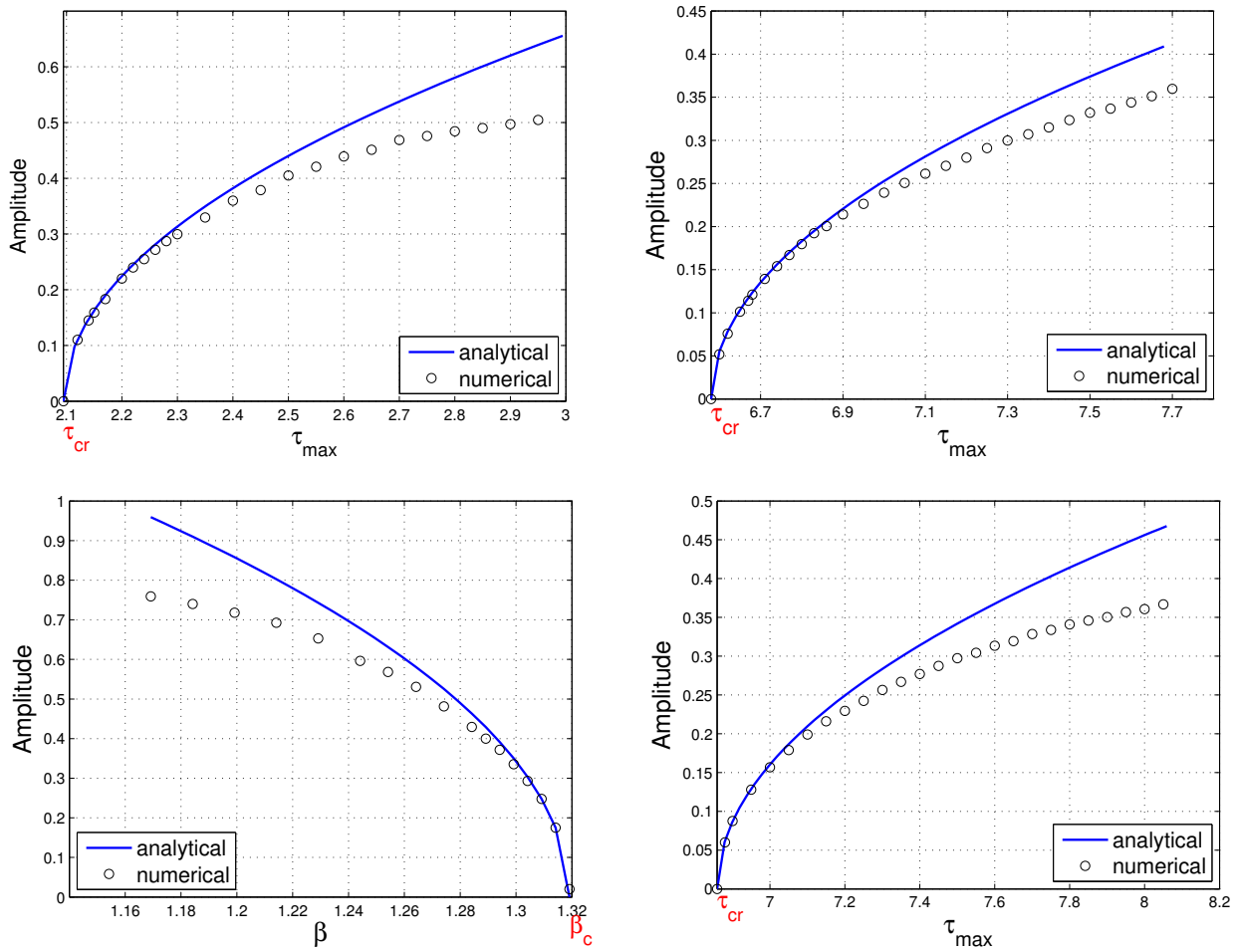


Figure 4.2: The amplitude of the bifurcating periodic solution near the Hopf bifurcation, where $a = -1.5$, $b = 3$, $c = 0$, and $d = 1.5$. Top-left, Dirac distribution. Top-right, Uniform distribution. Bottom-left, Gamma distribution with $k = 3$. Bottom-right, Discrete distribution with $p = 0.6$.

4.2 Numerical Simulations

In this section, we propose to compare the properties of the bifurcating periodic solution obtained with the perturbation method, with numerical results. We display in Fig. 4.2 the amplitude of the bifurcating limit cycle given in propositions 11-14 and the amplitude obtained numerically, for different delay distributions.

In all the cases, we observe that the amplitude predicted analytically and that obtained numerically coincide for the values of τ (or β) close to τ_{cr} (or β_c). This can be explained by the fact that near the critical delay, the approximation is good and as the bifurcation parameter moves away from the critical value, the approximation becomes weaker and consequently the predicted amplitude does not coincide with the true one. For example, for the uniform distribution case, the Hopf bifurcation occurs when $\tau_{max} = 6.57$ time units, and the amplitude gradually increases as τ_{max} gets further from this critical value. The predicted amplitude and the amplitude obtained with numerical simulations coincide for the values of τ_{max} reasonably close to the critical value. In the case of discrete delays, the critical delay is given by $\tau_{cr} = 6.86$ time units, and for the values of delays close to τ_{cr} the amplitude predicted analytically and the one obtained numerically coincide but the difference between them increases gradually until reaching 0.1 when τ_{max} equals 8.05 time units. For the Gamma distribution with $k = 3$, $\beta_c = 1.32$ and the critical mean delay is given by 2.28 time units. The bifurcation occurs for the values of β near and below β_c (recall that the mean is k/β), which explains the shape of the amplitude growth. Furthermore, the highest growth rate of the periodic solution's amplitude is observed with the Gamma distribution, and the lowest rate is observed with the uniform distribution.

4.3 Conclusion

In this paper, we considered the two-strategy replicator dynamics subject to uncertain delays. Taking as a bifurcation parameter the mean delay, we proved that the asymptotic stability of the mixed equilibrium may be lost at the Hopf bifurcation, in which case the replicator dynamics exhibits a stable periodic oscillation (limit cycle) in the proportion of strategies in the population. As the mean delay moves away from the critical value, the amplitude of the limit cycle grows gradually. Using a nonlinear Lindstedt's perturbation method and considering different probabilities distributions of delays, we approximated the bifurcating limit cycle and we determined analytically the growth rate of the radius of the limit cycle. Furthermore, we compared with numerical simulations. As a possible extension to this work, we plan to investigate the center manifold approach which is more rigorous and more complicated than Lindstedt's method.

Chapter 5

Evolutionary Stability in Interacting Communities

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In this chapter, we consider an heterogeneous population composed of several communities, with each community having its own set of possible strategies and payoff functions. Furthermore, we consider that the interactions between them are nonuniform. In contrast to the classic framework, the pairwise interactions between individuals in this work are stochastic and may occur with different probabilities. Moreover, the outcome of an interaction depends not only on the strategies used but also on the types of interacting individuals. The stochasticity or nonuniform feature of the interactions may be due to the inherent differences between the communities that make one player prefer to interact with an opponent from his own community, or due to the relative abundancy of each community in the whole population. In this context, we define new evolutionarily stable strategies with different levels of robustness against mutations, and we give analytical expressions of the ESSs in the two-community case in function of the interaction probabilities and payoffs.

The present chapter is structured as follows:

- First, we give a formal description of our new model. We define strong, weak, and intermediate ESS, and we analyze the relationship between them;
- Second, we study a two-community two-strategy case. We derive the conditions of existence of different types of ESS (fully pure, partially mixed, fully mixed) with different levels of stability (strong, weak, intermediate).

We start by a several community case, and then we study a two-community case where we succeed to have explicit formulae of the different ESSs.

5.1 Motivation and Related Work

Motivated by the observation that a population cannot be fully homogeneous, we consider here a framework of multiple communities. In fact, individuals are inherently different and have the tendency to form groups or clusters sharing some common features such as language, culture, or religion. Furthermore, the interactions between individuals are not uniform. For example, an individual would prefer to interact with a similar one. Therefore, the classic assumption of a well-mixed population is here relaxed to enable the understanding of the emergence of equilibria and the dynamics in heterogeneous populations.

In [104], the authors introduced nonuniform interaction rates in the framework of two-strategy games. In this work, the probability for any pair of players to meet and interact depends on their strategies. The authors defined three rates or probabilities r_1 , r_2 , and r_3 that denote the probability of an encounter between two A strategies, A and B strategies (mixed interaction), and two B strategies, respectively. In this model, the fitness functions are nonlinear and new conditions of evolutionarily stability are

consequently obtained. Moreover, this model exhibits new evolutionary outcomes that do not exist in the classic case of uniform interactions. For example, for the Prisoner's Dilemma game, the authors demonstrated the appearance of a stable interior equilibrium at which cooperators and defectors coexist under some conditions on the rates. This scenario does not exist in the classic setting of evolutionary games. For the snowdrift game, the proportion of cooperators at the equilibrium, in this new context, is increased relative to the classic case, when the players are more likely to interact with opponents of their kind (i.e. for the higher values of r_1 and r_3).

We notice that, in this work, the population is homogeneous and the probability of interactions depends only on the strategies. In contrast, we consider here an heterogeneous population where the outcome of an interaction depends also on the types of players.

The context of community or group was also considered in [105]. In this work, the author considered a population composed of two communities and analyzed the evolutionarily stable strategies and the replicator dynamics in this context. However, it is assumed that the interactions are uniform, that is an individual is equally likely to interact with an opponent from its community or from the other community. To the best of our knowledge, the nonuniform interaction feature between communities has not appeared in evolutionary game literature before our work.

In the next section, we present our model and we study the evolutionarily stable strategies with different levels of stability against mutations.

5.2 Evolutionarily Stable Strategies

We consider a large population of players or individuals divided into N communities and each community has its own set of strategies, payoff matrices, and interacting probabilities [106, 107]. Random matching occurs through pairwise interactions and may engage individuals from the same community or from different communities. Let $p = (p_1, \dots, p_N)$ where $p_i = (p_{i1}, \dots, p_{iN})$ be the vector describing the interaction probabilities of community i with other communities. Here p_{ij} denotes the probability that an individual in community i involved in an interaction, interacts with an individual in community j and $\sum_j p_{ij} = 1$ (Figure 5.1). We assume there are n_i pure strategies for

each community i and a strategy of an individual is a probability distribution over the pure strategies. We denote by $A^{ij} = (a_{kl}^{ij})_{k=1..n_i, l=1..n_j}$ the payoff matrix. If a player of community i using pure strategy k interacts with a player of community j using pure strategy l , its payoff is a_{kl}^{ij} . Let $\mathbf{s} = (s_1, \dots, s_N)$ be the population profile where s_i is the column vector describing the distribution of pure strategies in community i (s_{ik} is the frequency of the pure strategy k in community i). We denote by $U_i(k, \mathbf{s}, p)$ the expected payoff of pure strategy k in community i , which depends on the frequency of strategies

in community i and in the other communities. The payoff function U_i is given by:

$$U_i(k, \mathbf{s}, p) = \sum_{j=1}^N p_{ij} e_k A^{ij} s_j, \quad (5.1)$$

where e_k is a row vector corresponding to the k -th element of the canonical basis of \mathbb{R}^n . The expected payoff of a player from community i using a mixed strategy z , when the profile of the population is \mathbf{s} , is given by:

$$\bar{U}_i(z, \mathbf{s}, p) = \sum_{k=1}^{n_i} z_k U_i(k, \mathbf{s}, p). \quad (5.2)$$

Our model covers the many situations as:

- The probabilities of interactions may depend on the size of the communities. An individual is more likely to meet and interact with an opponent from the larger community.
- The probabilities of interactions depend on spatial aspects, in which case an individual is more likely to interact with individuals in his neighborhood.

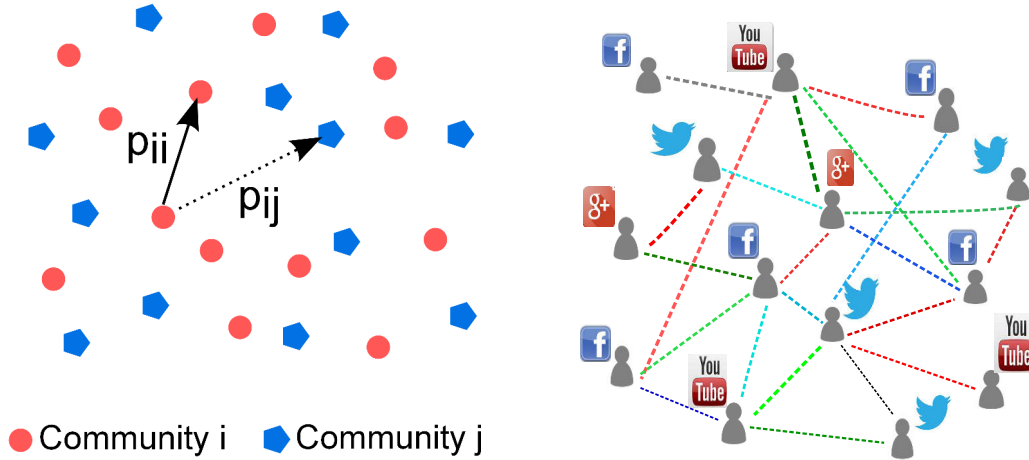


Figure 5.1: Left, spatial interactions between two communities. Right, interactions between users of different social networks.

As a remark, we note that the probabilities of interactions may depend only on the relative size of communities in the population. For example, if N_1 and N_2 are the number of individuals in communities 1 and 2 respectively, then the probability for a player from community 1 to meet and interact with an opponent, uniformly randomly picked from the same community is $p_{11} = \frac{N_1 - 1}{N_1 + N_2 - 1}$ while the probability to interact with a player from the other community is $p_{12} = \frac{N_2}{N_1 + N_2 - 1}$. Similarly, the probability for a player

from community 2 to interact with a player from community 2 is $p_{22} = \frac{N_2 - 1}{N_1 + N_2 - 1}$ and the probability to interact with a player from community 1 is $p_{21} = \frac{N_1}{N_1 + N_2 - 1}$. In this context, we observe that p_{12} is not equal to p_{21} .

More generally, let γ denote the interaction rate of an individual in both communities (i.e. the number of interactions per unit of time). Then the total rate of inter-community interactions, that is interactions between two individuals who are not in the same community, in community 1, is given by $N_1\gamma p_{12}$, where p_{12} is the probability of inter-community interaction in community 1. And the total rate of inter-community interactions in community 2 is given by $N_2\gamma p_{21}$. Since the total rates of inter-community interactions in the two communities should be equal (each time a player from community 1 interacts with a player from community 2, the converse is true), we have $p_{12} = p_{21} \frac{N_2}{N_1}$. This gives us the relationship between the probabilities of inter-community interactions in both communities. If $N_1 = N_2$, then the probabilities of inter-community interactions are the same for both communities, otherwise, p_{12} and p_{21} are not equal. For instance, if $N_1 \gg N_2$, then p_{12} is very small and p_{21} is larger. In the following, we present different ESS characterizations that differ in the stability level.

5.2.1 Strong ESS

A strong ESS is a strategy that, when adopted by the entire population, cannot be invaded by a sufficiently small group composed from all communities and using an alternative strategy. The incumbent players following the strong ESS, will get a strictly higher expected payoff when playing against the population composed of incumbents and mutants, than the mutants will get. The following definition can be stated:

Definition 1. A strategy \mathbf{s}^* is a strong ESS, if for all $\mathbf{s} \neq \mathbf{s}^*$, there exists an $\varepsilon(\mathbf{s}) > 0$ such that for all $i = 1, \dots, N$ and $\varepsilon \leq \varepsilon(\mathbf{s})$,

$$\bar{U}_i(s_i, \varepsilon \mathbf{s} + (1 - \varepsilon)\mathbf{s}^*, p) < \bar{U}_i(s_i^*, \varepsilon \mathbf{s} + (1 - \varepsilon)\mathbf{s}^*, p). \quad (5.3)$$

This strong ESS must in fact have a uniform invasion barrier [35] or threshold where any proportion of invaders using an alternative strategy is repelled. An alternative definition can be established as follows:

Definition 2. A strategy \mathbf{s}^* is a strong ESS if it meets two conditions for all i and for all $\mathbf{s} \neq \mathbf{s}^*$,

$$\bullet \bar{U}_i(s_i, \mathbf{s}^*, p) \leq \bar{U}_i(s_i^*, \mathbf{s}^*, p), \quad (5.4)$$

$$\bullet \text{if } \bar{U}_i(s_i, \mathbf{s}^*, p) = \bar{U}_i(s_i^*, \mathbf{s}^*, p), \text{ then } \bar{U}_i(s_i, \mathbf{s}, p) < \bar{U}_i(s_i^*, \mathbf{s}, p). \quad (5.5)$$

Proposition 15. The Definitions 1 and 2 are equivalent.

Proof. Let us first prove that definition 1 implies definition 2. Since the condition in Definition 1 holds for any sufficiently small ε , as $\varepsilon \rightarrow 0$, $\bar{U}_i(s_i, \varepsilon \mathbf{s} + (1 - \varepsilon) \mathbf{s}^*, p) < \bar{U}_i(s_i^*, \varepsilon \mathbf{s} + (1 - \varepsilon) \mathbf{s}^*, p)$ for all $i = 1, \dots, N$, implies $\bar{U}_i(s_i, \mathbf{s}^*, p) \leq \bar{U}_i(s_i^*, \mathbf{s}^*, p)$ for all $i = 1, \dots, N$. Therefore, the first condition in Definition 2 is established. Now we suppose there exists i such that $\bar{U}_i(s_i, \mathbf{s}^*, p) = \bar{U}_i(s_i^*, \mathbf{s}^*, p)$. Since the expected utility is linear in \mathbf{s} , the condition $\bar{U}_i(s_i, \varepsilon \mathbf{s} + (1 - \varepsilon) \mathbf{s}^*, p) < \bar{U}_i(s_i^*, \varepsilon \mathbf{s} + (1 - \varepsilon) \mathbf{s}^*, p)$ can be written as $\varepsilon \bar{U}_i(s_i, \mathbf{s}, p) + (1 - \varepsilon) \bar{U}_i(s_i, \mathbf{s}^*, p) < \varepsilon \bar{U}_i(s_i^*, \mathbf{s}, p) + (1 - \varepsilon) \bar{U}_i(s_i^*, \mathbf{s}^*, p)$. Since $\bar{U}_i(s_i, \mathbf{s}^*, p) = \bar{U}_i(s_i^*, \mathbf{s}^*, p)$, the last inequality can be written $\varepsilon \bar{U}_i(s_i, \mathbf{s}, p) < \varepsilon \bar{U}_i(s_i^*, \mathbf{s}, p)$; which yields $\bar{U}_i(s_i, \mathbf{s}, p) < \bar{U}_i(s_i^*, \mathbf{s}, p)$ since $\varepsilon > 0$. Therefore, the second condition in Definition 2 is established.

Let us now prove that definition 2 implies definition 1. We have for all i and for any $\mathbf{s} \neq \mathbf{s}^*$, $\bar{U}_i(s_i, \mathbf{s}^*, p) \leq \bar{U}_i(s_i^*, \mathbf{s}^*, p)$. If for some i , this inequality is strict, then the condition in Definition 1 is satisfied for $\varepsilon = 0$ and so for sufficiently small ε . If for some i , $\bar{U}_i(s_i, \mathbf{s}^*, p) = \bar{U}_i(s_i^*, \mathbf{s}^*, p)$, then the second condition in Definition 2 implies $\bar{U}_i(s_i, \mathbf{s}, p) < \bar{U}_i(s_i^*, \mathbf{s}, p)$. If we multiply this relation by ε and add $(1 - \varepsilon) \bar{U}_i(s_i, \mathbf{s}^*, p)$ to the left-hand side, and $(1 - \varepsilon) \bar{U}_i(s_i^*, \mathbf{s}^*, p)$ to the right-hand side, we get the condition in Definition 1. \square

A strong ESS yields a higher expected payoff than any alternative strategy when played against itself (condition (5.4)). If there is a strategy that yields the same payoff as the strong ESS when played against the ESS, then this strategy will yield a strictly lower expected payoff when played against itself than the ESS, and cannot spread in the population (condition (5.5)).

5.2.2 Weak ESS

In this subsection, we assume that mutants arise in one community and we introduce an alternative ESS version with a weaker stability condition. A weak ESS is a strategy that, when adopted by the entire population, then each community resists invasion by a sufficiently small group of mutants using an alternative strategy in that community. The definition of the weak ESS is given by:

Definition 3. A strategy \mathbf{s}^* is a weak ESS if for all $\mathbf{s} \neq \mathbf{s}^*$ and for all $i = 1, \dots, N$, there exists $\varepsilon_i(\mathbf{s}) > 0$ such that for all $\varepsilon_i \leq \varepsilon_i(\mathbf{s})$,

$$\bar{U}_i(s_i, (\varepsilon_i s_i + (1 - \varepsilon_i) s_i^*, \mathbf{s}_{-i}^*), p) < \bar{U}_i(s_i^*, (\varepsilon_i s_i + (1 - \varepsilon_i) s_i^*, \mathbf{s}_{-i}^*), p), \quad (5.6)$$

where $(\varepsilon_i s_i + (1 - \varepsilon_i) s_i^*, \mathbf{s}_{-i}^*)$ is the profile of the population where the i^{th} community is composed of the fraction ε_i of mutants using an alternative strategy s_i and the fraction $1 - \varepsilon_i$ of incumbent players using s_i^* , and the remaining of the population follows the ESS \mathbf{s}_{-i}^* .

An equivalent definition of the weak ESS can be stated as follows:

Definition 4. A strategy \mathbf{s}^* is a weak ESS if, for all i and for all $\mathbf{s} \neq \mathbf{s}^*$,

$$\bullet \bar{U}_i(s_i, \mathbf{s}^*, p) \leq \bar{U}_i(s_i^*, \mathbf{s}^*, p), \quad (5.7)$$

$$\bullet \text{ if } \bar{U}_i(s_i, \mathbf{s}^*, p) = \bar{U}_i(s_i^*, \mathbf{s}^*, p), \text{ then } \bar{U}_i(s_i, (s_i, \mathbf{s}_{-i}^*), p) < \bar{U}_i(s_i^*, (s_i, \mathbf{s}_{-i}^*), p), \quad (5.8)$$

where $\bar{U}_i(s_i, \mathbf{s}^*, p)$ is the expected payoff of a mutant in community i using s_i , and $\bar{U}_i(s_i^*, \mathbf{s}^*, p)$ is the expected payoff of an incumbent player in community i using s_i^* when the profile of the population is \mathbf{s}^* .

Proposition 16. The definitions 3 and 4 are equivalent.

Proof. First, let us prove that definition 3 implies definition 4. If we take $\varepsilon_i \rightarrow 0$ in definition 3, we get: $\bar{U}_i(s_i, \mathbf{s}^*, p) \leq \bar{U}_i(s_i^*, \mathbf{s}^*, p)$ for all i , and so condition (5.7). Now, to establish the condition (5.8) in definition 4, we suppose there exists i such that $\bar{U}_i(s_i^*, \mathbf{s}^*, p) = \bar{U}_i(s_i, \mathbf{s}^*, p)$, we need to prove that $\bar{U}_i(s_i, (s_i, \mathbf{s}_{-i}^*), p) < \bar{U}_i(s_i^*, (s_i, \mathbf{s}_{-i}^*), p)$. We can write condition (5.6) as follows:

$$\begin{aligned} & \bar{U}_i(s_i, (\varepsilon_i s_1^* + (1 - \varepsilon_i) s_1^*, \dots, \varepsilon_i s_i + (1 - \varepsilon_i) s_i^*, \dots, \varepsilon_i s_N^* + (1 - \varepsilon_i) s_N^*), p) \\ & < \bar{U}_i(s_i^*, (\varepsilon_i s_1^* + (1 - \varepsilon_i) s_1^*, \dots, \varepsilon_i s_i + (1 - \varepsilon_i) s_i^*, \dots, \varepsilon_i s_N^* + (1 - \varepsilon_i) s_N^*), p). \end{aligned}$$

By exploring the linearity of \bar{U}_i , we get:

$$\varepsilon_i \bar{U}_i(s_i, (s_1^*, \dots, s_i, \dots, s_N^*), p) + (1 - \varepsilon_i) \bar{U}_i(s_i, \mathbf{s}^*, p) < \varepsilon_i \bar{U}_i(s_i^*, (s_1^*, \dots, s_i, \dots, s_N^*), p) + (1 - \varepsilon_i) \bar{U}_i(s_i^*, \mathbf{s}^*, p).$$

Since we have $\varepsilon_i > 0$ and we suppose $\bar{U}_i(s_i^*, \mathbf{s}^*, p) = \bar{U}_i(s_i, \mathbf{s}^*, p)$, the above inequality yields:

$$\bar{U}_i(s_i, (s_i, \mathbf{s}_{-i}^*), p) < \bar{U}_i(s_i^*, (s_i, \mathbf{s}_{-i}^*), p),$$

and so condition (5.8).

Now we prove that definition 4 implies definition 3. We have for all i and for all $\mathbf{s} \neq \mathbf{s}^*$,

$$\bar{U}_i(s_i, \mathbf{s}^*, p) \leq \bar{U}_i(s_i^*, \mathbf{s}^*, p).$$

If this inequality is strict for all i , then condition (5.6) holds for $\varepsilon_i = 0$, and thus for sufficiently small ε_i . If there exists i such that the comparison in (5.7) is an equality, then we obtain $\bar{U}_i(s_i, (s_i, \mathbf{s}_{-i}^*), p) < \bar{U}_i(s_i^*, (s_i, \mathbf{s}_{-i}^*), p)$ (condition (5.8)). We multiply both sides by ε_i , and by observing that $\bar{U}_i(s_i, \mathbf{s}^*, p) = \bar{U}_i(s_i^*, \mathbf{s}^*, p)$, we add $(1 - \varepsilon_i) \bar{U}_i(s_i, \mathbf{s}^*, p)$ to the left side and $(1 - \varepsilon_i) \bar{U}_i(s_i^*, \mathbf{s}^*, p)$ to the right side, we get condition (5.6). \square

This ESS definition is different from that of Cressman, referred to as Cressman ESS in the literature [25, 105], which considers invasion of the communities by a fraction of mutants from all communities. For a state to be a Cressman ESS, it is enough that one community resist invasion from a mutant strategy. In our definition, we consider invasion of a single community by a small **local** group of mutants. In Section 5.3, we introduce a particular example with two communities and we show that a weak ESS cannot be a Cressman ESS in this case.

5.2.3 Intermediate ESS

In the intermediate ESS version [26, 35], we consider one fitness function which is the sum of fitnesses over all the communities. An intermediate ESS is a strategy that, when adopted by the entire population, then for any small group using a mutant strategy, the total expected payoff (fitness) of the incumbent strategies in all the communities is strictly higher than that of the mutant strategy. The formal definition of intermediate ESS is given by:

Definition 5. A strategy \mathbf{s}^* is an intermediate ESS if for all $\mathbf{s} \neq \mathbf{s}^*$, there exists an $\varepsilon(\mathbf{s}) > 0$ such that for all $\varepsilon \leq \varepsilon(\mathbf{s})$,

$$\sum_{i=1}^N \bar{U}_i(s_i, \varepsilon \mathbf{s} + (1 - \varepsilon) \mathbf{s}^*, p) < \sum_{i=1}^N \bar{U}_i(s_i^*, \varepsilon \mathbf{s} + (1 - \varepsilon) \mathbf{s}^*, p). \quad (5.9)$$

Equivalently, we have the following definition:

Definition 6. A strategy \mathbf{s}^* is an intermediate ESS if for all $\mathbf{s} \neq \mathbf{s}^*$,

$$\bullet \sum_{i=1}^N \bar{U}_i(s_i, \mathbf{s}^*, p) \leq \sum_{i=1}^N \bar{U}_i(s_i^*, \mathbf{s}^*, p), \quad (5.10)$$

$$\bullet \text{ if } \sum_{i=1}^N \bar{U}_i(s_i, \mathbf{s}^*, p) = \sum_{i=1}^N \bar{U}_i(s_i^*, \mathbf{s}^*, p), \text{ then } \sum_{i=1}^N \bar{U}_i(s_i, \mathbf{s}, p) < \sum_{i=1}^N \bar{U}_i(s_i^*, \mathbf{s}, p). \quad (5.11)$$

The condition (5.10) defines the best-reply requirement according to which a mutant strategy cannot yield a better total payoff than the ESS. When the comparison in this condition is an equality, i.e. in case of an alternative best-reply, the condition (5.11) guarantees that the population profile do not shift away from the ESS. It means that all the population have a positive incentive to remain at the ESS when there is a mutant strategy.

Proposition 17. The definitions 5 and 6 are equivalent.

Proof. The proof follows by carrying out exactly the same procedure as done in Proposition 16. \square

5.2.4 Relationship between The ESSs

In this section, we discuss the relationship between different concepts of ESS introduced earlier. We explain how these ESS concepts are overlapped one into another. Note that every ESS from strong to weak stability is a Nash equilibrium. However, a strict Nash equilibrium must be a strong ESS, and therefore the strict Nash equilibrium is an intermediate and weak ESS. We note that the second condition of the ESS (stability)

comes into play only in the case of alternative best replies. Hence with strict Nash equilibrium, there is no alternative to play another strategy that gets the same payoff.

Now, let us discuss the relationship between different ESSs based on their stability properties. The definition of the strong ESS makes it clear that strong ESS is an intermediate and also a weak ESS. Indeed, if we suppose there is a small fraction of mutants from all the communities using an alternative strategy, the strong ESS, when adopted by all the population, would resist this invasion because incumbent players would get a strictly higher expected payoff than mutants. The total expected payoff of the strong ESS in all the communities would also be strictly higher than that of the mutant strategy, and therefore the strong ESS is an intermediate ESS.

A similar argument explains why an intermediate ESS is also a weak ESS. In fact, if we suppose there is a small fraction of mutants in a single community, an intermediate ESS would resist this invasion by definition and also a weak ESS; therefore an intermediate ESS is a weak ESS. In the next section, we will show through the study of two communities, that (i) an intermediate ESS is not always a strong ESS, and (ii) a weak ESS is not always an intermediate ESS. We then have the following relationships between the different concepts of ESS considering interacting communities:

$$\text{Strong ESS} \Rightarrow \text{Intermediate ESS} \Rightarrow \text{Weak ESS}.$$

We note that all these definitions are obviously identical when there is a single community.

5.3 Two-community Two-strategy Model

For the sake of clarity, we consider only the case where there are two communities which interact in a nonuniform manner. All results obtained with two communities are still valid for more than two communities.

We consider two communities in which each individual from community $i = 1, 2$ involved in an interaction, may interact with an individual from the same community with probability p_i or with an individual from the other community with probability $1 - p_i$. In addition, we consider that each community i has two strategies $\{G_i, H_i\}$. Since there are two possible strategies in each community, the population profile can be defined by $\mathbf{s} = (s_1, s_2)$ where s_i is the frequency of strategy G_i in community i (so $1 - s_i$ is the frequency of strategy H_i). In fact, in the two-strategy setting, we have $s_{12} = 1 - s_{11}$ and $s_{22} = 1 - s_{21}$ and the population state can then be completely defined by the vector $\mathbf{s} = (s_1, s_2)$ where s_1 and s_2 are scalars. The pairwise interactions inside

Table 5.1: Parameters of the model.

Parameter	Value
L_1	$a_1 - b_1 - c_1 + d_1$
L_2	$a_2 - b_2 - c_2 + d_2$
L_{12}	$a_{12} - b_{12} - c_{12} + d_{12}$
L_{21}	$a_{21} - b_{21} - c_{21} + d_{21}$
K_1	$p_1(b_1 - d_1) + (1 - p_1)(b_{12} - d_{12})$
K_2	$p_2(b_2 - d_2) + (1 - p_2)(b_{21} - d_{21})$
Δ	$p_1 p_2 L_1 L_2 - (1 - p_1)(1 - p_2) L_{12} L_{21}$
Δ_1	$4p_1 p_2 L_1 L_2 - ((1 - p_1)L_{12} + (1 - p_2)L_{21})^2$

the communities are described by the matrices A and D :

$$A = \begin{matrix} & G_1 & H_1 \\ \begin{matrix} G_1 \\ H_1 \end{matrix} & \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \end{matrix}, \quad D = \begin{matrix} & G_2 & H_2 \\ \begin{matrix} G_2 \\ H_2 \end{matrix} & \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \end{matrix}.$$

The interactions between individuals from different communities are described by the following matrices:

$$B = \begin{matrix} & G_2 & H_2 \\ \begin{matrix} G_1 \\ H_1 \end{matrix} & \begin{pmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{pmatrix} \end{matrix}, \quad C = \begin{matrix} & G_1 & H_1 \\ \begin{matrix} G_2 \\ H_2 \end{matrix} & \begin{pmatrix} a_{21} & b_{21} \\ c_{21} & d_{21} \end{pmatrix} \end{matrix}.$$

The expected utility (fitness function) of a strategy depends on the distribution of strategies in both communities and the interaction probabilities. Using Eq. (5.1) and (5.2), we derive the utilities of strategies G_1 and H_1 in community 1 as follows:

$$\begin{aligned} U_1(G_1, \mathbf{s}, p) &= p_1(s_1 a_1 + (1 - s_1)b_1) + (1 - p_1)(s_2 a_{12} + (1 - s_2)b_{12}), \\ U_1(H_1, \mathbf{s}, p) &= p_1(s_1 c_1 + (1 - s_1)d_1) + (1 - p_1)(s_2 c_{12} + (1 - s_2)d_{12}). \end{aligned}$$

Similarly, the utility of an individual in community 2 using G_2 (resp. H_2) is given by:

$$\begin{aligned} U_2(G_2, \mathbf{s}, p) &= p_2(s_2 a_2 + (1 - s_2)b_2) + (1 - p_2)(s_1 a_{21} + (1 - s_1)b_{21}), \\ U_2(H_2, \mathbf{s}, p) &= p_2(s_2 c_2 + (1 - s_2)d_2) + (1 - p_2)(s_1 c_{21} + (1 - s_1)d_{21}). \end{aligned}$$

In addition, the expected utility of any individual from community i is given by:

$$\bar{U}_i(s_i, \mathbf{s}, p) = s_i U_i(G_i, \mathbf{s}, p) + (1 - s_i) U_i(H_i, \mathbf{s}, p).$$

In addition, we define in Table 5.1 the parameters which will be used to analyze the model. The parameters L_1 , L_2 , L_{12} , and L_{21} depend on the payoffs. The parameters K_1 , K_2 , Δ , and Δ_1 depend on the payoffs and the interaction probabilities.

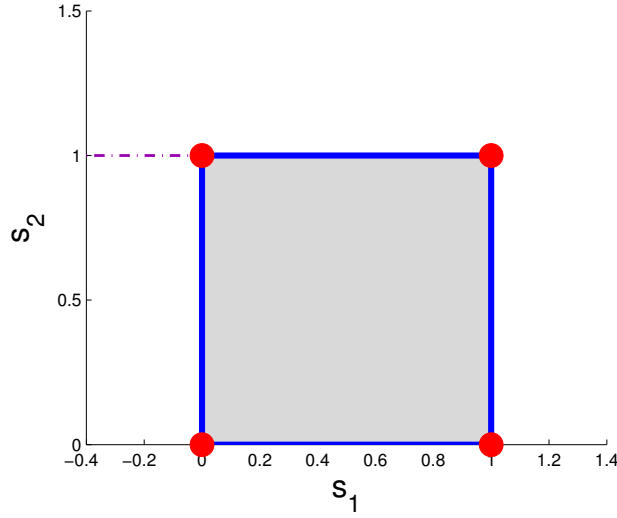


Figure 5.2: The shaded region corresponds to fully mixed ESS, the circles correspond to fully pure ESS, and the edges of the shaded region correspond to partially mixed ESS.

5.3.1 Dominant Strategies

First, we should point out as a remark that in this model, there are three types of ESS: fully mixed, partially mixed, and fully pure. An ESS can be fully mixed, in which case, all strategies are represented in all the communities, fully pure, where in each community all the agents use the same pure strategy, or partially-mixed, where at least one community follows a fully mixed ESS and one other community uses a fully pure ESS. We illustrate in Fig. 5.2 these types. An ESS is also characterized by its level of stability and can be strong, weak or intermediate. Therefore, we have nine different types ESSs.

In evolutionary games framework [108], the players get repeatedly involved in random pairwise interactions and at each interaction they get a payoff depending on the strategies used. Successful strategies increase in frequency in the population while less successful get less abundant. A dominant strategy will eventually thrive and displace all dominated strategies. From the model above, the strategy G_1 dominates the strategy H_1 in Community 1 if and only if $U_1(G_1, \mathbf{s}, p) \geq U_1(H_1, \mathbf{s}, p)$ for all $\mathbf{s} = (s_1, s_2) \in [0, 1]^2$; then, if and only if:

$$p_1 s_1 L_1 + (1 - p_1) s_2 L_{12} + K_1 \geq 0, \forall (s_1, s_2) \in [0, 1]^2. \quad (5.12)$$

Similarly, the strategy G_2 dominates the strategy H_2 in community 2 if and only if:

$$p_2 s_2 L_2 + (1 - p_2) s_1 L_{21} + K_2 \geq 0, \forall (s_1, s_2) \in [0, 1]^2. \quad (5.13)$$

We have the same kind of inequality for determining if H_i dominates G_i , $i = 1, 2$. Therefore, we can establish the following proposition.

Proposition 18. • *The strategy G_1 dominates the strategy H_1 if and only if:*

$$0 \leq K_1, 0 \leq p_1L_1 + K_1, 0 \leq (1 - p_1)L_{12} + K_1, \text{ and } 0 \leq p_1L_1 + (1 - p_1)L_{12} + K_1.$$

• *The strategy H_1 dominates the strategy G_1 if and only if:*

$$K_1 \leq 0, p_1L_1 + K_1 \leq 0, (1 - p_1)L_{12} + K_1 \leq 0, \text{ and } p_1L_1 + (1 - p_1)L_{12} + K_1 \leq 0.$$

• *The strategy G_2 dominates the strategy H_2 if and only if:*

$$0 \leq K_2, 0 \leq p_2L_2 + K_2, 0 \leq (1 - p_2)L_{21} + K_2 \text{ and } 0 \leq p_2L_2 + (1 - p_2)L_{21} + K_2.$$

• *The strategy H_2 dominates the strategy G_2 if and only if:*

$$K_2 \leq 0, p_2L_2 + K_2 \leq 0, (1 - p_2)L_{21} + K_2 \leq 0 \text{ and } p_2L_2 + (1 - p_2)L_{21} + K_2 \leq 0.$$

Proof. • The strategy G_1 dominates the strategy H_1 if and only if $\forall \mathbf{s} = (s_1, s_2) \in [0, 1]^2$, $U_1(G_1, \mathbf{s}, p) \geq U_1(H_1, \mathbf{s}, p)$ then as stated before, if and only if $p_1s_1L_1 + (1 - p_1)s_2L_{12} + K_1 \geq 0$, which is $K_1 \geq 0$, $p_1L_1 + K_1 \geq 0$, $(1 - p_1)L_{12} + K_1 \geq 0$ and $p_1L_1 + (1 - p_1)L_{12} + K_1 \geq 0$. The same procedure for determining if the strategy G_2 dominates the strategy H_2 in the second community.

• In the same way, strategy H_1 dominates the strategy G_1 if and only if $K_1 \leq 0$, $p_1L_1 + K_1 \leq 0$, $(1 - p_1)L_{12} + K_1 \leq 0$ and $p_1L_1 + (1 - p_1)L_{12} + K_1 \leq 0$.

□

In addition, we have the following result on pure ESSs.

Proposition 19. *If strategies $X_1 \in \{G_1, H_1\}$ and $X_2 \in \{G_2, H_2\}$ are strictly dominant in community 1 and 2, respectively, then (X_1, X_2) is a strong ESS.*

Proof. Let us prove that if G_1 and G_2 are strictly dominant in community 1 and 2 respectively, then $\mathbf{s}^* = (1, 1)$ is a strong ESS. In this case we have the following:

$$\begin{aligned} U_1(G_1, \mathbf{s}^*, p) &> U_1(H_1, \mathbf{s}^*, p), \\ U_2(G_2, \mathbf{s}^*, p) &> U_2(H_2, \mathbf{s}^*, p). \end{aligned}$$

The first condition in Definition 2 is a strict inequality, therefore \mathbf{s}^* is a strong ESS. Similarly, we prove that all pure strictly dominant strategies are strong ESSs. □

5.3.2 Fully Pure ESS

We study in this subsection the conditions of existence of fully pure ESSs under different levels of stability in the case of two interacting communities. An ESS $\mathbf{s}^* = (s_1, s_2)$ is fully pure if, in all the communities, only one strategy is used, i.e. $s_i \in \{0, 1\}$ for $i = 1, 2$. The results are given in the next proposition.

Proposition 20. • $\mathbf{s}^* = (1, 1)$ is a fully pure weak ESS if $p_1L_1 + (1 - p_1)L_{12} + K_1 > 0$ or if $(p_1L_1 + (1 - p_1)L_{12} + K_1 = 0$ and $L_1 < 0)$ and also if $p_2L_2 + (1 - p_2)L_{21} + K_2 > 0$ or if $(p_2L_2 + (1 - p_2)L_{21} + K_2 = 0$ and $L_2 < 0)$.

- $\mathbf{s}^* = (1, 1)$ is an intermediate ESS if it is a weak ESS and if $p_1L_1 + (1 - p_1)L_{12} + K_1 > 0$, $p_2L_2 + (1 - p_2)L_{21} + K_2 > 0$, or if $(p_1L_1 + (1 - p_1)L_{12} + K_1 = 0$, $p_2L_2 + (1 - p_2)L_{21} + K_2 = 0$, and $\Delta_1 > 0)$.
- $\mathbf{s}^* = (1, 1)$ is a strong ESS if it is an intermediate ESS and also if $p_1L_1 + (1 - p_1)L_{12} + K_1 > 0$ or $p_2L_2 + (1 - p_2)L_{21} + K_2 > 0$.
- $\mathbf{s}^* = (0, 1)$ is a weak ESS if $(1 - p_1)L_{12} + K_1 < 0$ or $((1 - p_1)L_{12} + K_1 = 0$ and $L_1 < 0)$; and also if $p_2L_2 + K_2 > 0$ or $(p_2L_2 + K_2 = 0$ and $L_2 < 0)$.
- $\mathbf{s}^* = (0, 1)$ is an intermediate ESS if it is a weak ESS and if $(1 - p_1)L_{12} + K_1 < 0$, $p_2L_2 + K_2 > 0$, or $((1 - p_1)L_{12} + K_1 = 0$, $p_2L_2 + K_2 = 0$ and $\Delta_1 > 0)$.
- $\mathbf{s}^* = (0, 1)$ is a strong ESS if it is an intermediate ESS and if $(1 - p_1)L_{12} + K_1 < 0$ or $p_2L_2 + K_2 > 0$.
- $\mathbf{s}^* = (1, 0)$ is a weak ESS if $p_1L_1 + K_1 > 0$ or if $(p_1L_1 + K_1 = 0$ and $L_1 < 0)$ and also if $(1 - p_2)L_{21} + K_2 < 0$ or $((1 - p_2)L_{21} + K_2 = 0$ and $L_2 < 0)$.
- $\mathbf{s}^* = (1, 0)$ is an intermediate ESS if it is a weak ESS and if $p_1L_1 + K_1 > 0$, $(1 - p_2)L_{21} + K_2 < 0$, or $(p_1L_1 + K_1 = 0$, $(1 - p_2)L_{21} + K_2 = 0$, and $\Delta_1 > 0)$.
- $\mathbf{s}^* = (1, 0)$ is a strong ESS if it is an intermediate ESS and if $p_1L_1 + K_1 > 0$ or $(1 - p_2)L_{21} + K_2 < 0$.
- $\mathbf{s}^* = (0, 0)$ is a weak ESS if $K_1 < 0$ or $(K_1 = 0$ and $L_1 < 0)$ and also if $K_2 < 0$ or $(K_2 = 0$ and $L_2 < 0)$.
- $\mathbf{s}^* = (0, 0)$ is an intermediate ESS if it is a weak ESS and if $K_1 < 0$, $K_2 < 0$, or $(K_1 = 0$, $K_2 = 0$, and $\Delta_1 > 0)$.
- $\mathbf{s}^* = (0, 0)$ is a strong ESS if it is an intermediate ESS and if $K_1 < 0$ or $K_2 < 0$.

Proof. • $\mathbf{s}^* = (1, 1)$ is a weak ESS if,

- $\bar{U}_1(s_1, (1, 1), p) \leq U_1(G_1, (1, 1), p)$, and,
- if $\bar{U}_1(s_1, (1, 1), p) = U_1(G_1, (1, 1), p)$ then $\bar{U}_1(s_1, (s_1, 1), p) < U_1(G_1, (s_1, 1), p)$,
- $\bar{U}_2(s_2, (1, 1), p) \leq \bar{U}_2(G_2, (1, 1), p)$, and,
- if $\bar{U}_2(s_2, (1, 1), p) = \bar{U}_2(G_2, (1, 1), p)$, then $\bar{U}_2(s_2, (1, s_2), p) < \bar{U}_2(G_2, (1, s_2), p)$.

The conditions above yield, for the first community $p_1L_1 + (1 - p_1)L_{12} + K_1 > 0$ or ($p_1L_1 + (1 - p_1)L_{12} + K_1 = 0$ and $L_1 < 0$); and for the second community we have $p_2L_2 + (1 - p_2)L_{21} + K_2 > 0$ or ($p_2L_2 + (1 - p_2)L_{21} + K_2 = 0$ and $L_2 < 0$).

Furthermore, \mathbf{s}^* is an intermediate ESS if $\bar{U}_1(s_1, \mathbf{s}^*, p) + \bar{U}_2(s_2, \mathbf{s}^*, p) \leq U_1(G_1, \mathbf{s}^*, p) + U_2(G_2, \mathbf{s}^*, p)$, and if there exists an \mathbf{s} for which this condition is an equality, then $\bar{U}_1(s_1, \mathbf{s}, p) + \bar{U}_2(s_2, \mathbf{s}, p) < U_1(G_1, \mathbf{s}, p) + U_2(G_2, \mathbf{s}, p)$.

These conditions yield: $p_1L_1 + (1 - p_1)L_{12} + K_1 > 0$ or ($p_1L_1 + (1 - p_1)L_{12} + K_1 = 0$ and $L_1 < 0$) and $p_2L_2 + (1 - p_2)L_{21} + K_2 > 0$ or ($p_2L_2 + (1 - p_2)L_{21} + K_2 = 0$ and $L_2 < 0$) or ($p_1L_1 + (1 - p_1)L_{12} + K_1 = 0$, $p_2L_2 + (1 - p_2)L_{21} + K_2 = 0$, $L_1 < 0$, $L_2 < 0$, and $\Delta_1 > 0$). Therefore, \mathbf{s}^* is an intermediate ESS if it is a weak ESS and if either ($p_1L_1 + (1 - p_1)L_{12} + K_1 > 0$ or $p_2L_2 + (1 - p_2)L_{21} + K_2 > 0$), or ($p_1L_1 + (1 - p_1)L_{12} + K_1 = 0$ and $p_2L_2 + (1 - p_2)L_{21} + K_2 = 0$, and $\Delta_1 > 0$).

Finally, \mathbf{s}^* is a strong ESS if for all $\mathbf{s} \neq \mathbf{s}^*$,

- $\bar{U}_1(s_1, \mathbf{s}^*, p) < U_1(G_1, \mathbf{s}^*, p)$, and,
- $\bar{U}_2(s_2, \mathbf{s}^*, p) < U_2(G_2, \mathbf{s}^*, p)$,

or if,

- $\bar{U}_1(s_1, \mathbf{s}^*, p) = U_1(G_1, \mathbf{s}^*, p)$ and, $\bar{U}_1(s_1, (s_1, 1), p) < U_1(G_1, (s_1, 1), p)$,
- $\bar{U}_2(s_2, \mathbf{s}^*, p) < U_2(G_2, \mathbf{s}^*, p)$,

or if,

- $U_1(s_1, \mathbf{s}^*, p) < U_1(G_1, \mathbf{s}^*, p)$, and,
- $U_2(s_2, \mathbf{s}^*, p) = U_2(G_2, \mathbf{s}^*, p)$ and $\bar{U}_2(s_2, (1, s_2), p) < \bar{U}_2(G_2, (1, s_2), p)$,

or if,

- $U_1(s_1, \mathbf{s}^*, p) = U_1(G_1, \mathbf{s}^*, p)$, $U_2(s_2, \mathbf{s}^*, p) = U_2(G_2, \mathbf{s}^*, p)$ and,
- $\bar{U}_2(s_2, (s_1, s_2), p) < \bar{U}_2(G_2, (s_1, s_2), p)$ and $\bar{U}_2(s_2, (s_1, s_2), p) < \bar{U}_2(G_2, (s_1, s_2), p)$.

The conditions above yield $p_1L_1 + (1 - p_1)L_{12} + K_1 > 0$ or ($p_1L_1 + (1 - p_1)L_{12} + K_1 = 0$ and $L_1 < 0$) and $p_2L_2 + (1 - p_2)L_{21} + K_2 > 0$ or ($p_2L_2 + (1 - p_2)L_{21} +$

$K_2 = 0$ and $L_2 < 0$). Therefore \mathbf{s}^* is strong ESS if it is an intermediate ESS and if either $p_1L_1 + (1 - p_1)L_{12} + K_1 > 0$ or $p_2L_2 + (1 - p_2)L_{21} + K_2 > 0$.

We can follow the same procedure for determining the conditions of existence of all fully pure ESSs. \square

The analytical results in this proposition corroborate the relationship between the ESSs explained earlier. Indeed, we clearly observe that a strong ESS is an intermediate ESS which is also a weak ESS.

5.3.3 Fully Mixed Nash Equilibrium and ESS

In this subsection, we characterize the existence of mixed ESSs under different stability conditions. We study the case of fully mixed ESSs and the case of partially mixed ESSs. At the fully mixed ESS, all strategies in both communities coexist, that is $0 < s_i^* < 1$ for $i = 1, 2$. The following proposition summarizes results on the existence of fully mixed ESSs.

Proposition 21. *Let $\mathbf{s}^* = (s_1^*, s_2^*)$ with*

$$s_1^* = \frac{(1 - p_1)L_{12}K_2 - p_2L_2K_1}{\Delta}, \text{ and } s_2^* = \frac{(1 - p_2)L_{21}K_1 - p_1L_1K_2}{\Delta}.$$

We have the following results on \mathbf{s}^ :*

- \mathbf{s}^* is a unique fully mixed Nash equilibrium, i.e. $0 < s_1^* < 1$ and $0 < s_2^* < 1$, if
 - $0 < \Delta$, $0 < (1 - p_1)L_{12}K_2 - p_2L_2K_1$, $(1 - p_1)L_{12}K_2 - p_2L_2K_1 < \Delta$, $0 < (1 - p_2)L_{21}K_1 - p_1L_1K_2$, and $(1 - p_2)L_{21}K_1 - p_1L_1K_2 < \Delta$, or
 - $\Delta < 0$, $(1 - p_1)L_{12}K_2 - p_2L_2K_1 < 0$, $\Delta < (1 - p_1)L_{12}K_2 - p_2L_2K_1$, $0 < (1 - p_2)L_{21}K_1 - p_1L_1K_2 < 0$, and $\Delta < (1 - p_2)L_{21}K_1 - p_1L_1K_2$.
- \mathbf{s}^* cannot be a strong ESS.
- \mathbf{s}^* is a weak ESS if $L_1 < 0$ and $L_2 < 0$.
- \mathbf{s}^* is an intermediate ESS if $L_1 < 0$, $L_2 < 0$ and $\Delta_1 > 0$.

Proof. • There exists a mixed Nash equilibrium strategy $\mathbf{s}^* = (s_1^*, s_2^*)$, when users from any community are indifferent from playing strategy G_i or H_i , i.e. all (pure) strategies are equally fit. At the equilibrium, we have the following system of equations:

$$\begin{cases} U_1(G_1, \mathbf{s}^*, p) = U_1(H_1, \mathbf{s}^*, p), \\ U_2(G_2, \mathbf{s}^*, p) = U_2(H_2, \mathbf{s}^*, p). \end{cases}$$

Thus, we obtain the following system:

$$\begin{cases} p_1 s_1^* L_1 + (1 - p_1) s_2^* L_{12} + K_1 = 0, & \text{(a)} \\ p_2 s_2^* L_2 + (1 - p_2) s_1^* L_{21} + K_2 = 0, & \text{(b)} \end{cases}$$

where $L_1 = a_1 - b_1 - c_1 + d_1$, $L_{12} = a_{12} - b_{12} - c_{12} + d_{12}$, $L_2 = a_2 - b_2 - c_2 + d_2$, $L_{21} = a_{21} - b_{21} - c_{21} + d_{21}$, $K_1 = p_1(b_1 - d_1) + (1 - p_1)(b_{12} - d_{12})$, $K_2 = p_2(b_2 - d_2) + (1 - p_2)(b_{21} - d_{21})$. The solution of this system is given by $\mathbf{s}^* = (s_1^*, s_2^*)$, with $s_1^* = \frac{(1 - p_1)L_{12}K_2 - p_2L_2K_1}{\Delta}$ and $s_2^* = \frac{(1 - p_2)L_{21}K_1 - p_1L_1K_2}{\Delta}$; where $\Delta = p_1p_2L_1L_2 - (1 - p_1)(1 - p_2)L_{12}L_{21}$. Clearly, $0 < s_i^* < 1$, $i = 1, 2$, if:

- $0 < \Delta$, $0 < (1 - p_1)L_{12}K_2 - p_2L_2K_1$, $(1 - p_1)L_{12}K_2 - p_2L_2K_1 < \Delta$, $0 < (1 - p_2)L_{21}K_1 - p_1L_1K_2$, and $(1 - p_2)L_{21}K_1 - p_1L_1K_2 < \Delta$, or,
 - $\Delta < 0$, $(1 - p_1)L_{12}K_2 - p_2L_2K_1 < 0$, $\Delta < (1 - p_1)L_{12}K_2 - p_2L_2K_1$, $0 < (1 - p_2)L_{21}K_1 - p_1L_1K_2 < 0$, and $\Delta < (1 - p_2)L_{21}K_1 - p_1L_1K_2$.
- Let us check for which conditions $\mathbf{s}^* = (s_1^*, s_2^*)$, if exists, is a strong ESS. Assume there is a small proportion of "mutants" that uses another strategy $\mathbf{s} = (s_1, s_2)$. Using the definition of the expected utility, we obtain:

$$\bar{U}_1(s_1^*, \mathbf{s}^*, p) - \bar{U}_1(s_1, \mathbf{s}^*, p) = (s_1^* - s_1)(p_1 s_1^* L_1 + (1 - p_1) s_2^* L_{12} + K_1) = 0.$$

Following the same procedure for community 2, we obtain,

$$\bar{U}_2(s_2^*, \mathbf{s}^*, p) - \bar{U}_2(s_2, \mathbf{s}^*, p) = 0.$$

From (5.5), \mathbf{s}^* is a strong ESS if $\bar{U}_i(s_i^*, \mathbf{s}, p) - \bar{U}_i(s_i, \mathbf{s}, p) > 0$ for $i = 1, 2$. But,

$$\begin{aligned} \bar{U}_1(s_1^*, \mathbf{s}, p) - \bar{U}_1(s_1, \mathbf{s}, p) &= (s_1^* - s_1)(p_1 s_1 L_1 + (1 - p_1) s_2 L_{12} + K_1), \\ \bar{U}_2(s_2^*, \mathbf{s}, p) - \bar{U}_2(s_2, \mathbf{s}, p) &= (s_2^* - s_2)(p_2 s_2 L_2 + (1 - p_2) s_1 L_{21} + K_2). \end{aligned}$$

We define f_i , $i=1,2$ as follows:

$$\begin{cases} f_1(s_1, s_2) = (s_1^* - s_1)(p_1 s_1 L_1 + (1 - p_1) s_2 L_{12} + K_1), \\ f_2(s_1, s_2) = (s_2^* - s_2)(p_2 s_2 L_2 + (1 - p_2) s_1 L_{21} + K_2). \end{cases}$$

We have $\nabla f_1^T = [2p_1L_1(s_1^* - s_1) + (1 - p_1)L_{12}(s_2^* - s_2), (1 - p_1)L_{12}(s_1^* - s_1)]$. Hence, $\frac{\partial^2 f_1}{\partial s_1^2} \frac{\partial^2 f_1}{\partial s_2^2} - \frac{\partial^2 f_1}{\partial s_1 \partial s_2} \frac{\partial^2 f_1}{\partial s_2 \partial s_1} = -(1 - p_1)^2 L_{12}^2 < 0$ at \mathbf{s}^* (if $p_1 \neq 1$). Consequently, \mathbf{s}^* is a saddle point. Since $f_1(\mathbf{s}^*) = 0$, f_1 changes the sign around \mathbf{s}^* . Therefore, the first community cannot resist invasions by mutants. Following the same procedure with f_2 , we find that (s_1^*, s_2^*) is a saddle point. Therefore, the condition of stability (5.5) does not hold and consequently, \mathbf{s}^* is not a strong ESS.

- Now, let us study for which condition $\mathbf{s}^* = (s_1^*, s_2^*)$ is a weak ESS. $\mathbf{s}^* = (s_1^*, s_2^*)$ is a weak ESS if $\bar{U}_1(s_1^*, (s_1, s_2^*), p) > \bar{U}_1(s_1, (s_1, s_2^*), p)$ and, $\bar{U}_2(s_2^*, (s_1^*, s_2), p) > \bar{U}_2(s_2, (s_1^*, s_2), p)$. But,

$$\bar{U}_1(s_1^*, (s_1, s_2^*), p) - \bar{U}_1(s_1, (s_1, s_2^*), p) = -p_1 L_1 (s_1^* - s_1)^2,$$

which is strictly positive if $L_1 < 0$. Following the same procedure with the second population, we get:

$$\bar{U}_2(s_2^*, (s_1^*, s_2), p) - \bar{U}_2(s_2, (s_1^*, s_2), p) = -p_2 L_2 (s_2^* - s_2)^2,$$

which is strictly positive when $L_2 < 0$. Therefore, if $L_1 < 0$ and $L_2 < 0$, then \mathbf{s}^* is a weak ESS.

- Finally, \mathbf{s}^* is an intermediate ESS if $\bar{U}_1(s_1, \mathbf{s}, p) + \bar{U}_2(s_2, \mathbf{s}, p) < \bar{U}_1(s_1^*, \mathbf{s}, p) + \bar{U}_2(s_2^*, \mathbf{s}, p)$.

Let $g(s_1, s_2) = \bar{U}_1(s_1^*, \mathbf{s}, p) + \bar{U}_2(s_2^*, \mathbf{s}, p) - \bar{U}_1(s_1, \mathbf{s}, p) - \bar{U}_2(s_2, \mathbf{s}, p)$, we have:

$$g(s_1, s_2) = (s_1^* - s_1)(p_1 s_1 L_1 + (1 - p_1) s_2 L_{12} + K_1) + (s_2^* - s_2)(p_2 s_2 L_2 + (1 - p_2) s_1 L_{21} + K_2).$$

The Hessian matrix of g is given by:

$$\mathcal{H}(g) = \begin{pmatrix} -2p_1 L_1 & -(1 - p_1) L_{12} - (1 - p_2) L_{21} \\ -(1 - p_1) L_{12} - (1 - p_2) L_{21} & -2p_2 L_2 \end{pmatrix}.$$

The determinant of $\mathcal{H}(g)$ is $\Delta_1 = 4p_1 p_2 L_1 L_2 - ((1 - p_1) L_{12} + (1 - p_2) L_{21})^2$. Hence, g is strictly positive for all $s_1 \neq s_1^*$, $s_2 \neq s_2^*$, if $L_1 < 0$, $L_2 < 0$ and $\Delta_1 > 0$.

□

Proposition 21 establishes that any fully mixed strong ESS does not exist. Indeed, the stability condition (5.5) in Definition 2 cannot be satisfied for a fully mixed equilibrium. In contrast, the fully mixed equilibrium can be an intermediate or a weak ESS under some conditions on the payoffs and the interaction probabilities. We also note that for a weak ESS to be an intermediate ESS, it is required that the condition $\Delta_1 > 0$ be satisfied. This condition cannot be always satisfied. As an example, we consider the following payoffs:

$$A = \begin{pmatrix} -1 & 8 \\ 0 & 4 \end{pmatrix}, D = \begin{pmatrix} -1 & 8 \\ 0 & 4 \end{pmatrix}, B = \begin{pmatrix} 0.5 & 8 \\ 0 & \frac{16}{3} \end{pmatrix}, C = \begin{pmatrix} -3 & 8 \\ 0 & \frac{8}{3} \end{pmatrix}.$$

When $p_1 = 0.9$ and $p_2 = 0.35$, there exists a unique fully mixed equilibrium given by $s_1^* = 0.85$ and $s_2^* = 0.14$. The conditions $L_1 < 0$ and $L_2 < 0$ are satisfied, therefore $\mathbf{s}^* = (s_1^*, s_2^*)$ is a weak ESS. However, Δ_1 is strictly negative and consequently \mathbf{s}^* is

not an intermediate ESS. When $p_1 = 0.75$ and $p_2 = 0.6$, then for the same values of payoffs we have $\mathbf{s}^* = (0.9, 0.51)$ and $\Delta_1 > 0$. Therefore, \mathbf{s}^* is a weak and intermediate ESS. In addition, \mathbf{s}^* cannot be a Cressman ESS because neither community would resist invasion from a small fraction of mutants composed of all the communities.

In this paragraph, we examine two limit cases where (i) $p_1 = p_2 = 1$, and (ii) $p_1 = p_2 = 0$. When $p_1 = p_2 = 1$, the two communities are completely independent, $s_1^* = -\frac{b_1 - d_1}{L_1}$, $s_2^* = -\frac{b_2 - d_2}{L_2}$, and $\Delta_1 = 4L_1L_2$. We find the classical case of a single community: if $L_1 < 0$ and $L_2 < 0$, then $\mathbf{s}^* = (s_1^*, s_2^*)$ is an evolutionarily stable strategy. When $p_1 = p_2 = 0$, the evolutionary game is completely asymmetric, $s_1^* = -\frac{b_{21} - d_{21}}{L_{21}}$, $s_2^* = -\frac{b_{12} - d_{12}}{L_{12}}$, and $\Delta_1 = -(L_{12} + L_{21})^2$ which is strictly negative. Therefore, $\mathbf{s}^* = (s_1^*, s_2^*)$ is neither an intermediate nor a weak ESS. In [90, 27], the authors show that no mixed evolutionarily stable strategy can exist in asymmetric games. In [27], the authors introduced the notion of Nash-Pareto pairs in asymmetric games, which is an equilibrium characterized by the concept of Pareto optimality: it is not possible for players from both communities to simultaneously profit from a deviation from the equilibrium. The mixed equilibrium \mathbf{s}^* is a Nash-Pareto pair if $L_{12}L_{21} < 0$ [27].

5.3.4 Partially Mixed ESS

We study in this section the existence of partially mixed ESS, that is, an ESS which is pure in one community and mixed in the other. More precisely, a partially mixed ESS $\mathbf{s}^* = (s_1^*, s_2^*)$ is characterized by $s_1^* = 0$ or $s_1^* = 1$ and $0 < s_2^* < 1$ (or the inverse). In the next proposition, we derive the conditions of existence of partially mixed ESSs.

Proposition 22. • $\mathbf{s}^* = (1, s_2^*)$ where $s_2^* = -\frac{(1-p_2)L_{21} + K_2}{p_2L_2}$ is a weak ESS if $p_1L_1 + (1-p_1)L_{12}s_2^* + K_1 > 0$ and $L_2 < 0$; or if $p_1L_1 + (1-p_1)L_{12}s_2^* + K_1 = 0$, $L_1 < 0$, and $L_2 < 0$.

- $\mathbf{s}^* = (1, s_2^*)$ is an intermediate ESS if it is weak and either $p_1L_1 + (1-p_1)L_{12}s_2^* + K_1 > 0$ or $\Delta_1 > 0$.
- $\mathbf{s}^* = (1, s_2^*)$ is a strong ESS if it is an intermediate ESS and if $p_1L_1 + (1-p_1)L_{12}s_2^* + K_1 > 0$.
- $\mathbf{s}^* = (0, s_2^*)$ where $s_2^* = -\frac{K_2}{p_2L_2}$ is a weak ESS if $(1-p_1)s_2^*L_{12} + K_1 < 0$ and $L_2 < 0$, or if $(1-p_1)s_2^*L_{12} + K_1 = 0$, $L_1 < 0$, and $L_2 < 0$.
- $\mathbf{s}^* = (0, s_2^*)$ \mathbf{s}^* is an intermediate ESS if it is a weak ESS and either $(1-p_1)s_2^*L_{12} + K_1 < 0$ or $\Delta_1 > 0$.

- $\mathbf{s}^* = (0, s_2^*)$ is a strong ESS if it is an intermediate ESS and if $(1 - p_1)s_2^*L_{12} + K_1 < 0$.
- $\mathbf{s}^* = (s_1^*, 1)$ where $s_1^* = -\frac{(1 - p_1)L_{12} + K_1}{p_1L_1}$ is a weak ESS if $p_2L_2 + (1 - p_2)L_{21}s_1^* + K_2 > 0$ and $L_1 < 0$; or if $p_2L_2 + (1 - p_2)L_{21}s_1^* + K_2 = 0$, $L_1 < 0$ and $L_2 < 0$.
- $\mathbf{s}^* = (s_1^*, 1)$ is an intermediate ESS if it is a weak ESS and either $p_2L_2 + (1 - p_2)L_{21}s_1^* + K_2 > 0$ or $\Delta_1 > 0$.
- $\mathbf{s}^* = (s_1^*, 1)$ is a strong ESS if it is an intermediate ESS and $p_2L_2 + (1 - p_2)L_{21}s_1^* + K_2 > 0$.
- $\mathbf{s}^* = (s_1^*, 0)$ with $s_1^* = -\frac{K_1}{p_1L_1}$ is a weak ESS if $(1 - p_2)L_{21}s_1^* + K_2 < 0$ and $L_1 < 0$ or if $(1 - p_2)L_{21}s_1^* + K_2 = 0$, $L_1 < 0$ and $L_2 < 0$.
- $\mathbf{s}^* = (s_1^*, 0)$ is an intermediate ESS if it is a weak ESS and either $(1 - p_2)L_{21}s_1^* + K_2 < 0$ or $\Delta_1 > 0$.
- $\mathbf{s}^* = (s_1^*, 0)$ is a strong ESS if it is an intermediate ESS and $(1 - p_2)L_{21}s_1^* + K_2 < 0$.

Proof. • $\mathbf{s}^* = (1, s_2^*)$ where $s_2^* = -\frac{(1 - p_2)L_{21} + K_2}{p_2L_2}$ is a weak ESS if either:

- $\bar{U}_1(s_1, \mathbf{s}^*, p) < U_1(G_1, \mathbf{s}^*, p)$; $\bar{U}_2(s_2, \mathbf{s}^*, p) = \bar{U}_2(s_2^*, \mathbf{s}^*, p)$, and $\bar{U}_2(s_2, (1, s_2), p) < \bar{U}_2(s_2^*, (1, s_2), p)$ for all $\mathbf{s} \neq \mathbf{s}^*$ (since the equilibrium is mixed in the second community); or,
- if $\bar{U}_1(s_1, \mathbf{s}^*, p) = U_1(G_1, \mathbf{s}^*, p)$ and $\bar{U}_1(s_1, (s_1, s_2^*), p) < U_1(G_1, (s_1, s_2^*), p)$ and $\bar{U}_2(s_2, \mathbf{s}^*, p) = \bar{U}_2(s_2^*, \mathbf{s}^*, p)$, and $\bar{U}_2(s_2, (1, s_2), p) < \bar{U}_2(s_2^*, (1, s_2), p)$.

The first set of conditions yields $p_1L_1 + (1 - p_1)L_{12}s_2^* + K_1 > 0$ and $L_2 < 0$.
 The second set of conditions yields $p_1L_1 + (1 - p_1)L_{12}s_2^* + K_1 = 0$, $L_1 < 0$, and $L_2 < 0$.

- $\mathbf{s}^* = (1, s_2^*)$ where $s_2^* = -\frac{(1 - p_2)L_{21} + K_2}{p_2L_2}$ is an intermediate ESS if:
 - $\bar{U}_1(s_1, (1, s_2^*), p) + \bar{U}_1(s_2, 1, s_2^*), p) \leq U_1(G_1, (1, s_2^*), p) + \bar{U}_2(s_2^*, (1, s_2^*), p)$.
 - If there exists \mathbf{s} for which the above condition is an equality, then $\bar{U}_1(s_1, \mathbf{s}, p) + \bar{U}_2(s_2, \mathbf{s}, p) < \bar{U}_1(s_1^*, \mathbf{s}, p) + \bar{U}_2(s_2^*, \mathbf{s}, p)$.

We conclude that the conditions of existence of the intermediate ESS are either $L_2 < 0$ and $p_1L_1 + (1 - p_1)L_{12}s_2^* + K_1 > 0$, or $p_1L_1 + (1 - p_1)L_{12}s_2^* + K_1 = 0$, $L_1 < 0$, $L_2 < 0$ and $\Delta_1 > 0$. Therefore, \mathbf{s}^* is an intermediate ESS if it is a weak ESS and either $p_1L_1 + (1 - p_1)L_{12}s_2^* + K_1 > 0$ or $\Delta_1 > 0$.

Table 5.2: Types of ESSs.

	Stability levels		
	Strong	Intermediate	Weak
Fully Pure	Proposition 20	Proposition 20	Proposition 20
Partially Mixed	Proposition 22	Proposition 22	Proposition 22
Fully Mixed	Never	Proposition 21	Proposition 21

- Finally, $\mathbf{s}^* = (1, s_2^*)$ is a partially-mixed strong ESS if,

$$\begin{aligned} \bar{U}_1(s_1, (1, s_2^*), p) &< \bar{U}_1(G_1, (1, s_2^*), p), \\ \bar{U}_2(s_2, (1, s_2^*), p) &= \bar{U}_2(s_2^*, (1, s_2^*), p) \text{ and } \bar{U}_2(s_2, (1, s_2), p) < \bar{U}_2(s_2^*, (1, s_2), p). \end{aligned}$$

Or if,

$$\begin{aligned} \bar{U}_1(s_1, (1, s_2^*), p) &= \bar{U}_1(G_1, (1, s_2^*), p) \text{ and } \bar{U}_1(s_1, (s_1, s_2), p) < \bar{U}_1(G_1, (s_1, s_2), p), \\ \bar{U}_2(s_2, (s_1, s_2), p) &< \bar{U}_2(s_2^*, (s_1, s_2), p). \end{aligned}$$

The first set of conditions yield $p_1 L_1 + (1 - p_1) L_{12} s_2^* + K_1 > 0$ and $L_2 < 0$. The second set of conditions cannot be satisfied (saddle point). We conclude that \mathbf{s}^* is a strong ESS if its an intermediate ESS and if $p_1 L_1 + (1 - p_1) L_{12} s_2^* + K_1 > 0$. In the same way, we can examine the existence of other partially-mixed ESS.

□

We summarize in Table 5.2 the different ESSs defined in our model and the propositions wherein defined.

5.4 Conclusion

In this chapter, we considered an heterogeneous population composed of communities which interact in a nonuniform manner. We believe that our work is the first to consider nonuniform interactions between communities. In this scenario, we established a complete static analysis. In particular, we defined three ESSs with different levels of stability against mutations and we examined their connection to each other: strong ESS which resists invasion from any sufficiently small fraction of mutants composed of all the communities, a weak ESS, where each community resists invasion from any sufficiently small and local group of mutants, and intermediate ESS which resists invasion from any sufficiently small group of mutants composed of agents from all the communities, when we consider as a fitness function the sum of utilities over all the communities (a single fitness is considered). We proved that a strong ESS is an intermediate ESS,

which is also a weak ESS, and the converse is not true. We gave counterexamples that corroborate this claim.

In the case of two communities, we succeeded to have closed-form results and we derived the conditions of existence of the ESSs in function of the interaction probabilities and the payoffs. We showed that no fully mixed strong ESS can exist and we derived the conditions of existence of all other types of ESS.

In the next chapter, we propose to extend our analysis by considering the replicator dynamics in the context of interacting communities.

Chapter 6

Replicator Dynamics in Interacting Communities

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In this chapter, we extend the classic replicator dynamics to the context of interacting groups or communities. Furthermore, we introduce two types of delays: *spatial* delays associated to the types of interactions and arise only in interactions between individuals from different communities, *strategic* delays associated to strategies, and *spatial strategic* delays which are a combination of the two types of delays. Spatial delays appear in mixed interactions only, that is interactions that involve individuals from different communities. For example, in a social network, individuals from the same community will share faster some content as there is some kind of confidence between them, whereas, a content coming from an individual from another community may yield to a careful behavior and then increases the outcome delay of the interaction. Strategic delay is the time interval between the moment that an individual uses a strategy and the moment it feels the impact of its strategy. In this work, we aim to study the effects of these delays on the stability of the replicator dynamics. We will show that the two different types of delays yield different expected utilities and consequences on the stability of the replicator dynamics.

The present chapter is structured as follows:

- First, in Section 6.1, we study the replicator dynamics without delays and we examine the connection between the ESSs and the stationary points of the replicator dynamics.
- In Section 6.2, we examine the stability of the replicator dynamics subject to strategic delays.
- In Section 6.3, we investigate the stability of the replicator dynamics in presence of spatial delays.
- Finally, we study in Section 6.4 the stability of the replicator dynamics when both types of delays are present.

6.1 Replicator Dynamics without Delay

In this section, we introduce the replicator dynamics which describe the evolution of the various strategies in the communities. In this dynamics, the proportion of a given strategy in a community grows at a rate equal to the difference between the expected payoff of that strategy and the average payoff in the considered community.

Let $s_i(t)$ denote the proportion of agents using strategy G_i (the first strategy) in community i and $1 - s_i(t)$ denotes the proportion of strategy H_i . The scalar $s_i(t)$ determines the profile or state of community i . The evolution of the state of the full population is then governed by a system of two ordinary differential equations (ODE) whose variables

are the scalars s_1 and s_2 . The replicator dynamics writes, for $i = 1, 2$:

$$\begin{aligned}\dot{s}_i(t) &= s_i(t) [U_i(G_i, \mathbf{s}(t), p) - \bar{U}_i(s_i(t), \mathbf{s}(t), p)], \\ &= s_i(t)(1 - s_i(t)) [U_i(G_i, \mathbf{s}(t), p) - U_i(H_i, \mathbf{s}(t), p)],\end{aligned}$$

where $\mathbf{s}(t) = (s_1(t), s_2(t))$, which yields the following pair of nonlinear ordinary differential equations:

$$\begin{aligned}\dot{s}_1(t) &= s_1(t)(1 - s_1(t)) [p_1 s_1(t) L_1 + (1 - p_1) s_2(t) L_{12} + K_1], \\ \dot{s}_2(t) &= s_2(t)(1 - s_2(t)) [p_2 s_2(t) L_2 + (1 - p_2) s_1(t) L_{21} + K_2].\end{aligned}\quad (6.1)$$

Note that s_1 and s_2 vary in the interval $[0, 1]$ and the domain of study is $[0, 1] \times [0, 1]$. In the system (6.1), there are nine stationary points at which both $\dot{s}_1 = 0$ and $\dot{s}_2 = 0$ simultaneously, which are: $(0, 0)$, $(1, 1)$, $(0, 1)$, $(1, 0)$, $(0, -\frac{K_2}{p_2 L_2})$, $(-\frac{K_1}{p_1 L_1}, 0)$, $(1, -\frac{(1 - p_2)L_{21} + K_2}{p_2 L_2})$, $(-\frac{(1 - p_1)L_{12} + K_1}{p_1 L_1}, 1)$, and the interior point \mathbf{s}^* defined in Proposition 21. Recall that the strict Nash equilibrium is locally asymptotically stable in the replicator dynamics and the stationary points are Nash equilibria. The asymptotic stability of corner and border stationary points are given in the Appendix. The dynamic property of \mathbf{s}^* is brought out in the next proposition.

Proposition 23. *The interior stationary point \mathbf{s}^* is asymptotically stable in the replicator dynamics if $L_1 < 0$, $L_2 < 0$, and $\Delta > 0$.*

- \mathbf{s}^* is unstable if $\Delta = p_1 p_2 L_1 L_2 - (1 - p_1)(1 - p_2)L_{12}L_{21} < 0$.

Proof. In order to examine the stability of the interior stationary point, we make a linearization of the system (6.1) around \mathbf{s}^* and observe how the linearized system behaves. We introduce a small perturbation around \mathbf{s}^* defined by $x_1(t) = s_1(t) - s_1^*$ and $x_2(t) = s_2(t) - s_2^*$. The replicator dynamics is then given by:

$$\begin{aligned}\dot{x}_1(t) &= (x_1(t) + s_1^*)(1 - x_1(t) - s_1^*)(p_1 x_1(t) L_1 + (1 - p_1) x_2(t) L_{12}), \\ \dot{x}_2(t) &= (x_2(t) + s_2^*)(1 - x_2(t) - s_2^*)(p_2 x_2(t) L_2 + (1 - p_2) x_1(t) L_{21}).\end{aligned}$$

Keeping only linear terms in x_1 and x_2 , we obtain a linearized system of the form $\dot{x}(t) = Ax(t)$ where $x^t = (x_1, x_2)$,

$$A = \begin{pmatrix} \gamma_1 p_1 L_1 & \gamma_1 (1 - p_1) L_{12} \\ \gamma_2 (1 - p_2) L_{21} & \gamma_2 p_2 L_2 \end{pmatrix},$$

$\gamma_1 = s_1^*(1 - s_1^*)$, and $\gamma_2 = s_2^*(1 - s_2^*)$. The linearized system is asymptotically stable if all the eigenvalues of A have negative real parts. The eigenvalues of A are the roots of the characteristic polynomial $\mathcal{X}_A = \lambda^2 - \text{tr}(A)\lambda + \det(A)$, with $\text{tr}(A) = \gamma_1 p_1 L_1 + \gamma_2 p_2 L_2$ and $\det(A) = \gamma_1 \gamma_2 (p_1 p_2 L_1 L_2 - (1 - p_1)(1 - p_2)L_{12}L_{21})$. We check that if $\Delta = p_1 p_2 L_1 L_2 - (1 - p_1)(1 - p_2)L_{12}L_{21} > 0$, $L_1 < 0$ and $L_2 < 0$, then the two eigenvalues of A have negative real parts and the stability follows. \square

Table 6.1: The fully mixed ESSs.

ESS type	Conditions of existence	Asymptotic stability in the RD
Strong ESS	does not exist	cannot be defined
Weak ESS	$L_1 < 0$ and $L_2 < 0$	asymptotically stable if $\Delta > 0$ and unstable otherwise
Intermediate ESS	$L_1 < 0$, $L_2 < 0$, and $\Delta_1 > 0$	asymptotically stable

The next corollary about the asymptotic stability of the fully mixed intermediate ESS follows.

Corollary 4. *The fully mixed intermediate ESS is locally asymptotically stable in the replicator dynamics.*

Proof. We aim to prove that the mixed intermediate ESS is asymptotically stable in the replicator dynamics. By virtue of Proposition 21, the interior equilibrium s^* is an intermediate ESS if $L_1 < 0$, $L_2 < 0$ and $\Delta_1 > 0$. In addition, from Proposition 23, s^* is asymptotically stable if $L_1 < 0$, $L_2 < 0$, and $\Delta > 0$. We can then prove that, if $\Delta_1 = 4p_1p_2L_1L_2 - ((1-p_1)L_{12} + (1-p_2)L_{21})^2 > 0$, then $\Delta = p_1p_2L_1L_2 - (1-p_1)(1-p_2)L_{12}L_{21} > 0$ (or equivalently $4\Delta > 0$). We have:

$$4p_1p_2L_1L_2 - ((1-p_1)L_{12} + (1-p_2)L_{21})^2 - 4(p_1p_2L_1L_2 - (1-p_1)(1-p_2)L_{12}L_{21}) = -((1-p_1)L_{12} - (1-p_2)L_{21})^2 < 0.$$

The proof follows. □

This result confirms that the ESS with the intermediate level of stability is asymptotically stable in the replicator dynamics. In contrast, for the weak ESS to be asymptotically stable, it is required that the condition $\Delta > 0$, which depends on the payoff matrices and the interaction probabilities, be satisfied. We can summarize results about the fully mixed ESSs and their asymptotic stability in the replicator dynamics in Table 6.1.

Remark 1. *We consider the numerical example in Section 5.3.3. For $p_1 = 0.9$ and $p_2 = 0.35$, there exists a unique fully mixed equilibrium given by $s^* = (0.85, 0.14)$ and it is not an intermediate ESS because we have $\Delta_1 < 0$ (Proposition 21). However, s^* is asymptotically stable in the replicator dynamics (conditions in Theorem 23 are satisfied). Therefore, an asymptotically stable point in the replicator dynamics is not necessarily an intermediate ESS.*

We display in Fig. 6.1 a numerical solution of the replicator dynamics, where the interaction probabilities are fixed to $p_1 = 0.4$, $p_2 = 0.6$, and the payoffs are set to $a_1 = -1.25$, $b_1 = 1.5$, $c_1 = 0$, $d_1 = 0.75$, $a_2 = -1.25$, $b_2 = 1.5$, $c_2 = 0$, $d_2 = 0.75$, $a_{12} = -0.25$, $b_{12} = 1.5$, $c_{12} = 0$, $d_{12} = 1.05$, $a_{21} = -1.75$, $b_{21} = 1.5$, $c_{21} = 0$, $d_{21} = 0.45$. In this configuration, the fully mixed intermediate ESS exists and is given by $s^* =$

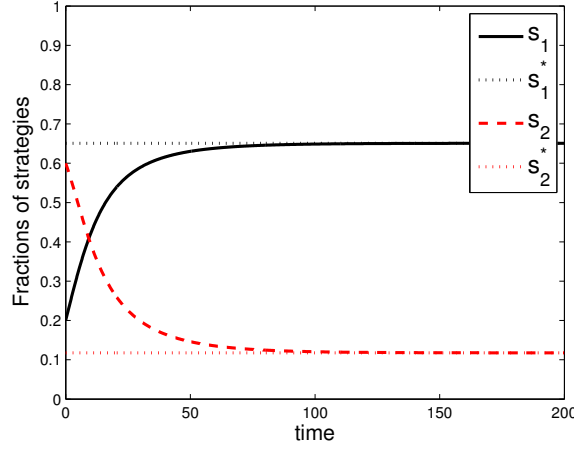


Figure 6.1: Convergence of the replicator dynamics to the fully mixed intermediate ESS (s_1^*, s_2^*) , where $a_1 = -1.25$, $b_1 = 1.5$, $c_1 = 0$, $d_1 = 0.75$, $a_2 = -1.25$, $b_2 = 1.5$, $c_2 = 0$, $d_2 = 0.75$, $a_{12} = -0.25$, $b_{12} = 1.5$, $c_{12} = 0$, $d_{12} = 1.05$, $a_{21} = -1.75$, $b_{21} = 1.5$, $c_{21} = 0$, $d_{21} = 0.45$, $s_1^* = 0.65$, $s_2^* = 0.12$.

$(0.65, 0.12)$, that is, at the intermediate evolutionarily stability, almost 65% of the first community uses the first strategy and 35% uses the second strategy, and for the second community, 12% uses the first strategy and 88% uses the second strategy. The initial population profile is arbitrarily chosen and is fixed to $\mathbf{s} = (0.2, 0.6)$. As predicted in Corollary 4, the replicator dynamics converges to \mathbf{s}^* .

In the next proposition, we study the asymptotic stability of partially mixed ESSs.

Proposition 24. *The partially-mixed strong ESSs are asymptotically stable in the replicator dynamics.*

Proof. Let us prove that the partially mixed strong ESS $\mathbf{s}^* = (1, s_2^*)$ with $s_2^* = \frac{(1-p_2)L_{21} + K_2}{p_2L_2}$ is asymptotically stable. From the section above, The Jacobian matrix obtained by linearisation of the replicator dynamics around \mathbf{s}^* is given by:

$$A = \begin{pmatrix} -p_1L_1 - (1-p_1)s_2^*L_{12} - K_1 & 0 \\ \gamma_2(1-p_2)L_{21} & \gamma_2p_2L_2 \end{pmatrix},$$

with $\gamma_2 = s_2^*(1-s_2^*)$. The eigenvalues of the triangular Jacobian matrix are the diagonal terms. Therefore, \mathbf{s}^* is asymptotically stable if $p_1L_1 + (1-p_1)s_2^*L_{12} + K_1 > 0$ and $L_2 < 0$. By virtue of Proposition 22, the strong ESS \mathbf{s}^* is asymptotically stable. Similarly, we can prove this result for all other partially mixed and fully pure strong ESSs. \square

This proposition establishes that all partially-mixed strong ESSs are asymptotically stable in the replicator dynamics. In the next section, we examine the stability of the

fully mixed intermediate ESS in the replicator dynamics with delays.

6.2 Replicator Dynamics with Strategic Delay

In this section, we examine the impact of time delays of strategies on the dynamics. An action taken today will have some effect after some time [6]. We assume the strategies take a delay τ_{st} . An equivalent interpretation of delays is as follows: the players repeatedly revise their strategies and imitate the strategies with higher payoffs. However, the agents have a delayed estimate of payoffs. Therefore, the utility of a given strategy in community i at instant t is a function of the delayed profile of the full population, i.e. $\mathbf{s}(t - \tau_{st}) = (s_1(t - \tau_{st}), s_2(t - \tau_{st}))$. The delayed replicator dynamics for the first community is then given by:

$$\dot{s}_1(t) = s_1(t)(1 - s_1(t)) [U_1(G_1, \mathbf{s}(t - \tau_{st}), p) - U_1(H_1, \mathbf{s}(t - \tau_{st}), p)].$$

Then, we get:

$$\dot{s}_1(t) = s_1(t)(1 - s_1(t)) [p_1 L_1 s_1(t - \tau_{st}) + (1 - p_1) L_{12} s_2(t - \tau_{st}) + K_1].$$

Doing the same with the second community, we get:

$$\dot{s}_2(t) = s_2(t)(1 - s_2(t)) [p_2 L_2 s_2(t - \tau_{st}) + (1 - p_2) L_{21} s_1(t - \tau_{st}) + K_2].$$

As a remark, we note that at the equilibrium of the delayed replicator dynamics, the stationary solutions satisfy $\dot{s}_1(t) = 0$, $\dot{s}_2(t) = 0$, $s_1^*(t) = s_1^*(t - \tau_{st})$ and $s_2^*(t) = s_2^*(t - \tau_{st})$. Indeed at the equilibrium, the values of s_1 and s_2 do not change in time, and therefore the delayed and nondelayed solutions are equal. We conclude that the solutions of the delayed replicator dynamics coincide with the solutions of the non-delayed replicator dynamics.

To examine the local stability of the fully mixed equilibrium, we introduce a small perturbation around \mathbf{s}^* defined by $x_1(t) = s_1(t) - s_1^*$ and $x_2(t) = s_2(t) - s_2^*$. We then make a linearization of the two previous equations around the interior equilibrium point $\mathbf{s}^* = (s_1^*, s_2^*)$ and we study the linearized system. We get the following system:

$$\begin{aligned} \dot{x}_1(t) &= \gamma_1 (p_1 L_1 x_1(t - \tau_{st}) + (1 - p_1) L_{12} x_2(t - \tau_{st})), \\ \dot{x}_2(t) &= \gamma_2 (p_2 L_2 x_2(t - \tau_{st}) + (1 - p_2) L_{21} x_1(t - \tau_{st})), \end{aligned}$$

with $\gamma_1 = s_1^*(1 - s_1^*)$ and $\gamma_2 = s_2^*(1 - s_2^*)$. Taking the Laplace transform of the system above, we obtain the following characteristic equation:

$$\lambda^2 - \lambda [p_1 \gamma_1 L_1 + p_2 \gamma_2 L_2] e^{-\lambda \tau_{st}} + \gamma_1 \gamma_2 [p_1 p_2 L_1 L_2 - (1 - p_1)(1 - p_2) L_{12} L_{21}] e^{-2\lambda \tau_{st}} = 0. \quad (6.2)$$

The zero solution is asymptotically stable if all roots of (6.2) have negative real parts [96]. Eq. (6.2) is typical for a linear system of two equations of the form $\dot{x}(t) = Ax(t - \tau_{st})$ which was studied by the authors in [109, 110]. Based on their results, we establish the following proposition on the asymptotic stability of the intermediate ESS in presence of symmetric strategic delays.

Proposition 25. *The fully mixed intermediate ESS is asymptotically stable in the delayed replicator dynamics if $\tau_{st} < \bar{\tau}_{st} = \min(\frac{\pi}{2|\lambda_+|}, \frac{\pi}{2|\lambda_-|})$, with $\lambda_{\pm} = \frac{p_1\gamma_1 L_1 + p_2\gamma_2 L_2 \pm \sqrt{D}}{2}$, and $D = [p_1\gamma_1 L_1 + p_2\gamma_2 L_2]^2 - 4\gamma_1\gamma_2 [p_1 p_2 L_1 L_2 - (1 - p_1)(1 - p_2)L_{12}L_{21}]$.*

Proof. We showed in the proof of Theorem 23, that the eigenvalues of A , which are solutions of $\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$ have negative real parts when $\Delta = p_1 p_2 L_1 L_2 - (1 - p_1)(1 - p_2)L_{12}L_{21} > 0$, $L_1 < 0$ and $L_2 < 0$; and the mixed intermediate ESS is asymptotically stable when $\tau_{st} = 0$ (Corollary 4). For the remaining of the proof that gives the bound on τ_{st} for which the stability is unaffected, the reader should refer to [109], pp.82, Theorem 3.4. \square

When there is a single population, we check that $\bar{\tau}_{st}$ given in the theorem above coincides with the critical value of delay given in [16, 86].

At the critical delay $\bar{\tau}_{st}$, a transition from stability to instability occurs in the replicator dynamics and a periodic oscillation is created with a frequency given by $w = \frac{\pi}{2\bar{\tau}_{st}}$. The frequency of oscillations is determined from the characteristic equation, and it equals the imaginary part of the pure imaginary root of (6.2). We clearly observe that $\bar{\tau}_{st}$ and w are functions of the payoffs and the interaction probabilities.

Proposition 25 gives an upper bound on strategic delays for which the intermediate ESS remains asymptotically stable in the population. Beyond this delay bound, the stability is lost and persistent oscillations around the ESS occur. In Fig. 6.2, we illustrate the consequences that strategic delays may have on the convergence to the equilibrium. We fixed the value of the strategic delay to 4.2 time units, and the critical delay value is given by 3.5 time units. The replicator dynamics exhibit permanent oscillations around the equilibrium in both communities. The strategic delay exceeds the critical value, and consequently the asymptotic stability is lost.

6.3 Replicator Dynamics with Spatial Delay

In this section, we assume that the delays are not associated with the strategy used by an individual but rather with the types of the interacting players. Spatial delays, which we denote by τ_{sp} , arise when two individuals from different communities get involved in an

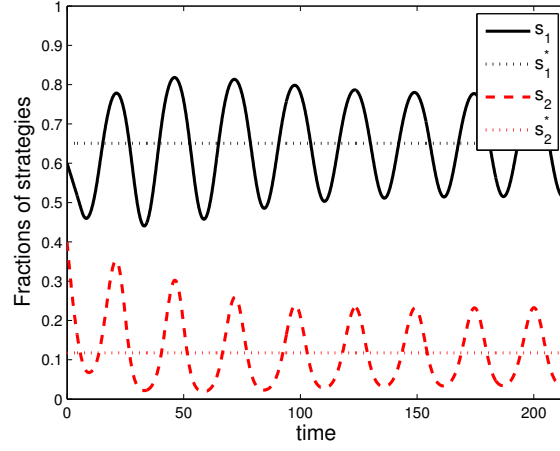


Figure 6.2: Oscillations around the fully mixed intermediate ESS (s_1^*, s_2^*) in presence of strategic delays, where $a_1 = -1.25$, $b_1 = 1.5$, $c_1 = 0$, $d_1 = 0.75$, $a_2 = -1.25$, $b_2 = 1.5$, $c_2 = 0$, $d_2 = 0.75$, $a_{12} = -0.25$, $b_{12} = 1.5$, $c_{12} = 0$, $d_{12} = 1.05$, $a_{21} = -1.75$, $b_{21} = 1.5$, $c_{21} = 0$, $d_{21} = 0.45$, $s_1^* = 0.65$, $s_2^* = 0.12$, $\bar{\tau}_{st} = 3.5$ time units.

interaction. In fact, in many real-world applications, we observe that similar individuals have precise and up-to-date information about each other, whereas individuals from different communities are more likely to have delayed information about each other. Thus, delays would appear in mixed interactions.

In this model, the expected utility of a strategy in community i at time t depends on the current profile of community i , i.e. $(s_i(t), (1 - s_i(t)))$, and on the delayed profile of the other community i.e. $(s_{-i}(t - \tau_{sp}), 1 - s_{-i}(t - \tau_{sp}))$. Therefore, we propose the following delayed fitness functions:

$$\begin{aligned} U_1(G_1, \mathbf{s}, p) &= p_1(s_1(t)a_1 + (1 - s_1(t))b_1) + (1 - p_1)(s_2(t - \tau_{sp})a_{12} + (1 - s_2(t - \tau_{sp}))b_{12}), \\ U_1(H_1, \mathbf{s}, p) &= p_1(s_1(t)c_1 + (1 - s_1(t))d_1) + (1 - p_1)(s_2(t - \tau_{sp})c_{12} + (1 - s_2(t - \tau_{sp}))d_{12}). \end{aligned}$$

Similarly, for the second community, we have:

$$\begin{aligned} U_2(G_2, \mathbf{s}, p) &= p_2(s_2(t)a_2 + (1 - s_2(t))b_2) + (1 - p_2)(s_1(t - \tau_{sp})a_{21} + (1 - s_1(t - \tau_{sp}))b_{21}), \\ U_2(H_2, \mathbf{s}, p) &= p_2(s_2(t)c_2 + (1 - s_2(t))d_2) + (1 - p_2)(s_1(t - \tau_{sp})c_{21} + (1 - s_1(t - \tau_{sp}))d_{21}). \end{aligned}$$

Consequently, the replicator dynamics with spatial delays is given by:

$$\begin{aligned} \dot{s}_1(t) &= s_1(t)(1 - s_1(t)) [U_1(G_1, (s_1(t), s_2(t - \tau_{sp})), p) - U_1(H_1, (s_1(t), s_2(t - \tau_{sp})), p)], \\ \dot{s}_2(t) &= s_2(t)(1 - s_2(t)) [U_2(G_2, (s_1(t - \tau_{sp}), s_2(t)), p) - U_2(H_2, (s_1(t - \tau_{sp}), s_2(t)), p)], \end{aligned}$$

which yields,

$$\dot{s}_1(t) = s_1(t)(1 - s_1(t)) [p_1 L_1 s_1(t) + (1 - p_1) L_{12} s_2(t - \tau_{sp}) + K_1], \quad (6.3)$$

$$\dot{s}_2(t) = s_2(t)(1 - s_2(t)) [p_2 L_2 s_2(t) + (1 - p_2) L_{21} s_1(t - \tau_{sp}) + K_2]. \quad (6.4)$$

Following the same procedure as in the previous sections, we get the following characteristic equation:

$$\lambda^2 - (p_1\gamma_1L_1 + p_2\gamma_2L_2)\lambda + \gamma_1\gamma_2p_1p_2L_1L_2 - \gamma_1\gamma_2(1-p_1)(1-p_2)L_{12}L_{21}e^{-2\lambda\tau_{sp}} = 0.$$

Or equivalently,

$$\lambda^2 + \alpha\lambda + \beta + \delta e^{-\lambda\tau} = 0, \quad (6.5)$$

where $\tau = 2\tau_{sp}$, $\alpha = -(p_1\gamma_1L_1 + p_2\gamma_2L_2)$, $\beta = \gamma_1\gamma_2p_1p_2L_1L_2$, $\delta = -\gamma_1\gamma_2(1-p_1)(1-p_2)L_{12}L_{21}$. Now, we summarize the stability property of the mixed ESS for the delayed replicator dynamics in the following theorem which is based on the results of the authors in [111] related to the location of the roots of the characteristic equation (6.5).

Proposition 26. *The fully mixed intermediate ESS is asymptotically stable in the replicator dynamics with spatial delays for any $\tau_{sp} \geq 0$.*

Proof. The proof of this theorem is based on that given by Freedman and Kuang [111] (Theorem 4.1, page 202), related to the location of roots of the characteristic equation (6.5), and stated as follows:

- If $\beta^2 < \delta^2$, \Rightarrow if \mathbf{s}^* is unstable for $\tau = 0$ then it is unstable for any $\tau \geq 0$; and if \mathbf{s}^* is stable at $\tau = 0$, then it remains stable for τ inferior than some $\tau_s \geq 0$. But, if \mathbf{s}^* is stable at $\tau = 0$, then $\Delta = p_1p_2L_1L_2 - (1-p_1)(1-p_2)L_{12}L_{21} > 0 \Rightarrow \beta^2 > \delta^2$. Therefore, this case is excluded.
- If $\beta^2 > \delta^2$, $2\beta - \alpha^2 > 0$, and $(2\beta - \alpha^2)^2 > 4(\beta^2 - \delta^2)$, then the stability of the stationary point can change a finite number of times at most as τ is increased, and eventually it becomes unstable. But,

$$\begin{aligned} 2\beta - \alpha^2 &= 2\gamma_1\gamma_2p_1p_2L_1L_2 - (p_1\gamma_1L_1 + p_2\gamma_2L_2)^2 \\ &= -p_1^2\gamma_1^2L_1^2 - p_2^2\gamma_2^2L_2^2 < 0. \end{aligned}$$

Therefore, this case is excluded in our model.

- Otherwise, (this is the only case when \mathbf{s}^* is stable at $\tau = 0$), the stability of the stationary point \mathbf{s}^* does not change for any $\tau \geq 0$. $\Rightarrow \mathbf{s}^*$ is asymptotically stable for any $\tau \geq 0$.

□

Remarkably, spatial delays do not affect the stability of the mixed ESS. Indeed, for any value of the delay τ_{sp} , the frequency of strategies in the population converges to the mixed intermediate ESS after some possible damped oscillations. In Fig. 6.3, we plot the trajectories of solutions of the replicator dynamics with the spatial delay fixed to 6 time units. As predicted analytically, the population profile converges to the mixed ESS.

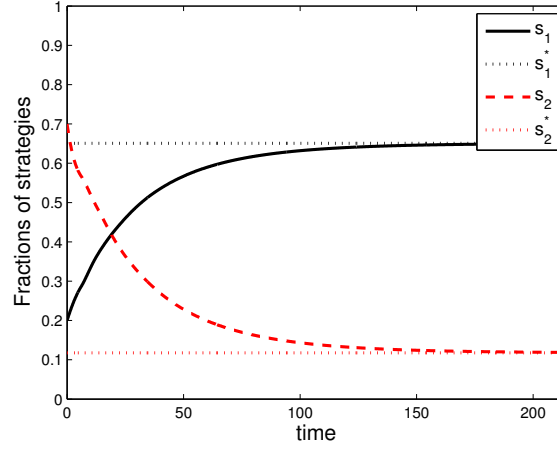


Figure 6.3: Convergence to the fully mixed intermediate ESS (s_1^*, s_2^*) in presence of spatial delays, where $a_1 = -1.25$, $b_1 = 1.5$, $c_1 = 0$, $d_1 = 0.75$, $a_2 = -1.25$, $b_2 = 1.5$, $c_2 = 0$, $d_2 = 0.75$, $a_{12} = -0.25$, $b_{12} = 1.5$, $c_{12} = 0$, $d_{12} = 1.05$, $a_{21} = -1.75$, $b_{21} = 1.5$, $c_{21} = 0$, $d_{21} = 0.45$, $s_1^* = 0.65$, $s_2^* = 0.12$.

6.4 Replicator Dynamics with Spatial Strategic Delays

In this section, we study the stability of the replicator dynamics with both strategic and spatial delays. In particular, we aim to study whether the spatial delay has a stabilizing effect on the replicator dynamics with strategic delay. We define the delays as follows:

- τ_{st} is the strategic delay, that is the delay associated with the strategies,
- τ_{sp} is the spatial delay associated with the inter-community interactions.

The expected payoffs of strategies G_1 and H_1 in community 1 then write:

$$U_1(G_1, (s_1(t - \tau_{st}), s_2(t - \tau_{st} - \tau_{sp})), p) = p_1 [s_1(t - \tau_{st})a_1 + (1 - s_1(t - \tau_{st}))b_1] + (1 - p_1) [s_2(t - \tau_{st} - \tau_{sp})a_{12} + (1 - s_2(t - \tau_{st} - \tau_{sp}))b_{12}].$$

And,

$$U_1(H_1, (s_1(t - \tau_{st}), s_2(t - \tau_{st} - \tau_{sp})), p) = p_1 [s_1(t - \tau_{st})c_1 + (1 - s_1(t - \tau_{st}))d_1] + (1 - p_1) [s_2(t - \tau_{st} - \tau_{sp})c_{12} + (1 - s_2(t - \tau_{st} - \tau_{sp}))d_{12}].$$

Hence, the equation governing the evolution of the proportion of players using strategy G_1 in the first community is given by:

$$\dot{s}_1(t) = s_1(t)(1 - s_1(t)) [p_1 L_1 s_1(t - \tau_{st}) + (1 - p_1) L_{12} s_2(t - \tau_{st} - \tau_{sp}) + K_1].$$

Doing the same with the second community, we obtain:

$$\dot{s}_2(t) = s_2(t)(1 - s_2(t)) [p_2 L_2 s_2(t - \tau_{st}) + (1 - p_2) L_{21} s_1(t - \tau_{st} - \tau_{sp}) + K_2].$$

Following the same procedure in the previous sections, we get the following characteristic equation with mixed delays:

$$\lambda^2 - \lambda[p_1\gamma_1L_1 + p_2\gamma_2L_2]e^{-\tau_{st}\lambda} + p_1p_2\gamma_1\gamma_2L_1L_2e^{-2\tau_{st}\lambda} - (1-p_1)(1-p_2)\gamma_1 \times \gamma_2L_{12}L_{21}e^{-2(\tau_{st}+\tau_{sp})\lambda} = 0. \quad (6.6)$$

When $\tau_{sp} = 0$, we find the characteristic equation (6.2) obtained when there is only a strategic delay. Eq. (6.6) can be solved numerically.

Eq. (6.6) can be simplified by making the assumption of small time delays. By substituting the exponential term with a Taylor series expansion and keeping only linear terms in τ_{st} and τ_{sp} in the equation above, we obtain the following second order equation:

$$(1 + A\tau_{st})\lambda^2 + \lambda(-A - 2B\tau_{st} + 2C(\tau_{st} + \tau_{sp})) + B - C = 0,$$

where $A = p_1\gamma_1L_1 + p_2\gamma_2L_2$, $B = p_1p_2\gamma_1\gamma_2L_1L_2$, and $C = (1-p_1)(1-p_2)\gamma_1\gamma_2L_{12}L_{21}$. We can establish the following proposition:

Proposition 27. *If $\tau_{st} < -\frac{1}{p_1\gamma_1L_1 + p_2\gamma_2L_2}$ and $\Delta\tau_{st} - C\tau_{sp} < -\frac{p_1\gamma_1L_1 + p_2\gamma_2L_2}{2}$, then the fully mixed intermediate ESS is asymptotically stable.*

Proof. The fully mixed intermediate ESS is asymptotically stable if all the roots of the characteristic equation have negative real parts. Since we have a second order equation, then the roots have negative real parts if their product is positive and their sum is negative, that is if (i) $\frac{B-C}{1+A\tau_{st}} = \frac{\Delta}{1+A\tau_{st}} > 0$ and (ii) $\frac{A+2B\tau_{st}-2C(\tau_{st}+\tau_{sp})}{1+A\tau_{st}} < 0$. By virtue of Corollary 4, we have $\Delta > 0$ and then condition (i) yields $\tau_{st} < -\frac{1}{p_1\gamma_1L_1 + p_2\gamma_2L_2}$; and condition (ii) yields $\Delta\tau_{st} - C\tau_{sp} < -\frac{p_1\gamma_1L_1 + p_2\gamma_2L_2}{2}$. \square

6.5 Conclusion

In this chapter, we extended the replicator dynamics to the context of two interacting communities. We examined the connection between the ESSs and the stationary points of the replicator dynamics and we proved that the mixed intermediate ESS is always asymptotically stable. In contrast, the weak ESS cannot be always asymptotically stable in the replicator dynamics, depending on the payoffs and the interaction probabilities.

Furthermore, we introduced two types of delays in the replicator dynamics: strategic delays and spatial delays. Strategic delays appear in all types of interactions and can be defined as the time interval a strategy takes to have a consequence on the fitness. Spatial delays appear in mixed interactions only, that is interactions between agents from different communities, and are caused by the heterogeneity of interactions. We

showed that spatial delays do not affect the stability of the fully mixed ESS whereas strategic delays may have destabilizing consequences.

In Chapter 7, we propose to illustrate these results with numerical examples inspired from the Hawk-Dove game.

Chapter 7

Applications

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In this chapter, we propose to apply the theoretical results previously obtained to the Hawk-Dove game. We notice that the Hawk-Dove game played between two groups of different levels of aggressiveness has been studied in [112] as well. However, the authors limited their analysis to the dynamical aspects of the game, that is the replicator dynamics. They did not examine the evolutionarily stable strategy. Moreover, they did not consider the nonuniform feature of interactions, neither considered the consequences of delays. In contrast, we propose here a complete static and dynamic analysis of the game.

This chapter is structured as follows:

- In Section 7.1, we study the classic Hawk-Dove game with random delays;
- In Section 7.2, we study the Hawk-Dove game in two communities with asymmetric levels of aggressiveness;
- In Section 7.3, we study the Hawk-Dove game on graphs;
- In Section 7.4, we conclude the chapter.

7.1 Hawk-Dove Game in a Single Population

7.1.1 Game Model

In the classical Hawk-Dove game [108, 71], two individuals compete for a scarce resource. A player may use a Hawk strategy (H) or a Dove strategy (D). The strategy H stands for an aggressive behavior that fights for the resource while the strategy D represents a peaceful behavior which never fights. The matrix that gives the outcome for such competition is given as follows:

$$\begin{array}{c} H \\ D \end{array} \begin{array}{cc} H & D \\ \left(\begin{array}{cc} \frac{V-C}{2} & V \\ 0 & \frac{V}{2} \end{array} \right), \end{array}$$

where $C > 0$ and $V > 0$. V represents the value of the resource for which the players compete and C represents the cost incurred by a hawk when fighting for the resource against a hawk. The coefficients of the payoff matrix can be interpreted as follows: If two doves meet, they share equally the resource and each one obtains as a payoff $\frac{V}{2}$. If two hawks meet, they fight until one of them gets injured and the other takes the whole resource. When a hawk and a dove meet, the dove withdraws and the hawk takes the whole resource. If $C < V$, then the strategy H is dominant and the entire population will

adopt the aggressive behavior. If $C > V$, there exists a mixed ESS given by $(\frac{V}{C}, 1 - \frac{V}{C})$, at which both behaviors coexist. Indeed, in this case the strategy H cannot be dominant since the cost of fighting is higher than the resource value. When a fully mixed ESS exists, the population can be:

- Polymorphic, in which case a proportion of the population equal to $\frac{V}{C}$ uses the Hawk strategy and the proportion $1 - \frac{V}{C}$ uses the Dove strategy;
- Monomorphic in which case each individual uses a Hawk strategy with probability $\frac{V}{C}$ and a Dove strategy with probability $1 - \frac{V}{C}$.

7.1.2 Replicator Dynamics with Random Delays

In this subsection, we apply some of the results obtained in Chapters 2 and 3 to the Hawk-Dove game. This part covers the replicator dynamics with random delays.

- **Uniform delay distribution:** When the delays are uniformly at random distributed in the interval $[0, \tau_{max}]$, we proved in Chapter 2, that the stability of the mixed ESS is lost at the critical value of τ_{max} given by $\frac{\pi^2}{2D}$ where $D = \delta s^*(1 - s^*)$. In the Hawk-Dove game, this critical value is given by $\pi^2 \frac{C}{V(C - V)}$, and so the critical value of the mean delay is $\pi^2 \frac{C}{2V(C - V)}$, and the critical frequency is given by $w_0 = \frac{V(C - V)}{\pi C}$. It is worth mentioning that, in the case of a fixed delay, the critical value of the delay is $\pi \frac{C}{V(C - V)}$ and the frequency of oscillations at the Hopf bifurcation is given by $\frac{V(C - V)}{2C}$.

Let $\rho = \frac{C}{V}$, which can be interpreted as the *cost per unit of benefit* [17]. The critical mean value of delay, τ_m can therefore be written in function of ρ as $\bar{\tau}_{max} = \pi^2 \frac{\rho}{2V(\rho - 1)}$, and the frequency of oscillations is given by $w_0 = \frac{V(\rho - 1)}{\pi \rho}$. We depict in the left subfigure of Fig. 7.1, the normalized critical value of the mean delay of the uniform distribution in function of ρ . We observe that the mean value of delays for which the stability of the mixed ESS is lost is larger than that in case of fixed delays. Therefore, random delays decrease the potential of instability. We also note that as the quotient $\frac{C}{V}$ increases, the critical mean delay decreases. In addition, we depict in the

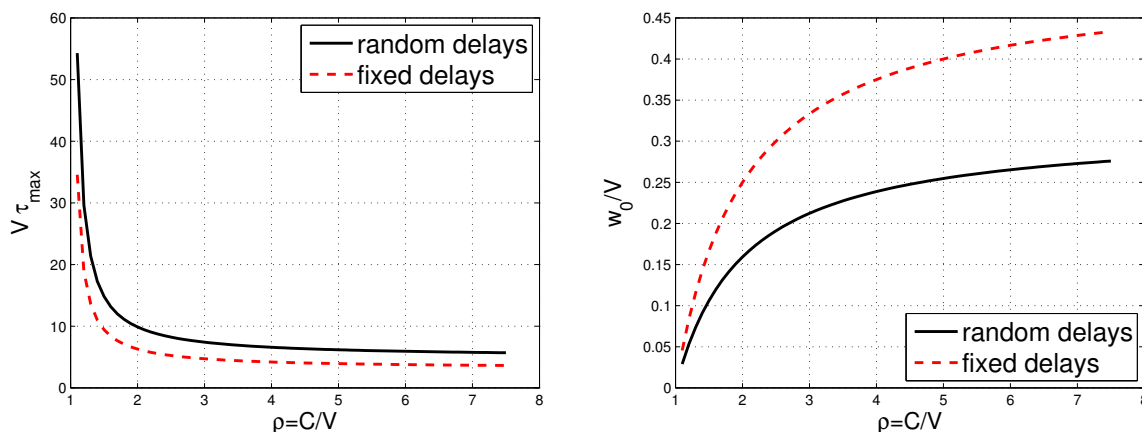


Figure 7.1: Left, normalized critical value of the mean delay $\tau_m V$ in the uniform delay case. Right, normalized frequency of oscillations w_0/V in the uniform delay case.

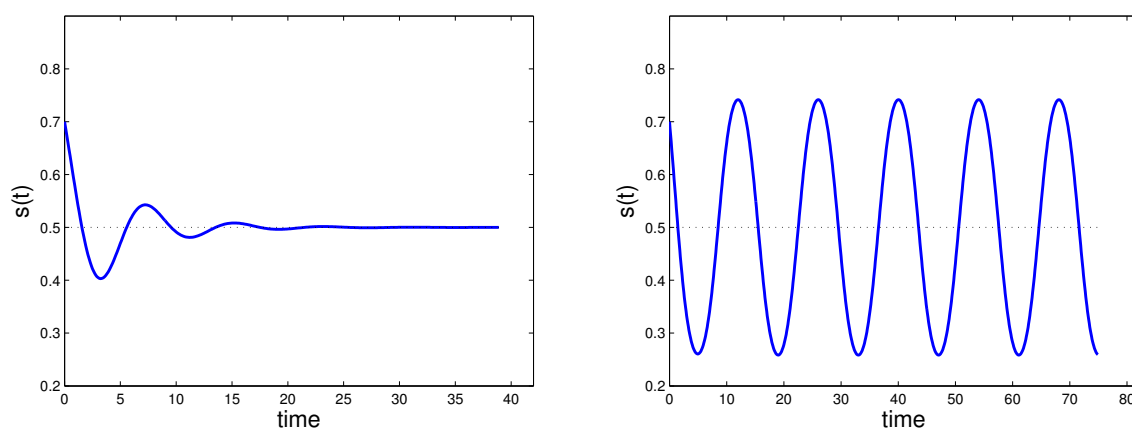


Figure 7.2: $V = 3$, $C = 6$. Left, the mean delay $= 1.5 < \tau_m = 3.25$ time units. Right, the mean delay $= 3.5 > \tau_m = 3.25$ time units (uniform delay).

right subfigure of Fig. 7.1, the normalized frequency of oscillations at the Hopf bifurcation. We clearly observe that the frequency of oscillations increases in function of ρ , which emphasizes the instability property. Also, the frequency of oscillations in the random delay case is clearly higher than that in the constant delay case. For example, when $\rho = 4$, the critical frequency of oscillations is equal to 0.24 in the uniform distribution case whereas this value increases to 0.38 in the constant delay case.

We illustrate in Fig. 7.2 the effects of increasing the mean delay on the convergence to the ESS. In the left subfigure, we considered a mean delay value smaller than the critical mean delay; whereas in the right subfigure, the mean delay is larger than the critical mean delay and persistent oscillations consequently appear.

- **Exponential delay distribution:** In the case of exponential delay distribution, we

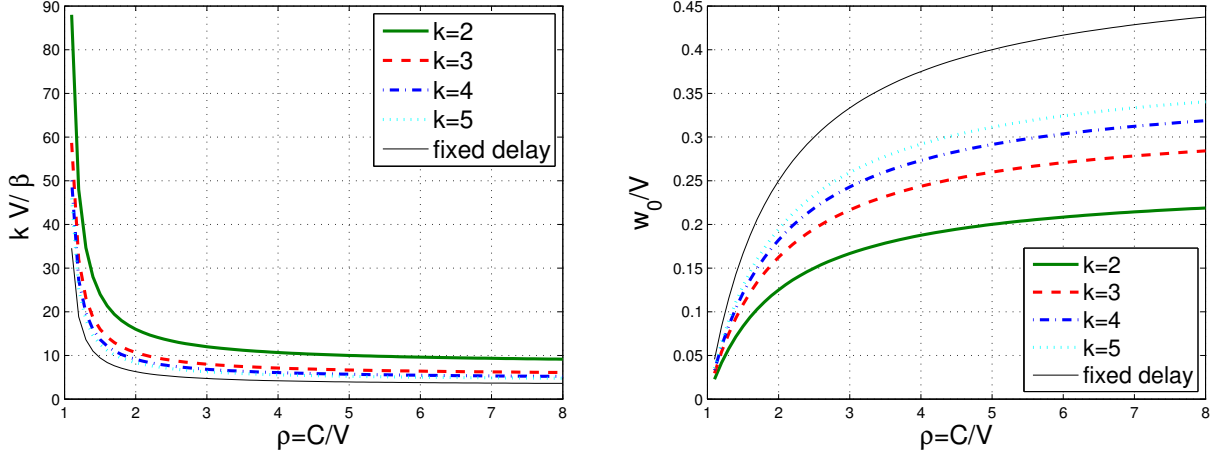


Figure 7.3: Left, the normalized critical value of the mean delay of the Gamma/Erlang distribution $\frac{kV}{\beta_c}$ in function of the rate parameter and ρ . Right, the normalized critical frequency $\frac{w_0}{V}$ in function of the rate parameter and ρ .

proved that the stability of the ESS cannot be lost, for any value of the mean delay, that is no Hopf bifurcation can occur in this model.

• **Erlang/Gamma delay distribution:** When the delays follow an Erlang or Gamma distribution, there exists a critical value of the mean delay at which the stability of the mixed ESS is lost. The critical value of the mean delay is given by $\frac{k}{\beta_c} = \frac{k}{D} \frac{\sin(\frac{\pi}{2k})}{\cos^{k+1}(\frac{\pi}{2k})}$ and the critical frequency of oscillations satisfies $w_0 = D \cos^k(\frac{\pi}{2k})$. Therefore, for the Hawk-Dove game, the critical mean value is given by:

$$\frac{k}{\beta_c} = \frac{2kC}{V(C-V)} \frac{\sin \frac{\pi}{2k}}{\cos^{k+1}(\frac{\pi}{2k})}, \quad (7.1)$$

which can be further written in function of $\rho = \frac{C}{V}$ as follows:

$$\frac{k}{\beta_c} = \frac{2k\rho}{V(\rho-1)} \frac{\sin \frac{\pi}{2k}}{\cos^{k+1}(\frac{\pi}{2k})}. \quad (7.2)$$

In addition, the critical frequency of oscillations is given by:

$$w_0 = V \frac{\rho-1}{2\rho} \cos^k(\frac{\pi}{2k}). \quad (7.3)$$

We depict in Fig. 7.3 the critical mean delay (left) and the critical frequency of oscillations (right) in function of the parameter ρ and for different values of k (parameter

of the Gamma distribution). As we can clearly see, the critical mean delay decreases as ρ grows, which means that the probability of instability increases as ρ gets higher. We recall that the asymptotic stability of the mixed ESS is lost when the value of the mean delay exceeds the critical mean delay value. Also, increasing the value of the parameter k results in the decrease of the critical mean delay, making the instability more probable. We could expect this result intuitively while examining the shape of the Gamma distribution. Finally, the critical mean delay is higher than that in the constant delay case. For instance, when $\rho = 3$, the critical mean delay value is 12 time units whereas this value reduces to 4.7 time units in the constant delay case.

• **One delayed strategy:** We consider here that there is only one delayed strategy that would take a delay τ with probability p or no delay with probability $1 - p$. This scenario was thoroughly studied in Chapter 3. The critical delay is given by:

$$\tau_{cr} = \frac{2}{V(C-V)} \sqrt{\left(\frac{C^3}{V - (1-2p)(C+V)}\right) \text{acos}\left(\frac{V-p(V+C)}{p(V+C)}\right)}.$$

The critical frequency is given by:

$$w_0 = \frac{V(C-V)}{2} \sqrt{\frac{1}{C^3}(V - (1-2p)(C+V))}.$$

Let $\rho = \frac{C}{V}$, we can write the normalized critical delay and frequency of oscillations near the Hopf bifurcation as:

$$\tau_{cr}V = \frac{2}{\rho-1} \sqrt{\frac{\rho^3}{1 - (1-2p)(1+\rho)}} \text{acos}\left(\frac{1 - (1-p)(\rho+1)}{p(\rho+1)}\right),$$

and,

$$\frac{w_0}{V} = \frac{\rho-1}{2\rho} \sqrt{\frac{1 - (1-2p)(\rho+1)}{\rho}}.$$

In order to better understand these results, we depict in Fig. 7.4 the critical value of delay and the critical frequency of oscillations for two different values of p (0.5 and 0.3). First, we observe that even in the case in which only one strategy is delayed, a Hopf bifurcation may occur and the asymptotic stability of the ESS can be lost. Furthermore, as the probability of the delay grows, (recall that p is the probability of a delayed strategy), the normalized critical delay decreases and thus the probability of the loss of stability increases. For example, when $\rho = 1.15$ and $p = 0.3$, the normalized critical delay is given by 39 time units and this value reduces to 26 time units when $p = 0.5$. Also, the critical frequency of oscillations increases as ρ grows.

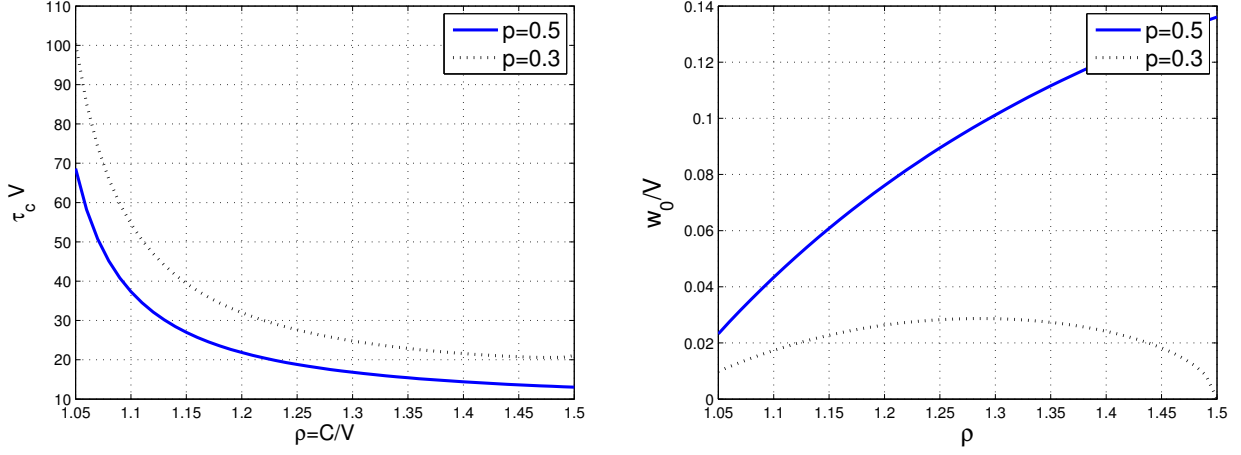


Figure 7.4: Left, the normalized critical value of delay in function of ρ . Right, the normalized critical frequency $\frac{w_0}{V}$ in function of ρ (in the case one strategy is delayed).

7.2 Hawk-Dove Game in Interacting Communities

In this section, we apply our model defined in Chapter 5 to the Hawk-Dove game played in an heterogeneous population composed of two communities of hawks and doves with different levels of aggressiveness and which interact in a nonuniform fashion. Let:

- p_1 (resp. p_2) be the probability that an individual from Community 1 (resp. 2), involved in an interaction, competes with an individual from the same community;
- $1 - p_1$ (resp. $1 - p_2$) be the probability that an individual from Community 1 (resp. 2) competes with an inter-community opponent.

Furthermore, the interactions inside the communities 1 and 2 are described by the matrices A and D :

$$\begin{matrix} H_1 & D_1 \\ H_1 \begin{pmatrix} \frac{V-C}{2} & V \\ 0 & \frac{V}{2} \end{pmatrix}, & H_2 \begin{pmatrix} \frac{V-C}{2} & V \\ 0 & \frac{V}{2} \end{pmatrix}. \\ D_1 & D_2 \end{matrix}$$

The inter-community interactions are described by the following matrices:

$$\begin{matrix} H_2 & D_2 \\ H_1 \begin{pmatrix} \frac{V-C_{SW}}{2} & V \\ 0 & \alpha V \end{pmatrix}, & H_2 \begin{pmatrix} \frac{V-C_{WS}}{2} & V \\ 0 & (1-\alpha)V \end{pmatrix}. \\ D_1 & D_2 \end{matrix}$$

We introduce the parameters C_{SW} , C_{WS} and α into the payoff matrices to incorporate

Table 7.1: Parameters of the model of Hawk-Dove game.

Parameter	Value
L_1	$-0.5C$
L_2	$-0.5C$
L_{12}	$-0.5C_{SW} + V(\alpha - 0.5)$
L_{21}	$-0.5C_{WS} - V(\alpha - 0.5)$
K_1	$0.5p_1V + (1 - \alpha)(1 - p_1)V$
K_2	$0.5p_2V + \alpha(1 - p_2)V$
Δ	$p_1p_20.25C^2 - (1 - p_1)(1 - p_2)(-0.5C_{SW} + V(\alpha - 0.5))(-0.5C_{WS} - V(\alpha - 0.5))$
Δ_1	$p_1p_2C^2 - ((1 - p_1)(-0.5C_{SW} + V(\alpha - 0.5)) + (1 - p_2)(-0.5C_{WS} - V(\alpha - 0.5)))^2$

the disparity in the levels of aggressiveness of the two communities. These parameters can be defined as follows:

- C is the fighting cost incurred by a hawk when fighting against a hawk from the same community;
- C_{SW} is the fighting cost incurred by a hawk from the more aggressive community when fighting against a hawk from the other community;
- C_{WS} is the fighting cost incurred by a hawk from the less aggressive community when fighting against a hawk from the other community;
- α is the resource part that takes a dove from the more aggressive community when competing with a dove from the other community.

The parameters satisfy: $0 < C_{SW} < C < C_{WS}$, and $0.5 \leq \alpha < 1$. When two hawks from different communities compete for a resource, they do not incur the same cost as when fighting against an intra-community hawk. Also, when two doves from different communities meet, they share the resource unevenly. In particular, with these parameters, the first community is more aggressive than the second community: hawks in community 1 provoke more injuries than hawks in the other community, and doves in community 1 take more resource than those in community 2. When $C_{SW} = C_{WS}$ and $\alpha = 0.5$, the two communities have the same degree of aggressiveness.

In Table 7.1, we write the parameters of the model in function of the resource value, the fighting costs, and the probabilities of interactions.

In the following, we aim to illustrate with numerical examples (i) the effect of the interaction probabilities and the game parameters on the existence of ESSs; and (ii) the impact of these parameters and delays on the convergence to the fully mixed intermediate ESS in the replicator dynamics.

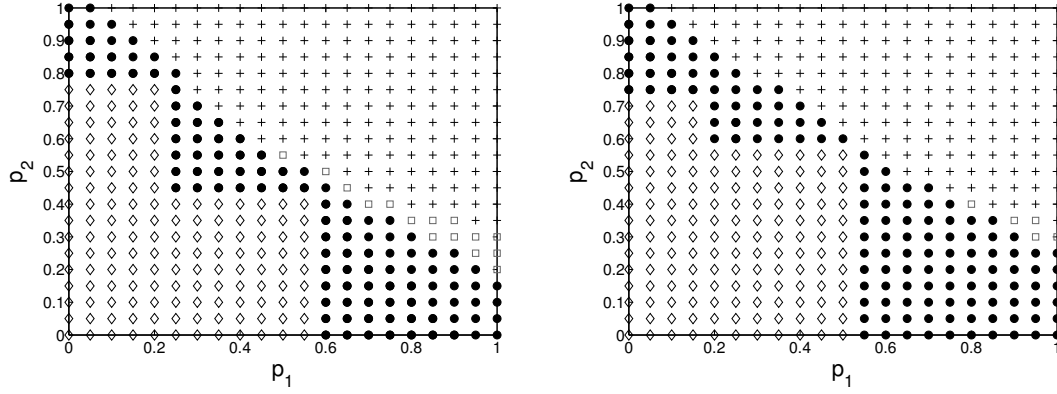


Figure 7.5: Regions of ESSs in the interaction probabilities plane (p_1, p_2) for the parameter values $C = 6$, $C_{SW} = 5$, $C_{WS} = 9$, $\alpha = 0.7$. The plus signs (+) represent fully mixed intermediate ESS, the squares (\square) represent fully mixed weak ESSs, the circles (\bullet) represent partially-mixed strong ESS, and the diamonds (\diamond) represent fully-pure strong ESS.

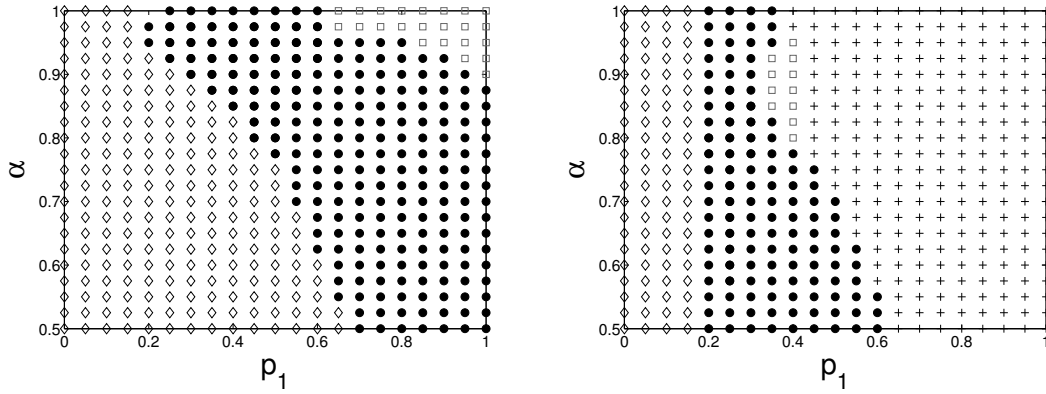


Figure 7.6: Regions of ESSs in the plane (p_1, α) for the parameter values $C = 6$, $C_{SW} = 5$, $C_{WS} = 9$, and $V = 4$. The plus signs (+) represent fully mixed intermediate ESS, the squares (\square) represent fully mixed weak ESSs and the circles (\bullet) represent partially mixed strong ESS.

7.2.1 ESSs in the Hawk-Dove Game

In Figure 7.5, we plot the region of ESSs in function of the interaction probabilities p_1 and p_2 for the parameter values $C = 6$, $C_{SW} = 5$, $C_{WS} = 9$, $\alpha = 0.7$ and for two different values of the resource: $V = 2$ in the left subfigure and $V = 4$ in the right subfigure. We observe that in both subfigures, the fully-mixed intermediate ESS exists for the higher values of p_1 and p_2 ; that is when the intra-community interactions are more probable than inter-community interactions in both communities. This region allows all strategies in both communities to co-exist. For the lower values of p_1 and p_2 , we observe in both subfigures, the existence of fully-pure strong ESS. The area between these two regions

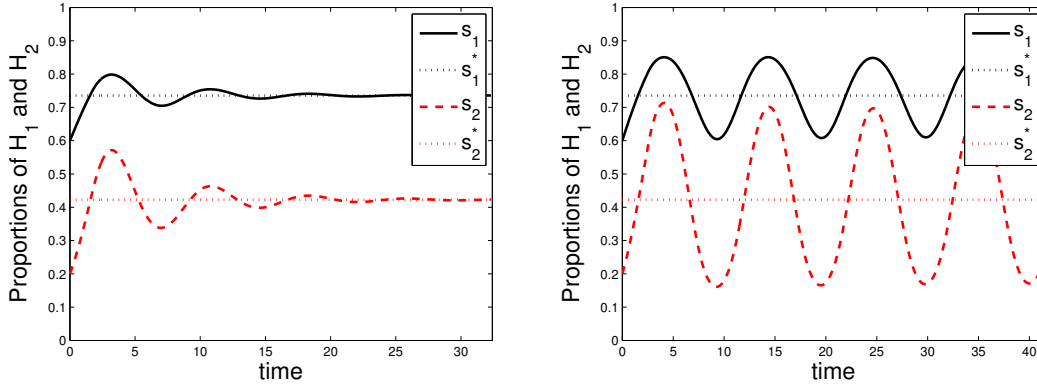


Figure 7.7: Replicator Dynamics with strategic delays $\bar{\tau}_{st} = 2.4$ time units, $C = 6$, $C_{SW} = 5$, $C_{WS} = 9$, $\alpha = 0.7$, $V = 4$, $p_1 = 0.7$, $p_2 = 0.6$.

is characterized by the existence of partially-mixed strong ESS, that is, an ESS which is pure in one community and mixed in the other. We also note the existence of fully-mixed weak ESS in a small region in both figures.

Furthermore, we plot in Fig. 7.6 the region of ESSs in function of p_1 and α for two values of p_2 : $p_2 = 0.2$ in the left subfigure and $p_2 = 0.6$ in the right subfigure. When $p_2 = 0.2$, we note that any fully-mixed intermediate ESS does not exist. We observe that fully-mixed weak ESSs exist for a small range of higher values of p_1 and α . The fully-pure and partially-mixed strong ESS exist for a large range of values of p_1 and α . In the right subfigure, when $p_2 = 0.6$, we note the appearance of fully-mixed intermediate ESSs for the higher values of p_1 .

7.2.2 Replicator Dynamics in the Hawk-Dove Game with Communities

In this subsection, we study the effects of delays and the game parameters on the convergence of the replicator dynamics to the fully-mixed intermediate ESS. For the parameter values $C = 6$, $C_{SW} = 5$, $C_{WS} = 9$, $\alpha = 0.7$, $V = 4$, $p_1 = 0.7$, $p_2 = 0.6$, we conclude from the right subfigure in Fig. 7.6, there exists a fully mixed intermediate ESS which is given by $\mathbf{s}^* = (0.73, 0.42)$. Furthermore, \mathbf{s}^* is asymptotically stable in the replicator dynamics (Corollary 4). We plot in Figure 7.7, the evolution of the proportion of hawks in both communities over time according to the replicator dynamics with strategic delays where the initial (arbitrary) population profile is given by $(0.6, 0.2)$. Each strategy has a delay of τ_{st} . We observe in the left subfigure the convergence to the ESS for $\tau_{st} = 1.6$ time units after some damped oscillations, whereas we observe permanent oscillations around the ESS in the right subfigure for $\tau_{st} = 2.5$ time units. From Proposition 25, the maximum value of the strategic delay for which the fully mixed intermediate ESS is

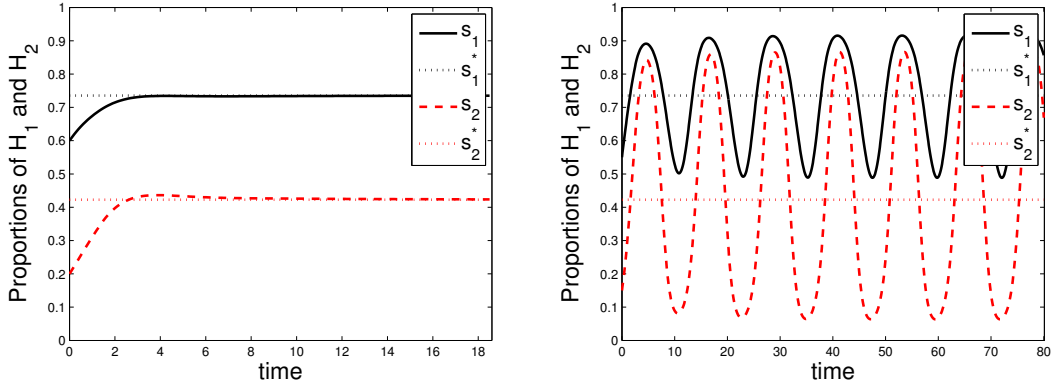


Figure 7.8: *Replicator Dynamics with purely spatial delays (left) and spatial-strategic delays (right).* $C = 6$, $C_{SW} = 5$, $C_{WS} = 9$, $\alpha = 0.7$, $V = 4$, $p_1 = 0.7$, $p_2 = 0.6$.

asymptotically stable is $\bar{\tau}_{st} = 2.4$ time units. Therefore, the asymptotic stability of the ESS is lost for $\tau_{st} = 2.5$ time units.

Furthermore, we study the effect of spatial delays on the convergence to the ESS in the replicator dynamics. Spatial delays appear in inter-community interactions only. In left subfigure of Fig. 7.8, we observe a convergence to the intermediate ESS. Indeed, from Proposition 26, the replicator dynamics converges to the fully-mixed intermediate ESS for any value of τ_{sp} . When both delays exist, spatial delay ($\tau_{sp} = 1.4$ time units) and strategic delay ($\tau_{st} = 2.5$), we observe in the right subfigure of Fig. 7.8 persistent oscillations.

In addition, we study the effect of the resource value on the convergence to the ESS. For a fixed strategic delay value of $\tau_{st} = 2.2$ time units, we observe in Fig. 7.9 oscillations around the fully mixed intermediate ESS when $V = 3$. For the same value of the strategic delay and when $V = 4$, we observe the convergence to the ESS (Fig. 7.7). We note that increasing the resource value had a stabilizing effect. Furthermore, we illustrate in Fig. 7.10 the effect of the intra-community fighting cost C on the convergence to the fully mixed ESS in the replicator dynamics. In the left subfigure, for $C = 6$, we observe the convergence to the mixed ESS whereas in the right subfigure, for $C = 8$, we note persistent oscillations around the ESS and the loss of stability. Therefore, increasing the intra-community fighting cost has a destabilizing effect.

7.3 Interacting Communities on Random Graphs

In this section, we are interested in studying the evolutionary games on random graphs. The application of evolutionary game theory in graphs or structured populations was considered in many works [47, 68, 69, 29, 30]. For example, in [69], the authors studied

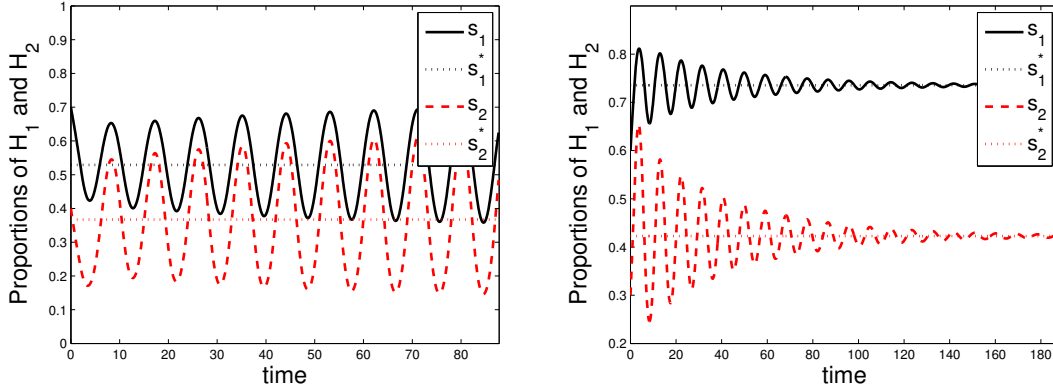


Figure 7.9: Effect of the resource value on the convergence to the ESS for the same strategic delay value $\tau_{st} = 2.2$ time units. $C = 6$, $C_{SW} = 5$, $C_{WS} = 9$, $\alpha = 0.7$, $p_1 = 0.7$, $p_2 = 0.6$.

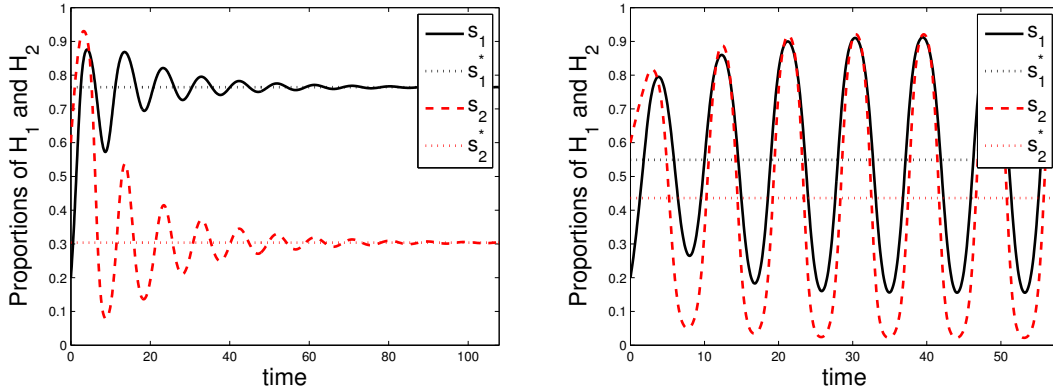


Figure 7.10: Effect of the intra-community fighting cost on the convergence to the ESS for the same strategic delay value $\tau_{st} = 2.2$ time units. $V = 4$, $C_{SW} = 5$, $C_{WS} = 10$, $\alpha = 0.7$, $p_1 = 0.7$, $p_2 = 0.6$.

evolutionary games in cycles considering three different update mechanisms for the evolutionary dynamics.

We consider a population, which is represented by an undirected graph $\mathcal{G} = (V, E)$ where V is the set of vertices representing the agents or individuals and E is the set of edges representing the pairwise interactions. A neighbor of an individual i in V is any other agent j such that there exists an edge ij in E . Moreover, we consider a complete graph where any individual may interact with any other individual. The payoff of a player is the sum of payoffs obtained when it interacts with all immediate neighbors [47, 69]. We consider the following probabilistic imitation update rule [41, 47]:

- At each iteration or time step a player i is randomly chosen to revise its strategy;
- The selected player will randomly choose a neighbor j from its community;

- If the payoff of player j is higher than that of player i , then player i will switch to player j 's strategy with a probability proportional to the difference between the payoffs of the two strategies; Otherwise, player i will keep its strategy.

Note that this update rule is different from the imitation rule in [47] as we consider here a non-homogeneous population composed of two-communities; and the outcome of an interaction depends not only on the strategies used but also on the types of players. Note also that an individual may interact with any other individual including inter-community opponents but for the imitation process, an individual may imitate only an opponent from the same community.

We make simulations to study the evolution of behavior of the population using the imitation update rule. We considered a population composed of $N_1 = 1000$ individuals from Community 1 and $N_2 = 4000$ individuals from Community 2. Moreover, we considered a Hawk-Dove game with the values of parameters (defined in the previous section) given by $V = 1.3$, $C = 4$, $C_{SW} = 2$, $C_{WS} = 6$, $\alpha = 0.8$. We plot in Fig. 7.11 (top left subfigure), the evolution of the population state using the imitation update rule, with an (arbitrary) initial state of the population given by $(0.3, 0.4)$. As we can clearly see, the population state converges to the mixed ESS given by $\mathbf{s}^* = (0.2, 0.6)$.

Furthermore, we introduced discrete delays: In Fig. 7.11 (top right subfigure), we considered a strategic delay of 14 time slots: the payoff of a player with the strategic delay depends on the actions of its opponents taken at some previous time (14 time slots). We observe oscillations around the equilibrium. In Fig. 7.11 (bottom), we considered a spatial delay fixed to 21 time slots: the payoff of player obtained when it interacts with an inter-community opponent depends on the action of the latter taken some time slots before. We note a convergence to the mixed ESS.

7.4 Conclusion

In this chapter, we applied our results to the Hawk-Dove game. First, we studied the replicator dynamics with heterogeneous and random delays in this context. We depicted the variation of the mean critical delay values in the cases of uniform and Gamma distribution, in function of the *cost per unit of benefit*. We also examined a case of discrete delays in which only one strategy is delayed. Furthermore, we studied the case in which the population is heterogeneous and composed of two communities of hawks and doves with asymmetric levels of aggressiveness. We illustrated in this context the regions of different types of ESS in function of the interaction probabilities and the parameters of the game. We also illustrated the effects of different types of delays (strategic and spatial delays), and the game parameters on the convergence to the fully mixed ESS in the replicator dynamics. At the end of this chapter, we gave an illustration of evolutionary games on graphs. Particularly, we illustrated the convergence to the fully mixed ESS

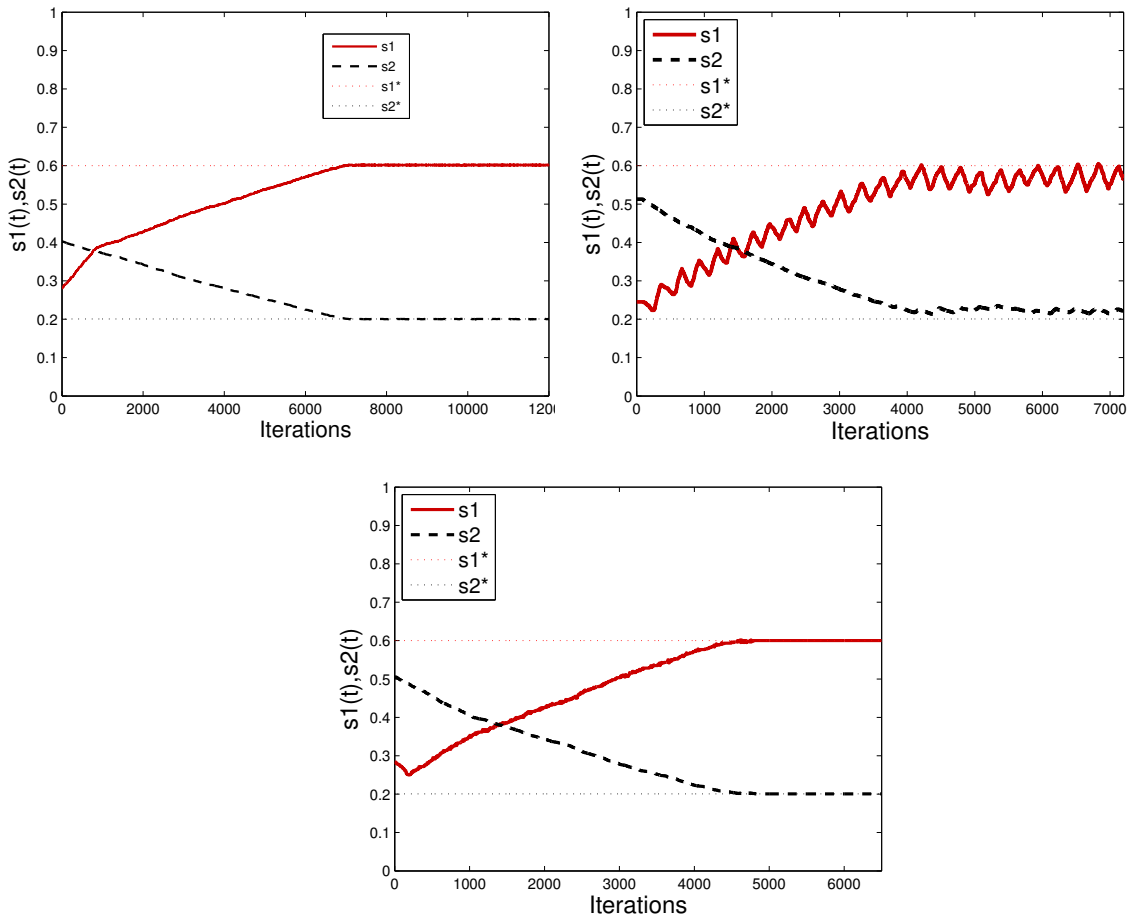


Figure 7.11: Evolution of the population state with the imitation update rule.

considering the imitation update rule on graphs.

Chapter 8

Conclusions and Perspectives

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8.1 Summary of Contributions

In this dissertation, we studied evolutionary games in an extended framework where we considered (i) the consequences of heterogeneous random delays on the stability of the replicator dynamics, (ii) a complete static and dynamic analysis of an heterogeneous population composed of communities that are interacting in a nonuniform manner, and (iii) an application to the Hawk-Dove Game.

8.1.1 Replicator Dynamics with Random Delays

In Chapter 2, we extended the classic replicator dynamics by introducing random delays. Indeed, in many real-world applications, the presence of uncertain delays is ubiquitous. We examined the effects of the distribution of delays on the stability of the mixed evolutionarily stable strategy. We proved that, for the exponential distribution, the stability of the mixed ESS is unaffected; whereas for the uniform and Gamma distribution we derived the critical value of the mean delay at which a Hopf bifurcation occurs. We demonstrated that, in all the cases, the critical mean delay value is higher than that in the case of constant delays.

In addition, in Chapter 3, we considered discrete stochastic delays and we proved that when the instantaneous term is dominant, that is when the probability of nondelayed interactions is sufficiently high, the stability of the mixed ESS is unaffected. In the case of two delays, we illustrated how the consequence on the stability of the mixed ESS can be changed by keeping the values of delays unchanged and varying only the probabilities of delays. When only one strategy is delayed, we established the critical delay value at which a Hopf bifurcation occurs and the stability of the ESS is lost.

Near the Hopf bifurcation, the replicator dynamics exhibits a stable periodic oscillation in the proportions of strategies in the population. In Chapter 4, we examined the properties of the bifurcating limit cycle in the replicator dynamics with random delays. Considering Dirac, uniform, Gamma, and discrete delay distributions, we approximated the stable bifurcating limit cycle, and we derived analytically the radius of the limit cycle in function of the value of the bifurcation parameter (the mean delay). Furthermore, we validated our theoretical results with numerical simulations.

8.1.2 Evolutionary Games in Interacting Communities

In Chapter 5, we studied an heterogeneous population composed of groups or communities that are interacting in a nonuniform manner. We defined for each community, its payoffs and interaction probabilities with other communities. In this context, the outcome of a pairwise interaction is a function not only of the strategies used, as in the

classical setting of evolutionary games, but also of the types (or communities) of the interacting players. We derived accordingly the utility functions for each group and we established a complete static analysis in this framework by considering three types of ESS with different levels of stability against mutations: strong ESS which resists invasion from a sufficiently small group composed of all the communities in the population, weak ESS, in which case each group resists invasion from a sufficiently small local group of mutants, and intermediate ESS in which case the fitness of the full population cannot be higher for any sufficiently small group composed of all the mutants. Indeed, for the intermediate ESS, we consider a single fitness of the full population, that is the sum of the fitness over all the communities, instead of the fitness of each group separately. We examined the relationship between the ESS and we proved that a strong ESS is an intermediate ESS which is a weak ESS, and the inverse cannot be always true. We gave a counterexample that corroborates our claim.

In the case of two communities, we succeeded to have closed form formulae of the ESS. Remarkably, we proved that any fully mixed equilibrium cannot be a strong ESS but can be intermediate or weak ESS under some conditions on the payoffs and interaction probabilities.

In Chapter 6, we introduced a dynamic analysis in this framework of heterogeneous population. In particular, we derived the replicator dynamics to model the time evolution of strategies in each group in function of the payoffs and the probabilities of interactions. We examined the relationship between the ESS and the rest points of the replicator dynamics, and we proved that in the case of two communities, the fully mixed intermediate ESS is asymptotically stable whereas the weak stability condition of the ESS does not ensure the asymptotic stability under the replicator dynamics.

Furthermore, we introduced two types of delays in the replicator dynamics: spatial delays and strategic delays. We proved that the spatial delay does not affect the stability of the fully mixed ESS in the replicator dynamics, whereas the strategic delay may result in permanent oscillations around the equilibrium.

8.1.3 Applications

In Chapter 7, we applied the results obtained in the previous chapters to the Hawk-Dove Game. We considered a population composed of two communities of hawks and doves with asymmetric fighting abilities and which interact in a nonuniform manner. In this context, we identified the regions of ESS in function of the game parameters and the interaction probabilities. We also examined the stability of the replicator dynamics in function of delays and the game parameters. In addition, we studied the Hawk-Dove game on random graphs.

8.2 Perspectives

We propose in this section possible extensions to the work done in this thesis. These extensions can be categorized in three parts, which are novel delay modeling, extension of the community framework, and an application to economics. Each part is detailed below.

8.2.1 Novel Delay Modeling

The work related to delays presented in this dissertation focuses on the destabilizing effects of delays, that is the possible appearance of fluctuations or bifurcating solutions around the equilibrium. However, time delays may have a stabilizing effect. As a perspective, we propose to investigate the stabilizing effects of delays in a class of games that admit an unstable interior equilibrium (coordination games) such as the Stag-Hunt Game.

Another perspective is to consider state-dependent delays: that is delays that depend on the state of the population (distribution of strategies in the population). For example, in congestion networks, where each agent has to choose which link to follow, the delay experienced by a user may depend on the number of users using the same link as himself.

Another related point is to consider time-varying delays. Indeed, in many situations, the delay cannot be constant but might depend on the time at which the action is taken by the players. Thus, the time interval after which the consequence of an action is observed can be modeled as a function of time. This is expected to yield novel results compared to the case of constant delays. However, it is more challenging to analyze the replicator dynamics with time variable delays since the classic approach of analyzing the corresponding characteristic equation cannot be used in this scenario [20].

Furthermore, it is possible to consider asymmetric delays across the strategies. This scenario is more realistic than that of symmetric delays. Most of the work done related to asymmetric delays have considered one delayed strategy and little is investigated when both the strategies have distinct and nonzero delays. It is possible to investigate further results in this context. Moreover, it would be interesting to analyze the Hopf bifurcation in evolutionary dynamics by taking one delay as a bifurcation parameter. Another interesting perspective is to consider asymmetric and interaction-dependent delays: in this scenario there is an asymmetry across the interaction types, that is interactions between agents using the same strategy would take a delay value, whereas interactions involving agents using different strategies would take another delay value. In addition, by taking one delay as a bifurcation parameter, it is possible to examine the Hopf bifurcation and make a comparison with results obtained when delays are asymmetric and strategy-dependent.

8.2.2 Extension of The Community Framework

When considering heterogeneous populations composed of communities, it is important to take account of special relations that may emerge among the communities. Indeed, the complex relationships entertained between the agents might change dramatically the evolutionary outcome. For instance, agents from the same community may exhibit altruistic behaviors towards each other, and consequently a novel equilibrium definition is suitable in this context, see for example the notion of equilibrium proposed in [113]. The appearance of coalition or cooperation between the communities is also expected.

Moreover, the communities can be linked with migration that may change the evolutionary outcome. As an extension to the work related to the interactions between groups or communities, it is interesting to introduce into the evolutionary dynamics the possible migration between the groups. Hence, the growth rate of a given strategy in a community may also depend on the rate at which agents or players in other groups migrate to the considered community. The replicator dynamics can be extended to allow the modeling of this phenomenon.

8.2.3 Economic Applications

Thanks to novel technology development, we are witnessing the appearance of a new phenomenon in economics, which is the ‘uberization’ phenomenon, a form of the sharing economy. This phenomenon has arisen in industrial sectors such as transportation, through the transportation network company ‘Uber’ or through ‘Blablacar’, hotel reservations and housing, through the company ‘Airbnb’, bringing new entrants to these sectors. In wireless industry, the uberization of wireless networks is also expected. Such phenomenon results in the emergence of novel types of actors and the modification of the value chain. For telecommunication operators, the main revenue is due to retail services (network access). Moreover, the operators are facing important investments in infrastructures for the bandwidth increase in the network access (such as 4G, 5G, and optical fiber). However, faced to the network uberization, the competition among the operators, and the regulation effects, their contribution to the value chain might be significantly reduced. Consequently, they have to revise their contribution to the value chain and review their business models. In particular, it becomes necessary for the operators to think about a possible collaboration with new actors. In this context, game theory and evolutionary game theory allow the modeling of the competition framework between the actors and identify, for each actor of this competitive context, the best-reply to this phenomenon.

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Chapter 9

Appendices

Appendix A : Stability of Corner and Border Stationary Points

9.1 Nondelayed Replicator Dynamics

In our model, there exist four corner equilibrium points, four border equilibrium points and a unique interior equilibrium. In order to study the stability of a stationary point \mathbf{s}^* , we make a linearization of the replicator dynamics around this point and we study the Jacobian matrix. If all the eigenvalues of the Jacobian matrix have negative real parts then the asymptotic stability follows. For the equilibrium point $(0,0)$, we obtain the following linear system: $\dot{x}(t) = Ax(t)$ where,

$$A = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}.$$

The eigenvalues of A (Jacobian matrix) are K_1 and K_2 . Thus, the origin is asymptotically stable if $K_1 < 0$ and $K_2 < 0$, and unstable if $K_1 > 0$ and $K_2 > 0$. Doing the same for the all the stationary points, we obtain the following result:

- The stationary point $(0,0)$ is stable if $K_1 < 0$ and $K_2 < 0$; and unstable if $K_1 > 0$ or $K_2 > 0$;
- The stationary point $(1,1)$ is asymptotically stable if $p_1L_1 + (1 - p_1)L_{12} + K_1 > 0$ and $p_2L_2 + (1 - p_2)L_{21} + K_2 > 0$; and unstable if $p_1L_1 + (1 - p_1)L_{12} + K_1 < 0$ or $p_2L_2 + (1 - p_2)L_{21} + K_2 < 0$;

The Jacobian matrix is given by:

$$\begin{pmatrix} -(p_1L_1 + (1 - p_1)L_{12} + K_1) & 0 \\ 0 & -(p_2L_2 + (1 - p_2)L_{21} + K_2) \end{pmatrix}.$$

- The stationary point $(0,1)$ is asymptotically stable if $(1 - p_1)L_{12} + K_1 < 0$ and $-(p_2L_2 + K_2) < 0$; and unstable if $(1 - p_1)L_{12} + K_1 > 0$ or $-(p_2L_2 + K_2) > 0$;

The Jacobian matrix is given by:

$$\begin{pmatrix} (1-p_1)L_{12} + K_1 & 0 \\ 0 & -(p_2L_2 + K_2) \end{pmatrix}.$$

- The stationary point $(1,0)$ is asymptotically stable if $p_1L_1 + K_1 > 0$ and $(1-p_2)L_{21} + K_2 < 0$, and unstable if $p_1L_1 + K_1 < 0$ and $(1-p_2)L_{21} + K_2 > 0$;

The Jacobian matrix is given by:

$$\begin{pmatrix} -(p_1L_1 + K_1) & 0 \\ 0 & (1-p_2)L_{21} + K_2 \end{pmatrix}.$$

- The stationary point $(0, s_2^*)$ where $s_2^* = -\frac{K_2}{p_2L_2}$ is asymptotically stable if $(1-p_1)L_{12}s_2^* + K_1 < 0$ and $L_2 < 0$. Indeed, the Jacobian matrix obtained by making a linearisation around this equilibrium point is given by:

$$\begin{pmatrix} (1-p_1)L_{12}s_2^* + K_1 & 0 \\ \gamma_2(1-p_2)L_{21} & \gamma_2p_2L_2 \end{pmatrix}.$$

The eigenvalues of this triangular matrix are the diagonal terms, that is $(1-p_1)L_{12}s_2^* + K_1$ and $\gamma_2p_2L_2$. The result follows.

- The stationary point $(s_1^*, 0)$ where $s_1^* = -\frac{K_1}{p_1L_1}$ is asymptotically stable if $(1-p_2)L_{21}s_1^* + K_2 < 0$ and $L_1 < 0$. The Jacobian matrix is given by:

$$\begin{pmatrix} \gamma_1p_1L_1 & \gamma_1(1-p_1)L_{12} \\ 0 & (1-p_2)L_{21}s_1^* + K_2 \end{pmatrix}.$$

- The stationary point $(1, s_2^*)$ where $s_2^* = -\frac{(1-p_2)L_{21} + K_2}{p_2L_2}$ is asymptotically stable if $-(p_1L_1 + (1-p_1)L_{12}s_2^* + K_1) < 0$ and $L_2 < 0$;

The Jacobian matrix is given by:

$$\begin{pmatrix} -(p_1L_1 + (1-p_1)L_{12}s_2^* + K_1) & 0 \\ \gamma_2(1-p_2)L_{21} & \gamma_2p_2L_2 \end{pmatrix}.$$

- The stationary point $(s_1^*, 1)$ where $s_1^* = -\frac{(1-p_1)L_{12} + K_1}{p_1L_1}$ is asymptotically stable if $L_1 < 0$ and $-(p_2L_2 + (1-p_2)L_{21}s_1^* + K_2) < 0$.

The Jacobian matrix is given by:

$$\begin{pmatrix} \gamma_1p_1L_1 & \gamma_1(1-p_1)L_{12} \\ 0 & -(p_2L_2 + (1-p_2)L_{21}s_1^* + K_2) \end{pmatrix}.$$

9.2 Delayed Replicator Dynamics

In this section, we complete our analysis of delayed replicator dynamics. We study the stability of corner and border stationary ESS in the presence of spatial delays.

9.2.1 Spatial Delays

We recall that the replicator dynamics with spatial delays is given by:

$$\dot{s}_1(t) = s_1(t)(1 - s_1(t)) [p_1 L_1 s_1(t) + (1 - p_1) L_{12} s_2(t - \tau_{sp}) + K_1], \quad (9.1)$$

$$\dot{s}_2(t) = s_2(t)(1 - s_2(t)) [p_2 L_2 s_2(t) + (1 - p_2) L_{21} s_1(t - \tau_{sp}) + K_2]. \quad (9.2)$$

The linearization around $(0, 0)$ gives the following linear system:

$$\dot{s}(t) = As(t), \quad (9.3)$$

We observe that the linearized replicator equations do not depend on τ_{sp} . Therefore, the spatial delay does not affect the stability of the fully pure ESS.

Similarly, we can prove this result for partially mixed ESSs. Let us consider $\mathbf{s}^* = (0, s_2^*)$ where $s_2^* = -\frac{K_2}{p_2 L_2}$. The linearization around this equilibrium yields the following system:

$$\dot{s}(t) = As(t) + Bs(t - \tau_{sp}), \quad (9.4)$$

where,

$$A = \begin{pmatrix} (1 - p_1)L_{12}s_2^* + K_1 & 0 \\ 0 & \gamma_2 p_2 L_2 \end{pmatrix}.$$

and

$$B = \begin{pmatrix} 0 & 0 \\ \gamma_2(1 - p_2)L_{21} & 0 \end{pmatrix}.$$

The characteristic equation is given by:

$$\lambda^2 - \lambda \left(\gamma_2 p_2 L_2 + (1 - p_1) L_{12} s_2^* + K_1 \right) + \gamma_2 p_2 L_2 \left((1 - p_1) L_{12} s_2^* + K_1 \right) = 0$$

The characteristic equation does not depend on τ_{sp} . Therefore, the spatial delay does not affect the stability of the partially mixed ESS.

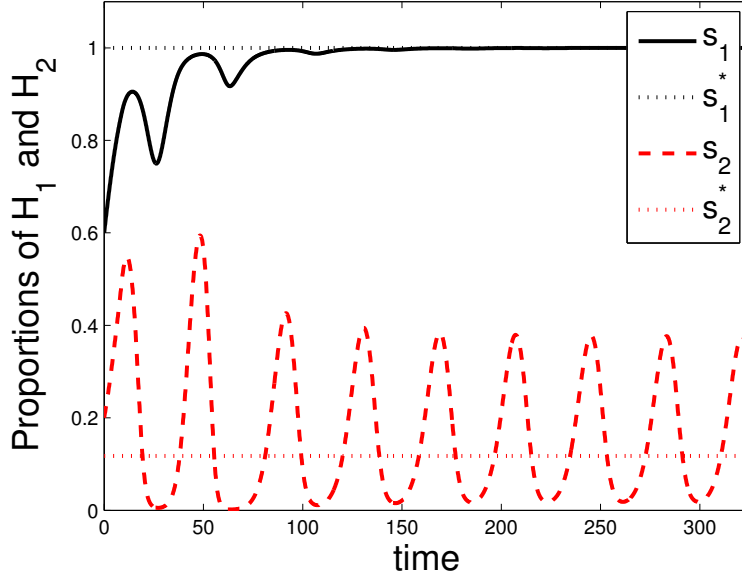


Figure 9.1: Oscillations in the proportion of strategies in the second group in presence of strategic delays.

9.2.2 Strategic Delays

The replicator dynamics with strategic delays is given by:

$$\dot{s}_1(t) = s_1(t)(1 - s_1(t)) [p_1 L_{11} s_1(t - \tau_{st}) + (1 - p_1) L_{12} s_2(t - \tau_{st}) + K_1], \quad (9.5)$$

$$\dot{s}_2(t) = s_2(t)(1 - s_2(t)) [p_2 L_{22} s_2(t - \tau_{st}) + (1 - p_2) L_{21} s_1(t - \tau_{st}) + K_2]. \quad (9.6)$$

The linearization of the delayed replicator dynamics around $\mathbf{s}^* = (0,0)$ yields to the following linear system $\dot{x}(t) = Ax(t)$ where,

$$A = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}.$$

In the characteristic equation associated to the linearized system, the delay τ_{st} disappears. Therefore, the fully pure ESS $\mathbf{s}^* = (0,0)$ is asymptotically stable for any τ_{st} . Similarly, we can prove the (local) asymptotic stability of all other fully pure ESSs for any delay τ_{st} .

However, for the partially-mixed ESS, the stability can be lost; this is illustrated in Fig. 9.1, where we observe permanent oscillations in the proportion of strategies in the second community.

Bibliography

- [1] J. Maynard Smith and G.R. Price. The logic of animal conflict. *Nature*, 246:15–18., Nov. 1973.
- [2] P. D. Taylor and L. B. Jonker. Evolutionarily stable strategies and game dynamics. *Mathematical Biosciences*, 40:145–156, 1978.
- [3] A. Rosas. Evolutionary game theory meets social science: Is there a unifying rule for human cooperation? *Journal of Theoretical Biology*, 264:450–456, May 2010.
- [4] T. C. Burnham and D. P. Johnson. The biological and evolutionary logic of human cooperation. *Analyse & Kritik*, 27:113–35, 2005.
- [5] H. Tembine, E. Altman, R. El-Azouzi, and Y. Hayel. Evolutionary games with random number of interacting players applied to access control. In *Proceedings of WiOpt*, pages 344–351, Berlin, April 2008.
- [6] H. Tembine, E. Altman, R. El-Azouzi, and Y. Hayel. Evolutionary games in wireless networks. *IEEE Trans. Syst. Man Cybern.*, 40:634 – 646, June 2010.
- [7] R. El-Azouzi, F. De Pellegrini, H. Sidi, and V. Kamble. Evolutionary forwarding games in delay tolerant networks: Equilibria, mechanism design and stochastic approximation. *Computer Networks*, 57:1003 – 1018, March 2013.
- [8] Q. Zhu, H. Tembine, and T. Başar. *Advances in Dynamic Games: Theory, Applications, and Numerical Methods for Differential and Stochastic Games*, chapter Evolutionary Games for Multiple Access Control, pages 39–71. Birkhäuser Boston, Boston, MA, 2013.
- [9] Q. Zhu, H. Tembine, and T. Basar. Evolutionary games for hybrid additive white gaussian noise multiple access control. In *Proceedings of IEEE Conference on Global Telecommunications, GLOBECOM 2009.*, pages 1–6, Honolulu, HI, Nov 2009.
- [10] H. Gaiech, R. El-Azouzi, M. Haddad, E. Altman, and I. Mabrouki. Evolutionary games for multiple access control: From egoism to altruism. In *Proceedings of*

- the 7th International Conference on NETWORK Games CONTROL and OPTimization (NETGCOOP 2014)*, Trento, Italy, Oct. 2014.
- [11] S. Shakkottai, E. Altman, and A. Kumar. The case for non-cooperative multi-homing of users to access points in IEEE 802.11 WLANs. In *Proceedings of IEEE Infocom*, pages 1–12, Barcelona, Spain, 2006.
- [12] Y. Zheng and Z. Feng. Evolutionary game and resources competition in the internet. In *Modern Communication Technologies, 2001. SIBCOM-2001. The IEEE-Siberian Workshop of Students and Young Researchers*, pages 51–54, 2001.
- [13] Q. Ma, L. Gao, Y. F. Liu, and J. Huang. A game-theoretic analysis of user behaviors in crowdsourced wireless community networks. In *13th International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOpt)*, pages 355–362, Mumbai, May 2015.
- [14] P. Coucheney and C. Touti. Replicator dynamics based adaptive algorithm for heterogeneous wireless systems. In *Proceedings of the 13th International Symposium on Dynamic Games and Applications (ISDG'2008)*, 2008.
- [15] H. Tembine, E. Altman, R. El-Azouzi, and Y. Hayel. Bio-inspired delayed evolutionary game dynamics with networking applications. *Telecommun. Syst.*, 47:137–152, 2011.
- [16] T. Yi and W. Zuwang. Effect of time delay and evolutionarily stable strategy. *J. Theor. Biol.*, 187(1):111–116, July 1997.
- [17] E. Wesson, R. Rand, and D. Rand. Hopf bifurcations in two-player delayed replicator dynamics. *International Journal of Bifurcation and Chaos*, 26(1), 2016.
- [18] J. Alboszta and J. Miekisz. Stability of evolutionarily stable strategies in discrete replicator dynamics with time delay. *J. Theor. Biol.*, 231(2):175–179, Nov. 2004.
- [19] J. Miekisz. *Multiscale Problems in the Life Sciences*, chapter Evolutionary Game Theory and Population Dynamics, pages 269–316. Springer, Berlin, 2008.
- [20] K. Gopalsamy. *Stability and Oscillations in Delay Differential Equations of Population Dynamics*. Kluwer Academic Publisher, London, 1992.
- [21] H. Tembine, E. Altman, and R. El-Azouzi. Asymmetric delay in evolutionary games. In *Proceedings of Valuetools*, Nantes, France, oct 2007.
- [22] J. von Neumann and O. Morgenstern. *Theory of games and economic behavior*. Princeton University Press, Princeton, 1944.
- [23] R. Gibbons. *Game Theory for Applied Economists*. Princeton University Press, 1992.

-
- [24] Dimitris E. Charilas and Athanasios D. Panagopoulos. A survey on game theory applications in wireless networks. *Computer Networks*, 54(18):3421 – 3430, 2010.
- [25] W. H. Sandholm. *Population Games and Evolutionary Dynamics*. MIT Press, 2010.
- [26] P. D. Taylor. Evolutionarily stable strategies with two types of player. *J. Appl. Prob.*, 16(1):76–83, Mars 1979.
- [27] J. Hofbauer and K. Sigmund. *Evolutionary Games and Population Dynamics*. Cambridge University Press, Cambridge, 1998.
- [28] P. Shakarian, P. Roos, and A. Johnson. A review of evolutionary graph theory with applications to game theory. *Biosystems*, 107(2):66–80, Feb. 2012.
- [29] B. Allen and M. A. Nowak. Evolutionary shift dynamics on a cycle. *Journal of Theoretical Biology*, 311:28–39, Oct. 2012.
- [30] E. Lieberman, C. Hauert, and M. A. Nowak. Evolutionary dynamics on graphs. *Nature*, 433:312–316, Jan. 2005.
- [31] X. Zhao and A. H. Sayed. Clustering via diffusion adaptation over networks. *Proc. Int. Workshop Cognitive Inf. Process*, pages 1–6, 2012.
- [32] M. Broom and J. Rychtar. A general framework for analysing multiplayer games in networks using territorial interactions as a case study. *Journal of Theoretical Biology*, 302:70–80, June 2012.
- [33] Hongyan, Nan Yao, Zi-Gang Huang, Junpyo Park, Younghae Do, and Ying-Cheng Lai. Mesoscopic interactions and species coexistence in evolutionary game dynamics of cyclic competitions. *Sci. Rep.*, 4:137–152, 12 2014.
- [34] J. Hofbauer and K. Sigmund. Evolutionary game dynamics. *Bull. Am. Math. Soc.*, 40(4):479–519, July 2003.
- [35] J. W. Weibull. *Evolutionary Game Theory*. MIT Press, Cambridge, 1995.
- [36] H. Gintis. *Game Theory Evolving: A Problem-Centered Introduction to Modeling Strategic Interaction*. Princeton Univ. Press, Princeton, 2000.
- [37] E. C. Zeeman. *Global Theory of Dynamical Systems: Proceedings of an International Conference Held at Northwestern University, Evanston, Illinois*, chapter Population dynamics from game theory, pages 471–497. Berlin, Heidelberg, 1980.
- [38] E.C. Zeeman. Dynamics of the evolution of animal conflicts. *Journal of Theoretical Biology*, 89(2):249 – 270, 1981.

- [39] J. Hofbauer, P. Schuster, and K. Sigmund. A note on evolutionary stable strategies and game dynamics. *Journal of Theoretical Biology*, 81(3):609–612, 1979.
- [40] W. H. Sandholm. Local stability under evolutionary game dynamics. *Theoretical Economics*, 5:27–50., 2010.
- [41] W. H. Sandholm. Population games and deterministic evolutionary dynamics. *Forthcoming in the Handbook of Game Theory and Economic Applications*, H. Petyon Young and Shmuel Zamir, eds, 4:703–778, 12 2014.
- [42] M. A. Nowak and R.M. May. Evolutionary games and spatial chaos. *Nature*, 359:829–829, 1992.
- [43] R. Durrett and S. A. Levin. Stochastic spatial models: A users guide to ecological applications. *Philos. Trans. R. Soc. Lond. B, Biol. Sci.*, 343:329–350, 1994.
- [44] C. Hauert and M. Doebeli. Spatial structure often inhibits the evolution of cooperation in the snow-drift game. *Nature*, 428:643–646, Apr. 2004.
- [45] G. Szabo and G. Fath. Evolutionary games on graphs. *Physical Report*, 446:97–216, 2007.
- [46] H. Ohtsuki and M. A. Nowak. Evolutionary stability on graphs. *Journal of Theoretical Biology*, 251:698–707, Apr. 2008.
- [47] H. Ohtsuki and M. Nowak. The replicator equation on graphs. *Journal Theoretical Biology*, 243(1):86–97, Nov. 2006.
- [48] H. Ohtsuki, C. Hauert, E. Lieberman, and M. A. Nowak. A simple rule for the evolution of cooperation on graphs and social networks. *Nature*, 441:502–505, May 2006.
- [49] F. C. Santos, M. D. Santos, and J. M. Pacheco. Social diversity promotes the emergence of cooperation in public goods games. *Nature*, 454:213–216, July 2008.
- [50] R. Iijima. Heterogeneous information lags and evolutionary stability. *Math. Soc. Sci.*, 61:83–85, 2011.
- [51] R. Iijima. On delayed discrete evolutionary dynamics. *Journal of Theoretical Biology*, 300:1–6, 2012.
- [52] T. Erneux. *Applied Delay Differential Equations*. Springer Science+Business Media, New York, 2009.
- [53] N. Sirghi, M. Neamtu, and D. Opris. Dynamical evolutionary games with delay. In *Proceedings of the 13th Int. Conf. on Mathematics and Computers in Business and Economics (MCBE)*, 2012.

-
- [54] B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan. *Theory and applications of Hopf bifurcation*. Cambridge University Press, Cambridge, 1981.
- [55] R. Rand and A. Verdugo. Hopf bifurcation formula for first order differential-delay equations. *Communications in Nonlinear Science and Numerical Simulation*, 12:859–864, 2007.
- [56] E. Wesson and R. Rand. Hopf bifurcations in delayed rock-paper-scissors replicator dynamics. *Journal of Dynamic Games and Applications*, 6:139 – 156, 2016.
- [57] J. A. Moreira, F. L. Pinheiro, A. Nunes, and J. M. Pacheco. Evolutionary dynamics of collective action when individual fitness derives from group decisions taken in the past. *Journal of Theoretical Biology*, 298:8–15, 2012.
- [58] Y. Wang and D. Cheng. Dynamics and stability of evolutionary games with time-invariant delay in strategies. In *27th Chinese Control and Decision Conference (CCDC), 2015*, pages 6427–6432, May 2015.
- [59] Jacek M. and S. Wesołowski. Stochasticity and time delays in evolutionary games. *Dynamic Games and Applications*, 1(3):440–448, 2011.
- [60] C. H. Hommesy and M. I. Ochea. Multiple steady states, limit and chaotic attractors in evolutionary games with logit dynamics. *Universiteit van Amsterdam, Center for Nonlinear Dynamics in Economics and Finance*, 2010.
- [61] H. Oaku. Evolution with delay. *Japanese Economic Review*, 53:114–133, 2002.
- [62] D. Foster and P. Young. Stochastic evolutionary game dynamics. *Theoretical Population Biology*, 38:219–232, 1990.
- [63] R. Cressman and G.T. Vickers. Spatial and density effects in evolutionary game theory. *Journal of Theoretical Biology*, 184(4):359 – 369, 1997.
- [64] V. C. L. Hutson and G. T. Vickers. Travelling waves and dominance of ESS's. *Journal of Mathematical Biology*, 30:457–471, April 1992.
- [65] J. Hofbauer. *Game Theory, Experience, Rationality: Foundations of Social Sciences, Economics and Ethics*. In *Honor of John C. Harsanyi*, chapter Equilibrium Selection Via Travelling Waves, pages 245–259. Springer Netherlands, Dordrecht, 1998.
- [66] Sung-Ha Hwang, Markos Katsoulakis, and Luc Rey-Bellet. Deterministic equations for stochastic spatial evolutionary games. *Theoretical Economics*, 8:829–874, Sep. 2013.
- [67] G. Szabo and C. Toke. Evolutionary prisoner s dilemma game on a square lattice. *Physical Review E*, 58, July 1998.

- [68] M. Tomassini and E. Pestelacci. Coordination games on dynamical networks. *Games*, 1:242–261, 2010.
- [69] H. Ohtsuki and M. A. Nowak. Evolutionary games on cycles. *Proc. R. Soc. B*, 273:2249–56, 2006.
- [70] J. M. Pacheco, A. Traulsen, and M. A. Nowak. Coevolution of strategy and structure in complex networks with dynamical linking. *Phys. Rev. Lett.*, 97, Dec 2006.
- [71] M. A. Nowak. *Evolutionary Dynamics - Exploring the Equations of Life*. Harvard University Press, Cambridge, 2006.
- [72] A. I. Houston and J. M. McNamara. Fighting for food: a dynamic version of the hawk-dove game. *Evolutionary Ecology*, 2(1):51–64, 1988.
- [73] J. Yearsley. Hawks and doves in heterogeneous environments. *Mathematical and Computer Modelling*, 27(4):99–108, 1998.
- [74] J. M. McNamara and A. I. Houston. If animals know their own fighting ability, the evolutionarily stable level of fighting is reduced. *Journal of Theoretical Biology*, 232(1):1–6, 2005.
- [75] E. Altman, J. Gaillard, M. Haddad, and P. Wiecek. *Bio-Inspired Models of Networks, Information, and Computing Systems: 6th International ICST Conference, BIONETICS 2011*, chapter Dynamic Hawk and Dove Games within Flocks of Birds, pages 115–124. Springer Berlin Heidelberg, Berlin, Heidelberg, 2012.
- [76] J. M. McNamara, S. Merad, and E. J. Collins. The hawk-dove game as an average-cost problem. *Advances in Applied Probability*, 23(4):667–682, 1991.
- [77] Y. Hayel, E. V. Belmega, and E. Altman. Hawks and doves in a dynamic framework. *Dynamic Games and Applications*, 3(1):24–37, 2012.
- [78] R. Cressman. *The Stability Concept of Evolutionary Game Theory*. Springer-Verlag Berlin Heidelberg, 1992.
- [79] P. Auger and D. Pontier. Fast game dynamics coupled to slow population dynamics: A single population with hawk-dove strategies. *Mathematical and Computer Modelling*, 27(4):81 – 88, 1998.
- [80] E. Altman and I. Brunetti. A markov decision evolutionary game for the study of a dynamic hawk and dove problem. In *ISDG Workshop*, Barcelona, 2013.
- [81] T. Killingback and M. Doebeli. Spatial evolutionary game theory: Hawks and doves revisited. *Proceedings of the Royal Society of London B: Biological Sciences*, 263(1374):1135–1144, 1996.

- [82] M. Haddad, E. Altman, J. Gaillard, and D. Fiems. *NETWORKING 2012: 11th International IFIP TC 6 Networking Conference, Prague, Czech Republic, May 21-25, 2012, Proceedings, Part II*, chapter A Semi-dynamic Evolutionary Power Control Game, pages 392–403. Springer Berlin Heidelberg, Berlin, Heidelberg, 2012.
- [83] M. Haddad, E. Altman, D. Fiems, and J. Gaillard. Paradoxes in semi-dynamic evolutionary power control game: When intuition fools you! *IEEE Transactions on Wireless Communications*, 12(11):5728–5739, Nov. 2013.
- [84] E. Altman, D. Fiems, M. Haddad, and J. Gaillard. Semi-dynamic hawk and dove game, applied to power control. In *Proceedings IEEE INFOCOM*, pages 2771–2775, March 2012.
- [85] E. Altman, R. El-Azouzi, Y. Hayel, and H. Tembine. The evolution of transport protocols: An evolutionary game perspective. *Computer Networks*, 53(10):1751 – 1759, 2009.
- [86] H. Tembine, E. Altman, and R. El-Azouzi. Delayed evolutionary game dynamics applied to medium access control. In *Proceedings of the IEEE 4th International Conference on Mobile Adhoc and Sensor Systems*, pages 1–6, Pisa, Italy, oct. 2007.
- [87] Matthias Hanauske, Jennifer Kunz, Steffen Bernius, and Wolfgang Konig. Doves and hawks in economics revisited: An evolutionary quantum game theory based analysis of financial crises. *Physica A: Statistical Mechanics and its Applications*, 389(21):5084–5102, 2010.
- [88] Y. Hayel and Q. Zhu. Evolutionary poisson games for controlling large population behaviors. In *Proceedings of IEEE Control and Decision Conference (CDC)*, pages 300 – 305, Osaka, Japan, Dec. 2015.
- [89] B. Roger Myerson. Population uncertainty and poisson games. *International Journal of Game Theory*, 27(3):375–392.
- [90] R. Selten. A note on evolutionarily stable strategies in asymmetric animal conflicts. *Journal of Theoretical Biology*, 84:93–101, May 1980.
- [91] D. S. Menasché, D. R. Figueiredo, and E. de Souza e Silva. An evolutionary game-theoretic approach to congestion control. *Performance Evaluation*, 62(1-4):295–312, oct 2005.
- [92] Y. Zheng and Z. Feng. Evolutionary game theoretic model and end-to-end flow control in the internet. In *Proceedings of the International Conference on Info-tech and Info-net (ICII)*, volume 4, pages 273–277, Beijing, 2001.

Bibliography

- [93] M. A. Nowak, J. B. Plotkin, and D. C. Krakauer. The evolutionary language game. *Journal of Theoretical Biology*, 200(2):147 – 162, 1999.
- [94] M. Enquist and S. Ghirlanda. Evolution of social learning does not explain the origin of human cumulative culture. *Journal of Theoretical Biology*, 246(1):129–135, 2007.
- [95] N. Ben Khalifa, R. El-Azouzi, and Y. Hayel. Random time delays in evolutionary game dynamics. In *Proceedings of IEEE Control and Decision Conference (CDC)*, pages 3840 – 3845, Osaka, Japan, Dec. 2015.
- [96] R. Bellman and K.L. Cooke. *Differential Difference Equations*. Academic Press, New York.
- [97] Y. Kuang. *Delay Differential Equations: With Applications in Population Dynamic*. Academic Press, New York, 1993.
- [98] J. K. Hale and W. Z. Huang. Global geometry of the stable regions for two delay differential equations. *J. Math. Anal. Appl.*, 178:344–362, 1993.
- [99] X. Li, S. Ruan, and J. Wei. Stability and bifurcation in delay-differential equations with two delays. *J. Math. Anal. Appl.*, 236:254–280, 1999.
- [100] L. Berezansky and E. Braverman. On stability of some linear and nonlinear delay differential equations. *J. Math. Anal. Appl.*, 314:391–411, 2006.
- [101] J. Bélair and S. A. Campbell. Stability and bifurcations of equilibria in a multiple-delayed differential equation. *SIAM J. Appl. Math.*, 54:1402–1424, 1994.
- [102] J. Chen and H.A. Latchman. Asymptotic stability independent of delays: simple necessary and sufficient conditions. In *Proceedings of The American Control Conference*, pages 1027–1031, Baltimore, Maryland, USA, June 1994.
- [103] L. Berezansky and E. Braverman. Explicit exponential stability conditions for linear differential equations with several delays. *J. Math. Anal. Appl.*, 332:246–264, 2007.
- [104] C. Taylor and M. A. Nowak. Evolutionary game dynamics with non-uniform interaction rates. *Theor. Popul. Biol.*, 69(3):243–252, May 2006.
- [105] R. Cressman. Evolutionary game theory with two groups of individuals. *Games and Economic Behavior*, 11(2):237–253, Nov. 1995.
- [106] N. Ben Khalifa, R. El-Azouzi, Y. Hayel, H. Sidi, and I. Mabrouki. Evolutionary stable strategies in interacting communities. In *Proceedings of Valuetools*, pages 214–222, Torino, Italy, Dec. 2013.

- [107] N. Ben Khalifa, R. El-Azouzi, Y. Hayel, and I. Mabrouki. Evolutionary games in interacting communities. *Dynamic Games and Applications*, 2016.
- [108] J. Maynard Smith. *Evolution and the theory of Games*. Cambridge University Press, UK.
- [109] T. Hara and J. Sugie. Stability region for systems of differential-difference equations. *Funkcialaj Ekvacioj*, 39(1):69–86, 1996.
- [110] M. Buslowicz. Simple stability criterion for a class of delay differential systems. *Inter. J. Syst. Sci.*, 18(5):993–995, 1987.
- [111] H. I. Freedman and Y. Kuang. Stability switches in linear scalar neutral delay equations. *Funkcialaj Ekvacioj*, 34(1):187–209, 1991.
- [112] P. Auger. Hawk-dove game and competition dynamics. *Mathl. Comput. Modelling*, 27:89–98, 1998.
- [113] I. Brunetti, R. El-Azouzi, and E. Altman. Altruism in groups: an evolutionary games approach. In *Proceedings of the 7th International Conference on NETWORK Games CONTROL and OPTimization (NETGCOOP 2014)*, Trento, Italy, Oct. 2014.

Bibliography

Publications

JOURNAL PAPERS

- Nesrine Ben Khalifa, Rachid El Azouzi, Yezekael Hayel, and Issam Mabrouki. "Evolutionary Games in Interacting Communities". Accepted in Journal of Dynamic Games and Applications, DGAA, 2016.
- Nesrine Ben Khalifa, Rachid El Azouzi, and Yezekael Hayel. "Discrete and Continuous random time delays in evolutionary dynamics". **Submitted** to Journal of Dynamics and Games.

CONFERENCE PAPERS

- Nesrine Ben Khalifa, Rachid El Azouzi, and Yezekael Hayel. "Hopf Bifurcations in Evolutionary Game Dynamics with Random Delays". Submitted.
- Nesrine Ben Khalifa, Rachid El Azouzi, and Yezekael Hayel. "Random time delays in evolutionary game dynamics". In Proceedings of IEEE Conference on Decision and Control, CDC, Osaka, Japan, December 15-18, 2015, pages 3840-3845.
- Nesrine Ben Khalifa, Rachid El Azouzi, and Yezekael Hayel. "Delayed Evolutionary game dynamics with non-uniform interactions in two communities". In Proceedings of IEEE Conference on Decision and Control, CDC, Los Angeles, CA, USA, December 15-17, 2014, pages 3809-3814.
- Nesrine Ben Khalifa, Rachid El Azouzi, Yezekael Hayel, Habib B. A. Sidi, and Issam Mabrouki. "Evolutionary stable strategies in interacting communities". In Proceedings of the International Conference on Performance Evaluation Methodologies and Tools, ValueTools '13, Torino, Italy, December 10-12, 2013, pages 214-222.

Bibliography
