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Université Pierre et Marie Curie

École doctorale de Sciences Mathématiques de Paris Centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Macarena Peche Irissarry

The reduction of G-ordinary crystalline representations with G-structure

---

dirigée par Christophe Cornut et Laurent Fargues

Soutenue le 15 novembre 2016 devant le jury composé de :

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Résumé

Résumé

Le foncteur D_{cris} de Fontaine nous permet d'obtenir des isocristaux à partir des représentations cristallines. Pour un groupe reductif G, on s'intéresse à étudier la réduction des réseaux dans un germe de représentations cristallines avec G-structure V, vers les réseaux (qui sont des cristaux) avec G-structure contenus dans $D_{\text{cris}}(V)$. En utilisant la théorie des modules de Kisin, on donne une description de cette réduction en termes du groupe G, dans le cas où la représentation est (G-)ordinaire. Pour cela, il faut d'abord généraliser la construction de la filtration de Harder-Narasimhan des groupes p-divisibles, donnée par Fargues, aux modules de Kisin.

Mots-clés

Modules de Kisin, Filtration de Harder-Narasimhan, Représentations cristallines, G-structure.

Abstract

Fontaine's D_{cris} functor allows us to associate an isocrystal to any crystalline representation. For a reductive group G, we study the reduction of lattices inside a germ of crystalline representations with G-structure V to lattices (which are crystals) with G-structure inside $D_{\text{cris}}(V)$. Using Kisin modules theory, we give a description of this reduction in terms of G, in the case when the representation V is (G-)ordinary. In order to do that, first we need to generalize Fargues' construction of the Harder-Narasimhan filtration for p-divisible groups to Kisin modules.

Keywords

Kisin modules, Harder-Narasimhan filtration, Crystalline representations, G-structure.

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Chapter 1

Introduction

1.1 The problem

Integral models of Shimura varieties have been constructed by Kisin in [26] and [27], and by Kisin and Pappas in [28]. Given an integral model, we can ask ourselves how to reduce special cycles of the Shimura variety at places of good reduction. This thesis was motivated by that question, in particular, by the study of 0-dimensional special cycles, which thus correspond to CM points of the Shimura variety. If γ is an isogeny class of such points, then γ may be described as a coset

$$\mathrm{Aut}(\gamma)\backslash\mathcal{X}^p(\gamma)\times\mathcal{X}_p(\gamma).$$

The sets $\mathcal{X}^p(\gamma)$ and $\mathcal{X}_p(\gamma)$ parameter the level structures prime to p and in p, respectively, and together they determine the position of a point in the isogeny class. Langlands made a conjecture in [33] about the mod p points of a Shimura variety. Afterwards, the conjecture was made more precise by Kottwitz ([30]) and then by Langlands and Rapoport ([34]). This conjecture has been proved in the case of Shimura varieties of abelian type by Kisin in [24]. The conjecture implies that the points of the Shimura variety in characteristic p may be described analogously as disjoint union of sets of the form

$$\mathrm{Aut}(\overline{\gamma})\backslash\mathcal{X}^p(\overline{\gamma})\times\mathcal{X}_p(\overline{\gamma}).$$

While the morphism $\mathrm{Aut}(\gamma)\to\mathrm{Aut}(\overline{\gamma})$ and the bijection $\mathcal{X}^p(\gamma)\simeq\mathcal{X}^p(\overline{\gamma})$ are well known, the map

$$\mathrm{red}\ :\ \mathcal{X}_p(\gamma)\to\mathcal{X}_p(\overline{\gamma})$$

remains quite mysterious. The aim is to describe the latter as concretely as possible.
In order to do that, we can use methods from integral p-adic Hodge theory and Bruhat-Tits buildings. In particular, we will be working with Kisin modules and various kinds of well-known filtrations in p-adic Hodge theory.

The D_{cris} functor and Bruhat-Tits theory

Let $G/\mathbb{Z}_p$ be the reductive group such that $G_{\mathbb{Q}_p}$ is the group associated to the Shimura variety. For the integral models constructed by Kisin, there is presumably (see the article by Milne [37]) a Tannakian description for the reduction map. An element $x\in\mathcal{X}_p(\gamma)$ corresponds to a $\otimes$-functor

$$x\ :\ \mathrm{Rep}_{\mathbb{Z}_p}G\to\mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Z}_p}\mathrm{Gal}_K$$

where $\mathrm{Rep}_{\mathbb{Z}_p}G$ is the category of $\mathbb{Z}_p$-linear representations of G and $\mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Z}_p}\mathrm{Gal}_K$ is the category of abelian crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ for some finite extension K of $K_0=W(\mathbb{F})[\frac{1}{p}]$ and $\mathbb{F}=\overline{\mathbb{F}}_p$. Similarly, an element $y\in\mathcal{X}_p(\overline{\gamma})$ corresponds to a $\otimes$-functor

$$y\ :\ \mathrm{Rep}_{\mathbb{Z}_p}G\to\mathrm{Mod}^{\sigma}_{W(\mathbb{F})}$$

where $\mathrm{Mod}^{\sigma}_{W(\mathbb{F})}$ is the category of finite free $W(\mathbb{F})$-modules with a Frobenius isomorphism $\sigma_M : \sigma^* M[\frac{1}{p}] \to M[\frac{1}{p}]$. Here σ is the lifting to $W(\mathbb{F})$ of the Frobenius on $\mathbb{F}$. Then, the reduction map is essentially given by

$$y = D_{\mathrm{cris}} \circ x$$

where D_{cris} is Kisin's integral functor

$$D_{\mathrm{cris}} \ : \ \mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Z}_p} \mathrm{Gal}_K \to \mathrm{Mod}^{\sigma}_{W(\mathbb{F})}$$

described in [6] or [25]. For the Tate module $T_p(X)$ of a p-divisible group over $\mathcal{O}_K$, $D_{\mathrm{cris}}(T_p(X))$ is the Dieudonné crystal of the reduction of X to the residue field $\mathbb{F}$.

Bruhat-Tits buildings provide yet another convenient description of the source and target sets of the reduction map. The source set $\mathcal{X}_p(\gamma) \simeq G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ is simply the $G(\mathbb{Q}_p)$-orbit of the hyperspecial point of the extended Bruhat-Tits building $\mathbf{B}^e(G_{\mathbb{Q}_p})$ which corresponds to $G/\mathbb{Z}_p$. The target set is usually more difficult to grasp, but it may still be embedded in the much larger Bruhat-Tits building $\mathbf{B}^e(G_{K_0})$ of G over K_0. Nevertheless, it also has a simple description under the ordinary condition on γ (that we shall explain later): in this case the target of our reduction map is easy to describe, it is the quotient $M(\mathbb{Q}_p)/M(\mathbb{Z}_p)$ where M is a Levi of a parabolic $P = U \rtimes M$ of G associated to the Newton type of y.
Moreover, $M(\mathbb{Q}_p)/M(\mathbb{Z}_p) \hookrightarrow G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ is a fundamental domain for the action of $U(\mathbb{Q}_p)$ on $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$. Thus, there is a natural retraction

$$G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \twoheadrightarrow M(\mathbb{Q}_p)/M(\mathbb{Z}_p)$$

where the source and target sets respectively correspond to $\mathcal{X}_p(\gamma)$ and $\mathcal{X}_p(\overline{\gamma})$. Our main result, Theorem 8.5.6, gives us:

Theorem 1.1.1. *The above map coincides with the reduction map* $\mathrm{red} : \mathcal{X}_p(\gamma) \to \mathcal{X}_p(\overline{\gamma})$.

The Harder-Narasimhan filtration of a G-Kisin module

The proof of the theorem uses the theory of buildings, Wintenberger's work in [48] on abelian crystalline representations and the factorization

$$D_{\mathrm{cris}} \ : \ \mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Z}_p} \mathrm{Gal}_K \xrightarrow{\mathfrak{M}} \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}} \xrightarrow{\mathrm{mod}\ u} \mathrm{Mod}^{\sigma}_{W(\mathbb{F})}$$

where $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$ is the category of Kisin modules, defined by Kisin in [25] together with the functor $\mathfrak{M}$. We thus have to study G-Kisin modules, i.e. $\otimes$-functors

$$M \ : \ \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$$

and their mod p^n-variants,

$$M_n \ : \ \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{t}}, \quad M_n = M/p^n M$$

where $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{t}}$ is the category of finitely generated $\mathfrak{S}$-modules killed by a power of p and with no u-torsion with a σ-linear morphism φ that becomes an isomorphism after inverting E (the minimal polynomial of an uniformizer of K). Inside this category, there is a full subcategory $\mathrm{Mod}^{\varphi,\mathrm{al}}_{\mathbb{F}[[u]],\mathrm{fr}}$ of the category of p-torsion Kisin modules composed by those called aligned (see Definition 1).

The key point of the proof is to lift the Newton filtration of $\mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}^{\sigma}_{W(\mathbb{F})}$ to a filtration of Harder-Narasimhan type on the G-Kisin module. Harder-Narasimhan filtrations have already been defined in different contexts such as:

- Fargues developed a Harder-Narasimhan theory for finite flat group schemes and for filtered isocrystals in [18] and [19].
- Fargues also defined the Harder-Narasimhan polygon for p-divisible groups in [18, Theorem 2].
- Moonen ([38]) and Shen ([45]) constructed the Hodge-Newton filtration for a p-divisible group with additional structures (which corresponds to the case of PEL Shimura varieties), under the ordinarity condition. This filtration lifts the Newton filtration of the isocrystal. Shen's result is more general, since it proves that whenever the Hodge and Newton filtration of the filtered isocrystal coincide in a break point, we can lift the sub-filtered isocrystal to a sub-p-divisible group.

1.2 The results

1.2.1 Settings

In chapter 2, we present basic notions about filtrations and lattices on vector spaces and the space of types of those filtrations (which are principal objects in this thesis). An important operator defined in this chapter is the relative position between two lattices, denoted by Pos. This operator will be used many times in order to define Harder-Narasimhan filtrations on categories of Kisin modules (more precisely, it will useful to define degree functions), and to define Hodge types on those categories, since Pos can be viewed as the type of a filtration. Also, we recall the Harder-Narasimhan formalism given by André in [2] and we give the tools from the theory of Bruhat-Tits buildings that we will need.

In chapter 3, we start by setting the p-adic Hodge theory framework in which we will be working. We recall some properties of modules over the ring $\mathfrak{S} = W(\mathbb{F})[[u]]$ for $\mathbb{F}$ a perfect field of characteristic $p > 0$ and $W(\mathbb{F})$ the ring of Witt vectors over $\mathbb{F}$. Let K be a finite extension of $K_0 = \operatorname{Frac} W(\mathbb{F})$, with uniformizer π_K. Recall that a Kisin module is a finite free $\mathfrak{S}$-module M together with a Frobenius morphism

$$\varphi_M \ : \ \varphi^* M[\tfrac{1}{E}] \xrightarrow{\sim} M[\tfrac{1}{E}]$$

where $E \in \mathfrak{S}$ is the minimal polynomial of π_K. These modules were defined by Kisin in [25]. In the same article, he also constructed a functor

$$\mathfrak{M} \ : \ \operatorname{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p} \operatorname{Gal}_K \to \operatorname{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$$

and such that $\mathfrak{M}(L)/u\mathfrak{M}(L)[\frac{1}{p}] \simeq D_{\mathrm{cris}}(L \otimes \mathbb{Q}_p)$ for any $L \in \operatorname{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p} \operatorname{Gal}_K$. Some other categories of Kisin modules that will appear are: the categories of isogeny classes of Kisin modules that we denote by $\operatorname{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$ and $\operatorname{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$, the category of p^∞-torsion Kisin modules that we denote by $\operatorname{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{t}}$ (formed by finitely generated $\mathfrak{S}$-modules killed by a power of p with no u^∞-torsion, endowed with a Frobenius) and the category of p-torsion Kisin modules denoted by $\operatorname{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ (formed by finite free $\mathbb{F}[[u]]$-modules endowed with a Frobenius). We also describe the category MHP of Hodge-Pink modules, defined by Genestier and Lafforgue in [22]. In their article, they also gave $\otimes$-functors

$$\operatorname{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}} \to \mathrm{MHP} \to \mathrm{MF}^{\sigma}_{K}$$

where MF^{σ}_{K} is the category of filtered isocrystals. Then, they proved that there are equivalence of categories between Hodge-Pink modules verifying Griffiths transversality condition and filtered isocrystals, and which sends weakly admissible Hodge-Pink modules (that they define) to weakly admissible filtered isocrystals.

All the following categories admit Harder-Narasimhan filtrations:

$$\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\mathrm{Gal}_K \overset{\simeq\ \mathrm{wa}}{\longleftarrow} \mathrm{MF}^{\sigma}_K \overset{\simeq}{\longleftarrow} \mathrm{MHP}^{\mathrm{wa,Gr}} \hookrightarrow \mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p \hookrightarrow \mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]},$$

where $\mathrm{MHP}^{\mathrm{wa,Gr}}$ denotes the category of weakly admissible Hodge-Pink modules verifying the Griffiths transversality condition. In particular, the Harder-Narasimhan filtration on $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\mathrm{Gal}_K$ is given by Fargues in [18], where he also showed that this filtration is compatible with tensor products and with the change of extension K. In Proposition 3.7.1, we show that all the filtrations defined in the categories above are compatible with the functors between the categories, thus the filtrations are all compatible with tensor products, by Fargues' result ([18, Corollaire 6]).
Motivated by the study of special points of Shimura varieties, we want to be able to change the extension K over K_0. For that reason, we define the germ of crystalline representations (Definition 7.3.8) that we can think of as pairs (V,ρ) such that $(V,\rho) \in \mathrm{Rep}_{\mathbb{Q}_p}\mathrm{Gal}_K$ for a sufficiently large finite extension K of K_0. We denote by $\mathrm{Rep}_{\mathbb{Q}_p}\{\mathrm{Gal}_{K_0}\}$ the category of germs of crystalline representations. In Proposition 3.8.1 we prove that the Fargues filtration on crystalline representations extends to a filtration on germs of crystalline representation. In Lemma 3.8.2, following Liu's results in [35], we have that the functor $\mathrm{Rep}_{\mathbb{Z}_p}\mathrm{Gal}_K \to \mathrm{Mod}^{\sigma}_{W(\mathbb{F})}$, constructed by Kisin, induces a $\otimes$-functor

$$\mathrm{Rep}_{\mathbb{Z}_p}\{\mathrm{Gal}_{K_0}\} \to \mathrm{Mod}^{\sigma}_{W(\mathbb{F})}.$$

1.2.2 Chapters 4, 5 and 6: Fargues filtrations on Kisin modules

Chapter 4 is divided principally in two parts: at the beginning we give a degree and rank function on objects in $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ by

$$\mathrm{rank}(M,\varphi_M) = \mathrm{rank}_{\mathbb{F}[[u]]} M \quad \text{and} \quad \deg(M,\varphi_M) = \mathrm{Pos}(M,\varphi_M\varphi^* M).$$

These functions define a slope function in the sense of André's formalism and thus Harder-Narasimhan filtration $\mathcal{F}_{\mathrm{F},1}(M)$ and its polygon $\mathbf{t}_{\mathrm{F},1}(M)$ on each object $M \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$. This is a generalization of the results by Fargues in [19], who gave this construction for finite flat group schemes (which are related to p^∞-torsion Kisin modules in the same way p-divisible groups corresponds to Kisin modules of height 1) and, for this reason, we call Fargues filtration the Harder-Narasimhan filtrations defined on categories of Kisin modules. In a second part, we define two types of p-torsion Kisin modules, those who are aligned and those who are flat. These two properties are given as conditions about the polygons associated to relative positions between M and the image of the n-th iteration of its Frobenius, but they also have a geometric interpretation in the Bruhat-Tits building associated to the $\mathbb{F}((u))$-vector space $M[\frac{1}{u}]$: we say that a module is aligned when for n large enough and $m \geq 1$, the modules $\varphi^{mn}M$ are in a same geodesic ray together with M, when we see these modules as points in the Bruhat-Tits building of G. We denote by $\mathrm{Mod}^{\varphi,\mathrm{al}}_{\mathbb{F}[[u]],\mathrm{fr}}$ the category of p-torsion Kisin modules which are aligned. The main result in this section is the following:

Theorem 1.2.1. *The Fargues filtration on the category* $\mathrm{Mod}^{\varphi,\mathrm{al}}_{\mathbb{F}[[u]],\mathrm{fr}}$ *is compatible with tensor products, exterior and symmetric powers.*

The proof of this theorem follows the ideas of Totaro's compatibility of semi-stable objects with tensor products, given in [47]. First, adapting the ideas in [12], we prove that an object is semi-stable of slope μ (for the slope function defined above) if and only if we have

$$\langle M, \varphi_V M; \Xi\rangle + \langle V(-\mu), \Xi\rangle \leq 0$$

for every $\mathbb{R}$-filtration Ξ defined on $V = M[\frac{1}{u}]$ which is φ_V-stable, $V(-\mu)$ the filtration with just one jump at $-\mu$, $\langle -, - \rangle$ is the scalar product between two filtrations (see section 2.6) and $\langle -, -; \Xi \rangle$ a certain scalar product that we will define in section 4.5. The second step is to prove that for $M = M_1 \otimes M_2$ with M_1 and M_2 semi-stable of slope, respectively μ_1 and μ_2, and for every $\Xi = \Xi_1 \otimes \Xi_2$ where Ξ_1 and Ξ_2 are φ-stable filtrations on $M_1[\frac{1}{u}]$ and $M_2[\frac{1}{u}]$, respectively, the condition above is verified if we take $\mu = \mu_1 + \mu_2$. Now, buildings get in the game. There is a convex projection p sending a φ-stable filtration Ξ on M to $p(\Xi) = \Xi_1 \otimes \Xi_2$ for some φ-stable filtrations Ξ_1, Ξ_2 on, respectively, M_1 and M_2. This projection verifies that

$$\langle \mathcal{F}, \Xi \rangle \leq \langle \mathcal{F}, p(\Xi) \rangle$$

for every φ-stable filtration Ξ and every filtration $\mathcal{F}$ which is already decomposed, i.e. the tensor product of two filtrations, see [11, 5.7.7]. In particular, it is true for

$$\mathcal{F} = V(-\mu) = V(-\mu_1) \otimes V(-\mu_2).$$

To finish, we can prove that for M an aligned p-torsion Kisin module, we have

$$\langle M, \varphi_V M; \Xi \rangle \leq \langle M, \varphi_V M; p(\Xi) \rangle$$

for every φ_V-stable filtration Ξ on M. The hypothesis aligned is needed in order to have a good behavior of the Frobenius iterates of M inside the building, with respect to the filtration Ξ.

Near the end of the work in this thesis, it came to our attention that an article on this subject has been published on arXiv, by Levin and Wang Erickson, see [17]. Their theorem seems more general than ours but their proof uses a different construction.

In chapter 5, we work with p^∞-torsion Kisin modules. We want to construct the Fargues filtration but we need to define our degree function in a different (but equivalent) way, as we cannot define relative positions on objects in p^∞-torsion $\mathfrak{S}$-modules in general. For a p^∞-torsion Kisin module (M, φ_M) we define the i-th twist of (M, φ_M) by

$$(M(i), \varphi_{M(i)}) = (M, u^i \varphi_M),$$

for every $i \in \mathbb{Z}$. We can thus reduce ourselves to an effective module and define the rank and degree functions of $(M, \varphi_M) \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$ by

$$\mathrm{rank}(M, \varphi_M) = \mathrm{length}_{W(\mathbb{F})}(M/uM) \quad \text{and} \quad \deg(M, \varphi_M) = \mathrm{length}_{\mathfrak{S}}(Q(i)) + i\,\mathrm{rank}(M, \varphi_M)$$

for i large enough and $Q(i) = \mathrm{coker}\,\varphi_{M(i)}$. The degree function does not depend on i for a sufficiently large $i \in \mathbb{Z}$. This construction gives the Fargues filtration $\mathcal{F}_{\mathrm{F,t}}(M)$ and its polygon $\mathbf{t}_{\mathrm{F,t}}(M)$ for every object of $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$.

In chapter 6, we follow the strategy given by Fargues in [18] and Shen in [45] to construct a Harder-Narasimhan filtration on p-divisible groups under some hypothesis. For a Kisin module $M = (M, \varphi_M) \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$, we use the polygons defined on the p^∞-torsion Kisin modules $M/p^n M$ to define a polygon $\mathbf{t}_{\mathrm{F},\infty}(M)$ on M, and we prove that this polygon does not depend on the isogeny class of M. Even though the category $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$ is not quasi-abelian, we have good degree and rank functions

$$\mathrm{rank}(M, \varphi_M) = \mathrm{rank}_{\mathfrak{S}}\, M \quad \text{and} \quad \deg(M, \varphi_M) = \mathrm{Pos}(M \otimes \hat{\mathfrak{S}}, \varphi_M(\varphi^* M \otimes \hat{\mathfrak{S}}))$$

for $\hat{\mathfrak{S}}$ the completion of $\mathfrak{S}[\frac{1}{p}]$ with respect to the ideal generated by E. Again, this functions only depend on the isogeny class of M and gives us an invariant μ_M. We can define a notion of

semi-stability using this invariant and we see that it coincides with the semi-stability on M/pM or M/p^nM. We define HN-type Kisin modules as the ones for which there exists a filtration by Kisin modules and whose graded pieces are semi-stable Kisin modules. In the rest of the chapter, we prove two important results:

1. Proposition 6.4.3: A Kisin module M is HN-type if and only if $\mathbf{t}_\infty(M) = \mathbf{t}_{\mathrm{F},1}(M)$,
2. Theorem 6.5.3: Every Kisin module is isogenous to a HN-type Kisin module.

Both proofs are made in the same spirit as the proofs given by Fargues and Shen for p-divisible groups.

1.2.3 Charpter 7: The reduction of germs of G-crystalline representations

In chapter 7, we will be working with objects with G-structure, for G a reductive group over $\mathbb{Z}_p$ (related to the reductive group in the Shimura datum). We start then by presenting the filtrations, graduations and space of types for fiber functors

$$\omega_R \ : \ \mathrm{Rep}_{\mathcal{O}}\, G \to \mathrm{Bun}_R$$

where $\mathcal{O} \in \{\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{F}_p\}$ and R is a ring. Then, we use the result given by Broshi in [7] to prove the following result, which will be a key tool for us:

Proposition 1.2.2. *Let R be a local strictly henselian and faithfully flat $\mathcal{O}$-algebra. Then, any exact and faithful $\otimes$-functor as above is $\otimes$-isomorphic to the trivial fiber function $\omega_{G,R}$ (the forgetful functor).*

Even if we start working with a trivial (or isomorphic to trivial) crystalline representation with G-structure, à priori, the functors that we will apply to it (Fontaine's D_{cris} functor, Kisin's $\mathfrak{M}$ functor, etc.) will change the functor to a non-trivial one. However, the proposition above will always allow us to reduce to the (isomorphic to the) trivial case. This is very important as the results we will be using from different authors are only stated for the trivial fiber functor. Another important result given in this section is a generalization of Haboush theorem, using Seshadri's results in [44]:

Proposition 1.2.3. *Let L be a field which is an $\mathcal{O}$-algebra. Let $\Gamma \subset \mathbb{R}$ be a subring. Suppose that a fiber functor $\omega_L : \mathrm{Rep}_{\mathcal{O}}\, G \to \mathrm{Vect}_L$ admits a factorization through an additive $\otimes$-functor*

$$\mathcal{F} \ : \ \mathrm{Rep}_{\mathcal{O}}\, G \to \mathrm{Fil}_L^{\Gamma}$$

which is compatible with exterior and symmetric powers. Then $\mathcal{F}$ is exact, thus a filtration on ω_L.

We use this proposition to prove that the Fargues filtrations and Hodge filtrations defined in previous sections can be generalized to filtrations on the object with G-structure.
We define the germs of crystalline representations as faithful $\otimes$-functors

$$V : \mathrm{Rep}_{\mathbb{Q}_p}\, G \to \mathrm{Rep}_{\mathbb{Q}_p}\{\mathrm{Gal}_{K_0}\}.$$

Using Fontaine's functor, we can associate to V a filtered isocrystal with G-structure, that we denote by D. Let $\mathbf{C}^{\mathbb{Q}}(G)$ be a closed Weyl chamber for the split group G over K_0, equipped with the dominance order (this is the set where the types of filtrations on objects with G-structure live). Then, we define the ordinary germs of crystalline representation with G-structure as those verifying

$$\mathbf{t}_{\mathrm{N}}(D) = \mathbf{t}_{\mathrm{H}}(D)^{\#}$$

in $\mathbf{C}^{\mathbb{Q}}(G)$, where $\mathbf{t}_{\mathrm{N}}(D)$ and $\mathbf{t}_{\mathrm{H}}(D)$ are the Newton and Hodge types associated to the Newton graduation and Hodge filtration of D, and $\#$ is the average of the Galois orbits of a type. In this case, we can already say something about the reduction map: in Corollary 7.3.24, we prove that for an ordinary V, there is a factorization of the reduction map $\mathrm{red} : \mathcal{L}(V) \to \mathcal{L}(D)$ by

$$\mathcal{L}(V) \to \mathcal{L}(D, \mathrm{t}^{\iota}_{\mathrm{H}}(D)) \hookrightarrow \mathcal{L}(D)$$

where $\mathcal{L}(V)$ and $\mathcal{L}(D)$ are the notations that we will use in the text for the sets corresponding to $\mathcal{X}_p(\gamma)$ and $\mathcal{X}_p(\overline{\gamma})$ in the case of points of Shimura varieties, and $\mathcal{L}(D, \mathrm{t}^{\iota}_{\mathrm{H}}(D))$ is the subset of lattices y in D such that $\mathrm{Pos}(y, \sigma_D y) = \mathbf{t}_{\mathrm{H}}(D)$. Note that this factorization is always true (without the ordinary condition) in the case coming from Shimura varieties, as the type $\mathbf{t}^{\iota}_{\mathrm{H}}(D)$ is minuscule in that situation. We also prove the following theorem (Theorem 7.3.25):

Theorem 1.2.4. *Suppose V is ordinary, let $x : \mathrm{Rep}_{\mathbb{Z}_p} \to \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p} \mathrm{Gal}_K$ for some K such that V is defined over it. Let $M = \mathfrak{M} \circ x$ and $\overline{M} = M/pM$. Then:*

1. *The p-torsion Kisin module with G-structure $\overline{M}$ is aligned.*
2. *The Kisin module M is HN-type. Therefore its Fargues filtration $\mathcal{F}_{\mathrm{F}}(M)$ exists.*

In the end of this chapter we define two operators $\Phi^s_{\acute{e}t}$ and Φ^s_{cris} on, respectively, lattices inside V and lattices inside D and prove in Proposition 7.3.27 that they commute with the reduction map red in the ordinary case. Then, we obtain the following result (Theorem 7.3.30):

Theorem 1.2.5. *For V an ordinary germ of crystalline representations with G-structure, the map* red *admits a factorization*

$$\begin{tikzcd} \mathcal{L}(V) \arrow[rr, "\mathrm{red}"] \arrow[dr, two heads, "\pi"] & & \mathcal{L}(D, \mathbf{t}^{\iota}_{\mathrm{H}}(D)). \\ & U(\mathbb{Q}_p)\backslash\mathcal{L}(V) \arrow[ur] & \end{tikzcd}$$

where U is the unipotent radical of the parabolic subgroup of G stabilizing the Fargues filtration of V.

1.2.4 Chapter 8: The abelian case

In this chapter we study the particular case when the germ of crystalline representation is abelian, which is the case that interested us originally (since it is the one coming from CM points in Shimura varieties). In [43, 2], Fontaine proves, building on Serre's results in [42], that there is an equivalence of categories

$$V^u_K \ : \ \mathrm{Rep}_{\mathbb{Q}_p} T_K \xrightarrow{\sim} \mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Q}_p} \mathrm{Gal}_K,$$

where $T_K = \varprojlim_{E \subset K} \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_{m,E})$. Then, in [48], Wintenberger constructs a $\otimes$-functor

$$D_{\pi_K} \ : \ \mathrm{Rep}_{\mathbb{Q}} T_K \to \mathrm{Mod}^{\sigma}_{W(\mathbb{F})}$$

and proves that when we restrict ourselves to a finitely generated tensor subcategory $\mathcal{V}$ of $\mathrm{Rep}_{\mathbb{Q}_p} T_K$, we have a isomorphism of $\otimes$-functors

$$D_{\pi_K|\mathcal{V}} \simeq (D_{\mathrm{cris}} \circ V^u_K)_{|\mathcal{V}}.$$

We generalize both constructions to germs of crystalline representation with G-structure, in Proposition 8.2.1 and section 8.3.3, and then give an explicit description, as cocharacters, of the Hodge and Newton types in Proposition 8.3.3. In Proposition 8.3.5, we prove that in this case,

the Fargues filtration defined on a germ of crystalline representation with G-structure coincides with the opposed Newton filtration of the G-isocrystal D associated to it.

For the rest of the chapter we restrict ourselves to the ordinary case. Using the explicit description of G-isocrystals given by Kottwitz in [29], [31] and [32], and by Rapoport and Richartz in [40], we are able to say a little bit more about the reduction map. Putting together the results in section 8.5, we obtain the main theorem (Theorem 8.5.6):

Theorem 1.2.6. *The reduction map*

$$\mathrm{red} \ : \ \mathcal{L}(V) \to \mathcal{L}(D, \mathbf{t}^{\iota}_{\mathrm{H}}(D))$$

factors through an $M(\mathbb{Q}_p)$-equivariant bijection

$$U_{\mathcal{F}_{\mathrm{F}}}(\mathbb{Q}_p)\backslash\mathcal{L}(V) \simeq \mathcal{L}(D, \mathbf{t}^{\iota}_{\mathrm{H}}(D)).$$

Chapter 2

Preliminaries

Most of the structures described in this section can be found in [2] and [11].

2.1 Space of types and the dominance order

Let Γ be a nonzero subring of $\mathbb{R}$ and $r \in \mathbb{N}$. We define the space of Γ-types of length r in three different but equivalent ways :

1. The cone $\Gamma^r_{\geq} = \{(\gamma_1, \ldots, \gamma_r) \in \Gamma^r \mid \gamma_1 \geq \ldots \geq \gamma_r\}$.

2. Consider first the group-ring $\mathbb{Z}[\Gamma] = \left\{\mathbf{t} = \sum_{\gamma \in \Gamma} n_\gamma \cdot e^\gamma \mid n_\gamma \in \mathbb{Z} \text{ and } \mathbf{t} \text{ has finite support}\right\}$. Inside the group-ring, we have the space of positive elements

$$\mathbb{N}[\Gamma] = \left\{\mathbf{t} = \sum_{\gamma \in \Gamma} n_\gamma \cdot e^\gamma \mid n_\gamma \in \mathbb{N} \text{ and } \mathbf{t} \text{ has finite support}\right\}$$

   and finally, the space of types of length r is given by

$$\mathbb{N}[\Gamma]^r = \left\{\mathbf{t} = \sum_{\gamma \in \Gamma} n_\gamma \cdot e^\gamma \mid \sum_{\gamma \in \Gamma} n_\gamma = r \text{ and } n_\gamma \in \mathbb{N}\right\},$$

   i.e. it is the subspace of positive elements of degree r.

3. Concave polygons : Continuous functions $\mathbf{t} : [0, r] \to \mathbb{R}$ with $f(0) = 0$, which are affine on $[i-1, i]$ for every $1 \leq i \leq r$ with slopes $\gamma_i \in \Gamma$ verifying $\gamma_1 \geq \ldots \geq \gamma_r$.

We can also define the entire space of Γ-types as the union of the spaces of types of length r, for all $r \in \mathbb{N}$. In the second case, the entire space of Γ-types corresponds to $\mathbb{N}[\Gamma]$. We will drop the notation Γ when there is no confusion about the indexation of the type. We recall some of the operations that we can consider in the space of types.

Degree

The degree of a type is defined by

1. The map

$$\begin{array}{rccc} \deg & : & \Gamma^r_{\geq} & \to & \Gamma \\ & & (\gamma_1, \ldots, \gamma_r) & \mapsto & \sum_{i=1}^r \gamma_i \end{array}$$

2. The map
$$\begin{array}{rccc} \deg & : & \mathbb{N}[\Gamma] & \to & \Gamma \\ & & \sum_{\gamma\in\Gamma} n_\gamma \cdot e^\gamma & \mapsto & \sum_{\gamma\in\Gamma} n_\gamma \cdot \gamma \end{array}$$
3. For a concave polygon, its degree corresponds to the y-coordinate of its ending point.

Degree functions will appear everywhere in this thesis so, in this section, we will show how the degree of a type behave with respect to other operators that will be defined.

Dominance order

We can define a partial order relation called the dominance order over the space of types of length r:

(1) In the cone, the order is given by $\gamma \leq \delta$ if and only if $\sum_{i=1}^k \gamma_i \leq \sum_{i=1}^k \delta_i$ for all $1 \leq k \leq r$ with equality when $k = r$, where $\gamma = (\gamma_1, \ldots, \gamma_r)$ and $\delta = (\delta_1, \ldots, \delta_r)$ are two elements of $\Gamma^r_\geq$.

(3) Viewed as concave polygons this partial order corresponds to saying that δ is above γ and they both have the same extremities.

If $\mathbf{t}_1 \leq \mathbf{t}_2$, then $\deg \mathbf{t}_1 = \deg \mathbf{t}_2$.

Addition

The addition of two types of length r is given by:

(1) The map
$$\begin{array}{rccc} + & \Gamma^r_\geq \times \Gamma^r_\geq & \to & \Gamma^r_\geq \\ & (\gamma, \delta) & \mapsto & \gamma + \delta = (\gamma_1 + \delta_1, \ldots, \gamma_r + \delta_r) \end{array}$$
where $\gamma = (\gamma_1, \ldots, \gamma_r)$ and $\delta = (\delta_1, \ldots, \delta_r)$.

(3) As concave polygons, the addition corresponds to the usual addition of functions.

From the formula above, it is easy to see that
$$\deg(\gamma + \delta) = \deg \gamma + \deg \delta.$$

Norm

We define the norm of a type as
$$\begin{array}{rccc} ||\cdot|| & : & \Gamma^r_\geq & \to & \mathbb{R}_+ \\ & & (\gamma_1, \ldots, \gamma_r) & \mapsto & \sqrt{\gamma_1^2 + \ldots + \gamma_r^2} \end{array}$$
and it verifies two properties.

1. We have $||\mathbf{t}_1 + \mathbf{t}_2|| \leq ||\mathbf{t}_1|| + ||\mathbf{t}_2||$ for all $\mathbf{t}_1, \mathbf{t}_2 \in \Gamma^r_\geq$, since $||\cdot||$ is the euclidean distance in $\mathbb{R}^r_+ \subset \mathbb{R}^r$.
2. For $\mathbf{t}_1, \mathbf{t}_2 \in \Gamma^r_\geq$, the inequality for the dominance order $\mathbf{t}_1 \leq \mathbf{t}_2$ implies an inequality for the respective norms $||\mathbf{t}_1|| \leq ||\mathbf{t}_2||$, moreover $||\mathbf{t}_1|| = ||\mathbf{t}_2||$ implies that $\mathbf{t}_1 = \mathbf{t}_2$. To prove that, let $(\gamma_1, \ldots, \gamma_r) \leq (\delta_1, \ldots, \delta_r)$, thus $\Gamma_i = \sum_{j\leq i} \gamma_j \leq \Delta_i = \sum_{j\leq i} \delta_j$, and the equality holds for $i = r$. Then
$$\begin{array}{rcl} \sum_{i=1}^r \delta_i^2 - \sum_{i=1}^r \gamma_i^2 & = & \sum_{i=1}^r (\delta_i - \gamma_i)(\delta_i + \gamma_i) \\ & = & \sum_{i=1}^r (\Delta_i - \Gamma_i)(\delta_i + \gamma_i) - \sum_{i=1}^{r-1} (\Delta_i - \Gamma_i)(\delta_{i+1} + \gamma_{i+1}) \\ & = & \sum_{i=1}^{r-1} (\Delta_i - \Gamma_i)(\delta_i - \delta_{i+1} + \gamma_i - \gamma_{i+1}) \\ & \geq & 0, \end{array}$$
and the equality holds if and only if $(\Delta_i - \Gamma_i)(\delta_i - \delta_{i+1} + \gamma_i - \gamma_{i+1}) = 0$ for $1 \leq i \leq r-1$, which means that $(\gamma_1, \ldots, \gamma_r) = (\delta_1, \ldots, \delta_r)$.

Multiplication by scalars

The multiplication by scalars is given by

1. For a type $(\gamma_1, \ldots, \gamma_r)$ and $c > 0$ an element in Γ, then $c \cdot (\gamma_1, \ldots, \gamma_r) = (c\gamma_1, \ldots, c\gamma_r)$.
2. For $\sum_{\gamma \in \Gamma} n_\gamma \cdot e^\gamma \in \mathbb{N}[\Gamma]$ and $c > 0$ an element in Γ, then $c \cdot \sum_{\gamma \in \Gamma} n_\gamma \cdot e^\gamma = \sum_{\gamma \in \Gamma} n_\gamma \cdot e^{c\gamma}$.

For the degree, we have $\deg(c\gamma) = c \deg \gamma$.

Involution

We define the involution of a type as:

(1) For a type $\gamma = (\gamma_1, \ldots, \gamma_r) \in \Gamma^r_{\geq}$, it is the map

$$\begin{array}{rcccc} \cdot^\iota & : & \Gamma^r_{\geq} & \to & \Gamma^r_{\geq} \\ & & \gamma & \mapsto & \gamma^\iota = (-\gamma_r, \ldots, -\gamma_1) \end{array}.$$

(2) For an element of $\mathbb{N}[\Gamma]$, it is the map

$$\begin{array}{rcccc} \cdot^\iota & : & \mathbb{N}[\Gamma] & \to & \mathbb{N}[\Gamma] \\ & & \sum_{\gamma \in \Gamma} n_\gamma \cdot e^\gamma & \mapsto & \sum_{\gamma \in \Gamma} n_\gamma \cdot e^{-\gamma} \end{array}$$

The degree verifies $\deg(\gamma^\iota) = -\deg \gamma$.

Concatenation

We define the concatenation of two types as

(1) In the cone, the map

$$\begin{array}{rcccc} * & : & \Gamma^r_{\geq} \times \Gamma^s_{\geq} & \to & \Gamma^{r+s}_{\geq} \\ & & (\gamma, \delta) & \mapsto & \gamma * \delta = (\epsilon_{\sigma(1)}, \ldots, \epsilon_{\sigma(r+s)}) \end{array}$$

where if $\gamma = (\gamma_1, \ldots, \gamma_r)$, $\delta = (\delta_1, \ldots, \delta_s)$, then $(\epsilon_1, \ldots, \epsilon_{r+s}) = (\gamma_1, \ldots, \gamma_r, \delta_1, \ldots, \delta_s)$ and $\sigma \in \mathfrak{S}_{r+s}$ is a permutation such that $\epsilon_{\sigma(1)} \geq \ldots \geq \epsilon_{\sigma(r+s)}$. From this description, we see that

$$c(\gamma * \delta) = c\gamma * c\delta$$

for every γ, δ in the space of types and every $c > 0$ in Γ.

(2) Inside $\mathbb{N}[\Gamma]$, it is the usual addition.

(3) As polygons, the concatenation is the smallest concave polygon above the collection of points $\{(i + j, \gamma(i) + \delta(j))\}_{\substack{0 \leq i \leq r \\ 0 \leq j \leq s \\ i,\, j \in \mathbb{N}}}$ or, equivalently, the polygon defined as the function

$$\begin{array}{rcccc} \gamma * \delta & : & [0, r+s] & \to & \mathbb{R} \\ & & x & \mapsto & \sup\limits_{\substack{a+b=x \\ 0 \leq a \leq r \\ 0 \leq b \leq s}} \gamma(a) + \delta(b) \end{array}$$

From the first formula, we get $\deg(\gamma * \delta) = \deg(\gamma) + \deg(\delta)$.

Tensor product

The tensor product of two types is given by

(1) In the cone, the map

$$\begin{array}{rccc} \otimes & : & \Gamma^r_{\geq} \times \Gamma^s_{\geq} & \to & \Gamma^{rs}_{\geq} \\ & & (\gamma, \delta) & \mapsto & \gamma \otimes \delta = (\epsilon_{\sigma(1)}, \ldots, \epsilon_{\sigma(rs)}) \end{array}$$

where if $\gamma = (\gamma_1, \ldots, \gamma_r)$, $\delta = (\delta_1, \ldots, \delta_s)$, then

$$(\epsilon_1, \ldots, \epsilon_{rs}) = (\gamma_1 + \delta_1, \ldots, \gamma_1 + \delta_s, \ldots, \gamma_r + \delta_1, \ldots, \gamma_r + \delta_s)$$

and $\sigma \in \mathfrak{S}_{rs}$ is a permutation such that $\epsilon_{\sigma(1)} \geq \ldots \geq \epsilon_{\sigma(rs)}$. From this description, we see that

$$c(\gamma \otimes \delta) = c\gamma \otimes c\delta$$

for every γ, δ in the space of types and every $c > 0$ in Γ.

(2) Inside $\mathbb{N}[\Gamma]$, it is given by the usual multiplication on the group-ring $\mathbb{Z}[\Gamma]$ ($\mathbb{N}[\Gamma]$ is stable under this multiplication).

Again, from the first formula, we have $\deg(\gamma \otimes \delta) = s \deg \gamma + r \deg \delta$, for γ and δ of length r and s, respectively.

Exterior powers

For $k \geq 1$, we define the k-th exterior power of a type by

$$\begin{array}{rccc} \Lambda^k & : & \Gamma^r_{\geq} & \to & \Gamma^s_{\geq} \\ & & (\gamma_1, \ldots, \gamma_r) & \mapsto & (\underline{\gamma}_{\sigma(1)}, \ldots, \underline{\gamma}_{\sigma(s)}) \end{array}$$

where $s = \binom{r}{k}$, $\{\underline{\gamma}_1, \ldots, \underline{\gamma}_s\} = \{\gamma_{i_1} + \ldots + \gamma_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq r\}$ and $\sigma \in \mathfrak{S}_s$ is a permutation such that $\underline{\gamma}_{\sigma(1)} \geq \ldots \geq \underline{\gamma}_{\sigma(s)}$. If we take k bigger than the length of the type, we will get a zero as its k-th exterior power.
A straightforward calculation gives us

$$\deg(\Lambda^k \gamma) = \binom{r-1}{k-1} \deg \gamma.$$

In particular, for $k = r$, we have $\deg(\Lambda^k \gamma) = \deg \gamma$.

Symmetric powers

For $k \geq 1$, we define the k-th symmetric power of a type by

$$\begin{array}{rccc} \mathrm{Sym}^k & : & \Gamma^r_{\geq} & \to & \Gamma^s_{\geq} \\ & & (\gamma_1, \ldots, \gamma_r) & \mapsto & (\underline{\gamma}_{\sigma(1)}, \ldots, \underline{\gamma}_{\sigma(s)}) \end{array}$$

where $s = \binom{r+k-1}{k}$, $\{\underline{\gamma}_1, \ldots, \underline{\gamma}_s\} = \{\gamma_{i_1} + \ldots + \gamma_{i_k} \mid 1 \leq i_1 \leq \ldots \leq i_k \leq r\}$ and $\sigma \in \mathfrak{S}_s$ is a permutation such that $\underline{\gamma}_{\sigma(1)} \geq \ldots \geq \underline{\gamma}_{\sigma(s)}$.
A straightforward calculation gives us

$$\deg(\mathrm{Sym}^k \gamma) = r \binom{r+k-1}{k-1} \deg \gamma.$$

2.2 Filtrations

Let Γ a non zero subgroup of $\mathbb{R}$, let K be a field and V a finite dimensional K-vector space. A Γ-filtration $\mathcal{F}$ on V is a collection of K-subspaces $(\mathcal{F}^{\geq\gamma}V)_{\gamma\in\Gamma}$ of V which is decreasing (i.e. $\mathcal{F}^{\geq\gamma'}V \subset \mathcal{F}^{\geq\gamma}V$ for $\gamma' \geq \gamma$), separated (i.e. $\mathcal{F}^{\geq\gamma} = 0$ for γ large enough), exhaustive (i.e. $\mathcal{F}^{\geq\gamma}V = V$ for γ small enough) and left continuous (i.e. for all $\gamma \in \Gamma$, there exists $\varepsilon > 0$ such that for every γ' with $\gamma - \epsilon \leq \gamma' \leq \gamma$, we have $\mathcal{F}^{\geq\gamma'}V = \mathcal{F}^{\geq\gamma}V$).

For a filtration $\mathcal{F}$ of V, we define

$$\mathcal{F}^{>\gamma}V := \bigcup_{\gamma'>\gamma} \mathcal{F}^{\geq\gamma'}V \quad \text{and} \quad \mathrm{Gr}^{\gamma}_{\mathcal{F}} V = \mathcal{F}^{\geq\gamma}V/\mathcal{F}^{>\gamma}V.$$

We thus have a short exact sequence of K-vector spaces

$$0 \to \mathcal{F}^{>\gamma}V \to \mathcal{F}^{\geq\gamma}V \to \mathrm{Gr}^{\gamma}_{\mathcal{F}} V \to 0.$$

This yields a Γ-graded K-vector space associated to $\mathcal{F}$,

$$\mathrm{Gr}_{\mathcal{F}} V := \bigoplus_{\gamma\in\Gamma} \mathrm{Gr}^{\gamma}_{\mathcal{F}} V = \mathrm{Gr}^{\gamma_1}_{\mathcal{F}} V \oplus \ldots \oplus \mathrm{Gr}^{\gamma_s}_{\mathcal{F}} V$$

where $\{\gamma_1, \ldots, \gamma_s\} = \{\gamma \mid \mathrm{Gr}^{\gamma}_{\mathcal{F}} V \neq 0\}$ are called the breaks of $\mathcal{F}$. The type of $\mathcal{F}$ (or, equivalently, the type of $\mathrm{Gr}_{\mathcal{F}}$) is defined by $\mathbf{t}(\mathcal{F}) = \mathbf{t}(\mathrm{Gr}_{\mathcal{F}}) = (\gamma_1, \ldots, \gamma_s)$ where each γ_i for $1 \leq i \leq s$ is written as many times as the dimension of $\mathrm{Gr}^{\gamma_i}_{\mathcal{F}} V$. This is an element in the space of types of length $r = \dim_K V$ defined in last subsection and we have

$$\mathbf{t}(\mathcal{F}) = \textstyle\sum_{\gamma\in\Gamma} \dim_K(\mathrm{Gr}^{\gamma}_{\mathcal{F}} V) \cdot e^{\gamma} \quad \text{in} \quad \mathbb{N}[\Gamma].$$

We define the degree of the filtration as the degree of $\mathbf{t}(\mathcal{F})$. Thus

$$\deg(\mathcal{F}) = \sum_{\gamma\in\Gamma} \dim_K(\mathrm{Gr}^{\gamma}_{\mathcal{F}} V) \cdot \gamma.$$

Example 2.2.1. For $\gamma \in \Gamma$, Denote by $V(\gamma)$ the filtration of V such that $\mathrm{Gr}^{\gamma}_{V(\gamma)} = V$ and $\mathrm{Gr}^{\gamma'}_{V(\gamma)} = 0$ if $\gamma' \neq \gamma$. For this filtration, we have

$$\begin{aligned} \mathbf{t}(V(\gamma)) &= \dim_K V \cdot e^{\gamma} \\ \deg(V(\gamma)) &= \dim_K V \cdot \gamma. \end{aligned}$$

Proposition 2.2.1. *Some important properties of the filtrations are the following:*

1. *Let*

$$0 \to W \to V \to V/W \to 0$$

be an exact sequence of K-vector spaces and let $\mathcal{F}_V$ be a filtration on V. We also consider the filtrations on W and V/W induced by $\mathcal{F}_V$ and denote them by $\mathcal{F}_W$ and $\mathcal{F}_{V/W}$ respectively. Then, the types verify

$$\mathbf{t}(\mathcal{F}_V) \leq \mathbf{t}(\mathcal{F}_W) * \mathbf{t}(\mathcal{F}_{V/W}).$$

2. *Given a filtration $\mathcal{F}_i$ on a K-vector space V_i, for $1 \leq i \leq n$, we can define a filtration $\mathcal{F} = \oplus_{i=1}^{n}\mathcal{F}_i$ on $V = \oplus_{i=1}^{n} V_i$ by*

$$\mathcal{F}^{\gamma}(V) = \oplus_{i=1}^{n}\mathcal{F}_i^{\gamma}(V_i)$$

and it verifies

$$\mathbf{t}(\mathcal{F}) = \mathbf{t}(\mathcal{F}_1) * \ldots * \mathbf{t}(\mathcal{F}_n).$$

3. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two filtrations on a K-vector space V. Then we have*

$$\mathbf{t}(\mathcal{F}_2) \leq \mathbf{t}(\mathrm{Gr}_{\mathcal{F}_1}(\mathcal{F}_2))$$

where $\mathrm{Gr}_{\mathcal{F}_1}(\mathcal{F}_2)$ is the filtration that $\mathcal{F}_2$ induces on $\mathrm{Gr}_{\mathcal{F}_1} V$. This property is a consequence of the last two properties.

4. *Given a filtration $\mathcal{F}_i$ on a K-vector space V_i, for $i = 1, 2$, we can define a filtration $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$ on $V = V_1 \otimes V_2$ by*

$$\mathcal{F}^\gamma(V) = \sum_{\substack{\gamma_1 + \gamma_2 = \gamma \\ \gamma_1, \gamma_2 \in \Gamma}} \mathcal{F}_1^{\gamma_1}(V_1) \otimes \mathcal{F}_2^{\gamma_2}(V_2).$$

Thus

$$\mathrm{Gr}_{\mathcal{F}}^\gamma(V) = \bigoplus_{\substack{\gamma_1 + \gamma_2 = \gamma \\ \gamma_1, \gamma_2 \in \Gamma}} \mathrm{Gr}_{\mathcal{F}_1}^{\gamma_1}(V_1) \otimes \mathrm{Gr}_{\mathcal{F}_2}^{\gamma_2}(V_2)$$

and

$$\mathbf{t}(\mathcal{F}) = \mathbf{t}(\mathcal{F}_1) \otimes \mathbf{t}(\mathcal{F}_2).$$

5. *Given a filtration $\mathcal{F}$ on a K-vector space V, we can define a filtration $\mathrm{Sym}^k \mathcal{F}$ on $\mathrm{Sym}^k V$ for every $k \geq 1$ by taking the image of $\mathcal{F}$ by $V^{\otimes k} \twoheadrightarrow \mathrm{Sym}^k V$, and we have*

$$\mathbf{t}(\mathrm{Sym}^k \mathcal{F}) = \mathrm{Sym}^k(\mathbf{t}(\mathcal{F})).$$

6. *Given a filtration $\mathcal{F}$ on a K-vector space V, we can define a filtration $\Lambda^k \mathcal{F}$ on $\Lambda^k V$ for every $k \geq 1$ by taking the image of $\mathcal{F}$ by $V^{\otimes k} \twoheadrightarrow \Lambda^k V$, and we have*

$$\mathbf{t}(\Lambda^k \mathcal{F}) = \Lambda^k(\mathbf{t}(\mathcal{F})).$$

2.3 Lattices

Suppose $\mathcal{O}_K$ is a discrete valuation ring with uniformizer u, fraction field K and residue field k. Let M_1 and M_2 be two $\mathcal{O}_K$-lattices inside the same finite dimensional K-vector space V.

Let $\overline{M}_1 = M_1/uM_1$. We can define a $\mathbb{Z}$-filtration $\mathcal{F}(M_1, M_2)$ on the k-vector space $\overline{M}_1$ by

$$\mathcal{F}^i(M_1, M_2) = \frac{u^i M_2 \cap M_1 + uM_1}{uM_1}$$

for every $i \in \mathbb{Z}$. The filtration $\mathcal{F}(M_1, M_2)$ allows us to define two operators over the lattices, by

$$\mathrm{Pos}(M_1, M_2) = \mathbf{t}(\mathcal{F}(M_1, M_2)) \quad \text{and} \quad \nu(M_1, M_2) = \deg(\mathcal{F}(M_1, M_2)).$$

We call $\mathrm{Pos}(M_1, M_2)$ the relative position of M_1 and M_2 and we are going to explain the reason why. There exists a basis $\{e_1, \ldots, e_r\}$ of V adapted to M_1 and M_2, meaning that

$$M_1 = \bigoplus_{i=1}^r \mathcal{O}_K \cdot e_i, \quad M_2 = \bigoplus_{i=1}^r \mathcal{O}_K \cdot u^{-a_i} e_i$$

for some $a_i \in \mathbb{Z}$ verifying $a_1 > \ldots > a_r$. Then $(a_1, \ldots, a_n)$ does not depend upon the chosen basis. In fact:

$$\mathrm{Pos}(M_1, M_2) = (a_1, \ldots, a_r) \quad \text{and} \quad \nu(M_1, M_2) = \sum_{i=1}^r a_i$$

Indeed,

$$\begin{array}{rcl} u^k M_2 \cap M_1 & = & \bigoplus_{i=1}^r \mathcal{O}_K \cdot u^{\max\{k-a_i,0\}} e_i \\ u^k M_2 \cap M_1 + uM_1 & = & \bigoplus_{i=1}^r \mathcal{O}_K \cdot u^{\min\{1,\max\{k-a_i,0\}\}} e_i \end{array}$$

thus

$$\mathcal{F}^k(M_1, M_2) = \bigoplus_{i=1}^r k \cdot \left(\begin{array}{ll} 0 & \text{if } k > a_i \\ \overline{e}_i & \text{if } k \leq a_i \end{array} \right),$$

where $\overline{e}_i$ is the image of e_i in $\overline{M}_1$, so

$$\mathrm{Gr}^k_{\mathcal{F}(M_1,M_2)} = \bigoplus_{i=1}^r k \cdot \left(\begin{array}{ll} 0 & \text{if } k \neq a_i \\ \overline{e}_i & \text{if } k = a_i \end{array} \right),$$

hence $\mathbf{t}(\mathcal{F}(M_1, M_2)) = (a_1, \ldots, a_r)$.

Remark 1. If $M_1 \subset M_2$, then $\mathrm{Pos}(M_1, M_2)$ corresponds to the invariant factors (given by the structure theorem for finitely generated torsion modules over a principal ideal domain) of the quotient $Q = M_2/M_1$ and $\nu(M_1, M_2)$ corresponds to the length of Q.

The relative position of two lattices is an element of the space of $\mathbb{Z}$-types. For two $\mathcal{O}_K$-lattices M_1, M_2 inside V, we can also define the operator

$$\mathrm{d}(M_1, M_2) = ||\,\mathrm{Pos}(M_1, M_2)||$$

where $||\cdot||$ is the norm of a type.

Lemma 2.3.1. *Let M_1, M_2, M_3 be $\mathcal{O}_K$-lattices in a K-vector space. Then:*

1. *The relative position verifies the triangular inequality, i.e.*

$$\mathrm{Pos}(M_1, M_3) \leq \mathrm{Pos}(M_1, M_2) + \mathrm{Pos}(M_2, M_3).$$

2. *The operator* d *verifies the triangular inequality*

$$\mathrm{d}(M_1, M_3) \leq \mathrm{d}(M_1, M_2) + \mathrm{d}(M_2, M_3),$$

thus d *is a distance.*

Proof. The triangular inequality is given in [11, 6.1]. For the second point, we have

$$\begin{array}{rcl} \mathrm{d}(M_1, M_3) & = & ||\,\mathrm{Pos}(M_1, M_3)|| \\ & \leq & ||\,\mathrm{Pos}(M_1, M_2) + \mathrm{Pos}(M_2, M_3)|| \\ & \leq & ||\,\mathrm{Pos}(M_1, M_2)|| + ||\,\mathrm{Pos}(M_2, M_3)|| \\ & = & \mathrm{d}(M_1, M_2) + \mathrm{d}(M_2, M_3). \end{array}$$

where the first and second inequalities are given, respectively, by the first and second properties of the norm listed in 2.1.

□

We study some properties of the filtration associated to two lattices in the next proposition and, afterwards, we will study the properties of the relative position.

Proposition 2.3.2. *The filtration $\mathcal{F}(M_1, M_2)$ verifies the following properties:*

1. *It is compatible with tensor products, i.e. for M_1, M_2 (resp. M_1', M_2') two $\mathcal{O}_K$-lattices in V (resp. V'), we have*

$$\mathcal{F}(M_1 \otimes M_1', M_2 \otimes M_2') = \mathcal{F}(M_1, M_2) \otimes \mathcal{F}(M_1', M_2').$$

2. *Let $0 \to W \to V \xrightarrow{\pi} V/W \to 0$ be an exact sequence of K-vector spaces, M_1, M_2 two lattices in V, and let $N_i = M_i \cap W$ and $Q_i = \pi(M_i)$ for $i = 1, 2$. Suppose there exists a basis adapted to M_1, M_2 and W. Then, there is an exact sequence*

$$0 \to \mathcal{F}^i(N_1, N_2) \to \mathcal{F}^i(M_1, M_2) \to \mathcal{F}^i(Q_1, Q_2) \to 0$$

for each $i \in \mathbb{Z}$.

3. *Symmetric powers: for M_1 and M_2 two $\mathcal{O}_K$-lattices in V, we have*

$$\mathcal{F}(\mathrm{Sym}^k M_1, \mathrm{Sym}^k M_2) = \mathrm{Sym}^k(\mathcal{F}(M_1, M_2))$$

for every $k \geq 1$.

4. *Exterior powers: for M_1 and M_2 two $\mathcal{O}_K$-lattices in V, we have*

$$\mathcal{F}(\Lambda^k M_1, \Lambda^k M_2) = \Lambda^k(\mathcal{F}(M_1, M_2))$$

for every $k \geq 1$.

5. *Direct sums: for M_1, M_2 (resp. M_1', M_2') two $\mathcal{O}_K$-lattices in V (resp. V'), we have*

$$\mathcal{F}^i(M_1 \oplus M_1', M_2 \oplus M_2') = \mathcal{F}^i(M_1, M_2) \oplus \mathcal{F}^i(M_1', M_2')$$

for every $i \in \mathbb{Z}$.

6. *Graduations: For a filtration $\mathcal{F}$ and an $\mathcal{O}_K$-lattice M on a K-vector space V, we define an $\mathcal{O}_K$-lattice in $\mathrm{Gr}^\gamma_{\mathcal{F}} V$ by $\mathrm{Gr}^\gamma_{\mathcal{F}} M = (M \cap \mathcal{F}^{\geq\gamma})/(M \cap \mathcal{F}^{>\gamma}) \subset \mathrm{Gr}^\gamma_{\mathcal{F}} V$. Then, for two $\mathcal{O}_K$-lattices M_1, M_2 in V such that there exists an adapted basis for M_1, M_2 and $\mathcal{F}$, we have*

$$\mathcal{F}(\mathrm{Gr}_{\mathcal{F}} M_1, \mathrm{Gr}_{\mathcal{F}} M_2) = \oplus_{\gamma\in\Gamma} \mathcal{F}(\mathrm{Gr}^\gamma_{\mathcal{F}} M_1, \mathrm{Gr}^\gamma_{\mathcal{F}} M_2)$$

Proof. 1. There is a natural map

$$\mathcal{F}(M_1, M_2) \otimes \mathcal{F}(M_1', M_2') \to \mathcal{F}(M_1'', M_2'')$$

for $(M_1'', M_2'') = (M_1 \otimes M_1', M_2 \otimes M_2')$, given by the inclusion

$$\sum_{i+j=l} \frac{M_1 \cap u^i M_2 + uM_1}{uM_1} \otimes \frac{M_1' \cap u^j M_2' + uM_1'}{uM_1'} \hookrightarrow \frac{M_1'' \cap u^l M_2'' + uM_1''}{uM_1''}.$$

It is easy to check this inclusion, since in $M_1''[\frac{1}{u}]$ we have

$$(M_1 \cap u^i M_2) \otimes (M_1' \cap u^j M_2') = M_1'' \cap u^l M_2'' \cap (M_1 \otimes u^j M_2') \cap (u^i M_2 \otimes M_1')$$

for some fixed $i + j = l$, because M_1 and M_1' are $\mathcal{O}_K$-lattices and the tensor product commute with intersections of $\mathcal{O}_K$-lattices. Taking basis adapted to (M_1, M_2) and (M_1', M_2') we get a basis adapted to (M_1'', M_2'') and we check easily that the natural inclusion is an isomorphism.

2. Let $\{e_1, \ldots, e_n\}$ be a basis adapted to M_1, M_2 and W, which means that

$$\begin{array}{rclrcl} M_1 & = & \langle u^{a_1} e_1, \ldots, u^{a_n} e_n \rangle & M_1 \cap W & = & \langle u^{a_1} e_1, \ldots, u^{a_d} e_d \rangle \\ M_2 & = & \langle e_1, \ldots, e_n \rangle & M_2 \cap W & = & \langle e_1, \ldots, e_d \rangle \end{array}$$

for some $a_i \in \mathbb{Z}$ for $1 \leq j \leq n$. Then,

$$\mathcal{F}^i(M_1, M_2) = \bigoplus_{j=1}^{n} k \cdot \frac{u^i e_j \cap u^{a_j} e_j + u^{a_j+1} e_j}{u^{a_j+1} e_j},$$

thus

$$\mathcal{F}^i(M_1, M_2) \cap W = \bigoplus_{j=1}^{d} k \cdot \frac{u^i e_j \cap u^{a_j} e_j + u^{a_j+1} e_j}{u^{a_j+1} e_j}$$

and

$$\pi(\mathcal{F}^i(M_1, M_2)) = \bigoplus_{j=d+1}^{n} k \cdot \frac{u^i e_j \cap u^{a_j} e_j + u^{a_j+1} e_j}{u^{a_j+1} e_j}.$$

On the other hand, we have a basis adapted to Q_1 and Q_2 given by $\{u^{a_{d+1}} e_{d+1}, \ldots, u^{a_n} e_n\}$ and $\{e_{d+1}, \ldots, e_n\}$. Using this basis and the adapted basis for $M_1 \cap W$ and $M_2 \cap W$, it is easy to see that

$$\begin{array}{rcl} \mathcal{F}^i(Q_1, Q_2) & = & \pi(\mathcal{F}^i(M_1, M_2)) \\ \mathcal{F}^i(N_1, N_2) & = & \mathcal{F}^i(M_1, M_2) \cap W \end{array}$$

for every $i \in \mathbb{Z}$.

3. By (2), we just need to find an adapted base to $M_1^{\otimes k}$, $M_2^{\otimes k}$ and the kernel of $M[\frac{1}{u}]^{\otimes k} \to \operatorname{Sym}^k M[\frac{1}{u}]$. Let $\{e_1, \ldots, e_r\}$ a basis of M_1 and $\{u^{a_1} e_1, \ldots, u^{a_r} e_r\}$ a basis of M_2, with $a_1 \geq \ldots \geq a_r$. Then, we have basis adapted to $M_1^{\otimes k}$ and $M_2^{\otimes k}$ given by

$$\{e_{\underline{i}}\}_{\underline{i}=(i_1,\ldots,i_k)\in[1,\ldots,r]^k}$$

$$\{u^{a(\underline{i})} e_{\underline{i}}\}_{\underline{i}=(i_1,\ldots,i_k)\in[1,\ldots,r]^k}$$

where $e_{\underline{i}} = e_{i_1} \otimes \ldots \otimes e_{i_k}$ and $a(\underline{i}) = a_{i_1} + \ldots + a_{i_k}$. Then, if we take $\sigma(\underline{i})$ the permutation of $\underline{i}$ such that $(\sigma(i_1) \leq \ldots \leq \sigma(i_k))$, the basis $\{e'_{\underline{i}}\}_{\underline{i}\in[1,\ldots,r]^k}$ given by

$$e'_{\underline{i}} = \begin{cases} e_{\underline{i}} & \text{if} \quad \underline{i} = \sigma(\underline{i}) \\ e_{\underline{i}} - e_{\sigma(\underline{i})} & \text{if} \quad \underline{i} \neq \sigma(\underline{i}) \end{cases}$$

we get a basis adapted to $M_1^{\otimes k}$, $M_2^{\otimes k}$ and the kernel of $M[\frac{1}{u}]^{\otimes k} \to \operatorname{Sym}^k M[\frac{1}{u}]$.

4. For Λ^k, the proof works exactly the same as the proof for the symmetric powers, by taking the adapted base $\{e'_{\underline{i}}\}_{\underline{i}\in[1,\ldots,r]^k}$ given by

$$e'_{\underline{i}} = \begin{cases} e_{\underline{i}} & \text{if} \quad i_1 < \ldots < i_k \\ e_{\underline{i}} - \epsilon(\sigma) e_{\sigma(\underline{i})} & \text{otherwise} \end{cases}$$

where σ is the permutation such that $\sigma(i_1) < \ldots < \sigma(i_k)$.

5. The direct sum commutes with intersections of $\mathcal{O}_K$-lattices, so we have $(u^i M_2 \cap M_1) \oplus (u^i M'_2 \cap M'_1) = u^i (M_2 \oplus M'_2) \cap (M_1 \oplus M'_1)$, thus an isomorphism

$$\frac{M_1 \cap u^i M_2 + u M_1}{u M_1} \oplus \frac{M'_1 \cap u^i M'_2 + u M'_1}{u M'_1} \simeq \frac{M''_1 \cap u^i M''_2 + u M''_1}{u M''_1}$$

for every $i \in \mathbb{Z}$, where $M''_1 = M_1 \oplus M'_1$ et $M''_2 = M_2 \oplus M''_2$.

6. It is a consequence of (2) and (5).

□

Proposition 2.3.3. *The relative position verifies the following properties:*

1. *The relative position is antisymmetric, i.e. for M_1, M_2 two $\mathcal{O}_K$-lattices, we have*

$$\mathrm{Pos}(M_2, M_1) = \mathrm{Pos}(M_1, M_2)^\iota$$

as types.

2. *Let $0 \to W \to V \xrightarrow{\pi} V/W \to 0$ be an exact sequence of K-vector spaces, M_1, M_2 two lattices in V, and let $N_i = M_i \cap W$ and $Q_i = \pi(M_i)$ for $i = 1, 2$. We have*

$$\mathrm{Pos}(N_1, N_2) * \mathrm{Pos}(Q_1, Q_2) \leq \mathrm{Pos}(M_1, M_2)$$

with an equality if there exists a basis adapted to M_1, M_2 and W.

3. *Let M_1, M_2 (respectively, M_1', M_2') be two $\mathcal{O}_K$-lattices inside a K-vector space V (respectively, V'). Then we have*

$$\mathrm{Pos}(M_1 \oplus M_1', M_2 \oplus M_2') = \mathrm{Pos}(M_1, M_2) * \mathrm{Pos}(M_1', M_2')$$

as types.

4. *Let M_1, M_2 (respectively, M_1', M_2') be two $\mathcal{O}_K$-lattices in the K-vector space V (respectively, V'). Then we have*

$$\mathrm{Pos}(M_1 \otimes M_1', M_2 \otimes M_2') = \mathrm{Pos}(M_1, M_2) \otimes \mathrm{Pos}(M_1', M_2')$$

as types.

5. *Let $\mathcal{F}$ a filtration on a K-vector space V and M a $\mathcal{O}_K$-lattice in V. Then, for M_1, M_2 two $\mathcal{O}_K$-lattices in V, we have*

$$\mathrm{Pos}(\mathrm{Gr}_{\mathcal{F}} M_1, \mathrm{Gr}_{\mathcal{F}} M_2) \leq \mathrm{Pos}(M_1, M_2)$$

as types, with equality if there exists a basis adapted to M_1, M_2 and $\mathcal{F}$.

Proof. 1. It is an easy calculation using adapted basis.

2. It can be found in [11, Proposition 99].

3. It is a consequence of the properties of the type of a filtration given in 2.2.1 and the properties of the filtration listed in 2.3.2.

4. It is also a consequence of 2.2.1 and 2.3.2.

5. It comes from (2) and (3).

□

Definition 2.3.1. Given a K-vector space V, a $\mathbb{Z}$-filtration $\mathcal{F}$ on V and an $\mathcal{O}_K$-lattice M in V, we can define another $\mathcal{O}_K$-lattice by

$$M + \mathcal{F} = \sum_{i \in \mathbb{Z}} u^{-i} M \cap \mathcal{F}^i V.$$

A basis adapted to an $\mathcal{O}_K$-lattice M and a $\mathbb{Z}$-filtration $\mathcal{F}$ on V is a basis $\langle a_0, \ldots, a_n \rangle$ of M such that

$$\mathcal{F}^i V = \bigoplus_{k=0}^{\dim_K \mathcal{F}^i V} K \cdot a_k.$$

Proposition 2.3.4. *The addition operator defined above verifies the following properties:*

1. *We have* $\mathrm{Pos}(M, M+\mathcal{F}) = \mathbf{t}(\mathcal{F})$.
2. *It is compatible with tensor products: we have*

$$(M_1 \otimes M_2) + (\mathcal{F}_1 \otimes \mathcal{F}_2) = (M_1 + \mathcal{F}_1) \otimes (M_2 + \mathcal{F}_2)$$

for $M_1 \subset V_1$, $M_2 \subset V_2$ $\mathcal{O}_K$*-lattices and* $\mathcal{F}_1, \mathcal{F}_2$ *filtrations on the* K*-vector spaces* V_1 *and* V_2*, respectively.*

Proof. These are both easy computations with adapted basis.

□

2.4 Quasi-abelian categories

Let $\mathcal{C}$ be a category with a null object 0, i.e. an object that is both initial and terminal. Then, there are also zero morphisms 0 in $\mathcal{C}$, which are the composition of the morphisms $M \to 0 \to N$ for any two objects M and N of $\mathcal{C}$.

Let $M \xrightarrow{f} N$ be a morphism in $\mathcal{C}$. A kernel $\ker f$ of f is a morphism with codomain M such that $f \circ \ker f = 0$ which is universal for this property. We also call kernel of f the domain of this morphism. A morphism f is called mono (or monic) if for any two morphisms g, h, we have that $f \circ g = f \circ h$ implies $g = h$. A kernel is always mono and it is called a strict mono. Its domain is called a strict subobject of its codomain.

Dualy, a cokernel $\operatorname{coker} f$ of f is a morphism with domain N such that $\operatorname{coker} f \circ f = 0$ which is universal for this property. We also call cokernel of f the codomain of this morphism. A morphism f is called epi if for any two morphisms g, h, we have that $g \circ f = h \circ f$ implies $g = h$. A cokernel is always epi and it is called a strict epi. Its codomain is called a strict quotient of its domain.

Definition 2.4.1. A pre-abelian category is an additive category such that any morphism has a kernel and a cokernel.

In a pre-abelian category C, we can define flags and exact sequences. A flag of length r on an object M of $\mathcal{C}$ is a finite sequence

$$F(M) \ : \ 0 = M_0 \hookrightarrow M_1 \hookrightarrow \ldots \hookrightarrow M_r = M$$

of strict subobjects of M such that $M_i \neq M_{i-1}$ for $1 \leq i \leq r$. A short exact sequence, denoted by

$$0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$$

is a pair of composable morphisms (f, g) such that $f = \ker g$ and $g = \operatorname{coker} f$.

Let $P \xrightarrow{f} Q \xleftarrow{g} N$ be a pair of morphisms with common codomain. A pull-back is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f'} & N \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ P & \xrightarrow{f} & Q, \end{array}$$

that is universal. We say that f' is the pull-back of f by g. Dually, for a pair $P \xleftarrow{g'} M \xrightarrow{f'} N$ of morphisms with a common domain, one has the dual notion of push-outs.

The pull-back of a strict mono always exists and is a strict mono. Dually, the push-out of a strict epi g' always exists and is a strict epi.

Definition 2.4.2. A quasi-abelian category is a pre-abelian category where every pull-back of a strict epi is a strict epi and every push-out of a strict mono is a strict mono.

For a pre-abelian category and a morphism $M \xrightarrow{f} N$, we set

$$\operatorname{coim} f = \operatorname{coker} \ker f \quad \text{and} \quad \operatorname{im} f = \ker \operatorname{coker} f \ .$$

For any morphism $M \xrightarrow{f} N$, there is a unique factorization

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, two heads, "\text{strict epi}"'] & N \\ \operatorname{coim} f \arrow[r, "\overline{f}"] & \operatorname{im} f \arrow[u, hook, "\text{strict mono}"'] \end{tikzcd}$$

and $\overline{f}$ is mono-epi if $\mathcal{C}$ is a quasi-abelian category.

In André's article, we can find an equivalent characterization of quasi-abelian category.

Proposition 2.4.1. *An additive category $\mathcal{C}$ is quasi-abelian if and only if it can be fully embedded in an Abelian category $\mathcal{A}$ with the following properties :*

1. *any object of $\mathcal{A}$ is a quotient of an object of $\mathcal{C}$,*
2. *there is a strictly full subcategory $\mathcal{T} \subset \mathcal{A}$ (the torsion subcategory) such that*
   - *any object $A \in \mathcal{A}$ sits in a unique (up to unique isomorphism) short exact sequence*

     $$0 \to A_{\text{tors}} \to A \to M \to 0$$

     where $A_{\text{tors}} \in Ob\mathcal{T}$ and $M \in Ob\mathcal{C}$.
   - *there are no nonzero morphisms from objects of $\mathcal{T}$ to objects of $\mathcal{C}$.*

Using this characterization, it is easy to give the description of kernel, cokernel, coimage and image. Let $M \xrightarrow{f} N$ a morphism in $\mathcal{C}$. Then

1. The object $\ker f$ correspond to the kernel viewed as a morphism in $\mathcal{A}$, since in the exact sequence $0 \to \ker f_{\text{tors}} \to \ker f \to Q \to 0$ for $\ker f$ an object in $\mathcal{A}$, we must have $\ker f_{\text{tors}} = 0$ because otherwise, there would be a mono $\ker f_{\text{tors}} \hookrightarrow M$ and there are no nonzero morphisms from $\mathcal{T}$ to $\mathcal{C}$.
2. For the cokernel, let Q be the cokernel of f viewed as a morphism in $\mathcal{A}$. Then, there is an exact sequence $0 \to Q_{\text{tors}} \to Q \to Q' \to 0$ with Q' an object in $\mathcal{C}$, and we have $\operatorname{coker} f = Q'$ in $\mathcal{C}$. Indeed, let $M \xrightarrow{f} N \xrightarrow{p} P$ in $\mathcal{C}$ such that $p \circ f = 0$, then in $\mathcal{A}$ we have a morphism $Q \to P$ which factors through Q' since $Q' = Q/Q_{\text{tors}}$ and there is no nonzero morphism $Q_{\text{tors}} \to P$.
3. The coimage $\operatorname{coim} f = M/\ker f$ since this is a subobject of N and thus in the exact sequence $0 \to (M/\ker f)_{\text{tors}} \to M/\ker f \to P \to 0$ we must have $(M/\ker f)_{\text{tors}} = 0$ so there is no nonzero morphism from $\mathcal{T}$ to $\mathcal{C}$.

4. For the image, we need to introduce the notion of saturation. Let M be an object of $\mathcal{A}$, then for any subobject M' of M, we denote by $(M')^{\mathrm{sat}}$ the subobject of M such that $(M')^{\mathrm{sat}}/M' = (M/M')_{\mathrm{tors}}$. It is called the $\mathcal{T}$-saturation of M' in M. Then, the image of f is given by $\operatorname{im} f = f(M)^{\mathrm{sat}}$, the $\mathcal{T}$-saturation of $f(M)$ in N, where $f(M)$ is the image of f computed in $\mathcal{A}$.

2.5 The Harder-Narasimhan formalism

In this section, we will introduce the Harder-Narasimhan formalism given by André in [2] and afterwards, we will add some extra conditions that will be verified for all the categories that will appear, and some properties of the Harder-Narasimhan filtration and polygon considering those extra conditions.

André's formalism

Let $\mathcal{C}$ be a quasi-abelian category and Γ a $\mathbb{Q}$-subspace of $\mathbb{R}$. We denote by $\operatorname{sk}\mathcal{C}$ the skeleton of $\mathcal{C}$, i.e. the isomorphism classes of objects of $\mathcal{C}$.

Definition 2.5.1. A *rank function* on $\mathcal{C}$ is a map

$$\operatorname{rank} \ : \ \operatorname{sk}\mathcal{C} \to \mathbb{N}$$

that is additive on short exact sequences and takes the value 0 only on the 0 object.

A *degree function* on $\mathcal{C}$, with values in Γ, is a map

$$\deg : \ \operatorname{sk}\mathcal{C} \to \Gamma$$

taking value 0 at the 0 object (but, unlike the case of the rank function, we do not require that the 0 object is the only element taking value 0), that satisfies the following two conditions:

1. It is additive on short exact sequences.
2. For any epi-monic $M_1 \to M_2$, one has $\deg(M_1) \leq \deg(M_2)$.

Given a degree and rank function, we can associate a slope function

$$\mu = \frac{\deg}{\operatorname{rank}} \ : \ \operatorname{sk}\mathcal{C}\backslash\{0\} \to \mathbb{Q}.$$

Two important properties of the slope function are :

- For any short exact sequence $0 \to M' \to M \to M'' \to 0$ of nonzero objects, one has

  $$\min\{\mu(M'), \mu(M'')\} \leq \mu(M) \leq \max\{\mu(M'), \mu(M'')\},$$

  both inequalities being strict unless $\mu(M') = \mu(M) = \mu(M'')$.

- More generally, for any flag $0 = M_0 \hookrightarrow M_1 \hookrightarrow \ldots \hookrightarrow M_r = M$ with nonzero quotients M_i/M_{i-1}, one has

  $$\min(\mu(M_i/M_{i-1}))_{1\leq i\leq r} \leq \mu(M) \leq \max(\mu(M_i/M_{i-1}))_{1\leq i\leq r},$$

  both inequalities being strict unless all the $\mu(M_i/M_{i-1})$ are equal to $\mu(M)$.

Definition 2.5.2. A non-zero object M is called μ-semistable (or just semistable when there is no ambiguity) when for every strict subobject $M' \subset M$, we have

$$\mu(M') \leq \mu(M).$$

We denote by $\mathcal{C}(\gamma)$ the full subcategory of objects of $\mathcal{C}$ which are μ-semistable of slope $\gamma \in \Gamma$, together with the zero object.

Remark 2.

1. The semi-stability condition is equivalent to ask that for any strict quotient Q of M, we have
$$\mu(M) \leq \mu(Q),$$
by the property of the slope function over exact sequences given above.

2. As a consequence, we have that if M is semi-stable of slope μ, then for every strict subobject N (resp. strict quotient M/N) of M such that $\mu(N) = \mu$ (resp. $\mu(M/N) = \mu$), we have that N (resp. M/N) is also semi-stable.

For the rest of the section, suppose we are given a rank function on $\mathcal{C}$. A Γ-filtration on $\mathcal{C}$ is a functor

$$\mathcal{F}^{\geq} \ : \ \Gamma^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

which sends any object (γ, M) to a strict subobject $\mathcal{F}^{\geq\gamma}M$ of M and verifying:

- It is decreasing, i.e. $\mathcal{F}^{\geq\gamma'}M \subset \mathcal{F}^{\geq\gamma}M$ for $\gamma' \geq \gamma$ for any M in $\mathcal{C}$.
- It is separated, i.e. $\varprojlim \mathcal{F}^{\geq\gamma}M = 0$ for any M in $\mathcal{C}$.
- It is exhaustive, i.e $\varinjlim \mathcal{F}^{\geq\gamma}M = M$ for any M in $\mathcal{C}$.
- It is left continuous, i.e. $\mathcal{F}^{\geq\gamma}M = \varprojlim_{\gamma'<\gamma} \mathcal{F}^{\geq\gamma'}M$ for any M in $\mathcal{C}$.

For a Γ-filtration $\mathcal{F}$, we define

$$\mathcal{F}^{>\gamma}M := \varinjlim_{\gamma'>\gamma} \mathcal{F}^{\geq\gamma'}M \quad \text{and} \quad \mathrm{Gr}^{\gamma}_{\mathcal{F}} M := \mathcal{F}^{\geq\gamma}M/\mathcal{F}^{>\gamma}M \ .$$

Then, we can define the type and degree associated to a Γ-filtration $\mathcal{F}$ as we did in 2.2, by

$$\begin{array}{rcl} \mathbf{t}(\mathcal{F}(M)) = \sum_{\gamma\in\Gamma} \mathrm{rank}(\mathrm{Gr}^{\gamma}_{\mathcal{F}} M) \cdot e^{\gamma} & \text{in} & \mathbb{N}[\Gamma] \\ \deg(M) = \sum_{\gamma\in\Gamma} \mathrm{rank}(\mathrm{Gr}^{\gamma}_{\mathcal{F}} M) \cdot \gamma & \text{in} & \Gamma \end{array}$$

for every object M of $\mathcal{C}$, and a function $\mu(M) = \frac{\deg M}{\mathrm{rank}\, M}$. Using these functions, we define the slope Γ-filtrations.

Definition 2.5.3. A slope Γ-filtration $\mathcal{F}^{\geq}$ on $\mathcal{C}$ is a Γ-filtration on $\mathcal{C}$ satisfying

1. for any γ, the filtration of $\mathcal{F}^{\geq\gamma}M$ (resp. $M/\mathcal{F}^{\geq\gamma}$) is induced by the filtration of M,
2. the associated function μ is a slope function in the sense of Definition 2.5.1.

Since the rank bounds the length of any flag on an object M of $\mathcal{C}$, it is easy to see that for any Γ-filtration, and any object M, there is a partition by intervals

$$I_r =]-\infty, \gamma_r], \ldots, I_2 =]\gamma_2, \gamma_1], I_1 =]\gamma_1, +\infty]$$

such that $\mathcal{F}^{\geq\gamma}M$ is constant on each interval, and a flag of length r

$$F(M) \ : \ 0 \hookrightarrow M_1 = \mathcal{F}^{\geq\gamma_1}M \hookrightarrow \ldots \hookrightarrow M_r = \mathcal{F}^{\geq\gamma_r}M = M.$$

Proposition 2.5.1. *Let $\mathcal{F}^{\geq\cdot}$ be a slope filtration on $\mathcal{C}$ and let M be a nonzero object of $\mathcal{C}$. The flag*

$$F(M) \ : \ 0 \hookrightarrow \mathcal{F}^{\geq\gamma_1}M = M_1 \hookrightarrow \ldots \hookrightarrow \mathcal{F}^{\geq\gamma_r}M = M_r = M$$

attached to $\mathcal{F}^{\geq\cdot}$ is the unique flag on M whose graded pieces M_i/M_{i-1} for $1 \leq i \leq r$ are semistable of decreasing slopes

$$\gamma_1 > \ldots > \gamma_r.$$

We can now state the main theorem, which gives us the Harder-Narasimhan filtration.

Theorem 2.5.2. *[2, 1.4.7] The construction $\mathcal{F} \mapsto \mu$ yields a bijection between slope filtrations on $\mathcal{C}$ and slope functions on $\mathcal{C}$.*

The slope Γ-filtration $\mathcal{F}^{\geq\cdot}$ of the theorem is called the Harder-Narasimhan filtration of M and the slopes $\gamma_1 > \ldots > \gamma_r$ are called the breaks of the filtration $\mathcal{F}$. Sometimes, we will refer to the flag associated to $\mathcal{F}^{\geq\cdot}$ as the Harder-Narasimhan filtration of M.

Proposition 2.5.3. *Let $\mathcal{F}_{\mathrm{HN}}$ be the Harder-Narasimhan filtration of an object M of $\mathcal{C}$ and $\mathbf{t}_{\mathrm{HN}}(M)$ its type viewed as a polygon. Then $(\operatorname{rk} M', \deg M')$ lies below $\mathbf{t}_{\mathrm{HN}}(M)$ for every strict subobject M' of M. Moreover, $\mathbf{t}_{\mathrm{HN}}(M)$ is the convex hull of such points.*

Proposition 2.5.4. *The Harder-Narasimhan filtration verifies the following properties:*

1. *Let*
$$0 \to M_1 \to M_2 \to M_3 \to 0$$
be an exact sequence of objects of $\mathcal{C}$. Then we have
$$\mathbf{t}_{\mathrm{HN}}(M_2) \leq \mathbf{t}_{\mathrm{HN}}(M_1) * \mathbf{t}_{\mathrm{HN}}(M_3).$$

2. *Let M_i be a finite family of objects in $\mathcal{C}$. Then the Harder-Narasimhan filtration of $M = M_1 \oplus \ldots \oplus M_n$ is given by*
$$\mathcal{F}^{\geq\gamma}(M) = \bigoplus_{i=1}^{n} \mathcal{F}^{\geq\gamma}(M_i)$$
for all $\gamma \in \Gamma$ and thus
$$\mathbf{t}_{\mathrm{HN}}(M) = \mathbf{t}_{\mathrm{HN}}(M_1) * \ldots * \mathbf{t}_{\mathrm{HN}}(M_n).$$

3. *Let $\mathcal{F}'$ be a flag of M. Then we have*
$$\mathbf{t}_{\mathrm{HN}}(M) \leq \mathbf{t}_{\mathrm{HN}}(\mathrm{Gr}_{\mathcal{F}'}(M)).$$
If $\mathcal{F}'$ is the flag attached to $\mathcal{F}$, then
$$\mathbf{t}_{\mathrm{HN}}(M) = \mathbf{t}_{\mathrm{HN}}(\mathrm{Gr}_{\mathcal{F}}\, M).$$

Further assumptions

Proposition 2.5.5. *Let $\mathcal{C}$ be a quasi-abelian category with rank and degree functions such that*

1. *For a mono-epi $M_1 \to M_2$, we have* $\operatorname{rank} M_1 = \operatorname{rank} M_2$.

2. *The function* rank *detects the mono-epi morphisms, i.e. it verifies that a mono (resp. an epi) $M_1 \to M_2$ in $\mathcal{C}$ is an epi (resp. a mono) if and only if* $\operatorname{rank} M_1 = \operatorname{rank} M_2$.

3. *The functions* deg *detects the isomorphisms, i.e. for a mono-epi $M_1 \xrightarrow{f} M_2$ we have* $\deg M_1 = \deg M_2$ *if and only if f is an isomorphism in $\mathcal{C}$.*

Then,

1. *The function μ also detects the isomorphisms.*
2. *The category $\mathcal{C}(\gamma)$ is abelian, for every $\gamma \in \Gamma$.*
3. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence in $\mathcal{C}$. Suppose that*
$$\mathbf{t}_{\mathrm{HN}}(M_2) = \mathbf{t}_{\mathrm{HN}}(M_1) * \mathbf{t}_{\mathrm{HN}}(M_3).$$
Then, for every $\gamma \in \Gamma$, we have an exact sequence
$$0 \to \mathcal{F}^{\geq\gamma}_{\mathrm{HN}} M_1 \to \mathcal{F}^{\geq\gamma}_{\mathrm{HN}} M_2 \to \mathcal{F}^{\geq\gamma}_{\mathrm{HN}} M_3 \to 0.$$

Proof. 1. It follows from the condition (1) since the equality for the rank gives us an equality for μ if and only if we have an equality for deg.

2. Let $f : M_1 \to M_2$ be a morphism in $\mathcal{C}(\gamma)$, where M_1 and M_2 are nonzero objects in $\mathcal{C}(\gamma)$. Then, the inclusion $\ker f \subset M_1$ gives us $\mu(\ker f) \leq \gamma$ and $\operatorname{im} f$, $\operatorname{coim} f \subset M_2$ gives us $\mu(\operatorname{coim} f) \leq \gamma$ and $\mu(\operatorname{im} f) \leq \gamma$. We have an exact sequence
$$0 \to \ker f \to M_1 \to \operatorname{coim} f \to 0,$$
so $\min\{\mu(\ker f), \mu(\operatorname{coim} f)\} \leq \gamma \leq \max\{\mu(\ker f), \mu(\operatorname{coim} f)\}$, thus we have an equality $\mu(\ker f) = \mu(\operatorname{coim} f) = \gamma$, so $\ker f$, $\operatorname{coim} f$ are semi-stable. Since $\operatorname{coim} f \hookrightarrow \operatorname{im} f$ is a mono-epi, we have $\gamma = \mu(\operatorname{coim} f) \leq \mu(\operatorname{im} f) \leq \gamma$, so $\mu(\operatorname{im} f) = \gamma$ and then $\operatorname{im} f$ is semi-stable. To finish, the exact sequence
$$0 \to \operatorname{im} f \to M_2 \to \operatorname{coker} f \to 0$$
gives us $\mu(\operatorname{coker} f) = \gamma$, so $\operatorname{coker} f$ is semi-stable.

3. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence in $\mathcal{C}$ with $\mathbf{t}_{\mathrm{HN}}(M_2) = \mathbf{t}_{\mathrm{HN}}(M_1) * \mathbf{t}_{\mathrm{HN}}(M_3)$. By induction on the rank of M_2, it suffices to prove the exactness of
$$0 \to \mathcal{F}^{\geq\gamma}_{\mathrm{HN}} M_1 \to \mathcal{F}^{\geq\gamma}_{\mathrm{HN}} M_2 \to \mathcal{F}^{\geq\gamma}_{\mathrm{HN}} M_3 \to 0.$$
for γ the maximal slope in $\mathbf{t}_{\mathrm{HN}}(M_2)$. We have a strict mono
$$\mathcal{F}^{\geq\gamma} M_1 \hookrightarrow M_1 \hookrightarrow M_2$$
so we can consider the quotients
$$0 \to M_1/\mathcal{F}^{\geq\gamma} M_1 \to M_2/\mathcal{F}^{\geq\gamma} M_1 \to M_3 \to 0.$$
Suppose first $\mathcal{F}^{\geq\gamma} M_1 \neq 0$, then $\operatorname{rank}(M_2/\mathcal{F}^{\geq\gamma} M_1) < \operatorname{rank} M_2$. It is easy to see that we have again
$$\mathbf{t}_{\mathrm{HN}}(M_2/\mathcal{F}^{\geq\gamma} M_1) = \mathbf{t}_{\mathrm{HN}}(M_1/\mathcal{F}^{\geq\gamma} M_1) * \mathbf{t}_{\mathrm{HN}}(M_3)$$
so, by induction, we get
$$0 \to \mathcal{F}^{\geq\mu}(M_1/\mathcal{F}^{\geq\gamma} M_1) \to \mathcal{F}^{\geq\mu}(M_2/\mathcal{F}^{\geq\gamma} M_1) \to \mathcal{F}^{\geq\mu} M_3 \to 0$$
for every $\mu \in \Gamma$. In particular, for $\mu = \gamma$, we have $\mathcal{F}^{\geq\gamma}(M_2)/\mathcal{F}^{\geq\gamma}(M_1) \simeq \mathcal{F}^{\geq\gamma}(M_3)$ as we wanted. Suppose then $\mathcal{F}^{\geq\gamma} M_1 = 0$, then since $\mathcal{F}^{\geq\gamma} M_2$ and $\mathcal{F}^{\geq\gamma} M_3$ are semi-stable of slope γ, the kernel of $\mathcal{F}^{\geq\gamma} M_2 \to \mathcal{F}^{\geq\gamma} M_3$ is also semi-stable of slope γ and contained in M_1, thus it is zero and the morphism is a mono. Now, $\mathcal{F}^{\geq\gamma} M_2$ and $\mathcal{F}^{\geq\gamma} M_3$ have the same rank by assumption, so by (2), we have that it is a strict epi. Then, $\mathcal{F}^{\geq\gamma} M_2 \to \mathcal{F}^{\geq\gamma} M_3$ is a mono-epi and $\deg(\mathcal{F}^{\geq\gamma} M_2) = \deg(\mathcal{F}^{\geq\gamma} M_3)$. It is then an isomorphism, by (3). It remains to prove that $\operatorname{im} f \simeq \operatorname{coim} f$ which is true since we have a mono-epi morphism $\operatorname{coim} f \to \operatorname{im} f$ with $\mu(\operatorname{coim} f) = \mu(\operatorname{im} f)$, and by our hypothesis 1 and 3, it is an isomorphism.

□

Remark 3. If $\mathcal{C}$ is an abelian category, then it verifies the properties of last proposition trivially, since mono-epi morphisms in abelian categories are isomorphisms.

We will see that all the categories that we will work with verify this condition for the rank and degree functions, so the Harder-Narasimhan formalisms that we consider include the properties of the proposition.

2.6 Bruhat-Tits theory

2.6.1 The vectorial Tits building $\mathbf{F}^{\mathbb{R}}(V)$

Let K be a field, V a K-vector space of dimension $r \in \mathbb{N}$. Denote by $\mathbf{F}^{\mathbb{R}}(V)$ the set of all $\mathbb{R}$-filtrations on V: this is the vectorial Tits building of $G = GL(V)$, also denoted by $\mathbf{F}^{\mathbb{R}}(G)$ or $\mathbb{F}^{\mathbb{R}}(G)(K)$ in [11].

Action of $G(K)$, facets

The group $G(K)$ acts on $\mathbf{F}^{\mathbb{R}}(V)$ by $(g \cdot \mathcal{F})^{\geq\gamma} = g \cdot \mathcal{F}^{\geq\gamma}$. The stabilizer of $\mathcal{F}$ is (the group of K-rational points of) a parabolic subgroup of G, denoted by $P_{\mathcal{F}}$. Two $\mathbb{R}$-filtrations $\mathcal{F}_1$ and $\mathcal{F}_2$ are in the same facet of $\mathbf{F}^{\mathbb{R}}(V)$ if and only if $P_{\mathcal{F}_1} = P_{\mathcal{F}_2}$. This yields a partition of $\mathbf{F}^{\mathbb{R}}(V)$ into facets indexed by the parabolic subgroups of G. On the other hand, the type map

$$\mathbf{t} : \mathbf{F}^{\mathbb{R}}(V) \to \mathbb{R}^r_{\geq}$$

yields a bijection $G(K)\backslash\mathbf{F}^{\mathbb{R}}(V) \simeq \mathbb{R}^r_{\geq}$.

Apartments

Definition 2.6.1. A line decomposition of V is a subset $\mathcal{D} \subset \mathbf{P}^1(V)$ such that

$$V = \oplus_{D\in\mathcal{D}} D$$

(thus $\mathcal{D}$ is finite and $\sharp\mathcal{D} = r$). We say that a line decomposition $\mathcal{D}$ of V is adapted to an $\mathbb{R}$-filtration $\mathcal{F}$ of V when

$$\mathcal{F}^{\geq\gamma} = \oplus_{D\in\mathcal{D}} \mathcal{F}^{\geq\gamma} \cap D$$

for every $\gamma \in \mathbb{R}$.

We denote by $\mathbf{F}^{\mathbb{R}}(\mathcal{D})$ the set of all filtrations adapted to $\mathcal{D}$: this is the apartment of $\mathbf{F}^{\mathbb{R}}(V)$ attached to $\mathcal{D}$, a disjoint union of finitely many facets. The map $\mathcal{D} \mapsto \mathbf{F}^{\mathbb{R}}(\mathcal{D})$ yields a $G(K)$-equivariant bijection between line decompositions of V and apartments of $\mathbf{F}^{\mathbb{R}}(V)$. If $\underline{e} = (e_1, \cdots, e_r)$ is a K-basis of V, we also denote by $\mathbf{F}^{\mathbb{R}}(\underline{e})$ the apartement corresponding to the line decomposition $\mathcal{D} = \{Ke_i\}$. Then $\mathbb{R}^r \simeq \mathbf{F}^{\mathbb{R}}(\underline{e})$ by $(\gamma_1, \cdots, \gamma_r) \mapsto \mathcal{F}$, with $\mathcal{F}^{\geq\gamma} = \oplus_{\gamma_i\geq\gamma} Ke_i$. The resulting structure of $\mathbb{R}$-vector space on $\mathbf{F}^{\mathbb{R}}(\mathcal{D})$ does not depend upon the base $\underline{e}$ such that $\mathcal{D} = \{Ke_i\}$. For every $(\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{F}^{\mathbb{R}}(V)$, there is an apartment $\mathbf{F}^{\mathbb{R}}(\mathcal{D})$ containing $\mathcal{F}_1$ and $\mathcal{F}_2$.

Scalar product

For $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}^{\mathbb{R}}(V)$, we set

$$\begin{aligned}
\langle \mathcal{F}_1, \mathcal{F}_2 \rangle &= \sum_{\gamma_1,\gamma_2} \gamma_1\gamma_2 \dim_K \frac{\mathcal{F}_1^{\geq\gamma_1} \cap \mathcal{F}_2^{\geq\gamma_2}}{\mathcal{F}_1^{\geq\gamma_1} \cap \mathcal{F}_2^{>\gamma_2} + \mathcal{F}_1^{>\gamma_1} \cap \mathcal{F}_2^{\geq\gamma_2}} \\
&= \sum_{\gamma_1} \gamma_1 \deg \mathrm{Gr}^{\gamma_1}_{\mathcal{F}_1}(\mathcal{F}_2) \\
&= \sum_{\gamma_2} \gamma_2 \deg \mathrm{Gr}^{\gamma_2}_{\mathcal{F}_2}(\mathcal{F}_1)
\end{aligned}$$

where $\mathrm{Gr}^{\gamma_i}_{\mathcal{F}_i}(\mathcal{F}_{3-i})$ is the filtration induced by $\mathcal{F}_{3-i}$ on $\mathrm{Gr}^{\gamma_i}_{\mathcal{F}_i}$. This yields a $G(K)$-invariant "scalar product"

$$\langle -, - \rangle : \mathbf{F}^{\mathbb{R}}(V) \times \mathbf{F}^{\mathbb{R}}(V) \to \mathbb{R}$$

whose restriction to any apartement is a scalar product in the usual sense.

Distance

The formulas $\|\mathcal{F}\| = \sqrt{\langle \mathcal{F}, \mathcal{F} \rangle}$ and

$$d(\mathcal{F}_1, \mathcal{F}_2) = \sqrt{\|\mathcal{F}_1\|^2 + \|\mathcal{F}_2\|^2 - 2\langle \mathcal{F}_1, \mathcal{F}_2 \rangle}$$

define a $G(K)$-invariant distance

$$d : \mathbf{F}^{\mathbb{R}}(V) \times \mathbf{F}^{\mathbb{R}}(V) \to \mathbb{R}$$

on $\mathbf{F}^{\mathbb{R}}(V)$, whose restriction to any apartment is Euclidean. The resulting metric space $(\mathbf{F}^{\mathbb{R}}(V), d)$ is a complete CAT(0)-space, see [11, 4.2.10].

Convex projection

For any closed and convex subset C of $\mathbf{F}^{\mathbb{R}}(V)$, there is a convex projection

$$p : \mathbf{F}^{\mathbb{R}}(V) \to C$$

such that for every $\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(V)$,

$$d(\mathcal{F}, p(\mathcal{F})) = \inf \{ d(\mathcal{F}, \mathcal{G}) : \mathcal{G} \in C \}.$$

We will use the following case: $V = V_1 \otimes V_2$ and C is the set of decomposed filtrations, by which we mean the image of

$$\mathbf{F}^{\mathbb{R}}(V_1) \times \mathbf{F}^{\mathbb{R}}(V_2) \to \mathbf{F}^{\mathbb{R}}(V_1 \otimes V_2) \quad (\mathcal{F}_1, \mathcal{F}_2) \mapsto \mathcal{F}_1 \otimes \mathcal{F}_2.$$

In this situation, $C = \mathbf{F}^{\mathbb{R}}(H)$ is the vectorial Tits building of the reductive group H over K defined by the sequence

$$GL(V_1) \times GL(V_2) \twoheadrightarrow H \hookrightarrow GL(V_1 \otimes V_2).$$

The projection $p : \mathbf{F}^{\mathbb{R}}(G) \to \mathbf{F}^{\mathbb{R}}(H)$ is closely related to the notion of Kempf filtrations, used for instance in [47]. For any $\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(V_1 \otimes V_2)$ and $\mathcal{H} \in \mathbf{F}^{\mathbb{R}}(H)$, we have

$$\langle \mathcal{F}, \mathcal{H} \rangle \leq \langle p(\mathcal{F}), \mathcal{H} \rangle$$

by [11, 5.7.7].

2.6.2 The Bruhat-Tits building $\mathbf{B}(V)$

Suppose that K is the fraction field of a complete discrete valuation ring $\mathcal{O}_K$, with non-trivial valuation

$$|-| : K \to \mathbb{R}_+.$$

Write $q > 1$ for the real number such that $q^{\mathbb{Z}} = |K^\times|$ and k for the residue field.

The space $\mathbf{B}(V)$ and its apartments

A K-norm on V is a function

$$\alpha : V \to \mathbb{R}_+$$

such that for every $u, v \in V$, $\lambda \in K$

1. $\alpha(v) = 0$ if and only if $v = 0$,
2. $\alpha(\lambda v) = |\lambda|\, \alpha(v)$,
3. $\alpha(u + v) \leq \max\{\alpha(u), \alpha(v)\}$.

The Bruhat-Tits building of $G = GL(V)$ is the set $\mathbf{B}(V)$ of all K-norms on V. The group $G(K)$ acts on it by $(g \cdot \alpha)(v) = \alpha(g^{-1}v)$. We say that a line decomposition $\mathcal{D}$ of V is adapted to $\alpha \in \mathbf{B}(V)$ if

$$\alpha(v) = \max\{\alpha(v_D) \mid D \in \mathcal{D}\}$$

for all $v = \sum v_D$ in $V = \oplus_{D\in\mathcal{D}} D$. We denote by $\mathbf{B}(\mathcal{D})$ the set of all K-norms adapted to $\mathcal{D}$: the apartment of $\mathcal{D}$ in $\mathbf{B}(V)$. The map $\mathcal{D} \mapsto \mathbf{B}(\mathcal{D})$ is a $G(K)$-equivariant bijection between line decompositions of V and apartments of $\mathbf{B}(V)$. If $\mathcal{D} = \{Ke_i\}$ for some K-basis $\underline{e} = (e_1, \cdots, e_r)$ of V, we also denote by $\mathbf{B}(\underline{e})$ the corresponding apartement. Thus $\mathbb{R}^r \simeq \mathbf{B}(\underline{e})$ by the map $(\gamma_1, \cdots, \gamma_r) \mapsto \alpha$ with

$$\alpha(\textstyle\sum \lambda_i e_i) = \max_{1\leq i\leq r}\{|\lambda_i|\, q^{-\gamma_i}\}.$$

For every $x, y \in \mathbf{B}(V)$, there is an apartement $\mathbf{B}(\mathcal{D})$ containing x and y.

The embedding $\mathcal{L}(V) \to \mathbf{B}(V)$

Write $\mathcal{L}(V)$ for the set of $\mathcal{O}_K$-lattices L in V. Then $G(K)$ acts on $\mathcal{L}(V)$ and there is a $G(K)$-equivariant embedding

$$\mathcal{L}(V) \hookrightarrow \mathbf{B}(V), \qquad L \mapsto \alpha_L$$

where α_L is the gauge norm of L, defined by

$$\alpha_L(v) = \inf\{|\lambda| \mid v \in \lambda L\}$$

The image of $\mathcal{L}(V)$ is the set of K-norms α on V such that

$$\alpha(V \setminus \{0\}) = \left|K^\times\right| = q^{\mathbb{Z}}.$$

We write $\mathbf{A}(\mathcal{D})$ for $\mathbf{B}(\mathcal{D}) \cap \mathcal{L}(V)$ and refer to it as the apartement of $\mathcal{D}$ in $\mathcal{L}(V)$:

$$\mathbf{A}(\mathcal{D}) = \{L \in \mathcal{L}(V) \mid L = \oplus_{D\in\mathcal{D}} L \cap D\}.$$

The action of $\mathbf{F}^{\mathbb{R}}(V)$

There is a $G(K)$-equivariant "action"

$$\mathbf{B}(V) \times \mathbf{F}^{\mathbb{R}}(V) \to \mathbf{B}(V) \quad (\alpha, \mathcal{F}) \mapsto \alpha + \mathcal{F}$$

defined by

$$(\alpha + \mathcal{F})(v) = \min\left\{\max\left\{q^{-\gamma}\alpha(v_\gamma) \mid \gamma \in \mathbb{R}\right\} \mid v = \sum v_\gamma,\, v_\gamma \in \mathcal{F}^{\geq\gamma}\right\}.$$

For any line decomposition $\mathcal{D}$ of V, this induces a map

$$\mathbf{B}(\mathcal{D}) \times \mathbf{F}^{\mathbb{R}}(\mathcal{D}) \to \mathbf{B}(\mathcal{D})$$

which turns $\mathbf{B}(\mathcal{D})$ into an affine space with underlying vector space $\mathbf{F}^{\mathbb{R}}(\mathcal{D})$. This operation yields a commutative diagram

$$\begin{array}{ccccc} \mathcal{L}(V) & \times & \mathbf{F}^{\mathbb{Z}}(V) & \to & \mathcal{L}(V) \\ \cap & & \cap & & \cap \\ \mathbf{B}(V) & \times & \mathbf{F}^{\mathbb{R}}(V) & \to & \mathbf{B}(V) \end{array}$$

where $\mathbf{F}^{\mathbb{Z}}(V)$ is the set of all $\mathbb{Z}$-filtrations on V and the top map is

$$(L, \mathcal{F}) \mapsto L + \mathcal{F} = \sum \pi_K^{-i} L \cap \mathcal{F}^{\geq i}$$

where π_K is a uniformizer (thus $|\pi_K| = q^{-1}$).

The vectorial and classical distances

For every $x, y \in \mathbf{B}(V)$, there is an $\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(V)$ such that $y = x + \mathcal{F}$. Such an $\mathcal{F}$ is not at all unique, but

$$\mathbf{d}(x, y) = \mathbf{t}(\mathcal{F}) \in \mathbb{R}^r_{\geq}$$

does not depend upon the choice of $\mathcal{F}$. This yields a $G(K)$-equivariant function

$$\mathbf{d} : \mathbf{B}(V) \times \mathbf{B}(V) \to \mathbb{R}^r_{\geq}$$

which is called the vectorial distance, see [11, 5.2.8]. We have

1. $\mathbf{d}(x, y) = 0$ if and only if $x = y$,
2. $\mathbf{d}(y, x) = \mathbf{d}(x, y)^{\iota}$
3. $\mathbf{d}(x, z) \leq \mathbf{d}(x, y) + \mathbf{d}(y, z)$

The formula (for $y = x + \mathcal{F}$)

$$d(x, y) = |\mathbf{d}(x, y)| = \|\mathcal{F}\| \in \mathbb{R}_+$$

thus defines a genuine $G(K)$-equivariant distance

$$d : \mathbf{B}(V) \times \mathbf{B}(V) \to \mathbb{R}_+.$$

Then $(\mathbf{B}(V), d)$ is yet another complete CAT(0)-space, see [11, 6.1 and 5.3.2].

Since K is complete, the map

$$\begin{array}{rcl} \mathbf{B}(V) \times \mathbf{F}^{\mathbb{R}}(V) & \to & \mathcal{C}^0\left(\mathbb{R}_+, \mathbf{B}(V)\right) \\ (\alpha, \mathcal{F}) & \mapsto & c(t) = \alpha + t\mathcal{F} \end{array}$$

identifies:

1. $\mathbf{B}(V) \times \mathbf{F}^{\mathbb{R}}(V)$ with the set of all (constant speed) geodesic rays in $\mathbf{B}(V)$ (allowing constant functions as geodesic rays with speed 0),
2. $\mathbf{F}^{\mathbb{R}}(V)$ with asymptotic classes of (constant speed) geodesic rays in $\mathbf{B}(V)$,
3. Elements of $\mathbf{F}^{\mathbb{R}}(V)$ with norm 1 with the visual boundary $\partial\mathbf{B}(V)$ of $\mathbf{B}(V)$ (defined in [5, II.8]).

This follows from [41, 2.3.8], see [11, 6.2.8]. With notations as above, the speed of the geodesic ray $c(t)$ is the norm of $\mathcal{F}$.

The following diagram is commutative:

$$\begin{array}{rccccc} \mathrm{Pos}: & \mathcal{L}(V) & \times & \mathcal{L}(V) & \to & \mathbb{Z}^r_{\geq} \\ & \cap & & \cap & & \cap \\ \mathbf{d}: & \mathbf{B}(V) & \times & \mathbf{B}(V) & \to & \mathbb{R}^r_{\geq} \end{array}$$

Indeed for $L, L' \in \mathcal{L}(V)$, pick a line decomposition $\mathcal{D}$ of V such that $L, L' \in \mathbf{A}(\mathcal{D})$, let $\mathcal{F}$ be the unique element of $\mathbf{F}^{\mathbb{R}}(\mathcal{D})$ such that $\alpha_{L'} = \alpha_L + \mathcal{F}$. Then $\mathcal{F}(L, L')$ is the filtration induced by $\mathcal{F}$ on the residue $\overline{L} = L \otimes k$ of L, thus

$$\mathrm{Pos}(L, L') = \mathbf{t}(\mathcal{F}(L, L')) = \mathbf{t}(\mathcal{F}) = \mathbf{d}(\alpha_L, \alpha_{L'})$$

and $\mathcal{F}$ belongs to $\mathbf{F}^{\mathbb{Z}}(\mathcal{D}) = \mathbf{F}^{\mathbb{Z}}(G) \cap \mathbf{F}^{\mathbb{R}}(\mathcal{D})$, the set of all $\mathbb{Z}$-filtrations split by $\mathcal{D}$. We may also define a $G(K)$-invariant function

$$\begin{array}{rcl} \nu : \mathbf{B}(V) \times \mathbf{B}(V) & \to & \mathbb{R} \\ (x, y) & \mapsto & \nu(x, y) = \deg(\mathbf{d}(x, y)). \end{array}$$

Thus $\nu(x, z) = \nu(x, y) + \nu(y, z)$ and $\nu(y, x) = -\nu(x, y)$.

Busemann functions

For $x, y \in \mathbf{B}(V)$ and $\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(V)$, we set

$$\langle \overrightarrow{xy}, \mathcal{F} \rangle = \|\mathcal{F}\| \cdot \lim_{t \to \infty} d(x, z + t\mathcal{F}) - d(y, z + t\mathcal{F})$$

where z is any point of $\mathbf{B}(V)$: the limit exists and does not depend upon z by [5, II.8.18-20]. This yields a well-defined, $G(K)$-equivariant "Busemann scalar product" [11, 5.5.8].

$$\mathbf{B}(V)^2 \times \mathbf{F}^{\mathbb{R}}(V) \to \mathbb{R}$$

whose restriction to any appartment is the obvious scalar product

$$\begin{array}{ccccc} \mathbf{B}(\mathcal{D})^2 \times \mathbf{F}^{\mathbb{R}}(\mathcal{D}) & \to & \mathbf{F}^{\mathbb{R}}(\mathcal{D}) \times \mathbf{F}^{\mathbb{R}}(\mathcal{D}) & \to & \mathbb{R} \\ (x, y, \mathcal{F}) & \mapsto & (\overrightarrow{xy}, \mathcal{F}) & \mapsto & \langle \overrightarrow{xy}, \mathcal{F} \rangle . \end{array}$$

For any $x, y, z \in \mathbf{B}(V)$ and $\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(V)$,

$$\langle \overrightarrow{xz}, \mathcal{F} \rangle = \langle \overrightarrow{xy}, \mathcal{F} \rangle + \langle \overrightarrow{yz}, \mathcal{F} \rangle \quad \text{and} \quad \langle \overrightarrow{yx}, \mathcal{F} \rangle = -\langle \overrightarrow{xy}, \mathcal{F} \rangle$$

Proposition 2.6.1. *Let $\mathcal{D}$ be a line decomposition of V. Suppose that a sequence $(x_n)_n \in \mathbf{B}(\mathcal{D})$ converges to some $\xi \in \partial\mathbf{B}(\mathcal{D}) \subset \mathbf{F}^{\mathbb{R}}(\mathcal{D})$. Then for every $x \in \mathbf{B}(V)$,*

$$\lim_{n \mapsto \infty} \frac{\langle \overrightarrow{xx_n}, \mathcal{F} \rangle}{d(x, x_n)} = \langle \xi, \mathcal{F} \rangle .$$

Proof. Since x_n is unbounded and

$$\frac{\langle \overrightarrow{xx_n}, \mathcal{F} \rangle}{d(x, x_n)} = \frac{\langle \overrightarrow{xy}, \mathcal{F} \rangle}{d(x, x_n)} + \frac{d(y, x_n)}{d(x, x_n)} \cdot \frac{\langle \overrightarrow{yx_n}, \mathcal{F} \rangle}{d(y, x_n)}$$

for every x and y in $\mathbf{B}(V)$, the left hand side limit exists and is independent of x if it exists for one single x. We may thus assume that x also belongs to the Euclidean affine space $\mathbf{B}(\mathcal{D})$, in which case the result is trivial.

□

Remark 4. If $(x_n)_n \in \mathbf{B}(V)$ converges to some $\xi \in \partial\mathbf{B}(V) \subset \mathbf{F}^{\mathbb{R}}(V)$ without being contained in any apartment, the following inequality still holds:

$$\liminf \frac{\langle \overrightarrow{xx_n}, \mathcal{F}\rangle}{d(x, x_n)} \geq \langle \xi, \mathcal{F}\rangle .$$

However, this is usually a strict inequality.

An explicit formula for the Busemann scalar product is given by

$$\langle \overrightarrow{xy}, \mathcal{F}\rangle = \sum_{\gamma\in\mathbb{R}} \gamma \cdot \nu \left(\mathrm{Gr}^{\gamma}_{\mathcal{F}} x, \mathrm{Gr}^{\gamma}_{\mathcal{F}} y\right)$$

where $\mathrm{Gr}^{\gamma}_{\mathcal{F}}(z)$ is the K-norm induced by z on $\mathrm{Gr}^{\gamma}_{\mathcal{F}}$, namely

$$\mathrm{Gr}^{\gamma}_{\mathcal{F}}(z)(\overline{v}_\gamma) = \inf\{z(v_\gamma) \mid v_\gamma \in \overline{v}_\gamma\} \quad \text{for } \overline{v}_\gamma \in \mathrm{Gr}^{\gamma}_{\mathcal{F}} = \mathcal{F}^{\geq\gamma}/\mathcal{F}^{>\gamma}.$$

Let us sketch the proof of this formula. We fix $\mathcal{F}$ and view both sides as functions of (x, y). For u and v in the unipotent radical $U_{\mathcal{F}}$ of $P_{\mathcal{F}}$, both sides are invariant under $(x, y) \mapsto (ux, vy)$. This is clear for the right hand side. As for the left hand side, we have

$$\begin{aligned}
\langle \overrightarrow{uxvy}, \mathcal{F}\rangle &= \|\mathcal{F}\| \lim_{t\to\infty} d(ux, z + t\mathcal{F}) - d(vy, z + t\mathcal{F}) \\
&= \|\mathcal{F}\| \lim_{t\to\infty} d(x, u^{-1}z + t\mathcal{F}) - d(y, v^{-1}z + t\mathcal{F}) \\
&= \|\mathcal{F}\| \lim_{t\to\infty} d(x, z + t\mathcal{F}) - d(y, z + t\mathcal{F})
\end{aligned}$$

because $u^{-1}z + t\mathcal{F} = z + t\mathcal{F} = v^{-1}z + t\mathcal{F}$ for $t \gg 0$ (this is the axiom UN^+ in [11, 5.4.6]). Using this invariance, we may assume that $x, y \in \mathbf{B}(\mathcal{D})$ and $\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(\mathcal{D})$ for some line decomposition $\mathcal{D}$ of V. Then $y = x + \mathcal{G}$ for some $\mathcal{G} \in \mathbf{F}^{\mathbb{R}}(\mathcal{D})$, which implies that also $\mathrm{Gr}^{\gamma}_{\mathcal{F}} x = \mathrm{Gr}^{\gamma}_{\mathcal{F}} y + \mathrm{Gr}^{\gamma}_{\mathcal{F}} \mathcal{G}$ in $\mathbf{B}(\mathrm{Gr}^{\gamma}_{\mathcal{F}})$ for every $\gamma \in \mathbb{R}$. Our claim follows:

$$\langle \overrightarrow{xy}, \mathcal{F}\rangle = \langle \mathcal{G}, \mathcal{F}\rangle = \sum_{\gamma\in\mathbb{R}} \gamma \cdot \deg \mathrm{Gr}^{\gamma}_{\mathcal{F}}(\mathcal{G}) = \sum_{\gamma\in\mathbb{R}} \gamma \cdot \nu \left(\mathrm{Gr}^{\gamma}_{\mathcal{F}} x, \mathrm{Gr}^{\gamma}_{\mathcal{F}} y\right).$$

Many apartments

Any one of the following pair of objects in $\mathbf{B}(V)$ is contained in some apartment:

1. Two points: axiom $R(s)$ of [11, 5.2.5], see [9, 1.26].
2. Two germs of segments: see [8, 7.4.18 (i)].
3. The germ of a segment and an asymptotic class of geodesic rays: axiom $L(s)^+$ of [11, 5.3.1].
4. The germ of a segment and the germ of a geodesic ray: see [41, 1.2.6].
5. Two germs of geodesic rays: see [41, 1.2.6].

Note that (4) $\Rightarrow$ (3) $\Rightarrow$ every geodesic ray is contained in an apartment, and (2) $\Rightarrow$ (1) $\Rightarrow$ every segment is contained in an apartment, and thus extends (in both directions) to a geodesic line. We use this in the following proposition:

Proposition 2.6.2. *Let $(L_n)_{n\geq 0}$ be a sequence of $\mathcal{O}_K$-lattices in V and let $\mathcal{F}$ be an $\mathbb{R}$-filtration on V. Suppose that*

$$d(L_0, L_n) = d(L_0, L_m) + d(L_m, L_n)$$

for every $0 \leq m \leq n$. Then:

1. *There exists a K-basis $\underline{e}$ of V which is adapted to all L_n's.*

2. *There exists a K-basis $\underline{e}$ of V which is adapted to $\mathcal{F}$ and all but finitely many L_n's.*

Proof. Let $x_n \in \mathbf{B}(V)$ be the point corresponding to $M_n \in \mathcal{L}(V)$. The assumption implies that for every $0 \leq m \leq n$, x_m lies on the geodesic segment $[x_0, x_n]$ of the complete CAT(0)-metric space $(\mathbf{B}(V), d)$. Then x_n converges to a point x_∞ in the compactification $\overline{\mathbf{B}}(V) = \mathbf{B}(V) \cup \partial\mathbf{B}(V)$ of $\mathbf{B}(V)$, all x_n's belong to $[x_0, x_\infty]$, and any germ of $[x_0, x_\infty]$ at x_∞ contains all but finitely many of the x_n's. Fix an apartment $\mathbf{B}(\mathcal{D})$ which contains $[x_0, x_\infty]$ (resp. which contains $\mathcal{F}$ in its boundary together with a germ of $[x_0, x_\infty]$ at x_∞), which exists by (1) or (3) (resp. (3) or (5)) depending upon $x_\infty \in \mathbf{B}(V)$ or $\partial\mathbf{B}(V)$ (i.e. x_n is bounded or unbounded). Then any K-basis $\underline{e}$ of V such that $\mathcal{D} = \{Ke_i\}$ is adapted to all the M_n's (resp. to $\mathcal{F}$ and all but finitely many M_n's). □

We will also use the following result:

Proposition 2.6.3. *For any apartment $\mathbf{B}(\mathcal{D})$ of $\mathbf{B}(V)$ and any $\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(V)$, there is a finite set of line decompositions $\{\mathcal{D}_1, \cdots, \mathcal{D}_n\}$ of V such that $\mathbf{B}(\mathcal{D}) \subset \cup \mathbf{B}_{i=1}^n(\mathcal{D}_i)$ and $\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(\mathcal{D}_i)$ for every $i \in \{1, \cdots, n\}$.*

Proof. This follows from the stronger statement in [41, 1.2.8], which implies that we may even replace the assympotic class of geodesic rays $\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(V)$ by the germ of any geodesic ray $c(t) = x + t\mathcal{F}$ in this class.

□

Chapter 3

Kisin's theory

3.1 Modules over $\mathfrak{S}$

The ring $\mathfrak{S}$

Let $\mathbb{F}$ be a perfect field of characteristic p and let $W(\mathbb{F})$ be the ring of Witt vectors over $\mathbb{F}$. The ring

$$\mathfrak{S} := W(\mathbb{F})[[u]]$$

is a complete, regular, local ring of Krull dimension 2, with maximal ideal $\mathfrak{m} = (p, u)$, so it is a unique factorization domain. Moreover, for all $f \in \mathfrak{m}$ nonzero, the ring $\mathfrak{S}[\frac{1}{f}]$ is a unique factorization domain of Krull dimension 1, therefore a principal ideal domain.

A nonconstant polynomial $P(u) \in \mathfrak{S}$ is called distinguished if

$$P(u) = u^n + a_{n-1}u^{n-1} + \ldots + a_0$$

where $p|a_i$ for $0 \leq i \leq n-1$. In [39, 5.3.7], we have a classification of the prime ideals of $\mathbb{Z}_p[[T]]$ that we can easily generalize to $\mathfrak{S}$.

Proposition 3.1.1. *The prime ideals of $\mathfrak{S}$ are 0, $\mathfrak{m}$, (p) and the ideals $(P(u))$ where $P(u)$ is an irreducible and distinguished polynomial.*

Classification of finitely generated $\mathfrak{S}$-modules

Let M be a finitely generated $\mathfrak{S}$-module. Then there is a unique exact sequence

$$0 \to M_{\text{tors}} \to M \to M_{\text{fr}} \to \overline{M} \to 0$$

where M_{tors} is a torsion module, M_{fr} is a free module and $\overline{M}$ is a finite length module. This dévissage of M is given by Bhatt-Morrow-Scholze in [4]. It is easy to give some descriptions of M_{fr} as

$$M_{\text{fr}} = M^{\vee\vee} = \bigcap_{\text{ht}(\mathfrak{p})=1} (M/M_{\text{tors}})_{\mathfrak{p}}$$

where $M^{\vee\vee}$ is the bidual of M, and the intersection is considered in $M \otimes \operatorname{Frac}\mathfrak{S}$.

Definition 3.1.1. A finitely generated $\mathfrak{S}$-module is called pseudo-null if it has finite length. A morphism between two $\mathfrak{S}$-modules is called a pseudo-isomorphism if its kernel and cokernel are pseudo-null.

The following theorem is the equivalent of the structure theorem for finitely generated modules over a PID. It is Iwasawa's structure theorem, in [39, 5.1.10], using Proposition 3.1.1.

Theorem 3.1.2 (Structure theorem of finitely generated $\mathfrak{S}$-modules)**.** *Let M be a finitely generated $\mathfrak{S}$-module. Then there exists a pseudo-isomorphism*

$$M \sim \mathfrak{S}^r \oplus \bigoplus_{i=0}^{s} \mathfrak{S}/(p^{n_i}) \oplus \bigoplus_{j=0}^{t} \mathfrak{S}/(P_j(u)^{m_j})$$

for some $r \geq 0$, s, $t \geq 0$, $n_i \geq 0$ for $0 \leq i \leq s$, and $m_j \geq 0$ and $P_j(u)$ distinguished irreducible $\mathfrak{S}$-polynomials for $0 \leq j \leq t$. These quantities are unique up to reordering.

The structure theorem gives us two important invariants:

$$\mu_{\mathrm{IW}}(M) = \sum_{i=0}^{r} n_i, \quad \lambda_{\mathrm{IW}}(M) = \sum_{j=0}^{t} m_j \deg(f_j).$$

Remark 5.

1. For a torsion $\mathfrak{S}$-module, after localization by (p), we get an isomorphism

$$M_{(p)} \simeq \bigoplus_{i=0}^{s} \mathfrak{S}_{(p)}/p^{n_i}$$

for $s \geq 0$ and $n_i \geq 1$ for $0 \leq i \leq s$ given by the structure theorem. Thus

$$\mu_{\mathrm{IW}}(M) = \mathrm{length}_{\mathfrak{S}_{(p)}}(M_{(p)}).$$

2. As a consequence, we have that μ_{IW} is additive on short exact sequences of torsion $\mathfrak{S}$-modules.

Cohomological properties

Let M be a finitely generated $\mathfrak{S}$-module. There is an exact sequence

$$0 \to M[\mathfrak{m}^\infty] \to M_{\mathrm{tors}} \to M_{\mathrm{t}} \to 0$$

where $M[\mathfrak{m}^\infty]$ is pseudo-null. We have a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & & & \\
 & M[\mathfrak{m}^\infty] \arrow[d] & & & & \\
0 \arrow[r] & M_{\mathrm{tors}} \arrow[r] \arrow[d] & M \arrow[r] & M_{\mathrm{fr}} \arrow[r] & \overline{M} \arrow[r] & 0 \\
 & M_{\mathrm{t}} \arrow[d] & & & & \\
 & 0 & & & &
\end{tikzcd}$$

Definition 3.1.2. We define the depth of M as

$$\mathrm{depth}(M) := \min\{i \mid \mathrm{Ext}^i(\mathbb{F}, M) \neq 0\}$$

or, equivalently, the maximal length of an M-regular $\mathfrak{m}$-sequence, i.e. a sequence of elements $x_1, \ldots, x_r \in \mathfrak{m}$ such that x_1 is not a zero divisor of M and x_i is not a zero divisor in $M/(x_1, \ldots, x_{i-1})M$ for $2 \leq i \leq r$.

The projective dimension of M is the minimal length of a projective resolution of M, i.e. the miniminal length a sequence

$$0 \to P_d \to \ldots \to P_2 \to P_1 \to P_0 \to M \to 0$$

where P_i is a projective module for $i \geq 0$.We write $\mathrm{pd}(M)$ for the projective dimension of M. In the case of modules over noetherian local rings, there is an interesting relation between the projective dimension and the Tor functor.

Proposition 3.1.3. *Let M be a finitely generated $\mathfrak{S}$-module. Then*

$$\mathrm{pd}(M) = \max\{i \mid \mathrm{Tor}^i(\mathbb{F}, M) \neq 0\}.$$

We can relate the depth and projective dimension with the Auslander-Buchsbaum theorem [3]:

Theorem 3.1.4. *Let M be a finitely generated $\mathfrak{S}$-module. Then*

$$\mathrm{pd}(M) + \mathrm{depth}(M) = 2$$

As a consequence of this theorem, we get some good properties for finite free and finitely generated torsion modules over $\mathfrak{S}$.

- For a finitely generated $\mathfrak{S}$-module M, the following conditions are equivalent: M is projective, M flat, M is free, M is reflexive and $\mathrm{Tor}^1(\mathbb{F}, M) = 0$.
- For a finitely generated torsion $\mathfrak{S}$-module M, the following conditions are equivalent: $\mathrm{pd}(M) = 1$, $\mathrm{depth}(M) = 1$ and $\mathrm{Ext}^0(\mathbb{F}, M) \simeq \mathrm{Tor}^2(\mathbb{F}, M) = M[\mathfrak{m}] = 0$.

Proposition 3.1.5. *Let*

$$0 \to K \to M \to Q \to 0$$

with M a finite free $\mathfrak{S}$-module and $\mathrm{pd}(Q) \leq 1$. Then K is a finite free $\mathfrak{S}$-module.

Proof. Since M is a finite free $\mathfrak{S}$-module, we know that $\mathrm{Tor}^i(\mathbb{F}, M) = 0$ for $i = 1, 2$, and thus $\mathrm{Tor}^1(\mathbb{F}, K) = \mathrm{Tor}^2(\mathbb{F}, Q) = 0$ since $\mathrm{pd}(Q) \leq 1$.

□

Another way to check if M is free is given by the following proposition.

Proposition 3.1.6. *Let M be a finitely generated $\mathfrak{S}$-module. Then M is free if and only if*

1. *The reduction M/p^nM has no u-torsion for every $n \geq 1$.*
2. *The μ_{IW} invariants of the modules M/p^nM are compatible, i.e.*

$$\mu_{\mathrm{IW}}(M/p^nM) = n\,\mu_{\mathrm{IW}}(M/pM)$$

for every $n \geq 1$.

Proof. First, we remark that the condition are necessary. Conversely, by hypothesis we know that the $\mathbb{F}[[u]]$-module M/pM is free. Let m be its rank. By Nakayama's lemma, we know that there is a system of m generators of M, giving us the exact sequence

$$0 \to \ker f \to \mathfrak{S}^m \xrightarrow{f} M \to 0.$$

Considering reductions modulo p^n for each $n \geq 1$ we get

$$0 \to M[p^n] \to \ker f/p^n \ker f \to (\mathfrak{S}/p^n\mathfrak{S})^m \to M/p^nM \to 0$$

that we can cut into two exact sequences:

$$0 \to M[p^n] \to \ker f/p^n \ker f \to K_n \to 0$$

$$0 \to K_n \to (\mathfrak{S}/p^n\mathfrak{S})^m \to M/p^nM \to 0.$$

Now, the second hypothesis tells us that

$$\mu_{\mathrm{IW}}((\mathfrak{S}/p^n\mathfrak{S})^m) - \mu_{\mathrm{IW}}(M/p^nM) = mn - n\mu_{\mathrm{IW}}(M/pM) = 0.$$

So by the additivity of μ_{IW} on short exact sequences of torsion $\mathfrak{S}$-modules seen in Remark 5 we get $\mu_{\mathrm{IW}}(K_n) = 0$. Being also killed by p^n, K_n has to be pseudo-null. Since $(\mathfrak{S}/p^n\mathfrak{S})^m$ contains no nonzero pseudo-null submodule, actually $K_n = 0$. Therefore $(\mathfrak{S}/p^n\mathfrak{S})^m \simeq M/p^nM$. Since $\mathfrak{S}$ and M are p-adically complete, it follows that $\mathfrak{S}^m \simeq M$.

□

Some categories of $\mathfrak{S}$-modules

Denote $\mathrm{Mod}_{\mathfrak{S}}$ the category of finitely generated $\mathfrak{S}$-modules and let $\mathcal{A}$ be the category of finitely generated $\mathfrak{S}$-modules killed by a power of p. It is a full subcategory of $\mathrm{Mod}_{\mathfrak{S}}$ which is abelian and stable by subobjects, quotients and extensions. Inside $\mathcal{A}$, we consider two subcategories:

- The full subcategory $\mathcal{T}$ of objects with finite length, i.e. the $\mathfrak{S}$-modules such that $M[\mathfrak{m}^\infty] = M$ (which is the same as $M[u^\infty] = M$ since the $\mathfrak{m}^\infty$-torsion and the u^∞-torsion coincide in this category).
- The full subcategory $\mathrm{Mod}_{\mathfrak{S},\mathrm{t}}$ of objects with projective dimension 1. This is a full subcategory of $\mathrm{Mod}_{\mathfrak{S}}$, it contains $\mathfrak{S}/p\mathfrak{S} = \mathbb{F}[[u]]$ and it is stable by subobjects and extensions, but not stable by quotients.

For any module M in $\mathcal{A}$, consider the short exact sequence

$$0 \to M[u^\infty] \to M \to M' \to 0$$

where $M[u^\infty]$ is an object in $\mathcal{T}$ and $M' = M/M[u^\infty]$ is an object in $\mathrm{Mod}_{\mathfrak{S},\mathrm{t}}$. There are no nonzero morphisms from $\mathcal{T}$ to $\mathrm{Mod}_{\mathfrak{S},\mathrm{t}}$ since the objects in $\mathrm{Mod}_{\mathfrak{S},\mathrm{t}}$ have no u^∞-torsion. This proves that $\mathrm{Mod}_{\mathfrak{S},\mathrm{t}}$ is a quasi-abelian category and we can give an explicit description of kernels, cokernels, images and coimages. Let $M \xrightarrow{f} N$ in $\mathrm{Mod}_{\mathfrak{S},\mathrm{t}}$, then:

- The object $\ker f$ corresponds to the usual kernel in the category $\mathrm{Mod}_{\mathfrak{S}}$, since it is a submodule of M, so it has no $\mathfrak{m}^\infty$-torsion and it already is an object in $\mathrm{Mod}_{\mathfrak{S},\mathrm{t}}$.
- For the cokernel, let $f(M)$ be the usual image in the category $\mathrm{Mod}_{\mathfrak{S}}$. We define the saturation of $f(M)$ by u^∞ in N as

  $$f(M)^{\mathrm{sat}} := \{n \in N \mid u^i n \in f(M) \text{ for some } i \geq 0\},$$

  then we have

  $$\operatorname{coker} f = N/f(M)^{\mathrm{sat}}.$$

  To prove it, suppose we have $M \xrightarrow{f} N \xrightarrow{q} Q$ with $q \circ f = 0$, then it suffices to show that $f(M)^{\mathrm{sat}}$ is in the kernel of q so we can factor through $N/f(M)^{\mathrm{sat}}$. For $n \in f(M)^{\mathrm{sat}}$, let $i \geq 0$ such that $u^i n \in f(M)$, then $q(u^i n) = u^i q(n) = 0$ since $q \circ f = 0$. Since Q has no u-torsion, $q(m)$ must be zero.
- For the image, we know that $\operatorname{im} f = \ker \operatorname{coker} f = f(M)^{\mathrm{sat}}$.

- For the coimage, we have $\operatorname{coim} f = \operatorname{coker}\ker f = M/\ker f$, since it is isomorphic to the submodule $f(M)$ of N, and thus has no u-torsion.

There is a rank function in the sense of the Harder-Narasimhan formalism on the abelian category $\mathcal{A}$, given by the Iwasawa invariant μ_{IW} and we can reinterpret it in the following way:

Proposition 3.1.7. *Let M be an object in $\mathcal{A}$, then*

$$\mu_{\mathrm{IW}}(M) = \operatorname{length}_{\mathfrak{S}_{(p)}}(M_{(p)}) = \operatorname{length}_{\mathfrak{S}[\frac{1}{u}]}(M[\tfrac{1}{u}])$$

and for M an object of the subcategory $\mathrm{Mod}_{\mathfrak{S},\mathrm{t}}$*, we have*

$$\mu_{\mathrm{IW}}(M) = \operatorname{length}_{\mathfrak{S}_{(p)}}(M_{(p)}) = \operatorname{length}_{\mathfrak{S}[\frac{1}{u}]}(M[\tfrac{1}{u}]) = \operatorname{length}_{W(\mathbb{F})}(M/uM).$$

Proof. We have already seen in Remark 5 the first equality. For the second one, let M be an object in $\mathcal{A}$, the structure theorem gives an exact sequence

$$0 \to K \to M \to M' \to Q \to 0$$

where $M' = \bigoplus_{i=0}^{r} \mathfrak{S}/p^{n_i}$ for some $r \geq 0$, $n_i \geq 1$, $Q[\mathfrak{m}^\infty] = Q$ and $K[\mathfrak{m}^\infty] = K$. Then, $K[\frac{1}{u}] = Q[\frac{1}{u}] = 0$, so

$$M[\tfrac{1}{u}] \simeq M'[\tfrac{1}{u}] = \textstyle\bigoplus_{i=0}^{r} \mathfrak{S}[\frac{1}{u}]/(p^{n_i})$$

and thus $\mu_{\mathrm{IW}}(M) = \operatorname{length}_{\mathfrak{S}[\frac{1}{u}]}(M)$.

Moreover, if M is an object in $\mathrm{Mod}_{\mathfrak{S},\mathrm{t}}$, we have $K = 0$ because there are no nonzero $\mathfrak{m}^\infty$-torsion submodules in M as an object in $\mathrm{Mod}_{\mathfrak{S},\mathrm{t}}$. The multiplication by u gives us an exact sequence

$$0 \to Q[u] \to M/uM \to M'/uM' \to Q/uQ \to 0$$

where all the terms are of finite length and then

$$\operatorname{length}_{W(\mathbb{F})}(M'/uM') - \operatorname{length}_{W(\mathbb{F})}(M/uM) = \operatorname{length}_{W(\mathbb{F})}(Q/uQ) - \operatorname{length}_{W(\mathbb{F})}(Q[u]) = 0.$$

As

$$M'/uM' \simeq \bigoplus_{i=0}^{r} W(\mathbb{F})/(p^{n_i}),$$

we have $\mu_{\mathrm{IW}}(M) = \sum_{i=0}^{r} n_i = \operatorname{length}_{W(\mathbb{F})}(M/uM)$. □

Frobenius

We define a Frobenius morphism over $\mathfrak{S}$ as

$$\begin{array}{cccc} \varphi & : & \mathfrak{S} & \to & \mathfrak{S} \\ & & \sum_{i\geq 0} a_i u^i & \mapsto & \sum_{i\geq 0} \sigma(a_i) u^{pi} \end{array}$$

where σ is the morphism on $W(\mathbb{F})$ lifting the Frobenius morphism of $\mathbb{F}$. For a $\mathfrak{S}$-module M, we define the module

$$\varphi^* M = M \otimes_{\varphi,\mathfrak{S}} \mathfrak{S}.$$

Lemma 3.1.8. *Let $M \in \mathcal{T}$. Then, we have*

$$\operatorname{length} \varphi^* M = p \operatorname{length} M.$$

Proof. Since $M \in \mathcal{T}$, there exists a flag

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

such that $M_i/M_{i-1} \simeq \mathfrak{S}/\mathfrak{m} \simeq \mathbb{F}[[u]]/u$ for $1 \leq i \leq r$, and we have $\operatorname{length} M = r$. Since φ is flat, we obtain a flag

$$0 = \varphi^* M_0 \subset \varphi^* M_1 \subset \ldots \subset \varphi^* M_r = \varphi^* M$$

with $\varphi^* M_i/\varphi^* M_{i-1} \simeq \varphi^*(M_i/M_{i-1}) \simeq \varphi^*(\mathbb{F}[[u]]/u) \simeq \mathbb{F}[[u]]/u^p$. Thus,

$$\operatorname{length} \varphi^* M = \sum_{i=1}^{r} \operatorname{length}(\varphi^* M_i/\varphi^* M_{i-1}) = \sum_{i=1}^{r} p = pr$$

since $\operatorname{length}(\mathbb{F}[[u]]/u^p) = p$.

□

3.2 Categories of Kisin modules

The category $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$ and its dévissage

Let $K_0 = \operatorname{Frac} W(\mathbb{F})$ and let K be a finite totally ramified extension of K_0, with ring of integers $\mathcal{O}_K$, uniformizer π and residue field $\mathbb{F}$. Let $E(u) \in \mathfrak{S}$ be the minimal polynomial of π, which is Eisenstein. Then, E is an irreductible distinguished polynomial, hence a prime element in $\mathfrak{S}$, and $\mathfrak{S}/E\mathfrak{S} \simeq \mathcal{O}_K$ by $u \mapsto \pi$.

We have already seen in last section that for every object in the category $\mathrm{Mod}_{\mathfrak{S}}$ of finitely generated $\mathfrak{S}$-modules, there is an exact sequence

$$0 \to M_{\mathrm{tors}} \to M \to M_{\mathrm{fr}} \to \overline{M} \to 0,$$

and that for M_{tors} in the sequence, we have another exact sequence

$$0 \to M[\mathfrak{m}^\infty] \to M_{\mathrm{tors}} \to M_{\mathrm{t}} \to 0.$$

Consider now the category $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$, whose objects are pairs (M, φ_M) with M a module in $\mathrm{Mod}_{\mathfrak{S}}$ and φ_M is a Frobenius isomorphism

$$\varphi_M \ : \ \varphi^* M[\tfrac{1}{E}] \xrightarrow{\sim} M[\tfrac{1}{E}],$$

where φ is the Frobenius over $\mathfrak{S}$. When there is no confusion, we will denote only by M the objects in $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$. A morphism $f : (M, \varphi_M) \to (N, \varphi_N)$ in $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$ is a morphism of $\mathfrak{S}$-modules $f : M \to N$ making the following diagram commute

$$\begin{tikzcd}
\varphi^* M[\frac{1}{E}] \arrow[r, "\varphi^* f"] \arrow[d, "\varphi_M"] & \varphi^* N[\frac{1}{E}] \arrow[d, "\varphi_N"] \\
M[\frac{1}{E}] \arrow[r, "f"] & N[\frac{1}{E}].
\end{tikzcd}$$

The category $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$ is abelian and there is a unique dévissage

$$\begin{tikzcd}
 & 0 \arrow[d] & & & & \\
 & (M[\mathfrak{m}^\infty], \varphi_{\mathfrak{m}^\infty}) \arrow[d] & & & & \\
0 \arrow[r] & (M_{\mathrm{tors}}, \varphi_{\mathrm{tors}}) \arrow[r] \arrow[d] & (M, \varphi_M) \arrow[r] & (M_{\mathrm{fr}}, \varphi_{\mathrm{fr}}) \arrow[r] & (\overline{M}, \varphi_{\overline{M}}) \arrow[r] & 0 \\
 & (M_{\mathrm{t}}, \varphi_{\mathrm{t}}) \arrow[d] & & & & \\
 & 0 & & & &
\end{tikzcd}$$

where M_{fr} is a free $\mathfrak{S}$-module, M_{tors} is killed by a power of p (i.e. it is an object in the category $\mathcal{A}$ from last section), M_{t} is an object in $\mathrm{Mod}_{\mathfrak{S},\mathrm{t}}$, and $\overline{M}$ and $M[\mathfrak{m}^\infty]$ are $\mathfrak{S}$-modules of finite length. The proof that M_{tors} is killed by a power of p is given by Bhatt, Morrow and Scholze in [4].

Remark 6.

1. Since $M[\mathfrak{m}^\infty]$ (resp. $\overline{M}$) has finite length, we have
$$M[\mathfrak{m}^\infty][\tfrac{1}{E}] = \varphi^* M[\mathfrak{m}^\infty][\tfrac{1}{E}] = 0$$
(resp. $\overline{M}[\frac{1}{E}] = \varphi^*\overline{M}[\frac{1}{E}] = 0$) and so the Frobenius $\varphi_{\mathfrak{m}^\infty}$ (resp. $\varphi_{\overline{M}}$) is just the zero morphism between zero objects.

2. For a module M in $\mathcal{A}$, we have $M[\frac{1}{E}] \simeq M[\frac{1}{u}]$. Indeed, $E(u) = u^{\deg E} + f(u)$, where $f(u)$ is nilpotent on M.

3. We have $M[\mathfrak{m}^\infty] = M[u^\infty] = M[E^\infty]$ for every $M \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi}$, since both $M[E^\infty]$ and $M[u^\infty]$ are submodules of M_{tors}, so they are also killed by a power of p.

4. The module $M[\frac{1}{p}]$ is free over $\mathfrak{S}[\frac{1}{p}]$, since $\overline{M}[\frac{1}{p}] = M_{\mathrm{tors}}[\frac{1}{p}] = 0$.

The decomposition above allows us to define the following categories that we will study in more detail later:

1. The category $\mathrm{Mod}_{\mathfrak{S},\mathrm{fr}}^{\varphi}$: It is the additive full subcategory of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$ whose objects are pairs (M, φ_M) with M a free module. We call its objects Kisin modules.

2. The category $\mathrm{Mod}_{\mathfrak{S},\mathrm{tors}}^{\varphi}$: It is the abelian full subcategory of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$ whose objects are pairs (M, φ_M) with M a module in $\mathcal{A}$. By point 2 in the remark above, we see that the objects in $\mathrm{Mod}_{\mathfrak{S},\mathrm{tors}}^{\varphi}$ do not depend on K, π or E.

Using the exact sequence

$$0 \to (M[\mathfrak{m}^\infty], \varphi_{\mathfrak{m}^\infty}) \to (M_{\mathrm{tors}}, \varphi_{\mathrm{tors}}) \to (M_{\mathrm{t}}, \varphi_{\mathrm{t}}) \to 0$$

above, we can define a torsion theory on $\mathrm{Mod}_{\mathfrak{S},\mathrm{tors}}^{\varphi}$, using the following two categories:

(3) The category $\mathrm{Mod}_{\mathfrak{S},\mathrm{t}}^{\varphi}$: It is the full subcategory of $\mathrm{Mod}_{\mathfrak{S},\mathrm{tors}}^{\varphi}$ whose objects are pairs (M, φ_M) with M a module in $\mathrm{Mod}_{\mathfrak{S},\mathrm{t}}$. We call its object p^∞-torsion Kisin modules. This will play the part of the torsion-free subcategory. It is then a quasi-abelian category.

(4) The category $\mathrm{Mod}_{\mathfrak{S},\mathrm{fl}}^{\varphi}$: It is the abelian full subcategory of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$ whose objects are pairs (M, φ_M) with M a finite length module, i.e. $M[\mathfrak{m}^\infty] = M$. We have seen that $\varphi_{\mathfrak{m}^\infty} = 0$ in Remark 5, so we have an equivalence of categories $\mathrm{Mod}_{\mathfrak{S},\mathrm{fl}}^{\varphi} \simeq \mathrm{Mod}_{\mathfrak{S},\mathrm{fl}}$. This will play the part of the torsion subcategory.

One last category will be studied, the p-torsion category:

(5) The category $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$: It is the additive full subcategory of $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$ whose objects are killed by p. This corresponds to finite free $\mathbb{F}[[u]]$-modules (since they do not have u-torsion, as $M[E] = M[u]$ for every $M \in \mathrm{Mod}_{\mathfrak{S}}$ killed by p and objects in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$ have no E-torsion) endowed with a Frobenius $\varphi_M : \varphi^* M[\frac{1}{u}] \xrightarrow{\sim} M[\frac{1}{u}]$. We call the objects in this category p-torsion Kisin modules. This is a quasi-abelian category, since it is fully embedded in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$ and it is stable by kernels and cokernels.

Definition 3.2.1. Let M be an object in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ such that $M[\mathfrak{m}^\infty] = 0$. Then $M \subset M[\frac{1}{E}]$ and $\varphi^* M \subset \varphi^* M[\frac{1}{E}]$. We say that M is effective when $\varphi_M(\varphi^* M) \subset M$. Thus $\varphi_M : \varphi^* M \to M$ is an injective morphism with cokernel killed by a power of E.

Lemma 3.2.1. *Suppose M is effective, then a subobject N of M is effective if M/N has no $\mathfrak{m}^\infty$-torsion and a quotient of M is effective if it has no $\mathfrak{m}^\infty$-torsion.*

Proof. Indeed, for a subobject N of M, we have that $\varphi_M \varphi^* N \subset N[\frac{1}{E}] \cap M$. Now, for M/N we have the exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & & & \\
 & (M/N)[\mathfrak{m}^\infty] \arrow[d] & & & & \\
0 \arrow[r] & (M/N)_{\mathrm{tors}} \arrow[r] \arrow[d] & M/N \arrow[r] & (M/N)_{\mathrm{fr}} \arrow[r] & \overline{M/N} \arrow[r] & 0 \\
 & (M/N)_{\mathrm{t}} \arrow[d] & & & & \\
 & 0 & & & &
\end{tikzcd}$$

so, if $(M/N)[\mathfrak{m}^\infty] = 0$, we have $(M/N)_{\mathrm{tors}} \simeq (M/N)_{\mathrm{t}}$ which is in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$, so it doesn't have E-torsion, and neither does M/N. An element $x \in N[\frac{1}{E}] \cap M$ has E-torsion image in M/N, so its image is zero, and $x \in N$.

For a quotient $\pi : M \twoheadrightarrow Q$, we have $\varphi^* Q = (\varphi^* \pi)(\varphi^* M)$, so

$$\varphi_Q(\varphi^* Q) = \varphi_Q \circ (\varphi^* \pi)(\varphi^* M) = \pi \circ \varphi_M(\varphi^* M) \subset \pi(M) = Q.$$

□

Isogenies classes of Kisin modules

Let $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$ be the category whose objects are the same objects as in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ and whose morphisms are morphisms in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ tensored by $\mathbb{Q}_p$. It is the isogeny category of Kisin modules. For (M, φ_M) in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$, we denote by $(M, \varphi_M) \otimes \mathbb{Q}_p$ the corresponding object in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$ (or $M \otimes \mathbb{Q}_p$ as an abuse of notation when there is no confusion).

Proposition 3.2.2. *There is an equivalence of categories*

$$\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p \simeq \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}} \otimes \mathbb{Q}_p.$$

Proof. It is induced by the adjoint functors

$$\begin{array}{ccc} \mathrm{Mod}^{\varphi}_{\mathfrak{S}} & \to & \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}} \\ M & \mapsto & M_{\mathrm{fr}} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}} & \to & \mathrm{Mod}^{\varphi}_{\mathfrak{S}} \\ M & \mapsto & M \end{array}$$

since $M_{\mathrm{tors}} \otimes \mathbb{Q}_p = \overline{M} \otimes \mathbb{Q}_p = 0$.

□

Both categories $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$ and $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}} \otimes \mathbb{Q}_p$ are abelian, since $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ is abelian.

Definition 3.2.2. Let $M_1, M_2 \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}}$. An isogeny between M_1 and M_2 is a morphism $f : M_1 \to M_2$ which becomes an isomorphism in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$. Equivalently, it is a morphism whose kernel and cokernel are objects in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{tors}}$. In particular, if M_1 and M_2 are free Kisin module, the kernel is trivial and the cokernel belongs to $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{t}}$.

We say that two objects are isogenous if there is an isogeny between them, i.e. if they become isomorphic in the category $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$.

There is a fully faithfully functor, studied in [22],

$$\begin{array}{ccc} \mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p & \to & \mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]} \\ (M \otimes \mathbb{Q}_p, \varphi_M) & \mapsto & (M[\frac{1}{p}], \varphi_M \otimes 1) \end{array}$$

where $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$ is the abelian category whose objects are finitely generated $\mathfrak{S}[\frac{1}{p}]$-modules N together with a Frobenius

$$\varphi_N : \varphi^* N[\tfrac{1}{E}] \xrightarrow{\sim} N[\tfrac{1}{E}],$$

and whose morphisms are the morphisms between modules compatible with the Frobenius, as above. When an object in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$ comes from an object M in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$, we will denote it by $M[\frac{1}{p}]$. By proceeding as in [4, Proposition 4.3], we can prove that all the modules in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$ are free modules.

Lemma 3.2.3. *The essential image of the functor* $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p \to \mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$ *given above is stable by subobjects and quotients. In particular, it is an abelian subcategory of* $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$.

Proof. It suffices to show that for any submodule N of $M[\frac{1}{p}]$ in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$, where $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$, there is a module N' in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ such that $N'[\frac{1}{p}] = N$. We can take $N' = N \cap M$ and it is easy to show that this is a module in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$, by the compatibily between φ_M, $\varphi_M[\frac{1}{p}]$ and φ_N. Then, each quotient is also included in the essential image, since the localization functor is exact, thus the essential image forms an abelian category. The lemma follows since the essential image of $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}} \to \mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$ equals the essential image of $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p \to \mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$.

□

Now, we want to show that there are Harder-Narasimhan filtrations in the two categories, and compare them. For $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$, we put

$$\begin{array}{cccc} \mathrm{rank} & : & \mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]} & \to & \mathbb{Z} \\ & & N & \mapsto & \mathrm{rank}_{\mathfrak{S}[\frac{1}{p}]} N \end{array}$$

and

$$\begin{array}{cccc} \deg & : & \mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]} & \to & \mathbb{Z} \\ & & N & \mapsto & \nu(N \otimes \hat{\mathfrak{S}}, \varphi_N \varphi^* N \otimes \hat{\mathfrak{S}}), \end{array}$$

where the operator ν was defined in section 2.3. In the next proposition, we show that they verify the conditions of a Harder-Narasimhan rank and degree function.

Proposition 3.2.4. *The degree function on* $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$ *is additive on short exact sequences and the rank function verifies the following properties:*

1. *If* $\mathrm{rank}(N) = 0$*, then* $N = 0$*.*
2. *It is additive on short exact sequences.*

Proof. The two properties for the rank function are obvious since every object in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$ is free, as we saw in 3.2, and the usual rank function is additive on short exact sequences of free modules. For the degree, we may assume the modules to be effective. A short exact sequence

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$ yields a diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \varphi^* N_1 \arrow[r] \arrow[d] & \varphi^* N_2 \arrow[r] \arrow[d] & \varphi^* N_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_1 \arrow[r] \arrow[d] & N_2 \arrow[r] \arrow[d] & N_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{coker} \varphi_{N_1} \arrow[r] \arrow[d] & \operatorname{coker} \varphi_{N_2} \arrow[r] \arrow[d] & \operatorname{coker} \varphi_{N_3} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and by localization at (E), we obtain

$$\begin{aligned}
\deg N_2 &= -\operatorname{length}_{\mathfrak{S}_{(E)}}(\operatorname{coker} \varphi_{N_2(E)}) \\
&= -\operatorname{length}_{\mathfrak{S}_{(E)}}(\operatorname{coker} \varphi_{N_1(E)}) - \operatorname{length}_{\mathfrak{S}_{(E)}}(\operatorname{coker} \varphi_{N_3(E)}) \\
&= \deg N_1 + \deg N_3.
\end{aligned}$$

□

The rank and degree functions above together with the fact the $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$ is an abelian category, define a Harder-Narasimhan filtration on objects of $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$ that we will denote by $\mathcal{F}_{\mathrm{F},\mathfrak{S}[\frac{1}{p}]}$.

For $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$, let

$$\begin{array}{rcccc}
\mathrm{rank} & : & \mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p & \to & \mathbb{Z} \\
 & & M \otimes \mathbb{Q}_p & \mapsto & \mathrm{rank}_{\mathfrak{S}\otimes\mathbb{Q}_p}(M \otimes \mathbb{Q}_p) = \mathrm{rank}_{\mathfrak{S}}(M)
\end{array}$$

and

$$\begin{array}{rcccc}
\deg & : & \mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p & \to & \mathbb{Z} \\
 & & M \otimes \mathbb{Q}_p & \mapsto & \nu(M \otimes \hat{\mathfrak{S}}, \varphi_M \varphi^* M \otimes \hat{\mathfrak{S}}).
\end{array}$$

Remark 7. The degree and rank function on $M \otimes \mathbb{Q}_p$ and $M[\frac{1}{p}]$ coincide.

Proposition 3.2.5. *The degree function on* $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}} \otimes \mathbb{Q}_p$ *is additive on short exact sequences and the rank function verifies the following properties:*

1. *If* $\mathrm{rank}(M \otimes \mathbb{Q}_p) = 0$, *then* $M \otimes \mathbb{Q}_p = 0$.
2. *It is additive on short exact sequences.*

Proof. It follows from the properties of the degree and rank function defined on $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$, since the functor $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p \to \mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$ is exact, and the degree and rank functions coincide in the two categories.

□

The degree and rank functions above together with the fact the $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$ is an abelian category, gives us a Harder-Narasimhan filtration on $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$ that we denote by $\mathcal{F}_{\mathrm{F},\circ}$ and whose polygon we will denote by $\mathbf{t}_{\mathrm{F},\circ}$. Next proposition proves that the functor $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p \to \mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$ is compatible with the Harder-Narasimhan filtrations given in each category.

Proposition 3.2.6. *Let* $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$, *then*

$$\mathcal{F}^{\geq\gamma}_{\mathrm{F},\mathfrak{S}[\frac{1}{p}]}(M[\tfrac{1}{p}]) = (\mathcal{F}^{\geq\gamma}_{\mathrm{F},\circ}(M))[\tfrac{1}{p}]$$

for every $\gamma \in \mathbb{Q}$. *In particular* $\mathbf{t}_{\mathrm{F},\mathfrak{S}[\frac{1}{p}]}(M[\frac{1}{p}]) = \mathbf{t}_{\mathrm{F},\circ}(M)$.

Proof. It suffices to show that the image of the graded spaces for $\mathcal{F}_{\mathrm{F},\circ}$ are semi-stable for the slope function defined on $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$. Let $M^{\gamma} = \mathrm{Gr}^{\gamma}_{\mathcal{F}_{\mathrm{F},\circ}}(M)$ and denote by $\mu_{\circ}$ (resp. $\mu_{\mathfrak{S}[\frac{1}{p}]}$) the slope function on $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$ (resp. on $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$). Then

$$\mu_{\mathfrak{S}[\frac{1}{p}]}(M^{\gamma}[\tfrac{1}{p}]) = \mu_{\circ}(M^{\gamma}) = \gamma$$

by definition. Now, for every subobject N of $M^{\gamma}[\frac{1}{p}]$, lemma 3.2.3 tells us that there exists a submodule N' of M such that $N'[\frac{1}{p}] = N$. Thus,

$$\mu_{\mathfrak{S}[\frac{1}{p}]}(N) = \mu_{\circ}(N') < \gamma$$

where the last inequality is due to the semi-stability of M^{γ} in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$. This proves that $M^{\gamma}[\frac{1}{p}]$ is semi-stable for every $\gamma \in \mathbb{Q}$.

□

3.3 Hodge-Pink modules

We recall in this section the definitions and some results about Hodge-Pink modules that can be found in the article of Genestier and Lafforgue [22]. First, we introduce the category of isocrystals, where the objects are pairs (D, σ_D) such that D is a finite dimensional K_0-vector space and

$$\sigma_D : \sigma^* D \xrightarrow{\sim} D$$

is an isomorphism of K_0-vector spaces, for σ the Frobenius on K_0. The morphisms are the morphisms $f : D \to D'$ of K_0-vector spaces which are compatible with the Frobenius σ_D and $\sigma_{D'}$. We denote by $\mathrm{Vect}^{\sigma}_{K_0}$ the category of isocrystals. Let K be a totally ramified finite extension of K_0 with uniformizer π_K, let $E \in \mathfrak{S}$ be the minimal polynomial of π_K and denote by $\hat{\mathfrak{S}}$ the completion of $\mathfrak{S}[\frac{1}{p}]$ with respect to the ideal generated by E.

Definition 3.3.1. For D a finite dimensional K_0-vector space, a Hodge-Pink structure on D is a free $\hat{\mathfrak{S}}$-module V_D which is a lattice in $\varphi^* D \otimes_{K_0} \hat{\mathfrak{S}}[\frac{1}{E}]$.

A Hodge-Pink module is a 3-tuple (D, φ_D, V_D) where:

- D is a finite dimensional K_0-vector space,
- $\sigma_D : \sigma^* D \to D$ is an isomorphism of K_0-vector spaces,
- V_D is a Hodge-Pink structure on D.

We denote by MHP the category of Hodge-Pink modules. A morphism of Hodge-Pink modules $D \to D'$ is a morphism f compatible with σ_D and $\sigma_{D'}$ such that $f(V_D) \subset V_{D'}$. This is a quasi-abelian category.

Remark 8. If (D, σ_D, V_D) is a Hodge-Pink module and $(D', \sigma_{D'})$ a subobject of the isocrystal (D, σ_D), we can endow D' with a Hodge-Pink structure by setting $V_{D'} = V_D \cap (\sigma^* D' \otimes_{K_0} \hat{\mathfrak{S}}[\frac{1}{E}])$.

We denote by D the Hodge-Pink module (D, σ_D, V_D) when there is no confusion. We define the Newton and Hodge types of D by

$$\mathbf{t}_{\mathrm{N}}(D, \sigma_D, V_D) = \mathbf{t}_{\mathrm{N}}(D, \sigma_D) \quad \text{and} \quad \mathbf{t}_{\mathrm{H}}(D, \sigma_D, V_D) = \mathrm{Pos}(\sigma^* D \otimes_{K_0} \hat{\mathfrak{S}}, V_D)$$

where $\mathbf{t}_{\mathrm{N}}(D, \sigma_D)$ denotes the type associated to the Newton polygon of the isocrystal (D, σ_D) given by the Dieudonné-Manin decomposition. We denote their degrees by

$$\mathrm{t}_{\mathrm{N}}(D) = \deg(\mathbf{t}_{\mathrm{N}}(D, \sigma_D, V_D)) \quad \text{and} \quad \mathrm{t}_{\mathrm{H}}(D) = \deg(\mathbf{t}_{\mathrm{H}}(D, \sigma_D, V_D)) = \nu(\sigma^* D \otimes_{K_0} \hat{\mathfrak{S}}, V_D).$$

The usual rank function and the degree function given by

$$\deg(D) = \mathrm{t}_{\mathrm{H}}(D) - \mathrm{t}_{\mathrm{N}}(D)$$

define a Harder-Narasimhan theory on MHP. Indeed, it is easy to see that the rank function verify the conditions. For the degree function, a mono-epi

$$f \ : \ (D, \sigma_D, V_D) \to (D', \sigma_{D'}, V_{D'}),$$

yields an isomorphism between the underlying isocrystals, so

$$\mathrm{t}_{\mathrm{N}}(D) = \mathrm{t}_{\mathrm{N}}(D').$$

Also, since $f(V_D) \subset V_{D'}$ and $f(\sigma^* D \otimes_{K_0} \hat{\mathfrak{S}}) = \sigma^* D' \otimes_{K_0} \hat{\mathfrak{S}}$, we obtain

$$\mathrm{t}_{\mathrm{H}}(D') - \mathrm{t}_{\mathrm{H}}(D) \geq 0$$

with equality if and only if $V_D \simeq V_{D'}$, i.e. f is an isomorphism of Hodge-Pink modules.

Definition 3.3.2. We say that a Hodge-Pink module D is weakly admissible when it is semi-stable of slope 0 for the Harder-Narasimhan theory defined above. We denote by $\mathrm{MHP}^{\mathrm{wa}}$ the full subcategory of weakly admissible Hodge-Pink modules. It is an abelian category.

We say that a Hodge-Pink module D verifies the Griffiths transversality condition when

$$1 \otimes u \frac{d}{du}(V_D) \subset E^{-1} V_D.$$

We denote by $\mathrm{MHP}^{\mathrm{Gr}}$ the full subcategory of Hodge-Pink modules verifying the Griffiths transversality condition, and $\mathrm{MHP}^{\mathrm{wa,Gr}}$ the full subcategory of weakly admissible Hodge-Pink modules verifying the Griffiths transversality condition.

Proposition 3.3.1. *The category* $\mathrm{MHP}^{\mathrm{wa,Gr}}$ *is a full subcategory of* $\mathrm{MHP}^{\mathrm{wa}}$ *stable by subobjects.*

Proof. Let $D = (D, \sigma_D, V_D)$ be an object in $\mathrm{MHP}^{\mathrm{wa,Gr}}$, $D' = (D', \sigma_{D'}, V_{D'})$ be a weakly admissible subobject of D and $D'' = (D', \sigma_{D'}, V'_{D'})$ the image of D' in the category MHP, i.e.

$$V'_{D'} = V_D \cap \sigma^* D' \otimes \hat{\mathfrak{S}}[\tfrac{1}{E}].$$

We have

$$\mathrm{t_H}(D') \leq \mathrm{t_H}(D'') \leq \mathrm{t_N}(D'') = \mathrm{t_N}(D') = \mathrm{t_H}(D')$$

where the first inequality is due to the definition of $\mathrm{t_H}$, the second inequality is given by the weakly admissibility of D and the last equality is given by the weakly admissibility of D'. Thus $\mathrm{t_H}(D') = \mathrm{t_H}(D'')$, so $V_{D'} = V'_{D'}$. Now, by hypothesis we have $u\frac{d}{du}(V_D) \subset E^{-1}V_D$, so

$$\begin{aligned} u\tfrac{d}{du}(V_{D'}) &= u\tfrac{d}{du}\left(V_D \cap \sigma^* D' \otimes_{K_0} \hat{\mathfrak{S}}[\tfrac{1}{E}]\right) \\ &\subset u\tfrac{d}{du}(V_D) \cap u\tfrac{d}{du}\left(\sigma^* D' \otimes_{K_0} \hat{\mathfrak{S}}[\tfrac{1}{E}]\right) \\ &\subset E^{-1}\left(V_D \cap \sigma^* D' \otimes_{K_0} \hat{\mathfrak{S}}[\tfrac{1}{E}]\right) \\ &= E^{-1}V_{D'}, \end{aligned}$$

therefore D' verifies the Griffiths transversality condition.

□

The following theorem gives us the relation between Hodge-Pink modules and Kisin modules. It can be found in [22, Theorem 0.4].

Theorem 3.3.2. *There is an equivalence of $\otimes$-categories*

$$\mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}} \otimes \mathbb{Q}_p \simeq \mathrm{MHP}^{\mathrm{wa}}.$$

The functor is given by

$$\begin{array}{ccc} \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}} & \to & \mathrm{MHP}^{\mathrm{wa}} \\ (M, \varphi_M) & \mapsto & (D, \sigma_D, V_D) \end{array}$$

where

$$\begin{aligned} D &= M \otimes_{\mathfrak{S}} K_0 = M[\tfrac{1}{p}] \otimes_{\mathfrak{S}[\frac{1}{p}]} K_0 = M[\tfrac{1}{p}]/uM[\tfrac{1}{p}] \\ \sigma_D &= \varphi_M \otimes 1 \end{aligned}$$

and in order to define V_D, we use the $\mathcal{O}[\frac{1}{\lambda}]$-linear isomorphism ξ in [22, Lemma 3.5] or [25, 1.2.6], making the following diagram commute

$$\begin{array}{ccc} D \otimes_{K_0} \mathcal{O}\left[\frac{1}{\lambda}\right] & \xrightarrow{\ \xi\ } & M \otimes_{\mathfrak{S}} \mathcal{O}\left[\frac{1}{\lambda}\right] \\ {\scriptstyle \sigma_D \otimes 1}\uparrow\simeq & & \simeq\uparrow{\scriptstyle \varphi_M \otimes 1} \\ \sigma^* D \otimes_{K_0} \mathcal{O}\left[\frac{1}{\lambda}\right] & \xrightarrow{\ \varphi^*(\xi)\ } & \varphi^* M \otimes_{\mathfrak{S}} \mathcal{O}\left[\frac{1}{\lambda}\right]. \end{array}$$

We can tensor this with the morphism $\mathcal{O} \to \hat{\mathfrak{S}}$ defined in [22, Lemma 3.2] (with $n = 0$), to obtain the diagram

$$\begin{array}{ccccc} D \otimes_{K_0} \hat{\mathfrak{S}}\left[\frac{1}{E}\right] & \xrightarrow{\ \xi\ } & M \otimes_{\mathfrak{S}} \hat{\mathfrak{S}}\left[\frac{1}{E}\right] & \supset & M \otimes_{\mathfrak{S}} \hat{\mathfrak{S}} \\ {\scriptstyle \sigma_D \otimes 1}\uparrow\simeq & & \simeq\uparrow{\scriptstyle \varphi_M \otimes 1} & & \\ \sigma^* D \otimes_{K_0} \hat{\mathfrak{S}}\left[\frac{1}{E}\right] & \xrightarrow{\ \varphi^*(\xi)\ } & \varphi^* M \otimes_{\mathfrak{S}} \hat{\mathfrak{S}}\left[\frac{1}{E}\right]. & & \end{array}$$

Then,

$$\begin{aligned} V_D &= (\sigma_D \otimes 1)^{-1}\xi^{-1}(M \otimes_{\mathfrak{S}} \hat{\mathfrak{S}}) \\ &= \varphi^*(\xi)^{-1} \circ (\varphi_M \otimes 1)^{-1}(M \otimes_{\mathfrak{S}} \hat{\mathfrak{S}}). \end{aligned}$$

The check that $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p \to \mathrm{MHP}^{\mathrm{wa}}$ is a $\otimes$-functor, it suffices to show that the construction of ξ is compatible with tensor products, and this is evident since ξ is unique.

Definition 3.3.3. We denote by $\mathrm{Mod}^{\varphi,\log}_{\mathfrak{S},\mathrm{fr}}$ the full subcategory of $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$ whose image in $\mathrm{MHP}^{\mathrm{wa}}$ lies in $\mathrm{MHP}^{\mathrm{wa,Gr}}$.

The following proposition gives us the relation between the Hodge types of the Kisin module and the Hodge-Pink module.

Proposition 3.3.3. *We have*

$$\mathbf{t}_{\mathrm{H}}(M) = \mathbf{t}_{\mathrm{H}}(D)^{\iota}.$$

Proof. We have

$$\begin{aligned}
\mathbf{t}_{\mathrm{H}}(M) &= \mathrm{Pos}\left(M \otimes_{\mathfrak{S}} \hat{\mathfrak{S}}, \varphi_M \otimes 1(\varphi^* M \otimes_{\mathfrak{S}} \hat{\mathfrak{S}})\right) \\
\mathbf{t}_{\mathrm{H}}(D) &= \mathrm{Pos}\left(\sigma^* D \otimes_{K_0} \hat{\mathfrak{S}}, V_D\right) \\
&= \mathrm{Pos}\left(\xi(D \otimes_{K_0} \hat{\mathfrak{S}}), M \otimes_{\mathfrak{S}} \hat{\mathfrak{S}}\right).
\end{aligned}$$

By [25, Lemma 1.2.6], we have

$$\xi(D \otimes_{K_0} \hat{\mathfrak{S}}) = \varphi_M \otimes 1\left(\varphi^* M \otimes_{\mathfrak{S}} \hat{\mathfrak{S}}\right),$$

hence

$$\mathbf{t}_{\mathrm{H}}(M) = \mathbf{t}_{\mathrm{H}}(D)^{\iota}.$$

□

3.4 Filtered isocrystals

Definition 3.4.1. A filtered isocrystal is a 3-tuple $(D, \sigma_D, \mathcal{F}_{\mathrm{H}})$ where:

- The pair (D, σ_D) is an isocrystal,
- $\mathcal{F}_{\mathrm{H}}$ is a $\mathbb{Z}$-filtration on $D_K = D \otimes_{K_0} K$ by K-subspaces.

We denote by MF^{σ}_K the category of filtered isocrystals (with the filtration defined on K), whose objects are filtered isocrystals and the morphisms are morphisms between isocrystals $f : D \to D'$ such that $f(\mathcal{F}^{\geq i}_{\mathrm{H}} D_K) \subset \mathcal{F}^{\geq i}_{\mathrm{H}} D'_K$ for every $i \in \mathbb{Z}$. This is a quasi-abelian category. If D' is a subobject of the underlying isocrystal of $D = (D, \sigma_D, \mathcal{F}_{\mathrm{H}})$, we can endow D' with a Hodge filtration given by $\mathcal{F}^{\geq i}_{\mathrm{H}} D'_K = \mathcal{F}^{\geq i}_{\mathrm{H}} D_K \cap (D' \otimes_{K_0} K)$.

We can define the Newton and Hodge type by

$$\mathbf{t}_{\mathrm{N}}(D, \sigma_D, \mathcal{F}_{\mathrm{H}}) = \mathbf{t}_{\mathrm{N}}(D, \sigma_D) \quad \text{and} \quad \mathbf{t}_{\mathrm{H}}(D, \varphi_D, \mathcal{F}_{\mathrm{H}}) = \mathbf{t}(\mathcal{F}_{\mathrm{H}}).$$

We denote their degrees by

$$\mathrm{t}_{\mathrm{N}}(D) = \deg(\mathbf{t}_{\mathrm{N}}(D, \sigma_D, \mathcal{F}_{\mathrm{H}})) \quad \text{and} \quad \mathrm{t}_{\mathrm{H}}(D) = \deg(\mathcal{F}_{\mathrm{H}}) = \textstyle\sum_{i \in \mathbb{Z}} i \cdot \dim_K(\mathrm{Gr}^i_{\mathcal{F}_{\mathrm{H}}}(D_K)).$$

It is easy to check (analogously to the Hodge-Pink modules case) that the usual rank function and the degree function given by

$$\deg(D) = \mathrm{t}_{\mathrm{H}}(D) - \mathrm{t}_{\mathrm{N}}(D)$$

define a Harder-Narasimhan theory on MF^{σ}_K.

Definition 3.4.2. We say that a filtered iscocrystal D is weakly admissible when it is semi-stable of slope 0 for the Harder-Narasimhan theory defined above. We denote by ${}^{\mathrm{wa}}\mathrm{MF}^{\sigma}_K$ the full subcategory of weakly admissible filtered isocrystals. It is an abelian category.

The category ${}^{\mathrm{wa}}\mathrm{MF}_K^{\sigma}$ is abelian and in [18], we see that there is a good Harder-Narasimhan formalism by taking for every object $D = (D, \varphi_D, \mathcal{F}_{\mathrm{H}})$

$$\deg D = -\mathrm{t}_{\mathrm{H}}(D) \quad \text{and} \quad \operatorname{rank} D = \dim_{K_0} D.$$

We denote by $\mathcal{F}_{\mathrm{F,wa}}$ the Harder-Narasimhan filtration given by these degree and rank functions, and $\mathbf{t}_{\mathrm{F,wa}}$ will be its polygon.

There is a functor

$$\begin{array}{ccc} \mathrm{MHP} & \to & \mathrm{MF}_K^{\sigma} \\ D_{\mathrm{HP}} = (D, \sigma_D, V_D) & \mapsto & D_{\mathrm{Fil}} = (D, \sigma_D, \mathcal{F}_{\mathrm{H}}) \end{array}$$

where

$$\mathcal{F}_{\mathrm{H}} = \mathcal{F}(D \otimes_{K_0} \hat{\mathfrak{S}}, \sigma_D \otimes 1(V_D))$$

since both arguments are $\hat{\mathfrak{S}}$-lattices in $D \otimes \hat{\mathfrak{S}}[\frac{1}{E}]$, so we obtain a filtration on the reduction of $D \otimes_{K_0} \hat{\mathfrak{S}}$, which is $D \otimes_{K_0} K$. It is clear that this is a $\otimes$-functor. The following theorem is given by Genestier and Lafforgue in [22, Lemma 1.3] and [22, Lemma 1.4].

Theorem 3.4.1. *This construction yields an equivalence of $\otimes$-categories*

$$\mathrm{MHP}^{\mathrm{Gr}} \xrightarrow{\sim} \mathrm{MF}_K^{\sigma}$$

inducing an equivalence of $\otimes$-categories

$${}^{\mathrm{wa}}\mathrm{MF}_K^{\sigma} \xrightarrow{\sim} \mathrm{MHP}^{\mathrm{wa,Gr}}.$$

The next proposition gives us the comparison between the Hodge types defined on the Hodge-Pink module and the filtered isocrystal.

Proposition 3.4.2. *Let D_{HP} be a Hodge-Pink module whose image by the functor above is D_{Fil}. We have*

$$\mathbf{t}_{\mathrm{H}}(D_{\mathrm{HP}}) = \mathbf{t}_{\mathrm{H}}(D_{\mathrm{Fil}}).$$

Proof. We have

$$\mathbf{t}_{\mathrm{H}}(D_{\mathrm{HP}}) = \mathrm{Pos}\left(\sigma^* D \otimes_{K_0} \hat{\mathfrak{S}}, V_D\right) = \mathrm{Pos}\left(D \otimes_{K_0} \hat{\mathfrak{S}}, \sigma_D \otimes 1(V_D)\right) = \mathbf{t}_{\mathrm{H}}(D_{\mathrm{Fil}}).$$

□

To summarize, we get a diagram

$$\begin{tikzcd}
\mathrm{Mod}_{\mathfrak{S}}^{\varphi} \otimes \mathbb{Q}_p \arrow[r, "\simeq"] & \mathrm{MHP}^{\mathrm{wa}} \arrow[r, hook] & \mathrm{MHP} \\
\mathrm{Mod}_{\mathfrak{S},\mathrm{fr}}^{\varphi,\log} \otimes \mathbb{Q}_p \arrow[u, hook] \arrow[r, "\simeq"] & \mathrm{MHP}^{\mathrm{wa,Gr}} \arrow[u, hook] \arrow[r, hook] \arrow[d, "\simeq"] & \mathrm{MHP}^{\mathrm{Gr}} \arrow[u, hook] \arrow[d, "\simeq"] \\
 & {}^{\mathrm{wa}}\mathrm{MF}_K^{\sigma} \arrow[r, hook] \arrow[ul, "\alpha", "\bar{\Theta}"'] & \mathrm{MF}_K^{\sigma}
\end{tikzcd}$$

where the category $\mathrm{Mod}_{\mathfrak{S},\mathrm{fr}}^{\varphi,\log}$ is defined in definition 3.3.3, and the functor

$$\alpha \ : \ \mathrm{Mod}_{\mathfrak{S},\mathrm{fr}}^{\varphi,\log} \otimes \mathbb{Q}_p \to {}^{\mathrm{wa}}\mathrm{MF}_K^{\sigma}$$

is described by associating to $(M, \varphi_M) \otimes \mathbb{Q}_p$ the filtered isocrystal $(M(0), \varphi_{M(0)}, \mathcal{F}_{\mathrm{H}})$ where

- The underlying module is $M(0) = (M/uM)[\frac{1}{p}]$,

- The Frobenius $\varphi_{M(0)}$ is the reduction of φ_M to $M(0)$,
- The filtration on $M(0)_K = M(0) \otimes_{K_0} K$ is given, via the isomorphism

$$M(0) \otimes_{K_0} K \xrightarrow{\sim} \varphi_M \varphi^* M \otimes_{\hat{\mathfrak{S}}} K,$$

by the filtration

$$\mathcal{F}(\varphi_M \varphi^* M \otimes \hat{\mathfrak{S}}, M \otimes \hat{\mathfrak{S}}) = \frac{\varphi_M \varphi^* M \cap E(u)^i M + E(u)\varphi_M \varphi^* M}{E(u)\varphi_M \varphi^* M} \subset \varphi_M \varphi^* M \otimes_{\hat{\mathfrak{S}}} K.$$

and

$$\tilde{\Theta} \ : \ {}^{\mathrm{wa}}\mathrm{MF}_K^{\sigma} \to \mathrm{Mod}_{\mathfrak{S},\mathrm{fr}}^{\varphi,\log} \otimes \mathbb{Q}_p$$

is its inverse, the functor defined in [25].

From the diagram above and the results for Hodge-Pink modules, we get:

Proposition 3.4.3. *The category* $\mathrm{Mod}_{\mathfrak{S},\mathrm{fr}}^{\varphi,\log} \otimes \mathbb{Q}_p$ *is a full subcategory of* $\mathrm{Mod}_{\mathfrak{S}}^{\varphi} \otimes \mathbb{Q}_p$ *stable by subobjects.*

As a consequence, we get the compatibility for the filtrations, as we did in 3.2.6.

Proposition 3.4.4. *Let* $D \in^{\mathrm{wa}} \mathrm{MF}_K^{\sigma}$*, then*

$$\mathcal{F}_{\mathrm{F},\circ}^{\geq\gamma}(\widetilde{\Theta}(D)) = \widetilde{\Theta}(\mathcal{F}_{\mathrm{F},\mathrm{wa}}^{\geq\gamma}(D))$$

for every $\gamma \in \mathbb{Q}$.

3.5 Fontaine's functors

Using the same notations from last section, let K be a finite totally ramified extension of K_0 contained in $\overline{K}_0$, π_K an uniformizer of K, $K_\infty = \cup_{n\geq 1} K(\sqrt[p^n]{\pi_K})$, $\mathrm{Gal}_K = \mathrm{Gal}(\overline{K}_0/K)$ and $\mathrm{Gal}_{K_\infty} = \mathrm{Gal}(\overline{K}_0/K_\infty)$. We denote by $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cr}}\,\mathrm{Gal}_K$ the category of crystalline Gal_K-representations, and by $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cr}}\,\mathrm{Gal}_K$ the category of Gal_K-stable $\mathbb{Z}_p$-lattices L such that $L \otimes \mathbb{Q}_p \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cr}}\,\mathrm{Gal}_K$.

For a crystalline representation V, recall that we have Fontaine functor

$$D_{\mathrm{cris}}(V) = (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}_K}$$

defined in [20, 2.3.3], where $D_{\mathrm{cris}}(V)$ is an object in MF_K^{φ}, the category of filtered isocrystals, and it is compatible with the change of the extension K of K_0. We get an equivalence of $\otimes$-categories

$$D_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cr}}\,\mathrm{Gal}_K \xrightarrow{\simeq^{\mathrm{wa}}} \mathrm{MF}_K^{\sigma},$$

whose inverse is denoted by V_{cris}. Then, we also get a functor

$$\mathfrak{N} = \tilde{\Theta} \circ D_{\mathrm{cris}} \ : \ \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cr}}\,\mathrm{Gal}_K \to \mathrm{Mod}_{\mathfrak{S}}^{\varphi} \otimes \mathbb{Q}_p.$$

By construction, there is a canonical isomorphism of $\otimes$-functors

$$\alpha \circ \mathfrak{N} \simeq D_{\mathrm{cris}} \ : \ \mathrm{Rep}_{\mathbb{Q}_p}\,\mathrm{Gal}_K \to^{\mathrm{wa}} \mathrm{MF}_K^{\sigma}.$$

Denote by $\mathcal{O}_{\mathcal{E}}$ the p-adic completion of $\mathfrak{S}_{(p)}$, a discrete valuation ring with fraction field $\mathcal{E} = \mathrm{Frac}\,\mathcal{O}_{\mathcal{E}}$ and residue field $\mathbb{F}((u))$. Let $\mathrm{Mod}_{\mathcal{E}}^{\varphi,\text{ét}}$ be the category whose objects are finite dimensional $\mathcal{E}$-vector spaces M together with a Frobenius isomorphism

$$\varphi_M : \varphi^* M \xrightarrow{\sim} M$$

and the morphisms are morphisms between $\mathcal{E}$-vector spaces compatible with the Frobenius. Also, let $\mathrm{Mod}^{\varphi,\text{ét}}_{\mathcal{O}_\mathcal{E}}$ be the category whose objects are finite free $\mathcal{O}_\mathcal{E}$-modules together with a Frobenius isomorphism as above. Then, Fontaine gives in [20] two equivalences of $\otimes$-categories

$$\mathrm{Mod}^{\varphi,\text{ét}}_{\mathcal{O}_\mathcal{E}} \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{Z}_p}\mathrm{Gal}_{K_\infty} \quad \text{and} \quad \mathrm{Mod}^{\varphi,\text{ét}}_{\mathcal{E}} \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{Q}_p}\mathrm{Gal}_{K_\infty}$$

which are exact. These two isomorphism induce a diagram

$$\begin{tikzcd}
\mathrm{Mod}^{\varphi,\log}_{\mathfrak{S},\mathrm{fr}} \arrow[r, "-\otimes\mathcal{O}_\mathcal{E}"] \arrow[d, "-\otimes\mathbb{Q}_p"] & \mathrm{Mod}^{\varphi,\text{ét}}_{\mathcal{O}_\mathcal{E}} \arrow[r, "\simeq"] & \mathrm{Rep}_{\mathbb{Z}_p}\mathrm{Gal}_{K_\infty} \arrow[d, "-\otimes\mathbb{Q}_p"] \\
\mathrm{Mod}^{\varphi,\log}_{\mathfrak{S},\mathrm{fr}}\otimes\mathbb{Q}_p \arrow[r, "-\otimes\mathcal{E}"] \arrow[d, equal] & \mathrm{Mod}^{\varphi,\text{ét}}_{\mathcal{E}} \arrow[r, "\simeq"] & \mathrm{Rep}_{\mathbb{Q}_p}\mathrm{Gal}_{K_\infty} \\
\mathrm{Mod}^{\varphi,\log}_{\mathfrak{S},\mathrm{fr}}\otimes\mathbb{Q}_p \arrow[r, "\alpha"] & {}^{\mathrm{wa}}\mathrm{MF}^{\sigma}_{K} \arrow[r, "\simeq"] & \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\mathrm{Gal}_K \arrow[u]
\end{tikzcd}$$

where the upper diagram is strictly commutative by construction and Kisin proves in [27, Proposition 2.1.5] that the lower diagram is also commutative, i.e. the two functors $\mathrm{Mod}^{\varphi,\log}_{\mathfrak{S},\mathrm{fr}}\otimes\mathbb{Q}_p \to \mathrm{Rep}_{\mathbb{Q}_p}\mathrm{Gal}_{K_\infty}$ are canonically isomorphic.

3.6 Kisin's functor

Using the Fontaine's functors from last subsection, we obtain an exact and fully faithful $\otimes$-functor

$$\mathrm{Mod}^{\varphi,\log}_{\mathfrak{S},\mathrm{fr}} \to \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p}(\mathrm{Gal}_K, \mathrm{Gal}_{K_\infty})$$

which is an equivalence of categories by [25, 2.1.15], where $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p}(\mathrm{Gal}_K, \mathrm{Gal}_{K_\infty})$ denotes the category whose objects are the couples (T, V) such that $T \in \mathrm{Rep}_{\mathbb{Z}_p}\mathrm{Gal}_{K_\infty}$ is a lattice in $V \in \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\mathrm{Gal}_K$. We denote by $\mathfrak{M}$ its inverse and also the fully faithful $\otimes$-functor

$$\mathfrak{M} \ : \ \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p}\mathrm{Gal}_K \to \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$$

which is obtained by pre and post composing this inverse with the fully faithful embeddings $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p}\mathrm{Gal}_K \hookrightarrow \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p}(\mathrm{Gal}_K, \mathrm{Gal}_{K_\infty})$ and $\mathrm{Mod}^{\varphi,\log}_{\mathfrak{S},\mathrm{fr}} \hookrightarrow \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$. The functor $\mathfrak{M}$ is compatible with the formation of symmetric and exterior powers, by [27, Theorem 1.2.1]. By construction,

- We have $\mathfrak{M}\otimes\mathbb{Q}_p \simeq \mathfrak{N}(-\otimes\mathbb{Q}_p)$. In particular, since $\mathfrak{N}$ and $-\otimes\mathbb{Q}_p$ are exact, so are $\mathfrak{M}\otimes\mathbb{Q}_p$ and $\mathfrak{M}[\frac{1}{p}]$.
- The functor $\mathfrak{M}\otimes\mathcal{O}_\mathcal{E}$ is isomorphic to Fontaine's functor

$$\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p}\mathrm{Gal}_K \to \mathrm{Rep}_{\mathbb{Z}_p}\mathrm{Gal}_{K_\infty} \to \mathrm{Mod}^{\varphi,\text{ét}}_{\mathcal{O}_\mathcal{E}}.$$

  In particular, it is exact and since $\mathfrak{S}_{(p)} \to \mathcal{O}_\mathcal{E}$ is faithfully flat, so is $\mathfrak{M}_{(p)} = \mathfrak{M}\otimes_{\mathfrak{S}}\mathfrak{S}_{(p)}$.

Remark 9. It follows that for every $\mathfrak{p} \in \operatorname{Spec}\mathfrak{S}\backslash\{\mathfrak{m}\}$, the localized functor $\mathfrak{M}_\mathfrak{p}$ is exact. For an exact sequence

$$0 \to L_1 \to L_2 \to L_3 \to 0$$

in $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p}\mathrm{Gal}_K$, the sequence

$$0 \to \mathfrak{M}(L_1) \to \mathfrak{M}(L_2) \to \mathfrak{M}(L_3)$$

is exact and the cokernel of $\mathfrak{M}(L_2) \to \mathfrak{M}(L_3)$ has finite length.

In the sequel, a $\mathbb{Z}$-filtration on an A-module M is a decreasing collection $\mathcal{F} = (\mathcal{F}^{\geq i})_{i\in\mathbb{Z}}$ of direct summands of M such that $\mathcal{F}^{\geq i} = M$ for $i \ll 0$ and $\mathcal{F}^{\geq i} = 0$ for $i \gg 0$. Any such filtration has a (non-unique) splitting: a $\mathbb{Z}$-graduation $M = \oplus_{i\in\mathbb{Z}}\mathcal{G}_i$ such that $\mathcal{F}^{\geq i} = \oplus_{j\geq i}\mathcal{G}_j$ for all $i \in \mathbb{Z}$.

For $M \in \mathrm{Mod}_{\mathfrak{S},\mathrm{fr}}$ and a $\mathbb{Z}$-filtration $\mathcal{F} = (\mathcal{F}^{\geq i})_{i\in\mathbb{Z}}$ on $N = M[\frac{1}{p}]$, we set

$$M + \mathcal{F} = \sum_{i\in\mathbb{Z}} p^{-i}M \cap \mathcal{F}^{\geq i}.$$

Since $M + \mathcal{F} = \sum_{i=-r}^{r} p^{-i}M \cap \mathcal{F}^{\geq i}$ for $r \gg 0$, this is a finitely generated $\mathfrak{S}$-submodule of N, and so is therefore also

$$M +_{\mathrm{fr}} \mathcal{F} = (M + \mathcal{F})_{\mathrm{fr}}.$$

The following conditions on $(M, \mathcal{F})$ are equivalent:

1. We have $\mathcal{F} \cap M = (\mathcal{F}^{\geq i} \cap M)_{i\in\mathbb{Z}}$ is a $\mathbb{Z}$-filtration on M,
2. There is a $\mathbb{Z}$-filtration $\mathcal{F}' = (\mathcal{F}'^{\geq i})_{i\in\mathbb{Z}}$ on M such that $\mathcal{F}'[\frac{1}{p}] = \mathcal{F}$.

Indeed (1) $\Rightarrow$ (2) with $\mathcal{F}' = \mathcal{F} \cap M$ and (2) $\Rightarrow$ (1) since then $\mathcal{F} \cap M = \mathcal{F}'$. They are also equivalent to:

3. For each $i \in \mathbb{Z}$, $M/\mathcal{F}^{\geq i} \cap M$ is free over $\mathfrak{S}$.

Then

$$M + \mathcal{F} = M +_{\mathrm{fr}} \mathcal{F}.$$

Indeed, let $M = \oplus\mathcal{G}_i$ be a splitting of $\mathcal{F} \cap M$. Then each $\mathcal{G}_i$ is a free $\mathfrak{S}$-module and

$$\begin{aligned}
p^{-i}M \cap \mathcal{F}^{\geq i} &= \left(\oplus_{j\in\mathbb{Z}} p^{-i}\mathcal{G}_j\right) \cap \left(\oplus_{j\geq i}\mathcal{G}_j[\tfrac{1}{p}]\right) \\
&= \oplus_{j\geq i} p^{-i}\mathcal{G}_j \\
\text{thus} \quad M + \mathcal{F} &= \sum_{i\in\mathbb{Z}} \left(\oplus_{j\geq i} p^{-i}\mathcal{G}_j\right) \\
&= \oplus_{j\in\mathbb{Z}} \left(\sum_{j\geq i} p^{-i}\mathcal{G}_j\right) \\
&= \oplus_{j\in\mathbb{Z}} p^{-j}\mathcal{G}_j
\end{aligned}$$

is already a free $\mathfrak{S}$-submodule of N.

If now $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$ and $\mathcal{F}$ is a φ_N-stable $\mathbb{Z}$-filtration on $N = M[\frac{1}{p}] \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$, then $M + \mathcal{F}$ and $(M + \mathcal{F})_{\mathrm{fr}}$ are φ_N-stable $\mathfrak{S}$-submodules of N. We may thus view them as objects in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ and $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$, respectively.

Proposition 3.6.1. *Let $L \in \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p}\mathrm{Gal}_K$, let $V = L \otimes \mathbb{Q}_p$, and let $\mathcal{F}$ be a Gal_K-stable $\mathbb{Z}$-filtration on V. Then we have*

$$\mathfrak{M}(L + \mathcal{F}) = \mathfrak{M}(L) +_{\mathrm{fr}} \mathfrak{N}(\mathcal{F}).$$

Proof. Plainly, $L + \mathcal{F}$ is a Gal_K-stable sub-lattice of V. Applying Kisin's functor $\mathfrak{M}$ and $\mathfrak{N}$ to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p^{-i}L \cap \mathcal{F}^{\geq i} & \longrightarrow & p^{-i}L & \longrightarrow & p^{-i}L/p^{-i}L \cap \mathcal{F}^{\geq i} & \longrightarrow & 0 \\
 & & \cap & & \cap & & \cap & & \\
0 & \longrightarrow & \mathcal{F}^{\geq i} & \longrightarrow & V & \longrightarrow & V/\mathcal{F}^{\geq i} & \longrightarrow & 0
\end{array}$$

we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{M}(p^{-i}L \cap \mathcal{F}^{\geq i}) & \longrightarrow & \mathfrak{M}(p^{-i}L) & \longrightarrow & \mathfrak{M}(p^{-i}L/p^{-i}L \cap \mathcal{F}^{\geq i}) & & \\ & & \cap & & \cap & & \cap & & \\ 0 & \longrightarrow & \mathfrak{N}(\mathcal{F}^{\geq i}) & \longrightarrow & \mathfrak{N}(V) & \longrightarrow & \mathfrak{N}(V/\mathcal{F}^{\geq i}) & \longrightarrow & 0 \end{array}$$

Since also $\mathfrak{M}\left(p^{-i}L\right) = p^{-i}\mathfrak{M}(L)$ in $\mathfrak{N}(V)$, we obtain

$$\begin{aligned} \mathfrak{M}\left(p^{-i}L \cap \mathcal{F}^{\geq i}\right) &= \mathfrak{M}\left(p^{-i}L\right) \cap \mathfrak{N}\left(\mathcal{F}^{\geq i}\right) \\ &= p^{-i}\mathfrak{M}\left(L\right) \cap \mathfrak{N}\left(\mathcal{F}^{\geq i}\right) \end{aligned}$$

inside $\mathfrak{N}(V)$. Fix $r \geq 0$ such that $\mathcal{F}^{\geq -r} = V$ and $\mathcal{F}^{\geq r} = 0$. Applying Kisin's (additive) functors $\mathfrak{M}$ and $\mathfrak{N}$ to the diagram

$$\begin{array}{ccccc} \oplus_{i=-r}^{r} p^{-i}L \cap \mathcal{F}^{\geq i} & \longrightarrow & L + \mathcal{F} & \longrightarrow & 0 \\ \cap & & \cap & & \\ \oplus_{i=-r}^{r} \mathcal{F}^{\geq i} & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

we thus obtain a commutative diagram

$$\begin{array}{ccccc} \oplus_{i=-r}^{r} p^{-i}\mathfrak{M}(L) \cap \mathfrak{N}(\mathcal{F}^{\geq i}) & \longrightarrow & \mathfrak{M}(L + \mathcal{F}) & & \\ \cap & & \cap & & \\ \oplus_{i=-r}^{r} \mathfrak{N}(\mathcal{F}^{\geq i}) & \longrightarrow & \mathfrak{N}(V) & \longrightarrow & 0 \end{array}$$

The image of the top map is the $\mathfrak{S}$-submodule $\mathfrak{M}(L) + \mathfrak{N}(\mathcal{F})$ of $\mathfrak{N}(V)$ and its cokernel has finite length, thus indeed

$$(\mathfrak{M}(L) +_{\mathrm{fr}} \mathfrak{N}(\mathcal{F})) = (\mathfrak{M}(L) + \mathfrak{N}(\mathcal{F}))_{\mathrm{fr}} = \mathfrak{M}(L + \mathcal{F}).$$

□

3.7 Harder-Narasimhan filtrations

We use the same notations as we did in 3.2, that is $\mathbb{F}$ a perfect field of characteristic p, $K_0 = \operatorname{Frac} W(\mathbb{F})$, K is a totally ramified extension of K_0, with ring of integers $\mathcal{O}_K$, uniformizer π and residue field $\mathbb{F}$. Then, $E(u) \in \mathfrak{S}$ is the minimal polynomial of π, which is an irreducible distinguished polynomial, hence a prime element in $\mathfrak{S}$. Denote by $C = \hat{\overline{K}}$, the completion of an algebraic closure $\overline{K}$ of K.

In [18], Fargues constructs a Harder-Narasimhan filtration $\mathcal{F}_{\mathrm{F,cr}}$ on $\operatorname{Rep}_{\mathbb{Q}_p}^{\mathrm{HT}} \operatorname{Gal}_K$, the abelian category of Hodge-Tate $\mathbb{Q}_p$-representations of Gal_K, by considering the dimension as rank function and the degree function defined by $\deg(V) = d$ where

$$\wedge^{\operatorname{rank} V} V_C \simeq C(d)$$

where $V_C = V \otimes_{\mathbb{Q}_p} C$. Then, he constructs a Harder-Narasimhan filtration on $\operatorname{Rep}_{\mathbb{Q}_p}^{\mathrm{cr}} \operatorname{Gal}_K$ via the fully faithful functor

$$\operatorname{Rep}_{\mathbb{Q}_p}^{\mathrm{cr}} \operatorname{Gal}_K \to \operatorname{Rep}_{\mathbb{Q}_p}^{\mathrm{HT}} \operatorname{Gal}_K,$$

since every subrepresentation of a crystalline representation is crystalline.

Now, consider the category $\mathrm{VectFil}_{C/\mathbb{Q}_p}$ of $\mathbb{Q}_p$-vector spaces endowed with a filtration on V_C. This is a quasi-abelian category admitting a Harder-Narasimhan formalism for the rank function defined as the dimension of the $\mathbb{Q}_p$-vector space and the degree function defined by

$$\begin{array}{rcccc} \deg & : & \mathrm{VectFil}_{C/\mathbb{Q}_p} & \to & \mathbb{N} \\ & & (V, \mathrm{Fil}^\bullet V_C) & \mapsto & \sum_{i\in\mathbb{Z}} i \cdot \dim_C(\mathrm{Gr}^i_{\mathrm{Fil}^\bullet} V_C). \end{array}$$

Moreover, the Harder-Narasimhan filtration obtained is compatible with tensor products. Fargues also shows that there is an exact and faithful $\otimes$-functor

$$\mathcal{G} \ : \ \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p} \mathrm{Gal}_K \to \mathrm{VectFil}_{C/\mathbb{Q}_p}$$

such that $\deg V = \deg(\mathcal{G}(V))$ and $\mathrm{rank}\, V = \mathrm{rank}(\mathcal{G}(V))$.

To summarize, we have the following relations between all the categories presented before:

$$\begin{tikzcd} \mathrm{MHP}^{\mathrm{wa,Gr}} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \mathrm{Mod}^{\varphi,\log}_{\mathfrak{S}} \otimes \mathbb{Q}_p \arrow[r, hook] & \mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p \arrow[r, hook] & \mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]} \\ {}^{\mathrm{wa}}\mathrm{MF}^{\sigma}_K \arrow[r, "\simeq"] & \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p} \mathrm{Gal}_K \arrow[r, hook] & \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p} \mathrm{Gal}_K \arrow[r, hook] & \mathrm{VectFil}_{C/\mathbb{Q}_p} \end{tikzcd}$$

where all the functors are exact and $\otimes$-functors. Also, there is a Harder-Narasimhan filtration defined in every category above (the filtration in $\mathrm{MHP}^{\mathrm{wa,Gr}}$ is defined by the equivalence of categories with $\mathrm{Mod}^{\varphi,\log}_{\mathfrak{S}} \otimes \mathbb{Q}_p$).

Proposition 3.7.1. *All the Harder-Narasimhan filtrations defined on the categories above are compatible.*

Proof. As we have seen in previous subsections, all the filtrations defined from $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$ to ${}^{\mathrm{wa}}\mathrm{MF}^{\sigma}_K$ are compatible. For the lower row, Fargues proves that via the equivalence of categories D_{cris}, the filtration on crystalline representations corresponds to the filtration defined on ${}^{\mathrm{wa}}\mathrm{MF}^{\sigma}_K$ for the degree function given above and then, in [18, Proposition 13], he proves that the filtration on $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p} \mathrm{Gal}_K$ is compatible with the filtration on $\mathrm{VectFil}_{C/\mathbb{Q}_p}$.

□

3.8 Germs of crystalline representations

Assume $\mathbb{F}$ algebraically closed.

Definition 3.8.1. For a $\mathbb{Q}_p$-vector space V, denote by $\mathrm{Cr}(V, K)$ the set of all morphisms

$$\rho : \mathrm{Gal}_K \to \mathrm{GL}(V)$$

such that (V, ρ) is a crystalline representation of Gal_K, and set

$$\mathrm{Cr}(V) := \varinjlim_{K_0 \subset K \subset \overline{K}_0} \mathrm{Cr}(V, K)$$

where the transition maps $\mathrm{Cr}(V, K) \to \mathrm{Cr}(V, K')$ are induced by the inclusion $\mathrm{Gal}_{K'} \hookrightarrow \mathrm{Gal}_K$, for finite (totally ramified) extensions $K_0 \subset K \subset K' \subset \overline{K}_0$. For an element $\rho \in \mathrm{Cr}(V)$, we denote by ρ_K the morphism $\rho_K : \mathrm{Gal}_K \to \mathrm{GL}(V)$, for K large enough.

We define a germ of crystalline representations of Gal_{K_0} as an object (V, ρ) where V is a finite dimensional $\mathbb{Q}_p$-vector space and $\rho \in \mathrm{Cr}(V)$. A morphism $f : (V, \rho) \to (V', \rho')$ of germs of

representations is a $\mathbb{Q}_p$-linear morphism $f : V \to V'$ verifying that for every K and K' such that $\rho = \rho_K$ and $\rho' = \rho_{K'}$, there exists a finite extension $K_0 \subset K'' \subset \overline{K}_0$ containing K and K' such that $f \circ \rho_{K|K''} = \rho_{K'|K''} \circ f$, where $\rho_{K|K''}$ and $\rho_{K'|K''}$ are the restrictions of ρ_K and $\rho_{K'}$ to K''. We define the germs of integral crystalline representations of Gal_{K_0} analogously.

The category of germs of crystalline representations and the category of germs of integral crystalline representations are denoted by $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\{\mathrm{Gal}_{K_0}\}$ and $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p}\{\mathrm{Gal}_{K_0}\}$, respectively.

The Harder-Narasimhan filtration on germs of crystalline representations

For an extension $K_0 \subset K \subset K' \subset \overline{K}_0$, we have a diagram

$$\begin{tikzcd}
{}^{\mathrm{wa}}\mathrm{MF}^{\sigma}_{K} \arrow[d, hook] & \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\mathrm{Gal}_K \arrow[l, "\simeq"'] \arrow[r, "\mathcal{G}"] \arrow[d, hook] & \mathrm{VectFil}_{\mathrm{C}/\mathbb{Q}_{\mathrm{p}}} \arrow[d, equal] \\
{}^{\mathrm{wa}}\mathrm{MF}^{\varphi}_{K'} \arrow[d, hook] & \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\mathrm{Gal}_{K'} \arrow[l, "\simeq"'] \arrow[r, "\mathcal{G}"] \arrow[d, hook] & \mathrm{VectFil}_{\mathrm{C}/\mathbb{Q}_{\mathrm{p}}} \arrow[d, equal] \\
{}^{\mathrm{wa}}\mathrm{MF}^{\varphi}_{\overline{K}} & \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\{\mathrm{Gal}_{K_0}\} \arrow[l, "\simeq"'] \arrow[r, "\mathcal{G}"] & \mathrm{VectFil}_{\mathrm{C}/\mathbb{Q}_{\mathrm{p}}}
\end{tikzcd}$$

We have Harder-Narasimhan filtrations in the first two rows and next proposition gives us the necessary compatibility between them in order to get a Harder-Narasimhan filtration on $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\{\mathrm{Gal}_K\}$.

Proposition 3.8.1. *The Harder-Narasimhan filtration defined on objects in* $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\mathrm{Gal}_K$ *is compatible with base change. In particular, it defines a Harder-Narasimhan filtration on objects in* $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\{\mathrm{Gal}_{K_0}\}$.

Proof. Proposition 13 in [18] tells us that after forgetting the action of Gal_{K_0}, the Harder-Narasimhan filtration on a crystalline representation V is given by applying the functor $\mathrm{VectFil}_{C/\mathbb{Q}_p} \to \mathrm{Vect}_{\mathbb{Q}_p}$ to the Harder-Narasimhan filtration on $\mathcal{G}(V)$. For every extension K of K_0, by the same argument and because C does not depend on K, after forgetting the action of Gal_K, we obtain the same filtration. The compatibility between the actions is evident, since $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\mathrm{Gal}_K \to \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\mathrm{Gal}_{K_0}$ is just given by $(V, \rho) \mapsto (V, \rho \circ i)$ for $i : \mathrm{Gal}_{K_0} \to \mathrm{Gal}_K$.

□

The D_{cris} functor on germs of crystalline representations

For a germ of crystalline representations (V, ρ), there always exists a large enough finite extension $K_0 \subset K \subset \overline{K}_0$ such that $(V, \rho_K) \in \mathrm{Rep}_{\mathbb{Q}_p}\mathrm{Gal}_K$. Then, we can consider Fontaine's D_{cris} functor to get a functor

$$\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\{\mathrm{Gal}_{K_0}\} \to \mathrm{Vect}^{\sigma}_{K_0}$$

(which is just the composition of D_{cris} with the functor forgetting the filtration) and by Fontaine's construction, this functor does not depend on K.

We would like to define the D_{cris} functor on germs of integral crystalline representations using Kisin's construction but, in order to do that, we have to prove that this construction is independent of the choice of K. For a finite extension $K_0 \subset K \subset \overline{K}_0$ with uniformizer π_K. Let η be the $\otimes$-isomorphism between Kisin's $\otimes$-functor

$$D^K_{\mathrm{cris}} \ : \ \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\mathrm{Gal}_K \xrightarrow{\mathfrak{N}} \mathrm{Mod}^{\varphi,\log}_{\mathfrak{S},\mathrm{fr}} \otimes \mathbb{Q}_p \xrightarrow{\alpha} {}^{\mathrm{wa}}\mathrm{MF}^{\sigma}_{K}$$

and Fontaine's functor

$$D_{\mathrm{cris}} \ : \ \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p} \mathrm{Gal}_K \to^{\mathrm{wa}} \mathrm{MF}^{\sigma}_K.$$

Define $\otimes$-functors

$$\begin{array}{rcccc} D^K_{\mathrm{cris}} & : & \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p} \mathrm{Gal}_K & \to & \mathrm{Mod}^{\sigma}_{W(\mathbb{F})} \\ & & L & \mapsto & \mathfrak{M}(L)/u\mathfrak{M}(L) \end{array}$$

and

$$\begin{array}{rcccc} D_{\mathrm{cris}} & : & \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p} \mathrm{Gal}_K & \to & \mathrm{Mod}^{\sigma}_{W(\mathbb{F})} \\ & & L & \mapsto & \eta_{L\otimes\mathbb{Q}_p}(D^K_{\mathrm{cris}}(L)) \end{array}$$

so that $D^K_{\mathrm{cris}}(L)$ is a lattice in $D^K_{\mathrm{cris}}(L \otimes \mathbb{Q}_p)$ and $D_{\mathrm{cris}}(L)$ is a lattice in $D_{\mathrm{cris}}(L \otimes \mathbb{Q}_p)$.

Lemma 3.8.2. *This construction induces a $\otimes$-functor*

$$D_{\mathrm{cris}} \ : \ \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p}\{\mathrm{Gal}_{K_0}\} \to \mathrm{Mod}^{\sigma}_{W(\mathbb{F})}\,.$$

Proof. It follows from the more general result given by Liu in [35, Proposition 2.2.4] for semi-stable representations, knowing that in the case of a crystalline representation we have $D_{\mathrm{st}}(V) = D_{\mathrm{cris}}(V)$.

□

Chapter 4

The Fargues filtration on $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$

4.1 The category $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$

We have already defined the category $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ in 3.2, but we can give another interpretation of this categories as lattices inside vector spaces.

Denote by $\mathrm{Mod}^{\varphi}_{\mathbb{F}((u))}$ the abelian category whose objects are finite dimensional $\mathbb{F}((u))$-vector spaces V together with a Frobenius isomorphism $\varphi_V : \varphi^* V \xrightarrow{\sim} V$ and the morphisms are the linear morphisms of vector spaces $f : V \to V'$ which are compatible with the isomorphisms φ_V and $\varphi_{V'}$. For an object V in $\mathrm{Mod}^{\varphi}_{\mathbb{F}((u))}$, denote by $\mathcal{L}(V)$ the set of lattices of V. The category $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ is then equivalent to the category of all pairs (V, M) where $V \in \mathrm{Mod}^{\varphi}_{\mathbb{F}((u))}$ and $M \in \mathcal{L}(V)$.

We have already seen that this is a quasi-abelian category. Now, we give explicitely the kernel, cokernel, image and coimage in this category. For a morphism $f : M \to N$ between two objects in $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$, the kernel $(\ker f, \varphi_{\ker f})$ is given by:

- The underlying module is $\ker f = \{m \in M \mid f(m) = 0\}$, which is a free module since it is a submodule of a free module over a PID.
- The morphism $\varphi_{\ker f} = \varphi_{M|\ker f}$ since from the compatibility of f with φ_M and φ_N, the image of $\varphi^* \ker f[\frac{1}{u}]$ by φ_M is contained in $\ker f[\frac{1}{u}]$.

The cokernel of f is given by $(\operatorname{coker} f, \varphi_{\operatorname{coker} f})$ where

- The underlying module is $\operatorname{coker} f = N/f(M)^{\mathrm{sat}}$, where $f(M)^{\mathrm{sat}}$ is the saturation of $f(M)$ in N.
- The morphism $\varphi_{\operatorname{coker} f} = \overline{\varphi}_N$ where $\overline{\varphi}_N$ is the morphism induced by φ_N on the quotient $\operatorname{coker} f$, since φ_N sends $\varphi^* f(M)^{\mathrm{sat}}$ to $f(M)^{\mathrm{sat}}[\frac{1}{u}] \simeq f(M)[\frac{1}{u}]$.

The image of f is given by $(\operatorname{im} f, \varphi_{\operatorname{im} f})$ where

- The underlying module is $\operatorname{im} f = \ker \operatorname{coker} f = f(M)^{\mathrm{sat}}$.
- The morphism $\varphi_{\operatorname{im} f} = \varphi_{N|\operatorname{im} f}$ is the restriction of φ_N to $\operatorname{im} f$.

To give a mono-epi in this category, is the same as to give a monomorphism $M_1 \hookrightarrow M_2$ such that $M_1\left[\frac{1}{u}\right] \xrightarrow{\sim} M_2\left[\frac{1}{u}\right]$, which is the same as to say that we have an exact sequence in the category of $\mathbb{F}[[u]]$-modules.

$$0 \to M_1 \to M_2 \to Q \to 0$$

with Q a torsion $\mathbb{F}[[u]]$-module.

Some operators we can consider on $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ are

1. Tensor products : For two p-torsion Kisin modules (M_1, φ_{M_1}) and (M_2, φ_{M_2}), define

$$(M_1 \otimes M_2, \varphi_{M_1 \otimes M_2}) = (M_1 \otimes_{\mathbb{F}[[u]]} M_2, \varphi_{M_1} \otimes \varphi_{M_2}).$$

   This definition works since we have $\varphi^*(M_1 \otimes_{\mathbb{F}[[u]]} M_2)[\frac{1}{u}] = (\varphi^* M_1[\frac{1}{u}]) \otimes_{\mathbb{F}((u))} (\varphi^* M_2[\frac{1}{u}])$ and $(M_1 \otimes_{\mathbb{F}[[u]]} M_2)[\frac{1}{u}] = (M_1[\frac{1}{u}]) \otimes_{\mathbb{F}((u))} (M_2[\frac{1}{u}])$. The identity object for the tensor product is the p-torsion Kisin module $\mathbf{1} = (\mathbb{F}[[u]], \varphi_{\mathbb{F}[[u]]})$ where $\varphi_{\mathbb{F}[[u]]}$ is the identity on $\varphi^*\mathbb{F}[[u]] \simeq \mathbb{F}[[u]]$.

2. Twist: We define the i-twist of a p-torsion Kisin module as

$$\begin{array}{rccc} -(i) & : & \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}} & \to & \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}} \\ & & (M, \varphi_M) & \mapsto & (M(i), \varphi_{M(i)}) = (M, u^i \varphi_M) \end{array}$$

   For any p-torsion Kisin module, there exists $i \geq 0$ such that $M(i)$ is effective. We have $M(i) = M \otimes \mathbf{1}(i)$ for $i \in \mathbb{Z}$.

3. Exterior powers: We define the exterior powers of a p-torsion Kisin module by

$$\begin{array}{rccc} \Lambda^k & : & \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}} & \to & \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}} \\ & & (M, \varphi_M) & \mapsto & (\Lambda^k M, \Lambda^k \varphi_M) \end{array}$$

   where $\Lambda^k \varphi_M$ is defined as the composition of the morphism $\Lambda^k(\varphi_M) : \Lambda^k \varphi^* M \to \Lambda^k M$ with the isomorphism $\varphi^* \Lambda^k M \simeq \Lambda^k \varphi^* M$, for all $1 \leq k \leq \mathrm{rank}_{\mathbb{F}[[u]]} M$.

4. Symmetric powers: We define the symmetric power of a p-torsion Kisin module by

$$\begin{array}{rccc} \mathrm{Sym}^k & : & \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}} & \to & \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}} \\ & & (M, \varphi_M) & \mapsto & (\mathrm{Sym}^k M, \mathrm{Sym}^k \varphi_M) \end{array}$$

   where $\mathrm{Sym}^k \varphi_M$ is defined as the composition of the morphism $\mathrm{Sym}^k(\varphi_M) : \mathrm{Sym}^k \varphi^* M \to \mathrm{Sym}^k M$ with the isomorphism $\varphi^* \mathrm{Sym}^k M \simeq \mathrm{Sym}^k \varphi^* M$, for all $k \geq 1$.

5. Duality : For a p-torsion Kisin module (M, φ_M), we define the dual of (M, φ_M) as $(M^\vee, \varphi_{M^\vee})$ where $M^\vee = \mathrm{Hom}_{\mathbb{F}[[u]]}(M, \mathbb{F}[[u]])$ and

$$\begin{array}{rccc} \varphi_{M^\vee} & : & (\varphi^* M^\vee)[\frac{1}{u}] & \to & M^\vee[\frac{1}{u}] \\ & & f & \mapsto & \varphi \circ f \circ \varphi_M^{-1} \end{array}$$

   since $\varphi^* \mathrm{Hom}(M, \mathbb{F}[[u]])[\frac{1}{u}] \simeq \mathrm{Hom}(\varphi^* M, \varphi^* \mathbb{F}[[u]])[\frac{1}{u}] \simeq \mathrm{Hom}(\varphi^* M[\frac{1}{u}], \varphi^* \mathbb{F}[[u]][\frac{1}{u}])$.

6. Internal homomorphisms: Since we have defined the tensor product and duality, we can define an internal Hom by

$$\underline{\mathrm{Hom}}((M_1, \varphi_{M_1}), (M_2, \varphi_{M_2})) = (M_1, \varphi_{M_1})^\vee \otimes (M_2, \varphi_{M_2})$$

   for all pair of objects $(M_1, \varphi_1), (M_2, \varphi_{M_2})$ in $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$.

4.2 The filtration

For a Harder-Narasimhan filtration, we need a degree and a rank function. For the rank function, we will take the usual rank of the module

$$\begin{array}{rccc} \mathrm{rank} & : & \mathrm{sk}\,\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}} & \to & \mathbb{N} \\ & & M & \mapsto & \mathrm{rank}(M) = \mathrm{rank}_{\mathbb{F}[[u]]}(M) \end{array}.$$

Remark 10. We see that another description of the rank function is given by

$$\mathrm{rank}(M) = \mu_{\mathrm{IW}}(M)$$

for $M \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ viewed as an $\mathfrak{S}$-module.
We define the degree function by

$$\begin{array}{rccc} \deg & : & \mathrm{sk}\,\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}} & \to & \mathbb{Z} \\ & & M & \mapsto & \nu(M, \varphi_M(\varphi^* M)) \end{array}$$

for ν the operator defined in 2.3. When M is effective, we have $\deg M = -\mathrm{length}_{\mathbb{F}[[u]]} Q$ for $Q = M/\varphi_M(\varphi^* M)$ the cokernel of φ_M in the category of $\mathbb{F}[[u]]$-modules. Also, define

$$\mu(M) = \frac{\deg M}{\mathrm{rank}\, M}.$$

We show how the rank and degree functions behave with respect to the operators defined on $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$. All the formulas for the degree function (except the formula for the dual) come directly from 2.1 and Proposition 2.3.3, since the degree function is the degree of a type (the relative position $\mathrm{Pos}(M, \varphi_M(\varphi^* M))$:

1. Tensor product: Let M_1, $M_2 \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ and $M = M_1 \otimes M_2$, by a direct calculation we get
$$\begin{array}{rcl} \mathrm{rank}\, M & = & \mathrm{rank}\, M_1 \,\mathrm{rank}\, M_2 \\ \deg M & = & \deg M_1 \,\mathrm{rank}\, M_2 + \deg M_2 \,\mathrm{rank}\, M_1 \\ \mu(M) & = & \mu(M_1) + \mu(M_2). \end{array}$$

2. Twist: For $M \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$, we have trivially $\mathrm{rank}(M(i)) = \mathrm{rank}\, M$ for all $i \in \mathbb{Z}$. For the degree we have
$$\begin{array}{rcl} \deg(M(i)) & = & \nu(M(i), \varphi_{M(i)}\varphi^* M(i)) \\ & = & \nu(M, u^i \varphi_M \varphi^* M) \\ & = & \nu(M, \varphi_M \varphi^* M) - i\,\mathrm{rank}\, M \\ & = & \deg M - i\,\mathrm{rank}\, M \end{array}$$
for all $i \in \mathbb{Z}$. Thus $\mu(M(i)) = \mu(M) - i$ for all $i \in \mathbb{Z}$.

3. Exterior power: Let $M \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$, $r = \mathrm{rank}\, M$, then
$$\begin{array}{rcl} \deg(\Lambda^k M) & = & \binom{r-1}{k-1} \deg M \\ \mathrm{rank}(\Lambda^k M) & = & \binom{r}{k} \\ \mu(\Lambda^k M) & = & k\mu(M) \end{array}$$
for all $k \geq 1$.

4. Symmetric power: Let $M \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$, $r = \mathrm{rank}\, M$, then
$$\begin{array}{rcl} \deg(\mathrm{Sym}^k M) & = & r\binom{r+k-1}{k-1} \deg M \\ \mathrm{rank}(\mathrm{Sym}^k M) & = & \binom{r+k-1}{k} \\ \mu(\mathrm{Sym}^k M) & = & k\mu(M) \end{array}$$
for all $k \geq 1$.

5. Duality: Let $M \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$, then

$$\begin{aligned} \deg M^{\vee} &= -\deg M \\ \operatorname{rank} M^{\vee} &= \operatorname{rank} M \\ \mu(M^{\vee}) &= -\mu(M) \end{aligned}$$

Let us prove the formula for the degree in (5). We start by verifying that $\varphi_{M^{\vee}}(\varphi^*(M^{\vee})) \simeq (\varphi_M(\varphi^* M))^{\vee}$. Let $\{e_i\}_{1\leq i\leq r}$ be a basis of M. Then, a basis of $\varphi_{M^{\vee}}(\varphi^*(M^{\vee}))$ is given by $\{\varphi_{M^{\vee}}(e_i^* \otimes 1)\}_{1\leq i\leq r}$, where $\{e_i^*\}_{1\leq i\leq r}$ is the dual basis of $M^{\vee}$, and a basis of $(\varphi_M(\varphi^* M))^{\vee}$ is given by $\{(\varphi_M(e_i \otimes 1))^*\}_{1\leq i\leq r}$, the basis dual to $\{\varphi_M(e_i \otimes 1)\}_{1\leq i\leq r}$. For the basis $\{\varphi_M(e_i \otimes 1)\}_{1\leq i\leq r}$ of $\varphi_M \varphi^* M$, we have

$$\begin{aligned} (\varphi_{M^{\vee}}(e_i^* \otimes 1))(\varphi_M(e_j \otimes 1)) &= \varphi \circ (e_i^* \otimes 1) \circ \varphi_M^{-1})(\varphi_M(e_j \otimes 1)) \\ &= (\varphi \circ (e_i^* \otimes 1))(e_j \otimes 1) \\ &= \varphi(\delta_{ij} \otimes 1) \\ &= \delta_{ij} \end{aligned}$$

for $1 \leq i, j \leq r$, so both basis coincide and $\varphi_{M^{\vee}}(\varphi^*(M^{\vee}) \simeq (\varphi_M \varphi^* M)^{\vee}$. Thus,

$$\mathrm{Pos}(M^{\vee}, \varphi_M(\varphi^*(M^{\vee}))) = \mathrm{Pos}(M^{\vee}, (\varphi_M(\varphi^* M))^{\vee}) = \mathrm{Pos}(M, \varphi_M(\varphi^* M))^{\iota}$$

where the last equality is calculated using adapted basis as follows: Let $\{u^{\gamma_i} \cdot e_i\}_{1\leq i\leq r}$ and $\{e_i\}_{1\leq i\leq r}$ be an adapted basis of M and $\varphi_M(\varphi^* M)$. Then, an adapted basis of $M^{\vee}$ and $(\varphi_M(\varphi^* M))^{\vee}$ is given by $\{u^{-\gamma_i} \cdot e_i^*\}_{1\leq i\leq r}$ and $\{e_i^*\}_{1\leq i\leq r}$, since $u^{-\gamma_i} \cdot e_i^* = (u^{-\gamma_i} \cdot e_i)^*$ for $1 \leq i \leq r$. The equalities between the relative positions give us $\deg(M^{\vee}) = -\deg M$.

The following proposition shows that rank and deg are a rank and degree function in the sense of the Harder-Narasimhan formalism.

Proposition 4.2.1. *The function* rank *verifies the following properties:*

1. *It is additive on short exact sequence of objects in* $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$.
2. *If* $\operatorname{rank} M = 0$, *then* $M = 0$.
3. *If* $f : M_1 \to M_2$ *is a mono-epi, then* $\operatorname{rank} M_1 = \operatorname{rank} M_2$.

The function deg *verifies the following properties:*

1. *It is additive on short exact sequences of objects in* $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$.
2. *For* $f : M_1 \to M_2$ *a mono-epi, we have* $\deg M_1 \leq \deg M_2$. *Moreover, the equality holds if and only if* f *is an isomorphism.*

Proof. The statements about the rank function are trivial. For the degree function, we can reduce to the effective case by the additivity of rank and the formula for the degree of a twisted module given in 2. The exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

and the flatness of φ then gives rise to an exact sequence

$$0 \to \mathrm{coker}\varphi_{M_1} \to \mathrm{coker}\varphi_{M_2} \to \mathrm{coker}\varphi_{M_3} \to 0$$

so

$$\begin{aligned}\deg M_2 &= -\operatorname{length}_{\mathbb{F}[[u]]}(\operatorname{coker}\varphi_{M_2})\\ &= -\operatorname{length}_{\mathbb{F}[[u]]}(\operatorname{coker}\varphi_{M_1}) - \operatorname{length}_{\mathbb{F}[[u]]}(\operatorname{coker}\varphi_{M_3})\\ &= \deg M_1 + \deg M_3.\end{aligned}$$

For the second part, we have seen that f mono-epi means that there is an exact sequence in the category of $\mathbb{F}[[u]]$-modules

$$0 \to M_1 \xrightarrow{f} M_2 \to Q \to 0$$

with Q a torsion $\mathbb{F}[[u]]$-module. Again, we can consider M_1 and M_2 to be effective since $\operatorname{rank} M_1 = \operatorname{rank} M_2$. By the flatness of φ, we get a diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & \ker\varphi_Q \arrow[d] & \\
0 \arrow[r] & \varphi^*M_1 \arrow[r] \arrow[d, "\varphi_{M_1}"] & \varphi^*M_2 \arrow[r] \arrow[d, "\varphi_{M_2}"] & \varphi^*Q \arrow[r] \arrow[d, "\varphi_Q"] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker}\varphi_{M_1} \arrow[d] & \operatorname{coker}\varphi_{M_2} \arrow[d] & \operatorname{coker}\varphi_Q \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in the category of $\mathbb{F}[[u]]$-modules. By the snake lemma, we have an exact sequence

$$0 \to \ker\varphi_Q \to \operatorname{coker}\varphi_{M_1} \to \operatorname{coker}\varphi_{M_2} \to \operatorname{coker}\varphi_Q \to 0.$$

From this, we get that

$$\begin{aligned}\deg M_2 - \deg M_1 &= -\operatorname{length}_{\mathbb{F}[[u]]} \operatorname{coker}\varphi_{M_2} + \operatorname{length}_{\mathbb{F}[[u]]} \operatorname{coker}\varphi_{M_1}\\ &= -\operatorname{length}_{\mathbb{F}[[u]]} Q + \operatorname{length}_{\mathbb{F}[[u]]}(\varphi^*Q)\\ &= (p-1)\cdot \operatorname{length}_{\mathbb{F}[[u]]} Q\\ &\geq 0\end{aligned}$$

where the last equality is proved in 3.1.8. We also see that the equality holds if and only if $\operatorname{length} Q = 0$, which implies that $Q = 0$ and f is an isomorphism.

□

The deg and rank functions define a Harder-Narasimhan filtration on every object of $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$. This filtration was already announced by Carl Wang-Erickson and Brandon Levin in [17] and was originally inspired by Fargues' theory [19] of Harder-Narasimhan filtrations for finite flat group schemes. This is why, from now on, we will refer to our filtration as the the Fargues filtrations. We will denote by $\mathcal{F}_{\mathrm{F},1}$ the Fargues filtration and by $\mathbf{t}_{\mathrm{F},1}$, the polygon associated to it.

4.3 Hodge types

Let $M \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ with $\operatorname{rank} M = r$, and let $V = M[\frac{1}{u}]$. Define the type

$$\mathrm{Pos}(M, M_n) \quad \text{in} \quad \mathbb{Z}^r_{\geq},$$

where $M_0 = M$, $M_1 = \varphi_M(\varphi^* M)$ and for $n \geq 2$, we have

$$M_n = (\varphi_M \circ \varphi^*\varphi_M \circ \ldots \circ \varphi^{(n-1)*}\varphi_M)(\varphi^{(n)*}M)$$

where $\varphi^{(k)*}M$ is the pullback of M by $\varphi^{(n)}$ and

$$\varphi^{(k)*}\varphi_M \ : \ \varphi^{(k)*}(\varphi^*\varphi_M) = (\varphi \circ \varphi^{(k)})^* M = \varphi^{(k+1)*}M \to \varphi^{(k)*}M$$

for $k \geq 2$. There is an alternative definition using the isomorphism $\varphi_V \ : \ \varphi^* V \xrightarrow{\sim} V$. Consider the composition of morphisms

$$\mathcal{L}(V) \xrightarrow{\varphi^*} \mathcal{L}(\varphi^* V) \xrightarrow{\varphi_V} \mathcal{L}(V)$$

sending a lattice M to $\varphi_V(\varphi^* M)$. As an abuse of notation, we denote this composition by φ_V. We put $M_0 = M$ and $M_n = \varphi_V(M_{n-1}) = \varphi_V^n(M_0)$ for every $n \geq 1$. This interpretation of the lattices M_n will be very useful, since the following lemma tells us how the relative position of lattices interacts with $\varphi_V : \mathcal{L}(V) \to \mathcal{L}(V)$.

Lemma 4.3.1. *Let M_1, M_2 be two $\mathbb{F}[[u]]$-lattices in V. Then we have*

$$\mathrm{Pos}(\varphi_V(M_1),\, \varphi_V(M_2)) = p\,\mathrm{Pos}(M_1, M_2).$$

Proof. Let $\{u^{n_i} \cdot e_i\}_{1 \leq i \leq r}$ and $\{e_i\}_{1 \leq i \leq r}$ be adapted basis of M_1 and M_2. Then, basis of $\varphi_V(M_1)$ and $\varphi_V(M_2)$ are given by $\{\varphi_V(u^{n_i} \cdot e_i \otimes 1)\}_{1 \leq i \leq r}$ and $\{\varphi_V(e_i \otimes 1)\}_{1 \leq i \leq r}$, respectively. Since $\varphi_V(u^{n_i} e_i \otimes 1) = \varphi_V(e_i \otimes u^{pn_i}) = u^{pn_i}\varphi_V(e_i \otimes 1)$, the lemma follows.

□

As a consequence, we have:

Corollary 4.3.2. *Let M_1, M_2 be two $\mathbb{F}[[u]]$-lattices in V. Then*

$$\mathrm{d}(\varphi_V M_1, \varphi_V M_2) = p\,\mathrm{d}(M_1, M_2).$$

In particular, for a p-torsion Kisin module, we have

$$\begin{aligned} \mathrm{Pos}(M_i, M_j) &= p^i\,\mathrm{Pos}(M, M_{j-i}) \\ \mathrm{d}(M_i, M_j) &= p^i\,\mathrm{d}(M, M_{j-i}) \end{aligned}$$

for $1 \leq i \leq j$. Next proposition gives some other properties of the modules M_n.

Proposition 4.3.3. *The modules M_n verify:*

1. *For all $n \geq 1$ and M_1, M_2 two $\mathbb{F}[[u]]$-lattices in two objects V and V' of $\mathrm{Mod}^{\varphi}_{\mathbb{F}((u))}$, we have $(M_1 \otimes M_2)_n = M_{1,n} \otimes M_{2,n}$.*
2. *For all $n, k \geq 1$, we have $\Lambda^k(M_n) = (\Lambda^k M)_n$.*
3. *For all $n, k \geq 1$, we have $\mathrm{Sym}^k(M_n) = (\mathrm{Sym}^k M)_n$.*
4. *For all $n \geq 1$ and M_1, M_2 two $\mathbb{F}[[u]]$-lattices in $V, V' \in \mathrm{Mod}^{\varphi}_{\mathbb{F}((u))}$, we have $(M_1 \oplus M_2)_n = M_{1,n} \oplus M_{2,n}$.*

5. *For an exact sequence* $0 \to W \to V \xrightarrow{\pi} V/W \to 0$ *in* $\mathrm{Mod}^{\varphi}_{\mathbb{F}((u))}$, *and any* $\mathbb{F}[[u]]$*-lattice* M *in* V, *we have* $(M_n \cap W) = (M \cap W)_n$ *and* $(\pi(M))_n = \pi(M_n)$.

6. *For every* φ_V*-stable flag* F *of* V, *we have* $\mathrm{Gr}_F(M_n) = (\mathrm{Gr}_F M)_n$.

Proof. 1. We prove it by induction on n since

$$\varphi_{V\otimes V'}(M_1 \otimes M_2) = \varphi_V M_1 \otimes \varphi_{V'} M_2 = M_{1,1} \otimes M_{2,1}$$

and if the statement is true for $n-1$, then

$$\begin{aligned} \varphi_{V\otimes V'}(M_1 \otimes M_2)_{n-1} &= \varphi_{V\otimes V'}(M_{1,n-1} \otimes M_{2,n-1}) \\ &= \varphi_V M_{1,n-1} \otimes \varphi_{V'} M_{2,n-1} \\ &= M_{1,n} \otimes M_{2,n}. \end{aligned}$$

2. We prove it by induction on n, since $\Lambda^k(\varphi_V M) = \varphi_{\Lambda^k V}(\Lambda^k M)$ and if we suppose that $\Lambda^k M_{n-1} = (\Lambda^k M)_{n-1}$, then

$$\Lambda^k(M_n) = \Lambda^k(\varphi_V M_{n-1}) = \varphi_{\Lambda^k V}(\Lambda^k M_{n-1}) = \varphi_{\Lambda^k V}((\Lambda^k M)_{n-1}) = (\Lambda^k M)_n.$$

3. The proof is the analogous to the one for the exterior power.

4. We prove it by induction on n, since $\varphi_{V\oplus V'}(M_1 \oplus M_2) = \varphi_V(M_1) \oplus \varphi_{V'}(M_2)$ and if we suppose that $(M_1 \oplus M_2)_{n-1} = M_{1,n-1} \oplus M_{2,n-1}$, then

$$(M_1 \oplus M_2)_n = \varphi_{V\oplus V'}((M_1 \oplus M_2)_{n-1}) = \varphi_{V\oplus V'}(M_{1,n-1} \oplus M_{2,n-1}) = M_{1,n} \oplus M_{2,n}.$$

5. Let W be a φ_V-stable subspace of V. Since φ is flat, it commutes with intersections, so we have $\varphi^*(M \cap W) = \varphi^* M \cap \varphi^* W$. Since φ_V is an isomorphism, $\varphi_V(\varphi^* M \cap \varphi^* W) = \varphi_V(\varphi^* M) \cap \varphi_V(\varphi^* W)$. Thus $\varphi_W(M \cap W) = \varphi_V(M) \cap W$, since W is φ_V-stable. If we suppose $(M \cap W)_{n-1} = M_{n-1} \cap W$, we get

$$(M \cap W)_n = \varphi_W(M \cap W)_{n-1} = \varphi_W(M_{n-1} \cap W) = \varphi_V(M_{n-1}) \cap W = M_n \cap W.$$

Let $V \xrightarrow{\pi} V/W$ be the projection. Then $\pi(\varphi_V M) = \varphi_{V/W}(\pi(M))$ again using the flatness of φ and the bijectivity of φ_V, and then if we consider the statement true for $n-1$, we have

$$\pi(M_n) = \pi(\varphi_V M_{n-1}) = \varphi_{V/W}(\pi(M_{n-1})) = \varphi_{V/W}(\pi(M))_{n-1} = (\pi(M))_n.$$

6. It is a consequence of (4) and (5).

□

Definition 4.3.1. We define the n-th Hodge filtration of a p-torsion Kisin module M by

$$\mathcal{F}_{\mathrm{H},n}(M) = \mathcal{F}(M, M_n),$$

i.e. the filtration induced by M_n on M/uM.

Proposition 4.3.4. *The Hodge filtrations verify the following properties:*

1. *For* $M_1, M_2 \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$, *let* $M = M_1 \otimes M_2$. *Then*

$$\mathcal{F}_{\mathrm{H},n}(M) = \mathcal{F}_{\mathrm{H},n}(M_1) \otimes \mathcal{F}_{\mathrm{H},n}(M_2)$$

for all $n \in \mathbb{N}$.

2. *For* $M \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ *and for every* $k, n \in \mathbb{N}$*, we have*

$$\mathcal{F}_{\mathrm{H},n}(\Lambda^k M) = \Lambda^k(\mathcal{F}_{\mathrm{H},n}(M)).$$

3. *For* $M \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ *and for every* $k, n \in \mathbb{N}$*, we have*

$$\mathcal{F}_{\mathrm{H},n}(\mathrm{Sym}^k M) = \mathrm{Sym}^k(\mathcal{F}_{\mathrm{H},n}(M)).$$

4. *For* $M_1, M_2 \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$*, let* $M = M_1 \oplus M_2$*. Then*

$$\mathcal{F}_{\mathrm{H},n}(M) = \mathcal{F}_{\mathrm{H},n}(M_1) \oplus \mathcal{F}_{\mathrm{H},n}(M_2)$$

for all $n \in \mathbb{N}$*.*

Proof. It follows directly from Proposition 2.3.2 and Proposition 4.3.3.

□

Definition 4.3.2. We define the n-th Hodge type of a p-torsion Kisin module M by

$$\mathbf{t}_{\mathrm{H},n}(M) = p_n^{-1}\mathbf{t}(\mathcal{F}_{\mathrm{H},n}(M)) = p_n^{-1}\,\mathrm{Pos}(M, M_n) \quad \text{in} \quad \mathbb{Q}^r_{\geq}$$

for $p_n = \frac{p^n-1}{p-1}$ and $r = \mathrm{rank}\, M$.

In the next lemma, we prove that the Hodge types are decreasing for the divisibility in $\mathbb{N}$.

Lemma 4.3.5. *For all* $n \geq 1$ *and* $m \geq 1$*, we have*

$$\mathbf{t}_{\mathrm{H},nm}(M) \leq \mathbf{t}_{\mathrm{H},n}(M) \quad \textit{in} \quad \mathbb{Q}^r_{\geq}.$$

Proof. For all n, $m \geq 1$, we have

$$\begin{array}{rcl} \mathrm{Pos}(M, M_{nm}) & \leq & \sum_{i=0}^{m-1} \mathrm{Pos}(M_{ni}, M_{n(i+1)}) \\ & = & \sum_{i=0}^{m-1} p^{ni}\,\mathrm{Pos}(M, M_n) \\ & = & \frac{p_{mn}}{p_n}\,\mathrm{Pos}(M, M_n) \end{array}$$

for all $m \geq 1$, thus $\mathbf{t}_{\mathrm{H},nm} \leq \mathbf{t}_{\mathrm{H},n}$.

□

Remark 11. In particular, all the Hodge types $\mathbf{t}_{\mathrm{H},n}$ have the same ending point $(\mathrm{rank}\, M, \deg M)$. We consider the lower limit of the collection $(\mathbf{t}_{\mathrm{H},n}(M))_{n\geq 1}$ for a p-torsion Kisin module M:

$$\mathbf{t}_{\mathrm{H},\infty}(M) = \lim \mathbf{t}_{\mathrm{H},n}(M) \quad \text{in} \quad \mathbb{R}^r_{\geq}$$

where the limit is taken with respect to divisibility. This limit exists since the Hodge types can be interpreted as piecewise affine concave polygons with the same starting and ending points, and those are lower bounded by the linear polygon starting and ending at the same points as the $\mathbf{t}_{\mathrm{H},n}(M)$.

Proposition 4.3.6. *Some properties of the Hodge types are:*

1. *For* $M_1, M_2 \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$*, let* $M = M_1 \otimes M_2$*. Then*

$$\mathbf{t}_{\mathrm{H},n}(M) = \mathbf{t}_{\mathrm{H},n}(M_1) \otimes \mathbf{t}_{\mathrm{H},n}(M_2)$$

for all $n \in \mathbb{N} \cup \{\infty\}$*.*

2. *For $M \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ and every $n, k \geq 1$, we have*

$$\mathbf{t}_{\mathrm{H},n}(\Lambda^k M) = \Lambda^k(\mathbf{t}_{\mathrm{H},n}(M)).$$

3. *For $M \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ and every $n, k \geq 1$, we have*

$$\mathbf{t}_{\mathrm{H},n}(\mathrm{Sym}^k M) = \mathrm{Sym}^k(\mathbf{t}_{\mathrm{H},n}(M)).$$

4. *For $M_1, M_2 \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$, let $M = M_1 \oplus M_2$. Then*

$$\mathbf{t}_{\mathrm{H},n}(M) = \mathbf{t}_{\mathrm{H},n}(M_1) \oplus \mathbf{t}_{\mathrm{H},n}(M_2)$$

for all $n \in \mathbb{N} \cup \{\infty\}$.

5. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of p-torsion Kisin modules, then we have*

$$\mathbf{t}_{\mathrm{H},n}(M_1) * \mathbf{t}_{\mathrm{H},n}(M_3) \leq \mathbf{t}_{\mathrm{H},n}(M_2)$$

for all $n \in \mathbb{N} \cup \{\infty\}$.

6. *Let $M \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ and $\mathcal{F}$ be a flag on $V = M[\frac{1}{u}]$ which is φ_V-stable, then*

$$\mathbf{t}_{\mathrm{H},n}(\mathrm{Gr}_{\mathcal{F}} M) \leq \mathbf{t}_{\mathrm{H},n}(M)$$

for all $n \in \mathbb{N} \cup \{\infty\}$.

Proof.(1)-(4) It follows from Proposition 4.3.4 and Proposition 2.2.1.

(5) It follows from 4.3.3 and 2.3.3.

(6) We have $\mathbf{t}_{\mathrm{H},n}(\mathrm{Gr}_{\mathcal{F}} M) = p_n^{-1}\,\mathrm{Pos}(\mathrm{Gr}_{\mathcal{F}} M, (\mathrm{Gr}_{\mathcal{F}} M)_n) = p_n^{-1}\,\mathrm{Pos}(\mathrm{Gr}_{\mathcal{F}} M, \mathrm{Gr}_{\mathcal{F}}(M_n))$, where the last equality is given in 4.3.3. Then, by 2.3.3, we have $p_n^{-1}\,\mathrm{Pos}(\mathrm{Gr}_{\mathcal{F}} M, \mathrm{Gr}_{\mathcal{F}}(M_n)) \leq p_n^{-1}\,\mathrm{Pos}(M, M_n) = \mathbf{t}_{\mathrm{H},n}(M)$ as we wanted.

□

Proposition 4.3.7. *Let $M \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$. Then we have*

$$\mathbf{t}_{\mathrm{F},1}(M) \leq \mathbf{t}_{\mathrm{H},\infty}(M) \quad \text{in} \quad \mathbb{R}^r_{\geq}.$$

Proof. In Proposition 2.5.4, we have seen that $\mathbf{t}_{\mathrm{F}}(M) = \mathbf{t}_{\mathrm{F}}(\mathrm{Gr}_{\mathcal{F}_{\mathrm{F}}} M) = \mathbf{t}_{\mathrm{F}}(\mathrm{Gr}^{\gamma_1}_{\mathcal{F}_{\mathrm{F}}} M) * \ldots * \mathbf{t}_{\mathrm{F}}(\mathrm{Gr}^{\gamma_r}_{\mathcal{F}_{\mathrm{F}}} M)$ where $\{\gamma_1, \ldots, \gamma_r\}$ are the breaks in the Fargues filtration of M. Now, for each $n \geq 1$ and for each $1 \leq i \leq r$, we have $\mathbf{t}_{\mathrm{F}}(\mathrm{Gr}^{\gamma_i}_{\mathcal{F}_{\mathrm{F}}} M) \leq \mathbf{t}_{\mathrm{H},n}(\mathrm{Gr}^{\gamma_i}_{\mathcal{F}_{\mathrm{F}}} M)$ since both are polygons with the same terminal points and $\mathrm{Gr}^{\gamma_i}_{\mathcal{F}_{\mathrm{F}}} M$ is semi-stable, so its Fargues polygons only have one slope. Thus, also $\mathbf{t}_{\mathrm{F}}(\mathrm{Gr}^{\gamma_i}_{\mathcal{F}_{\mathrm{F}}} M) \leq \mathbf{t}_{\mathrm{H},\infty}(\mathrm{Gr}^{\gamma_i}_{\mathcal{F}_{\mathrm{F}}} M)$ for all $1 \leq i \leq r$ and

$$\mathbf{t}_{\mathrm{F}}(\mathrm{Gr}^{\gamma_1}_{\mathcal{F}_{\mathrm{F}}} M) * \ldots * \mathbf{t}_{\mathrm{F}}(\mathrm{Gr}^{\gamma_r}_{\mathcal{F}_{\mathrm{F}}} M) \leq \mathbf{t}_{\mathrm{H},\infty}(\mathrm{Gr}^{\gamma_1}_{\mathcal{F}_{\mathrm{F}}} M) * \ldots * \mathbf{t}_{\mathrm{H},\infty}(\mathrm{Gr}^{\gamma_r}_{\mathcal{F}_{\mathrm{F}}} M).$$

Therefore

$$\mathbf{t}_{\mathrm{F}}(M) = *^r_{i=1}\mathbf{t}_{\mathrm{F}}(\mathrm{Gr}^{\gamma_i}_{\mathcal{F}_{\mathrm{F}}} M) \leq *^r_{i=1}\mathbf{t}_{\mathrm{H},\infty}(\mathrm{Gr}^{\gamma_i}_{\mathcal{F}_{\mathrm{F}}} M) = \mathbf{t}_{\mathrm{H},\infty}(\mathrm{Gr}_{\mathcal{F}_{\mathrm{F}}} M) \leq \mathbf{t}_{\mathrm{H},\infty}(M)$$

where the last inequality is given by Proposition 4.3.6.

□

4.4 Aligned and flat modules

Definition 4.4.1. 1. We say that the p-torsion Kisin module M is aligned if we have

$$\mathbf{t}_{\mathrm{H},\infty} = \mathbf{t}_{\mathrm{H},n}$$

for some $n \in \mathbb{N}$. We denote by $\mathrm{Mod}^{\varphi,\mathrm{al}}_{\mathbb{F}[[u]],\mathrm{fr}}$ the full subcategory of p-torsion Kisin modules which are aligned.

2. We say that a p-torsion Kisin module M is flat if there exists an apartment A that contains M_n for infinitely many $n \in \mathbb{N}$. We denote by $\mathrm{Mod}^{\varphi,\mathrm{flat}}_{\mathbb{F}[[u]]}$ the full subcategory of p-torsion Kisin modules which are flat.

Proposition 4.4.1. *Aligned and flat modules verify the following properties:*

1. *Let M be an aligned p-torsion Kisin module, then M is also flat.*
2. *Let M_1, M_2 be two aligned $\mathbb{F}[[u]]$-lattices in V_1 and V_2. Then $M_1 \otimes M_2$ is an aligned p-torsion Kisin module.*
3. *Let M be an aligned p-torsion Kisin module, then $\Lambda^k M$ and $\mathrm{Sym}^k M$ are aligned for every $k \geq 1$.*
4. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence in $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$. Then, M_2 flat implies that M_1 and M_3 are also flat.*
5. *Any strict subquotient of a flat p-torsion Kisin module is flat.*
6. *The Fargues filtration $\mathcal{F}_{\mathrm{F},1}$ of a flat module is formed by flat submodules.*

Proof. 1. Suppose M is aligned, then there exists $n \geq 1$ such that $\mathbf{t}_{\mathrm{H},\infty} = \mathbf{t}_{\mathrm{H},n} = \mathbf{t}_{\mathrm{H},nm}$ for all $m \geq 1$. By the triangular inequality, we have

$$\begin{array}{rcl} \mathrm{d}(M, M_{nm}) & \leq & \sum_{i=0}^{m-1} \mathrm{d}(M_{ni}, M_{n(i+1)}) \\ & = & \sum_{i=0}^{m-1} p^{ni}\, \mathrm{d}(M, M_n) \\ & = & \frac{p_{mn}}{p_n}\, \mathrm{d}(M, M_n) \end{array}$$

for all $m \geq 1$, thus

$$p_{nm}||\mathbf{t}_{\mathrm{H},nm}|| = \mathrm{d}(M, M_{nm}) \leq \frac{p_{nm}}{p_n}\, \mathrm{d}(M, M_n) = p_{nm}||\mathbf{t}_{\mathrm{H},n}|| = p_{nm}||\mathbf{t}_{\mathrm{H},nm}||$$

where the last equality is due to our hypothesis. Then 2.6.2 proves that there is an apartment containing M_{nm} for all $m \geq 0$, and M is flat.

2. Recall that, by 4.3.6, we have

$$\begin{array}{rcl} \mathbf{t}_{\mathrm{H},n}(M_1 \otimes M_2) & = & \mathbf{t}_{\mathrm{H},n}(M_1) \otimes \mathbf{t}_{\mathrm{H},n}(M_2) \\ \mathbf{t}_{\mathrm{H},\infty}(M_1 \otimes M_2) & = & \mathbf{t}_{\mathrm{H},\infty}(M_1) \otimes \mathbf{t}_{\mathrm{H},\infty}(M_2). \end{array}$$

If M_1 and M_2 are aligned, there exists $n_1, n_2 \in \mathbb{Z}$ such that $\mathbf{t}_{\mathrm{H},n_i}(M_i) = \mathbf{t}_{\mathrm{H},\infty}(M_i)$ for $i = 1, 2$, thus also $\mathbf{t}_{\mathrm{H},n_1n_2}(M_i) = \mathbf{t}_{\mathrm{H},\infty}(M_i)$ for $i = 1, 2$ and we have

$$\begin{array}{rcl} \mathbf{t}_{\mathrm{H},\infty}(M_1 \otimes M_2) & = & \mathbf{t}_{\mathrm{H},\infty}(M_1) \otimes \mathbf{t}_{\mathrm{H},\infty}(M_2) \\ & = & \mathbf{t}_{\mathrm{H},n_1n_2}(M_1) \otimes \mathbf{t}_{\mathrm{H},n_1n_2}(M_2) \\ & = & \mathbf{t}_{\mathrm{H},n_1n_2}(M_1 \otimes M_2), \end{array}$$

so $M_1 \otimes M_2$ is aligned.

3. By hypothesis, there exists $n \geq 1$ such that $\mathbf{t}_{\mathrm{H},n}(M) = \mathbf{t}_{\mathrm{H},\infty}(M)$. Then, by Proposition 4.3.6, we have

$$\mathbf{t}_{\mathrm{H},n}(\Lambda^k M) = \Lambda^k \mathbf{t}_{\mathrm{H},n}(M) = \Lambda^k \mathbf{t}_{\mathrm{H},\infty}(M) = \mathbf{t}_{\mathrm{H},\infty}(\Lambda^k M)$$

and

$$\mathbf{t}_{\mathrm{H},n}(\mathrm{Sym}^k M) = \mathrm{Sym}^k \mathbf{t}_{\mathrm{H},n}(M) = \mathrm{Sym}^k \mathbf{t}_{\mathrm{H},\infty}(M) = \mathbf{t}_{\mathrm{H},\infty}(\mathrm{Sym}^k M)$$

for every $k \geq 1$, so $\Lambda^k M$ and $\mathrm{Sym}^k M$ are aligned for $k \geq 1$.

4. Consider the exact sequences

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, hook] & M_2 \arrow[r] \arrow[d, hook] & M_3 \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & V_1 \arrow[r] & V_2 \arrow[r, "\pi"] & V_3 \arrow[r] & 0 \end{tikzcd}$$

for V_i objects in $\mathrm{Mod}^{\varphi}_{\mathbb{F}((u))}$ for $i = 1, 2, 3$. If M_2 is flat, there exists an apartment $\mathcal{A}_2$ of $\mathcal{L}(V_2)$ containing $M_{2,n}$ for infinitely many $n \in \mathbb{N}$. Then, Proposition 2.6.3 tells us that there exists a finite family of apartments of $\mathcal{L}(V_2)$ adapted to the exact sequence such that $\mathcal{A}_2 \subset \cup_{i=1}^r \mathcal{A}_{2,i}$, thus there exists $i \in \{1, \ldots, r\}$ such that $\mathcal{A}_{2,i}$ contains $M_{2,n}$ for infinitely many integers $n \in \mathbb{N}$. We denote by S the subset of such integers. Let $\mathcal{D}$ be the line decomposition of V_2 defining $\mathcal{A}_{2,i}$. Since $\mathcal{A}_{2,i}$ is adapted to the exact sequence, we get line decompositions of V_1 and V_3 by taking $\mathcal{D}_1 = \mathcal{D} \cap V_1$ and $\mathcal{D}_3 = \pi(\mathcal{D})$, respectively. Then, for every $n \in S$, we have

$$\begin{aligned} M_{1,n} = M_{2,n} \cap V_1 &= (\oplus_{D \in \mathcal{D}}(M_{2,n} \cap D)) \cap (\oplus_{D \in \mathcal{D}_1} D) \\ &= \oplus_{D \in \mathcal{D}}(M_{2,n} \cap D \cap V_1) \\ &= \oplus_{D \in \mathcal{D}}(M_{1,n} \cap (D \cap V_1)) \\ &= \oplus_{D \in \mathcal{D}_1}(M_{1,n} \cap D) \end{aligned}$$

so $M_{1,n} \in \mathcal{A}(\mathcal{D}_1)$ for infinitely many $n \in \mathbb{Z}$, thus M_1 is flat. We can do the same for M_3 since

$$M_{3,n} = \pi(M_{2,n}) = \pi\left(\oplus_{D \in \mathcal{D}}(M_{2,n} \cap D)\right) = \oplus_{D \in \mathcal{D}_3}(\pi(M_{2,n}) \cap \pi(D)) = \oplus_{D \in \mathcal{D}_3}(M_{3,n} \cap \pi(D)),$$

so the apartment $A(\mathcal{D}_3)$ contains $M_{3,n}$ for every $n \in S$. The points (5) and (6) are a consequence of (4).

□

4.5 Compatibility with tensor products

In this subsection, V will always refer to an object in $\mathrm{Mod}^{\varphi}_{\mathbb{F}((u))}$. We denote by $\mathbf{F}^{\mathbb{R}}(V, \varphi_V)$ the set of $\mathbb{R}$-filtrations of V which are φ_V-stable.

For every filtration $\mathcal{F}$ on V, and two $\mathbb{F}[[u]]$-lattices in V, we define the scalar product with respect to M_1 and M_2 as

$$\langle M_1, M_2; \mathcal{F} \rangle = \sum_{\gamma \in \Gamma} \gamma \cdot \nu(\mathrm{Gr}^{\gamma}_{\mathcal{F}}(M_1), \mathrm{Gr}^{\gamma}_{\mathcal{F}}(M_2)).$$

In particular, for a $\mathbb{F}[[u]]$-lattice M inside V, we have

$$\langle M, \varphi_V M; \mathcal{F} \rangle = \sum_{\gamma \in \Gamma} \gamma \cdot \deg(\mathrm{Gr}^{\gamma}_{\mathcal{F}} M).$$

Remark 12. If we view the lattices M_1, M_2 as points x_{M_1}, x_{M_2} inside the building $\mathbf{B}(V)$, we see that the scalar product defined above corresponds to the Busemann function $\langle \overrightarrow{x_{M_1} x_{M_2}}, \mathcal{F}\rangle$ defined in 2.6.2.

Lemma 4.5.1. *Let M be an $\mathbb{F}[[u]]$-lattice in V. Then M is semi-stable of slope μ for the Fargues filtration if and only if we have*

$$\langle M, \varphi_V M; \Xi\rangle + \langle V(-\mu), \Xi\rangle \leq 0$$

for every $\Xi \in \mathbf{F}^{\mathbb{R}}(V, \varphi_V)$, where $V(-\mu)$ is the filtration of V such that $\mathrm{Gr}^{-\mu}_{V(-\mu)} = V$ and $\mathrm{Gr}^{\gamma}_{V(-\mu)} = 0$ if $\gamma \neq -\mu$.

Proof. Suppose first the inequality above is true. We have to show that for every φ_V-stable subspace W of V, we have $\mu(M \cap W) \leq \mu(M)$. For all $a, b \in \mathbb{R}$ with $a \leq b$ and for every φ_V-stable subspace W of V, we define a filtration $\Xi_{W,a,b}$, given by

$$\Xi^{\gamma}_{W,a,b} = \begin{cases} V & \text{if} \quad \gamma \leq a \\ W & \text{if} \quad a < \gamma \leq b \\ 0 & \text{if} \quad b < \gamma \end{cases}.$$

Set $M' = M \cap W$ and $M'' = \pi(M)$ for the projection $\pi : V \to V/W$. This filtration verifies

$$\begin{aligned} \langle M, \varphi_V M; \Xi^{\gamma}_{W,a,b}\rangle &= a \deg M'' + b \deg M' \\ &= a \deg M + (b-a) \deg M' \end{aligned}$$

$$\begin{aligned} \langle V(-\mu), \Xi^{\gamma}_{W,a,b}\rangle &= -\mu \cdot \deg(\mathrm{Gr}^{-\mu}_{V(-\mu)} \Xi_{W,a,b}) \\ &= -\mu \cdot (a \dim_{\mathbb{F}((u))}(V/W) + b \dim_{\mathbb{F}((u))} W) \\ &= -\mu \cdot (a \dim_{\mathbb{F}((u))} V + (b-a) \dim_{\mathbb{F}((u))} W). \end{aligned}$$

By hypothesis, we have $\langle M, \varphi_V M; \Xi_{W,a,b}\rangle + \langle V(-\mu), \Xi_{W,a,b}\rangle \leq 0$ because $\Xi_{W,a,b} \in \mathbf{F}^{\mathbb{R}}(V, \varphi_V)$, i.e.

$$a(\deg M - \mu \dim_{\mathbb{F}((u))} V) + (b-a)(\deg M' - \mu \dim_{\mathbb{F}((u))} W) \leq 0.$$

Taking $b = a$ in $\mathbb{R}$, we find that $\deg M = \mu \dim_{\mathbb{F}((u))} V$, i.e. $\mu(M) = \mu$. Taking $a = 0$ and $b = 1$, we find that $\deg M' - \mu \dim_{\mathbb{F}((u))} W \leq 0$, i.e. $\mu(M') \leq \mu$. Hence

$$\mu(M) = \frac{\deg M}{\dim_{\mathbb{F}((u))} V} = \mu$$

and for every strict p-torsion Kisin submodule $M' \subset M$, we get $\mu(M') \leq \mu(M) = \mu$ by taking $W = M'[\frac{1}{u}]$ and $b = 1, a = 0$ in the inequality. Hence, M is semi-stable of slope μ.

Conversely, suppose that M is semi-stable of slope μ. Then, for any $\Xi \in \mathbf{F}^{\mathbb{R}}(V, \varphi_V)$, we have

$$\begin{aligned} \langle V(-\mu), \Xi\rangle &= -\mu \deg \Xi \\ &= -\mu \sum_{i=1}^{n} \gamma_i \cdot (\dim_{\mathbb{F}((u))} \Xi^{\gamma_i} - \dim_{\mathbb{F}((u))} \Xi^{\gamma_{i+1}}) \\ &= -\mu \left(\sum_{i=2}^{n} (\gamma_i - \gamma_{i-1}) \dim_{\mathbb{F}((u))} \Xi^{\gamma_i} + \gamma_1 \dim_{\mathbb{F}((u))} V \right) \end{aligned}$$

where $\{\gamma_1 < \ldots < \gamma_n\} = \{\gamma \in \mathbb{R} \mid \mathrm{Gr}^{\gamma}_{\Xi} \neq 0\}$ (which means that $\Xi^{\gamma_{n+1}} = 0$), and

$$\begin{aligned} \langle M, \varphi_V M; \Xi\rangle &= \sum_{i=1}^{n} \gamma_i \cdot \deg(\mathrm{Gr}^{\gamma_i}_{\Xi}(M)) \\ &= \sum_{i=1}^{n} \gamma_i \cdot (\deg(\Xi^{\gamma_i} \cap M) - \deg(\Xi^{\gamma_{i+1}} \cap M)) \\ &= \sum_{i=2}^{n} (\gamma_i - \gamma_{i-1}) \deg(\Xi^{\gamma_i} \cap M) + \gamma_1 \deg M. \end{aligned}$$

If M is semi-stable of slope μ, we have

$$\mu(\Xi^{\gamma_i} \cap M) = \frac{\deg(\Xi^{\gamma_i} \cap M)}{\dim_{\mathbb{F}((u))} \Xi^{\gamma_i}} \leq \mu$$

for $1 \leq i \leq n$ and $\deg(M) = \mu \dim_{\mathbb{F}((u))} V$, so the inequality holds.

□

Lemma 4.5.2. *Let V_1 and V_2 be two $\mathbb{F}((u))$-vector spaces. Then:*

1. *For $\mathcal{F}_1, \mathcal{G}_1$ (resp. $\mathcal{F}_2$, $\mathcal{G}_2$) two filtrations on V_1 (resp. V_2), we have*

$$\begin{aligned} \langle \mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{G}_1 \otimes \mathcal{G}_2 \rangle &= \langle \mathcal{F}_1, \mathcal{G}_1 \rangle \dim_{\mathbb{F}((u))} V_2 \\ &+ \langle \mathcal{F}_2, \mathcal{G}_2 \rangle \dim_{\mathbb{F}((u))} V_1 \\ &+ \deg \mathcal{F}_1 \deg \mathcal{G}_2 \\ &+ \deg \mathcal{F}_2 \deg \mathcal{G}_1. \end{aligned}$$

2. *For $\mathcal{G}_1$ (resp. $\mathcal{G}_2$) a filtration and M_1, M_1' (resp. M_2, M_2') two $\mathbb{F}[[u]]$-lattices on V_1 (resp. on V_2), we have*

$$\begin{aligned} \langle M_1 \otimes M_2, M_1' \otimes M_2'; \mathcal{G}_1 \otimes \mathcal{G}_2 \rangle &= \langle M_1, M_1'; \mathcal{G}_1 \rangle \dim_{\mathbb{F}((u))} V_2 \\ &+ \langle M_2, M_2'; \mathcal{G}_2 \rangle \dim_{\mathbb{F}((u))} V_1 \\ &+ \nu(M_1, M_1') \deg \mathcal{G}_2 \\ &+ \nu(M_2, M_2') \deg \mathcal{G}_1. \end{aligned}$$

Proof. (2) Let $M = M_1 \otimes M_2$, $M' = M_1' \otimes M_2'$ and $\mathcal{G} = \mathcal{G}_1 \otimes \mathcal{G}_2$. By the properties given for the relative position in Proposition 2.3.3 and the formulas given for degree of a type in 2.1 and 2.1, we have:

$$\begin{aligned} \langle M, M'; \mathcal{G} \rangle &= \sum_{\gamma \in \mathbb{R}} \gamma \cdot \nu(\mathrm{Gr}^{\gamma}_{\mathcal{G}} M, \mathrm{Gr}^{\gamma}_{\mathcal{G}} M') \\ &= \sum_{\gamma \in \mathbb{R}} \gamma \cdot \nu \left(\bigoplus_{\gamma_1 + \gamma_2 = \gamma} \mathrm{Gr}^{\gamma_1}_{\mathcal{G}_1}(M_1) \otimes \mathrm{Gr}^{\gamma_2}_{\mathcal{G}_2}(M_2), \bigoplus_{\gamma_1 + \gamma_2 = \gamma} \mathrm{Gr}^{\gamma_1}_{\mathcal{G}_1}(M_1') \otimes \mathrm{Gr}^{\gamma_2}_{\mathcal{G}_2}(M_2') \right) \\ &= \sum_{\substack{\gamma = \gamma_{1,i} + \gamma_{2,i} \\ \gamma_{1,i} \in \Gamma_1 \\ \gamma_{2,i} \in \Gamma_2}} \gamma \cdot \left(d_{2,i} \nu(\mathrm{Gr}^{\gamma_{1,i}}_{\mathcal{G}_1}(M_1), \mathrm{Gr}^{\gamma_{1,i}}_{\mathcal{G}_1}(M_1')) + d_{1,i} \nu(\mathrm{Gr}^{\gamma_{2,i}}_{\mathcal{G}_2}(M_2), \mathrm{Gr}^{\gamma_{2,i}}_{\mathcal{G}_2}(M_2')) \right) \\ &= \sum_{\gamma_{1,i} \in \Gamma_1} \gamma_{1,i} \nu(\mathrm{Gr}^{\gamma_{1,i}}_{\mathcal{G}_1}(M_1), \mathrm{Gr}^{\gamma_{1,i}}_{\mathcal{G}_1}(M_1')) \sum_{\gamma_{2,i}} d_{2,i} \\ &+ \sum_{\gamma_{2,i} \in \Gamma_2} \gamma_{2,i} \nu(\mathrm{Gr}^{\gamma_{2,i}}_{\mathcal{G}_2}(M_2), \mathrm{Gr}^{\gamma_{2,i}}_{\mathcal{G}_2}(M_2')) \sum_{\gamma_{1,i}} d_{1,i} \\ &+ \sum_{\gamma_{1,i} \in \Gamma_1} \gamma_{1,i} \cdot d_{1,i} \sum_{\gamma_{2,i}} \nu(\mathrm{Gr}^{\gamma_{2,i}}_{\mathcal{G}_2}(M_2), \mathrm{Gr}^{\gamma_{2,i}}_{\mathcal{G}_2}(M_2')) \\ &+ \sum_{\gamma_{2,i} \in \Gamma_2} \gamma_{2,i} \cdot d_{2,i} \sum_{\gamma_{1,i}} \nu(\mathrm{Gr}^{\gamma_{1,i}}_{\mathcal{G}_1}(M_1), \mathrm{Gr}^{\gamma_{1,i}}_{\mathcal{G}_1}(M_1')) \end{aligned}$$

where $d_{k,i} = \mathrm{rank}_{\mathbb{F}((u))} \mathrm{Gr}^{\gamma_i}_{\mathcal{G}_k} M_k$ and $\Gamma_k = \{\gamma \in \mathbb{Z} \mid \mathrm{Gr}^{\gamma}_{\mathcal{G}_k} \neq 0\}$ for $k = 1, 2$. We know that $\sum_{\gamma_k, i} d_{k,i} = \dim_{\mathbb{F}((u))} V_k$ and $\sum_{\gamma_{k,i}} \gamma_{k,i} \cdot d_{k,i} = \deg \mathcal{G}_k$, for $k = 1, 2$. Also, by point (5) in Proposition 2.3.3, we have

$$\mathrm{Pos}(\mathrm{Gr}_{\mathcal{G}_k}(M_k), \mathrm{Gr}_{\mathcal{G}_k}(M_k')) \leq \mathrm{Pos}(M_k, M_k')$$

for $k = 1, 2$. In particular,

$$\sum_{\gamma_{k,i}} \nu(\mathrm{Gr}^{\gamma_{k,i}}_{\mathcal{G}_k}(M_k), \mathrm{Gr}^{\gamma_{k,i}}_{\mathcal{G}_k}(M_k')) = \nu(\mathrm{Gr}_{\mathcal{G}_k}(M_k), \mathrm{Gr}_{\mathcal{G}_k}(M_k')) = \nu(M_k, M_k'),$$

thus

$$\begin{aligned}\langle M_1 \otimes M_2, M_1' \otimes M_2'; \mathcal{G}_1 \otimes \mathcal{G}_2\rangle &= \langle M_1, M_1'; \mathcal{G}_1\rangle \dim_{\mathbb{F}((u))} V_2 \\ &+ \langle M_2, M_2'; \mathcal{G}_2\rangle \dim_{\mathbb{F}((u))} V_1 \\ &+ \nu(M_1, M_1') \deg \mathcal{G}_2 \\ &+ \nu(M_2, M_2') \deg \mathcal{G}_1.\end{aligned}$$

(1) The proof is analogous to (2).

□

For the rest of the section, we fix M_1, $M_2 \in \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ two p-torsion Kisin modules modules, and note $M = M_1 \otimes M_2$, $V_1 = M_1[\frac{1}{u}]$, $V_2 = M_2[\frac{1}{u}]$ and $V = V_1 \otimes V_2$.

Corollary 4.5.3. *Suppose M_1, M_2 semi-stable of slope μ_1 and μ_2 respectively. Then, for every $\Xi \in \mathbf{F}^{\mathbb{R}}(V, \varphi_V)$ which is decomposed, i.e*

$$\Xi = \Xi_1 \otimes \Xi_2 \quad \textit{with} \quad \Xi_i \in \mathbf{F}^{\mathbb{R}}(V_i, \varphi_i),$$

we have

$$\langle M, \varphi M; \Xi\rangle + \langle V(-\mu), \Xi\rangle \leq 0$$

where $\mu = \mu_1 + \mu_2$.

Proof. We apply Proposition 4.5.2 with $M_i' = \varphi_{V_i} M_i$, $\mathcal{G}_i = \Xi_i$ and $\mathcal{F}_i = V_i(-\mu_i)$, for $i = 1, 2$, and we get

$$\begin{aligned}\langle M, \varphi M; \Xi\rangle &= \langle M_1, \varphi_1 M_1; \Xi_1\rangle \dim_{\mathbb{F}((u))} V_2 \\ &+ \langle M_2, \varphi_2 M_2; \Xi_2\rangle \dim_{\mathbb{F}((u))} V_1 \\ &+ \deg M_1 \deg \Xi_2 \\ &+ \deg M_2 \deg \Xi_1.\end{aligned}$$

On the other hand, using the fact that $V(-(\mu_1 + \mu_2)) = V_1(-\mu_1) \otimes V_2(-\mu_2)$ and an analogous calculation, we get

$$\begin{aligned}\langle V(-(\mu_1 + \mu_2)), \Xi\rangle &= \langle V_1(-\mu_1), \Xi_1\rangle \dim_{\mathbb{F}((u))} V_2 \\ &+ \langle V_2(-\mu_2), \Xi_2\rangle \dim_{\mathbb{F}((u))} V_1 \\ &- \mu_1 \dim_{\mathbb{F}((u))} V_1 \deg \Xi_2 \\ &- \mu_2 \dim_{\mathbb{F}((u))} V_2 \deg \Xi_1.\end{aligned}$$

Then, if M_1 and M_2 are semi-stable of slope μ_1 and μ_2 respectively, we obtain using 4.5.1,

$$\langle M, \varphi M; \Xi\rangle + \langle V(-(\mu_1 + \mu_2)), \Xi\rangle \leq 0$$

for all $\Xi \in \mathbf{F}^{\mathbb{R}}(M_1[\frac{1}{u}], \varphi_1) \otimes \mathbf{F}^{\mathbb{R}}(M_2[\frac{1}{u}], \varphi_2)$.

□

Now, for every $\Xi \in \mathbf{F}^{\mathbb{R}}(V, \varphi_V)$, the projection

$$p \ : \ \mathbf{F}^{\mathbb{R}}(V) \to \mathrm{im}(\mathbf{F}^{\mathbb{R}}(V_1) \times \mathbf{F}^{\mathbb{R}}(V_2))$$

given in section 2.6.1 gives us $p(\Xi) = \Xi_1 \otimes \Xi_2$ for some $\Xi_i \in \mathbf{F}^{\mathbb{R}}(V_i, \varphi_i)$ for $i = 1, 2$. Indeed,

$$\mathbf{d}(\Xi, \varphi_V(p(\Xi))) = \mathbf{d}(\varphi_V(\Xi), \varphi_V(p(\Xi))) = \mathbf{d}(\Xi, p(\Xi))$$

where the first equality is true since Ξ is fixed by φ_V, so $\varphi_V(p(\Xi)) = p(\Xi)$, as one property of p is that it is the only map such that $\mathbf{d}(\Xi, p(\Xi))$ is minimal. Thus

$$\langle M, \varphi_V M; p(\Xi)\rangle + \langle V(-\mu), p(\Xi)\rangle \leq 0.$$

Moreover, we know that $\langle \mathcal{F}, \Xi\rangle \leq \langle \mathcal{F}, p(\Xi)\rangle$ for every $\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(V)$, in particular for

$$\mathcal{F} = V(-\mu) = V_1(-\mu_1) \otimes V_2(-\mu_2).$$

Hence, to prove that M is semi-stable of slope μ, it suffices to prove that $\langle M, \varphi_V M; \Xi\rangle \leq \langle M, \varphi_V M; p(\Xi)\rangle$ for every $\Xi \in \mathbf{F}^{\mathbb{R}}(V, \varphi_V)$ and to be able to do it, we need some preliminary results.

Lemma 4.5.4. *For $M, N \in \mathcal{L}(V)$, let $M_n = \varphi_V^n M$ and $N_n = \varphi_V^n N$. Let Ξ be a φ_V-stable filtration on V. Then*

1. *For every $n \geq 1$, we have $\langle M_n, N_n; \Xi\rangle = p^n \langle M, N; \Xi\rangle$.*
2. *For every $n \geq 1$, we have $\langle M, M_n; \Xi\rangle = p_n \langle M, \varphi_V M; \Xi\rangle$, where $p_n = \frac{p^n-1}{p-1}$.*

Proof. 1. We have

$$\begin{aligned}
\langle M_n, N_n; \Xi\rangle &= \sum_{\gamma\in\mathbb{R}} \gamma \cdot \nu(\mathrm{Gr}^{\gamma}_{\Xi}(M_n), \mathrm{Gr}^{\gamma}_{\Xi}(N_n)) \\
&= \sum_{\gamma\in\mathbb{R}} \gamma \cdot \nu((\mathrm{Gr}^{\gamma}_{\Xi}(M))_n, (\mathrm{Gr}^{\gamma}_{\Xi}(N))_n) \\
&= \sum_{\gamma\in\mathbb{R}} \gamma \cdot p^n \nu(\mathrm{Gr}^{\gamma}_{\Xi} M, \mathrm{Gr}^{\gamma}_{\Xi} N) \\
&= p^n \langle M, N; \Xi\rangle
\end{aligned}$$

for every $\Xi \in \mathbf{F}^{\mathbb{R}}(V, \varphi_V)$ and every $n \geq 0$, where the second equality is given by Proposition 4.3.3 and the third equality is given by Proposition 4.3.1.

2. This is a consequence of (1), since

$$\langle M, M_n; \Xi\rangle = \sum_{i=0}^{n-1} \langle M_i, M_{i+1}; \Xi\rangle = \sum_{i=0}^{n-1} p^i \langle M, \varphi_V M; \Xi\rangle = p_n \langle M, \varphi_V M; \Xi\rangle,$$

for $p_n = \frac{p^n-1}{p-1}$.

□

Proposition 4.5.5. *Suppose M flat. Then, for every $\Xi \in \mathbf{F}^{\mathbb{R}}(V, \varphi_V)$, we have*

$$\langle M, \varphi_V M; \Xi\rangle \leq \langle M, \varphi_V M; p(\Xi)\rangle$$

for p the projection given in 2.6.1.

Proof. First, we can suppose that there is no unbounded sub-family in M_n because if there exists a bounded subsequence, we have

$$\langle M, \varphi_V M; \Xi\rangle = \langle M, \varphi_V M; p(\Xi)\rangle = 0$$

for every φ_V-stable filtration Ξ on V. Indeed, let $\{M_{b(n)}\}_{n\geq 0}$ be a bounded subsequence. Then, also $\{\mathrm{Gr}^{\gamma}_{\Xi} M_{b(n)}\}_{n\geq 0}$ is bounded for every $\gamma \in \mathbb{R}$, thus

$$p_{b(n)} \langle M, \varphi_V M; \Xi\rangle = \langle M, M_{b(n)}; \Xi\rangle = \sum_{\gamma\in\mathbb{R}} \gamma \cdot \nu(\mathrm{Gr}^{\gamma}_{\Xi} M, \mathrm{Gr}^{\gamma}_{\Xi} M_{b(n)})$$

is bounded. Therefore, $\langle M, \varphi_V M; \Xi\rangle = 0$.

Since M is flat, we know that there is a sub-family of modules $\{M_{a(n)}\}_{n\geq 0}$ contained in an appartment. Inside this family, we take a sub-family of modules $\{M_{b(n)}\}_{n\geq 0}$ converging to a point in the boundary of the apartment, that we denote by ξ. By Proposition 2.6.1, we have

$$\frac{\langle M, M_{b(n)}; \Xi\rangle}{\mathrm{d}(M, M_{b(n)})} \to \langle \xi, \Xi\rangle$$

and the same goes for $p(\Xi)$. Then, for every ε, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$\frac{\langle M, M_{b(n)}; \Xi\rangle}{\mathrm{d}(M, M_{b(n)})} - \varepsilon \leq \langle \xi, \Xi\rangle \leq \langle \xi, p(\Xi)\rangle \leq \frac{\langle M, M_{b(n)}; p(\Xi)\rangle}{\mathrm{d}(M, M_{b(n)})} + \varepsilon,$$

where the central inequality is given in section 2.6.1, thus

$$\langle M, M_{b(n)}; \Xi\rangle \leq \langle M, M_{b(n)}M; p(\Xi)\rangle + 2\varepsilon\, \mathrm{d}(M, M_{b(n)})$$

and therefore by Lemma 4.5.4,

$$\langle M, \varphi_V M; \Xi\rangle \leq \langle M, \varphi_V M; p(\Xi)\rangle + 2\varepsilon \frac{\mathrm{d}(M, M_{b(n)})}{p_{b(n)}}.$$

The triangular inequality for the distance given in Proposition 2.3.1 gives us

$$\begin{aligned} \mathrm{d}(M, M_{b(n)}) &\leq \sum_{i=1}^{b(n)} \mathrm{d}(M_{i-1}, M_i) \\ &= \sum_{i=1}^{b(n)} p^{i-1}\, \mathrm{d}(M, \varphi_V M) \\ &= p_{b(n)} \mathrm{d}(M, \varphi_V M) \end{aligned}$$

where $p_{b(n)} = \frac{p^{b(n)}-1}{p-1}$, so

$$\langle M, \varphi_V M; \Xi\rangle \leq \langle M, \varphi_V M; p(\Xi)\rangle + 2\varepsilon\, \mathrm{d}(M, \varphi_V M)$$

for every $\varepsilon \geq 0$, hence

$$\langle M, \varphi_V M; \Xi\rangle \leq \langle M, \varphi_V M; p(\Xi)\rangle.$$

□

To summarize, we have the following corollary.

Corollary 4.5.6. *Let M_1, M_2 be two semi-stable p-torsion Kisin modules of slope μ_1 and μ_2 respectively, as above, and suppose $M = M_1 \otimes M_2$ is flat. Then M is semi-stable of slope $\mu = \mu_1 + \mu_2$.*

Proof. For every $\Xi \in \mathbf{F}^{\mathbb{R}}(V, \varphi_V)$, we have

$$\langle M, \varphi_V M; \Xi\rangle + \langle (V(-\mu), \Xi\rangle \leq \langle M, \varphi_V M; p(\Xi)\rangle + \langle (V(-\mu), p(\Xi)\rangle \leq 0$$

where the first inequality is due to last proposition and the second one is given by Corollary 4.5.3, since M_1 and M_2 are semi-stable. Thus M is semi-stable of slope μ by Lemma 4.5.1.

□

As a consequence, we get the main theorem of the section.

Theorem 4.5.7. *The Fargues filtration on the category $\mathrm{Mod}^{\varphi,\mathrm{al}}_{\mathbb{F}[[u]],\mathrm{fr}}$ is compatible with tensor products, exterior powers and symmetric powers.*

Proof. Let M_1 and M_2 be two aligned p-torsion Kisin modules, and $M = M_1 \otimes M_2$. We have to verify that

$$\mathcal{F}_{\mathrm{F},1}(M) = \mathcal{G}(M)$$

where $\mathcal{G}(M) = \mathcal{F}_{\mathrm{F},1}(M_1) \otimes \mathcal{F}_{\mathrm{F},1}(M_2)$, i.e. to verify that for every $\gamma \in \Gamma$,

$$\mathrm{Gr}^{\gamma}_{\mathcal{G}}(M) = \bigoplus_{\substack{\gamma_1+\gamma_2=\gamma \\ \gamma_1,\gamma_2\in\Gamma}} \mathrm{Gr}^{\gamma_1}_{\mathcal{F}_{\mathrm{F},1}}(M_2) \otimes \mathrm{Gr}^{\gamma_2}_{\mathcal{F}_{\mathrm{F},1}}(M_2)$$

is semi-stable of slope γ and, since the category $\mathcal{C}(\gamma)$ is abelian, we only need to prove if for $\mathrm{Gr}^{\gamma_1}_{\mathcal{F}_{\mathrm{F},1}}(M_2) \otimes \mathrm{Gr}^{\gamma_2}_{\mathcal{F}_{\mathrm{F},1}}(M_2)$, knowing that $\mathrm{Gr}^{\gamma_i}_{\mathcal{F}_{\mathrm{F},1}}(M_i)$ is semi-stable of slope γ_i for $i = 1, 2$. If M_1 and M_2 are aligned, then, by Proposition 4.4.1, the module M is aligned and then flat by the same proposition. Since flat modules are stable by subobject and quotients, we have $\mathrm{Gr}^{\gamma}_{\mathcal{G}}(M)$ flat, and again by stability of quotients, we have that $\mathrm{Gr}^{\gamma_1}_{\mathcal{F}_{\mathrm{F},1}}(M_1) \otimes \mathrm{Gr}^{\gamma_2}_{\mathcal{F}_{\mathrm{F},1}}(M_2)$ is also flat. We apply the corollary above and get $\mathrm{Gr}^{\gamma_1}_{\mathcal{F}_{\mathrm{F},1}}(M_1) \otimes \mathrm{Gr}^{\gamma_2}_{\mathcal{F}_{\mathrm{F},1}}(M_2)$ semi-stable of slope $\gamma_1 + \gamma_2 = \gamma$.

There are natural maps $(M, \varphi_M)^{\otimes k} \twoheadrightarrow \mathrm{Sym}^k(M, \varphi_M)$ and $(M, \varphi_M)^{\otimes k} \twoheadrightarrow \Lambda^k(M, \varphi_M)$ giving us maps $\mathcal{F}_{\mathrm{F},1}(M, \varphi_M)^{\otimes k} \to \mathcal{F}_{\mathrm{F},1}(\mathrm{Sym}^k(M, \varphi_M))$ and $\mathcal{F}_{\mathrm{F},1}(M, \varphi_M)^{\otimes k} \to \mathcal{F}_{\mathrm{F},1}(\Lambda^k(M, \varphi_M))$, since we just proved that $\mathcal{F}_{\mathrm{F},1}$ commutes with tensor products for aligned modules. We get mono-epi morphisms

$$\mathrm{Sym}^k \mathcal{F}_{\mathrm{F},1}(M, \varphi_M) \to \mathcal{F}_{\mathrm{F},1}(\mathrm{Sym}^k(M, \varphi_M))$$

and

$$\Lambda^k \mathcal{F}_{\mathrm{F},1}(M, \varphi_M) \to \mathcal{F}_{\mathrm{F},1}(\Lambda^k(M, \varphi_M)).$$

We can calculate the slopes to see that they are isomorphisms. For the left side, we obtain

$$\mu(\mathrm{Sym}^k \mathcal{F}_{\mathrm{F},1}(M, \varphi_M)) = k\mu(\mathcal{F}_{\mathrm{F},1}(M, \varphi_M)) = k\mu(M, \varphi_M)$$

by the formulas given in Proposition 4.2. For the right side, we have that $\mathcal{F}_{\mathrm{F},1}$ and $\mathcal{F}_{\mathrm{H},1}$ have the same total slope by definition, so we do the calculation for $\mathcal{F}_{\mathrm{H},1}$. Since we already know that $\mathrm{Sym}^k \mathcal{F}_{\mathrm{H},1}(M, \varphi_M) \simeq \mathcal{F}_{\mathrm{H},1}(\mathrm{Sym}^k(M, \varphi_M))$ by Proposition 4.3.4, we have

$$\begin{aligned}
\mu(\mathcal{F}_{\mathrm{F},1}(\mathrm{Sym}^k(M, \varphi_M))) &= \mu(\mathcal{F}_{\mathrm{H},1}(\mathrm{Sym}^k(M, \varphi_M))) \\
&= \mu(\mathrm{Sym}^k(\mathcal{F}_{\mathrm{H},1}(M, \varphi_M))) \\
&= k\mu(\mathcal{F}_{\mathrm{H},1}(M, \varphi_M)) \\
&= k\mu(M, \varphi_M)
\end{aligned}$$

where the third equality is given by the formulas for the degree in 2.1. Same goes for Λ^k.

□

Even though we do not know if the alignment hypothesis is always verified, it is verified whenever the Kisin module comes from an ordinary germ of crystalline representations, which is the case studied in this thesis.

Chapter 5

The Fargues filtration on $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$

5.1 The category $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$

We have already defined the category $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$ in 3.2. It is a quasi-abelian category and we are going to give explicitely the kernel, cokernel, image and coimage of a morphism. We have already seen existence of kernels and cokernels for the underlying module in 3.1, so we only need to study the Frobenius which works in the same way as it did in $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ as we will see. For a morphism $f : M \to N$ between two objects in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$, the kernel $(\ker f, \varphi_{\ker f})$ is given by:

- The underlying module is $\ker f = \{m \in M \mid f(m) = 0\}$.
- The morphism $\varphi_{\ker f} = \varphi_{M|\ker f}$ since from the compatibility of f with φ_M and φ_N, the image of $\varphi^* \ker f[\frac{1}{u}]$ by φ_M is contained in $\ker f[\frac{1}{u}]$.

The cokernel of f is given by $(\operatorname{coker} f, \varphi_{\operatorname{coker} f})$ where

- The underlying module is $\operatorname{coker} f = N/f(M)^{\mathrm{sat}}$, where $f(M)^{\mathrm{sat}}$ is the saturation of $f(M)$ in N, i.e. $f(M)^{\mathrm{sat}}/f(M) = (N/f(M))[\mathfrak{m}^{\infty}]$.
- The morphism $\varphi_{\operatorname{coker} f} = \overline{\varphi}_N$ where $\overline{\varphi}_N$ is the morphism induced by φ_N on the quotient $\operatorname{coker} f$, since φ_N sends $\varphi^* f(M)^{\mathrm{sat}}$ to $f(M)^{\mathrm{sat}}[\frac{1}{u}] \simeq f(M)[\frac{1}{u}]$.

The image of f is given by $(\operatorname{im} f, \varphi_{\operatorname{im} f})$ where

- The underlying module is $\operatorname{im} f = f(M)^{\mathrm{sat}}$.
- The morphism $\varphi_{\operatorname{im} f} = \varphi_{N|\operatorname{im} f}$ is the restriction of φ_N to $\operatorname{im} f$.

The coimage of f is given by $(\operatorname{coim} f, \varphi_{\operatorname{coim} f})$ where

- The underlying module is $\operatorname{coim} f = M/\ker f = f(M)$.
- The morphism $\varphi_{\operatorname{coim} f} = \overline{\varphi}_M$ is the morphism induced by φ_M on the quotient $\operatorname{coim} f$, since we have already seen that φ_M sends $\varphi^* \ker f$ to $\ker f[\frac{1}{u}]$.

To give a mono-epi in this category, is the same as to give a monomorphism $M_1 \hookrightarrow M_2$ such that $M_1\left[\frac{1}{u}\right] \xrightarrow{\sim} M_2\left[\frac{1}{u}\right]$, which is the same as to say that we have an exact sequence in the category $\mathcal{A}$ defined in 3.1 of torsion $\mathfrak{S}$-modules

$$0 \to M_1 \to M_2 \to Q \to 0$$

with Q an object in $\mathcal{T}$, i.e. a finite length $\mathfrak{S}$-module.

We define the i-twist of a p^∞-torsion Kisin module as

$$\begin{array}{rcccc} -(i) & : & \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}} & \to & \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}} \\ & & (M,\varphi_M) & \mapsto & (M(i),\varphi_{M(i)}) = (M, u^i\varphi_M) \end{array}$$

and we have that for all p^∞-torsion Kisin modules, there exists $i \geq 0$ such that $M(i)$ is effective.

5.2 The filtration

Now we need to define a slope function. First, we start by defining the rank function as

$$\begin{array}{rcccc} \mathrm{rank} & : & \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}} & \to & \mathbb{Z} \\ & & M & \mapsto & \mathrm{length}_{W(\mathbb{F})}(M/uM) \end{array}.$$

Proposition 5.2.1. *Let $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$. We have the following properties :*

1. *The rank function is additive on short exact sequences of torsion Kisin modules.*
2. *We have an equality* $\mathrm{rank}(M) = \mathrm{length}_{\mathfrak{S}[\frac{1}{u}]}(M[\frac{1}{u}]) = \mu_{\mathrm{IW}}(M)$.

Proof. The additivity follows from the fact that M has no torsion other than p^∞, so multiplication by u is injective and from the exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

we get

$$0 \to M_1/uM_1 \to M_2/uM_2 \to M_3/uM_3 \to 0.$$

The second point is proved in 3.1.7.

□

We cannot use relative position in this context to define the degree function since we do not have a DVR anymore, but recall that the degree function of a p-torsion Kisin module corresponded to the length of a quotient when the p-torsion Kisin module was effective. Since the twist turns a module effective for a sufficiently large integer, we define the degree function by

$$\begin{array}{rcccc} \deg & : & \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}} & \to & \mathbb{Z} \\ & & M & \mapsto & -\mathrm{length}_{\mathfrak{S}}(Q(i)) + i\,\mathrm{rank}\,M \end{array}$$

for i large enough and $Q(i) = M(i)/(\varphi_{M(i)}\varphi^*M(i))$, i.e. the cokernel of $\varphi_{M(i)}$ viewed as a morphism in the category of $\mathfrak{S}$-modules.

Proposition 5.2.2. *The function* deg *is independent of the choice of a large enough $i \in \mathbb{N}$.*

Proof. First, remark that by definition $Q(i) = M/(u^i\varphi_M\varphi^*M)$ for all i large enough. The exact sequence

$$0 \to K(i) \to M/(u^{i+1}\varphi_M\varphi^*M) \to M/(u^i\varphi_M\varphi^*M) \to 0$$

gives us

$$\mathrm{length}_{\mathfrak{S}}(Q(i+1)) = \mathrm{length}_{\mathfrak{S}}(Q(i)) + \mathrm{length}_{\mathfrak{S}}(K(i)).$$

Since $K(i) \simeq (u^i\varphi_M\varphi^*M)/(u^{i+1}\varphi_M\varphi^*M)$ and multiplication by u is injective because $\varphi_M\varphi^*M$ has no u-torsion, we get $K(i) \simeq (\varphi_M\varphi^*M)/(u\varphi_M\varphi^*M)$ and

$$\begin{array}{rcl} \mathrm{length}_{\mathfrak{S}}(K(i)) & = & \mathrm{length}_{W(\mathbb{F})}((\varphi_M\varphi^*M)/(u\varphi_M\varphi^*M)) \\ & = & \mathrm{length}_{\mathfrak{S}[\frac{1}{u}]}(\varphi_M\varphi^*M[\frac{1}{u}]) \\ & = & \mathrm{length}_{\mathfrak{S}[\frac{1}{u}]}(M[\frac{1}{u}]) \\ & = & \mathrm{rank}\,M \end{array}$$

where the second equality is given by 3.1.7. Using the formula above, we have

$$-\operatorname{length}_{\mathfrak{S}}(Q(i+1)) + (i+1)\operatorname{rank} M = -\operatorname{length}_{\mathfrak{S}}(Q(i)) + i\operatorname{rank} M$$

so the degree function does not depend on i, for i large enough.

□

Proposition 5.2.3. *The degree function verifies the following properties :*

1. *It is additive on short exact sequences of p^∞-torsion Kisin modules.*
2. *If $f : M_1 \to M_2$ is a mono-epi, then $\deg(M_1) \leq \deg(M_2)$. Moreover, the equality holds if and only if f is an isomorphism.*
3. *For p-torsion Kisin modules, it coincides with the degree function defined in last section.*

Proof. 1. The exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ and the fact that φ is flat and φ_M is injective for every effective p^∞-torsion Kisin module M, give rises to an exact sequence

$$0 \to Q_1(i) \to Q_2(i) \to Q_3(i) \to 0$$

for a large enough i and $Q_k(i) = Q_k(i) = M_k(i)/(\varphi_{M_k(i)}\varphi^* M_k(i))$ for $k \in \{1,2,3\}$. Thus,

$$\operatorname{length}_{\mathfrak{S}}(Q_2(i)) = \operatorname{length}_{\mathfrak{S}}(Q_1(i)) + \operatorname{length}_{\mathfrak{S}}(Q_3(i))$$

and since we have already seen the additivity of the rank, we have $\deg M_2 = \deg M_1 + \deg M_3$.

2. Up to twist, we may assume that the modules are effective. Let $f : M_1 \to M_2$ be a mono-epi with cokernel $Q \in \mathcal{T}$, so that we have an exact sequence

$$0 \to M_1 \to M_2 \to Q \to 0$$

with $Q = Q[\mathfrak{m}^\infty]$. This goes into a diagram in the category of $\mathfrak{S}$-modules:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & \varphi^* M_1 \arrow[r] \arrow[d, "\varphi_{M_1}"] & \varphi^* M_2 \arrow[r] \arrow[d, "\varphi_{M_2}"] & \varphi^* Q \arrow[r] \arrow[d, "\varphi_Q"] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
 & Q_1 \arrow[d] & Q_2 \arrow[d] & Q/(\varphi_Q \varphi^* Q) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $Q_i = M_i/(\varphi_{M_i}\varphi^* M_i)$ for $i = 1,2$ and φ_Q is the projection of φ_{M_2} to $\varphi^* Q$. By the snake lemma, we have an exact sequence

$$0 \to K \to Q_1 \to Q_2 \to Q/(\varphi_Q \varphi^* Q) \to 0.$$

From this, we get that

$$\operatorname{length}_{\mathfrak{S}} Q_2 - \operatorname{length}_{\mathfrak{S}} Q_1 = \operatorname{length}_{\mathfrak{S}} Q - \operatorname{length}_{\mathfrak{S}}(\varphi^* Q) = -(p-1)\operatorname{length}_{\mathfrak{S}} Q \leq 0$$

by Lemma 3.1.8, therefore $\deg M_2 - \deg M_2 \geq 0$. The equality holds if and only if $Q = 0$. $\square$

The deg and rank functions define a Harder-Narasimhan filtration on every object of $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$. We will denote by $\mathcal{F}_{\mathrm{F,t}}(M)$ the Fargues filtration of a p^{∞}-torsion Kisin module M and by $\mathbf{t}_{\mathrm{F,t}}(M)$, the type associated to $\mathcal{F}_{\mathrm{F,t}}(M)$.

Chapter 6

The Fargues filtration on $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$

6.1 The category $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$

Some operators we defined for objects in $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$ can be generalized to objects in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$:

1. Tensor products : For two Kisin modules (M_1, φ_{M_1}) and (M_2, φ_{M_2}), define

$$(M_1 \otimes M_2, \varphi_{M_1 \otimes M_2}) = (M_1 \otimes_{\mathfrak{S}} M_2, \varphi_{M_1} \otimes \varphi_{M_2}).$$

   This definition works since we have $\varphi^*(M_1 \otimes_{\mathfrak{S}} M_2)[\frac{1}{E}] = (\varphi^* M_1[\frac{1}{E}]) \otimes_{\mathfrak{S}[\frac{1}{E}]} (\varphi^* M_2[\frac{1}{E}])$ and $(M_1 \otimes_{\mathfrak{S}} M_2)[\frac{1}{E}] = (M_1[\frac{1}{E}]) \otimes_{\mathfrak{S}[\frac{1}{E}]} (M_2[\frac{1}{E}])$. The identity object for the tensor product is the Kisin module $\mathbf{1} = (\mathfrak{S}, \varphi)$ where φ is the identity on $\varphi^*\mathfrak{S} \simeq \mathfrak{S}$.

2. Duality : For a Kisin module (M, φ_M), we define the dual of (M, φ_M) as $(M^\vee, \varphi_{M^\vee})$ where $M^\vee = \mathrm{Hom}_{\mathfrak{S}}(M, \mathfrak{S})$ and

$$\begin{array}{rccc} \varphi_{M^\vee} & : & (\varphi^* M^\vee)[\frac{1}{E}] & \to & M^\vee[\frac{1}{E}] \\ & & f & \mapsto & \varphi \circ f \circ \varphi_M^{-1} \end{array}$$

   since $\varphi^* \mathrm{Hom}_{\mathfrak{S}}(M, \mathfrak{S})[\frac{1}{E}] \simeq \mathrm{Hom}_{\varphi^*\mathfrak{S}}(\varphi^* M, \varphi^*\mathfrak{S})[\frac{1}{E}] \simeq \mathrm{Hom}_{\varphi^*\mathfrak{S}[\frac{1}{E}]}(\varphi^* M[\frac{1}{E}], \varphi^*\mathfrak{S}[\frac{1}{E}])$.

3. Internal homomorphisms: Since we have defined the tensor product and duality, we can define an internal Hom by

$$\underline{\mathrm{Hom}}((M_1, \varphi_{M_1}), (M_2, \varphi_{M_2})) = (M_1, \varphi_{M_1})^\vee \otimes (M_2, \varphi_{M_2})$$

   for all pair of objects $(M_1, \varphi_1), (M_2, \varphi_{M_2})$ in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$.

4. Twist: We define the i-twist of a Kisin module as

$$\begin{array}{rccc} -(i) & : & \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{t}} & \to & \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{t}} \\ & & (M, \varphi_M) & \mapsto & (M(i), \varphi_{M(i)}) = (M, E^i \varphi_M) \end{array}$$

   and we have that for all Kisin modules, there exists $i \geq 0$ such that $M(i)$ is effective. We have $M(i) = M \otimes \mathbf{1}(i)$ for $i \in \mathbb{Z}$.

Let $\mathrm{Mod}^{\sigma}_{W(\mathbb{F})}$ denote the category of crystals, whose objects are finite free $W(\mathbb{F})$-modules D together with a morphism $\sigma_D : D[\frac{1}{p}] \xrightarrow{\sim} D[\frac{1}{p}]$. We can always associate a crystal to a Kisin module, considering the functor

$$\begin{array}{c} \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}} \to \mathrm{Mod}^{\sigma}_{W(\mathbb{F})} \\ (M, \varphi_M) \mapsto (D, \sigma_D) \end{array}$$

where $D = M/uM$ and σ_D is the reduction modulo u of φ_M.

For a Kisin module $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$, there are natural rank and degree functions that we can define in the exact same way we defined the degree for the isogeny category, i.e. for a Kisin module M, we define a rank function by

$$\mathrm{rank}(M) = \mathrm{rank}_{\mathfrak{S}} M.$$

Also, we can define two Hodge filtrations and their associated types on M by

$$\begin{array}{rcl} \mathcal{F}_{\mathrm{H}}(M) = \mathcal{F}(M_{(E)}, \varphi_M \varphi^* M_{(E)}) & \text{and} & \mathcal{F}_{\mathrm{H},u}(M) = \mathcal{F}(M/uM, \sigma_M \sigma^*(M/uM)), \\ \mathbf{t}_{\mathrm{H}}(M) = \mathrm{Pos}(M_{(E)}, \varphi_M \varphi^* M_{(E)}) & \text{and} & \mathbf{t}_{\mathrm{H},u}(M) = \mathrm{Pos}(M/uM, \sigma_M \sigma^*(M/uM)) \end{array}$$

in $\mathbb{Z}^{\mathrm{rank}\, M}_{\geq}$, where σ is the Frobenius on $\mathfrak{S}/u\mathfrak{S} = W(\mathbb{F})$ and σ_M is the Frobenius on M/uM. Then, we can also define a degree function given by

$$\deg(M) = \deg(\mathbf{t}_{\mathrm{H}}(M)) = \nu(M_{(E)}, \varphi_M \varphi^* M_{(E)}).$$

Suppose M is effective, then there is an exact sequence in the category of $\mathfrak{S}$-modules

$$0 \to \varphi^* M \to M \to Q \to 0$$

where Q is killed by a power of E. In particular, there is a pseudo-isomorphism

$$Q \sim \bigoplus_{i=1}^{m} \mathfrak{S}/(E(u)^{n_i})$$

with $n_i \geq 0$ for $1 \leq i \leq m$, for $m = \mathrm{rank}\, M$. Then, we have $\mathbf{t}_{\mathrm{H}}(M) = (-n_1 \geq \cdots \geq -n_m)$ (up to reordering the n_i) and $\deg(M) = -\mathrm{length}_{\mathfrak{S}_{(E)}}(Q_{(E)}) = -\sum_{i=1}^{m} n_i$.

If M is an object in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$, then $M/p^n M$ is an object in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$ for every $n \geq 1$, the Frobenius being the reduction of the Frobenius on M. In particular M/pM is an object in $\mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$.

Proposition 6.1.1. *Let $M \in \mathrm{Mod}_{\mathfrak{S},\mathrm{fr}}$ and $\overline{M} = M/pM$. Then*

$$\begin{array}{rcl} \mathbf{t}_{\mathrm{H},1}(\overline{M}) & \leq & e \cdot \mathbf{t}_{\mathrm{H}}(M) \\ \mathbf{t}_{\mathrm{H},u}(M) & \leq & \mathbf{t}_{\mathrm{H}}(M). \end{array}$$

Proof. We can reduce to the effective case, since

$$\begin{array}{rcl} \mathbf{t}_{\mathrm{H}}(M(r)) & = & \mathbf{t}_{\mathrm{H}}(M) - (r, \ldots, r) \\ \mathbf{t}_{\mathrm{H},1}(\overline{M}(r)) & = & \mathbf{t}_{\mathrm{H},1}(\overline{M}) - e \cdot (r, \ldots, r) \\ \mathbf{t}_{\mathrm{H},u}(M(r)) & = & \mathbf{t}_{\mathrm{H},u}(M) - (r, \ldots, r) \end{array}$$

for every $r \in \mathbb{Z}$. So, let M be an effective Kisin module and $Q = \mathrm{coker}\, \varphi_M$ and $m = \mathrm{rank}\, M$. We fix a pseudo-isomorphism $Q \sim Q' = \bigoplus_{i=1}^{m} \mathfrak{S}/(E(u)^{n_i})$, which becomes an isomorphism after localizing by (E), with $n_1 \geq \ldots \geq n_m \geq 0$ (completing by 0 the invariant factors of Q). For $N = n_1$, consider the filtration

$$0 = Q'_0 \subsetneq Q'_{\frac{1}{N}} \subsetneq \ldots \subsetneq Q'_{\frac{N}{N}} = Q'$$

with $Q'_{\frac{i}{N}} = Q'[E^i]$. Let $Q_{\frac{i}{N}} = Q \cap Q'_{\frac{i}{N}} = Q[E^i]$, so we have a filtration

$$0 = Q_0 \subsetneq Q_{\frac{1}{N}} \subsetneq \ldots \subsetneq Q_{\frac{N}{N}} = Q.$$

Also, an exact sequence

$$0 \to Q/Q_{\frac{i}{N}} \hookrightarrow Q'/Q'[E^i] \twoheadrightarrow Q'/(Q + Q'[E^i]) \to 0$$

and an isomorphism

$$Q'/Q'[E^i] \simeq \bigoplus_{n_j > i} \mathfrak{S}/E^{n_j - i}\mathfrak{S}$$

for $1 \leq i \leq N$, so $\mathrm{Tor}^2\left(Q/Q_{\frac{i}{N}}, \mathbb{F}\right) = \left(Q/Q_{\frac{i}{N}}\right)[\mathfrak{m}] = 0$. Let $\varphi^* M \subset M_{\frac{i}{N}} \subset M$ with

$$M_{\frac{i}{N}}/\varphi^* M = Q_{\frac{i}{N}} \subset Q = M/\varphi^* M.$$

Since $M/M_{\frac{i}{N}} = Q/Q_{\frac{i}{N}}$, we have $\mathrm{Tor}^2\left(M/M_{\frac{i}{N}}, \mathbb{F}\right) = 0$, $\mathrm{Tor}^1(M_{\frac{i}{N}}, \mathbb{F}) = 0$ and $M_{\frac{i}{N}}$ is free. So we have a filtration

$$\varphi^* M \subsetneq M_{\frac{1}{N}} \subsetneq \ldots \subsetneq M_{\frac{N}{N}} = M$$

with quotients

$$M_{\frac{i}{N}}/M_{\frac{i-1}{N}} = Q_{\frac{i}{N}}/Q_{\frac{i-1}{N}} = Q[E^i]/Q[E^{i-1}] = R_i,$$

which are contained in

$$R'_i = Q'[E^i]/Q'[E^{i-1}] \simeq \bigoplus_{n_j \geq i} \mathfrak{S}/E\mathfrak{S}$$

and with cokernel of finite length. Since $\mathfrak{S}/E\mathfrak{S}$ is a DVR, we also have

$$R_i \simeq \bigoplus_{n_j \geq i} \mathfrak{S}/E\mathfrak{S}.$$

For $\overline{M}$, we have the sequence

$$\overline{\varphi^* M} \subsetneq \overline{M}_{\frac{1}{N}} \subsetneq \ldots \subsetneq \overline{M}_{\frac{N}{N}} = \overline{M},$$

with torsion $\mathbb{F}[[u]]$-modules as quotients,

$$\overline{R}_i = \bigoplus_{n_j \geq i} \mathbb{F}[u]/E\mathbb{F}[u].$$

The triangle inequality for relative positions tells us

$$\mathrm{Pos}(\overline{\varphi^* M}, \overline{M}) \leq \sum_{i=1}^{N} \mathrm{Pos}(\overline{M}_{\frac{i-1}{N}}, \overline{M}_{\frac{i}{N}})$$

and we know that

$$\mathrm{Pos}(\overline{M}_{\frac{i-1}{N}}, \overline{M}_{\frac{i}{N}}) = (e, \ldots, e, 0, \ldots, 0) \in \mathbb{Z}^m_{\geq}$$

where $e = \deg E$ and the multiplicity of e is $\#\{j \ : \ n_j \geq i\}$. So we get

$$\mathrm{Pos}(\overline{\varphi^* M}, \overline{M}) \leq e \cdot (n_1, n_2, \ldots, n_m) = e \cdot \mathrm{Pos}(\varphi^* M_{(E)}, M_{(E)}).$$

For M/uM, we have the sequence

$$\sigma^*(M/uM) \subsetneq (M/uM)_{\frac{1}{N}} \subsetneq \ldots \subsetneq (M/uM)_{\frac{N}{N}} = M/uM,$$

with torsion $W(\mathbb{F})$-modules as quotients,

$$R_i/uR_i = \bigoplus_{n_j \geq i} \mathbb{F}.$$

The triangle inequality for relative positions tells us

$$\mathrm{Pos}(\sigma^*(M/uM), M/uM) \leq \sum_{i=1}^{N} \mathrm{Pos}((M/uM)_{\frac{i-1}{N}}, (M/uM)_{\frac{i}{N}})$$

and we know that

$$\mathrm{Pos}((M/uM)_{\frac{i-1}{N}}, (M/uM)_{\frac{i}{N}}) = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{Z}^m_{\geq}$$

where the multiplicity of 1 is $\#\{j \ : \ n_j \geq i\}$. So we get

$$\mathrm{Pos}(\sigma^*(M/uM), M/uM) \leq (n_1,\, n_2,\, \ldots,\, n_m) = \mathrm{Pos}(\varphi^* M_{(E)}, M_{(E)}).$$

□

Proposition 6.1.2. *Let $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$. Then :*

1. *For $n \geq 1$, we have*
$$\begin{array}{rcl} \deg(M/p^n M) & = & en \deg(M) \\ \mathrm{rank}(M/p^n M) & = & n\,\mathrm{rank}(M), \end{array}$$
where $e = \deg E$.

2. *For the isogeny class, we have $\deg(M \otimes \mathbb{Q}_p) = \deg(M)$ and $\mathrm{rank}(M \otimes \mathbb{Q}_p) = \mathrm{rank}(M)$.*

Proof. Up to twist, we may assume that M is effective and consider the exact sequence

$$0 \to \varphi^* M \to M \to Q \to 0.$$

Then, (1) follows from the previous proposition, using

$$0 \to M/p^n M \xrightarrow{p} M/p^{n+1} M \to M/pM \to 0$$

and (2) is obvious. □

This way, we get an invariant

$$\mu_M = \frac{\deg M}{\mathrm{rank}_{\mathfrak{S}} M}$$

which only depends on the isogeny class of M.

We want to get rid of the coefficient e in the last proposition. For that, it suffices to define a new degree function on $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$ by putting

$$\deg_{\mathrm{new}}(M) = \tfrac{1}{e} \deg_{\mathrm{old}}(M)$$

for every $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$. The new degree function verifies the same properties as the old one and gives us a new slope function, Fargues filtration and Fargues polygon verifying

$$\begin{array}{rcl} \mu_{\mathrm{new}}(M) & = & \frac{1}{e}\mu_{\mathrm{old}}(M) \\ \mathcal{F}^{\geq\gamma}_{\mathrm{F,t,new}}(M) & = & \mathcal{F}^{\geq\frac{\gamma}{e}}_{\mathrm{F,t,old}}(M) \\ \mathbf{t}_{\mathrm{F,t,new}}(M) & = & \frac{1}{e}\mathbf{t}_{\mathrm{F,t,old}}(M) \end{array}$$

for every $\gamma \in \mathbb{Q}$. From now on, we will only work with the new degree function, filtration and polygon and drop it from the notation.

Corollary 6.1.3. *Let $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$. Then*

$$\begin{array}{rcl} n \deg(M) & = & \deg(M/p^n M) \\ n\,\mathrm{rank}(M) & = & \mathrm{rank}(M/p^n M) \\ \mu_M & = & \mu(M/p^n M) \end{array}$$

for all $n \in \mathbb{N}$.

6.2 Minimal slopes

Definition 6.2.1. For $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$, let $M^{\min}$ be the last quotient of the Harder-Narasimhan filtration of M. We define the minimal slope of M as

$$\mu_{\min}(M) := \mu(M^{\min}) = \max\{\lambda \mid \mathcal{F}^{\geq\lambda}M = M\}.$$

It is easy to see that then M is semi-stable if and only if $\mu(M) = \mu_{\min}(M)$.

Proposition 6.2.1. *Let $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$. Then for every nonzero strict quotient $M \twoheadrightarrow Q$ in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$, we have*

$$\mu(Q) \geq \mu_{\min}(Q) \geq \mu_{\min}(M)$$

and the equality $\mu(Q) = \mu_{\min}(M)$ holds if and only if Q is a semi-stable quotient of $M^{\min}$.

Proof. 1. The filtration $\mathcal{F}_{\mathrm{F,t}}$ is functorial, so have morphisms $\mathcal{F}^{\geq\lambda}_{\mathrm{F,t}}M \to \mathcal{F}^{\geq\lambda}_{\mathrm{F,t}}Q$ for every $\lambda \in \mathbb{R}$. In particular, for $\lambda = \mu_{\min}(M)$, we get

$$\begin{tikzcd} M = \mathcal{F}^{\geq\mu_{\min}(M)}_{\mathrm{F,t}}M \arrow[rr] \arrow[dr, two heads] & & \mathcal{F}^{\geq\mu_{\min}(M)}_{\mathrm{F,t}}Q \arrow[dl, hook] \\ & Q & \end{tikzcd}$$

so $\mathcal{F}^{\geq\mu_{\min}(M)}_{\mathrm{F,t}}Q = Q$. Since, by definition, we have $\mu_{\min}(Q) = \max\{\lambda \in \mathbb{R} \mid \mathcal{F}^{\geq\lambda}_{\mathrm{F,t}}Q = Q\}$ it follows that $\mu_{\min}(M) \leq \mu_{\min}(Q)$. Also, we know that $\mu_{\min}(Q) \leq \mu(Q)$ by concavity of $\mathbf{t}_{\mathrm{F,t}}(Q)$.

2. Suppose that $\mu(Q) = \mu_{\min}(M)$, thus

$$\mu_{\min}(M) = \mu_{\min}(Q) = \mu(Q).$$

Then Q is semi-stable and $M \twoheadrightarrow Q$ maps $\mathcal{F}^{>\mu_{\min}(M)}_{\mathrm{F,t}}M$ to $\mathcal{F}^{>\mu_{\min}(M)}_{\mathrm{F,t}}Q = 0$, i.e the morphsim $M \twoheadrightarrow Q$ factors through $M \twoheadrightarrow M^{\min} \twoheadrightarrow Q$ and Q is a semi-stable quotient of $M^{\min}$.

□

For $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$, we set $\mu_{\min}(M) := \mu_{\min}(M/pM)$.

Lemma 6.2.2. *Let $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$ nonzero and $M_n = M/p^nM$. Then:*

1. *For every $n \geq 1$, we have $\mu_{\min}(M) = \mu_{\min}(M_n) = \mu(M_n^{\min})$.*
2. *For $1 \leq m \leq n$, we have*
$$p^m M_n^{\min} = \mathrm{im}(M_{n-m}^{\min} \to M_n^{\min}).$$
3. *For $1 \leq m \leq n$, we have*
$$M_n^{\min}/p^m M_n^{\min} = M_m^{\min}.$$
4. *The system $(N)_{n\geq 1}$, for $N_n = \ker(M_n \to M_n^{\min})$, is Mittag-Leffler surjective.*

Proof. 1. Let $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$ and consider the exact sequence

$$0 \to M_n^{\min}[p^m] \to M_n^{\min} \xrightarrow{\times p^m} M_n^{\min} \to M_n^{\min}/p^m M_n^{\min} \to 0$$

for all $1 \leq m \leq n$. By the properties of our Harder-Narasimhan formalism, seen in 2.5.5, which are verified by torsion Kisin modules as we saw in 5.2.1 and 5.2.3, we know that $M_n^{\min}/p^m M_n^{\min}$ is semi-stable and

$$\mu(M_n^{\min}) = \mu(M_n^{\min}/p^m M_n^{\min}).$$

Since $\mu_{\min}(M/p^n M) = \mu(M_n^{\min})$ (and $\mu_{\min}(M/p^m M) = \mu(M_m^{\min})$ respectively), and we have the quotients

$$M/p^n M \twoheadrightarrow M/p^m M \twoheadrightarrow M_m^{\min}$$

and

$$M/p^m M \simeq M_n/p^m M_n \twoheadrightarrow M_n^{\min}/p^m M_n^{\min}$$

we get the following inequalities

$$\mu(M_n^{\min}) \leq \mu(M_m^{\min}) \leq \mu(M_n^{\min}/p^m M_n^{\min})$$

which must be equalities in order to have $\mu(M_n^{\min}/p^m M_n^{\min}) = \mu(M_n^{\min})$. In particular $\mu(M_n^{\min}) = \mu(M_1^{\min}) = \mu_{\min}(M)$, for every $n \geq 1$.

2. We have

$$\begin{aligned} \mathrm{im}(M_{n-m}^{\min} \to M_n^{\min}) &= \mathrm{im}(M_{n-m} \twoheadrightarrow M_{n-m}^{\min} \to M_n^{\min}) \\ &= \mathrm{im}(M_{n-m} \hookrightarrow M_n \twoheadrightarrow M_n^{\min}) \\ &= \mathrm{im}(p^m M_n \hookrightarrow M_n \twoheadrightarrow M_n^{\min}) \\ &= p^m M_n^{\min}. \end{aligned}$$

3. If we apply last proposition to the surjection $M_m \simeq M_n/p^m M_n \twoheadrightarrow M_n^{\min}/p^m M_n^{\min}$, we get

$$M_m^{\min} \twoheadrightarrow M_n^{\min}/p^m M_n^{\min}.$$

On the other hand, applying last proposition to the surjection $M_n \twoheadrightarrow M_m \twoheadrightarrow M_m^{\min}$ we obtain that $M_n^{\min} \twoheadrightarrow M_m^{\min}$ and it is obvious that it factors through $M_n^{\min}/p^m M_n^{\min}$, giving us a surjection

$$M_n^{\min}/p^m M_n^{\min} \twoheadrightarrow M_m^{\min},$$

which ends the proof.

4. Denote by $N_n^m = \ker(N_n \to N_m)$ and $Q_n^m = \ker(M_n^{\min} \to M_m^{\min})$, for $n \geq m \geq 1$, so we have a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N_n^m \arrow[r] \arrow[d] & N_n \arrow[r] \arrow[d] & N_m \arrow[d] & \\
0 \arrow[r] & M_{n-m} \arrow[r] \arrow[d] & M_n \arrow[r] \arrow[d] & M_m \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q_n^m \arrow[r] & M_n^{\min} \arrow[r] \arrow[d] & M_m^{\min} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Let $C = \mathrm{coker}(N_n \to N_m)$. Then, we have $N_{n-m} \subset M_{n-m} \cap N_n = N_n^m$ and a diagram

$$\begin{tikzcd} 0 \arrow[r] & N_{n-m} \arrow[r] \arrow[d, hook] & M_{n-m} \arrow[r] \arrow[d, equal] & M_{n-m}^{\min} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N_n^m \arrow[r] & M_{n-m} \arrow[r] & Q_n^m \arrow[r] & C \end{tikzcd}$$

which gives us

$$0 \to N_n^m/N_{n-m} \to M_{n-m}^{\min} \to Q_n^m \to C \to 0.$$

Since both $M_n^{\min}$ and $M_m^{\min}$ are semi-stable of slope $\mu_{\min}(M)$, then Q_n^m is semi-stable of slope $\mu_{\min}(M)$. Suppose $C \neq 0$, then C is semi-stable of slope $\mu_{\min}(M)$ and also a quotient of N_m, so

$$\mu_{\min}(M) = \mu(C) \geq \mu(N_m) > \mu(M_m^{\min}) = \mu_{\min}(M)$$

which is a contradiction, therefore $C = 0$ and $(N_n)_{n\geq 1}$ is Mittag-Leffler surjective.

□

6.3 A polygon on $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$

Even though we have rank and degree functions defined on Kisin modules, $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$ is not a quasi-abelian category, so we cannot use André's formalism in order to endow this category with a Harder-Narasimhan filtration. However, we can consider the projective limit of Fargues polygons on torsion Kisin modules to try to construct a polygon on a Kisin module, as follows.

For M a Kisin module, we can renormalize the polygons $\mathbf{t}_{\mathrm{F},\mathrm{t}}(M/p^nM)$ by considering the concave functions $\mathbf{t}_{\mathrm{F},n}(M)$ defined as

$$\mathbf{t}_{\mathrm{F},n}(M)(x) = \tfrac{1}{n}\mathbf{t}_{\mathrm{F},\mathrm{t}}(M/p^nM)(nx)$$

for every $x \in [0, \mathrm{rank}\, M]$, for every $n \geq 1$. Such functions are concave polygons with end point at $(\mathrm{rank}\, M, \deg M)$ but whose break points may not have integer abscissas, so we cannot consider them as types, but we can extend all the properties for types to these concave functions. In particular, the dominance order still makes sense for them. Moreover, the next lemma shows that they form a decreasing system.

Lemma 6.3.1. *For every Kisin module M, we have.*

$$\mathbf{t}_{\mathrm{F},nm}(M) \leq \mathbf{t}_{\mathrm{F},n}(M)$$

for every $m, n \geq 1$.

Proof. By induction using the exact sequences

$$0 \to M/p^nM \to M/p^{mn}M \to M/p^{m(n-1)}M \to 0$$

we get

$$\mathbf{t}_{\mathrm{F},\mathrm{t}}(M/p^{mn}M) \leq \mathbf{t}_{\mathrm{F},\mathrm{t}}(M/p^nM)^{*m}.$$

Now, the polygon $\mathbf{t}_{\mathrm{F},\mathrm{t}}(M/p^nM)^{*m}$ is just an homotethy of $\mathbf{t}_{\mathrm{F},\mathrm{t}}(M/p^nM)$ and we have

$$\mathbf{t}_{\mathrm{F},\mathrm{t}}(M/p^nM)^{*m}(x) = m\mathbf{t}_{\mathrm{F},\mathrm{t}}(M/p^nM)(\tfrac{x}{m})$$

for $0 \leq x \leq mn\,\mathrm{rank}\, M$. Thus

$$\tfrac{1}{mn}\mathbf{t}_{\mathrm{F},\mathrm{t}}(M/p^{mn}M)(mnx) \leq \tfrac{1}{n}\mathbf{t}_{\mathrm{F},\mathrm{t}}(M/p^nM)(nx)$$

for $0 \leq x \leq \mathrm{rank}\, M$ and we obtain $\mathbf{t}_{\mathrm{F},mn}(M) \leq \mathbf{t}_{\mathrm{F},n}(M)$.

□

Then, we obtain:

Theorem 6.3.2. *Let $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$. Let $r = \mathrm{rank}\, M$ and $d = \deg M$. The sequence of functions*

$$\begin{array}{ccc} [0,r] & \to & \mathbb{R} \\ x & \mapsto & \mathbf{t}_{\mathrm{F},n}(M) \end{array}$$

converges uniformly for the divisibility order to a continuous concave function

$$\mathbf{t}_{\mathrm{F},\infty}(M) \ : \ [0,r] \to \mathbb{R}$$

which is equal to

$$\inf_{n\geq 1} \mathbf{t}_{\mathrm{F},n}(M)$$

and moreover, verifying $\mathbf{t}_{\mathrm{F},\infty}(M)(0) = 0$ and $\mathbf{t}_{\mathrm{F},\infty}(M)(r) = d$. We call this function the Fargues polygon of M.

Proof. Denote by $f_n = \mathbf{t}_{\mathrm{H},n}$ and f its limit. The limit exists since the functions f_n are decreasing and lower bounded. For $n \geq 1$ and $0 \leq x < y$ in $[0,r]$,

$$f'_{1,r}(0) \leq \frac{f_n(y) - f_n(x)}{y - x} \leq f'_{1,l}(r),$$

so $\{f_n\} \cup \{f\}$ is equi-Lispchitz over $[0,r]$, i.e. there exists an $M > 0$ independent of x, y and n, such that for every $n \geq 1$ and $x < y$,

$$\left| \frac{f_n(y) - f_n(x)}{y - x} \right| \leq M.$$

Let $\varepsilon > 0$, $\frac{Mr}{C} \leq \varepsilon$ such that $x_i = \frac{ir}{C}$, N such that if $N \mid n$, then $|f_n(x_i) - f(x_i)| \leq \varepsilon$. Then, for every $x \in [x_i, x_{i+1}]$, we have

$$\begin{array}{rcl} |f_n(x) - f(x)| & \leq & |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(x)| \\ & \leq & M|x - x_i| + \varepsilon + M|x_i - x| \\ & \leq & 3\varepsilon \end{array}$$

so the convergence is uniform over $[0,r]$. In particular, f is continuous and the theorem follows. □

The Fargues polygon of a Kisin module only depends on the isogeny class of the module, as we can see in the next proposition.

Proposition 6.3.3. *Let M_1, $M_2 \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$ such that $M_1 \otimes \mathbb{Q}_p \simeq M_2 \otimes \mathbb{Q}_p$ in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$. Then*

$$\mathbf{t}_{\mathrm{F},\infty}(M_1) = \mathbf{t}_{\mathrm{F}.\infty}(M_2).$$

Proof. The proof is analogous to the one in [18, Proposition 3]. By reflexivity of the isogeny relation, it suffices to prove one inequality. Fix an isogeny $M_1 \to M_2$ giving rise to an exact sequence

$$0 \to M_1 \to M_2 \to Q \to 0$$

in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ with $p^n Q = 0$ for n large enough. By multiplication by p^n, we obtain an exact sequence

$$0 \to Q \to M_1/p^n M_1 \to M_2/p^n M_2 \to Q \to 0$$

in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$. This yields two exact sequences

$$0 \to Q \to M_1/p^n M_1 \to N_n \to 0$$

$$0 \to N_n \to M_2/p^n M_2 \to Q \to 0$$

in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$, with $nr = r_n + q$, for $r = \operatorname{rank} M$, $r_n = \operatorname{rank} N_n$ and $q = \operatorname{rank} Q$. Then $\frac{r_n}{n} = r - \frac{q}{n}$. For n large enough, let $x \in [0, \frac{r_n}{n}] = [0, r - \frac{q}{n}]$, then the second exact sequence gives us

$$\mathbf{t}_{\mathrm{F,t}}(N_n)(nx) \leq \mathbf{t}_{\mathrm{F,t}}(M_2/p^n M_2)(nx)$$

and the first exact sequence implies

$$\begin{aligned} \mathbf{t}_{\mathrm{F,t}}(M_1/p^n M_1)(nx) &\leq (\mathbf{t}_{\mathrm{F,t}}(N_n) * \mathbf{t}_{\mathrm{F,t}}(Q))(nx) \\ &= \mathbf{t}_{\mathrm{F,t}}(N_n)(nx - \delta_n) + \mathbf{t}_{\mathrm{F,t}}(Q)(\delta_n) \end{aligned}$$

for some $0 \leq \delta_n \leq nx, q$, by definition of the concatenation. By concavity of $\mathbf{t}_{\mathrm{F,t}}(N_n)$, we have

$$\mathbf{t}_{\mathrm{F,t}}(N_n)(nx - \delta_n) \leq \mathbf{t}_{\mathrm{F,t}}(N_n)(nx) - \mu_{\min}(N_n) \cdot \delta_n.$$

Since N_n is a quotient of $M_1/p^n M_1$ by proposition 6.2.1, we obtain

$$\mu_{\min}(N_n) \geq \mu_{\min}(M_1/p^n M_1) = \mu_{\min}(M_1),$$

thus

$$\begin{aligned} \mathbf{t}_{\mathrm{F,t}}(M_1/p^n M_1)(nx) &\leq \mathbf{t}_{\mathrm{F,t}}(N_n)(nx) + \mathbf{t}_{\mathrm{F,t}}(Q)(\delta_n) - \mu_{\min}(M_1) \cdot \delta_n \\ &\leq \mathbf{t}_{\mathrm{F,t}}(N_n)(nx) + C \end{aligned}$$

for

$$C = \max\{\mathrm{t}_{\mathrm{F,t}}(Q)(\delta) - \mu_{\min}(M_1) \cdot \delta \mid \delta \in [0, q]\}$$

which is independent of n. Combining all the inequalities above, we have

$$\mathbf{t}_{\mathrm{F,t}}(M_1/p^n M_1)(nx) \leq \mathbf{t}_{\mathrm{F,t}}(N_n)(nx)) + C \leq \mathbf{t}_{\mathrm{F,t}}(M_2/p^n M_2)(nx) + C,$$

so

$$\mathbf{t}_{\mathrm{F},n}(M_1)(x) \leq \mathbf{t}_{\mathrm{F},n}(M_2)(x) + \frac{C}{n}$$

for n large enough and $x \in [0, r - \frac{q}{n}]$, therefore

$$\mathbf{t}_{\mathrm{F},\infty}(M_1)(x) \leq \mathbf{t}_{\mathrm{F},\infty}(M_2)(x)$$

for every $x \in [0, r)$ and we already know the equality for $x = r$.

□

6.4 Semi-stability and type HN

Even though there is not a good Harder-Narasimhan formalism, we can define some sort of semi-stability on objects in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$.

Proposition 6.4.1. *Let $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$. The following conditions are equivalent:*

1. *The module M/pM is semi-stable (as a p-torsion Kisin module),*
2. *For every $n \geq 1$, the module $M/p^n M$ is semi-stable (as a p^∞-torsion Kisin module),*
3. *For every p^∞-torsion Kisin module Q which is a quotient of M we have $\mu_M \leq \mu(Q)$,*

Proof. It is easy to see that (1) implies (2), since the category $\mathcal{C}(\mu)$ of semi-stable objects of slope μ is stable by extension. For (2) implies (3), we have that for any p^∞-torsion quotient Q of M, there exists a $n \geq 1$, such that we have a factorization $M \twoheadrightarrow M/p^n M \twoheadrightarrow Q$, thus $\mu_M = \mu(M/p^n M) \leq \mu(Q)$. For (3) implies (1), we have that any quotient Q of M/pM is a p^∞-torsion quotient of M therefore $\mu(M/pM) = \mu_M \leq \mu(Q)$ and M/pM is semi-stable.

□

Definition 6.4.1. We say that a Kisin module M is semi-stable if it verifies one of the conditions above.

Remark 13. If a Kisin module M is semi-stable, then $\mathbf{t}_{\mathrm{F},\infty}(M)$ is isoclinic of slope μ_M.

In the next proposition, we compare the semi-stability we just defined on $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$ with the semi-stability on $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$.

Proposition 6.4.2. *Let $M, M' \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$ isogenous and M' semi-stable in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$. Then for every subobject $N \neq 0$ of $M \otimes \mathbb{Q}_p$, we have $\mu_N \leq \mu_M$, i.e. $M \otimes \mathbb{Q}_p$ is semi-stable in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$.*

Proof. As we have already discussed, an object in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}} \otimes \mathbb{Q}_p$ is semi-stable of slope λ if and only if its image in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$ is semi-stable of slope λ. We use this characterization. Let $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$ and $N = M[\frac{1}{p}]$ its image in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$. An exact sequence $0 \to N_1 \to N \to N_2 \to 0$ in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$ induces an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$, where $M_1 = M \cap N_1$ and M_2 is the image of M in N_2. The cokernel of $M_2 \hookrightarrow M_{2,\mathrm{fr}}$ is killed by p^m for some $m \geq 1$, so multiplication by p^m gives us an isogeny $f : M_{2,\mathrm{fr}} \hookrightarrow M_2$. We have thus a diagram

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "i"] & M \arrow[r] \arrow[d, "i"] & M_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_{1,\mathrm{fr}} \arrow[r] \arrow[d, equal] & M' \arrow[r] & M_2 \arrow[r] & 0 \\
0 \arrow[r] & M_{1,\mathrm{fr}} \arrow[r] & M'' \arrow[r] \arrow[u, "f"] & M_{2,\mathrm{fr}} \arrow[r] \arrow[u, "f"] & 0
\end{tikzcd}$$

by push out of $i : M_1 \to M_{1,\mathrm{fr}}$ and pull-back by $f : M_{2,\mathrm{fr}} \to M_2$. Since M'' is an extension of two free modules, it is itself free and we have an exact sequence

$$0 \to M_{1,\mathrm{fr}}/p^n M_{1,\mathrm{fr}} \to M''/p^n M'' \to M_{2,\mathrm{fr}}/p^n M_{2,\mathrm{fr}} \to 0$$

for every $n \geq 1$.

Suppose M is semi-stable of slope λ. Then, M/pM is semi-stable of slope λ, so $\mathbf{t}_{\mathrm{F},1}(M)$ is isoclinic of slope λ, which implies that $\mathbf{t}_{\mathrm{F},\infty}(M)$ is also isoclinic of slope λ, since $\mathbf{t}_{\mathrm{F},\infty}(M) \leq \mathbf{t}_{\mathrm{F},1}(M)$. Now, $\mathbf{t}_{\mathrm{F},\mathfrak{S}[\frac{1}{p}]}(N)$ has the same starting and ending point, so

$$\mathbf{t}_{\mathrm{F},\infty}(M) \leq \mathbf{t}_{\mathrm{F},\mathfrak{S}[\frac{1}{p}]}(N).$$

Let $\lambda_{\max}$ be the maximal slope of $\mathbf{t}_{\mathrm{F},\mathfrak{S}[\frac{1}{p}]}(N)$. Then, $\mathbf{t}_{\mathrm{F},\mathfrak{S}[\frac{1}{p}]}(N)$ is isoclinic of slope λ if and only if $\lambda_{\max} = \lambda$, if and only if $\lambda_{\max} \leq \lambda$, since we already have the other inequality. Let N_1 be the first submodule in the Fargues filtration of N, such that $\mu(N_1) = \lambda_{\max}$ and $r_1 = \operatorname{rank} N_1$. We have that

$$r_1 \lambda_{\max} = \deg N_1 = \deg M_{1,\mathrm{fr}} = \mathbf{t}_{\mathrm{F},n}(M_{1,\mathrm{fr}})(r_1) \leq \mathbf{t}_{\mathrm{F},n}(M'')(r_1)$$

so $r_1 \lambda_{\max} \leq \mathbf{t}_{\mathrm{F},\infty}(M'')(r_1)$. In 6.3.3, we have seen that $\mathbf{t}_{\mathrm{F},\infty}$ is invariant by isogeny, so we have $\mathbf{t}_{\mathrm{F},\infty}(M'') = \mathbf{t}_{\mathrm{F},\infty}(M'_{1,\mathrm{fr}}) = \mathbf{t}_{\mathrm{F},\infty}(M)$ which implies that

$$r_1 \lambda_{\max} \leq \mathbf{t}_{\mathrm{F},\infty}(M'')(r_1) = \mathbf{t}_{\mathrm{F},\infty}(M)(r_1) = r_1 \lambda,$$

hence $\lambda_{\max} \leq \lambda$ and

$$\mathbf{t}_{\mathrm{F},\mathfrak{S}[\frac{1}{p}]}(N) = \mathbf{t}_{\mathrm{F},\circ}(M \otimes \mathbb{Q}_p) = \mathbf{t}_{\mathrm{F},\infty}(M)$$

is isoclinic of slope λ.

$\square$

We cannot use André's formalism to give a Harder-Narasimhan filtration on Kisin modules, but some of them may have a Harder-Narasimhan filtration for the semi-stable definition given above.

Proposition 6.4.3. *Let $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$. The following statements are equivalent:*

1. *The Kisin module M admits a Harder-Narasimhan filtration by Kisin submodules or, equivalently, an increasing flag*
$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$
of Kisin submodules such that, for $1 \leq i \leq r$, the modules M_i/M_{i-1} are free and semi-stable of slopes μ_i verifying
$$\mu_1 > \ldots > \mu_r.$$
2. *We have $\mathbf{t}_{\mathrm{F},\infty}(M) = \mathbf{t}_{\mathrm{F},1}(M)$.*

Moreover, if the conditions above are verified, we have:

(a) The flag is uniquely determined by $M_i = \varprojlim_n \mathcal{F}^{\mu_i}_{\mathrm{F},n}(M)$ for $1 \leq i \leq r$ and we have $\mathbf{t}_{\mathrm{F},\infty}(M) = (\mu_1, \ldots, \mu_r)$ (with multiplicity).

(b) The reduction of the flag modulo p^n gives us the Fargues filtration of M/p^nM.

(c) The flag is compatible with the one obtained for the isogeny categories, i.e. we have
$$M_i \otimes \mathbb{Q}_p = \mathcal{F}^{\mu_i}_{\mathrm{F},\circ}(M \otimes \mathbb{Q}_p) \quad \text{and} \quad M_i[\tfrac{1}{p}] = \mathcal{F}^{\mu_i}_{\mathrm{F},\mathfrak{S}[\frac{1}{p}]}(M[\tfrac{1}{p}])$$
for $1 \leq i \leq r$.

Proof. It is easy to see that (1) implies (2) since the semi-stability condition is preserved by reducing modulo p^n, so we obtain a flag for M_n whose quotients are semi-stable, and the Fargues filtration is unique. Moreover, the slopes are also preserved by reduction and by the compatibility of the rank and slopes, we get that the polygons modulo p^n are constant and we have $\mathbf{t}_{\mathrm{F},n}(M) = (\mu_1, \ldots, \mu_r)$, thus $\mathbf{t}_{F,\infty}(M) = \lim \mathbf{t}_{\mathrm{F},n}(M) = (\mu_1, \ldots, \mu_r)$.

Conversely, suppose that $\mathbf{t}_{\mathrm{F},\infty}(M) = \mathbf{t}_{\mathrm{F},1}(M)$. We consider the sequences
$$0 \to M_n \to M_{m+n} \to M_m \to 0,$$
for $n, m \geq 1$. Our hypothesis implies that all the polygons $\mathbf{t}_{\mathrm{F},n}(M)$ coincide for $n \geq 1$, so we have
$$\mathbf{t}_{\mathrm{F},\mathrm{t}}(M/p^{n+m}M) = \mathbf{t}_{\mathrm{F},\mathrm{t}}(M/p^nM) * \mathbf{t}_{\mathrm{F},\mathrm{t}}(M/p^mM)$$
for $n, m \geq 1$. Then, we can apply the point (3) in 2.5.5 to get a diagram
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F}^{>\gamma}_{\mathrm{F},\mathrm{t}} M_n \arrow[r] \arrow[d] & \mathcal{F}^{>\gamma}_{\mathrm{F},\mathrm{t}} M_{m+n} \arrow[r] \arrow[d] & \mathcal{F}^{>\gamma}_{\mathrm{F},\mathrm{t}} M_m \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}^{\geq\gamma}_{\mathrm{F},\mathrm{t}} M_n \arrow[r] \arrow[d] & \mathcal{F}^{\geq\gamma}_{\mathrm{F},\mathrm{t}} M_{m+n} \arrow[r] \arrow[d] & \mathcal{F}^{\geq\gamma}_{\mathrm{F},\mathrm{t}} M_m \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Gr}^{\geq\gamma}_{\mathcal{F}_{\mathrm{F},\mathrm{t}}} M_n \arrow[r] \arrow[d] & \mathrm{Gr}^{\gamma}_{\mathcal{F}_{\mathrm{F},\mathrm{t}}} M_{m+n} \arrow[r] \arrow[d] & \mathrm{Gr}^{\gamma}_{\mathcal{F}_{\mathrm{F},\mathrm{t}}} M_m \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

for all $\gamma \in \mathbb{Q}$. In particular $(\mathcal{F}^{>\gamma}_{\mathrm{F,t}} M_n)_{n\geq 1}$, $(\mathcal{F}^{\geq\gamma}_{\mathrm{F,t}} M_n)_{n\geq 1}$ and $(\mathrm{Gr}^{\gamma}_{\mathcal{F}_{\mathrm{F,t}}} M_n)_{n\geq 1}$ are Mittag-Leffler surjective. Call $\mathcal{F}^{>\gamma}_{\mathrm{F}} M$, $\mathcal{F}^{\geq\gamma}_{\mathrm{F}} M$ and $\mathrm{Gr}^{\gamma}_{\mathcal{F}_{\mathrm{F}}} M$ their projective limit. Since $(\mathcal{F}^{>\gamma}_{\mathrm{F,t}} M)_{n\geq 1}$ is Mittag-Leffler surjective, we obtain an exact sequence

$$0 \to \mathcal{F}^{>\gamma}_{\mathrm{F}} M \to \mathcal{F}^{\geq\gamma}_{\mathrm{F}} M \to \mathrm{Gr}^{\gamma}_{\mathcal{F}_{\mathrm{F}}} M \to 0.$$

On the other hand, an exact sequence

$$0 \to X_n \xrightarrow{\times p^m} X_{m+n} \to X_m \to 0$$

for all $n, m \geq 1$, induces an exact sequence

$$0 \to X \xrightarrow{\times p^m} X \to X_m \to 0$$

where $X = \varprojlim_n X_n \simeq \varprojlim_n X_{n+m}$. We apply this result to

- The family $(X_n)_{n\geq 1} = (\mathcal{F}^{>\gamma}_{\mathrm{F,t}} M_n)_{n\geq 1}$ to obtain that $\mathcal{F}^{>\gamma}_{\mathrm{F}} M / p^m \mathcal{F}^{>\gamma}_{\mathrm{F}} M \simeq \mathcal{F}^{>\gamma}_{\mathrm{F,t}} M_m$ for every $m \geq 1$, so $\mathcal{F}^{>\gamma}_{\mathrm{F}} M$ is free of rank $\max\{i \mid \mu_i > \gamma\}$.
- The family $(X_n)_{n\geq 1} = (\mathcal{F}^{\geq\gamma}_{\mathrm{F,t}} M_n)_{n\geq 1}$ to obtain that $\mathcal{F}^{\geq\gamma}_{\mathrm{F}} M / p^m \mathcal{F}^{\geq\gamma}_{\mathrm{F}} M \simeq \mathcal{F}^{\geq\gamma}_{\mathrm{F,t}} M_m$ for every $m \geq 1$, so $\mathcal{F}^{\geq\gamma}_{\mathrm{F}} M$ is free of rank $\max\{i \mid \mu_i \geq \gamma\}$.
- The family $(X_n)_{n\geq 1} = (\mathrm{Gr}^{\gamma}_{\mathcal{F}_{\mathrm{F,t}}} M_n)_{n\geq 1}$ to obtain that $\mathrm{Gr}^{\gamma}_{\mathcal{F}_{\mathrm{F}}} M / p^m \mathrm{Gr}^{\gamma}_{\mathcal{F}_{\mathrm{F}}} M \simeq \mathrm{Gr}^{\gamma}_{\mathcal{F}_{\mathrm{F,t}}} M_m$ for every $m \geq 1$, so $\mathrm{Gr}^{\gamma}_{\mathcal{F}_{\mathrm{F}}} M$ is free of rank $\max\{i \mid \mu_i = \gamma\}$.

In particular, $\mathrm{Gr}_{\mathcal{F}_{\mathrm{F}}} M$ is semi-stable of slope γ, for $\gamma \in \{\mu_1, \ldots, \mu_r\}$, and the zero object for γ otherwise. Since $\mathcal{F}^{\gamma}_{\mathrm{F}} M = M$ for $\gamma \leq \mu_r$, this proves (1).

Assertions (a) and (b) have already been established in the proof of (1) if and only if (2), and (3) follows from Proposition 6.4.2.

□

Definition 6.4.2. Let $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$. We say that M is HN-type if it verifies one of the conditions above.

6.5 The algorithm

The aim of this subsection is to prove that every Kisin module is isogenous to a HN-type Kisin module.

We would like to define the last submodule in the Harder Narasimhan filtration of $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$ as

$$\theta M = \ker(M \to M^{\min}) = \varprojlim_i (\ker(M_i \twoheadrightarrow M_i^{\min}))$$

for

$$M^{\min} = \varprojlim_i M_i^{\min}$$

and $M_i^{\min}$ defined as above. The problem is that we cannot ensure that $M/\theta M$ is a Kisin module. In the next proposition, we find a Kisin module M' isogenous to M and such that $M'/\theta M$ is also a Kisin module.

Proposition 6.5.1. *Let $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$. Then:*

1. *The inclusion $\theta M \to M$ factors (not canonically) through*

$$\theta M \to M' \to M$$

where θM, $M' \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$, $M'/\theta M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$ is either zero or semi-stable of slope $\mu_{\min}(M)$ and $M' \to M$ is an isogeny with cokernel in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$ which is semi-stable of slope $\mu_{\min}(M)$.

2. *If $\theta M \neq 0$, then $\mu_{\min}(\theta M) > \mu_{\min}(M)$.*

3. *If M is effective, then θM is effective.*

4. *We have*

$$\theta(M(r)) = (\theta M)(r)$$

for every $r \geq 1$.

Proof. 1. Let $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{fr}}$. We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker \pi_{k-1} \arrow[r] \arrow[d, "{[p]}"] & M_k^{\min} \arrow[r, "\pi_{k-1}"] \arrow[d, "{[p]}"] & M_{k-1}^{\min} \arrow[r] \arrow[d, "{[p]}"] & 0 \\
0 \arrow[r] & \ker \pi_k \arrow[r] & M_{k+1}^{\min} \arrow[r, "\pi_k"] & M_k^{\min} \arrow[r] & 0
\end{tikzcd}$$

where the morphism π_k is the reduction modulo p^k and $[p]$ is multiplication by p, coming from

$$M/p^k M \xrightarrow{p} pM/p^{k+1}M \hookrightarrow M/p^{k+1}M.$$

Multiplication by p restricted to the kernels is surjective. Indeed, let $x \in \ker \pi_k = p^k M_{k+1}^{\min}$, so there exist an $y \in M_{k+1}^{\min}$ such that $x = p^k y$. Consider $y' = \pi_k(y)$, and $x' = p^{k-1}y'$ which is in $p^{k-1}M_k^{\min} = \ker \pi_{k-1}$. Since $([p] \circ \pi_k)(y) = py$, we get that $[p]\,(x') = x$.

This way, for

$$a_k = \mathrm{rank}(\ker \pi_k),$$

we get a decreasing family of integers $(a_k)_{k\geq 1}$ that will stabilize at some point. Let $k_0 \geq 1$ such that $a_{k+k_0} = a_{k_0}$ for all $k \geq 0$. This means that $\ker \pi_{k+k_0} \simeq \ker \pi_{k_0}$ for all $k \geq 0$, a fact that we will use later. In the case that $a_{k_0} = 0$ we get that $M_k^{\min} \simeq M_{k_0}^{\min}$ for all $k \geq k_0$ and thus $M^{\min} = M_{k_0}^{\min}$.

Let $K_i = \ker(M_{i+k_0}^{\min} \twoheadrightarrow M_{k_0}^{\min})$ for such a $k_0 \geq 1$. Thus,

$$K_i = p^{k_0} M_{i+k_0}^{\min} = \mathrm{im}(M_i^{\min} \to M_{i+k_0}^{\min})$$

by Lemma 6.2.2. We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_{i+1} \arrow[r] \arrow[d, two heads, "\pi_{i+k_0}"] & M_{i+1+k_0}^{\min} \arrow[r] \arrow[d, two heads, "\pi_{i+k_0}"] & M_{k_0}^{\min} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_i \arrow[r] & M_{i+k_0}^{\min} \arrow[r] & M_{k_0}^{\min} \arrow[r] & 0
\end{tikzcd}$$

so $(K_i)_{i\geq 1}$ is Mittag-Leffler surjective and we can prove by induction that $\mathrm{rank}(K_i) = i\,\mathrm{rank}(\ker \pi_{k_0})$. We would like to prove that $K = \varprojlim K_i$ is a Kisin module using Proposition 3.1.6. It only remains to show that $K/p^nK \simeq K_n$ for $n \geq 1$, since K_n has no u-torsion and the ranks are compatible.

First, we remark that for a fixed $n \geq 1$ and for all $i \geq n$, we have

$$\begin{aligned} \ker(K_i \twoheadrightarrow K_n) &= \ker(M^{\min}_{i+k_0} \twoheadrightarrow M^{\min}_{n+k_0}) \\ &= p^{n+k_0} M^{\min}_{i+k_0} \\ &= p^n K_i \end{aligned}$$

since $K_i = p^{k_0} M^{\min}_{i+k_0}$ for every $i \geq 1$, so $K_n \simeq K_i/p^nK_i$. Moreover, using the properties of the last quotient studied in 6.2.2, we have

$$K_i = \ker(M^{\min}_{i+k_0} \to M_{k_0}) = p^{k_0} M^{\min}_{i+k_0} \simeq M^{\min}_i \simeq p^{n+k_0} M^{\min}_{i+n+k_0} \simeq p^n K_{i+n}$$

for every $i, n \geq 1$, so there are exact sequences

$$0 \to K_n \xrightarrow{\times p^i} K_{i+n} \to K_i \to 0,$$

for every $i, n \geq 1$, which gives us

$$\varprojlim_i K_{i+n} \simeq \varprojlim_i K_i$$

and

$$0 \to K \xrightarrow{\times p^n} K \to K_n \to 0,$$

thus

$$K/p^nK = K_n$$

for all $n \geq 1$, hence K is free. It is semi-stable of slope $\mu_{\min}(M)$ since so is each $K_n = K/p^nK$ being the kernel of $M^{\min}_{n+k_0} \to M^{\min}_{k_0}$, both semi-stable of slope $\mu_{\min}(M)$.

Now, for every $i \geq k_0$, let $M'_i = \ker(M_i \twoheadrightarrow M^{\min}_{k_0})$ and $(\theta M)_i = \ker(M_i \to M^{\min}_i)$. There is a diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (\theta M)_i \arrow[r] \arrow[d, equal] & M'_i \arrow[r] \arrow[d] & K_{i-k_0} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\theta M)_i \arrow[r] & M_i \arrow[r] \arrow[d] & M^{\min}_i \arrow[r] \arrow[d] & 0 \\
 & & M^{\min}_{k_0} \arrow[r, equal] \arrow[d] & M^{\min}_{k_0} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where $(M_i)_{i\geq k_0}$, $(M'_i)_{i\geq k_0}$ and $(K_{i-k_0})_{i\geq k_0}$ are Mittag-Leffler surjective. Moreover, we have seen in Lemma 6.2.2 that $((\theta M)_i)_{i\geq 1}$ is also Mittag-Leffler surjective. Then, taking limits in the above diagram, we obtain

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \theta M \arrow[r] \arrow[d, equal] & M' \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \theta M \arrow[r] & M \arrow[r] \arrow[d] & M^{\min} \arrow[r] \arrow[d] & 0 \\
 & & M^{\min}_{k_0} \arrow[r, equal] \arrow[d] & M^{\min}_{k_0} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Using Proposition 3.1.5 on the isogeny $0 \to M' \to M \to M^{\min}_{k_0} \to 0$, we have that $M' \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$ and the same proposition for $0 \to \theta M \to M' \to K \to 0$ tells us that $\theta M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$, since we have already proved that $K \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$. Moreover, $M'/\theta M \simeq K \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$ and it is semi-stable of slope $\mu_{\min}(M)$.

2. If Q is a nonzero quotient of θM in $\mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{t}}$, there exists $i \geq 1$ such that Q is a quotient of $(\theta M)_i$, therefore
$$\mu(Q) \geq \mu(\theta M_i) > \mu_{\min}(M_i) \geq \mu_{\min}(M).$$
Thus, if $\theta M \neq 0$,
$$\mu_{\min}(\theta M) = \mu_{\min}(\theta M/p\theta M) > \mu_{\min}(M).$$

3. By Lemma 3.2.1, we have that M' is effective since it is a submodule of M such that $M/M' = M^{\min}_{k_0} \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{t}}$ has no $\mathfrak{m}^{\infty}$-torsion. Now, it suffices to show that $M'/\theta M$ has no $\mathfrak{m}^{\infty}$-torsion, but $M'/\theta M = K \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$, so θM is effective.

4. We have
$$\begin{aligned}
\theta(M(r)) &= \ker((M, E^r\varphi_M) \to \varprojlim_n ((M, E^r\varphi_M)_n)^{\min}) \\
&= \ker((M, E^r\varphi_M) \to \varprojlim_n (M_n, E^r\varphi_{M_n})^{\min}) \\
&= \ker((M, E^r\varphi_M) \to \varprojlim_n (M^{\min}_n, E^r\varphi_{M^{\min}_n})) \\
&= (\ker(M \to \varprojlim_n M^{\min}_n), E^r\varphi_M) \\
&= (\theta M)(r)
\end{aligned}$$
where the third equality is verified since twisting by E^r does not change the Harder-Narasimhan filtration on M_n, it just change the slopes.

□

We get a family of Kisin modules $(M^{(i)})_{i\geq 0}$, where $M^{(i)} = \theta^{(i)}M$, together with inclusions
$$\iota_i \ : \ M^{(i+1)} \hookrightarrow M^{(i)}$$
verifying the properties of last proposition. We say that this algorithm stops when $M^{(i)} = 0$ for some $i \geq 0$.

Proposition 6.5.2. *Let $M \in \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$. The algorithm above stops in a finite number of steps.*

Proof. We can reduce to the case when M is an effective Kisin module for all $i \geq 1$. Suppose that the algorithm does not stop, then, by point (2) in last proposition $\mu_i = \mu_{\min}(\theta^{(i)}M)$ is strictly increasing for $i \geq 1$. However
$$\mu_{\min}(\theta^{(i)}M) = \frac{\deg(\theta^{(i)}M/p\theta^{(i)}M)^{\min}}{\mathrm{rank}(\theta^{(i)}M/p\theta^{(i)}M)^{\min}} \in \frac{-\frac{1}{e}\mathbb{N}}{\mathrm{rank}(\theta^{(i)}M)} \subset -\frac{1}{e\cdot \mathrm{rank}\, M!}\mathbb{N}$$

where $e = \deg E$, since $(\theta^{(i)}M/p\theta^{(i)}M)^{\min}$ is effective by hypothesis. This is a discrete set, so we have a contradiction.

□

Theorem 6.5.3. *Every Kisin module is isogenuous to a HN-type Kisin module.*

Proof. Suppose we have three families $(M^{(i)})_{0\leq i\leq r}$, $(M'^{(i)})_{0\leq i\leq r-1}$ and $(Q^{(i)})_{0\leq i\leq r-1}$ of Kisin modules verifying

- $M^{(0)} = M$,
- There is an isogeny between $M'^{(i)}$ and $M^{(i)}$ for all $0 \leq i \leq r-1$,
- For $0 \leq i \leq r-1$ we have exact sequences
$$0 \to M^{(i+1)} \to M'^{(i)} \to Q^{(i)} \to 0,$$
- For $0 \leq i \leq r-1$, the Kisin module $Q^{(i)}$ is semi-stable of slope μ_i and we have
$$\mu_0 < \ldots < \mu_{r-1}.$$

Then we can prove by induction on r that M is isogenuous to a HN-type Kisin module. Indeed, suppose that $f : M^{(2)} \to N$ is an isogeny with N a HN-type Kisin module. So we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & M^{(2)} \arrow[r] \arrow[d, "f"] & M'^{(1)} \arrow[r] \arrow[d] & Q^{(1)} \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & N \arrow[r] & N' \arrow[r] & Q^{(1)} \arrow[r] & 0
\end{tikzcd}$$

where N' is the pushout of $M^{(2)} \to M'^{(1)}$ and $M^{(2)} \to N$. The isogeny f gives us an isogeny between $M'^{(1)}$ and N' and so an isogeny between $M = M^{(1)}$ and N'. If we put $\mathrm{Fil}_r N' = N'$, $\mathrm{Fil}_{r-1} N' = N$ and $\mathrm{Fil}_i N' = \iota(\mathrm{Fil}_i N)$ for all $1 \leq i \leq r-1$, where $\mathrm{Fil}_\bullet N$ is the HN flag defined on N, we obtain a HN-type Kisin module isogenuous to M.

It remains to show that such three families of modules exist for any Kisin module M, but this is equivalent to show that the algorithm given before stops, since that way we can take $M^{(i)} = \theta^{(i)}M$, $M'^{(i)}$ corresponds to the M' given by 6.5.1 for $M^{(i)}$ and $Q^{(i)} = M'^{(i)}/\theta^{(i+1)}M$. By proposition 6.5.2, the algorithm stops.

□

As a consequence, we have

Corollary 6.5.4. *Notations as above, we have*

$$\mathbf{t}_{\mathrm{F},\infty}(M) = \mathbf{t}_{\mathrm{F},\circ}(M) \quad \textit{in} \quad \mathbb{Q}^r_{\geq}$$

To finish this section, the following lemma gives us all the inequalities and equalities between the polygons presented before.

Lemma 6.5.5. *Let M be a Kisin module. Then, we have inequalities of types*

$$\mathbf{t}_{\mathrm{F},\circ}(M) = \mathbf{t}_{\mathrm{F},\infty}(M) \leq \mathbf{t}_{\mathrm{F},n}(M) \leq \mathbf{t}_{\mathrm{F},1}(M) \leq \mathbf{t}_{\mathrm{H},\infty}(M/pM) \leq \mathbf{t}_{\mathrm{H},1}(M/pM) \leq \mathbf{t}_{\mathrm{H}}(M)$$

for every $n, m \geq 1$.

Proof. Last proposition together with 6.4.3 gives us the first equality. The inequality $\mathbf{t}_{\mathrm{F},\infty} \leq \mathbf{t}_{\mathrm{F},n}$ is given by 6.3.1 and 6.3.2. The inequality $\mathbf{t}_{\mathrm{F},1}(M) \leq \mathbf{t}_{\mathrm{H},1}(M/pM)$ is given in 4.3.7 and the last inequality is given by 6.1.1.

□

Chapter 7

Crystalline representations with G-structure

7.1 The partially ordered commutative monoid $\mathbb{C}^\Gamma(G)$

For a reductive group G over a base scheme S and a totally ordered commutative group

$$\Gamma = (\Gamma, +, \leq) \neq 0,$$

Cornut constructs a sequence of S-schemes

$$\mathbb{G}^\Gamma(G) \xrightarrow{\text{Fil}} \mathbb{F}^\Gamma(G) \xrightarrow{t} \mathbb{C}^\Gamma(G)$$

in [11, 2]. The construction is compatible with base change on S and covariantly functorial in G and Γ. On $\mathbb{G}^\Gamma(G)$ there is an involution ι, compatible with the order and addition, inducing an involution on $\mathbb{C}^\Gamma(G)$. Moreover, $\mathbb{C}^\Gamma(G)$ is an étale, partially ordered commutative monoid over S. The partial order is the weak dominance order of [11, 2.2.12]. It is compatible with the functoriality in G and Γ, but the monoid structure is only compatible with the functoriality in Γ. The functors represented by these schemes are related to Γ-graduations and Γ-filtrations on various fiber functors, as explained in [11, 3] – see also section 7.2.1 below.

Suppose that S is a connected normal scheme, Γ is (uniquely) divisible and T is a connected S-scheme. Then $\mathbb{C}^\Gamma(G)(S) \to \mathbb{C}^\Gamma(G)(T)$ is a monomorphism, with a canonical additive retraction $\sharp : \mathbb{C}^\Gamma(G)(T) \to \mathbb{C}^\Gamma(G)(S)$, functorial in the connected S-scheme T, by [11, 3.11.8]. If $T = s$ is a geometric point of S, corresponding to the fundamental group $\pi(S, s)$, then $\pi(S, s)$ acts on $\mathbb{C}^\Gamma(G)(s)$ with finite orbits, $\mathbb{C}^\Gamma(G)(S)$ is the fixed point set of this action and the retraction $\sharp : \mathbb{C}^\Gamma(G)(s) \to \mathbb{C}^\Gamma(G)(S)$ maps x to the average of its orbit in the (uniquely) divisible monoid $\mathbb{C}^\Gamma(G)(s)$.

If $\text{Spec}(R)$ is an affine S-scheme, we set

$$\begin{aligned}
\mathbf{G}^\Gamma(G_R) &= \mathbb{G}^\Gamma(G)(R) \\
\mathbf{F}^\Gamma(G_R) &= \mathbb{F}^\Gamma(G)(R) \\
\mathbf{C}^\Gamma(G_R) &= \text{image of } \mathbb{F}^\Gamma(G)(R) \to \mathbb{C}^\Gamma(G)(R)
\end{aligned}$$

Suppose that R is local, as in [11, 4.1]. Then $\mathbf{C}^\Gamma(G_R)$ is a (partially ordered) commutative submonoid of $\mathbb{C}^\Gamma(G)(R)$. Moreover,

$$G_R \text{ is quasi-split} \iff \mathbf{C}^\Gamma(G_R) = \mathbb{C}^\Gamma(G)(R).$$

This follows from [11, 4.1.8] and [1, XXVI 3.8]. If R is a valuation ring with fraction field K, then $\mathbf{C}^\Gamma(G_R) = \mathbf{C}^\Gamma(G_K)$ by [11, 4.1.18]. If R is Henselian with residue field k, then $\mathbf{C}^\Gamma(G_R) = \mathbf{C}^\Gamma(G_k)$ by [11, 4.1.17], and G_R is quasi-split if and only if G_k is quasi-split. This is for instance the case if k is finite (by Lang's theorem) or algebraically closed. If G_R is split, then $\mathbb{C}^\Gamma(G_R)$ is the constant partially ordered commutative monoid $\mathbf{C}^\Gamma(G_R)_{\mathrm{Spec}(R)}$ ([11, 2.2.11]). Then for any morphism of local S-algebras $R \to R'$, the base change map $\mathbf{C}^\Gamma(G_R) \to \mathbf{C}^\Gamma(G_{R'})$ is an isomorphism. If $G = GL_n$, then $\mathbf{C}^\Gamma(G_R) = \Gamma^n_{\geq}$.

In the sequel, we will have a reductive group G over $\mathbb{Z}_p$. Thus G is quasi-split over $\mathbb{Z}_p$ and split over a finite unramified extension $\mathbb{Z}_{p^N}$ of $\mathbb{Z}_p$. In particular, applying $\mathbf{C}^\Gamma(G_-)$ to the diagram

$$\begin{tikzcd}
 & W(\overline{\mathbb{F}}_p) \arrow[r, two heads] \arrow[d] & \overline{\mathbb{F}}_p \arrow[d] \\
K_0 \arrow[dd] & W(\mathbb{F}) \arrow[l] \arrow[r, two heads] \arrow[d] & \mathbb{F} \arrow[d] \\
 & \mathfrak{S} \arrow[r, two heads] \arrow[d] & \mathbb{F}[[u]] \\
K & \mathfrak{S}_{(E)} \arrow[l] &
\end{tikzcd}$$

all the arrows become isomorphisms. We will drop the index relative to the ring and note it simply by $\mathbf{C}^\Gamma(G)$ (this is an abuse of notation, but we will denote by $\mathbf{C}^\Gamma(G_{\mathbb{Z}_p})$ when we want to make clear that we are considering the group defined over $\mathbb{Z}_p$). The Frobenius σ acts on it, with fixed point set $\mathbf{C}^\Gamma(G_{\mathbb{Z}_p}) = \mathbf{C}^\Gamma(G_{\mathbb{F}_p})$ and $\sigma^N \equiv \mathrm{Id}$ on $\mathbf{C}^\Gamma(G)$. The partial order on $\mathbf{C}^\Gamma(G)$ then has the following Tannakian caracterisation, which can be found in [11, 3.11.8], see also [14, 9.4.2].

Proposition 7.1.1. *Suppose Γ is divisible and G is defined over $\mathbb{Z}_p$. For every connected $\mathbb{Z}_p$-scheme* $\mathrm{Spec}\, R$ *and $\mathbf{t}_1, \mathbf{t}_2 \in \mathbf{C}^\Gamma(G_R)$, consider the following conditions:*

1. $\mathbf{t}_1 \leq \mathbf{t}_2$ *in* $\mathbf{C}^\Gamma(G_R)$,
2. *For every* $\tau \in \mathrm{Rep}_R\, G$, $\mathbf{t}_1(\tau) \leq \mathbf{t}_2(\tau)$ *in* $\mathbf{C}^\Gamma(G_R)(\mathrm{GL}(V(\tau)) = \Gamma^{\mathrm{rank}_R(\tau)}_{\geq}$,
3. $\mathbf{t}_1^\# \leq \mathbf{t}_2^\#$ *in* $\mathbf{C}^\Gamma(G_{\mathbb{Z}_p})$,
4. *For every* $\tau \in \mathrm{Rep}_{\mathbb{Z}_p}\, G$, $\mathbf{t}_1(\tau_R) \leq \mathbf{t}_2(\tau_R)$ *in* $\mathbf{C}^\Gamma(G)(\mathrm{GL}(V(\tau_R)) = \Gamma^{\mathrm{rank}_{\mathbb{Z}_p}(\tau)}_{\geq}$.

Then, we have (1) $\iff$ (2), (4) $\Longrightarrow$ (3) *and we have* (3) $\iff$ (4) *if* $\mathbf{t}_1 = \mathbf{t}_1^\#$.

7.2 Fiber functors

In this section, let G be a reductive group defined over $\mathcal{O}$, for $\mathcal{O} \in \{\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{F}_p\}$. For a scheme X, we define the category Bun_X as the category of finite locally free sheaves of $\mathcal{O}_X$-modules. In particular, for $X = \mathrm{Spec}\, R$, it corresponds to the category of finite projective R-modules and we denote this category by Bun_R. We define the category $\mathrm{Rep}_{\mathcal{O}}\, G$ as the category of algebraic representations of G on finite free $\mathcal{O}$-modules. We say that a functor

$$\omega_X \ : \mathrm{Rep}_{\mathcal{O}}\, G \to \mathrm{Bun}_X$$

is a fiber functor when ω_X is an exact $\otimes$-functor, for a scheme over $\mathrm{Spec}\, \mathcal{O}$. We denote by $\mathrm{Aut}^\otimes(\omega_X)$ the group of $\otimes$-automorphisms of ω_X.

7.2.1 Graduations and filtrations

Definition 7.2.1. Let Γ be a subring of $\mathbb{R}$. A Γ-graduation $\mathcal{G}$ on ω_R is a factorization

$$\mathrm{Rep}_{\mathcal{O}} G \xrightarrow{\mathcal{G}} \mathrm{Gr}^{\Gamma} \mathrm{Bun}_R \xrightarrow{f} \mathrm{Bun}_R$$

where f is the forgetful functor and $\mathrm{Gr}^{\Gamma} \mathrm{Bun}_R$ is the category of finite projective R-modules endowed with a Γ-graduation, such that if $\mathcal{G}^{\gamma}$ is the γ-component of $\mathcal{G}$,

(G1) For every $\tau_1, \tau_2 \in \mathrm{Rep}_{\mathcal{O}} G$ and $\gamma \in \Gamma$, we have

$$\mathcal{G}^{\gamma}(\tau_1 \otimes \tau_2) = \oplus_{\gamma_1+\gamma_2=\gamma} \mathcal{G}^{\gamma_1}(\tau_1) \otimes \mathcal{G}^{\gamma_2}(\tau_2).$$

(G2) For the trivial representation τ of G on $L \in \mathrm{Bun}_{\mathcal{O}}$, we have

$$\mathcal{G}^0(\tau) = \omega_R(\tau) \quad \text{and} \quad \mathcal{G}^{\gamma}(\tau) = 0 \text{ if } \gamma \neq 0.$$

A Γ-filtration $\mathcal{G}$ on ω_R is a factorization

$$\mathrm{Rep}_{\mathcal{O}} G \xrightarrow{\mathcal{G}} \mathrm{Fil}^{\Gamma} \mathrm{Bun}_R \xrightarrow{f} \mathrm{Bun}_R$$

where f is the forgetful functor and $\mathrm{Fil}^{\Gamma} \mathrm{Bun}_R$ is the category of finite projective R-modules endowed with a Γ-filtration by direct summands as defined in [11, 3.3.1], such that if $\mathcal{F}^{\gamma}$ is the γ-component of $\mathcal{F}$,

(F1) For every $\tau_1, \tau_2 \in \mathrm{Rep}_{\mathcal{O}} G$ and $\gamma \in \Gamma$, we have

$$\mathcal{F}^{\gamma}(\tau_1 \otimes \tau_2) = \sum_{\gamma_1+\gamma_2=\gamma} \mathcal{F}^{\gamma_1}(\tau_1) \otimes \mathcal{F}^{\gamma_2}(\tau_2).$$

(F2) For the trivial representation τ of G on $L \in \mathrm{Bun}_{\mathcal{O}}$, we have

$$\mathcal{F}^{\gamma}(\tau) = \omega_R(\tau) \text{ if } \gamma \leq 0 \quad \text{and} \quad \mathcal{F}^{\gamma}(\tau) = 0 \text{ if } \gamma > 0.$$

(F3) For every $\gamma \in \Gamma$, $\mathcal{F}^{\gamma}$ is exact.

We denote by $\mathbf{G}^{\Gamma}(\omega_R)$ and $\mathbf{F}^{\Gamma}(\omega_R)$ the set of Γ-graduations and Γ-filtrations on ω_R, respectively.

For every $\mathcal{O}$-algebra R and every Γ subring of $\mathbb{R}$, there is an exact $\otimes$-functor

$$\mathrm{Fil} \ : \ \mathrm{Gr}^{\Gamma} \mathrm{Bun}_R \to \mathrm{Fil}^{\Gamma} \mathrm{Bun}_R$$

sending a graduation $\mathcal{G}$ to the filtration $\mathrm{Fil}(\mathcal{G}) = \mathcal{F}$ defined by $\mathcal{F}^{\geq\gamma} = \oplus_{\eta\geq\gamma} \mathcal{G}^{\eta}$. It induces an $\mathrm{Aut}^{\otimes}(\omega_R)$-equivariant map

$$\mathrm{Fil} \ : \ \mathbf{G}^{\Gamma}(\omega_R) \to \mathbf{F}^{\Gamma}(\omega_R).$$

We define the types on ω_R by

$$\mathbf{C}^{\Gamma}(\omega_R) := \mathrm{Aut}^{\otimes}(\omega_R) \backslash \mathbf{F}^{\Gamma}(\omega_R).$$

The three sets $\mathbf{G}^{\Gamma}(\omega_R)$, $\mathbf{F}^{\Gamma}(\omega_R)$ and $\mathbf{C}^{\Gamma}(\omega_R)$ are compatible with change of the $\mathcal{O}$-algebra R, so we can define three presheaves $\mathbf{G}^{\Gamma}(\omega)$, $\mathbf{F}^{\Gamma}(\omega)$ and $\mathbf{C}^{\Gamma}(\omega)$ together with maps

$$\mathbf{G}^{\Gamma}(\omega_R) \xrightarrow{\mathrm{Fil}} \mathbf{F}^{\Gamma}(\omega_R) \xrightarrow{\mathbf{t}} \mathbf{C}^{\Gamma}(\omega_R).$$

Now, consider the standard fiber functor

$$\omega_{G,R} \ : \ \mathrm{Rep}_{\mathcal{O}} G \to \mathrm{Bun}_R$$

given by the composition of the forgetting functor $\omega_G : \mathrm{Rep}_{\mathcal{O}} G \to \mathrm{Bun}_{\mathcal{O}}$ with the tensorization by R. This fiber functor is called the trivial fiber functor. The next proposition gives a criterion for the existence of a $\otimes$-isomorphism $\omega_{G,R} \simeq \omega_R$.

Proposition 7.2.1. *Let R be a local strictly henselian and faithfully flat $\mathcal{O}$-algebra. Then, any exact and faithful $\otimes$-functor*

$$\omega_R \ : \ \mathrm{Rep}_{\mathcal{O}}\, G \to \mathrm{Bun}_R$$

is $\otimes$-isomorphic to the standard fiber functor $\omega_{G,R}$.

Proof. We can use the results in [7] since our base scheme $\mathrm{Spec}(\mathcal{O})$ is a Dedekind scheme for the cases considered. In [7], Broshi proved that there is an equivalence of categories between exact and faithful $\otimes$-functors taking values in Bun_R and G-torsors over $\mathrm{Spec}(R)$. Now, G-torsors over $\mathrm{Spec}(R)$ are classified by the étale cohomology group $H^1(\mathrm{Spec}(R), G)$. Using [1, Exposé XXIV, Proposition 8.1], we have that

$$H^1(\mathrm{Spec}(R), G) \simeq H^1(\mathbb{F}, G_{\mathbb{F}})$$

under our hypothesis. Then, in [46, Theorem 1.9], we see that

$$H^1(\mathbb{F}, G_{\mathbb{F}}) = 0,$$

thus, all G-torsors over R are trivial, and $\omega_R \simeq \omega_{G,R}$ as wanted, by the equivalence of categories given by Broshi.

□

The choice of a $\otimes$-isomorphism $\omega_R \simeq \omega_{G,R}$ (when it exists) yields isomorphisms

$$\begin{array}{rcrcl} \mathbf{G}^{\Gamma}(\omega_R) & \simeq & \mathbf{G}^{\Gamma}(\omega_{G,R}) & =: & \mathbf{G}^{\Gamma}(G_R) \\ \mathbf{F}^{\Gamma}(\omega_R) & \simeq & \mathbf{F}^{\Gamma}(\omega_{G,R}) & =: & \mathbf{F}^{\Gamma}(G_R) \\ \mathbf{C}^{\Gamma}(\omega_R) & \simeq & \mathbf{C}^{\Gamma}(\omega_{G,R}) & =: & \mathbf{C}^{\Gamma}(G_R) \end{array}$$

so, in this case, we can use the results in [11]. The first two isomorphisms are not canonical, since they depend on the chosen isomorphism $\omega \simeq \omega_{G,R}$. Let η, η' be two different isomorphisms $\omega \simeq \omega_{G,R}$, then we have $\eta' = \kappa\eta$ for some $\kappa \in \mathrm{Aut}^{\otimes}(\omega_{G,R}) = G(R)$. Then, the induced isomorphisms

$$\eta, \eta' \ : \ \mathbf{C}^{\Gamma}(\omega_R) = \mathrm{Aut}^{\otimes}(\omega_R)\backslash\mathbf{G}^{\Gamma}(\omega_R) \simeq G(R)\backslash\mathbf{G}^{\Gamma}(G_R) = \mathbf{C}^{\Gamma}(G_R)$$

are equal, so $\mathbf{C}^{\Gamma}(\omega)$ is canonically isomorphic to $\mathbf{C}^{\Gamma}(G_R)$.

The definition of a filtration on a fiber functor implies that the filtration is compatible with exterior and symmetric powers. The next proposition shows that the converse is also sometimes true.

Proposition 7.2.2. *Let L be a field which is an $\mathcal{O}$-algebra. Suppose that a fiber functor*

$$\omega_L \ : \ \mathrm{Rep}_{\mathcal{O}}\, G \to \mathrm{Vect}_L$$

admits a factorization through an additive $\otimes$-functor

$$\mathcal{F} \ : \ \mathrm{Rep}_{\mathcal{O}}\, G \to \mathrm{Fil}_L^{\Gamma}$$

which is compatible with Λ and Sym. *Then $\mathcal{F}$ is exact, thus $\mathcal{F}$ is a filtration of ω_L.*

Proof. We say that $\mathcal{F}$ is exact if it verifies the following equivalent conditions. For every exact sequence

$$0 \to \tau_1 \to \tau_2 \xrightarrow{\pi} \tau_3 \to 0$$

with $\tau_1, \tau_2, \tau_3 \in \mathrm{Rep}_{\mathcal{O}}\, G$:

- The sequence
$$0 \to \mathcal{F}^\gamma(\tau_1) \to \mathcal{F}^\gamma(\tau_2) \to \mathcal{F}^\gamma(\tau_3) \to 0$$
is exact, for every $\gamma \in \Gamma$.
- We have $\mathcal{F}^\gamma(\tau_1) = \mathcal{F}^\gamma(\tau_2) \cap \omega_L(\tau_1)$, and $\mathcal{F}^\gamma(\tau_3) = \pi(\mathcal{F}^\gamma(\tau_2))$, for every $\gamma \in \Gamma$.
- The functor $\mathcal{F}$ transforms strict monomorphisms (respectively strict epimorphisms) into strict monomorphisms (respectively, strict epimorphisms).

We will use the last characterization of an exact functor. By [16, II, Proposition 1.9], an additive $\otimes$-functor between two rigid categories is also compatible with duality, i.e. $\mathcal{F}(\tau)^\vee = \mathcal{F}(\tau^\vee)$. This means that we only need to check that $\mathcal{F}$ transforms strict monomorphisms into strict monomorphisms.

Let $\tau_1 \hookrightarrow \tau_2$ be a strict monomorphism. Set $\mathcal{G}^\gamma(\tau_1) = \mathcal{F}^\gamma(\tau_2) \cap \omega_L(\tau_1)$. Since $\mathcal{F}$ is a functor, we have a commutative diagram in Fil_L^Γ

$$\begin{tikzcd} \mathcal{F}(\tau_1) \arrow[rr, "\iota"] \arrow[dr, "i_1"] & & \mathcal{F}(\tau_2) \\ & \mathcal{G}(\tau_1) \arrow[ur, "i_2"] & \end{tikzcd}$$

where i_1 is a mono-epi and i_2 is a strict mono. We want to show that $\mathcal{F}(\tau_1) = \mathcal{G}(\tau_1)$, i.e. that i_1 is an isomorphism. First, we prove that we can reduce to the case where $\mathrm{rank}_{\mathcal{O}}\, \tau_1 = 1$. Indeed, let $d = \mathrm{rank}_{\mathcal{O}}\, \tau_1$ and consider $\Lambda^d \tau_1 \hookrightarrow \Lambda^d \tau_2$, applying $\mathcal{F}$, and since $\mathcal{F}$ is compatible with exterior powers, we get a diagram

$$\begin{tikzcd} \Lambda^d \mathcal{F}(\tau_1) \arrow[rr, "\Lambda^d \iota"] \arrow[dr, "\Lambda^d i_1"] & & \Lambda^d \mathcal{F}(\tau_2) \\ & \Lambda^d \mathcal{G}(\tau_1) \arrow[ur, "\Lambda^d i_2"] & \end{tikzcd}$$

where $\Lambda^d i_1$ is again a mono-epi by the properties of the exterior power, and by a direct calculation considering a splitting, we can check that $\Lambda^d i_2$ is a strict monomorphism. Since

$$\mu(\Lambda^d \mathcal{F}(\tau_1)) = d\mu(\mathcal{F}(\tau_1)) \quad \text{and} \quad \mu(\Lambda^d \mathcal{G}(\tau_1)) = d\mu(\mathcal{G}(\tau_1))$$

we obtain that i_1 is an isomorphism if and only if $\Lambda^d i_1$ is an isomorphism.

Now, we have $\tau_1 \hookrightarrow \tau_2$ where τ_1 is a character of G. Since $\mathcal{F}$ is compatible with tensor products and duality, twisting the initial sequence by the inverse of that character, we may assume that $\tau_1 = \mathbf{1}$.

Let thus $\mathbf{1} \hookrightarrow \tau$ be a strict monomorphism. The generalization of Haboush's theorem given by Seshadri in [44, Theorem 1] tells us that there is an $r \geq 1$ such that $\mathbf{1} = \mathrm{Sym}^r \mathbf{1} \hookrightarrow \mathrm{Sym}^r(\tau)$ is split. This implies that $\mathcal{F}(\mathbf{1}) \to \mathcal{F}(\mathrm{Sym}^r(\tau)) = \mathrm{Sym}^r(\mathcal{F}(\tau))$ is a strict monomorphism. Since $\mathrm{Sym}^r i_2$ is also a strict mono, the mono-epi

$$\mathrm{Sym}^r i_1 \ : \ \mathrm{Sym}^r \mathcal{F}(\mathbf{1}) \to \mathrm{Sym}^r \mathcal{G}(\mathbf{1})$$

is an isomorphism. Since

$$\mu(\mathrm{Sym}^r \mathcal{F}(\tau_1)) = r\mu(\mathrm{Sym}^r(\tau_1)) \quad \text{and} \quad \mu(\mathrm{Sym}^r \mathcal{G}(\tau_1)) = r\mu(\mathcal{G}(\tau_1))$$

we have $\mathcal{F}(\mathbf{1}) \simeq \mathcal{G}(\mathbf{1})$.

$\square$

Remark 14. In the case where $\mathcal{O} = \mathbb{Q}_p$, every exact sequence in $\mathrm{Rep}_{\mathcal{O}} G$ is split, in which case the exactness of $\mathcal{F}$ follows from its additivity.

7.2.2 Lattices

Let $\mathcal{O}_L$ be an $\mathcal{O}$-algebra which is a discrete valuation ring, L its fraction field, π_L a uniformizer and l its residue field. For G a reductive group over $\mathcal{O}$, let

$$V \ : \ \mathrm{Rep}_{\mathcal{O}} G \to \mathrm{Bun}_L$$

be a $\otimes$-functor. We denote by $\mathcal{L}'(V)$ the set of $\otimes$-functors

$$x \ : \ \mathrm{Rep}_{\mathcal{O}} G \to \mathrm{Bun}_{\mathcal{O}_L}$$

such that

1. For every $\tau \in \mathrm{Rep}_{\mathcal{O}} G$, its image $x(\tau)$ is an $\mathcal{O}_L$-lattice in $V(\tau)$,
2. For every $\tau_1 \xrightarrow{f} \tau_2$, the morphism $x(f) : x(\tau_1) \to x(\tau_2)$ is induced by $V(f) : V(\tau_1) \to V(\tau_2)$.

Then, there is a canonical $\otimes$-isomorphism $x_L \simeq V$, where

$$x_L \ : \ \mathrm{Rep}_{\mathcal{O}} G \xrightarrow{x} \mathrm{Bun}_{\mathcal{O}_L} \xrightarrow{\otimes L} \mathrm{Bun}_L \, .$$

Conversely, to give an element of $\mathcal{L}'(V)$ amounts to give an isomorphism class of pairs (x', η) where $x' : \mathrm{Rep}_{\mathcal{O}} G \to \mathrm{Bun}_{\mathcal{O}_L}$ is a $\otimes$-functor and η is an isomorphism of $\otimes$-functors $x'_L \simeq V$: we associate to (x', η) the functor $x(\tau) = \eta_\tau(x'(\tau))$.

We denote by $\mathcal{L}^{\mathrm{ex}}(V) \subset \mathcal{L}'(V)$ the subset of lattices x which are exact. Then $\mathcal{L}^{\mathrm{ex}}(V) \neq \emptyset$ implies that V is exact. We suppose V exact henceforth, i.e. $V : \mathrm{Rep}_{\mathcal{O}} G \to \mathrm{Bun}_L$ is a fiber functor. Denote by $\mathcal{L}(V) \subset \mathcal{L}^{\mathrm{ex}}(V)$ the subset of lattices x which are isomorphic to the standard fiber functor $\omega_{G,\mathcal{O}_L}$ (they are exact since the latter is exact). Then, if $\mathcal{L}(V) \neq \emptyset$, we have $V \simeq \omega_{G,L}$ and, conversely, if $V \simeq \omega_{G,L}$, then we have $\omega_{G,\mathcal{O}_L} \in \mathcal{L}(\omega_{G,L}) \simeq \mathcal{L}(V) \neq \emptyset$. We suppose $V \simeq \omega_{G,L}$ henceforth.

Remark 15. In the case when $\mathcal{O}_L$ is a local strictly henselian faithfully flat $\mathcal{O}$-algebra and

$$V \ : \ \mathrm{Rep}_{\mathcal{O}} G \to \mathrm{Bun}_L$$

is faithful, we get $\mathcal{L}(V) = \mathcal{L}^{\mathrm{ex}}(V)$, using Proposition 7.2.1, since V faithful implies x faithful for every $x \in \mathcal{L}'(V)$.

Let $\eta : V_1 \to V_2$ be a $\otimes$-isomorphism. Then η induces a bijection $\mathcal{L}^*(V_1) \xrightarrow{\sim} \mathcal{L}^*(V_2)$, by $x \mapsto \eta x$ where $\eta x(\tau) = \eta_\tau(x(\tau))$ (so η is also a $\otimes$-isomorphism $x \simeq \eta x$). In particular, the group $\mathrm{Aut}^{\otimes}(V)$ acts on $\mathcal{L}(V) \subset \mathcal{L}^{\mathrm{ex}}(V) \subset \mathcal{L}'(V)$. The action is transitive on $\mathcal{L}(V)$, since two elements $x, y \in \mathcal{L}(V)$ are isomorphic as functors, through a $\otimes$-isomorphism $\eta : x \to y$ inducing a $\otimes$-isomorphism $\eta_L : V \to V$, i.e. an element $g \in \mathrm{Aut}^{\otimes}(V)$, which sends x to y by definition. Thus for any $x \in \mathcal{L}(V)$ with stabilizer $\mathrm{Aut}^{\otimes}(x)$ in $\mathrm{Aut}^{\otimes}(V)$, the map $g \mapsto g \cdot x$ yields an $\mathrm{Aut}^{\otimes}(V)$-equivariant bijection

$$\mathrm{Aut}^{\otimes}(V) / \mathrm{Aut}^{\otimes}(x) \simeq \mathcal{L}(V).$$

Remark 16. 1. For $V = \omega_{G,L}$ and $x = \omega_{G,\mathcal{O}_L}$, we obtain

$$G(L)/G(\mathcal{O}_L) \simeq \mathcal{L}(V)$$

since $\mathrm{Aut}^{\otimes}(x) = G(\mathcal{O}_L)$ inside $\mathrm{Aut}^{\otimes}(V) = G(L)$.

2. For $\tilde{G} = G_{\mathcal{O}_L}$, $V = \omega_{G,L}$ and $\tilde{V} = \omega_{\tilde{G},L}$, the map $\mathcal{L}(\tilde{V}) \to \mathcal{L}(V)$ defined by $\tilde{x} \mapsto x$, where

$$x \ : \ \mathrm{Rep}_{\mathcal{O}} G \xrightarrow{\otimes \mathcal{O}_L} \mathrm{Rep}_{\mathcal{O}_L} \tilde{G} \xrightarrow{\tilde{x}} \mathrm{Bun}_{\mathcal{O}_L}$$

is $G(L) = \mathrm{Aut}^{\otimes}(\tilde{V}) = \mathrm{Aut}^{\otimes}(V)$-equivariant, thus a bijection since

$$\mathcal{L}(\tilde{V}) \simeq G(L)/G(\mathcal{O}_L) \simeq \mathcal{L}(V).$$

3. For $\mathcal{O} = \mathcal{O}_L$ Henselian and $V = \omega_{G,L}$, there is a $G(L)$-equivariant embedding

$$\mathcal{L}(V) \hookrightarrow \mathbf{B}^e(\omega_G, L)$$

mapping x to the gauge norms $\alpha_x \in \mathbf{B}^e(\omega_G, L)$ defined by

$$\alpha_x(\tau)(v) = \inf\{|\lambda| \mid \lambda \in L, \ v \in \lambda x(\tau)\}$$

for $\tau \in \mathrm{Rep}_{\mathcal{O}} G$ and $v \in \omega_{G,L}(\tau)$. Here, $\mathbf{B}^e(\omega_G, L)$ is the space of L-norms on ω_G defined in [11, 6.4].

Definition 7.2.2. Let $\mathbf{F}^{\mathbb{Z}}(V) = \mathbf{F}^{\mathbb{Z}}(\omega(V))$. We define an addition operator between lattices and filtrations

$$\begin{array}{rcccc} + & : & \mathcal{L}(V) \times \mathbf{F}^{\mathbb{Z}}(V) & \to & \mathcal{L}(V) \\ & & (x, \mathcal{F}) & \mapsto & x + \mathcal{F} \end{array}$$

where, for every $\tau \in \mathrm{Rep}_{\mathbb{Z}_p} G$, we have

$$(x + \mathcal{F})(\tau) = x(\tau) + \mathcal{F}(\tau \otimes \mathbb{Q}_p) = \sum_{i \in \mathbb{Z}} \pi_L^{-1} x(\tau) \cap \mathcal{F}^{\geq i}(\tau \otimes \mathbb{Q}_p).$$

Proposition 7.2.3. *The operator defined above is well-defined.*

Proof. We may assume that V is trivial. Fix $x \in \mathcal{L}(V)$ and $\mathcal{F} \in \mathbf{F}^{\mathbb{Z}}(V)$. We know that $x + \mathcal{F}$ is functorial and, by Proposition 2.3.4, compatible with tensor products. We have then an element $x + \mathcal{F} \in \mathcal{L}'(V)$, so it suffices to see that it is isomorphic to the trivial lattice. Up to multiplication with an element of $G(L)$, we may assume that $x = \omega_{G,\mathcal{O}_L}$. Since $\mathcal{F} \in \mathbf{F}^{\mathbb{Z}}(G_L) = \mathbf{F}^{\mathbb{Z}}(G_{\mathcal{O}_L})$, there exists a cocharacter $\chi : \mathbb{G}_{m,\mathcal{O}_L} \to G_{\mathcal{O}_L}$ splitting $\mathcal{F}$. For every $\tau \in \mathrm{Rep}_{\mathbb{Z}_p} G$, let $x(\tau) = \oplus_{i \in \mathbb{Z}} x(\tau)_i$ be the weight decomposition of $(\tau \otimes \mathcal{O}_L) \circ \chi : \mathbb{G}_{m,\mathcal{O}_L} \to \mathrm{GL}_{\mathcal{O}_L}(x(\tau))$. Then $\mathcal{F}^{\geq j}(\tau \otimes \mathbb{Q}_p) = \oplus_{i \geq j} V(\tau)_j$ where $V(\tau)_j = x(\tau)_j \otimes L$, so

$$\begin{array}{rcl} x(\tau) + \mathcal{F}^{\geq j}(\tau \otimes \mathbb{Q}_p) & = & \sum_j {\pi_L}^{-j} x(\tau) \cap \mathcal{F}^{\geq j}(\tau \otimes \mathbb{Q}_p) \\ & = & \sum_j \oplus_{i \geq j} {\pi_L}^{-j} x(\tau)_i \\ & = & \bigoplus_i \sum_{j \geq i} {\pi_L}^{-j} x(\tau)_i \\ & = & \bigoplus_i {\pi_L}^{-i} x(\tau)_i \\ & = & \tau(\chi(\pi_L^{-1})) \cdot x(\tau) \\ & = & (\chi(\pi_L^{-1}) \cdot x)(\tau), \end{array}$$

and $x + \mathcal{F} = \chi(\pi_L^{-1}) \cdot x$, which indeed belongs to $\mathcal{L}(V)$.

$\square$

7.2.3 A variant: From $\mathrm{Rep}_{\mathbb{Q}_p} G$ to $\mathrm{Rep}_{\mathbb{Z}_p} G$

For the rest of the chapter, we will use a slightly different framework. If $\mathcal{O} = \mathbb{Z}_p$, G a reductive group over $\mathbb{Z}_p$ and

$$V \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G_{\mathbb{Q}_p} \to \mathrm{Bun}_L$$

is a fiber functor, then we denote

$$\mathcal{L}(V) = \mathcal{L}(V')$$

where

$$V' \ : \ \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Bun}_L$$

is the fiber functor induced by precomposition of V with $\mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Rep}_{\mathbb{Q}_p} G$. We have

$$\begin{aligned} \mathcal{L}(V) \neq \emptyset \ &\Leftrightarrow \ V \text{ is isomorphic to the trivial fiber functor } \omega_{G_{\mathbb{Q}_p},L} \\ &\Leftrightarrow \ V' \text{ is isomorphic to the trivial fiber functor } \omega_{G,L}. \end{aligned}$$

Indeed, we already know that $\mathcal{L}(V) \neq \emptyset$ if and only if $\mathcal{L}(V') \neq \emptyset$ if and only if $V' \simeq \omega_{G,L}$. Suppose $V \simeq \omega_{G_{\mathbb{Q}_p},L}$, then it is obvious that $V \simeq \omega_{G,L}$. Conversely, suppose we have an element $a : V' \simeq \omega_{G,L}$ in $\mathrm{Iso}^{\otimes}(V', \omega_{G,L})$. By [15, Theorem 1.12 and Remark 1.13], there exists a finite Galois extension L'/L such that there exists an isomorphism $b : V_{L'} \simeq \omega_{G_{\mathbb{Q}_p},L'}$. Consider the diagram

$$\begin{tikzcd} \mathrm{Iso}^{\otimes}(V, \omega_{G_{\mathbb{Q}_p},L}) \arrow[r] \arrow[d] & \mathrm{Iso}^{\otimes}(V', \omega_{G,L}) \ni a \arrow[d] \\ b \in \mathrm{Iso}^{\otimes}(V_{L'}, \omega_{G_{\mathbb{Q}_p},L'}) \arrow[r] & \mathrm{Iso}^{\otimes}(V'_{L'}, \omega_{G,L'}). \end{tikzcd}$$

Changing b by an element $g \in G_{\mathbb{Q}_p}(L') = G(L') = \mathrm{Aut}^{\otimes}(\omega_{G,L'})$, we may assume that b and a have the same image in $\mathrm{Iso}^{\otimes}(V'_{L'}, \omega_{G,L'})$. Then b is fixed by the action of $\mathrm{Gal}(L'/L)$ on $\mathrm{Iso}^{\otimes}(V_{L'}, \omega_{G_{\mathbb{Q}_p},L'})$, thus $b \in \mathrm{Iso}^{\otimes}(V, \omega_{G_{\mathbb{Q}_p},L})$ and indeed $V \simeq \omega_{G_{\mathbb{Q}_p},L}$.

7.2.4 Vectorial distance

For $x, y \in \mathcal{L}'(V)$, we denote by

$$\mathcal{F}(x,y) \ : \mathrm{Rep}_{\mathcal{O}} G \to \mathrm{Fil}^{\mathbb{Z}} \mathrm{Bun}_l$$

the functor given by

$$\mathcal{F}(x,y)(\tau) = \mathcal{F}(x(\tau), y(\tau)),$$

with underlying fiber functor

$$\overline{x} \ : \mathrm{Rep}_{\mathcal{O}} G \xrightarrow{x} \mathrm{Bun}_{\mathcal{O}_L} \xrightarrow{\otimes l} \mathrm{Bun}_l \, .$$

It is clear that $\overline{x}$ is a $\otimes$-functor, and so is $\mathcal{F}(x,y)$, by Proposition 2.3.2.

Proposition 7.2.4. *Suppose $x, y \in \mathcal{L}^{\mathrm{ex}}(V)$, then the $\otimes$-functor $\mathcal{F}(x,y)$ is exact, i.e.*

$$\mathcal{F}(x,y) \in \mathbf{F}^{\mathbb{Z}}(\overline{x}).$$

In particular, it is true for every $x, y \in \mathcal{L}(V)$.

Proof. Let $x, y \in \mathcal{L}^{\mathrm{ex}}(V)$. Then, $\overline{x}$ is exact and, by Proposition 7.2.2, it suffices to show that $\mathcal{F}(x,y)$ is compatible with exterior and symmetric powers. We check it for exterior powers. We have

$$\Lambda^r \mathcal{F}(x(\tau), y(\tau)) = \mathcal{F}(\Lambda^r x(\tau), \Lambda^r y(\tau)) = \mathcal{F}(x(\Lambda^r \tau), y(\Lambda^r \tau))$$

for every $\tau \in \mathrm{Rep}_{\mathcal{O}} G$ and every $r \geq 0$, where the first equality is given by Proposition 2.3.2 and the second equality follows from the exactness of x and y. Thus $\mathcal{F}(x,y)$ is exact.

□

Using this filtration, we define the vectorial distance (also called relative position) of $x, y \in \mathcal{L}(V)$, as

$$\mathrm{Pos}(x,y) = \mathbf{t}(\mathcal{F}(x,y)) \in \mathbf{C}^{\mathbb{Z}}(\overline{x}) = \mathbf{C}^{\mathbb{Z}}(G_l).$$

(where the equality $\mathbf{C}^{\mathbb{Z}}(\overline{x}) = \mathbf{C}^{\mathbb{Z}}(G_l)$ is due to the fact that $\overline{x}$ is isomorphic to the trivial fiber functor).

Example 7.2.1. Let $V = \omega_{G,L}$, $x = \omega_{G,\mathcal{O}_L}$ and $y = \mu(\pi_L) \cdot \omega_{G,\mathcal{O}_L}$, where

$$\mu \ : \ \mathbb{G}_{m,\mathcal{O}_L} \to G_{\mathcal{O}_L}.$$

We also denote by μ the type of the graduation given by μ. Then, we have

$$\mathrm{Pos}(x,y) = \mu^{\iota} \quad \text{in} \quad \mathbf{C}^{\mathbb{Z}}(G_l).$$

Remark 17. If $\mathcal{O}_L$ is Henselian, with the proper normalization of the multiplicative valuation on L, there is a commutative diagram

$$\begin{array}{ccccc} \mathcal{L}(\omega_{G,L}) & \times & \mathcal{L}(\omega_{G,L}) & \xrightarrow{\mathrm{Pos}} & \mathbf{C}^{\mathbb{Z}}(G_l) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{B}^e(\omega_{\tilde{G}}, L) & \times & \mathbf{B}^e(\omega_{\tilde{G}}, L) & \xrightarrow{\mathbf{d}} & \mathbf{C}^{\mathbb{R}}(G_L) \end{array}$$

where $\tilde{G} = G_{\mathcal{O}_L}$, $\tilde{V} = \omega_{\tilde{G},L}$ and $\mathcal{L}(V) \leftarrow \mathcal{L}(\tilde{V}) \hookrightarrow \mathbf{B}^e(\omega_{\tilde{G}}, L)$ are as above and $\mathbf{C}^{\mathbb{Z}}(G_l) \simeq \mathbf{C}^{\mathbb{Z}}(G_{\mathcal{O}_L}) \simeq \mathbf{C}^{\mathbb{Z}}(G_L) \subset \mathbf{C}^{\mathbb{R}}(G_L)$, by [11, 4.1.17] and [11, 4.1.18]. The vectorial distance $\mathbf{d}$ is introduced in [11, 5.2.8], using [11, 6.2] and [11, 6.4]. We explain the diagram a bit more in detail. For the calculation of relative positions, we may assume that $\mathcal{O} = \mathcal{O}_L$ and $V = \omega_{G,L}$. Both maps $\mathcal{L}(\omega_{G,L})^2 \to \mathbf{C}^{\mathbb{R}}(G_L)$ are $G(L)$-equivariant, so we may fix the first component of $(x,y) \in \mathcal{L}(\omega_{G,L})^2$ to $x = \omega_{G,\mathcal{O}_L}$, which maps to $\alpha^{\circ}_{G,L}$ in $\mathbf{B}^{\circ}(\omega_G, L)$. By [11, 6.4.8], it suffices to check the commutativity of

$$\begin{array}{rccc} \mathrm{Pos}(\omega_{G,\mathcal{O}_L}, -): & \mathcal{L}(\omega_{G,L}) & \to & \mathbf{F}^{\mathbb{Z}}(G_l) \\ & \downarrow & & \downarrow \\ \mathrm{loc}: & \mathbf{B}^e(\omega_G, L) & \to & \mathbf{F}^{\mathbb{R}}(G_l) \end{array}$$

The two maps $\mathcal{L}(\omega_{G,L}) \to \mathbf{F}^{\mathbb{R}}(G_l)$ on the diagram send $y \in \mathcal{L}(\omega_{G,L})$ to the filtration defined by

$$\begin{aligned} \mathcal{F}_1^i(\tau) &= \frac{\omega_{G,\mathcal{O}_L}(\tau) \cap \pi_L^i y(\tau) + \pi_L \omega_{G,\mathcal{O}_L}(\tau)}{\pi_L \omega_{G,\mathcal{O}_L}} \subset \omega_l(\tau) \\ \mathcal{F}_2^{\gamma}(\tau) &= \frac{\omega_{G,\mathcal{O}_L}(\tau) \cap \overline{B}(\alpha_y(\tau), \gamma) + \pi_L \omega_{G,\mathcal{O}_L}(\tau)}{\pi_L \omega_{G,\mathcal{O}_L}} \subset \omega_l(\tau) \end{aligned}$$

for every $i \in \mathbb{Z}$, every $\gamma \in \mathbb{R}$ and every $\tau \in \mathrm{Rep}_{\mathbb{Z}_p} G$, where

$$\overline{B}(\alpha_y(\tau), \gamma) = \{v \mid \alpha_y(\tau)(v) \leq \exp(-\gamma)\}.$$

If $|L^{\times}| = \exp(\mathbb{Z})$, i.e. $|\pi_L| = \exp(-1)$, then $\mathcal{F}_2 \in \mathbf{F}^{\mathbb{Z}}(G_l)$ and

$$\overline{B}(\alpha_y(\tau), i) = \{v \mid \alpha_y(\tau)(v) \leq \exp(-i)\} = \pi_L^i y(\tau),$$

therefore $\mathcal{F}_1 = \mathcal{F}_2$.

Proposition 7.2.5. *The relative position defined above verifies*

1. *The triangular inequality: for every $x, y, z \in \mathcal{L}(V)$, we have*
$$\mathrm{Pos}(x,z) \leq \mathrm{Pos}(x,y) + \mathrm{Pos}(y,z),$$
in the partially ordered commutative monoid $\mathbf{C}^{\mathbb{Z}}(G_l)$.

2. *Compatibility with the involution: For every $x, y \in \mathcal{L}(V)$, we have*
$$\mathrm{Pos}(y,x) = \mathrm{Pos}(x,y)^{\iota}.$$

Proof. We may assume that $V = \omega_{G,L}$. We can now use all the results in [11] as follows: as we have seen above, the functor sending a lattice to its Gauge norm embeds $\mathcal{L}(V)$ in $\mathbf{B}(\omega_{\tilde{G}}, L)$ and in the last remark we have seen that for a suitable normalizaton of the valuation on L, the operator Pos corresponds to the vectorial distance $\mathbf{d}$ defined on $\mathbf{B}(\omega_{\tilde{G}}, L)$ via 6.4.10 and 6.2 in [11], which verifies itself the triangular inequality. The compatibility with the involution is an obvious property of the vectorial distance $\mathbf{d}$ given in [11, 5.2.8].

□

Another property that will be useful is given by the next proposition:

Proposition 7.2.6. *Let $\varphi : L \to L'$ be a finite extension with ramification index e and set*
$$V' = V \otimes L' \ : \ \mathrm{Rep}_{\mathcal{O}} G \to \mathrm{Bun}_L \to \mathrm{Bun}_{L'}.$$
Then we have
$$\mathcal{F}(x', y') = e \cdot \mathcal{F}(x,y) \otimes l' \quad \textit{in} \quad \mathbf{F}^{\mathbb{Z}}(\overline{x}') = \mathbf{F}^{\mathbb{Z}}(\overline{x}_{l'})$$
for every $x, y \in \mathcal{L}(V)$, $x' = x \otimes \mathcal{O}_{L'}$, $y' = y \otimes \mathcal{O}_{L'}$ in $\mathcal{L}(V')$ and l' the residue field of $\mathcal{O}_{L'}$. As a consequence, we have
$$\mathrm{Pos}(x', y') = e \cdot \varphi \, \mathrm{Pos}(x,y) \quad \textit{in} \quad \mathbf{C}^{\mathbb{Z}}(G_{l'}).$$

Proof. It suffices to prove that $\mathcal{F}(x'(\tau), y'(\tau)) = e \cdot \mathcal{F}(x(\tau), y(\tau)) \otimes l'$ for every $\tau \in \mathrm{Rep}_{\mathcal{O}} G$. Let $x(\tau) = \mathcal{O}_L e_1 \oplus \ldots \oplus \mathcal{O}_L e_r$, $y(\tau) = \mathcal{O}_L \pi_L^{-n_1} e_1 \oplus \ldots \oplus \mathcal{O}_L \pi_L^{-n_r} e_r$, where π_L and $\pi_{L'}$ are the uniformizers of $\mathcal{O}_L$ and $\mathcal{O}_{L'}$, respectively. Then
$$\begin{array}{rcl} x(\tau) \cap \pi_L^n y(\tau) & = & \oplus_{i=1}^r \mathcal{O}_L \pi_L^{\max\{0, n-n_i\}} e_i \\ x(\tau) \cap \pi_L^n y(\tau) + \pi_L x(\tau) & = & \oplus_{i=1}^r \mathcal{O}_L \pi_L^{\min\{1, \max\{0, n-n_i\}\}} e_i \\ \mathcal{F}^n(x(\tau), y(\tau)) & = & \oplus_{n_i \leq n} l \overline{e}_i. \end{array}$$
Now, we have $x'(\tau) = \mathcal{O}_{L'} e_1 \oplus \ldots \oplus \mathcal{O}_{L'} e_r$, $y'(\tau) = \mathcal{O}_{L'} \pi_{L'}^{-en_1} e_1 \oplus \ldots \oplus \mathcal{O}_{L'} \pi_{L'}^{-en_r} e_r$, so
$$\begin{array}{rcl} \mathcal{F}^n(x'(\tau), y'(\tau)) & = & \oplus_{en_i \leq n} l' \overline{e}_i \\ & = & \oplus_{n_i \leq \frac{n}{e}} l' \overline{e}_i \\ & = & \mathcal{F}^{\frac{n}{e}}(x(\tau), y(\tau)) \otimes l' \\ & = & e \cdot \mathcal{F}^n(x(\tau), y(\tau)) \otimes l'. \end{array}$$

□

Remark 18. We can apply the last proposition in the case where $L' = L$ and φ is a morphism which may be ramified, with ramification index e. Suppose there is an isomorphism $\varphi_V : \varphi^* V \to V$ of $\otimes$-functors, where $\varphi^* V = V \otimes_{L,\varphi} L$. As an abuse of notation, we denote by
$$\varphi_V \ : \ \mathcal{L}(V) \xrightarrow{\varphi} \mathcal{L}(\varphi^* V) \xrightarrow{\mathcal{L}(\varphi_V)} \mathcal{L}(V)$$
the induced morphism on lattices. Then we have
$$\mathrm{Pos}(\varphi_V(x), \varphi_V(y)) = e\varphi \, \mathrm{Pos}(x,y) \quad \text{in} \quad \mathbf{C}^{\mathbb{Z}}(G_l),$$
where l is the residue field of L, for every $x, y \in \mathcal{L}(V)$.

7.3 Crystalline representations with G-structure

Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 0$, $W(\mathbb{F})$ the ring of Witt vectors over $\mathbb{F}$, $K_0 = \operatorname{Frac} W(\mathbb{F})$ and we fix an algebraic closure of K_0 denoted by $\overline{K}_0$. We denote by

$$\mathrm{Gal}_K = \mathrm{Gal}(\overline{K}_0/K)$$

for every (totally ramified) extension $K_0 \subset K \subset \overline{K}_0$. Let G be a reductive group over $\mathbb{Z}_p$.

The aim of this section is to define and study the diagram

$$\begin{tikzcd}
\mathcal{L}(V,K) \arrow[r] \arrow[d, hook] & \mathcal{L}(N) \arrow[r] & \mathcal{L}(D', \leq \mathbf{t}_{\mathrm{H}}) \arrow[d, "\simeq"] \\
\mathcal{L}(V) \arrow[rr] & & \mathcal{L}(D, \leq \mathbf{t}_{\mathrm{H}})
\end{tikzcd}$$

for V, N, D and D' some $\otimes$-functors on $\mathrm{Rep}_{\mathbb{Q}_p} G$ that will be defined later.

7.3.1 Isocrystals with G-structure

An isocrystal with G-structure (or G-isocrystal) is an exact and faithful $\otimes$-functor

$$D \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Mod}^{\sigma}_{K_0} .$$

We denote by $\omega(D)$ its underlying fiber functor

$$\omega(D) : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Bun}_{K_0} .$$

We say that D is trivial when $\omega(D) = \omega_{G,K_0}$.

Lemma 7.3.1. *There is a canonical correspondance between trivial G-isocrystals and elements of $G(K_0)$.*

Proof. A trivial G-isocrystal is a $\otimes$-isomorphism

$$\sigma_D \ : \ \sigma^* \omega_{G,K_0} \simeq \omega_{G,K_0}.$$

Since $\sigma^*\omega_{G,K_0}$ is canonically isomorphic to ω_{G,K_0}, a trivial G-isocrystal corresponds to an element in $\mathrm{Aut}^{\otimes}(\omega_{G,K_0}) = G(K_0)$.

□

Composing D with the Newton slope graduation $\mathrm{Mod}^{\sigma}_{K_0} \to \mathrm{Gr}^{\mathbb{Q}}_{K_0}$, we obtain a $\mathbb{Q}$-graduation

$$\mathcal{G}_{\mathrm{N}} \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Gr}^{\mathbb{Q}}_{K_0} .$$

It lives in $\mathbf{G}^{\mathbb{Q}}(\omega(D))$, and so does the opposite graduation, defined by ${\mathcal{G}^{\iota}}^{\gamma}_{\mathrm{N}} = \mathcal{G}^{-\gamma}_{\mathrm{N}}$, for every $\gamma \in \mathbb{Q}$. We get two filtrations $\mathcal{F}_{\mathrm{N}} = \mathrm{Fil}(\mathcal{G}_{\mathrm{N}})$ and $\mathcal{F}^{\iota}_{\mathrm{N}} = \mathrm{Fil}(\mathcal{G}^{\iota}_{\mathrm{N}})$, living in $\mathbf{F}^{\mathbb{Q}}(\omega(D))$, called the Newton filtrations. We denote by $\mathbf{t}_{\mathrm{N}}(D)$ and $\mathbf{t}^{\iota}_{\mathrm{N}}(D)$ the types of these filtrations, which are elements in $\mathbf{C}^{\mathbb{Q}}(\omega(D))$.

Proposition 7.3.2. *Suppose D is isomorphic to a trivial G-isocrystal, then the types*

$$\mathbf{t}_{\mathrm{N}}(D), \mathbf{t}^{\iota}_{\mathrm{N}}(D) \in \mathbf{C}^{\mathbb{Q}}(G_{K_0})$$

are fixed by σ, i.e.

$$\mathbf{t}_{\mathrm{N}}(D) = \mathbf{t}_{\mathrm{N}}(D)^{\#} \quad \textit{and} \quad \mathbf{t}^{\iota}_{\mathrm{N}}(D) = \mathbf{t}^{\iota}_{\mathrm{N}}(D)^{\#} \quad \textit{in} \quad \mathbf{C}^{\mathbb{Q}}(G_{K_0}).$$

Proof. This is well known, see [29, 4.4]. The Frobenius of D induces an isomorphism of fiber functors $\sigma_D : \sigma^*\omega(D) \to \omega(D)$. It gives rises to a commutative diagram

$$\begin{array}{ccccc} \mathbf{F}^{\mathbb{Q}}(\omega(D)) & \xrightarrow{\sigma} & \mathbf{F}^{\mathbb{Q}}(\sigma^*\omega(D)) & \xrightarrow{\sigma_D} & \mathbf{F}^{\mathbb{Q}}(\omega(D)) \\ \downarrow t & & \downarrow t & & \downarrow t \\ \mathbf{C}^{\mathbb{Q}}(G_{K_0}) & \xrightarrow{\sigma} & \mathbf{C}^{\mathbb{Q}}(G_K) & \xrightarrow{\mathrm{Id}} & \mathbf{C}^{\mathbb{Q}}(G_K) \end{array}$$

Since $\mathcal{G}_{\mathrm{N}}$ is a graduation by sub-isocrystals, the top map fixes $\mathcal{F}_{\mathrm{N}}$ and $\mathcal{F}^{\iota}_{\mathrm{N}}$. Thus $\sigma\mathbf{t}_{\mathrm{N}}(D) = \mathbf{t}_{\mathrm{N}}(D)$ and $\sigma\mathbf{t}^{\iota}_{\mathrm{N}}(D) = \mathbf{t}^{\iota}_{\mathrm{N}}(D)$.

□

Set $\mathcal{L}(D) = \mathcal{L}(\omega(D))$. Thus $\mathcal{L}(D) \neq \emptyset$ if and only if D is isomorphic to a trivial G-isocrystal, by section 7.2.3, which we assume from now on. As we have seen in Remark 18, the Frobenius on D induces a bijection

$$\mathcal{L}(D) = \mathcal{L}(\omega(D)) \xrightarrow{\sigma} \mathcal{L}(\sigma^*\omega(D)) \xrightarrow{\sigma_D} \mathcal{L}(\omega(D)) = \mathcal{L}(D)$$

which we simply denote by σ_D. By Remark 18, we have

$$\mathrm{Pos}(\sigma_D x, \sigma_D y) = \sigma\,\mathrm{Pos}(x, y)$$

in $\mathbf{C}^{\mathbb{Z}}(G_{\mathbb{F}})$, for every $x, y \in \mathcal{L}(D)$. For any $y \in \mathcal{L}(D)$, we define the Hodge filtration by

$$\mathcal{F}_{\mathrm{H}}(y) = \mathcal{F}(y, \sigma_D y).$$

Proposition 7.2.4 shows that it lives in $\mathbf{F}^{\mathbb{Z}}(G_{\mathbb{F}})$. We denote by

$$\mathbf{t}_{\mathrm{H}}(y) = \mathrm{Pos}(y, \sigma_D y)$$

its type in $\mathbf{C}^{\mathbb{Z}}(G_{\mathbb{F}}) = \mathbf{C}^{\mathbb{Z}}(G_{K_0})$, which verifies Mazur's inequality given by the next proposition.

Proposition 7.3.3. *The types above verify the inequality*

$$\mathbf{t}^{\iota}_{\mathrm{N}}(D) \leq \mathbf{t}_{\mathrm{H}}(y)^{\#} \quad \textit{in} \quad \mathbf{C}^{\mathbb{Q}}(G_{K_0})$$

for any $y \in \mathcal{L}(D)$.

Proof. This is well-known, see [40, Theorem 4.2], [32, 4.1 and 4.10] or [13, 4.2]. The inequality $\mathbf{t}^{\iota}_{\mathrm{N}}(D)(\tau) \leq \mathbf{t}_{\mathrm{H}}(y)(\tau)$ for every $\tau \in \mathrm{Rep}_{\mathbb{Z}_p} G$ and every $y \in \mathcal{L}(D)$ is a well-known inequality, stated by Mazur in [36] and whose proof can be found in [23, Theorem 1.4.1]. Thus, by proposition 7.1.1 and since $\mathbf{t}_{\mathrm{N}}(D) = \mathbf{t}_{\mathrm{N}}(D)^{\#}$, we get $\mathbf{t}^{\iota}_{\mathrm{N}}(D) \leq \mathbf{t}_{\mathrm{H}}(y)^{\#}$.

□

We define two subsets of lattices in $\mathcal{L}(D)$ by

$$\begin{array}{rcl} \mathcal{L}(D, \leq \mu) & = & \{y \in \mathcal{L}(D) \mid \mathbf{t}_{\mathrm{H}}(y) \leq \mu\} \\ \mathcal{L}(D, \mu) & = & \{y \in \mathcal{L}(D) \mid \mathbf{t}_{\mathrm{H}}(y) = \mu\} \end{array}$$

for any $\mu \in \mathbf{C}^{\mathbb{Z}}(G_{K_0})$.

These objects have already been studied by various authors, in particular we can give an improved version of Mazur's inequality. First, Kottwitz in [31], and Rapoport and Richartz in [40], classify trivial isocrystals with G-structure, i.e. they describe the set $B(G)$ of σ-conjugacy classes in $G(K_0)$. The key ingredient for the classification is a map

$$\begin{array}{rcl} B(G) & \to & \mathbf{C}^{\mathbb{Q}}(G_{K_0}) \times \pi_1(G)_{\Gamma} \\ [b] & \mapsto & (\nu_G[b], \kappa_G[b]) \end{array}$$

which is shows to be injective in [31, 4.13], see also [40]. The first component is our Newton type, constructed by Kottwitz in [29, 3]: if D_b is the G-isocrystal corresponding to $b \in G(K_0)$, then $\nu_G[b] = \mathbf{t}_{\mathrm{N}}(D_b)$. The second component is the Kottwitz map, defined in [29, Proposition 5.6] or [40, 1.13], with values in the Γ-coinvariants of the algebraic fundamental group $\pi_1(G)$, there $\Gamma = \mathrm{Gal}_{\mathbb{Q}_p}$. In our quasi-split case, $\mathbf{C}^{\mathbb{Q}}(G)^{\Gamma} = \mathbf{C}^{\mathbb{Q}}(G_{\mathbb{Q}_p})$ and for any Borel pair (T, B) in $G_{\mathbb{Q}_p}$, there are canonical Γ-equivariant isomorphisms

$$\mathbf{C}^{\mathbb{Z}}(G) \simeq X_*(T)^{B-\mathrm{dom}}, \quad \mathbf{C}^{\mathbb{Q}}(G) \simeq (X_*(T) \otimes \mathbb{Q})^{B-\mathrm{dom}} \quad \text{and} \quad \pi_1(G) \simeq X_*(T)/\mathbb{Z} \cdot R_G^{\vee}$$

where $X_*(T)^{B-\mathrm{dom}}$ and $(X_*(T) \otimes \mathbb{Q})^{B-\mathrm{dom}}$ are the cones of B-dominant elements in $X_*(T)$ and $X_*(T) \otimes \mathbb{Q}$, respectively, while $\mathbb{Z} \cdot R_G^{\vee}$ is the subgroup of $X_*(T)$ spanned by the coroots $R_G^{\vee}$ of T in G. Moreover, the maps

$$[-]_G : \mathbf{C}^{\mathbb{Z}}(G) \twoheadrightarrow \pi_1(G) \quad \text{and} \quad [-]_{G,\Gamma} : \mathbf{C}^{\mathbb{Z}}(G) \twoheadrightarrow \pi_1(G)_{\Gamma}$$

induced by these isomorphisms, given in [40, 4.1] do not depend upon the chosen Borel pair.

Let (T, B) be a Borel pair in $G = G_{\mathbb{Z}_p}$. For $\mu \in X_*(T)^{B-\mathrm{dom}} \simeq \mathbf{C}^{\mathbb{Z}}(G_{K_0})$ and $b \in G(K_0)$, set

$$X_{\mu}^G(b) = \{x \in G(K_0)/G(W(\mathbb{F})) \mid x^{-1} b \sigma(x) \in G(W(\mathbb{F}))\mu(p)G(W(\mathbb{F}))\}.$$

The following improved version of Mazur's inequality was established in [40, Theorem 4.2], see also Theorem 4.1 and section 4.4 of [32]:

$$X_{\mu}^G(b) \neq \emptyset \implies \begin{cases} \nu_G[b] \leq \mu^{\#} & \text{in } \mathbf{C}^{\mathbb{Q}}(G) \quad \text{and} \\ \kappa_G[b] = [\mu]_{G,\Gamma} & \text{in } \pi_1(G)_{\Gamma}. \end{cases}$$

The last equality is also explained in [13, 4.2]. The converse implication was established by Gashi in [21], building on a strategy proposed by Kottwitz in [32]. Thus:

$$X_{\mu}^G(b) \neq \emptyset \iff \begin{cases} \nu_G[b] \leq \mu^{\#} & \text{in } \mathbf{C}^{\mathbb{Q}}(G) \quad \text{and} \\ \kappa_G[b] = [\mu]_{G,\Gamma} & \text{in } \pi_1(G)_{\Gamma}. \end{cases}$$

On the other hand, the $G(K_0)$-equivariant bijection

$$\begin{array}{rcl} G(K_0)/G(W(\mathbb{F})) = \mathrm{Aut}^{\otimes}(\omega_{G,K_0})/\mathrm{Aut}^{\otimes}(\omega_{G,W(\mathbb{F})}) & \to & \mathcal{L}(\omega_{G,K_0}) = \mathcal{L}(D_b) \\ g & \mapsto & g \cdot \omega_{G,W(\mathbb{F})} \end{array}$$

induces a bijection

$$X_{\mu}^G(b) \simeq \mathcal{L}(D_b, \mu^{\iota}),$$

as we can check using Example 7.2.1. Thus,

$$\mathcal{L}(D, \mu) \neq \emptyset \iff \begin{cases} \mathbf{t}_{\mathrm{N}}^{\iota}(D) \leq \mu^{\#} & \text{in } \mathbf{C}^{\mathbb{Q}}(G_{K_0}) \quad \text{and} \\ \kappa(D) = -[\mu]_{G,\Gamma} & \text{in } \pi_1(G)_{\Gamma} \end{cases}$$

for every G-isocrystal D.

Definition 7.3.1. For $\mu \in \mathbf{C}^{\mathbb{Z}}(G_{K_0})$, we say that D is μ-ordinary when

$$\mathcal{L}(D, \mu) \neq \emptyset \quad \text{and} \quad \mathbf{t}_{\mathrm{N}}^{\iota}(D) = \mu^{\#}.$$

Proposition 7.3.4. *Up to isomorphism, there is a unique D which is μ-ordinary.*

Proof. We will prove the existence later. For the unicity, if D is μ-ordinary, $\mathcal{L}(D) \neq \emptyset$ so we may assume that D is trivial. From the discussion above, we obtain that then there is at most one $[b] \in B(G)$ with $X_{\mu}(b) \neq \emptyset$ and $\nu_G[b] = \mu^{\#}$, the one associated to the element

$$(\mu^{\#}, -[\mu]_{G,\Gamma}) \in \mathbf{C}^{\mathbb{Q}}(G_{K_0}) \times \pi_1(G)_{\Gamma}.$$

□

7.3.2 Torsion Kisin modules with G-structure

Definition 7.3.2. An exact $\otimes$-functor

$$M_{\text{tors}} \ : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$$

is called a torsion Kisin module with G-structure. We denote by $\omega(M_{\text{tors}})$ its underlying fiber functor and we say that M_{tors} is trivial when $\omega(M_{\text{tors}}) = \omega_{G,\mathbb{F}[[u]]}$. An exact and faithful $\otimes$-functor

$$M_{\mathbb{F}_p} \ : \ \mathrm{Rep}_{\mathbb{F}_p} G_{\mathbb{F}_p} \to \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}$$

is called a Kisin module with $G_{\mathbb{F}_p}$-structure. We denote by $\omega(M_{\mathbb{F}_p})$ its underlying fiber functor and we say that $M_{\mathbb{F}_p}$ is trivial when $\omega(M_{\mathbb{F}_p}) = \omega_{G_{\mathbb{F}_p},\mathbb{F}[[u]]}$.

Note 1. Torsion Kisin modules with G-structure should be called p-torsion Kisin modules with G-structure to be completely coherent with the notations of the sections before. However, since we will not work with functors $\mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}^{\varphi}_{\mathfrak{S},\mathrm{t}}$, we can drop p from the notation.

Remark 19. A Kisin module with $G_{\mathbb{F}_p}$-structure extends canonically to a torsion Kisin module with G-structure, by considering the composition

$$\begin{tikzcd} \mathrm{Rep}_{\mathbb{Z}_p} G \arrow[r] \arrow[d] & \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}} \\ \mathrm{Rep}_{\mathbb{F}_p} G_{\mathbb{F}_p} \arrow[ur] & \end{tikzcd}$$

Proposition 7.3.5. *There are canonical correspondences between trivial torsion Kisin modules with G-structure and elements of $G(\mathbb{F}((u)))$ and trivial Kisin modules with $G_{\mathbb{F}_p}$-structure and elements of $G(\mathbb{F}((u)))$.*

Proof. A trivial torsion Kisin module with G-structure M is nothing but a $\otimes$-isomorphism

$$\varphi \ : \ \varphi^* \omega_{G,\mathbb{F}((u))} \simeq \omega_{G,\mathbb{F}((u))}$$

where $\omega_{G,\mathbb{F}((u))} : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Bun}_{\mathbb{F}((u))}$ is the trivial fiber functor. Since $\varphi^*\omega_{G,\mathbb{F}((u))}$ is canonically isomorphic to $\omega_{G,\mathbb{F}((u))}$, a trivial torsion Kisin module M with G-structure corresponds to an element of $\mathrm{Aut}^{\otimes}(\omega_{G,\mathbb{F}((u))}) = G(\mathbb{F}((u)))$. The same proof works for a trivial Kisin module with $G_{\mathbb{F}_p}$-structure, by replacing G by $G_{\mathbb{F}_p}$.

□

Remark 20. In particular, the last proposition proves that there is a canonical correspondence between trivial torsion Kisin modules with G-structure and trivial Kisin modules with $G_{\mathbb{F}_p}$-structure, given by the extension procedure of Remark 19.

Proposition 7.3.6. *Every Kisin module with $G_{\mathbb{F}_p}$-structure is isomorphic to a trivial Kisin module with $G_{\mathbb{F}_p}$-structure.*

Proof. It is a consequence of 7.2.1, since $\mathbb{F}[[u]]$ is a strictly henselian faithfully flat $\mathbb{F}_p$-algebra.

□

Definition 7.3.3. An isogeny class of torsion Kisin modules with G-structure is an exact $\otimes$-functor

$$\overline{X} \ : \ \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}^{\varphi}_{\mathbb{F}((u))} .$$

We denote by $\omega(\overline{X})$ its underlying fiber functor and we say what $\overline{X}$ is trivial when $\omega(\overline{X}) = \omega_{G,\mathbb{F}((u))}$.

Set $\mathcal{L}(\overline{X}) = \mathcal{L}(\omega(\overline{X}))$. Thus $\mathcal{L}(\overline{X}) \neq \emptyset$ if and only if $\overline{X}$ is isomorphic to a trivial isogeny class of torsion Kisin module with G-structure, which we assume from now on. Any $z \in \mathcal{L}(\overline{X})$ has a unique factorization through a $\otimes$-functor

$$\mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}},$$

i.e. yields a torsion Kisin module with G-structure.

As explained in Remark 18, the Frobenius morphism induces a map

$$\varphi_{\overline{X}}\ ;\ \mathcal{L}(\overline{X}) = \mathcal{L}(\omega(\overline{X})) \xrightarrow{\varphi^*} \mathcal{L}(\varphi^*\omega(\overline{X})) \xrightarrow{\varphi_{\overline{X}}} \mathcal{L}(\omega(\overline{X})) = \mathcal{L}(\overline{X}).$$

For every $z \in \mathcal{L}(\overline{X})$ and $n \geq 1$, we set

$$\mathcal{F}_{\mathrm{H},n}(z) = \mathcal{F}(z, \varphi^n_{\overline{X}}(z)) \quad \text{in} \quad \mathbf{F}^{\mathbb{Z}}(\omega(z \otimes \mathbb{F})),$$

by Proposition 7.2.4. Its type lives in $\mathbf{C}^{\mathbb{Z}}(G_{\mathbb{F}}) = \mathbf{C}^{\mathbb{Z}}(G_{K_0})$. However, we will be working with the renormalized Hodge types, defined as

$$\mathbf{t}_{\mathrm{H},n}(z) = p_n^{-1}\mathbf{t}(\mathcal{F}_{\mathrm{H},n}(z))$$

for $p_n = \frac{p^n-1}{p-1}$. These types live in $\mathbf{C}^{\mathbb{Q}}(G_{\mathbb{F}})$.

Proposition 7.3.7. *Let $z \in \mathcal{L}(\overline{X})$. If $\varphi^N = 1$ on $\mathbf{C}^{\mathbb{Q}}(G)$, then for every $N|n$ and $m \geq 1$, we have*

$$\mathbf{t}_{\mathrm{H},nm}(z) \leq \mathbf{t}_{\mathrm{H},n}(z) \quad \textit{in} \quad \mathbf{C}^{\mathbb{Q}}(G_{\mathbb{F}}).$$

Proof. Let $N|n$. We have

$$\begin{array}{rcl} \mathbf{t}(\mathcal{F}_{\mathrm{H},nm}(z)) & = & \mathrm{Pos}(z, \varphi^{nm}_{\overline{X}}(z)) \\ & \leq & \sum_{i=0}^{m-1} \mathrm{Pos}(\varphi^{ni}_{\overline{X}}(z), \varphi^{n(i+1)}_{\overline{X}}(z)) \\ & = & \sum_{i=0}^{m-1} p^{ni}\varphi^{ni}\,\mathrm{Pos}(z, \varphi^{n}_{\overline{X}}(z)), \end{array}$$

where the last equality is given by Remark 18, taking $L = \mathbb{F}((u))$ and for φ we have $e = p$. Since $\varphi^n = 1$ on $\mathbf{C}^{\mathbb{Q}}(G)$, we obtain $\mathbf{t}(\mathcal{F}_{\mathrm{H},nm}(z)) \leq \frac{p_{nm}}{p_n}\mathbf{t}(\mathcal{F}_{\mathrm{H},n}(z))$, thus $\mathbf{t}_{\mathrm{H},nm}(z) \leq \mathbf{t}_{\mathrm{H},n}(z)$.

□

Definition 7.3.4. A torsion Kisin module $z \in \mathcal{L}(\overline{X})$ with G-structure is called aligned if for every $\mathbb{Z}_p$-representation τ of G, the torsion Kisin module $z(\tau)$ is aligned.

The following proposition gives us a sufficient condition for a p-torsion Kisin module with G-structure to be aligned.

Proposition 7.3.8. *Suppose $\mathbf{t}_{\mathrm{H},nm}(z)^{\#} = \mathbf{t}_{\mathrm{H},n}(z)^{\#}$ for some $N|n \geq 0$ and all $m \geq 1$, then z is aligned.*

Proof. We have just seen that $\mathbf{t}_{\mathrm{H},mn}(z) \leq \mathbf{t}_{\mathrm{H},n}(z)$ for every $m \geq 1$. Suppose that the inequality is strict, then $\sigma^i\mathbf{t}_{\mathrm{H},mn}(z) < \sigma^i\mathbf{t}_{\mathrm{H},n}(z)$ for $i \geq 1$, thus $\mathbf{t}^{\#}_{\mathrm{H},mn}(z) < \mathbf{t}^{\#}_{\mathrm{H},n}(z)$ which is a contradiction. So,

$$\mathbf{t}_{\mathrm{H},mn}(z) = \mathbf{t}_{\mathrm{H},n}(z),$$

thus $\mathbf{t}_{\mathrm{H},mn}(z)(\tau) = \mathbf{t}_{\mathrm{H},n}(z)(\tau)$ for every $m \geq 1$ and every $\tau \in \mathrm{Rep}_{\mathbb{Z}_p} G$, which implies that $\mathbf{t}_{\mathrm{H},\infty}(z)(\tau) = \mathbf{t}_{\mathrm{H},n}(z)(\tau)$ for every $\tau \in \mathrm{Rep}_{\mathbb{Z}_p} G$, so $z(\tau)$ is aligned for every $\tau \in \mathrm{Rep}_{\mathbb{Z}_p} G$, i.e. z is aligned.

□

Suppose now that $z \in \mathcal{L}(\overline{X})$ is aligned. The filtration $\mathcal{F}_{\mathrm{F},1}$ over $\mathrm{Mod}^{\varphi,\mathrm{al}}_{\mathbb{F}[[u]],\mathrm{fr}}$ defines a $\otimes$-functor

$$\begin{array}{rcccc} \mathcal{F}_{\mathrm{F},1}(z) = \mathcal{F}_{\mathrm{F},1} \circ z & : & \mathrm{Rep}_{\mathbb{Z}_p} G & \to & \mathrm{Fil}^{\mathbb{Q}}_{\mathbb{F}[[u]]} \\ & & \tau & \mapsto & \mathcal{F}_{\mathrm{F},1}(z(\tau)) \end{array}$$

and the next proposition shows that this is a filtration in $\mathbf{F}^{\mathbb{Q}}(\omega(z))$.

Proposition 7.3.9. *The functor*

$$\mathcal{F}_{\mathrm{F},1}(z) = \mathcal{F}_{\mathrm{F},1} \circ z \ : \ \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Fil}^{\mathbb{Q}}_{\mathbb{F}[[u]]},$$

factoring the fiber functor $\omega(z)$, is an exact $\otimes$-functor.

Proof. We have to show that $\mathcal{F}_{\mathrm{F},1}(z)$ is exact. Since it is a filtration by strict subobjects, it is sufficient to show that

$$\mathcal{F}_{\mathrm{F},1}(z) \otimes \mathbb{F}((u)) : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Fil}^{\mathbb{Q}}_{\mathbb{F}((u))}$$

is exact. It follows from Proposition 4.5.7, using Proposition 7.2.2.

□

Definition 7.3.5. If $z \in \mathcal{L}(\overline{X})$ is aligned, we define the Fargues type $\mathbf{t}_{\mathrm{F},1}(z)$ as the type associated to the filtration $\mathcal{F}_{\mathrm{F},1}$ of z. It lives in $\mathbf{C}^{\mathbb{Q}}(G_{\mathbb{F}[[u]]}) = \mathbf{C}^{\mathbb{Q}}(G)$.

Proposition 7.3.10. *If $x \in \mathcal{L}(\overline{X})$ is aligned, then the types defined above on $z \in \mathcal{L}(\overline{X})$ verify the following inequalities:*

1. $\mathbf{t}_{\mathrm{F},1}(z) = \mathbf{t}_{\mathrm{F},1}(z)^{\#}$
2. $\mathbf{t}_{\mathrm{F},1}(z) \leq \mathbf{t}_{\mathrm{H},n}(z)^{\#}$ *for every* $n \geq 1$.

Proof. 1. The commutative diagram

$$\begin{array}{ccccc} \mathbf{F}^{\mathbb{Q}}(\omega(\overline{X})) & \xrightarrow{\varphi} & \mathbf{F}^{\mathbb{Q}}(\varphi^*\omega(\overline{X})) & \xrightarrow{\varphi_{\overline{X}}} & \mathbf{F}^{\mathbb{Q}}(\omega(\overline{X})) \\ \downarrow t & & \downarrow t & & \downarrow t \\ \mathbf{C}^{\mathbb{Q}}(G_{\mathbb{F}((u))}) & \xrightarrow{\sigma} & \mathbf{C}^{\mathbb{Q}}(G_{\mathbb{F}((u))}) & \xrightarrow{\mathrm{Id}} & \mathbf{C}^{\mathbb{Q}}(G_{\mathbb{F}((u))}) \end{array}$$

gives us

$$\mathbf{t}_{\mathrm{F},1}(z) = \mathbf{t}\left(\mathcal{F}_{\mathrm{F},1}(z) \otimes \mathbb{F}((u))\right) = \mathbf{t}(\varphi_{\overline{X}}\varphi^*(\mathcal{F}_{\mathrm{F},1}(z) \otimes \mathbb{F}((u)))) = \sigma \cdot \mathbf{t}\left(\mathcal{F}_{\mathrm{F},1}(z) \otimes \mathbb{F}((u))\right) = \sigma \cdot \mathbf{t}_{\mathrm{F},1}(z)$$

since $\mathcal{F}_{\mathrm{F},1}(z) \otimes \mathbb{F}((u))$ is stable under the upper row map.

2. We have $\mathbf{t}_{\mathrm{F},1}(z)(\tau) \leq \mathbf{t}_{\mathrm{H},n}(z)(\tau)$ for every $\tau \in \mathrm{Rep}_{\mathbb{Z}_p} G$, by Proposition 4.3.7, so by proposition 7.1.1 and the previous point, we have $\mathbf{t}_{\mathrm{F},1}(z) \leq \mathbf{t}_{\mathrm{H},n}(z)^{\#}$ for every $n \geq 1$.

□

7.3.3 Kisin modules with G-structure

We fix a finite extension K of K_0, an uniformizer π_K of K, $E \in \mathfrak{S}$ the minimal polynomial of π_K. Let $e = [K : K_0]$. Let $\hat{\mathfrak{S}}$ be the completion of $\mathfrak{S}_{(E)}$.

Definition 7.3.6. An exact and faithful $\otimes$-functor

$$M \ : \ \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}}$$

is called a Kisin module with G-structure. Let $\omega(M)$ be its underlying fiber functor. We say that M is trivial if $\omega(N) = \omega_{G,\mathfrak{S}}$. An exact and faithful $\otimes$-functor

$$N \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Mod}^{\varphi}_{\mathfrak{S}[\frac{1}{p}]}$$

is called an isogeny class of Kisin modules with G-structure. We denote by $\omega(N)$ its underlying fiber functor. We say that N is trivial if $\omega(M) = \omega_{G,\mathfrak{S}[\frac{1}{p}]}$.

Proposition 7.3.11. *There are canonical correspondences between trivial Kisin modules with G-structure and elements of $G(\mathfrak{S}[\frac{1}{E}])$, and between trivial isogeny classes of Kisin modules with G-structure and elements of $G(\mathfrak{S}[\frac{1}{pE}])$.*

Proof. A trivial Kisin module with G-structure M is nothing but a $\otimes$-isomorphism

$$\varphi \ : \ \varphi^*\omega_{G,\mathfrak{S}[\frac{1}{E}]} \simeq \omega_{G,\mathfrak{S}[\frac{1}{E}]}$$

where $\omega_{G,\mathfrak{S}[\frac{1}{E}]} \ : \ \mathrm{Rep}\, G \to \mathrm{Bun}_{\mathfrak{S}[\frac{1}{E}]}$ is the trivial fiber functor. Since $\varphi^*\omega_{G,\mathfrak{S}[\frac{1}{E}]}$ is canonically isomorphic to $\omega_{G,\mathfrak{S}[\frac{1}{E}]}$, a trivial Kisin module M with G-structure corresponds to an element of $\mathrm{Aut}^{\otimes}(\omega_{G,\mathfrak{S}[\frac{1}{E}]}) = G(\mathfrak{S}[\frac{1}{E}])$. We proof for a trivial isogeny class of Kisin modules with G-structure is analogous.

□

Remark 21. To any Kisin module with G-structure M, we can associate a torsion Kisin module with G-structure $\overline{M}$, obtained by composition

$$\overline{M} \ : \ \mathrm{Rep}_{\mathbb{Z}_p} G \xrightarrow{M} \mathrm{Mod}^{\varphi}_{\mathfrak{S},\,\mathrm{fr}} \xrightarrow{\mathrm{mod}\ p} \mathrm{Mod}^{\varphi}_{\mathbb{F}[[u]],\mathrm{fr}}\,.$$

We fix an isogeny class of Kisin modules with G-structure N. We define the set of lattices inside N, and denote it by $\mathcal{L}(N)$, as the set of Kisin modules with G-structure M such that $M[\frac{1}{p}]$ corresponds to the $\otimes$-functor N' given by precomposition of N with $\mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Rep}_{\mathbb{Q}_p} G$.

Proposition 7.3.12.

1. *Every Kisin module with G-structure is isomorphic to a trivial Kisin module with G-structure.*
2. *If $\mathcal{L}(N) \neq \emptyset$, then N is isomorphic to a trivial isogeny class of Kisin modules with G-structure*

Proof.

1. This follows from Proposition 7.2.1.
2. Let $M \in \mathcal{L}(N)$. Then $\omega(M) \simeq \omega_{G,\mathfrak{S}}$, thus $\omega(N') = \omega(M) \otimes \mathfrak{S}[\frac{1}{p}] \simeq \omega_{G,\mathfrak{S}[\frac{1}{p}]}$. This also implies that $\omega(N) \simeq \omega_{G_{\mathbb{Q}_p},\mathfrak{S}[\frac{1}{p}]}$, as in 7.2.3.

□

We assume that $\mathcal{L}(N) \neq \emptyset$ from now on. We can construct a functor

$$\mathcal{F}_{\mathrm{H}}(N) = \mathcal{F}(N \otimes \hat{\mathfrak{S}}, \varphi_N \varphi^* N \otimes \hat{\mathfrak{S}}) \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Fil}(\omega(N)_K).$$

Then, Proposition 7.2.4 tells us that it is a filtration in $\mathbf{F}^{\mathbb{Z}}(\omega_K(N))$. Its type it is denoted by $\mathbf{t}_{\mathrm{H}}(N)$ and it lives in $\mathbf{C}^{\mathbb{Z}}(G_K) = \mathbf{C}^{\mathbb{Z}}(G_{K_0})$.

We can associate an isocrystal with G-structure to N, by setting

$$D' = N/uN : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Mod}^{\sigma}_{K_0},$$

which is a faithful functor since its faithfulness only depends on the fiber functor $\omega(D')$ which is faithful as it is isomorphic to the trivial fiber functor ω_{G,K_0} by assumption. We thus get a map

$$\begin{array}{ccc} \mathcal{L}(N) & \to & \mathcal{L}(D') \\ M & \mapsto & y' \end{array}$$

where y' is the exact $\otimes$-functor

$$y' = M/uM \ : \ \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}^{\sigma}_{W(\mathbb{F})}.$$

Proposition 7.3.13. *We have*

$$\mathbf{t}_{\mathrm{H}}(y') \leq \mathbf{t}_{\mathrm{H}}(N),$$

i.e. the map $\mathcal{L}(N) \to \mathcal{L}(D')$ factors through $\mathcal{L}(D', \leq \mathbf{t}_{\mathrm{H}}(N))$.

Proof. We can suppose that M is trivial, so the Frobenius on M is given by an element $g \in G(\mathfrak{S}[\frac{1}{E}])$. Then

$$\mathbf{t}_{\mathrm{H}}(y') = \mathrm{Pos}(\omega_{G,W(\mathbb{F})}, g(0) \cdot \omega_{G,W(\mathbb{F})}) \quad \text{and} \quad \mathbf{t}_{\mathrm{H}}(N) = \mathrm{Pos}(\omega_{G,\hat{\mathfrak{S}}}, g \cdot \omega_{G,\hat{\mathfrak{S}}})$$

in, respectively, $\mathbf{C}^{\mathbb{Z}}(G_{\mathbb{F}}) = \mathbf{C}^{\mathbb{Z}}(G)$ and $\mathbf{C}^{\mathbb{Z}}(G_K) = \mathbf{C}^{\mathbb{Z}}(G)$. Thus, by 7.1.1, it suffices to show that for every $\tau \in \mathrm{Rep}_{\mathfrak{S}} G_{\mathfrak{S}}$ (or $\mathrm{Rep}_{W(\mathbb{F})} G_{W(\mathbb{F})}$), we have

$$\mathrm{Pos}(X/uX, Y/uY) \leq \mathrm{Pos}(X \otimes \hat{\mathfrak{S}}, Y \otimes \hat{\mathfrak{S}})$$

where $X = M(\tau)$ and $Y = \tau(g)M(\tau)$. This follows from the proof of 6.1.1.

□

Another important inequality concerning the Hodge type of N is the following:

Proposition 7.3.14. *Let $M \in \mathcal{L}(N)$ and $\overline{M}$ the torsion Kisin module with G-structure associated to M. We have*

$$\mathbf{t}_{\mathrm{H},1}(\overline{M}) \leq e \cdot \mathbf{t}_{\mathrm{H}}(N).$$

Proof. We can suppose that M is trivial, so the Frobenius on M is given by an element $g \in G(\mathfrak{S}[\frac{1}{E}])$. Then

$$\mathbf{t}_{\mathrm{H},1}(\overline{M}) = \mathrm{Pos}(\omega_{G,\mathbb{F}[[u]]}, \overline{g} \cdot \omega_{G,\mathbb{F}[[u]]}) \quad \text{and} \quad \mathbf{t}_{\mathrm{H}}(N) = \mathrm{Pos}(\omega_{G,\hat{\mathfrak{S}}}, g \cdot \omega_{G,\hat{\mathfrak{S}}})$$

in, respectively, $\mathbf{C}^{\mathbb{Z}}(G_{\mathbb{F}}) = \mathbf{C}^{\mathbb{Z}}(G)$ and $\mathbf{C}^{\mathbb{Z}}(G_K) = \mathbf{C}^{\mathbb{Z}}(G)$. Thus, by 7.1.1, it suffices to show that for every $\tau \in \mathrm{Rep}_{\mathfrak{S}} G_{\mathfrak{S}}$ (or $\mathrm{Rep}_{\mathbb{F}[[u]]} G_{\mathbb{F}[[u]]}$), we have

$$\mathrm{Pos}(X/pX, Y/pY) \leq e \cdot \mathrm{Pos}(X \otimes \hat{\mathfrak{S}}, Y \otimes \hat{\mathfrak{S}})$$

where $X = M(\tau)$ and $Y = \tau(g)M(\tau)$. This follows from the proof of 6.1.1.

□

Definition 7.3.7. A Kisin module with G-structure is said to be HN-type if for every $\mathbb{Z}_p$-representation τ of G, the Kisin module $M(\tau)$ is HN-type.

Under the type HN-type hypothesis, we can define a Fargues type on M.

Proposition 7.3.15. *Suppose M is HN-type and $\overline{M}$ is aligned, then the functor $\mathcal{F}_{\mathrm{F}}(M)$ sending a $\mathbb{Z}_p$-representation τ of G to the Fargues filtration of the HN-type Kisin module $M(\tau)$ is in $\mathbf{F}^{\mathbb{Q}}(\omega(M))$.*

Proof. It is a filtration by direct summands, so we need to check the exactness and compatibility with tensor products. For exactness, let

$$0 \to \tau_1 \to \tau_2 \to \tau_3 \to 0$$

be an exact sequence of representations of G. Then, we have a commutative diagram where the second row is exact

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{\mathrm{F}}^{\gamma}(\tau_1) \arrow[r] \arrow[d] & \mathcal{F}_{\mathrm{F}}^{\gamma}(\tau_2) \arrow[r, "f"] \arrow[d] & \mathcal{F}_{\mathrm{F}}^{\gamma}(\tau_3) \arrow[d] & \\
0 \arrow[r] & M(\tau_1) \arrow[r] & M(\tau_2) \arrow[r] & M(\tau_3) \arrow[r] & 0
\end{tikzcd}$$

Reducing the first sequence modulo p, we obtain the exact sequence

$$0 \to \mathcal{F}_{\mathrm{F},1}^{\gamma}(\tau_1) \to \mathcal{F}_{\mathrm{F},1}^{\gamma}(\tau_2) \to \mathcal{F}_{\mathrm{F},1}^{\gamma}(\tau_3) \to 0$$

so by Nakayama's lemma, the map $\mathcal{F}_{\mathrm{F}}^{\gamma}(\tau_2) \to \mathcal{F}_{\mathrm{F}}^{\gamma}(\tau_3)$ is surjective. Then,

$$\ker f = \ker(\mathcal{F}_{\mathrm{F}}^{\gamma}(\tau_2) \to \mathcal{F}_{\mathrm{F}}^{\gamma}(\tau_3))$$

is a free $\mathfrak{S}$-module, and we also have $\mathcal{F}_{\mathrm{F}}^{\gamma}(\tau_1) \hookrightarrow \ker f$, which becomes an isomorphism modulo p, so again by Nakayama's lemma, we have $\mathcal{F}_{\mathrm{F}}^{\gamma}(\tau_1) \simeq \ker f$.

For the compatibility with tensor products, let $\mathcal{G}(\tau_1 \otimes \tau_2) = \mathcal{F}_{\mathrm{F}}(\tau_1) \otimes \mathcal{F}_{\mathrm{F}}(\tau_2)$. Then, we have to show that the Kisin module

$$X = \mathrm{Gr}_{\mathcal{G}}^{\gamma}(\tau_1 \otimes \tau_2) = \bigoplus_{\gamma_1+\gamma_2=\gamma} \mathrm{Gr}_{\mathrm{F}}^{\gamma_1}(M(\tau_1)) \otimes \mathrm{Gr}_{\mathrm{F}}^{\gamma_2}(M(\tau_2))$$

is semi-stable of slope γ. We can reduce modulo p to obtain

$$\begin{aligned}
\overline{X} = \mathrm{Gr}_{\overline{\mathcal{G}}}^{\gamma}(\tau_1 \otimes \tau_2) &= \bigoplus_{\gamma_1+\gamma_2=\gamma} \mathrm{Gr}_{\mathrm{F}}^{\gamma_1}(\overline{M}(\tau_1)) \otimes \mathrm{Gr}_{\mathrm{F}}^{\gamma_2}(\overline{M}(\tau_2)) \\
&= \mathrm{Gr}_{\mathcal{F}_{\mathrm{F},1}}^{\gamma}(\tau_1 \otimes \tau_2)
\end{aligned}$$

Then $\overline{X}$ is semi-stable of slope $\gamma = \gamma_1 + \gamma_2$ and so is therefore also X.

□

For M a HN-type Kisin module such that $\overline{M}$ is aligned, we can define the Fargues type $\mathbf{t}_{\mathrm{F}}(M)$ associated to the filtration $\mathcal{F}_{\mathrm{F}}(M)$. It is an element in $\mathbf{C}^{\mathbb{Q}}(G_{\mathfrak{S}})$.

7.3.4 Germs of crystalline representations with G-structure

Definition 7.3.8. A germ of crystalline representations with G-structure is a faithful $\otimes$-functor

$$V : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cr}}\{\mathrm{Gal}_{K_0}\}.$$

We denote by $\omega(V)$ its underlying fiber functor. We say that V is trivial when $\omega(V) \simeq \omega_{G,\mathbb{Q}_p}$.

Let V be a germ of crystalline representations with G-structure. We can define a Fargues filtration on V by

$$\begin{array}{rcccc} \mathcal{F}_{\mathrm{F}}(V) & : & \mathrm{Rep}_{\mathbb{Q}_p} G & \to & \mathrm{Fil}^{\mathbb{Q}}_{\mathbb{Q}_p} \\ & & \tau & \mapsto & \mathcal{F}_{\mathrm{F,cr}}(V(\tau)) \end{array}$$

for $\mathcal{F}_{\mathrm{F,cr}}$ the filtration defined on crystalline representation in section 3.7. This functor a $\otimes$-functor by [18, Corollary 6] and exact since it is defined over $\mathbb{Q}_p$ and every exact sequence in $\mathrm{Rep}_{\mathbb{Q}_p} G$ is split, so we have $\mathcal{F}_{\mathrm{F}}(V) \in \mathbf{F}^{\mathbb{Q}}(\omega(V))$. We denote its type by $\mathbf{t}_{\mathrm{F}}(V)$ and it lives in $\mathbf{C}^{\mathbb{Q}}(G_{\mathbb{Q}_p})$.

Remark 22. The type $\mathbf{t}_{\mathrm{F}}(V)$ is invariant under $\#$, since it is defined in $\mathbf{C}^{\mathbb{Q}}(G_{\mathbb{Q}_p})$ and σ acts trivially on this set.

The functor V induces a filtered isocrystal with G-structure by taking

$$D = D_{\mathrm{cr}} \circ V \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \to^{\mathrm{wa}} \mathrm{MF}^{\sigma}_{\overline{K}_0} \, .$$

We (still) denote by

$$D \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Mod}^{\sigma}_{K_0}$$

the underlying G-isocrystal, by

$$\omega(D) \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Bun}_{K_0}$$

the underlying fiber functor, and by

$$\mathcal{F}_{\mathrm{H}}(D) \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \ \to \ \mathrm{Fil}^{\mathbb{Z}}_{\overline{K}_0}$$

the Hodge filtration of the filtered isocrystal, which is an exact $\otimes$-functor by construction, i.e. an element of $\mathbf{F}^{\mathbb{Z}}(\omega(D) \otimes_{K_0} \overline{K}_0)$. Since $\overline{K}_0$ is algebraically closed, $\omega(D) \otimes_{K_0} \overline{K}_0 \simeq \omega_{G,\overline{K}_0}$ and the type $\mathbf{t}_{\mathrm{H}}(D)$ of $\mathcal{F}_{\mathrm{H}}(D)$ lives in $\mathbf{C}^{\mathbb{Z}}(G_{\overline{K}_0}) = \mathbf{C}^{\mathbb{Z}}(G_{K_0})$. We can also define a Fargues filtration on weakly admissible filtered isocrystals with G-structure by

$$\begin{array}{rcccc} \mathcal{F}_{\mathrm{F}}(D) & : & \mathrm{Rep}_{\mathbb{Q}_p} G & \to & \mathrm{Fil}^{\mathbb{Q}}_{K_0} \\ & & \tau & \mapsto & \mathcal{F}_{\mathrm{F,wa}}(D(\tau)) \end{array}$$

where $\mathcal{F}_{\mathrm{F,wa}}$ is the Fargues filtration on weakly admissible filtered isocrystals. This is an exact $\otimes$-functor by [12, Theorem 12], i.e. an element in $\mathbf{F}^{\mathbb{Q}}(\omega(D))$. If $\omega(D)$ is isomorphic to the trivial fiber functor, we obtain a type $\mathbf{t}_{\mathrm{F}}(D) \in \mathbf{C}^{\mathbb{Q}}(G_{K_0})$.

Proposition 7.3.16. *Suppose that D is isomorphic to a trivial G-isocrystal. We have*

1. *The equalities*
$$\mathbf{t}_{\mathrm{F}}(D) = \mathbf{t}_{\mathrm{F}}(D)^{\#} \quad \textit{and} \quad \mathbf{t}_{\mathrm{F}}(V) = \mathbf{t}_{\mathrm{F}}(D) \ .$$
2. *The inequality $\mathbf{t}_{\mathrm{F}}(V) \leq \mathbf{t}^{\iota}_{\mathrm{N}}(D)$.*
3. *If $\mathbf{t}_{\mathrm{N}}(D) = \mathbf{t}_{\mathrm{H}}(D)^{\#}$, then $\mathcal{F}^{\iota}_{\mathrm{N}}(D) = \mathcal{F}_{\mathrm{F}}(D)$ and $\mathbf{t}^{\iota}_{\mathrm{N}}(D) = \mathbf{t}_{\mathrm{F}}(D)$.*

Proof. 1. The commutative diagram

$$\begin{array}{ccccc} \mathbf{F}^{\mathbb{Q}}(\omega(D)) & \xrightarrow{\sigma} & \mathbf{F}^{\mathbb{Q}}(\sigma^*\omega(D)) & \xrightarrow{\sigma_D} & \mathbf{F}^{\mathbb{Q}}(\omega(D)) \\ \downarrow t & & \downarrow t & & \downarrow t \\ \mathbf{C}^{\mathbb{Q}}(G_{K_0}) & \xrightarrow{\sigma} & \mathbf{C}^{\mathbb{Q}}(G_{K_0}) & \xrightarrow{\mathrm{Id}} & \mathbf{C}^{\mathbb{Q}}(G_{K_0}) \end{array}$$

gives us

$$\mathbf{t}_{\mathrm{F}}(D) = \mathbf{t}(\mathcal{F}_{\mathrm{F}}(D)) = \mathbf{t}(\sigma_D \sigma^*_{K_0} \mathcal{F}_{\mathrm{F}}(D)) = \sigma \cdot \mathbf{t}(\mathcal{F}_{\mathrm{F}}(D)) = \sigma \cdot \mathbf{t}_{\mathrm{F}}(D)$$

since $\mathcal{F}_{\mathrm{F}}(D)$ is stable under the upper row map.

For the other equality, we have $\mathbf{t}_{\mathrm{F}}(V)(\tau) = \mathbf{t}_{\mathrm{F}}(D)(\tau)$ for every $\tau \in \mathrm{Rep}_{\mathbb{Z}_p} G$, by Proposition 3.7.1, so by proposition 7.1.1, we have

$$\mathbf{t}_{\mathrm{F}}(V) = \mathbf{t}_{\mathrm{F}}(V)^{\#} = \mathbf{t}_{\mathrm{F}}(D)^{\#} = \mathbf{t}_{\mathrm{F}}(D).$$

2. The inequality $\mathbf{t}_{\mathrm{F}}(V)(\tau) \leq \mathbf{t}^{\iota}_{\mathrm{N}}(D)(\tau)$ is given by Fargues in [18, Theorem 6], for every $\tau \in \mathrm{Rep}_{\mathbb{Q}_p} G$, thus
$$\mathbf{t}_{\mathrm{F}}(V) = \mathbf{t}_{\mathrm{F}}(V)^{\#} \leq \mathbf{t}^{\iota}_{\mathrm{N}}(D)^{\#} = \mathbf{t}^{\iota}_{\mathrm{N}}(D)$$
by Proposition 7.1.1 and Remark 22.

3. We may assume D is trivial and so, we can use the results in [12]. We drop D from the notation. We have to show that $\mathbf{t}_{\mathrm{N}} = \mathbf{t}^{\#}_{\mathrm{H}}$ implies $\mathcal{F}^{\iota}_{\mathrm{N}} = \mathcal{F}_{\mathrm{F}}$. In [12], we see that if we equip $\mathbf{F}^{\mathbb{R}}(G_{K_0})$ with a φ-invariant CAT(0)-distance as defined in [11, Corollary 88], then

$$\begin{array}{rcl} \mathbf{F}^{\mathbb{R}}(G_{K_0}, \varphi) & = & \{\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(G_{K_0}) \mid \mathcal{F}^{\gamma}(\tau) \in \mathrm{Mod}^{\sigma}_{K_0} \text{ for every } \gamma \in \mathbb{R} \text{ and every } \tau \in \mathrm{Rep}_{\mathbb{Q}_p} G\} \\ \mathbf{F}^{\mathbb{R}}(G_{K_0}, \varphi, \mathcal{F}_{\mathrm{H}}) & = & \{\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(G_{K_0}) \mid \mathcal{F}^{\gamma}(\tau) \in^{\mathrm{wa}} \mathrm{MF}^{\sigma}_{\overline{K}_0} \text{ for every } \gamma \in \mathbb{R} \text{ and every } \tau \in \mathrm{Rep}_{\mathbb{Q}_p} G\}. \end{array}$$

are closed convex subsets of $\mathbf{F}^{\mathbb{R}}(G_{K_0})$ and there is a convex projection

$$\mathbf{F}^{\mathbb{R}}(G_{K_0}, \varphi) \to \mathbf{F}^{\mathbb{R}}(G_{K_0}, \varphi, \mathcal{F}_{\mathrm{H}}).$$

In [12, Proposition 13], we see that $\mathcal{F}_{\mathrm{F}}$ is the convex projection of $\mathcal{F}^{\iota}_{\mathrm{N}}$, so it suffices to prove that $\mathcal{F}^{\iota}_{\mathrm{N}} \in \mathbf{F}^{\mathbb{R}}(G_{K_0}, \varphi, \mathcal{F}_{\mathrm{H}})$. In [12, Lemma 11], we see that

$$\begin{array}{rcl} \mathbf{F}^{\mathbb{R}}(G_{K_0}, \varphi, \mathcal{F}_{\mathrm{H}}) & = & \{\Xi \in \mathbf{F}^{\mathbb{R}}(G_{K_0}, \varphi) \mid \langle \mathcal{F}_{\mathrm{H}}, \Xi \rangle = \langle \mathcal{F}_{\mathrm{N}}, \Xi \rangle\} \\ & = & \{\Xi \in \mathbf{F}^{\mathbb{R}}(G_{K_0}, \varphi) \mid \langle \mathcal{F}_{\mathrm{H}}, \Xi \rangle \geq \langle \mathcal{F}_{\mathrm{N}}, \Xi \rangle\} \end{array}$$

so we need to prove that, under the assumption $\mathbf{t}_{\mathrm{N}}(D) = \mathbf{t}_{\mathrm{H}}(D)^{\#}$, we have

$$\langle \mathcal{F}_{\mathrm{H}}, \mathcal{F}^{\iota}_{\mathrm{N}} \rangle \geq \langle \mathcal{F}_{\mathrm{N}}, \mathcal{F}^{\iota}_{\mathrm{N}} \rangle.$$

The operator

$$\langle x, y \rangle^{\mathrm{tr}} = \inf\{\langle X, Y \rangle \mid \mathbf{t}(X) = x, \mathbf{t}(Y) = y \; X, Y \in \mathbf{F}^{\mathbb{R}}(G_{K_0})\}$$

defined in [11, 4.2.5] verifies that for two opposed filtrations $\mathcal{F}$ and $\mathcal{F}^{\iota}$, we have

$$\langle \mathbf{t}(\mathcal{F}), \mathbf{t}(\mathcal{F}^{\iota}) \rangle^{\mathrm{tr}} = \langle \mathcal{F}, \mathcal{F}^{\iota} \rangle.$$

Thus $\langle \mathbf{t}_{\mathrm{N}}, \mathbf{t}^{\iota}_{\mathrm{N}} \rangle^{\mathrm{tr}} = \langle \mathcal{F}_{\mathrm{N}}, \mathcal{F}^{\iota}_{\mathrm{N}} \rangle$ and $\langle \mathcal{F}_{\mathrm{H}}, \mathcal{F}^{\iota}_{\mathrm{N}} \rangle \geq \langle \mathbf{t}_{\mathrm{H}}, \mathbf{t}^{\iota}_{\mathrm{N}} \rangle^{\mathrm{tr}}$. Therefore,

$$\begin{array}{rcl} \langle \mathcal{F}_{\mathrm{H}}, \mathcal{F}^{\iota}_{\mathrm{N}} \rangle & \geq & \langle \mathbf{t}_{\mathrm{H}}, \mathbf{t}^{\iota}_{\mathrm{N}} \rangle^{\mathrm{tr}} \\ & = & \langle \mathbf{t}^{\#}_{\mathrm{H}}, \mathbf{t}^{\iota}_{\mathrm{N}} \rangle^{\mathrm{tr}} \\ & = & \langle \mathbf{t}_{\mathrm{N}}, \mathbf{t}^{\iota}_{\mathrm{N}} \rangle^{\mathrm{tr}} \\ & = & \langle \mathcal{F}_{\mathrm{N}}, \mathcal{F}^{\iota}_{\mathrm{N}} \rangle \end{array}$$

where the first equality is true because the operator $\langle \cdot, \cdot \rangle^{\mathrm{tr}}$ is invariant under the action of Galois and additive by [11, 4.2.7], and the second equality is given by our hypothesis. Thus, $\mathcal{F}^{\iota}_{\mathrm{N}} = \mathcal{F}_{\mathrm{F}}$.

□

Remark 23. 1. We will show later, in Proposition 24, that if V is isomorphic to a trivial germ of crystalline representations with G-structure, then D is isomorphic to a trivial isocrystal with G-structure.

2. The filtration $\mathcal{F}_{\mathrm{F}}(D)$ is the image of $\mathcal{F}_{\mathrm{F}}(V)$ by D_{cris}. However, since D_{cris} is not induced by an isomorphism between fiber functors, we cannot deduce directly that $\mathbf{t}_{\mathrm{F}}(D) = \mathbf{t}_{\mathrm{F}}(V)$.

Set $\mathcal{L}(V) = \mathcal{L}(\omega(V))$. Thus $\mathcal{L}(V) \neq \emptyset$ if and only if V is isomorphic to a trivial germ of crystalline representations with G-structure, by Remark 7.2.3, which we assume from now on. Any x in $\mathcal{L}(V)$ has a unique factorization through an exact and faithful $\otimes$-functor

$$\mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p}\{\mathrm{Gal}_{K_0}\}$$

that we also denote by x. We call such a $\otimes$-functor a germ of integral crystalline representations with G-structure.

Definition 7.3.9. 1. A finite extension $K \subset \overline{K}_0$ of K_0 is a field of definition of V if V factors through the full subcategory $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p} \mathrm{Gal}_K$ of $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\{\mathrm{Gal}_{K_0}\}$.

2. A finite extension $K \subset \overline{K}_0$ of K_0 is a field of definition of $x \in \mathcal{L}(V)$ if x factors through the full subcategory $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p} \mathrm{Gal}_K$ of $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p}\{\mathrm{Gal}_{K_0}\}$. We denote by $\mathcal{L}(V, K)$ the set of lattices in $\mathcal{L}(V)$ having K as field of definition.

Lemma 7.3.17. *There exists a field of definition of V.*

Proof. Let τ be a $\otimes$-generator of $\mathrm{Rep}_{\mathbb{Q}_p} G$. Then $V(\tau) \in \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p} \mathrm{Gal}_K$ for some large enough K, in which case also $V(\tau') \in \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p} \mathrm{Gal}_K$ for every $\tau' \in \mathrm{Rep}_{\mathbb{Q}_p} G$. Then V factors through the full-subcategory $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p} \mathrm{Gal}_K$ of $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\{\mathrm{Gal}_{K_0}\}$.

□

Let K be a field of definition of V. Then Gal_K acts on $\mathcal{L}(V)$ by

$$(g \cdot x)(\tau) = g \cdot x(\tau)$$

for $g \in \mathrm{Gal}_K$, $x \in \mathcal{L}(V)$ and $\tau \in \mathrm{Rep}_{\mathbb{Z}_p} G$. Plainly, $\mathcal{L}(V, K') = \mathcal{L}(V)^{\mathrm{Gal}_{K'}}$ for every finite extension K' of K.

Lemma 7.3.18. *We have*

$$\mathcal{L}(V) = \cup \mathcal{L}(V, K')$$

for $K' \subset \overline{K}_0$ running through the finite extensions of K. In particular, every $x \in \mathcal{L}(V)$ has a field of definition.

Proof. We may assume that V is trivial. Then $V : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p} \mathrm{Gal}_K$ is induced by a continuous morphism $\mathrm{Gal}_K \to \mathrm{Aut}^{\otimes}(V) = G(\mathbb{Q}_p)$. Since any $x \in \mathcal{L}(V) \simeq G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ has an open stabilizer in $G(\mathbb{Q}_p)$, it also has an open stabilizer in Gal_K, which proves the lemma.

□

Fix K, a field of definition of V such that $\mathcal{L}(V, K) \neq \emptyset$, with uniformizer π_K and E_K the minimal polynomial of π_K. Then, V induces an isogeny class of Kisin modules with G-structure N, defined by the exact and faithful $\otimes$-functor

$$N = \mathfrak{N} \circ V \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Mod}^{\varphi, E_K}_{\mathfrak{S}[\frac{1}{p}]} .$$

Let

$$D' = N/uN : \mathrm{Rep}_{\mathbb{Q}_p} G \to^{\mathrm{wa}} \mathrm{MF}^{\sigma}_K$$

the weakly admissible filtered G-isocrystal associated to N. We also denote by D' the underlying G-isocrystal.

Note 2. Until now, we have avoided the superscript E_K on categories of Kisin modules, to ease the notations, as we were working with a fixed field K. Since we will be changing K, depending on the field of definition of germs of integral crystalline representations with G-structure, we use now the rigorous notation, including E_K.

Given an integral crystalline representation with G-structure $x \in \mathcal{L}(V, K)$, we can associate a Kisin module with G-structure M, with isogeny class N, by composition with the Kisin functor $\mathfrak{M}$. Next proposition tells us that M is an exact functor (we already know that it is a faithful $\otimes$-functor) giving us a map

$$\begin{array}{ccc} \mathcal{L}(V,K) & \to & \mathcal{L}(N) \\ x & \mapsto & \mathfrak{M} \circ x. \end{array}$$

Proposition 7.3.19. *Let $x : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p} \mathrm{Gal}_K$ be an integral crystalline representation with G-structure. Then the functor*

$$M = \mathfrak{M}(x) \ : \ \mathrm{Rep}_{\mathbb{Z}_p} G \xrightarrow{x} \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Z}_p} \mathrm{Gal}_K \xrightarrow{\mathfrak{M}} \mathrm{Mod}^{\varphi, E_K}_{\mathfrak{S},\mathrm{fr}}$$

is exact.

Proof. We view $\omega(M)$ as a faithful $\otimes$-functor

$$\omega \ : \ \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Bun}_{\mathfrak{S}} \, .$$

Let $\omega_U : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Bun}_U$ be the restriction of $\omega(M)$ to the open set $U = \mathrm{Spec}\, \mathfrak{S} \backslash \{\mathfrak{m}\} \subset \mathrm{Spec}\, \mathfrak{S}$. Then ω_U is a faithful $\otimes$-functor which is exact by the exactness of x and the properties of the Kisin functor $\mathfrak{M}$. By Broshi [7], such functors are classified by $H^1(U, G)$. By [10, Theorem 6.13], we have

$$H^1(U, G) = H^1(\mathfrak{S}, G)$$

and we have seen in the proof of 7.2.1 that $H^1(\mathfrak{S}, G) = 0$. Thus, $\omega_U \simeq \omega_{G,U}$. But then, we have

$$\omega = \Gamma(U, -) \circ \omega_U \simeq \Gamma(U, -) \circ \omega_{G,U} \simeq \omega_{G,\mathfrak{S}},$$

so $\omega \simeq \omega_{G,\mathfrak{S}}$ is exact.

□

Remark 24. In particular, $\mathcal{L}(N) \neq \emptyset$ since $\mathcal{L}(V, K) \neq \emptyset$. Thus $\omega(N) \simeq \omega_{G_{\mathbb{Q}_p}, \mathfrak{S}[\frac{1}{p}]}$ by Proposition 7.3.12. Therefore,

$$\omega(D') = \omega(N) \otimes K_0 \simeq \omega_{G_{\mathbb{Q}_p}, \mathfrak{S}[\frac{1}{p}]} \otimes K_0 = \omega_{G_{\mathbb{Q}_p}, K_0}$$

so D' is isomorphic to a trivial G-isocrystal. By Kisin's construction, there is an isomorphism $\eta : D \to D'$, so D is isomorphic to a trivial G-isocrystal, as claimed in last Remark.

Proposition 7.3.20. *We have*

$$\mathbf{t}_{\mathrm{H}}(N) = \mathbf{t}_{\mathrm{H}}(D')^{\iota} = \mathbf{t}_{\mathrm{H}}(D)^{\iota}.$$

Proof. The isomorphism $D \simeq D'$ gives $\mathbf{t}_{\mathrm{H}}(D) = \mathbf{t}_{\mathrm{H}}(D')$. On the other hand, the $\otimes$-isomorphism

$$\omega(D') \otimes_{K_0} K \to \omega(\varphi_N \varphi^* N) \otimes_{\mathfrak{S}} K$$

maps $\mathcal{F}_{\mathrm{H}}(D')$ to

$$\mathcal{F}^{\iota}_{\mathrm{H}}(N) = \mathcal{F}(\omega(\varphi_N \varphi^* N) \otimes \hat{\mathfrak{S}}, \omega(N) \otimes \hat{\mathfrak{S}}).$$

as we have seen in section 3.4. Therefore,

$$\begin{aligned}\mathbf{t}_{\mathrm{H}}(D') = \mathbf{t}(\mathcal{F}_{\mathrm{H}}(D') & \\ &= \mathbf{t}(\mathcal{F}^{\iota}_{\mathrm{H}}(N)) \\ &= \mathrm{Pos}\left(\omega(\varphi_N\varphi^*N)\otimes\hat{\mathfrak{S}}, \omega(N)\otimes\hat{\mathfrak{S}}\right) \\ &= \mathrm{Pos}\left(\omega(N)\otimes\hat{\mathfrak{S}}, \omega(\varphi_N\varphi^*N)\otimes\hat{\mathfrak{S}}\right)^{\iota} \\ &= \mathbf{t}^{\iota}_{\mathrm{H}}(N)\end{aligned}$$

□

The next proposition gives us the relation between the Fargues type defined on V and the types defined on the torsion Kisin module with G-structure $\overline{M}$, for $x \in \mathcal{L}(V,K)$ and $M = \mathfrak{M}\circ x \in \mathcal{L}(N)$.

Proposition 7.3.21. *We have*

1. *The inequality* $e\cdot\mathbf{t}_{\mathrm{F}}(V) \leq \mathbf{t}_{\mathrm{H},n}(\overline{M})^{\#}$.
2. *If* $\overline{M}$ *is aligned, then* $e\cdot\mathbf{t}_{\mathrm{F}}(V) \leq \mathbf{t}_{\mathrm{F},1}(\overline{M})$. *If, moreover,* $e\cdot\mathbf{t}_{\mathrm{F}}(V) = \mathbf{t}_{\mathrm{F},1}(\overline{M})$, *then,* M *is HN-type.*

Proof. 1. In section 4 we have seen that for all $\tau \in \mathrm{Rep}_{\mathbb{Z}_p} G$, there are inequalities

$$e\cdot\mathbf{t}_{\mathrm{F}}(V)(\tau) = e\cdot\mathbf{t}_{\mathrm{F}}(N(\tau)) \leq \mathbf{t}_{\mathrm{F},1}(\overline{M})(\tau) \leq \mathbf{t}_{\mathrm{H},\infty}(\overline{M}(\tau)) \leq \mathbf{t}_{\mathrm{H},n}(\overline{M})(\tau)$$

for every $n \geq 1$ (the coefficient e appears since we have not renormalized the types defined on $\overline{M}$ as we did in section 5), since proposition 6.1.1 gives us $\mathbf{t}_{\mathrm{F},1}(\overline{M}(\tau)) \leq \mathbf{t}_{\mathrm{H},n}(\overline{M}(\tau))$ for all τ and every $n \geq 1$. Using proposition 7.1.1, we get

$$e\cdot\mathbf{t}_{\mathrm{F}}(V) = e\cdot\mathbf{t}_{\mathrm{F}}(V)^{\#} \leq \mathbf{t}_{\mathrm{H},n}(\overline{M})^{\#}$$

for every $n \geq 1$, the first equality given by Remark 22.

2. For every $\tau \in \mathrm{Rep}_{\mathbb{Z}_p} G$, we have

$$e\cdot\mathbf{t}_{\mathrm{F}}(V)(\tau) = e\cdot\mathbf{t}_{\mathrm{F}}(N(\tau)) \leq \mathbf{t}_{\mathrm{F},1}(\overline{M})(\tau),$$

thus $e\cdot\mathbf{t}_{\mathrm{F}}(V) \leq \mathbf{t}_{\mathrm{F},1}(\overline{M})$ by Proposition 7.1.1, since both types are invariant by # by Remark 22 and Proposition 7.3.10, respectively. The equality implies that

$$e\cdot\mathbf{t}_{\mathrm{F}}(V)(\tau) = e\cdot\mathbf{t}_{\mathrm{F}}(N)(\tau) = \mathbf{t}_{\mathrm{F},1}(\overline{M})(\tau)$$

for every $\tau \in \mathrm{Rep}_{\mathbb{Z}_p} G$, so $M(\tau)$ is HN-type by Proposition 6.4.3, and M is HN-type.

□

For every finite extension $K_0 \subset K \subset \overline{K}_0$, we can construct maps

$$\begin{array}{rcccc} D^K_{\mathrm{cris}} & : & \mathcal{L}(V,K) & \to & \mathcal{L}(D) \\ & & x & \mapsto & y \end{array}$$

where $y = \eta(y')$ for $y' \in \mathcal{L}(D')$ the image of $M = \mathfrak{M}\circ x$ for the map $\mathcal{L}(N) \to \mathcal{L}(D')$ constructed in last section, and $\eta : \mathcal{L}(D') \simeq \mathcal{L}(D)$ induced by the isomorphism between Fontaine's D_{cris} and Kisin's functor $\mathfrak{M}\otimes K_0$. This construction extends to a map

$$\mathrm{red} \ : \ \mathcal{L}(V) \to \mathcal{L}(D)$$

thanks to Liu's results in 3.8.2 which gives us compatibility of D_{cris} with base change for K' a totally ramified extension of K, and Proposition 7.3.18 which gives us $\mathcal{L}(V) = \cup\mathcal{L}(V,K)$.

We can still say a bit more about the image of this functor.

Proposition 7.3.22. *There is a factorization*

$$\begin{tikzcd} \mathcal{L}(V) \arrow[rr, "\mathrm{red}"] \arrow[dr, "\mathrm{red}"'] & & \mathcal{L}(D) \\ & \mathcal{L}(D, \leq \mathbf{t}_{\mathrm{H}}^{\iota}(D)) \arrow[ur, hook] & \end{tikzcd}$$

where the maps are equivariant under the action of $\mathrm{Aut}^{\otimes}(V) \to \mathrm{Aut}^{\otimes}(D)$.

Proof. We have $\mathcal{L}(D) \simeq \mathcal{L}(D')$ and $\mathcal{L}(D, \leq \mathbf{t}_{\mathrm{H}}^{\iota}(D)) \simeq \mathcal{L}(D', \mathbf{t}_{\mathrm{H}}^{\iota}(D'))$, by Proposition 7.3.20. We conclude by the analogous result for D' given in Proposition 7.3.13, since $\mathbf{t}_{\mathrm{H}}(N) = \mathbf{t}_{\mathrm{H}}(D)^{\iota}$ by Proposition 7.3.20.

□

7.3.5 The ordinary case

Fix a germ of crystalline representations with G-structure

$$V \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\{\mathrm{Gal}_{K_0}\}$$

inducing a filtered G-isocrystal

$$D = D_{\mathrm{cris}} \circ V \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Mod}^{\sigma}_{K_0} \, .$$

Proposition 7.3.23. *Suppose V isomorphic to a trivial germ of crystalline representations with G-structure, then D is isomorphic to a trivial filtered G-isocrystal and we have*

$$\mathbf{t}_{\mathrm{N}}(D) \leq \mathbf{t}_{\mathrm{H}}(D)^{\#} \quad \textit{in} \quad \mathbf{C}^{\mathbb{Q}}(G_{K_0}).$$

Proof. We have V isomorphic to a trivial germ of crystalline representations with G-structure if and only if $\mathcal{L}(V) \neq \emptyset$, which implies that $\mathcal{L}(D) \neq \emptyset$, so D is isomorphic to a trivial filtered G-isocrystal. The inequality is obtained assembling Proposition 7.3.3, Proposition 7.3.13 and Proposition 7.3.20.

□

Definition 7.3.10. We say that V is ordinary when

$$\mathbf{t}_{\mathrm{N}}(D) = \mathbf{t}_{\mathrm{H}}(D)^{\#} \quad \text{in} \quad \mathbf{C}^{\mathbb{Q}}(G_{K_0}).$$

Under the ordinarity condition, the more general proposition in last subsection becomes:

Corollary 7.3.24. *Suppose that V is ordinary, then there is a factorization*

$$\begin{tikzcd} \mathcal{L}(V) \arrow[rr, "\mathrm{red}"] \arrow[dr, "\mathrm{red}"'] & & \mathcal{L}(D). \\ & \mathcal{L}(D, \mathbf{t}_{\mathrm{H}}^{\iota}(D)) \arrow[ur, hook] & \end{tikzcd}$$

In particular, D is $\mathbf{t}_{\mathrm{H}}^{\iota}(D)$-ordinary.

Proof. Let $x \in \mathcal{L}(V)$ with image $\mathrm{red}(x) = y \in \mathcal{L}(D)$. From $\mathbf{t}_{\mathrm{N}}^{\iota}(D) \leq \mathbf{t}_{\mathrm{H}}(y)^{\#} \leq \mathbf{t}_{\mathrm{H}}^{\iota}(D)^{\#}$ and the ordinary hypothesis $\mathbf{t}_{\mathrm{N}}^{\iota}(D) = \mathbf{t}_{\mathrm{H}}^{\iota}(D)^{\#}$, we get the equality $\mathbf{t}_{\mathrm{H}}(y)^{\#} = \mathbf{t}_{\mathrm{H}}^{\iota}(D)^{\#}$. Moreover, we have seen that $\mathbf{t}_{\mathrm{H}}(y) \leq \mathbf{t}_{\mathrm{H}}^{\iota}(D)$. Thus actually, $\mathbf{t}_{\mathrm{H}}(y) = \mathbf{t}_{\mathrm{H}}^{\iota}(D)$.

□

Theorem 7.3.25. *Suppose V is ordinary, with field of definition K. Given $x \in \mathcal{L}(V, K)$ and N as before, we have that for every $M \in \mathcal{L}(N)$ (for instance $M = \mathfrak{M} \circ x$),*

1. *The torsion Kisin module $\overline{M}$ with G-structure is aligned.*

2. *M is HN-type. Therefore $\mathcal{F}_{\mathrm{F}}(M)$ exists.*

Proof. 1. Under the ordinary condition we have (with $e = [K : K_0]$)

$$e \cdot \mathbf{t}_{\mathrm{N}}^{\iota}(D) = e \cdot \mathbf{t}_{\mathrm{F}}(D) = e \cdot \mathbf{t}_{\mathrm{F}}(V) \leq \mathbf{t}_{\mathrm{H},mn}(\overline{M})^{\#} \leq \mathbf{t}_{\mathrm{H},n}(\overline{M})^{\#} \leq e \cdot \mathbf{t}_{\mathrm{H}}(N)^{\#} = e \cdot \mathbf{t}_{\mathrm{N}}^{\iota}(D),$$

where the inequalities are given by Propositions 7.3.16, 7.3.21, 7.3.7, 7.3.14 and 7.3.20. Thus $\mathbf{t}_{\mathrm{H},mn}(\overline{M})^{\#} = \mathbf{t}_{\mathrm{H},\,n}(\overline{M})^{\#}$ for every $m \geq 1$ and $\overline{M}$ is aligned, by Proposition 7.3.8.

2. We use the fact that $\overline{M}$ is aligned, which means that $\mathcal{F}_{\mathrm{F},1}$ is an exact $\otimes$-functor, so a filtration in $\mathbf{F}^{\mathbb{Q}}(G_{\mathbb{F}[[u]]})$ and we can consider its type $\mathbf{t}_{\mathrm{F},1}$ as an element in $\mathbf{C}^{\mathbb{Q}}(G_{\mathbb{F}[[u]]})$. Then, we get inequalities

$$e \cdot \mathbf{t}_{\mathrm{F}}(V) \leq \mathbf{t}_{\mathrm{F},1}(\overline{M}) \leq \mathbf{t}_{\mathrm{H},1}(\overline{M})^{\#} \leq e \cdot \mathbf{t}_{\mathrm{H}}(N)^{\#}$$

by Proposition 7.3.21, Proposition 7.3.10 and Proposition 7.3.14. Again, by ordinarity, since $\mathbf{t}_{\mathrm{F}}(V) = \mathbf{t}_{\mathrm{F}}(D) = \mathbf{t}_{\mathrm{H}}^{\iota}(D)^{\#} = \mathbf{t}_{\mathrm{H}}(N)^{\#}$, we get

$$e \cdot \mathbf{t}_{\mathrm{F}}(V) = \mathbf{t}_{\mathrm{F},1}(\overline{M})$$

and so M is HN-type, by Proposition 7.3.21.

□

Next proposition allows us to associate a $\mathbb{Z}$-filtration to $\mathcal{F}_{\mathrm{F}}(V)$, which will be used to define some operators:

Lemma 7.3.26. *There exists an integer $s > 0$ such that $s\mathcal{F}_{\mathrm{F}}(V) \in \mathbf{F}^{\mathbb{Z}}(\omega(V))$.*

Proof. We may suppose that V is trivial. Then $\mathcal{F}_{\mathrm{F}}(V)$ is (non-canonically) split by a morphism $\mathcal{G} : \mathbb{D}(\mathbb{Q}) \to G_{\mathbb{Q}_p}$. The kernel of this morphism is $\mathbb{D}(\mathbb{Q}/N)$, where N is a finitely generated subgroup of $\mathbb{Q}$, i.e. there exists an integer $s > 0$ such that $sN \subset \mathbb{Z}$. Thus $s\mathcal{G}$ factors through

$$\begin{tikzcd} \mathbb{D}(\mathbb{Q}) \arrow[rr, "s\mathcal{G}"] \arrow[dr] & & G_{\mathbb{Q}_p} \\ & \mathbb{D}(\mathbb{Z}) \arrow[ur] & \end{tikzcd}$$

and $\mathrm{Fil}(s\mathcal{G}) = s\,\mathrm{Fil}(\mathcal{G}) = s\mathcal{F}_{\mathrm{F}}(V)$ is a $\mathbb{Z}$-filtration.

□

We fix $s > 0$ such that $s\mathcal{F}_{\mathrm{F}}(V) \in \mathbf{F}^{\mathbb{Z}}(\omega(V))$. We define the étale and crystalline operators by

$$\begin{array}{rccc} \Phi_{\text{ét}}^{s} & : & \mathcal{L}(V) & \to & \mathcal{L}(V) \\ & & x & \mapsto & \Phi_{\text{ét}}^{s}(x) = x + s\mathcal{F}_{\mathrm{F}}(V) \end{array}$$

and

$$\begin{array}{rccc} \Phi_{\text{cris}}^{s} & : & \mathcal{L}(D) & \to & \mathcal{L}(D) \\ & & y & \mapsto & \Phi_{\text{cris}}^{s}(y) = y + s\mathcal{F}_{\mathrm{N}}^{\iota}(D), \end{array}$$

where the addition of a lattice and a filtration is given in Definition 7.2.2. The following proposition gives us the compatibility between the two operators.

Proposition 7.3.27. *Suppose that V is ordinary and let $x \in \mathcal{L}(V)$. Then,*

$$\mathrm{red}(\Phi_{\text{ét}}^{s}(x)) = \Phi_{\text{cris}}^{s}(\mathrm{red}(x)).$$

Proof. Fix a field of definition K for x, so that $x \in \mathcal{L}(V, K)$. Then also

$$x' = \Phi^s_{et}(x) \in \mathcal{L}(V, K)$$

since $\mathcal{F}_F(V)$ is a Gal_K-stable filtration on V. Fix also a uniformizer π_K of K, giving rise to $N = \mathfrak{N}(V)$, $M = \mathfrak{M}(x)$ and $M' = \mathfrak{M}(x')$ in $\mathcal{L}(N)$, as well as their images y and y' in $\mathcal{L}(D')$, where D' is the (weakly admissible filtered) G-isocrystal obtained from N by reduction modulo u. We have to show that

$$y' = \Phi^s_{\mathrm{cris}}(y) \quad \text{in} \quad \mathcal{L}(D')$$

where $\Phi^s_{\mathrm{cris}}(y) = y + s\mathcal{F}^\iota_N(D')$: the isomorphism of G-isocrystals $\eta : D' \to D$ transports this equality in $\mathcal{L}(D')$ to the desired equality in $\mathcal{L}(D)$.

Since V is ordinary, M is HN-type and $\mathcal{F}_F(M)$ exists. By compatibility of the various Fargues filtrations,

$$\mathcal{F}_F(N) := \mathcal{F}_F(M)[\tfrac{1}{p}] = \mathfrak{N}\left(\mathcal{F}_F(V)\right)$$

and this reduces to $\mathcal{F}_F(D')$, which equals $\mathcal{F}^\iota_N(D')$ by Proposition 7.3.16. In particular, multiplying any of these $\mathbb{Q}$-filtrations by s yields $\mathbb{Z}$-filtrations. Let

$$\omega(M) = \oplus_{a\in\mathbb{Z}}\omega_a(M)$$

be a splitting of $\mathcal{F}_F(M)$. Note that such a splitting exists by Proposition 7.3.12 and [11, 3.11.3]. It gives rise to splittings

$$\omega(N) = \oplus_{a\in\mathbb{Z}}\omega_a(N) \quad \text{and} \quad \omega(D') = \oplus_{a\in\mathbb{Z}}\omega_a(D')$$

of $\mathcal{F}_F(N)$ and $\mathcal{F}_F(D') = \mathcal{F}^\iota_N(D')$. Consider now some $\tau \in \mathrm{Rep}_{\mathbb{Z}_p}G$. By Proposition 3.6.1 and the computations preceeding it,

$$\begin{aligned} M'(\tau) &= M(\tau) +_{\mathrm{fr}} s\mathcal{F}_F(N)(\tau) \\ &= M(\tau) + s\mathcal{F}_F(N)(\tau) \end{aligned}$$

with underlying $\mathfrak{S}$-module

$$\omega(M')(\tau) = \oplus_{a\in\mathbb{Z}}p^{-a}\omega_a(M)(\tau).$$

This reduces modulo u to the $W(\mathbb{F})$-lattice

$$y'(\tau) = \oplus_{a\in\mathbb{Z}}p^{-a}\omega_a(y)(\tau)$$

in $D'(\tau)$, where $y = \oplus_{a\in\mathbb{Z}}\omega_a(y)$ is the reduction of $\omega(M) = \oplus_{a\in\mathbb{Z}}\omega_a(M)$, so that also $\omega_a(y)[\frac{1}{p}] = \omega_a(D')$ for every $a \in \mathbb{Z}$. On the other hand,

$$\begin{aligned} \Phi^s_{\mathrm{cris}}(y)(\tau) &= y(\tau) + s\mathcal{F}^\iota_N(D')(\tau) \\ &= \textstyle\sum_{i\in\mathbb{Z}}p^{-i}y(\tau) \cap \left(s\mathcal{F}^\iota_N(D')\right)^i(\tau) \\ &= \textstyle\sum_{i\in\mathbb{Z}}\left(\oplus_a p^{-i}\omega_a(y)(\tau)\right) \cap \left(\oplus_{a\geq i}\omega_a(D')(\tau)\right) \\ &= \textstyle\sum_{i\in\mathbb{Z}}\left(\oplus_{a\geq i}p^{-i}\omega_a(y)(\tau)\right) \\ &= \oplus_{a\in\mathbb{Z}}\textstyle\sum_{i\leq a}p^{-i}\omega_a(y)(\tau) \\ &= \oplus_{a\in\mathbb{Z}}p^{-a}\omega_a(y)(\tau). \end{aligned}$$

Thus $\Phi^s_{\mathrm{cris}}(y)(\tau) = y'(\tau)$ for every $\tau \in \mathrm{Rep}_{\mathbb{Z}_p}(G)$ and $\Phi^s_{\mathrm{cris}}(y) = y'$.

□

Lemma 7.3.28. *Let $x, y \in \mathcal{L}(V)$ and let*

$$\pi : \mathcal{L}(V) \to U_{\mathcal{F}_F}(\mathbb{Q}_p)\backslash\mathcal{L}(V)$$

be the usual projection, for $\mathcal{F}_F$ the Fargues filtration on V and $U_{\mathcal{F}_F}$ the unipotent radical of the parabolic subgroup of G associated to $\mathcal{F}_F$. Then, we have $\pi(x) = \pi(y)$ if and only if

$$(\Phi^s_{\text{ét}})^{(n)}(x) = (\Phi^s_{\text{ét}})^{(n)}(y)$$

for n large enough.

Proof. We may assume V is trivial. We use the results in [11] as follows: In 6.2.7, we see that $\mathbf{B}^e(\omega_G)$ is a tight building, so verifies the axiom ST. It also verifies $L(s)$, thus it verifies UN^+, by Lemma 114. Then, the lemma holds by 5.6.2.

□

Lemma 7.3.29. *The operator Φ^s_{cris} is bijective on the set $\mathcal{L}(D, \mathbf{t}^\iota_{\text{H}}(D))$.*

Proof. As we will see in the next chapter (see Remark 25), the lemma is true in the abelian case. We can construct an abelian germ of crystalline representations V' such that $D' = D_{\text{cris}}(V')$ is $\mathbf{t}^\iota_{\text{H}}(D)$-ordinary: fix a Borel pair (T_G, B_G) in G and lift $\mathbf{t}_{\text{H}}(D)$ to a B_G-dominant cocharacter $\tilde{\mathbf{t}}_{\text{H}}(D) : \mathbb{G}_{m,K_0} \to T_{G,K_0}$. Let V' be the abelian germ of crystalline representations associated to this cocharacter by Proposition 8.4.1. By Proposition 8.4.3, V' is ordinary and by Proposition 7.3.24, D' is $\mathbf{t}^\iota_{\text{H}}(D)$-ordinary.

Now, by Proposition 7.3.4, there is a unique $\mathbf{t}^\iota_{\text{H}}(D)$-ordinary G-isocrystal up to isomorphism, we get $D' \simeq D$ and the lemma follows.

□

As a consequence of the three previous results, we obtain:

Theorem 7.3.30. *Notations as above. For V an ordinary germ of crystalline representations with G-structure, the map* red *admits a factorization*

$$\begin{tikzcd} \mathcal{L}(V) \arrow[rr, "\text{red}"] \arrow[dr, two heads, "\pi"] & & \mathcal{L}(D, \mathbf{t}^\iota_{\text{H}}(D)). \\ & U_{\mathcal{F}_{\text{F}}}(\mathbb{Q}_p)\backslash\mathcal{L}(V) \arrow[ur] & \end{tikzcd}$$

Proof. Suppose we have $\pi(x) = \pi(y)$ for $x, y \in \mathcal{L}(V)$. By last lemma, we have

$$(\Phi^s_{\text{ét}})^{(n)}(x) = (\Phi^s_{\text{ét}})^{(n)}(y)$$

for n large enough. We apply the corollary above to obtain

$$(\Phi^s_{\text{cris}})^{(n)}(\text{red}(x)) = \text{red}((\Phi^s_{\text{ét}})^{(n)}(x)) = \text{red}((\Phi^s_{\text{ét}})^{(n)}(y)) = (\Phi^s_{\text{cris}})^{(n)}(\text{red}(y))$$

for n large enough, and the theorem follows by Lemma 7.3.29.

□

Chapter 8

A particular case: the abelian case

The aim of this section is to describe more precisely the results obtained in last section in the case where the germ of crystalline representations with G-structure is abelian. We use the same notations from last section. Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 0$, $K_0 = \operatorname{Frac} W(\mathbb{F})$, we fix an algebraic closure $\overline{K}_0$ of K_0 and an embedding $\iota_0 : \overline{\mathbb{Q}}_p \subset \overline{K}_0$. We shall use the notation E for a finite extension $\mathbb{Q}_p \subset E \subset \overline{\mathbb{Q}}_p$, with ring of integers $\mathcal{O}_E$ and uniformizer π_E, and the notation K for a finite extension $K_0 \subset K \subset \overline{K}_0$, with ring of integers $\mathcal{O}_K$ and uniformizer π_K.

8.1 Preliminaries

8.1.1 The pro-tori T_K and T

Denote by

$$T_E = \operatorname{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_{m,E})$$

the torus over $\mathbb{Q}_p$ associated to the group $E^\times$, i.e. for a $\mathbb{Q}_p$-algebra R, we define $T_E(R)$ as the group of units of $E \otimes_{\mathbb{Q}_p} R$. For $E \subset E' \subset K$, the norm $N_{E'/E}$ induces a morphism $T_{E'} \to T_E$. We define two pro-tori by

$$T_K = \varprojlim_{E \subset K} T_E \quad \text{and} \quad T = \varprojlim_{E \subset \overline{K}_0} T_E$$

where the norms $N_{E'/E}$ are the transition maps. For every $E \subset K$, there is a continuous morphism

$$\chi_{K,E} \ : \ \operatorname{Gal}_K \to \operatorname{Gal}(E^{\mathrm{ab}}/E^{\mathrm{ur}}) \simeq \mathcal{O}_E^\times \subset E^\times = T_E(\mathbb{Q}_p)$$

and since these morphisms are compatible with the norm maps, they define a continuous morphism

$$\chi_K \ : \ \operatorname{Gal}_K \to T_K(\mathbb{Q}_p) = \varprojlim_{E \subset K} T_E(\mathbb{Q}_p).$$

The maps $T \to T_K \to T_E$ give us maps between the characters groups

$$X^*(T_E) \to X^*(T_K) \to X^*(T)$$

and the cocharacters groups

$$X_*(T) \to X_*(T_K) \to X_*(T_E).$$

For a finite extension E of $\mathbb{Q}_p$, we have

$$X^*(T_E) \simeq \mathcal{C}(\operatorname{Hom}(E, \overline{\mathbb{Q}}_p), \mathbb{Z}),$$

the set of functions $f : \operatorname{Hom}(E, \overline{\mathbb{Q}}_p) \to \mathbb{Z}$. The character $\chi_f : T_{E|\overline{\mathbb{Q}}_p} \to \mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ corresponding to f induces

$$\begin{array}{ccc} T_E(\overline{\mathbb{Q}}_p) & \xrightarrow{\chi_f} & \mathbb{G}_m(\overline{\mathbb{Q}}_p) \\ \| & & \| \\ (E\otimes\overline{\mathbb{Q}}_p)^\times & \longrightarrow & \overline{\mathbb{Q}}_p^\times \\ x & \longmapsto & \prod_{\iota:E\hookrightarrow\overline{\mathbb{Q}}_p}(\iota\otimes\mathrm{id})(x)^{f(\iota)}. \end{array}$$

The action of $\mathrm{Gal}_{\mathbb{Q}_p}$ on $X^*(T_E)$ is given by

$$(\sigma\cdot f)(\iota) = f(\sigma^{-1}\circ\iota), \quad \text{for } \sigma\in\mathrm{Gal}_{\mathbb{Q}_p},\ \iota : E\hookrightarrow\overline{\mathbb{Q}}_p.$$

Since $T = \varprojlim T_E$, for E running through all finite extensions of $\mathbb{Q}_p$ contained in $\overline{K}_0$,

$$X^*(T) = \varinjlim X^*(T_E) \simeq \mathcal{C}(\mathrm{Gal}_{\mathbb{Q}_p},\mathbb{Z}),$$

the set of locally constant functions $f : \mathrm{Gal}_{\mathbb{Q}_p}\to\mathbb{Z}$ with the Galois action

$$(\sigma\cdot f)(\sigma') = f(\sigma^{-1}\sigma') \quad \text{for } \sigma,\sigma'\in\mathrm{Gal}_{\mathbb{Q}_p}\ .$$

Similarly, for $T_K = \varprojlim_{E\subset K} T_E$,

$$X^*(T_K) = \varinjlim_{E\subset K} X^*(T_E) \simeq \mathcal{C}(\mathrm{Gal}_{\mathbb{Q}_p}/\mathrm{Gal}_{K\cap\overline{\mathbb{Q}}_p},\mathbb{Z}),$$

the set of locally constant, right $\mathrm{Gal}_{K\cap\overline{\mathbb{Q}}_p}$-invariant functions $f : \mathrm{Gal}_{\mathbb{Q}_p}\to\mathbb{Z}$.

8.1.2 The Hodge cocharacters

We define the Hodge cocharacter

$$\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p}\to T_{|\overline{\mathbb{Q}}_p}$$

as the cocharacter corresponding to the evaluation at 1 in the dual group

$$X_*(T) = \mathrm{Hom}_{\mathrm{Gr}}(X^*(T),\mathbb{Z}) = \mathrm{Hom}_{\mathrm{Gr}}(\mathcal{C}(\mathrm{Gal}_{\mathbb{Q}_p},\mathbb{Z}),\mathbb{Z}),$$

i.e. $\langle\cdot,\mu\rangle = \mathrm{ev}_1$, where

$$\begin{array}{rccc} \mathrm{ev}_1 & : & \mathcal{C}(\mathrm{Gal}_{\mathbb{Q}_p},\mathbb{Z}) & \to & \mathbb{Z} \\ & & f & \mapsto & f(1). \end{array}$$

Thus, for any $x\in\overline{\mathbb{Q}}_p^\times = \mathbb{G}_m(\overline{\mathbb{Q}}_p)$ and $f\in\mathcal{C}(\mathrm{Gal}_{\mathbb{Q}_p},\mathbb{Z})$,

$$\chi_f\circ\mu(x) = x^{\langle\chi_f,\mu\rangle} = x^{f(1)}.$$

We denote by $\mu_K\in X^*(T_K)$ and $\mu_E\in X^*(T_E)$ the corresponding Hodge cocharacters of T_K and T_E, which are respectively defined over $K\cap\overline{\mathbb{Q}}_p$ and E:

$$\mu_K : \mathbb{G}_{m,K\cap\overline{\mathbb{Q}}_p}\to T_{K|K\cap\overline{\mathbb{Q}}_p} \quad \text{and } \mu_E : \mathbb{G}_{m,E}\to T_{E|E}.$$

For every E-algebra R, the canonical decomposition $E\otimes E\simeq E\times E'$ where $E' = \ker m$ for

$$\begin{array}{ccc} E\otimes E & \xrightarrow{m} & E \\ x\otimes y & \mapsto & xy, \end{array}$$

yields a canonical decomposition $E\otimes R\simeq R\times R'$, and

$$\begin{array}{ccc} \mathbb{G}_{m,E}(R) & \xrightarrow{\mu_E} & T_E(R) \\ \| & & \| \\ R^\times & \longrightarrow & R^\times\times R'^\times \\ x & \longmapsto & (x,1). \end{array}$$

In the dual group $X_*(T_E)$ of $X^*(T_E) = \mathcal{C}(\mathrm{Hom}(E,\overline{\mathbb{Q}}_p),\mathbb{Z})$, the cocharacter μ_E corresponds to evaluation at the given embedding $E\hookrightarrow\overline{\mathbb{Q}}_p$.

8.1.3 The Newton cocharacter

Similarly, integration against the Haar measure μ_{Haar} of $\mathrm{Gal}_{\mathbb{Q}_p}$ defines a $\mathrm{Gal}_{\mathbb{Q}_p}$-invariant morphism

$$\begin{array}{rccc} i_{\mathrm{Haar}} & : & \mathcal{C}(\mathrm{Gal}_{\mathbb{Q}_p}, \mathbb{Z}) & \to & \mathbb{Q} \\ & & f & \mapsto & \int_{\mathrm{Gal}_{\mathbb{Q}_p}} f d\mu_{\mathrm{Haar}} \end{array}$$

which yields the Newton cocharacter, defined over $\mathbb{Q}_p$,

$$\nu \ : \ \mathbb{D}(\mathbb{Q}) \to T.$$

We denote by

$$\nu_K : \mathbb{D}(\mathbb{Q}) \to T_K \quad \text{and} \quad \nu_E : \mathbb{D}(\mathbb{Q}) \to T_E$$

the corresponding Newton cocharacters for T_K and T_E. Inside

$$\mathrm{Hom}\left(\mathbb{D}(\mathbb{Q})_{|\overline{\mathbb{Q}}_p}, T_{E|\overline{\mathbb{Q}}_p}\right) = \mathrm{Hom}\left(\mathcal{C}(\mathrm{Hom}(E, \overline{\mathbb{Q}}_p), \mathbb{Q}\right)$$

the Newton cocharacter ν_E corresponds to the morphism

$$\begin{array}{ccc} \mathcal{C}(\mathrm{Hom}(E, \overline{\mathbb{Q}}_p), \mathbb{Z}) & \to & \mathbb{Q} \\ f & \to & \dfrac{1}{[E : \mathbb{Q}_p]} \displaystyle\sum_{\iota : E \hookrightarrow \overline{\mathbb{Q}}_p} f(\iota). \end{array}$$

8.2 The Fontaine-Serre functor

Definition 8.2.1. We say that a crystalline representation $(V, \rho) \in \mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p} \mathrm{Gal}_K$ is abelian when its image $\rho(\mathrm{Gal}_K) \subset \mathrm{GL}(V)$ is abelian. We denote by $\mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Q}_p} \mathrm{Gal}_K$ the category of abelian crystalline representations of Gal_K.

Denote by $\mathrm{CrAb}(V, K) \subset \mathrm{Cr}(V, K)$ the set of all morphisms $\rho : \mathrm{Gal}_K \to \mathrm{GL}(V)$ such that (V, ρ) is an abelian crystalline representation of Gal_K, and set

$$\mathrm{CrAb}(V) := \varinjlim_{K_0 \subset_f K \subset \overline{K}_0} \mathrm{CrAb}(V, K) \subset \mathrm{Cr}(V).$$

We say that a germ of crystalline representations (V, ρ) is abelian when $\rho \in \mathrm{CrAb}(V)$. We denote by $\mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Q}_p}\{\mathrm{Gal}_{K_0}\}$ the category of germs of abelian crystalline representations, a full Tannakian subcategory of $\mathrm{Rep}^{\mathrm{cr}}_{\mathbb{Q}_p}\{\mathrm{Gal}_{K_0}\}$.

In [43, 2], Fontaine proves, building on Serre's results in [42], that there is an equivalence of categories

$$V^u_K \ : \ \mathrm{Rep}_{\mathbb{Q}_p} T_K \xrightarrow{\sim} \mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Q}_p} \mathrm{Gal}_K$$

given by the composition of χ_K with a representation $T_K(\mathbb{Q}_p) \to \mathrm{GL}(V)$.

Proposition 8.2.1. *The equivalences of neutralized Tannakian categories*

$$V^u_K \ : \ \mathrm{Rep}_{\mathbb{Q}_p} T_K \xrightarrow{\sim} \mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Q}_p} \mathrm{Gal}_K,$$

for every finite extension $K_0 \subset K \subset \overline{K}_0$, extends to a diagram

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}_p} T_K & \hookrightarrow & \mathrm{Rep}_{\mathbb{Q}_p} T \\ \downarrow{\scriptstyle V^u_K} & & \downarrow{\scriptstyle V^u} \\ \mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Q}_p} \mathrm{Gal}_K & \hookrightarrow & \mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Q}_p}\{\mathrm{Gal}_{K_0}\} \end{array}$$

where V^u is an equivalence of neutralized Tannakian categories.

Proof. Let $(V,\rho) \in \mathrm{Rep}_{\mathbb{Q}_p} T$. For a large enough finite extension K of K_0, we have $(V,\rho) \in \mathrm{Rep}_{\mathbb{Q}_p} T_K$, thus $(V, \rho \circ \chi_K) \in \mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Q}_p} \mathrm{Gal}_K$. For a finite extension $K \subset K'$, we have $(V,\rho) \in \mathrm{Rep}_{\mathbb{Q}_p} T_K \subset \mathrm{Rep}_{\mathbb{Q}_p} T_{K'}$ and the commutative diagram

$$\begin{tikzcd} \mathrm{Gal}_K \arrow[r, "\chi_K"] & T_K(\mathbb{Q}_p) \\ \mathrm{Gal}_{K'} \arrow[u, "\mathrm{Res}"] \arrow[r, "\chi_{K'}"] & T_{K'}(\mathbb{Q}_p) \arrow[u, "N_{K'/K}"'] \end{tikzcd}$$

induces another commutative diagram

$$\begin{tikzcd} \mathrm{Rep}_{\mathbb{Q}_p} T_K \arrow[r] \arrow[d] & \mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Q}_p} \mathrm{Gal}_K \arrow[d] \\ \mathrm{Rep}_{\mathbb{Q}_p} T_{K'} \arrow[r] & \mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Q}_p} \mathrm{Gal}_{K'} \end{tikzcd}$$

thus $\rho \circ \chi_K \in \mathrm{CrAb}(V)$ does not depend upon the chosen K, so V is a well-defined $\otimes$-functor. It is plainly compatible with fiber functors and faithful. The diagram is commutative, and any object or arrow on the bottom right category comes from an object or arrow on the bottom left category for a sufficiently large K, so V is also full and essentially surjective.

□

8.3 Wintenberger's functor

8.3.1 Universal norms

Let K_0^{a} be the composite of K_0 and $\overline{\mathbb{Q}}_p$ in $\overline{K}_0$. First we need to prove that there exists a norm compatible system of uniformizers $\pi = (\pi_K) \in \varprojlim K^\times$, for $K \subset K_0^{\mathrm{a}}$. For each finite extension $K_0 \subset K \subset \overline{K}_0$, let $v_K : K^\times \to \mathbb{Z}$ be the normalized valuations of K. For $K_0 \subset K_1 \subset K_2 \subset \overline{K}_0$, the norm map yields a morphism of exact sequences

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{O}_{K_2}^\times \arrow[r] \arrow[d] & K_2^\times \arrow[r, "v_{K_2}"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\ 1 \arrow[r] & \mathcal{O}_{K_1}^\times \arrow[r] & K_1^\times \arrow[r, "v_{K_1}"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

It is known by [39, 6.5.6 and 6.5.8] that the norm maps $N_{K_2/K_1} : \mathcal{O}_{K_2}^\times \to \mathcal{O}_{K_1}^\times$ are all surjective. Every extension of K_0 contained in K_0^{a} can be written as K_0E, for $E \subset \overline{\mathbb{Q}}_0$ a finite extension of $\mathbb{Q}_p$. There are finitely many finite extensions $E \subset \overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ of a given degree, so K_0 has only finitely many finite extensions of any given degree. Writing K_n for the composite of all finite extensions of K_0 whose degree is less than n, we thus obtain a totally ordered and countable cofinal subset of $\{K \mid K_0 \subset K \subset K_0^{\mathrm{a}}\}$, so we can use the Mittag-Leffler condition on this system to get an exact sequence

$$1 \to \varprojlim_{K_0 \subset K \subset K_0^{\mathrm{a}}} \mathcal{O}_K^\times \to \varprojlim_{K_0 \subset K \subset K_0^{\mathrm{a}}} K^\times \xrightarrow{v} \mathbb{Z} \to 0..$$

We fix $\pi = (\pi_K) \in \varprojlim_{K_0 \subset K \subset K_0^{\mathrm{a}}} K^\times$ such that $v(\pi) = 1$, i.e. a norm compatible system of uniformizers. From now on, we also require that $\pi_{K_0} = p$.

8.3.2 The element $b \in T(K_0)$

There is a morphism

$$\varprojlim_{K_0 \subset K \subset K_0^{\mathrm{a}}} K^\times \to T(K_0)$$

which maps

$$\{x_K\} \in \varprojlim_{K_0 \subset K \subset K_0^{\mathrm{a}}} K^\times$$

to the element

$$\{y_E\} \in T(K_0) = \varprojlim_{\mathbb{Q}_p \subset E \subset \overline{\mathbb{Q}}_p} T_E(K_0) = \varprojlim_{\mathbb{Q}_p \subset E \subset \overline{\mathbb{Q}}_p} (E \otimes K_0)^\times$$

defined as follows. Let $E_0 = E \cap K_0 = E \cap \mathbb{Q}_p^{\mathrm{nr}} \subset E$ be the maximal unramified extension of $\mathbb{Q}_p$ in E, and let $K \simeq E \otimes_{E_0} K_0$ be the compositum of E and K_0 in K_0^{a}. Then

$$E \otimes K_0 = E \otimes_{E_0} (E_0 \otimes_{\mathbb{Q}_p} K_0) = \prod_{i=0}^{[E_0:\mathbb{Q}_p]-1} E \otimes_{E_0,\sigma^i} K_0.$$

For $i = 0$, we have $E \otimes_{E_0} K_0 \simeq K$ by $e \otimes b \mapsto eb$. This construction allows us to define an inclusion

$$K^\times \simeq (E \otimes_{E_0} K_0)^\times \subset (E \otimes K_0)^\times = T_E(K_0).$$

We set y_E as the image of x_K under this inclusion.

Lemma 8.3.1. *This construction yields a well-defined morphism*

$$\varprojlim_{K_0 \subset K \subset K_0^{\mathrm{a}}} K^\times \to T(K_0).$$

Proof. Fix $E \subset E'$, giving $K \subset K'$ and $E_0 \subset E_0'$. We have to show that the norm

$$N_{E'/E} : T_{E'}(K_0) \to T_E(K_0)$$

maps $y_{E'}$ to y_E. Considering the decomposition

$$E \otimes K_0 = \prod_i L_i$$

where $L_i = E \otimes_{E_0,\sigma^i} K_0$ for $0 \le i \le a-1$ with $a = [E_0 : \mathbb{Q}_p]$, so that

$$\begin{array}{rcl} E' \otimes K_0 & = & E' \otimes_E (E \otimes K_0) \\ & = & \prod_i E' \otimes_E (E \otimes_{E_0,\sigma^i} K_0) \\ & = & \prod_i E' \otimes_{E_0,\sigma^i} K_0 \\ & = & \prod_i E' \otimes_{E_0'} (E_0' \otimes_{E_0,\sigma^i} K_0) \\ & = & \prod_i E' \otimes_{E_0',\sigma^{aj+i}} K_0 \\ & = & \prod_i \prod_j L_{i,j} \end{array}$$

with $L_{i,j} = E' \otimes_{E_0',\sigma^{aj+i}} K_0$ for $0 \le j \le b-1$ with $b = [E_0' : E]$. We obtain

$$N_{E'/E}(x_{i,j}) = (\prod_j N_{E'/E}(x_{i,j}))_i.$$

For $y_{E'}$, all $x_{i,j} = 1$ except $x_{0,0} = \pi_{K'}$, so $N_{E'/E}(x_{i,j})$ has all components 1, except at $i = 0$, where we get $\prod_j N_{E'/E}(x_{0,j}) = N_{E'/E}(x_{0,0}) = N_{E'/E}(\pi_{K'}) = \pi_K$. Thus $N_{E'/E}(y_{E'}) = y_E$.

□

We define b as the image of π in $T(K_0)$ by this morphism. Let b_E and b_K be the images of b in $T_E(K_0)$ and $T_K(K_0)$, respectively.

Lemma 8.3.2. *We have $b_K = N_{K/K_0}(\mu_K(\pi_K))$ in $T_K(K_0)$, for $K \subset K_0^{\mathrm{a}}$.*

Proof. We look at the images of both elements in $T_E(K_0)$ for a E large enough such that $K = E \cdot K_0$, using

$$\begin{aligned} E \otimes K_0 &= \prod_{i=0}^{[E_0:\mathbb{Q}_p]-1} E \otimes_{E_0,\sigma^i} K_0 = \prod_{i=0}^{[E_0:\mathbb{Q}_p]-1} L_i \\ E \otimes K &= \prod_{i=0}^{[E_0:\mathbb{Q}_p]-1} L_i \otimes_{K_0} K \end{aligned}$$

Then, b maps to $b_E = (\pi_K, 1, \ldots, 1)$ in $T_E(K_0) = \prod L_i^\times$, $\mu_K(\pi_K)$ maps to $(x, 1, \ldots, 1)$ in $T_E(K) = \prod (L_i \otimes_{K_0} K)^\times$ and $N_{K/K_0}(\mu_K(\pi_K))$ maps to $(N_{K/K_0}(x), 1, \ldots, 1)$ in $\prod L_i^\times$, where $x = (\pi_K, 1)$ in $L_0 \otimes K = K \times K'$, thus $N_{K/K_0}(x) = \pi_K$ in $L_0 = K$, i.e. b_K and $N_{K/K_0}(\mu_K(\pi_K))$ have the same image in $T_E(K_0)$ for every $E \subset K$ and the lemma holds.

□

8.3.3 Wintenberger's functor

For $K \subset \overline{K}_0$, there is a strictly commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}_p} T_K & \hookrightarrow & \mathrm{Rep}_{\mathbb{Q}_p} T \\ \downarrow{\scriptstyle D_{\pi_K}} & & \downarrow{\scriptstyle D_\pi} \\ {}^{\mathrm{wa}}\mathrm{MF}_K^\sigma & \hookrightarrow & {}^{\mathrm{wa}}\mathrm{MF}_{\overline{K}_0}^\sigma \end{array}$$

of $\otimes$-categories. The two horizontal arrows are the obvious $\otimes$-functors. The left hand side of the diagram is given by Wintenberger's $\otimes$-functor

$$D_{\pi_K} \ : \ \mathrm{Rep}_{\mathbb{Q}_p} T_K \to^{\mathrm{wa}} \mathrm{MF}_K^\sigma$$

defined by taking, for a representation $\tau = (V, \rho) \in \mathrm{Rep}_{\mathbb{Q}_p} T_K$, the filtered isocrystal $D_{\pi_K}(\tau)$ given by:

- The underlying K_0-vector space $D_{\pi_K}(V) = V \otimes_{\mathbb{Q}_p} K_0$,
- The Frobenius morphism $\sigma_{D_{\pi_K}(V)} = \rho(b_K) \otimes \sigma$, for $b_K \in T_K(K_0)$ defined as above.
- The Hodge filtration given by $\mathcal{F}^i_{\mathrm{H}}(D_{\pi_K}(V)_K) = \oplus_{j \geq i} V_{K,j}$ for every $i \in \mathbb{Z}$, where $V \otimes_{\mathbb{Q}_p} K = \oplus_{i \in \mathbb{Z}} V_{K,i}$ is the weight decomposition attached to

$$\rho \circ \mu_K \ : \mathbb{G}_{m,K} \to T_K \to \mathrm{GL}(V)_K.$$

The right hand side of the diagram is given by the $\otimes$-functor

$$D_\pi \ : \ \mathrm{Rep}_{\mathbb{Q}_p} T \to^{\mathrm{wa}} \mathrm{MF}_{\overline{K}}^\sigma$$

defined as follows: for every $(V, \rho) \in \mathrm{Rep}_{\mathbb{Q}_p} T$,

- The underlying module $D_\pi(V) = V \otimes_{\mathbb{Q}_p} K_0$,
- The Frobenius on $D_\pi(V)$ given by $\sigma_{D_\pi(V)} = \rho(b) \otimes \sigma$, for $b \in T(K_0)$ defined as above.

- The Hodge filtration on $D_\pi(V)_{\overline{K}_0}$ defined by taking

$$\mathcal{F}^i_{\mathrm{H}}(D_\pi(V)_{\overline{K}_0}) = \oplus_{j\geq i} V_{\overline{K}_0,j}$$

for every $i \in \mathbb{Z}$, where $V \otimes \overline{K}_0 = \oplus_{i\in\mathbb{Z}} V_{\overline{K}_0,i}$ is the weight decomposition attached to

$$\rho \circ \mu \ : \mathbb{G}_{m,\overline{K}_0} \to T_{\overline{K}_0} \to \mathrm{GL}(V)_{\overline{K}_0}.$$

We may view D_π as a trivial filtered isocrystal with T-structure.

Proposition 8.3.3. *The Hodge filtration*

$$\mathcal{F}_H \circ D_\pi : \mathrm{Rep}_{\mathbb{Q}_p} T \to \mathrm{Fil}^{\mathbb{Z}}_{\overline{K}_0}$$

is split by $\mu : \mathbb{G}_{m,\overline{K}_0} \to T_{\overline{K}_0}$. *The Newton graduation*

$$\mathcal{G}_{\mathrm{N}} \circ D_\pi : \mathrm{Rep}_{\mathbb{Q}_p} T \to \mathrm{Gr}^{\mathbb{Q}}_{K_0}$$

is given by the cocharacter $\nu : \mathbb{D}(\mathbb{Q}) \to T$.

Proof. The statement about the Hodge filtration is obvious from the definition. For the newton graduation, we need to prove that for every $\tau = (V, \rho) \in \mathrm{Rep}_{\mathbb{Q}_p} T$, the weight decomposition $V = \oplus_{\lambda\in\mathbb{Q}} V_\lambda$ given by $\rho \circ \nu$ is the Newton graduation of $D_\pi(\tau)$, for every $\lambda \in \mathbb{Q}$.

We may assume that τ factors through T_E for a finite extension E. By definition, ν_E is the average for the Galois orbits of μ_E. On the other hand, Kottwitz proves in [29, 2.8.1] that the average map has a factorization

$$X_*(T_E) \twoheadrightarrow X_*(T_E)_\Gamma \simeq B(T_E) \hookrightarrow (X_*(T_E) \otimes \mathbb{Q})^\Gamma$$

where $\Gamma = \mathrm{Gal}_{\mathbb{Q}_p}$, $X_*(T_E)^\Gamma$ and $X_*(T_E)_\Gamma$ are, respectively, the Γ-invariants and Γ-coinvariants of $X_*(T_E)$, and $B(T_E)$ is the set of σ-conjugacy classes in $T_E(K_0)$ and the last map is the Newton map (with values in $(X_*(T_E) \otimes \mathbb{Q})^\Gamma = \mathbf{C}^{\mathbb{Q}}(T_E)^\Gamma$). So it suffices to prove that the image of μ_E in $B(T_E)$ under the first map is σ-conjugated to b_E. In [29, 2.5], we see that the image of μ_E in $B(T_E)$ is given by the σ-conjugation class of $b'_E = N_{E/E_0}(\mu_E(\pi'_E))$ for π'_E an uniformizer of E. Now, the image of b'_E in $T_E(K_0) = \prod_{i=0}^{s-1} L_i^\times$, where $L_i = E \otimes_{E_0,\sigma^i} K_0$, is $(\pi'_E, 1, \ldots, 1)$ while the image of b_E is $(\pi_K, 1 \ldots, 1)$.

Let $s = [E : \mathbb{Q}_p]$. It remains to prove that there exists $x = (x_0, \ldots, x_{s-1}) \in T_E(K_0)$ such that $(\sigma - 1)(x)(b'_E) = b_E$. Since we have

$$(\sigma - 1)(x_0, \ldots, x_{s-1}) = \left(\tfrac{\sigma x_{s-1}}{x_0}, \tfrac{\sigma x_0}{x_1}, \ldots, \tfrac{\sigma x_{s-2}}{x_{s-1}}\right) \quad \text{in} \quad T_E(K_0),$$

then it suffices to find $x_0 \in K_0$ such that $(\sigma^s - 1)x_0 = u$ where $u \in \mathcal{O}^\times_{K_0}$ such that $u\pi'_E = \pi_E$. Since $(\sigma^s - 1)$ is surjective on $\mathcal{O}^\times_{K_0}$, this element $x_0 \in K_0$ exist and we can take

$$x = (x_0, \sigma x_0, \ldots, \sigma^{s-1} x_0) \in T_E(K_0)$$

to obtain the σ-conjugation between b'_E and b_E.

□

The next proposition gives us the relation between Wintenberger's functor

$$D_\pi \ : \ \mathrm{Rep}_{\mathbb{Q}_p} T \to^{\mathrm{wa}} \mathrm{MF}^\sigma_{\overline{K}_0}$$

and Fontaine's functor

$$D_{\mathrm{cris}} \circ V^u \ : \ \mathrm{Rep}_{\mathbb{Q}_p} T \to^{\mathrm{wa}} \mathrm{MF}^\sigma_{\overline{K}_0}.$$

Proposition 8.3.4. *For any finitely generated tensor subcategory $\mathcal{V}$ of $\mathrm{Rep}_{\mathbb{Q}_p} T$, there is an isomorphism of $\otimes$-functors*

$$(D_{\mathrm{cris}} \circ V^u)_{|\mathcal{V}} \simeq D_{\pi|\mathcal{V}}.$$

Proof. This immediately follows from the main result of [48]: any such $\mathcal{V}$ is contained in $\mathrm{Rep}_{\mathbb{Q}_p} T_K$ for a sufficiently large finite extension $K \subset K_0^{\mathrm{a}}$ of K_0.

□

As a consequence, we get the following result about the Fargues filtration of V^u.

Proposition 8.3.5. *The filtration $\mathcal{F}_{\mathrm{F}}(V^u)$ is split by $\nu^\iota = \nu^{-1}$ and*

$$\mathcal{F}_{\mathrm{F}}(D_{\mathrm{cris}} \circ V^u) = \mathcal{F}_{\mathrm{N}}^\iota(D_{\mathrm{cris}} \circ V^u).$$

Proof. Write $V = V^u$. Fix $\tau \in \mathrm{Rep}_{\mathbb{Q}_p} T$. Since T is commutative, we have a decomposition $\tau = \oplus_{\lambda \in \mathbb{Q}} \tau_\lambda$ according to ν^ι-weights. We first want to show $\mathcal{F}_{\mathrm{F}}^\gamma(V(\tau)) = \oplus_{\lambda \geq \gamma} V(\tau_\lambda)$. We obviously have

$$\mathcal{F}_{\mathrm{F}}^\gamma(V(\tau)) = \oplus_{\lambda \in \mathbb{Q}} \mathcal{F}_{\mathrm{F}}^\gamma(V(\tau_\lambda)),$$

so we need to show $\mathcal{F}_{\mathrm{F}}^\gamma(V(\tau_\lambda)) = V(\tau_\lambda)$ if $\lambda \geq \gamma$ and 0 otherwise, i.e. we want to show that $V(\tau_\lambda)$ is semi-stable of slope λ. This is equivalent to prove that the weakly admissible filtered isocrystal $D_{\mathrm{cris}}(V(\tau_\lambda))$ is semi-stable of slope λ. Now, we have $D_{\mathrm{cris}}(V(\tau_\lambda)) \simeq D_\pi(\tau_\lambda)$ which is Newton isoclinic of slope $-\lambda$, by Proposition 8.3.3. Since $\mathbf{t}_{\mathrm{F}}(D_{\mathrm{cris}}(V(\tau_\lambda))) \leq \mathbf{t}_{\mathrm{N}}^\iota(D_{\mathrm{cris}}(V(\tau_\lambda)))$, the equality must hold and $D_{\mathrm{cris}}(V(\tau_\lambda))$ is indeed semi-stable of slope λ.

We next want to show that $\mathcal{F}_{\mathrm{F}}(D_{\mathrm{cris}} \circ V(\tau)) = \mathcal{F}_{\mathrm{N}}^\iota(D_{\mathrm{cris}} \circ V(\tau))$ and we know that

$$\mathcal{F}_{\mathrm{F}} \circ D_{\mathrm{cris}} = D_{\mathrm{cris}} \circ \mathcal{F}_{\mathrm{F}},$$

so it suffices to show that $\mathcal{F}_{\mathrm{N}}^{\iota,\gamma}(D_{\mathrm{cris}} \circ V(\tau)) = \oplus_{\lambda \geq \gamma} D_{\mathrm{cris}} \circ V(\tau_\lambda)$. Let $\mathcal{V}$ be the $\otimes$-category generated by τ and τ_λ for each λ appearing in the decomposition of τ (since there is a finite number of such τ_λ's, $\mathcal{V}$ is finitely generated). By last proposition, we can replace $D_{\mathrm{cris}} \circ V$ by D_π and it suffices to show that $\mathcal{F}_{\mathrm{N}}^{\iota,\gamma}(D_\pi(\tau)) = \oplus_{\lambda \geq \gamma} D_\pi(\tau_\lambda)$, which is obvious since $\mathcal{F}_{\mathrm{N}}^\iota \circ D_\pi$ is split by ν^ι.

□

8.4 Germs of abelian crystalline representations with G-structure

We want to add a G-structure to the germs of abelian crystalline representation, as we did in the sections before for general germs of crystalline representations. Let G be a reductive group over $\mathbb{Z}_p$. A germ of abelian crystalline representation with G-structure is an exact and faithful $\otimes$-functor

$$V \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cr,ab}}\{\mathrm{Gal}_{K_0}\}.$$

We say that V is trivial when $\omega(V) = \omega_{G,\mathbb{Q}_p}$.

Proposition 8.4.1. *The following data are equivalent:*

1. *A morphism $x : T \to G$,*
2. *A trivial germ of abelian crystalline representation with G-structure V_x,*
3. *A cocharacter $\mu_x \ : \ \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ such that the Mumford-Tate group $\mathrm{MT}(\mu_x)$ is a torus.*

Moreover, K is a field definition of V_x if and only if x factors tough T_K, if and only if μ_x is defined over $K \cap \overline{\mathbb{Q}}_p$.

Proof. (1) $\Leftrightarrow$ (2): From a morphism $x : T \to G$ we get

$$V_x = V^u \circ x \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Rep}_{\mathbb{Q}_p} T \simeq \mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Q}_p}\{\mathrm{Gal}_{K_0}\}.$$

Given V_x, we obtain a map $\mathrm{Aut}^{\otimes}(\omega(V_x)) \to G$. On the other hand, Fontaine gives an isomorphism $\mathrm{Aut}^{\otimes}(\omega(V_x)) \to T$. Composing the inverse of the latter with the first map, we obtain $x : T \to G$.

(1) $\Leftrightarrow$ (3): Now, given x, we obtain $\mu_x = x \circ \mu$. Then, $\mathrm{MT}(\mu_x) \subset T$ since $\mathrm{MT}(\mu_x)$ is defined as the smallest algebraic subgroup of G, defined over $\mathbb{Q}_p$ such that μ_x factors through it. Moreover, $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ is a connected group, so we have a factorization

$$\mathbb{G}_{m,\overline{\mathbb{Q}}_p} \xrightarrow{\mu_x} \mathrm{MT}(\mu_x)^{\circ}_{\overline{\mathbb{Q}}_p} \subset \mathrm{MT}(\mu_x)_{\overline{\mathbb{Q}}_p}.$$

Then $\mathrm{MT}(\mu_x)^{\circ}_{\overline{\mathbb{Q}}_p} = \mathrm{MT}(\mu_x)_{\overline{\mathbb{Q}}_p}$ and $\mathrm{MT}(\mu_x)$ is connected. Over $\mathbb{Q}_p$, every connected subgroup of a torus is a torus, so $\mathrm{MT}(\mu_x)$ is a torus. Conversely, starting with μ_x defined over E, pick a finite extension $E \subset \overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ such that μ_x is defined over E. Then μ_x yields an element of

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}[\mathrm{Gal}_E]}(X^*(\mathrm{MT}(\mu_x)), \mathbb{Z}) &= \mathrm{Hom}_{\mathbb{Z}[\mathrm{Gal}_{\mathbb{Q}_p}]}(X^*(\mathrm{MT}(\mu_x)), \mathcal{C}(\mathrm{Hom}(E, \overline{\mathbb{Q}}_p), \mathbb{Z})) \\ &= \mathrm{Hom}(T_E, \mathrm{MT}(\mu_x)) \end{aligned}$$

hence $x : T \to T_E \to \mathrm{MT}(\mu_x) \subset G$. In the display, the first map is $f \mapsto \tilde{f}$, $\tilde{f}(a)(\iota) = f(\sigma a)$ if $\sigma\iota = \iota_0$.

The statement about the field of definition is obvious by construction of the equivalences. □

The next proposition gives us the Hodge and Newton types for a trivial germ of abelian crystalline representations with G-structure.

Proposition 8.4.2. *Let V_x be a trivial germ of abelian crystalline representation with G-structure, with associated morphism $x : T \to G$. Then, we obtain the Hodge, Newton and Fargues types by*

$$\begin{aligned} \mathbf{t}_{\mathrm{H}}(V_x) &= [x \circ \mu] \quad \text{in} \quad \mathbf{C}^{\mathbb{Z}}(G_{\overline{\mathbb{Q}}_p}) \\ \mathbf{t}_{\mathrm{N}}(V_x) &= [x \circ \nu] \quad \text{in} \quad \mathbf{C}^{\mathbb{Q}}(G_{\mathbb{Q}_p}). \\ \mathbf{t}_{\mathrm{F}}(V_x) &= [x \circ \nu^{\iota}] \quad \text{in} \quad \mathbf{C}^{\mathbb{Q}}(G_{\mathbb{Q}_p}) \end{aligned}$$

Therefore, V_x is ordinary when $[x \circ \nu] = [x \circ \mu]^{\#}$.

Proof. As an abuse of notation, let $x : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Rep}_{\mathbb{Q}_p} T$ and let $\mathcal{V}_x$ be the essential image of this map. It is a finitely generated $\otimes$-subcategory of $\mathrm{Rep}_{\mathbb{Q}_p} T$, thus $D_{\mathrm{cris}} \circ V_x = D_{\mathrm{cris}} \circ V^u \circ x \simeq D_{\pi} \circ x$, by Proposition 8.3.4. The functor $D_{\pi} \circ x$ is trivial, and we know that $\mathcal{F}_{\mathrm{H}}$ is split by $x \circ \mu$ and $\mathcal{G}_{\mathrm{N}}$ is given by $x \circ \nu$, by Proposition 8.3.3 and that $\mathcal{F}_{\mathrm{F}}$ is split by $x \circ \nu^{\iota}$ by Proposition 8.3.5. □

Since G is defined over $\mathbb{Z}_p$, it is quasi-split over $\mathbb{Z}_p$ and $\mathbb{Q}_p$: there are Borel pairs (T_0, B) in G and they are all conjugated. In particular, we obtain a conjugation class of distinguished maximal tori in G, formed by the ones contained in a Borel pair. Let (T_0, B) be a Borel pair in G. A cocharacter $\mu \in X_*(T_0)$ is called B-dominant if for every positive root α of T_0 in B we have $\langle \alpha, \mu \rangle \geq 0$, where $\langle \cdot, \cdot \rangle$ is the pairing between $X_*(T_0)$ and $X^*(T_0)$. We denote by $X_*(T_0)^{\mathrm{dom}}$ the set of B-dominant cocharacters of T_0.

Proposition 8.4.3. *If the image of x is contained in a distinguished maximal torus, then V_x is ordinary. In this case, the definition field is an unramified extension of $\mathbb{Q}_p$ and we have $x(b) = \mu_x(p)$.*

Proof. By assumption, there is a Borel pair $(T_0, B) \subset G$ defined over $\mathbb{Q}_p$ such that $\mu_x = x \circ \mu$ factors through a B-dominant cocharacter of T_0, i.e. $\mu_x \in X_*(T_0)^{\mathrm{dom}}$. The morphism $T_0 \hookrightarrow G$ induces a $\mathrm{Gal}_{\mathbb{Q}_p}$-equivariant map

$$X_*(T_0) \otimes \mathbb{Q} = \mathbf{C}^{\mathbb{Q}}(T_{0,\overline{\mathbb{Q}}_p}) \to \mathbf{C}^{\mathbb{Q}}(G_{\overline{\mathbb{Q}}_p})$$

whose restriction to the B-dominant cone is also compatible with the monoid structures: it is a $\mathrm{Gal}_{\mathbb{Q}_p}$-equivariant isomorphism of monoids

$$(X_*(T_0) \otimes \mathbb{Q})^{\mathrm{dom}} \to \mathbf{C}^{\mathbb{Q}}(G_{\overline{\mathbb{Q}}_p}).$$

Plainly, $(x \circ \mu)^{\#} = x \circ \nu$ on the left hand side, thus also $[x \circ \mu]^{\#} = [x \circ \nu]$ on the right hand side, i.e. $\mathbf{t}_{\mathrm{H}}(V_x) = \mathbf{t}_{\mathrm{N}}(V_x)$ by proposition 8.4.2 and V_x is ordinary.

In this case, μ_x is a cocharacter of T_0, which is split over K_0, so μ_x is defined over an unramified extension E of $\mathbb{Q}_p$. Thus x factors through T_{K_0} by construction and

$$x(b) = x_{K_0}(b_{K_0}) = x_{K_0} \circ \mu_{K_0}(\pi_{K_0}) = x_{K_0} \circ \mu_{K_0}(p) = (x \circ \mu)(p) = \mu_x(p),$$

where $x_{K_0} : T_{K_0} \to T_0 \hookrightarrow G$ is the factorization of x.

□

8.5 The reduction map

For the rest of the section, we fix a trivial germ of abelian crystalline representations

$$V_x : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Rep}^{\mathrm{cr,ab}}_{\mathbb{Q}_p}\{\mathrm{Gal}_{K_0}\}$$

with G-structure such that $\mathrm{MT}(\mu_x)$ is contained in T_0, for (T_0, B) a Borel pair of $G_{\mathbb{Z}_p}$ with μ_x dominant. We set

$$\nu_x = x \circ \nu \ : \ \mathbb{D}(\mathbb{Q})_{|\mathbb{Q}_p} \to T_{|\mathbb{Q}_p} \to G_{|\mathbb{Q}_p}$$

and $M_x = Z_G(\nu_x)$. The cocharacter ν_x is defined over $\mathbb{Z}_p$, by [11, Proposition 3 and Lemma 4], so M_x is also defined over $\mathbb{Z}_p$. By Proposition 8.4.2, M_x is a Levi subgroup of the parabolic subgroup $P_x = P_{\mathcal{F}}$ stabilizing $\mathcal{F}_{\mathrm{F}}(V)$. Note also that $B \subset P_{\mathcal{F}}$ since ν_x is B-dominant, since μ_x and all its conjugates are.

As we have seen in last section, the map red has a factorization

$$\begin{tikzcd}
\mathcal{L}(V_x) \arrow[rr, "\mathrm{red}"] \arrow[dr, two heads] & & \mathcal{L}(D_x, \mathbf{t}^{\iota}_{\mathrm{H}}(D_x)) \\
& U_{\mathcal{F}_F}(\mathbb{Q}_p)\backslash\mathcal{L}(V_x) \arrow[ur] &
\end{tikzcd}$$

where $D_x = D_{\mathrm{cris}}(V_x)$, $\mathcal{F}_F$ is the Fargues filtration on the germ of G-crystalline representation V_x, and the map red is compatible with the morphism $\mathrm{Aut}^{\otimes}(V_x) \to \mathrm{Aut}^{\otimes}(D_x)$. The aim of the rest of the section is to give a more explicit description of the diagram above.

8.5.1 The source

Since V_x is trivial, we have a canonical isomorphism

$$\mathcal{L}(V_x) \simeq G(\mathbb{Q}_p)/G(\mathbb{Z}_p),$$

since $\mathcal{L}(V_x) = \mathcal{L}(\omega_{G,\mathbb{Q}_p})$ and $\mathrm{Aut}^{\otimes}(\omega_{G,\mathbb{Q}_p}) = G(\mathbb{Q}_p)$ acts transitively on $\mathcal{L}(\omega_{G,\mathbb{Q}_p})$, with stabilizer $\mathrm{Aut}^{\otimes}(\omega_{G,\mathbb{Z}_p}) = G(\mathbb{Z}_p)$, for $\omega_{G,\mathbb{Z}_p} \in \mathcal{L}(V_x)$ the trivial lattice. Next proposition describes the automorphism group of V_x as a subgroup of $\mathrm{Aut}^{\otimes}(\omega(V_x)) = G(\mathbb{Q}_p)$.

Proposition 8.5.1. *For V_x as above, we have*

$$\mathrm{Aut}^{\otimes}(V_x) = M_x(\mathbb{Q}_p).$$

Proof. For $g \in G(\mathbb{Q}_p)$, we denote by $\mathrm{Int}(g)$ the inner automorphism defined by g. Then, an element $g \in \mathrm{Aut}^{\otimes}(\omega(V_x))$ defines an isomorphism $V_x \to V_{\mathrm{Int}(g)\cdot x}$, since

$$\begin{array}{rcl} \tau(g)(\sigma \cdot_x v) & = & \tau(g)(\tau(x \circ \chi_K(\sigma)) \cdot v) \\ & = & \tau(g \cdot x \circ \chi_K(\sigma))(v) \\ & = & \tau(g \cdot x \circ \chi_K(\sigma) \cdot g^{-1} \cdot g)(v) \\ & = & \tau(\mathrm{Int}(g)(x) \circ \chi_K(\sigma) \cdot g)(v) \\ & = & \tau(\mathrm{Int}(g)(x) \circ \chi_K(\sigma))(\tau(g) \cdot v) \\ & = & \sigma \cdot_{\mathrm{Int}(g)(x)} \tau(g)v \end{array}$$

for $\tau \in \mathrm{Rep}_{\mathbb{Q}_p} G$, $v \ni \omega_G(\tau)$, K a field of definition of V_x and $\sigma \in \mathrm{Gal}_K$ (we may take $K = K_0$). By definition, $\mathrm{Aut}^{\otimes}(V_x)$ is the set of $g \in \mathrm{Aut}^{\otimes}(\omega(V_x))$ inducing an isomorphism of V_x, i.e. the set of $g \in G(\mathbb{Q}_p)$ such that $V_x = V_{\mathrm{Int}(g)(x)}$, and by the classification of trivial germs of abelian crystalline representations with G-structure given in 8.4.1, this is equivalent to the set of $g \in G(\mathbb{Q}_p)$ such that $x = \mathrm{Int}(g)(x)$ or, with respect to μ_x, the set of $g \in G(\mathbb{Q}_p)$ such that $\mu_x = \mathrm{Int}(g)(\mu_x)$, i.e.

$$\begin{array}{rcl} \mathrm{Aut}^{\otimes}(V_x) & = & \{g \in G(\mathbb{Q}_p) \mid g \cdot x \cdot g^{-1} = x\} \\ & = & \{g \in G(\mathbb{Q}_p) \mid g \cdot \mu_x \cdot g^{-1} = \mu_x\}. \end{array}$$

Now, the faithful $\otimes$-functor D_π is compatible with fiber functors and induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}^{\otimes}(V_x) & \longrightarrow & \mathrm{Aut}^{\otimes}(D_\pi(V_x)) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Aut}^{\otimes}(\omega(V_x)) & \longrightarrow & \mathrm{Aut}^{\otimes}(\omega(D_\pi(V_x))) \\ \| & & \| \\ G(\mathbb{Q}_p) & \longrightarrow & G(K_0). \end{array}$$

If $g \in \mathrm{Aut}^{\otimes}(V_x)$, $g \in G(K_0)$ preserves in particular the underlying isocrystal associated to $D_\pi(V_x)$, so it also preserves the Newton graduation, which is given by ν_x, thus g normalizes ν_x, i.e.

$$g \in M_x(K_0) \cap G(\mathbb{Q}_p) = M_x(\mathbb{Q}_p).$$

On the other hand, we have a factorization (all groups over $\overline{\mathbb{Q}}_p$)

$$\begin{array}{ccccc} \mathbb{G}_m & \xrightarrow{\mu_x} & T_0 & \longrightarrow & G \\ & \searrow & \searrow & \nearrow & \\ & Z_G(M_x) & \hookrightarrow & M_x & \end{array}$$

Indeed, ν_x factors through T_0, which is commutative, so $T_0 \subset M_x$, thus μ_x factors through M_x since it factors through $T_0 \subset M_x$. Let $\mathcal{M} = \mathrm{Lie}(M_x)$. Let $\Phi(T_0, M_x)$ be the system of roots of T_0 in M_x, i.e. characters such that $\mathcal{M} = \mathcal{M}^0 \oplus \oplus_{\alpha \in \Phi(T_0, M_x)} \mathcal{M}^\alpha$, where $\mathcal{M}^0 = \mathrm{Lie}\, T_0$. Let $\Phi^+(T_0, M_x) = \Phi(T_0, M_x) \cap \Phi^+(T_0, G)$ be the roots of T_0 in $\mathcal{M}$ which are positive with respect to $B \cap M_x$, a Borel subgroup of M_x with Levi T_0. For every $\alpha \in \Phi^+(T_0, M_x)$ and μ' in the $\mathrm{Gal}_{\mathbb{Q}_p}$-orbit $O(\mu_x)$ of μ_x, we have $\langle \alpha, \mu' \rangle \geq 0$, but $\sum_{O(\mu_x)} \langle \alpha, \mu' \rangle = \langle \alpha, \sum_{O(\mu_x)} \mu' \rangle = \#O(\mu_x) \langle \alpha, \nu_x \rangle = 0$ since ν_x is central in M_x, thus $\langle \alpha, \mu' \rangle = 0$ for every $\mu' \in O(\mu_x)$. In particular $\langle \alpha, \mu_x \rangle = 0$ for every $\alpha \in \Phi^+(T_0, M_x)$, which implies that μ_x is central in M_x. Thus $M_x(\mathbb{Q}_p) \subset \mathrm{Aut}^{\otimes}(V_x)$ which proves that $M_x(\mathbb{Q}_p) = \mathrm{Aut}^{\otimes}(V_x)$.

□

The last result can be reformulated as follows: since x factors through T_0, it factors through M_x and we have

$$V_x \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Rep}_{\mathbb{Q}_p} M_x \xrightarrow{V_x^{M_x}} \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cr,ab}}\{\mathrm{Gal}_{K_0}\}$$

which induces a diagram

$$\begin{array}{ccc}
\mathrm{Aut}^{\otimes}(V_x^{M_x}) & \xrightarrow{\simeq} & \mathrm{Aut}^{\otimes}(V_x) \\
\| & & \downarrow \\
\mathrm{Aut}^{\otimes}(\omega(V_x^{M_x})) & \hookrightarrow & \mathrm{Aut}^{\otimes}(\omega(V_x)) \\
\| & & \| \\
M_x(\mathbb{Q}_p) & \hookrightarrow & G(\mathbb{Q}_p).
\end{array}$$

Note that $V_x^{M_x}$ is also ordinary by Proposition 8.4.3, since the corresponding morphism $x : T \to M_x$ factors through the distinguished torus T_0 of M_x. Moreover, we have an $M_x(\mathbb{Q}_p)$-equivariant diagram

$$\begin{array}{ccccc}
\mathcal{L}(V_x^{M_x}) & \longrightarrow & \mathcal{L}(V_x) & \longrightarrow & U_{\mathcal{F}_{\mathrm{F}}}(\mathbb{Q}_p)\backslash\mathcal{L}(V_x) \\
\| & & \| & & \| \\
M_x(\mathbb{Q}_p)/M_x(\mathbb{Z}_p) & \longrightarrow & G(\mathbb{Q}_p)/G(\mathbb{Z}_p) & \longrightarrow & U_{\mathcal{F}_{\mathrm{F}}}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p).
\end{array}$$

The next proposition gives us a description of the quotient:

Proposition 8.5.2. *The top map $\mathcal{L}(V_x^{M_x}) \to U_{\mathcal{F}_{\mathrm{F}}}(\mathbb{Q}_p)\backslash\mathcal{L}(V_x)$ bijective.*

Proof. It is well known. The Iwasawa decomposition $G(\mathbb{Q}_p) = B(\mathbb{Q}_p)G(\mathbb{Z}_p)$ gives us

$$G(\mathbb{Q}_p) = P_{\mathcal{F}_{\mathrm{F}}}(\mathbb{Q}_p)G(\mathbb{Z}_p) = U_{\mathcal{F}_{\mathrm{F}}}(\mathbb{Q}_p)M_x(\mathbb{Q}_p)G(\mathbb{Z}_p)$$

as $B(\mathbb{Q}_p) \subset P_{\mathcal{F}_{\mathrm{F}}}(\mathbb{Q}_p) = U_{\mathcal{F}_{\mathrm{F}}}(\mathbb{Q}_p) \rtimes M_x(\mathbb{Q}_p)$. Thus we have the surjectivity of the bottom map. The injectivity is given by [11, 6.3.3].

□

8.5.2 The target

It remains to study the description of both the set $\mathcal{L}(D_x, \mathbf{t}_{\mathrm{H}}^{\iota}(D_x))$ and the action of the automorphism group $\mathrm{Aut}^{\otimes}(D_x)$. We may assume that D_x is isomorphic to a trivial G-isocrystal, since we can fix an isomorphism $D_x = D_{\mathrm{cris}} \circ V_x \simeq D_\pi \circ x$ and use Wintenberger's results in [48]. Applying Fontaine's functor D_{cris} to

$$V_x \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Rep}_{\mathbb{Q}_p} M_x \xrightarrow{V_x^{M_x}} \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cr,ab}}\{\mathrm{Gal}_{K_0}\}$$

we obtain

$$D_x \ : \ \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Rep}_{\mathbb{Q}_p} M_x \xrightarrow{D_x^{M_x}} \mathrm{Mod}_{K_0}^{\sigma}.$$

Proposition 8.5.3. *The functor D_{cris} induces a bijection*

$$\mathrm{Aut}^{\otimes}(V_x) \simeq \mathrm{Aut}^{\otimes}(D_x).$$

Proof. We may replace D_x by $D_\pi \circ x$. As we have seen, the $\otimes$-automorphisms of $D_\pi \circ x$ preserve the Newton cocharacter, so their image is in $M_x(K_0)$ and we get

$$\mathrm{Aut}^{\otimes}(D_\pi \circ x) = \{m \in M_x(K_0) \mid mx(b) = x(b)\sigma(m)\}.$$

Now, by ordinarity, we know by Proposition 8.4.3 that $x(b) = x(\mu(p)) = \mu_x(p)$, which is central in $M_x(K_0)$, thus

$$\mathrm{Aut}^{\otimes}(D_\pi(V_x)) = \{m \in M_x(K_0) \mid m = \sigma(m)\} = M_x(\mathbb{Q}_p).$$

Our claim now follows from Proposition 8.5.1.

□

The previous result and Proposition 8.5.1 together imply that D_{cris} and the embedding $M_x \hookrightarrow G$ yield group isomorphisms

$$\begin{array}{ccc} \mathrm{Aut}^{\otimes}(V_x^{M_x}) & \xrightarrow{\simeq} & \mathrm{Aut}^{\otimes}(D_x^{M_x}) \\ \downarrow\simeq & & \downarrow\simeq \\ \mathrm{Aut}^{\otimes}(V_x) & \xrightarrow{\simeq} & \mathrm{Aut}^{\otimes}(D_x). \end{array}$$

A similar result holds for lattices. We will use the script M_x or G to indicate if we are working with the vectorial distance defined on $\mathbf{B}^e(M_{xK})$ or $\mathbf{B}^e(G_K)$, respectively.

Proposition 8.5.4. *The functor D_{cris} and the embedding $M_x \hookrightarrow G$ yield a commutative diagram*

$$\begin{array}{ccc} \mathcal{L}(V_x^{M_x}) & \xrightarrow{\simeq} & \mathcal{L}\left(D_x^{M_x}, \mathbf{t}_{\mathrm{H}}^{\iota,M_x}(D_x^{M_x})\right) \\ \cap & & \downarrow\simeq \\ \mathcal{L}(V_x) & \xrightarrow{\mathrm{red}} & \mathcal{L}\left(D_x, \mathbf{t}_{\mathrm{H}}^{\iota,G}(D_x)\right). \end{array}$$

Proof. To compute the right hand side map, we may again replace D_x by $D_\pi \circ x$, that we call D to easy the notations. Also, we set $V = V_x$, $M = M_x$, $\mathbf{t}_M = \mathbf{t}_{\mathrm{H}}^{\iota,M_x}(D_x^{M_x})$ in $\mathbf{C}^{\mathbb{Z}}(M_x)$ and $\mathbf{t}_{\mathrm{G}} = \mathbf{t}_{\mathrm{H}}^{\iota,G}(D_x)$ in $\mathbf{C}^{\mathbb{Z}}(G)$. We have to show that the embedding $\mathcal{L}(V^M) \subset \mathcal{L}(V)$ yields a bijection $\mathcal{L}(D^M, \mathbf{t}_M) \simeq \mathcal{L}(D, \mathbf{t}_G)$. This is well-known result, given by Kottwitz in [32, Theorem 1.1]. We propose here an alternative proof. We have a diagram

$$\begin{array}{ccc} \mathcal{L}(D^M, \mathbf{t}_M) & \hookrightarrow & \mathcal{L}(D, \mathbf{t}_G) \\ \downarrow & & \downarrow \\ \mathbf{B}(\omega_M, K_0) & \hookrightarrow & \mathbf{B}(\omega_G, K_0) \\ \simeq\uparrow & & \simeq\uparrow \\ \mathbf{B}^e(M_{K_0}) & \hookrightarrow & \mathbf{B}^e(G_{K_0}) \end{array}$$

which is cartesian, i.e. $\mathcal{L}(D^M, \mathbf{t}_M) = \mathcal{L}(D, \mathbf{t}_G) \cap \mathbf{B}^e(G_{K_0})$. From [13, Theorem 7], we thus already obtain $\mathcal{L}(D, \mathbf{t}_G) \subset \mathcal{L}(D^M)$, therefore

$$\mathcal{L}(D, \mathbf{t}_G) = \coprod_{\mathbf{t}'_M \mapsto \mathbf{t}_G} \mathcal{L}(D^M, \mathbf{t}'_M)$$

where $\mathbf{t}'_M$ runs through the fiber $\mathbf{C}^{\mathbb{Z}}(M) \twoheadrightarrow \mathbf{C}^{\mathbb{Z}}(G)$ over $\mathbf{t}_G$. So, it suffices to show that $\mathcal{L}(D^M, \mathbf{t}'_M) = \emptyset$ whenever $\mathbf{t}'_M \neq \mathbf{t}_M$. Note that $\mathcal{L}(D^M, \mathbf{t}_M)$ is indeed non empty, since the M-isocrystal D^M is $\mathbf{t}_M$-ordinary by 7.3.24, it is therefore sufficient to establish that $\mathbf{t}_M$ is the only point in the intersection of the relevant fibers of the Kottwitz map $[-]_{M,\Gamma} : \mathbf{C}^{\mathbb{Z}}(M) \to \pi_1(M)_\Gamma$

and the projection $\mathbf{C}^{\mathbb{Z}}(M) \to \mathbf{C}^{\mathbb{Z}}(G)$.

Let $W_G(T_0) = N_G(T_0)/Z_G(T_0)$ be the Weyl group of T_0 in G, denote by $R_G^\vee = \Phi^\vee(T_0, G)$ the system of coroots of T_0 in G, Δ_G the simple coroots of T_0 in G and $X_*(T_0)^{G-\mathrm{dom}}$ the cone of dominant cocharacters defined by B. We use analogous notation for the corresponding objects defined with respect to M and $M \cap B$. Finally, denote by X_Γ the coinvariants of $\Gamma = \mathrm{Gal}_{\mathbb{Q}_p}$ acting on a group X. We then have the following commutative diagram, with exact second and third rows (compare with [32]):

$$\begin{tikzcd}
 & & \mathbf{C}^{\mathbb{Z}}(M) \simeq W_{M_x}(T_0)\backslash X_*(T_0) \arrow[r] \arrow[d, two heads] & W_G(T_0)\backslash X_*(T_0) \simeq \mathbf{C}^{\mathbb{Z}}(G) \arrow[d, two heads] & \\
0 \arrow[r] & \bigoplus_{\delta\in\Delta_G\backslash\Delta_M} \mathbb{Z}\cdot\delta \arrow[r] \arrow[d] & X_*(T_0)/\mathbb{Z}\cdot R_M^\vee \arrow[r] \arrow[d, two heads] & X_*(T_0)/\mathbb{Z}\cdot R_G^\vee \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \bigoplus_{\Gamma\cdot\delta\in\Gamma\backslash(\Delta_G\backslash\Delta_M)} \mathbb{Z}\cdot\Gamma\cdot\delta \arrow[r] & (X_*(T_0)/\mathbb{Z}\cdot R_M^\vee)_\Gamma \arrow[r] & (X_*(T_0)/\mathbb{Z}\cdot R_G^\vee)_\Gamma \arrow[r] & 0
\end{tikzcd}$$

By construction $\mathbf{t}_M = \mu_x$ in $X_*(T_0)^{G-\mathrm{dom}} \subset X_*(T_0)^{M-\mathrm{dom}}$. Let $\mathbf{t}'_M \in X_*(T_0)^{M-\mathrm{dom}}$ be any other element in the fiber of $\mathbf{C}^{\mathbb{Z}}(M) \twoheadrightarrow \mathbf{C}^{\mathbb{Z}}(G)$ over $\mathbf{t}_G$. Then, $\mathbf{t}'_{M_x} = w \cdot \mathbf{t}_{M_x}$ for some $w \in W_G(T_0)$, thus $\mathbf{t}_M - \mathbf{t}'_M \in X_*(T_0)$ is a linear combination of elements of Δ_G with non-negative coefficients. In the second line of our diagram, we thus obtain

$$\mathbf{t}_{M_x} - \mathbf{t}'_{M_x} = \textstyle\sum_{\delta\in\Delta_G\backslash\Delta_{M_x}} n_\delta \cdot \delta \in \displaystyle\bigoplus_{\delta\in\Delta_G\backslash\Delta_{M_x}} \mathbb{Z}\cdot\delta, \quad \text{with } n_\delta \geq 0.$$

The map

$$\bigoplus_{\delta\in\Delta_G\backslash\Delta_{M_x}} \mathbb{Z}\cdot\delta \to \bigoplus_{\Gamma\cdot\delta\in\Gamma\backslash(\Delta_G\backslash\Delta_{M_x})} \mathbb{Z}\cdot\Gamma\cdot\delta$$

is given by

$$\sum_{\delta\in\Delta_G\backslash\Delta_{M_x}} n_\delta \cdot \delta \mapsto \sum_{\delta\in\Delta_G\backslash\Delta_{M_x}} \left(\sum_{\delta'\in\Gamma\cdot\delta} n_{\delta'}\right) \cdot \delta.$$

Thus if also $[\mathbf{t}_M]_{M,\Gamma} = [\mathbf{t}'_M]_{M,\Gamma}$ in $(X_*(T_0)/\mathbb{Z}\cdot R_M^\vee)_\Gamma$ (which is the case if $\mathcal{L}(D^M, \mathbf{t}'_M) \neq \emptyset$), then actually $\mathbf{t}'_M = \mathbf{t}_M$ as was to be shown, since all the n_δ must be 0.

We still have to show that $\mathcal{L}(V^M) \hookrightarrow \mathcal{L}(D^M)$ induces a bijection $\mathcal{L}(V^M) \simeq \mathcal{L}(D^M, \mathbf{t}_M)$. Inside $\mathcal{L}(D^M) \simeq M(K_0)/M(W(\mathbb{F}))$, we have

$$\mathcal{L}(D^M, \mathbf{t}_M) = \{m \in M(K_0)/M(W(\mathbb{F})) \mid m^{-1}\mu_x(p)\sigma(m) \in G(W(\mathbb{F})\mu_x(p)G(W(\mathbb{F})\}$$

which equals

$$\{m \in M(K_0)/M(W(\mathbb{F})) \mid m^{-1}\sigma(m) \in G(W(\mathbb{F})\}$$

since $\mu_x(p)$ is central in M. In other words,

$$\mathcal{L}(D^M, \mathbf{t}_M) = \{y \in \mathcal{L}(D^M) \mid \sigma y = y\} = \mathcal{L}(V^x)$$

using, for instance, [13, 2.5.5] for the last equality.

□

Let s be the order of the Γ-orbit of μ_x in $X_*(T_0)$. Then $s\nu_x \in X_*(T_0)$ and $\sigma^s(\mu_x) = \mu_x$, thus

$$\sigma^s \mathbf{t}^\iota_{\mathrm{H}}(D_x) = \mathbf{t}^\iota_{\mathrm{H}}(D_x) \quad \text{in} \quad \mathbf{C}^{\mathbb{Z}}(G).$$

It follows that the s-power of the Frobenius of D_x acts on $\mathcal{L}(D_x, \mathbf{t}^\iota_{\mathrm{H}}(D_x))$, giving an automorphism

$$F \ : \ \mathcal{L}(D_x, \mathbf{t}^\iota_{\mathrm{H}}(D_x)) \to \mathcal{L}(D_x, \mathbf{t}^\iota_{\mathrm{H}}(D_x)).$$

It is related to the operator Φ^s_{cris} of section 7.3.5 as follows:

Proposition 8.5.5. *We have $F = \Phi^s_{\mathrm{cris}}$ on $\mathcal{L}(D_x, \mathbf{t}^\iota_{\mathrm{H}}(D_x))$.*

Proof. We may as usual replace D_x by $D_\pi \circ x$ and call it D to ease the notations. Let also $V = V_x$ and $M = M_x$. Then F acts on

$$\mathcal{L}(D) \simeq \mathcal{L}(\omega_{G,K_0}) \simeq G(K_0)/G(\mathbb{Z}_p)$$

by the element

$$(\mu_x(p), \sigma)^s = (s\nu_x(p), \sigma^s) \in G(K_0) \rtimes \langle \sigma \rangle^{\mathbb{Z}}.$$

It thus acts on

$$\mathcal{L}(D, \mathbf{t}^\iota_{\mathrm{H}}(D)) \simeq \mathcal{L}(V^M) \simeq M(\mathbb{Q}_p)/M(\mathbb{Z}_p)$$

by $s\nu_x(p) \in M(\mathbb{Q}_p)$. For any $y \in \mathcal{L}(D, \mathbf{t}^\iota_{\mathrm{H}}(D))$ and $\tau \in \mathrm{Rep}_{\mathbb{Z}_p} G$ with weight decomposition $\tau|M = \oplus_{a\in\mathbb{Z}} \tau_a$ with respect to the central cocharacter $s\nu_x : \mathbb{G}_{m,\mathbb{Z}_p} \to M$, we thus have

$$F(y)(\tau) = \oplus_{a\in\mathbb{Z}} p^a \cdot y(\tau_a).$$

On the other hand, since ν_x^{-1} splits $\mathcal{F}^\iota_{\mathrm{N}}(D)$,

$$\begin{array}{rcl} \Phi^s_{\mathrm{cris}}(y)(\tau) & = & \sum_{i\in\mathbb{Z}} p^{-i} y(\tau) \cap (s\mathcal{F}^\iota_{\mathrm{N}}(D))^i(\tau) \\ & = & \sum_{i\in\mathbb{Z}} \left(\oplus_{a\in\mathbb{Z}} p^{-i} y(\tau_a)\right) \cap (\oplus_{a\leq -i} y(\tau_a) \otimes K_0) \\ & = & \oplus_{a\in\mathbb{Z}} p^a \cdot y(\tau_a). \end{array}$$

Thus indeed $F = \Phi^s_{\mathrm{cris}}$ on $\mathcal{L}(D_x, \mathbf{t}^\iota_{\mathrm{H}}(D_x))$.

□

Remark 25. This also proves that Φ^s_{cris} is bijective on $\mathcal{L}(D_x, \mathbf{t}^\iota_{\mathrm{H}}(D_x))$, as we claimed in the proof of 7.3.30.

8.5.3 The main result

Putting all the results in this section together, we have obtained a commutative diagram

$$\begin{array}{ccccc} M_x(\mathbb{Q}_p)/M_x(\mathbb{Z}_p) & \xrightarrow{\simeq} & \mathcal{L}(V_x^{M_x}) & \xrightarrow{\simeq} & \mathcal{L}(D_x^{M_x}, \mathbf{t}^\iota_{\mathrm{H}}(D_x^{M_x})) \\ \cap & & \cap & & \downarrow\simeq \\ G(\mathbb{Q}_p)/G(\mathbb{Z}_p) & \xrightarrow{\simeq} & \mathcal{L}(V_x) & \xrightarrow{\mathrm{red}} & \mathcal{L}(D_x, \mathbf{t}^\iota_{\mathrm{H}}(D_x)) \end{array}$$

which is equivariant with respect to the actions of

$$\begin{array}{ccccccc} M_x(\mathbb{Q}_p) & \xrightarrow{\simeq} & \mathrm{Aut}^\otimes(\omega(V_x^{M_x})) & \xrightarrow{\simeq} & \mathrm{Aut}^\otimes(V_x^{M_x}) & \xrightarrow{\simeq} & \mathrm{Aut}^\otimes(D_x^{M_x}) \\ \cap & & \cap & & \downarrow\simeq & & \downarrow\simeq \\ G(\mathbb{Q}_p) & \xrightarrow{\simeq} & \mathrm{Aut}^\otimes(\omega(V_x)) & \supset & \mathrm{Aut}^\otimes(V_x) & \xrightarrow{\simeq} & \mathrm{Aut}^\otimes(D_x). \end{array}$$

In particular:

Theorem 8.5.6. *The reduction map*

$$\mathrm{red} \ : \ \mathcal{L}(V_x) \to \mathcal{L}(D_x, \mathbf{t}^\iota_{\mathrm{H}}(D_x))$$

factors through an $M_x(\mathbb{Q}_p)$-equivariant bijection

$$U_{\mathcal{F}_{\mathrm{F}}}(\mathbb{Q}_p)\backslash\mathcal{L}(V_x) \simeq \mathcal{L}(D_x, \mathbf{t}^\iota_{\mathrm{H}}(D_x)).$$

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