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Stochastic differential equations under G-expectation and applications

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**Équations différentielles
stochastiques
sous G -espérances
et applications**

Thèse soutenue à Rennes

le jeudi 04 mai 2017

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À ma famille

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Résumé

Depuis la publication de l'ouvrage de Choquet (1955), la théorie d'espérance non linéaire a attiré le grand intérêt des chercheurs pour ses applications potentielles dans les problèmes d'incertitude, les mesures de risque et le super-hedging en finance. Shige Peng a construit une sorte d'espérance entièrement non linéaire dynamiquement cohérente par l'approche des EDP. Un cas important d'espérance non linéaire cohérente en temps est la G -espérance, dans laquelle le processus canonique correspondant $(B_t)_{t \geq 0}$ est appelé G -mouvement brownien et joue un rôle analogue au processus de Wiener classique.

L'objectif de cette thèse est d'étudier, dans le cadre de la G -espérance, certaines équations différentielles stochastiques rétrogrades (G -EDSR) à croissance quadratique avec applications aux problèmes de maximisation d'utilité robuste avec incertitude sur les modèles, certaines équations différentielles stochastiques (G -EDS) réfléchies et équations différentielles stochastiques rétrogrades réfléchies avec générateurs lipschitziens.

On considère d'abord des G -EDSRs à croissance quadratique. Dans le Chapitre 2 nous fournissons un résultat d'existence et unicité pour des G -EDSRs à croissance quadratique. D'une part, nous établissons des estimations a priori en appliquant le théorème de type Girsanov, d'où l'on en déduit l'unicité. D'autre part, pour prouver l'existence de solutions, nous avons d'abord construit des solutions pour des G -EDSR discrètes en résolvant des EDPs non-linéaires correspondantes, puis des solutions pour les G -EDSRs quadratiques générales dans les espaces de Banach. Dans le Chapitre 3 nous appliquons les G -EDSRs quadratiques aux problèmes de maximisation d'utilité robuste. Nous donnons une caractérisation de la fonction valeur et une stratégie optimale pour les fonctions d'utilité exponentielle, puissance et logarithmique.

Dans le Chapitre 4, nous traitons des G -EDS réfléchies multidimensionnelles. Nous examinons d'abord la méthode de pénalisation pour résoudre des problèmes de Skorokhod déterministes dans des domaines non convexes et établissons des estimations pour des fonctions α -Hölder continues. A l'aide de ces résultats obtenus pour des problèmes déterministes, nous définissons le G -mouvement Brownien réfléchi et prouvons son existence et son unicité dans un espace de Banach. Ensuite, nous prouvons l'existence et l'unicité de solution pour les G -EDSRs multidimensionnelles réfléchies via un argument de point fixe.

Dans le Chapitre 5, nous étudions l'existence et l'unicité pour les équations différentielles stochastiques rétrogrades réfléchies dirigées par un G -mouvement brownien lorsque la barrière S est un processus de G -Itô.

Mots-clés : G -espérance, G -mouvement brownien, équations différentielles stochastiques, équations différentielles stochastiques rétrogrades, croissance quadratique, EDPs non-linéaire, maximisation d'utilité robuste, problèmes de Skorokhod, méthode de pénalisation, α -Hölder continues, domaines non convexes, barrière.

Abstract

Since the publication of Choquet's (1955) book, the theory of nonlinear expectation has attracted great interest from researchers for its potential applications in uncertainty problems, risk measures and super-hedging in finance. Shige Peng has constructed a kind of fully nonlinear expectation dynamically coherent by the PDE approach. An important case of time-consistent nonlinear expectation is G -expectation, in which the corresponding canonical process $(B_t)_{t \geq 0}$ is called G -Brownian motion and plays a similar role to the classical Wiener process.

The objective of this thesis is to study, in the framework of the G -expectation, some backward stochastic differential equations (G -BSDE) under a quadratic growth condition on their coefficients with applications to robust utility maximization problems with uncertainty on models, Reflected stochastic differential equations (reflected G -SDE) and reflected backward stochastic differential equations with Lipschitz coefficients (reflected G -BSDE).

We first consider G -BSDE with quadratic growth. In Chapter 2 we provide a result of existence and uniqueness for quadratic G -BSDEs. On the one hand, we establish a priori estimates by applying the Girsanov-type theorem, from which we deduce the uniqueness. On the other hand, to prove the existence of solutions, we first constructed solutions for discrete G -BSDEs by solving corresponding nonlinear PDEs, then solutions for the general quadratic G -BSDEs in the spaces of Banach. In Chapter 3 we apply quadratic G -BSDE to robust utility maximization problems. We give a characterization of the value function and an optimal strategy for exponential, power and logarithmic utility functions.

In Chapter 4, we discuss multidimensional reflected G -SDE. We first examine the penalization method to solve deterministic Skorokhod problems in non-convex domains and establish estimates for continuous α -Hölder functions. Using these results for deterministic problems, we define the reflected G -Brownian motion and prove its existence and its uniqueness in a Banach space. Then we prove the existence and uniqueness of the solution for the multidimensional reflected G -SDE via a fixed point argument.

In Chapter 5, we study the existence and uniqueness of the reflected backward stochastic differential equations driven by a G -Brownian motion when the obstacle S is a G -Itô process.

Keywords: G -expectation, G -Brownian motion, stochastic differential equations, backward stochastic differential equations, quadratic growth, nonlinear PDEs, robust utility maximization, Skorokhod problem, penalization method, α -Hölder continuity, non-convex domains, obstacle.

Contents

RESUME EN FRANCAIS	1
0.1 EDSRs	1
0.1.1 EDSRs quadratiques	1
0.1.2 EDSs et EDSRs réfléchies	4
0.2 G -espérance	6
0.3 Nouveaux résultats	12
0.3.1 G -EDSRs quadratiques	12
0.3.2 Maximisation d'utilité robuste	15
0.3.3 G -EDSs réfléchies	18
0.3.4 G -EDSRs réfléchies	21
1 Preliminaries in G-framework	25
1.1 G -Brownian motion and G -expectation	26
1.2 G -stochastic integral	30
2 Quadratic BSDEs driven by G-Brownian motion	33
2.1 Introduction	34
2.2 The formulation of G -BSDEs	37
2.3 G -Girsanov theorem and estimates for GBSDEs	38
2.3.1 The Girsanov type theorem	38
2.3.2 A priori estimates for GBSDEs	40
2.4 The existence of solutions to discrete GBSDEs	43
2.4.1 Discrete PDEs	43
2.4.2 The solution of the discrete GBSDE	45
2.5 Existence of solutions for general quadratic GBSDEs	47
3 Robust utility maximization with G-BSDE	55
3.1 Market model	56
3.2 Exponential utility	56
3.3 Power utility	61
3.4 Logarithmic utility	64
4 Reflected G-SDEs in non-convex domains	67
4.1 Introduction	68
4.2 Preliminaries	70
4.2.1 Conditions on the domain	70
4.2.2 Deterministic Skorohod problem	71

4.3	Main results	71
4.3.1	Reflected G -Brownian motion	71
4.3.2	Reflected stochastic differential equations driven by G -Brownian motion	72
4.4	Proofs	73
4.4.1	Estimates for the deterministic Skorohod problem	73
4.4.2	The existence and uniqueness for the reflected G -Brownian motion	77
4.4.3	The existence and uniqueness for the RGSDE	81
5	Reflected BSDEs driven by G-Brownian motion	87
5.1	Introduction	88
5.2	Formulation of the reflected GBSDEs	91
5.3	Some a priori estimations	93
5.4	Existence of the solution of the reflected G -BSDE	102

RESUME EN FRANCAIS

L'objectif de cette thèse est d'étudier, dans le cadre de la G -espérance, certaines équations différentielles stochastiques rétrogrades à croissance quadratique (G -EDSRs quadratique en abrégé) avec applications aux problèmes de maximisation d'utilité robuste avec incertitude sur les modèles, certaines équations différentielles stochastiques réfléchies (G -EDS réfléchies en abrégé) et équations différentielles stochastiques rétrogrades réfléchies avec générateurs lipschitziens (G -EDSRs réfléchies en abrégé). Dans cette introduction, nous présentons les notions d'EDSR en Section 1, de G -espérance en Section 2. La Section 3 présente une synthèse des principaux résultats obtenus dans cette thèse. Ceux-ci sont l'objet des chapitres suivants.

0.1 EDSRs

On se place sur un espace probabilisé complet $(\Omega, \mathbb{F}, \mathbb{P})$, sur lequel est construit un mouvement brownien d -dimensionnel $(W_t)_{t \geq 0}$ dont on note $(\mathcal{F}_t)_{t \geq 0}$ la filtration naturelle augmentée. Sur cet espace, une équation différentielle stochastique rétrograde (EDSR) d'horizon $T > 0$ qui peut être déterministe ou aléatoire (temps d'arrêt) est une équation de la forme :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \quad (0.1)$$

La valeur terminale ξ est une variable aléatoire \mathcal{F}_T -mesurable à valeurs dans \mathbb{R}^n et le générateur $f : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^n$ est une fonction progressivement mesurable par rapport à la filtration $(\mathcal{B}([0, t]) \times \mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^{n \times d}))_{0 \leq t \leq T}$. Les inconnues de l'EDSR (0.1) sont les processus Y et Z , à qui on impose d'être adaptés par rapport à la filtration brownienne. Les premières études sur les EDSRs furent les conditions d'existence et d'unicité de la solution dans le cas où l'horizon T est déterministe. Les EDSRs ont été introduites d'abord dans le cas où le générateur f est linéaire par Bismut en 1973 dans [9]. Dans le cas où le générateur f est une fonction uniformément lipschitzienne en y et z , Pardoux et Peng donnent dans [75] un résultat d'existence et d'unicité de la solution sous les hypothèses $\xi \in L^2(\mathcal{F}_T; R)$ et $\{f(t, 0, 0)\}_{0 \leq t \leq T}$ est un processus de carré intégrable (voir le théorème 3.1 [75]). Depuis, la théorie des EDSR s'est considérablement développée : de nombreuses autres études ont été menées avec pour objectif la recherche de conditions minimales sur f et ξ qui garantissent néanmoins l'existence et l'unicité de la solution.

0.1.1 EDSRs quadratiques

Le premier résultat concernant l'existence et l'unicité de solution pour les EDSRs à croissance quadratique a été obtenu en 2000, par Kobylanski dans [53], dans le cas où la valeur terminale est bornée. Il a construit des solutions pour les EDSRs unidimensionnelles dont les générateurs sont

à croissance quadratique en z : pour tout $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$|f(t, y, z)| \leq \alpha_t + \beta |y| + \frac{\gamma}{2} |z|^2, \quad \text{uniformément en } (t, \omega), \quad (0.2)$$

où β et γ sont deux constantes positives, et α est un processus adapté positif satisfaisant la propriété d'intégrabilité suivante : il existe une constante positive C , telle que

$$\int_0^T \alpha_t dt \leq C, \quad \mathbb{P} - p.s..$$

Pour prouver l'existence d'une solution à (0.1) dans ce cadre, l'auteur construit dans [53] une suite de solutions des EDSRs dont les générateurs vérifient une hypothèse quadratique et sont croissants (resp. décroissants) et minorés (resp. majorés) par une fonction linéaire en (y, z) . De plus, en appliquant un changement de variable exponentiel, ces EDSRs se transforment en des équations à coefficients lipschitziens. Par conséquent, l'existence de solutions pour ces EDSRs est assurée par le résultat de Pardoux et Peng [75]. De plus, on peut montrer par un principe de comparaison que la suite de solutions est monotone. Par une technique de convergence faible empruntée aux EDPs, un théorème de stabilité monotone pour les EDSRs quadratiques est prouvé dans le même article [53] : il démontre que si une suite de $(Y_s)_{s \in [0, T]}$ converge uniformément sur $[0, T]$ trajectoire par trajectoire, la suite de $(Z_s)_{s \in [0, T]}$ correspondants converge dans $H^2(\mathbb{R}^d)$ pour la topologie forte. Grâce à ce théorème, une solution maximale (resp. minimale) à l'EDSR (0.1) peut être construite dans $\mathcal{S}^\infty(\mathbb{R}) \times H^2(\mathbb{R}^d)$ comme limite de la suite ci-dessus. Nous présentons ici une version légèrement généralisée de ce théorème de stabilité monotone (cf. Briand et Hu [13]) :

Théorème 0.1 Soient $\{\xi^n\}_{n \in \mathbb{N}}$ une suite de variables aléatoires \mathcal{F}_T -mesurables bornées, et $\{f^n\}_{n \in \mathbb{N}}$ une suite de générateurs continus en (y, z) . Supposons que $\xi^n \rightarrow \xi$, \mathbb{P} -p.s., $f^n \rightarrow f$ localement uniformément en (y, z) , et

- $\sup_{n \in \mathbb{N}} \|\xi^n\|_{L^\infty} < +\infty$;
- $\sup_{n \in \mathbb{N}} |f^n(t, y, z)|$ vérifie l'inégalité (0.2).

Nous supposons de plus que pour chaque $n \in \mathbb{N}$, l'EDSR correspondant aux paramètres (ξ^n, f^n) admet une solution (Y^n, Z^n) dans $\mathcal{S}^\infty(\mathbb{R}) \times H^2(\mathbb{R}^d)$, telle que la suite $\{Y^n\}_{n \in \mathbb{N}}$ est croissante (resp. décroissante). Alors, la suite $\{Y^n\}_{n \in \mathbb{N}}$ converge vers $Y_t := \sup_{n \in \mathbb{N}} Y_t^n$ (resp. $\inf_{n \in \mathbb{N}} Y_t^n$) uniformément sur $[0, T]$, \mathbb{P} -p.s.. De plus, la suite $\{Z^n\}_{n \in \mathbb{N}}$ converge vers un certain Z dans $H^2(\mathbb{R}^d)$ et le couple (Y, Z) est une solution de l'EDSR correspondant aux paramètres (ξ, f) .

Notons que le résultat d'existence obtenu par Kobylanski dans [53] a été amélioré : par exemple, Lepeltier et San Martín fournissent dans [58] un résultat dans le cas où la croissance du générateur f n'est plus linéaire en y ; Briand et Hu considèrent dans [13] des EDSRs dont les générateurs vérifient (0.2), mais dont les valeurs terminales ne sont plus bornées. Dans ces articles, l'approximation du générateur initial f est facile. Elle est donnée par :

$$f^n(t, y, z) := \sup_{(p, q) \in \mathbb{Q}^{1+d}} \{f(t, p, q) - n|p - y| - n|q - z|\}, \quad \text{pour chaque } n \in \mathbb{N}. \quad (0.3)$$

Si f vérifiant (0.2) est majorée par une fonction linéaire en (y, z) , les générateurs f^n , $n \in \mathbb{N}$, définis en (0.3) sont lipschitziens, et la suite est décroissante. D'autre part, si f est minorée par une telle fonction, une suite croissante peut être également définie par inf-convolution. Dans le domaine des EDSRs, cette idée de construction par convolution est issue de l'article de Lepeltier et San Martín [57] pour obtenir l'existence de solutions à des EDSRs dont les générateurs sont continus et à croissance linéaire par rapport à y , mais lipschitziens en z .

Pour l'unicité, Kobylanski fournit dans [53] un résultat sous certaines hypothèses techniques, en supposant que le générateur f est lipschitzien en y mais seulement localement lipschitzien en z , Hu et al. prouvent dans [44] que si la valeur terminale est bornée, Z est un générateur de martingale à oscillation moyenne bornée (martingale OMB en abrégé), et ainsi ils montrent l'unicité de la solution dans $\mathcal{S}^\infty(\mathbb{R}) \times H_{BMO}^2(\mathbb{R}^d)$. L'hypothèse technique mentionnée ci-dessus est de la forme suivante : pour tout $(y, z, z') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ et une certaine constante $K > 0$,

$$|g(t, y, z) - g(t, y, z')| \leq K |z - z'| (1 + |z| + |z'|), \quad \text{uniformément en } (\omega, t, y). \quad (0.4)$$

Par ailleurs, Briand et Hu [14] ainsi que Delbean et al. [22] traitent le même sujet, mais dans le cas où le générateur est convexe et la valeur terminale est non bornée.

Signalons que les EDSRs à croissance quadratique en z sont utiles pour la résolution de problème de maximisation sous contraintes de l'utilité d'un portefeuille en finance, qui est de la forme suivante :

$$V(x) := \sup_{\pi \in \tilde{\mathcal{A}}} E^{\mathbb{P}} [U(X_T^{x, \pi})]. \quad (0.5)$$

Le premier résultat pour ce problème via la technique des EDSRs quadratiques est obtenu par El Karoui et Rouge dans [34], lorsque la fonction d'utilité est exponentielle et est donnée par $U(x) := -c \exp(-x)$, $c > 0$, et la contrainte est convexe. Un problème dual du pricing est établi et la résolution de ce problème dual est donnée par la solution d'une EDSR quadratique. Ce résultat est amélioré dans l'article [44] de Hu et al., où le problème initial est directement traité sans hypothèse de convexité sur la contrainte.

Considérons qu'il y a une seule obligation et n actifs sur le marché financier ($n \leq d$). Le taux d'intérêt de cette obligation est zéro et les processus de prix des actifs suivent les EDSs suivantes : pour un certain processus b borné et un certain processus σ tel que $rg(\sigma) = n$ et $\sigma\sigma^{Tr}$ est uniformément elliptique,

$$\frac{dS_t^i}{S_t^i} = b_t^i dt + \sigma_t^i dW_t, \quad 0 \leq t \leq T, \quad \mathbb{P} - p.s., \quad i = 1, \dots, n. \quad (0.6)$$

Soit $\tilde{\mathcal{A}}$ un ensemble de processus π , qui sont \mathcal{F} -progressivement mesurables, à valeurs dans une contrainte \tilde{C} , tels que $\int_0^T |\pi_t \sigma_t| dt < +\infty$, \mathbb{P} -p.s., et $\{\exp(-cX_\tau^\pi)\}_{\tau \in \mathcal{T}_0^T}$ est une famille uniformément intégrable. Ici, π désigne une stratégie de l'investisseur, où π_t^i désigne le montant investi dans l'actif i au temps t . Alors, la fonction de valeur s'écrit :

$$V(x) = \sup_{\pi \in \tilde{\mathcal{A}}} \{\mathbb{E}[U(X_T^\pi - \xi)]\} = \sup_{\pi \in \tilde{\mathcal{A}}} \mathbb{E} \left[-\exp \left(-c \left(x + \int_0^T \pi_t \frac{dS_t}{S_t} - \xi \right) \right) \right],$$

où la variable aléatoire \mathcal{F}_T -mesurable et bornée ξ désigne un actif contingent autre que le portefeuille, qui arrive à la date T . Le but de l'investisseur est de choisir une des meilleures stratégies π appartenant à $\tilde{\mathcal{A}}$ qui optimise l'utilité espérée au temps T , i.e. la valeur de V . Dans [44], une résolution de ce problème est obtenue grâce à la solution d'une EDSR quadratique. Plus précisément, la fonction de valeur peut être représentée par

$$V(x) = -\exp(-c(x - Y_0)),$$

où Y_0 est la solution de l'EDSR donnée par

$$Y_t = \xi + \int_t^T f(s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dW_s, \quad t \in [0, T], \quad (0.7)$$

et où pour chaque $(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d$,

$$f(\omega, t, z) := \frac{c}{2} \left\{ \text{dist} \left(z + \frac{1}{c} \theta_t(\omega), \tilde{C}_t \sigma(\omega) \right) \right\}^2 - z^{Tr} \theta_t(\omega) - \frac{1}{2c} |\theta_t(\omega)|^2,$$

avec $\theta_t = \sigma_t^{tr} (\sigma_t \sigma_t^{tr})^{-1} b_t$ et dist est la fonction distance. Un calcul simple permet de montrer que f vérifie (0.2), (0.4) et les conditions usuelles pour les générateurs d'EDSRs. L'existence et l'unicité de la solution sont alors assurées pour (0.7).

Par ailleurs, les EDSRs quadratiques ont été étudiées dans un cadre non brownien (citons par exemple, Morlais [70]), et aussi dans le cadre d'une filtration discontinue.

0.1.2 EDSs et EDSRs réfléchies

Les procesus de diffusion réfléchis ont été indroduits d'abord par Skorohod dans [102, 103], où l'étude est fondée sur les problèmes de Skorohod. Depuis, des solutions réfléchies aux équations différentielles stochastiques (EDS) ont été étudiées par de nombreux auteurs. Dans le cas unidimensionnel, El Karoui [29], El Karoui et Chaleyat-Maurel [30] et Yamada [115] ont étudié les EDS sur une demi-droite. Le cas multidimensionnel remonte à Stroock et Varadhan [111] qui ont établi l'existence de solutions faibles à des EDS réfléchies sur la frontière d'un domaine lisse. Par la suite, Tanaka [112] a prouvé l'existence et l'unicité des solutions dans le cas de domaine convexe avec une réflexion normale par une approche directe fondée sur la solution au problème de Skorohod. Ces résultats ont été généralisés, en particulier, par Lions et Sznitman [65] et Saisho [97] aux domaines $D \subset \mathbb{R}^d$, non convexe satisfaisant les conditions suivantes :

CONDITION (A). (Condition de la sphère extérieure uniforme) Il existe $r_0 > 0$ tel que $\mathcal{N}_x = \mathcal{N}_{x, r_0} \neq \emptyset$, $\forall x \in \partial D$, où pour $x \in \partial D$, on note

$$\begin{aligned} \mathcal{N}_{x, r} &= \left\{ \mathbf{n} \in \mathbb{R}^d, |\mathbf{n}| = 1 \text{ et } B(x - r\mathbf{n}, r) \cap D = \emptyset \right\} \\ \mathcal{N}_x &= \bigcup_{r > 0} \mathcal{N}_{x, r}. \end{aligned}$$

CONDITION (B). Il existe $\delta > 0$, $\beta \in [1, \infty)$ tels que l'on ait : $\forall x \in \partial D$ il existe un vecteur unitaire l_x tel que

$$\langle l_x, \mathbf{n} \rangle \geq \frac{1}{\beta} \quad \text{pour tout } \mathbf{n} \in \bigcup_{y \in B(x, \delta) \cap \partial D} \mathcal{N}_y.$$

Si un domaine $D \subset \mathbb{R}^d$ satisfait ces conditions, alors pour tout $x \in \mathbb{R}^d$ tel que $\text{dist}(x, \overline{D}) < r_0$, il existe un unique (projeté) $\bar{x} \in \overline{D}$ vérifiant $|x - \bar{x}| = \text{dist}(x, \overline{D})$ et une fonction positive $U \in \mathcal{C}^1(\mathbb{R}^d)$, de gradient ∇U borné et lipschitzien, telle que $U(x) = |x - \bar{x}|^2 = d_D^2(x)$ si $\text{dist}(x, \overline{D}) \leq r_0/2$.

En imposant une condition d'admissibilité sur le domaine, Lions et Sznitman ont prouvé dans [65] l'existence et l'unicité des solutions aux EDSs réfléchies dans deux cas différents. Ils considèrent d'abord le cas d'une réflexion normale sur des domaines satisfaisant les conditions (A) et (B), puis le cas de directions obliques de réflexion sur des domaines lisses. En outre, pour des directions de réflexion "variant régulièrement", des résultats d'existence et d'unicité ont été obtenus dans le cas particulier où le cône de réflexion oblique peut être transformé en cône normal par multiplication par une fonction matrice lisse. En fait, Saisho a montré plus tard dans [97] que dans le cas de réflexion normale, la condition d'admissibilité n'est pas nécessaire et peut être supprimée. Notons que dans le cas d'un orthant avec des directions constantes de réflexion oblique sur les

côtés, Harrison et Reiman [46] ont trouvé des conditions suffisantes pour l'existence et l'unicité des solutions au problème de Skorohod ainsi que pour la continuité de l'application de réflexion de Skorohod. Dans ce contexte, nous mentionnons également que Bernard et El Kharroubi [8] ont fourni des conditions nécessaires et suffisantes pour l'existence de solutions au problème de Skorohod dans un orthant avec des directions de réflexion constantes sur chaque face.

Jusqu'ici, les résultats les plus généraux concernant l'existence de solutions au problème de Skorohod avec la réflexion oblique ont été fournis par Costantini dans [17]. Par exemple, Dupuis et Ishii ont obtenu dans [24] l'unicité pour un polyèdre convexe avec des directions de réflexion constantes sur les faces en supposant l'existence d'un certain ensemble convexe, défini en termes des directions normales et des directions de réflexion. Dupuis et Ishii ont ensuite étendu ce résultat dans [25, 26] à des domaines lisses par morceaux avec des directions de réflexion variables sur chaque face. En outre, nous mentionnerons ici les travaux [27, 28] de Dupuis et Ramanan fondés sur des techniques de dualité convexe. En particulier, dans [28], la dualité convexe est utilisée pour transformer la condition de Dupuis et Ishii dans [24] en une condition qui est beaucoup plus facile à vérifier. D'autres auteurs ont traité le cas du problème de Skorohod dans des domaines dépendant du temps, par exemple, Costantini et al ont prouvé dans [18] l'existence et l'unicité des solutions au problème de Skorohod avec une réflexion normale dans des domaines lisses et dépendants du temps puis Nystrom et Onskog avec réflexion oblique [73].

Soulignons que la littérature consacrée aux problèmes de Skorohod, à leurs extensions et applications est beaucoup plus riche que ce qui est indiqué ici, beaucoup plus de chercheurs ont contribué à ce vaste champ. En particulier, les champs d'application du problèmes de Skorohod comprennent l'analyse du trafic intense, des réseaux de files d'attente (voir par exemple [1, 94, 96]), la théorie du contrôle, la théorie des jeux et l'économie mathématique (voir par exemple [2, 54]), le traitement d'image (voir par exemple [10]) et la dynamique moléculaire (voir par exemple [98, 99, 100]).

Les EDSRs réfléchies ont été introduites par El Karoui et al. dans [31] dans le cas unidimensionnel. Il s'agit de chercher un triplet de processus progressivement mesurables (Y, Z, K) , où le processus K est non décroissant et tel que

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, & 0 \leq t \leq T; \\ Y_t \geq S_t, & 0 \leq t \leq T; \\ \int_0^T (Y_t - S_t) dK_t = 0. \end{cases}$$

Ici, S est un processus progressivement mesurable, qui jouera le rôle d'une barrière. Le rôle du processus K ici est de pousser le processus Y vers le haut pour le maintenir au-dessus de la barrière S . La dernière condition est connue sous le nom de condition de Skorohod et garantit que le processus K agit de manière minimale, c'est-à-dire seulement lorsque le processus Y atteint la barrière inférieure S . Dans [31], les auteurs ont prouvé l'unicité et l'existence à la fois par un argument de point fixe et par une approximation par pénalisation. Sur la base de ces résultats, Cvitanic et Karatzas ont ensuite introduit dans [19] des EDSRs réfléchies avec deux barrières, ils cherchent alors une solution à une EDSR dont la composante Y est forcée de rester entre deux processus progressivement mesurables L et U ($L \leq U$). L'EDSR multidimensionnelle réfléchie a été étudié par Gegout-Petit et Pardoux [37], dans un domaine convexe et une direction de réflexion normale et dans le cas de réflexion oblique par Ramasubramanian [92] puis Hu et Tang [47].

0.2 G -espérance

Depuis la publication de l'ouvrage [16] de Choquet (1955), la théorie d'espérance non linéaire a attiré avec grand intérêt des chercheurs pour ses applications potentielles dans les problèmes d'incertitude, les mesures de risque et le super-hedging en finance. Peng a construit dans [78, 79] une sorte d'espérance entièrement non linéaire dynamiquement cohérente par l'approche des EDPs. Un cas important d'espérance non linéaire cohérente en temps est la G -espérance, dans laquelle le processus canonique correspondant $(B_t)_{t \geq 0}$ est appelé G -mouvement brownien et est l'analogie du processus de Wiener classique, voir [80, 81, 82, 83, 84]. Dans le cadre de la G -espérance, le calcul stochastique correspondant du type d'Itô a été établi dans [81, 82, 83].

Dans cette section, nous passons en revue les notations et les résultats de base, dans le cadre de la G -espérance, qui concernent la formulation du G -mouvement brownien et le G -calcul stochastique.

Soit Ω un espace métrique séparable complet, et soit \mathcal{H} un espace vectoriel de fonctions réelles définies sur Ω satisfaisant : si $X_i \in \mathcal{H}$, $i = 1, \dots, n$, alors

$$\varphi(X_1, X_2, \dots, X_n) \in \mathcal{H}, \quad \forall \varphi \in \mathcal{C}_{l,Lip}(\mathbb{R}^n),$$

où $\mathcal{C}_{l,Lip}(\mathbb{R}^n)$ est l'espace de toutes les fonctions continues à valeurs réelles définies sur \mathbb{R}^n telle que pour certains $C > 0$ et $k \in \mathbb{N}$ dépendant de φ ,

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \quad \forall x, y \in \mathbb{R}^n.$$

Définition 0.2 (Espace d'espérance sous linéaire) Une espérance sous linéaire $\mathbb{E}[\cdot]$ est une fonction $\mathbb{E} : \mathcal{H} \rightarrow \mathbb{R}$ satisfaisant les propriétés suivantes : pour tout $X, Y \in \mathcal{H}$, nous avons

- (a) Monotonie : si $X \geq Y$, alors $\mathbb{E}[X] \geq \mathbb{E}[Y]$;
- (b) Conservation de constante : $\mathbb{E}[c] = c$, $c \in \mathbb{R}$;
- (c) Sous-additivité : $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$;
- (d) Homogénéité positive : $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, pour tout $\lambda \geq 0$.

Le triplet $(\Omega, \mathcal{H}, \mathbb{E})$ s'appelle alors espace d'espérance sous linéaire.

Dans la suite, fixons un espace d'espérance sous linéaire $(\Omega, \mathcal{H}, \mathbb{E})$.

Définition 0.3 (Indépendance) Une variable aléatoire $Y \in \mathcal{H}$ est dite indépendante de $X_1, X_2, \dots, X_n \in \mathcal{H}$, si pour tout $\varphi \in \mathcal{C}_{l,Lip}(\mathbb{R}^{n+1})$,

$$\mathbb{E}[\varphi(X_1, X_2, \dots, X_n, Y)] = \mathbb{E}\left[\mathbb{E}[\varphi(x_1, x_2, \dots, x_n, Y) \mid (x_1, x_2, \dots, x_n) = (X_1, X_2, \dots, X_n)]\right].$$

Nous introduisons maintenant la notion de distribution G -normale.

Définition 0.4 (Distribution G -normale) On dit qu'une variable aléatoire $X \in \mathcal{H}$ suit une distribution G -normale, noté par $X \sim \mathcal{N}(0, [\underline{\sigma}, \bar{\sigma}])$, $0 \leq \underline{\sigma} \leq \bar{\sigma}$, si pour toute fonction $\varphi \in \mathcal{C}_{l,Lip}(\mathbb{R})$, la fonction u définie par $u(t, x) := \mathbb{E}[\varphi(x + \sqrt{t}X)]$, $(t, x) \in [0, +\infty) \times \mathbb{R}$, est une solution de viscosité de la G -équation de la chaleur :

$$\partial_t u - G(D_x^2 u) = 0; \quad u(0, x) = \varphi(x),$$

où

$$G(a) := \frac{1}{2} (\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-).$$

Dans la suite, nous ne considérons que le cas non dégénéré, i.e., $\sigma > 0$. Nous fixons maintenant $\Omega := \mathcal{C}([0, +\infty))$, qui est équipé de la filtration \mathcal{F} engendrée par le processus canonique $(B_t)_{t \geq 0}$, i.e., $B_t(\omega) = \omega_t$, pour $(t, \omega) \in [0, \infty) \times \Omega$. Considérons les espaces fonctionnels définis par

$$Lip(\Omega_T) := \left\{ \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}) : \right. \\ \left. 0 \leq t_1 \leq t_2 \leq \dots, t_n \leq T, \varphi \in \mathcal{C}_{l,Lip}(\mathbb{R}^n) \right\}, \quad \text{pour } T > 0,$$

et

$$Lip(\Omega) := \bigcup_{n=1}^{\infty} Lip(\Omega_n).$$

Définition 0.5 (G-espérance, G-mouvement Brownien) Soit $G(\cdot) : \mathbb{S}_d \rightarrow \mathbb{R}$ une fonction monotone et sous-linéaire donnée.

1. Une G-espérance est une espérance sous-linéaire définie sur $Lip(\Omega)$ par

$$\mathbb{E}[X] = \tilde{\mathbb{E}} \left[\varphi \left(\sqrt{t_1} \xi_1, \sqrt{t_2 - t_1} \xi_2, \dots, \sqrt{t_n - t_{n-1}} \xi_n \right) \right],$$

pour tout $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})$ avec $0 = t_0 < t_1 < \dots < t_n < \infty$, où $\xi_1, \xi_2, \dots, \xi_n$ sont de distribution G-normale indistinctement distribués dans un espace d'espérance sous-linéaire $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$ tels que ξ_{i+1} est indépendant de (ξ_1, \dots, ξ_i) pour tout $i = 1, \dots, n-1$.

2. Sur l'espace d'espérance sous linéaire $(\Omega, Lip(\Omega), \mathbb{E})$, le processus canonique $(B_t)_{t \geq 0}$ est appelé G-mouvement Brownien et on a, les propriétés suivantes :

(a) $B_0 = 0$;

(b) pour chaque $t, s \geq 0$, l'accroissement $B_{t+s} - B_t \sim \mathcal{N}(0, [\sqrt{s}\sigma, \sqrt{s}\bar{\sigma}])$ et est indépendant de $(B_{t_1}, B_{t_2}, \dots, B_{t_n})$, pour $0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq t$.

Définition 0.6 (G-espérance conditionnelle) Pour une variable aléatoire $\xi \in Lip(\Omega_T)$ de la forme suivante :

$$\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}), \quad \varphi \in \mathcal{C}_{l,Lip}(\mathbb{R}^n),$$

la G-espérance conditionnelle $\mathbb{E}_{t_i}[\cdot]$, $i = 1, \dots, n$, est définie comme suit

$$\mathbb{E}_{t_i}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})] = \tilde{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_i} - B_{t_{i-1}}),$$

où

$$\tilde{\varphi}(x_1, \dots, x_i) = \mathbb{E} \left[\varphi(x_1, \dots, x_i, B_{t_{i+1}} - B_{t_i}, \dots, B_{t_n} - B_{t_{n-1}}) \right].$$

Si $t \in (t_i, t_{i+1})$, alors la G-espérance conditionnelle $\mathbb{E}_t[\xi]$ peut être définie en reformulant ξ comme

$$\xi = \hat{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_t - B_{t_i}, B_{t_{i+1}} - B_t, \dots, B_{t_n} - B_{t_{n-1}}), \quad \hat{\varphi} \in \mathcal{C}_{l,Lip}(\mathbb{R}^{n+1}).$$

Pour $\xi \in Lip(\Omega_T)$ et $p \geq 1$, nous considérons la norme $\|\xi\|_{L_G^p} = (\mathbb{E}[|\xi|^p])^{1/p}$. Désignons par $L_G^p(\Omega_T)$ le complété de Banach de $Lip(\Omega_T)$ sous la norme $\|\cdot\|_{L_G^p}$. Il est facile de vérifier que la G-espérance conditionnelle $\mathbb{E}_t[\cdot] : Lip(\Omega_T) \rightarrow Lip(\Omega_t)$ est une application continue et peut donc être étendue à $\mathbb{E}_t : L_G^p(\Omega_T) \rightarrow L_G^p(\Omega_t)$.

Définition 0.7 (G-martingale) Un processus $(M_t)_{t \in [0, T]}$ avec $M_t \in L_G^1(\Omega_t)$, $0 \leq t \leq T$, est une G-martingale si $\mathbb{E}_s[M_t] = M_s$, pour tout $0 \leq s \leq t \leq T$. Le processus $(M_t)_{t \in [0, T]}$ est appelé G-martingale symétrique si de plus $-M$ est aussi une G-martingale.

Selon Denis et al. [20], nous avons le théorème de représentation suivant de la G-espérance sur $L_G^1(\Omega_T)$. Dans la suite, nous notons \mathbb{P}_0 la mesure de Wiener, sous laquelle le processus canonique $(B_t)_{t \geq 0}$ est un \mathbb{P}_0 -mouvement Brownien.

Théorème 0.8 (Représentation de la G-espérance) La G-espérance peut être représentée par l'espérance supérieure sur une collection de mesures de martingale, i.e., pour $\xi \in L_G^1(\Omega_T)$, nous avons

$$\mathbb{E}[\xi] = \sup_{\mathbb{P} \in \mathcal{P}_G} E^{\mathbb{P}}[\xi],$$

où

$$\mathcal{P}_G = \left\{ \mathbb{P}_h : \mathbb{P}_h = \mathbb{P}_0 \circ X^{-1}, X_t = \int_0^t h_s dB_s, h \in \mathbb{H}_{\mathbb{P}_0}^2(0, T), \right. \\ \left. h_t \in [\underline{\sigma}, \bar{\sigma}], \mathbb{P}_0 - p.s., 0 \leq t \leq T \right\}.$$

Grâce au théorème ci-dessus, la G-espérance peut être étendue à un domaine plus grand, i.e., pour toute fonction \mathcal{F}_T -mesurable X , $\mathbb{E}[X] := \sup_{\mathbb{P} \in \mathcal{P}_G} E^{\mathbb{P}}[X]$. Il est également prouvé dans [20] que \mathcal{P}_G est relativement faiblement compact et donc son complété $\bar{\mathcal{P}}_G$ est faiblement compact. Par conséquent, nous pouvons naturellement définir la capacité de Choquet $\bar{C}(\cdot)$ en posant $\bar{C}(A) := \sup_{P \in \bar{\mathcal{P}}_G} P(A)$, $A \in \mathcal{B}(\Omega_T)$ et introduire la notion de quasi-sure.

Définition 0.9 (Quasi-sure) Un ensemble $A \in \mathcal{B}(\Omega_T)$ est dit polaire si $\bar{C}(A) = 0$. Une propriété est vraie «quasi-sûrement» (q.s.) si elle est vraie en dehors d'un ensemble polaire.

La proposition suivante permet de comprendre la correspondance entre les G-EDSRs et les 2EDSRs.

Proposition 0.10 Supposons que X et $Y \in L_G^1$, alors les assertions suivantes sont équivalentes :

- (a) pour tout $\mathbb{P} \in \mathcal{P}_G$, $X = Y$, \mathbb{P} -p.s. ;
- (b) $\mathbb{E}[|X - Y|] = 0$;
- (c) $\bar{C}(\{X \neq Y\}) = 0$.

Nous définissons l'espace $L_G^\infty(\Omega_T)$ comme le complété de $Lip(\Omega_T)$ sous la norme

$$\|\xi\|_{L_G^\infty} := \inf \{M \geq 0 : |\xi| \leq M \text{ q.s.}\}.$$

Une caractéristique importante de la théorie des G-espérances est que la variation quadratique du G-mouvement Brownien $\langle B \rangle_{t \geq 0}$ n'est plus un processus déterministe ; elle est donnée par

$$\langle B \rangle_t = \lim_{\mu(\pi_t^N) \rightarrow 0} \sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2 = B_t^2 - 2 \int_0^t B_s dB_s,$$

où les π_t^N , $N = 1, 2, \dots$, sont des partitions de $[0, t]$ et $\mu(\pi_t^N)$ désigne leur pas. Par Peng [83], pour tout $t, s \geq 0$, on a $\langle B \rangle_{t+s} - \langle B \rangle_t \in [s\underline{\sigma}^2, s\bar{\sigma}^2]$, q.s.

Dans ce qui suit, nous discutons des intégrales stochastiques de type Itô par rapport au G -mouvement brownien et sa variation quadratique définies suivant une procédure usuelle, c'est-à-dire la définissant d'abord pour des intégrands simples puis en complétant en suite les espaces d'intégrands sous la norme induite par l'espérance supérieure liée à \mathcal{P}_G .

Définition 0.11 Soit $\mathcal{H}_G^0(0, T)$ l'ensemble des processus simples de la forme :

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{1}_{[t_j, t_{j+1})}(t), \quad (0.8)$$

où $\pi_T = \{0 = t_0 < t_1 < \dots < t_N = T\}$ est une partition donnée de $[0, T]$ et $\xi_i \in \text{Lip}(\Omega_{t_i})$, pour tout $i = 0, 1, 2, \dots, N-1$. Pour $p \geq 1$ et $\eta \in \mathcal{H}_G^0(0, T)$, posons $\|\eta\|_{\mathcal{H}_G^p} = \{\mathbb{E}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}$ et $\|\eta\|_{M_G^p} = \left(\mathbb{E}\left[\frac{1}{T} \int_0^T |\eta_s|^p ds\right]\right)^{1/p}$. Nous désignons par $\mathcal{H}_G^p(0, T)$ et $M_G^p(0, T)$ les complétés respectifs de $\mathcal{H}_G^0(0, T)$ sous les normes $\|\cdot\|_{\mathcal{H}_G^p}$ et $\|\cdot\|_{M_G^p}$.

Définition 0.12 (G-intégrales stochastiques) Pour $\eta \in \mathcal{H}_G^0(0, T)$ de la forme (0.8), l'intégrale Itô par rapport au G -mouvement brownien est définie par l'application linéaire $\mathcal{I} : \mathcal{H}_G^0(0, T) \rightarrow L_G^2(\Omega_T)$

$$\mathcal{I}(\eta) := \int_0^T \eta_t dB_t = \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}),$$

qui peut être continûment étendue à $\mathcal{I} : \mathcal{H}_G^1(0, T) \rightarrow L_G^p(\Omega_T)$. D'autre part, l'intégrale stochastique par rapport à $(\langle B \rangle_t)_{t \geq 0}$ est définie par l'application linéaire $\mathcal{Q} : \mathcal{H}_G^0(0, T) \rightarrow L_G^1(\Omega_T)$

$$\mathcal{Q}(\eta) := \int_0^T \eta_t d\langle B \rangle_t = \sum_{j=0}^{N-1} \xi_j(\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}),$$

qui peut également être continûment étendue à $\mathcal{Q} : \mathcal{H}_G^1(0, T) \rightarrow L_G^1(\Omega_T)$.

La norme $\|\cdot\|_{\mathcal{H}_G^p}$ étant beaucoup plus forte que dans le cas classique, l'espace des intégrands est plus petit que le classique. Nous avons cependant quelques propriétés des intégrales de type G -Itô.

Proposition 0.13 (Inégalité du type BDG) Pour $\eta \in \mathcal{H}_G^\alpha(0, T)$, $\alpha \geq 1$ et $p > 0$, nous avons,

$$\begin{aligned} \underline{\sigma}^p c_p \mathbb{E} \left[\left(\int_0^T |\eta_s|^2 ds \right)^{p/2} \right] &\leq \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t \eta_s dB_s \right|^p \right] \\ &\leq \bar{\sigma}^p C_p \mathbb{E} \left[\left(\int_0^T |\eta_s|^2 ds \right)^{p/2} \right], \end{aligned}$$

où $0 < c_p < C_p < \infty$ sont des constantes indépendantes de η , $\underline{\sigma}$ et $\bar{\sigma}$.

Proposition 0.14 Pour tout $\eta, \theta \in \mathcal{H}_G^\alpha(0, T)$, $\alpha \geq 1$, avec une variable aléatoire bornée $\xi \in L_G^1(\Omega_t)$, nous avons

$$\begin{aligned} \mathbb{E}_t \left[\int_t^T \eta_s dB_s \right] &= 0; \\ \int_t^T (\xi \eta_s + \theta_s) dB_s &= \xi \int_t^T \eta_s dB_s + \int_t^T \theta_s dB_s. \end{aligned}$$

Enfin, nous définissons l'espace $\mathcal{S}_G^p(0, T)$. Posons

$$\mathcal{S}_G^0(0, T) := \left\{ h(t, B_{t_1 \wedge t}, B_{t_2 \wedge t} - B_{t_1 \wedge t}, \dots, B_{t_n \wedge t} - B_{t_{n-1} \wedge t}) : \right. \\ \left. 0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq T, h \in \mathcal{C}_{b, Lip}(\mathbb{R}^{n+1}) \right\},$$

où $\mathcal{C}_{b, Lip}(\mathbb{R}^{n+1})$ est l'ensemble de toutes les fonctions bornées et lipschitziennes sur \mathbb{R}^{n+1} . Pour $p \geq 1$ et $\eta \in \mathcal{S}_G^0(0, T)$, on pose $\|\eta\|_{\mathcal{S}_G^p} = \{\mathbb{E}[\sup_{t \in [0, T]} |\eta_t|^p]\}^{1/p}$. Nous désignons alors par $\mathcal{S}_G^p(0, T)$ le complété de $\mathcal{S}_G^0(0, T)$ sous la norme $\|\cdot\|_{\mathcal{S}_G^p}$.

En utilisant ces notions de calcul stochastique dans le cadre de la G -espérance, on peut obtenir des résultats d'existence et d'unicité pour certains types d'EDS dirigées par le G -mouvement brownien (G -EDS). Comme dans le cas classique, les G -EDSs sont bien définies dans le sens quasi-sûr et leur résolution peut être obtenue par le théorème de point fixe sous les hypothèses de Lipschitz (cf. Peng [83], Gao [35] et Lin et Bai [4]). Cependant, une certaine hypothèse de régularité supplémentaire devrait être imposée aux intégrandes pour s'assurer que les intégrales sont bien définies. Dans [83, 35, 4], les auteurs ont étudié des G -EDSs en supposant la condition suivante sur les coefficients des équations : pour tout $x \in \mathbb{R}$,

$$f(\cdot, x), g(\cdot, x) \in M_G^2([0, T]).$$

À ce prix, tous les résultats dans ces travaux pour les G -EDSs énumérés ci-dessus sont vraies dans le sens "quasi-sûre" (q.s.), c'est-à-dire en dehors d'un ensemble polaire, et tous les processus sont immédiatement "agrégés". Récemment, Lin [62] a prouvé l'existence et l'unicité de la solution des équations différentielles stochastiques réfléchies dirigées par le G -mouvement brownien dans le cas unidimensionnel, et dans sa thèse [61] il considère le problème de G -EDS multidimensionnelles dans un domaine convexe.

Comme dans le cas classique, le théorème de représentation de G -martingale est la clé pour la formulation des G -équations différentielles stochastiques rétrogrades (G -EDSRs). Pour une famille dense de G -martingales, Peng [80] a obtenu le résultat suivant : une G -martingale M se décompose comme suit

$$M_t = M_0 + \overline{M}_t + K_t, \quad (0.9)$$

où

$$\overline{M}_t = \int_0^t Z_s dB_s \quad \text{et} \quad K_t = \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds. \quad (0.10)$$

Contrairement à la représentation de martingale classique, la G -martingale M est décomposée en deux parties : la partie de type G -intégrale d'Itô $\overline{M} = \int Z dB$, qui est appelée G -martingale symétrique, (au sens où $-\overline{M}$ est encore une G -martingale) la partie G -martingale décroissante K , qui absente dans la théorie classique, joue cependant un rôle important dans ce nouveau contexte. Que le processus K admette la représentation unique sous la forme (0.10) est une question complexe. La première réponse positive est donnée par Peng dans [80] pour les G -martingale avec valeur terminale $M_T \in Lip(\Omega_T)$, comme une fonction lisse et fini-dimensionnelle de trajectoire. Il convient également de mentionner qu'une série de travaux successifs par Soner et al. [106] et Song [109] affirment l'existence et l'unicité de la décomposition (0.9) pour $M_T \in L_G^p(\Omega)$, $p > 1$. Enfin, avec la norme introduite dans [110] par Song, un théorème de représentation complète de G -martingale a été obtenu par Peng et al. [86] sur un sous-espace métrique complet de $L_G^p(\Omega)$, $p > 1$. Prenons en considération le théorème de représentation de G -martingale, une G -EDSR se formule naturellement comme suit, où la G -martingale décroissante K apparaît dans la dynamique :

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds + \int_t^T h(s, Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t). \quad (0.11)$$

Sous les hypothèses de Lipschitz sur les générateurs, Hu et al. ont étudié dans [39] l'existence et l'unicité du triplet (Y, Z, K) dans des espaces de Banach propres satisfaisant cette équation. Ils ont commencé avec des EDSRs avec des générateurs bornés et lisses et des valeurs terminales markoviennes et des solutions construites par des solutions classiques d'EDPs entièrement non linéaires (cf. les résultats de Krylov dans [55]). Ensuite, ils ont utilisé le théorème de partition de l'unité dans [39] pour procéder par une approximation de type de Galerkin des solutions des G -EDSRs avec des paramètres généraux. En outre, l'unicité a été déduit dans [39] sur la base d'estimations a priori. En particulier, l'unicité de la G -martingale décroissante K est imposée par les estimations de G -martingale dans [109]. Les résultats de [39] ouvrent une nouvelle voie dans la théorie des G -espérances. Dans le document d'accompagnement [40], Hu et al. ont discuté des propriétés fondamentales des G -EDSRs : le théorème de comparaison, la formule de Feynman-Kac entièrement non linéaire et la transformation de Girsanov. De plus, la correspondance entre les G -EDSRs et les solutions de type Sobolev des EDPs non linéaires à trajectoires dépendantes est examinée dans [85].

Nous comparons maintenant le résultat de [39] avec les œuvres profondes [107, 108] de Soner et al., dans lesquelles les équations différentielles stochastiques rétrogrades de second ordre (2EDSRs) sont étudiées en détail. Ce type d'équation est fortement lié aux G -EDSRs et il est défini sur l'espace de Wiener comme suit :

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s, \hat{a}_s) ds - \int_t^T Z_s dB_s + (K_T - K_t), \quad \mathbb{P} - \text{p.s.}, \forall \mathbb{P} \in \mathcal{P}_H,$$

où B est le processus canonique, le processus \hat{a} est la densité de $\langle B \rangle$ et \mathcal{P}_H est une collection de mesures martingales semblable à \mathcal{P}_G (\mathcal{P}_H pourrait être encore plus grande que \mathcal{P}_G). Cette équation est une EDSR renforcée dans le sens où elle est vraie \mathbb{P} -p.s. pour tout $\mathbb{P} \in \mathcal{P}_H$ de plus, la famille des $K := \{K^{\mathbb{P}}\}_{\mathbb{P} \in \mathcal{P}_H}$ doit satisfaire une condition minimale (alors $-K$ vérifie la contrainte de G -martingale dans le contexte G -EDSR, voir [106]) :

$$K_t^{\mathbb{P}} = \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H(t, \mathbb{P})} E_t^{\mathbb{P}'} [K_T^{\mathbb{P}'}], \quad \mathbb{P} - \text{p.s.}, \text{ pour tout } \mathbb{P} \in \mathcal{P}_H, \quad 0 \leq t \leq T.$$

Sous les hypothèses de Lipschitz, l'unicité pour une 2EDSR est prouvée dans [107] en observant que la solution de la 2EDSR peut être représentée comme le supremum essentiel de solutions d'une classe de EDSR dirigée par des martingales. Pour l'existence, la preuve implique une construction délicate : Le processus Y est défini par trajectoires par des solutions de EDSRs sur de changement d'espaces. Ce processus vérifie un principe critique d'optimalité et donc, la structure de 2EDSR pourrait être dérivée de la décomposition de g -supermartingale (cf. [77]), où la famille des processus K peut être a posteriori agrégée une fois que la partie intégrale stochastique est agrégée par Nutz [71]. Pour se débarrasser du problème de mesurabilité au cours de la construction des solutions, Soner et al. supposent la condition technique que ξ et F sont uniformément continus en ω , alors que cette condition est supprimée dans le travail récent [87] de Possamaï et al.. Dans le cadre des 2EDSRs, les résultats [107, 108] sont généralisés par Possamaï et Zhou [88] et par Lin [63] au cas quadratique. Par ailleurs, Matoussi et al. [69] ont appliqué les 2EDSRs quadratiques pour résoudre les problèmes de maximisation de l'utilité de [44] dans le contexte des modèles non dominés. On peut voir que les G -EDSR (0.11) correspondent aux 2EDSRs définies avec

$$F(t, y, z, a) = g(t, y, z) + h(t, y, z)a.$$

Cependant, la G -EDSR exige plus de conditions de structure sur le coefficient et la valeur terminale afin que la solution puisse être trouvée avec plus de régularité adaptée à l'exigence de l'espace de processus dans le cadre G -espérance.

0.3 Nouveaux résultats

On décrit maintenant les principales contributions de cette thèse.

0.3.1 G -EDSRs quadratiques

Le Chapitre 2 de cette thèse traite des équations différentielles stochastiques rétrogrades dirigées par le G -mouvement brownien (G -EDSR) sous des hypothèses quadratiques sur le générateur. L'objectif principal de ce chapitre est de fournir le résultat d'existence et d'unicité pour les G -EDSRs à croissance quadratique de type suivant :

$$Y_t = \xi + \int_t^T h(s, \omega_{\cdot \wedge s}, Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - K_T + K_t, \quad (0.12)$$

où la valeur terminale ξ et le générateur h satisfont les conditions suivantes :

Hypothèse 0.15 *Supposons que la valeur terminale $\xi \in L_G^\infty$ et le générateur $h : [0, T] \times \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfont :*

(H0) *Pour chaque $(t, \omega) \in [0, T] \times \Omega$,*

$$|h(t, \omega, 0, 0)| + |\xi(\omega)| \leq M_0;$$

(Hc) *De plus, $h(\cdot, \cdot, y, z)$ est uniformément continue en (t, ω) et le module de continuité est indépendant de (y, z) , i.e., pour chaque $(y, z) \times \mathbb{R}^2$,*

$$|h(t_1, \omega^1, y, z) - h(t_2, \omega^2, y, z)| \leq w^h (|t_1 - t_2| + \|\omega_1 - \omega_2\|_\infty);$$

(Hq) *La fonction h est uniformément lipschitzienne en y et uniformément localement lipschitzienne en z , i.e., pour chaque $(t, \omega) \in [0, T] \times \Omega$,*

$$|h(t, \omega, y^1, z^1) - h(t, \omega, y^2, z^2)| \leq L_y |y^1 - y^2| + L_z (1 + |z^1| + |z^2|) |z^1 - z^2|.$$

Pour prouver l'existence de solutions de (0.12), nous considérons d'abord les G -EDSRs à croissance quadratique auxiliaires suivantes, dans lesquelles le générateur h est remplacé par une fonction discrète f et une valeur terminale du type fonctionnel. Fixant $N \in \mathbb{N}$ et une partition $\pi^N := \{0 = t_0 < t_1 < \dots < t_N = T\}$ de $[0, T]$, nous considérons l'équation discrète

$$Y_t = \xi + \int_t^T f(s, B_{t_1 \wedge s}, B_{t_2 \wedge s} - B_{t_1 \wedge s}, \dots, B_{t_N \wedge s} - B_{t_{N-1} \wedge s}, Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - K_T + K_t, \quad (0.13)$$

où la valeur terminale $\xi = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_N} - B_{t_{N-1}})$ et le générateur f vérifient l'hypothèse suivante :

Hypothèse 0.16 *Nous supposons que la fonction $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$ et le générateur $f : [0, T] \times \mathbb{R}^N \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfont :*

(H0') *Pour chaque $(t, x_1, x_2, \dots, x_N) \in [0, T] \times \mathbb{R}^N$,*

$$|f(t, x_1, x_2, \dots, x_N, 0, 0)| + |\varphi(x_1, x_2, \dots, x_N)| \leq M_0;$$

(Hc') De plus, $f(t, x_1, x_2, \dots, x_N, y, z)$ est uniformément continue en $(t, x_1, x_2, \dots, x_N)$ et le module de continuité est indépendant de (y, z) , i.e., pour chaque $(y, z) \in \mathbb{R}^2$,

$$|f(t_1, x_1^1, x_2^1, \dots, x_N^1, y, z) - f(t_2, x_1^2, x_2^2, \dots, x_N^2, y, z)| \leq w^f \left(|t_1 - t_2| + \sum_{i=1}^N |x_i^1 - x_i^2| \right);$$

(Hq') La fonction f est uniformément lipschitzienne en y et uniformément localement lipschitzienne en z , i.e., pour chaque $(t, x_1, x_2, \dots, x_N) \in [0, T] \times \mathbb{R}^N$,

$$\begin{aligned} & |f(t, x_1, x_2, \dots, x_N, y^1, z^1) - f(t, x_1, x_2, \dots, x_N, y^2, z^2)| \\ & \leq L_y |y^1 - y^2| + L_z (1 + |z^1| + |z^2|) |z^1 - z^2|. \end{aligned}$$

Pour $p \geq 2$, on définit une solution de la G -EDSR (0.12) (respectivement (0.13)) comme étant un triplet de processus (Y, Z, K) , avec $Y \in \mathcal{S}_G^p(0, T)$, $Z \in \mathcal{H}_G^p(0, T)$ et K est une G -martingale décroissante telle que $K_0 = 0$ et $K_T \in L_G^p(\Omega_T)$ et pour $0 \leq t \leq T$, (Y, Z, K) satisfait (0.12) (respectivement (0.13)).

Si (Y, Z, K) est solution de (0.12) ou (0.13), alors par le Lemme 3.1 et le Théorème 3.2 de [88], on obtient les estimations a priori suivantes, pour Y et la norme G -BMO de Z ,

$$\left\| \sup_{0 \leq t \leq T} |Y_t| \right\|_{L_G^\infty} + \|Z\|_{BMO_G} \leq \hat{C} := C(M_0, L_y, L_z).$$

Cette borne assure que Z est un générateur de martingale G -BMO et pour $p \geq 1$, on a

$$\mathbb{E}[|K_t|^p] \leq \tilde{C}_p := C(p, M_0, L_y, L_z), \quad 0 \leq t \leq T.$$

Considérant deux triplets de solutions (Y^1, Z^1, K^1) et (Y^2, Z^2, K^2) avec paramètre respectif (h^1, ξ^1) et (h^2, ξ^2) , on pose :

$$\hat{b}_s^\varepsilon = (1 - l(\hat{Z}_s)) \frac{h^1(s, Y_s^2, Z_s^1) - h^1(s, Y_s^2, Z_s^2)}{|\hat{Z}_s|^2} \hat{Z}_s \mathbf{1}_{\{|\hat{Z}_s| > 0\}},$$

où l est une fonction lipschitzienne telle que $\mathbf{1}_{[-\varepsilon, \varepsilon]}(x) \leq l(x) \leq \mathbf{1}_{[-2\varepsilon, 2\varepsilon]}(x)$. Alors le processus $\hat{b}^\varepsilon \in \mathcal{H}_G^2(0, T)$ est un générateur de martingale G -BMO. Ainsi, par une transformation de type Girsanov, nous pourrions définir une nouvelle G -espérance $\hat{\mathbb{E}}[\cdot]$ avec $\mathcal{E}(-\hat{b}^\varepsilon)$, telle que $\hat{B} := B - \int \hat{b}^\varepsilon d\langle B \rangle$ soit un G -mouvement brownien sous $\hat{\mathbb{E}}[\cdot]$. Avec un argument de linéarisation, cela nous permet d'établir le résultat suivant de stabilité pour les G -EDSR (0.12) et (0.13) :

Proposition 0.17 *Considérons deux G -EDSRs quadratiques (0.12) avec paramètres (ξ^1, h^1) et (ξ^2, h^2) , où (ξ^i, h^i) , $i = 1, 2$, satisfont (H0) et (Hq) avec les mêmes constantes M_0, L_y et L_z . Supposons que pour $p \geq 2$, (Y^i, Z^i, K^i) sont des solutions correspondant à ces paramètres. Alors, pour tous $0 \leq t \leq T$,*

$$|Y_t^1 - Y_t^2| \leq C \left(\|\xi^1 - \xi^2\|_{L_G^\infty} + \hat{\mathbb{E}}_t \left[\int_t^T |h^1(s, Y_s^2, Z_s^2) - h^2(s, Y_s^2, Z_s^2)| d\langle B \rangle_s \right] \right),$$

et pour $1 \leq p'/2 < p$,

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^T |Z_t^1 - Z_t^2|^2 dt \right)^{p'/2} \right] & \leq C(p, \bar{\sigma}, \sigma, M_0, L_y, L_z) \left(\|\xi^1 - \xi^2\|_{L_G^\infty}^{p'} \right. \\ & \left. + \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^1 - Y_t^2|^p \right]^{p'/2p} \right). \end{aligned}$$

Le résultat principal de ce chapitre est le théorème suivant :

Théorème 0.18 *Supposons que les paramètres (ξ, h) de la G -EDSR (0.12) satisfont l'Hypothèse 0.15. Alors, il existe un unique triplet (Y, Z, K) solution de la G -EDSR (0.12).*

L'unicité de la solution est déduite de la Proposition 0.17. Pour construire la solution de la G -EDSR discrète (0.13), nous suivons la méthode de Hu et Ma dans [43] en considérant d'abord l'EDP non linéaire, discrète suivante : pour $k = 1, 2, \dots, N, t \in [t_{k-1}, t_k]$

$$\begin{aligned} \partial_t u^k(t, \mathbf{x}^{(k-1)}, x_k) + G(D_{x_k x_k}^2 u^k(t, \mathbf{x}^{(k-1)}, x_k) \\ + 2f(t, \mathbf{x}^{(k-1)}, x_k, \underbrace{0, \dots, 0}_{N-k}, u^k(t, \mathbf{x}^{(k-1)}, x_k), D_{x_k} u^k(t, \mathbf{x}^{(k-1)}, x_k))) = 0, \end{aligned} \quad (0.14)$$

avec la condition

$$u^k(t_k, \mathbf{x}^{(k-1)}, x_k) := u^{k+1}(t_k, \mathbf{x}^{(k-1)}, x_k, 0), \quad \text{si } k = 1, 2, \dots, N-1$$

et

$$u(T, x_1, x_2, \dots, x_N) = \varphi(x_1, x_2, \dots, x_N) \in \mathcal{C}_{b, lip}(\mathbb{R}^N),$$

où le générateur $f : [0, T] \times \mathbb{R}^N \times \mathbb{R}^2$ satisfait l'hypothèse supplémentaire suivante :

(Hd') La fonction f est au moins de classe \mathcal{C}^1 en $x_1, x_2, \dots, x_N, y, z$, différentiable en t et deux fois différentiable en $x_1, x_2, \dots, x_N, y, z$, avec les dérivées secondes de f en $x_1, x_2, \dots, x_N, y, z$ bornées sur l'ensemble $[0, T] \times \mathbb{R}^N \times [-M_y, M_y] \times [-M_z, M_z]$, pour tout $M_y, M_z > 0$.

Procédant comme dans l'appendice §B-4 de Peng [83], en appliquant les estimations de Krylov pour les EDP entièrement non linéaires (les résultats de l'Exemple 6.1.8 et du Théorème 6.4.3 dans Krylov [55]) on montre alors que l'EDP (0.14) admet une solution classique u bornée par $M := M(M_0, L_y)$ avec des dérivées premières en x_1, x_2, \dots, x_N bornées. Ainsi posant pour $t \in [t_{k-1}, t_k], k = 1, 2, \dots, N$:

$$\mathbf{B}_t^k := (B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_{k-1}} - B_{t_{k-2}}, B_t - B_{t_{k-1}});$$

$$Y_t := Y_t^k := u^k(t, \mathbf{B}_t^k);$$

$$Z_t := Z_t^k := D_{x_k} u^k(t, \mathbf{B}_t^k);$$

et

$$\begin{aligned} K_t &:= K_t^k \\ &= K_{t_{k-1}}^{k-1} + \\ &\int_{t_{k-1}}^t \left(D_{x_k x_k}^2 u^k(s, \mathbf{B}_s^k) + 2f(s, \mathbf{B}_s^k, \underbrace{0, \dots, 0}_{N-k}, u^k(s, \mathbf{B}_s^k), D_{x_k} u^k(s, \mathbf{B}_s^k)) \right) d\langle B \rangle_s \\ &- \int_{t_{k-1}}^t G \left(D_{x_k x_k}^2 u^k(s, \mathbf{B}_s^k) + 2f(s, \mathbf{B}_s^k, \underbrace{0, \dots, 0}_{N-k}, u^k(s, \mathbf{B}_s^k), D_{x_k} u^k(s, \mathbf{B}_s^k)) \right) ds, \end{aligned}$$

nous prouvons que le triplet (Y, Z, K) est une solution de la G -EDSR discrète (0.13) avec des paramètres satisfaisant les Hypothèses 0.16 et **(Hd')**. Par régularisations successives, nous obtenons l'existence de la solution de (0.13) sous l'Hypothèse 0.16. Pour le cas général de la G -EDSR quadratique (0.12), on suppose que la valeur terminale $\xi \in L_G^\infty$ est approximée par la suite suivante

$$\xi^n := \varphi^n(B_{t_1^n}, B_{t_2^n} - B_{t_1^n}, \dots, B_{t_{N(n)}^n} - B_{t_{N(n)-1}^n}) \in Lip(\Omega_T),$$

où pour chaque $n \in \mathbb{N}$, $0 = t_0^n \leq t_1^n \leq \dots \leq t_{N(n)}^n = T$, $\mu^n := \max_{k=1,2,\dots,N(n)} |t_k^n - t_{k-1}^n| \leq 1/2^n$, et pour $n \in \mathbb{N}$ fixé, nous construisons la fonction \bar{f}^n de $\mathbf{x}(n) = (x_1, x_2, \dots, x_{N(n)})$ par discrétisation de la fonction h , en suivant des procédures de linéarisation par morceaux des trajectoires arrêtées à l'instant t en posant

$$\bar{f}^n(t, x_1, x_2, \dots, x_{N(n)}, y, z) := h(t, \omega^{\mathbf{x}(n),t}, y, z),$$

avec :

- $\omega_{t_0}^{\mathbf{x}(n),t} = 0$;
- $\omega_{t_1}^{\mathbf{x}(n),t} = x_1$;
- $\omega_{t_2}^{\mathbf{x}(n),t} = x_1 + x_2$;
- ... ;
- $\omega_{t_{k-1}}^{\mathbf{x}(n),t} = \sum_{i=1}^{k-1} x_i$;
- $\omega_t^{\mathbf{x}(n),t} = \omega_{t_k}^{\mathbf{x}(n),t} = \omega_{t_{k+1}}^{\mathbf{x}(n),t} = \omega_{t_{k+2}}^{\mathbf{x}(n),t} = \dots = \omega_{t_{N(n)}}^{\mathbf{x}(n),t} = \sum_{i=1}^k x_i$;
- $\omega^{\mathbf{x}(n),t}$ est une fonction linéaire de t sur $[t_{i-1}, t_i]$, pour $i = 1, 2, \dots, k$, et est également linéaire sur $[t_k, t]$ et prend une valeur constante sur $[t, T]$.

Alors la fonction \bar{f}^n satisfait l'Hypothèse 0.16 et donc la G -EDSR (0.13) de paramètres (\bar{f}^n, ξ^n) admet une solution $(\bar{Y}^n, \bar{Z}^n, \bar{K}^n)$. On montre ensuite que $\{\bar{Y}^n\}_{n \in \mathbb{N}}$ est une suite de Cauchy dans $\mathcal{S}_G^p(0, T)$ et que $\{\bar{Z}^n\}_{n \in \mathbb{N}}$ est une suite de Cauchy dans $\mathcal{H}_G^p(0, T)$. Ainsi, il existe un processus K tel que $K_0 = 0$ et $K_T \in L_G^p(\Omega_T)$, et comme pour $0 \leq t \leq T$,

$$\mathbb{E} [|\bar{K}_t^n - K_t|^p] \longrightarrow 0, \quad \text{quand } n \rightarrow \infty,$$

le processus K est une G -martingale décroissante.

0.3.2 Maximisation d'utilité robuste

Le Chapitre 3 de cette thèse est une application des G -EDSRs quadratiques étudiées dans le premier chapitre aux problèmes de maximisation robuste. Considérons un marché financier composé d'une seule obligation à taux d'intérêt zéro et n actifs sur le marché financier, $d \leq m$. Le processus des prix de l'action i évolue selon la G -EDS :

$$\frac{dS_t^i}{S_t^i} = b^i d \langle B \rangle_t + dB_t, \quad i = 1, \dots, d, \quad (0.15)$$

où b^i est une constante. Si pour une stratégie π , les processus π_t^i , $1 \leq i \leq d$, décrivent le montant d'argent investi en actions i au temps t , le processus de richesse X^π pour la stratégie π avec un capital initial x satisfait l'équation

$$X_t^{\pi, x} = x + \sum_{i=1}^d \int_0^t \frac{\pi_s^i}{S_s^i} dS_s^i = x + \int_0^t \pi_u (dB_u + b d \langle B \rangle_u), \quad t \in [0, T].$$

L'objectif de l'investisseur dans ce marché financier est de trouver une des meilleures stratégies π qui optimise l'utilité espérée au temps T . Alors, si U désigne la fonction d'utilité de l'investisseur, la fonction valeur V du problème de maximisation est donnée par :

$$V(x) = \sup_{\pi \in \tilde{\mathcal{A}}} \inf_{\mathbb{P} \in \mathcal{P}_G} \mathbb{E}^{\mathbb{P}} [U(X_T^\pi - F)] = - \inf_{\pi \in \tilde{\mathcal{A}}} \mathbb{E} [-U(X_T^\pi - F)],$$

où $\tilde{\mathcal{A}}$ désigne l'ensemble de toutes les stratégies admissibles.

Utilité exponentielle : Pour $\alpha > 0$, la fonction d'utilité exponentielle est définie par :

$$U(x) = -\exp(-\alpha x), \quad x \in \mathbb{R}.$$

On suppose que les stratégies sont à valeurs dans un ensemble fermé, c'est-à-dire, $\pi_t(\omega) \in \tilde{C} \subseteq \mathbb{R}^{1 \times d}$. L'ensemble des stratégies admissibles \mathcal{A} se compose de tous les processus d -dimensionnels π tels que $\pi = (\pi_t)_{0 \leq t \leq T} \in \tilde{C}$ soit un générateur de martingale G -BMO. Le problème de maximisation est alors :

$$V(x) = - \inf_{\pi \in \tilde{\mathcal{A}}} \mathbb{E} \left[\exp \left(-\alpha \left(x + \int_0^T \pi_t (dB_t + b d\langle B \rangle_t) - F \right) \right) \right].$$

Pour trouver la fonction valeur et une stratégie optimale, on pose :

$$\begin{aligned} R_t^{(\pi)} &= \exp(-\alpha(x - Y_0)) \exp \left(\alpha K_t - \int_0^t \alpha(\pi_s - Z_s) dB_s \right. \\ &\quad \left. - \alpha \int_0^t (b\pi_s + f(s, Z_s)) d\langle B \rangle_s \right), \end{aligned}$$

où (Y, Z, K) est l'unique solution de la G -EDSR :

$$Y_t = F + \int_t^T f(Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad t \in [0, T], \quad (0.16)$$

dont le générateur est donné par :

$$f(z) = \frac{\alpha}{2} \left\{ \text{dist} \left(z + \frac{1}{\alpha} b, C \right) \right\}^2 - zb - \frac{1}{2\alpha} |b|^2,$$

qui satisfait les hypothèses (H0), (Hc) et (Hq) du Chapitre 1. Nous prouvons que pour toute stratégie admissible π , le processus $R^{(\pi)}$ est une G -sous martingale et pour une stratégie π^* telle que $p_t^* \in \Pi_{C_t}(Z_t + \frac{1}{\alpha} b)$, $t \in [0, T]$, le processus $R^{(\pi^*)}$ soit une G -martingale. Donc pour tout $t \in [0, T]$, on a :

$$\mathbb{E} [R_t^{(\pi^*)}] = R_0 \leq \mathbb{E} [R_t^{(\pi)}], \quad \text{pour tout } \pi \in \tilde{\mathcal{A}}.$$

On peut alors comparer les utilités espérées pour les stratégies $\pi \in \tilde{\mathcal{A}}$ et $\pi^* \in \tilde{\mathcal{A}}$ par

$$-\mathbb{E} \left[\exp \left(-\alpha(X_T^{(\pi)} - F) \right) \right] \leq -R_0 = -\mathbb{E} \left[\exp \left(-\alpha(X_T^{(\pi^*)} - F) \right) \right] = V(x).$$

Finalement, la fonction valeur du problème d'optimisation est donnée par

$$V(x) = U_\alpha(x - Y_0) = -\exp(-\alpha(x - Y_0)),$$

où Y_0 est l'unique solution la G -EDSR (0.15) et π^* est une stratégie optimale.

Utilité puissance : Pour la fonction d'utilité puissance :

$$U_\gamma(x) = -\frac{1}{\gamma}x^{-\gamma}, \quad x > 0, \quad \gamma > 0,$$

l'investisseur maximise l'utilité attendue à l'instant final T sans passif additionnel ($F = 0$).

L'ensemble des stratégies admissibles $\tilde{\mathcal{A}}$ se compose de tous les processus d -dimensionnels, $\rho = (\rho_t)_{0 \leq t \leq T}$ tels que $\rho_t(\omega) \in C$ et ρ soit un générateur de G -BMO dans $M_G^2(0, T)$. Alors pour une stratégie ρ , le processus de richesse $X^{(\rho)}$ peut s'écrire sous la forme :

$$X_t^{(\rho)} = x \exp \left(\int_0^t \rho_s (b - \frac{1}{2} \rho_s) d\langle B \rangle_s + \int_0^t \rho_s dB_s \right), \quad t \in [0, T].$$

L'investisseur est alors confronté au problème de maximisation

$$\tilde{V}(x) = - \inf_{\rho \in \tilde{\mathcal{A}}} \mathbb{E} \left[\frac{x^{-\gamma}}{\gamma} \exp \left(- \int_0^T \gamma \rho_s dB_s - \int_0^T \gamma \rho_s b d\langle B \rangle_s + \frac{1}{2} \int_0^T \gamma |\rho_s|^2 d\langle B \rangle_s \right) \right].$$

Pour trouver la fonction valeur et une stratégie optimale, on pose pour $t \in [0, T]$

$$\begin{aligned} \tilde{R}_t^{(\rho)} &= \frac{x^{-\gamma} e^{Y_0}}{\gamma} \exp \left(K_t + \int_0^t (\gamma \rho_s + Z_s) dB_s - \frac{1}{2} \int_0^t |\gamma \rho_s + Z_s|^2 d\langle B \rangle_s \right) \\ &\quad \times \exp \left(\int_0^t \nu_s^{(\rho)} d\langle B \rangle_s \right), \end{aligned}$$

où (Y, Z, K) est la solution de la G -EDSR quadratique :

$$Y_t = 0 + \int_t^T f(Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad t \in [0, T],$$

de générateur

$$f(z) = \frac{\gamma(1+\gamma)}{2} \text{dist}^2 \left(\frac{1}{1+\gamma}(z+b), C \right) - \frac{\gamma|z+b|^2}{2(1+\gamma)} + \frac{|z|^2}{2}.$$

Comme dans le cas de la fonction d'utilité exponentielle, pour toute stratégie admissible ρ , le processus $R^{(\rho)}$ est une G -sous martingale et pour une stratégie ρ^* telle que $\rho_t^* \in \Pi_C \left(\frac{1}{1+\gamma}(Z_t + b) \right)$, $t \in [0, T]$, le processus $R^{(\rho^*)}$ est une G -martingale, alors

$$-\mathbb{E} \left[-U \left(X_T^{(\rho, x)} \right) \right] \leq -\tilde{R}_0^{(x)} = -\frac{x^{-\gamma}}{\gamma} \exp(Y_0), \quad \forall \rho \in \mathcal{A}.$$

Utilité logarithmique : Pour la fonction d'utilité logarithmique,

$$U(x) = \log(x), \quad x > 0,$$

les stratégies et le processus de richesse ont la même signification que dans le cas de l'utilité puissance. Le problème d'optimisation est donné par

$$V(x) = \log(x) - \inf_{\rho \in \mathcal{A}_t} \mathbb{E} \left[- \int_0^T \rho_s dB_s - \int_0^T \left(\rho_s b - \frac{1}{2} |\rho_s|^2 \right) d\langle B \rangle_s \right].$$

Pour $t \in [0, T]$, on pose :

$$R_t^{(\rho)} = Y_0 + \int_0^t (Z_s - \rho_s) dB_s + \int_0^t \left(\frac{1}{2} |\rho_s - b|^2 - \frac{1}{2} |b|^2 - f(s) \right) d\langle B \rangle_s + K_t,$$

où (Y, Z, K) est la solution de la G -EDSR linéaire :

$$Y_t = 0 + \int_t^T f(s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad t \in [0, T],$$

de générateur f :

$$f(t) = \frac{1}{2} \text{dist}^2(b_t, C_t) - \frac{1}{2} |b|^2, \quad t \in [0, T].$$

Le processus $R^{(\rho)}$ est une G -sous martingale pour tout $\rho \in \mathcal{A}_l$. Une stratégie optimale ρ^* satisfait $\rho_t^* \in \Pi_C(b)$ et on a :

$$V(x) = \log(x) - R_0^{\rho^*}(x) = \log(x) + \mathbb{E} \left[\int_0^T f(s) d\langle B \rangle_s \right].$$

0.3.3 G -EDSs réfléchies

Le Chapitre 4 traite des G -EDS réfléchies multidimensionnelles. Récemment, Lin [62] a prouvé l'existence et l'unicité de la solution des équations différentielles stochastiques réfléchies dirigées par le G -mouvement brownien dans le cas unidimensionnel, et dans sa thèse [61] il considère le cas des G -EDS multidimensionnelles dans un domaine convexe. L'objectif de ce chapitre est d'étudier la résolution de ces G -EDS réfléchies multidimensionnelles lorsque le domaine est non convexe. La G -EDS réfléchie multidimensionnelle que nous considérons est définie comme suit :

$$\begin{aligned} X_t &= x_0 + \int_0^t f(s, X_s) ds + \int_0^t h^{ij}(s, X_s) d\langle B^i, B^j \rangle_s \\ &\quad + \int_0^t g^j(s, X_s) dB_s^j + K_t, \quad 0 \leq t \leq T, \quad \text{q.s.}; \end{aligned} \quad (0.17)$$

sous les conditions

$$\begin{aligned} X_t &\in \bar{D}, \quad 0 \leq t \leq T, \quad \text{q.s.}; \\ |K|_t &= \int_0^t \mathbf{1}_{\{X_s \in \partial \bar{D}\}} d|K|_s, \quad \text{q.s.}; \\ K_t &= \int_0^t \xi_s d|K|_s, \quad \text{q.s. avec } \xi_s \in \mathbf{n}(X_s), \end{aligned}$$

où $\langle B, B \rangle$ est le processus de matrice de covariation du G -mouvement brownien d -dimensionnel B , D est un domaine de \mathbb{R}^d et le processus croissant K est minimal dans le sens où il croît seulement lorsque X est à la frontière du domaine D , et quand il croît, c'est suivant la normale à D en X .

Nous dirons qu'un couple de processus (X, K) est solution de la G -EDS réfléchie (0.17) s'il existe un ensemble polaire A , tel que :

- (i) $(X, K) \in M_G^2([0, T]; \mathbb{R}^d) \times M_G^2([0, T]; \mathbb{R}^d)$ vérifie (0.17);
- (ii) $\forall \omega \in A^c$, $X(\omega)$ est continue à valeurs dans \bar{D} ;

(iii) $\forall \omega \in A^c$, $K(\omega)$ est continue à variation bornée sur $[0, T]$ avec $K_0(\omega) = 0$ et pour tout $t \in [0, T]$, on a

$$\begin{aligned} |K|_t(\omega) &= \int_0^t \mathbf{1}_{\{X_s(\omega) \in \partial D\}} d|K|_s(\omega); \\ K_t(\omega) &= \int_0^t \mathbf{n}_s(\omega) d|K|_s(\omega) \end{aligned}$$

où $\mathbf{n}_s(\omega) \in \mathcal{N}_{X_s(\omega)}$, si $X_s(\omega) \in \partial D$.

Nous considérons tout d'abord le G -mouvement brownien réfléchi, c'est-à-dire lorsque $f = h^{ij} = 0$ et $g^j = 1$, dans ce cas nous suivons la méthode de Saisho et Tanaka dans [101], pour établir le résultat suivant :

Théorème 0.19 *Supposons que le domaine $D \subset \mathbb{R}^d$ satisfait les conditions (A) et (B) et $x_0 \in \bar{D}$. Alors, il existe une unique paire de processus $(X, K) \in M_G^2([0, T], \mathbb{R}^d) \times M_G^2([0, T], \mathbb{R}^d)$, vérifiant le problème de Skorohod*

$$X_t = x_0 + B_t + K_t, \quad 0 \leq t \leq T, \quad q.s.$$

avec quasi sûrement

(i) X à valeurs dans \bar{D} ,

(ii) $K_0 = 0$, K est à variation bornée et

$$|K|_t = \int_0^t \mathbf{1}_{\partial D}(X_s) d|K|_s, \quad K_t = \int_0^t \mathbf{n}(X_s) d|K|_s.$$

La preuve consiste à utiliser le problème de Skorohod déterministe pour montrer l'existence trajectorielle et une méthode de pénalisation pour montrer que cette solution trajectorielle est la limite d'une suite d'éléments de $M_G^2([0, T])$. De là, on déduit le corollaire :

Corollaire 0.20 *Supposons que le domaine $D \subset \mathbb{R}^d$ satisfait les conditions (A) et (B) et $x_0 \in \bar{D}$. Alors il existe une unique paire $(X, K) \in M_G^2([0, T]; \mathbb{R}^d) \times M_G^2([0, T]; \mathbb{R}^d)$ satisfaisant le problème de Skorohod :*

$$X_t = x_0 + \int_0^t \alpha_s ds + \int_0^t \eta_s^{ij} d\langle B^i, B^j \rangle_s + \int_0^t \beta_s^j dB_s^j + K_t, \quad 0 \leq t \leq T, \quad q.s.,$$

sous les conditions

$$\begin{aligned} X_t &\in \bar{D}, \quad 0 \leq t \leq T, \quad q.s.; \\ |K|_t &= \int_0^t \mathbf{1}_{\{X_s \in \partial \bar{D}\}} d|K|_s, \quad q.s.; \\ K_t &= \int_0^t \xi_s d|K|_s, \quad q.s. \text{ avec } \xi_s \in \mathbf{n}(X_s), \end{aligned}$$

où α , η^{ij} , β^j sont des éléments bornés de $M_G^2([0, T], \mathbb{R}^d)$.

Dans un deuxième temps, nous supposons que les fonctions $f, h^{ij}, g^j : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ vérifient les hypothèses :

(H1) Pour tout $x \in \mathbb{R}^n$, les processus $f(\cdot, x), h^{ij}(\cdot, x), g^j(\cdot, x) \in M_G^2([0, T], \mathbb{R}^d)$;

(H2) Les fonctions f, h^{ij}, g^j sont uniformément bornées sur $[0, T] \times \mathbb{R}^d$ et satisfont la condition de Lipschitz uniforme en $x : \forall (\omega, t) \in \Omega \times [0, T], \forall x, y \in \mathbb{R}^d$

$$\|f(t, x) - f(t, y)\| + \|h^{ij}(t, x) - h^{ij}(t, y)\| + \|g^j(t, x) - g^j(t, y)\| \leq L_0 \|x - y\|.$$

Nous supposons aussi que le domaine D satisfait la condition supplémentaire suivante :

Condition (C) : il existe une fonction $\Psi \in C_b^2(\mathbb{R}^d)$ telle que ses dérivées partielles première et seconde soient bornées et une constante $\gamma > 0$ telle que :

$$\forall x \in \partial D, \forall y \in \bar{D}, \forall \mathbf{n} \in \mathcal{N}_x \langle y - x, \mathbf{n} \rangle + \frac{1}{\gamma} \langle \nabla \Psi(x), \mathbf{n} \rangle |y - x|^2 \geq 0.$$

Nous établissons alors l'estimation a priori :

Proposition 0.21 *Supposons que le domaine $D \subset \mathbb{R}^d$ satisfait aux conditions (A), (B) et (C) et pour $i = 1, 2$ les fonctions g^i, f^i vérifient les hypothèses (H1) et (H2). Pour $x_0 \in \bar{D}$, posons*

$$\begin{aligned} \tilde{X}_t^i &= x_0 + \int_0^t f^i(s, X_s^i) ds + \int_0^t g^i(s, X_s^i) dB_s + K_t^i, \quad i = 1, 2, \\ |K^i|_t &= \int_0^t \mathbf{1}_{\{\tilde{X}_s^i \in \partial \bar{D}\}} d|k^i|_s \text{ et } K_t^i = \int_0^t \mathbf{n}_s^i d|K^i|_s \text{ avec } \mathbf{n}_s^i \in \mathcal{N}_{\tilde{X}_s^i}. \end{aligned}$$

Alors il existe une constante positive C ne dépendant que des paramètres $\Gamma, d, L_\Psi, \delta$ et L_0 telle que :

$$\begin{aligned} &\mathbb{E} \left[\sup_{0 \leq s \leq t} |\tilde{X}_s^1 - \tilde{X}_s^2|^4 \right] + \mathbb{E} \left[\sup_{0 \leq s \leq t} |K_s^1 - K_s^2|^4 \right] \\ &\leq C \int_0^t \left(\mathbb{E} \left[\sup_{0 \leq u \leq s} |X_u^1 - X_u^2|^4 \right] + \mathbb{E} \left[\sup_{0 \leq u \leq s} |\hat{f}_u|^4 \right] + \mathbb{E} \left[\sup_{0 \leq u \leq s} |\hat{g}_u|^4 \right] \right) ds, \end{aligned}$$

où $\hat{g}_s = g^1(s, X_s^2) - g^2(s, X_s^2)$ et $\hat{f}_s = f^1(s, X_s^2) - f^2(s, X_s^2)$.

On déduit le resultat d'existence et d'unicité de la solution de (0.17) :

Théorème 0.22 *Supposons que le domaine $D \subset \mathbb{R}^d$ satisfait aux conditions (A), (B) et (C) et que les fonctions f, h^{ij}, g^j vérifient les hypothèses (H1) et (H2). Alors pour tout $x_0 \in \bar{D}$, l'équation (0.17) admet une unique paire de solution $(X, K) \in M_G^2([0, T], \mathbb{R}^d) \times M_G^2([0, T], \mathbb{R}^d)$.*

L'unicité de la solution de (0.17) se déduit directement de la Proposition 0.21. Pour prouver l'existence, par le Corollaire 0.20 on peut construire une suite $\{(X^m, K^m)\}_{m \in \mathbb{N}^*}$ par une itération de type Picard commençant par $X^0 \equiv K^0 = 0$,

$$\begin{aligned} Y_t^{m+1} &= \int_0^t f(s, X_s^m) ds + \int_0^t g(s, X_s^m) dB_s, \quad X_t^{m+1} = x_0 + Y_t^{m+1} + K_t^{m+1}, \\ |K^{m+1}|_t &= \int_0^t \mathbf{1}_{\{X_s^{m+1} \in \partial D\}} d|K^{m+1}|_s \text{ et } K_t^{m+1} = \int_0^t \mathbf{n}_s^{m+1} d|K^i|_s \text{ avec } \mathbf{n}_s^{m+1} \in \mathcal{N}_{X_s^{m+1}}, \end{aligned}$$

nous établissons alors, les convergences suivantes quand $m \rightarrow +\infty$:

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t^m - X_t|^2 \right] + \mathbb{E} \left[\sup_{0 \leq t \leq T} |K_t^m - K_t|^2 \right] \longrightarrow 0, \\ & \mathbb{E} \left[\sup_{0 \leq t \leq T} \left| Y_t^m - \int_0^t f(s, X_s) ds - \int_0^t g(s, X_s) dB_s \right|^2 \right] \longrightarrow 0. \end{aligned}$$

Le Chapitre 4 est organisé de la manière suivante. Dans la Section 2, nous introduisons le problème de Skorohod déterministe. Dans la Section 3, nous présentons nos principaux résultats et la Section 4 est consacrée aux preuves des résultats.

0.3.4 G -EDSRs réfléchies

Dans le Chapitre 5, nous étudions les équations différentielles stochastiques rétrogrades réfléchies dirigées par un G -mouvement brownien du type

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s + N_T - N_t, & 0 \leq t \leq T, \\ Y_t \geq S_t, & 0 \leq t \leq T. \end{cases} \quad (0.18)$$

lorsque la barrière S est un processus de G -Itô de la forme

$$S_t = S_0 + \int_0^t a_s ds + \int_0^t b_s d\langle B \rangle_s + \int_0^t c_s dB_s,$$

où les processus a et b sont des éléments de $M_G^\beta([0, T])$ et c est un élément de $H_G^\beta([0, T])$, pour $\beta > 2$. La valeur terminale $\xi \in L_G^\beta(\Omega_T)$ et le générateur $f(t, \omega, y, z) : [0, T] \times \Omega_T \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfait les hypothèses suivantes :

(H1) pour tout $(y, z) \in \mathbb{R}^2$, $f(\cdot, \cdot, y, z) \in M_G^\beta(0, T)$;

(H2) f est uniformément lipschitzienne en y et z , i.e., $\exists L > 0$ tel que $\forall (t, \omega) \in [0, T] \times \Omega$, $\forall (y, z), (y', z') \in \mathbb{R}^2$,

$$|f(t, \omega, y, z) - f(t, \omega, y', z')| \leq L (|y - y'| + |z - z'|).$$

Définition 0.23 Un triplet de processus (Y, Z, N) est appelé solution de l'équation (0.18) si pour $1 < \alpha \leq \beta$ les propriétés suivantes sont vraies :

(a) $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T)$ et $N \in M_G^\alpha(0, T)$ tel que

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s + N_T - N_t;$$

(b) Le processus $N \in M_G^\alpha(0, T)$ est quasi-sûrement continu et croissant, avec

$$N_0 = 0 \text{ et } N_T \in L_G^\alpha(\Omega_T), \text{ q.s.};$$

- (c) Le processus K définie par $K_t = \int_0^t (Y_s - S_s) dN_s$ est une G -martingale décroissante ;
(d)

$$Y_t \geq S_t, \quad 0 \leq t \leq T, \quad q.s. .$$

Notons que la décomposition du processus $N = R - K$ n'est pas unique, donc on peut avoir plusieurs quadruplets (Y, Z, K, R) vérifiant l'équation (0.18) au sens de la Définition 0.23. Par exemple, considérons la G -EDSR réfléchie

$$Y_t = \int_t^T (\underline{\sigma}^2 - \bar{\sigma}^2) ds - (K_T - K_t) + R_T - R_t, \quad (0.19)$$

avec pour obstacle $S = 0$. Alors

$$(0, 0, 0, (\bar{\sigma}^2 - \underline{\sigma}^2) t)$$

et

$$(0, 0, \underline{\sigma}^2 t - \langle B \rangle_t, \bar{\sigma}^2 t - \langle B \rangle_t)$$

vérifient l'équation (0.19).

La condition $\left(-\int_0^t (Y_s - S_s) dN_s\right)_{0 \leq t \leq T}$ est une G -martingale décroissante est l'équivalent de la condition minimale de Skorokhod $\int_0^T (Y_t - S_t) dR_t = 0$ qui signifie que le processus R agit seulement lorsque la contrainte est saturée, c'est-à-dire quand $Y_t = S_t$. Notons que dans un cadre déterministe, cela correspond au problème de Skorohod. Par conséquent, comme l'intégrale $\int_0^T (Y_t - S_t) dN_t$ est une intégrale de Stieltjes, on pourrait raisonner ω par ω et appliquer la propriété du problème de Skorohod. Le processus réflexion R peut être écrit comme un supremum. Plus précisément, soit $(Y, Z, N = R - K)$ une solution de la G -EDSR réfléchie (0.18), en appliquant le lemme de Skorohod avec

$$x_t = \left(\xi + \int_{T-t}^T f(s, Y_s, Z_s) ds - \int_{T-t}^T (Z_s, dB_s) - (K_T - K_{T-t}) - S_{T-t} \right) (\omega),$$

$$r_t = (R_T - R_{T-t}) (\omega),$$

et

$$y_t = (Y_{T-t} - S_{T-t}) (\omega),$$

nous avons que $(Y_{T-t}(\omega) - S_{T-t}(\omega), R_{T-t}(\omega) - R_T(\omega), 0 \leq t \leq T)$ est la solution du problème de Skorohod :

$$y_t = \xi(\omega) + x_t + r_t, \quad 0 \leq t \leq T.$$

Nous avons donc, quasi-sûrement, pour $t \in [0, T]$,

$$R_T - R_t = \sup_{t \leq u \leq T} \left(\xi + \int_u^T f(s, Y_s, Z_s) ds - \int_u^T (Z_s, dB_s) - (K_T - K_u) - S_u \right)^-.$$

Nous établissons le résultat d'existence et d'unicité suivant :

Théorème 0.24 *Supposons que la fonction f satisfait les hypothèses (H1) et (H2) pour $\beta > 2$ et $\xi \in L_G^\beta(\Omega_T)$. Alors, il existe un unique triplet (Y, Z, N) , solution de la G -EDSR réfléchie (0.18) au sens de la Définition 0.23.*

L'unicité de la solution est déduite des estimations suivantes :

Proposition 0.25 Soient (ξ^i, f^i, S^i) pour $i = 1, 2$, deux triplets satisfaisant les hypothèses ci-dessus pour $\beta > 1$. Supposons que (Y^i, Z^i, N^i) est une solution de la G -EDSR réfléchie associée à (ξ^i, f^i, S^i) pour $\beta \geq \alpha > 1$. Posons

$$\hat{\xi} = \xi^1 - \xi^2; \quad \hat{S} = S^1 - S^2; \quad \hat{Y} = Y^1 - Y^2; \quad \hat{Z} = Z^1 - Z^2; \quad \hat{N} = N^1 - N^2.$$

Alors il existe une constante positive $C_\alpha = C(\alpha, T, \underline{\sigma}, \bar{\sigma}, L)$ telle que :

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} |\hat{Y}_t|^\alpha \right] \\ & \leq C_\alpha \mathbb{E} \left[\sup_{0 \leq t \leq T} \mathbb{E}_t \left[|\hat{\xi}|^\alpha + \sup_{0 \leq s \leq T} |\hat{S}_s|^\alpha + \int_0^T (|\hat{a}_s|^\alpha + |\hat{b}_s|^\alpha) ds \right. \right. \\ & \quad \left. \left. + \left(\int_0^T |\hat{c}_s|^2 ds \right)^{\frac{\alpha}{2}} + \int_0^T |f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)|^\alpha ds \right] \right], \end{aligned}$$

et

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T |\hat{Z}_s|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{2}} \right] \\ & \leq C_\alpha \left\{ \mathbb{E} \left[\sup_{0 \leq t \leq T} |\hat{Y}_t|^\alpha \right] + \mathbb{E} \left[\left(\int_0^T |f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)| ds \right)^\alpha \right] \right. \\ & \quad + \left(\mathbb{E} \left[\sup_{0 \leq t \leq T} |\hat{Y}_t|^\alpha \right] \right)^{1/2} \sum_{i=1}^2 \left(\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^i|^\alpha \right] \right)^{1/2} \\ & \quad \left. + \left(\mathbb{E} \left[\int_0^T |f^i(s, 0, 0)|^\alpha ds \right] \right)^{1/2} \right\}. \end{aligned}$$

Nous montrons l'existence de la solution réfléchie en suivant une méthode de pénalisation introduite par El Karoui et al dans [31]. Pour tout $n \in \mathbb{N}$, $t \in [0, T]$ et $(y, z) \in \mathbb{R}^2$, posons $f_n(t, y, z) = f(t, y, z) + n(y - S_t)^-$. Alors pour toute fonction f_n satisfaisant les hypothèses **(H1)** et **(H2)**, la G -EDSR lipschitzienne non réfléchie

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n) ds - \int_t^T Z_s^n dB_s - (K_T^n - K_t^n) \quad (0.20)$$

admet une unique solution (Y^n, Z^n, K^n) telle que, pour $1 < \alpha < \beta$, nous avons :

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n|^\alpha \right] < +\infty.$$

Posons

$$R_0^n = 0 \text{ et } R_t^n - R_t^n = n \int_t^T (Y_s^n - S_s)^- ds, \quad 0 \leq t \leq T \quad (0.21)$$

et

$$N_T^n = R_T^n - K_T^n = n \int_0^T (Y_s^n - S_s)^- ds - K_t^n, \quad 0 \leq t \leq T. \quad (0.22)$$

Le processus R^n pénalisera la solution de (0.20) quand elle est inférieure à la barrière S et forcera Y^n de plus en plus fortement à rester au-dessus de la barrière S . Nous avons alors l'estimation uniforme en n , pour les quadriplets (Y^n, Z^n, N^n) :

Lemme 0.26 Pour $n \in \mathbb{N}$, soit (Y^n, Z^n, K^n) l'unique solution de la G -EDSR non réfléchie (0.20) et N^n le processus défini par (0.22). Alors pour tout $1 < \alpha < \beta$, il existe une constante C_α , qui ne dépend pas de $n \in \mathbb{N}$, telle que

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n|^\alpha + \left(\int_0^T |Z_s^n|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{2}} + |N_T^n|^\alpha \right] \leq C_\alpha.$$

Comme pour tout $n \in \mathbb{N}$, $f_n \leq f_{n+1}$, il résulte du théorème de comparaison (théorème 3.6 dans [40]) que $Y_t^n \leq Y_t^{n+1}$, $0 \leq t \leq T$, q.s., alors

$$Y_t^n \uparrow Y_t, \quad 0 \leq t \leq T \quad \text{q.s.},$$

et par le lemme de Fatou, nous avons

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^\alpha \right] \leq C_\alpha.$$

Nous montrons alors que $\{Y^n\}_{n \in \mathbb{N}^*}$ est une suite de Cauchy dans $S_G^\alpha(0, T)$, $\{Z^n\}_{n \in \mathbb{N}^*}$ est une suite de Cauchy dans $H_G^\alpha(0, T)$ et $\{N^n\}_{n \in \mathbb{N}^*}$ est une suite de Cauchy dans $S_G^\alpha(0, T)$.

Le Chapitre 5 est organisé ainsi : Dans la Section 2, nous introduisons quelques préliminaires dans le cadre de la G -espérance qui sont nécessaires pour la suite. Dans la Section 3, nous établissons les estimations a priori et l'unicité et la Section 4 est consacrée à l'existence de la solution.

Chapter 1

Preliminaries in G -framework

In this introductory chapter, we review notations and basic results in the framework of G -expectation, which concern the formulation of G -Brownian motion and related G -stochastic integral. The readers interested in more details on this topic are referred to [83, 35, 60, 62].

1.1 G -Brownian motion and G -expectation

To introduce G -expectation, we need to give a description of the sublinear expectation. Let Ω be a complete separable metric space, and let \mathcal{H} be a linear space of real-valued functions defined on Ω , satisfying: if $X_i \in \mathcal{H}$, $i = 1, \dots, n$, then

$$\varphi(X_1, X_2, \dots, X_n) \in \mathcal{H}, \quad \forall \varphi \in \mathcal{C}_{l,Lip}(\mathbb{R}^n),$$

where $\mathcal{C}_{l,Lip}(\mathbb{R}^n)$ is the space of all continuous real-valued functions defined on \mathbb{R}^n such that for some $C > 0$ and $k \in \mathbb{N}$ depending on φ ,

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \quad \forall x, y \in \mathbb{R}^n.$$

Definition 1.1 (Sublinear expectation space) A sublinear expectation \mathbb{E} on \mathcal{H} is a functional $\mathbb{E} : \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

- (a) *Monotonicity*: if $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;
- (b) *Constant preservation*: $\mathbb{E}[c] = c$, $c \in \mathbb{R}$;
- (c) *Sub-additivity*: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$;
- (d) *Positive homogeneity*: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, for all $\lambda \geq 0$.

We call the triple $(\Omega, \mathcal{H}, \mathbb{E})$ sublinear expectation space.

Remark 1.2 If the inequality in (c) becomes equality, then \mathbb{E} is a linear expectation.

Definition 1.3 (Independence) Fix the sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. A random variable $Y \in \mathcal{H}$ is said to be independent of $X_1, X_2, \dots, X_n \in \mathcal{H}$ under \mathbb{E} , if for all $\varphi \in \mathcal{C}_{l,Lip}(\mathbb{R}^{n+1})$,

$$\mathbb{E}[\varphi(X_1, X_2, \dots, X_n, Y)] = \mathbb{E}\left[\mathbb{E}[\varphi(x_1, x_2, \dots, x_n, Y)] \Big|_{(x_1, x_2, \dots, x_n) = (X_1, X_2, \dots, X_n)}\right].$$

Definition 1.4 Let X_1 and X_2 be two n -dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$. They are called identically distributed, denoted by $X_1 \stackrel{d}{=} X_2$, if $\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)]$, for all $\varphi \in \mathcal{C}_{l,Lip}(\mathbb{R}^n)$.

Now we introduce the definition of G -normal distribution.

Definition 1.5 (G -normal distribution) A n -dimensional random vector $X = (X_1, \dots, X_n)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called G -normally distributed if $\mathbb{E}[|X|^3] < \infty$ and for all $a, b \geq 0$,

$$aX + b\bar{X} \stackrel{d}{=} \sqrt{a^2 + b^2}X,$$

where \bar{X} is an independent copy of X , i.e., $\bar{X} \stackrel{d}{=} X$ and \bar{X} independent of X . Here the letter G denotes the function

$$G(A) := \frac{1}{2}\mathbb{E}[\langle AX, X \rangle] : \mathbb{S}_n \rightarrow \mathbb{R},$$

where \mathbb{S}_n denotes the collection of $n \times n$ symmetric matrices.

The G -normal distribution is characterized by a nonlinear heat equation as follows [83]:

Proposition 1.6 *A n -dimensional random vector $X = (X_1, \dots, X_n)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is G -normally distributed if and only if for each $\varphi \in \mathcal{C}_{l,Lip}(\mathbb{R}^n)$, the function u defined by*

$$u(t, x) := \mathbb{E}[\varphi(x + \sqrt{t}X)], (t, x) \in [0, \infty) \times \mathbb{R}^n,$$

is the unique viscosity solution of the following G -heat equation:

$$\begin{cases} \partial_t u - G(D_x^2 u) = 0; \\ u(0, x) = \varphi(x), \end{cases}$$

where $D_x^2 u = \left(\frac{\partial^2}{\partial x^i \partial x^j} \right)_{i,j=1}^n$ is the Hessian matrix of u and the function $G(\cdot) : \mathbb{S}_n \rightarrow \mathbb{R}$ is a monotonic, sublinear mapping on \mathbb{S}_n , which implies that there exists a bounded, convex and closed subset $\Sigma \subset \mathbb{S}_n^+$ such that

$$G(A) = \frac{1}{2} \sup_{B \in \Sigma} \text{tr}[AB],$$

\mathbb{S}_n^+ denotes the collection of nonnegative elements in \mathbb{S}_n .

We now fix $\Omega := \mathcal{C}_0([0, \infty); \mathbb{R}^n)$, the space of \mathbb{R}^n valued continuous paths $(\omega_t)_{t \geq 0}$ vanishing at the origin, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{N=1}^{\infty} 2^{-N} \left(\max_{t \in [0, N]} |\omega_t^1 - \omega_t^2| \wedge 1 \right).$$

The canonical process $(B_t)_{t \geq 0}$ is defined by $B_t(\omega) = \omega_t$, for $(t, \omega) \in [0, \infty) \times \Omega$. and $(B_t)_{t \geq 0}$ be the canonical process, i.e., $B_t(\omega) := \omega_t$. For each $t \in [0, \infty)$, we list the following notations:

- $\Omega_t := \{\omega_{\cdot \wedge t} : \omega \in \Omega\}$; $\mathcal{F}_t := \mathcal{B}(\Omega_t)$;
- $L^0(\Omega)$: the space of all $\mathcal{B}(\Omega)$ -measurable real functions;
-

$$\begin{aligned} Lip(\Omega_T) := \{ & \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}) : \\ & 0 \leq t_1 \leq t_2 \leq \dots, t_n \leq T, \varphi \in \mathcal{C}_{l,Lip}(\mathbb{R}^{d \times n}) \}, \text{ for } T > 0; \end{aligned}$$

-

$$Lip(\Omega) := \bigcup_{n=1}^{\infty} Lip(\Omega_n).$$

Let $G(\cdot) : \mathbb{S}_d \rightarrow \mathbb{R}$ be a given monotonic and sublinear function. G -expectation is a sublinear expectation on $Lip(\Omega)$ defined by

$$\mathbb{E}[X] = \tilde{\mathbb{E}} \left[\varphi \left(\sqrt{t_1} \xi_1, \sqrt{t_2 - t_1} \xi_2, \dots, \sqrt{t_n - t_{n-1}} \xi_n \right) \right],$$

for all $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})$ with $0 = t_0 < t_1 < \dots < t_n < \infty$, where $\xi_1, \xi_2, \dots, \xi_n$ are identically distributed d -dimensional G -normally distributed random vectors in a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$ such that ξ_{i+1} is independent of (ξ_1, \dots, ξ_i) for every $i = 1, \dots, n-1$.

Definition 1.7 (G -Brownian motion and G -expectation) On the sublinear expectation space $(\Omega, Lip(\Omega), \mathbb{E})$, the canonical process $(B_t)_{t \geq 0}$ is called G -Brownian motion if the following properties are verified:

(a) $B_0 = 0$;

(b) for each $t, s \geq 0$, the increment $B_{t+s} - B_t$ in G -normally distributed random vector and is independent from $(B_{t_1}, B_{t_2}, \dots, B_{t_n})$, for $0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq t$.

Moreover, the sublinear expectation $\mathbb{E}[\cdot]$ is called G -expectation.

Definition 1.8 (Conditional G -expectation) For each random variable $\xi \in Lip(\Omega_T)$ of the following form:

$$\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}), \quad \varphi \in \mathcal{C}_l, Lip(\mathbb{R}^n),$$

the conditional G -expectation $\mathbb{E}_{t_i}[\cdot]$, $i = 1, \dots, n$, is defined as follows

$$\mathbb{E}_{t_i}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})] = \tilde{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_i} - B_{t_{i-1}}),$$

where

$$\tilde{\varphi}(x_1, \dots, x_i) = \mathbb{E}[\varphi(x_1, \dots, x_i, B_{t_{i+1}} - B_{t_i}, \dots, B_{t_n} - B_{t_{n-1}})].$$

If $t \in (t_i, t_{i+1})$, then the conditional G -expectation $\mathbb{E}_t[\xi]$ could be defined by reformulating ξ as

$$\xi = \hat{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_t - B_{t_i}, B_{i+1} - B_t, \dots, B_{t_n} - B_{t_{n-1}}), \quad \hat{\varphi} \in \mathcal{C}_l, Lip(\mathbb{R}^{n+1}).$$

For $\xi \in Lip(\Omega_T)$ and $p \geq 1$, we consider the norm $\|\xi\|_{L_G^p} = (\mathbb{E}[|\xi|^p])^{1/p}$. We denote by $L_G^p(\Omega_T)$ the Banach completion of $Lip(\Omega_T)$ under $\|\cdot\|_{L_G^p}$. The G -expectation $\mathbb{E}[\cdot]$ can be continuously extended to a sublinear expectation on $(\Omega, L_G^1(\Omega_T))$ still denoted by $\mathbb{E}[\cdot]$. For each given $t \in [0, T]$, the conditional G -expectation $\mathbb{E}_t[\cdot] : Lip(\Omega_T) \rightarrow Lip(\Omega_t)$ can be also extended as a mapping $\mathbb{E}_t[\cdot] : L_G^1(\Omega_T) \rightarrow L_G^1(\Omega_t)$ and satisfies the following properties:

- (i) If $\xi, \eta \in L_G^1(\Omega_t)$, $\xi \leq \eta$, then $\mathbb{E}_s[\xi] \leq \mathbb{E}_s[\eta]$ for all $s \leq t$;
- (ii) If $\xi \in L_G^1(\Omega_t)$ and $\eta \in L_G^1(\Omega)$ then $\mathbb{E}_t[\xi + \eta] = \xi + \mathbb{E}_t[\eta]$;
- (iii) $\mathbb{E}_t[\xi + \eta] \leq \mathbb{E}_t[\xi] + \mathbb{E}_t[\eta]$;
- (iv) If $\xi \in L_G^0(\Omega_t)$ is bounded, $\eta \in L_G^1(\Omega)$, then $\mathbb{E}_t[\xi\eta] = \xi^+ \mathbb{E}_t[\eta] + \xi^- \mathbb{E}_t[-\eta]$;
- (v) If $\xi \in L_G^1(\Omega)$ then $\mathbb{E}_s[\mathbb{E}_t[\xi]] = \mathbb{E}_{s \wedge t}[\xi]$.

Definition 1.9 (G -martingale) A process $(M_t)_{t \in [0, T]}$ with $M_t \in L_G^1(\Omega_t)$, $0 \leq t \leq T$, is called a G -martingale (respectively, G -supermartingale, G -submartingale) if for all $0 \leq s \leq t \leq T$, we have

$$\mathbb{E}_s[M_t] = M_s \quad (\text{respectively, } \leq M_s, \geq M_s).$$

The process $(M_t)_{t \in [0, T]}$ is called symmetric G -martingale if $-M$ is also a G -martingale.

From now on, we suppose the function G non-degenerate, i.e., there exist two constants $0 < \underline{\sigma}^2 \leq \bar{\sigma}^2 < +\infty$, such that for all $A, B \in \mathbb{S}_d$:

$$\frac{1}{2}\underline{\sigma}^2 \text{tr}(A - B) \leq G(A) - G(B) \leq \frac{1}{2}\bar{\sigma}^2 \text{tr}(A - B).$$

In the case that $\underline{\sigma} = \bar{\sigma}$, the function G is linear, so G -framework is the classical Wiener case.

To give a description of elements in $L_G^p(\Omega)$, Denis et al. [20] gave the following representation theorem of the G -expectation $\mathbb{E}[\cdot]$ on $L_G^1(\Omega_T)$. In the sequel, we denote by \mathbb{P}_0 the Wiener measure, under which the canonical process $(B_t)_{t \geq 0}$ is a \mathbb{P}_0 -Brownian motion.

Theorem 1.10 (Representation of G -expectation) Assume Γ_G is a bounded, convex and closed subset of $\mathbb{R}^{d \times d}$, which represents function G , i.e.,

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma_G} \text{tr} [A\gamma\gamma'], \quad \text{for } A \in \mathbb{S}_d.$$

Denote the Wiener measure by \mathbb{P}_0 . Then, for any time sequence $0 = t_0 < t_1 < \dots < t_n$, the G -expectation has the following representation:

$$\begin{aligned} \mathbb{E} [\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})] &= \sup_{a \in \mathcal{A}^{\Gamma_G}} E^{\mathbb{P}_0} \left[\varphi \left(\int_0^{t_1} a_s dB_s, \dots, \int_{t_{n-1}}^{t_n} a_s dB_s \right) \right] \\ &= \sup_{\mathbb{P} \in \mathcal{P}_G} E^{\mathbb{P}} [\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})], \end{aligned}$$

where \mathcal{A}^{Γ} is the set of progressively measurable processes with values in Γ_G and \mathcal{P}_G is the set of laws of $\int_0^\cdot a_s dB_s$ with $a \in \mathcal{A}^{\Gamma_G}$ under Wiener measure. Furthermore, \mathcal{P}_G is tight.

By the theorem above, the G -expectation can be extended to a larger domain, i.e., for all \mathcal{F}_T measurable function X , $\mathbb{E}[X] := \sup_{\mathbb{P} \in \mathcal{P}_G} E^{\mathbb{P}}[X]$. It is also proved in [20, 49] that \mathcal{P}_G is relatively weakly compact and thus its completion $\overline{\mathcal{P}}_G$ is weakly compact. Therefore, we can naturally define the Choquet capacity $\overline{C}(\cdot)$ by $\overline{C}(A) := \sup_{P \in \overline{\mathcal{P}}_G} P(A)$, $A \in \mathcal{B}(\Omega_T)$ and introduce the notion of quasi-sure.

Definition 1.11 (Quasi-sure) A set $A \in \mathcal{B}(\Omega_T)$ is a \overline{C} -polar if $\overline{C}(A) = 0$. A property is said to hold “quasi-surely” (q.s.) with respect to \overline{C} , if it holds true outside a \overline{C} -polar set.

The following proposition helps to understand the correspondence between GBSDEs and 2BS-DEs.

Proposition 1.12 Suppose X and $Y \in L_G^1$, then the following statements are equivalent:

- (a) for each $\mathbb{P} \in \mathcal{P}_G$, $X = Y$, \mathbb{P} -a.s.;
- (b) $\mathbb{E}[|X - Y|] = 0$;
- (c) $\overline{C}(\{X \neq Y\}) = 0$.

By the definition of L_G^1 , we do not distinguish two random variables if they are equal outside a polar set.

Definition 1.13 1. A mapping X on Ω_T with values in \mathbb{R} is said to be quasi-continuous if for any $\varepsilon > 0$, there exists an open set O , with $\overline{C}(O) < \varepsilon$ such that X is continuous in O^c .

- 2. One says that $Y : \Omega_T \rightarrow \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function X , such that $X = Y$, q.s..

Theorem 1.14 One has the following representation for L_G^1 :

$$L_G^1(\Omega_T) = \left\{ X \in \mathcal{B}(\Omega_T) : X \text{ has a quasi-continuous version, } \lim_{n \rightarrow \infty} \mathbb{E}[|X| \mathbf{1}_{\{|X| > n\}}] = 0 \right\}.$$

Proposition 1.15 Assume that $(X_n)_{n \geq 1}$ is a sequence of random variables, and converges to X in the sense of $\|\cdot\|_{L_G^p}$. Then the convergence holds in the sense of capacity, i.e., for any $\varepsilon > 0$,

$$\overline{C}(|X_n - X| > \varepsilon) \xrightarrow{n} 0.$$

Furthermore, there exists a subsequence $(X_{n_k})_{k \geq 1}$ converging to X quasi-surely.

Remark 1.16 *It is vital to point out that though the above proposition holds true in the G -framework, even sup linear expectation framework, the dominated convergence theorem (the quasi-surely version), and the claim that quasi-surely convergence implies convergence in capacity, all fail in G -framework.*

For the terminal value of quadratic GBSDE, we define the space $L_G^\infty(\Omega_T)$ as the completion of $Lip(\Omega_T)$ under the norm

$$\|\xi\|_{L_G^\infty} := \inf \{M \geq 0 : |\xi| \leq M \text{ q.s.}\}.$$

Notice that the following Markov's inequality still holds in the context of the upper expectation and related Choquet capacity (Lemma 13 in [20]).

Lemma 1.17 (Markov's inequality) *Let $X \in L^0(\Omega)$ satisfying $\mathbb{E}[|X|^p] < \infty$, for $p > 0$. Then, for each $a > 0$,*

$$\bar{C}(\{|X| > a\}) \leq \frac{\mathbb{E}[|X|^p]}{a^p}.$$

We also have a generalized Fatou's lemma (Lemma 2.11 in Bai and Lin [4]) in the G -framework.

Lemma 1.18 (Fatou's lemma) *Assume that $\{X^n\}_{n \in \mathbb{N}}$ is a sequence in $L^0(\Omega)$ and for a $Y \in L^0(\Omega)$ satisfying $\mathbb{E}[|Y|] < \infty$ and all $n \in \mathbb{N}$, $X^n \geq Y$, q.s., then*

$$\mathbb{E}[\liminf_{n \rightarrow \infty} X^n] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[X^n].$$

By Kolmogorov criterion, Theorem. 36 in [20], we have.

Lemma 1.19 *Let $p > 0$ and a process $(X_t)_{t \in [0, T]}$ with $X_t \in L_G^1(\Omega_t)$, $0 \leq t \leq T$. Assume that there exist positive constants c and ε such that for $s, t \in [0, T]$*

$$\mathbb{E}[|X_t - X_s|^p] \leq c|t - s|^{1+\varepsilon}.$$

Then X admits a modification \tilde{X} such that

$$\mathbb{E} \left[\sup_{s \neq t} \left(\frac{|\tilde{X}_t - \tilde{X}_s|}{|t - s|^\alpha} \right)^p \right] < \infty,$$

for every $\alpha \in [0, \varepsilon/p)$. As a consequence, paths of \tilde{X} are quasi-surely α -Hölder continuous for every $\alpha < \varepsilon/p$ in the sense that there exists a polar set N such that for all $\omega \in N^c$, the map $t \mapsto \tilde{X}_t(\omega)$ is α Hölder continuous for every $\alpha < \varepsilon/p$.

1.2 G -stochastic integral

In what follows, we discuss the stochastic integrals with respect to the G -Brownian motion and its quadratic variation. In [83], Peng introduce the Itô type stochastic integral with respect to the G -Brownian motion by first considering the simple process space:

$$\mathcal{H}_G^0(0, T) = \left\{ \eta : \eta_t(\omega) = \sum_{i=0}^{N-1} \xi_i(\omega) \mathbf{1}_{[t_i, t_{i+1})}(t), \right.$$

$$\left. \forall N \in \mathbb{N}^*, 0 = t_0 < \dots < t_N = T, \xi_i \in Lip(\Omega_{t_i}), i = 0, \dots, N-1 \right\}.$$

Definition 1.20 For $p \geq 1$ and $\eta \in \mathcal{H}_G^0(0, T)$, define $\|\eta\|_{\mathcal{H}_G^p} := \left\{ \mathbb{E} \left[\left(\int_0^T |\eta_s|^2 ds \right)^{p/2} \right] \right\}^{1/p}$, $\|\eta\|_{M_G^p} := \left\{ \mathbb{E} \left[\frac{1}{T} \int_0^T |\eta_s|^p ds \right] \right\}^{1/p}$ and denote by $\mathcal{H}_G^p(0, T)$, $M_G^p(0, T)$ the completion of $\mathcal{H}_G^0(0, T)$ under the norms $\|\cdot\|_{\mathcal{H}_G^p}$, $\|\cdot\|_{M_G^p}$ respectively.

Here below is the definition of the G -Itô integral. In the sequel, $B^{\mathbf{a}}$ denotes the inner product of $\mathbf{a} \in \mathbb{R}^d$ and B , which is a 1-dimensional $G_{\mathbf{a}}$ -Brownian motion, where

$$G_{\mathbf{a}}(\alpha) = \frac{1}{2} (\sigma_{\mathbf{a}^t \mathbf{a}}^2 \alpha^+ - \sigma_{-\mathbf{a}^t \mathbf{a}}^2 \alpha^-), \quad \sigma_{\mathbf{a}^t \mathbf{a}}^2 = 2G(\mathbf{a}^t \mathbf{a}), \quad \sigma_{-\mathbf{a}^t \mathbf{a}}^2 = -2G(-\mathbf{a}^t \mathbf{a}).$$

Definition 1.21 (G -stochastic integrals) For $\eta \in \mathcal{H}_G^0(0, T)$ of the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{1}_{[t_j, t_{j+1})}(t), \quad (1.1)$$

the Itô integral with respect to G -Brownian motion is defined by the linear mapping $\mathcal{I} : \mathcal{H}_G^0(0, T) \rightarrow L_G^2(\Omega_T)$

$$\mathcal{I}_{[0, T]}(\eta) := \int_0^T \eta_t dB_t^{\mathbf{a}} := \sum_{k=0}^{N-1} \xi_k (B_{t_{k+1}}^{\mathbf{a}} - B_{t_k}^{\mathbf{a}}).$$

Then, the linear mapping $\mathcal{I}_{[0, T]}$ on $\mathcal{H}_G^0(0, T)$ can be continuously extended to $\mathcal{I}_{[0, T]} : M_G^2(0, T) \rightarrow L_G^2(\Omega_T)$ and for each $\eta \in M_G^2(0, T)$, we define $\int_0^T \eta_t dB_t^{\mathbf{a}} := \mathcal{I}_{[0, T]}(\eta)$.

We have moreover some properties of the G -Itô type integrals.

Proposition 1.22 For all $\eta, \theta \in M_G^p(0, T)$, $p \geq 2$, with a bounded random variable $\xi \in L_G^1(\Omega_t)$, we have

$$\mathbb{E}_t \left[\int_t^T \eta_s dB_s^{\mathbf{a}} \right] = 0;$$

$$\int_t^T (\xi \eta_s + \theta_s) dB_s^{\mathbf{a}} = \xi \int_t^T \eta_s dB_s^{\mathbf{a}} + \int_0^T \theta_s dB_s^{\mathbf{a}}.$$

We emphasize that Gao [35] proves the G -Itô integral $X_t = \int_0^t \eta_s dB_s^{\mathbf{a}}$ has a continuous \bar{C} -modification, for any $\eta \in M_G^2(0, T)$.

A important feature of the G -expectation framework is that the quadratic variation $\langle B^{\mathbf{a}} \rangle$ of the G -Brownian motion is no longer a deterministic process, which is given by

$$\langle B^{\mathbf{a}} \rangle_t = \lim_{\mu(\pi_t^N) \rightarrow 0} \sum_{j=0}^{N-1} (B_{t_{j+1}}^{\mathbf{a}} - B_{t_j}^{\mathbf{a}})^2 = (B_t^{\mathbf{a}})^2 - 2 \int_0^t B_s^{\mathbf{a}} dB_s^{\mathbf{a}},$$

where π_t^N is a partition of $[0, t]$, i.e., $\pi_t^N = \{t_0, t_1, \dots, t_N\}$ such that $0 = t_0 < t_1 < \dots < t_N = t$, and $\mu(\pi_t^N) := \max_{1 \leq i \leq N} |t_i^N - t_{i-1}^N|$. For two given vectors $\mathbf{a}, \bar{\mathbf{a}} \in \mathbb{R}^d$, the mutual variation process of $B^{\mathbf{a}}$ and $B^{\bar{\mathbf{a}}}$ is defined by

$$\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}} \rangle_t := \frac{1}{4} (\langle B^{\mathbf{a}+\bar{\mathbf{a}}} \rangle_t - \langle B^{\mathbf{a}-\bar{\mathbf{a}}} \rangle_t).$$

By Corollary 5.7 in Chapter III of Peng [83], for each $0 \leq s \leq t \leq T$,

$$\langle B^{\mathbf{a}} \rangle_t - \langle B^{\mathbf{a}} \rangle_s \leq \sigma_{\mathbf{a}^t \mathbf{a}}^2 (t - s). \quad (1.2)$$

Let B^i denote the i th coordinate of the G -Brownian motion B and set $\langle B, B \rangle_t = (\langle B^i, B^j \rangle_t)_{i,j=1,\dots,d}$. $\langle B, B \rangle$ shares properties of independent stationary increment just as G -Brownian motion. Moreover, the path of $\langle B, B \rangle$ quasi-surely has a bounded density and indeed the stochastic integral on $M_G^1([0, T]; \mathbb{R})$ with respect to $\langle B^{\mathbf{a}} \rangle$ could be defined pathwisely.

Definition 1.23 For $\eta \in \mathcal{H}_G^0(0, T)$ of the form (1.1), the stochastic integral with respect to $(\langle B^{\mathbf{a}} \rangle_t)_{t \geq 0}$ is defined by the linear mapping $\mathcal{Q} : \mathcal{H}_G^0(0, T) \rightarrow L_G^1(\Omega_T)$

$$\mathcal{Q}(\eta) := \int_0^T \eta_t d\langle B^{\mathbf{a}} \rangle_t = \sum_{j=0}^{N-1} \xi_j (\langle B^{\mathbf{a}} \rangle_{t_{j+1}} - \langle B^{\mathbf{a}} \rangle_{t_j}),$$

which can also be continuously extended to $\mathcal{Q} : M_G^1(0, T) \rightarrow L_G^1(\Omega_T)$.

Moreover, we have the following BDG type inequality (Theorem 2.1 in [35]). Define

Lemma 1.24 (BDG type inequality) Let $p \geq 2$, $\mathbf{a} \in \mathbb{R}^d$, $\eta \in M_G^p([0, T])$ and $0 \leq s \leq t \leq T$. Then,

$$c_p \sigma_{-\mathbf{a}^t \mathbf{a}}^p \mathbb{E} \left[\left(\int_s^t |\eta_u|^2 du \right)^{p/2} \right] \leq \mathbb{E} \left[\sup_{s \leq u \leq t} \left| \int_s^u \eta_r dB_r^{\mathbf{a}} \right|^p \right] \leq C_p \sigma_{\mathbf{a}^t \mathbf{a}}^p \mathbb{E} \left[\left(\int_s^t |\eta_u|^2 du \right)^{p/2} \right],$$

where $0 < c_p < C_p < \infty$ are constants independent of \mathbf{a} , η and Γ_G .

Finally, we define the space $\mathcal{S}_G^p(0, T)$ for solutions of quadratic GBSDEs. Let

$$\mathcal{S}_G^0(0, T) := \left\{ h(t, B_{t_1 \wedge t}, B_{t_2 \wedge t} - B_{t_1 \wedge t}, \dots, B_{t_n \wedge t} - B_{t_{n-1} \wedge t}) : \right. \\ \left. 0 \leq t_1 \leq t_2 \leq \dots, t_n \leq T, h \in \mathcal{C}_{b, Lip}(\mathbb{R}^{n+1}) \right\},$$

where $\mathcal{C}_{b, Lip}(\mathbb{R}^{n+1})$ is the collection of all bounded and Lipschitz functions on \mathbb{R}^{n+1} . For $p \geq 1$ and $\eta \in \mathcal{S}_G^0(0, T)$, we set $\|\eta\|_{\mathcal{S}_G^p} = \{\mathbb{E}[\sup_{t \in [0, T]} |\eta_t|^p]\}^{1/p}$. We denote by $\mathcal{S}_G^p(0, T)$ the completion of $\mathcal{S}_G^0(0, T)$ under the norm $\|\cdot\|_{\mathcal{S}_G^p}$.

Chapter 2

Quadratic BSDEs driven by G -Brownian motion

Abstract: In this chapter, we consider backward stochastic differential equations driven by G -Brownian motion (GBSDEs) under quadratic assumptions on coefficients. We prove the existence and uniqueness of solution for such equations. On the one hand, a priori estimates are obtained by applying the Girsanov type theorem in the G -framework, from which we deduce the uniqueness. On the other hand, to prove the existence of solutions, we first construct solutions for discrete GB-SDEs by solving corresponding fully nonlinear PDEs, and then approximate solutions for general quadratic GBSDEs in Banach spaces.

Key words. Backward stochastic differential equations, quadratic growth, G -Brownian motion, discretization, fully nonlinear PDEs.

AMS subject classifications. 60H10; 60H30

2.1 Introduction

The first existence and uniqueness result for nonlinear backward stochastic differential equations (BSDEs for short) of the following form is provided by Pardoux and Peng in [75]:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

where the generator f is uniformly Lipschitz and the terminal value ξ is square integrable. Since then, BSDEs have been studied with great interest and moreover, these equations are found to have strong connections with different mathematical fields, such as mathematical finance, stochastic control and partial differential equations. In particular, many efforts have been made to relax the assumption on the generator. For instance, Kobylanski [53] was the first to investigate the BSDE with a generator having quadratic growth in Z and a bounded terminal value. She used an exponential transformation in order to come back to the framework of linear growth generator. This seminal work of quadratic BSDE has been extended by many authors. Since a complete review of these literatures is too extensive, we only concentrate on those of immediate interest. For the existence, Briand and Hu [13] observed that the existence of exponential moments of the terminal condition is sufficient to construct a solution of quadratic BSDEs; uniqueness is proved in [14] under the additional assumption that the generator is convex (or concave). On the other hand, Hu et al. discovered in [44] that for certain type of f locally Lipschitz in Z , if the solution Y is bounded, the solution Z is bounded in BMO norm, and thus the uniqueness could be proved by applying linearization of the generator. Afterwards, Tevzadze improved the methodology of [44] and he gave a direct proof in [113] for the solvability of quadratic BSDEs by standard fixed point arguments, whereas the terminal value was assumed to be sufficient small. To get rid of this technical assumption on ξ , Briand and Elie exhibited a priori estimates in the light of [3, 11] and approximated bounded terminal values with Malliavin differentiable ones in [12]. In more general situations, Morlais reconsidered the problem of [53] with continuous martingale driver in [70], while recently Barrieu and El Karoui obtained similar results in [5] but by a completely different forward method.

Motivated by mathematical finance problems with Knightian uncertainty, Peng established systematically in [81, 82, 83, 84] a framework of time-consistent sublinear expectation, called G -expectation. In particular, this sublinear expectation is associated with a new type of Brownian motion $(B_t)_{t \geq 0}$, i.e., G -Brownian motion, which has independent, stationary and G -normally distributed increments. This process and its quadratic variation $\langle B \rangle$ play center roles in the related nonlinear stochastic analysis. Indeed, the stochastic integrals with respect to G -Brownian motion and its quadratic variation have been first introduced by Peng in his pioneer work [81], which are initially defined on the simple process space and later extended as linear operator on Banach completions. Thereafter, the G -stochastic calculus is further developed, for example, in [83, 35, 60, 62]. Another important feature of the G -expectation is found by Denis et al. in [20], namely, the G -expectation can be represented by the upper expectation over a collection of mutually singular martingale measures \mathcal{P}_G . Moreover, the notion of quasi-sure with respect to the associated Choquet capacity is introduced by Denis et al. to the G -framework.

As their classical counterparts, stochastic differential equations driven by G -Brownian motion (GSDEs) are well defined in the quasi-sure sense and their solvability can be established by the contracting mapping theorem under Lipschitz assumptions (cf. [83] and [35]). However, the

challenging problem of wellposedness for backward GSDEs (GBSDEs) remained open until a complete theorem has been proved by Hu et al. [39].

Similarly to the classical case, the G -martingale representation theorem is heuristic to the formulation of GBSDEs, which reads as follows

$$M_t = M_0 + \overline{M}_t + K_t, \quad (2.1)$$

where

$$\overline{M}_t = \int_0^t Z_s dB_s \quad \text{and} \quad K_t = \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds. \quad (2.2)$$

In contrast to the classical martingale representation, the G -martingale M is decomposed into two parts: the G -Itô type integral part $\overline{M} = \int Z dB$, which is called symmetric G -martingale, in the sense that $-\overline{M}$ is still a G -martingale; the decreasing G -martingale part K , which vanishes in the classical theory, however, plays a significant role in this new context. Whether the process K admits a unique representation in the form (2.2) is a sophisticated question. The first positive answer is given by Peng in [80] for the G -martingale associated with a terminal value $M_T \in Lip(\Omega_T)$, which reads as smooth and finitely dimensional path function. It is also worth mentioning that a series of successive works by Soner et al. [106] and Song [109] affirm the existence and uniqueness of the first level decomposition (2.1) for $M_T \in L_G^p(\Omega)$, $p > 1$, which is the Banach completion of $Lip(\Omega_T)$. Finally, with the help of the norm creatively introduced in Song [110], a complete theorem for G -martingale representation has been obtained by Peng et al. [86] on a complete metric subspace of $L_G^p(\Omega)$, $p > 1$.

We take into consideration of the G -martingale representation theorem and naturally, we can formulate GBSDE as follows, where the decreasing G -martingale K appears in the dynamics:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds + \int_t^T h(s, Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t). \quad (2.3)$$

Under Lipschitz assumptions on the generators, Hu et al. investigated in [39] the existence and uniqueness of the triple (Y, Z, K) in proper Banach spaces satisfying the above equation. They started with BSDEs with bounded and smooth generators and Markovian terminal values and constructed solutions by classical solutions of fully nonlinear PDEs (cf. Krylov's results in [55]). Then, the partition of unity theorem was employed in [39] to proceed a type of Galerkin approximation to solutions of GBSDEs with general parameters. Besides, the uniqueness was deduced in [39] based on a priori estimates. In particular, the uniqueness of K is impressive in the light of G -martingale estimates found in [109]. The results in [39] breaks new ground in the G -expectation theory. In the accompanying paper [40], Hu et al. discussed fundamental properties of the above GBSDE: the comparison theorem, the fully nonlinear Feynman-Kac formula and the related Girsanov transformation. Moreover, the correspondence between GBSDEs and Sobolev type solutions of nonlinear path-dependent PDEs is examined in [85].

We now compare the result of [39] with the profound works [107, 108] by Soner et al., in which the so-called second order backward stochastic differential equations (2BSDEs) are deeply studied. This type of equation is highly related to the GBSDE and it is defined on the Wiener space as follows:

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s, \widehat{a}_s) ds - \int_t^T Z_s dB_s + (K_T - K_t), \quad \mathbb{P} - \text{a.s.}, \text{ for all } \mathbb{P} \in \mathcal{P}_H,$$

where B is the canonical process, the process \hat{a} is the density of $\langle B \rangle$ and \mathcal{P}_H is a collection of martingale measures similar to \mathcal{P}_G (could be even larger). This equation is a reinforced BSDE in the sense that it holds true \mathbb{P} -a.s. for all $\mathbb{P} \in \mathcal{P}_H$ and moreover, the family of $K := \{K^{\mathbb{P}}\}_{\mathbb{P} \in \mathcal{P}_H}$ should satisfy a minimum condition (then $-K$ verifies the G -martingale constraint in the GBSDE context, see [106]):

$$K_t^{\mathbb{P}} = \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H(t, \mathbb{P})}^{\mathbb{P}} E_t^{\mathbb{P}'} [K_T^{\mathbb{P}'}], \quad \mathbb{P} - \text{a.s.}, \text{ for all } \mathbb{P} \in \mathcal{P}_H, \quad 0 \leq t \leq T.$$

Under Lipschitz assumptions, the uniqueness of the 2BSDE is proved in [107] by observing that the solution to the 2BSDE can be represented as the (essential) supremum of a class of martingale-driven BSDEs solutions. For the existence, the proof involves a delicate pathwise construction: the process Y is defined pathwisely by solutions of BSDEs on shift spaces. This process verifies a critical principle of optimality and thus, the structure of 2BSDE could be derived from the g -supermartingale decomposition (cf. [77]), where the family of processes K can be a posteriori aggregated once the stochastic integral part is aggregated by Nutz [71]. To get rid of the measurability problem during the construction of solutions, Soner et al. assume the technical condition that both ξ and F is uniformly continuous in ω , whereas this assumption is removed in the recent work of Possamaï et al. [87]. In the framework of 2BSDEs, the results in [107, 108] are generalized by Possamaï and Zhou [88] and by Lin [63] to the quadratic case and furthermore, Matoussi et al. [69] applied quadratic 2BSDEs to solve the utility maximization problems from [44] in the context with non-dominated models. One could see that the GBSDE (2.3) actually corresponds to the 2BSDE defined with

$$F(t, y, z, a) = g(t, y, z) + h(t, y, z)a,$$

however, the GBSDE requires more structure conditions on the coefficient and the terminal value so that the solution can be found with more regularity adapted to the requirement of process space in the G -framework.

The main objective of this chapter is to provide the existence and uniqueness result for scalar-valued quadratic GBSDEs adapted to the setting of [39]. Without loss of generality, we consider only the following type of GBSDE:

$$Y_t = \xi + \int_t^T h(s, B_{\cdot \wedge s}(\cdot), Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t),$$

where ξ is an element in the L_G^∞ completion of $Lip(\Omega_T)$ and h is Lipschitz in y and locally Lipschitz in z , similarly to [44] and [70] in the classical framework. Moreover, we require that h is uniformly continuous with respect to ω , which is a technical condition for the successive approximation. This assumption is stronger than the corresponding structure condition in [39] and how to weaken such technical assumption is postponed to future research.

This chapter is organized as follows. Section 1.2 is dedicated to formulation of quadratic GBSDEs. In Section 1.3, we introduce the space of G -BMO martingale generators and deduce a priori estimates for quadratic GBSDEs through the G -Girsanov transformation. Meanwhile, we obtain the uniqueness straightforwardly by the a priori estimates. In Section 1.4, we consider GBSDEs with discrete generators and terminal values of the following functional type:

$$\begin{aligned} \xi &= \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_N} - B_{t_{N-1}}); \\ h(t, B_{\cdot \wedge t}(\cdot), Y_t, Z_t) &= f(t, B_{t_1 \wedge t}, B_{t_2 \wedge t} - B_{t_1 \wedge t}, \dots, B_{t_N \wedge t} - B_{t_{N-1} \wedge t}, Y_t, Z_t), \end{aligned}$$

where $0 = t_0 \leq t_1 \leq \dots \leq t_{N-1} \leq t_N = T$ is a given partition of $[0, T]$. We proceed with the argument of Hu and Ma in [43] to construct solutions of such GBSDEs, where Krylov's estimates for fully nonlinear PDEs are applied as in [39]. The last section present the technics of discretization and regularization, moreover, the existence result for general quadratic GBSDEs shall then be proved by successive approximation.

2.2 The formulation of G-BSDEs

In this chapter, we shall consider the following type of equation:

$$Y_t = \xi + \int_t^T h(s, \omega_{\cdot \wedge s}, Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - K_T + K_t, \quad (2.4)$$

where the terminal value ξ and the generator h satisfies the following conditions.

Assumption 2.1 *Assume that the generator $h : [0, T] \times \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies the following conditions:*

(H0) *For each $(t, \omega) \in [0, T] \times \Omega$, $|h(t, \omega, 0, 0)| + |\xi(\omega)| \leq M_0$;*

(Hc) *Moreover, $h(\cdot, \cdot, y, z)$ is uniformly continuous in (t, ω) and the modulus of continuity is independent of (y, z) , i.e., for each $(y, z) \times \mathbb{R}^2$,*

$$|h(t_1, \omega^1, y, z) - h(t_2, \omega^2, y, z)| \leq w^h (|t_1 - t_2| + \|\omega_1 - \omega_2\|_\infty);$$

(Hq) *The function h is uniformly Lipschitz in y and uniformly locally Lipschitz in z , i.e., for each $(t, \omega) \in [0, T] \times \Omega$,*

$$|h(t, \omega, y^1, z^1) - h(t, \omega, y^2, z^2)| \leq L_y |y^1 - y^2| + L_z (1 + |z^1| + |z^2|) |z^1 - z^2|.$$

Remark 2.2 *From Theorem 4.7 in [48], we have for each $(y, z) \times \mathbb{R}^2$, $h(\cdot, \cdot, y, z) \in \mathcal{H}_G^2(0, T)$. Due to the boundedness, in fact, for any $p \geq 2$, $h(\cdot, \cdot, y, z) \in \mathcal{H}_G^p(0, T)$. Therefore, if $Y \in \mathcal{S}_G^p(0, T)$, $Z \in \mathcal{H}_G^{2p}(0, T)$, $h(\cdot, \cdot, Y, Z) \in \mathcal{H}_G^p(0, T)$.*

Remark 2.3 *One can always find a concave and sub-additive modulus w in (Hc).*

Definition 2.4 *For $p \geq 2$, a triple of processes (Y, Z, K) belongs to $\mathfrak{G}_G^p(0, T)$, if $Y \in \mathcal{S}_G^p(0, T)$, $Z \in \mathcal{H}_G^p(0, T)$ and K is a decreasing G-martingale with $K_0 = 0$ and $K_T \in L_G^p(\Omega_T)$. The triple (Y, Z, K) is said solution of GBSDE (2.4), if $(Y, Z, K) \in \mathfrak{G}_G^p(0, T)$, and for $0 \leq t \leq T$, it satisfies (2.4).*

Remark 2.5 *All results in this paper hold for the GBSDE in a more general form, by trivially generalizing the argument:*

$$Y_t = \xi + \int_t^T g(s, \omega_{\cdot \wedge s}, Y_s, Z_s) ds + \int_t^T h(s, \omega_{\cdot \wedge s}, Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - K_T + K_t,$$

where the generator g also satisfies Assumption 2.1.

To prove the existence of solutions to (2.4), we shall study the following auxiliary GBSDEs, in which the generator h is replaced by a discrete function f , and correspondingly, a discrete terminal

value is introduced here. Fixing $N \in \mathbb{N}$ and a partition $\pi^N := \{0 = t_0, t_1, \dots, t_N = T\}$ on $[0, T]$, we consider

$$\begin{aligned} Y_t &= \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_N} - B_{t_{N-1}}) \\ &\quad + \int_t^T f(s, B_{t_1 \wedge s}, B_{t_2 \wedge s} - B_{t_1 \wedge s}, \dots, B_{t_N \wedge s} - B_{t_{N-1} \wedge s}, Y_s, Z_s) d\langle B \rangle_s \\ &\quad - \int_t^T Z_s dB_s - K_T + K_t. \end{aligned} \quad (2.5)$$

Similarly to Assumption 2.1, we introduce the following conditions on the terminal value $\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_N} - B_{t_{N-1}})$ and the generator f .

Assumption 2.6 We assume that the generator $f : [0, T] \times \mathbb{R}^N \times \mathbb{R}^2$ satisfies the following conditions:

(H0') For each $(t, x_1, x_2, \dots, x_N) \in [0, T] \times \mathbb{R}^N$, $|f(t, x_1, x_2, \dots, x_N, 0, 0)| + |\varphi(x_1, x_2, \dots, x_N)| \leq M_0$;

(Hc') Moreover, $f(\cdot, \cdot, \dots, \cdot, y, z)$ is uniformly continuous in $(t, x_1, x_2, \dots, x_N)$ and the modulus of continuity is independent of (y, z) , i.e., for each $(y, z) \in \mathbb{R}^2$,

$$|f(t_1, x_1^1, x_2^1, \dots, x_N^1, y, z) - f(t_2, x_1^2, x_2^2, \dots, x_N^2, y, z)| \leq w^f \left(|t_1 - t_2| + \sum_{i=1}^N |x_i^1 - x_i^2| \right);$$

(Hq') The function f is uniformly Lipschitz in y and uniformly locally Lipschitz in z , i.e., for each $(t, x_1, x_2, \dots, x_N) \in [0, T] \times \mathbb{R}^N$,

$$|f(t, x_1, x_2, \dots, x_N, y^1, z^1) - f(t, x_1, x_2, \dots, x_N, y^2, z^2)| \leq L_y |y^1 - y^2| + L_z (1 + |z^1| + |z^2|) |z^1 - z^2|.$$

2.3 G -Girsanov theorem and estimates for GBSDEs

In this section, we first exhibit the Girsanov type theorem in the G -framework by introducing the notion of G -BMO martingale generators. And then, we deduce a priori estimates for solutions of GBSDEs under quadratic assumptions.

2.3.1 The Girsanov type theorem

Similarly to Possamaï and Zhou [88], we can generalize the definition of BMO-martingale generators to the G -framework:

Definition 2.7 Assuming $Z \in \mathcal{H}_G^2(0, T)$, we say that Z is a G -BMO martingale generator if

$$\|Z\|_{BMO_G}^2 := \sup_{\mathbb{P} \in \mathcal{P}_G} \left[\sup_{\tau \in \mathcal{T}_0^T} \left| E_\tau^{\mathbb{P}} \left[\int_\tau^T |Z_t|^2 d\langle B \rangle_t \right] \right| \right]_{\infty} < +\infty,$$

where \mathcal{T}_0^T denotes the collection of all \mathcal{F} -stopping times taking values in $[0, T]$.

Lemma 2.8 Suppose that $Z \in \mathcal{H}_G^2(0, T)$ is a G -BMO martingale generator, then

$$\mathcal{E} \left(\int Z dB \right) := \exp \left(\int Z dB - \frac{1}{2} \int |Z|^2 d\langle B \rangle \right)$$

is a symmetric G -martingale.

Proof: First, we verify that $\mathcal{E}_t(\int Z dB) \in L_G^1(\Omega_t)$ by Theorem 54 in Denis et al. [20]. As mentioned in Lemma 2.1 in [88], if for some $q > 1$ such that $\|Z\|_{BMO_G} \leq \Phi(q)$ (see Theorem 3.1 in Kazamaki [52]), then

$$\mathbb{E} \left[\left(\mathcal{E} \left(\int_0^\cdot Z_s dB_s \right) \right)_t \right]^q < +\infty.$$

Fixing $t \in [0, T]$, we have for all $N > 0$, $\mathbb{E} [\mathcal{E}(Z)_t \mathbf{1}_{\{\mathcal{E}(Z)_t \geq N\}}] \leq \frac{1}{N^{q-1}} \mathbb{E} [(\mathcal{E}(Z)_t)^q]$, which implies

$$\lim_{N \rightarrow \infty} \mathbb{E} [\mathcal{E}(Z)_t \mathbf{1}_{\{\mathcal{E}(Z)_t \geq N\}}] = 0.$$

Moreover, the quasi-continuity of $\mathcal{E}(\int Z dB)_t$ is inherited from $\int_0^t Z_s dB_s$ and $\int_0^t |Z|^2 d\langle B \rangle_s$. Thus, $\mathcal{E}(\int Z dB)_t \in L_G^1(\Omega_t)$.

From the well known results in [52], for G -BMO martingale generator Z , the process $\mathcal{E}(\int Z dB)$ is a martingale under each $\mathbb{P} \in \mathcal{P}_1$. By the representation of G -conditional expectation (Proposition 3.4 in Soner et al. [106]), we can deduce the desired result. \square

Following the procedure introduced in [114], we can define a new G -expectation on the space $Lip(\Omega_T)$ with $\mathcal{E}(Z)$ by

$$\tilde{\mathbb{E}}[X] := \sup_{\mathbb{P} \in \mathcal{P}_G} E^{\mathbb{P}}[\mathcal{E}(Z)_T X]. \quad (2.6)$$

Then, complete $Lip(\Omega_T)$ under $\tilde{\mathbb{E}}[\cdot]$ and obtain $\tilde{L}_G^1(\Omega_T)$. If $X \in L_G^p(\Omega_T)$ with $p > \frac{q}{q-1}$, $X \in \tilde{L}_G^1(\Omega_T)$.

The conditional expectation $\tilde{\mathbb{E}}_t[\cdot]$ thus can be first defined on $Lip(\Omega_T)$ then on $\tilde{L}_G^1(\Omega_T)$. Obviously, for G -BMO martingale generator Z and any $p \geq 1$, $\|Z\|_{\mathcal{H}_G^p(0,T)} < \infty$, and thus

$$\left(\int_0^t Z_s d\langle B \rangle_s \right)^p \in L^1(\Omega_t) \quad \text{and} \quad \left(\int_0^t Z_s dB_s \right)^p \in \tilde{L}^1(\Omega_t).$$

Notice that Xu et al. [114] assumed the reinforced Novikov condition on Z (Assumption 2.1 in [114]) and develop the G -Girsanov type theory. This condition is mostly used for the proof of Lemma 2.2 in [114] (corresponding to Lemma 2.8 in the present paper) and thus, substituting this condition by a BMO one will not alter the theory in [114]. In particular, we have

Lemma 2.9 *Suppose that Z is a G -BMO martingale generator. We define a new G -expectation $\tilde{\mathbb{E}}[\cdot]$ by $\mathcal{E}(Z)$. Then, the process $B - \int Z d\langle B \rangle$ is a G -Brownian motion under $\tilde{\mathbb{E}}[\cdot]$.*

The following result shows that a decreasing G -martingale under $\mathbb{E}[\cdot]$ is still a decreasing G -martingale under $\tilde{\mathbb{E}}[\cdot]$, if it satisfies a sufficient integrability condition.

Lemma 2.10 *Assume the same as in the above lemma. Suppose that K is a decreasing G -martingale such that $K_0 = 0$ and for some $p > \frac{q}{q-1}$, $K_t \in L_G^p(\Omega_t)$, $0 \leq t \leq T$, where q is the order in the reverse Hölder inequality for $\mathcal{E}(Z)$. Then K is a decreasing G -martingale under the new G -expectation defined by (2.6).*

Proof : The integrability condition on K ensures that $\tilde{\mathbb{E}}_s[K_t]$, $0 \leq s \leq t \leq T$, is well defined in the space $\tilde{L}_G^1(\Omega_s)$. To prove this lemma, it suffices to verify the martingale property. Indeed, we

recall Proposition 3.4 in Soner et al. [106] and deduce for some $0 < \alpha < 1$ such that $p > \frac{\alpha q}{\alpha q - 1}$, and for $\mathbb{P} \in \mathcal{P}_G$, \mathbb{P} -a.s.,

$$\begin{aligned}
0 &\geq \tilde{\mathbb{E}}_t [K_T - K_t] \\
&= \mathbb{E}_t \left[\frac{\mathcal{E}(Z)_T}{\mathcal{E}(Z)_t} (K_T - K_t) \right] \\
&= \operatorname{esssup}_{\mathbb{P}' \in \mathcal{P}_G(t, \mathbb{P})} E_t^{\mathbb{P}'} \left[\frac{\mathcal{E}(Z)_T}{\mathcal{E}(Z)_t} (K_T - K_t) \right] \\
&= - \operatorname{essinf}_{\mathbb{P}' \in \mathcal{P}_G(t, \mathbb{P})} E_t^{\mathbb{P}'} \left[\frac{\mathcal{E}(Z)_T}{\mathcal{E}(Z)_t} (K_t - K_T) \right] \\
&\geq - \operatorname{essinf}_{\mathbb{P}' \in \mathcal{P}_G(t, \mathbb{P})} \left(E_t^{\mathbb{P}'} \left[\left(\frac{\mathcal{E}(Z)_T}{\mathcal{E}(Z)_t} \right)^q \right]^{\frac{1}{q}} E_t^{\mathbb{P}'} \left[(K_t - K_T)^{\frac{\alpha q}{\alpha q - 1}} \right]^{\frac{\alpha q - 1}{q}} E_t^{\mathbb{P}'} [K_t - K_T]^{1 - \alpha} \right) \\
&\geq \operatorname{esssup}_{\mathbb{P}' \in \mathcal{P}_G(t, \mathbb{P})} E_t^{\mathbb{P}'} \left[\left(\frac{\mathcal{E}(Z)_T}{\mathcal{E}(Z)_t} \right)^q \right]^{\frac{1}{q}} \operatorname{esssup}_{\mathbb{P}' \in \mathcal{P}_G(t, \mathbb{P})} E_t^{\mathbb{P}'} \left[(K_t - K_T)^{\frac{\alpha q}{\alpha q - 1}} \right]^{\frac{\alpha q - 1}{q}} \\
&\quad \times \left(- \operatorname{essinf}_{\mathbb{P}' \in \mathcal{P}_G(t, \mathbb{P})} E_t^{\mathbb{P}'} [K_t - K_T]^{1 - \alpha} \right) \\
&= \mathbb{E}_t \left[\left(\frac{\mathcal{E}(Z)_T}{\mathcal{E}(Z)_t} \right)^q \right]^{\frac{1}{q}} \mathbb{E}_t \left[(K_t - K_T)^{\frac{\alpha q}{\alpha q - 1}} \right]^{\frac{\alpha q - 1}{q}} \mathbb{E}_t [K_T - K_t]^{1 - \alpha} = 0,
\end{aligned}$$

where $\mathcal{P}_G(t, \mathbb{P}) := \{\mathbb{P}' : \mathbb{P}' \in \mathcal{P}_G, \mathbb{P}'|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}\}$. We apply the result of Proposition 1.12 to end the proof. \square

2.3.2 A priori estimates for GBSDEs

Consider the solution triple (Y, Z, K) for either (2.4) or (2.5). Applying Lemma 3.1 and Theorem 3.2 in [88], we could easily obtain an upper bound for Y and the G -BMO norm of Z , i.e.,

$$\left\| \sup_{0 \leq t \leq T} |Y_t| \right\|_{L_G^\infty} + \|Z\|_{BMO_G} \leq \hat{C} := C(M_0, L_y, L_z), \quad (2.7)$$

which implies that Z is a G -BMO martingale generator. Moreover, for $p \geq 1$,

$$\mathbb{E}[|K_t|^p] \leq \tilde{C}_p := C(p, M_0, L_y, L_z), \quad 0 \leq t \leq T. \quad (2.8)$$

Now we establish the following stability results for both (2.4) and (2.5) (compare with Theorem 3.2 in [88]).

Proposition 2.11 *Consider two quadratic GBSDEs (2.4) with parameter (ξ^1, h^1) and (ξ^2, h^2) , where (ξ^i, h^i) satisfies (H0) and (Hq) with the same constants M_0, L_y and L_z . Suppose that $(Y^i, Z^i, K^i) \in \mathfrak{G}_G^p(0, T)$, $p \geq 2$, are solutions corresponding to these parameters. Then, for all $0 \leq t \leq T$,*

$$|Y_t^1 - Y_t^2| \leq C \left(\|\xi^1 - \xi^2\|_{L_G^\infty} + \tilde{\mathbb{E}}_t \left[\int_t^T |h^1(s, Y_s^2, Z_s^2) - h^2(s, Y_s^2, Z_s^2)| d\langle B \rangle_s \right] \right),$$

where $\tilde{\mathbb{E}}[\cdot]$ is the new G -expectation under the Girsanov transform induced by $\mathcal{E}(-b^\varepsilon)$ and b^ε is defined in (2.9).

Proof: Let $\hat{Y} := Y^1 - Y^2$, $\hat{Z} := Z^1 - Z^2$, $\hat{K} := K^1 - K^2$ and $\hat{\xi} := \xi^1 - \xi^2$. For $0 \leq t \leq T$, we have

$$\begin{aligned} \hat{Y}_t &= \hat{\xi} + \int_t^T (h^1(s, Y_s^1, Z_s^1) - h^2(s, Y_s^2, Z_s^2)) d\langle B \rangle_s \\ &\quad - \int_t^T \hat{Z}_s dB_s - (K_T^1 - K_t^1) + (K_T^2 - K_t^2). \end{aligned}$$

We employ a linearization argument similar to the proof of Theorem 3.6 in [40] by setting for $0 \leq s \leq T$,

$$\begin{aligned} \hat{a}_s^\varepsilon &= (1 - l(\hat{Y}_s)) \frac{h^1(s, Y_s^1, Z_s^1) - h^1(s, Y_s^2, Z_s^1)}{\hat{Y}_s} \mathbf{1}_{\{|\hat{Y}_s| > 0\}}; \\ \hat{b}_s^\varepsilon &= (1 - l(\hat{Z}_s)) \frac{h^1(s, Y_s^2, Z_s^1) - h^1(s, Y_s^2, Z_s^2)}{|\hat{Z}_s|^2} \hat{Z}_s \mathbf{1}_{\{|\hat{Z}_s| > 0\}}; \\ \hat{m}_s^\varepsilon &= l(\hat{Y}_s) (h^1(s, Y_s^1, Z_s^1) - h^1(s, Y_s^2, Z_s^1)) + l(\hat{Z}_s) (h^1(s, Y_s^2, Z_s^1) - h^1(s, Y_s^2, Z_s^2)); \\ \hat{h}_s &= h^1(s, Y_s^2, Z_s^2) - h^2(s, Y_s^2, Z_s^2), \end{aligned} \tag{2.9}$$

where l is a Lipschitz function such that $\mathbf{1}_{[-\varepsilon, \varepsilon]}(x) \leq l(x) \leq \mathbf{1}_{[-2\varepsilon, 2\varepsilon]}(x)$. So we have

$$\hat{Y}_t = \hat{\xi} + \int_t^T (\hat{h}_s + \hat{m}_s^\varepsilon + \hat{a}_s^\varepsilon \hat{Y}_s + \hat{b}_s^\varepsilon \hat{Z}_s) d\langle B \rangle_s - \int_t^T \hat{Z}_s dB_s - \int_t^T dK_s^1 + \int_t^T dK_s^2,$$

and

$$\begin{aligned} |\hat{a}_s^\varepsilon| &\leq L_y, |\hat{b}_s^\varepsilon| \leq L_z (1 + |Z_s^1| + |Z_s^2|), \\ |\hat{m}_s^\varepsilon| &\leq 2\varepsilon (L_y + L_z (1 + 2\varepsilon + 2|Z_s^1|)). \end{aligned}$$

From the proof of Theorem 3.6 in Hu et al. [40], we know the process \hat{a}^ε belongs to $\mathcal{H}_G^2(0, T)$ and furthermore due to the boundedness, \hat{a}^ε and $e^{\int \hat{a}^\varepsilon d\langle B \rangle}$ belong to $\mathcal{H}_G^p(0, T)$, for any $p \geq 2$.

Applying Itô's formula to $e^{\int \hat{a}^\varepsilon d\langle B \rangle} \hat{Y}$, we have

$$\begin{aligned} e^{\int_0^t \hat{a}_s^\varepsilon d\langle B \rangle_s} \hat{Y}_t &= e^{\int_0^t \hat{a}_t^\varepsilon d\langle B \rangle_t} \hat{\xi} + \int_t^T e^{\int_0^s \hat{a}_u^\varepsilon d\langle B \rangle_u} (\hat{h}_s + \hat{m}_s^\varepsilon + \hat{b}_s^\varepsilon \hat{Z}_s) d\langle B \rangle_s \\ &\quad - \int_t^T e^{\int_0^s \hat{a}_u^\varepsilon d\langle B \rangle_u} \hat{Z}_s dB_s - \int_t^T e^{\int_0^s \hat{a}_u^\varepsilon d\langle B \rangle_u} dK_s^1 + \int_t^T e^{\int_0^s \hat{a}_u^\varepsilon d\langle B \rangle_u} dK_s^2. \end{aligned}$$

Furthermore,

$$\begin{aligned} e^{\int_0^t \hat{a}_s^\varepsilon d\langle B \rangle_s} \hat{Y}_t + \int_t^T e^{\int_0^s \hat{a}_u^\varepsilon d\langle B \rangle_u} dK_s^1 &\leq e^{\int_0^T \hat{a}_t^\varepsilon d\langle B \rangle_t} \hat{\xi} + \int_t^T e^{\int_0^s \hat{a}_u^\varepsilon d\langle B \rangle_u} (\hat{h}_s + \hat{m}_s^\varepsilon + \hat{b}_s^\varepsilon \hat{Z}_s) d\langle B \rangle_s \\ &\quad - \int_t^T e^{\int_0^s \hat{a}_u^\varepsilon d\langle B \rangle_u} \hat{Z}_s dB_s. \end{aligned}$$

Thanks to the lemma below, \hat{b}^ε belongs to $\mathcal{H}_G^2(0, T)$. Moreover, \hat{b}^ε is a *G*-BMO martingale generator. Thus, we could define a new *G*-expectation $\hat{\mathbb{E}}[\cdot]$ by $\mathcal{E}(\hat{b}^\varepsilon)$, such that $\hat{B} := B - \int \hat{b}^\varepsilon d\langle B \rangle$ is a *G*-Brownian motion under $\hat{\mathbb{E}}[\cdot]$. From (2.8), we know that K_t is *p*-integrable, for any $p \geq 1$, thus we could assume without loss of generality that $p > \frac{q}{q-1}$ and q satisfies $L_z(1 + 2\hat{C}) < \Phi(q)$. Since $e^{\int_0^t \hat{a}_t^\varepsilon d\langle B \rangle_t}$ is a positive process, from Lemma 3.4 in Hu et al. [39], we know

$\int_0^\cdot e^{\int_0^s \hat{a}_u^\varepsilon d\langle B \rangle_u} dK_s^1$ and $\int_0^\cdot e^{\int_0^s \hat{a}_u^\varepsilon d\langle B \rangle_u} dK_s^2$ is a decreasing G -martingale under both $\mathbb{E}[\cdot]$ and $\hat{\mathbb{E}}[\cdot]$.

Taking the conditional G -expectation $\hat{\mathbb{E}}_t[\cdot]$ on both sides, it is easy to see that

$$\begin{aligned} e^{\int_0^t \hat{a}_s^\varepsilon d\langle B \rangle_s} \hat{Y}_t &\leq \hat{\mathbb{E}}_t \left[e^{\int_0^T \hat{a}_t^\varepsilon d\langle B \rangle_t} \hat{\xi} + \int_t^T e^{\int_0^s \hat{a}_u^\varepsilon d\langle B \rangle_u} \left(\hat{h}_s + \hat{m}_s^\varepsilon \right) d\langle B \rangle_s \right] \\ &\leq e^{\bar{\sigma}^2 L_y T} \|\hat{\xi}\|_{L_G^\infty} + e^{\bar{\sigma}^2 L_y T} \left(\hat{\mathbb{E}}_t \left[\int_t^T |\hat{h}_s| d\langle B \rangle_s \right] \right) \\ &\quad + 2\varepsilon e^{\bar{\sigma}^2 L_y T} \left(\hat{\mathbb{E}}_t \left[\int_t^T (L_y + L_z (1 + 2\varepsilon + 2|Z_s^1|)) d\langle B \rangle_s \right] \right), \end{aligned}$$

which implies

$$\begin{aligned} \hat{Y}_t &\leq e^{2\bar{\sigma}^2 L_y T} \|\hat{\xi}\|_{L_G^\infty} + e^{2\bar{\sigma}^2 L_y T} \left(\hat{\mathbb{E}}_t \left[\int_t^T |\hat{h}_s| d\langle B \rangle_s \right] \right) \\ &\quad + 2\varepsilon e^{2\bar{\sigma}^2 L_y T} \left(\hat{\mathbb{E}}_t \left[\int_t^T (L_y + L_z (1 + 2\varepsilon + 2|Z_s^1|)) d\langle B \rangle_s \right] \right). \end{aligned}$$

Sending $\varepsilon \rightarrow 0$, we have

$$\hat{Y}_t \leq e^{2\bar{\sigma}^2 L_y T} \|\hat{\xi}\|_{L_G^\infty} + e^{2\bar{\sigma}^2 L_y T} \left(\hat{\mathbb{E}}_t \left[\int_t^T |\hat{h}_s| d\langle B \rangle_s \right] \right).$$

Moreover, we deduce in the same way that

$$-\hat{Y}_t \leq e^{2\bar{\sigma}^2 L_y T} \|\hat{\xi}\|_{L_G^\infty} + e^{2\bar{\sigma}^2 L_y T} \left(\hat{\mathbb{E}}_t \left[\int_t^T |\hat{h}_s| d\langle B \rangle_s \right] \right).$$

This completes the proof. \square

Lemma 2.12 *The process defined in (2.9) belongs to $\mathcal{H}_G^2(0, T)$ and is a G -BMO martingale generator.*

Proof: For each $n \in \mathbb{N}$, define h_n^i as follows

$$h_n^1(s, y, z) = h^1 \left(s, y, \frac{|z| \wedge n}{|z|} z \right),$$

which is Lipschitz in z with the Lipschitz constant $L_z(1 + 2n)$. For each $n \in \mathbb{N}$, the process $\hat{b}^{\varepsilon, n}$ is defined by

$$\hat{b}_s^{\varepsilon, n} = (1 - l(\hat{Z}_s)) \frac{h_n^1(s, Y_s^2, Z_s^1) - h_n^1(s, Y_s^2, Z_s^2)}{|\hat{Z}_s|^2} \hat{Z}_s \mathbf{1}_{\{|\hat{Z}_s| > 0\}}, \quad 0 \leq s \leq T,$$

which belongs to $\mathcal{H}_G^2(0, T)$ according to [40]. On the other hand,

$$\begin{aligned} |\hat{b}_s^{\varepsilon, n} - \hat{b}_s^\varepsilon| &\leq (1 - l(\hat{Z}_s)) \frac{|h^1(s, Y_s^2, Z_s^1) - h_n^1(s, Y_s^2, Z_s^1)| + |h^1(s, Y_s^2, Z_s^2) - h_n^1(s, Y_s^2, Z_s^2)|}{|\hat{Z}_s|} \mathbf{1}_{\{|\hat{Z}_s| > 0\}} \\ &\leq \frac{L_z}{\varepsilon} \left((|Z_s^1| - n)(1 + n + |Z_s^1|) \mathbf{1}_{\{|Z_s^1| > n\}} + (|Z_s^2| - n)(1 + n + |Z_s^2|) \mathbf{1}_{\{|Z_s^2| > n\}} \right) \\ &\leq \frac{CL_z}{\varepsilon} \left(|Z_s^1|^2 \mathbf{1}_{\{|Z_s^1| > n\}} + |Z_s^2|^2 \mathbf{1}_{\{|Z_s^2| > n\}} \right), \end{aligned}$$

where C is independent of n . By Proposition 2.9 in Li and Peng [60], we conclude that \hat{b}^ε belongs to $\mathcal{H}_G^2(0, T)$. From the estimate for \hat{b}^ε , we deduce that $\|\hat{b}^\varepsilon\|_{BMO_G} \leq L_z(1 + 2\hat{C})$. Thus, \hat{b}^ε is a G -BMO martingale generator. \square

Proposition 2.13 Consider two quadratic GBSDEs (2.4) with parameter (ξ^1, h^1) and (ξ^2, h^2) , where (ξ^i, h^i) satisfies (H0) and (Hq) with the same constants M_0, L_y and L_z . Suppose $(Y^i, Z^i, K^i) \in \mathfrak{G}_G^p(0, T)$, $p \geq 2$, are solutions corresponding to these parameters. Then, for $1 \leq p'/2 < p$,

$$\mathbb{E} \left[\left(\int_0^T |Z_t^1 - Z_t^2|^2 dt \right)^{p'/2} \right] \leq C(p, \bar{\sigma}, \underline{\sigma}, M_0, L_y, L_z) \left(\|\xi^1 - \xi^2\|_{L_G^\infty}^{p'} + \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^1 - Y_t^2|^p \right]^{p'/2p} \right).$$

Proof: We keep the notations in the proof of Proposition 2.11. Indeed, due to the boundedness of the G -BMO norm of Z^1 and Z^2 , $Z^1, Z^2 \in \mathcal{H}_G^p(0, T)$, for any $p \geq 2$. Applying Itô's formula to $|\hat{Y}|^2$, we have

$$\begin{aligned} 2|\hat{Y}_0|^2 + \int_0^T |\hat{Z}_t|^2 d\langle B \rangle_t &\leq 2|\hat{\xi}|^2 + 4 \int_0^T \hat{Y}_t (F^1(t, Y_t^1, Z_t^1) - F^2(t, Y_t^2, Z_t^2)) d\langle B \rangle_t \\ &\quad - 4 \int_0^T \hat{Y}_t \hat{Z}_t dB_t - 4 \int_0^T \hat{Y}_t d\hat{K}_t. \end{aligned}$$

Then, for $1 \leq p'/2 < p$,

$$\begin{aligned} &\mathbb{E} \left[\left(\int_0^T |\hat{Z}_t|^2 dt \right)^{p'/2} \right] \leq C(p, \bar{\sigma}, \underline{\sigma}) \left(\|\hat{\xi}\|_{L_G^\infty}^{p'} \right. \\ &+ \mathbb{E} \left[\left(\sup_{0 \leq t \leq T} |\hat{Y}_t| \int_0^T (2M_0 + L_y (|Y_t^1| + |Y_t^2|) + 2L_z (1 + |Z_t^1|^2 + |Z_t^2|^2)) d\langle B \rangle_t \right)^{p'/2} \right] \\ &+ \mathbb{E} \left[\left(\sup_{0 \leq t \leq T} |\hat{Y}_t|^2 \int_0^T |\hat{Z}_t|^2 d\langle B \rangle_t \right)^{p'/4} \right] + \mathbb{E} \left[\left(\sup_{0 \leq t \leq T} |\hat{Y}_t| |K_T^1| \right)^{p'/2} \right] \\ &\left. + \mathbb{E} \left[\left(\sup_{0 \leq t \leq T} |\hat{Y}_t| |K_T^2| \right)^{p'/2} \right] \right). \end{aligned}$$

From (2.7) and (2.8) and by Hölder's inequality, we have

$$\mathbb{E} \left[\left(\int_0^T |\hat{Z}_t|^2 dt \right)^{p'/2} \right] \leq C(p, \bar{\sigma}, \underline{\sigma}, M_0, L_y, L_z) \left(\|\hat{\xi}\|_{L_G^\infty}^{p'} + \mathbb{E} \left[\sup_{0 \leq t \leq T} |\hat{Y}_t|^p \right]^{p'/2p} \right).$$

□

Remark 2.14 The uniqueness for the quadratic GBSDE can be derived from Proposition 2.11 and Proposition 2.13, or by regarding it as a quadratic 2BSDEs studied in [88].

2.4 The existence of solutions to discrete GBSDEs

In this section, we prove the existence of solutions to the equation (2.5), which are constructed by solutions of the corresponding discrete PDEs.

2.4.1 Discrete PDEs

In this subsection, we follow Hu and Ma [43] to consider discrete PDEs and deduce the boundedness of the first derivatives of the solution u in x .

First, we introduce the following fully nonlinear PDE on $[t_{N-1}, T]$:

$$\partial_t u + G(D_{x_N x_N}^2 u + 2f(t, x_1, x_2, \dots, x_N, u, D_{x_N} u)) = 0, \quad (2.10)$$

with $u(T, x_1, x_2, \dots, x_N) = \varphi(x_1, x_2, \dots, x_N) \in \mathcal{C}_{b, \text{lip}}(\mathbb{R}^N)$, where the terminal value φ and the generator f satisfies Assumption 2.6 and the following assumption.

Assumption 2.15 *We assume that the generator $f : [0, T] \times \mathbb{R}^N \times \mathbb{R}^2$ satisfies moreover the following conditions:*

(Hd') *The function f is at least C^1 in $x_1, x_2, \dots, x_N, y, z$, differentiable in t and twice differentiable in $x_1, x_2, \dots, x_N, y, z$, where the second derivative of f in $x_1, x_2, \dots, x_N, y, z$ are bounded on the set $[0, T] \times \mathbb{R}^N \times [-M_y, M_y] \times [-M_z, M_z]$, for any $M_y, M_z > 0$.*

Remark 2.16 *From (Hc') and (Hq'), we could conclude that the first derivative of f in $t, x_1, x_2, \dots, x_N, y, z$ are bounded on the set $[0, T] \times \mathbb{R}^N \times \mathbb{R} \times [-M_z, M_z]$, for any $M_z > 0$.*

Proposition 2.17 *The PDE (2.10) admits a classical solution bounded by $M := M(M_0, L_y)$, and for arbitrary small κ , there exists a constant $\alpha := \alpha(\kappa)$, such that*

$$\|u\|_{\mathcal{C}^{1+\alpha/2, 2+\alpha}([t_{N-1}, T-\kappa] \times \mathbb{R})} < \infty.$$

Remark 2.18 *The proof of this proposition is not difficult by proceeding a similar argument as Appendix §B-4 in Peng [83], where the results from Example 6.1.8 and Theorem 6.4.3 in Krylov [55] play very important roles.*

Fix $N \in \mathbb{N}$ and a partition π^N on $[0, T]$. Denote by L^φ the Lipschitz constant of φ , and by L_x^f and L_y^f the Lipschitz constants of f in x_1, x_2, \dots, x_N and in y .

Denote $\mathbf{x}^{(k)} := (x_1, x_2, \dots, x_k)$, $k = 1, 2, \dots, N$. We rewrite (2.10) into the following form:

$$\begin{aligned} \partial_t u^N(t, \mathbf{x}^{(N-1)}, x_N) + G(D_{x_N x_N}^2 u^N(t, \mathbf{x}^{(N-1)}, x_N) \\ + 2f(t, \mathbf{x}^{(N-1)}, x_N, u^N(t, \mathbf{x}^{(N-1)}, x_N), D_{x_N} u^N(t, \mathbf{x}^{(N-1)}, x_N))) = 0, \end{aligned}$$

with $u^N(T, \mathbf{x}^{(N-1)}, x) = \varphi(\mathbf{x}^{(N-1)}, x)$.

To estimate the first derivative $\frac{\partial u}{\partial x_N}$, we proceed the same as Step 1 of proof for Theorem 4.1 in Hu et al. [39] and obtain

$$\left| D_{x_N} u^N(t, \mathbf{x}^{(N-1)}, x_N) \right| \leq \left(L^\varphi + \frac{L_x^f}{L_y^f} \right) \exp(\bar{\sigma}^2 L_y^f (T - t_{N-1})) - \frac{L_x^f}{L_y^f} := L^N.$$

Remark 2.19 *We remark here for proving the above result, we shall recall a general comparison result in Buckdhan and Li [15] (Theorem 6.1). Note that the function f we consider is only local Lipschitz, while Theorem 6.1 in [15] requires the Lipschitz assumption. This has little matter, since we could eventually see that $D_{x_N} u$ is bounded by a constant $M_z := M_z(N, L^\varphi, L_x, L_y)$, then a standard truncation technique may apply here.*

In a similar way, we have moreover, for $k = 1, 2, \dots, N$, $t \in [t_{N-1}, T]$,

$$\left| D_{x_k} u^N(t, \mathbf{x}^{(N-1)}, x_N) \right| \leq L^N.$$

Then, we could define the following PDE on $[t_{N-2}, t_{N-1}]$:

$$\begin{aligned} \partial_t u^{N-1}(t, \mathbf{x}^{(N-2)}, x_{N-1}) + G(D_{x_{N-1}x_{N-1}}^2 u^{N-1}(t, \mathbf{x}^{(N-2)}, x_{N-1}) \\ + 2f(t, \mathbf{x}^{(N-2)}, x_{N-1}, 0, u^{N-1}(t, \mathbf{x}^{(N-2)}, x_{N-1}), D_x u^{N-1}(t, \mathbf{x}^{(N-2)}, x_{N-1}))) = 0, \end{aligned}$$

with the terminal condition

$$u^{N-1}(t_{N-1}, \mathbf{x}^{(N-2)}, x_{N-1}) := u^N(t_{N-1}, \mathbf{x}^{(N-2)}, x_{N-1}, 0).$$

From the estimate above, we know that the Lipschitz constant of u^N in x_{N-1} is L^N , then for $k = 1, 2, \dots, N-1, t \in [t_{N-2}, t_{N-1}]$,

$$\left| D_{x_k} u^{N-1}(t, \mathbf{x}^{(N-2)}, x_{N-1}) \right| \leq \left(L^N + \frac{L_x^f}{L_y^f} \right) \exp(\bar{\sigma}^2 L_y^f (t_{N-1} - t_{N-2})) - \frac{L_x^f}{L_y^f} := L^{N-1}.$$

By recurrence, we consider the following PDE on $[t_{k-1}, t_k]$:

$$\begin{aligned} \partial_t u^k(t, \mathbf{x}^{(k-1)}, x_k) + G(D_{x_k x_k}^2 u^k(t, \mathbf{x}^{(k-1)}, x_k) \\ + 2f(t, \mathbf{x}^{(k-1)}, x_k, \underbrace{0, \dots, 0}_{N-k}, u^k(t, \mathbf{x}^{(k-1)}, x_k), D_{x_k} u^k(t, \mathbf{x}^{(k-1)}, x_k))) = 0, \end{aligned} \quad (2.11)$$

with the terminal condition

$$u^k(t_k, \mathbf{x}^{(k-1)}, x_k) := u^{k+1}(t_k, \mathbf{x}^{(k-1)}, x_k, 0).$$

For each $i = 0, 1, 2, \dots, k, t \in [t_{k-1}, t_k]$,

$$\left| D_{x_i} u^k(t, \mathbf{x}^{(k-1)}, x_k) \right| \leq \left(L^{k+1} + \frac{L_x^f}{L_y^f} \right) \exp(\bar{\sigma}^2 L_y^f (t_k - t_{k-1})) - \frac{L_x^f}{L_h^f} := L^k.$$

2.4.2 The solution of the discrete GBSDE

In this subsection, we construct the solution of the discrete GBSDE (2.5) satisfying Assumption 2.6 and 2.15. Fix $N \in \mathbb{N}$ and the partition π^N of $[0, T]$. We note

$$\mathbf{B}_t^k := (B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_{k-1}} - B_{t_{k-2}}, B_t - B_{t_{k-1}}), \quad t \in [t_{k-1}, t_k], \quad k = 1, 2, \dots, N.$$

Then, the solution of the GBSDE (2.5) is defined in the following way: for $t \in [t_{k-1}, t_k]$,

$$\begin{aligned} Y_t &:= Y_t^k := u^k(t, \mathbf{B}_t^k); \\ Z_t &:= Z_t^k := D_{x_k} u^k(t, \mathbf{B}_t^k); \end{aligned}$$

and

$$\begin{aligned} K_t &:= K_t^k = K_{t_{k-1}}^{k-1} + \int_{t_{k-1}}^t \left(D_{x_k x_k}^2 u^k(s, \mathbf{B}_s^k) + 2f(s, \mathbf{B}_s^k, \underbrace{0, \dots, 0}_{N-k}, u^k(s, \mathbf{B}_s^k), D_{x_k} u^k(s, \mathbf{B}_s^k)) \right) d\langle B \rangle_s \\ &\quad - \int_{t_{k-1}}^t G \left(D_{x_k x_k}^2 u^k(s, \mathbf{B}_s^k) + 2f(s, \mathbf{B}_s^k, \underbrace{0, \dots, 0}_{N-k}, u^k(s, \mathbf{B}_s^k), D_{x_k} u^k(s, \mathbf{B}_s^k)) \right) ds, \end{aligned}$$

where u^k , $k = 1, 2, \dots, N$, is the solution to the corresponding PDE (2.11) on $[t_{k-1}, t_k]$.

From the estimates of the corresponding PDEs, we can find the following bounds:

$$|Y| \leq M_y := M_y(N, M_0, L_y^f)$$

and

$$|Z| \leq M_z := M_z(N, L^\varphi, L_x^f, L_y^f). \quad (2.12)$$

If $N = 1$, then following the proof of Theorem 4.1 in [39], we can verify that for each $\kappa \in (0, 1)$, (Y, Z, K) is a solution of the following GBSDE on $[0, T - \kappa]$:

$$Y_t = \varphi(B_{T-\kappa}) + \int_t^{T-\kappa} f(t, B_s, Y_s, Z_s) d\langle B \rangle_s - \int_t^{T-\kappa} Z_s dB_s - (K_{T-\kappa} - K_t),$$

where K is a decreasing G -martingale on $[0, T - \kappa]$ with $K_0 = 0$. Indeed, the function u and the derivative Du and D^2u is bounded and α -Hölder continuous, and thus $D_{xx}^2 u(\cdot, B) + 2f(\cdot, B, u(\cdot, B), D_x u(\cdot, B)) \in \mathcal{H}_G^p(0, T - \kappa)$, for any $p \geq 2$.

Similarly to (4.3) in [39], we could obtain for $0 < \tilde{t} \leq \hat{t} < T$ and $\tilde{x}, \hat{x} \in \mathbb{R}$ and some positive constant L ,

$$|u(\tilde{t}, \tilde{x}) - u(\hat{t}, \hat{x})| \leq L \left(\sqrt{|\tilde{t} - \hat{t}|} + |\tilde{x} - \hat{x}| \right),$$

which implies that (Y, Z, K) is the solution of (2.5) on $[0, T]$, and K is a decreasing G -martingale with $K_0 = 0$ and closed by $K_T \in L_G^p(\Omega_T)$, for any $p \geq 1$, which can be defined as the quasi-sure limit of the decreasing sequence $K_{T-\frac{1}{p}}$. We remark also that $Y \in \mathcal{S}_G^p(0, T)$ and $Z \in \mathcal{H}_G^p(0, T)$, for any $p \geq 2$, which could be deduced by the same procedure as [39].

For the case that $N > 1$, it suffices to prove without loss of generality that $(Y, Z, K) \in \mathfrak{G}_G^p(0, T)$ is the solution of (2.5) on $[0, T]$ when $N = 2$.

On $[0, t_1]$, it follows from the previous case $N = 1$ that the triple (Y, Z, K) defined by

$$Y := u^1(\cdot, B), \quad Z := D_{x_1} u^1(\cdot, B),$$

$$\begin{aligned} K := & \int_0^\cdot (D_{x_1 x_1}^2 u^1(s, B_s) + 2f(s, B_s, 0, u^1(s, B_s), D_{x_1} u^1(s, B_s))) d\langle B \rangle_s \\ & - \int_0^\cdot G(D_{x_1 x_1}^2 u^1(s, B_s) + 2f(s, B_s, 0, u^1(s, B_s), D_{x_1} u^1(s, B_s))) ds, \end{aligned}$$

solves the following GBSDE:

$$Y_t = u^2(t_1, B_{t_1}, 0) + \int_t^{t_1} f(s, B_s, 0, Y_s, Z_s) d\langle B \rangle_s - \int_t^{t_1} Z_s dB_s - (K_{t_1} - K_t).$$

Now, it suffice to verify that

$$Y_t^2 := u^2(t, B_{t_1}, B_t - B_{t_1}), \quad Z_t^2 := D_{x_2} u^2(t, B_{t_1}, B_t - B_{t_1}),$$

$$\begin{aligned}
K_t := & K_{t_1} + \int_{t_1}^t D_{x_2 x_2}^2 u^2(s, B_{t_1}, B_s - B_{t_1}) \\
& + 2f(s, B_{t_1}, B_s - B_{t_1}, u^2(s, B_{t_1}, B_s - B_{t_1}), D_{x_2} u^2(s, B_{t_1}, B_s - B_{t_1})) d\langle B \rangle_s \\
& - \int_{t_1}^t G(D_{x_2 x_2}^2 u^2(s, B_{t_1}, B_s - B_{t_1}) \\
& + 2f(s, B_{t_1}, B_s - B_{t_1}, u^2(s, B_{t_1}, B_s - B_{t_1}), D_{x_2} u^2(s, B_{t_1}, B_s - B_{t_1}))) ds,
\end{aligned}$$

defines a solution on $[t_1, T]$ of the following GBSDE:

$$Y_t = \varphi(B_{t_1}, B_T - B_{t_1}) + \int_t^T f(s, B_{t_1}, B_{s \wedge t_1} - B_{t_1}, Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t).$$

This can be achieved by first proving a generalized Itô's formula for $u(t, B_{t_1}, B_t - B_{t_1})$ on $[t_1, T - \kappa]$ as §III-6 in [83] and then letting $\kappa \rightarrow 0$.

2.5 Existence of solutions for general quadratic GBSDEs

In this section, we shall prove the existence result for the general quadratic GBSDE (2.4) under Assumption 2.1. Indeed, we start by considering the solution to the discrete GBSDE (2.5) under assumptions weaker than (Hd') and construct solutions to (2.4) by successive approximation.

Step 1: Fix $N \in \mathbb{N}$ and the partition π^N on $[0, T]$. We consider the GBSDE (2.5) with the generator \hat{f} satisfying Assumption 2.6 and what follows.

Assumption 2.20 We assume that the generator $\hat{f} : [0, T] \times \mathbb{R}^N \times \mathbb{R}^2$ satisfies moreover the following conditions:

(Hd'') The first derivative of f in t , the first and the second derivatives of f in x_1, x_2, \dots, x_N are bounded on the set $[0, T] \times \mathbb{R}^N \times [-M_y, M_y] \times [-M_z, M_z]$, for any $M_y, M_z > 0$.

In what follows, we shall regularize \hat{h} in y and z : for each $(t, x_1, x_2, \dots, x_N) \in [0, T] \times \mathbb{R}^N$, $(y, z) \in \mathbb{R}^2$, we define

$$f^n(t, x_1, x_2, \dots, x_N, y, z) := \int_{\mathbb{R}^2} \hat{f}(t, x_1, x_2, \dots, x_N, y - \tilde{y}, z - \tilde{z}) \rho_n(\tilde{y}, \tilde{z}) d\tilde{y}d\tilde{z},$$

where ρ_n is a positive smooth function such that its support is contained in a $\frac{1}{n}$ -ball in \mathbb{R}^2 and $\int_{\mathbb{R}^2} \rho_n = 1$. Without loss of generality, we assume that, for each $n \in \mathbb{N}$, f^n satisfies (Hq') with the same Lipschitz constants.

Obviously, f^n has bounded first derivatives in y and z on the set $[0, T] \times \mathbb{R}^N \times [-M_y, M_y] \times [-M_z, M_z]$, for any $M_y, M_z > 0$, because that \hat{f} satisfies (Hq'). We now calculate the second derivative of f^n in y by

$$\frac{\partial^2 f^n}{\partial y^2} = \int_{\mathbb{R}^2} \hat{f}(t, x_1, x_2, \dots, x_N, \tilde{y}, \tilde{z}) \frac{\partial^2 \rho_n}{\partial y^2}(y - \tilde{y}, z - \tilde{z}) d\tilde{y}d\tilde{z}.$$

Since on the set $[0, T] \times \mathbb{R}^N \times [-M_y - \frac{1}{n}, M_y + \frac{1}{n}] \times [-M_z - \frac{1}{n}, M_z + \frac{1}{n}]$,

$$\left| \hat{f}(t, x_1, x_2, \dots, x_N, y, z) \right| \leq M_0 + L_y \left(M_y + 1 \right) + L_z \left(M_z + 2 \right)^2 := M_{yz},$$

we have

$$\left| \frac{\partial^2 f^n}{\partial y^2} \right| \leq C(n) M_{yz} \left\| \frac{\partial^2 \rho_n}{\partial y^2} \right\|_{\infty},$$

which means that f^n has bounded second derivative in y on the set $[0, T] \times \mathbb{R}^N \times [-M_y, M_y] \times [-M_z, M_z]$, for any $M_y, M_z > 0$. Similar result can be obtained for the second derivative of f^n in z .

From the results in the last section, we know that, for any $p \geq 2$, the GBSDE (2.5) with the coefficient f^n admits a solution $(Y^n, Z^n, K^n) \in \mathfrak{G}_G^p(0, T)$. We now verify that $\{Y^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}_G^p(0, T)$. For simplicity, we denote $f_t^n(y, z) := f^n(t, B_{t_1 \wedge t}, B_{t_2 \wedge t} - B_{t_1 \wedge t}, \dots, B_{t_N \wedge t} - B_{t_{N-1} \wedge t}, y, z)$ and $\hat{f}_t(y, z) := \hat{f}(t, B_{t_1 \wedge t}, B_{t_2 \wedge t} - B_{t_1 \wedge t}, \dots, B_{t_N \wedge t} - B_{t_{N-1} \wedge t}, y, z)$. Indeed, for $n, m \in \mathbb{N}, n \geq m, 0 \leq t \leq T$,

$$\begin{aligned} |f_t^n(Y_t^n, Z_t^n) - f_t^m(Y_t^m, Z_t^m)| &\leq |f_t^n(Y_t^n, Z_t^n) - \hat{f}_t(Y_t^n, Z_t^n)| + |f_t^m(Y_t^m, Z_t^m) - \hat{f}_t(Y_t^m, Z_t^m)| \\ &\leq \int_{\mathbb{R}^2} \left| \hat{f}_t(Y_t^n, Z_t^n) - \hat{f}_t(Y_t^n - \tilde{y}, Z_t^n - \tilde{z}) \right| \rho_n(\tilde{y}, \tilde{z}) d\tilde{y} d\tilde{z} \\ &\quad + \int_{\mathbb{R}^2} \left| \hat{f}_t(Y_t^m, Z_t^m) - \hat{f}_t(Y_t^m - \tilde{y}, Z_t^m - \tilde{z}) \right| \rho_m(\tilde{y}, \tilde{z}) d\tilde{y} d\tilde{z} \\ &\leq \frac{2}{m} (L_y + 2L_z(M_z + 1)), \end{aligned} \tag{2.13}$$

where M_z is the bounded of all the solutions Z^n defined in (2.12), which is independent of n . Then, from Proposition 2.11, we have $\{Y^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}_G^p(0, T)$, so that there exists a $\hat{Y} \in \mathcal{S}_G^p(0, T)$, such that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n - \hat{Y}_t|^p \right] \longrightarrow 0. \tag{2.14}$$

Furthermore, from Proposition 2.13, we have, for $1 \leq p'/2 < p$, $\{Z^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_G^{p'}(0, T)$. Then, there exists a $\hat{Z} \in \mathcal{H}_G^{p'}(0, T)$, such that

$$\mathbb{E} \left[\left(\int_0^T |Z_t^n - \hat{Z}_t|^2 dt \right)^{p'/2} \right] \longrightarrow 0. \tag{2.15}$$

We define

$$\hat{K}_t = \hat{Y}_t - \hat{Y}_0 + \int_0^t \hat{f}(s, B_{t_1 \wedge s}, B_{t_2 \wedge s} - B_{t_1 \wedge s}, \dots, B_{t_N \wedge s} - B_{t_{N-1} \wedge s}, \hat{Y}_s, \hat{Z}_s) d\langle B \rangle_s - \int_0^t \hat{Z}_s dB_s.$$

We proceed to prove that \hat{K} is a decreasing G -martingale starting from $K_0 = 0$ and $\hat{K}_T \in L_G^{p'/2}(\Omega_T)$. it suffices to prove that

$$\mathbb{E} \left[\left(\int_0^T |f_t^n(Y_t^n, Z_t^n) - \hat{f}_t(\hat{Y}_t, \hat{Z}_t)| dt \right)^{p'/2} \right] \longrightarrow 0,$$

which can be easily deduced by recalling (2.13), (2.14), (2.15) and (Hq'). Thus, we obtain for $0 \leq t \leq T$,

$$\mathbb{E} \left[|K_t^n - \hat{K}_t|^{p'/2} \right] \longrightarrow 0,$$

from which have that, for $0 \leq s \leq t \leq T$, $\mathbb{E}_s[\hat{K}_t] = \hat{K}_s$ by applying Proposition 2.5 in Hu et al. [39] and $\hat{K}_0 = 0$. Note that p could be arbitrary large, which is ensured by the uniform boundedness of Y^n and the G -BMO norm of Z^n . Therefore, for any $p \geq 2$, we could find a triple $(\hat{Y}, \hat{Z}, \hat{K}) \in \mathfrak{G}_G^p(0, T)$, which is a solution to the GBSDE (2.5) under Assumption 2.6 and 2.20.

Step 2: Fix $N \in \mathbb{N}$ and the partition π^N on $[0, T]$. We consider the GSDE (2.5) with a generator \bar{f} satisfying Assumption 2.6. In this step, we shall construct the solution to such a GBSDE by regularizing \bar{f} in t and x_1, x_2, \dots, x_N : for each $(y, z) \in \mathbb{R}^2$, $(t, x_1, x_2, \dots, x_N) \in [0, T] \times \mathbb{R}^N$, we define

$$\begin{aligned} \hat{f}^n(t, x_1, x_2, \dots, x_N, y, z) &:= \int_{\mathbb{R}^{N+1}} \bar{f}(t - \tilde{t}, x_1 - \tilde{x}_1, x_2 - \tilde{x}_2, \dots, x_N - \tilde{x}_N, y, z) \\ &\quad \times \rho_n(\tilde{t}, \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_N) d\tilde{t} d\tilde{x}_1 d\tilde{x}_2 \dots d\tilde{x}_N, \end{aligned}$$

where ρ_n is a positive smooth function such that its support is contained in a $\frac{1}{n}$ -ball in \mathbb{R}^{N+1} and $\int_{\mathbb{R}^{N+1}} \rho_n = 1$. In addition, we define the extension of the function \bar{f} on \mathbb{R}_- , i.e., if $t < 0$, $\bar{f}(t, \cdot, \cdot, \dots, \cdot, \cdot, \cdot) := \bar{f}(0, \cdot, \cdot, \dots, \cdot, \cdot, \cdot)$.

Proceeding the same argument as in the last step, we can show that the first derivative of \hat{f}^n in t , the first and second derivatives of \hat{f}^n in x_i , $i = 1, 2, \dots, N$, are bounded on the set $[0, T] \times \mathbb{R}^N \times [-M_y, M_y] \times [-M_z, M_z]$, for any $M_y, M_z > 0$. Therefore, recalling the result in the last step, we obtain that, for any $p \geq 2$, the GBSDE (2.5) with the coefficient \hat{f}^n admits a solution $(\hat{Y}^n, \hat{Z}^n, \hat{K}^n) \in \mathfrak{G}_G^p(0, T)$.

For $n, m \in \mathbb{N}$, $n \geq m$, $0 \leq t \leq T$, by the definition of \hat{f}^n and (Hc'),

$$\begin{aligned} &|\hat{f}^n(t, B_{t_1 \wedge t}, B_{t_2 \wedge t} - B_{t_1 \wedge t}, \dots, B_{t_N \wedge t} - B_{t_{N-1} \wedge t}, \hat{Y}_t^n, \hat{Z}_t^n) \\ &\quad - \hat{f}^m(t, B_{t_1 \wedge t}, B_{t_2 \wedge t} - B_{t_1 \wedge t}, \dots, B_{t_N \wedge t} - B_{t_{N-1} \wedge t}, \hat{Y}_t^m, \hat{Z}_t^m)| \\ &\leq (N+1)w^{\bar{f}}\left(\frac{1}{n}\right) \leq (N+1)w^{\bar{f}}\left(\frac{1}{m \wedge n}\right), \end{aligned}$$

from which we could deduce that $\{\hat{Y}^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence, and similarly to (2.14), there exists $\bar{Y} \in \mathcal{S}_G^p(0, T)$, such that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |\hat{Y}_t^n - \bar{Y}_t|^p \right] \longrightarrow 0.$$

We could conclude in a similar way as in the previous step that for any $p \geq 2$, there exists a triple $(\bar{Y}, \bar{Z}, \bar{K}) \in \mathfrak{G}_G^p(0, T)$, which solves the GBSDE (2.5) under Assumption 2.6.

To study the more general GBSDE (2.4), we need the following assumption on the terminal value ξ .

Assumption 2.21 $\xi \in L_G^\infty$.

We are now ready to introduce the main result of this paper.

Theorem 2.22 Consider the GBSDE (2.4) satisfying Assumption 2.1 and 2.21. It admits at least a solution $(Y, Z, K) \in \mathfrak{G}_G^2(0, T)$.

Proof: Without loss of generality, we assume ξ is approximated by the following sequence

$$\xi^n := \varphi^n(B_{t_1^n}, B_{t_2^n} - B_{t_1^n}, \dots, B_{t_{N(n)}^n} - B_{t_{N(n)-1}^n}) \in Lip(\Omega_T),$$

where for each $n \in \mathbb{N}$, $0 = t_0^n \leq t_1^n \leq \dots \leq t_{N(n)}^n = T$, $\mu^n := \max_{k=1,2,\dots,N(n)} |t_k^n - t_{k-1}^n| \leq 1/2^n$. Assume moreover that for each $n \geq m$, $\{0 = t_0^m, t_1^m, t_2^m, \dots, t_{N(m)}^m = T\} =: \pi^m \subset \pi^n := \{0 = t_0^n, t_1^n, t_2^n, \dots, t_{N(n)}^n = T\}$.

Fix $n \in \mathbb{N}$. We construct the function \bar{f}^n in terms of h by discretization. For simplicity of notation, we omit the superscript n for t_k^n , $k = 0, 1, \dots, N(n)$. Denote by $\mathbf{x}(n)$ the vector $(x_1, x_2, \dots, x_{N(n)}) \in \mathbb{R}^{N(n)}$. Let $t \in [t_{k-1}, t_k]$, where $k = 1, 2, \dots \leq N(n)$. We define by the following procedures a piecewisely linear path stopped at time t in terms of $\mathbf{x}(n)$, noted by $\omega^{\mathbf{x}(n),t}$.

- $\omega_{t_0}^{\mathbf{x}(n),t} = 0$;
- $\omega_{t_1}^{\mathbf{x}(n),t} = x_1$;
- $\omega_{t_2}^{\mathbf{x}(n),t} = x_1 + x_2$;
- ...;
- $\omega_{t_{k-1}}^{\mathbf{x}(n),t} = \sum_{i=1}^{k-1} x_i$;
- $\omega_t^{\mathbf{x}(n),t} = \omega_{t_k}^{\mathbf{x}(n),t} = \omega_{t_{k+1}}^{\mathbf{x}(n),t} = \omega_{t_{k+2}}^{\mathbf{x}(n),t} = \dots = \omega_{t_{N(n)}}^{\mathbf{x}(n),t} = \sum_{i=1}^k x_i$;
- $\omega^{\mathbf{x}(n),t}$ is a linear function in t on $[t_{i-1}, t_i]$, for $i = 1, 2, \dots, k$. It is also linear on $[t_k, t]$ and takes a constant value on $[t, T]$.

Define

$$\bar{f}^n(t, x_1, x_2, \dots, x_{N(n)}, y, z) := h(t, \omega^{\mathbf{x}(n),t}, y, z).$$

We can verify that for each $t \in [0, T]$, $\mathbf{x}^1(n) := (x_1^1, x_2^1, \dots, x_{N(n)}^1)$, $\mathbf{x}^2(n) := (x_1^2, x_2^2, \dots, x_{N(n)}^2)$ and $(y, z) \in \mathbb{R}^2$,

$$\begin{aligned} & \left| \bar{f}^n(t, x_1^1, x_2^1, \dots, x_{N(n)}^1, y, z) - \bar{f}^n(t, x_1^2, x_2^2, \dots, x_{N(n)}^2, y, z) \right| \\ & \leq w^h \left(\left\| \omega^{\mathbf{x}^1(n),t} - \omega^{\mathbf{x}^2(n),t} \right\|_\infty \right) \\ & \leq w^h \left(\sum_{k=1,2,\dots,N(n)} |x_k^1 - x_k^2| \right), \end{aligned}$$

which implies that \bar{f} is uniformly continuous with modulus w^h in $x_1, x_2, \dots, x_{N(n)}$, where the modulus is independent of y and z . From the result in Step 2, we know that, for any $p \geq 2$, the GBSDE (2.5) with the parameters (\bar{f}^n, ξ^n) admits a solution $(\bar{Y}^n, \bar{Z}^n, \bar{K}^n) \in \mathfrak{G}_G^p(0, T)$.

For $n, m \in \mathbb{N}$, $n \geq m$, we denote

$$\begin{aligned} \eta^{n,m} & := \int_0^T \bar{f}^n(t, B_{t_1 \wedge t}, B_{t_2 \wedge t} - B_{t_1 \wedge t}, \dots, B_{t_{N(n)} \wedge t} - B_{t_{N(n)-1} \wedge t}, \bar{Y}_t^n, \bar{Z}_t^n) d\langle B \rangle_t \\ & - \int_0^T \bar{f}^m(t, B_{t_1 \wedge t}, B_{t_2 \wedge t} - B_{t_1 \wedge t}, \dots, B_{t_{N(m)} \wedge t} - B_{t_{N(m)-1} \wedge t}, \bar{Y}_t^n, \bar{Z}_t^n) d\langle B \rangle_t. \end{aligned}$$

By Proposition 2.11, we have

$$|Y_t^n - Y_t^m| \leq C \left(\|\xi^n - \xi^m\|_{L_G^\infty} + \tilde{\mathbb{E}}_t[\eta^{n,m}] \right) \leq C \left(\|\xi^1 - \xi^2\|_{L_G^\infty} + \mathbb{E}_t \left[\frac{\mathcal{E}(b^{n,m})_T}{\mathcal{E}(b^{n,m})_t} \eta^{n,m} \right] \right),$$

where $b^{n,m}$ is defined by (2.9) in terms of \bar{Z}^n and \bar{Z}^m . Fortunately, its G -BMO norm is dominated by a constant C_b independent of n and m . Thus, we could find a uniform order $q > 1$ and a uniform constant C_q for the reverse Hölder inequality for $\mathcal{E}(b^{n,m})$, for all $n, m \in \mathbb{N}$. Since the function Φ is decreasing (see Theorem 3.1 in [52]), we assume without loss of generality that $q < 2$. Then, we obtain

$$|Y_t^n - Y_t^m| \leq C \left(\|\xi^1 - \xi^2\|_{L_G^\infty} + C_q^{1/q} \mathbb{E}_t [(\eta^{n,m})^p]^{1/p} \right),$$

where $1/q + 1/p = 1$. Furthermore, applying Theorem 2.8 in [39], we have, for $2 < p < p''$ and $1 < \gamma < p''/p$,

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} |\bar{Y}_t^n - \bar{Y}_t^m|^p \right] &\leq C \left(\|\xi^1 - \xi^2\|_{L_G^\infty}^p + \mathbb{E} \left[\sup_{0 \leq t \leq T} \mathbb{E}_t [(\eta^{n,m})^p] \right] \right) \\ &\leq C \left(\|\xi^1 - \xi^2\|_{L_G^\infty}^p + \mathbb{E} \left[(\eta^{n,m})^{p''} \right]^{p/p''} + \mathbb{E} \left[(\eta^{n,m})^{p''} \right]^{1/\gamma} \right), \end{aligned}$$

where the constant C varies from line to line, nevertheless, is independent of n and m .

To verify that $\{\bar{Y}^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}_G^p(0, T)$, it suffices to have

$$\mathbb{E} \left[(\eta^{n,m})^{p''} \right] \longrightarrow 0, \quad \text{as } m, n \rightarrow \infty. \quad (2.16)$$

For simplicity, note $B^t := B_{\cdot, t}$. For each $\omega \in \Omega$, we define

$$\mathbf{x}^{B^t(\omega)}(n) := (B_{t_1}(\omega), B_{t_2}(\omega) - B_{t_1}(\omega), \dots, B_t(\omega) - B_{t_{k-1}}(\omega), \underbrace{0, \dots, 0}_{N(n)-k}), \quad t \in [t_{k-1}, t_k]$$

and a mapping $B^{n,t} : \mathcal{C}(0, T) \rightarrow \mathcal{C}(0, T)$,

$$B^{n,t}(\omega) := \omega^{\mathbf{x}^{B^t(\omega)}(n), t},$$

Then,

$$\begin{aligned} \eta^{n,m} &= \int_0^T \left| \bar{f}^n(t, \mathbf{x}^{B^t(\cdot)}(n), \bar{Y}_t^n, \bar{Z}_t^n) - \bar{f}^m(t, \mathbf{x}^{B^t(\cdot)}(m), \bar{Y}_t^m, \bar{Z}_t^m) \right| d\langle B \rangle_t \\ &= \int_0^T |h(t, B^{n,t}(\cdot), \bar{Y}_t^n, \bar{Z}_t^n) - h(t, B^{m,t}(\cdot), \bar{Y}_t^m, \bar{Z}_t^m)| d\langle B \rangle_t \\ &\leq \int_0^T w^h (\|B^{n,t}(\cdot) - B^{m,t}(\cdot)\|_\infty) dt \\ &\leq \int_0^T w^h (\|B^{n,t}(\cdot) - B^t(\cdot)\|_\infty) dt + \int_0^T w^h (\|B^{m,t}(\cdot) - B^t(\cdot)\|_\infty) dt, \end{aligned}$$

where the last inequality is from the sub-additivity of w^h .

Now, our aim is to prove that

$$\mathbb{E} \left[\int_0^T w^h (\|B^{n,t}(\cdot) - B^t(\cdot)\|_\infty) dt \right] \longrightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Then, due to the boundedness of w^h , we have

$$\begin{aligned} \mathbb{E} \left[(\eta^{n,m})^{p''} \right] &\leq C_{p''} \left(\mathbb{E} \left[\left(\int_0^T w^h (\|B^{n,t}(\cdot) - B^t(\cdot)\|_\infty) dt \right)^{p''} \right] \right. \\ &\quad \left. + \mathbb{E} \left[\left(\int_0^T w^h (\|B^{m,t}(\cdot) - B^t(\cdot)\|_\infty) dt \right)^{p''} \right] \right) \longrightarrow 0, \text{ as } m, n \rightarrow \infty. \end{aligned} \quad (2.17)$$

Indeed, since w^h is a concave function, by Lemma 2.12 in Bai and Lin [4], we have

$$\begin{aligned} \mathbb{E} \left[\int_0^T w^h (\|B^{n,t}(\cdot) - B^t(\cdot)\|_\infty) dt \right] &\leq w^h \left(\mathbb{E} \left[\int_0^T \|B^{n,t}(\cdot) - B^t(\cdot)\|_\infty dt \right] \right) \\ &\leq w^h \left(\int_0^T \mathbb{E} [\|B^{n,t}(\cdot) - B^t(\cdot)\|_\infty] dt \right). \end{aligned} \quad (2.18)$$

Then, to eventually prove (2.16), it suffices to have

$$\mathbb{E} [\|B^{n,t}(\cdot) - B^t(\cdot)\|_\infty] \longrightarrow 0, \text{ as } n \rightarrow \infty, \quad (2.19)$$

and then we could apply the Lebesgue dominated convergence theorem to deduce the convergence of (2.18), which yields (2.17). For $t \in [t_{k-1}, t_k]$, $k = 1, 2, \dots, N(n)$, we calculate (2.19) and obtain

$$\begin{aligned} &\|B^{n,t}(\omega) - B^t(\omega)\|_\infty \\ &\leq \max_{i=1,2,\dots,k-1} \left(\sup_{s \in [t_{i-1}, t_i]} B_s(\omega) - \inf_{s \in [t_{i-1}, t_i]} B_s(\omega) \right) \vee \left(\sup_{s \in [t_{k-1}, t]} B_s(\omega) - \inf_{s \in [t_{k-1}, t]} B_s(\omega) \right) \\ &\leq 2 \left(\max_{i=1,2,\dots,k-1} \left(\sup_{s \in [t_{i-1}, t_i]} |B_s(\omega) - B_{t_{i-1}}(\omega)| \right) \vee \sup_{s \in [t_{k-1}, t]} |B_s(\omega) - B_{t_{k-1}}(\omega)| \right). \end{aligned}$$

Thus, for $\alpha > 2$,

$$\mathbb{E} [\|B^{n,s}(\cdot) - B^s(\cdot)\|_\infty^\alpha] \leq C_{\bar{\sigma}} 2^{-n(\frac{\alpha}{2}-1)} \longrightarrow 0, \text{ as } n \rightarrow \infty,$$

which implies (2.19).

We conclude from (2.16) that, for $p > 2$, there exists a process $Y \in \mathcal{S}_G^p(0, T)$ such that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |\bar{Y}_t^n - Y_t|^p \right] \longrightarrow 0, \text{ as } n \rightarrow \infty.$$

Then, proceeding the same argument as in Step 1, we obtain, for $2 \leq p'/2 < p$, there exists $Z \in \mathcal{H}_G^{p'}(0, T)$, such that

$$\mathbb{E} \left[\left(\int_0^T |\bar{Z}_t^n - Z_t|^2 dt \right)^{p'/2} \right] \longrightarrow 0, \text{ as } n \rightarrow \infty,$$

and

$$\mathbb{E} \left[\left(\int_0^T \left| \bar{f}^n \left(t, \mathbf{x}^{B^t(\cdot)}(n), \bar{Y}_t^n, \bar{Z}_t^n \right) - h \left(t, B_{\cdot \wedge t}(\cdot), Y_t, Z_t \right) \right| dt \right)^{p'/2} \right] \longrightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus, there exists process K with $K_0 = 0$ and $K_T \in L_G^{p'/2}(\Omega_T)$, and due to the fact that for $0 \leq t \leq T$,

$$\mathbb{E} \left[|\bar{K}_t^n - K_t|^{p'/2} \right] \longrightarrow 0, \quad \text{as } n \rightarrow \infty,$$

which implies that K is a decreasing G -martingale. In conclusion, since p and p' can be arbitrarily large, we could finally find the triple $(Y, Z, K) \in \mathfrak{G}_G^2(0, T)$ which solves (2.4). We complete the proof. \square

Chapter 3

Robust utility maximization with G -BSDE

Abstract. In this chapter, we consider applications of backward stochastic differential equations driven by G -Brownian motion (GBSDEs) under quadratic assumptions on coefficients to robust utility maximization problems.

Key words. Backward stochastic differential equations, quadratic growth, G -Brownian motion, robust utility maximization.

AMS subject classifications. 60H10; 60H30

3.1 Market model

We assume that the financial market is composed of a bond with zero interest rate and $d \leq m$ of stocks. The process of price of the share i evolves according to equation

$$\frac{dS_t^i}{S_t^i} = b^i d\langle B \rangle_t + dB_t, \quad i = 1, \dots, d, \quad (3.1)$$

where b^i is a constant. We suppose that the G -expectation satisfying the uniformly elliptic condition, that is, $\underline{\sigma}^2 > 0$, and for any A and B

$$A \geq B \Rightarrow G(A) - G(B) \geq \underline{\sigma}^2 \text{tr}[A - B].$$

A process $\pi = (\pi_t)_{0 \leq t \leq T}$ is a strategy if $\int \pi \frac{dS}{S}$ is well defined. For $1 \leq i \leq d$, the process π_t^i describe the amount of money invested in shares i at time t . The wealth process X^π for a strategy π with initial capital x satisfies equation

$$X_t^{\pi, x} = x + \int_0^t \pi_u (dB_u + b d\langle B \rangle_u), \quad t \in [0, T].$$

Trading strategies are self-financing. Gains or losses are obtained only by trade with the stock.

Suppose the investor has an obligation F at time T . This random variable F is assumed to be bounded. He tries to find an optimal strategy.

The problem of the investor in this financial market is to maximize the utility expected under the uncertainty of the model of its terminal wealth $X_T - F$. This obligation could represent the value of an option or contract that expires at time T .

Let us denote by U the utility function of the investor, the value function V of the maximization problem is therefore given by :

$$\begin{aligned} V(x) &= \sup_{\pi \in \tilde{\mathcal{A}}} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} [U(X_T^\pi - F)] \\ &= \sup_{\pi \in \tilde{\mathcal{A}}} \left[- \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} [-U(X_T^\pi - F)] \right] \\ &= - \inf_{\pi \in \tilde{\mathcal{A}}} \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} [-U(X_T^\pi - F)] \\ &= - \inf_{\pi \in \tilde{\mathcal{A}}} \mathbb{E} [-U(X_T^\pi - F)], \end{aligned}$$

where X_T^π Is the terminal value of the wealth process associated with a strategy $\pi \in \tilde{\mathcal{A}}$ the set of all admissible strategies.

In the case where \mathcal{P} , The set of probability measures representing the G -expectance contains only a probability measure \mathbb{P} , the problem is reduces to the classical maximization problem:

$$V(x) = \sup_{\pi \in \tilde{\mathcal{A}}} \mathbb{E}^{\mathbb{P}} [U(X_T^\pi - F)].$$

3.2 Exponential utility

Here we assume that the investor must pay an obligation F on the horizon T . Then he seeks to maximize his expected utility by using the exponential utility function of his total wealth $X_T^\pi - F$. For $\alpha > 0$, The exponential utility function is defined by:

$$U(x) = - \exp(-\alpha x), \quad x \in \mathbb{R}.$$

We assume that the strategies have values in a closed convex set, that is to say, $\pi_t(\omega) \in \tilde{C}$, with $\tilde{C} \subseteq \mathbb{R}^{1 \times d}$.

Definition 3.1 (*Eligible strategies with constraints*). Let \tilde{C} be a closed convex subset of $\mathbb{R}^{1 \times d}$. The set of eligible strategies \mathcal{A} consists of all the d -dimensional processes π such as $\pi = (\pi_t)_{0 \leq t \leq T} \in \tilde{C}$ and π is a G -BMO martingale generator.

Remark 3.2 Since $\pi \in M_G^2(0, T)$, we have

$$X_T^{\pi, x} = X_t^{\pi, x} + \int_t^T \pi_s dB_s + b \int_t^T \pi_s d\langle B \rangle_s, \quad \forall t \in [0, T].$$

Then, we have $(X_t^{\pi, x}, \pi_t, 0)_{t \in [0, T]}$ is the unique solution of the linear G -BSDE :

$$X_t^{\pi, x} = X_T^{\pi, x} - \int_t^T b \pi_s d\langle B \rangle_s - \int_t^T \pi_s dB_s - (K_T - K_t), \quad \forall t \in [0, T].$$

The objective of the investor is then to solve the problem of maximization:

$$\begin{aligned} V(x) &= \sup_{p \in \mathcal{A}} \left\{ -\mathbb{E} \left[-U \left(X_T^p - F \right) \right] \right\} \\ &= \sup_{p \in \mathcal{A}} \left\{ -\mathbb{E} \left[\exp \left(-\alpha \left(x + \int_0^T p_t \frac{dS_t}{S_t} - F \right) \right) \right] \right\}, \end{aligned}$$

where x is its initial capital and V is the value function. The problem of maximization is equivalent to the problem:

$$\begin{aligned} V(x) &= -\inf_{p \in \mathcal{A}} \mathbb{E} \left[\exp \left(-\alpha \left(x + \int_0^T p_t \frac{dS_t}{S_t} - F \right) \right) \right] \\ &= -\inf_{p \in \mathcal{A}} \mathbb{E} \left[\exp \left(-\alpha \left(x + \int_0^T p_t (dB_t + b d\langle B \rangle_t) - F \right) \right) \right]. \end{aligned} \quad (3.2)$$

To find the value function and an optimal strategy, we build a family of stochastic processes $R^{(p)}$ with the following properties:

- $R_T^{(p)} = -U(X_T^p - F) = \exp(-\alpha(X_T^p - F)) \forall p \in \mathcal{A}$,
- $R_0^{(p)} = R_0$ is constant for all $p \in \mathcal{A}$,
- For all $p \in \mathcal{A}$ the process $R^{(p)}$ is a G -submartingale and there is a strategy $p^* \in \mathcal{A}$ such as the process $R^{(p^*)}$ is a G -martingale.

The process $R^{(p)}$ and its initial value R_0 depend on the initial capital x . For processes with these properties, we have

$$\mathbb{E} \left[R_T^{(p^*)} \right] = R_0 \leq \mathbb{E} \left[R_T^{(p)} \right], \quad \forall p \in \mathcal{A},$$

then, we can compare the expected utility for the strategies $p \in \mathcal{A}$ and $p^* \in \mathcal{A}$ by

$$-\mathbb{E} \left[\exp \left(-\alpha(X_T^{(p)} - F) \right) \right] \leq -R_0 = -\mathbb{E} \left[\exp \left(-\alpha(X_T^{(p^*)} - F) \right) \right] = V(x). \quad (3.3)$$

Then we notice that p^* is the optimal strategy. To build this family, we set

$$R_t^{(p)} := \exp(-\alpha(X_t^p - Y_t)), \quad t \in [0, T], \quad p \in \mathcal{A},$$

where (Y, Z, K) is the unique solution of the G -BSDE:

$$Y_t = F + \int_t^T f(s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad t \in [0, T].$$

Under these conditions, we must choose a function f for which $R^{(p)}$ is a G -submartingale for any strategy $p \in \mathcal{A}$ and there exists $p^* \in \mathcal{A}$ such that $R^{(p^*)}$ be a G -martingale. This function f must also depend on the set of constraints (C_t) , where (p_t) takes its values. We have:

$$\begin{aligned} R_t^{(p)} &= \exp(-\alpha(x - Y_0)) \exp\left(\alpha K_t - \int_0^t \alpha(p_s - Z_s) dB_s\right. \\ &\quad \left. - \alpha \int_0^t (bp_s + f(s, Z_s)) d\langle B \rangle_s\right). \end{aligned}$$

To determine the generator f , we write the process $R^{(p)}$ as the product of a G -martingale $M^{(p)}$ and an increasing process $A^{(p)}$ which is constant for a certain strategy $p^* \in \mathcal{A}$. For $t \in [0, T]$, set :

$$\begin{aligned} M_t^{(p)} &= \exp(-\alpha(x - Y_0)) \exp\left(\alpha K_t - \int_0^t \alpha(p_s - Z_s) dB_s\right. \\ &\quad \left. - \frac{1}{2} \int_0^t \alpha^2 (p_s - Z_s)^2 d\langle B \rangle_s\right), \end{aligned}$$

and

$$\tilde{A}_t^{(p)} = \exp\left(\int_0^t v(s, p_s, Z_s) d\langle B \rangle_s\right), \quad t \in [0, T],$$

with

$$v(s, p, z) = -\alpha bp - \alpha f(s, z) + \frac{1}{2} \alpha^2 |p - z|^2.$$

For the process $A^{(p)}$ to be increasing, it is sufficient to choose f such that:

$$v(t, p_t, Z_t) \geq 0 \quad \forall p \in \mathcal{A}$$

and

$$v(t, p_t^*, Z_t) = 0$$

For some particular $p^* \in \mathcal{A}$. For $t \in [0, T]$, we have:

$$\begin{aligned} \frac{1}{\alpha} v(t, p_t, Z_t) &= \frac{\alpha}{2} |p_t|^2 - \alpha p_t \left(Z_t + \frac{1}{\alpha} b\right) + \frac{\alpha}{2} |Z_t|^2 - f(t, Z_t) \\ &= \frac{\alpha}{2} \left|p_t - \left(Z_t + \frac{1}{\alpha} b\right)\right|^2 - \frac{\alpha}{2} \left|Z_t + \frac{1}{\alpha} b\right|^2 + \frac{\alpha}{2} Z_t^2 - f(t, Z_t) \\ &= \frac{\alpha}{2} \left|p_t - \left(Z_t + \frac{1}{\alpha} b\right)\right|^2 - bZ_t - \frac{1}{2\alpha} |b|^2 - f(t, Z_t). \end{aligned}$$

Now let's pose

$$f(t, z) = \frac{\alpha}{2} \text{dist}^2\left(z + \frac{1}{\alpha} b, \tilde{C}\right) - zb - \frac{1}{2\alpha} |b|^2.$$

For this choice of the generator, we obtain that $v(t, p, z) \geq 0$ for all $p \in \mathcal{A}$ and for

$$p_t^* \in \Pi_{\tilde{C}} \left(Z_t + \frac{1}{\alpha} b \right), \quad t \in [0, T],$$

we have $v(\cdot, p^*, Z) = 0$.

Theorem 3.3 *The value function of the optimization problem is given by*

$$V(x) = U_\alpha(x - Y_0) = -\exp(-\alpha(x - Y_0)),$$

where Y_0 is defined by the unique solution (Y, Z, K) of the quadratic GBSDE:

$$Y_t = F + \int_t^T f(Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad t \in [0, T], \quad (3.4)$$

with the generator defined by:

$$f(z) = \frac{\alpha}{2} \text{dist}^2 \left(z + \frac{1}{\alpha} b, \tilde{C} \right) - zb - \frac{1}{2\alpha} |b|^2.$$

There is an optimal strategy $p^* \in \mathcal{A}$, with

$$p_t^* \in \Pi_{\tilde{C}} \left(Z_t + \frac{1}{\alpha} b \right), \quad t \in [0, T]. \quad (3.5)$$

PROOF. Let us check that the generator f , defined by $f(z) = \frac{\alpha}{2} \text{dist}^2 \left(z + \frac{1}{\alpha} b, \tilde{C} \right) - zb - \frac{1}{2\alpha} |b|^2$, satisfies the hypotheses (H0), (Hc) and (Hq).

For $s \in [0, T]$, $z^1, z^2 \in \mathbb{R}^m$, we have:

$$\begin{aligned} & f(z^1) - f(z^2) \\ &= \frac{\alpha}{2} \left[\text{dist}^2 \left(z^1 + \frac{1}{\alpha} b, \tilde{C} \right) - \text{dist}^2 \left(z^2 + \frac{1}{\alpha} b, \tilde{C} \right) \right] - (z^1 - z^2) b. \end{aligned}$$

By the Lipschitz property of the distance function, one obtains the estimate

$$\begin{aligned} |f(z^1) - f(z^2)| &\leq c_1 |z^1 - z^2| + c_2 (|z^1| + |z^2|) |z^1 - z^2| \\ &\leq c_3 (1 + |z^1| + |z^2|) |z^1 - z^2|. \end{aligned}$$

Then the generator f is quadratic grown in z . The G -GSDE

$$Y_t = F + \int_t^T f(Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad t \in [0, T]$$

is quadratic and therefore admits a unique solution (Y, Z, K) where $Y \in S_G^2(0, T)$, $Z \in H_G^2(0, T)$ $\int_0^\cdot Z_s dB_s$ is a G -BMO martingale and K is a decreasing G -martingale such that $K_0 = 0$ et $K_T \in L_G^p(\Omega_T)$ for every $p \geq 1$.

Let p^* be the process constructed in Lemma 11 of [44] for $a = Z + \frac{1}{\alpha} b$. Then we have:

$$|p^*| \leq \left| Z + \frac{1}{\alpha} b \right| + \left| p^* - \left(Z + \frac{1}{\alpha} b \right) \right|.$$

By the definition of p^* there exist constants k_1, k_2 such that:

$$|p^*| \leq 2|Z_t| + \frac{2}{\alpha}|b| + k_1 \leq 2|Z_t| + k_2, \quad t \in [0, T].$$

For every $\mathbb{P} \in \mathcal{P}$ and for every stopping time $\tau \leq T$, we have:

$$\mathbb{E}^{\mathbb{P}} \left[\int_{\tau}^T |p_t^*|^2 d\langle B \rangle_s \mid \mathcal{F}_{\tau} \right] \leq \mathbb{E}^{\mathbb{P}} \left[\left(\int_{\tau}^T 8|Z_t|^2 dt + 2Tk_2^2 \right) \mid \mathcal{F}_{\tau} \right].$$

This implies that p^* is a $\text{BMO}(\mathbb{P})$ generator. Then p^* is a generator of G -BMO, from where $p^* \in \mathcal{A}$ and $v(t, p_t^*, Z_t) = 0$, so $\tilde{A}_t^{(p^*)} = 1$.

For any $p \in \mathcal{A}$, we have:

$$\begin{aligned} M_t^{(p,x)} &= \exp(-\alpha(x - Y_0)) \exp \left(\alpha K_t - \int_0^t \alpha(p_s - Z_s) dB_s \right. \\ &\quad \left. - \frac{1}{2} \int_0^t \alpha^2 (p_s - Z_s)^2 d\langle B \rangle_s \right) \\ &= \exp(-\alpha(x - Y_0)) \exp \left(- \int_0^t \alpha(p_s - Z_s) dB_s \right. \\ &\quad \left. - \frac{\alpha^2}{2} \int_0^t (p_s - Z_s)^2 d\langle B \rangle_s \right) \exp(\alpha K_t). \end{aligned}$$

Since $\int_0^\cdot (p_s - Z_s) dB_s$ is a \mathbb{P} -BMO martingale, for each $\mathbb{P} \in \mathcal{P}$, the process $N^{(p,x)}$ defined for $t \in [0, T]$ by:

$$\begin{aligned} N_t^{(p,x)} &: = \exp(-\alpha(x - Y_0)) \\ &\quad \times \exp \left(- \int_0^t \alpha(p_s - Z_s) dB_s - \frac{\alpha^2}{2} \int_0^t (p_s - Z_s)^2 d\langle B \rangle_s \right) \end{aligned}$$

is a positive symmetric G -martingale (see Lemma 2.8).

On the other hand $\mathbb{E}[e^{\alpha K_t}] \leq 1$ since K is decreasing and then by Jensen's inequality, we have that

$$\mathbb{E}[e^{\alpha K_t}] \geq e^{\mathbb{E}[\alpha K_t]} = 1.$$

This implies that $e^{\alpha K_t}$ is a G -martingale. Moreover, it belongs to L_G^p for each $p > 1$. Then by Lemma 2.10, $M^{(p,x)} = N^{(p,x)} e^K$ is a G -martingale.

Let us now show that $R^{(p)} = \tilde{A}^{(p)} M^{(p,x)}$ is a G -sub-martingale for any $p \in \mathcal{A}$. As with any $p \in \mathcal{A}$, $v(t, p_t, Z_t) \geq 0$, the process

$$\tilde{A}_t^{(p)} = \exp \left(\int_0^t v(s, p_s, Z_s) d\langle B \rangle_s \right), \quad t \in [0, T],$$

is increasing and positive, for all $s, t \in [0, T]$ $s < t$, we have

$$\frac{1}{\tilde{A}_s^{(p)}} \mathbb{E}_s \left[R_t^{(p)} \right] = \mathbb{E}_s \left[\frac{\tilde{A}_t^{(p)}}{\tilde{A}_s^{(p)}} M_t^{(p,x)} \right] \geq \mathbb{E}_s \left[M_t^{(p,x)} \right] = M_s^{(p,x)},$$

which implies that the process $R^{(p)} = \tilde{A}^{(p)} M^{(p,x)}$ is a G -submartingale for any $p \in \mathcal{A}$.

As with any $t \in [0, T]$ $\tilde{A}_t^{(p^*)} = 1$ and $M^{(p^*,x)}$ is a G -martingale, $R^{(p^*,x)} = M^{(p^*,x)}$ is a G -martingale. Hence for all $t \in [0, T]$ we have :

$$R_0(x) = \exp(-\alpha(x - Y_0)) = \mathbb{E} \left[M_t^{(p^*,x)} \right] = \mathbb{E} \left[\exp \left(-\alpha \left(X_T^{p^*} - F \right) \right) \right].$$

■

3.3 Power utility

In this section we are interested in the power utility:

$$U_\gamma(x) = -\frac{1}{\gamma}x^{-\gamma}, \quad x > 0, \quad \gamma > 0.$$

This time, the investor maximizes the expected utility of his wealth at the final time T without additional liabilities. Trading strategies are forced to take values in a closed and convex set $C \subseteq \mathbb{R}^d$. We use a somewhat different notion for strategy: $\rho = (\rho^i)_{i=1,\dots,d}$ is the share of wealth invested in action i . The number of shares of stock i is given by $\frac{\rho^i X_t}{S_t^i}$. The wealth process is defined by:

$$X_t^{(\rho)} = x + \int_0^t \sum_{i=1}^d \frac{X_s^{(\rho)} \rho_{i,s}}{S_{i,s}} dS_{i,s} = x + \int_0^t X_s^{(\rho)} \rho_s (dB_s + bd \langle B \rangle_s), \quad (3.6)$$

And the initial capital x is positive.

In order to formulate the optimization problem, let us first define the set of admissible strategies.

Definition 3.4 *The set of admissible strategies \mathcal{A} consists of all the d -dimensional stochastic processes, $\rho = (\rho_t)_{0 \leq t \leq T}$ such as $\rho_t(\omega) \in C$ and ρ is a G -BMO generator in $M_G^2(0, T)$.*

Since the wealth process is defined by the linear G-EDS:

$$X_t = x + \int_0^t \rho_s b X_s d \langle B \rangle_s + \int_0^t \rho_s X_s dB_s, \quad t \in [0, T],$$

where $x > 0$ is given, and $\rho \in M_G^2(0, T)$ is a G -BMO martingale generator. The wealth process $X^{(\rho)}$ can be written as:

$$\begin{aligned} X_t^{(\rho)} &= x \exp \left(\int_0^t \rho_s \left(b - \frac{1}{2} \rho_s \right) d \langle B \rangle_s + \int_0^t \rho_s dB_s \right) \\ &= x \mathcal{E}_t \left(\int_0^t \rho_s (dB_s + bd \langle B \rangle_s) \right), \quad t \in [0, T]. \end{aligned}$$

We assume that there is no obligation ($F = 0$). The investor is then confronted with the problem of optimization

$$\begin{aligned} \tilde{V}(x) &= \sup_{\rho \in \tilde{\mathcal{A}}} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[U \left(X_T^{(\rho)} \right) \right] \\ &= \sup_{\rho \in \tilde{\mathcal{A}}} \left[-\mathbb{E} \left[-U \left(X_T^{(\rho)} \right) \right] \right] \\ &= - \inf_{\rho \in \tilde{\mathcal{A}}} \mathbb{E} \left[-U \left(X_T^{(\rho)} \right) \right]. \end{aligned} \quad (3.7)$$

To find the value function and an optimal strategy for the problem, we apply the same method as for the exponential utility function. So we must construct a stochastic process $\tilde{R}^{(\rho)}$ with terminal value

$$\tilde{R}_T^{(\rho)} = -U \left(x + \int_0^T X_s^{(\rho)} \rho_s \frac{dS_s}{S_s} \right),$$

and of initial value $\tilde{R}_0^{(\rho)} = \tilde{R}_0^x$ which does not depend on ρ . $\tilde{R}^{(\rho)}$ must be a G -submartingale for any $\rho \in \tilde{\mathcal{A}}$ and a G -martingale for a $\rho^* \in \tilde{\mathcal{A}}$. Then the process ρ^* is the optimal strategy and the value function is given by $\tilde{V}(x) = -\tilde{R}_0^x$. The application of the power utility function to the wealth process gives us: $\forall t \in [0, T]$

$$\begin{aligned} U(X_t^{(\rho)}) &= -\frac{(X_t^{\rho,x})^{-\gamma}}{\gamma} \\ &= -\frac{x^{-\gamma}}{\gamma} \exp\left(-\int_0^t \gamma \rho_s dB_s - \int_0^t \gamma \rho_s b d\langle B \rangle_s + \frac{1}{2} \int_0^t \gamma |\rho_s|^2 d\langle B \rangle_s\right). \end{aligned}$$

This equation suggests, then the following choice of process $\tilde{R}^{(\rho)}$:

$$\tilde{R}_t^{(\rho)} = \frac{x^{-\gamma}}{\gamma} \exp\left(-\int_0^t \gamma \rho_s dB_s - \int_0^t \gamma \rho_s b d\langle B \rangle_s + \frac{1}{2} \int_0^t \gamma |\rho_s|^2 d\langle B \rangle_s + Y_t\right), \quad (3.8)$$

where (Y, Z, K) is the solution of the G -EDSR:

$$Y_t = 0 + \int_t^T f(s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad t \in [0, T]. \quad (3.9)$$

Then

$$\begin{aligned} \tilde{R}_t^{(\rho)} &= \frac{x^{-\gamma} e^{Y_0}}{\gamma} \exp\left(\int_0^t (Z_s - \gamma \rho_s) dB_s \right. \\ &\quad \left. + \int_0^t \left(\frac{1}{2} \gamma |\rho_s|^2 - f(s, Z_s) - \gamma b \rho_s\right) d\langle B \rangle_s + K_t\right) \\ &= \frac{x^{-\gamma} e^{Y_0}}{\gamma} \exp\left(\int_0^t (Z_s - \gamma \rho_s) dB_s - \frac{1}{2} \int_0^t |Z_s - \gamma \rho_s|^2 d\langle B \rangle_s + K_t\right) \\ &\quad \times \exp\left(\int_0^t \left(\frac{1}{2} \gamma |\rho_s|^2 + \frac{1}{2} |Z_s - \gamma \rho_s|^2 - f(s, Z_s) - \gamma b \rho_s\right) d\langle B \rangle_s\right). \end{aligned}$$

In order to obtain the G -submartingale property of the process $\tilde{R}^{(\rho)}$, we must construct the generator f of the G -BSDE (3.9) such that, for $t \in [0, T]$,

$$\frac{1}{2} \gamma |\rho_s|^2 + \frac{1}{2} |Z_s - \gamma \rho_s|^2 - f(s, Z_s) - \gamma b \rho_s \geq 0, \quad \forall \rho \in \tilde{\mathcal{A}}. \quad (3.10)$$

The process $\tilde{R}^{(\rho^*)}$ Will be a G -martingale if we have equality for strategy $\rho^* \in \tilde{\mathcal{A}}$. This is equivalent to

$$f(t, Z_t) \leq \frac{\gamma(1+\gamma)}{2} \left| \rho_t - \frac{Z_t + b}{1+\gamma} \right|^2 - \frac{\gamma |Z_t + b|^2}{2(1+\gamma)} + \frac{|Z_t|^2}{2},$$

Therefore, the choice of the function f is:

$$f(t, z) = \frac{\gamma(1+\gamma)}{2} \text{dist}^2\left(\frac{1}{1+\gamma}(z+b), C\right) - \frac{\gamma |z+b|^2}{2(1+\gamma)} + \frac{|z|^2}{2},$$

and a candidate for optimal strategy must satisfy

$$\rho_t^* \in \Pi_C\left(\frac{1}{1+\gamma}(Z_t + b)\right), \quad t \in [0, T].$$

We have the following theorem for the value function and the optimal strategy.

Theorem 3.5 *The value function of the optimization problem is given by*

$$V(x) = -\frac{x^{-\gamma}}{\gamma} \exp(Y_0), \text{ where } x > 0,$$

Y_0 is defined by the unique solution (Y, Z, K) of the quadratic G -BSDE:

$$Y_t = 0 + \int_t^T f(s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad t \in [0, T], \quad (3.11)$$

such as $Y \in S_G^2(0, T)$, $Z \in M_G^2(0, T)$, $\int_0^T Z_s dB_s$ is a G -BMO martingale and K is a decreasing G -martingale such that $K_0 = 0$ and $K_T \in L_G^2(\Omega_T)$, with

$$f(t, z) = \frac{\gamma(1+\gamma)}{2} \text{dist}^2 \left(\frac{1}{1+\gamma}(z+b), C \right) - \frac{\gamma|z+b|^2}{2(1+\gamma)} + \frac{|z|^2}{2}.$$

There is an optimal strategy $\rho^* \in \tilde{\mathcal{A}}$ satisfying

$$\rho_t^* \in \Pi_C \left(\frac{1}{1+\gamma}(Z_t + b) \right). \quad (3.12)$$

PROOF. Since the generator $(f(t, z))_{t \in [0, T]}$ is with quadratic growth in z , the G -BSDE (3.11) admits a unique solution (Y, Z, K) where $Y \in S_G^2(0, T)$, $Z \in H_G^2(0, T)$, $\int_0^T Z_s dB_s$ is a G -BMO martingale and the process K is a decreasing G -martingale such that $K_0 = 0$ and $K_T \in L_G^p(\Omega_T)$ for $p \geq 1$.

By the same method as in the case of exponential utility we have $\rho^* \in \tilde{\mathcal{A}}$. So, for all $\rho \in \tilde{\mathcal{A}}$, we have the decomposition of the process $\tilde{R}^{(\rho)}$:

$$\begin{aligned} \tilde{R}_t^{(\rho)} &= \frac{x^{-\gamma} e^{Y_0}}{\gamma} \exp \left(K_t + \int_0^t (\gamma \rho_s + Z_s) dB_s - \frac{1}{2} \int_0^t |\gamma \rho_s + Z_s|^2 d\langle B \rangle_s \right) \\ &\quad \times \exp \left(\int_0^t \nu_s^{(\rho)} d\langle B \rangle_s \right). \end{aligned}$$

where

$$\nu_s^{(\rho)} = \frac{\gamma(1+\gamma)}{2} \left[\left| \frac{1}{1+\gamma}(Z_s + b) - \rho_s \right|^2 - \text{dist}^2 \left(\frac{1}{1+\gamma}(Z_s + b), C \right) \right].$$

We have for each $\rho \in \tilde{\mathcal{A}}$ the process M defined by:

$$M_t = \frac{x^{-\gamma}}{\gamma} e^{Y_0} \exp \left(K_t + \int_0^t (\gamma \rho_s + Z_s) dB_s - \frac{1}{2} \int_0^t |\gamma \rho_s + Z_s|^2 d\langle B \rangle_s \right)$$

is a G -martingale.

Since $\nu^{(\rho^*)} \equiv 0$, $\tilde{R}^{(\rho^*)}$ is a G -martingale and we have:

$$\tilde{R}_0 = \frac{x^{-\gamma}}{\gamma} e^{Y_0} = \mathbb{E} \left[\tilde{R}_t^{(\rho^*)} \right], \quad \forall t \in [0, T].$$

On the other hand, for all $\rho \in \tilde{\mathcal{A}}$, we have $\nu_s^{(\rho)} \geq 0$, therefore the process defined by

$$A_t = \exp \left(\int_0^t \nu_s^{(\rho)} d\langle B \rangle_s \right)$$

is increasing. So for each $\rho \in \tilde{\mathcal{A}}$, $\tilde{R}^{(\rho)}$ is a G -submartingale, hence the result:

$$-\mathbb{E} \left[-U \left(X_T^{(\rho, x)} \right) \right] \leq -\tilde{R}_0^{(x)} = -\frac{x^{-\gamma}}{\gamma} \exp(Y_0) \quad \forall \rho \in \tilde{\mathcal{A}}.$$

■

3.4 Logarithmic utility

In this section we will consider the logarithmic utility function. As in the previous section, the agent has no obligation at time T . The strategies and the wealth process have the same meaning as in the case of utility power. The wealth process is defined by:

$$X_t^{(\tilde{\rho})} = x + \int_0^t \sum_{i=1}^d \frac{X_s^{(\tilde{\rho})} \tilde{\rho}_{i,s}}{S_{i,s}} dS_{i,s} = x + \int_0^t X_s^{(\tilde{\rho})} \tilde{\rho}_s (dB_s + b_s d\langle B \rangle_s),$$

and an initial capital x is positive. So for $\tilde{\rho} \in M_G^2(0, T)$, the process of wealth $X^{(\tilde{\rho})}$ can be written as:

$$\begin{aligned} X_t^{(\tilde{\rho})} &= x \exp \left(\int_0^t \tilde{\rho}_s (b_s - \frac{1}{2} \tilde{\rho}_s) d\langle B \rangle_s + \int_0^t \tilde{\rho}_s dB_s \right) \\ &= x \mathcal{E}_t \left(\int_0^t \tilde{\rho}_s (dB_s + b_s d\langle B \rangle_s) \right), \quad t \in [0, T]. \end{aligned}$$

Strategies ρ are constrained to take their values in a closed and convex set $\tilde{C} \subset \mathbb{R}^d$. In order to compare the logarithmic utility of the terminal richness of two strategies, we impose a weak condition of integrability on ρ . Recall that $\rho^i > 1$ means that the investor must borrow money to buy shares i and if $\rho^i < 0$, then the investor has a negative number of share i . Thus, an integrability condition on ρ is not restrictive.

Definition 3.6 *The set of admissible strategies \mathcal{A}_l consists of all the processes $\rho \in H_G^2(0, T, \mathbb{R}^d)$ such as $\rho_t \in \tilde{C}$ q.s.*

For the logarithmic utility function,

$$U(x) = \log(x), \quad x > 0,$$

we obtain a relatively simple G -BSDE which leads to the value function and the optimal strategy. The problem of optimization is given by

$$\begin{aligned} V(x) &= \sup_{\rho \in \mathcal{A}_l} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\log \left(X_T^{(\rho)} \right) \right] \\ &= - \inf_{\rho \in \mathcal{A}_l} \mathbb{E} \left[- \log \left(X_T^{(\rho)} \right) \right] \\ &= \log(x) - \inf_{\rho \in \mathcal{A}_l} \mathbb{E} \left[- \int_0^T \rho_s dB_s - \int_0^T \left(\rho_s b_s - \frac{1}{2} |\rho_s|^2 \right) d\langle B \rangle_s \right], \end{aligned} \tag{3.13}$$

Where the initial capital x is positive. As in the case of power utility, we determine a process $R^{(\rho)}$ with $R_T^{(\rho)} = \log(x) - \log(X_T^{(\rho)})$ and an initial value that does not depend on ρ . In addition, $R^{(\rho)}$ is a G -submartingale for each $\rho \in \mathcal{A}_l$, and there is a strategy $\rho^* \in \mathcal{A}_l$ such as $R^{(\rho^*)}$ is a G -martingale. The strategy ρ^* is then the optimal strategy and $\log(x) - R_0^{\rho^*}$ is the value function of the optimization problem (3.13).

We can choose, for $t \in [0, T]$,

$$\begin{aligned} R_t^{(\rho)} &= Y_t - \int_0^t \rho_s dB_s - \int_0^t \left(\rho_s b_s - \frac{1}{2} |\rho_s|^2 \right) d\langle B \rangle_s \\ &= Y_0 + \int_0^t (Z_s - \rho_s) dB_s - \int_0^t \left(\rho_s b_s - \frac{1}{2} |\rho_s|^2 + f(s) \right) d\langle B \rangle_s + K_t \\ &= Y_0 + \int_0^t (Z_s - \rho_s) dB_s + \int_0^t \left(\frac{1}{2} |\rho_s - b_s|^2 - \frac{1}{2} |b_s|^2 - f(s) \right) d\langle B \rangle_s + K_t, \end{aligned}$$

where (Y, Z, K) is the solution of the linear G -BSDE:

$$Y_t = 0 + \int_t^T f(s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad t \in [0, T], \quad (3.14)$$

of generator f defined by:

$$f(t) = \frac{1}{2} \text{dist}^2(b_t, \tilde{C}) - \frac{1}{2} |b_t|^2, \quad t \in [0, T].$$

For $0 \leq t_1 \leq t_2 \leq T$, we have

$$\begin{aligned} \mathbb{E}_{t_1} [R_{t_2}^{(\rho)}] &= \mathbb{E}_{t_1} \left[Y_0 + \int_0^{t_2} (Z_s - \rho_s) dB_s + K_{t_2} + \frac{1}{2} \int_0^{t_2} (|\rho_s - \theta_s|^2 - \text{dist}^2(\theta_s, \tilde{C})) d\langle B \rangle_s \right] \\ &\geq \mathbb{E}_{t_1} \left[Y_0 + \int_0^{t_2} (Z_s - \rho_s) dB_s + K_{t_2} + \frac{1}{2} \int_0^{t_1} (|\rho_s - \theta_s|^2 - \text{dist}^2(\theta_s, \tilde{C})) d\langle B \rangle_s \right] \\ &= Y_0 + \int_0^{t_1} (Z_s - \rho_s) dB_s + K_{t_1} + \frac{1}{2} \int_0^{t_1} (|\rho_s - \theta_s|^2 - \text{dist}^2(\theta_s, \tilde{C})) d\langle B \rangle_s \\ &= R_{t_1}^{(\rho)}. \end{aligned}$$

Hence, $R^{(\rho)}$ is a G -submartingale for all $\rho \in \mathcal{A}_t$. An optimal strategy ρ^* , which satisfies $\rho_t^* \in \Pi_{\tilde{C}}(b_t)$, may be constructed by means of Lemma 11 of [44]. The initial value Y_0 of the linear G -BSDE (3.14), satisfied:

$$Y_0 = -\mathbb{E} \left[\int_0^T f(s) d\langle B \rangle_s \right].$$

consequently,

$$V(x) = \log(x) - R_0^{\rho^*}(x) = \log(x) + \mathbb{E} \left[\int_0^T f(s) d\langle B \rangle_s \right].$$

In particular, ρ^* depends only on b and the set \tilde{C} describing the constraints on trading strategies.

Chapter 4

Reflected G -SDEs in non-convex domains

Abstract: In this chapter, we first review the penalization method for solving deterministic Skorokhod problems in non-convex domains and establish estimates for problems with α -Hölder continuous functions. With the help of these results obtained previously for deterministic problems, we pathwisely define the reflected G -Brownian motion and prove its existence and uniqueness in a Banach space. Finally, multi-dimensional reflected stochastic differential equations driven by G -Brownian motion are investigated via a fixed-point argument.

Key words. Hölder continuity, Skorokhod problem, Penalization method, G -Brownian motion, Stochastic differential equations, non-convex reflecting boundaries.

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4.1 Introduction

Skorokhod [102, 103] first introduced diffusion processes with reflecting boundaries in the 1960s. Since then, reflected solutions to stochastic differential equations (SDEs) have been investigated by many authors. For the one-dimensional case, El Karoui [29] and Yamada [115] studied reflected SDEs on a half-line. The multidimensional case, goes back to Stroock and Varadhan [111] which have established the existence of weak solutions to reflected SDEs on a smooth domain. Subsequently, Tanaka [112] proved the existence and uniqueness of solutions in convex domain with normal reflection by a direct approach based on the solution to the Skorokhod problem. Furthermore, these results were generalized to non-convex domain by, in particular, Lions and Sznitman [65] and Saisho [97]. By imposing an admissibility condition on the domain, Lions and Sznitman [65] proved the existence and uniqueness of solutions of the above SDE in two different cases. They first consider the case of normal reflection on domains satisfying a uniform exterior sphere condition. Moreover, they consider oblique directions of reflection on smooth domains. In addition, for smoothly varying directions of reflection on domains satisfying a uniform exterior sphere condition, existence and uniqueness results were obtained in the special case when the oblique reflection cone can be transformed into the normal cone by multiplication by a smooth matrix function. Saisho [97] later showed that in the case of normal reflection, the admissibility condition is not necessary and can be removed. Moreover, we note that in the case of an orthant with constant directions of oblique reflection on the sides, Harrison and Reiman [46] found sufficient conditions for the existence and uniqueness of solutions to the Skorokhod problem as well as for continuity of the reflection map. In this context we also mention that Bernard and El Kharroubi [8] provided necessary and sufficient conditions for the existence of solutions to the Skorokhod problem in an orthant with constant directions of reflection on each face. The most general results so far concerning the existence of solutions to the Skorokhod problem with oblique reflection were derived by Costantini [17]. We emphasize that the literature devoted to Skorokhod problems, their extensions and applications is much larger than what is conveyed above and, in fact, many more researcher have contributed to this rich field.

Since the publication of the work of Choquet (1955) [16], the theory of nonlinear expectation has attracted the researchers' great interest for its potential applications in uncertainty problems, risk measures and the superhedging in finance. Peng [78, 79] constructed a kind of dynamically consistent fully nonlinear expectations through PDE approach. An important case, of time-consistent nonlinear expectation, is the G -expectation, in which the corresponding canonical process $(B_t)_{t \geq 0}$ is called G -Brownian motion analogous to the classical Wiener process see [80, 81, 82, 83, 84]. Under the G -expectation framework, the corresponding stochastic calculus of Itô's type was established in [81, 82, 83]. In order to solve the super-replication problem in an uncertainty volatility model, Denis and Martini [23] independently introduced a notion of upper expectation and the related capacity theory. Moreover, a stochastic integral of Itô's type under a class of non-dominated probability measures was formulated. Recently, Denis et al. [20] found there is a strong link that connects these two frameworks, that is, the G -expectation $\mathbb{E}[\cdot]$ can be represented by a concrete weakly compact family \mathcal{P}_G of probability measures on $(\Omega, \mathcal{B}(\Omega))$:

$$\mathbb{E}[X] = \sup_{\mathbb{P} \in \mathcal{P}_G} E^{\mathbb{P}}[X], \text{ for } X \in L_G^1(\Omega).$$

Then, a Choquet capacity $\bar{C}(\cdot)$ can be naturally introduced to the G -framework:

$$\bar{C}(A) := \sup_{\mathbb{P} \in \mathcal{P}_G} \mathbb{P}(A), \quad A \in \mathcal{B}(\Omega),$$

by which we can have the following definition to the concept of “quasi-surely”, similar to the one in Denis and Martini [23]: A set $A \subset \Omega$ is polar if $\bar{C}(A) = 0$; and a property holds “quasi-surely” (q.s. for short) if it holds outside a polar set. Moreover, in [20] they gave a characterization of the space $L_G^p(\Omega)$ and proved that every random variable in $L_G^p(\Omega)$ is quasi-continuous. In this G -framework, a stochastic integral of Itô type is defined following a usual procedure, that is, giving a definition first for some simple integrands and then completing the spaces of integrands under the norm induced by the upper expectation related to \mathcal{P}_G . This norm is much stronger than that in the classical case and thus, the space of integrands is smaller than the classical one. In other words, some additional regularity assumption should be imposed on the integrands to ensure that the integrals are well defined. Using these notions of stochastic calculus in the G -framework, the existence and uniqueness results for some types of SDEs driven by G -Brownian motion (GSDEs) can be obtained (cf. Peng [83], Gao [35] and Lin and Bai [4]). For the reason stated above, the authors who studied GSDEs always assumed the following condition on the coefficients of the equations: for each $x \in \mathbb{R}$,

$$f(x), g(x) \in M_G^2([0, T]).$$

At this price, all results in the works for GSDEs listed above hold in the “quasi-surely” (q.s.) sense, i.e., outside a polar set, and all the processes are immediately aggregated. Recently, Lin [62] proved the existence and uniqueness of the solution of the scalar valued stochastic differential equations driven by G -Brownian motion with reflecting boundary conditions, and in [61] he considered multidimensional reflected problems in a convex domain.

The aim of this chapter is to study the solvability of multidimensional stochastic differential equations driven by G -Brownian motion with reflecting boundary conditions when the domain is non-convex. The multidimensional RGSDE that we consider is defined as following:

$$\begin{aligned} X_t &= x_0 + \int_0^t f(s, X_s) ds + \int_0^t h^{ij}(s, X_s) d\langle B^i, B^j \rangle_s \\ &\quad + \int_0^t g^j(s, X_s) dB_s^j + K_t, \quad 0 \leq t \leq T, \text{ q.s.}; \end{aligned} \quad (4.1)$$

$$\begin{aligned} X_t &\in \bar{D}, \quad 0 \leq t \leq T, \text{ q.s.}; \\ |K|_t &= \int_0^t \mathbf{1}_{(X_s \in \partial \bar{D})} d|K|_s, \text{ q.s.}; \\ K_t &= \int_0^t \xi_s d|K|_s, \text{ q.s. with } \xi_s \in \mathbf{n}(X_s), \end{aligned}$$

where $\langle B, B \rangle$ is the covariation matrix process of the d -dimensional G -Brownian motion B , D is a domain of \mathbb{R}^d and the increasing process K is minimal in the sense that it grows only when X is at the boundary of the domain D and when it grows, it is normal to D in X . We first consider the G -Brownian motion, that is to say when $f = h = 0$ and $g = 1$, in this case we follow the method of Saisho and Tanaka in [101], which consists in using the deterministic Skorohod problem To show the trajectorial existence and a method of penalization to show that the trajectorial solution limits a sequence of elements of $M_G^2([0, T])$. From this, We deduce the existence of a unique pair $(X, K) \in M_G^2([0, T]; \bar{D}) \times M_G^2([0, T]; \mathbb{R}^d)$ solution of the Skorohod problem

$$X_t = x_0 + \int_0^t \alpha_s ds + \int_0^t \eta_s^{ij} d\langle B^i, B^j \rangle_s + \int_0^t \beta_s^j dB_s^j + K_t,$$

where α , η^{ij} , β^j are bounded element of $M_G^2([0, T]; \mathbb{R}^d)$. Then we establish a priori estimates, from which we deduce the uniqueness of the solution of (4.1), whose existence is proved by the point method by the iteration of Picard.

The chapter is organized as follows. In Section 3.2, we introduce some preliminaries of the deterministic Skorohod problem. In Section 3.3 we present our main results and Section 3.4 is devoted to proves of main results.

4.2 Preliminaries

In this section, we shall briefly discuss the deterministic Skorohod problem in non-convex domains and the sufficient conditions for its solvability.

4.2.1 Conditions on the domain

In order to investigate the reflected G -Brownian motion in this chapter, we shall first recall the results in [101] for the deterministic Skorohod problem in a domain $D \subset \mathbb{R}^d$, $d \in \mathbb{N}^*$. In that chapter, the following conditions are assumed:

CONDITION (A). For $x \in \partial D$, we denote

$$\mathcal{N}_{x,r} = \{\mathbf{n} \in \mathbb{R}^d : |\mathbf{n}| = 1, B(x - r\mathbf{n}, r) \cap D = \emptyset\}, \quad r > 0 \quad \text{and} \quad \mathcal{N}_x = \bigcup_{r>0} \mathcal{N}_{x,r},$$

where $B(z, r) := \{y \in \mathbb{R}^d : |y - z| < r\}$, for $z \in \mathbb{R}^d$. We assume that there exists a constant $r_0 > 0$ such that $\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset$, for all $x \in \partial D$.

CONDITION (B). Assume that there exist constants $\delta > 0$ and $\beta \in [1, \infty)$ such that for any $x \in \partial D$, we can find a unit vector l_x such that

$$\langle l_x, \mathbf{n} \rangle \geq \frac{1}{\beta}, \quad \text{for all } \mathbf{n} \in \bigcup_{y \in B(x, \delta) \cap \partial D} \mathcal{N}_y,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in \mathbb{R}^d .

Throughout this chapter, we consider a domain $D \subset \mathbb{R}^d$ satisfying both Condition (A) and (B). It is easy to see for each $x \in \mathbb{R}^d$ such that $\text{dist}(x, \overline{D}) < r_0$, there exists a unique $\bar{x} \in \overline{D}$ with $|x - \bar{x}| = \text{dist}(x, \overline{D})$. If $x \notin \overline{D}$ we have $\bar{x} \in \partial D$ and $\frac{\bar{x} - x}{|\bar{x} - x|} \in \mathcal{N}_{\bar{x}}$ (see e.g. Remark 1.3 in [97]). We keep this notation \bar{x} for the projection of x on D in the remainder of this chapter.

To consider the solvability of reflected multi-dimensional Skorohod stochastic differential equations on the domain D , we assume furthmore

CONDITION (C). There exists a bounded function $\Psi \in C_b^2(\mathbb{R}^d)$ whose first and second derivatives are also bounded, and there exists a $\delta' > 0$, such that

$$\forall x \in \partial D, \forall y \in \overline{D}, \forall \mathbf{n} \in \mathcal{N}_x, \langle y - x, \mathbf{n} \rangle + \frac{1}{\delta'} \langle \nabla \Psi(x), \mathbf{n} \rangle |y - x|^2 \geq 0.$$

We note the bound of Ψ and its derivatives by L_Ψ . This condition is critical for proving the a priori estimate (4.22), which is similar to its analogue in §3 of [65].

4.2.2 Deterministic Skorohod problem

We recall here the result of deterministic Skorohod problems on the domain D satisfying Conditions (A) and (B), which are proved in [97]. This result is our starting point of this chapter.

Assume that ϕ is a continuous function taking values in \mathbb{R}^d and that ϕ is of bounded variation over each finite interval. We denote by $|\phi|_t$ the total variation of ϕ over $[0, t]$, i.e.,

$$|\phi|_t = \sup_{0=t_1 < t_2 < \dots < t_n = t, n \in \mathbb{N}} \sum_{k=1}^n |\phi_{t_k} - \phi_{t_{k-1}}|.$$

We also note $|\phi|_t^s = |\phi|_t - |\phi|_s$, $0 \leq s \leq t$.

For a continuous function w defined on $[0, T]$, $T > 0$, taking values in \mathbb{R}^d with $w(0) = 0$ and for $x_0 \in \overline{D}$, we consider the Skorohod equation below:

$$\xi_t = x_0 + w_t + \phi_t, \quad t \in [0, T]. \quad (4.2)$$

Definition 4.1 We call a couple of functions (ξ, ϕ) solution of (4.2), if it satisfies (4.2) and the following conditions:

1. The function ξ is continuous and takes values in \overline{D} ;
2. The function ϕ is continuous and takes values in \mathbb{R}^d with $\phi(0) = 0$. Moreover, it is of bounded variation over $[0, T]$ and for all $t \in [0, T]$,

$$\begin{aligned} \phi_t &= \int_0^t \mathbf{n}_s d|\phi|_s; \\ |\phi|_t &= \int_0^t 1_{\{\xi_s \in \partial D\}} d|\phi|_s, \end{aligned}$$

where $\mathbf{n}_s \in \mathcal{N}_{\xi_s}$, if $\xi_s \in \partial D$.

Theorem 4.2 (Theorem 4.1 in [97]) Suppose that the domain $D \subset \mathbb{R}^d$ is open and satisfies Conditions (A) and (B). Then there exists a unique solution (ξ, ϕ) for the deterministic Skorohod problem (4.2).

4.3 Main results

In this section, we present our main results on the reflected G -Brownian motion and on reflected stochastic differential equations driven by G -Brownian motion.

4.3.1 Reflected G -Brownian motion

We replace the deterministic function w in the Skorohod problem (4.2) by the G -Brownian motion B and establish the following equation in the “quasi-sure” sense:

$$X_t = x_0 + B_t + K_t, \quad x_0 \in \overline{D}, \quad 0 \leq t \leq T. \quad (4.3)$$

Definition 4.3 We call a couple of processes (X, K) solves the Skorohod problem for the G -Brownian motion (4.3), if there exists a polar set A , such that

1. The processes X et K belong to $M_G^2([0, T]; \mathbb{R}^d)$, and for all $\omega \in A^c$,

$$X_t(\omega) = x_0 + B_t(\omega) + K_t(\omega), \quad 0 \leq t \leq T;$$

2. For all $\omega \in A^c$, $X(\omega)$ is continuous and takes values in \bar{D} ;

3. For all $\omega \in A^c$, $K(\omega)$ is continuous and takes values in \mathbb{R}^d with $K_0(\omega) = 0$. Moreover, $K(\omega)$ is of bounded variation over $[0, T]$ and for all $t \in [0, T]$,

$$\begin{aligned} K_t(\omega) &= \int_0^t \mathbf{n}_s(\omega) d|K|_s(\omega); \\ |K|_t(\omega) &= \int_0^t \mathbf{1}_{\{X_s(\omega) \in \partial D\}} d|K|_s(\omega), \end{aligned}$$

where $\mathbf{n}_s(\omega) \in \mathcal{N}_{X_s(\omega)}$, if $X_s(\omega) \in \partial D$.

In addition, we call X reflected G -Brownian motion on the domain D .

We have the following existence and uniqueness theorem for the reflected G -Brownian motion. The proof of this theorem is postponed to the next section.

Theorem 4.4 *Suppose that the domain $D \subset \mathbb{R}^d$ satisfies Conditions (A) and (B). Then there exists a couple $(X, K) \in (M_G^2([0, T]; \mathbb{R}^d) \times M_G^2([0, T]; \mathbb{R}^d))$ which solves the Skorohod problem (4.3) whenever $x_0 \in \bar{D}$. Moreover, if the problem (4.3) admits two solutions (X, K) and (X', K') , then there exists a polar set \tilde{A} , such that for all $\omega \in \tilde{A}^c$,*

$$X(\omega) = X'(\omega) \quad \text{and} \quad K(\omega) = K'(\omega), \quad 0 \leq t \leq T.$$

4.3.2 Reflected stochastic differential equations driven by G -Brownian motion

In addition to the reflected G -Brownian motion, we shall study reflected stochastic differential equations driven by G -Brownian motion, which is formulated as

$$\begin{aligned} X_t = x_0 + \int_0^t f(s, X_s) ds + \int_0^t h^{ij}(s, X_s) d\langle B^i, B^j \rangle_s \\ + \int_0^t g^j(s, X_s) dB_s^j + K_t, \quad 0 \leq t \leq T, \quad \text{q.s..} \end{aligned} \quad (4.4)$$

Here we adopt the Einstein summation convention. In (4.4), the process $\langle B, B \rangle$ is the covariation matrix of the d -dimensional G -Brownian motion B . In what follows, we assume that the functions f, h, g satisfy the following conditions:

Assumption 4.5 *The functions $f, h^{ij}, g^j : \Omega \times [0, T] \times \bar{D} \rightarrow \mathbb{R}^d$, $i, j = 1, 2, \dots, d$, are functions such that*

(H1) *For all $x \in \bar{D}$, the processes $f(\cdot, x), h^{ij}(\cdot, x), g^j(\cdot, x)$ belong to $M_G^2([0, T]; \mathbb{R}^d)$;*

(H2) *The functions f, h^{ij}, g^j are uniformly bounded by L_0 and uniformly L_0 -Lipschitz, i.e., there exists a constant $L_0 > 0$ such that for all $(\omega, t) \in \Omega \times [0, T]$,*

$$\|f(t, x) - f(t, y)\| + \|h^{ij}(t, x) - h^{ij}(t, y)\| + \|g^j(t, x) - g^j(t, y)\| \leq L_0 \|x - y\|, \quad \forall x, y \in \bar{D}.$$

where $\|\cdot\|$ denotes the Hilbert-Schmidt norm for matrices.

Definition 4.6 We call a couple of processes (X, K) solves the Skorohod stochastic differential equation (4.4), if there exists a polar set A , such that

1. The processes X and K belong to $M_G^2([0, T]; \mathbb{R}^d)$ and satisfies (4.4);
2. For all $\omega \in A^c$, $X(\omega)$ takes values in \bar{D} ;
3. For all $\omega \in A^c$, $K(\omega)$ takes values in \mathbb{R}^d with $K_0(\omega) = 0$. Moreover, $K(\omega)$ is of bounded variation over $[0, T]$ and for all $t \in [0, T]$,

$$\begin{aligned} K_t(\omega) &= \int_0^t \mathbf{n}_s(\omega) d|K|_s(\omega); \\ |K|_t(\omega) &= \int_0^t \mathbf{1}_{\{X_s(\omega) \in \partial D\}} d|K|_s(\omega), \end{aligned}$$

where $\mathbf{n}_s(\omega) \in \mathcal{N}_{X_s(\omega)}$, if $X_s(\omega) \in \partial D$.

Using a fixed point type argument, we shall prove in the next section the following existence and uniqueness theorem for the Skorohod stochastic differential equation (4.4).

Theorem 4.7 Suppose that the domain $D \subset \mathbb{R}^d$ satisfies Conditions (A) and (B). Then there exists a unique couple $(X, K) \in (M_G^2([0, T]; \mathbb{R}^d) \times M_G^2([0, T]; \mathbb{R}^d))$ which solves the Skorohod stochastic differential equation (4.4) whenever $x_0 \in \bar{D}$ and the coefficients f, h, g satisfy Assumptions (H1) and (H2).

4.4 Proofs

In this section, we shall prove Theorem 4.4 and 4.7. First, we recall the results for deterministic Skorohod problem in Saisho and Tanaka [101] and provide an estimate when the function w is α -Hölder continuous, $\alpha \in (0, 1/2)$.

4.4.1 Estimates for the deterministic Skorohod problem

In (4.2), we assume in addition that w is an α -Hölder continuous function on $[0, T]$, where $\alpha \in (0, 1/2)$, i.e.,

$$\|w\|_\alpha = \sup_{0 \leq s < t \leq T} \frac{|w_t - w_s|}{|t - s|^\alpha} < \infty.$$

Set

$$\Delta_{s,t}(w) := \sup \{|w_{t_2} - w_{t_1}| : s \leq t_1 < t_2 \leq t\},$$

and $\|w\|_T := \sup\{|w|_t : 0 \leq t \leq T\}$.

We recall the penalization method in [65, 101] and define a sequence of equations: for $m \in \mathbb{N}^*$,

$$\xi_t^m = x_0 + w_t - \frac{m}{2} \int_0^t \nabla U(\xi_s^m) ds, \quad (4.5)$$

where $x_0 \in \bar{D}$ and U is a function satisfying:

- (a) $U \in C^1(\mathbb{R}^d)$ and $U \geq 0$;
- (b) $U(x) = |x - \bar{x}|^2$, if $\text{dist}(x, \bar{D}) \leq r_0/2$ (r_0 is from Condition (A));

(c) ∇U is bounded and Lipschitz.

Indeed, the existence of such function U is ensured by Condition (A) and (B). We denote by $2L$ the Lipschitz constant of U . The equation (4.5) admits a unique solution ξ^m which is continuous on $[0, T]$.

For each $m \in \mathbb{N}^*$, we define

$$\varepsilon_m^\alpha(w) := \frac{12e^L}{m^\alpha} \|w\|_\alpha. \quad (4.6)$$

For $\varepsilon > 0$, we note

$$D_\varepsilon := \left\{ x \in \mathbb{R}^d, \text{dist}(x, \bar{D}) < \varepsilon \right\}.$$

Additionally, set

$$\tilde{\xi}_t^m := \xi_{t/m}^m,$$

it is easy to verify that

$$\tilde{\xi}_t^m = x_0 + \tilde{w}_{t/m} - \frac{1}{2} \int_0^t \nabla U(\tilde{\xi}_s^m) ds, \quad 0 \leq t \leq mT.$$

Similar to Lemma 4.2 in [101], we have the lemma below.

Lemma 4.8 *Assume that the domain $D \subset \mathbb{R}^d$ satisfies Conditions (A) and (B). For each $m \in \mathbb{N}^*$ such that $0 < \varepsilon_m^\alpha(w) \leq r_0/2$ and for all $m' \geq m$, if there exists $u \in (0, m'T)$ such that $\tilde{\xi}_u^{m'} \in \partial D_{\varepsilon_m^\alpha(w)/2}$, then $\{\tilde{\xi}_t^{m'}\}_{u \leq t \leq T}$ hits $\partial D_{\varepsilon_m^\alpha(w)/3}$ before hitting $\partial D_{\varepsilon_m^\alpha(w)}$.*

Proof. This lemma can be proved by slightly modifying the proof of Lemma 4.2 in [101]. Precisely, we could verify that for the given ε_m^α , the number m itself is large enough to ensure Lemma 4.2 in [101] if $\varepsilon_m^\alpha(w)$ is defined as (4.6). For the convenience of the reader, we briefly prove this lemma.

Fix $m' \geq m$ and suppose that $\tilde{\xi}_u^{m'} \in \partial D_{\varepsilon_m^\alpha(w)/2}$. Consider the auxiliary equation:

$$\eta_t = \tilde{\xi}_u^{m'} - \frac{1}{2} \int_u^t \nabla U(\eta_s) ds, \quad t \geq u.$$

From Lemma 4.1 in [101], the above equation is solved by

$$\eta_t = \overline{\tilde{\xi}_u^{m'}} - \left(\overline{\tilde{\xi}_u^{m'}} - \tilde{\xi}_u^{m'} \right) \exp\{-(t-u)\},$$

and the function η satisfies that for all $t \geq u$,

$$|\overline{\eta}_t - \eta_t| = \frac{\varepsilon_m^\alpha(w)}{2} \exp\{-(t-u)\}.$$

Then, we denote

$$u' := \inf \{t > u : \eta_t \in \partial D_{\varepsilon_m^\alpha(w)/4}\},$$

and it is obvious that $u' = u + \log 2 < u + 1$.

On the other hand, for $u \leq t \leq m'T$, we have

$$\tilde{\xi}_t^{m'} - \eta_t = w_{t/m'} - w_{u/m'} - \frac{1}{2} \int_u^t \left(\nabla U(\tilde{\xi}_s^{m'}) - \nabla U(\eta_s) \right) ds,$$

which implies

$$\left| \tilde{\xi}_t^{m'} - \eta_t \right| \leq |w_{t/m'} - w_{u/m'}| + L \int_u^t \left| \tilde{\xi}_s^{m'} - \eta_s \right| ds. \quad (4.7)$$

- If $u' \leq m'T$, then for all $u \leq t \leq u'$, we can deduce that

$$|w_{t/m'} - w_{u/m'}| \leq \frac{\|w\|_\alpha}{m'^\alpha} \leq \frac{\|w\|_\alpha}{m^\alpha} = \frac{\varepsilon_m^\alpha(w)}{12e^L}.$$

We apply Gronwall's lemma to (4.7) and obtain

$$\left| \tilde{\xi}_t^{m'} - \eta_t \right| \leq \frac{\varepsilon_m^\alpha(w)}{12e^L} e^{L(t-u)} < \frac{\varepsilon_m^\alpha(w)}{12}, \quad u \leq t \leq u'.$$

Therefore, for $u \leq t \leq u'$,

$$\left| \overline{\tilde{\xi}_t^{m'}} - \tilde{\xi}_t^{m'} \right| \leq |\overline{\eta}_t - \eta_t| + \left| \tilde{\xi}_t^{m'} - \eta_t \right| < \frac{\varepsilon_m^\alpha(w)}{2} + \frac{\varepsilon_m^\alpha(w)}{12} < \varepsilon_m^\alpha(w),$$

whereas

$$\left| \overline{\tilde{\xi}_{u'}^{m'}} - \tilde{\xi}_{u'}^{m'} \right| \leq |\overline{\eta}_{u'} - \eta_{u'}| + \left| \tilde{\xi}_{u'}^{m'} - \eta_{u'} \right| < \frac{\varepsilon_m^\alpha(w)}{4} + \frac{\varepsilon_m^\alpha(w)}{12} = \frac{\varepsilon_m^\alpha(w)}{3},$$

which implies that $\{\tilde{\xi}_t^{m'}\}_{u \leq t \leq u'}$ hits $\partial D_{\varepsilon_m^\alpha(w)/3}$ before hitting $\partial D_{\varepsilon_m^\alpha(w)}$.

- If $u' > m'T$, then we could repeat the procedure above to prove that for $u \leq t \leq m'T$, $\left| \overline{\tilde{\xi}_t^{m'}} - \tilde{\xi}_t^{m'} \right| < \varepsilon_m^\alpha(w)$, which implies that $\{\tilde{\xi}_t^{m'}\}_{u \leq t \leq m'T}$ never hits $\partial D_{\varepsilon_m^\alpha(w)}$.

■

We now give a proposition which is a straightforward corollary of Lemma 4.8. The proof of this proposition is omitted and we refer the reader to Proposition 4.1 in [101].

Proposition 4.9 *Assume that the domain $D \subset \mathbb{R}^d$ satisfies Conditions (A) and (B). For each $m \in \mathbb{N}^*$ such that $0 < \varepsilon_m^\alpha(w) < r_0/2$ and for all $m' \geq m$,*

$$\xi_t^{m'} \in D_{\varepsilon_m^\alpha(w)}, \quad 0 \leq t \leq T.$$

In particular, the results in the remainder of this subsection is based on the fact that if $m \in \mathbb{N}^*$ is large enough such that $0 < \varepsilon_m^\alpha(w) < r_0/2$ then

$$\xi_t^m \in D_{\varepsilon_m^\alpha(w)}, \quad 0 \leq t \leq T.$$

For $m, n \in \mathbb{N}^*$, set

$$\begin{aligned} T_{m,0} &= \inf \{t \geq 0 : \overline{\xi}_t^m \in \partial D\}; \\ t_{m,n} &= \inf \left\{ t > T_{m,n-1} : \left| \overline{\xi}_t^m - \overline{\xi}_{T_{m,n-1}}^m \right| \geq \delta/2 \right\}; \\ T_{m,n} &= \inf \{t \geq t_{m,n} : \overline{\xi}_t^m \in \partial D\}, \end{aligned}$$

where the constant δ is from Condition (B). Moreover, denote by

$$\phi_t^m := -\frac{m}{2} \int_0^t \nabla U(\xi_s^m) ds, \quad 0 \leq t \leq T.$$

In the remainder of this subsection, we shall provide an estimate of $|\phi^m|_T^0$. First, we prove that for sufficient large m , there is a lower bounded for $T_{m,n} - T_{m,n-1}$.

For simplicity, note $\gamma := \frac{2\kappa^2(r_0/2)^\beta}{r_0}$ and $\lambda(w) := \exp\{\gamma(\|w\|_T + \delta)\}$, where $\kappa(r_0/2)$ is a constant from Lemma 2.1 in [101] such that for all $|x - \bar{x}| < r_0/2$ and $|y - \bar{y}| < r_0/2$, $|\bar{x} - \bar{y}| \leq \kappa(r_0/2)|x - y|$. Obviously, $\lambda > 1$.

Lemma 4.10 *Assume that the domain $D \subset \mathbb{R}^d$ satisfies Conditions (A) and (B). For each $m \in \mathbb{N}^*$ such that $0 < \varepsilon_m^\alpha(w) < \frac{\delta}{180\beta\lambda(w)} \wedge r_0/2$ and for each $n \geq 1$ such that $T_{m,n} < \infty$,*

$$|T_{m,n} - T_{m,n-1}| \geq h := \left(\frac{\delta}{36\beta\lambda(w)\|w\|_\alpha} \right)^{1/\alpha},$$

Proof. From Lemma 5.3 in [101], we have

$$\Delta_{s,t}(\xi^m) \leq (8\beta\lambda(w) + 1) (\Delta_{s,t}(w) + \varepsilon_m^\alpha(w)). \quad (4.8)$$

Then, from Proposition 4.9 and the definitions of $t_{m,n}$ and $T_{m,n}$, we have

$$\begin{aligned} \frac{\delta}{2} - 2\varepsilon_m^\alpha(w) &= \left| \overline{\xi_{t_{m,n}}^m} - \overline{\xi_{T_{m,n-1}}^m} \right| - 2\varepsilon_m^\alpha(w) \\ &\leq \left| \xi_{t_{m,n}}^m - \xi_{T_{m,n-1}}^m \right| \\ &\leq 9\beta\lambda(w) (\Delta_{T_{m,n}, T_{m,n-1}}(w) + \varepsilon_m^\alpha(w)), \end{aligned}$$

which implies

$$\Delta_{T_{m,n}, T_{m,n-1}}(w) \geq \frac{\delta - 4\varepsilon_m^\alpha}{18\beta\lambda(w)} - \varepsilon_m^\alpha \geq \frac{\delta}{18\beta\lambda(w)} - 5\varepsilon_m^\alpha \geq \frac{\delta}{36\beta\lambda(w)}.$$

Furthermore, we deduce the desired result by the definition of $\|w\|_\alpha$. ■

Proposition 4.11 *Assume that the domain $D \subset \mathbb{R}^d$ satisfies Conditions (A) and (B). For each $m \in \mathbb{N}^*$ such that $0 < \varepsilon_m^\alpha(w) < \frac{\delta}{180\beta\lambda(w)} \wedge r_0/2$, we have*

$$|\phi^m|_T^0 \leq C_0 \left(\|w\|_\alpha^{1+1/\alpha} + \|w\|_\alpha \right) \exp \{ \gamma (1 + 1/\alpha) \|w\|_T \}, \quad (4.9)$$

where C_0 depends only on α, β, L, δ and T .

Proof. For any s, t , such that $T_{m,n-1} \leq s \leq t \leq T_{m,n}$, we know from Lemma 5.1 in [101] that

$$|\phi^m|_t^s \leq \beta (\Delta_{s,t}(\xi^m) + \Delta_{s,t}(w)),$$

We combine this inequality with (4.8) and deduce

$$|\phi^m|_t^s \leq 10\beta^2 \exp \{ \gamma (\|w\|_T + \delta) \} (\Delta_{s,t}(w) + \varepsilon_m^\alpha(w)).$$

Thus,

$$\begin{aligned} |\phi^m|_T^0 &\leq 10 \left(\frac{T}{h} + 1 \right) \beta^2 \exp \{ \gamma (\|w\|_T + \delta) \} (\Delta_{0,T}(w) + \varepsilon_m^\alpha(w)) \\ &\leq 10 \left(\frac{T}{h} + 1 \right) \beta^2 \exp \{ \gamma (\|w\|_T + \delta) \} \left(T^\alpha \|w\|_\alpha + \frac{12e^L}{m^\alpha} \|w\|_\alpha \right). \end{aligned}$$

By definition, $\exp \{ \gamma \|w\|_T \} \geq 1$, then we can complete the proof by recalling the definition of h .

■

4.4.2 The existence and uniqueness for the reflected G -Brownian motion

By Theorem 4.2, for each $\omega \in \Omega$, there exists a pair $(X(\omega), K(\omega))$ that solves the deterministic Skorohod problem for $B(\omega)$, i.e.,

$$X(\omega) = x_0 + B(\omega) + K(\omega), \quad x_0 \in \bar{D}. \quad (4.10)$$

It is easy to see that the pair of process (X, K) verify (i), (ii) and (iii) in Definition 4.3. Thus, to prove Theorem 4.4, it suffices to show that both X and K belong to $M_G^2([0, T]; \mathbb{R}^d)$.

Lemma 4.12 *Assume that the domain $D \subset \mathbb{R}^d$ satisfies Conditions (A) and (B). For each $\omega \in \Omega$, we define the pair of processes (X, K) by the unique solution of the deterministic Skorohod problem*

$$X(\omega) = x_0 + B(\omega) + K(\omega), \quad x_0 \in \bar{D},$$

where B is a G -Brownian motion. Then, X and K belong to $M_G^2([0, T]; \mathbb{R}^d)$.

The main idea of prove this lemma is to construct a sequence of $\{(X^m, K^m)\}_{m \in \mathbb{N}^*}$ formed by elements from $(M_G^p([0, T]; \mathbb{R}^d) \times M_G^p([0, T]; \mathbb{R}^d))$ and then to show that the following convergences hold

$$X^m \rightarrow X, \quad K^m \rightarrow K, \quad \text{in } M_G^p([0, T]; \mathbb{R}^d), \quad \text{for } p \geq 2.$$

Similar to the previous subsection, we define for each $m \in \mathbb{N}^*$,

$$X_t^m = x_0 + B_t - \frac{m}{2} \int_0^t \nabla U(X_s^m) ds, \quad 0 \leq t \leq T,$$

which is a stochastic differential equations driven by G -Brownian motion with bounded Lipschitz coefficients. For each $m \in \mathbb{N}^*$, the above equation admits a unique solution in $M_G^p([0, T]; \mathbb{R}^d)$, $p \geq 2$. (cf. Theorem 4.2 in [35]) Moreover, one can find version of X^m , denoted still by X^m , such that there exists a polar set A^m , for all $\omega \in (A^m)^c$, X^m is continuous and

$$X_t^m(\omega) = x_0 + B_t(\omega) - \frac{m}{2} \int_0^t \nabla U(X_s^m(\omega)) ds, \quad 0 \leq t \leq T.$$

We note $A := \cup_{m \in \mathbb{N}^*} A^m$, which is still a polar set, and we note

$$K_t^m(\omega) := -\frac{m}{2} \int_0^t \nabla U(X_s^m(\omega)) ds.$$

In what follows, we shall find a bounded for $\mathbb{E} [\sup_{0 \leq t \leq T} |X_t^m|^p]$ and $\mathbb{E} [(|K_t^m|_T^0)^p]$ which is uniform in m .

First, by Kolmogorov's Criterion (cf. Theorem 36 in Denis et al. [20]), for any $\alpha \in (0, 1/2)$, there exists a polar set A' such that for all $\omega \in A'^c$, the path of G -Brownian motion $B(\omega)$ is α -Hölder continuous. Moreover, for any $p > 0$, $\alpha \in (0, 1/2)$,

$$\mathbb{E} [\|B\|_\alpha^p] = \mathbb{E} \left[\left(\sup_{0 \leq s < t \leq T} \frac{|B_t - B_s|}{|t - s|^\alpha} \right)^p \right] < \infty. \quad (4.11)$$

For each $m \in \mathbb{N}^*$, define

$$\bar{A}^m := \left(\left\{ \omega \in \Omega : \varepsilon_m^\alpha(B(\omega)) < \frac{\delta}{180\beta\lambda(B(\omega))} \wedge \frac{r_0}{2} \right\} \cap A^c \cap A'^c \right)^c.$$

The following lemma gives an estimate for $\bar{C}(\bar{A}^m)^c$.

Lemma 4.13 Fix $\alpha \in (0, 1/2)$. For $m \in \mathbb{N}^*$,

$$\bar{C}(\bar{A}^m) \leq \frac{C_{\alpha,p}}{m^p}, \quad p \geq 2,$$

where $C_{\alpha,p}$ depends on $T, \alpha, p, r_0, \delta, L$ and β .

Proof. It is clear that

$$\begin{aligned} \bar{C}(\bar{A}^m) &\leq \bar{C}\left(\left\{\omega \in \Omega : \varepsilon_m^\alpha(B.(\omega)) \geq \frac{r_0}{2}\right\}\right) \\ &\quad + \bar{C}\left(\left\{\omega \in \Omega : \varepsilon_m^\alpha(B.(\omega)) \geq \frac{\delta}{180\beta\lambda(B.(\omega))}\right\}\right). \end{aligned}$$

We calculate by Markov's inequality, for any $p \geq 2$,

$$\begin{aligned} \bar{C}\left(\left\{\omega \in \Omega : \varepsilon_m^\alpha(B.(\omega)) \geq \frac{r_0}{2}\right\}\right) &= \bar{C}\left(\left\{\omega \in \Omega : \|B.(\omega)\|_\alpha \geq \frac{r_0 e^{-L}}{24} m^\alpha\right\}\right) \\ &\leq \frac{C_1 \mathbb{E}\left[\|B\|_\alpha^{p/\alpha}\right]}{m^p}, \end{aligned}$$

where $C_1 > 0$ depends on r_0 and L . On the other hand,

$$\begin{aligned} &\bar{C}\left(\left\{\omega \in \Omega : \varepsilon_m^\alpha(B.(\omega)) \geq \frac{\delta}{180\beta\lambda(B.(\omega))}\right\}\right) \\ &= \bar{C}\left(\left\{\omega \in \Omega : \|B.(\omega)\|_\alpha \exp(\gamma\|B.(\omega)\|_T) \geq \frac{\delta e^{-L}}{2160\beta \exp\{\gamma\delta\}} m^\alpha\right\}\right) \\ &\leq \frac{C_2 \mathbb{E}\left[\|B\|_\alpha^{p/\alpha} \exp\left\{\frac{\gamma p}{\alpha}\|B\|_T\right\}\right]}{m^p} \\ &\leq \frac{C_2 \mathbb{E}\left[\|B\|_\alpha^{2p/\alpha}\right]^{1/2} \mathbb{E}\left[\exp\left\{\frac{2\gamma p}{\alpha}\|B\|_T\right\}\right]^{1/2}}{m^p}, \end{aligned}$$

where $C_2 > 0$ depends on r_0, δ, L and β . By Theorem 3.3 in Luo and Wang [66], we have

$$\mathbb{E}\left[\exp\left\{\frac{2\gamma p}{\alpha}\|B\|_T\right\}\right] \leq C',$$

where C' depends on T, p, α and γ . We combine this with (4.11) to conclude the desired result. ■

Proposition 4.14 Assume that the domain $D \subset \mathbb{R}^d$ satisfies Conditions (A) and (B). Fix $\alpha \in (0, \frac{1}{2})$, then we have

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t^m|^p\right] + \mathbb{E}\left[\left(|K^m|_T^0\right)^p\right] \leq C'_{\alpha,p},$$

where $C'_{\alpha,p}$ depends on $\alpha, p, r_0, \delta, L$ and β .

Proof. We denote by Λ an upper bound of $|\nabla U|$, then

$$\mathbb{E}\left[\left(|K^m|_T^0\right)^p \mathbf{1}_{\bar{A}^m}\right] \leq \left(\frac{m}{2} T \Lambda\right)^p \bar{C}(\bar{A}^m) = C_{\alpha,p} \left(\frac{T \Lambda}{2}\right)^p. \quad (4.12)$$

From (4.9), we have, for each $\omega \in (\bar{A}^m)^c$,

$$|K^m|_T^0(\omega) \leq C_0 \left(\|B \cdot(\omega)\|_\alpha^{1+1/\alpha} + \|B \cdot(\omega)\|_\alpha \right) \exp \left\{ \gamma \left(1 + \frac{1}{\alpha} \right) \|B \cdot(\omega)\|_T \right\}, \quad (4.13)$$

where C_0 is the constant from (4.9). Thus,

$$\mathbb{E} \left[\left(|K^m|_T^0 \right)^p \mathbf{1}_{(A^m)^c} \right] \leq C_0 \left(\mathbb{E} \left[\|B\|_\alpha^{2p(1+\alpha)/\alpha} \right]^{1/2} + \mathbb{E} \left[\|B\|_\alpha^{2p} \right]^{1/2} \right) \mathbb{E} \left[\exp \left\{ \frac{2\gamma p(1+\alpha)}{\alpha} \|B\|_T \right\} \right]^{1/2}.$$

Recall that for some $C_p > 0$, which depends only on p ,

$$|X_t^m|^p \leq C_p (|x_0|^p + |B_t|^p + |K_t^m|^p),$$

we can deduce that

$$\sup_{0 \leq t \leq T} |X_t^m|^p \leq C_p \left(|x_0|^p + \sup_{0 \leq t \leq T} |B_t|^p + \left(|K^m|_T^0 \right)^p \right),$$

We take the G -expectation on both sides and apply the BDG type inequality, (4.12) and (4.13) to conclude the desired result. ■

Now we are ready to prove Lemma 4.12.

Proof of Lemma 4.12. From (6.4) in [101], we have for $\omega \in (A \cup A')^c$,

$$X_t^m(\omega) \longrightarrow X_t(\omega), \text{ uniform on } [0, T],$$

and for each $\omega \in (\bar{A}^m)^c$,

$$\begin{aligned} \sup_{0 \leq s \leq t} |X_s^m(\omega) - X_s(\omega)|^2 &\leq 4\varepsilon_m^\alpha(\omega) \left(|K^m(\omega)|_t^0 + |K(\omega)|_t^0 \right) \\ &\quad + \frac{\gamma}{2\beta} \int_0^t \sup_{0 \leq u \leq s} (|X_u^m(\omega) - X_u(\omega)|^2) d \left(|K^m(\omega)|_s^0 + |K(\omega)|_s^0 \right), \end{aligned}$$

which implies

$$\sup_{0 \leq t \leq T} |X_t^m(\omega) - X_t(\omega)|^2 \leq 4\varepsilon_m^\alpha(\omega) \left(|K^m(\omega)|_T^0 + |K(\omega)|_T^0 \right) \exp \left\{ \frac{\gamma}{2} \left(|K^m(\omega)|_T^0 + |K(\omega)|_T^0 \right) \right\}.$$

Now we shall prove that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t^m - X_t|^2 \right] \longrightarrow 0, \text{ as } m \longrightarrow \infty. \quad (4.14)$$

For $\epsilon > 0$, by Markov's inequality, we could first fix a constant $M_0 > 0$, such that

$$\left(2C'_{\alpha,4} \right)^{\frac{1}{2}} \mathbb{E} \left[\mathbf{1}_{\|B \cdot\|_\alpha \geq M_0} \right] \leq \frac{\left(2C'_{\alpha,4} \right)^{\frac{1}{2}} \mathbb{E} [\|B \cdot\|_\alpha]}{M_0} \leq \frac{\epsilon}{2}, \quad (4.15)$$

where $C'_{\alpha,4}$ is the constant from Proposition 4.14. Then, we choose $m_0 \in \mathbb{N}^*$ sufficiently large such that

$$\frac{12e^L}{m_0^\alpha} M_0 \leq \frac{r_0}{2} \text{ and } \frac{12e^L}{m_0^\alpha} M_0 \leq \frac{\delta}{180\beta \exp \{ \gamma (\delta + M_0 T^\alpha) \}}; \quad (4.16)$$

$$4 \frac{12e^L}{m_0^\alpha} M_0 \left(4C_0 M_0^{1+1/\alpha} \exp \left\{ \gamma \left(1 + \frac{1}{\alpha} \right) M_0 T^\alpha \right\} \right) \\ \times \exp \left\{ \frac{\gamma C_0}{\beta} M_0^{1+1/\alpha} \exp \left\{ \gamma \left(1 + \frac{1}{\alpha} \right) M_0 T^\alpha \right\} \right\} \leq \frac{\epsilon}{2}, \quad (4.17)$$

where C_0 is the constant from Proposition 4.11. From (4.16), we know for $\omega \in \{\|B\|_\alpha < M_0\} \cap A^c \cap A'^c$ and $m \geq m_0$,

$$\varepsilon_m^\alpha(B(\omega)) < \frac{\delta}{180\beta\lambda(B(\omega))} \wedge \frac{r_0}{2}.$$

It follows that for $m \geq m_0$,

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t^m - X_t|^2 \right] \leq \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t^m - X_t|^2 \mathbf{1}_{\{\|B\|_\alpha < M_0\} \cap A^c \cap A'^c} \right] \\ + \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t^m - X_t|^2 \mathbf{1}_{\{\|B\|_\alpha \geq M_0\} \cup A \cup A'} \right] \\ \leq \frac{\epsilon}{2} + \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t^m - X_t|^4 \right]^{\frac{1}{2}} \mathbb{E} [\mathbf{1}_{\{\|B\|_\alpha \geq M_0\}}] \\ \leq \epsilon,$$

where the last inequality is deduced from (4.15) and (4.17). Therefore, (4.14) holds true. From (4.14), it is obvious that,

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |K_t^m - K_t|^2 \right] \longrightarrow 0, \quad \text{as } m \longrightarrow \infty.$$

We end the proof. ■

Proof of Theorem 4.4. We define pathwisely a couple (X, K) by the solution of the deterministic problem (4.10). Then, we apply Lemma 4.12 to prove that X and K belong to $M_G^2([0, T]; \mathbb{R}^d)$. Therefore, (X, K) is a couple satisfying Definition 4.3. The uniqueness of the solution is inherited from the pathwise uniqueness.

Instead of the G -Brownian motion, if we consider a G -Itô process as

$$Y_t = \int_0^t \alpha_s ds + \int_0^t \eta_s^{ij} d\langle B^i, B^j \rangle_s + \int_0^t \beta_s^j dB_s^j, \quad 0 \leq t \leq T, \quad (4.18)$$

where $\alpha, \eta^{ij}, \beta^j : \Omega \times [0, T] \longrightarrow \mathbb{R}^d, i, j = 1, 2, \dots, d$, are bounded functions in $M_G^2([0, T]; \mathbb{R}^d)$, then a similar result holds due to the fact that for any $p \geq 2$,

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^p \right] \leq C_p,$$

which can be easily obtained by the BDG type inequality. ■

Corollary 4.15 Suppose that the domain $D \subset \mathbb{R}^d$ satisfies Conditions (A) and (B). Then there exists a couple $(X, K) \in (M_G^2([0, T]; \mathbb{R}^d) \times M_G^2([0, T]; \mathbb{R}^d))$ which solves the Skorohod problem

$$X_t = x_0 + Y_t + K_t, \quad 0 \leq t \leq T, \quad (4.19)$$

whenever $x_0 \in \bar{D}$ and Y is defined by (4.18). Moreover, if the above problem admits two solutions (X, K) and (X', K') , then there exists a polar set A , such that for all $\omega \in A^c$,

$$X(\omega) = X'(\omega) \quad \text{and} \quad K(\omega) = K'(\omega), \quad 0 \leq t \leq T.$$

Remark 4.16 *Indeed, thanks to Proposition 4.14, we could have a stronger convergence instead of (4.14), that is, for any $p \geq 2$,*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t^m - X_t|^p \right] \longrightarrow 0, \quad \text{as } m \longrightarrow \infty.$$

Thus, the couple of solution (X, K) in both Theorem 4.4 and Corollary 4.15 could be found in $(M_G^p([0, T]; \mathbb{R}^d) \times M_G^p([0, T]; \mathbb{R}^d))$.

4.4.3 The existence and uniqueness for the RGSDE

Without loss of generality, we consider in this subsection the following equation instead of (4.4),

$$X_t = x_0 + \int_0^t f(s, X_s) ds + \int_0^t g(s, X_s) dB_s + K_t, \quad 0 \leq t \leq T, \quad \text{q.s.} \quad (4.20)$$

However, all result here holds for the more general case (4.4) due to the boundedness of the density of the process $\langle B, B \rangle$ (see §III-4 in [83]).

If the coefficients f and g satisfy Assumptions (H1) and (H2) and (4.20) admits a solution couple (X, K) , then (X, K) can be regarded as the solution couple of the Skorohod problem (4.19) in the domain D for

$$Y_t = \int_0^t f(s, X_s) ds + \int_0^t g^j(s, X_s) dB_s^j, \quad 0 \leq t \leq T.$$

Then, it is straightforward that for any $p \geq 2$ there exists a constant $C'_p > 0$ such that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t|^p \right] + \mathbb{E} \left[\left(|K_T^0| \right)^p \right] \leq C'_p.$$

Proposition 4.17 *Suppose that the domain $D \subset \mathbb{R}^d$ satisfies Conditions (A), (B) and (C). For $i = 1, 2$, the couple (\tilde{X}^i, K^i) are solutions of the following Skorohod problems*

$$\begin{aligned} \tilde{X}_t^i &= x_0 + \int_0^t f^i(s, X_s^i) ds + \int_0^t g^i(s, X_s^i) dB_s + K_t^i, \quad i = 1, 2, \\ |K^i|_t &= \int_0^t \mathbf{1}_{\{\tilde{X}_s^i \in \partial D\}} d|K^i|_s \quad \text{and} \quad K_t^i = \int_0^t \mathbf{n}_s^i d|K^i|_s \quad \text{with} \quad \mathbf{n}_s^i \in \mathcal{N}_{\tilde{X}_s^i}, \end{aligned}$$

where the coefficients f^i and g^i satisfy Assumptions (H1) and (H2). Then, there exists a constant $C > 0$ that depends on $\Gamma, d, \delta', L_\Psi$ and L_0 ,

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq s \leq t} \left| \tilde{X}_s^1 - \tilde{X}_s^2 \right|^4 \right] + \mathbb{E} \left[\sup_{0 \leq s \leq t} \left| K_s^1 - K_s^2 \right|^4 \right] & \quad (4.21) \\ & \leq C \int_0^t \left(\mathbb{E} \left[\sup_{0 \leq u \leq s} \left| X_u^1 - X_u^2 \right|^4 \right] + \mathbb{E} \left[\sup_{0 \leq u \leq s} \left| \hat{f}_u \right|^4 \right] + \mathbb{E} \left[\sup_{0 \leq u \leq s} \left| \hat{g}_u \right|^4 \right] \right) ds, \end{aligned} \quad (4.22)$$

where $\hat{f}_s := f^1(s, X_s^2) - f^2(s, X_s^2)$ and $\hat{g}_s := g^1(s, X_s^2) - g^2(s, X_s^2)$.

Proof. The proof is similar to the one of Lemma 3.1 in [65], so we only display the key steps for the convenience of the readers. First, we have

$$\begin{aligned}
|\tilde{X}_t^1 - \tilde{X}_t^2|^2 &= 2 \int_0^t \langle \tilde{X}_s^1 - \tilde{X}_s^2, f^1(s, X_s^1) - f^1(s, X_s^2) + \hat{f}_s \rangle ds \\
&\quad + 2 \int_0^t (\tilde{X}_s^1 - \tilde{X}_s^2) (g^1(s, X_s^1) - g^1(s, X_s^2) + \hat{g}_s) dB_s \\
&\quad + 2 \int_0^t \langle \tilde{X}_s^1 - \tilde{X}_s^2, \mathbf{n}_s^1 \rangle d|K^1|_s - 2 \int_0^t \langle \tilde{X}_s^1 - \tilde{X}_s^2, \mathbf{n}_s^2 \rangle d|K^2|_s \\
&\quad + \int_0^t \text{tr} [(g^1(s, X_s^1) - g^1(s, X_s^2) + \hat{g}_s) d\langle B, B \rangle_s^t (g^1(s, X_s^1) - g^1(s, X_s^2) + \hat{g}_s)],
\end{aligned}$$

and

$$\begin{aligned}
\Psi(\tilde{X}_t^i) &= \Psi(x_0) + \int_0^t \langle \nabla \Psi(\tilde{X}_s^i), f^i(s, X_s^i) \rangle ds + \int_0^t (\nabla \Psi(\tilde{X}_s^i)) g^i(s, X_s^i) dB_s \\
&\quad + \int_0^t \langle \nabla \Psi(\tilde{X}_s^i), \mathbf{n}_s^i \rangle d|K^i|_s \\
&\quad + \frac{1}{2} \int_0^t \text{tr} [\mathbf{H}(\Psi(\tilde{X}_s^i)) (\tilde{X}_s^i) g^i(s, X_s^i) d\langle B, B \rangle_s^t (g^i(s, X_s^i))] .
\end{aligned}$$

Then,

$$\begin{aligned}
& \exp \left\{ -\frac{1}{\delta'} \left(\Psi \left(\tilde{X}_t^1 \right) + \Psi \left(\tilde{X}_t^2 \right) \right) \right\} \times \left| \tilde{X}_s^1 - \tilde{X}_s^2 \right|^2 \\
&= 2 \int_0^t \exp \left\{ -\frac{1}{\delta'} \left(\Psi \left(\tilde{X}_s^1 \right) + \Psi \left(\tilde{X}_s^2 \right) \right) \right\} \\
&\quad \times \left\{ \left\langle \tilde{X}_s^1 - \tilde{X}_s^2, f^1 \left(s, X_s^1 \right) - f^1 \left(s, X_s^2 \right) + \hat{f}_s \right\rangle ds \right. \\
&\quad \quad + {}^t \left(\tilde{X}_s^1 - \tilde{X}_s^2 \right) \left(g^1 \left(s, X_s^1 \right) - g^1 \left(s, X_s^2 \right) + \hat{g}_s \right) dB_s \\
&\quad \quad \left. + \left\langle \tilde{X}_s^1 - \tilde{X}_s^2, \mathbf{n}_s^1 \right\rangle d|K^1|_s - \left\langle \tilde{X}_s^1 - \tilde{X}_s^2, \mathbf{n}_s^2 \right\rangle d|K^2|_s \right\} \\
&+ \int_0^t \exp \left\{ -\frac{1}{\delta'} \left(\Psi \left(\tilde{X}_s^1 \right) + \Psi \left(\tilde{X}_s^2 \right) \right) \right\} \\
&\quad \times \operatorname{tr} \left[\left(g^1 \left(s, X_s^1 \right) - g^1 \left(s, X_s^2 \right) + \hat{g}_s \right) d\langle B, B \rangle_s {}^t \left(g^1 \left(s, X_s^1 \right) - g^1 \left(s, X_s^2 \right) + \hat{g}_s \right) \right] \\
&- \frac{1}{\delta'} \int_0^t \exp \left\{ -\frac{1}{\delta'} \left(\Psi \left(\tilde{X}_s^1 \right) + \Psi \left(\tilde{X}_s^2 \right) \right) \right\} \times \left| \tilde{X}_s^1 - \tilde{X}_s^2 \right|^2 \\
&\quad \times \left\{ \left\langle \nabla \Psi \left(\tilde{X}_s^1 \right), f^1 \left(s, X_s^1 \right) \right\rangle + \left\langle \nabla \Psi \left(\tilde{X}_s^2 \right), f^2 \left(s, X_s^2 \right) \right\rangle ds \right. \\
&\quad \quad + \left({}^t \left(\nabla \Psi \left(\tilde{X}_s^1 \right) \right) g^1 \left(s, X_s^1 \right) + {}^t \left(\nabla \Psi \left(\tilde{X}_s^2 \right) \right) g^2 \left(s, X_s^2 \right) \right) dB_s \\
&\quad \quad \left. + \left\langle \nabla \Psi \left(\tilde{X}_s^1 \right), \mathbf{n}_s^1 \right\rangle d|K^1|_s + \left\langle \nabla \Psi \left(\tilde{X}_s^2 \right), \mathbf{n}_s^2 \right\rangle d|K^2|_s \right\} \\
&- \frac{1}{2\delta'} \int_0^t \exp \left\{ -\frac{1}{\delta'} \left(\Psi \left(\tilde{X}_s^1 \right) + \Psi \left(\tilde{X}_s^2 \right) \right) \right\} \times \left| \tilde{X}_s^1 - \tilde{X}_s^2 \right|^2 \\
&\quad \times \operatorname{tr} \left[\mathbf{H} \left(\Psi \left(\tilde{X}_s^1 \right) \right) g^1 \left(s, X_s^1 \right) d\langle B, B \rangle_s {}^t \left(g^1 \left(s, X_s^1 \right) \right) \right. \\
&\quad \quad \left. + \mathbf{H} \left(\Psi \left(\tilde{X}_s^2 \right) \right) g^2 \left(s, X_s^2 \right) d\langle B, B \rangle_s {}^t \left(g^2 \left(s, X_s^2 \right) \right) \right] \\
&+ \frac{1}{2} \left(\frac{1}{\delta'} \right)^2 \int_0^t \exp \left\{ -\frac{1}{\delta'} \left(\Psi \left(\tilde{X}_s^1 \right) + \Psi \left(\tilde{X}_s^2 \right) \right) \right\} \left| \tilde{X}_s^1 - \tilde{X}_s^2 \right|^2 \\
&\quad \times \left({}^t \left(\nabla \Psi \left(\tilde{X}_s^1 \right) \right) g^1 \left(s, X_s^1 \right) + {}^t \left(\nabla \Psi \left(\tilde{X}_s^2 \right) \right) g^2 \left(s, X_s^2 \right) \right) \\
&\quad \quad \times d\langle B, B \rangle_s \left(\nabla \Psi \left(\tilde{X}_s^1 \right) {}^t \left(g^1 \left(s, X_s^1 \right) \right) + \nabla \Psi \left(\tilde{X}_s^2 \right) {}^t \left(g^2 \left(s, X_s^2 \right) \right) \right) \\
&- \frac{2}{\delta'} \int_0^t \exp \left\{ -\frac{1}{\delta'} \left(\Psi \left(\tilde{X}_s^1 \right) + \Psi \left(\tilde{X}_s^2 \right) \right) \right\} \\
&\quad \times {}^t \left(\tilde{X}_s^1 - \tilde{X}_s^2 \right) \left(g^1 \left(s, X_s^1 \right) - g^1 \left(s, X_s^2 \right) + \hat{g}_s \right) \\
&\quad \quad \times d\langle B, B \rangle_s \left(\nabla \Psi \left(\tilde{X}_s^1 \right) {}^t \left(g^1 \left(s, X_s^1 \right) \right) + \nabla \Psi \left(\tilde{X}_s^2 \right) {}^t \left(g^2 \left(s, X_s^2 \right) \right) \right).
\end{aligned}$$

Thanks to Condition (C), we know that the integrals with respect to $d|K|$ are negative. Since the

set Γ , the function Ψ and its derivatives, the functions f^i and g^i , $i = 1, 2$, are bounded, we have

$$\begin{aligned} & \exp\left\{-\frac{2M}{\delta'}\right\} \times \left|\tilde{X}_s^1 - \tilde{X}_s^2\right|^2 \\ & \leq 2 \int_0^t \exp\left\{-\frac{1}{\delta'}\left(\Psi\left(\tilde{X}_s^1\right) + \Psi\left(\tilde{X}_s^2\right)\right)\right\} {}^t\left(\tilde{X}_s^1 - \tilde{X}_s^2\right) \left(g^1\left(s, X_s^1\right) - g^1\left(s, X_s^2\right) + \hat{g}_s\right) dB_s \\ & \quad - \frac{1}{\delta'} \int_0^t \exp\left\{-\frac{1}{\delta'}\left(\Psi\left(\tilde{X}_s^1\right) + \Psi\left(\tilde{X}_s^2\right)\right)\right\} \times \left|\tilde{X}_s^1 - \tilde{X}_s^2\right|^2 \\ & \quad \times \left({}^t\left(\nabla\Psi\left(\tilde{X}_s^1\right)\right) g^1\left(s, X_s^1\right) + {}^t\left(\nabla\Psi\left(\tilde{X}_s^2\right)\right) g^2\left(s, X_s^2\right)\right) dB_s \\ & \quad + C \int_0^t \left(\left|\tilde{X}_s^1 - \tilde{X}_s^2\right|^2 + \left|X_s^1 - X_s^2\right|^2 + \left|\hat{f}_s\right|^2 + \left|\hat{g}_s\right|^2\right) ds, \end{aligned}$$

where $C > 0$ is a constant that depends on Γ , d , δ' , L_Ψ and L_0 , which may vary from line to line in the sequel. We square both sides and apply the BDG type inequality to obtain

$$\begin{aligned} & \mathbb{E}\left[\sup_{0 \leq s \leq t} \left|\tilde{X}_s^1 - \tilde{X}_s^2\right|^4\right] \\ & \leq C \int_0^t \left(\mathbb{E}\left[\sup_{0 \leq u \leq s} \left|\tilde{X}_u^1 - \tilde{X}_u^2\right|^4\right] + \mathbb{E}\left[\sup_{0 \leq u \leq s} \left|X_u^1 - X_u^2\right|^4\right] + \mathbb{E}\left[\sup_{0 \leq u \leq s} \left|\hat{f}_u\right|^4\right] + \mathbb{E}\left[\sup_{0 \leq u \leq s} \left|\hat{g}_u\right|^4\right]\right) ds. \end{aligned}$$

The desired result follows from the Gronwall inequality. ■

Proof of Theorem 4.7. The uniqueness of solution is straightforward by Proposition 4.17.

We now turn to prove the existence. Indeed, by Corollary 4.15 one can construct a sequence $\{(X^m, K^m)\}_{m \in \mathbb{N}^*}$ by the Picard type iteration starting with $X^0 \equiv K^0 = 0$,

$$\begin{aligned} Y_t^{m+1} &= \int_0^t f(s, X_s^m) ds + \int_0^t g(s, X_s^m) dB_s, \\ X_t^{m+1} &= x_0 + Y_t^{m+1} + K_t^{m+1}, \\ |K_t^{m+1}|_t &= \int_0^t \mathbf{1}_{\{X_s^{m+1} \in \partial D\}} d|K_t^{m+1}|_s \end{aligned}$$

and

$$K_t^{m+1} = \int_0^t \mathbf{n}_s^{m+1} d|K_t^{m+1}|_s \text{ with } \mathbf{n}_s^{m+1} \in \mathcal{N}_{X_s^{m+1}}.$$

Thank to the a priori estimate (4.22), we can proceed a similar argument as the proof of Theorem 2.1 and 4.1 in [35], and find a couple of processes (X, K) such that

$$\begin{aligned} & \mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t^m - X_t|^2\right] + \mathbb{E}\left[\sup_{0 \leq t \leq T} |K_t^m - K_t|^2\right] \longrightarrow 0, \\ & \mathbb{E}\left[\sup_{0 \leq t \leq T} \left|Y_t^m - \int_0^t f(s, X_s) ds - \int_0^t g(s, X_s) dB_s\right|^2\right] \longrightarrow 0, \quad \text{as } m \longrightarrow \infty, \end{aligned}$$

and there exists a polar set A and subsequence $\{(X^{m_k}, K^{m_k})\}_{k \in \mathbb{N}^*}$, such that for each $\omega \in A^c$, $(X^{m_k}(\omega), K^{m_k}(\omega))$ is the solution couple for the deterministic Skorohod problem with $(Y^{m_k}(\omega), D)$, and

$$\begin{aligned} & \sup_{0 \leq t \leq T} |X_t^{m_k}(\omega) - X_t(\omega)| + \sup_{0 \leq t \leq T} |K_t^{m_k}(\omega) - K_t(\omega)| \longrightarrow 0, \\ & \sup_{0 \leq t \leq T} \left|Y^{m_k}(\omega) - \left(\int_0^t f(s, X_s) ds - \int_0^t g(s, X_s) dB_s\right)(\omega)\right| \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \end{aligned}$$

Obviously, $(X, K) \in (M_G^2([0, T]; \mathbb{R}^d) \times M_G^2([0, T]; \mathbb{R}^d))$. Besides, for each $\omega \in A^c$, $(X(\omega), K(\omega))$ verified (i) (ii) (iii) of Definiton 4.3, which can be proved by the last step of the proof to Theorem 4.1 in [97]. We complete the proof. ■

Chapter 5

Reflected BSDEs driven by G -Brownian motion

Abstract: In this chapter, we study reflected solution of one-dimensional backward stochastic differential equations driven by G -Brownian motion in the following form:

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s + N_T - N_t \\ Y_t \geq S_t \end{cases} .$$

Where N is a continuous increasing process.

Key words. Skorokhod problem, Penalization method, G -Brownian motion, Reflected backward stochastic differential equation, Comparison theorem.

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5.1 Introduction

Backward stochastic differential equations (BSDEs for short) is by now an established field of research. The solution to a classical BSDE, driven by a Wiener process W , is a pair of processes (Y, Z) such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

where ξ is a random variable that becomes known, with certainty, only at time T and the g enerator f is a progressively measurable function. In this setting $Y_t \in \mathbb{R}^d$, $d \geq 1$, and in the following we refer to the case $d = 1$ as the one-dimensional case and to the case $d > 1$ as the multi-dimensional case. The first result of nonlinear BSDE was introduced by Pardoux and Peng, in [75], who proved the existence and uniqueness of adapted solutions under suitable assumptions. Classical BSDEs have turned out important in many areas of mathematics including mathematic finance, see [33] and the long list of references therein, stochastic control theory and stochastic game theory, see, e.g., [19] and [41], as well as in the connection to partial differential equations, see, e.g., [6] and [76].

In [31] a notion of reflected backward stochastic differential equation was introduced. A solution to a one dimensional reflected BSDE is a triple of processes (Y, Z, R) satisfying

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s + R_T - R_t, & 0 \leq t \leq T, \\ Y_t \geq S_t, \end{cases}$$

where the barrier S is a given (one-dimensional) stochastic process. R is a continuous increasing process, with $R_0 = 0$, pushing the process Y upwards in order to keep it above the barrier. This is done with minimal energy in the sense that

$$\int_0^T (Y_s - S_s) dR_s = 0,$$

and consequently R increases only when Y is at the boundary of the space-time domain $\{(t, s) : s > S_t\}$. This type of reflected BSDE has important applications in the context of American options, optimal stopping and obstacle problem, see [31] and [32] as well as in the context of stochastic game problems, see [19].

In the multi-dimensional case there are at least two different type of reflected BSDEs studied in the literature.

The first type of multi-dimensional reflected BSDE was first studied in [37] where the authors considered reflected BSDEs of the form

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s + R_T - R_t, & 0 \leq t \leq T, \\ Y_t \in D, & 0 \leq t \leq T, \end{cases}$$

where $D \subset \mathbb{R}^d$. In this case $R_0 = 0$ and

$$R_t = \int_0^t \gamma_s d|R|_s, \quad \gamma_s \in \mathbf{n}(Y_s), \quad d|R|(\{t \in [0, T] : Y_t \in D\}) = 0,$$

where $\mathbf{n}(Y_s)$ is the unit inner normal to D at Y_s . In particular, the process R is of bounded total variation $|R|$ and it increases only when Y is at the boundary of D . To be more precise, when Y

is at the boundary it is pushed into the domain along $\gamma \in \mathbf{n}(Y)$. In [37] existence and uniqueness for this problem is established and we note that this problem, and its analysis, is inspired by the corresponding theory for reflected stochastic differential equations, see, e.g., [112], [65] and [97]. The second type of multi-dimensional reflected BSDEs occurs in the study of optimal switching problems and stochastic games, see, e.g., [21], [47], [50], and references therein. In the generic optimal switching problem a production facility is considered and it is assumed that the production can run in $d > 2$ different production modes. Furthermore, there is a stochastic process $X = (X_t)_{t \geq 0}$ which stands for the market price of the underlying commodities and other financial parameters that influence the production. When the facility is in mode i , the revenue per unit time is $f_i(t, X_t)$ and the cost of switching from mode i to mode j , at time t , is $c_{ij}(t, X_t)$. Let (Y_t^1, \dots, Y_t^d) be the value function associated with the optimal switching problem, on the time interval $[t, T]$, i.e., Y_t^i stand for the optimal expected profit if, at time t , the production is in mode i . In this case, one can prove, under various assumptions, see [21] and [47], that (Y_t^1, \dots, Y_t^d) solves the reflected BSDE

$$\begin{cases} Y_t^i = \xi_i + \int_t^T f_i(s, X_s) ds - \int_t^T Z_s^i dW_s + R_T^i - R_t^i, \\ Y_t^i \geq \max_{j \in A_i} (Y_t^j - c_{ij}(t, X_t)), \\ \int_0^T \left(Y_t^i - \max_{j \in A_i} (Y_t^j - c_{ij}(t, X_t)) \right) dR_t^i = 0, \end{cases}$$

where $i \in \{1, \dots, d\}$, $0 \leq t \leq T$, and $A_i = \{1, \dots, d\} \setminus \{i\}$. In this case the reflected BSDE evolves in the closure of the time-dependent domain

$$D = \left\{ (t, y) = (t, y_1, \dots, y_d) \in [0, T] \times \mathbb{R}^d, y_i \geq \max_{j \in A_i} (y_j - c_{ij}(t, X_t)), \forall i \in \{1, \dots, d\} \right\}.$$

Motivated by mathematical finance problems with Knightian uncertainty, Peng established systematically in [81, 82, 83, 84] a framework of time-consistent sub-linear expectation, called G -expectation. In particular, this sub-linear expectation is associated with a new type of Brownian motion $(B_t)_{t \geq 0}$, i.e., G -Brownian motion, which has independent, stationary and G -normally distributed increments. This process and its quadratic variation $\langle B \rangle$ play center roles in the related nonlinear stochastic analysis. Indeed, the stochastic integrals with respect to G -Brownian motion and its quadratic variation have been first introduced by Peng in his pioneer work [81], which are initially defined on the simple process space and later extended as linear operator on Banach completions. Thereafter, the G -stochastic calculus is further developed, for example, in [83, 35, 60, 62]. Another important feature of the G -expectation is found by Denis et al. in [20], namely, the G -expectation can be represented by the upper expectation over a collection of mutually singular martingale measures \mathcal{P}_G . Moreover, the notion of quasi-sure with respect to the associated Choquet capacity is introduced by Denis et al. to the G -framework.

As their classical counterparts, stochastic differential equations driven by G -Brownian motion (GSDEs) are well defined in the quasi-sure sense and their solvability can be established by the contracting mapping theorem under Lipschitz assumptions (cf. [83] and [35]). However, the challenging problem of wellposedness for backward GSDEs (GBSDEs) remained open until a complete theorem has been proved by Hu et al. [39].

Similarly to the classical case, the G -martingale representation theorem is heuristic to the formulation of GBSDEs, which reads as follows

$$M_t = M_0 + \overline{M}_t + K_t, \quad (5.1)$$

where

$$\overline{M}_t = \int_0^t Z_s dB_s \quad \text{and} \quad K_t = \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds. \quad (5.2)$$

In contrast to the classical martingale representation, the G -martingale M is decomposed into two parts: the G -Itô type integral part $\overline{M} = \int Z dB$, which is called symmetric G -martingale, in the sense that $-\overline{M}$ is still a G -martingale; the decreasing G -martingale part K , which vanishes in the classical theory, however, plays a significant role in this new context. Whether the process K admits a unique representation in the form (5.2) is a sophisticated question. The first positive answer is given by Peng in [80] for the G -martingale associated with a terminal value $M_T \in Lip(\Omega_T)$, which reads as smooth and finitely dimensional path function. It is also worth mentioning that a series of successive works by Soner et al. [106] and Song [109] affirm the existence and uniqueness of the first level decomposition (5.1) for $M_T \in L_G^p(\Omega)$, $p > 1$, which is the Banach completion of $Lip(\Omega_T)$. Finally, with the help of the norm creatively introduced in Song [110], a complete theorem for G -martingale representation has been obtained by Peng et al. [86] on a complete metric subspace of $L_G^p(\Omega)$, $p > 1$.

We take into consideration of the G -martingale representation theorem and naturally, we can formulate GBSDE as follows, where the decreasing G -martingale K appears in the dynamics:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds + \int_t^T h(s, Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t). \quad (5.3)$$

Under Lipschitz assumptions on the generators, Hu et al. investigated in [39] the existence and uniqueness of the triple (Y, Z, K) in proper Banach spaces satisfying the above equation. They started with BSDEs with bounded and smooth generators and Markovian terminal values and constructed solutions by classical solutions of fully nonlinear PDEs (cf. Krylov's results in [55]). Then, the partition of unity theorem was employed in [39] to proceed a type of Galerkin approximation to solutions of GBSDEs with general parameters. Besides, the uniqueness was deduced in [39] based on a priori estimates. In particular, the uniqueness of K is impressive in the light of G -martingale estimates found in [109]. The results in [39] breaks new ground in the G -expectation theory. In the accompanying paper [40], Hu et al. discussed fundamental properties of the above GBSDE: the comparison theorem, the fully nonlinear Feynman-Kac formula and the related Girsanov transformation. Moreover, the correspondence between GBSDEs and Sobolev type solutions of nonlinear path-dependent PDEs is examined in [85].

We now compare the result of [39] with the profound works [107, 108] by Soner et al., in which the so-called second order backward stochastic differential equations (2BSDEs) are deeply studied. This type of equation is highly related to the GBSDE and it is defined on the Wiener space as follows:

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s, \hat{a}_s) ds - \int_t^T Z_s dB_s + (K_T - K_t), \quad \mathbb{P} - \text{a.s.}, \text{ for all } \mathbb{P} \in \mathcal{P}_H,$$

where B is the canonical process, the process \hat{a} is the density of $\langle B \rangle$ and \mathcal{P}_H is a collection of martingale measures similar to \mathcal{P}_G (could be even larger). This equation is a reinforced BSDE in

the sense that it holds true \mathbb{P} -a.s. for all $\mathbb{P} \in \mathcal{P}_H$ and moreover, the family of $K := \{K^{\mathbb{P}}\}_{\mathbb{P} \in \mathcal{P}_H}$ should satisfy a minimum condition (then $-K$ verifies the G -martingale constraint in the GBSDE context, see [106]):

$$K_t^{\mathbb{P}} = \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H(t, \mathbb{P})} E_t^{\mathbb{P}'} [K_T^{\mathbb{P}'}], \quad \mathbb{P} - \text{a.s.}, \text{ for all } \mathbb{P} \in \mathcal{P}_H, \quad 0 \leq t \leq T.$$

Under Lipschitz assumptions, the uniqueness of the 2BSDE is proved in [107] by observing that the solution to the 2BSDE can be represented as the (essential) supremum of a class of martingale-driven BSDEs solutions. For the existence, the proof involves a delicate pathwise construction: the process Y is defined pathwisely by solutions of BSDEs on shift spaces. This process verifies a critical principle of optimality and thus, the structure of 2BSDE could be derived from the g -supermartingale decomposition (cf. [77]), where the family of processes K can be a posteriori aggregated once the stochastic integral part is aggregated by Nutz [71]. To get rid of the measurability problem during the construction of solutions, Soner et al. assume the technical condition that both ξ and F is uniformly continuous in ω , whereas this assumption is removed in the recent work of Possamaï et al. [87]. In the framework of 2BSDEs, the results in [107, 108] are generalized by Possamaï and Zhou [88] and by Lin [63] to the quadratic case and furthermore, Matoussi et al. [69] applied quadratic 2BSDEs to solve the utility maximization problems from [44] in the context with non-dominated models. One could see that the GBSDE (5.3) actually corresponds to the 2BSDE defined with

$$F(t, y, z, a) = g(t, y, z) + h(t, y, z)a,$$

however, the GBSDE requires more structure conditions on the coefficient and the terminal value so that the solution can be found with more regularity adapted to the requirement of process space in the G -framework.

The chapter is organized as follows. In Section 4.2, we introduce the formulation of the reflected GBSDEs. In Section 4.3 we present some a priori estimations and Section 3.4 is devoted the existence of solution of the reflected G -BSDE by penalizing method.

5.2 Formulation of the reflected GBSDEs

We consider the G -expectation space $(\Omega_T, L_G^1(\Omega_T), \widehat{\mathbb{E}})$ with $\Omega_T = C_0([0, T], \mathbb{R})$ and

$$0 < \underline{\sigma}^2 = -\widehat{\mathbb{E}}[-B_1^2] \leq \widehat{\mathbb{E}}[B_1^2] = \bar{\sigma}^2.$$

B is a 1 dimensional G -Brownian motion .

In this paper, one considers the reflected backward stochastic differential equations driven by G -Brownian motion of the following type :

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s + N_T - N_t; \\ Y_t \geq S_t. \end{cases} \quad (5.4)$$

The obstacle S is a G -Itô process of the form

$$S_t = S_0 + \int_0^t a_s ds + \int_0^t b_s d\langle B \rangle_s + \int_0^t c_s dB_s,$$

processes a and b are elements of $M_G^\beta([0, T])$ and c is an element of $H_G^\beta([0, T])$, $\beta > 2$. The function $f(t, \omega, y, z) : [0, T] \times \Omega_T \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies the following assumptions: There is some real $\beta > 2$ such that

(H1) for any $(y, z) \in \mathbb{R}^2$, $f(\cdot, \cdot, y, z) \in M_G^\beta(0, T)$;

(H2) The function f is uniformly Lipschitz in y and z , i.e., $\exists L > 0$ such as $\forall (t, \omega) \in [0, T] \times \Omega$, $\forall (y, z), (y', z') \in \mathbb{R}^2$,

$$|f(t, \omega, y, z) - f(t, \omega, y', z')| \leq L(|y - y'| + |z - z'|).$$

Definition 5.1 Let $\xi \in L_G^\beta(\Omega_T)$ with $\beta > 2$ and f satisfies assumptions **(H1)** and **(H2)**. A triple of process (Y, Z, N) is called a solution of the equation (5.4) if for some $1 < \alpha \leq \beta$ the following properties hold:

(a) $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T)$ and $N \in M_G^\alpha(0, T)$ such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s + N_T - N_t;$$

(b) The process $N \in M_G^\alpha(0, T)$ is quasi-surely continuous and increasing, with

$$N_0 = 0 \text{ and } N_T \in L_G^\alpha(\Omega_T), \text{ q.s.};$$

(c) The process K' define by

$$K'_t = - \int_0^t (Y_s - S_s) dN_s, \quad 0 \leq t \leq T$$

is a decreasing G -martingale;

(d) $Y_t \geq S_t$, $0 \leq t \leq T$, q.s..

Remark 5.2 In the formulation of the reflected G -BSDE, we do not need to use an increasing process R and a decreasing G -martingale K , as the decomposition of the increasing process N into $N = R - K$ is not unique, we have not the uniqueness of solution if we look for a quadruplet solution (Y, Z, K, R) satisfying:

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s - (K_T - K_t) + R_T - R_t, \\ Y_t \geq S_t \end{cases}$$

as shown in the following example.

Example 5.3 Consider the reflected G -BSDE

$$Y_t = \int_t^T (\underline{\sigma}^2 - \bar{\sigma}^2) ds - (K_T - K_t) + R_T - R_t, \quad (5.5)$$

with an obstacle $S = 0$. Then

$$(0, 0, 0, (\bar{\sigma}^2 - \underline{\sigma}^2)t)$$

and

$$(0, 0, \underline{\sigma}^2 t - \langle B \rangle_t, \bar{\sigma}^2 t - \langle B \rangle_t)$$

are both verify the equation (5.5).

The condition $\left(-\int_0^t (Y_s - S_s) dN_s\right)_{0 \leq t \leq T}$ is a decreasing G -martingale is the equivalent of the minimal condition of Skorokhod $\int_0^T (Y_s - S_s) dN_s = 0$ in the classical case, which says that the push is minimal in the sense that we push only when the constraint is saturated, that is, when $Y_t = S_t$. Note that in a deterministic framework, this corresponds to the Skorokhod problem. Therefore, as the integral $\int_0^t (Y_s - S_s) dN_s$ is an Stieltjes integral, one could reason ω by ω and we will be able to apply some well-known properties of the problem Skorokhod.

Recall the Skorokhod Lemma.

Lemma 5.4 *Let x be a real-valued continuous function on $[0, +\infty[$ such that $x_0 \geq 0$. There exists a unique pair (y, r) of functions on $[0, +\infty[$ such that :*

(a) $y = x + r$,

(b) y is positive

(c) $\{r_t\}$ is continuous and increasing, $r_0 = 0$ and $\int_0^{+\infty} y_t dr_t = 0$.

The pair (y, r) is said to be the solution of the Skorokhod problem. The function r is moreover given by

$$r_t = \sup_{s \leq t} x_s^-.$$

Now our problem involves a Skorokhod problem and, therefore, the increasing process N can be written as a supremum. More precisely, let (Y, Z, N) be a solution of the reflected G -BSDE (5.4), Applying the Skorokhod lemma with

$$x_t = \left(\xi + \int_{T-t}^T f(s, Y_s, Z_s) ds - \int_{T-t}^T (Z_s, dB_s) - S_{T-t} \right) (\omega),$$

$$r_t = (N_T - N_{T-t}) (\omega)$$

and

$$y_t = (Y_{T-t} - S_{T-t}) (\omega),$$

we have that $(Y_{T-t}(\omega) - S_{T-t}(\omega), N_{T-t}(\omega) - N_T(\omega), 0 \leq t \leq T)$ is the solution of a problem Skorokhod. So we have, quasi-surely, for all $t \in [0, T]$,

$$N_T - N_t = \sup_{t \leq u \leq T} \left(\xi + \int_u^T f(s, Y_s, Z_s) ds - \int_u^T (Z_s, dB_s) - S_u \right)^-.$$

5.3 Some a priori estimations

In this section, we give a priori estimates for solutions of reflected GBSDEs of form (5.4).

Proposition 5.5 (Proposition 3.5 in [39]) *Let the function f satisfies assumptions (H1) and (H2) for $\beta > 2$ and $\xi \in L_G^\beta(\Omega_T)$. Assume*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s + N_T - N_t.$$

where $Y \in S^\alpha(0, T)$, $Z \in H^\alpha(0, T)$, N is a increasing process with $N_0 = 0$ and $N_T \in L^\alpha(\Omega_T)$ for some $1 < \alpha \leq \beta$. Then there exists a constant $C_\alpha := C(\alpha, T, \underline{\sigma}, L) > 0$ such that

$$\mathbb{E} \left[\left(\int_0^T |Z_s|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{2}} \right] + \mathbb{E} [|N_T|^\alpha] \leq C_\alpha \left\{ \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^\alpha \right] + \mathbb{E} \left[\int_0^T |f(s, 0, 0)|^\alpha ds \right] \right\}.$$

Let us give now, a priori estimates for of Y in equation (5.4).

Proposition 5.6 *Let $\xi \in L_G^\beta(\Omega_T)$ for some $\beta > 2$ and f satisfy **(H1)** and **(H2)**. Assume that (Y, Z, N) is a solution of the equation (5.4) for $1 < \alpha < \beta$. Then there exists a constant $C_\alpha = C(\alpha, T, \underline{\sigma}, \bar{\sigma}, L) > 0$ such that*

$$|Y_t|^\alpha \leq C_\alpha \mathbb{E}_t \left\{ |\xi|^\alpha + \sup_{0 \leq s \leq T} |S_s|^\alpha + \int_0^T (|f_s|^\alpha + |a_s|^\alpha + |b_s|^\alpha) ds + \left(\int_0^T |c_s|^2 ds \right)^{\alpha/2} \right\},$$

and

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^\alpha \right] \\ & \leq C_\alpha \mathbb{E} \left\{ \sup_{0 \leq t \leq T} \mathbb{E}_t \left\{ |\xi|^\alpha + \sup_{0 \leq s \leq T} |S_s|^\alpha + \int_0^T (|f_s|^\alpha + |a_s|^\alpha + |b_s|^\alpha) ds + \left(\int_0^T |c_s|^2 ds \right)^{\alpha/2} \right\} \right\}. \end{aligned}$$

case $\alpha \leq 2$: For $\gamma > 0$, $\varepsilon > 0$, set $\tilde{Y}_t = |Y_t - S_t|^2 + \varepsilon_\alpha$, where $\varepsilon_\alpha = \varepsilon(1 - \frac{\alpha}{2})$, applying Itô's formula to $\tilde{Y}_t^{\alpha/2} e^{\gamma t}$, we have :

$$\begin{aligned}
& \tilde{Y}_t^{\alpha/2} e^{\gamma t} + \gamma \int_t^T e^{\gamma s} \tilde{Y}_s^{\alpha/2} ds \\
= & (|\xi - S_T|^2 + \varepsilon_\alpha)^{\frac{\alpha}{2}} e^{\gamma T} + \alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} (Y_s - S_s) a_s ds \\
& + \alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} (Y_s - S_s) b_s d\langle B \rangle_s + \alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} (Y_s - S_s) f(s) ds \\
& - \int_t^T \alpha e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} (Y_s - S_s) (Z_s - c_s) dB_s \\
& + \alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} (Y_s - S_s) dN_s - \frac{\alpha}{2} \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} |Z_s - c_s|^2 d\langle B \rangle_s \\
& + \alpha \left(1 - \frac{\alpha}{2}\right) \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-2} |Y_s - S_s|^2 |Z_s - c_s|^2 d\langle B \rangle_s \\
\leq & (|\xi - S_T|^2 + \varepsilon_\alpha)^{\frac{\alpha}{2}} e^{\gamma T} + \alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha-1}{2}} |a_s| ds \\
& + \alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha-1}{2}} |b_s| d\langle B \rangle_s + \alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} |Y_s - S_s| |f(s)| ds \\
& - (M_T - M_t) + \frac{\alpha}{2} (1 - \alpha) \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} |Z_s - c_s|^2 d\langle B \rangle_s,
\end{aligned} \tag{5.6}$$

where we set $f(s) = f(s, Y_s, Z_s)$ and

$$\begin{aligned}
M_t &= \alpha \int_0^t e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} (Y_s - S_s) (Z_s - c_s) dB_s \\
&\quad - \alpha \int_0^t e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} (Y_s - S_s) dN_s.
\end{aligned}$$

From the assumption of the function f and Young's inequality, we have (setting $f_s = f(s, 0, 0)$)

$$\begin{aligned}
& \alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} |Y_s - S_s| \times |f(s)| ds \\
\leq & \alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} \left(|Y_s - S_s| \times |f_s| + L |Y_s - S_s|^2 + L |Y_s - S_s| \times |Z_s - c_s| \right. \\
& \left. + L |Y_s - S_s| \times |S_s| + L |Y_s - S_s| \times |c_s| \right) ds \\
\leq & \alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha-1}{2}} |f_s| ds + \left(\alpha L + \frac{\alpha L^2}{\underline{\sigma}^2 (\alpha - 1)} \right) \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}} ds \\
& + \frac{\alpha}{4} (\alpha - 1) \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} |Z_s - c_s|^2 d\langle B \rangle_s \\
& + L\alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha-1}{2}} |S_s| ds + L\alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha-1}{2}} |c_s| ds \\
\leq & \int_t^T e^{\gamma s} (|f_s|^\alpha + L |S_s|^\alpha) ds + \frac{\alpha}{4} (\alpha - 1) \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} |Z_s - c_s|^2 d\langle B \rangle_s \\
& + \left(\alpha L + \frac{\alpha L^2}{\underline{\sigma}^2 (\alpha - 1)} + (\alpha - 1) (1 + L) \right) \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}} ds \\
& + L\alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha-1}{2}} |c_s| ds.
\end{aligned}$$

Since for $\alpha \leq 2$, we have $\frac{\alpha}{2(\alpha-1)} \geq 1$, by the inequalities of Hölder and Young, we obtain:

$$\begin{aligned}
& L\alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha-1}{2}} |c_s| ds \\
& \leq L\alpha \left(\int_t^T e^{\frac{2\gamma s(\alpha-1)}{\alpha}} \tilde{Y}_s^{\alpha-1} ds \right)^{1/2} \left(\int_t^T e^{\frac{2\gamma s}{\alpha}} |c_s|^2 ds \right)^{1/2} \\
& \leq L(\alpha-1) \left(\int_t^T e^{\frac{2\gamma s(\alpha-1)}{\alpha}} \tilde{Y}_s^{\alpha-1} ds \right)^{\frac{\alpha}{2(\alpha-1)}} + L \left(\int_t^T e^{\frac{2\gamma s}{\alpha}} |c_s|^2 ds \right)^{\alpha/2} \\
& \leq L(\alpha-1) \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}} ds + L \left(\int_t^T e^{\frac{2\gamma s}{\alpha}} |c_s|^2 ds \right)^{\alpha/2},
\end{aligned}$$

and by replacing the previous inequality we have

$$\begin{aligned}
& \alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} |Y_s - S_s| |f(s)| ds \\
& \leq \int_t^T e^{\gamma s} (|f_s|^\alpha + L|S_s|^\alpha) ds + L \left(\int_t^T e^{\frac{2\gamma s}{\alpha}} |c_s|^2 ds \right)^{\alpha/2} \\
& \quad + \frac{\alpha}{4} (\alpha-1) \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} |Z_s - c_s|^2 d\langle B \rangle_s \\
& \quad + \left(\alpha L + \frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)} + (\alpha-1)(1+2L) \right) \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}} ds.
\end{aligned} \tag{5.7}$$

Similarly, we have

$$\begin{aligned}
& \alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha-1}{2}} |a_s| ds + \alpha \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha-1}{2}} |b_s| d\langle B \rangle_s \\
& \leq \int_t^T e^{\gamma s} (|a_s|^\alpha + \bar{\sigma}^2 |b_s|^\alpha) ds + (\alpha-1)(1+\bar{\sigma}^2) \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}} ds.
\end{aligned} \tag{5.8}$$

While replacing the inequalities (5.7) and (5.8) in the inequality (5.6), we get

$$\begin{aligned}
& \tilde{Y}_t^{\alpha/2} e^{\gamma t} + (M_T - M_t) \\
& \leq (|\xi - S_T|^2 + \varepsilon_\alpha)^{\frac{\alpha}{2}} e^{\gamma T} + \int_t^T e^{\gamma s} (|f_s|^\alpha + L|S_s|^\alpha + |a_s|^\alpha + \bar{\sigma}^2 |b_s|^\alpha) ds \\
& \quad + \left(\alpha L + \frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)} + (\alpha-1)(2+2L+\bar{\sigma}^2) - \gamma \right) \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}} ds \\
& \quad + L \left(\int_t^T e^{\frac{2\gamma s}{\alpha}} |c_s|^2 ds \right)^{\alpha/2} - \frac{\alpha}{4} (\alpha-1) \int_t^T e^{\gamma s} \tilde{Y}_s^{\frac{\alpha}{2}-1} |Z_s - c_s|^2 d\langle B \rangle_s.
\end{aligned}$$

For

$$\gamma \geq \alpha L + \frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)} + (\alpha-1)(2+2L+\bar{\sigma}^2),$$

we have

$$\begin{aligned}
& \tilde{Y}_t^{\alpha/2} e^{\gamma t} + (M_T - M_t) \\
& \leq (|\xi - S_T|^2 + \varepsilon_\alpha)^{\frac{\alpha}{2}} e^{\gamma T} + L \left(\int_t^T e^{\frac{2\gamma s}{\alpha}} |c_s|^2 ds \right)^{\alpha/2} \\
& \quad + \int_t^T e^{\gamma s} (|f_s|^\alpha + L|S_s|^\alpha + |a_s|^\alpha + \bar{\sigma}^2 |b_s|^\alpha) ds.
\end{aligned}$$

Then, since M is a G -martingale, we obtain

$$\begin{aligned} \tilde{Y}_t^{\alpha/2} e^{\gamma t} &\leq \mathbb{E}_t \left[(|\xi - S_T|^2 + \varepsilon_\alpha)^{\frac{\alpha}{2}} e^{\gamma T} + L \left(\int_t^T e^{\frac{2\gamma s}{\alpha}} |c_s|^2 ds \right)^{\alpha/2} \right. \\ &\quad \left. + \int_t^T e^{\gamma s} (|f_s|^\alpha + L |S_s|^\alpha + |a_s|^\alpha + \bar{\sigma}^2 |b_s|^\alpha) ds \right] \\ &\leq C_\alpha \mathbb{E}_t \left[(|\xi - S_T|^2 + \varepsilon_\alpha)^{\frac{\alpha}{2}} + \left(\int_0^T |c_s|^2 ds \right)^{\alpha/2} \right. \\ &\quad \left. + \sup_{0 \leq s \leq T} |S_s|^\alpha + \int_0^T (|f_s|^\alpha + |a_s|^\alpha + |b_s|^\alpha) ds \right]. \end{aligned}$$

Letting $\varepsilon \downarrow 0$, we get

$$\begin{aligned} |Y_t - S_t|^\alpha &\leq C_\alpha \mathbb{E}_t \left[|\xi - S_T|^\alpha + \left(\int_0^T |c_s|^2 ds \right)^{\alpha/2} \right. \\ &\quad \left. + \sup_{0 \leq t \leq T} |S_t|^\alpha + \int_0^T (|f_s|^\alpha + |a_s|^\alpha + |b_s|^\alpha) ds \right], \end{aligned}$$

From which we deduce that

$$\begin{aligned} |Y_t|^\alpha &\leq C_\alpha \mathbb{E}_t \left[|\xi|^\alpha + \left(\int_0^T |c_s|^2 ds \right)^{\alpha/2} \right. \\ &\quad \left. + \sup_{0 \leq s \leq T} |S_s|^\alpha + \int_0^T (|f_s|^\alpha + |a_s|^\alpha + |b_s|^\alpha) ds \right], \end{aligned} \quad (5.9)$$

and

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^\alpha \right] &\leq C_\alpha \mathbb{E} \left[\sup_{0 \leq t \leq T} \mathbb{E}_t \left[|\xi|^\alpha + \left(\int_0^T |c_s|^2 ds \right)^{\alpha/2} \right. \right. \\ &\quad \left. \left. + \sup_{0 \leq s \leq T} |S_s|^\alpha + \int_0^T (|f_s|^\alpha + |a_s|^\alpha + |b_s|^\alpha) ds \right] \right] \\ &\leq C_\alpha \mathbb{E} \left[\sup_{0 \leq t \leq T} \mathbb{E}_t \left[|\xi|^\alpha + \left(\int_0^T |c_s|^2 ds \right)^{\alpha/2} \right. \right. \\ &\quad \left. \left. + \sup_{0 \leq s \leq T} |S_s|^\alpha + \int_0^T (|f_s|^\alpha + |a_s|^\alpha + |b_s|^\alpha) ds \right] \right]. \end{aligned}$$

case $\alpha > 2$: we take $\alpha = 2$ in the inequality (5.9) we obtain taking

$$\begin{aligned} |Y_t|^2 &\leq C \mathbb{E}_t \left[|\xi|^2 + \int_0^T |c_s|^2 ds + \sup_{0 \leq s \leq T} |S_s|^2 \right. \\ &\quad \left. + \int_0^T (|f_s|^2 + |a_s|^2 + |b_s|^2) ds \right], \end{aligned}$$

then, by the inequality of Jensen, we have

$$\begin{aligned} |Y_t|^\alpha &\leq C_\alpha \mathbb{E}_t \left[|\xi|^\alpha + \left(\int_0^T |c_s|^2 ds \right)^{\alpha/2} + \sup_{0 \leq s \leq T} |S_s|^\alpha \right. \\ &\quad \left. + \int_0^T (|f_s|^\alpha + |a_s|^\alpha + |b_s|^\alpha) ds \right], \end{aligned}$$

The result holds. \square

We can now estimate the variation of the solution caused by a variation in the data.

Proposition 5.7 (Proposition 3.8 in [39]) *Let (ξ^i, f^i, S^i) for $i = 1; 2$, be two triplets satisfying the above assumptions for some $\beta > 2$. Suppose that*

$$Y_t^i = \xi^i + \int_t^T f^i(s, Y_s^i, Z_s^i) ds - \int_t^T Z_s^i dB_s + N_T^i - N_t^i.$$

where $Y^i \in S^2(0, T)$, $Z^i \in H^2(0, T)$, N^i is a increasing process with $N_0^i = 0$ and $N_T^i \in L^2(\Omega_T)$. Set

$$\begin{aligned} \hat{\xi} &= \xi^1 - \xi^2; \quad \hat{S} = S^1 - S^2; \quad \hat{Y} = Y^1 - Y^2; \\ \hat{Z} &= Z^1 - Z^2; \quad \hat{N} = N^1 - N^2. \end{aligned}$$

Then there exists a constant C_α such that

$$\begin{aligned} &\mathbb{E} \left[\left(\int_0^T |\hat{Z}_s|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{2}} \right] \\ &\leq C_\alpha \left\{ \mathbb{E} \left[\sup_{0 \leq t \leq T} |\hat{Y}_t|^\alpha \right] + \mathbb{E} \left[\left(\int_0^T |f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)| ds \right)^\alpha \right] \right. \\ &\quad + \left(\mathbb{E} \left[\sup_{0 \leq t \leq T} |\hat{Y}_t|^\alpha \right] \right)^{1/2} \sum_{i=1}^2 \left(\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^i|^\alpha \right] \right)^{1/2} \\ &\quad \left. + \left(\mathbb{E} \left[\int_0^T |f^i(s, 0, 0)|^\alpha ds \right] \right)^{1/2} \right\}. \end{aligned}$$

Proof : By Ito's formula to $|\hat{Y}_t|^2$, we have :

$$\begin{aligned} |\hat{Y}_0|^2 + \int_0^T |\hat{Z}_s|^2 d\langle B \rangle_s &= |\hat{\xi}|^2 + 2 \int_0^T \hat{Y}_s (f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)) ds \\ &\quad - 2 \int_0^T \hat{Y}_s \hat{Z}_s dB_s + 2 \int_0^T \hat{Y}_s d\hat{N}_s. \end{aligned}$$

From the assumption of f , the Young's inequality and some majorisations, we obtain:

$$\begin{aligned} &\int_0^T |\hat{Z}_s|^2 d\langle B \rangle_s \\ &\leq 2|\hat{\xi}|^2 + 4 \sup_{0 \leq t \leq T} |\hat{Y}_t| \int_0^T |\hat{f}_s| ds + 4TL \left(1 + \frac{L}{\underline{\sigma}^2} \right) \sup_{0 \leq t \leq T} |\hat{Y}_t|^2 \\ &\quad - 4 \int_0^T \hat{Y}_s \hat{Z}_s dB_s + 4 \left(\sup_{0 \leq t \leq T} |\hat{Y}_t| \right) (|N_T^1| + |N_T^2|). \end{aligned}$$

Then there exists a positive constant C_α such that

$$\begin{aligned} & \left(\int_0^T |\hat{Z}_s|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{2}} \\ & \leq C_\alpha \left\{ \sup_{0 \leq t \leq T} |\hat{Y}_t|^\alpha + \left| \int_0^T \hat{Y}_s \hat{Z}_s dB_s \right|^{\frac{\alpha}{2}} \right. \\ & \quad \left. + \sup_{0 \leq t \leq T} |\hat{Y}_t|^{\frac{\alpha}{2}} \left(\left(\int_0^T |\hat{f}_s| ds \right)^{\frac{\alpha}{2}} + |N_T^1|^{\frac{\alpha}{2}} + |N_T^2|^{\frac{\alpha}{2}} \right) \right\}, \end{aligned}$$

and using the inequalities of type of BDG and Hölder, we have:

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T |\hat{Z}_s|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{2}} \right] \\ & \leq C_\alpha \left\{ \mathbb{E} \left[\sup_{0 \leq t \leq T} |\hat{Y}_t|^\alpha \right] + \left(\mathbb{E} \left[\sup_{0 \leq t \leq T} |\hat{Y}_t|^\alpha \right] \right)^{1/2} \left(\mathbb{E} \left[\left(\int_0^T |\hat{Z}_s|^2 ds \right)^{\frac{\alpha}{2}} \right] \right)^{1/2} \right. \\ & \quad + \left(\mathbb{E} \left[\sup_{0 \leq t \leq T} |\hat{Y}_t|^\alpha \right] \right)^{1/2} \left(\mathbb{E} \left[\left(\int_0^T |\hat{f}_s| ds \right)^\alpha \right] \right)^{1/2} \\ & \quad \left. + \left(\mathbb{E} \left[\sup_{0 \leq t \leq T} |\hat{Y}_t|^\alpha \right] \right)^{1/2} \left((\mathbb{E} [|N_T^1|^\alpha])^{1/2} + (\mathbb{E} [|N_T^2|^\alpha])^{1/2} \right) \right\}. \end{aligned}$$

Hence the result holds. \square

Proposition 5.8 Let (ξ^i, f^i, S^i) for $i = 1; 2$, be two triplets satisfying the above assumptions. suppose that (Y^i, Z^i, N^i) is a solution of the reflected G-BSDE associated with (ξ^i, f^i, S^i) for $\beta \geq \alpha > 1$. Let

$$\begin{aligned} \hat{\xi} &= \xi^1 - \xi^2; \quad \hat{S} = S^1 - S^2; \quad \hat{Y} = Y^1 - Y^2; \\ \hat{Z} &= Z^1 - Z^2; \quad \hat{N} = N^1 - N^2. \end{aligned}$$

So there is a constant C_α such that

$$\begin{aligned} |\hat{Y}_t|^\alpha & \leq C_\alpha \mathbb{E}_t \left[|\hat{\xi}|^\alpha + \sup_{t \leq s \leq T} |\hat{S}_s|^\alpha + \int_t^T (|\hat{a}_s|^\alpha + |\hat{b}_s|^\alpha) ds \right. \\ & \quad \left. + \left(\int_t^T |\hat{c}_s|^2 ds \right)^{\alpha/2} + \int_t^T |f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)|^\alpha ds \right], \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} |\hat{Y}_t|^\alpha \right] \\ & \leq C_\alpha \mathbb{E} \left[\sup_{0 \leq t \leq T} \mathbb{E}_t \left[|\hat{\xi}|^\alpha + \sup_{0 \leq s \leq T} |\hat{S}_s|^\alpha + \int_0^T (|\hat{a}_s|^\alpha + |\hat{b}_s|^\alpha) ds \right. \right. \\ & \quad \left. \left. + \left(\int_0^T |\hat{c}_s|^2 ds \right)^{\frac{\alpha}{2}} + \int_0^T |f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)|^\alpha ds \right] \right]. \end{aligned}$$

Proof. Let

$$\begin{aligned} A_t^i &= Y_t^i - S_t^i \\ &= \xi^i - S_T^i + \int_t^T (a_s^i + f^i(s, Y_s^i, Z_s^i)) ds \\ &\quad + \int_t^T b_s^i d\langle B \rangle_s - \int_t^T (Z_s^i - c_s^i) dB_s + N_T^i - N_t^i, \end{aligned}$$

and

$$\begin{aligned} \hat{A}_t &= \hat{Y}_t - \hat{S}_t \\ &= \hat{\xi} - \hat{S}_T + \int_t^T [(f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)) + \hat{a}_s] ds \\ &\quad + \int_t^T \hat{b}_s d\langle B \rangle_s - \int_t^T (\hat{Z}_s - \hat{c}_s) dB_s + \hat{N}_T - \hat{N}_t. \end{aligned}$$

Case $\alpha \leq 2$: For any $\gamma, \varepsilon > 0$, applying the Itô formula to $\left(|\hat{A}_t|^2 + \varepsilon_\alpha\right)^{\alpha/2} e^{\gamma t}$, where $\varepsilon_\alpha = \varepsilon(1 - \alpha/2)$, we have:

$$\begin{aligned} &\left(|\hat{A}_t|^2 + \varepsilon_\alpha\right)^{\alpha/2} e^{\gamma t} \\ &= \left(|\hat{A}_T|^2 + \varepsilon_\alpha\right)^{\alpha/2} e^{\gamma T} - \gamma \int_t^T e^{\gamma s} \left(|\hat{A}_s|^2 + \varepsilon_\alpha\right)^{\alpha/2} ds \\ &\quad + \alpha \int_t^T e^{\gamma s} \left(|\hat{A}_s|^2 + \varepsilon_\alpha\right)^{\frac{\alpha}{2}-1} \hat{A}_s \hat{b}_s d\langle B \rangle_s \\ &\quad + \alpha \int_t^T e^{\gamma s} \left(|\hat{A}_s|^2 + \varepsilon_\alpha\right)^{\frac{\alpha}{2}-1} \hat{A}_s [(f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)) - \hat{a}_s] ds \\ &\quad - \alpha \int_t^T e^{\gamma s} \left(|\hat{A}_s|^2 + \varepsilon_\alpha\right)^{\frac{\alpha}{2}-1} \hat{A}_s (\hat{Z}_s - \hat{c}_s) dB_s \\ &\quad + \alpha \int_t^T e^{\gamma s} \left(|\hat{A}_s|^2 + \varepsilon_\alpha\right)^{\frac{\alpha}{2}-1} \hat{A}_s d\hat{N}_s \\ &\quad - \frac{\alpha}{2} \int_t^T e^{\gamma s} \left(|\hat{A}_s|^2 + \varepsilon_\alpha\right)^{\frac{\alpha}{2}-1} |\hat{Z}_s - \hat{c}_s|^2 d\langle B \rangle_s \\ &\quad + \alpha \left(1 - \frac{\alpha}{2}\right) \int_t^T e^{\gamma s} \left(|\hat{A}_s|^2 + \varepsilon_\alpha\right)^{\frac{\alpha}{2}-2} |\hat{A}_s|^2 |\hat{Z}_s - \hat{c}_s|^2 d\langle B \rangle_s. \end{aligned}$$

Since

$$\hat{A}_s d\hat{N}_s = A_s^1 dN_s^1 + A_s^2 dN_s^2 - A_s^2 dN_s^1 - A_s^1 dN_s^2$$

and

$$-A_s^2 dN_s^1 - A_s^1 dN_s^2 \leq 0,$$

we set

$$\begin{aligned} \hat{f}_s &= (f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)) \\ M_t &= \alpha \int_0^t e^{\gamma s} \left(|\hat{A}_s|^2 + \varepsilon_\alpha\right)^{\frac{\alpha}{2}-1} \left[\hat{A}_s (\hat{Z}_s - \hat{c}_s) dB_s - A_s^1 dN_s^1 - A_s^2 dN_s^2\right], \end{aligned}$$

then, using the Lipschitz property of f^1 , Young's inequality and some majorisations, we obtain:

$$\begin{aligned}
& \left(|\hat{A}_t|^2 + \varepsilon_\alpha \right)^{\alpha/2} e^{\gamma t} + (M_T - M_t) \\
\leq & \left(|\hat{A}_T|^2 + \varepsilon_\alpha \right)^{\alpha/2} e^{\gamma T} + \int_t^T e^{\gamma s} \left(|\hat{f}_s|^\alpha + |\hat{a}_s|^\alpha + \bar{\sigma}^2 |\hat{b}_s|^\alpha + L |\hat{S}_s|^\alpha \right) ds \\
& + \left(\alpha L + 2(L+1)(\alpha-1) + \frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)} - \gamma \right) \int_t^T e^{\gamma s} \left(|\hat{A}_s|^2 + \varepsilon_\alpha \right)^{\alpha/2} ds \\
& + L \left(\int_t^T e^{\frac{2\gamma s}{\alpha}} |\hat{c}_s|^2 ds \right)^{\frac{\alpha}{2}} - \frac{\alpha}{4}(\alpha-1) \int_t^T e^{\gamma s} \left(|\hat{A}_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} |\hat{Z}_s - \hat{c}_s|^2 d\langle B \rangle_s.
\end{aligned}$$

Choosing

$$\gamma \geq \alpha L + 2(L+1)(\alpha-1) + \frac{\alpha L^2}{\underline{\sigma}^2(\alpha-1)},$$

we have

$$\begin{aligned}
& \left(|\hat{A}_t|^2 + \varepsilon_\alpha \right)^{\alpha/2} e^{\gamma t} + (M_T - M_t) \\
\leq & \left(|\hat{A}_T|^2 + \varepsilon_\alpha \right)^{\alpha/2} e^{\gamma T} + L \left(\int_t^T e^{\frac{2\gamma s}{\alpha}} |\hat{c}_s|^2 ds \right)^{\frac{\alpha}{2}} \\
& + \int_t^T e^{\gamma s} \left(|\hat{f}_s|^\alpha + |\hat{a}_s|^\alpha + \bar{\sigma}^2 |\hat{b}_s|^\alpha + L |\hat{S}_s|^\alpha \right) ds,
\end{aligned}$$

and since M is a G -martingale, taking the conditional G -expectation we get:

$$\begin{aligned}
& \left(|\hat{A}_t|^2 + \varepsilon_\alpha \right)^{\alpha/2} e^{\gamma t} \\
\leq & \mathbb{E}_t \left[\left(|\hat{A}_T|^2 + \varepsilon_\alpha \right)^{\alpha/2} e^{\gamma T} + L \left(\int_t^T e^{\frac{2\gamma s}{\alpha}} |\hat{c}_s|^2 ds \right)^{\frac{\alpha}{2}} \right. \\
& \left. + \int_t^T e^{\gamma s} \left(|\hat{f}_s|^\alpha + |\hat{a}_s|^\alpha + \bar{\sigma}^2 |\hat{b}_s|^\alpha + L |\hat{S}_s|^\alpha \right) ds \right].
\end{aligned}$$

Letting $\varepsilon \downarrow 0$, we could find a positive constant C_α , such that

$$\begin{aligned}
|\hat{A}_t|^\alpha & \leq C_\alpha \mathbb{E}_t \left[|\hat{A}_T|^\alpha + \left(\int_t^T |\hat{c}_s|^2 ds \right)^{\frac{\alpha}{2}} \right. \\
& \left. + \int_t^T \left(|\hat{f}_s|^\alpha + |\hat{a}_s|^\alpha + |\hat{b}_s|^\alpha + |\hat{S}_s|^\alpha \right) ds \right].
\end{aligned}$$

Consequently

$$\begin{aligned}
|\hat{Y}_t|^\alpha & \leq C_\alpha \mathbb{E}_t \left[|\hat{\xi}|^\alpha + \sup_{t \leq s \leq T} |\hat{S}_s|^\alpha + \left(\int_0^T |\hat{c}_s|^2 ds \right)^{\frac{\alpha}{2}} \right. \\
& \left. + \int_0^T \left(|\hat{f}_s|^\alpha + |\hat{a}_s|^\alpha + |\hat{b}_s|^\alpha \right) ds \right].
\end{aligned}$$

Case $\alpha > 2$: From the first case, we have

$$\begin{aligned} |\hat{Y}_t|^2 &\leq C \mathbb{E}_t \left[|\hat{\xi}|^2 + \sup_{t \leq s \leq T} |\hat{S}_s|^2 + \int_0^T |\hat{c}_s|^2 ds \right. \\ &\quad \left. + \int_0^T \left(|\hat{f}_s|^2 + |\hat{a}_s|^2 + |\hat{b}_s|^2 \right) ds \right], \end{aligned}$$

then, by the inequality of Jensen, we get

$$\begin{aligned} |\hat{Y}_t|^\alpha &\leq C_\alpha \mathbb{E}_t \left[|\hat{\xi}|^\alpha + \sup_{t \leq s \leq T} |\hat{S}_s|^\alpha + \left(\int_0^T |\hat{c}_s|^2 ds \right)^{\frac{\alpha}{2}} \right. \\ &\quad \left. + \int_0^T \left(|\hat{f}_s|^\alpha + |\hat{a}_s|^\alpha + |\hat{b}_s|^\alpha \right) ds \right]. \end{aligned}$$

□

5.4 Existence of the solution of the reflected G -BSDE

In this section, we will show the existence of solution of the reflected G -BSDE by penalizing method. In what follows, C denote a constant whose value can vary from line to line.

Theorem 5.9 *Assume f satisfies the assumptions **(H1)** and **(H2)** for some $\beta > 2$ and $\xi \in L_G^\beta(\Omega_T)$. Then there exists a unique triplet (Y, Z, N) , solution of the reflected G -BSDE:*

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s + N_T - N_t, \\ Y_t \geq S_t, \end{cases}$$

in the sense of the definition 5.1. Moreover, for any $1 < \alpha < \beta$ we have $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T)$ and $N_T \in L_G^\alpha(\Omega_T)$.

Assume f satisfies the assumptions **(H1)** and **(H2)** for some $\beta > 2$ and $\xi \in L_G^\beta(\Omega_T)$. For each $n \in \mathbb{N}$, let (Y^n, Z^n, K^n) the unique solution of the Lipschitz G -BSDE

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n) ds + n \int_t^T (Y_s^n - S_s)^- ds - \int_t^T Z_s^n dB_s - (K_T^n - K_t^n). \quad (5.10)$$

We set

$$R_0^n = 0 \quad \text{and} \quad R_T^n - R_t^n = n \int_t^T (Y_s^n - S_s)^- ds, \quad (5.11)$$

and

$$N_T^n - N_t^n = n \int_t^T (Y_s^n - S_s)^- ds - (K_T^n - K_t^n), \quad 0 \leq t \leq T.$$

The process R^n will penalize the solution when it is below S and force more and more strongly to stay above S . We will show that the triplet (Y^n, Z^n, N^n) converges as n tends to infinity towards

the solution of our reflected G -BSDE. According to the result of existence of Lipschitz BSDE driven by G -Brownian motion, for all $n \in \mathbb{N}$ and $1 < \alpha < \beta$, we have:

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n|^\alpha \right] < +\infty.$$

We will begin by establishing some estimates, uniform in n , for (Y^n, Z^n, N^n) .

Lemma 5.10 *Let $\beta > 2$, $\xi \in L_G^\beta(\Omega_T)$ and f satisfy assumptions (H1) and (H2). For $n \in \mathbb{N}$, let (Y^n, Z^n, K^n) be the unique solution of the non reflected G -BSDE (5.10) and R^n the penalization process defined by (5.11). Then for all $1 < \alpha < \beta$, there exists a constant C_α which does not depend on $n \in \mathbb{N}$, such that*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n|^\alpha + \left(\int_0^T |Z_s^n|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{2}} + |N_T^n|^\alpha \right] \leq C_\alpha.$$

Proof. If $\alpha \leq 2$, for $\gamma > 0$, $\varepsilon > 0$, by Ito's formula applied to $(|Y_t^n - S_t|^2 + \varepsilon_\alpha)^{\alpha/2} e^{\gamma t}$, where $\varepsilon_\alpha = \varepsilon \left(1 - \frac{\alpha}{2}\right)$, we have:

$$\begin{aligned} & \left(|Y_t^n - S_t|^2 + \varepsilon_\alpha \right)^{\alpha/2} e^{\gamma t} + \gamma \int_t^T e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\alpha/2} ds \\ = & \left(|\xi - S_T|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}} e^{\gamma T} + \alpha \int_t^T e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^n - S_s) a_s ds \\ & + \alpha \int_t^T e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^n - S_s) b_s d\langle B \rangle_s \\ & + \alpha \int_t^T e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^n - S_s) f(s) ds \\ & - \int_t^T \alpha e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^n - S_s) ((Z_s^n - c_s) dB_s + dK_s^n) \\ & + n\alpha \int_t^T e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^n - S_s) (Y_s^n - S_s)^- ds \\ & - \frac{\alpha}{2} \int_t^T e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} |Z_s^n - c_s|^2 d\langle B \rangle_s \\ & + \alpha \left(1 - \frac{\alpha}{2}\right) \int_t^T e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-2} |Y_s^n - S_s|^2 |Z_s^n - c_s|^2 d\langle B \rangle_s. \end{aligned}$$

Since

$$n\alpha \int_t^T e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^n - S_s) (Y_s^n - S_s)^- ds \leq 0$$

and

$$\alpha \left(1 - \frac{\alpha}{2}\right) \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-2} |Y_s^n - S_s|^2 \leq \alpha \left(1 - \frac{\alpha}{2}\right) \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1},$$

by the Lipschitz property of the function f , we obtain:

$$\begin{aligned}
& \left(|Y_t^n - S_t|^2 + \varepsilon_\alpha \right)^{\alpha/2} e^{\gamma t} + (M_T^n - M_t^n) \\
\leq & \left(|\xi - S_T|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}} e^{\gamma T} + \alpha \int_t^T e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha-1}{2}} |a_s| ds \\
& + \alpha \int_t^T e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha-1}{2}} |b_s| d\langle B \rangle_s \\
& + \alpha \int_t^T e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} |Y_s^n - S_s| (|f_s| + L|Y_s^n - S_s| + L|S_s| \\
& + L|Z_s^n - c_s| + L|c_s|) ds - \gamma \int_t^T e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\alpha/2} ds \\
& - \frac{\alpha}{2} (\alpha - 1) \int_t^T e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} |Z_s^n - c_s|^2 d\langle B \rangle_s
\end{aligned}$$

Here we set

$$M_t^n = 2 \int_0^t e^{\gamma s} (Y_s^n - S_s) (Z_s^n - c_s) dB_s + 2 \int_0^t e^{\gamma s} (Y_s^n - S_s)^+ dK_s^n.$$

Using Young's inequality, we get:

$$\begin{aligned}
& \left(|Y_t^n - S_t|^2 + \varepsilon_\alpha \right)^{\alpha/2} e^{\gamma t} + (M_T^n - M_t^n) \\
\leq & \left(|\xi - S_T|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}} e^{\gamma T} + \int_t^T e^{\gamma s} (|f_s|^\alpha + L|S_s|^\alpha + |a_s|^\alpha + \bar{\sigma}^2 |b_s|^\alpha) ds \\
& + \left((\alpha - 1) (2 + 2L + \bar{\sigma}^2) + \alpha L + \frac{\alpha L^2}{\underline{\sigma}^2 (\alpha - 1)} - \gamma \right) \int_t^T e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\alpha/2} ds \\
& + L \left(\int_t^T e^{\frac{2\gamma s}{\alpha}} |c_s|^2 ds \right)^{\alpha/2} - \frac{\alpha (\alpha - 1)}{4} \int_t^T e^{\gamma s} \left(|Y_s^n - S_s|^2 + \varepsilon_\alpha \right)^{\frac{\alpha-1}{2}} |Z_s^n - c_s|^2 d\langle B \rangle_s.
\end{aligned}$$

Then since M^n is a G -martingale, for

$$\gamma \geq (\alpha - 1) (2 + 2L + \bar{\sigma}^2) + \alpha L + \frac{\alpha L^2}{\underline{\sigma}^2 (\alpha - 1)},$$

by taking the conditional G -expectation, we obtain

$$\begin{aligned}
& \left(|Y_t^n - S_t|^2 + \varepsilon_\alpha \right)^{\alpha/2} \\
\leq & e^{\gamma T} \mathbb{E}_t \left[\left(|\xi - S_T|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}} + \int_t^T (|f_s|^\alpha + |a_s|^\alpha + \bar{\sigma}^2 |b_s|^\alpha) ds \right. \\
& \left. + LT \sup_{t \leq s \leq T} |S_s|^\alpha + L \left(\int_t^T e^{\frac{2\gamma s}{\alpha}} |c_s|^2 ds \right)^{\alpha/2} \right].
\end{aligned}$$

Letting $\varepsilon \downarrow 0$, we have the existence of a positive constant C_α (which does not depend on n) such that:

$$\begin{aligned}
|Y_t^n|^\alpha & \leq C_\alpha \mathbb{E}_t \left[|\xi|^\alpha + \sup_{t \leq s \leq T} |S_s|^\alpha + \int_t^T (|f_s|^\alpha + |a_s|^\alpha + |b_s|^\alpha) ds \right. \\
& \left. + \left(\int_t^T |c_s|^2 ds \right)^{\alpha/2} \right]. \tag{5.12}
\end{aligned}$$

For $2 \geq \alpha$, since by (5.12) we have

$$\begin{aligned} |Y_t^n|^2 \leq & C \mathbb{E}_t \left[|\xi|^2 + \sup_{t \leq s \leq T} |S_s|^2 + \int_t^T (|f_s|^2 + |a_s|^2 + |b_s|^2) ds \right. \\ & \left. + \int_t^T |c_s|^2 ds \right], \end{aligned}$$

we get

$$\begin{aligned} |Y_t^n|^\alpha & \leq C_\alpha \mathbb{E}_t \left[|\xi|^\alpha + \sup_{t \leq s \leq T} |S_s|^\alpha + \left(\int_t^T (|f_s|^2 + |a_s|^2 + |b_s|^2) ds \right)^{\alpha/2} \right. \\ & \quad \left. + \left(\int_t^T |c_s|^2 ds \right)^{\alpha/2} \right] \\ & \leq C_\alpha \mathbb{E}_t \left[|\xi|^\alpha + \sup_{t \leq s \leq T} |S_s|^\alpha + \int_t^T (|f_s|^\alpha + |a_s|^\alpha + |b_s|^\alpha) ds \right. \\ & \quad \left. + \left(\int_t^T |c_s|^2 ds \right)^{\alpha/2} \right]. \end{aligned} \tag{5.13}$$

so from (5.12) and (5.13), we obtain:

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n|^\alpha \right] & \leq C_\alpha \mathbb{E} \sup_{0 \leq t \leq T} \mathbb{E}_t \left[\left[|\xi|^\alpha + \sup_{0 \leq s \leq T} |S_s|^\alpha + \left(\int_t^T |c_s|^2 ds \right)^{\alpha/2} \right. \right. \\ & \quad \left. \left. + \int_0^T (|f_s|^\alpha + |a_s|^\alpha + |b_s|^\alpha) ds \right] \right]. \end{aligned}$$

Moreover applying Ito's formula to $|Y_t^n|^2$, we get:

$$\begin{aligned} & |Y_0^n|^2 + \int_0^T |Z_s^n|^2 d\langle B \rangle_s \\ = & |\xi|^2 + 2 \int_0^T Y_s^n f(s, Y_s^n, Z_s^n) ds - 2 \int_0^T Y_s^n Z_s^n dB_s + 2 \int_0^T Y_s^n dN^n \\ \leq & |\xi|^2 + 2 \int_0^T (|Y_s^n| |f_s| + L |Y_s^n|^2 + L |Y_s^n| |Z_s^n|) ds \\ & - 2 \int_0^T Y_s^n Z_s^n dB_s + 2 \sup_{0 \leq t \leq T} |Y_t^n| |N_T^n| \\ \leq & |\xi|^2 + \int_0^T |f_s|^2 ds + \left(1 + 2L + \frac{2L^2}{\sigma^2} \right) T \sup_{0 \leq t \leq T} |Y_t^n|^2 \\ & + \frac{1}{2} \int_0^T |Z_s^n|^2 d\langle B \rangle_s - 2 \int_0^T Y_s^n Z_s^n dB_s + 2 \sup_{0 \leq t \leq T} |Y_t^n| |N_T^n|. \end{aligned}$$

Then there exists a positive constant C_α such that:

$$\begin{aligned}
& \left(\int_0^T |Z_s^n|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{2}} \\
& \leq C_\alpha \left\{ \left| \int_0^T |f_s|^2 ds \right|^{\frac{\alpha}{2}} + \sup_{0 \leq t \leq T} |Y_t^n|^\alpha \right. \\
& \quad \left. + \left| \int_0^T Y_s^n Z_s^n dB_s \right|^{\frac{\alpha}{2}} + 2 \sup_{0 \leq t \leq T} |Y_t^n|^{\frac{\alpha}{2}} |N_T^n|^{\frac{\alpha}{2}} \right\} \\
& \leq C_\alpha \left\{ \int_0^T |f_s|^\alpha ds + \left(1 + \frac{1}{\varepsilon}\right) \sup_{0 \leq t \leq T} |Y_t^n|^\alpha \right. \\
& \quad \left. + \left| \int_0^T Y_s^n Z_s^n dB_s \right|^{\frac{\alpha}{2}} + \varepsilon |N_T^n|^\alpha \right\}. \tag{5.14}
\end{aligned}$$

Since

$$N_T^n = R_T^n - K_T^n = Y_0^n - \xi - \int_0^T f(s, Y_s^n, Z_s^n) ds + \int_0^T Z_s^n dB_s,$$

there exists a positive constant C_α , which does not depend on n , such that:

$$\begin{aligned}
|N_T^n|^\alpha & \leq C_\alpha \left\{ \int_0^T |f_s|^\alpha ds + \sup_{0 \leq t \leq T} |Y_t^n|^\alpha + \left(\int_0^T |Z_s^n|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{2}} \right. \\
& \quad \left. + \left| \int_0^T Z_s^n dB_s \right|^\alpha \right\}. \tag{5.15}
\end{aligned}$$

Then, substituting in the inequality (5.14), we obtain:

$$\begin{aligned}
& \left(\int_0^T |Z_s^n|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{2}} \\
& \leq C_\alpha \left\{ (1 + \varepsilon) \int_0^T |f_s|^\alpha ds + \left(1 + \frac{1}{\varepsilon} + \varepsilon\right) \sup_{0 \leq t \leq T} |Y_t^n|^\alpha \right. \\
& \quad \left. + \left| \int_0^T Y_s^n Z_s^n dB_s \right|^{\frac{\alpha}{2}} + \varepsilon \left(\int_0^T |Z_s^n|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{2}} + \varepsilon \left| \int_0^T Z_s^n dB_s \right|^\alpha \right\}.
\end{aligned}$$

Taking the G -expectation yields:

$$\begin{aligned}
& \mathbb{E} \left[\left(\int_0^T |Z_s^n|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{2}} \right] \\
& \leq C_\alpha \left\{ (1 + \varepsilon) \mathbb{E} \left[\int_0^T |f_s|^\alpha ds \right] + \left(1 + \frac{1}{\varepsilon} + \varepsilon\right) \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n|^\alpha \right] \right. \\
& \quad \left. + C_1 \mathbb{E} \left[\left(\int_0^T |Y_s^n Z_s^n|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{4}} \right] + \varepsilon (1 + C_2) \mathbb{E} \left[\left(\int_0^T |Z_s^n|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{2}} \right] \right\} \\
& \leq C_\alpha \left\{ (1 + \varepsilon) \mathbb{E} \left[\int_0^T |f_s|^\alpha ds \right] + \left(1 + \frac{1}{\varepsilon} + \varepsilon + \frac{1}{2\nu}\right) \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n|^\alpha \right] \right\} \\
& \quad + C_\alpha \left(\frac{\nu C_1}{2} + \varepsilon + \varepsilon C_2 \right) \mathbb{E} \left[\left(\int_0^T |Z_s^n|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{2}} \right].
\end{aligned}$$

we choose ν and ε positive as $C_\alpha \left(\frac{\nu C_1}{2} + \varepsilon + \varepsilon C_2 \right) < 1$, we get

$$\mathbb{E} \left[\left(\int_0^T |Z_s^n|^2 d\langle B \rangle_s \right)^{\frac{\alpha}{2}} \right] \leq C_\alpha \left\{ \mathbb{E} \left[\int_0^T |f_s|^\alpha ds \right] + \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n|^\alpha \right] \right\}. \quad (5.16)$$

By inequalities (5.15) and (5.16), we obtain:

$$\mathbb{E} [|N_T^n|^\alpha] \leq C_\alpha \left\{ \mathbb{E} \left[\int_0^T |f_s|^\alpha ds \right] + \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n|^\alpha \right] \right\}.$$

Remark 5.11 *If we define*

$$f_n(t, y, z) = f(t, y, z) + n(y - S_t)^-,$$

for each $n \in \mathbb{N}^*$, f_n satisfies the assumptions **(H1)** and **(H2)** and $f_n \leq f_{n+1}$ then it follows from the comparison theorem (theorem 3.6 [40]) that $Y_t^n \leq Y_t^{n+1}$, $0 \leq t \leq T$, q.s. Therefore

$$Y_t^n \uparrow Y_t, \quad 0 \leq t \leq T \quad \text{q.s.}$$

and Fatou's lemma, we have

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^\alpha \right] \leq C_\alpha.$$

Lemma 5.12 *Let $1 < \alpha < \beta$, there is a positive constant C_α not dependent on n , such that*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |(Y_t^n - S_t)^-|^\alpha \right] \leq C_\alpha \left\{ \frac{1}{n^{\alpha-1}} + \frac{1}{n^{\alpha/2}} \right\}.$$

Proof. For $n \in \mathbb{N}$, $s \in [0, T]$ and $y, z \in \mathbb{R}$, set

$$\begin{aligned} \tilde{f}^{n,1}(s, y, z) &= f(s, Y_s^n, Z_s^n) + n(y - S_s)^-, \\ \tilde{f}^{n,2}(s, y, z) &= f(s, 0, 0) - L|Y_s^n| - L|Z_s^n| - n(y - S_s). \end{aligned}$$

For $i = 1, 2$, let $(\tilde{Y}^{n,i}, \tilde{Z}^{n,i}, \tilde{K}^{n,i})$ be the unique solution of the G -BSDE

$$\tilde{Y}_t^{n,i} = \xi + \int_t^T \tilde{f}^{n,i}(s, \tilde{Y}_s^{n,i}, \tilde{Z}_s^{n,i}) ds - \int_t^T \tilde{Z}_s^{n,i} dB_s - (\tilde{K}_T^{n,i} - \tilde{K}_t^{n,i}).$$

Since the functions $\tilde{f}^{n,i}$ satisfy assumptions (H1) and (H2) and $\tilde{f}^{n,1} \geq \tilde{f}^{n,2}$, according to the comparison theorem for G -BSDE (cf theorem 3.6 [40]), we have q.s.

$$\tilde{Y}_t^{n,1} \geq \tilde{Y}_t^{n,2}, \quad 0 \leq t \leq T, \quad n \in \mathbb{N}.$$

So

$$\tilde{Y}^{n,1} - S \geq \tilde{Y}^{n,2} - S,$$

which implies

$$\left(\tilde{Y}^{n,1} - S \right)^- \leq \left(\tilde{Y}^{n,2} - S \right)^-, \quad \text{q.s..}$$

On the other hand $(\tilde{Y}^{n,2} - S, \tilde{Z}^{n,2} - c, \tilde{K}^{n,2})$ being the unique solution of the linear G -BSDE

$$\begin{aligned} \tilde{Y}_t^{n,2} - S_t &= \xi - S_T + \int_t^T \left\{ f_s - L|Y_s^n| - L|Z_s^n| - n(\tilde{Y}_s^{n,2} - S_s) + a_s \right\} ds \\ &\quad + \int_t^T b_s d\langle B \rangle_s - \int_t^T (\tilde{Z}_s^{n,2} - c_s) dB_s - (\tilde{K}_T^{n,2} - \tilde{K}_t^{n,2}), \end{aligned}$$

Therefore, according to Theorem 3.2. and Remark 3.3. of [40], we have

$$\begin{aligned} &\tilde{Y}_t^{n,2} - S_t \\ &= \mathbb{E}_t \left[e^{-n(T-t)} (\xi - S_T) + \int_t^T e^{-n(s-t)} \{ f_s - L|Y_s^n| - L|Z_s^n| + a_s \} ds \right. \\ &\quad \left. + \int_t^T e^{-n(s-t)} b_s d\langle B \rangle_s \right]. \end{aligned}$$

Since $S_T \leq \xi$ q.s., we have

$$\begin{aligned} \tilde{Y}_t^{n,2} - S_t &\geq \mathbb{E}_t \left[\int_t^T e^{-n(s-t)} \{ f_s - L|Y_s^n| - L|Z_s^n| + a_s \} ds \right. \\ &\quad \left. + \int_t^T e^{-n(s-t)} b_s d\langle B \rangle_s \right]. \end{aligned}$$

Then there exists a positive constant C not depending on n such that for all $t \in [0, T]$, $1 < \alpha < \beta$

$$\begin{aligned} &\left| (\tilde{Y}_t^{n,2} - S_t) \right|^\alpha \\ &\leq \left| \mathbb{E}_t \left[\int_t^T e^{-n(s-t)} \{ f_s - L|Y_s^n| - L|Z_s^n| + a_s \} ds + \int_t^T e^{-n(s-t)} b_s d\langle B \rangle_s \right] \right|^\alpha \\ &\leq C \left\{ \mathbb{E}_t \left[\left| \int_t^T e^{-n(s-t)} \{ f_s - |Y_s^n| + a_s \} ds + \int_t^T e^{-n(s-t)} b_s d\langle B \rangle_s \right|^\alpha \right] \right. \\ &\quad \left. + \mathbb{E}_t \left[\left| \int_t^T e^{-n(s-t)} |Z_s^n| ds \right|^\alpha \right] \right\} \\ &\leq C \left\{ \mathbb{E}_t \left[\left(\int_t^T e^{-n(s-t)} \{ |f_s| + |Y_s^n| + |a_s| + |b_s| \} ds \right)^\alpha \right] \right. \\ &\quad \left. + \mathbb{E}_t \left[\left(\int_t^T e^{-n(s-t)} |Z_s^n| ds \right)^\alpha \right] \right\} \\ &\leq C \left\{ \left(\frac{\alpha - 1}{n\alpha} \left(1 - e^{-n\alpha(T-t)/(\alpha-1)} \right) \right)^{\alpha-1} \mathbb{E}_t \left[\int_t^T \{ |f_s|^\alpha + |Y_s^n|^\alpha + |a_s|^\alpha + |b_s|^\alpha \} ds \right] \right. \\ &\quad \left. + \left(\frac{1 - e^{-2n(T-t)}}{2n} \right)^{\alpha/2} \mathbb{E}_t \left[\left(\int_t^T |Z_s^n|^2 ds \right)^{\alpha/2} \right] \right\} \\ &\leq C \left\{ \left(\frac{\alpha - 1}{n\alpha} \right)^{\alpha-1} \mathbb{E}_t \left[\int_0^T \{ |f_s|^\alpha + |Y_s^n|^\alpha + |a_s|^\alpha + |b_s|^\alpha \} ds \right] \right. \\ &\quad \left. + \left(\frac{1}{2n} \right)^{\alpha/2} \mathbb{E}_t \left[\left(\int_0^T |Z_s^n|^2 ds \right)^{\alpha/2} \right] \right\}. \end{aligned}$$

Hence by estimates of Lemma 5.10, there is a positive constant C not dependent to n such that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \left(\tilde{Y}_t^{n,1} - S_t \right)^- \right|^\alpha \right] \leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \left(\tilde{Y}_t^{n,2} - S_t \right)^- \right|^\alpha \right] \leq C \left\{ \frac{1}{n^{\alpha-1}} + \frac{1}{n^{\alpha/2}} \right\}.$$

Since $\tilde{Y}^{n,1} = Y$, we have the result. \square

Proof of Theorem 5.9: For $n, p \in \mathbb{N}$ ($n < p$), it follows from Ito's formula applied to $\left(|Y_t^n - Y_t^p|^2 + \varepsilon_\alpha \right)^{\alpha/2} e^{\gamma t}$, which $\gamma, \varepsilon > 0$ and $\varepsilon_\alpha = \varepsilon(1 - \alpha/2)^+$, that

$$\begin{aligned} & \left(|Y_t^n - Y_t^p|^2 + \varepsilon_\alpha \right)^{\alpha/2} e^{\gamma t} \\ = & (\varepsilon_\alpha)^{\alpha/2} e^{\gamma T} - \gamma \int_t^T e^{\gamma s} \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\alpha/2} ds \\ & + \alpha \int_t^T e^{\gamma s} \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^n - Y_s^p) (f(s, Y_s^n, Z_s^n) - f(s, Y_s^p, Z_s^p)) ds \\ & - \alpha \int_t^T e^{\gamma s} \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^n - Y_s^p) (Z_s^n - Z_s^p) dB_s \\ & - \alpha \int_t^T e^{\gamma s} \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^n - Y_s^p) d(K_s^n - K_s^p) \\ & + \alpha \int_t^T e^{\gamma s} \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^n - Y_s^p) d(R_s^n - R_s^p) \\ & - \frac{\alpha}{2} \int_t^T e^{\gamma s} \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} |Z_s^n - Z_t^p|^2 d\langle B \rangle_s \\ & + \alpha \left(1 - \frac{\alpha}{2} \right) \int_t^T e^{\gamma s} \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-2} |Y_s^n - Y_s^p|^2 |Z_s^n - Z_t^p|^2 d\langle B \rangle_s \\ \leq & (\varepsilon_\alpha)^{\alpha/2} e^{\gamma T} + \left(\alpha L + \frac{\alpha L^2}{\sigma^2(\alpha-1)} - \gamma \right) \int_t^T e^{\gamma s} \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\alpha/2} ds \\ & - (M_T - M_t) + \alpha \int_t^T e^{\gamma s} \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^n - S_s)^- dR_s^p \\ & + \alpha \int_t^T e^{\gamma s} \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^p - S_s)^- dR_s^n \\ & - \frac{\alpha(\alpha-1)}{4} \int_t^T e^{\gamma s} \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} |Z_s^n - Z_t^p|^2 d\langle B \rangle_s. \end{aligned}$$

Then as

$$\begin{aligned} & \int_t^T e^{\gamma s} \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^p - S_s)^- dR_s^n \\ = & n \int_t^T e^{\gamma s} \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^p - S_s)^- (Y_s^n - S_s)^- ds \\ \leq & p \int_t^T e^{\gamma s} \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^p - S_s)^- (Y_s^n - S_s)^- ds \\ = & \int_t^T e^{\gamma s} \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^n - S_s)^- dR_s^p, \end{aligned}$$

for $\gamma \geq \alpha L + \frac{\alpha L^2}{\sigma^2(\alpha-1)}$, there exists a positive constant C_α such that:

$$\begin{aligned}
& \left(|Y_t^n - Y_t^p|^2 + \varepsilon_\alpha \right)^{\alpha/2} \\
& \leq C_\alpha \left\{ (\varepsilon_\alpha)^{\alpha/2} + \mathbb{E}_t \left[\int_t^T \left(|Y_s^n - Y_s^p|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^n - S_s)^- dR_s^p \right] \right\} \\
& \leq C_\alpha \left\{ (\varepsilon_\alpha)^{\alpha/2} + \mathbb{E}_t \left[\int_t^T \left(|(Y_s^n - S_s)^-|^2 + \varepsilon_\alpha \right)^{\frac{\alpha-1}{2}} dR_s^p \right] \right. \\
& \quad \left. + \mathbb{E}_t \left[\int_t^T \left(|(Y_s^p - S_s)^-|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^n - S_s)^- dR_s^p \right] \right\} \\
& \leq C_\alpha \left\{ (\varepsilon_\alpha)^{\alpha/2} + \mathbb{E}_t \left[\sup_{0 \leq s \leq T} \left(|(Y_s^n - S_s)^-|^2 + \varepsilon_\alpha \right)^{\frac{\alpha-1}{2}} R_T^p \right] \right. \\
& \quad \left. + \mathbb{E}_t \left[\sup_{0 \leq s \leq T} \left(|(Y_s^p - S_s)^-|^2 + \varepsilon_\alpha \right)^{\frac{\alpha}{2}-1} (Y_s^n - S_s)^- R_T^p \right] \right\} \\
& \leq C_\alpha \left\{ (\varepsilon_\alpha)^{\alpha/2} + \mathbb{E}_t \left[\sup_{0 \leq s \leq T} \left(|(Y_s^n - S_s)^-|^2 + \varepsilon_\alpha \right)^{\frac{\alpha-1}{2}} N_T^p \right] \right\}.
\end{aligned}$$

Letting $\varepsilon \downarrow 0$, we obtain:

$$\begin{aligned}
|Y_t^n - Y_t^p|^\alpha & \leq C_\alpha \mathbb{E}_t \left[\sup_{0 \leq s \leq T} |(Y_s^n - S_s)^-|^{\alpha-1} N_T^p \right] \\
& \leq C_\alpha \left(\mathbb{E}_t \left[\sup_{0 \leq s \leq T} |(Y_s^n - S_s)^-|^{(\alpha-1)\alpha'^*} \right] \right)^{\frac{1}{\alpha'^*}} \times \left(\mathbb{E}_t \left[|N_T^p|^{\alpha'} \right] \right)^{\frac{1}{\alpha'}},
\end{aligned}$$

where $1 < \alpha < \alpha' < \beta$, and $\alpha'^* = \frac{\alpha'}{\alpha'-1}$. So

$$\begin{aligned}
& \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^\alpha \right] \\
& \leq C_\alpha \mathbb{E} \left[\sup_{0 \leq t \leq T} \left(\mathbb{E}_t \left[\sup_{0 \leq s \leq T} |(Y_s^n - S_s)^-|^{(\alpha-1)\alpha'^*} \right] \right)^{\frac{1}{\alpha'^*}} \times \sup_{0 \leq t \leq T} \left(\mathbb{E}_t \left[|N_T^p|^{\alpha'} \right] \right)^{\frac{1}{\alpha'}} \right] \\
& \leq C_\alpha \left\{ \mathbb{E} \left[\sup_{0 \leq t \leq T} \mathbb{E}_t \left[\sup_{0 \leq s \leq T} |(Y_s^n - S_s)^-|^{(\alpha-1)\alpha'^*} \right] \right] \right\}^{\frac{1}{\alpha'^*}} \times \left\{ \mathbb{E} \left[\sup_{0 \leq t \leq T} \mathbb{E}_t \left[|N_T^p|^{\alpha'} \right] \right] \right\}^{\frac{1}{\alpha'}} \\
& \leq C_\alpha C \left\{ \left(\mathbb{E} \left[\sup_{0 \leq s \leq T} |(Y_s^n - S_s)^-|^\alpha \right] \right)^{\frac{\alpha-1}{\alpha}} + \left(\mathbb{E} \left[\sup_{0 \leq s \leq T} |(Y_s^n - S_s)^-|^\alpha \right] \right)^{\frac{1}{\alpha'^*}} \right\} \\
& \leq C_\alpha \left\{ \left(\frac{1}{n^{\alpha-1}} + \frac{1}{n^{\alpha/2}} \right)^{\frac{\alpha-1}{\alpha}} + \left(\frac{1}{n^{\alpha-1}} + \frac{1}{n^{\alpha/2}} \right)^{\frac{\alpha'-1}{\alpha'}} \right\} \\
& \leq C_\alpha \left(\frac{1}{n^{\alpha-1}} + \frac{1}{n^{\alpha/2}} \right)^{\frac{\alpha-1}{\alpha}}. \tag{5.17}
\end{aligned}$$

Hence the sequence $\{Y^n\}_{n \in \mathbb{N}^*}$ is a Cauchy sequence in S_G^α .

On the other hand we have

$$\begin{aligned}
& |Y_0^n - Y_0^p|^2 + \int_0^T e^{\gamma s} |Z_s^n - Z_s^p|^2 d\langle B \rangle_s \\
&= 2 \int_0^T e^{\gamma s} (Y_s^n - Y_s^p) (f(s, Y_s^n, Z_s^n) - f(s, Y_s^p, Z_s^p)) ds \\
&\quad - 2 \int_0^T e^{\gamma s} (Y_s^n - Y_s^p) (Z_s^n - Z_s^p) dB_s - \gamma \int_0^T e^{\gamma s} |Y_s^n - Y_s^p|^2 ds \\
&\quad + 2 \int_0^T e^{\gamma s} (Y_s^n - Y_s^p) d(N_s^n - N_s^p) \\
&\leq \left(2L + \frac{2L^2}{\underline{\sigma}^2} - \gamma \right) \int_0^T e^{\gamma s} |Y_t^n - Y_t^p|^2 ds + \frac{1}{2} \int_0^T e^{\gamma s} |Z_s^n - Z_s^p|^2 d\langle B \rangle_s \\
&\quad - 2 \int_0^T e^{\gamma s} (Y_s^n - Y_s^p) (Z_s^n - Z_s^p) dB_s \\
&\quad + 2e^{\gamma T} \sup_{0 \leq s \leq T} |Y_s^n - Y_s^p| (|N_T^n| + |N_T^p|).
\end{aligned}$$

For $\gamma \geq 2L + \frac{2L^2}{\underline{\sigma}^2}$, one can find a constant C_α such that:

$$\begin{aligned}
& \mathbb{E} \left[\left(\int_t^T |Z_s^n - Z_s^p|^2 ds \right)^{\frac{\alpha}{2}} \right] \\
&\leq C_\alpha \left\{ \mathbb{E} \left[\sup_{0 \leq s \leq T} |Y_s^n - Y_s^p|^{\frac{\alpha}{2}} \left(\int_0^T |Z_s^n - Z_s^p|^2 ds \right)^{\frac{\alpha}{4}} \right] \right. \\
&\quad \left. + \mathbb{E} \left[\sup_{0 \leq s \leq T} |Y_s^n - Y_s^p|^{\frac{\alpha}{2}} (|K_T^n| + |K_T^p| + |R_T^n| + |R_T^p|)^{\frac{\alpha}{2}} \right] \right\} \\
&\leq C_\alpha \left(\mathbb{E} \left[\sup_{0 \leq s \leq T} |Y_s^n - Y_s^p|^\alpha \right] \right)^{1/2} \left\{ \left(\mathbb{E} \left[\left(\int_0^T |Z_s^n|^2 ds \right)^{\frac{\alpha}{2}} \right] \right)^{1/2} \right. \\
&\quad \left. + \left(\mathbb{E} \left[\left(\int_0^T |Z_s^p|^2 ds \right)^{\frac{\alpha}{2}} \right] \right)^{1/2} + (\mathbb{E} [|N_T^n|^\alpha])^{1/2} + (\mathbb{E} [|N_T^p|^\alpha])^{1/2} \right\} \\
&\leq C_\alpha \left(\frac{1}{n^{\alpha-1}} + \frac{1}{n^{\alpha/2}} \right)^{\frac{\alpha-1}{2\alpha}}. \tag{5.18}
\end{aligned}$$

We deduce that the quence $\{Z^n\}_{n \in \mathbb{N}^*}$ is a Cauchy sequence in H_G^α .

Now, we have

$$\begin{aligned}
Y_t^n - Y_t^p &= Y_0^n - Y_0^p - \int_0^t [f(s, Y_s^n, Z_s^n) - f(s, Y_s^p, Z_s^p)] ds \\
&\quad + \int_0^t (Z_s^n - Z_s^p) dB_s - (N_t^n - N_t^p).
\end{aligned}$$

Then by the Lipschitz condition of the function f , one obtains

$$\begin{aligned}
\sup_{0 \leq t \leq T} |N_t^n - N_t^p|^\alpha &\leq C_\alpha \left\{ \sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^\alpha + \left(\int_0^T |Z_s^n - Z_s^p|^2 ds \right)^{\alpha/2} \right. \\
&\quad \left. + \sup_{0 \leq t \leq T} \left| \int_0^t (Z_s^n - Z_s^p) dB_s \right|^\alpha \right\},
\end{aligned}$$

and taking the G -expectation, the BDG type inequality gives us

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} |N_t^n - N_t^p|^\alpha \right] \\ & \leq C_\alpha \left\{ \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^\alpha \right] + \mathbb{E} \left[\left(\int_0^T |Z_s^n - Z_s^p|^2 ds \right)^{\alpha/2} \right]^\alpha \right\}. \end{aligned}$$

Hence by the inequalities (5.17) and (5.18), $\{N^n\}_{n \in \mathbb{N}^*}$ is a Cauchy sequence under the norm $\|\cdot\|_{S_G^\alpha}$. The limit (Y, Z, N) satisfies points **(a)**, **(b)** and **(d)** of the definition 5.1.

To prove that the process $\left(-\int_0^t (Y_s - S_s) dN_s\right)_{0 \leq t \leq T}$ is a decreasing G -martingale, for each $n \in \mathbb{N}^*$ we set

$$\begin{aligned} N_t^n &= -\int_0^t (Y_s^n - S_s) dN_s^n \\ &= \int_0^t (Y_s^n - S_s) dK_s^n - \int_0^t (Y_s^n - S_s)^- dR_s^n \\ &= K_t^n - R_t^n. \end{aligned}$$

Then, for all $n \in \mathbb{N}^*$, K^n is a decreasing G -martingale and

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} [|R_T^n|] &= \lim_{n \rightarrow \infty} \mathbb{E} \left[\left| \int_0^T (Y_s^n - S_s)^- dR_s^n \right| \right] \\ &\leq \lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{0 \leq s \leq T} |(Y_s^n - S_s)^-| |N_T^n| \right] \\ &\leq C \lim_{n \rightarrow \infty} \left(\mathbb{E} \left[\sup_{0 \leq s \leq T} |(Y_s^n - S_s)^-|^2 \right] \right)^{1/2} = 0. \end{aligned} \quad (5.19)$$

Moreover, since N^n converges to N in S_G^α , we have for each step process u

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left| \int_0^T u_s dN_s^n - \int_0^T u_s dN_s \right| \right] = 0,$$

then since step process is dense in S_G^α , there exists a sequence $\{u^n\}$ of simple process such that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{0 \leq s \leq T} |u_s^n - (Y_s - S_s)|^\alpha \right] = 0.$$

Therefore, we get

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \mathbb{E} \left[\left| \int_0^T (Y_s^n - S_s) dN_s^n - \int_0^T (Y_s - S_s) dN_s \right| \right] \\
& \leq \lim_{n \rightarrow \infty} \left\{ \mathbb{E} \left[\left| \int_0^T (Y_s^n - Y_s) dN_s^n \right| \right] + \mathbb{E} \left[\left| \int_0^T ((Y_s - S_s) - u_s^n) dN_s^n \right| \right] \right. \\
& \quad \left. + \mathbb{E} \left[\left| \int_0^T u_s^n d(N_s^n - N_s) \right| \right] + \mathbb{E} \left[\left| \int_0^T ((Y_s - S_s) - u_s^n) dN_s \right| \right] \right\} \\
& \leq \lim_{n \rightarrow \infty} \left\{ \mathbb{E} \left[\left| \int_0^T (Y_s^n - Y_s) dN_s^n \right| \right] + \mathbb{E} \left[\left| \int_0^T ((Y_s - S_s) - u_s^n) dN_s^n \right| \right] \right. \\
& \quad \left. + \mathbb{E} \left[\left| \int_0^T ((Y_s - S_s) - u_s^n) dN_s \right| \right] \right\} \\
& \leq \lim_{n \rightarrow \infty} \left\{ \left(\mathbb{E} \left[\sup_{0 \leq s \leq T} |Y_s^n - Y_s|^2 \right] \right)^{1/2} \left(\mathbb{E} [|N_T^n|^2] \right)^{1/2} \right. \\
& \quad + \left(\mathbb{E} \left[\sup_{0 \leq s \leq T} |u_s^n - (Y_s - S_s)|^2 \right] \right)^{1/2} \left(\mathbb{E} [|N_T^n|^2] \right)^{1/2} \\
& \quad \left. + \left(\mathbb{E} \left[\sup_{0 \leq s \leq T} |u_s^n - (Y_s - S_s)|^2 \right] \right)^{1/2} \left(\mathbb{E} [|N_T|^2] \right)^{1/2} \right\} \\
& = 0.
\end{aligned} \tag{5.20}$$

So, if we set $K'_t = - \int_0^t (Y_s - S_s) dN_s$, Using inequalities (5.19) and (5.20), we get

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathbb{E} [|K_t'^n - K_t'|] &= \lim_{n \rightarrow \infty} \mathbb{E} \left[\left| K_t'^n + \int_0^t (Y_s - S_s) dN_s \right| \right] \\
&= \lim_{n \rightarrow \infty} \mathbb{E} \left[\left| R_t'^n - \int_0^t (Y_s^n - S_s) dN_s^n + \int_0^t (Y_s - S_s) dN_s \right| \right] \\
&\leq \lim_{n \rightarrow \infty} \mathbb{E} [|R_T'^n|] + \lim_{n \rightarrow \infty} \mathbb{E} \left[\left| \int_0^t (Y_s^n - S_s) dN_s^n + \int_0^t (Y_s - S_s) dN_s \right| \right] \\
&= 0.
\end{aligned}$$

We have for each $0 \leq s < t \leq T$,

$$\begin{aligned}
\mathbb{E} [|\mathbb{E}_s [K_t'] - K_s'|] &= \mathbb{E} [|\mathbb{E}_s [K_t'] - \mathbb{E}_s [K_t'^n] + K_s'^n - K_s'|] \\
&\leq \mathbb{E} [|\mathbb{E}_s [K_t'] - \mathbb{E}_s [K_t'^n]|] + \mathbb{E} [|K_s'^n - K_s'|] \\
&\leq \mathbb{E} [|\mathbb{E}_s [K_t' - K_t'^n]|] + \mathbb{E} [|K_s'^n - K_s'|] \\
&\leq \mathbb{E} [|K_t' - K_t'^n|] + \mathbb{E} [|K_s'^n - K_s'|] \rightarrow 0.
\end{aligned}$$

Thus the process K' is a decreasing G -martingale. We complete the proof \square

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Résumé

Equations différentielles stochastiques sous G -espérances et applications

Depuis la publication de l'ouvrage de Choquet (1955), la théorie d'espérance non linéaire a attiré le grand intérêt des chercheurs pour ses applications potentielles dans les problèmes d'incertitude, les mesures de risque et le super-hedging en finance. Shige Peng a construit une sorte d'espérance entièrement non linéaire dynamiquement cohérente par l'approche des EDP. Un cas important d'espérance non linéaire cohérente en temps est la G -espérance, dans laquelle le processus canonique correspondant $(B_t)_{t \geq 0}$ est appelé G -mouvement brownien et joue un rôle analogue au processus de Wiener classique.

L'objectif de cette thèse est d'étudier, dans le cadre de la G -espérance, certaines équations différentielles stochastiques rétrogrades (G -EDSR) à croissance quadratique avec applications aux problèmes de maximisation d'utilité robuste avec incertitude sur les modèles, certaines équations différentielles stochastiques (G -EDS) réfléchies et équations différentielles stochastiques rétrogrades réfléchies avec générateurs lipschitziens.

On considère d'abord des G -EDSRs à croissance quadratique. Dans le Chapitre 2 nous fournissons un résultat d'existence et unicité pour des G -EDSRs à croissance quadratique. D'une part, nous établissons des estimations a priori en appliquant le théorème de type Girsanov, d'où l'on en déduit l'unicité. D'autre part, pour prouver l'existence de solutions, nous avons d'abord construit des solutions pour des G -EDSR discrètes en résolvant des EDPs non-linéaires correspondantes, puis des solutions pour les G -EDSRs quadratiques générales dans les espaces de Banach. Dans le Chapitre 3 nous appliquons les G -EDSRs quadratiques aux problèmes de maximisation d'utilité robuste. Nous donnons une caractérisation de la fonction valeur et une stratégie optimale pour les fonctions d'utilité exponentielle, puissance et logarithmique.

Dans le Chapitre 4, nous traitons des G -EDS réfléchies multidimensionnelles. Nous examinons d'abord la méthode de pénalisation pour résoudre des problèmes de Skorokhod déterministes dans des domaines non convexes et établissons des estimations pour des fonctions α -Hölder continues. À l'aide de ces résultats obtenus pour des problèmes déterministes, nous définissons le G -mouvement Brownien réfléchi et prouvons son existence et son unicité dans un espace de Banach. Ensuite, nous prouvons l'existence et l'unicité de solution pour les G -EDSRs multidimensionnelles réfléchies via un argument de point fixe.

Dans le Chapitre 5, nous étudions l'existence et l'unicité pour les équations différentielles stochastiques rétrogrades réfléchies dirigées par un G -mouvement brownien lorsque la barrière S est un processus de G -Itô.

Mots-clés : G -espérance, G -mouvement brownien, équations différentielles stochastiques, équations différentielles stochastiques rétrogrades, croissance quadratique, EDPs non-linéaire, maximisation d'utilité robuste, problèmes de Skorokhod, méthode de pénalisation, α -Hölder continues, domaines non convexes, barrière.

Abstract

Stochastic Differential Equations under G -expectations and Applications

Since the publication of Choquet's (1955) book, the theory of nonlinear expectation has attracted great interest from researchers for its potential applications in uncertainty problems, risk measures and super-hedging in finance. Shige Peng has constructed a kind of fully nonlinear expectation dynamically coherent by the PDE approach. An important case of time-consistent nonlinear expectation is G -expectation, in which the corresponding canonical process $(B_t)_{t \geq 0}$ is called G -Brownian motion and plays a similar role to the classical Wiener process.

The objective of this thesis is to study, in the framework of the G -expectation, some backward stochastic differential equations (G -BSDE) under a quadratic growth condition on their coefficients with applications to robust utility maximization problems with uncertainty on models, Reflected stochastic differential equations (reflected G -SDE) and reflected backward stochastic differential equations with Lipschitz coefficients (reflected G -BSDE).

We first consider G -BSDE with quadratic growth. In Chapter 2 we provide a result of existence and uniqueness for quadratic G -BSDEs. On the one hand, we establish a priori estimates by applying the Girsanov-type theorem, from which we deduce the uniqueness. On the other hand, to prove the existence of solutions, we first constructed solutions for discrete G -BSDEs by solving corresponding nonlinear PDEs, then solutions for the general quadratic G -BSDEs in the spaces of Banach. In Chapter 3 we apply quadratic G -BSDE to robust utility maximization problems. We give a characterization of the value function and an optimal strategy for exponential, power and logarithmic utility functions.

In Chapter 4, we discuss multidimensional reflected G -SDE. We first examine the penalization method to solve deterministic Skorokhod problems in non-convex domains and establish estimates for continuous α -Hölder functions. Using these results for deterministic problems, we define the reflected G -Brownian motion and prove its existence and its uniqueness in a Banach space. Then we prove the existence and uniqueness of the solution for the multidimensional reflected G -SDE via a fixed point argument.

In Chapter 5, we study the existence and uniqueness of the reflected backward stochastic differential equations driven by a G -Brownian motion when the obstacle S is a G -Itô process.

Keywords: G -expectation, G -Brownian motion, stochastic differential equations, backward stochastic differential equations, quadratic growth, nonlinear PDEs, robust utility maximization, Skorokhod problem, penalization method, α -Hölder continuity, non-convex domains, obstacle.