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# Fonctionnelles de processus de Lévy et diffusions en milieux aléatoires

Grégoire Véchambre

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Discipline/ Spécialité : Mathématiques

FONCTIONNELLES DE PROCESSUS DE LÉVY ET  
DIFFUSIONS EN MILIEUX ALÉATOIRES

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**THÈSE DE DOCTORAT DE  
L'UNIVERSITÉ D'ORLÉANS**

Spécialité

**Mathématiques**

Présentée par

**Grégoire VÉCHAMBRE**

Pour obtenir le grade de  
**DOCTEUR de l'UNIVERSITÉ D'ORLÉANS**

Sujet de la thèse :

**Fonctionnelles de processus de Lévy et diffusions en  
milieux aléatoires**



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# Chapitre 1

## INTRODUCTION

Dans cette thèse nous nous sommes intéressés à trois types de problèmes :

- Étude de la convergence en loi du supremum du temps local d'une diffusion transiente en milieu brownien ou Lévy spectralement négatif.
- Étude du comportement presque sûr du supremum du temps local d'une telle diffusion.
- Étude des fonctionnelles exponentielles de certains processus de Lévy conditionnés à rester positif.

Bien que le troisième problème ait été traité séparément et semble peu lié au deux autres, il joue en réalité un rôle crucial pour l'étude de ces derniers. Nous montrerons notamment qu'il existe un lien très explicite entre le comportement presque sûr du supremum du temps local que nous étudions et les propriétés que nous établissons sur les fonctionnelles exponentielles des processus de Lévy conditionnés à rester positif.

- **Chapitre 2** : Nous étudions la convergence en loi du supremum du temps local et du point favori d'une diffusion transiente dans environnement brownien  $\kappa$ -drifté  $W_\kappa$  où  $0 < \kappa < 1$ . Les lois limites obtenues sont exprimées comme des fonctionnelles d'un subordonneur  $\kappa$ -stable bidimensionnel. Notre étude consiste en particulier à mettre en évidence une structure de renouvellement pour la diffusion.
- **Chapitre 3** : Nous étudions les propriétés des fonctionnelles exponentielles  $\int_0^{+\infty} e^{-X^\uparrow(t)} dt$  où  $X^\uparrow$  est un processus de Lévy spectralement négatif ou positif conditionné à rester positif. Nous avons en particulier étudié leur finitude, leur auto-décomposabilité, l'existence de moments exponentiels, leur queue en 0, l'existence et la régularité de leur densité.
- **Chapitre 4** : Nous étudions la convergence en loi du supremum du temps local et du point favori d'une diffusion transiente en environnement Lévy spectralement négatif. Il s'agit en grande partie d'une généralisation des résultats du Chapitre 2 à des environnements plus généraux. Ce travail passe en particulier par l'étude des  $h$ -vallées de l'environnement et de la répartition asymptotique des  $h$ -minima.

- **Chapitre 5** : Nous étudions le comportement presque sûr du supremum du temps local d'une diffusion transiente en environnement Lévy spectralement négatif. Nous montrons en particulier que ce comportement est lié à la queue en 0 de la fonctionnelle exponentielle de l'environnement conditionné à rester positif.

La suite de cette introduction présente les différents objets étudiés ainsi que les principaux résultats obtenus.

## 1.1 Diffusion en milieu aléatoire

La notion de marche ou de diffusion dans un environnement aléatoire généralise la notion de processus Markovien dans le sens où une telle marche ou diffusion est un *mélange* de processus Markoviens. Ces marches ou diffusions modélisent le problème d'un déplacement aléatoire dans un milieu qui est lui-même aléatoire et qui présente, selon le modèle choisi, une certaine hétérogénéité qui influe grandement sur le comportement global du processus. Dans le cas discret cela revient à considérer une marche aléatoire au plus proche voisin dont les probabilités de transitions sont elles-même choisies aléatoirement. De telles marches s'appellent *marches aléatoires en milieu aléatoire* (abrégié en RWRE pour le terme anglais *Random Walk in Random environment*) et ont été introduites en 1967 par Chernov [23] pour modéliser la réplication de l'A.D.N. Plus récemment, Lubensky et Nelson [49] font aussi usage de ces marches pour modéliser d'autres phénomènes en génétique. Notons également Temkin [71] pour des applications de ces marches à la métallurgie.

### 1.1.1 Définition et propriétés de la diffusion

Dans le cas continu auquel nous nous sommes intéressés, on considère le processus de diffusion  $(X_t)_{t \geq 0}$  qui se déplace dans un potentiel aléatoire  $(V(x), x \in \mathbb{R})$ , c'est-à-dire la solution de l'équation différentielle stochastique suivante :

$$\begin{cases} dX_t = -\frac{1}{2}V'(X_t)dt + dB_t \\ X_0 = 0 \end{cases} \quad (1.1.1)$$

où  $B$  est un mouvement brownien indépendant de  $V$ . Dans ce qui suit nous prenons comme potentiel  $V$  un mouvement brownien drifté ou non, ou plus généralement un processus de Lévy (indexé par  $\mathbb{R}$  et nul en 0 par convention). Pour de tels potentiels qui ne sont pas dérivables l'équation (1.1.1) est purement formelle et la diffusion  $X$  doit être définie conditionnellement à  $V$  par son générateur :

$$A_V = \frac{1}{2}e^{V(x)} \frac{d}{dx} e^{-V(x)} \frac{d}{dx}.$$

Sous des hypothèses de régularité du potentiel, le calcul stochastique permet d'exprimer une diffusion ayant ce générateur comme un mouvement brownien changé en

temps et en espace, ce qui autorise des calculs explicites sur la diffusion. Plus précisément, soit  $B$  un mouvement brownien indépendant de  $V$ , définissons  $S_V(x) := \int_0^x e^{V(u)} du$  et pour  $0 \leq s \leq \tau(B, \int_0^{+\infty} e^{V(u)} du)$ , où  $\tau(B, a)$  est le temps d'atteinte de  $a$  par  $B$ , posons

$$T_V(s) := \int_0^s e^{-2V(S_V^{-1}(B(u)))} du.$$

On montre qu'en posant

$$X(t) := S_V^{-1}(B(T_V^{-1}(t))),$$

on définit une diffusion qui, conditionnellement à  $V$ , a pour générateur  $A_V$ .

Pour l'étude d'une telle diffusion comme pour celle des RWRE il faut tenir compte d'une part de l'aléa du au milieu et d'autre part de celui du au déplacement aléatoire, cela implique de travailler avec plusieurs mesures de probabilité. On note en général  $P^V$  pour la loi de  $X$  conditionnellement à l'environnement  $V$ . Sous cette loi, appelée communément loi *quenched*, la diffusion  $X$  est un processus Markovien. On note  $\mathbb{P}$  pour la loi de probabilité dite *annealed* qui est la moyennisation de la loi quenched sur tous les environnements :  $\mathbb{P} := P^V(\cdot)P(dV)$  (où  $P(dV)$  désigne ici la loi de l'environnement). Le caractère Markovien de  $X$  est perdu sous la loi annealed mais cette dernière bénéficie des éventuelles propriétés d'invariance que présente la loi de l'environnement. L'étude d'un processus en environnement aléatoire nécessite de bien choisir la mesure de probabilité utilisée, on parle selon le cas de *méthodes quenched* ou de *méthodes annealed*.

Si  $V = W$  est mouvement brownien sans drift (dans ce cas la diffusion est appelé processus de Brox) la diffusion  $X$  est récurrente et converge en loi :

**Theorem 1.1.1** (Brox [17], 1986). *Pour le processus de Brox,*

$$(\log(t))^{-2} X(t) \xrightarrow{\mathcal{L}} m_\infty,$$

où  $m_\infty$  est une variable aléatoire non dégénérée dépendant de l'environnement.

Notons que le problème analogue pour le cas discret a aussi été étudié par Sinai [63] qui, en 1982, avait obtenu la convergence en loi des RWRE récurrentes avec la même renormalisation. La distribution limite des RWRE récurrentes a été explicitée indépendamment par Kesten [47] et Golosov [42] en 1986, il s'agit de la même distribution limite que pour le processus de Brox.

La lenteur du processus de Brox et des RWRE récurrentes par rapport respectivement au mouvement brownien et à la marche aléatoire simple classique (qui ont une vitesse en  $\sqrt{t}$ ) vient de l'hétérogénéité des environnements aléatoires. Ceux-ci présentent en effet des puits de potentiel au fond desquels les diffusions (ou les RWRE) resteront piégées un certain temps avant d'en sortir, ce qui va considérablement les ralentir. Les puits de potentiels jouent un rôle extrêmement important pour étudier la localisation des processus en milieu aléatoire et déterminer les points qu'ils ont le plus visités. La preuve du Théorème 1.1.1 se base d'ailleurs sur la localisation de la diffusion au fond d'un certain puits de potentiel et la loi limite  $m_\infty$  correspond à la distribution asymptotique du fond de ce puits.

Si  $V(x) = W_\kappa(x) := W(x) - \frac{\kappa}{2}x$  est mouvement brownien drifté, la diffusion  $X$  est transiente et son comportement, qui dépend de la valeur de  $\kappa$ , a été étudié par Kawazu et Tanaka [45] :

**Theorem 1.1.2** (Kawazu, Tanaka [45], 1995). *On suppose que  $V = W_\kappa$  :*

- si  $0 < \kappa < 1$  alors  $t^{-\kappa}X(t)$  converge en loi vers une distribution de Mittag-Leffler d'indice  $\kappa$ ,
- si  $\kappa = 1$  alors  $(t/\log(t))^{-1}X(t)$  converge en probabilité vers  $1/4$ ,
- si  $\kappa > 1$  alors  $t^{-1}X(t)$  converge presque sûrement vers  $(\kappa - 1)/4$ .

Ces résultats ont par la suite été précisés par Kawazu et Tanaka [46], Tanaka [70] et Hu, Shi et Yor [44] qui, lorsque  $\kappa > 1$ , exhibent plusieurs comportements possibles et prouvent des théorèmes de type central limite. Notons aussi que, comme pour le Théorème 1.1.1, le Théorème 1.1.2 a également été précédé d'un résultat analogue pour les RWRE prouvé en 1975 par Kesten, Kozlov et Spitzer [48]. Ils observent, pour les RWRE transientes, les mêmes trois types de régimes possibles en fonction d'un paramètre  $\kappa$ , dépendant de l'environnement et qui joue un rôle analogue à celui du cas continu. Dans le cas des RWRE transientes à vitesse nulle, leur résultat a été récemment retrouvé et explicité par Enriquez, Sabot et Zindy [35] avec d'autres méthodes.

Le comportement de la diffusion a également été étudié dans le cas où le potentiel est un processus de Lévy par Carmona [18], Cheliotis [21] et Singh [66], [64], [65]. Dans [66], Singh généralise notamment les résultats de Kawazu et Tanaka au cas où l'environnement  $V$  est un processus de Lévy spectralement négatif.

Notons également que la diffusion en milieu aléatoire a été étudiée en dimension supérieure à 1 par Tanaka [68] et Mathieu [50], [51].

### 1.1.2 Temps local et point favori

Pour la diffusion  $X$  que nous venons de définir, il existe un phénomène de localisation dans les puits ou *vallées* du potentiel. Le taux d'occupation d'un point  $x$  de l'espace par la diffusion avant le temps  $t$  est "mesuré" par  $\mathcal{L}_X(t, x)$  où  $\mathcal{L}_X$  est le processus *temps local* de  $X$ , c'est-à-dire la version continue en temps et càd-làg en espace de la densité de la mesure d'occupation de la diffusion :

$$\forall t \geq 0, \forall f \in L^\infty, \int_0^t f(X_s)ds = \int_{\mathbb{R}} f(y)\mathcal{L}_X(t, y)dy.$$

Cette relation implique en particulier l'approximation

$$\mathcal{L}_X(t, x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{X(s) \in [x, x+\epsilon]} ds$$

qui donne du sens à l'affirmation selon laquelle  $\mathcal{L}_X(t, x)$  "mesure" le taux d'occupation de  $x$  jusqu'au temps  $t$ , et qui peut être utilisée comme définition alternative du temps local.

Nous nous sommes intéressés plus particulièrement à  $\mathcal{L}_X^*$ , le supremum du temps local au temps  $t$  :

$$\mathcal{L}_X^*(t) = \sup_{x \in \mathbb{R}} \mathcal{L}_X(t, x).$$

Dans le cas où le potentiel  $V$  de la diffusion est mouvement brownien sans drift (c'est-à-dire dans le cas du processus de Brox) Shi [61] a étudié le comportement presque sûr de  $\mathcal{L}_X^*$  et montré que

$$\mathbb{P}\text{-p.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t \log(\log(\log(t)))} \geq \frac{1}{32}. \quad (1.1.2)$$

Plus tard Andreatti et Diel [5] ont établi la convergence en loi de  $\mathcal{L}_X^*(t)/t$  :

**Theorem 1.1.3** (Andreatti, Diel [5], 2010). *Pour le processus de Brox,*

$$\frac{\mathcal{L}_X^*(t)}{t} \xrightarrow[t \rightarrow +\infty]{\mathcal{L}} \frac{1}{\int_{-\infty}^{+\infty} e^{-R(x)} dx},$$

où  $\xrightarrow{\mathcal{L}}$  désigne la convergence en loi sous la probabilité annealed  $\mathbb{P}$  et où  $R(x) := R_1(-x)\mathbb{1}_{x \leq 0} + R_2(x)\mathbb{1}_{x \geq 0}$ ,  $R_1$  et  $R_2$  étant deux processus de Bessel de dimension 3, issus de 0 et indépendants.

Diel [29] a ensuite poursuivi l'étude de Shi [61] en obtenant les renormalisations exactes pour les limites supérieures et inférieures ainsi qu'un encadrement de ces limites :

**Theorem 1.1.4** (Diel [29], 2011). *Pour le processus de Brox,*

$$\mathbb{P}\text{-p.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t \log(\log(\log(t)))} \leq \frac{e^2}{2} \text{ et } \frac{j_0^2}{64} \leq \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t / \log(\log(\log(t)))} \leq \frac{e^2 \pi^2}{4},$$

où  $j_0$  est la plus petite racine strictement positive de la fonction de Bessel  $J_0$ .

Dans le cas où le potentiel  $V$  de la diffusion est  $W_\kappa$ , le mouvement brownien  $\kappa$ -drifté, Devulder [28] a obtenu par des méthodes annealed la convergence en loi et le comportement presque sûr de  $\mathcal{L}_X^*$  dans les cas  $\kappa = 1$  et  $\kappa > 1$  :

**Theorem 1.1.5** (Devulder [28], 2016). *Si  $V = W_\kappa$  alors, pour les convergences en loi :*

$$\text{si } \kappa = 1, \mathcal{L}_X^*(t)/t^{1/\kappa} \xrightarrow[t \rightarrow +\infty]{\mathcal{L}} \mathcal{F}(1, 1/2), \quad (1.1.3)$$

$$\text{si } \kappa > 1, \mathcal{L}_X^*(t)/t^{1/\kappa} \xrightarrow[t \rightarrow +\infty]{\mathcal{L}} \mathcal{F}(\kappa, 4(\kappa^2(\kappa - 1)/8)^{1/\kappa}), \quad (1.1.4)$$

où, pour  $\alpha, s > 0$ ,  $\mathcal{F}(\alpha, s)$  est la distribution de Fréchet de paramètres  $\alpha$  et  $s$ , c'est-à-dire la loi de fonction de répartition

$$\mathcal{F}(\alpha, s)([0, t]) = e^{-(s/t)^\alpha}. \quad (1.1.5)$$



Pour les convergences presque sûres : pour  $a$  une fonction positive croissante, si  $\kappa > 1$  on a

$$\sum_{n=1}^{+\infty} \frac{1}{na(n)} \begin{cases} < +\infty \\ = +\infty \end{cases} \Leftrightarrow \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{(ta(t))^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \mathbb{P}\text{-p.s.} \quad (1.1.6)$$

$$\text{et } \mathbb{P}\text{-p.s. } \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{(t/\log(\log(t)))^{1/\kappa}} = 4(\kappa^2(\kappa - 1)/8)^{1/\kappa}. \quad (1.1.7)$$

$$\text{Si } \kappa = 1, \mathbb{P}\text{-p.s. } \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t/\log(t)\log(\log(t))} \leq 1/2. \quad (1.1.8)$$

Dans le cas où  $0 < \kappa < 1$  les méthodes annealed sont beaucoup moins efficaces et Devulder [28] obtient des résultats partiels pour le comportement presque sûr du temps local. Il prouve notamment que la renormalisation de la  $\limsup$  doit être plus grande que  $t$  :

$$\mathbb{P}\text{-a.s. } \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t} = +\infty, \quad (1.1.9)$$

et que la renormalisation de la  $\liminf$  est au plus  $t/\log(\log(t))$  et plus grande que  $t/(\log(t))^{1/\kappa}(\log(\log(t)))^{(2/\kappa)+\epsilon}$  (pour tout  $\epsilon > 0$ ) :

$$\mathbb{P}\text{-a.s. } \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t/\log(\log(t))} \leq C(\kappa), \quad (1.1.10)$$

$$\forall \epsilon > 0, \mathbb{P}\text{-a.s. } \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t/(\log(t))^{1/\kappa}(\log(\log(t)))^{(2/\kappa)+\epsilon}} = +\infty, \quad (1.1.11)$$

où  $C(\kappa)$  est une constante positive non explicite.

Peu de choses sont connus pour des environnements plus généraux. Notons tout de même Diel et Voisin [30] qui généralisent les résultats de [5] (et donc en particulier la convergence en loi du Théorème 1.1.3) au cas où l'environnement  $V$  est un processus de Lévy  $\alpha$ -stable avec  $\alpha \in [1, 2]$ .

Dans le cas des RWRE le temps local au point  $x$  et à l'instant  $n$ , noté  $\xi(n, x)$ , correspond tout simplement au nombre de visite du point  $x$  jusqu'à l'instant  $n$ . Le supremum du temps local à l'instant  $n$  est noté  $\xi^*$ . Pour les RWRE récurrentes, la convergence en loi de  $\xi^*(n)/n$  a été établie par Gantert, Peres et Shi [40] qui obtiennent également la  $\limsup$  :

$$\mathbb{P}\text{-p.s. } \limsup_{n \rightarrow +\infty} \xi^*(n)/n = C, \quad (1.1.12)$$

où  $C$  est une constante explicite dépendant de la loi de l'environnement. La  $\liminf$  a ensuite été étudiée par Dembo, Gantert, Peres et Shi [26] qui obtiennent un résultat analogue à celui du Théorème 1.1.4 (établi par Diel [29] pour le cas continu récurrent) :

$$\mathbb{P}\text{-p.s. } 0 < \liminf_{n \rightarrow +\infty} \frac{\xi^*(n)}{n/\log(\log(\log(n)))} < +\infty.$$

**Remark 1.1.6.** *On constate une différence entre le résultat (1.1.12) et celui du cas continu récurrent donné par (1.1.2) et par le Théorème 1.1.4. Cela provient du fait que le puits de potentiel contenant la diffusion au temps  $t$  peut être beaucoup plus abrupte dans le cas continu que dans le cas discret, ce qui autorise de plus grandes valeurs pour le supremum du temps local et induit cette différence de renormalisation.*

Pour les RWRE transientes, le comportement presque sûr du supremum du temps local, plus précisément la  $\limsup$ , a été étudié par Gantert et Shi [41]. Ils distinguent deux cas :  $0 < \kappa \leq 1$  et  $\kappa > 1$ , où  $\kappa$  dépend de la loi de l'environnement et est défini comme dans [48] (où est prouvé l'analogie du Théorème 1.1.2 pour les RWRE transientes). Pour  $\kappa > 1$  ils obtiennent un résultat analogue à (1.1.6) (établi par Devulder [28]) : soit  $a$  une fonction positive croissante, on a

$$\sum_{n=1}^{+\infty} \frac{1}{na(n)} \begin{cases} < +\infty \\ = +\infty \end{cases} \Leftrightarrow \limsup_{t \rightarrow +\infty} \frac{\xi^*(n)}{(na(n))^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \mathbb{P}\text{-p.s.} \quad (1.1.13)$$

Lorsque  $0 < \kappa \leq 1$  leur résultat est similaire à celui du cas récurrent :

$$\mathbb{P}\text{-p.s. } 0 < \limsup_{n \rightarrow +\infty} \xi^*(n)/n < +\infty. \quad (1.1.14)$$

Pour en revenir au cas de la diffusion continue, un problème très lié à la fois au comportement asymptotique de  $\mathcal{L}_X^*(t)$  et à celui de la localisation de la diffusion est de déterminer le comportement du *point favori au temps  $t$* , c'est-à-dire le point  $F^*(t)$  en lequel est atteint le supremum du temps local au temps  $t$  :

$$F^*(t) := \inf\{x \in \mathbb{R}, \mathcal{L}_X(t, x) \vee \mathcal{L}_X(t, x-) = \mathcal{L}_X^*(t)\}. \quad (1.1.15)$$

La localisation des diffusions se fait généralement autour de minima locaux du potentiel. On appelle  *$h$ -minimum* la position du fond d'un puits de potentiel de hauteur au moins  $h$  (voir le Chapitre 2, un peu avant le Théorème 2.1.2 pour une définition rigoureuse des  $h$ -minima), et le puits de potentiel associé est appelé  *$h$ -vallée*. La notion de  $h$ -minimum du mouvement brownien a été introduite et étudiée par Neveu et Pitman [53]. Pour le processus de Brox, Cheliotis [22] a montré que le point favori  $F^*(t)$  est proche en probabilité du minimum de la  $\log(t)$ -vallée contenant 0, qui est également le point près duquel est localisée la diffusion au temps  $t$ . Ce résultat est généralisé par Diel et Voisin dans [30] pour la diffusion en milieu Lévy  $\alpha$ -stable avec  $\alpha \in [1, 2]$ . Notons que le problème du point favori a aussi été étudié pour les RWRE récurrentes par Hu et Shi [43].

Notre travail se situe dans la continuité des résultats présentés dans cette section. Nous étudions des problèmes de convergence en loi du supremum du temps local et du point favori aux Chapitres II et IV, et nous étudions des problèmes de comportement presque sûr au Chapitre 5 où, notamment, nous améliorons les résultats partiels (1.1.9), (1.1.10) et (1.1.11) dans un contexte plus général.

## 1.2 Processus de Lévy et fonctionnelles exponentielles

### 1.2.1 Processus de Lévy

Un *processus de Lévy réel* est un processus stochastique valant 0 à l'origine, à accroissements indépendants et stationnaires, dont les trajectoires sont presque sûrement càd-làg. Nous nous référons notamment aux livres de Bertoin [8] et Sato [59] pour une vision d'ensemble sur ces processus et pour les preuves des propriétés mentionnées dans cette section. Il est connu que la loi d'un processus de Lévy réel ( $Y(t)$ ,  $t \geq 0$ ) est caractérisée par les lois 1-dimensionnelles de celui-ci. Pour  $t \geq 0$ , la fonction caractéristique de  $Y(t)$  peut être mise sous la forme

$$\mathbb{E} [e^{i\xi Y(t)}] = e^{t\psi_Y(\xi)},$$

où  $\psi_Y$  est appelé *exposant caractéristique* de  $Y$ .  $\psi_Y$  est donné par la formule de Lévy-Khintchine :

$$\psi_Y(\xi) = -\frac{Q}{2}\xi^2 + i\gamma\xi + \int_{\mathbb{R}} (e^{i\xi x} - 1 - i\xi x \mathbf{1}_{|x|<1})\nu(dx), \quad (1.2.1)$$

où  $Q \geq 0$ ,  $\gamma \in \mathbb{R}$  et  $\nu$  est une *mesure de Lévy* sur  $\mathbb{R}$ , c'est-à-dire une mesure supportée par  $\mathbb{R} \setminus \{0\}$  telle que  $\int (1 \wedge x^2)\nu(dx) < +\infty$ . Un tel triplet  $(Q, \gamma, \nu)$  est unique et est appelé *triplet générateur* de  $Y$ . Réciproquement, à chaque triplet  $(Q, \gamma, \nu)$  vérifiant les propriétés ci-dessus correspond un (unique) processus de Lévy réel. Pour  $x \in \mathbb{R}$ ,  $Y_x$  désigne en général le *processus  $Y$  issu de  $x$* , qui est égal en loi à  $x + Y$ .

Les exemples les plus simples de processus de Lévy sont le mouvement brownien (éventuellement drifté) et les processus de Poisson composés. Dans la formule (1.2.1), le premier terme correspond à une composante Gaussienne, le deuxième à une composante de drift et le troisième à une composante de sauts compensée par un drift. Mouvement brownien et processus de Poisson composés sont donc en quelque sorte les ingrédients de base à partir desquels peuvent être construits tous les processus de Lévy. Une telle représentation des processus de Lévy est en effet donnée par la décomposition de Lévy-Ito qui illustre (1.2.1) en décomposant  $Y$  comme une somme de deux processus indépendants : un mouvement brownien (éventuellement drifté) et une limite de processus de Poisson composés (dont les petits sauts sont éventuellement compensés par un drift).

Du fait de ses accroissements indépendants et stationnaires, un processus de Lévy est généralement considéré comme l'analogie continu d'une marche aléatoire. Si en particulier  $\mathbb{E}[|Y(1)|] < +\infty$  alors  $Y$  satisfait à la loi des grand nombres :

$$Y(t)/t \xrightarrow[t \rightarrow +\infty]{p.s.} \mathbb{E}[Y(1)].$$

Plus généralement un processus de Lévy réel non nul possède, comme pour les marches aléatoires réelles, trois types de comportements asymptotiques possibles :

$$\lim_{t \rightarrow +\infty} Y(t) = -\infty, \quad \liminf_{t \rightarrow +\infty} Y(t) = -\infty \text{ et } \limsup_{t \rightarrow +\infty} Y(t) = +\infty, \quad \lim_{t \rightarrow +\infty} Y(t) = +\infty,$$

où toutes les limites sont ici des limites presque sûres. Dans le deuxième cas on dit généralement que  $Y$  *oscille*. Un critère général pour déterminer le comportement de  $Y$  est donné par la valeur des intégrales

$$I_- := \int_1^{+\infty} t^{-1} \mathbb{P}(Y(t) < 0) dt \quad \text{et} \quad I_+ := \int_1^{+\infty} t^{-1} \mathbb{P}(Y(t) > 0) dt.$$

Si  $I_- < +\infty$  alors  $Y$  converge vers  $-\infty$ , si  $I_+ < +\infty$  alors  $Y$  converge vers  $+\infty$ , et si  $I_- = I_+ = +\infty$  alors  $Y$  oscille.

### 1.2.2 Processus de Lévy spectralement négatif

Nous considérons le cas d'un processus de Lévy réel  $V$  ne possédant presque sûrement aucun saut positif. Si  $(Q, \gamma, \nu)$  désigne son triplet générateur, l'absence de sauts positifs se traduit par le fait que  $\nu(]0, +\infty[) = 0$ . Un tel processus  $V$  est dit *spectralement négatif*. L'exemple le plus simple de processus de Lévy spectralement négatif est celui du mouvement brownien, drifté ou non. Il est possible de montrer que  $V(t)$  admet une transformée de Laplace sur le demi-plan complexe positif  $\{\lambda \in \mathbb{C}, \Re(\lambda) \geq 0\}$  et dont la forme est la suivante :

$$\forall t, \lambda \geq 0, \quad \mathbb{E} [e^{\lambda V(t)}] = e^{t\Psi_V(\lambda)}.$$

$\Psi_V$  est appelé *exposant de Laplace* de  $V$  et est un prolongement de  $\psi_V$  sur le demi-plan complexe positif :  $\psi_V(\xi) = \Psi_V(i\xi)$ . Par prolongement analytique, l'expression de  $\Psi_V$  se déduit de (1.2.1) :

$$\Psi_V(\lambda) = \frac{Q}{2} \lambda^2 + \gamma \lambda + \int_{-\infty}^0 (e^{\lambda x} - 1 - \lambda x \mathbb{1}_{|x| < 1}) \nu(dx). \quad (1.2.2)$$

Comme cela est habituellement le cas dans l'étude des processus de Lévy spectralement négatifs, nous excluons le cas où  $V$  est monotone. La restriction de  $\Psi_V$  à la demi-droite des nombre réels positifs a alors la propriété de tendre vers l'infini en  $+\infty$ , de plus elle est convexe et nulle en 0. Il est donc possible de définir pour tout  $q$  positif :  $\Phi_V(q) := \sup\{\lambda \geq 0, \Psi_V(\lambda) = q\}$ . On montre (voir [8]) que  $V$  tend vers  $-\infty$ , oscille ou tend vers  $+\infty$  selon que  $\Psi'_V(0+) < 0$ ,  $\Psi'_V(0+) = 0$  ou  $\Psi'_V(0+) > 0$ . Dans le cas où  $V$  tend vers  $-\infty$ , le supremum du processus  $V$  sur  $[0, +\infty[$  suit une loi exponentielle de paramètre  $\Phi_V(0)$ .

Pour  $A$  un borélien, définissons le premier temps de passage de  $V$  dans  $A$  :

$$\tau(V, A) := \inf\{t \geq 0, V(t) \in A\},$$

et nous notons plus simplement  $\tau(V, y)$  pour  $\tau(V, \{y\})$ . Comme nous venons de le voir, le comportement asymptotique des processus de Lévy spectralement négatifs se détermine assez simplement. L'intérêt de ces processus vient souvent du fait qu'ils atteignent chaque niveau positif continûment :  $\tau(V, [y, +\infty[) = \tau(V, y)$ , ce qui permet d'appliquer facilement la propriété de Markov en  $\tau(V, [y, +\infty[)$  et confère de bonnes propriétés au processus des temps d'atteinte ( $\tau(V, [y, +\infty[)$ ,  $y \geq 0$ ) qui est un subordonateur (i.e. un processus de Lévy croissant) dont la transformée de Laplace s'exprime en fonction de  $\Phi_V$  :

$$\forall y, \lambda \geq 0, \mathbb{E} [e^{-\lambda\tau(V, [y, +\infty[)}] = e^{-y\Phi_V(\lambda)}. \quad (1.2.3)$$

Le fait de pouvoir appliquer facilement la propriété de Markov en  $\tau(V, [y, +\infty[)$  permet également de montrer que les processus de Lévy spectralement négatifs possèdent une *fonction d'échelle*,  $W$ , qui vérifie

$$\forall y > x > 0, \mathbb{P}(\tau(V_x, [y, +\infty[) < \tau(V_x, ]-\infty, 0]) = W(x)/W(y).$$

Cette fonction joue un rôle important pour conditionner le processus  $V$  à rester positif.

Comme cela a été mentionné à la fin de la Section 1.1.1, Singh [66] a généralisé les résultats de Kawazu et Tanaka [45] et Hu, Shi et Yor [44] pour la diffusion dans un environnement Lévy spectralement négatif. De même qu'au Théorème 1.1.2, plusieurs régimes distincts peuvent se présenter. Le  $\kappa$  dans ce théorème, qui représente le drift du mouvement brownien et détermine dans quel régime on se trouve, doit, dans le cas d'un environnement Lévy spectralement négatif, être remplacé par  $\kappa := \Phi_V(0)$  qui joue un rôle similaire. Comme nous le verrons aux Chapitres IV et V,  $\kappa := \Phi_V(0)$  joue également un rôle similaire au drift de l'environnement brownien pour ce qui est du comportement du temps local d'une diffusion en environnement Lévy spectralement négatif.

### 1.2.3 Conditionnement à rester positif

Nous introduisons ici la notion de conditionnement à rester positif pour les processus décrits à la section précédente. Nous renvoyons le lecteur à [8] pour les preuves ou pour plus de détails. Soit  $V$  un processus de Lévy spectralement négatif et non monotone. On définit un semi-groupe Markovien  $p_t^\uparrow$  par la relation

$$\forall t \geq 0, x, y > 0, p_t^\uparrow(x, dy) := W(y)\mathbb{P}\left(V_x(t) \in dy, \inf_{[0,t]} V_x > 0\right) / W(x).$$

Pour  $x > 0$ , on note  $V_x^\uparrow$  pour le processus Markovien issu de  $x$ , à valeur dans  $]0, +\infty[$  et dont le semi-groupe de transition est  $p_t^\uparrow$ .  $V_x^\uparrow$  est communément appelé processus *V issu de x conditionné à rester positif*. Cette appellation est justifiée par le fait que pour tous  $y > x > 0$  le processus  $(V_x^\uparrow(t), 0 \leq t \leq \tau(V_x^\uparrow, y))$  est égal en loi au

processus  $(V_x(t), 0 \leq t \leq \tau(V_x, y))$  conditionnellement à  $\{\tau(V_x, y) < \tau(V_x, ]-\infty, 0])\}$ . En d'autres termes, pour les processus tués en leur temps d'atteinte de  $y$ ,  $V_x^\uparrow$  a même loi que  $V_x$  conditionné (dans le sens usuel) à rester positif. Dans le cas où  $V$  converge vers  $+\infty$  la même relation reste vraie en remplaçant le temps d'atteinte de  $y$  par  $+\infty$ . Dans les cas où  $V$  oscille ou tend vers  $-\infty$ , il n'est pas possible de conditionner  $V_x$  à rester positif au sens usuel pendant un temps infini, puisque l'événement par rapport auquel on veut conditionner est de probabilité nulle. Il est toutefois notable que même dans ces cas le processus  $V_x^\uparrow$  soit défini et ait un temps de vie infini. Il est d'ailleurs possible de montrer que dans tous les cas, le processus  $V_x^\uparrow$  converge presque sûrement vers  $+\infty$ .

On peut montrer qu'il existe un processus  $V_0^\uparrow$ , noté plus simplement  $V^\uparrow$ , issu de 0 qui est la limite en loi des processus  $V_x^\uparrow$  lorsque  $x$  tend vers 0.  $V^\uparrow$  est appelé processus  $V$  conditionné à rester positif, c'est un processus de Feller dont la restriction à  $]0, +\infty[$  des lois de transition est encore donnée par le semi-groupe  $p_t^\uparrow$ .

Le processus  $V^\uparrow$  apparait de façon fondamentale dans la loi des excursions du processus de Lévy  $V$  réfléchi en son infimum, c'est cette propriété qui le fait apparaitre dans l'étude des diffusions en milieu aléatoire. En effet, ces diffusions ont tendance à rester piégées un certain temps autour des minima locaux du potentiel et il est donc indispensable de connaître la loi du potentiel à ces endroits.

Pour ce qui est du cas brownien drifté, il est connu que les processus  $W_\kappa^\uparrow$  et  $W_{-\kappa}^\uparrow$  ont même loi. Ceci peut se voir par exemple à partir de l'EDS satisfaite par  $W_\kappa^\uparrow$  (pour la preuve voir par exemple [37], Lemme 6) :

$$dX_t = dB_t + \kappa \coth(\kappa X_t) dt, \quad X_0 = 0.$$

Dans l'étude de  $W_\kappa^\uparrow$  il est donc suffisant de se restreindre au cas où  $\kappa > 0$ .

### 1.2.4 Fonctionnelles exponentielles

La fonctionnelle exponentielle d'un processus de Lévy réel  $Y$  est définie par :

$$I(Y) := \int_0^{+\infty} e^{-Y(t)} dt$$

et a été intensément étudiée, en particulier par Bertoin et Yor ([10], [11]) ou encore Carmona et al. ([19], [20]). Une monographie assez complète sur le sujet a été écrite par Bertoin et Yor [12]. Cette fonctionnelle intervient dans de nombreux domaines des probabilités : l'étude des processus Markoviens auto-similaires, les mathématiques financières, et également l'étude des diffusions en milieu aléatoire. En effet, lorsque  $V$  est un processus de Lévy réel indexé par  $\mathbb{R}$ , les valeurs en respectivement  $+\infty$  et  $-\infty$  de la fonction  $S_V$  (dite fonction d'échelle de la diffusion en milieu  $V$ , et définie en Section 1.1.1) ont même loi que respectivement  $I(-V)$  et  $-I(V)$ . La connaissance des conditions de finitude pour  $I(Y)$  et de ses queues de distribution

est donc essentielle pour l'étude des diffusions en milieu aléatoire. La finitude de  $I(Y)$  est reliée au comportement asymptotique de  $Y$  décrit à la fin de la Section 1.2.1.

**Theorem 1.2.1** (voir par exemple [12]). *Soit  $Y$  un processus de Lévy réel.*

- *Si  $Y$  tend vers  $+\infty$  alors  $I(Y)$  est finie presque sûrement.*
- *Si  $Y$  tend vers  $-\infty$  ou oscille alors  $I(Y)$  est infinie presque sûrement.*

Pour les queues de distributions de  $I(Y)$ , un résultat connu et particulièrement utile pour l'étude des diffusions en milieu aléatoire est le suivant :

**Theorem 1.2.2** (Rivero [58], 2005, voir aussi [12]). *Soit  $Y$  un processus de Lévy réel non arithmétique (c'est-à-dire qu'il n'existe aucun  $r > 0$  tel que presque sûrement  $Y(1) \in r\mathbb{Z}$ ). On suppose qu'il existe  $\theta > 0$  tel que  $\mathbb{E}[e^{-\theta Y(1)}] = 1$  et  $\mathbb{E}[|Y(1)|e^{-\theta Y(1)}] < +\infty$ . Il existe alors une constante positive  $\mathcal{C}$  telle que*

$$\mathbb{P}(I(Y) > x) \underset{x \rightarrow +\infty}{\sim} \mathcal{C}x^{-\theta}.$$

Une étude plus générale de la queue à droite de la fonctionnelle  $I(Y)$  a été effectuée par Maulik et Zwart [52], notamment dans le cas où la condition de Cramér  $\mathbb{E}[e^{-\theta Y(1)}] = 1$  n'est pas vérifiée. D'autres choses ont été étudiées sur les fonctionnelles exponentielles des processus de Lévy, notamment le calcul des moments [19], [11] (voir aussi [12]), l'absolue continuité [20], [9], et les propriétés de la densité [20], [54].

Dans le cas d'une diffusion dans un potentiel Lévy  $V$  qui tend vers  $-\infty$ , le Théorème 1.2.1 permet de montrer que la diffusion est presque sûrement transiente vers  $+\infty$ . Le Théorème 1.2.2 permet d'étudier le temps de sortie d'une vallée et de montrer par exemple que ce dernier est dans le domaine d'attraction d'une loi stable, de sorte que, dans certains cas, les temps d'atteintes de la diffusion convergent vers une loi stable (ceci peut fournir une approche pour prouver ou généraliser le premier point du Théorème 1.1.2).

Comme nous l'avons mentionné dans la section précédente,  $V^\uparrow$  apparaît lorsqu'on considère la loi du potentiel  $V$  autour de ses minima locaux. De plus, l'expression du temps passé par la diffusion à l'intérieur d'un puits de potentiel fait intervenir l'intégrale (prise dans le puits) de  $\exp(-\tilde{V})$ , où  $\tilde{V}$  est le potentiel re-centré au fond du puits. Ceci fait donc apparaître la fonctionnelle exponentielle de  $V^\uparrow$  dans l'étude de la diffusion en milieu  $V$ . Pour l'étude précise du temps local d'une diffusion en milieu  $V$  nous avons donc besoin de connaître certaines propriétés de la fonctionnelle exponentielle de  $V^\uparrow$ . Soit donc  $V$  un processus de Lévy spectralement négatif et non monotone. Au Chapitre 3 nous nous sommes intéressés à la fonctionnelle

$$I(V^\uparrow) := \int_0^{+\infty} e^{-V^\uparrow(t)} dt.$$

Un cas particulièrement simple d'une telle fonctionnelle est celui où  $V$  est un mouvement brownien drifté. On rappelle que  $W_\kappa$  désigne le mouvement brownien

$\kappa$ -drifté et que  $W_\kappa^\uparrow$  et  $W_{-\kappa}^\uparrow$  ont même loi de sorte qu'on peut supposer  $\kappa$  positif. Il est connu que  $I(W_\kappa^\uparrow)$  est fini presque sûrement et sa transformée de Laplace est connue explicitement :

$$\mathbb{E}[e^{-\lambda I(W_\kappa^\uparrow)}] = \frac{(2\sqrt{2\lambda})^\kappa}{2^\kappa \Gamma(1 + \kappa) I_\kappa(2\sqrt{2\lambda})}, \quad (1.2.4)$$

où  $I_\kappa$  est une fonction de Bessel modifiée. Cette expression de la transformée de Laplace de  $I(W_\kappa^\uparrow)$  a été démontrée et utilisée par Andreatti et Devulder [3] pour l'étude de la diffusion en milieu brownien drifté (voir le Lemme 4.2 dans [3], voir aussi le Lemme 2.6.6 au Chapitre 2). L'étude de cette transformée de Laplace (voir par exemple Section 3.1.2 au Chapitre 3) permet de déduire l'existence de moments exponentiels pour  $I(W_\kappa^\uparrow)$ , l'existence d'une densité de classe  $\mathcal{C}^\infty$ , ainsi que de déterminer sa queue à gauche :

$$-\log(\mathbb{P}(I(W_\kappa^\uparrow) \leq x)) \underset{x \rightarrow 0}{\sim} \frac{2}{x}. \quad (1.2.5)$$

Dans la suite, nous comparons nos résultats généraux obtenus pour  $I(V^\uparrow)$  avec ceux mentionnés ici pour  $I(W_\kappa^\uparrow)$ . Nous verrons en particulier qu'il existe, dans le cas général, une grande variété de comportements pour la queue à gauche de  $I(V^\uparrow)$ , et que cette queue détermine très précisément le comportement presque sûr du temps local d'une diffusion en environnement  $V$ .

## 1.3 Description des résultats obtenus

### 1.3.1 Chapitre 2 : Convergence en loi en milieu brownien drifté

Ce travail est issu d'une collaboration avec Pierre Andreatti et Alexis Devulder et a fait l'objet d'un article [4] accepté pour publication dans la revue ALEA.

L'objectif de ce chapitre est d'établir la convergence en loi du supremum du temps local  $\mathcal{L}_X^*$  dans le cas d'une diffusion en milieu aléatoire dont le potentiel est  $W_\kappa$ , le mouvement brownien  $\kappa$ -drifté avec  $0 < \kappa < 1$ .

Avant d'énoncer les résultats obtenus il nous faut au préalable introduire quelques objets.

Soit  $\mathcal{R}_\kappa$  une variable aléatoire ayant même loi que la somme de deux copies indépendantes de la variable  $I(W_\kappa^\uparrow)$  définie en Section 1.2.4. Soit  $(\mathcal{Y}_1, \mathcal{Y}_2)$  le subordonateur bidimensionnel  $\kappa$ -stable dont la mesure de Lévy  $\nu$  est donnée par :

$$\forall x > 0, y > 0, \nu([x, +\infty[ \times [y, +\infty[) = \frac{\mathcal{C}_2}{y^\kappa} \mathbb{E} \left[ \mathcal{R}_\kappa^\kappa \mathbf{1}_{\mathcal{R}_\kappa \leq \frac{y}{x}} \right] + \frac{\mathcal{C}_2}{x^\kappa} \mathbb{P} \left( \mathcal{R}_\kappa > \frac{y}{x} \right), \quad (1.3.1)$$

où  $\mathcal{C}_2$  est une constante positive.



### 1.3. DESCRIPTION DES RÉSULTATS OBTENUS

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Pour  $Z$  un processus croissant et càd-làg, et  $s \geq 0$ , on pose respectivement  $Z(s-)$ ,  $Z^\natural(s)$  et  $Z^{-1}(s)$  pour respectivement la limite à gauche de  $Z$  en  $s$ , le plus grand saut de  $Z$  avant  $s$ , et l'inverse généralisée de  $Z$  en  $s$  :

$$Z(s-) = \lim_{r \nearrow s} Z(r), \quad Z^\natural(s) := \sup_{0 \leq r \leq s} (Z(r) - Z(r-)), \quad Z^{-1}(s) := \inf\{u \geq 0, Z(u) > s\},$$

où  $\inf \emptyset = +\infty$  par convention. On définit le couple de variables aléatoires  $(\mathcal{I}_1, \mathcal{I}_2)$  par :

$$\mathcal{I}_1 := \mathcal{Y}_1^\natural(\mathcal{Y}_2^{-1}(1)-), \quad \mathcal{I}_2 := (1 - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)-)) \times \frac{\mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)-)}{\mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)-)},$$

Le résultat que nous obtenons sur la convergence en loi de  $\mathcal{L}_X^*$  peut alors s'écrire :

**Theorem 1.3.1.** *Pour la diffusion dans le potentiel  $W_\kappa$  avec  $0 < \kappa < 1$  on a,*

$$\mathcal{L}_X^*(t)/t \xrightarrow[t \rightarrow +\infty]{\mathcal{L}} \mathcal{I} := \max(\mathcal{I}_1, \mathcal{I}_2).$$

La preuve de ce résultat repose sur la décomposition de l'environnement en  $h_t$ -vallées (définies à la fin de la Section 1.1.2), c'est-à-dire en puits de potentiel d'une hauteur donnée  $h_t$  (dépendante du temps  $t$ ). Ces  $h_t$ -vallées sont indépendantes et visitées successivement par la diffusion sans retour en arrière, elles possèdent chacune un fond (le  $h_t$ -minimum qui leur est associé) près duquel la diffusion va rester piégée un certain temps avant de s'échapper de la vallée. Chacune des  $h_t$ -vallées visitées contient donc un pic de temps local situé en son minimum, et elle correspond à une dépense de temps de la part de la diffusion. Les contributions des vallées successives au temps local et au temps dépensé par la diffusion forment une suite *iid* dont on montre que la somme renormalisée par  $t$  des termes converge vers un le subordonateur  $\kappa$ -stable  $(\mathcal{Y}_1, \mathcal{Y}_2)$ . Il s'agit alors d'exprimer la quantité d'intérêt (le supremum du temps local au temps  $t$ ) comme une fonctionnelle de la suite *iid*, de montrer que cette fonctionnelle est continue puis d'utiliser le *continous mapping theorem* pour en déduire la convergence de la quantité d'intérêt vers cette fonctionnelle appliquée à  $(\mathcal{Y}_1, \mathcal{Y}_2)$ .

La forme de la loi limite peut alors s'expliquer de la façon suivante :  $\mathcal{I}_1$  est définie comme le plus grand saut de  $\mathcal{Y}_1$  avant le saut qui fait que  $\mathcal{Y}_2$  dépasse 1, elle représente donc le plus grand pic de temps local avant l'entrée de la diffusion dans la vallée qui la contient à l'instant  $t$ . Pour obtenir la loi limite il faut également tenir compte de la contribution de la dernière vallée, celle qui contient la diffusion à l'instant  $t$ . Le pic de temps local dans cette dernière vallée est représenté par  $\mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)-)$  (la première composante du saut de  $(\mathcal{Y}_1, \mathcal{Y}_2)$  qui fait que  $\mathcal{Y}_2$  dépasse 1), or il ne faut garder qu'une certaine proportion de ce pic de temps local : la partie qui date d'avant l'instant  $t$ . La proportion de temps passé avant l'instant  $t$  par la diffusion dans la dernière vallée est représentée par  $(1 - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)-))/(\mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)-))$ . En multipliant cette proportion par  $\mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)-)$  qui représente le pic

de temps local on obtient  $\mathcal{I}_2$  qui représente alors la contribution de la dernière vallée au temps local avant l'instant  $t$ . Il est alors naturel que la loi limite soit le maximum des variables  $\mathcal{I}_1$  et  $\mathcal{I}_2$ .

Il se trouve que la suite *iid* des contributions des vallées successives au temps local et au temps dépensé par la diffusion contient des informations sur d'autres quantités intéressantes. Il y a notamment  $F^*(t)$ , la position au temps  $t$  du point favori de la diffusion (défini en (1.1.15)). Il est donc possible d'appliquer la méthode que nous avons développée pour étudier ces autres quantités. On a en particulier le théorème suivant sur la convergence en loi du point favori :

**Theorem 1.3.2.** *Pour la diffusion dans le potentiel  $W_\kappa$  avec  $0 < \kappa < 1$  on a,*

$$F^*(t)/X(t) \xrightarrow[t \rightarrow +\infty]{\mathcal{L}} \mathcal{B} \mathcal{U}_{[0,1]} + 1 - \mathcal{B},$$

où  $\mathcal{B}$  est une variable de Bernoulli de paramètre  $\mathbb{P}(\mathcal{I}_1 < \mathcal{I}_2)$ , et  $\mathcal{U}_{[0,1]}$  est une variable aléatoire de loi uniforme sur  $[0, 1]$ , indépendante de  $\mathcal{B}$ .

La loi limite du Théorème 1.3.2 peut aussi s'expliquer intuitivement : avec une probabilité d'environ  $\mathbb{P}(\mathcal{I}_1 < \mathcal{I}_2)$  le supremum du temps local est atteint avant la dernière vallée et il correspond à un maximum de variables *iid*, la position de ce maximum est alors uniforme. Sur l'événement complémentaire, le supremum du temps local est atteint dans la dernière vallée, celle qui par définition contient la diffusion au temps  $t$ ,  $F^*(t)/X(t)$  est alors proche de 1. La forme de la loi limite obtenue illustre le rôle prépondérant des vallées en ce qui concerne le comportement du temps local de la diffusion, ceci explique que les méthodes annealed utilisées dans [28] soient peu adaptées au cas  $0 < \kappa < 1$ , malgré leur grande efficacité pour le cas  $\kappa > 1$ .

Au Chapitre 2, nous présentons également les convergences d'autres quantités qui nous semblent intéressantes : le supremum du temps local juste avant l'entrée dans la dernière vallée et le supremum du temps local juste après la sortie de celle-ci.

### 1.3.2 Chapitre 3 : Fonctionnelles exponentielles

Ce travail a fait l'objet d'un article [72] actuellement en révision.

L'objectif principal de ce chapitre est l'étude de la fonctionnelle exponentielle  $I(V^\uparrow)$  définie en Section 1.2.4, ceci afin d'établir les propriétés nécessaires pour la généralisation des résultats du Chapitre 2 au cas d'une diffusion dans un potentiel Lévy spectralement négatif. De plus, comme nous le verrons au Chapitre 5, le comportement presque sûr du supremum du temps local de la diffusion est étroitement lié à la queue à gauche de la variable  $I(V^\uparrow)$  qu'il convient donc d'étudier précisément.

L'extension au cas de certains processus de Lévy conditionnés à rester positif des résultats connus pour les fonctionnelles exponentielles des processus de Lévy est également une motivation à part entière de ce travail. Nous nous intéressons donc,

dans ce chapitre, à d'autres propriétés telles que l'existence et la régularité de la densité pour la fonctionnelle  $I(V^\uparrow)$ .

Dans ce chapitre nous supposons que les hypothèses habituelles sur  $V$  faites en Section 1.2.4 ( $V$  est spectralement négatif et non monotone) sont satisfaites.

La première question qui se pose dans l'étude de la fonctionnelle  $I(V^\uparrow)$  est sa finitude. De plus, pour l'étude du temps local de la diffusion en milieu  $V$  il est utile d'établir l'existence de moments exponentiels pour  $I(V^\uparrow)$ . Nous prouvons le résultat suivant :

**Theorem 1.3.3.** *La variable aléatoire  $I(V^\uparrow)$  est finie presque sûrement, elle admet une espérance finie  $\mathbb{E}[I(V^\uparrow)]$  et*

$$\forall \lambda < 1/\mathbb{E}[I(V^\uparrow)], \quad \mathbb{E} \left[ e^{\lambda I(V^\uparrow)} \right] < +\infty. \quad (1.3.2)$$

Un point clé dans l'étude de  $I(V^\uparrow)$  est de la voir comme la solution d'une équation affine aléatoire : on montre que pour tout  $y > 0$  il existe une variable aléatoire  $A^y$  telle que

$$I(V^\uparrow) \stackrel{\mathcal{L}}{=} A^y + e^{-y} I(V^\uparrow), \quad (1.3.3)$$

où les deux termes du membre de droite sont indépendants. Cela montre en particulier que  $I(V^\uparrow)$  est une variable aléatoire positive auto-décomposable et que par conséquent elle est absolument continue et unimodale.

À  $y > 0$  fixé, des itérations infinies de la relation (1.3.3) permettent d'écrire  $I(V^\uparrow)$  comme la somme d'une série entière aléatoire

$$I(V^\uparrow) \stackrel{\mathcal{L}}{=} \sum_{k \geq 0} e^{-ky} A_k^y, \quad (1.3.4)$$

où les coefficients  $A_k^y$  sont *iid* de même loi que  $A^y$ . Cette décomposition est particulièrement utile pour l'étude des propriétés  $I(V^\uparrow)$ , puisqu'elle permet de se ramener à l'étude de la variable  $A^y$ .

Les résultats suivants sont fondamentaux pour l'application aux diffusions en milieu aléatoire : ils décrivent la queue à gauche de la fonctionnelle  $I(V^\uparrow)$ . L'idée est de relier cette queue au comportement asymptotique de  $\Psi_V$ .

**Theorem 1.3.4.** *Supposons qu'il existe  $\alpha > 1$  et une constante positive  $C$  tels que pour tout  $\lambda$  suffisamment grand on ait  $\Psi_V(\lambda) \leq C\lambda^\alpha$ . Alors pour tout  $\delta \in ]0, 1[$ , on a pour  $x$  suffisamment petit*

$$\mathbb{P} (I(V^\uparrow) \leq x) \leq \exp \left( -\delta(\alpha - 1)/(Cx)^{1/(\alpha-1)} \right). \quad (1.3.5)$$

*Supposons qu'il existe  $\alpha > 1$  et une constante positive  $c$  tels que pour tout  $\lambda$  suffisamment grand on ait  $\Psi_V(\lambda) \geq c\lambda^\alpha$ . Alors pour tout  $\delta > 1$ , on a pour  $x$  suffisamment petit*

$$\mathbb{P} (I(V^\uparrow) \leq x) \geq \exp \left( -\delta\alpha^{\alpha/(\alpha-1)}/(cx)^{1/(\alpha-1)} \right). \quad (1.3.6)$$

Avant d'énoncer d'autres résultats il nous faut définir une façon de quantifier le comportement asymptotique de  $\Psi_V$ . En nous inspirant de ce qui se fait pour l'étude, par exemple, de la dimension de Hausdorff des trajectoires des processus de Lévy, nous définissons :

$$\sigma := \sup \left\{ \alpha \geq 0, \lim_{\lambda \rightarrow +\infty} \lambda^{-\alpha} \Psi_V(\lambda) = \infty \right\},$$

$$\beta := \inf \left\{ \alpha \geq 0, \lim_{\lambda \rightarrow +\infty} \lambda^{-\alpha} \Psi_V(\lambda) = 0 \right\}.$$

Si  $\Psi_V$  est à variation  $\alpha$ -régulière pour un  $\alpha \in [1, 2]$  on a  $\sigma = \beta = \alpha$ . Rappelons que  $Q$  désigne la composante gaussienne de  $V$  dans son triplet générateur. Il est possible de voir sur l'expression (1.2.2) que, lorsque  $Q > 0$ ,  $\Psi_V(\lambda) \sim Q\lambda^2/2$  et donc que  $\Psi_V$  est à variation 2-régulière et, lorsque  $Q = 0$ ,  $1 \leq \sigma \leq \beta \leq 2$ . Nos résultats sur les queues à gauche de  $I(V^\uparrow)$  peuvent être énoncés comme suit :

**Theorem 1.3.5.**

$$\forall \beta' > \beta, \lim_{x \rightarrow 0} x^{1/(\beta'-1)} \log (\mathbb{P} (I(V^\uparrow) \leq x)) = -\infty, \quad (1.3.7)$$

$$\text{si } \sigma > 1, \forall \sigma' \in ]1, \sigma[, \lim_{x \rightarrow 0} x^{1/(\sigma'-1)} \log (\mathbb{P} (I(V^\uparrow) \leq x)) = 0. \quad (1.3.8)$$

Le théorème précédent donne pour  $\mathbb{P}(I(V^\uparrow) \leq x)$  une borne inférieure impliquant  $\sigma$  et une borne supérieure impliquant  $\beta$ . Dans le cas où  $\Psi_V$  est à variation  $\alpha$ -régulière pour un  $\alpha \in ]1, 2]$  on sait que  $\sigma = \beta = \alpha$ , ce qui renforce la précision du résultat. En renforçant les hypothèse sur la régularité de  $\Psi_V$  on peut encore préciser ce résultat :

**Theorem 1.3.6.** *On suppose qu'il existe une constante positive  $C$  et  $\alpha \in ]1, 2]$  tels que  $\Psi_V(\lambda) \sim_{\lambda \rightarrow +\infty} C\lambda^\alpha$ , alors*

$$-\log (\mathbb{P} (I(V^\uparrow) \leq x)) \underset{x \rightarrow 0}{\sim} \frac{\alpha - 1}{(Cx)^{\frac{1}{\alpha-1}}}.$$

Ce théorème est vrai en particulier lorsque, pour un  $\alpha \in ]1, 2]$ ,  $V$  est un processus de Lévy  $\alpha$ -stable sans sauts positifs (avec adjonction ou non d'un drift). En particulier, la queue donnée dans l'expression (1.2.5) est un cas particulier du Théorème 1.3.6. En effet, lorsque  $V = W_\kappa$ , sa composante gaussienne vaut 1 et donc  $\Psi_V(\lambda) \sim \lambda^2/2$ . L'équivalent donné par le Théorème 1.3.6 est alors le même que l'équivalent (1.2.5).

**Remark 1.3.7.** *Puisque  $\Psi_V(\lambda)/\lambda^2$  a toujours une limite finie en  $+\infty$ , le Théorème 1.3.4 implique l'existence d'une constante positive  $K$  (dépendante de  $V$ ) telle que pour tout  $x$  suffisamment petit*

$$\mathbb{P} (I(V^\uparrow) \leq x) \leq e^{-K/x}.$$

**Remark 1.3.8.** *Le Théorème 1.3.5 s'applique lorsque  $\beta = 1$ , et  $\beta'$  peut alors être choisi de façon à ce que  $1/(\beta' - 1)$  soit aussi grand qu'on veut. Lorsque  $V$  est à variation bornée on peut prouver un résultat plus précis :  $\mathbb{P}(I(V^\uparrow) \leq x)$  est nulle pour  $x$  suffisamment petit.*

**Remark 1.3.9.** *On rappelle que  $I(V^\uparrow)$  est unimodale. Si 0 était un mode, alors on aurait  $\mathbb{P}(I(V^\uparrow) \leq x) \geq cx$  pour une certaine constante positive  $c$  et tout  $x$  suffisamment petit, ce qui contredirait (1.3.7). Ainsi la densité de  $I(V^\uparrow)$  est croissante sur un voisinage de 0. Ceci implique que les Théorèmes 1.3.4, 1.3.5, 1.3.6 et les Remarques 1.3.7, 1.3.8 sont vrais si on remplace la fonction de répartition  $\mathbb{P}(I(V^\uparrow) \leq \cdot)$  par la densité de  $I(V^\uparrow)$ .*

Lorsque  $V$  tend vers  $+\infty$  nous établissons que la queue à gauche de  $I(V^\uparrow)$  est la même que la queue à gauche de  $I(V)$ . Ceci implique que tous nos résultats sur la queue à gauche de  $I(V^\uparrow)$  s'appliquent aussi à la queue à gauche de  $I(V)$  :

**Proposition 1.3.10.** *Lorsque  $V$  tend vers  $+\infty$ , les Théorèmes 1.3.4, 1.3.5, 1.3.6 et les Remarques 1.3.7, 1.3.8 sont vraies si on remplace  $I(V^\uparrow)$  par  $I(V)$ .*

La dernière proposition illustre le fait que l'étude de la fonctionnelle exponentielle d'un processus de Lévy conditionné à rester positif permet de déduire des résultats pour la fonctionnelle exponentielle du processus de Lévy correspondant.

Nous savons déjà, grâce à l'auto-décomposabilité, que  $I(V^\uparrow)$  est absolument continue. Une question intéressante, bien qu'elle ne se pose pas pour l'étude de la diffusion en milieu aléatoire, est de connaître la régularité de cette densité. Le résultat suivant est une condition sur  $\Psi_V$  pour la régularité de la densité :

**Theorem 1.3.11.** *Si  $\sigma > 1$  et  $\beta$  sont tels que*

$$2\beta^2 - 3\sigma\beta + \sigma + \beta - 1 < 0, \quad (1.3.9)$$

*alors la densité de  $I(V^\uparrow)$  appartient à l'espace de Schwartz. Ses dérivées à tout ordre convergent vers 0 en  $+\infty$  et en 0.*

Lorsque  $I(V)$  est finie, le théorème précédent permet également d'étudier la régularité de sa densité. On a en effet le corollaire suivant :

**Corollary 1.3.12.** *Lorsque  $V$  tend vers  $+\infty$  et est tel que  $\sigma > 1$  et que (1.3.9) est satisfaite, alors la densité de  $I(V)$  est de classe  $\mathcal{C}^\infty$  et ses dérivées à tout ordre convergent vers 0 en  $+\infty$  et en 0.*

**Remark 1.3.13.** *Si  $\Psi_V$  est à variation  $\alpha$ -régulière pour un  $\alpha \in ]1, 2]$  alors  $\sigma = \beta = \alpha$  et la condition (1.3.9) devient  $-(\alpha - 1)^2 < 0$ , ce qui est toujours vrai pour  $\alpha > 1$ . On peut donc appliquer le Théorème 1.3.11 et le Corollaire 1.3.12. En d'autres termes, le fait d'être à variation  $\alpha$ -régulière pour  $\Psi_V$  implique la régularité des densités de  $I(V^\uparrow)$  (et de  $I(V)$  si cette fonctionnelle est finie) lorsque  $\alpha > 1$ .*

Nous venons de présenter des résultats concernant  $I(V^\uparrow)$ . Lorsque  $V$  converge vers  $-\infty$  certains de ces résultats sont fondamentaux pour l'étude, aux Chapitres IV et V, du temps local de la diffusion dans le potentiel  $V$ . Bien qu'elle joue un rôle moins déterminant, la fonctionnelle exponentielle de  $-V$  conditionné à rester positif intervient également dans cette étude et il nous faut donc aussi étudier les fonctionnelles exponentielles des processus de Lévy spectralement positifs conditionnés à rester positif.

Dans la suite de cette section,  $Z$  est un processus de Lévy spectralement positif qui converge vers  $+\infty$ , la définition de  $Z$  conditionné à rester positif, noté  $Z^\uparrow$ , est donnée au début du Chapitre 3. Nous étudions la fonctionnelle exponentielle  $I(Z^\uparrow)$ . Le premier résultat concerne sa finitude et l'existence de moments exponentiels :

**Theorem 1.3.14.** *La variable aléatoire  $I(Z^\uparrow)$  est finie presque sûrement et admet des moments exponentiels, c'est-à-dire,*

$$\exists \lambda > 0, \mathbb{E} \left[ e^{\lambda I(Z^\uparrow)} \right] < +\infty.$$

La présence de sauts positifs pour  $Z$  et  $Z^\uparrow$  alourdit les queues à gauche de  $I(Z)$  et  $I(Z^\uparrow)$  par rapport à celle obtenue pour  $I(V^\uparrow)$ . On a en effet :

**Theorem 1.3.15.** *Si  $Z$  est à variation non bornée et que sa mesure de Lévy est non nulle, alors il existe une constante positive  $c$  telle que*

$$e^{-c(\log(x))^2} \leq \mathbb{P}(I(Z) \leq x) \leq \mathbb{P}(I(Z^\uparrow) \leq x).$$

*La minoration pour  $\mathbb{P}(I(Z) \leq x)$  ne nécessite pas l'hypothèse de la variation non bornée.*

**Remark 1.3.16.** *Si la mesure de Lévy de  $Z$  est nulle, on voit sur la formule de Lévy-Khintchine (1.2.1) que  $Z$  est un mouvement brownien drifté. La queue en 0 de  $I(Z^\uparrow)$  est alors donnée par le Théorème 1.3.6 (qui s'applique avec  $\alpha = 2$ ) et elle est plus fine que celle donnée par le Théorème 1.3.15 dans le cas où  $Z$  a des sauts. La présence ou l'absence de sauts joue donc un rôle déterminant pour la queue à gauche de la fonctionnelle exponentielle.*

Il y a deux raisons pour lesquelles l'étude du cas spectralement positif est moins poussée que celle du cas spectralement négatif. La première est que nous n'avons pas, dans le cas spectralement positif, de décomposition du type (1.3.3) pour la fonctionnelle  $I(Z^\uparrow)$ , or presque toute notre étude du cas spectralement négatif repose sur une telle décomposition. La deuxième raison est que nous n'avons pas besoin, pour les applications aux diffusions en milieu aléatoire, de résultats aussi précis dans le cas spectralement positif que dans le cas spectralement négatif. En effet, lorsqu'on étudie la diffusion dans un environnement Lévy spectralement négatif  $V$  qui tend vers  $-\infty$ , il apparaît une variable aléatoire  $\mathcal{R}$  dont la loi est la convolution de celles de  $I(V^\uparrow)$  et de  $I((-V)^\uparrow)$ . Les résultats précédents montrent que pour certaines choses le comportement de  $I(V^\uparrow)$  est dominant par rapport à celui de  $I((-V)^\uparrow)$ , lorsque  $V$  a des sauts. En particulier, la queue à gauche de  $\mathcal{R}$  est la même que celle de  $I(V^\uparrow)$ .

### 1.3.3 Chapitre 4 : Convergence en loi en milieu Lévy spectralement négatif

Ce travail a fait l'objet d'un article [74] actuellement soumis.

L'objectif de ce chapitre est d'établir des résultats de convergence en loi du supremum du temps local  $\mathcal{L}_X^*(t)$  et du point favori  $F^*(t)$  dans le cas général d'une diffusion en milieu aléatoire dont le potentiel  $V$  est un processus de Lévy spectralement négatif et non monotone qui converge vers  $-\infty$  en  $+\infty$ . Comme cela a été mentionné à la Section 1.2.2, il est connu que pour de tels potentiels  $\kappa = \Phi_V(0) = \sup\{\lambda \geq 0, \Psi_V(\lambda) = 0\}$  joue un rôle similaire, pour le comportement de la diffusion, à celui du drift du mouvement brownien  $W_\kappa$  (voir notamment [66]).

Lorsque  $\kappa > 1$ , nous utilisons une égalité en loi entre le temps local et un processus d'Ornstein-Uhlenbeck généralisé, selon une méthode inspirée de [66]. Ceci permet de généraliser la convergence (1.1.4) (obtenue par Devulder [28] dans le cas d'un potentiel brownien drifté). Soient  $K$  et  $m$  définis comme dans [66] :

$$K := \mathbb{E} \left[ \left( \int_0^{+\infty} e^{V(t)} dt \right)^{\kappa-1} \right] \quad \text{et} \quad m := \frac{-2}{\Psi_V(1)} > 0. \quad (1.3.10)$$

Lorsque  $\kappa > 1$ , nous obtenons la convergence en loi suivante pour le supremum du temps local :

**Theorem 1.3.17.** *Si  $\kappa > 1$ ,*

$$\mathcal{L}_X^*(t)/t^{1/\kappa} \xrightarrow[t \rightarrow +\infty]{\mathcal{L}} \mathcal{F}(\kappa, 2(\Gamma(\kappa)\kappa^2 K/m)^{1/\kappa}),$$

où  $\mathcal{F}(\cdot, \cdot)$  désigne la loi de Fréchet définie en (1.1.5).

**Remark 1.3.18.** *Si  $V = W_\kappa$  pour un  $\kappa > 1$ , alors  $K = 2^{\kappa-1}/\Gamma(\kappa)$  (voir Exemple 1.1 dans [66]) et  $m = 4/(\kappa-1)$ . La loi limite donné par le Théorème 1.3.17 est alors exactement la même qu'en (1.1.4).*

L'intérêt de la méthode utilisée est qu'elle permet aussi d'étudier la convergence en loi du point favori de la diffusion :

**Theorem 1.3.19.** *Si  $\kappa > 1$ ,*

$$mF^*(t)/t \xrightarrow[t \rightarrow +\infty]{\mathcal{L}} \mathcal{U}_{[0,1]},$$

où  $\mathcal{U}_{[0,1]}$  désigne une loi uniforme sur  $[0, 1]$ .

Lorsque  $0 < \kappa < 1$ , la loi limite que nous exhibons pour  $\mathcal{L}_X^*(t)/t$  est proche de celle obtenue au Chapitre 2. Rappelons que la finitude des fonctionnelles  $I(V^\uparrow)$  et  $I((-V)^\uparrow)$  est assurée par les Théorèmes 1.3.3 et 1.3.14 de la Section 1.3.2. Soit  $\mathcal{R}$  une variable aléatoire dont la loi est la convolution des lois de  $I(V^\uparrow)$  et de  $I((-V)^\uparrow)$ .

$\mathcal{R}$  est l'analogue de  $\mathcal{R}_\kappa$  définie à la Section 1.3.1 (en effet, si  $V = W_\kappa$  les lois de  $I(V^\uparrow)$  et de  $I((-V)^\uparrow)$  coïncident). Rappelons que  $\mathcal{C}$  la constante apparaissant dans le Théorème 1.2.2 qui donne la queue à droite de la fonctionnelle  $I(-V)$ , et posons  $\mathcal{C}' := (\mathcal{C}/2) \int_0^{+\infty} u^\kappa e^{-u/2} du$ . Soit  $\mathcal{Y}_1$  le subordonateur  $\kappa$ -stable dont l'exposant de Laplace est  $\mathcal{C}'\Gamma(1-\kappa)\lambda^\kappa$  :

$$\forall t, \lambda \geq 0, \mathbb{E} [e^{-\lambda \mathcal{Y}_1(t)}] = e^{-t\mathcal{C}'\Gamma(1-\kappa)\lambda^\kappa}.$$

On considère le processus de Lévy  $(\mathcal{Y}_1, \mathcal{Y}_2)$  où la composante  $\mathcal{Y}_2$  est définie en multipliant chaque saut de  $\mathcal{Y}_1$  par une copie indépendante de  $\mathcal{R}$ .  $(\mathcal{Y}_1, \mathcal{Y}_2)$  peut également être définie par sa mesure de Lévy comme à la Section 1.3.1 :

$$\forall x > 0, y > 0, \nu([x, +\infty[ \times ]y, +\infty[) = \frac{\mathcal{C}'}{y^\kappa} \mathbb{E} \left[ \mathcal{R}^\kappa \mathbf{1}_{\mathcal{R} \leq \frac{y}{x}} \right] + \frac{\mathcal{C}'}{x^\kappa} \mathbb{P} \left( \mathcal{R} > \frac{y}{x} \right).$$

Il est possible de voir que ces deux définitions du processus  $(\mathcal{Y}_1, \mathcal{Y}_2)$  sont équivalentes et que ce dernier a pour transformée de Laplace

$$\forall t, \alpha, \beta \geq 0, \mathbb{E} [e^{-\alpha \mathcal{Y}_1(t) - \beta \mathcal{Y}_2(t)}] = e^{-t\mathcal{C}'\Gamma(1-\kappa)\mathbb{E}[(\alpha + \beta \mathcal{R})^\kappa]}.$$

On a alors la convergence en loi de  $\mathcal{L}_X^*(t)/t$  et la loi limite s'exprime grâce à la même fonctionnelle qu'à la Section 1.3.1. On définit le couple aléatoire  $(\mathcal{I}_1, \mathcal{I}_2)$  par

$$\mathcal{I}_1 := \mathcal{Y}_1^{\natural}(\mathcal{Y}_2^{-1}(1)-), \quad \mathcal{I}_2 := (1 - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)-)) \times \frac{\mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)-)}{\mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)-)},$$

et la convergence en loi dont est l'objet le supremum du temps local s'écrit

**Theorem 1.3.20.** *Si  $0 < \kappa < 1$ ,  $V$  est à variation non bornée et  $V(1) \in L^p$  pour un  $p > 1$  alors*

$$\mathcal{L}_X^*(t)/t \xrightarrow[t \rightarrow +\infty]{\mathcal{L}} \mathcal{I} := \max(\mathcal{I}_1, \mathcal{I}_2).$$

Ce théorème inclut le cas où  $V = W_\kappa$  et est donc une généralisation du Théorème 1.3.1. La différence entre le cas d'un environnement brownien drifté et le cas plus général d'un environnement Lévy spectralement négatif vient de la variable  $\mathcal{R}$  dont la queue à gauche peut grandement varier selon les cas. Ceci induit une multitude de possibilités pour le comportement presque sûr du temps local. Nous verrons en particulier au Chapitre 5 comment la renormalisation de  $\mathcal{L}_X^*(t)$  dépend de la loi de la variable  $\mathcal{R}$  en ce qui concerne le comportement presque sûr.

La preuve du Théorème 1.3.20 nécessite une étude précise de l'environnement  $V$  afin de généraliser les arguments de la preuve du Théorème 1.3.1 qui repose sur des estimés bien connus et explicites pour le mouvement brownien drifté  $W_\kappa$ . Il nous faut en particulier généraliser la décomposition du potentiel en  $h$ -vallées et étudier les propriétés de ces vallées, la distance entre deux  $h$ -minima consécutifs est par exemple un élément important à connaître. La hauteur  $h$  des vallées considérées



doit être adaptée à l'échelle de temps et notamment tendre vers l'infini lorsque le temps  $t$  tend vers l'infini. Nous avons donc besoin d'étudier la distribution asymptotique des  $h$ -minima. Notons  $(m_1, m_2, \dots)$  la suite des  $h$ -minima, nous déterminons le comportement asymptotique de cette suite dans le théorème suivant :

**Theorem 1.3.21.** *Lorsque  $h$  tend vers l'infini, la suite renormalisée  $e^{-\kappa h}(m_1, m_2, \dots)$  converge en loi vers la suite des temps de sauts d'un processus de Poisson standard de paramètre  $q$  (qui dépend explicitement de la loi de  $V$ ). Si  $V = W_\kappa$  alors  $q = \kappa^2/2$ .*

L'intérêt de ce théorème est de nous informer sur la distance typique entre deux  $h$ -minima consécutifs. Ceci permet en particulier de donner une explication heuristique au fait que les méthodes et résultats obtenus sont différents dans les cas  $0 < \kappa < 1$  et  $\kappa > 1$ .

Comme le montre le Théorème 1.3.21, la distance entre deux  $h$ -minima consécutifs est de l'ordre de  $e^{\kappa h}$ . Lorsque  $0 < \kappa < 1$ , cette distance est telle qu'on peut négliger ce qui se passe en dehors des fonds des  $h$ -vallées, si bien que les principales contributions au temps local et au temps dépensé par la diffusion sont localisées au fond des  $h$ -vallées et fortement corrélées. Ceci se traduit par l'apparition du sous-ordinateur  $(\mathcal{Y}_1, \mathcal{Y}_2)$  qui ne s'accroît que par sauts (chacun représentant une vallée) et dont les deux composantes sont corrélées (l'une représente les pics de temps local dans les vallées et l'autre les dépenses de temps associées à ces vallées). Lorsque  $\kappa > 1$ , la distance entre deux  $h$ -minima consécutifs est si grande que le temps passé par la diffusion entre deux  $h$ -minima n'est pas négligeable comparée au temps passé à proximité de ces  $h$ -minima. En particulier il existe également des pics de temps local dans la zone entre deux  $h$ -minima. Il est alors impossible d'utiliser les puits de potentiel pour localiser les pics du temps local, contrairement au cas  $0 < \kappa < 1$ . Ceci explique la distribution asymptotiquement uniforme du point favori donnée au Théorème 1.3.19. De plus, le cas  $\kappa > 1$  correspond au cas où  $X$  est à vitesse positive. Le temps local au temps  $t$  est alors proche du temps local en un temps d'atteinte qui lui-même est égal en loi au processus d'Ornstein-Uhlenbeck généralisé, introduit dans [66]. Le supremum du temps local peut alors être représenté comme la hauteur maximale dans un ensemble *iid* d'excursions d'un processus Markovien. Ceci explique la loi de Fréchet au Théorème 1.3.17, puisqu'une telle loi est classiquement l'une des trois limites possibles pour les maxima de suites *iid*.

### 1.3.4 Chapitre 5 : Comportement presque sûr en milieu Lévy spectralement négatif

Ce travail a fait l'objet d'un article [73] qui sera prochainement soumis.

L'objectif de ce chapitre est d'étudier le comportement presque sûr du supremum du temps local  $\mathcal{L}_X^*(t)$  pour une diffusion en milieu aléatoire dont le potentiel est  $V$ , un processus de Lévy spectralement négatif et non monotone qui converge vers  $-\infty$  en  $+\infty$ . Ici encore nous posons  $\kappa := \Phi_V(0) = \sup\{\lambda \geq 0, \Psi_V(\lambda) = 0\}$ . Un cas

particulièrement intéressant est celui des valeurs extrêmement grandes de  $\mathcal{L}_X^*(t)$  dans le cas  $0 < \kappa < 1$ . Ces dernières sont reliées aux propriétés de la variable  $\mathcal{R}$  qui intervient dans la loi limite du Théorème 1.3.20 et qui peut être étudiée grâce aux résultats du Chapitre 3.

Nous caractérisons le comportement presque sûr de  $\mathcal{L}_X^*(t)$  lorsque  $0 < \kappa < 1$  et  $\kappa > 1$ . En particulier, la restriction de nos résultats au cas où  $V = W_\kappa$  avec  $0 < \kappa < 1$  améliore les résultats partiels (1.1.9), (1.1.10) et (1.1.11) de Devulder [28] en donnant la renormalisation exacte ainsi que la valeur exacte de la constante pour la lim sup. Pour la lim inf on obtient aussi la renormalisation exacte et un majorant explicite pour la constante.

Commençons par énoncer les résultats obtenus lorsque  $\kappa > 1$ . L'étude de ce cas est assez classique et utilise la même méthode qu'au Chapitre 4, c'est-à-dire l'exploitation des propriétés du processus d'Ornstein-Uhlenbeck généralisé introduit par Singh dans [66]. Pour la lim sup on a :

**Theorem 1.3.22.** *Soit  $f$  une fonction positive et décroissante. Lorsque  $\kappa > 1$ , on a*

$$\int_1^{+\infty} \frac{(f(t))^\kappa}{t} dt \begin{cases} < +\infty \\ = +\infty \end{cases} \Leftrightarrow \limsup_{t \rightarrow +\infty} \frac{f(t)\mathcal{L}_X^*(t)}{t^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \mathbb{P}\text{-p.s.}$$

Ce résultat est l'analogue, pour la diffusion en milieu Lévy, de (1.1.13) qui a été montré par Gantert et Shi [41] pour les RWRE transientes. Dans le cas où  $V = W_\kappa$  (pour  $\kappa > 1$ ), notre résultat est une reformulation de (1.1.6) établi par Devulder [28] (dont il est donc une généralisation).

Pour la lim inf, il y a une valeur positive explicite. Soient  $K$  et  $m$  définis en (1.3.10). On a :

**Theorem 1.3.23.** *Lorsque  $\kappa > 1$ , on a  $\mathbb{P}$ -presque sûrement*

$$\liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{(t/\log(\log(t)))^{1/\kappa}} = 2(\Gamma(\kappa)\kappa^2 K/m)^{1/\kappa}.$$

**Exemple :** Si  $V = W_\kappa$  (pour  $\kappa > 1$ ) alors on rappelle que  $K = 2^{\kappa-1}/\Gamma(\kappa)$  et  $m = 4/(\kappa - 1)$  (voir Remarque 1.3.18). La limite du Théorème 1.3.23 est alors  $4(\kappa^2(\kappa - 1)/8)^{1/\kappa}$ , ce qui coïncide précisément avec (1.1.7).

Le cas  $0 < \kappa < 1$  est beaucoup plus subtil et nécessite de pousser encore plus loin les méthodes utilisées aux Chapitres II et IV. La loi limite de  $\mathcal{L}_X^*(t)/t$  est donnée par le Théorème 1.3.20 et elle dépend implicitement de la loi de la variable  $\mathcal{R}$ , définie comme la convolution des lois de  $I(V^\uparrow)$  et de  $I((-V)^\uparrow)$ . Dans l'étude de la lim sup de  $\mathcal{L}_X^*(t)$ , on relie le comportement asymptotique presque sûr de  $\mathcal{L}_X^*(t)$  avec la queue à gauche de  $I(V^\uparrow)$  (ou de  $\mathcal{R}$  dans le cas où  $V = W_\kappa$ ). Cette queue est reliée au comportement asymptotique de  $\Psi_V$  par les résultats du Chapitre 3. La proposition suivante permet de relier précisément les queues des fonctionnelles exponentielles trouvées au Chapitre 3 avec la lim sup de  $\mathcal{L}_X^*(t)$ .

**Theorem 1.3.24.** — *On suppose que  $0 < \kappa < 1$ ,  $V$  est à variation non bornée,  $V(1) \in L^p$  pour un  $p > 1$  et  $V$  possède des sauts négatifs.*

*S'il existe  $\gamma > 1$  et  $C > 0$  tels que pour  $x$  suffisamment petit*

$$\mathbb{P}(I(V^\uparrow) \leq x) \leq \exp\left(-\frac{C}{x^{\frac{1}{\gamma-1}}}\right), \quad (1.3.11)$$

*alors on a  $\mathbb{P}$ -presque sûrement*

$$\limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\gamma-1}} \leq C^{1-\gamma}. \quad (1.3.12)$$

*S'il existe  $\gamma > 1$  et  $C > 0$  tels que pour  $x$  suffisamment petit*

$$\mathbb{P}(I(V^\uparrow) \leq x) \geq \exp\left(-\frac{C}{x^{\frac{1}{\gamma-1}}}\right), \quad (1.3.13)$$

*alors on a  $\mathbb{P}$ -presque sûrement*

$$\limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\gamma-1}} \geq C^{1-\gamma}. \quad (1.3.14)$$

— *On suppose maintenant que  $V = W_\kappa$  avec  $0 < \kappa < 1$ , alors les implications précédentes ((1.3.11)  $\Rightarrow$  (1.3.12) et (1.3.13)  $\Rightarrow$  (1.3.14)) sont vraies, mais avec  $\mathcal{R}$  à la place de  $I(V^\uparrow)$ .*

La proposition précédente fait la distinction entre le cas où  $V$  possède des sauts négatifs et celui où  $V$  est un mouvement brownien drifté  $W_\kappa$ . Notons que ces deux cas sont complémentaires : la formule (1.2.2) permet de voir que si  $V$  n'admet pas de sauts négatifs alors  $V$  est un mouvement brownien drifté qui s'écrit  $\sqrt{Q}W_\kappa$  où  $Q$ , comme dans la la formule (1.2.2), est la composante gaussienne de  $V$ . Par un changement d'échelle, ce cas se ramène à celui où  $V = W_\kappa$ , d'où l'alternative considérée dans la proposition. La différence du résultat entre les deux cas vient de la présence ou non de symétrie pour l'environnement. En effet, la loi de la variable  $\mathcal{R}$  est la convolution des lois de  $I(V^\uparrow)$  et de  $I((-V)^\uparrow)$ . Si  $V$  possède des sauts négatifs seule la queue de  $I(V^\uparrow)$  est à prendre en compte dans la queue à gauche de  $\mathcal{R}$  (comme le montrent les Théorèmes 1.3.5 et 1.3.15). Si  $V = W_\kappa$ , notons que  $(-W_\kappa)^\uparrow$  a même loi que  $W_{-\kappa}^\uparrow$  qui a lui-même même loi que  $W_\kappa^\uparrow$  (voir Section 1.2.3), ainsi  $(-V)^\uparrow$  et  $V^\uparrow$  ont même loi, et la loi de  $\mathcal{R}$  est alors la convolution de deux lois identiques dont aucune ne peut être négligée.

**Remark 1.3.25.** *Les lim sup dans la proposition précédente sont presque sûrement égales à des constantes appartenant à  $[0, +\infty]$  et les inégalités (1.3.12) et (1.3.14) sont des inégalités pour ces constantes. Il en est de même pour tous les résultats à suivre : toutes les lim sup et lim inf considérées sont presque sûrement égales à des constantes. Ce fait sera justifié à la fin du Chapitre 5.*

En combinant le Théorème 1.3.24 avec ce que le Chapitre 3 nous apprend sur la queue à gauche de  $I(V^\uparrow)$ , plus précisément le Théorème 1.3.5, on obtient :

**Theorem 1.3.26.** *Si  $0 < \kappa < 1$ ,  $V$  est à variation non bornée et  $V(1) \in L^p$  pour un  $p > 1$  alors on a  $\mathbb{P}$ -presque sûrement*

$$\forall \beta' > \beta, \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\beta'-1}} = 0, \quad (1.3.15)$$

et

$$\text{si de plus } \sigma > 1 \text{ alors } \forall \sigma' \in ]1, \sigma[, \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\sigma'-1}} = +\infty. \quad (1.3.16)$$

Le Théorème 1.3.24 est très précis dans le sens où la connaissance exacte de la queue à gauche de  $I(V^\uparrow)$  (ou de  $\mathcal{R}$  si  $V = W_\kappa$ ) entraîne la connaissance de la renormalisation exacte de  $\mathcal{L}_X^*(t)$  et de la valeur exacte de la  $\limsup$ . En se plaçant sous les hypothèses des Théorèmes 1.3.4 et 1.3.6 on obtient des résultats plus précis pour la  $\limsup$  :

**Theorem 1.3.27.** — *On suppose que  $0 < \kappa < 1$ ,  $V$  est à variation non bornée,  $V(1) \in L^p$  pour un  $p > 1$  et  $V$  possède des sauts négatifs.*

*S'il existe deux constantes positives  $c < C$  et  $\alpha \in ]1, 2]$  tels que  $c\lambda^\alpha \leq \Psi_V(\lambda) \leq C\lambda^\alpha$  pour  $\lambda$  suffisamment grand. On a alors  $\mathbb{P}$ -presque sûrement*

$$\frac{c}{\alpha^\alpha} \leq \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\alpha-1}} \leq \frac{C}{(\alpha-1)^{\alpha-1}}.$$

*Si, plus précisément, il existe une constante positive  $C$  et  $\alpha \in ]1, 2]$  tels que  $\Psi_V(\lambda) \sim C\lambda^\alpha$  pour  $\lambda$  grand, on a alors  $\mathbb{P}$ -presque sûrement*

$$\limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\alpha-1}} = \frac{C}{(\alpha-1)^{\alpha-1}}.$$

— *On suppose maintenant que  $V = W_\kappa$  avec  $0 < \kappa < 1$ , on a alors  $\mathbb{P}$ -presque sûrement*

$$\limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))} = \frac{1}{8}.$$

**Remark 1.3.28.** *En combinant le Théorème 1.3.24 et la Remarque 1.3.7 on a que si  $0 < \kappa < 1$ ,  $V$  est à variation non bornée et  $V(1) \in L^p$  pour un  $p > 1$ , alors on a toujours*

$$\limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))} < +\infty.$$

*En d'autres termes,  $t(\log(\log(t)))$  est la renormalisation maximale parmi toutes les autres possibles pour la  $\limsup$ .*

**Remark 1.3.29.** *De même que cela avait été constaté par Shi [61] et Diel [29] dans le cas récurrent (voir Remarque 1.1.6), on constate ici aussi une différence entre la renormalisation des RWRE transientes à vitesse nulle donnée par (1.1.14) et celles des diffusions transientes à vitesse nulle données par les résultats précédents. Cette différence peut s'expliquer de la même façon que dans le cas récurrent, c'est-à-dire par le fait que les vallées peuvent être beaucoup plus abrupte dans le cas continu avec un potentiel à variation non bornée que dans le cas discret, et donc les pics de temps local peuvent potentiellement être beaucoup plus grands dans le premier cas.*

Les résultats précédents montrent que la renormalisation de  $\mathcal{L}_X^*(t)$  pour la  $\limsup$  dépend directement du comportement asymptotique de  $\Psi_V$ . En particulier, le Théorème 1.3.27 montre que pour un environnement  $\alpha$ -stable drifté (sans sauts positifs et avec  $\alpha > 1$ ), la renormalisation de  $\mathcal{L}_X^*(t)$  est  $t(\log(\log(t)))^{\alpha-1}$ . On voit qu'il existe, pour des environnements Lévy spectralement négatifs, une plus grande variété de comportements presque sûrs qu'avec des environnements browniens driftés. Même si, pour des raisons techniques, les résultats précédents ne s'appliquent pas dans le cas où l'environnement  $V$  est à variation bornée, on peut légitimement conjecturer que le comportement de  $\mathcal{L}_X^*(t)$  reste lié à la queue à gauche de  $I(V^\uparrow)$  qui est donnée par la Remarque 1.3.8. Ceci implique la conjecture suivante :

**Conjecture 1.3.30.** *Si  $V$  est à variation bornée, on a  $\mathbb{P}$ -presque sûrement*

$$0 < \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t} < +\infty.$$

Si cette conjecture est vraie on aurait, pour des environnements à variation bornée, la même renormalisation que dans le cas discret transient donné par (1.1.14). Ceci ne serait pas surprenant compte tenu du fait que le cas discret ne génère que des potentiels à variation bornée. De plus, si  $V$  est un processus de Lévy spectralement négatif, non monotone, à variation bornée, alors  $V$  est la différence d'un drift positif et d'un subordonateur. Les vallées ne peuvent alors pas être plus abrupte que le drift ce qui, conformément à l'explication heuristique de la Remarque 1.3.29, impose que la renormalisation de  $\mathcal{L}_X^*(t)$  soit la même que dans le cas discret.

Pour la  $\liminf$ , il n'y a qu'une seule renormalisation possible. Nous prouvons le résultat suivant :

**Theorem 1.3.31.** *Si  $0 < \kappa < 1$ ,  $V$  est à variation non bornée et  $V(1) \in L^p$  pour un  $p > 1$  alors on a  $\mathbb{P}$ -presque sûrement*

$$0 < \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t/\log(\log(t))} \leq \frac{1 - \kappa}{\kappa(\mathbb{E}[I(V^\uparrow)] + \mathbb{E}[I((-V)^\uparrow)])}. \quad (1.3.17)$$

**Remark 1.3.32.** *Les espérances  $\mathbb{E}[I(V^\uparrow)]$  et  $\mathbb{E}[I((-V)^\uparrow)]$  sont finies puisque  $I(V^\uparrow)$  et  $I((-V)^\uparrow)$  admettent des moments exponentiels finis en vertu des Théorèmes 1.3.3 et 1.3.14.*

**Exemple :** L'expression (1.2.4) permet de calculer le premier moment de  $I(W_\kappa^\uparrow)$  :  $\mathbb{E}[I(W_\kappa^\uparrow)] = 2/(1+\kappa)$ . On rappelle de plus que  $(-W_\kappa)^\uparrow$  a même loi que  $W_\kappa^\uparrow$ , si bien que  $\mathbb{E}[I(W_\kappa^\uparrow)] + \mathbb{E}[I((-W_\kappa)^\uparrow)] = 4/(1+\kappa)$ . Ainsi, pour la diffusion dans l'environnement  $V = W_\kappa$  (avec  $0 < \kappa < 1$ ), la borne supérieure du Théorème 1.3.31 pour la  $\liminf$  devient  $(1 - \kappa^2)/4\kappa$ . En mettant ceci en relation avec les résultats obtenus par Devulder [28], on voit que l'application du Théorème 1.3.31 dans le cas spécial d'un environnement brownien drifté amélioré (1.1.11) et complète (1.1.10) en donnant une majoration explicite de la  $\liminf$ .

Il est intéressant de se demander pourquoi il existe, selon l'environnement choisi, une multitude de renormalisations possibles pour la  $\limsup$ , mais seulement une pour la  $\liminf$ . Nous proposons l'explication heuristique suivante : dans chaque vallée, le temps passé est environ égal au pic de temps local au fond de la vallée multiplié par une fonctionnelle exponentielle du fond de la vallée (qui est proche de  $\mathcal{R}$ ). La  $\limsup$  est liée aux grandes valeurs prises par le temps local à un temps fixé  $t$ , elle est approchée lorsque le pic de temps local au fond d'une vallée est très grand tandis que le temps passé dans cette vallée est de l'ordre au plus de  $t$ , une telle chose arrive quand la fonctionnelle exponentielle du fond d'une certaine vallée est très petite. Un lien très précis entre la  $\limsup$  et les petites valeurs d'une fonctionnelle exponentielle est d'ailleurs fait rigoureusement par le Théorème 1.3.24. La  $\liminf$  est quant à elle liée aux petites valeurs prises par le temps local à un temps fixé  $t$ , elle est approchée lorsque les pics de temps local au fond des premières vallées sont très petits tandis que la somme des temps passés dans ces vallées est de l'ordre au moins de  $t$ , une telle chose arrive quand les fonctionnelles exponentielles des fonds de quelques-unes de ces vallées sont grandes. On voit alors que la différence de comportement entre la  $\limsup$  et la  $\liminf$  vient de la différence entre la queue à gauche et la queue à droite de  $\mathcal{R}$ . La queue à gauche est principalement celle de  $I(V^\uparrow)$  qui dépend du comportement asymptotique de  $\Psi_V$  d'après les Théorèmes 1.3.4-1.3.5-1.3.6, et il y a plusieurs possibilités pour le comportement de  $\Psi_V$ , selon le choix de l'environnement. D'un autre côté, la queue à droite est toujours exponentielle : elle est au plus exponentielle d'après les Théorèmes 1.3.3-1.3.14 et il sera justifié au Chapitre 3 (Remarque 3.3.5) qu'elle est au moins exponentielle. Ceci explique la différence des comportements entre les  $\limsup$  et  $\liminf$ .

En comparant les Théorèmes 1.3.22, 1.3.26 et 1.3.27 on constate que la renormalisation de la  $\limsup$  est plus grande dans le cas transient à vitesse nulle ( $0 < \kappa < 1$ ) que dans le cas transient à vitesse positive ( $\kappa > 1$ ), ce qui est en accord avec l'intuition. Il est cependant surprenant de constater que la renormalisation du cas transient à vitesse nulle est également plus grande que la renormalisation du cas récurrent donnée par (1.1.2) et par le Théorème 1.1.4. Nous interprétons ce phénomène de la façon suivante : dans le cas récurrent la diffusion est piégée au fond d'une vallée assez large, tandis que dans le cas transient à vitesse nulle la diffusion se retrouve successivement piégée au fond de vallées qui sont beaucoup plus étroites. Cette différence de largeur des vallées explique que les valeurs extrêmement grandes du temps local ont tendance à être plus grandes dans le second cas, malgré la transience.

Pour la  $\liminf$  les comportements mis en valeurs dans les différents cas sont en accord avec l'intuition : en comparant les Théorèmes 1.1.4, 1.3.31 et 1.3.23 on voit que la renormalisation de la  $\liminf$  dans le cas transient à vitesse positive est plus petite que la renormalisation du cas transient à vitesse nulle qui est elle-même plus petite que la renormalisation du cas recurrent.

## 1.4 Quelques perspectives

Nous présentons ici quelques travaux en cours et perspectives de recherche qui s'inscrivent dans la continuité des travaux présentés dans cette thèse.

Il serait intéressant de poursuivre l'étude effectuée au Chapitre 3, sur la fonctionnelle exponentielle  $I(V^\dagger)$ , notamment en tentant d'étendre à ce problème les méthodes d'analyse complexe (transformée de Mellin) utilisées pour l'étude des fonctionnelles exponentielles des processus de Lévy non conditionnés.

Pour la diffusion en milieu aléatoire, il serait intéressant d'utiliser les techniques que nous avons développées dans cette thèse pour étudier le problème du *temps semi-local*, c'est-à-dire déterminer la taille du plus petit intervalle où une proportion  $\beta \in ]0, 1[$  du temps a été passée par la diffusion. Cela permettrait de mettre en évidence des comportements spéciaux tels que la concentration dans un intervalle arbitrairement petit ou au contraire un étalement dans l'espace des points les plus occupés. Pour  $\beta \in ]0, 1[$ , définissons  $S(t, \beta)$ , la taille du plus petit intervalle où la diffusion a passé un temps plus grand que  $\beta t$  :

$$S(t, \beta) := \inf \left\{ \alpha > 0, \sup_{x \in \mathbb{R}} \int_{x-\alpha}^{x+\alpha} \mathcal{L}_X(t, y) dy \geq \beta t \right\}.$$

Nous nous proposons d'étudier la convergence en loi et le comportement presque sûr de  $S(t, \beta)$ .

Lorsque  $0 < \kappa < 1$ , notre étude du supremum du temps local exclu le cas où l'environnement est à variation bornée. Il serait intéressant de faire l'étude du temps local de la diffusion dans ce cas. Dans ce sens nous pourrions par exemple prouver la Conjecture 1.3.30 ou étudier l'existence, pour ces diffusions, d'un principe d'invariance, c'est-à-dire déterminer s'il existe ou non une proximité entre le temps local de la diffusion et celui d'une RWRE associée.

# Chapitre 2

## RENEWAL STRUCTURE AND LOCAL TIME FOR DIFFUSIONS IN RANDOM ENVIRONMENT

This work comes from a collaboration with Pierre Andreatti and Alexis Devulder and has been the object of an article [4] published in the journal ALEA.

### 2.1 Introduction

#### 2.1.1 Presentation of the model

Let  $(X(t), t \geq 0)$  be a diffusion in a random càdlàg potential  $(V(x), x \in \mathbb{R})$ , defined informally by  $X(0) = 0$  and

$$dX(t) = d\beta(t) - \frac{1}{2}V'(X(t))dt,$$

where  $(\beta(s), s \geq 0)$  is a Brownian motion independent of  $V$ . Rigorously,  $X$  is defined by its conditional generator given  $V$ ,

$$\frac{1}{2}e^{V(x)} \frac{d}{dx} \left( e^{-V(x)} \frac{d}{dx} \right).$$

We put ourselves in the case where  $V$  is a negatively drifted Brownian motion :  $V(x) = W_\kappa(x) := W(x) - \frac{\kappa}{2}x$ ,  $x \in \mathbb{R}$ , with  $0 < \kappa < 1$  and  $(W(x), x \in \mathbb{R})$  is a two sided Brownian motion. We explain at the end of Section 2.1.2 what should be done to extend our results to a more general Lévy potential.

We denote by  $P$  the probability measure associated to  $W_\kappa(\cdot)$ . The probability conditionally on the potential  $W_\kappa$  is denoted by  $\mathbb{P}^{W_\kappa}$  and is called the *quenched probability*. We also define the *annealed probability* as

$$\mathbb{P}(\cdot) := \int \mathbb{P}^{W_\kappa}(\cdot) P(W_\kappa \in d\omega).$$



We denote respectively by  $\mathbb{E}^{W_\kappa}$ ,  $\mathbb{E}$ , and  $E$  the expectations with regard to  $\mathbb{P}^{W_\kappa}$ ,  $\mathbb{P}$  and  $P$ . In particular,  $X$  is a Markov process under  $\mathbb{P}^{W_\kappa}$  but not under  $\mathbb{P}$ .

This diffusion  $X$  has been introduced by [60]. It is generally considered as a continuous time analogue of random walks in random environment (RWRE). We refer e.g. to [76] for general properties of RWRE.

In our case, since  $\kappa > 0$ , the diffusion  $X$  is a.s. transient and its asymptotic behavior was first studied by Kawazu and Tanaka : if  $H(r)$  is the hitting time of  $r \in \mathbb{R}$  by  $X$ ,

$$H(r) := \inf\{s > 0, X(s) = r\}, \quad (2.1.1)$$

[45] proved that, for  $0 < \kappa < 1$  under the annealed probability  $\mathbb{P}$ ,  $H(r)/r^{1/\kappa}$  converges in law as  $r \rightarrow +\infty$  to a  $\kappa$ -stable distribution (see also [44], and [69]). Here we are interested in the local time of  $X$ , which is the jointly continuous process  $(\mathcal{L}(t, x), t > 0, x \in \mathbb{R})$  satisfying, for any positive measurable function  $f$ ,

$$\int_0^t f(X(s))ds = \int_{-\infty}^{+\infty} f(x)\mathcal{L}(t, x)dx, \quad t > 0.$$

One quantity of particular interest is the supremum of the local time of  $X$  at time  $t$ , defined as

$$\mathcal{L}^*(t) := \sup_{x \in \mathbb{R}} \mathcal{L}(t, x), \quad t > 0.$$

For Brox's diffusion, that is, for the diffusion  $X$  in the recurrent case  $\kappa = 0$ , it is proved in [5] that the local time process until time  $t$  re-centered at the localization coordinate  $b_t$  (see [17]) and renormalized by  $t$  converges in law under the annealed probability  $\mathbb{P}$ . This allows the authors of [5] to derive the limit law of the supremum of the local time at time  $t$  as  $t \rightarrow +\infty$ . We recall their result below in order to compare it with the results of the present paper. To this aim, we introduce for every  $\kappa \geq 0$ ,

$$\mathcal{R}_\kappa := \int_0^{+\infty} e^{-W_\kappa^\uparrow(x)} dx + \int_0^{+\infty} e^{-\widetilde{W}_\kappa^\uparrow(x)} dx, \quad (2.1.2)$$

where  $(W_\kappa^\uparrow(x), x \geq 0)$  and  $(\widetilde{W}_\kappa^\uparrow, x \geq 0)$  are two independent copies of the process  $(W_\kappa(x), x \geq 0)$  Doob-conditioned to remain positive.

**Theorem 2.1.1.** ([5]) *If  $\kappa = 0$ , then*

$$\frac{\mathcal{L}^*(t)}{t} \xrightarrow{\mathcal{L}} \frac{1}{\mathcal{R}_\kappa},$$

where  $\xrightarrow{\mathcal{L}}$  denotes convergence in law under the annealed probability  $\mathbb{P}$  as  $t \rightarrow +\infty$ .

Extending their approach, and following the results of [61], [29] obtains the non-trivial normalizations for the almost sure behavior of the limsup and the liminf of  $\mathcal{L}^*(t)$  as  $t \rightarrow +\infty$  when  $\kappa = 0$ . Notice that corresponding results have been previously established in [26] and [40] for the discrete analogue of  $X$  in the recurrent case  $\kappa = 0$ , the recurrent RWRE generally called Sinai's random walk.

One of our aims in this paper is to extend the study of the local time of  $X$  in the case  $0 < \kappa < 1$ , and deduce from that the weak asymptotic behavior of  $\mathcal{L}^*(t)$  suitably renormalized as  $t \rightarrow +\infty$ .

Before going any further, let us recall to the reader what is known for the slow transient cases. For transient RWRE in the case  $0 < \kappa \leq 1$  (see [48] for the seminal paper), a result of [41] states the almost sure behavior for the limsup of the supremum of the local time  $\mathcal{L}_S^*(n)$  of these random walks (denoted by  $S$ ) at time  $n$ : there exists a constant  $c > 0$  such that  $\limsup_{n \rightarrow +\infty} \mathcal{L}_S^*(n)/n = c > 0$   $\mathbb{P}$  almost surely. Contrarily to the recurrent case ([40]) their method, based on a relationship between the RWRE  $S$  and a branching process in random environment, cannot be exploited to determine the limit law of  $\mathcal{L}_S^*(n)/n$ .

For the transient diffusion  $X$  considered here, the only paper dealing with  $\mathcal{L}^*(t)$  is [27], in which it is proved, among other results, that when  $0 < \kappa < 1$ ,  $\limsup_{t \rightarrow +\infty} \mathcal{L}^*(t)/t = +\infty$  almost surely. But once again his method cannot be used to characterize the limit law of  $\mathcal{L}^*(t)/t$  in the case  $0 < \kappa < 1$ .

Our motivation here is twofold, first we prove that our approach enables to characterize the limit law of  $\mathcal{L}^*(t)/t$  and open a way to determine the correct almost sure behavior of  $\mathcal{L}^*(t)$  as was done for Brox's diffusion by [61] and [29]. Second we make a first step on a specific way to study the local time which could be used in estimation problems in random environment, see [1], [2], [6], [7], [24], [25], [38].

The method we develop here is an improvement of the one used in [3] about the localization of  $X(t)$  for large  $t$ .

Before recalling the main result of this paper [3], we need to introduce some new objects. We start with the notion of  $h$ -extrema, with  $h > 0$ , introduced by [53] and studied more specifically in our case of drifted Brownian motions by [37]. For  $h > 0$ , we say that  $x \in \mathbb{R}$  is an  $h$ -minimum for a given continuous function  $f, \mathbb{R} \rightarrow \mathbb{R}$ , if there exist  $u < x < v$  such that  $f(y) \geq f(x)$  for all  $y \in [u, v]$ ,  $f(u) \geq f(x) + h$  and  $f(v) \geq f(x) + h$ . Moreover,  $x$  is an  $h$ -maximum for  $f$  iff  $x$  is an  $h$ -minimum for  $-f$ . Finally,  $x$  is an  $h$ -extremum for  $f$  iff it is an  $h$ -maximum or an  $h$ -minimum for  $f$ .

As we are interested in the diffusion  $X$  until time  $t$  for large  $t$ , we only focus on the  $h_t$ -extrema of  $W_\kappa$ , where

$$h_t := \log t - \phi(t), \quad \text{with } 0 < \phi(t) = o(\log t), \quad \log \log t = o(\phi(t)),$$

and  $t \mapsto \phi(t)$  is an increasing function, as in [3]. It is known (see [37]) that almost surely, the  $h_t$ -extrema of  $W_\kappa$  form a sequence indexed by  $\mathbb{Z}$ , unbounded from below and above, and that the  $h_t$ -minima and  $h_t$ -maxima alternate. We denote respectively by  $(m_j, j \in \mathbb{Z})$  and  $(M_j, j \in \mathbb{Z})$  the increasing sequences of  $h_t$ -minima and of  $h_t$ -maxima of  $W_\kappa$ , such that  $m_0 \leq 0 < m_1$  and  $m_j < M_j < m_{j+1}$  for every  $j \in \mathbb{Z}$ .

Define

$$N_t := \max \left\{ k \in \mathbb{N}, \sup_{0 \leq s \leq t} X(s) \geq m_k \right\}, \quad (2.1.3)$$

the number of (positive)  $h_t$ -minima on  $\mathbb{R}_+$  visited by  $X$  until time  $t$ . We have the following result.

**Theorem 2.1.2.** ([3]) *Assume  $0 < \kappa < 1$ . There exists a constant  $\mathcal{C}_1 > 0$ , such that*

$$\lim_{t \rightarrow +\infty} \mathbb{P} \left( |X(t) - m_{N_t}| \leq \mathcal{C}_1 \phi(t) \right) = 1.$$

This result proves that before time  $t$ , the diffusion  $X$  visits the  $N_t$  leftmost positive  $h_t$ -minima, and then gets stuck in a very small neighborhood of an ultimate one, which is  $m_{N_t}$ . An analogous result was proved for transient RWRE in the zero speed regime  $0 < \kappa < 1$  by [34]. This phenomenon is due to two facts : the first one is the appearance of a renewal structure which is composed of the times it takes the process to move from one  $h_t$ -minimum to the following one. The second is the fact that like in Brox's case  $\kappa = 0$ , the process is trapped a significant amount of time in the neighborhood of the local minimum  $m_{N_t}$ .

It is the extension of this renewal structure to the sequence of local times at the  $h_t$ -minima that we study here. We now detail our results.

## 2.1.2 Results

Let us introduce some notation involved in the statement of our results. Assume that  $0 < \kappa < 1$ . Denote by  $(D([0, +\infty), \mathbb{R}^2), J_1)$  the space of càdlàg functions  $[0, +\infty) \rightarrow \mathbb{R}^2$  with  $J_1$ -Skorokhod topology and denote by  $\xrightarrow{\mathcal{L}_S}$  the convergence in law for this topology. On this space, define a 2-dimensional Lévy process  $(\mathcal{Y}_1, \mathcal{Y}_2)$  taking values in  $\mathbb{R}_+ \times \mathbb{R}_+$ , which is a pure positive jump process with  $\kappa$ -stable Lévy measure  $\nu$  given by

$$\forall x > 0, \forall y > 0, \quad \nu([x, +\infty[ \times [y, +\infty[) = \frac{\mathcal{C}_2}{y^\kappa} \mathbb{E} \left[ (\mathcal{R}_\kappa)^\kappa \mathbf{1}_{\mathcal{R}_\kappa \leq \frac{y}{x}} \right] + \frac{\mathcal{C}_2}{x^\kappa} \mathbb{P} \left( \mathcal{R}_\kappa > \frac{y}{x} \right), \quad (2.1.4)$$

where  $\mathcal{R}_\kappa$  is defined in (2.1.2) and  $\mathcal{C}_2$  is a positive constant (see Lemma 2.4.1). The Laplace transform of  $\mathcal{R}_\kappa$  is given by

$$E(e^{-\gamma \mathcal{R}_\kappa}) = \left( \frac{(2\gamma)^{\kappa/2}}{\kappa \Gamma(\kappa) I_\kappa(2\sqrt{2\gamma})} \right)^2 \quad \gamma > 0, \quad (2.1.5)$$

as proved in Lemma 2.6.6 below, where  $I_\kappa$  is the modified Bessel function of the first kind of index  $\kappa$ . Moreover,  $\mathcal{R}_\kappa$  admits moments of any positive order (see also Lemma 2.6.6). In particular  $\mathbb{E}[(\mathcal{R}_\kappa)^\kappa]$  is finite and  $\nu$  is well defined.

For a given càdlàg function  $f$  in  $D([0, +\infty), \mathbb{R})$ , define for any  $s > 0, a > 0$  :

$$f^{\natural}(s) := \sup_{0 \leq r \leq s} (f(r) - f(r^-)), \quad f^{-1}(a) := \inf\{x \geq 0, f(x) > a\},$$

where  $f(r^-)$  denotes the left limit of  $f$  at  $r$ . In words,  $f^{\natural}(s)$  is the largest jump of  $f$  before time  $s$ , whereas  $f^{-1}(a)$  is the first time  $f$  is strictly larger than  $a$ . We also introduce the couple of random variables  $(\mathcal{I}_1, \mathcal{I}_2)$  as follows,

$$\mathcal{I}_1 := \mathcal{Y}_1^{\natural}(\mathcal{Y}_2^{-1}(1)^-), \quad \mathcal{I}_2 := (1 - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)^-)) \times \frac{\mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)^-)}{\mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)^-)}. \quad (2.1.6)$$

We recall that  $\xrightarrow{\mathcal{L}}$  denotes convergence in law under the annealed probability  $\mathbb{P}$  as  $t \rightarrow +\infty$ . We are now ready to state our first result.

**Theorem 2.1.3.** *Assume  $0 < \kappa < 1$ . We have,*

$$\frac{\mathcal{L}^*(t)}{t} \xrightarrow{\mathcal{L}} \mathcal{I} =: \max(\mathcal{I}_1, \mathcal{I}_2).$$

Contrary to the recurrent case  $\kappa = 0$ , we have no scaling property for the potential, and the diffusion  $X$  cannot be localized in a single valley as we can see in Theorem 2.1.2. However in the transient case we can make appear and use a renewal structure.

We now give an intuitive interpretation of this theorem, explaining the appearance of the Lévy process  $(\mathcal{Y}_1, \mathcal{Y}_2)$ .

First for any  $s > 0$ ,  $\mathcal{Y}_1(s)$  is the limit of the sum of the first  $\lfloor se^{\kappa\phi(t)} \rfloor$  normalized (by  $t$ ) local times taken specifically at the  $\lfloor se^{\kappa\phi(t)} \rfloor$  first  $h_t$ -minima (see Proposition 2.1.4 below). Similarly,  $\mathcal{Y}_2(s)$  is the limit of the sum of the exit times of the  $\lfloor se^{\kappa\phi(t)} \rfloor$  first  $h_t$ -valleys, normalized (by  $t$ ), where an  $h_t$ -valley is a large neighborhood of an  $h_t$ -minimum. For a rigorous definition of these  $h_t$ -valleys, see Section 2.2.2 and Figure 2.1.

So, by definition,  $\mathcal{I}_1$  is the largest jump of the process  $\mathcal{Y}_1$  before the first time  $\mathcal{Y}_2$  is larger than 1. It can be interpreted as the largest (re-normalized) local time among the local times at the  $h_t$ -minima visited by  $X$  until time  $t$  and from which  $X$  has already escaped. That is to say,  $\mathcal{I}_1$  is the limit of the random variable  $\sup_{k \leq N_t-1} \mathcal{L}(m_k, t)/t$ .  $\mathcal{I}_2$  is a product of two factors : the first one,  $(1 - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)^-))$ , corresponds to the (re-normalized) amount of time left to the diffusion  $X$  before time  $t$  after it has reached the ultimate visited  $h_t$ -minimum  $m_{N_t}$ , that is, to  $(t - H(m_{N_t}))/t$ . The second factor corresponds to the local time of  $X$  at this ultimate  $h_t$ -minimum  $m_{N_t}$ , that is to say  $\mathcal{I}_2$  is the limit of  $\mathcal{L}(t, m_{N_t})/t$ . Intuitively  $\mathcal{Y}_2$  is built from  $\mathcal{Y}_1$  by multiplying each of its jumps by an independent copy of the variable  $\mathcal{R}_\kappa$ . Therefore this second factor can be seen as an independent copy of  $1/\mathcal{R}_\kappa$  taken at the instant of the overshoot of  $\mathcal{Y}_2$  which makes it larger than 1. Notice that this variable  $\mathcal{R}_\kappa$  plays a similar role as  $\mathcal{R}_0$  of Theorem 2.1.1. Indeed as in the case  $\kappa = 0$ , the diffusion  $X$  is prisoner in the neighborhood of the last  $h_t$ -minimum visited before time  $t$ .

We prove Theorem 2.1.3 by showing first that portions of the trajectory of  $X$  re-centered at the local  $h_t$ -minima, until time  $t$ , are made (in probability) with independent parts. This has been partially proved in [3] but we have to improve their results and add simultaneously the study of the local time.

Second, we prove that the supremum of the local time is, mainly, a function of the sum of these independent parts, which converges to a Lévy process. We now provide some details about this.

Recall that  $(W_\kappa^\uparrow(s), s \geq 0)$  is defined as a continuous process, taking values in  $\mathbb{R}_+$ , with infinitesimal generator given for every  $x > 0$  by

$$\frac{1}{2} \frac{d^2}{dx^2} + \frac{\kappa}{2} \coth\left(\frac{\kappa}{2}x\right) \frac{d}{dx}.$$

This process  $W_\kappa^\uparrow$  can be thought of as a  $(-\kappa/2)$ -drifted Brownian motion  $W_\kappa$  Doob-conditioned to stay positive, with the terminology of [8], which is called Doob conditioned to reach  $+\infty$  before 0 in [37] (for more details, see Section 2.1 in [3], where  $W_\kappa^\uparrow$  is denoted by  $R$ ). We call  $\text{BES}(3, \kappa/2)$  the law of  $(W_\kappa^\uparrow(s), s \geq 0)$ . That is,  $(W_\kappa^\uparrow(s), s \geq 0)$  is a 3-dimensional  $(\kappa/2)$ -drifted Bessel process starting from 0. For any process  $(U(t), t \in \mathbb{R}_+)$ , we denote by

$$\tau^U(a) := \inf\{t > 0, U(t) = a\},$$

the first time this process hits  $a$ , with the convention  $\inf \emptyset = +\infty$ . For  $a < b$ ,  $(W_\kappa^b(s), 0 \leq s \leq \tau^{W_\kappa^b}(a))$  is defined as a  $(-\kappa/2)$ -drifted Brownian motion starting from  $b$  and killed when it first hits  $a$ . We now introduce some functionals of  $W_\kappa$  and  $W_\kappa^\uparrow$ , which already appeared in ([3], Section 4.1) :

$$F^\pm(x) := \int_0^{\tau^{W_\kappa^\uparrow}(x)} \exp(\pm W_\kappa^\uparrow(s)) ds, \quad x > 0, \quad (2.1.7)$$

$$G^\pm(a, b) := \int_0^{\tau^{W_\kappa^b}(a)} \exp(\pm W_\kappa^b(s)) ds, \quad a < b. \quad (2.1.8)$$

Let  $0 < \delta < 1$ , define

$$n_t := \lfloor e^{\kappa\phi(t)(1+\delta)} \rfloor, \quad t > 0,$$

which is, with large probability, an upper bound for  $N_t$  as stated in Lemma 2.3.1.

Let  $(S_j, R_j, \mathbf{e}_j, j \leq n_t)$  be a sequence of i.i.d. random variables depending on  $t$ , with  $S_j, R_j$  and  $\mathbf{e}_j$  independent,  $S_1 \stackrel{\mathcal{L}}{=} F^+(h_t) + G^+(h_t/2, h_t)$ ,  $R_1 \stackrel{\mathcal{L}}{=} F^-(h_t/2) + \tilde{F}^-(h_t/2)$  and  $\mathbf{e}_1 \stackrel{\mathcal{L}}{=} \mathcal{E}(1/2)$  (an exponential random variable with parameter 1/2), where  $\tilde{F}^-$  is an independent copy of  $F^-$  and  $F^+$  is independent of  $G^+$ , and  $\stackrel{\mathcal{L}}{=}$  denotes equality in law. Define  $\ell_j := \mathbf{e}_j S_j$  and  $\mathcal{H}_j := \ell_j R_j$ . Note that to simplify the notation, we do not make appear the dependence in  $t$  in the sequel. Intuitively,  $\ell_j$  plays the role of the local time at the  $j$ -th positive  $h_t$ -minimum  $m_j$  if  $X$  escapes from the  $j$ -th  $h_t$ -valley before time  $t$ , that is, if  $j < N_t$ . Similarly,  $\mathcal{H}_j$  plays the role of the time  $X$  spends in the  $j$ -th  $h_t$ -valley before escaping from it.

Define the family of processes  $(Y_1, Y_2)^t$  indexed by  $t$ , by

$$\forall s \geq 0, \quad (Y_1, Y_2)_s^t = (Y_1^t(s), Y_2^t(s)) := \frac{1}{t} \sum_{j=1}^{\lfloor se^{\kappa\phi(t)} \rfloor} (\ell_j, \mathcal{H}_j).$$

Recall that  $\xrightarrow{\mathcal{L}_1^S}$  denotes convergence in law under  $J_1$ -Skorokhod topology. Here is our next result.

**Proposition 2.1.4.** *Assume  $0 < \kappa < 1$ . We have under  $\mathbb{P}$ , as  $t \rightarrow +\infty$ ,*

$$(Y_1, Y_2)^t \xrightarrow{\mathcal{L}_1^S} (\mathcal{Y}_1, \mathcal{Y}_2).$$

Once this is proved, we check that we can approximate, in law, the renormalized local time  $\mathcal{L}^*(t)/t$  by a function of  $(Y_1, Y_2)^t$ . We obtain such an expression in Proposition 2.5.1. Then to obtain the limit claimed in Theorem 2.1.3, we prove the continuity (in  $J_1$ -topology) of the involved mapping and apply a continuous mapping Theorem (see Section 2.4.3).

It appears that with this method we can also obtain some other asymptotics. Indeed, we obtain in the following theorem the convergence in law of the supremum of the local time of  $X$  before  $X$  hits the last  $h_t$ -minimum  $m_{N_t}$  visited before time  $t$ , of the supremum of the local time of  $X$  before  $X$  leaves the last  $h_t$ -valley visited before time  $t$  (the one around  $m_{N_t}$ ) approximately at time  $H(m_{N_t+1})$ , and of the position of the favorite site.

**Theorem 2.1.5.** *Assume  $0 < \kappa < 1$ . We have the following convergences in law under  $\mathbb{P}$  as  $t \rightarrow +\infty$ ,*

$$\frac{\mathcal{L}^*(H(m_{N_t+1}))}{t} \xrightarrow{\mathcal{L}} \mathcal{Y}_1^\sharp(\mathcal{Y}_2^{-1}(1)), \quad (2.1.9)$$

$$\frac{\mathcal{L}^*(H(m_{N_t}))}{t} \xrightarrow{\mathcal{L}} \mathcal{Y}_1^\sharp(\mathcal{Y}_2^{-1}(1)^-) = \mathcal{I}_1. \quad (2.1.10)$$

Let us call  $F_t^*$  the position of the first favorite site, that is,  $F_t^* := \inf\{x \in \mathbb{R}, \mathcal{L}(t, x) = \mathcal{L}^*(t)\}$ . Then,

$$\frac{F_t^*}{X(t)} \xrightarrow{\mathcal{L}} \mathcal{B}U_{[0,1]} + 1 - \mathcal{B}, \quad (2.1.11)$$

where  $\mathcal{B}$  is a Bernoulli random variable with parameter  $\mathbb{P}(\mathcal{I}_1 < \mathcal{I}_2)$ , and  $U_{[0,1]}$  is a uniform random variable on  $[0, 1]$ , independent of  $\mathcal{B}$ .

We remark that with probability one there is at most one point  $x$  such that  $\mathcal{L}(t, x) = \mathcal{L}^*(t)$  so  $F_t^*$  is actually the favorite site. Note that similar questions about favorite points for  $X$  have been studied in the recurrent case  $\kappa = 0$  by [22].

One question we may ask here is : what happens in the discrete case (that is, for RWRE), or with a more general Lévy potential?

For RWRE, we expect a very similar behavior because the renewal structures which appear in both cases (RWRE and our diffusion  $X$ ) are very similar (see [34]). The main difference comes essentially from the functional  $\mathcal{R}_\kappa$ , which should be replaced by a sum of exponentials of simple random walks conditioned to remain positive (see [36], [34]).

For a more general Lévy potential, we have in mind for example a spectrally negative Lévy process (diffusions in such potentials have been studied by [66]). More work needs to be done, especially for the potential. First, to obtain a specific decomposition of the Lévy's path (similar to what is done for the drifted Brownian motion in [37]), and also to study the more complicated functional  $\mathcal{R}_\kappa$  which is less known than in the Brownian case. This is a work in preparation by [74].

The rest of the paper is organized as follows.

In Section 2.2, we recall the results of Faggionato on the path decomposition of the trajectories of  $W_\kappa$ . Also we recall from [3] the construction of specific  $h_t$ -minima which plays an important role in the appearance of independence, under  $\mathbb{P}$ , on the path of  $X$  before time  $t$ .

In Section 2.3, we study the joint process of the hitting times of the  $h_t$ -minima  $m_j$ ,  $1 \leq j \leq n_t$  and of local times at these  $m_j$ . We show that parts of the trajectory of  $X$  are not important for our study, that is, we prove that the time spent outside the  $h_t$ -valleys, and the supremum of the local time outside the  $h_t$ -valleys are negligible compared to  $t$ . We then prove the main result of this section : Proposition 2.3.5. It shows that the joint process (exit times, local times) can be approximated in probability by *i.i.d* random variables (which are the  $\mathcal{H}_j$  and  $\ell_j$ ). This part makes use of some technical results inspired from [3], they are summarized in Section 6.

In Section 2.4, we prove Proposition 2.1.4, and study the continuity of certain functionals of  $(\mathcal{Y}_1, \mathcal{Y}_2)$  which appear in the expression of the limit law  $\mathcal{I}$ . This section is independent of the other ones, we essentially prove a basic functional limit theorem and prepare to the application of continuous mapping theorem.

Section 2.5 is where we make appear the renewal structure in the problem we want to solve. In particular we show how the distribution of the supremum of the local time can be approximated by the distribution of some function of the couple  $(Y_1, Y_2)^t$ , the main step being Proposition 2.5.1.

Section 2.6 is a reminder of some key results and their extensions extracted from [3]. For some of these results, sketch of proofs or complementary proofs are added in order for this paper to be more self-contained.

Finally, Section 2.7 is a reminder of some estimates on Brownian motion, Bessel processes, and functionals of both of these processes.

### 2.1.3 Notation

In this section we introduce typical notation and tools for the study of diffusions in a random potential.

For any process  $(U(t), t \in \mathbb{R}_+)$  we denote by  $\mathcal{L}_U$  a bicontinuous version of the local time of  $U$  when it exists. Notice that for our main process  $X$  we simply write  $\mathcal{L}$  for its local time. The inverse of the local time for every  $x \in \mathbb{R}$  is denoted by  $\sigma_U(t, x) := \inf\{s > 0, \mathcal{L}_U(s, x) \geq t\}$  and in the same way  $\sigma(t, x) := \sigma_X(t, x)$ . We also denote by  $U^a$  the process  $U$  starting from  $a$ , and by  $P^a$  the law of  $U^a$ , with the

notation  $U = U^0$ . Now, let us introduce the following functional of  $W_\kappa$ ,

$$A(r) := \int_0^r e^{W_\kappa(x)} dx, \quad r \in \mathbb{R}.$$

We recall that since  $\kappa > 0$ ,  $A_\infty := \lim_{r \rightarrow +\infty} A(r) < \infty$  a.s. As in [17], there exists a Brownian motion  $(B(s), s \geq 0)$ , independent of  $W_\kappa$ , such that  $X(t) = A^{-1}[B(T^{-1}(t))]$  for every  $t \geq 0$ , where

$$T(r) := \int_0^r \exp\{-2W_\kappa[A^{-1}(B(s))]\} ds, \quad 0 \leq r < \tau^B(A_\infty). \quad (2.1.12)$$

The local time of the diffusion  $X$  at location  $x$  and time  $t$ , simply denoted by  $\mathcal{L}(t, x)$ , can be written as (see [61], eq. (2.5))

$$\mathcal{L}(t, x) = e^{-W_\kappa(x)} \mathcal{L}_B(T^{-1}(t), A(x)), \quad t > 0, x \in \mathbb{R}. \quad (2.1.13)$$

With this notation, we recall the following expression of the hitting times of  $X$ ,

$$H(r) = T[\tau^B(A(r))] = \int_{-\infty}^r e^{-W_\kappa(u)} \mathcal{L}_B[\tau^B(A(r)), A(u)] du, \quad r \geq 0. \quad (2.1.14)$$

## 2.2 Path decomposition and Valleys

### 2.2.1 Path decomposition in the neighborhood of the $h_t$ -minima

$m_i$

We first recall some results for  $h_t$ -extrema of  $W_\kappa$ . Let

$$V^{(i)}(x) := W_\kappa(x) - W_\kappa(m_i), \quad x \in \mathbb{R}, i \in \mathbb{N}^*,$$

which is the potential  $W_\kappa$  translated so that it is 0 at the local minimum  $m_i$ . We also define

$$\tau_i^-(h) := \sup\{s < m_i, V^{(i)}(s) = h\}, \quad h > 0, \quad (2.2.1)$$

$$\tau_i(h) := \inf\{s > m_i, V^{(i)}(s) = h\}, \quad h > 0. \quad (2.2.2)$$

The following result has been proved by [37] [for **(i)** and **(ii)**], and the last fact comes from the strong Markov property (see also ([3], Fact 2.1) and its proof).

**Fact 2.2.1.** (*path decomposition of  $W_\kappa$  around the  $h_t$ -minima  $m_i$* )

**(i)** *The truncated trajectories  $(V^{(i)}(m_i - s), 0 \leq s \leq m_i - \tau_i^-(h_t))$ ,  $(V^{(i)}(m_i + s), 0 \leq s \leq \tau_i(h_t) - m_i)$ ,  $i \geq 1$  are independent.*

**(ii)** *Let  $(W_\kappa^\uparrow(s), s \geq 0)$  be a process with law  $BES(3, \kappa/2)$ . All the truncated trajectories  $(V^{(i)}(m_i - s), 0 \leq s \leq m_i - \tau_i^-(h_t))$  for  $i \geq 2$  and  $(V^{(j)}(m_j + s), 0 \leq s \leq \tau_j(h_t) - m_j)$  for  $j \geq 1$  are equal in law to  $(W_\kappa^\uparrow(s), 0 \leq s \leq \tau^{W_\kappa^\uparrow}(h_t))$ .*

**(iii)** *For  $i \geq 1$ , the truncated trajectory  $(V^{(i)}(s + \tau_i(h_t)), s \geq 0)$  is independent of  $(W_\kappa(s), s \leq \tau_i(h_t))$  and is equal in law to  $(W_\kappa^{h_t}(s), s \geq 0)$ , that is, to a  $(-\kappa/2)$ -drifted Brownian motion starting from  $h_t$ .*



### 2.2.2 Definition of $h_t$ -valleys and of standard $h_t$ -minima $\tilde{m}_j$ , $j \in \mathbb{N}^*$

We are interested in the potential around the  $h_t$ -minima  $m_i$ ,  $i \in \mathbb{N}^*$ , in fact intervals containing at least  $[\tau_i^-( (1 + \kappa)h_t), M_i]$ . However, these valleys could intersect. In order to define valleys which are well separated and i.i.d., we introduce the following notation. This notation is used to define valleys of the potential around some  $\tilde{m}_i$ , which are thanks to Lemma 2.2.1 equal to the  $m_i$  for  $1 \leq i \leq n_t$  with large probability.

Let

$$h_t^+ := (1 + \kappa + 2\delta)h_t.$$

As in [3], we define  $\tilde{L}_0^+ := 0$ ,  $\tilde{m}_0 := 0$ , and recursively for  $i \geq 1$  (see Figure 2.1),

$$\begin{aligned} \tilde{L}_i^\# &:= \inf\{x > \tilde{L}_{i-1}^+, W_\kappa(x) \leq W_\kappa(\tilde{L}_{i-1}^+) - h_t^+\}, \\ \tilde{\tau}_i(h_t) &:= \inf\{x \geq \tilde{L}_i^\#, W_\kappa(x) - \inf_{[\tilde{L}_i^\#, x]} W_\kappa \geq h_t\}, \\ \tilde{m}_i &:= \inf\{x \geq \tilde{L}_i^\#, W_\kappa(x) = \inf_{[\tilde{L}_i^\#, \tilde{\tau}_i(h_t)]} W_\kappa\}, \\ \tilde{L}_i^+ &:= \inf\{x > \tilde{\tau}_i(h_t), W_\kappa(x) \leq W_\kappa(\tilde{\tau}_i(h_t)) - h_t - h_t^+\}. \end{aligned} \quad (2.2.3)$$

We also introduce the following random variables for  $i \in \mathbb{N}^*$  :

$$\begin{aligned} \tilde{M}_i &:= \inf\{s > \tilde{m}_i, W_\kappa(s) = \max_{\tilde{m}_i \leq u \leq \tilde{L}_i^+} W_\kappa(u)\}, \\ \tilde{L}_i^* &:= \inf\{x > \tilde{\tau}_i(h_t), W_\kappa(x) - W_\kappa(\tilde{m}_i) = 3h_t/4\}, \\ \tilde{L}_i &:= \inf\{x > \tilde{\tau}_i(h_t), W_\kappa(x) - W_\kappa(\tilde{m}_i) = h_t/2\}, \end{aligned} \quad (2.2.4)$$

$$\tilde{\tau}_i(h) := \inf\{s > \tilde{m}_i, W_\kappa(x) - W_\kappa(\tilde{m}_i) = h\}, \quad h > 0, \quad (2.2.5)$$

$$\tilde{\tau}_i^-(h) := \sup\{s < \tilde{m}_i, W_\kappa(x) - W_\kappa(\tilde{m}_i) = h\}, \quad h > 0, \quad (2.2.6)$$

$$\tilde{L}_i^- := \tilde{\tau}_i^-(h_t^+).$$

We stress that these random variables depend on  $t$ , which we do not write as a subscript to simplify the notation. Notice also that  $\tilde{\tau}_i(h_t)$  is the same in definitions (2.2.3) and (2.2.5) with  $h = h_t$ . Moreover by continuity of  $W_\kappa$ ,  $W_\kappa(\tilde{\tau}_i(h_t)) = W_\kappa(\tilde{m}_i) + h_t$ . Thus, the  $\tilde{m}_i$ ,  $i \in \mathbb{N}^*$ , are  $h_t$ -minima, since  $W_\kappa(\tilde{m}_i) = \inf_{[\tilde{L}_{i-1}^+, \tilde{\tau}_i(h_t)]} W_\kappa$ ,  $W_\kappa(\tilde{\tau}_i(h_t)) = W_\kappa(\tilde{m}_i) + h_t$  and  $W_\kappa(\tilde{L}_{i-1}^+) \geq W_\kappa(\tilde{m}_i) + h_t$ . In addition,

$$\tilde{L}_{i-1}^+ < \tilde{L}_i^\# \leq \tilde{m}_i < \tilde{\tau}_i(h_t) < \tilde{L}_i^* < \tilde{L}_i < \tilde{L}_i^+, \quad i \in \mathbb{N}^*, \quad (2.2.7)$$

$$\tilde{L}_{i-1}^+ \leq \tilde{L}_i^- < \tilde{m}_i < \tilde{\tau}_i(h_t) < \tilde{M}_i < \tilde{L}_i^+, \quad i \in \mathbb{N}^*. \quad (2.2.8)$$

Also by induction, the random variables  $\tilde{L}_i^\#$ ,  $\tilde{\tau}_i(h_t)$  and  $\tilde{L}_i^+$ ,  $i \in \mathbb{N}^*$  are stopping times for the natural filtration of  $(W_\kappa(x), x \geq 0)$ , and so  $\tilde{L}_i$ ,  $\tilde{L}_i^*$ ,  $i \in \mathbb{N}^*$ , are also stopping times. Moreover by induction,

$$\begin{aligned} W_\kappa(\tilde{L}_i^\#) &= \inf_{[0, \tilde{L}_i^\#]} W_\kappa, & W_\kappa(\tilde{m}_i) &= \inf_{[0, \tilde{\tau}_i(h_t)]} W_\kappa, \\ W_\kappa(\tilde{L}_i^+) &= \inf_{[0, \tilde{L}_i^+]} W_\kappa = W_\kappa(\tilde{m}_i) - h_t^+, \end{aligned} \quad (2.2.9)$$

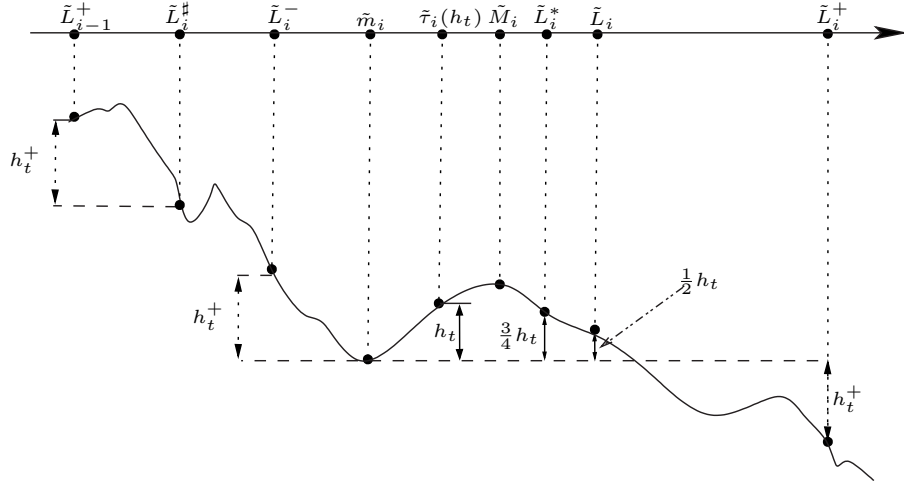


FIGURE 2.1 – Schema of the potential between  $\tilde{L}_{i-1}^+$  and  $\tilde{L}_i^+$ , in the case  $\tilde{L}_i^\sharp < \tilde{L}_i^-$ .

for  $i \in \mathbb{N}^*$ . We also introduce the analogue of  $V^{(i)}$  for  $\tilde{m}_i$  as follows :

$$\tilde{V}^{(i)}(x) := W_\kappa(x) - W_\kappa(\tilde{m}_i), \quad x \in \mathbb{R}, \quad i \in \mathbb{N}^*.$$

We call  $i$  th  $h_t$ -valley the translated truncated potential  $(\tilde{V}^{(i)}(x), \tilde{L}_i^- \leq x \leq \tilde{L}_i)$ , for  $i \geq 1$ .

The following lemma states that, with a very large probability, the first  $n_t + 1$  positive  $h_t$ -minima  $m_i$ ,  $1 \leq i \leq n_t + 1$ , coincide with the random variables  $\tilde{m}_i$ ,  $1 \leq i \leq n_t + 1$ . We introduce the corresponding event  $\mathcal{V}_t := \cap_{i=1}^{n_t+1} \{m_i = \tilde{m}_i\}$ .

**Lemma 2.2.1.** *Assume  $0 < \delta < 1$ . There exists a constant  $C_1 > 0$  such that for  $t$  large enough,*

$$P(\overline{\mathcal{V}}_t) \leq C_1 n_t e^{-\kappa h_t / 2} = e^{[-\kappa/2 + o(1)]h_t}.$$

Moreover, the sequence  $\left( (\tilde{V}^{(i)}(x + \tilde{L}_{i-1}^+), 0 \leq x \leq \tilde{L}_i^+ - \tilde{L}_{i-1}^+), i \geq 1 \right)$ , is i.i.d.

**Proof :** This lemma is proved in [3] : Lemma 2.3. □

The following remark is used several times in the rest of the paper.

**Remark 2.2.1.** *On  $\mathcal{V}_t$ , we have for every  $1 \leq i \leq n_t$ ,  $m_i = \tilde{m}_i$ , and as a consequence,  $\tilde{V}^{(i)}(x) = V^{(i)}(x)$ ,  $x \in \mathbb{R}$ ,  $\tau_i^-(h) = \tilde{\tau}_i^-(h)$  and  $\tau_i(h) = \tilde{\tau}_i(h)$  for  $h > 0$ . Moreover,  $\tilde{M}_i = M_i$ . Indeed,  $\tilde{M}_i$  is an  $h_t$ -maximum for  $W_\kappa$ , which belongs to  $[\tilde{m}_i, \tilde{m}_{i+1}] = [m_i, m_{i+1}]$  on  $\mathcal{V}_t$ , and there is exactly one  $h_t$ -maximum in this interval since the  $h_t$ -maxima and minima alternate, which we defined as  $M_i$ , so  $\tilde{M}_i = M_i$ . So in the following, on  $\mathcal{V}_t$ , we can write these random variables with or without tilde.*

## 2.3 Contributions for hitting and local times

### 2.3.1 Negligible parts for hitting times

In the following lemma we recall results of [3] which say, roughly speaking, that the time spent by the diffusion  $X$  outside the  $h_t$ -valleys is negligible compared to the amount of time spent by  $X$  inside the  $h_t$ -valleys. This lemma also gives an upper bound for the number of  $h_t$ -valleys visited before time  $t$ . Finally, it tells us that with large probability, up to time  $t$ , after first hitting the bottom  $\tilde{m}_j$  of each  $h_t$ -valley  $[\tilde{L}_j^-, \tilde{L}_j]$ ,  $X$  leaves this  $h_t$ -valley on its right, that is on  $\tilde{L}_j$ , and that  $X$  never backtracks in a previously visited  $h_t$ -valley. We define  $H_{x \rightarrow y} := \inf\{s > H(x), X(s) = y\} - H(x)$  for any  $x \geq 0$  and  $y \geq 0$ , which is equal to  $H(y) - H(x)$  if  $x < y$ . Let

$$U_0 := 0, \quad U_i := H(\tilde{L}_i) - H(\tilde{m}_i) = H_{\tilde{m}_i \rightarrow \tilde{L}_i}, \quad i \geq 1,$$

$$\mathcal{B}_1(m) := \bigcap_{k=1}^m \left\{ 0 \leq H(\tilde{m}_k) - \sum_{i=1}^{k-1} U_i < \tilde{v}_t \right\}, \quad m \geq 1,$$

where  $\tilde{v}_t := 2t / \log h_t$  and  $\sum_{i=1}^0 U_i = 0$  by convention. Finally, we introduce

$$\mathcal{B}_2(m) := \bigcap_{j=1}^m \left\{ H_{\tilde{m}_j \rightarrow \tilde{L}_j} < H_{\tilde{m}_j \rightarrow \tilde{L}_j^-}, H_{\tilde{L}_j \rightarrow \tilde{m}_{j+1}} < H_{\tilde{L}_j \rightarrow \tilde{L}_j^*} \right\}, \quad m \geq 1.$$

**Lemma 2.3.1.** *For any  $\delta > 0$  small enough, we have for all large  $t$ ,*

$$\mathbb{P}[H(\tilde{m}_1) \leq \tilde{v}_t] \geq \mathbb{P}[\mathcal{B}_1(n_t)] \geq 1 - C_2 v_t, \quad (2.3.1)$$

with  $v_t := n_t \cdot (\log h_t) e^{-\phi(t)} = o(1)$  and  $C_2 > 0$ . Moreover, there exists  $C_3 > 0$  such that for large  $t$ ,

$$\mathbb{P}(\mathcal{B}_2(n_t)) \geq 1 - C_3 n_t e^{-\delta \kappa h_t}, \quad (2.3.2)$$

$$\mathbb{P}(N_t < n_t) \geq 1 - e^{-\phi(t)}. \quad (2.3.3)$$

**Proof :** The first statement is Lemma 3.7 in [3]. The second one follows directly from Lemmata 3.2 and 3.3 in [3]. For the proof of (2.3.3) see Lemma 2.6.1.  $\square$

### 2.3.2 Negligible parts for local times

We now provide estimates for the local time of  $X$  at time  $t$ . We first prove that the local time of  $X$  outside the first  $n_t$   $h_t$ -valleys is negligible compared to  $t$ . Second, we prove that for every  $1 \leq j \leq n_t$  the local time of  $X$  inside the  $h_t$ -valley  $[\tilde{L}_j^-, \tilde{L}_j]$  but outside a small neighborhood of  $\tilde{m}_j$  is also negligible compared to  $t$ .

### 2.3.2.1 Supremum of the local time outside the valleys

The aim of this subsection is to prove that at time  $t$ , the maximum of the local time outside the  $h_t$ -valleys is negligible compared to  $t$ . More precisely, let  $f(t) := te^{[\kappa(1+3\delta)-1]\phi(t)}$  and, for  $m \geq 1$ ,

$$\begin{aligned} \mathcal{B}_3^1(m) &:= \left\{ \sup_{x \in [0, \tilde{m}_1]} \mathcal{L}(H(\tilde{m}_1), x) \leq f(t) \right\} \\ &\quad \cap \bigcap_{j=1}^{m-1} \left\{ \sup_{x \in [\tilde{L}_j, \tilde{m}_{j+1}]} \mathcal{L}(H(\tilde{m}_{j+1}), x) \leq f(t) \right\}, \\ \mathcal{B}_3^2(m) &:= \bigcap_{j=1}^{m-1} \left\{ \sup_{x \leq \tilde{L}_j} \left( \mathcal{L}(H(\tilde{m}_{j+1}), x) - \mathcal{L}(H(\tilde{L}_j), x) \right) \leq f(t) \right\}, \\ \mathcal{B}_3(m) &:= \mathcal{B}_3^1(m) \cap \mathcal{B}_3^2(m). \end{aligned}$$

This section is devoted to the proof of the following lemma.

**Lemma 2.3.2.** *Assume that  $\delta$  is small enough such that  $\kappa(1+3\delta) < 1$ . There exists  $C_5 > 0$  such that for any large  $t$*

$$\mathbb{P}(\mathcal{B}_3(n_t)) \geq 1 - C_5 w_t,$$

with  $w_t := e^{-\kappa\delta\phi(t)}$ .

Its proof is based on Lemma 2.3.3 below, for which we introduce the following notation, depending only on the potential  $W_\kappa$  :

$$\begin{aligned} \tau_1^*(h) &:= \inf\{u \geq 0, W_\kappa(u) - \inf_{[0,u]} W_\kappa \geq h\}, & h > 0, \\ m_1^*(h) &:= \inf\{y \geq 0, W_\kappa(y) = \inf_{[0,\tau_1^*(h)]} W_\kappa\}, & h > 0. \end{aligned}$$

Throughout the paper,  $C_+$  (resp.  $c_-$ ) denotes a positive constant that may grow (resp. decrease) from line to line.

**Lemma 2.3.3.** *Assume that  $\kappa(1+3\delta) < 1$ . For large  $t$ ,*

$$\mathbb{P}\left(\sup_{x \in [0, m_1^*(h_t)]} \mathcal{L}[H(\tau_1^*(h_t)), x] > te^{[\kappa(1+3\delta)-1]\phi(t)}\right) \leq \frac{C_+}{n_t e^{\kappa\delta\phi(t)}}. \quad (2.3.4)$$

**Proof of Lemma 2.3.3 :** Thanks to (2.1.13) and (2.1.14) there exists a Brownian motion  $(B(s), s \geq 0)$ , independent of  $W_\kappa$ , such that

$$\mathcal{L}[H(\tau_1^*(h_t)), x] = e^{-W_\kappa(x)} \mathcal{L}_B[\tau^B(A(\tau_1^*(h_t))), A(x)], \quad x \in \mathbb{R}. \quad (2.3.5)$$

By the first Ray–Knight theorem (see e.g. [57], chap. XI), for every  $\alpha > 0$ , there exists a Bessel processes  $Q_2$  of dimension 2 starting from 0, such that  $\mathcal{L}_B(\tau^B(\alpha), x)$  is equal to  $Q_2^2(\alpha - x)$  for every  $x \in [0, \alpha]$ . Consequently, using (2.3.5) and the independence of  $B$  and  $W_\kappa$ , there exists a 2-dimensional Bessel process  $Q_2$  such that

$$\mathcal{L}[H(\tau_1^*(h_t)), x] = e^{-W_\kappa(x)} Q_2^2[A(\tau_1^*(h_t)) - A(x)] \quad 0 \leq x \leq \tau_1^*(h_t). \quad (2.3.6)$$

In order to evaluate this quantity, the idea is to say that loosely speaking,  $Q_2^2$  grows almost linearly. More formally, we consider the functions  $k(t) := e^{2\kappa^{-1}\phi(t)}$ ,  $a(t) := 4\phi(t)$  and  $b(t) := 6\kappa^{-1}\phi(t)e^{\kappa h_t}$ , and define the following events

$$\begin{aligned} \mathcal{A}_0 &:= \left\{ A_\infty := \int_0^{+\infty} e^{W_\kappa(u)} du \leq k(t) \right\}, \\ \mathcal{A}_1 &:= \left\{ \forall u \in (0, k(t)], Q_2^2(u) \leq 2eu[a(t) + 4 \log \log[ek(t)/u]] \right\}, \\ \mathcal{A}_2 &:= \left\{ \inf_{[0, \tau_1^*(h_t)]} W_\kappa \geq -b(t) \right\}. \end{aligned}$$

We know that  $P(A_\infty \geq y) \leq C_+ y^{-\kappa}$  for  $y > 0$  since  $2/A_\infty$  is a gamma variable of parameter  $(\kappa, 1)$  (see [32], or [15] IV.48 p. 78), having a density equal to  $e^{-x} x^{\kappa-1} \mathbb{1}_{\mathbb{R}_+}(x)/\Gamma(\kappa)$ , so  $P(\overline{\mathcal{A}}_0) \leq C_+ k(t)^{-\kappa} = C_+ e^{-2\phi(t)}$ . Moreover,  $\mathbb{P}(\overline{\mathcal{A}}_1) \leq C_+ \exp[-a(t)/2] = C_+ e^{-2\phi(t)}$  by Lemma 2.7.5. Also we know that  $-\inf_{[0, \tau_1^*(h)]} W_\kappa$ , denoted by  $-\beta$  in ([37], eq. (2.2)) is exponentially distributed with mean  $2\kappa^{-1} \sinh(\kappa h/2) e^{\kappa h/2}$  ([37], eq. (2.4)). So for large  $t$ ,

$$\begin{aligned} P(\overline{\mathcal{A}}_2) &= P[-\inf_{[0, \tau_1^*(h_t)]} W_\kappa > b(t)] \\ &= \exp[-b(t)\kappa/(2 \sinh(\kappa h_t/2) e^{\kappa h_t/2})] \\ &\leq e^{-2\phi(t)}. \end{aligned}$$

Now, assume we are on  $\mathcal{A}_0 \cap \mathcal{A}_1 \cap \mathcal{A}_2$ . Due to (2.3.6), we have for every  $0 \leq x < \tau_1^*(h_t)$ , since  $0 < A(\tau_1^*(h_t)) - A(x) \leq A_\infty \leq k(t)$ ,

$$\begin{aligned} &\mathcal{L}[H(\tau_1^*(h_t)), x] \\ &\leq e^{-W_\kappa(x)} 2e[A(\tau_1^*(h_t)) - A(x)] \{a(t) + 4 \log \log [ek(t)/[A(\tau_1^*(h_t)) - A(x)]]\}. \end{aligned} \quad (2.3.7)$$

We now introduce

$$f_i := \inf\{u \geq 0, W_\kappa(u) \leq -i\} = \tau^{W_\kappa}(-i), \quad i \in \mathbb{N},$$

and let  $0 \leq x < \tau_1^*(h_t)$ . There exists  $i \in \mathbb{N}$  such that  $f_i \leq x < f_{i+1}$ . Moreover, we are on  $\mathcal{A}_2$ , so  $i \leq b(t)$ . Furthermore,  $x < f_{i+1}$ , so  $W_\kappa(x) \geq -(i+1)$  and then  $e^{-W_\kappa(x)} \leq e^{i+1} = e^{-W_\kappa(f_i)+1}$ . All this leads to

$$\begin{aligned} e^{-W_\kappa(x)} [A(\tau_1^*(h_t)) - A(x)] &= e^{-W_\kappa(x)} \int_x^{\tau_1^*(h_t)} e^{W_\kappa(u)} du \\ &\leq e \int_{f_i}^{\tau_1^*(h_t)} e^{W_\kappa(u) - W_\kappa(f_i)} du. \end{aligned} \quad (2.3.8)$$

To bound this, we introduce the event

$$\mathcal{A}_3 := \bigcap_{i=0}^{\lfloor b(t) \rfloor} \left\{ \int_{f_i}^{\tau_1^*(h_t)} e^{W_\kappa(u) - W_\kappa(f_i)} du \leq e^{(1-\kappa)h_t} b(t) n_t e^{\kappa\delta\phi(t)} \right\}.$$

We now consider  $\tau_1^*(u, h_t) := \inf\{y \geq u, W_\kappa(y) - \inf_{[u, y]} W_\kappa \geq h_t\} \geq \tau_1^*(h_t)$  for  $u \geq 0$ . We have

$$E\left(\int_{f_i}^{\tau_1^*(h_t)} e^{W_\kappa(u) - W_\kappa(f_i)} du\right) \leq E\left(\int_{f_i}^{\tau_1^*(f_i, h_t)} e^{W_\kappa(u) - W_\kappa(f_i)} du\right) = \beta_0(h_t),$$

by the strong Markov property applied at stopping time  $f_i$ , where we define  $\beta_0(h) := E\left(\int_0^{\tau_1^*(h)} e^{W_\kappa(u)} du\right)$ . By (2.6.15),  $\beta_0(h) \leq C_+ e^{(1-\kappa)h}$  for large  $h$ . Hence for large  $t$  by Markov inequality,

$$\begin{aligned} P(\overline{\mathcal{A}}_3) &\leq \sum_{i=0}^{\lfloor b(t) \rfloor} P\left(\int_{f_i}^{\tau_1^*(h_t)} e^{W_\kappa(u) - W_\kappa(f_i)} du > e^{(1-\kappa)h_t} b(t) n_t e^{\kappa\delta\phi(t)}\right) \\ &\leq \frac{[b(t) + 1]\beta_0(h_t)}{e^{(1-\kappa)h_t} b(t) n_t e^{\kappa\delta\phi(t)}} \leq \frac{C_+}{n_t e^{\kappa\delta\phi(t)}}. \end{aligned}$$

Now, on  $\cap_{j=0}^3 \mathcal{A}_j$ , (2.3.7) and (2.3.8) lead to

$$\begin{aligned} &\mathcal{L}[H(\tau_1^*(h_t)), x] \\ &\leq 2e^{2+(1-\kappa)h_t} b(t) n_t e^{\kappa\delta\phi(t)} \left\{ a(t) + 4 \log \log \left[ ek(t) / [A(\tau_1^*(h_t)) - A(x)] \right] \right\}. \end{aligned} \quad (2.3.9)$$

We now consider only  $0 \leq x \leq m_1^*(h_t)$ . By definition of  $\mathcal{A}_2$ ,  $\inf_{[0, \tau_1^*(h_t)]} W_\kappa \geq -b(t)$ , such that

$$\begin{aligned} A(\tau_1^*(h_t)) - A(x) &= \int_x^{\tau_1^*(h_t)} e^{W_\kappa(u)} du \\ &\geq \int_{m_1^*(h_t)}^{\tau_1^*(h_t)} e^{W_\kappa(u)} du \\ &\geq e^{-b(t)} [\tau_1^*(h_t) - m_1^*(h_t)] \\ &\geq e^{-b(t)} \end{aligned}$$

on the event  $\cap_{i=0}^4 \mathcal{A}_i$  with  $\mathcal{A}_4 := \{\tau_1^*(h_t) - m_1^*(h_t) \geq 1\}$ . Since  $m_1 = m_1^*(h_t)$  and  $\tau_1(h_t) = \tau_1^*(h_t)$  on  $\{M_0 \leq 0\}$  by definition of  $h_t$ -extrema, we have

$$\begin{aligned} P(\overline{\mathcal{A}}_4) &\leq P(0 < M_0 < m_1) + P[\tau_1(h_t) - m_1 < 1] \\ &\leq C_+ h_t e^{-\kappa h_t} + P[\tau^{W_\kappa^\uparrow}(h_t) - \tau^{W_\kappa^\uparrow}(h_t/2) < 1] \\ &\leq C_+ h_t e^{-\kappa h_t} + C_+ \exp[-(c_-)h_t^2] \end{aligned}$$

due to ([3], eq. (2.8), coming from [37]), Fact 2.2.1 (ii) and (2.7.4).

Now, we have  $ek(t)/[A(\tau_1^*(h_t)) - A(x)] \leq ek(t)e^{b(t)}$  on  $\cap_{i=0}^4 \mathcal{A}_i$ , and then, on this event, (2.3.9) leads to

$$\begin{aligned} \mathcal{L}[H(\tau_1^*(h_t)), x] &\leq 2e^{2+(1-\kappa)h_t} b(t) n_t e^{\kappa\delta\phi(t)} \{a(t) + 4 \log \log [ek(t)e^{b(t)}]\} \\ &\leq C_+ t \phi(t) e^{[\kappa(1+\delta)-1]\phi(t)} e^{\kappa\delta\phi(t)} h_t, \end{aligned}$$

since  $\phi(t) = o(\log t)$ ,  $h_t = \log t - \phi(t)$  and  $n_t = \lfloor e^{\kappa(1+\delta)\phi(t)} \rfloor$ . We notice that for large  $t$ ,  $C_+ \phi(t) h_t \leq e^{\kappa\delta\phi(t)}$  since  $\log \log t = o(\phi(t))$ . Hence, for large  $t$ ,

$$\mathcal{L}[H(\tau_1^*(h_t)), x] \leq t e^{[\kappa(1+3\delta)-1]\phi(t)},$$

on  $\cap_{i=0}^4 \mathcal{A}_i$  for every  $0 \leq x \leq m_1^*(h_t)$ . This gives for large  $t$ ,

$$\mathbb{P} \left( \sup_{x \in [0, m_1^*(h_t)]} \mathcal{L}[H(\tau_1^*(h_t)), x] \leq t e^{[\kappa(1+3\delta)-1]\phi(t)} \right) \geq \mathbb{P}(\cap_{i=0}^4 \mathcal{A}_i) \geq 1 - \frac{C_+}{n_t e^{\kappa\delta\phi(t)}},$$

due to the previous bounds for  $\mathbb{P}(\mathcal{A}_i)$ ,  $0 \leq i \leq 4$ . This proves the lemma.  $\square$

With the help of the previous lemma, we can now prove Lemma 2.3.2.

**Proof of Lemma 2.3.2 :** The method is to do a coupling, similarly as in the proof of Lemma 3.7 of [3]. Recall the definition of  $\tilde{L}_i^* < \tilde{L}_i < \tilde{L}_{i+1}^\sharp$  just above (2.2.5). Also, let

$$\begin{aligned} \tilde{\tau}_{i+1}^*(h_t) &:= \inf \{u \geq \tilde{L}_i^*, W_\kappa(u) - \inf_{[\tilde{L}_i^*, u]} W_\kappa \geq h_t\} \leq \tilde{\tau}_{i+1}(h_t), \quad i \geq 1, \\ \tilde{m}_{i+1}^*(h_t) &:= \inf \{u \geq \tilde{L}_i^*, W_\kappa(u) = \inf_{[\tilde{L}_i^*, \tilde{\tau}_{i+1}^*(h_t)]} W_\kappa\}, \quad i \geq 1, \\ \mathcal{A}_5 &:= \cap_{i=1}^{n_t-1} \{\tilde{\tau}_{i+1}^*(h_t) = \tilde{\tau}_{i+1}(h_t)\}, \\ X_i(u) &:= X(u + H(\tilde{L}_i)), \quad X_i^*(u) := X(u + H(\tilde{L}_i^*)), \quad u \geq 0, i \geq 1. \end{aligned} \tag{2.3.10}$$

Let  $i \geq 1$ . By the strong Markov property,  $X_i$  and  $X_i^*$  are diffusions in the potential  $W_\kappa$ , starting respectively from  $\tilde{L}_i$  and  $\tilde{L}_i^*$ . We denote respectively by  $\mathcal{L}_{X_i}$ ,  $\mathcal{L}_{X_i^*}$ ,  $H_{X_i}$  and  $H_{X_i^*}$  the local times and hitting times of  $X_i$  and  $X_i^*$ . We have for every  $x \geq \tilde{L}_i^*$ ,

$$\begin{aligned} \mathcal{L}(H(\tilde{m}_{i+1}), x) - \mathcal{L}(H(\tilde{L}_i), x) &\leq \mathcal{L}(H(\tilde{m}_{i+1}), x) - \mathcal{L}(H(\tilde{L}_i^*), x) \\ &= \mathcal{L}_{X_i^*}(H_{X_i^*}(\tilde{m}_{i+1}), x). \end{aligned}$$

Consequently, on  $\mathcal{A}_5 \cap \mathcal{A}_6$  with  $\mathcal{A}_6 := \cap_{j=1}^{n_t-1} \{H_{X_j}(\tilde{m}_{j+1}) < H_{X_j}(\tilde{L}_j^*)\}$ , for  $1 \leq i \leq n_t - 1$ ,

$$\begin{aligned} &\sup_{x \in \mathbb{R}} \left( \mathcal{L}(H(\tilde{m}_{i+1}), x) - \mathcal{L}(H(\tilde{L}_i), x) \right) \\ &= \sup_{\tilde{L}_i^* \leq x \leq \tilde{m}_{i+1}} \left( \mathcal{L}(H(\tilde{m}_{i+1}), x) - \mathcal{L}(H(\tilde{L}_i), x) \right) \\ &\leq \sup_{\tilde{L}_i^* \leq x \leq \tilde{m}_{i+1}} \mathcal{L}_{X_i^*}(H_{X_i^*}(\tilde{m}_{i+1}), x) \\ &\leq \sup_{\tilde{L}_i^* \leq x \leq \tilde{m}_{i+1}} \mathcal{L}_{X_i^*}(H_{X_i^*}(\tilde{\tau}_{i+1}^*(h_t)), x), \end{aligned} \tag{2.3.11}$$

since  $\tilde{m}_{i+1}^* = \tilde{m}_{i+1} \leq \tilde{\tau}_{i+1}(h_t) = \tilde{\tau}_{i+1}^*(h_t)$  on  $\mathcal{A}_5$ . Now, notice that the right hand side of (2.3.11) is the supremum of the local times of  $X_i^* - \tilde{L}_i^*$ , up to its first hitting time of  $\tilde{\tau}_{i+1}^*(h_t) - \tilde{L}_i^*$ , over all locations in  $[0, \tilde{m}_{i+1}^* - \tilde{L}_i^*]$ . Since  $X_i^* - \tilde{L}_i^*$  is a diffusion in the potential  $(W_\kappa(\tilde{L}_i^* + x) - W_\kappa(\tilde{L}_i^*), x \in \mathbb{R})$ , which has on  $[0, +\infty)$  the same law as  $(W_\kappa(x), x \geq 0)$  because  $\tilde{L}_i^*$  is a stopping time for  $W_\kappa$ , the right hand side of (2.3.11) has the same law, under the annealed probability  $\mathbb{P}$ , as  $\sup_{x \in [0, m_1^*(h_t)]} \mathcal{L}[H(\tau_1^*(h_t)), x]$ . Consequently,

$$\begin{aligned} & \mathbb{P}\left(\bigcup_{i=1}^{n_t-1} \left\{ \sup_{x \in \mathbb{R}} \left( \mathcal{L}(H(\tilde{m}_{i+1}), x) - \mathcal{L}(H(\tilde{L}_i), x) \right) > te^{[\kappa(1+3\delta)-1]\phi(t)} \right\}\right) \\ & \leq n_t \left[ \mathbb{P}\left(\sup_{x \in [0, m_1^*(h_t)]} \mathcal{L}[H(\tau_1^*(h_t)), x] > te^{[\kappa(1+3\delta)-1]\phi(t)}\right) + \mathbb{P}(\overline{\mathcal{A}}_5) + \mathbb{P}(\overline{\mathcal{A}}_6) \right] \\ & \leq C_+ e^{-\kappa\delta\phi(t)} \end{aligned} \quad (2.3.12)$$

by Lemma 2.3.3, since  $\mathbb{P}(\overline{\mathcal{A}}_5) \leq C_+ n_t h_t e^{-\kappa h_t}$  by (2.6.9),  $\mathbb{P}(\overline{\mathcal{A}}_6) \leq \mathbb{P}(\overline{\mathcal{B}}_2(n_t)) \leq C_3 n_t e^{-\delta\kappa h_t}$  by (2.3.2) and since  $\phi(t) = o(\log t)$ . Notice that, as before,  $\tilde{m}_1 = m_1 = m_1^*(h_t)$  on  $\mathcal{V}_t \cap \{M_0 \leq 0\}$ . Finally,

$$\begin{aligned} \mathbb{P}\left(\sup_{x \in [0, \tilde{m}_1]} \mathcal{L}(H(\tilde{m}_1), x) > te^{[\kappa(1+3\delta)-1]\phi(t)}\right) & \leq \frac{C_+}{e^{\kappa\delta\phi(t)}} + P(\overline{\mathcal{V}}_t) + P(0 < M_0 < m_1) \\ & \leq \frac{C_+}{e^{\kappa\delta\phi(t)}} \end{aligned}$$

also by Lemma 2.3.3, Lemma 2.2.1, and since  $P(0 < M_0 < m_1) \leq C_+ h_t e^{-\kappa h_t}$  due to (2.6.8). This and (2.3.12) prove the lemma.  $\square$

### 2.3.2.2 Local time inside the valley $[\tilde{L}_j^-, \tilde{L}_j]$ but far from $\tilde{m}_j$

We introduce for  $t > 0$  and  $j \geq 1$ ,

$$r_t := C_0 \phi(t), \quad \mathcal{D}_j := [\tilde{m}_j - r_t, \tilde{m}_j + r_t], \quad (2.3.13)$$

where  $C_0 > 0$  is a constant that can be chosen as large as needed. We also define

$$\mathcal{B}_4(m) := \bigcap_{j=1}^m \left\{ \sup_{x \in \overline{\mathcal{D}}_j \cap [\tilde{\tau}_j^-(h_t^+), \tilde{L}_j]} \left( \mathcal{L}(H(\tilde{L}_j), x) - \mathcal{L}(H(\tilde{m}_j), x) \right) < te^{-2\phi(t)} \right\}$$

for  $m \geq 1$ , where  $\overline{\mathcal{D}}_j$  is the complementary of  $\mathcal{D}_j$ . Moreover, we recall that  $\tilde{L}_j^- = \tilde{\tau}_j^-(h_t^+)$ .

**Lemma 2.3.4.** *There exists  $C_6 > 0$  such that if  $C_0$  is large enough, for large  $t$ ,*

$$\mathbb{P}[\mathcal{B}_4(n_t)] \geq 1 - C_6 n_t e^{-2\phi(t)}.$$



**Proof :** Let  $j \in [1, n_t]$ . Throughout the rest of the paper, for  $y \in \mathbb{R}$ , we denote by  $\mathbb{P}_y^{W_\kappa}$  the law of  $X$  starting from  $y$  instead of 0, conditionally on  $W_\kappa$ . As we are interested in the local time at  $x$  after  $X$  reaches  $\tilde{m}_j$  we work under  $\mathbb{P}_{\tilde{m}_j}^{W_\kappa}$ . So first, thanks to (2.1.13) and (2.1.14), under  $\mathbb{P}_{\tilde{m}_j}^{W_\kappa}$ , there exists a Brownian motion  $(B(s), s \geq 0)$ , independent of  $\tilde{V}^{(j)}$ , such that

$$\mathcal{L}[H(\tilde{L}_j), x] = e^{-\tilde{V}^{(j)}(x)} \mathcal{L}_B[\tau^B(A^j(\tilde{L}_j)), A^j(x)], \quad x \in \mathbb{R},$$

where  $A^j(x) := \int_{\tilde{m}_j}^x e^{\tilde{V}^{(j)}(s)} ds$ . Let  $\tilde{B}^j(\cdot) := B^j((A^j(\tilde{L}_j))^2)/A^j(\tilde{L}_j)$ . By scaling, and because  $B$  is independent from  $W_\kappa$ , we notice that conditionally to  $W_\kappa$ ,  $\tilde{B}^j$  is a standard Brownian motion. Therefore, even if  $W_\kappa$  appears in the expression of  $\tilde{B}^j$ ,  $\tilde{B}^j$  is (probabilistically) independent of  $W_\kappa$ . We still denote it by  $B$  in the sequel to simplify the notation. With this notation, we have

$$\mathcal{L}[H(\tilde{L}_j), x] = e^{-\tilde{V}^{(j)}(x)} A^j(\tilde{L}_j) \mathcal{L}_B[\tau^B(1), A^j(x)/A^j(\tilde{L}_j)], \quad x \in \mathbb{R}. \quad (2.3.14)$$

In order to bound the factors  $\mathcal{L}_B[\tau^B(1), \cdot]$  and  $A^j(\tilde{L}_j)$  in (2.3.14), we first introduce

$$\mathcal{A}_1 := \{ \sup_{u \in \mathbb{R}} \mathcal{L}_B[\tau^B(1), u] \leq e^{2\phi(t)} \}, \quad \mathcal{A}_2 := \{ A^j(\tilde{L}_j) \leq 2e^{h_t + 2\phi(t)/\kappa} \}. \quad (2.3.15)$$

We have  $\mathbb{P}(\overline{\mathcal{A}}_1) \leq 5e^{-2\phi(t)}$  for large  $t$  by Lemma 2.7.4 eq. (2.7.12) and (2.7.13). Moreover on  $\mathcal{V}_t$ , we have by Remark 2.2.1 and Fact 2.2.1 (ii) and (iii),

$$\begin{aligned} A^j(\tilde{L}_j) &\leq [\tilde{\tau}_j(h_t) - \tilde{m}_j] e^{h_t} + \int_{\tilde{\tau}_j(h_t)}^{\tilde{L}_j} e^{\tilde{V}^{(j)}(s)} ds \\ &= [\tau_j(h_t) - m_j] e^{h_t} + \int_{\tau_j(h_t)}^{L_j} e^{V^{(j)}(s)} ds \\ &\stackrel{\mathcal{L}}{=} e^{h_t} \tau^{W_\kappa^\uparrow}(h_t) + G^+(h_t/2, h_t), \end{aligned}$$

where  $W_\kappa^\uparrow$  has law  $BES(3, \kappa/2)$  and is independent of  $G^+(h_t/2, h_t)$ , which is defined in (2.1.8), and with  $\tilde{L}_j = \inf\{s > \tilde{\tau}_j(h_t), \tilde{V}^{(j)}(s) = h_t/2\}$  as defined in (2.2.4), and  $L_j := \inf\{s > \tau_j(h_t), V^{(j)}(s) = h_t/2\}$ . Consequently,

$$\begin{aligned} P(\overline{\mathcal{A}}_2) &\leq P(\tau^{W_\kappa^\uparrow}(h_t) > e^{2\phi(t)/\kappa}) + P(G^+(h_t/2, h_t) > e^{h_t + 2\phi(t)/\kappa}) + P(\overline{\mathcal{V}}_t) \\ &\leq C_+ e^{-2\phi(t)} \end{aligned}$$

for large  $t$  by Lemma 2.7.2 eq. (2.7.5), Lemma 2.7.3 eq. (2.7.10) and Lemma 2.2.1, and since  $\phi(t) = o(\log t)$  and  $\log \log t = o(\phi(t))$ .

Now, we would like to bound the factor  $e^{-\tilde{V}^{(j)}(x)}$  that appears in (2.3.14). To this aim, let

$$\begin{aligned} \mathcal{A}_3 &:= \{ \tilde{\tau}_j[\kappa C_0 \phi(t)/8] \leq \tilde{m}_j + C_0 \phi(t) \}, \\ \mathcal{A}_4 &:= \left\{ \inf_{[\tau_j[\kappa C_0 \phi(t)/8], \tau_j(h_t)]} V^{(j)} \geq \kappa C_0 \phi(t)/16 \right\}, \end{aligned}$$

with  $\tilde{\tau}_j$  and  $\tilde{\tau}_j^-$  defined in (2.2.5) and (2.2.6), and  $\tau_j$  and  $\tau_j^-$  in (2.2.1) and (2.2.2). First, using (2.6.12),  $P(\overline{\mathcal{A}}_3) \leq C_+ e^{-[\kappa^2 C_0 \phi(t)]/(16\sqrt{2})} \leq e^{-2\phi(t)}$  if we choose  $C_0$  large enough. Moreover Fact 2.2.1 together with (2.7.3) (applied with  $h = C_0 \phi(t)$ ,  $\alpha = \kappa/8$ ,  $\gamma = \kappa/16$  and  $\omega = h_t/(C_0 \phi(t))$ ), see also the remark at the end of Lemma 2.7.2) give  $P(\overline{\mathcal{A}}_4) \leq 2e^{-\kappa^2 C_0 \phi(t)/16} \leq e^{-2\phi(t)}$  for large  $t$ .

We notice that  $\inf_{[\tilde{m}_j + C_0 \phi(t), \tilde{\tau}_j(h_t)]} \tilde{V}^{(j)} \geq \kappa C_0 \phi(t)/16$  on  $\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{V}_t$ , since  $\tau_j = \tilde{\tau}_j$  and  $V^{(j)} = \tilde{V}^{(j)}$  on  $\mathcal{V}_t$  thanks to Remark 2.2.1. We prove similarly that

$$P(\overline{\mathcal{A}}_5) \leq C_+ e^{-\kappa^2 C_0 \phi(t)/(16\sqrt{2})} + P(\overline{\mathcal{V}}_t) \leq 2e^{-2\phi(t)},$$

where

$$\begin{aligned} \mathcal{A}_5 &:= \left\{ \inf_{[\tilde{\tau}_j^-(h_t), \tilde{m}_j - C_0 \phi(t)]} \tilde{V}^{(j)} \geq \kappa C_0 \phi(t)/16 \right\}, \\ \mathcal{A}_6 &:= \left\{ \inf_{[\tilde{\tau}_j^-(h_t^+), \tilde{\tau}_j^-(h_t)]} \tilde{V}^{(j)} \geq h_t/2 \right\}. \end{aligned}$$

Also by (2.6.10),  $P(\overline{\mathcal{A}}_6) \leq e^{-\kappa h_t/8}$ . We also know that  $\tilde{V}^{(j)}(x) \geq h_t/2 \geq \kappa C_0 \phi(t)/16$  for all  $\tilde{\tau}_j(h_t) \leq x \leq \tilde{L}_j$  by definition of  $\tilde{L}_j$ , uniformly for large  $t$ . Consequently on  $\cap_{i=3}^6 \mathcal{A}_i \cap \mathcal{V}_t$ , for all  $x \in \overline{\mathcal{D}}_j \cap [\tilde{\tau}_j^-(h_t^+), \tilde{L}_j]$ , we have  $e^{-\tilde{V}^{(j)}(x)} \leq e^{-\kappa C_0 \phi(t)/16}$ .

Hence on  $\cap_{i=1}^6 \mathcal{A}_i \cap \mathcal{V}_t$ , we have under  $\mathbb{P}_{\tilde{m}_j}^{W_\kappa}$ , by (2.3.14) and (2.3.15),

$$\sup_{x \in \overline{\mathcal{D}}_j \cap [\tilde{\tau}_j^-(h_t^+), \tilde{L}_j]} \mathcal{L}[H(\tilde{L}_j), x] \leq 2te^{(1+2/\kappa)\phi(t)} e^{-\kappa C_0 \phi(t)/16} < te^{-2\phi(t)},$$

if we choose  $C_0$  large enough. So, conditioning by  $W_\kappa$  and applying the strong Markov property at time  $H(\tilde{m}_j)$ , we get

$$\begin{aligned} &\mathbb{P} \left[ \sup_{x \in \overline{\mathcal{D}}_j \cap [\tilde{\tau}_j^-(h_t^+), \tilde{L}_j]} (\mathcal{L}[H(\tilde{L}_j), x] - \mathcal{L}[H(\tilde{m}_j), x]) < te^{-2\phi(t)} \right] \\ &\geq \mathbb{E}(\mathbb{P}_{\tilde{m}_j}^{W_\kappa}(\cap_{i=1}^6 \mathcal{A}_i \cap \mathcal{V}_t)) \geq 1 - C_+ e^{-2\phi(t)} \end{aligned}$$

uniformly for large  $t$  due to the previous estimates and thanks to Lemma 2.2.1. This proves the lemma.  $\square$

### 2.3.3 Approximation of the main contributions

In this section we give an approximation of the exit time of each  $h_t$ -valley  $[\tilde{L}_j^-, \tilde{L}_j]$  and of the local time at the bottom  $\tilde{m}_j$  of this  $h_t$ -valley for every  $1 \leq j \leq n_t$ . More precisely, we make a link between the family  $((U_j := H(\tilde{L}_j) - H(\tilde{m}_j), \mathcal{L}(H(\tilde{L}_j), \tilde{m}_j)), 1 \leq j \leq n_t)$ , and the i.i.d. sequence  $((\mathcal{H}_j, \ell_j), 1 \leq j \leq n_t)$  described in the introduction.

In the following,  $F_1^+(h_t)$ ,  $G^+(h_t/2, h_t)$ ,  $F_2^-(h_t/2)$  and  $F_3^-(h_t/2)$  denote independent r.v. with law respectively  $F^+(h_t)$ ,  $G^+(h_t/2, h_t)$ ,  $F^-(h_t/2)$  and  $F^-(h_t/2)$ , defined in (2.1.7) and (2.1.8).

**Proposition 2.3.5.** *For  $\delta > 0$  small enough (recall that  $\delta$  appears in the definitions of  $n_t$  and  $h_t^+$ ), there exist  $d_1 = d_1(\delta, \kappa) > 0$  and  $D_1(d_1) > 0$  such that for large  $t$ , possibly on an enlarged probability space, there exists a sequence  $((S_j, R_j, \mathbf{e}_j)$ ,  $1 \leq j \leq n_t$ ) of i.i.d. random variables depending on  $t$ , with  $S_j$ ,  $R_j$  and  $\mathbf{e}_j$  independent for every  $j$  and  $S_j \stackrel{\mathcal{L}}{=} F_1^+(h_t) + G^+(h_t/2, h_t)$ ,  $R_j \stackrel{\mathcal{L}}{=} F_2^-(h_t/2) + F_3^-(h_t/2)$  and  $\mathbf{e}_j \stackrel{\mathcal{L}}{=} \mathcal{E}(1/2)$  (exponential variable with mean 2) such that*

$$\mathbb{P} \left( \bigcap_{j=1}^{n_t} \left\{ |U_j - \mathcal{H}_j| \leq \varepsilon_t \mathcal{H}_j, |\mathcal{L}(H(\tilde{L}_j), \tilde{m}_j) - \ell_j| \leq \varepsilon_t \ell_j \right\} \right) \geq 1 - e^{-D_1 h_t}, \quad (2.3.16)$$

where  $\ell_j := S_j \mathbf{e}_j$ ,  $\mathcal{H}_j := R_j \ell_j$  and  $\varepsilon_t := e^{-d_1 h_t}$ .

The proof of the above proposition, which is in the spirit of the proofs of Propositions 3.4 and 4.4 in [3], makes use of the following lemma :

**Lemma 2.3.6.** *For  $\delta > 0$  small enough, there exist constants  $d_- > 0$  and  $D_- > 0$ , possibly depending on  $\kappa$  and  $\delta$ , such that the two following statements are true for  $t > 0$  large enough.*

(i) *There exists a sequence  $(\mathbf{e}_j, 1 \leq j \leq n_t)$  of i.i.d. random variables with exponential law of mean 2 and independent of  $W_\kappa$ , such that*

$$\mathbb{P} \left( \bigcap_{j=1}^{n_t} \left\{ |U_j - \tilde{\mathbb{H}}_j| \leq e^{-(d_-)h_t} \tilde{\mathbb{H}}_j, \mathcal{L}(H(\tilde{L}_j), \tilde{m}_j) = \mathbb{L}_j \right\} \right) \geq 1 - e^{-(D_-)h_t}, \quad (2.3.17)$$

where  $\mathbb{L}_j := \mathbf{e}_j \int_{\tilde{m}_j}^{\tilde{L}_j} e^{\tilde{V}^{(j)}(x)} dx$ ,  $\tilde{R}_j := \int_{\tilde{r}_j^-(h_t/2)}^{\tilde{r}_j^+(h_t/2)} e^{-\tilde{V}^{(j)}(x)} dx$  and  $\tilde{\mathbb{H}}_j := \mathbb{L}_j \tilde{R}_j$  for all  $1 \leq j \leq n_t$ . Moreover the random variables  $(\mathbb{L}_j, \tilde{\mathbb{H}}_j)$ ,  $1 \leq j \leq n_t$ , are i.i.d.

(ii) *Possibly on an enlarged probability space, there exist random variables  $R_j$  and  $S_j$ ,  $1 \leq j \leq n_t$ , such that all the random variables  $R_j$ ,  $S_j$ ,  $\mathbf{e}_j$ ,  $1 \leq j \leq n_t$  are independent, with  $S_j \stackrel{\mathcal{L}}{=} F_1^+(h_t) + G^+(h_t/2, h_t)$ , and  $R_j \stackrel{\mathcal{L}}{=} F_2^-(h_t/2) + F_3^-(h_t/2)$  for every  $1 \leq j \leq n_t$ , such that*

$$P \left( \bigcap_{j=1}^{n_t} \left\{ \left| \int_{\tilde{m}_j}^{\tilde{L}_j} e^{\tilde{V}^{(j)}(x)} dx - S_j \right| \leq e^{-(d_-)h_t} S_j, \tilde{R}_j = R_j \right\} \right) \geq 1 - e^{-(D_-)h_t}, \quad (2.3.18)$$

**Proof of Lemma 2.3.6 :** *We start with (i).* Recall that  $\tilde{m}_j < \tilde{L}_j < \tilde{m}_{j+1}$  for every  $j \geq 1$ , e.g. by (2.2.7). By the strong Markov property applied under  $\mathbb{P}^{W_\kappa}$  at stopping times  $H(\tilde{m}_j)$ , the random variables  $(U_j, \mathcal{L}[H(\tilde{L}_j), \tilde{m}_j])$ ,  $1 \leq j \leq n_t$ , are independent under  $\mathbb{P}^{W_\kappa}$ . By the same Markov property and formulas (2.1.13) and (2.1.14), the sequence  $(U_j, \mathcal{L}[H(\tilde{L}_j), \tilde{m}_j])$ ,  $1 \leq j \leq n_t$  is equal to the sequence

$(H_j(\tilde{L}_j), \mathcal{L}_j[H_j(\tilde{L}_j), \tilde{m}_j], 1 \leq j \leq n_t)$ , where

$$\begin{aligned} H_j(\tilde{L}_j) &:= \int_{-\infty}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} \mathcal{L}_{B^j} [\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)] du, \\ \mathcal{L}_j[H_j(\tilde{L}_j), \tilde{m}_j] &= \mathcal{L}_{B^j} [\tau^{B^j}(A^j(\tilde{L}_j)), 0], \quad A^j(u) := \int_{\tilde{m}_j}^u e^{\tilde{V}^{(j)}(x)} dx, \quad u \in \mathbb{R}, \end{aligned} \quad (2.3.19)$$

with  $(B^j, 1 \leq j \leq n_t)$  a sequence of independent standard Brownian motions independent of  $W_\kappa$ , such that  $B^j$  starts at  $A^j(\tilde{m}_j) = 0$  and is killed when it first hits  $A^j(\tilde{L}_j)$ . Recall that  $\mathcal{L}_{B^j}$  denotes the local time of  $B^j$ . Define  $\mathcal{A}_j := \{\max_{u < \tilde{L}_j^-} \mathcal{L}_{B^j} [\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)] = 0\}$ ,  $1 \leq j \leq n_t$ . By (2.6.6), there exists  $c_- > 0$  (possibly depending on  $\kappa$  and  $\delta$ ) such that  $\mathbb{P}(\cap_{j=1}^{n_t} \mathcal{A}_j) \geq 1 - e^{-(c_-)h_t}$  for large  $t$ . So for large  $t$ ,

$$\mathbb{P}\left(\bigcap_{j=1}^{n_t} \{H_j(\tilde{L}_j) = \tilde{h}_j\}\right) \geq 1 - e^{-(c_-)h_t}, \quad (2.3.20)$$

where

$$\tilde{h}_j := \int_{\tilde{L}_j^-}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} \mathcal{L}_{B^j} [\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)] du, \quad 1 \leq j \leq n_t.$$

We also notice that for every  $1 \leq j \leq n_t$ ,  $(\tilde{h}_j, \mathcal{L}_j[H_j(\tilde{L}_j), \tilde{m}_j])$  is measurable with respect to the  $\sigma$ -field generated by  $(\tilde{V}^{(j)}(x + \tilde{L}_{j-1}^+), 0 \leq x < \tilde{L}_j^+ - \tilde{L}_{j-1}^+)$  and  $B^j$ , where by (2.2.7) and (2.2.8),  $\tilde{L}_{j-1}^+ < \tilde{L}_j^- < \tilde{m}_j < \tilde{L}_j < \tilde{L}_j^+$ . Hence, the random variables  $(\tilde{h}_j, \mathcal{L}_j[H_j(\tilde{L}_j), \tilde{m}_j])$ ,  $1 \leq j \leq n_t$  are i.i.d under  $\mathbb{P}$  by the second fact of Lemma 2.2.1. For the same reason,  $(\tilde{R}_j, A^j(\tilde{L}_j))$ ,  $1 \leq j \leq n_t$  are also i.i.d. For  $1 \leq j \leq n_t$ , let  $\tilde{B}^j(\cdot) := B^j((A^j(\tilde{L}_j))^2 \cdot) / A^j(\tilde{L}_j)$ . Notice that

$$\mathcal{L}_{B^j} [\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)] = A^j(\tilde{L}_j) \mathcal{L}_{\tilde{B}^j} [\tau^{\tilde{B}^j}(1), A^j(u)/A^j(\tilde{L}_j)], \quad \tilde{L}_j^- \leq u \leq \tilde{L}_j. \quad (2.3.21)$$

Moreover by scaling and because  $B^j$  is independent from  $W_\kappa$ ,  $\tilde{B}^j$  is, conditionally to  $W_\kappa$ , a standard Brownian motion starting from 0 and killed when it first hits 1. Furthermore, even if  $W_\kappa$  appears in the expression of  $\tilde{B}^j$ ,  $\tilde{B}^j$  is independent of  $W_\kappa$ . Then, let

$$\mathbf{e}_j := \mathcal{L}_{\tilde{B}^j} [\tau^{\tilde{B}^j}(1), 0] = \mathcal{L}_{B^j} [\tau^{B^j}(A^j(\tilde{L}_j)), 0] / A^j(\tilde{L}_j). \quad (2.3.22)$$

Notice that by the first Ray-Knight theorem,  $\mathbf{e}_j$  is exponentially distributed with mean 2. Since  $\tilde{B}^j$  is independent of  $W_\kappa$ ,  $\mathbf{e}_j$  is also independent of  $W_\kappa$ . Also, the sequence  $\mathbf{e}_j$ ,  $1 \leq j \leq n_t$  is i.i.d. because the  $B^j$  are independent and the  $(\tilde{R}_j, A^j(\tilde{L}_j))$  are i.i.d., so  $(\mathbb{L}_j, \mathbb{H}_j)$ ,  $1 \leq j \leq n_t$ , are also i.i.d. Moreover, (2.3.21) leads to

$$\mathcal{L}_j[H_j(\tilde{L}_j), \tilde{m}_j] = A^j(\tilde{L}_j) \mathcal{L}_{\tilde{B}^j} [\tau^{\tilde{B}^j}(1), 0] = A^j(\tilde{L}_j) \mathbf{e}_j = \mathbb{L}_j. \quad (2.3.23)$$

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Now, for small  $\varepsilon > 0$ , thanks to Lemma 2.6.3, we have for large  $t$ ,

$$\begin{aligned} \mathbb{P}\left(\bigcap_{j=1}^{n_t} \left\{ \left| \tilde{h}_j - A^j(\tilde{L}_j) \mathbf{e}_j \tilde{R}_j \right| \leq 2e^{-(1-3\varepsilon)h_t/6} A^j(\tilde{L}_j) \mathbf{e}_j \tilde{R}_j \right\}\right) &\geq 1 - \frac{C_+ n_t}{e^{(c_-)\varepsilon h_t}} \\ &\geq 1 - \frac{C_+}{e^{(c_-/2)\varepsilon h_t}}, \end{aligned}$$

since  $n_t = e^{o(1)h_t}$ . Finally, this, together with (2.3.20), (2.3.23) and the equality of sequences at the start of this proof show (2.3.17) for some  $D_- > 0$  and  $d_- > 0$ . So (i) is proved.

We now prove (ii). The r.v.  $\tilde{A}_j(\tilde{L}_j) = \int_{\tilde{m}_j}^{\tilde{L}_j} e^{\tilde{V}^{(j)}(x)} dx$  and  $\tilde{R}_j$  are not independent, so we want to replace them by r.v. having better independence properties. Applying Lemma 2.6.4 with subscript 2 replaced by  $j$  for  $1 \leq j \leq n_t$  gives the existence of  $R_j$  and  $S_j$ , independent and independent of  $\mathbf{e}_j$ , having the law claimed in (ii) and satisfying (2.6.5) with 2 replaced by  $j$ . This gives (2.3.18) since  $n_t = e^{o(1)h_t}$ . The fact that we can build these  $R_j$  and  $S_j$  with the claimed independence properties follows from the fact that  $(\mathbf{e}_j, \tilde{R}_j, \tilde{A}^j(\tilde{L}_j)), 1 \leq j \leq n_t$  are i.i.d.  $\square$

**Proof of Proposition 2.3.5 :** The existence and the law of the  $\mathbf{e}_j$  come from Lemma 2.3.6 (i). The existence and the law of the  $R_j$  and  $S_j$ , and the independence of  $R_j, S_j, \mathbf{e}_j, 1 \leq j \leq n_t$  come from Lemma 2.3.6 (ii). Moreover, by Lemma 2.3.6 (i) and (ii), there exist  $d_1 > 0$  and  $D_1 > 0$  such that for large  $t$ ,

$$\begin{aligned} &\mathbb{P}\left(\bigcap_{j=1}^{n_t} \left\{ |U_j - \mathbf{e}_j S_j R_j| \leq \varepsilon_t \mathbf{e}_j S_j R_j, \quad |\mathcal{L}(H(\tilde{L}_j), \tilde{m}_j) - \mathbf{e}_j S_j| \leq \varepsilon_t \mathbf{e}_j S_j \right\}\right) \\ &\geq 1 - e^{-D_1 h_t}, \end{aligned}$$

which proves (2.3.16). So Proposition 2.3.5 is proved.  $\square$

## 2.4 Convergence toward the Lévy process $(\mathcal{Y}_1, \mathcal{Y}_2)$ and continuity

### 2.4.1 Preliminaries

We begin this section by the convergence of certain repartition functions. These key results are in the same spirit as the second part of Lemma 5.1 in [3].

**Lemma 2.4.1.** *Recall from Proposition 2.3.5 that  $\ell_1 := \mathbf{e}_1 S_1$  and  $\mathcal{H}_1 := \mathbf{e}_1 S_1 R_1$ . Then for any  $\varepsilon \in (0, 1/3)$ ,*

$$\lim_{t \rightarrow +\infty} \sup_{x \in [e^{-(1-2\varepsilon)\phi(t)}, +\infty[} |x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\ell_1/t > x) - \mathcal{C}_2| = 0, \quad (2.4.1)$$

$$\lim_{t \rightarrow +\infty} \sup_{y \in [e^{-(1-3\varepsilon)\phi(t)}, +\infty[} |y^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathcal{H}_1/t > y) - \mathcal{C}_2 \mathbb{E}[(\mathcal{R}_\kappa)^\kappa]| = 0, \quad (2.4.2)$$

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with  $\mathcal{C}_2$  a positive constant (see below (2.4.10)).

Moreover, for any  $\alpha > 0$ ,  $e^{\kappa\phi(t)}\mathbb{P}(\ell_1/t \geq x, \mathcal{H}_1/t \geq y)$  converges uniformly when  $t$  goes to infinity on  $[\alpha, +\infty[ \times [\alpha, +\infty[$  to  $\nu([x, +\infty[ \times [y, +\infty[)$ , where  $\nu$  is defined in (2.1.4).

**Proof :** Let  $\varepsilon \in (0, 1/3)$ .

**Proof of (2.4.1) :** We first prove that, as  $t \rightarrow +\infty$ ,  $x^\kappa e^{\kappa\phi(t)}\mathbb{P}(S_1/t > x)$  converges uniformly in  $x \in [e^{-(1-\varepsilon)\phi(t)}, +\infty[$  to a constant  $c$ , that is, we prove that

$$\lim_{t \rightarrow +\infty} \sup_{x \in [e^{-(1-\varepsilon)\phi(t)}, +\infty[} |x^\kappa e^{\kappa\phi(t)}\mathbb{P}(S_1/t > x) - c| = 0. \quad (2.4.3)$$

For that, with the change of variables  $y = e^{(1-\varepsilon)\phi(t)}x$ , we just have to prove that

$$\lim_{t \rightarrow +\infty} \sup_{y \in [1, +\infty[} |y^\kappa e^{\kappa\varepsilon\phi(t)}\mathbb{P}(S_1/e^{h_t+\varepsilon\phi(t)} > y) - c| = 0, \quad (2.4.4)$$

but this is equivalent to prove that for any function  $f : ]0, +\infty[ \rightarrow [1, +\infty[$ ,

$$\lim_{t \rightarrow +\infty} (f(t))^\kappa e^{\kappa\varepsilon\phi(t)}\mathbb{P}(S_1/e^{h_t+\varepsilon\phi(t)} > f(t)) = c. \quad (2.4.5)$$

First by definition (see Proposition 2.3.5),  $S_1$  can be written as the sum of two independent random variables, that we denote by  $F_1^+(h_t)$  and  $G^+(h_t/2, h_t)$  for simplicity. That is,

$$S_1/t = (F_1^+(h_t) + G^+(h_t/2, h_t)) / t = e^{-\phi(t)} (e^{-h_t} F_1^+(h_t) + e^{-h_t} G^+(h_t/2, h_t)). \quad (2.4.6)$$

Since we know the asymptotic behavior of the Laplace transforms of  $F^+(h_t)/e^{h_t}$  and  $G^+(h_t/2, h_t)/e^{h_t}$ , the proof of (2.4.5) is similar to the proof of a Tauberian theorem. First by (2.7.1) and (2.7.2) we have, using the independence of  $F_1^+(h_t)$  and  $G^+(h_t/2, h_t)$ ,

$$\begin{aligned} \forall \gamma > 0, \quad \omega_{f,t}(\gamma) &:= \frac{1}{\gamma} \left( 1 - \mathbb{E} \left[ e^{-\gamma S_1 / (f(t) e^{h_t + \varepsilon \phi(t)})} \right] \right) \\ &\underset{t \rightarrow +\infty}{\sim} c' \gamma^{\kappa-1} (f(t))^{-\kappa} e^{-\kappa\varepsilon\phi(t)}, \end{aligned} \quad (2.4.7)$$

where  $c' = \Gamma(1 - \kappa)2^\kappa / \Gamma(1 + \kappa)$ . Note that by Fubini,  $\omega_{f,t}$  is the Laplace transform of the measure  $dU_{f,t}(z) := \mathbf{1}_{\mathbb{R}_+}(z)\mathbb{P}(S_1/(f(t)e^{h_t+\varepsilon\phi(t)}) > z) dz$ , that is,  $\omega_{f,t}(\gamma) = \int_0^\infty e^{-\gamma z} dU_{f,t}(z)$ . From (2.4.7), we have

$$\forall \gamma > 0, \quad \frac{\omega_{f,t}(\gamma)}{\omega_{f,t}(1)} \xrightarrow{t \rightarrow +\infty} \gamma^{\kappa-1}.$$

We can now follow the same line as in the proof of a classical Tauberian theorem, making the link between a Laplace transform and the repartition function, (see for example [39] volume 2, section XIII.5, Theorem 1, page 442), we can deduce that

$$\forall z > 0, \quad \frac{U_{f,t}([0, z])}{\omega_{f,t}(1)} \xrightarrow{t \rightarrow +\infty} \frac{z^{1-\kappa}}{\Gamma(2 - \kappa)}.$$

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Then, e.g. as in the proof of Theorem 4 of the same reference page 446, or using inequalities similar to those at the end of the proof of Lemma 5.1 in [3], we deduce from the monotony of the densities of measures  $U_{f,t}$  that

$$\forall z > 0, \quad \frac{\mathbb{P}(S_1/(f(t)e^{h_t+\varepsilon\phi(t)}) > z)}{\omega_{f,t}(1)} \xrightarrow{t \rightarrow +\infty} z^{-\kappa} \frac{1-\kappa}{\Gamma(2-\kappa)}.$$

Considering this convergence with  $z = 1$  we get exactly (2.4.5) for  $c = c'(1-\kappa)/\Gamma(2-\kappa) = 2^\kappa/\Gamma(1+\kappa)$ , so (2.4.3) follows.

Now, let  $a_t := e^{\varepsilon\phi(t)}$ . For any  $x > 0$ ,

$$x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x, \mathbf{e}_1 < a_t) = 2^{-1} \int_0^{a_t} (x/u)^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1/t > x/u) u^\kappa e^{-u/2} du,$$

because  $\mathbf{e}_1$  has law  $\mathcal{E}(1/2)$  and is independent of  $S_1$ .

Taking  $x$  arbitrary in  $[e^{-(1-2\varepsilon)\phi(t)}, +\infty[$ , we have  $x/u \in [e^{-(1-\varepsilon)\phi(t)}, +\infty[$  for every  $u \in ]0, a_t]$ , so thanks to (2.4.3) we get

$$\lim_{t \rightarrow +\infty} \sup_{x \in [e^{-(1-2\varepsilon)\phi(t)}, +\infty[} \left| x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x, \mathbf{e}_1 < a_t) - \frac{c}{2} \int_0^{+\infty} \frac{u^\kappa}{e^{u/2}} du \right| = 0. \quad (2.4.8)$$

Now for  $t$  large enough such that  $\forall y \geq 1, y^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1/t > y) < 2c$  (see (2.4.3)), we have for any  $x > 0$ ,

$$\begin{aligned} & \left| x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x, \mathbf{e}_1 < a_t) - x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) \right| \\ &= x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x, \mathbf{e}_1 \geq a_t) \\ &= 2^{-1} \int_{a_t}^{+\infty} x^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1/t > x/u) e^{-u/2} du \\ &= 2^{-1} \int_{a_t}^{+\infty} u^\kappa (x/u)^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1/t > x/u) \mathbf{1}_{x \leq u} e^{-u/2} du \\ &+ 2^{-1} \int_{a_t}^{+\infty} u^\kappa (x/u)^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1/t > x/u) \mathbf{1}_{x > u} e^{-u/2} du \\ &\leq 2^{-1} e^{\kappa\phi(t)} \int_{a_t}^{+\infty} u^\kappa e^{-u/2} du + c \int_{a_t}^{+\infty} u^\kappa e^{-u/2} du. \end{aligned} \quad (2.4.9)$$

For the second term in the inequality we used the fact that

$$(x/u)^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1/t > x/u) < 2c$$

when  $x \geq u$ . Since  $a_t = e^{\varepsilon\phi(t)}$ , the right hand side of (2.4.9) converges to 0 when  $t$  goes to infinity. Combining this with (2.4.8), we get

$$\lim_{t \rightarrow +\infty} \sup_{x \in [e^{-(1-2\varepsilon)\phi(t)}, +\infty[} \left| x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) - 2^{-1} c \int_0^{+\infty} u^\kappa e^{-u/2} du \right| = 0, \quad (2.4.10)$$

and this is exactly (2.4.1) with  $\mathcal{C}_2 := 2^{-1} c \int_0^{+\infty} u^\kappa e^{-u/2} du = 2^\kappa \Gamma(\kappa + 1) c = 4^\kappa$ .

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**Proof of (2.4.2) :** Let  $\mu_{R_1}$  be the distribution of  $R_1$ . For any  $y > 0$ ,  $a > 0$  and  $t > 0$ , we have by independence of  $\mathbf{e}_1 S_1$  and  $R_1$ ,

$$y^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1 R_1 / t > y, R_1 < a) = \int_0^a (y/u)^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1 / t > y/u) u^\kappa \mu_{R_1}(du).$$

Taking  $a = a_t = e^{\varepsilon\phi(t)}$  and  $y$  arbitrary in  $[e^{-(1-3\varepsilon)\phi(t)}, +\infty[$ , we have  $y/u \in [e^{-(1-2\varepsilon)\phi(t)}, +\infty[$  for all  $u \in ]0, a_t]$ , so thanks to (2.4.10), we get

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \sup_{y \in [e^{-(1-3\varepsilon)\phi(t)}, +\infty[} \left| y^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1 R_1 / t > y, R_1 < a_t) - \mathcal{C}_2 \int_0^{a_t} u^\kappa \mu_{R_1}(du) \right| \\ & = 0, \end{aligned}$$

where we used  $\int_0^\infty u^\kappa \mu_{R_1}(du) = \mathbb{E}[(R_1)^\kappa] \leq \mathbb{E}[(\mathcal{R}_\kappa)^\kappa] < \infty$ , as explained in the following lines. By definition (see before Proposition 2.3.5 and (2.1.7)) and with  $\widetilde{W}_\kappa^\uparrow$  an independent copy of  $W_\kappa^\uparrow$ ,  $R_1$  is equal in law to  $\int_0^{\tau_{W_\kappa^\uparrow}(h_t/2)} e^{-W_\kappa^\uparrow(x)} dx + \int_0^{\tau_{\widetilde{W}_\kappa^\uparrow}(h_t/2)} e^{-\widetilde{W}_\kappa^\uparrow(x)} dx$ , which itself converges almost surely to  $\mathcal{R}_\kappa$  (defined in (2.1.2)) when  $t$  goes to infinity. This also shows that for each  $t$ ,  $R_1$  is stochastically inferior to  $\mathcal{R}_\kappa$ , which admits finite moments of any positive order by Lemma 2.6.6. In particular the family  $(R_1)_{t>0}$  is bounded in all  $L^p$  spaces, and more precisely,  $\mathbb{E}[(R_1)^p] \leq \mathbb{E}[(\mathcal{R}_\kappa)^p] < \infty$  for every  $p \in \mathbb{R}_+$ . So by the dominated convergence theorem,  $\int_0^{+\infty} u^\kappa \mu_{R_1}(du)$  converges to  $\mathbb{E}[(\mathcal{R}_\kappa)^\kappa]$  when  $t$  goes to infinity. Hence,

$$\lim_{t \rightarrow +\infty} \sup_{y \in [e^{-(1-3\varepsilon)\phi(t)}, +\infty[} \left| y^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1 R_1 / t > y, R_1 < a_t) - \mathcal{C}_2 \mathbb{E}[(\mathcal{R}_\kappa)^\kappa] \right| = 0.$$

Finally, as the family  $(R_1)_{t>0}$  is bounded in all  $L^p$  spaces,  $e^{\kappa\phi(t)} \int_{a_t}^\infty u^\kappa \mu_{R_1}(du)$  converges to 0 as  $t \rightarrow +\infty$ . So we can proceed as before (as in (2.4.9), integrating with respect to  $R_1$  instead of  $\mathbf{e}_1$  and using (2.4.1) instead of (2.4.3)) to remove the event  $R_1 < a_t$  and we thus get

$$\lim_{t \rightarrow +\infty} \sup_{y \in [e^{-(1-3\varepsilon)\phi(t)}, +\infty[} \left| y^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1 R_1 / t > y) - \mathcal{C}_2 \mathbb{E}[(\mathcal{R}_\kappa)^\kappa] \right| = 0, \quad (2.4.11)$$

which is (2.4.2).



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We now prove the last assertion. For any  $x > 0$ ,  $y > 0$ ,  $a > 0$  and  $t > 0$ , we have

$$\begin{aligned}
& e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x, \mathbf{e}_1 S_1 R_1/t > y, R_1 < a) \\
&= \int_0^a e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x, \mathbf{e}_1 S_1/t > y/u) \mu_{R_1}(du) \\
&= \int_0^{a \wedge (y/x)} e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > y/u) \mu_{R_1}(du) \\
&\quad + \int_{a \wedge (y/x)}^a e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) \mu_{R_1}(du), \\
&= \frac{1}{y^\kappa} \int_0^{a \wedge (y/x)} e^{\kappa\phi(t)} (y/u)^\kappa \mathbb{P}(\mathbf{e}_1 S_1/t > y/u) u^\kappa \mu_{R_1}(du) \\
&\quad + \frac{1}{x^\kappa} \int_{a \wedge (y/x)}^a e^{\kappa\phi(t)} x^\kappa \mathbb{P}(\mathbf{e}_1 S_1/t > x) \mu_{R_1}(du).
\end{aligned}$$

Taking  $a = a_t = e^{\varepsilon\phi(t)}$  and  $x, y$  arbitrary in  $[\alpha, +\infty[$  (for some  $\alpha > 0$ ), we have  $(y/u, x) \in [e^{-(1-2\varepsilon)\phi(t)}, +\infty[^2, \forall u \in ]0, a_t]$  whenever  $t$  is large enough, so, thanks to (2.4.10) we get that  $e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x, \mathbf{e}_1 S_1 R_1/t > y, R_1 < a_t)$  converges uniformly in  $(x, y) \in [\alpha, +\infty[ \times [\alpha, +\infty[$  toward

$$\mathcal{C}_2 x^{-\kappa} \mathbb{P}(\mathcal{R}_\kappa > y/x) + \mathcal{C}_2 y^{-\kappa} \mathbb{E}((\mathcal{R}_\kappa)^\kappa \mathbf{1}_{\mathcal{R}_\kappa \leq y/x}) = \nu([x, +\infty[ \times [y, +\infty[).$$

Then as before we can remove the event  $\{R_1 < a_t\}$  since  $e^{\kappa\phi(t)} \mathbb{P}(R_1 \geq a_t) \rightarrow 0$  as  $t \rightarrow +\infty$  because the family  $(R_1)_{t>0}$  is bounded in all  $L^p$  spaces, which gives the last assertion of Lemma 2.4.1.  $\square$

### 2.4.2 Proof of Proposition 2.1.4

We start with the finite dimensional convergence. We recall that  $(Y_1, Y_2)_s^t$  is defined just before Proposition 2.1.4, and  $(\mathcal{Y}_1, \mathcal{Y}_2)$  before (2.1.4). We sometimes use the notation  $(\mathcal{Y}_1, \mathcal{Y}_2)_s = (\mathcal{Y}_1(s), \mathcal{Y}_2(s))$  and  $(Y_1, Y_2)_s^t = (Y_1^t(s), Y_2^t(s))$ .

**Lemma 2.4.2.** *For any  $k \in \mathbb{N}$  and  $s_i > 0, i \leq k$ ,  $((Y_1, Y_2)_{s_i}^t, i \leq k)$  converges in law as  $t$  goes to infinity to  $((\mathcal{Y}_1, \mathcal{Y}_2)_{s_i}, i \leq k)$ .*

**Proof :** The proof is basic here, however we give some details as we deal with a two dimensional walk which increments depend on  $t$  itself. As  $Y_1^t(s)$  and  $Y_2^t(s)$  are sums of i.i.d sequences we only have to prove the convergence in law for the couple  $(Y_1, Y_2)_s^t$  for any  $s > 0$ . For  $b \geq 0$ , we define  $(Y_1^{>b}, Y_2^{>b})$ , obtained from  $(Y_1, Y_2)_s^t$  by keeping only the increments larger than  $b$ , that is,  $Y_1^{>b}(s) := \frac{1}{t} \sum_{j=1}^{\lfloor se^{\kappa\phi(t)} \rfloor} \ell_j \mathbf{1}_{\ell_j/t > b}$  and  $Y_2^{>b}(s) := \frac{1}{t} \sum_{j=1}^{\lfloor se^{\kappa\phi(t)} \rfloor} \mathcal{H}_j \mathbf{1}_{\mathcal{H}_j/t > b}$  for every  $s \geq 0$  and  $t > 0$ . Also let  $Y_i^{\leq b}(s) := Y_i^t(s) - Y_i^{>b}(s)$  for  $i \in \{1, 2\}$ . We first prove that for any  $s > 0$ ,

$$\lim_{\varepsilon \rightarrow 0} \limsup_{t \rightarrow +\infty} \mathbb{P}(\|(Y_1^{\leq \varepsilon}, Y_2^{\leq \varepsilon})_s^t\| > \varepsilon^{1-\kappa(2-\kappa)}) = 0, \quad (2.4.12)$$

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where for any  $a = (a_1, a_2) \in \mathbb{R}^2$ ,  $\|a\| := \max(|a_1|, |a_2|)$ , with  $(Y_1^{\leq \varepsilon}, Y_2^{\leq \varepsilon})_s^t = (Y_1^{\leq \varepsilon}(s), Y_2^{\leq \varepsilon}(s))$  and  $1 - \kappa(2 - \kappa) > 0$  since  $\kappa < 1$ .

Let  $\varepsilon > 0$  and  $s > 0$ . We now give an upper bound for the first moments of  $Y_1^{\leq \varepsilon}(s)$  and  $Y_2^{\leq \varepsilon}(s)$ . Let  $\eta > 0$  be such that  $\kappa - (1 - 3\eta) < 0$ . Applying Fubini, we have for large  $t$ ,

$$\begin{aligned}
& e^{\kappa\phi(t)} \mathbb{E} \left( \frac{\ell_1}{t} \mathbf{1}_{\ell_1/t \leq \varepsilon} \right) \\
&= e^{\kappa\phi(t)} \mathbb{E} \left[ \frac{\mathbf{e}_1 S_1}{t} \mathbf{1}_{\mathbf{e}_1 S_1/t \leq \varepsilon} \right] \\
&\leq \int_0^\varepsilon e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) dx \\
&= \int_0^{e^{-(1-2\eta)\phi(t)}} e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) dx + \int_{e^{-(1-2\eta)\phi(t)}}^\varepsilon e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) dx \\
&\leq e^{(\kappa - (1-2\eta))\phi(t)} + \int_{e^{-(1-2\eta)\phi(t)}}^\varepsilon x^{-\kappa} x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) dx. \tag{2.4.13}
\end{aligned}$$

The first term in (2.4.13) converges to 0 when  $t$  goes to infinity because  $\kappa - (1 - 2\eta) < -\eta < 0$ . Moreover, according to (2.4.1), for  $t$  large enough, we have

$$\forall x \geq e^{-(1-2\eta)\phi(t)}, \quad x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) \leq 2\mathcal{C}_2.$$

For such  $t$ , the second term in (2.4.13) is less than

$$2\mathcal{C}_2 \int_0^\varepsilon x^{-\kappa} dx = 2\mathcal{C}_2 \frac{\varepsilon^{1-\kappa}}{1-\kappa}.$$

So, we get for large  $t$ ,

$$e^{\kappa\phi(t)} \mathbb{E} \left( \frac{\ell_1}{t} \mathbf{1}_{\ell_1/t \leq \varepsilon} \right) \leq e^{(\kappa - (1-2\eta))\phi(t)} + C_+ \varepsilon^{1-\kappa}. \tag{2.4.14}$$

Using the same method and applying this time (2.4.2), we get for large  $t$ ,

$$e^{\kappa\phi(t)} \mathbb{E} \left( \frac{\mathcal{H}_1}{t} \mathbf{1}_{\mathcal{H}_1/t \leq \varepsilon} \right) \leq e^{(\kappa - (1-3\eta))\phi(t)} + C_+ \varepsilon^{1-\kappa}. \tag{2.4.15}$$

We thus obtain

$$\mathbb{E}(Y_1^{\leq \varepsilon}(s)) \leq s e^{(\kappa - (1-2\eta))\phi(t)} + C_+ s \varepsilon^{1-\kappa}, \tag{2.4.16}$$

$$\mathbb{E}(Y_2^{\leq \varepsilon}(s)) \leq s e^{(\kappa - (1-3\eta))\phi(t)} + C_+ s \varepsilon^{1-\kappa}, \tag{2.4.17}$$

then a Markov inequality leads to (2.4.12) since  $\kappa - (1 - 3\eta) < 0$ .

The next step is to prove that  $(Y_1^{> \varepsilon}, Y_2^{> \varepsilon})_s^t$  can be written as the integral of a point process which converges to the desired limit. We have

$$(Y_1^{> \varepsilon}, Y_2^{> \varepsilon})_s^t = (Y_1^{> \varepsilon}(s), Y_2^{> \varepsilon}(s)) = \left( \int_{x>\varepsilon} \int_0^s x \mathcal{P}_t^1(dx, dv), \int_{x>\varepsilon} \int_0^s x \mathcal{P}_t^2(dx, dv) \right)$$

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where the measures  $\mathcal{P}_t^1$  and  $\mathcal{P}_t^2$  are defined by  $\mathcal{P}_t^1 := \sum_{i=1}^{+\infty} \delta_{(t^{-1}\ell_i, e^{-\kappa\phi(t)i})}$  and similarly  $\mathcal{P}_t^2 := \sum_{i=1}^{+\infty} \delta_{(t^{-1}\mathcal{H}_i, e^{-\kappa\phi(t)i})}$ . Recall that  $\mathcal{P}_t^1$  and  $\mathcal{P}_t^2$  are not independent. We now prove that  $(\mathcal{P}_t^1, \mathcal{P}_t^2)$  converges to a Poisson point measure. For that just use Lemma 2.4.1 together with Proposition 3.1 in [56] after discretization, it implies that  $(\mathcal{P}_t^1, \mathcal{P}_t^2)$  converges weakly to the Poisson random measure denoted by  $(\mathcal{P}^1, \mathcal{P}^2)$  with intensity measure given by  $ds \times \nu$ .

Then using that for any  $\varepsilon > 0$ , and  $T < +\infty$ , on  $[0, T) \times (\varepsilon, +\infty) \times (\varepsilon, +\infty)$   $ds \times \nu$  is finite, we have that  $(Y_1^{>\varepsilon}, Y_2^{>\varepsilon})_s^t$  converges weakly to

$$(\mathcal{Y}_1^{>\varepsilon}, \mathcal{Y}_2^{>\varepsilon})_s := \left( \int_{x>\varepsilon} \int_0^s x \mathcal{P}^1(dx, dv), \int_{x>\varepsilon} \int_0^s x \mathcal{P}^2(dx, dv) \right).$$

We are left to prove that  $(\mathcal{Y}_1^{>\varepsilon}, \mathcal{Y}_2^{>\varepsilon})$  converges to  $(\mathcal{Y}_1, \mathcal{Y}_2)$  when  $\varepsilon \downarrow 0$ . This is a straightforward computation, that we detail for completeness. Let  $\nu_1([x, +\infty[) := \nu([x, +\infty[ \times \mathbb{R}_+) = C_2/x^\kappa$ , we have

$$\mathbb{E} \left( \int_{x \leq \varepsilon} \int_0^s x \mathcal{P}^1(dx, dv) \right) = s \int_{x \leq \varepsilon} x \nu_1(dx) = C\varepsilon^{1-\kappa},$$

Then a Markov inequality proves that for any  $s > 0$ , the process  $\int_{x \leq \varepsilon} \int_0^s x \mathcal{P}^1(dx, dv)$  converges to zero (when  $\varepsilon$  goes to zero) in probability. The same is true for  $\int_{x \leq \varepsilon} \int_0^s x \mathcal{P}^2(dx, dv)$ , so we obtain that  $(\mathcal{Y}_1^{>\varepsilon}, \mathcal{Y}_2^{>\varepsilon})_s$  converges in probability to  $(\mathcal{Y}_1, \mathcal{Y}_2)_s$  when  $\varepsilon \rightarrow 0$ .  $\square$

We now prove the tightness of  $(\mathcal{D}(Y_1, Y_2)^t)_t$ , the family of measures induced by processes  $(Y_1, Y_2)^t$ .

**Lemma 2.4.3.** *The family of laws  $(\mathcal{D}(Y_1, Y_2)^t)_t$  is tight on  $(D([0, +\infty), \mathbb{R}^2), J_1)$ .*

**Proof :** We only have to prove that the family law of the restriction of the process to the interval  $[0, T]$ ,  $((Y_1, Y_2)^t|_{[0, T]})_t$  is tight. To prove this we use the following restatement of Theorem 1.8 in [13] using Aldous's tightness criterion (see Condition 1, and equation (16.22) page 176 in [13]) also used in [16] page 100. We have to check the two following statements :

1) for any  $\varepsilon > 0$ , there exists  $a$  such that for any  $t$  large enough,

$$\mathbb{P} \left( \sup_{s \in [0, T]} \|(Y_1, Y_2)_s^t\| \geq a \right) \leq \varepsilon.$$

2) for any  $\varepsilon > 0$ , and  $\eta > 0$  there exists  $\delta, 0 < \delta < T$  and  $t_0 > 0$  such that for  $t > t_0$ ,

$$\mathbb{P}[\omega((Y_1, Y_2)^t, \delta, T) \geq \eta] \leq \varepsilon,$$

with  $\omega((Y_1, Y_2)^t, \delta, T) := \sup_{0 \leq r \leq T} \omega((Y_1, Y_2)^t, \delta, T, r)$ , and

$$\begin{aligned} & \omega((Y_1, Y_2)^t, \delta, T, r) \\ := & \sup_{0 \vee (r-\delta) \leq u_1 < u < u_2 \leq (r+\delta) \wedge T} \left( \min \left\{ \|(Y_1, Y_2)_{u_2}^t - (Y_1, Y_2)_u^t\|, \right. \right. \\ & \left. \left. \|(Y_1, Y_2)_u^t - (Y_1, Y_2)_{u_1}^t\| \right\} \right). \end{aligned}$$

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Also

$$\mathbb{P}(v((Y_1, Y_2)^t, 0, \delta, T) \geq \eta) \leq \varepsilon, \text{ and } \mathbb{P}(v((Y_1, Y_2)^t, T, \delta, T) \geq \eta) \leq \varepsilon,$$

where  $v((Y_1, Y_2)^t, u, \delta, T) := \sup_{(u-\delta) \vee 0 \leq u_1 \leq u_2 \leq (u+\delta) \wedge T} \{ \|(Y_1, Y_2)_{u_1}^t - (Y_1, Y_2)_{u_2}^t\| \}$ .

We first check 1) since the process is monotone increasing,

$$\mathbb{P}\left(\sup_{s \in [0, T]} \|(Y_1, Y_2)_s^t\| \geq a\right) = \mathbb{P}(\|(Y_1, Y_2)_T^t\| \geq a) \leq \mathbb{P}(Y_1(T) \geq a) + \mathbb{P}(Y_2(T) \geq a). \quad (2.4.18)$$

Recall that  $Y_1^{>b}$  is obtained from  $Y_1$  where we remove the increments  $\ell_j/t$  smaller than  $b$  and  $Y_1^{\leq b} = Y_1 - Y_1^{>b}$ . Define  $N_u^{>b} := \sum_{i=1}^{\lfloor ue^{\kappa\phi(t)} \rfloor} \mathbf{1}_{\ell_j/t > b}$ . Let  $0 < \delta_1 < 1$ . A Markov inequality yields

$$\begin{aligned} \mathbb{P}(Y_1^t(T) \geq a) &\leq \mathbb{P}\left(Y_1^{\leq 1}(T) \geq \frac{a}{2}\right) + \mathbb{P}\left(Y_1^{>1}(T) \geq \frac{a}{2}\right) \\ &\leq \frac{2}{a} \mathbb{E}[Y_1^{\leq 1}(T)] + \frac{1}{a^{\delta_1}} \mathbb{E}(N_T^{>1}) + \mathbb{P}\left(Y_1^{>1}(T) \geq \frac{a}{2}, N_T^{>1} \leq a^{\delta_1}\right). \end{aligned} \quad (2.4.19)$$

On  $\{N_T^{>1} \leq a^{\delta_1}\}$  there is at most  $a^{\delta_1}$  terms in the sum  $Y_1^{>1}(T)$  so

$$\begin{aligned} \mathbb{P}\left(Y_1^{>1}(T) > a/2, N_T^{>1} \leq a^{\delta_1}\right) &\leq \sum_{1 \leq i \leq a^{\delta_1}} \mathbb{P}(\ell_i/t \geq (a^{1-\delta_1}/2) | \ell_i/t \geq 1) \\ &\leq a^{\delta_1} \mathbb{P}(\ell_1/t \geq (a^{1-\delta_1}/2) | \ell_1/t \geq 1) \\ &\leq a^{\delta_1} 2 \frac{\mathcal{C}_2 e^{-\kappa\phi(t)} a^{-\kappa(1-\delta_1)} 2^\kappa}{\mathcal{C}_2 e^{-\kappa\phi(t)}} \\ &= 2^{1+\kappa} a^{\delta_1 - \kappa(1-\delta_1)}, \end{aligned} \quad (2.4.20)$$

for all  $t$  large enough thanks to (2.4.1) and  $\delta_1$  such that  $\delta_1 - \kappa(1 - \delta_1) < 0$ .

Also, notice that for any  $b > 0$ ,  $N_T^{>b}$  follows a binomial law with parameters  $(\lfloor T e^{\kappa\Phi(t)} \rfloor, \mathbb{P}(\ell_1/t > b))$ . So, using (2.4.1) again and (2.4.16), we obtain for  $t$  large enough,

$$\mathbb{E}(N_T^{>b}) \leq 2\mathcal{C}_2 T b^{-\kappa}, \quad \mathbb{E}\left[Y_1^{\leq b}(T)\right] \leq 2\mathcal{C}_2 T b^{1-\kappa}. \quad (2.4.21)$$

Collecting (2.4.20), (2.4.21) and (2.4.19), we get the existence of  $t_1 > 0$  such that

$$\lim_{a \rightarrow +\infty} \sup_{t \geq t_1} \mathbb{P}(Y_1(T) \geq a) = 0. \quad (2.4.22)$$

The same arguments holds for  $Y_2$  (using (2.4.2) instead of (2.4.1) and (2.4.17) instead of (2.4.16)) so (2.4.22) also holds for  $Y_2$  instead of  $Y_1$ . We conclude the proof

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of 1) by putting (2.4.22) and its analogous for  $Y_2$  in (2.4.18).

We now check 2) We first write, as usual,

$$\begin{aligned} & \{\omega((Y_1, Y_2)^t, \delta, T) \geq \eta\} \\ \subset & \{\omega((Y_1^{\leq b}, Y_2^{\leq b})^t, \delta, T) \geq \eta/2\} \cup \{\omega((Y_1^{> b}, Y_2^{> b})^t, \delta, T) \geq \eta/2\}. \end{aligned}$$

For  $Y^{\leq b}$ , we have

$$\mathbb{P}[\omega((Y_1^{\leq b}, Y_2^{\leq b})^t, \delta, T) \geq \eta/2] \leq \mathbb{P}[\omega(Y_1^{\leq b}, \delta, T) \geq \eta/2] + \mathbb{P}[\omega(Y_2^{\leq b}, \delta, T) \geq \eta/2].$$

Moreover, by positivity of the increments,

$$\begin{aligned} & \mathbb{P}\left(\omega(Y_1^{\leq b}, \delta, T) \geq \eta/2\right) \\ \leq & \mathbb{P}\left(\bigcup_{k \leq \lfloor T/2\delta \rfloor} \left\{Y_1^{\leq b}((k+1)2\delta) - Y_1^{\leq b}(k2\delta) \geq \eta/4\right\}\right) \\ \leq & \sum_{k \leq \lfloor T/\delta \rfloor} \mathbb{P}\left(Y_1^{\leq b}((k+1)2\delta) - Y_1^{\leq b}(k2\delta) \geq \eta/4\right). \end{aligned} \quad (2.4.23)$$

For any  $k$ ,  $Y_1^{\leq b}((k+1)2\delta) - Y_1^{\leq b}(k2\delta)$  is the sum of at most  $\lfloor 2\delta e^{\kappa\Phi(t)} \rfloor + 1$  i.i.d. random variables having the same law as  $\ell_1/t$ . We get that for any integer  $k$

$$\mathbb{P}\left(Y_1^{\leq b}((k+1)2\delta) - Y_1^{\leq b}(k2\delta) \geq \eta/4\right) \leq \mathbb{P}\left(Y_1^{\leq b}(3\delta) \geq \eta/4\right) \leq 8\mathcal{C}_2\delta b^{1-\kappa}/\eta,$$

where the first inequality holds for  $t$  large enough so that  $2\delta e^{\kappa\Phi(t)} \geq 1$  and the second from the second expression in (2.4.21) (replacing  $T$  by  $2\delta$ ). Combining with (2.4.23) we get for large  $t$

$$\mathbb{P}\left(\omega(Y_1^{\leq b}, \delta, T) \geq \eta/2\right) \leq 24\mathcal{C}_2T(1+2\delta)b^{1-\kappa}/\eta, \quad (2.4.24)$$

[note that  $\delta$  will be chosen later (and will be less than 1)].  $T$  and  $\eta$  are fixed so we choose  $b$  small enough so that the right hand side of (2.4.23) is less than  $\varepsilon/4$ . A similar estimate can be proved for  $\mathbb{P}(\omega(Y_2^{\leq b}, \delta, T) \geq \eta/2)$ .

For  $Y^{> b}$ , we have again

$$\mathbb{P}(\omega((Y_1^{> b}, Y_2^{> b})^t, \delta, T) \geq \eta/2) \leq \mathbb{P}(\omega(Y_1^{> b}, \delta, T) \geq \eta/2) + \mathbb{P}(\omega(Y_2^{> b}, \delta, T) \geq \eta/2).$$

Since  $Y_1^{> b}$  is piecewise constant with jumps larger than  $b$ ,  $\{\omega(Y_1^{> b}, \delta, T) > \eta/2\}$  implies that two jumps larger than  $b$  for  $Y_1^t$  occur in an interval smaller than  $2\delta$ . That is  $\{\omega(Y_1^{> b}, \delta, T) > \eta/2\} \subset \bigcup_{j=1}^{\lfloor Te^{\kappa\phi(t)} \rfloor} \bigcup_{i>j, (i-j)/e^{\kappa\phi(t)} \leq 2\delta} \{\ell_j \wedge \ell_i/t > b\}$ . Applying (2.4.1) for  $t$  large enough,

$$\mathbb{P}\left(\bigcup_{j=1}^{\lfloor Te^{\kappa\phi(t)} \rfloor} \bigcup_{i>j, (i-j)/e^{\kappa\phi(t)} \leq 2\delta} \{\ell_j \wedge \ell_i/t > b\}\right) \leq 8\mathcal{C}_2^2\delta T b^{-2\kappa},$$

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which can be small choosing this time  $\delta = \delta(b)$  properly. Again the same argument can be used for  $\omega(Y_2^{>b}, \delta, T)$ . To finish the proof, we have to deal with  $v()$ , as again our processes are increasing,

$$\mathbb{P}(v((Y_1, Y_2)^t, 0, \delta, T) \geq \eta) \leq \mathbb{P}(\|(Y_1, Y_2)_\delta^t\| \geq \eta)$$

we can then proceed as for 1) decreasing the value of  $\delta$  if needed, this also applies to  $\mathbb{P}(v((Y_1, Y_2)^t, T, \delta, T) \geq \eta)$ .  $\square$

Putting together the two preceding lemmata we obtain Proposition 2.1.4.

### 2.4.3 Continuity of some functionals of $(\mathcal{Y}_1, \mathcal{Y}_2)$ in $J_1$ topology

In this section, we study the continuity of some functionals which will be applied later to  $(Y_1, Y_2)^t$  and to the Lévy processes  $(\mathcal{Y}_1, \mathcal{Y}_2)$ .

For our purpose, we are interested in the following mappings. We have already mentioned the first two in the introduction :

$$J : \begin{array}{ccc} D(\mathbb{R}_+, \mathbb{R}) & \longrightarrow & D(\mathbb{R}_+, \mathbb{R}) \\ f & \longmapsto & f^\natural \end{array} \quad I : \begin{array}{ccc} (D(\mathbb{R}_+, \mathbb{R}), J_1) & \longrightarrow & (D(\mathbb{R}_+, \mathbb{R}), U) \\ f & \longmapsto & f^{-1} \end{array}$$

where  $U$  denotes uniform convergence on every compact subset of  $\mathbb{R}_+$ . Then we also need the compositions of these two : for any positive  $a$ , let

$$J_{I,a} : \begin{array}{ccc} D(\mathbb{R}_+, \mathbb{R}^2) & \longrightarrow & \mathbb{R} \\ f = (f_1, f_2) & \longmapsto & f_1^\natural(f_2^{-1}(a)), \end{array} \quad J_{I,a}^- : \begin{array}{ccc} D(\mathbb{R}_+, \mathbb{R}^2) & \longrightarrow & \mathbb{R} \\ f = (f_1, f_2) & \longmapsto & f_1^\natural(f_2^{-1}(a)^-), \end{array}$$

$J_{I,a}$  (respectively  $J_{I,a}^-$ ) produces the largest jump of  $f_1$ , between 0 and the time just after (respectively before)  $f_2$  first reaches  $(a, +\infty)$ . We also define  $K_{I,a}$ ,  $K_{I,a}^-$ ,  $\tilde{K}_{I,a}$  and  $\tilde{K}_{I,a}^-$  as follows.

$$\begin{aligned} K_{I,a} : \begin{array}{ccc} D(\mathbb{R}_+, \mathbb{R}^2) & \longrightarrow & \mathbb{R} \\ f = (f_1, f_2) & \longmapsto & f_1(f_2^{-1}(a)), \end{array} \\ K_{I,a}^- : \begin{array}{ccc} D(\mathbb{R}_+, \mathbb{R}^2) & \longrightarrow & \mathbb{R} \\ f = (f_1, f_2) & \longmapsto & f_1(f_2^{-1}(a)^-), \end{array} \\ \tilde{K}_{I,a} : \begin{array}{ccc} D(\mathbb{R}_+, \mathbb{R}^2) & \longrightarrow & \mathbb{R} \\ f = (f_1, f_2) & \longmapsto & f_2(f_2^{-1}(a)), \end{array} \\ \tilde{K}_{I,a}^- : \begin{array}{ccc} D(\mathbb{R}_+, \mathbb{R}^2) & \longrightarrow & \mathbb{R} \\ f = (f_1, f_2) & \longmapsto & f_2(f_2^{-1}(a)^-). \end{array} \end{aligned} \tag{2.4.25}$$

Finally, with  $\Delta f_1(s) := f_1(s) - f_1(s^-)$ , define  $F^*$  by

$$F^* : \begin{array}{ccc} D(\mathbb{R}_+, \mathbb{R}^2) & \longrightarrow & \mathbb{R} \\ f = (f_1, f_2) & \longmapsto & \inf \left\{ s \in (0, f_2^{-1}(1)), \Delta f_1(s) = f_1^\natural(f_2^{-1}(1)^-) \right\}. \end{array}$$

We need this functional  $F^*$  for the characterization of the favorite site.

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**Lemma 2.4.4.**  *$J$  is continuous in the  $J_1$  topology.*

**Proof :** This fact is basic. However, we have not found a proof in the literature, so we give some details. To prove the continuity on  $D(\mathbb{R}_+, \mathbb{R})$ , we only have to prove it for every compact subset of  $\mathbb{R}_+$ , (see [75] Theorem 12.9.1). So let  $f \in D(\mathbb{R}_+, \mathbb{R})$  and  $T > 0$  at which  $f$  is continuous, let us prove that  $J_T$  defined by

$$J_T : \begin{array}{ccc} D([0, T], \mathbb{R}) & \longrightarrow & D([0, T], \mathbb{R}) \\ g & \longmapsto & g^\natural \end{array}$$

is continuous at the restriction  $f|_{[0, T]}$ . Let  $\varepsilon > 0$  and  $g \in D([0, T], \mathbb{R})$  such that  $d_T(f|_{[0, T]}, g) \leq \frac{\varepsilon}{2}$ .  $d_T$  is the usual metric  $d$  of the  $J_1$ -topology restricted to the interval  $[0, T]$ . By definition of  $d_T$  there exists a strictly increasing continuous mapping of  $[0, T]$  onto itself,  $e : [0, T] \longrightarrow [0, T]$  such that

$$\sup_{s \in [0, T]} |e(s) - s| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{s \in [0, T]} |g(e(s)) - f|_{[0, T]}(s)| \leq \frac{\varepsilon}{2}.$$

So for every  $s \in [0, T]$  we have

$$\begin{aligned} |\Delta g(e(s)) - \Delta f|_{[0, T]}(s)| &= |(g(e(s)) - g(e(s)-)) - (f|_{[0, T]}(s) - f|_{[0, T]}(s-))| \\ &\leq |g(e(s)) - f|_{[0, T]}(s)| + |g(e(s)-) - f|_{[0, T]}(s-)| \\ &\leq 2 \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

where  $\Delta h(s) = h(s) - h(s^-)$ . This implies  $d_T(J_T(f|_{[0, T]}), J_T(g)) \leq \varepsilon$ . □

**Lemma 2.4.5.** *Fix  $a > 0$ . The mappings  $J_{I, a}^-$ ,  $J_{I, a}$ ,  $K_{I, a}^-$ ,  $K_{I, a}$ ,  $\tilde{K}_{I, a}^-$  and  $\tilde{K}_{I, a}$  are continuous for  $J_1$ -topology at every couple  $(f^1, f^2) \in D(\mathbb{R}_+, \mathbb{R}^2)$  such that*

- i. For any  $\varepsilon > 0$ ,  $f^1$  and  $f^2$  have a finite number of jumps greater than  $\varepsilon$  on every compact subset of  $\mathbb{R}_+^*$ ,*
- ii.  $f^2$  is strictly increasing, with a limit equal to  $+\infty$ ,*
- iii.  $f^2(0) = 0$ ,*
- iv.  $f^2$  has a jump at  $I(f^2)(a)$  and  $f^2(I(f^2)(a)-) < a < f^2(I(f^2)(a))$ .*

**Proof :** This fact may also be known as we are looking at randomly stopped process, but once again we did not find what we need in the literature ([62], [75]).

Let  $(f_n^1, f_n^2)_n$  be a sequence of elements of  $D(\mathbb{R}_+, \mathbb{R})$  which converges to  $(f^1, f^2)$  for the  $J_1$  topology. To prove continuity, we prove that the sequence  $(J_{I, a}^-(f_n^1, f_n^2))_n$  converges to  $J_{I, a}^-(f^1, f^2)$ , and the equivalent for  $J_{I, a}$ .

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The first hypothesis guaranties that there exist neighborhoods of  $I(f^2)(a)$  for which  $f^1$  makes no jump greater than  $1/4$  times its higher previous jump, that is to say there exists  $\delta \in ]0, I(f^2)(a)[$  (notice that  $I(f^2)(a)$  exists thanks to (2) and is positive thanks to (3)) such that  $f^1$  makes no jump greater than  $J(f^1)(I(f^2)(a) - \delta)/4$  on  $[I(f^2)(a) - \delta, I(f^2)(a)[$  and on  $]I(f^2)(a), I(f^2)(a) + \delta]$ . Note also that  $J(f^1)$  is constant on  $[I(f^2)(a) - \delta, I(f^2)(a)[$  and on  $]I(f^2)(a), I(f^2)(a) + \delta]$ .

Also  $\delta$  can be made smaller (if needed) in such a way that  $I(f^2)(a) + \delta$  is a point of continuity of  $(f^1, f^2)$  and  $(f_n^1, f_n^2)_n$  for every  $n \in \mathbb{N}$ . By hypothesis  $d((f_n^1, f_n^2), (f^1, f^2)) \xrightarrow{n \rightarrow +\infty} 0$  so

$$d_n := d_{[0, I(f^2)(a) + \delta]}((f_n^1, f_n^2)|_{[0, I(f^2)(a) + \delta]}, (f^1, f^2)|_{[0, I(f^2)(a) + \delta]}) \xrightarrow{n \rightarrow +\infty} 0,$$

where  $|_{[0, I(f^2)(a) + \delta]}$  in index means restriction to  $[0, I(f^2)(a) + \delta]$ . Also by continuity of  $J$  (see Lemma 2.4.4) we also have  $d(J(f_n^1), J(f^1)) \xrightarrow{n \rightarrow +\infty} 0$  and therefore

$$d'_n := d_{[0, I(f^2)(a) + \delta]}((J(f_n^1))|_{[0, I(f^2)(a) + \delta]}, (J(f^1))|_{[0, I(f^2)(a) + \delta]}) \xrightarrow{n \rightarrow +\infty} 0.$$

Let  $h^-$  (respectively  $h^+$ ) be the largest jump of  $f^1$  just before (resp. just after)  $I(f^2)(a)$ . By definition of  $\delta$  we have

$$h^- = J(f^1)(I(f^2)(a) - \delta), h^+ = J(f^1)(I(f^2)(a) + \delta).$$

We have two cases, either  $J(f^1)$  is continuous at  $I(f^2)(a)$  or it makes a jump.

*Case  $J(f^1)$  makes a jump*, in this case the size of the jump is  $h^+ - h^- > 0$ .

Let  $\alpha = 8^{-1} \min(h^-, \delta, 1 - f^2(I(f^2)(a)^-), f^2(I(f^2)(a)) - 1)$ , and  $n_0 \in \mathbb{N}$  be such that for any  $n \geq n_0$ ,  $d_n < \alpha$  and  $d'_n < \alpha$ .  $T = I(f^2)(a) + \delta$ , there exist two homeomorphisms  $e_n, e'_n : [0, T] \rightarrow [0, T]$  such that :

- $\sup_{s \in [0, T]} |e_n(s) - s| \leq d_n$ ,
- $\sup_{s \in [0, T]} \left\| \left( (f_n^1(e_n(s)), f_n^2(e_n(s)))|_{[0, I(f^2)(a) + \delta]} - (f^1(s), f^2(s))|_{[0, I(f^2)(a) + \delta]} \right) \right\|_\infty \leq d_n$ .
- $\sup_{s \in [0, T]} |e'_n(s) - s| \leq d'_n$ ,
- $\sup_{s \in [0, T]} \left| (J(f_n^1))|_{[0, I(f^2)(a) + \delta]}(e'_n(s)) - (J(f^1))|_{[0, I(f^2)(a) + \delta]}(s) \right| \leq d'_n$ .

The second inequality implies that for any  $n \geq n_0$ ,

$$f_n^2(e_n(I(f^2)(a)^-)) < a < f_n^2(e_n(I(f^2)(a))),$$

so as we also have  $f_n^2(I(f_n^2)(a)^-) \leq a \leq f_n^2(I(f_n^2)(a))$  we get

$$I(f_n^2)(a) = e_n(I(f^2)(a)). \tag{2.4.26}$$

The fourth point implies that for any  $n \geq n_0$ ,

$$J(f_n^1)\left(e'_n\left(I(f^2)(a) - \frac{1}{2}\delta\right)\right) \geq J(f^1)\left(I(f^2)(a) - \frac{1}{2}\delta\right) - \alpha = h^- - \alpha > \frac{1}{2}h^-. \tag{2.4.27}$$



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The second point and the argument of the previous proof imply that for any  $n \geq n_0$ , each jump of  $f_n^1$  on  $[e_n(I(f^2)(a) - \delta), e_n(I(f^2)(a))]$  is  $2\alpha$ -close to a jump of  $f^1$  on  $[I(f^2)(a) - \delta, I(f^2)(a)]$ , but such jumps are less than  $h^-/4$  because of the definition of  $\delta$ . Thus,  $f_n^1$  makes no jump larger than  $h^-/2$  on the interval  $[e_n(I(f^2)(a) - \delta), e_n(I(f^2)(a))]$ . Moreover, the increases of  $e'_n$  and the first and third points imply that

$$e_n(I(f^2)(a) - \delta) \leq e'_n(I(f^2)(a) - \delta/2) \leq e_n(I(f^2)(a)).$$

So, combining this with (2.4.27), we get that  $J(f_n^1)$  is constant on the interval  $[e'_n(I(f^2)(a) - \delta/2), e_n(I(f^2)(a))]$ .

Now by definition of  $J_{I,a}^-$ , with (2.4.26) and then collecting what have just done above yields

$$\begin{aligned} \forall n \geq n_0, \quad J_{I,a}^-((f_n^1, f_n^2)) &= J(f_n^1)(I(f_n^2)(a)^-) = J(f_n^1)(e_n(I(f^2)(a))^-) \\ &= J(f_n^1)(e'_n(I(f^2)(a) - \delta/2)). \end{aligned} \quad (2.4.28)$$

From definition of  $J_{I,a}^-$  and the constantness of  $J(f^1)$  on  $[I(f^2)(a) - \delta, I(f^2)(a)]$  we also have

$$J_{I,a}^-(f^1, f^2) := J(f^1)(I(f^2)(a)^-) = J(f^1)(I(f^2)(a) - \delta/2). \quad (2.4.29)$$

Combining (2.4.28), (2.4.29) and the fourth point gives that, as  $n$  goes to infinity,  $J_{I,a}^-((f_n^1, f_n^2))$  converges to  $J_{I,a}^-((f^1, f^2))$ .

For  $J_{I,a}$ , we prove in a similar way as above that  $J(f_n^1)$  is constant on  $[e_n(I(f^2)(a)), e'_n(I(f^2)(a) + \delta/2)]$  so, as in (2.4.28) we have for  $n$  large enough

$$J_{I,a}((f_n^1, f_n^2)) = J(f_n^1)(e'_n(I(f^2)(a) + \delta/2)),$$

which, combined with the analogous of (2.4.29)

$$J_{I,a}(f^1, f^2) = J(f^1)(I(f^2)(a) + \delta/2)$$

allows us to conclude, using the fourth point, that  $J_{I,a}((f_n^1, f_n^2))$  converges to  $J_{I,a}((f^1, f^2))$  as  $n$  goes to infinity. Therefore, both  $J_{I,a}^-$  and  $J_{I,a}$  are continue at  $(f^1, f^2)$ . The continuity of the other functionals are proved similarly.  $\square$

**Lemma 2.4.6.** *For any  $(f^1, f^2)$  in  $D(\mathbb{R}_+, \mathbb{R}^2)$  that satisfy the hypothesis of lemma 2.4.5 and such that the sizes of the jumps of  $f^1$  are all distinct,  $F^*$  is continuous at  $(f^1, f^2)$  in the  $J_1$  topology.*

**Proof :** The proof follows mainly the steps of Lemma 2.4.5, we keep the same notation. The jump which takes place at the instant  $F^*(f^1, f^2)$  has value  $h^-$ . With the additional hypothesis that the values of the jumps for  $f^1$  are all different we have unicity for the value  $h^-$ . Let us define  $h'$ , the second highest jump  $f^1$  before

instant  $I(f^2)(1)$ . With the additional condition that  $\alpha < \frac{1}{8}(h^- - h')$  we have with the same arguments as in the proof of the continuity of  $J$  that for any  $n \geq n_0$ ,  $f_n^1$  effectuates at  $e_n(F^*(f^1, f^2))$  a jump larger than  $h^- - 2\alpha$ , and larger than all the other jumps of  $f_n^1$  before  $e_n(I(f^2)(1)-) = I(f_n^2)(1)$  which are smaller than  $h' + 2\alpha$ . So for  $n \geq n_0$ , the largest jump of  $f^1$  before  $I(f_n^2)(1)$  is obtained for  $e_n(F^*(f^1, f^2))$ , that is to say for any  $n \geq n_0$ ,

$$F^*((f_n^1, f_n^2)) = e_n(F^*(f^1, f^2)),$$

this implies  $F^*((f_n^1, f_n^2)) \xrightarrow{n \rightarrow \infty} F^*(f^1, f^2)$ .  $\square$

## 2.5 Supremum of the Local time - and other functionals

### 2.5.1 Supremum of the local time (proof of Theorem 2.1.3)

First, notice that since the diffusion  $X$  is almost surely transient to the right, the random variable  $\sup_{x < 0} \mathcal{L}(+\infty, x)$  is  $\mathbb{P}$ -almost surely finite. So almost surely,

$$\lim_{t \rightarrow +\infty} \sup_{x < 0} \mathcal{L}(t, x)/t = 0.$$

As a consequence, we only have to study the asymptotic behavior of  $\sup_{x \geq 0} \mathcal{L}(t, x)/t$  as  $t \rightarrow +\infty$ .

We start with the proof of the following proposition, which makes a link between the supremum of the local time and the process  $(Y_1, Y_2)^t$ .

**Proposition 2.5.1.** *Let  $\alpha > 0$ . For any  $\varepsilon > 0$  and large  $t$ ,*

$$\mathcal{P}_1^- - v(\varepsilon, t) \leq \mathbb{P} \left( \sup_{x \geq 0} \mathcal{L}(t, x)/t \leq \alpha \right) \leq \mathcal{P}_1^+ + v(\varepsilon, t),$$

where

$$\mathcal{P}_1^\pm := \mathbb{P} \left[ \left( 1 - \bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon} - 1} \right) \frac{\bar{\ell}_{\mathcal{N}_t^{2\varepsilon}} - \bar{\ell}_{\mathcal{N}_t^{2\varepsilon} - 1}}{(\bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}} - \bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon} - 1})} \leq \alpha_t^\pm, \max_{1 \leq j \leq \mathcal{N}_t^{2\varepsilon} - 1} \frac{\ell_j}{t} \leq \alpha_t^\pm \right],$$

and with  $\bar{\mathcal{H}}_k := Y_2^t(k e^{-\kappa\phi(t)}) = \frac{1}{t} \sum_{i=1}^k \mathcal{H}_i$ ,  $\bar{\ell}_k := Y_1^t(k e^{-\kappa\phi(t)}) = \frac{1}{t} \sum_{i=1}^k \ell_i$  for any  $k \in \mathbb{N}$ ,  $\mathcal{N}_t^{2\varepsilon} := \inf \{ m \geq 1, \bar{\mathcal{H}}_m > 1 - 2\varepsilon \}$ ,  $\alpha_t^\pm := \alpha(1 \pm (\log \log t)^{-1/2})$ , and  $v$  is a positive function such that  $\lim_{t \rightarrow +\infty} v(\varepsilon, t) \leq \text{const} \times \varepsilon^{\kappa \wedge (1 - \kappa)}$ .

The proof of this proposition relies on the three following lemmata. The first one deals with the local time at the  $h_t$ -minima for which the diffusion  $X$  already escaped before time  $t$ . The second deals with the local time at the last  $h_t$ -minimum  $m_{N_t}$  in the remaining time before time  $t$ . Finally the last one is a technical point.

**Lemma 2.5.2.** *For any large  $t > 0$ ,  $2 \leq k \leq n_t$ , any  $x > 0$  and  $\gamma > 0$  possibly depending on  $t$ , define the repartition function*

$$F_\gamma(x) := \mathbb{P} \left( \max_{1 \leq j \leq k-1} \mathcal{L}(H(\tilde{L}_j), \tilde{m}_j) \leq \gamma t, \sum_{i=1}^{k-1} U_i \leq xt \right).$$

Then for large  $t$ , for all  $2 \leq k \leq n_t$ ,  $x > 0$  and  $\gamma > 0$ ,

$$F_\gamma^-(x) - e^{-D_1 h t} \leq F_\gamma(x) \leq F_\gamma^+(x) + e^{-D_1 h t},$$

where  $F_\gamma^\pm(x) := \mathbb{P} \left( \max_{1 \leq j \leq k-1} \ell_j \leq \gamma_t^\pm t, \sum_{i=1}^{k-1} \mathcal{H}_i \leq x_t^\pm t \right)$  with  $\gamma_t^\pm := \gamma(1 \pm 2\varepsilon_t)$ ,  $x_t^\pm := x(1 \pm 2\varepsilon_t)$ ,  $\varepsilon_t$  and  $D_1$  are given in Proposition 2.3.5.

**Lemma 2.5.3.** *For any  $t > 0$ , define for every  $\gamma > 0$  and  $0 < x < 1$  possibly depending on  $t$ ,*

$$f_\gamma(x) := E \left[ \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \mathcal{L}_{X'}(t(1-x), \tilde{m}_1) \leq \gamma t, H'(\tilde{L}_1) > t(1-x), H'(\tilde{L}_1) < H'(\tilde{L}_1^-) \right) \right].$$

For such  $t$ ,  $\gamma$  and  $x$ , we also introduce

$$\begin{aligned} & \tilde{f}_\gamma(x) \\ & := E \left( \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \sup_{y \in \mathcal{D}_1} \mathcal{L}_{X'}[t(1-x), y] \leq \gamma t, H'(\tilde{L}_1) > t(1-x), H'(\tilde{L}_1) < H'(\tilde{L}_1^-) \right) \right), \end{aligned}$$

with  $\mathcal{D}_1$  defined in (2.3.13). Here  $X'$  is an independent copy of  $X$  starting at  $\tilde{m}_1$ , and the definition of  $H'$  for  $X'$  is the same as the definition of  $H$  for  $X$ . Let  $\varepsilon \in (0, 1/2)$ . There exists  $c_2 > 0$  such that for large  $t$ , for every  $x \in [\varepsilon, 1 - \varepsilon]$ ,

$$f_\gamma^-(x) - o(n_t^{-1}) \leq \tilde{f}_\gamma(x) \leq f_\gamma(x) \leq f_\gamma^+(x) + o(n_t^{-1}), \quad (2.5.1)$$

with  $f_\gamma^\pm(x) := \mathbb{P} \left( \frac{1}{R_1} \leq \frac{\gamma}{1-x}(1 \pm \varepsilon'_t), \mathcal{H}_1 > t(1-x)(1 \mp \varepsilon'_t) \right)$  and  $\varepsilon'_t = e^{-c_2 h t}$ .

**Lemma 2.5.4.** *For any  $0 < a < 1/4$ , we have for any  $t > 0$ ,*

$$\sum_{1 \leq k \leq n_t} \mathbb{P} [\bar{\mathcal{H}}_k > 1 - a/2, 1 - 2a < \bar{\mathcal{H}}_{k-1} \leq 1 - 3a/4] \leq s(a, t), \quad (2.5.2)$$

with  $s(a, t)$  such that  $\lim_{t \rightarrow +\infty} s(a, t) = \text{const} \times a^{1-\kappa}$ . For any  $\varepsilon \in (0, 1/2)$ ,

$$\forall t > 0, \quad \mathbb{P} [\varepsilon t \leq H(m_{N_t}) \leq (1 - \varepsilon)t] \geq 1 - \tilde{s}(\varepsilon, t), \quad (2.5.3)$$

with  $\tilde{s}(\varepsilon, t)$  such that  $\lim_{t \rightarrow +\infty} \tilde{s}(\varepsilon, t) = \text{const} \times \varepsilon^{(1-\kappa) \wedge \kappa}$ .

We postpone the proof of these lemmata after the proof of Proposition 2.5.1.

**Proof of Proposition 2.5.1 :** Recall from (2.1.3) that  $N_t$  is the largest index such that  $\sup_{s \leq t} X(s) \geq m_{N_t}$ . In particular,  $H(\tilde{L}_j) \leq H(\tilde{m}_{j+1}) = H(m_{j+1}) \leq H(m_{N_t}) \leq t$  for every  $1 \leq j \leq N_t - 1$  on  $\mathcal{V}_t \cap \{N_t \leq n_t\}$ . The main idea is to use the fact that the supremum of the local time at time  $t$  is achieved in the neighborhood of the  $h_t$ -minima  $m_i$ ,  $1 \leq i \leq n_t$ .

**We start with the upper bound.** Let  $\alpha > 0$  and  $0 < \varepsilon < 1/2$ . Notice that  $\mathbb{P}(N_t = 0, \mathcal{V}_t) \leq \mathbb{P}[H(\tilde{m}_1) > t] \leq C_2 v_t$  by (2.3.1). Using (2.3.1), (2.3.2), (2.3.3), (2.5.3), Lemma 2.2.1 and Remark 2.2.1, we have for  $t$  large enough,

$$\begin{aligned} & \mathbb{P}\left(\sup_{x \in \mathbb{R}} \mathcal{L}(t, x) \leq \alpha t\right) \leq E\left[\mathbb{P}^{W_\kappa}\left(\max_{1 \leq j \leq N_t} \mathcal{L}(t, m_j) \leq \alpha t\right)\right] \\ & \leq E\left[\mathbb{P}^{W_\kappa}\left(\max_{1 \leq j \leq N_t-1} \mathcal{L}[H(\tilde{L}_j), \tilde{m}_j] \leq \alpha t, \mathcal{L}(t, \tilde{m}_{N_t}) \leq \alpha t, \mathcal{Q}, \mathcal{B}_1(n_t), \mathcal{B}_2(n_t), \mathcal{V}_t\right)\right] \\ & \quad + \bar{s}(\varepsilon, t). \end{aligned} \quad (2.5.4)$$

with  $\mathcal{Q} := \{\varepsilon t \leq H(m_{N_t}) \leq (1-\varepsilon)t, 1 \leq N_t \leq n_t\}$  and  $\bar{s}$  satisfying  $\lim_{t \rightarrow +\infty} \bar{s}(\varepsilon, t) \leq C_{+\varepsilon}^{(1-\kappa) \wedge \kappa}$ . We will introduce in what follows different measures denoted by the letter  $\nu$ ; they depend on  $k$  but we do not write  $k$  as a subscript to simplify the notation. First, define two measures  $\nu_1^{W_\kappa}$  and  $\nu_2^{W_\kappa}$  on  $(0, 1)$  by, for every  $0 < y < 1$ ,

$$\begin{aligned} \nu_1^{W_\kappa}(y) & := \nu_1^{W_\kappa}([0, y]) \\ & := \mathbb{P}^{W_\kappa}\left(\max_{1 \leq j \leq k-1} \mathcal{L}[H(\tilde{L}_j), \tilde{m}_j] \leq \alpha t, H(\tilde{m}_k) - \sum_{i=1}^{k-1} U_i < \tilde{v}_t, H(\tilde{m}_k) \leq yt\right), \\ \nu_2^{W_\kappa}(y) & := \nu_2^{W_\kappa}([0, y]) \\ & := \mathbb{P}_{\tilde{m}_k}^{W_\kappa}\left(\mathcal{L}_{X'}[t(1-y), \tilde{m}_k] \leq \alpha t, H'(\tilde{m}_{k+1}) > t(1-y), \right. \\ & \quad \left. H'(\tilde{m}_{k+1}) < H'(\tilde{L}_k^-), H'(\tilde{m}_{k+1}) - H'(\tilde{L}_k) \leq \tilde{v}_t\right), \end{aligned}$$

with  $X'$  a diffusion starting from  $\tilde{m}_k$  independent of  $X$  (conditionally on  $W_\kappa$ ), and  $H'$  has the same definition as  $H$  (see (2.1.1)) but for  $X'$ . Partitioning on the values of  $N_t$ , and  $H(\tilde{m}_k)$ , we obtain by the strong markov property (applied at time  $H(\tilde{m}_k)$  under  $\mathbb{P}^{W_\kappa}$ ), that the probability  $E[\mathbb{P}^{W_\kappa}(\cdot)]$  in the line below (2.5.4) is smaller than

$$\sum_{1 \leq k \leq n_t} \int_\varepsilon^{1-\varepsilon} E(\nu_2^{W_\kappa}(x) d\nu_1^{W_\kappa}(x)) = \sum_{1 \leq k \leq n_t} E\left[\int_\varepsilon^{1-\varepsilon} \nu_2^{W_\kappa}(x) d\nu_1^{W_\kappa}(x)\right]. \quad (2.5.5)$$

The next step is to prove that the previous expectation can be approximated by a product of expectations. First notice that both  $y \rightarrow \nu_1^{W_\kappa}(y)$  and  $y \rightarrow \nu_2^{W_\kappa}(y)$  are

positive increasing. So integrating by parts

$$\begin{aligned}
 \int_{\varepsilon}^{1-\varepsilon} \nu_2^{W_\kappa}(x) d\nu_1^{W_\kappa}(x) &= [\nu_2^{W_\kappa}(x) \nu_1^{W_\kappa}(x)]_{\varepsilon}^{1-\varepsilon} - \int_{\varepsilon}^{1-\varepsilon} \nu_1^{W_\kappa}(x) d\nu_2^{W_\kappa}(x) \\
 &\leq [\nu_2^{W_\kappa}(x) \nu_1^{W_\kappa}(x)]_{\varepsilon}^{1-\varepsilon} - \int_{\varepsilon}^{1-\varepsilon} \tilde{\nu}_1^{W_\kappa}(x) d\nu_2^{W_\kappa}(x) \\
 &= \left[ \nu_2^{W_\kappa}(x) \left( \nu_1^{W_\kappa}(x) - \tilde{\nu}_1^{W_\kappa}(x) \right) \right]_{\varepsilon}^{1-\varepsilon} + \mathcal{I}_1, \tag{2.5.6}
 \end{aligned}$$

with  $\tilde{\nu}_1^{W_\kappa}(x) := \mathbb{P}^{W_\kappa} \left( \mathcal{G}_1, H(\tilde{m}_k) - \sum_{i=1}^{k-1} U_i < \tilde{v}_t, \sum_{i=1}^{k-1} U_i + \tilde{v}_t \leq xt \right) \leq \nu_1^{W_\kappa}(x)$  and  $\mathcal{G}_1 := \{ \max_{1 \leq j \leq k-1} \mathcal{L}(H(\tilde{L}_j), \tilde{m}_j) \leq \alpha t \}$  and

$$\begin{aligned}
 \mathcal{I}_1 &:= \int_{\varepsilon}^{1-\varepsilon} \nu_2^{W_\kappa}(x) d\tilde{\nu}_1(x) \leq \int_{\varepsilon}^{1-\varepsilon} \nu_2^{W_\kappa}(x) d\nu_3^{W_\kappa}(x) =: \mathcal{I}'_1, \\
 \nu_3^{W_\kappa}(x) &:= \mathbb{P}^{W_\kappa} \left( \mathcal{G}_1, \sum_{i=1}^{k-1} U_i + \tilde{v}_t \leq xt \right).
 \end{aligned}$$

**First**, we deal with what is going to be a negligible part, that is to say the first term in (2.5.6). As  $\nu_1^{W_\kappa}(x) \leq \mathbb{P}^{W_\kappa} \left( \mathcal{G}_1, H(\tilde{m}_k) - \sum_{i=1}^{k-1} U_i < \tilde{v}_t, \sum_{i=1}^{k-1} U_i \leq xt \right)$  because by definition  $\sum_{i=1}^{k-1} U_i < H(\tilde{m}_k)$ , we have, for  $\varepsilon < x < 1 - \varepsilon$ ,

$$\left| \nu_1^{W_\kappa}(x) - \tilde{\nu}_1^{W_\kappa}(x) \right| \leq \mathbb{P}^{W_\kappa} \left( xt - \tilde{v}_t < \sum_{i=1}^{k-1} U_i \leq xt \right) =: h_k(x).$$

so  $[\nu_2^{W_\kappa}(x) (\nu_1^{W_\kappa}(x) - \tilde{\nu}_1^{W_\kappa}(x))]_{\varepsilon}^{1-\varepsilon} \leq \nu_2^{W_\kappa}(1 - \varepsilon) h_k(1 - \varepsilon) + \nu_2^{W_\kappa}(\varepsilon) h_k(\varepsilon)$ . Notice that  $\sum_{i=1}^{k-1} U_i$  is measurable with respect to  $\sigma(X(s), 0 \leq s \leq H(\tilde{L}_{k-1}^-); W_\kappa(x), x \leq \tilde{L}_{k-1}^+)$ , since  $\tilde{L}_{k-1}^- \leq \tilde{L}_{k-1}^+$ , whereas the event in the definition of  $\nu_2^{W_\kappa}$  belongs to

$$\sigma(X'(s), 0 \leq s \leq \min(H(\tilde{L}_k^-), H(\tilde{m}_{k+1})); W_\kappa(x) - W_\kappa(\tilde{m}_k), \tilde{L}_{k-1}^+ \leq x \leq \tilde{L}_{k+1}^+),$$

with  $X'$  an independent copy of  $X$  starting at  $\tilde{m}_k$ .

So independence of  $X$  and  $X'$ , and independence of the two portions of the environment involved (see Lemma 2.2.1) imply independence between  $\nu_2^{W_\kappa}$  and  $h_k$ . Hence,

$$\begin{aligned}
 &E \left( \left[ \nu_2^{W_\kappa}(x) \left( \nu_1^{W_\kappa}(x) - \tilde{\nu}_1^{W_\kappa}(x) \right) \right]_{\varepsilon}^{1-\varepsilon} \right) \\
 &\leq E \left[ \nu_2^{W_\kappa}(1 - \varepsilon) \right] E[h_k(1 - \varepsilon)] + E \left[ \nu_2^{W_\kappa}(\varepsilon) \right] E[h_k(\varepsilon)] \\
 &= E \left[ \tilde{\nu}_2^{W_\kappa}(1 - \varepsilon) \right] E[h_k(1 - \varepsilon)] + E \left[ \tilde{\nu}_2^{W_\kappa}(\varepsilon) \right] E[h_k(\varepsilon)]. \tag{2.5.7}
 \end{aligned}$$

with for any  $x$ ,

$$\begin{aligned}
 \tilde{\nu}_2^{W_\kappa}(x) &:= \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \mathcal{L}_X[t(1-x), \tilde{m}_1] \leq \alpha t, H(\tilde{m}_2) > t(1-x), \right. \\
 &\quad \left. H(\tilde{m}_2) < H(\tilde{L}_1^-), H(\tilde{m}_2) - H(\tilde{L}_1) \leq \tilde{v}_t \right).
 \end{aligned}$$

As  $E(\tilde{\nu}_2^{W_\kappa}(x)) \leq \mathbb{P}[U_1 > t(1-x) - \tilde{v}_t]$  and for every small  $\varepsilon > 0$  and  $t$  large enough  $h_k(x) \leq \mathbb{P}^{W_\kappa}\left((x-\varepsilon)t < \sum_{i=1}^{k-1} U_i \leq xt\right)$  we can apply Proposition 2.3.5, we get

$$\begin{aligned} & E[h_k(1-\varepsilon)]E[\tilde{\nu}_2^{W_\kappa}(1-\varepsilon)] \\ & \leq \mathbb{P}\left(\frac{1-2\varepsilon}{1+\varepsilon_t} < \sum_{i=1}^{k-1} \frac{\mathcal{H}_i}{t} \leq \frac{1-\varepsilon}{1-\varepsilon_t}\right) \mathbb{P}\left(\mathcal{H}_1 > \frac{t\varepsilon - \tilde{v}_t}{1+\varepsilon_t}\right) + 3e^{-D_1 h_t}. \end{aligned}$$

By (2.4.2) and the first part of Lemma 2.6.2, for any  $0 < a < 1$  and  $b > 0$ ,

$$\begin{aligned} \lim_{t \rightarrow +\infty} \sum_{1 \leq k \leq n_t} \mathbb{P}\left(1-a < \sum_{i=1}^{k-1} \frac{\mathcal{H}_i}{t} \leq 1\right) \mathbb{P}(\mathcal{H}_1 > tb) &= \frac{\text{const}}{b^\kappa} \int_{1-a}^1 y^{\kappa-1} dy \\ &\leq \frac{\text{const}}{b^\kappa} (1 - (1-a)^\kappa). \end{aligned} \quad (2.5.8)$$

Therefore, we obtain

$$\sum_{1 \leq k \leq n_t} E[\tilde{\nu}_2^{W_\kappa}(1-\varepsilon)]E[h_k(1-\varepsilon)] \leq C_+ \cdot u(t, \varepsilon)$$

with  $u$  a positive function such that  $\lim_{t \rightarrow +\infty} u(t, \varepsilon) = \max(\varepsilon^{1-\kappa}, \varepsilon^\kappa)$ . A similar argument also works for the second term in (2.5.7), which yields

$$\sum_{1 \leq k \leq n_t} E\left[\left[\nu_2^{W_\kappa}(x) \left(\nu_1^{W_\kappa}(x) - \tilde{\nu}_1^{W_\kappa}(x)\right)\right]_\varepsilon^{1-\varepsilon}\right] \leq 2C_+ \cdot u(t, \varepsilon). \quad (2.5.9)$$

**We now deal with  $\mathcal{I}'_1$ .** By independence between  $X$  and  $X'$ , and the independent parts of the potential  $W_\kappa$  involved in  $\nu_2^{W_\kappa}(x)$  and  $\nu_3^{W_\kappa}(x)$ ,

$$E(\mathcal{I}'_1) = \int_\varepsilon^{1-\varepsilon} \nu_2(x) d\nu_3(x), \quad (2.5.10)$$

with  $\nu_2(x) := E(\nu_2^{W_\kappa}(x)) = E(\tilde{\nu}_2^{W_\kappa}(x))$  and  $\nu_3(x) := E(\nu_3^{W_\kappa}(x))$ .

By the lower bound in Lemma 2.5.2, we have  $\nu_3(x) = F_\alpha(x - \tilde{v}_t/t) \geq F_\alpha^-(x - \tilde{v}_t/t) - e^{-D_1 h_t}$  for every  $x > \varepsilon$  for large  $t$ . So, again since  $y \rightarrow \nu_2(y)$  is positive increasing and  $\nu_3$  is a repartition function, integrating by parts twice as in (2.5.6) gives with the change of variables  $u = x - \tilde{v}_t/t$ ,

$$\begin{aligned} \int_\varepsilon^{1-\varepsilon} \nu_2(x) d\nu_3(x) &\leq \int_{\varepsilon - \tilde{v}_t/t}^{1-\varepsilon - \tilde{v}_t/t} \nu_2(x + \tilde{v}_t/t) dF_\alpha^-(x) + e^{-D_1 h_t} \\ &\quad + \left[ \left( F_\alpha(x) - F_\alpha^-(x) \right) \nu_2(x + \tilde{v}_t/t) \right]_{\varepsilon - \tilde{v}_t/t}^{1-\varepsilon - \tilde{v}_t/t}. \end{aligned} \quad (2.5.11)$$

Recall (see before Lemma 2.3.1) that  $\tilde{v}_t/t = 2/\log(h_t) = o(1)$  as  $t \rightarrow +\infty$ . Then we can prove in a similar way we have obtained (2.5.9) that :

$$\sum_{1 \leq k \leq n_t} \left[ \left( F_\alpha(x) - F_\alpha^-(x) \right) \nu_2(x + \tilde{v}_t/t) \right]_{\varepsilon - \tilde{v}_t/t}^{1 - \varepsilon - \tilde{v}_t/t} \leq C_+ \cdot u(t, \varepsilon), \quad (2.5.12)$$

with as usual a possibly enlarged  $C_+$ . Indeed by Lemma 2.5.2,  $-(F_\alpha(\varepsilon - \tilde{v}_t/t) - F_\alpha^-(\varepsilon - \tilde{v}_t/t))\nu_2(\varepsilon) \leq e^{-D_1 t} = o(n_t^{-1})$  for every  $1 \leq k \leq n_t$  for large  $t$ , and  $(F_\alpha - F_\alpha^-)(1 - \varepsilon - \tilde{v}_t/t) \leq (F_\alpha^+ - F_\alpha^-)(1 - \varepsilon - \tilde{v}_t/t) + e^{-D_1 t} \leq \mathbb{P}(\max_{1 \leq j \leq k-1} \ell_j \in [\gamma_t^- t, \gamma_t^+ t]) + \mathbb{P}(\sum_{i=1}^{k-1} \mathcal{H}_i \in [x_t^- t, x_t^+ t]) + e^{-D_1 t}$  for every  $k \leq n_t$  for large  $t$ , with  $\gamma = \alpha$  and  $x = 1 - \varepsilon - \tilde{v}_t/t$ . The first probability is less than  $n_t \mathbb{P}(S_1 \mathbf{e}_1 \in [\gamma_t^- t, \gamma_t^+ t]) = n_t \mathbb{E}(\int_{\gamma_t(1-2\varepsilon_t)/S_1}^{\gamma_t(1+2\varepsilon_t)/S_1} e^{-u/2} du / 2) \leq 8n_t \varepsilon_t \sup_{v \geq 0} (v e^{-v}) = o(1/n_t)$ , whereas the second one is treated as (2.5.8), which leads to (2.5.12).

So the important term in the right hand side of inequality (2.5.11) comes from the integral. We now work on  $\nu_2(x)$ . We have,

$$\begin{aligned} \nu_2(x) &\leq E(\mathbb{P}_{\tilde{m}_1}^{W_\kappa} [\mathcal{L}_X(t(1-x), \tilde{m}_1) \leq \alpha t, H(\tilde{L}_1) > t(1-x) - \tilde{v}_t, H(\tilde{L}_1) < H(\tilde{L}_1^-)]) \\ &\leq E(\mathbb{P}_{\tilde{m}_1}^{W_\kappa} [\mathcal{L}_X(t(1-x) - \tilde{v}_t, \tilde{m}_1) \leq \alpha t, \\ &\quad H(\tilde{L}_1) > t(1-x) - \tilde{v}_t, H(\tilde{L}_1) < H(\tilde{L}_1^-)]) = f_\alpha(x + \tilde{v}_t/t), \end{aligned}$$

as defined in Lemma 2.5.3. Then, as  $F_\alpha^-(x)$  is positive and increasing in  $x$ , using Lemma 2.5.3 with  $\gamma = \alpha$ , we obtain

$$\int_{\varepsilon - \tilde{v}_t/t}^{1 - \varepsilon - \tilde{v}_t/t} \nu_2(x + \tilde{v}_t/t) dF_\alpha^-(x) \leq \int_{\varepsilon - \tilde{v}_t/t}^{1 - \varepsilon - \tilde{v}_t/t} f_\alpha^+(x + 2\tilde{v}_t/t) dF_\alpha^-(x) + o(n_t^{-1}). \quad (2.5.13)$$

Now, as  $f_\alpha^+(x + 2\tilde{v}_t/t)$  can be written (since  $\mathcal{H}_k = \ell_k R_k$ , see Proposition 2.3.5),

$$f_\alpha^+(x + 2\tilde{v}_t/t) = \mathbb{P} \left( (1 - x - 2\tilde{v}_t/t) \frac{\ell_k}{\mathcal{H}_k} \leq \alpha(1 + \varepsilon'_t), \mathcal{H}_k > t(1 - x - 2\tilde{v}_t/t)(1 - \varepsilon'_t) \right),$$

we get by independence of the random variables  $((\ell_j, \mathcal{H}_j), j \leq n_t)$ ,

$$\begin{aligned} &\int_{\varepsilon - \tilde{v}_t/t}^{1 - \varepsilon - \tilde{v}_t/t} f_\alpha^+(x + 2\tilde{v}_t/t) dF_\alpha^-(x) \\ &\leq \mathbb{P} \left[ (1 - \bar{\mathcal{H}}_{k-1}) \frac{\bar{\ell}_k - \bar{\ell}_{k-1}}{\bar{\mathcal{H}}_k - \bar{\mathcal{H}}_{k-1}} \leq \alpha + \tilde{\varepsilon}_t(k), \bar{\mathcal{H}}_k \geq 1 - \delta'_t, \right. \\ &\quad \left. \max_{1 \leq j \leq k-1} \frac{\ell_j}{t} \leq \alpha, \bar{\mathcal{H}}_{k-1} \leq 1 - \varepsilon + \delta'_t \right], \quad (2.5.14) \end{aligned}$$

with  $\delta'_t := 3\tilde{v}_t/t$ ,  $\tilde{\varepsilon}_t(k) := (\alpha + \ell_k/\mathcal{H}_k) \delta'_t$ .

The idea now is to make appear the event  $\{\mathcal{N}_t^{2\varepsilon} = k\}$  in the above probability (recall

the definition of  $\mathcal{N}_t^{2\varepsilon}$  given in Proposition 2.5.1) and then sum over  $k$ .

We first prove that the sum over  $k \leq n_t$ , of the above probability is small if we intersect its event with the event  $\{\mathcal{N}_t^{2\varepsilon} \neq k\}$ . In other words, let us prove that

$$\sum_1 := \sum_{1 \leq k \leq n_t} \mathbb{P} [\bar{\mathcal{H}}_k \geq 1 - \delta'_t, \bar{\mathcal{H}}_{k-1} \leq 1 - \varepsilon + \delta'_t, \mathcal{N}_t^{2\varepsilon} \neq k] \quad (2.5.15)$$

is small. As  $\{\mathcal{N}_t^{2\varepsilon} \neq k\} = \{\bar{\mathcal{H}}_k \leq 1 - 2\varepsilon\} \cup \{\bar{\mathcal{H}}_{k-1} > 1 - 2\varepsilon\}$ , and since for  $t$  large enough,  $\{\bar{\mathcal{H}}_k \geq 1 - \delta'_t\} \cap \{\bar{\mathcal{H}}_k \leq 1 - 2\varepsilon\} = \emptyset$ , we have

$$\sum_1 \leq \sum_{1 \leq k \leq n_t} \mathbb{P} [\bar{\mathcal{H}}_k \geq 1 - \delta'_t, 1 - 2\varepsilon < \bar{\mathcal{H}}_{k-1} \leq 1 - \varepsilon + \delta'_t].$$

Therefore, for  $t$  large enough, with  $s(\varepsilon, t)$  defined in Lemma 2.5.4,

$$\sum_1 \leq \sum_{1 \leq k \leq n_t} \mathbb{P} [\bar{\mathcal{H}}_k > 1 - \varepsilon/2, 1 - 2\varepsilon < \bar{\mathcal{H}}_{k-1} \leq 1 - 3\varepsilon/4] \leq s(\varepsilon, t) \quad (2.5.16)$$

by (2.5.2). Finally, combining equations from (2.5.10) to (2.5.16) leads to

$$\begin{aligned} & \sum_{1 \leq k \leq n_t} E(\mathcal{I}_1) \\ & \leq \mathbb{P} \left[ (1 - \bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon-1}}) \frac{\bar{\ell}_{\mathcal{N}_t^{2\varepsilon}} - \bar{\ell}_{\mathcal{N}_t^{2\varepsilon-1}}}{\bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}} - \bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon-1}}} \leq \alpha + \tilde{\varepsilon}_t(\mathcal{N}_t^{2\varepsilon}), \max_{1 \leq j \leq \mathcal{N}_t^{2\varepsilon-1}} \frac{\ell_j}{t} \leq \alpha \right] \\ & \quad + s(\varepsilon, t) + C_+ u(t, \varepsilon) + o(1). \end{aligned} \quad (2.5.17)$$

To finish we have to deal with  $\tilde{\varepsilon}_t(\mathcal{N}_t^{2\varepsilon})$ , a basic computation partitioning on the values of  $\mathcal{N}_t^{2\varepsilon}$ , shows that  $\mathbb{P}[\tilde{\varepsilon}_t(\mathcal{N}_t^{2\varepsilon}) \geq \alpha\sqrt{\delta'_t}/6] \leq C_+ \mathbb{P}(R_1 \leq \sqrt{\delta'_t}) = o(1)$  as  $R_1$  converges in distribution to  $\mathcal{R}_\kappa$  which is almost surely positive. Collecting this last fact, (2.5.4), (2.5.5), (2.5.6), (2.5.9) and (2.5.17) finish the proof of the upper bound.

### **Proof of the lower bound :**

The proof here follows the same line as the upper bound. The main difference comes from the fact that we can no longer use the inequality  $\sup_{x \in \mathbb{R}} \mathcal{L}(t, x) \geq \sup_{1 \leq j \leq N_t} \mathcal{L}(t, m_j)$ . So for this part of the proof we stress on what is different from the upper bound, and refer to the previous computations when very few changes occur.

Assume for the moment that

$$\mathbb{P} \left( \left\{ \sup_{x \in \mathbb{R}} \mathcal{L}(t, x) \geq 2\tilde{w}_t \right\} =: \mathcal{E}_2 \right) \geq 1 - o(1), \quad (2.5.18)$$

with  $\tilde{w}_t := t e^{(\kappa(1+3\delta)-1)\phi(t)}$ , and recall that  $\delta$  is chosen small enough such that  $\kappa(1+3\delta) < 1$  (see Lemma 2.3.2). This fact (2.5.18) is a direct consequence of the upper-bound of  $\mathbb{P}(\sup_{x \in \mathbb{R}} \mathcal{L}(t, x) \leq \alpha t)$  (see at the beginning of the proof of Theorem 2.1.3



for a proof of (2.5.18)). Recall (2.3.13), and define for any  $\ell \geq 1$ ,

$$\begin{aligned} \mathcal{E}_3(\ell) &:= \mathcal{E}_3^1(\ell) \cap \mathcal{E}_3^2(\ell), \quad \text{with} \\ \mathcal{E}_3^1(\ell) &:= \bigcap_{j=1}^{\ell-1} \left\{ \sup_{x \in \mathcal{D}_j} \left[ \mathcal{L}(H(\tilde{L}_j), x) - \mathcal{L}(H(\tilde{m}_j), x) \right] \leq t\tilde{\alpha}_t \right\}, \\ \mathcal{E}_3^2(\ell) &:= \left\{ \sup_{x \in \mathcal{D}_\ell} \left[ \mathcal{L}(t, x) - \mathcal{L}(H(\tilde{m}_\ell), x) \right] \leq t\tilde{\alpha}_t \right\}, \end{aligned}$$

with  $\tilde{\alpha}_t := (\alpha t - 2\tilde{w}_t)/t$ . Recall the definitions of the events  $\mathcal{B}_i$ ,  $1 \leq i \leq 4$  in Sections 2.3.1 and 2.3.2. We have for large  $t$ ,

$$\begin{aligned} & \left\{ \sup_{x \in \mathbb{R}_+} \mathcal{L}(t, x) \leq \alpha t \right\} \cap \mathcal{V}_t \cap \mathcal{E}_2 \cap \{N_t \leq n_t\} \cap \bigcap_{i=1}^4 \mathcal{B}_i(n_t) \\ & \supseteq \mathcal{E}_3(N_t) \cap \mathcal{V}_t \cap \mathcal{E}_2 \cap \{N_t \leq n_t\} \cap \bigcap_{i=1}^4 \mathcal{B}_i(n_t). \end{aligned}$$

Indeed,  $\mathcal{L}(t, x) \leq \tilde{w}_t$  for every  $x \in (\mathbb{R}_+ - \bigcup_{j=1}^{n_t} [\tilde{L}_j^-, \tilde{L}_j])$  on  $\mathcal{B}_2(n_t) \cap \mathcal{B}_3(n_t) \cap \mathcal{V}_t \cap \{N_t \leq n_t\}$ , and on the same event intersected with  $\mathcal{B}_4(n_t)$ ,  $\mathcal{L}(t, x) \leq \tilde{w}_t + te^{-2\phi(t)} < 2\tilde{w}_t$  for every  $x \in \bigcup_{j=1}^{n_t} ([\tilde{L}_j^-, \tilde{L}_j] \cap \overline{\mathcal{D}_j})$ , whereas for  $x \in \mathcal{D}_j$ ,  $\mathcal{L}(H(\tilde{m}_j), x) \leq \tilde{w}_t$  if  $j \leq n_t$  and  $\mathcal{L}(t, x) - \mathcal{L}(H(\tilde{L}_j), x) \leq \tilde{w}_t$  if  $j < N_t$ . Notice that by Lemmata 2.2.1, 2.3.1, 2.3.2, 2.3.4 and the above assumption (2.5.18),

$$\mathbb{P}(\mathcal{V}_t \cap \mathcal{E}_2 \cap \{N_t \leq n_t\} \cap \bigcap_{i=1}^4 \mathcal{B}_i(n_t)) \geq 1 - o(1).$$

We now deal with  $\mathbb{P}(\mathcal{E}_3(N_t) \cap \mathcal{B}_1(N_t) \cap \mathcal{B}_2(n_t) \cap \mathcal{V}_t \cap \{N_t \leq n_t\})$ . Using Lemma 2.2.1, the fact that  $H(\tilde{L}_k) \leq H(\tilde{m}_{k+1})$  and the strong Markov property with respect to  $\mathbb{P}^{W_\kappa}$ , we obtain

$$\begin{aligned} & \mathbb{P}(\mathcal{E}_3(N_t) \cap \mathcal{B}_1(N_t) \cap \mathcal{B}_2(n_t) \cap \mathcal{V}_t \cap \mathcal{Q}) \\ & \geq \sum_{k=1}^{n_t} E \left( \int_{\varepsilon}^{1-\varepsilon} \nu_4^{W_\kappa}(y) \mathbb{P}^{W_\kappa}(\mathcal{E}_3^1(k), \mathcal{B}_1(k), \mathcal{B}_2(k-1), H(\tilde{m}_k)/t \in dy) \right) - o(1) \end{aligned}$$

with

$$\begin{aligned} & \nu_4^{W_\kappa}(y) \\ & := \mathbb{P}_{\tilde{m}_k}^{W_\kappa} \left( \sup_{x \in \mathcal{D}_k} \mathcal{L}_X(t(1-y), x) \leq t\tilde{\alpha}_t, H(\tilde{L}_k) > t(1-y), H(\tilde{L}_k) < H(\tilde{L}_k^-) \right), \end{aligned}$$

Now, by computations similar to the ones giving the upper bounds in (2.5.9) and (2.5.10), we have

$$\begin{aligned} & \mathbb{P}(\mathcal{E}_3(N_t) \cap \mathcal{B}_1(N_t) \cap \mathcal{B}_2(n_t) \cap \mathcal{V}_t, \mathcal{Q}) \\ & \geq \sum_{k=1}^{n_t} \int_{\varepsilon}^{1-\varepsilon} E(\nu_4^{W_\kappa}(y) d\nu_5^{W_\kappa}(y)) - o(1) = \sum_{k=1}^{n_t} \int_{\varepsilon}^{1-\varepsilon} \nu_4(y) d\nu_5(y) - o(1). \end{aligned}$$

with  $\nu_5^{W_\kappa}(y) := \mathbb{P}^{W_\kappa} \left( \mathcal{E}_3^1(k), \mathcal{B}_1(k), \mathcal{B}_2(k-1), \sum_{i=1}^{k-1} U_i/t \leq y \right)$ ,  $\nu_4(y) := E(\nu_4^{W_\kappa}(y))$  and  $\nu_5(y) := E(\nu_5^{W_\kappa}(y))$ . The next step is to remove  $\mathcal{B}_1(k)$  in the above expression. For that, we only have to prove that

$$\sum_{k=1}^{n_t} \int_{\varepsilon}^{1-\varepsilon} E \left( \nu_4^{W_\kappa}(y) \mathbb{P}^{W_\kappa} \left( \mathcal{E}_3^1(k), \bar{\mathcal{B}}_1(k), \mathcal{B}_2(k-1), \sum_{i=1}^{k-1} U_i/t \in dy \right) \right)$$

is negligible, one can check that this quantity is smaller than

$$\begin{aligned} & \sum_{k=1}^{n_t} \int_{\varepsilon}^{1-\varepsilon} E \left[ \mathbb{P}_{\tilde{m}_k}^{W_\kappa} \left( H(\tilde{L}_k) < H(\tilde{L}_k^-), H(\tilde{L}_k) > t(1-y) \right) \right] \\ & \quad \mathbb{P} \left( \bar{\mathcal{B}}_1(k), \mathcal{B}_2(k-1), \sum_{i=1}^{k-1} U_i/t \in dy \right) \\ & \leq \sum_{k=1}^{n_t} \mathbb{P} \left( \sum_{i=1}^{k-1} U_i/t \leq 1, \sum_{i=1}^k U_i/t > 1, \bar{\mathcal{B}}_1(k) \right) \\ & \leq \mathbb{P}(\bar{\mathcal{B}}_1(n_t)) \leq C_2 v_t = o(1), \end{aligned}$$

where the last inequality comes from (2.3.1). Therefore, collecting the above computations yields

$$\mathbb{P} \left( \sup_{x \in \mathbb{R}} \mathcal{L}(t, x) \leq \alpha \right) \geq \sum_{k=1}^{n_t} \int_{\varepsilon}^{1-\varepsilon} \nu_4(y) d\tilde{\nu}_5(y) - o(1),$$

with  $\tilde{\nu}_5(y) := \mathbb{P} \left( \mathcal{E}_3^1(k), \mathcal{B}_2(k-1), \sum_{i=1}^{k-1} U_i/t \leq y \right)$ .

We start with an estimation of the repartition function  $\tilde{\nu}_5(y)$ . Recall that like in the proof of Lemma 2.3.6, by the strong Markov property, the occupation time formula (2.1.13) and (2.1.14) the sequence  $(U_j, \{\mathcal{L}(H(\tilde{L}_j), x) - \mathcal{L}(H(\tilde{m}_j), x), x \in \mathcal{D}_j\}, j \leq n_t)$  under  $\mathcal{B}_2(n_t)$  is equal to a sequence  $(H_j(\tilde{L}_j), \{\mathcal{L}_j(H_j(\tilde{L}_j), x), x \in \mathcal{D}_j\}, j \leq n_t)$ , with this time

$$\begin{aligned} H_j(\tilde{L}_j) & := A^j(\tilde{L}_j) \int_{\tilde{L}_j^-}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} \mathcal{L}_{B^j}[\tau^{B^j}(1), A^j(u)/A^j(\tilde{L}_j)] du, \\ \mathcal{L}_j(H_j(\tilde{L}_j), x) & := A^j(\tilde{L}_j) e^{-\tilde{V}^{(j)}(x)} \mathcal{L}_{B^j}[\tau^{B^j}(1), A^j(x)/A^j(\tilde{L}_j)], \end{aligned}$$

where  $A^j(u) = \int_{\tilde{m}_j}^u e^{\tilde{V}^{(j)}(x)} dx$ . Using Remark 2.2.1, Lemma 2.2.1, Fact 2.2.1 (ii), and then (2.7.5) and (2.7.6), we have for large  $t$  for any  $1 \leq j \leq n_t$  since  $\phi(t) = o(\log t)$ ,

$$\begin{aligned} P[\tilde{\tau}_j(\kappa r_t/8) \leq \tilde{m}_j + r_t \leq \tilde{\tau}_j(r_t)] & \geq 1 - C_+ e^{-(c_-)r_t}, \\ P[\tilde{\tau}_j^-(r_t) \leq \tilde{m}_j - r_t \leq \tilde{\tau}_j^-(\kappa r_t/8)] & \geq 1 - C_+ e^{-(c_-)r_t}. \end{aligned} \tag{2.5.19}$$

with  $c_- > 0$ . Therefore for any  $j$ ,  $P(\mathcal{D}_j \subset [\tilde{\tau}_j^-(r_t), \tilde{\tau}_j(r_t)]) \geq 1 - 2C_+e^{-(c_-)r_t}$ . Then on  $\{\mathcal{D}_j \subset [\tilde{\tau}_j^-(r_t), \tilde{\tau}_j(r_t)]\}$ , for any  $x \in \mathcal{D}_j$ ,

$$\mathcal{L}_j(H_j(\tilde{L}_j), x) \leq A^j(\tilde{L}_j)\mathcal{L}_{B^j}[\tau^{B^j}(1), A^j(x)/A^j(\tilde{L}_j)].$$

Also with probability  $\geq 1 - 2C_+e^{-(c_-)r_t}$ ,  $\mathcal{D}_j \subset [\tilde{\tau}_j^-(r_t), \tilde{\tau}_j(r_t)]$  so for any  $x \in \mathcal{D}_j$ ,

$$A^j(\tilde{\tau}_j^-(r_t)) \leq A^j(x) \leq A^j(\tilde{\tau}_j(r_t)). \quad (2.5.20)$$

With Remark 2.2.1, Lemma 2.2.1, Fact 2.2.1 and (2.7.8), we obtain with a probability larger than  $1 - e^{-(c_-)r_t}$ ,

$$\begin{aligned} -e^{-h_t/4} &\leq -e^{2r_t}e^{-(1-1/2)h_t} \leq \frac{A^j(\tilde{\tau}_j^-(r_t))}{A^j(\tilde{L}_j)} \leq \frac{A^j(\tilde{\tau}_j(r_t))}{A^j(\tilde{L}_j)} \\ &\leq e^{2r_t}e^{-(1-1/2)h_t} \leq e^{-h_t/4}. \end{aligned} \quad (2.5.21)$$

Therefore, applying (2.7.11) (with  $\delta = e^{-h_t/4}$  and  $\varepsilon = \delta^{1/3}$ ), we obtain with a probability larger than  $1 - e^{-(c_-)r_t}$ ,

$$\sup_{x \in \mathcal{D}_j} A^j(\tilde{L}_j)\mathcal{L}_{B^j}(\tau^{B^j}(1), A^j(x)/A^j(\tilde{L}_j)) \leq A^j(\tilde{L}_j)\mathcal{L}_{B^j}(\tau^{B^j}(1), 0)(1 + e^{-h_t/12}). \quad (2.5.22)$$

Collecting the different estimates we then obtain,

$$\tilde{\nu}_5(y) \geq \mathbb{P} \left( \max_{1 \leq j \leq k-1} \mathcal{L}_j(H_j(\tilde{L}_j), \tilde{m}_j) \leq t\bar{\alpha}_t, \sum_{j=1}^{k-1} \frac{H_j(\tilde{L}_j)}{t} \leq y \right) - C_+e^{-(c_-)r_t},$$

with  $\bar{\alpha}_t := \tilde{\alpha}_t(1 + e^{-h_t/12})^{-1}$ . We can then inverse the equality in law we have used above, and then obtain

$$\tilde{\nu}_5(y) \geq F_{\bar{\alpha}_t}(y) - C_+e^{-(c_-)r_t},$$

with  $F_{\bar{\alpha}_t}$  defined in Lemma 2.5.2. Then we can follow the same lines as for the upper bound (especially computations after (2.5.9)), and obtain via Lemma 2.5.2 and by choosing  $C_0$  large enough in such a way that  $(c_-)r_t/\phi(t) = (c_-)C_0 > \kappa(1 + \delta)$  :

$$\int_{\varepsilon}^{1-\varepsilon} \nu_4(y) d\tilde{\nu}_5(y) \geq \int_{\varepsilon}^{1-\varepsilon} \nu_4(y) dF_{\bar{\alpha}_t}^+(y) - o(n_t^{-1}).$$

Remark also that (2.5.22) implies the concentration of the local time at the  $h_t$ -minima : with probability larger than  $1 - C_+e^{-(c_-)r_t}$ ,

$$\left| \sup_{y \in \mathcal{D}_j} \mathcal{L}_j(H_j(\tilde{L}_j), y) - \mathcal{L}_j(H_j(\tilde{L}_j), \tilde{m}_j) \right| \leq e^{-h_t/12} \mathcal{L}_j(H_j(\tilde{L}_j), \tilde{m}_j). \quad (2.5.23)$$

We now work on  $\nu_4(y)$ . By the second part of Lemma 2.2.1 it is equal to

$$E \left( \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \sup_{z \in \mathcal{D}_1} \mathcal{L}_{X'}(t(1-y), z) \leq t\tilde{\alpha}_t, H'(\tilde{L}_1) > t(1-y), H'(\tilde{L}_1) < H'(\tilde{L}_1^-) \right) \right) \\ =: \tilde{\nu}_4(y),$$

and by Lemma 2.5.3,  $\tilde{\nu}_4(y) \geq f_{\tilde{\alpha}_t}^-(y) - o(n_t^{-1})$ . Therefore

$$\int_\varepsilon^{1-\varepsilon} \nu_4(y) d\tilde{\nu}_5(y) \geq \int_\varepsilon^{1-\varepsilon} f_{\tilde{\alpha}_t}^-(y) dF_{\tilde{\alpha}_t}^+(y) - o(n_t^{-1}).$$

From now on, the computations are very close from that of the upper bound (see (2.5.13) and below) and we do not give more details.  $\square$

### Proof of Lemmata 2.5.2, 2.5.3 and 2.5.4.

**Proof of Lemma 2.5.2 :** This is a direct consequence of Proposition 2.3.5.  $\square$

**Proof of Lemma 2.5.3 :** To obtain the result we use a similar method than in [5]. That is to say, we study the inverse of the local time at  $\tilde{m}_1$ , and use our knowledge about  $H(\tilde{L}_1)$ . From the definitions of  $f_\gamma$  and  $\tilde{f}_\gamma$  we have easily  $\tilde{f}_\gamma(x) \leq f_\gamma(x)$  for all  $x$ . So, to prove (2.5.1), we only need to prove the upper bound for  $f_\gamma$  and the lower bound for  $\tilde{f}_\gamma$ . We fix  $\varepsilon \in (0, 1/2)$ .

• *Upper bound for  $f_\gamma(x)$ .* Recall that  $\sigma(u, \tilde{m}_1) = \inf\{s > 0, \mathcal{L}(s, \tilde{m}_1) \geq u\}$ ,  $u \geq 0$ . First, notice that for  $0 < x < 1$ ,  $f_\gamma(x)$  is equal to

$$E \left[ \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \mathcal{L}(t(1-x), \tilde{m}_1) \leq \gamma t, H(\tilde{L}_1) > t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-) \right) \right] \\ = E \left[ \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \sigma(\gamma t, \tilde{m}_1) \geq t(1-x), H(\tilde{L}_1) > t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-) \right) \right] \quad (2.5.24)$$

$$= E \left[ \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( H(\tilde{L}_1) > \sigma(\gamma t, \tilde{m}_1) \geq t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-) \right) \right] \quad (2.5.25)$$

$$+ E \left[ \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \sigma(\gamma t, \tilde{m}_1) > H(\tilde{L}_1) > t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-) \right) \right]. \quad (2.5.26)$$

Let us first study the expectation in (2.5.25). On  $\{H(\tilde{L}_1) > \sigma(\gamma t, \tilde{m}_1), H(\tilde{L}_1) < H(\tilde{L}_1^-)\}$  under  $\mathbb{P}_{\tilde{m}_1}^{W_\kappa}$ ,  $X$  remains between  $\tilde{L}_1^-$  and  $\tilde{L}_1$  until time  $\sigma(\gamma t, \tilde{m}_1)$  which is finite. On this event and under  $\mathbb{P}_{\tilde{m}_1}^{W_\kappa}$ , considering (2.1.13) and (2.1.14) as in ([61] p. 248), the inverse of the local time can be written for  $X$  starting at  $\tilde{m}_1$  as

$$\sigma(\gamma t, \tilde{m}_1) = \int_{\tilde{L}_1^-}^{\tilde{L}_1} e^{-\tilde{V}^{(1)}(z)} \mathcal{L}_B(\sigma_B(\gamma t, 0), A^1(z)) dz =: I, \quad (2.5.27)$$

where  $A^1(z) = \int_{\tilde{m}_1}^z e^{\tilde{V}^{(1)}(y)} dy$  and  $B$  is a standard Brownian motion independent of  $W_\kappa$ , such that  $B$  starts at  $A^1(\tilde{m}_1) = 0$  and is killed when it first hits  $A^1(\tilde{L}_1)$ . In

(2.5.27), we integrate only between  $\tilde{L}_1^-$  and  $\tilde{L}_1$  because under  $\mathbb{P}_{\tilde{m}_1}^{W_\kappa}$ ,  $e^{-\tilde{V}^{(1)}(z)} \mathcal{L}_B(\sigma_B(\gamma t, 0), A^1(z)) = \mathcal{L}(\sigma(\gamma t, \tilde{m}_1), z) = 0$  for  $z \notin [\tilde{L}_1^-, \tilde{L}_1]$  as explained after (2.5.26). We have

$$I = \gamma t \int_{\tilde{L}_1^-}^{\tilde{L}_1} e^{-\tilde{V}^{(1)}(z)} \mathcal{L}_{\tilde{B}}(\sigma_{\tilde{B}}(1, 0), \tilde{a}(z)) dz,$$

with  $\tilde{a}(z) := (\gamma t)^{-1} A^1(z) = (\gamma t)^{-1} \int_{\tilde{m}_1}^z e^{\tilde{V}^{(1)}(y)} dy$  and where  $\tilde{B} := B((\gamma t)^2) / (\gamma t)$ . By scale invariance  $\tilde{B}$  is also a standard Brownian motion that we still denote by  $B$  in the sequel. Also, recall that  $\sigma_U(r, y) := \inf\{s > 0, \mathcal{L}_U(s, y) > r\}$  for  $r > 0, y \in \mathbb{R}$  is the inverse of the local time of the process  $U$ . Since we consider  $X$  starting at  $\tilde{m}_1$ , we have  $H(\tilde{L}_1) = H(\tilde{L}_1) - H(\tilde{m}_1) = U_1$ , for which Proposition 2.3.5 gives

$$\mathbb{E}\left(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}\left\{|H(\tilde{L}_1) - \mathcal{H}_1| \leq \varepsilon_t \mathcal{H}_1\right\}\right) = \mathbb{P}(\{|U_1 - \mathcal{H}_1| \leq \varepsilon_t \mathcal{H}_1\} =: \mathcal{G}_1) \geq 1 - e^{-D_1 h_t}, \quad (2.5.28)$$

with  $\varepsilon_t := e^{-\delta h_t}$ , if  $\delta > 0$  is chosen small enough. This will explain the appearance of  $\mathcal{H}_1$  in  $f_\gamma^\pm(x)$ . So, we now deal with  $I$ . Notice that  $(\gamma t)^{-1} I$  can be split into two terms  $(\gamma t)^{-1} I = I_1 + I_2$ , with

$$I_1 := \int_{\tilde{\tau}_1^-(h_t/2)}^{\tilde{\tau}_1^+(h_t/2)} e^{-\tilde{V}^{(1)}(z)} \mathcal{L}_B(\sigma_B(1, 0), \tilde{a}(z)) dz,$$

and  $I_2 := (\gamma t)^{-1} I - I_1 \geq 0$ . We now prove that the main contribution in  $(\gamma t)^{-1} I$  comes from  $I_1$  and obtain its approximation in probability. Let  $\varepsilon \in (0, 1/100)$ . First, using the second part of Lemma 2.2.1, followed by Remark 2.2.1, Fact 2.2.1 (ii) (for which we need  $i \geq 2$ ), (2.7.8) and finally the first part of Lemma 2.2.1, we get

$$\begin{aligned} & P\left[|A^1(\tilde{\tau}_1^-(h_t/2))| \leq e^{h_t(1+\varepsilon)/2}, |A^1(\tilde{\tau}_1^+(h_t/2))| \leq e^{h_t(1+\varepsilon)/2}\right] \\ &= P\left[|A^2(\tilde{\tau}_2^-(h_t/2))| \leq e^{h_t(1+\varepsilon)/2}, |A^2(\tilde{\tau}_2^+(h_t/2))| \leq e^{h_t(1+\varepsilon)/2}\right] \\ &\geq 1 - 2P[F^+(h_t/2) > e^{h_t(1+\varepsilon)/2}] - P[\mathcal{V}_t] \geq 1 - C_+ e^{-\kappa \varepsilon h_t/4}. \end{aligned} \quad (2.5.29)$$

Therefore, since  $\tilde{a}(\tilde{\tau}_1^-(h_t/2)) \leq \tilde{a}(z) \leq \tilde{a}(\tilde{\tau}_1^+(h_t/2))$  for all  $z \in [\tilde{\tau}_1^-(h_t/2), \tilde{\tau}_1^+(h_t/2)]$ ,

$$P(\forall z \in [\tilde{\tau}_1^-(h_t/2), \tilde{\tau}_1^+(h_t/2)], |\tilde{a}(z)| \leq e^{-(\log t)(1-3\varepsilon)/2}) \geq 1 - C_+ e^{-\kappa \varepsilon h_t/4}. \quad (2.5.30)$$

Also, using (2.7.15) and the second Ray-Knight theorem (see before (2.7.15)), we have

$$\mathbb{P}\left(\sup_{|u| \leq e^{-(\log t)(1-3\varepsilon)/2}} |\mathcal{L}_B(\sigma_B(1, 0), u) - 1| \geq \hat{\varepsilon}_t\right) \leq e^{-t^\varepsilon/16}. \quad (2.5.31)$$

with  $\hat{\varepsilon}_t := t^{-(1-5\varepsilon)/4}$ . So we obtain

$$\mathbb{E}\left[\mathbb{P}_{\tilde{m}_1}^{W_\kappa}(|I_1 - \tilde{R}_1| \leq \hat{\varepsilon}_t \tilde{R}_1)\right] \geq 1 - C_+ e^{-\kappa \varepsilon h_t/4}, \quad (2.5.32)$$

with  $\tilde{R}_1 := \int_{\tilde{\tau}_1^-(h_t/2)}^{\tilde{\tau}_1^+(h_t/2)} e^{-\tilde{V}^{(1)}(z)} dz$ . We now prove that  $I_2$  is negligible compared to the integral  $\tilde{R}_1$  which appears in the previous equation, and then compared to  $I_1$ . First thanks to (2.7.16) and the second Ray-Knight theorem, we have

$$\mathbb{E} \left[ \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \sup_{z \in [\tilde{L}_1^-, \tilde{L}_1]} \mathcal{L}_B[\sigma_B(1, 0), \tilde{a}(z)] > e^{\varepsilon \log t} \right) \right] \leq 2e^{-\varepsilon \log t}.$$

So with probability larger than  $1 - 2e^{-\varepsilon \log t}$ , we have

$$I_2 \leq e^{\varepsilon \log t} \left( \int_{\tilde{L}_1^-}^{\tilde{\tau}_1^-(h_t/2)} e^{-\tilde{V}^{(1)}(z)} dz + \int_{\tilde{\tau}_1^+(h_t/2)}^{\tilde{L}_1} e^{-\tilde{V}^{(1)}(z)} dz \right) =: e^{\varepsilon \log t} I_3.$$

By Lemma 2.6.8, with a probability larger than  $1 - 2e^{-(c_-)\varepsilon h_t}$  for large  $t$ ,

$$I_3 \leq C_+ h_t^2 e^{-(1-\varepsilon)h_t/2}.$$

Also, by Lemma 2.3.6, with probability larger  $1 - e^{-(D_-)h_t}$ ,  $\tilde{R}_1 = R_1$  (which is the same  $R_1$  as in (2.5.28)), which law is given by the sum of two independent copies of  $F^-(h_t/2)$ . So using (2.7.9), with a probability larger than  $1 - 2e^{-(D_-)h_t}$ ,

$$\tilde{R}_1 = R_1 \geq e^{-\varepsilon h_t/2}.$$

We deduce from the last three inequalities that with a probability larger than  $1 - e^{-(c_-)\varepsilon h_t}$ ,

$$I_2 < R_1 e^{-(1-5\varepsilon)h_t/2} = \tilde{R}_1 e^{-(1-5\varepsilon)h_t/2}. \quad (2.5.33)$$

Finally, using  $(\gamma t)^{-1}I = I_1 + I_2$  together with (2.5.32) and (2.5.33), we get

$$\begin{aligned} & E \left[ \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( |I - \gamma t R_1| \geq 2t^{-(1-5\varepsilon)/4} (\gamma t) R_1, \right. \right. \\ & \quad \left. \left. H(\tilde{L}_1) > \sigma(\gamma t, \tilde{m}_1), H(\tilde{L}_1) < H(\tilde{L}_1^-) \right) \right] \\ & \leq C_+ e^{-\varepsilon(c_-)h_t}. \end{aligned} \quad (2.5.34)$$

We recall that by (2.5.27),  $\sigma(\gamma t, \tilde{m}_1) = I$  on  $\{H(\tilde{L}_1) > \sigma(\gamma t, \tilde{m}_1), H(\tilde{L}_1) < H(\tilde{L}_1^-)\}$  under  $\mathbb{P}_{\tilde{m}_1}^{W_\kappa}$ . Hence, combining (2.5.34) with (2.5.28) gives for large  $t$  for every  $x \in [\varepsilon, 1 - \varepsilon]$ ,

$$\begin{aligned} & \left\{ H(\tilde{L}_1) > \sigma(\gamma t, \tilde{m}_1) \geq t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-) \right\} \\ & \subset \left\{ \frac{1}{R_1} \leq \frac{\gamma}{1-x} (1 + \varepsilon'_t), \mathcal{H}_1 > t(1-x)(1 - \varepsilon'_t), H(\tilde{L}_1) > \sigma(\gamma t, \tilde{m}_1) \right\} \cup \mathcal{E}_\varepsilon^1, \end{aligned} \quad (2.5.35)$$

where  $\mathcal{E}_\varepsilon^1$  is such that  $E[\mathbb{P}_{\tilde{m}_1}^{W_\kappa}(\mathcal{E}_\varepsilon^1)] \leq C_+ e^{-(\varepsilon c_-)h_t} + e^{-D_1 h_t}$  and where, as defined in the statement of the lemma,  $\varepsilon'_t = e^{-c_2 h_t}$  with  $c_2 > 0$  chosen small enough.

Now, let us study (2.5.26). On the event inside the probability in (2.5.26),  $\sigma(\gamma t, \tilde{m}_1)$  might be infinite. We work under  $\mathbb{P}_{\tilde{m}_1}^{W_\kappa}$ . There exists a Brownian motion  $B$  such that, with  $T^1$  playing under  $\mathbb{P}_{\tilde{m}_1}^{W_\kappa}$  the same role as  $T$  does under  $\mathbb{P}$  (see (2.1.13)),  $H(\tilde{L}_1) = T^1(\tau^B(A^1(\tilde{L}_1)))$  and  $\sigma(\gamma t, \tilde{m}_1) = T^1(\sigma_B(\gamma t, 0))$  (as in (2.5.27) and in [61] p. 248). Also by (2.1.13), notice for further use that under  $\mathbb{P}_{\tilde{m}_1}^{W_\kappa}$ ,

$$\mathcal{L}(\sigma(\gamma t, \tilde{m}_1), z) = e^{-\tilde{V}^{(1)}(z)} \mathcal{L}_B(\sigma_B(\gamma t, 0), A^1(z)), \quad z \in \mathbb{R}, y \in (0, +\infty). \quad (2.5.36)$$

So, we have

$$\begin{aligned} \sigma(\gamma t, \tilde{m}_1) > H(\tilde{L}_1) &\Leftrightarrow \sigma_B(\gamma t, 0) > \tau^B(A^1(\tilde{L}_1)) \\ &\Leftrightarrow \mathcal{L}_B[\sigma_B(\gamma t, 0), 0] = \gamma t > \mathcal{L}_B[\tau^B(A^1(\tilde{L}_1)), 0]. \end{aligned}$$

Now, note that, as in (2.3.22) in the proof of Lemma 2.3.6,  $\mathcal{L}_B[\tau^B(A^1(\tilde{L}_1)), 0] = A^1(\tilde{L}_1) \mathcal{L}_{\tilde{B}}(\tau^{\tilde{B}}(1), 0)$ , where  $\tilde{B} := B((A^1(\tilde{L}_1))^2)/A^1(\tilde{L}_1)$ . Also, by definition of  $\mathbf{e}_1$  given in (2.3.22), we have  $\mathcal{L}_{\tilde{B}}(\tau^{\tilde{B}}(1), 0) = \mathbf{e}_1$ . As a consequence,

$$\sigma(\gamma t, \tilde{m}_1) > H(\tilde{L}_1) \Leftrightarrow \gamma t > A^1(\tilde{L}_1) \mathbf{e}_1 \Leftrightarrow \gamma t R_1 > A^1(\tilde{L}_1) \mathbf{e}_1 R_1.$$

Then, according to (2.3.18), we have  $A^1(\tilde{L}_1) \geq (1 - e^{-(d_-)h_t}) S_1$  with probability greater than  $1 - e^{-(D_-)h_t}$ . Moreover, according to (2.5.28) and to the fact that under  $\mathbb{P}_{\tilde{m}_1}^{W_\kappa}$  the diffusion  $X$  starts at  $\tilde{m}_1$ , we have  $\mathcal{H}_1 = \mathbf{e}_1 S_1 R_1 \geq (1 + \varepsilon_t)^{-1} H(\tilde{L}_1)$  with probability greater than  $1 - e^{-(D_-)h_t}$ . As a consequence,

$$\sigma(\gamma t, \tilde{m}_1) > H(\tilde{L}_1) \Rightarrow \gamma t R_1 > (1 - e^{-(d_-)h_t})(1 + \varepsilon_t)^{-1} H(\tilde{L}_1), \quad (2.5.37)$$

except on an event which probability  $E[\mathbb{P}_{\tilde{m}_1}^{W_\kappa}(\cdot)]$  is less than  $2e^{-(D_-)h_t}$ . Combining this with (2.5.28) we get for large  $t$  for every  $x \in [\varepsilon, 1 - \varepsilon]$ ,

$$\begin{aligned} &\left\{ \sigma(\gamma t, \tilde{m}_1) > H(\tilde{L}_1) > t(1 - x), H(\tilde{L}_1) < H(\tilde{L}_1^-) \right\} \\ &\subset \left\{ \frac{1}{R_1} \leq \frac{\gamma}{1 - x} (1 + \varepsilon'_t), \mathcal{H}_1 > t(1 - x)(1 - \varepsilon'_t), \sigma(\gamma t, \tilde{m}_1) > H(\tilde{L}_1) \right\} \cup \mathcal{E}_\varepsilon^2, \end{aligned} \quad (2.5.38)$$

where  $\mathcal{E}_\varepsilon^2$  is such that  $E[\mathbb{P}_{\tilde{m}_1}^{W_\kappa}(\mathcal{E}_\varepsilon^2)] \leq 2e^{-(D_-)h_t} + e^{-D_1 h_t}$  and where, as before,  $\varepsilon'_t = e^{-c_2 h_t}$  with  $c_2 > 0$  possibly smaller than before.

Combining (2.5.25), (2.5.26) (2.5.35) and (2.5.38) with the strong Markov property, we get for large  $t$  for every  $x \in [\varepsilon, 1 - \varepsilon]$ , since  $\phi(t) = o(\log t)$ ,

$$\begin{aligned} f_\gamma(x) &\leq \mathbb{P} \left( \frac{1}{R_1} \leq \frac{\gamma}{1 - x} (1 + \varepsilon'_t), \mathcal{H}_1 > t(1 - x)(1 - \varepsilon'_t) \right) + o(n_t^{-1}). \\ &= f_\gamma^+(x) + o(n_t^{-1}). \end{aligned}$$

• *Lower bound for  $\tilde{f}_\gamma$ .* Let  $\tilde{\gamma} := \gamma(1 + e^{-h_t/12})^{-1}$  and  $y := (1 - x)/[R_1(1 - 4\hat{\varepsilon}_t)]$ . We have to distinguish the cases  $H(\tilde{L}_1) > \sigma(\tilde{\gamma}t, \tilde{m}_1)$  and  $\sigma(\tilde{\gamma}t, \tilde{m}_1) > H(\tilde{L}_1)$ . We work under  $\mathbb{P}_{\tilde{m}_1}^{W_\kappa}$ . On  $\{y \leq \tilde{\gamma}, H(\tilde{L}_1) > \sigma(\tilde{\gamma}t, \tilde{m}_1) \geq t(1 - x), H(\tilde{L}_1) < H(\tilde{L}_1^-)\}$ , we can express the local time of  $X$  at the inverse of its local time in  $\tilde{m}_1$  at time  $yt$  in terms of the standard Brownian motion driving the diffusion. More precisely by (2.5.36) and by scale invariance, there exists a Brownian motion  $B$  such that for any  $z \in \mathcal{D}_1$ ,

$$\mathcal{L}(\sigma(yt, \tilde{m}_1), z) = (yt)e^{-\tilde{V}^{(1)}(z)}\mathcal{L}_B(\sigma_B(1, 0), \hat{a}(z)) \quad (2.5.39)$$

with  $\hat{a}(z) := (yt)^{-1} \int_{\tilde{m}_1}^z e^{\tilde{V}^{(1)}(u)} du = A^1(z)/(yt)$ . Notice that by (2.1.7),  $F^-(h_t/2) \leq \tau^{W_\kappa^\uparrow}(h_t/2)$  in law, so  $P[R_1 > 8h_t/\kappa] \leq 2P[F^-(h_t/2) > 4h_t/\kappa] \leq 2P[\tau^{W_\kappa^\uparrow}(h_t/2) > 4h_t/\kappa] \leq e^{-(c_-)h_t}$  for large  $t$ . Moreover, we prove with the same method used to prove (2.5.19) that  $\tilde{\tau}^-(h_t/2) \leq \tilde{m}_1 - r_t \leq \tilde{m}_1 + r_t \leq \tilde{\tau}(h_t/2)$  with probability at least  $1 - C_+e^{-(c_-)h_t}$ . This and (2.5.29) give  $-e^{h_t(1+\varepsilon)/2} \leq A^1[\tilde{\tau}^-(h_t/2)] \leq A^1(z) \leq A^1[\tilde{\tau}(h_t/2)] \leq e^{h_t(1+\varepsilon)/2}$  for any  $z \in \mathcal{D}_1$  with probability  $\geq 1 - e^{-(c_-)\varepsilon h_t}$ . So, for large  $t$  for every  $x \in [\varepsilon, 1 - \varepsilon]$ ,  $|\hat{a}(z)| \leq e^{h_t(1+\varepsilon)/2} R_1/(t(1 - x)) \leq e^{-(\log t)(1-3\varepsilon)/2}$  for these  $z$  with such probability. Hence with the same method we used to prove (2.5.32) from (2.5.30) and (2.5.31), we get for large  $t$  for every  $x \in [\varepsilon, 1 - \varepsilon]$ ,

$$E \left( \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \sup_{z \in \mathcal{D}_1} \left| \mathcal{L}_B(\sigma_B(1, 0), \hat{a}(z)) - 1 \right| \leq \hat{\varepsilon}_t \right) \right) \geq 1 - 2e^{-(c_-)\varepsilon h_t}.$$

The above inequality together with (2.5.39) imply that for large  $t$  for every  $x \in [\varepsilon, 1 - \varepsilon]$ ,

$$\begin{aligned} & E \left( \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \left\{ \exists z \in \mathcal{D}_1, \left| \mathcal{L}(\sigma(yt, \tilde{m}_1), z) - yte^{-\tilde{V}^{(1)}(z)} \right| \geq 2yte^{-\tilde{V}^{(1)}(z)}\hat{\varepsilon}_t, \right. \right. \right. \\ & \quad \left. \left. \left. y \leq \tilde{\gamma}, H(\tilde{L}_1) > \sigma(\tilde{\gamma}t, \tilde{m}_1) \geq t(1 - x), H(\tilde{L}_1) < H(\tilde{L}_1^-) \right\} \right) \right) \\ & \leq 2e^{-(c_-)\varepsilon h_t}. \end{aligned} \quad (2.5.40)$$

On  $\{y \leq \tilde{\gamma}, H(\tilde{L}_1) > \sigma(\tilde{\gamma}t, \tilde{m}_1) \geq t(1 - x), H(\tilde{L}_1) < H(\tilde{L}_1^-)\}$ , if  $t(1 - x) > \sigma(yt, \tilde{m}_1)$ , then  $\sigma(yt, \tilde{m}_1) - ytR_1 < -4tyR_1\hat{\varepsilon}_t$ , and by (2.5.34) (applied with  $\gamma$  replaced by  $y$ ), this has on the previous event a probability  $E(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}(\cdot))$  less than  $C_+e^{-(c_-)\varepsilon h_t}$ . Thus on the previous event, we have  $t(1 - x) \leq \sigma(yt, \tilde{m}_1)$ , except on a sub event of probability smaller than  $C_+e^{-(c_-)\varepsilon h_t}$ . This is true for every  $x \in [\varepsilon, 1 - \varepsilon]$  for large  $t$ .

Then since the local time is increasing in time, we have on the previous event for any  $z \in \mathcal{D}_1$ ,  $\mathcal{L}(t(1 - x), z) \leq \mathcal{L}(\sigma(yt, \tilde{m}_1), z)$ , which is according to (2.5.40) less than  $yte^{-\tilde{V}^{(1)}(z)}(1 + 2\hat{\varepsilon}_t) \leq yt(1 + 2\hat{\varepsilon}_t)$  for every  $z \in \mathcal{D}_1$  with probability  $E(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}(\cdot))$  at least  $1 - 2e^{-(c_-)\varepsilon h_t}$ . Combining this and the definition of our  $y$  gives for large  $t$ , for every  $x \in [\varepsilon, 1 - \varepsilon]$ ,

$$\begin{aligned} & E \left( \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \left\{ \frac{\sup_{z \in \mathcal{D}_1} \mathcal{L}(t(1 - x), z)}{t} > \frac{(1 - x)}{R_1} \frac{1 + 2\hat{\varepsilon}_t}{1 - 4\hat{\varepsilon}_t} \right\} =: \bar{\mathcal{G}}_2 \right. \right. \\ & \quad \left. \left. \cap \left\{ y \leq \tilde{\gamma}, H(\tilde{L}_1) > \sigma(\tilde{\gamma}t, \tilde{m}_1) \geq t(1 - x), H(\tilde{L}_1) < H(\tilde{L}_1^-) \right\} \right) \right) \\ & \leq (2 + C_+)e^{-(c_-)\varepsilon h_t}. \end{aligned} \quad (2.5.41)$$



As a consequence, for  $t$  large enough so that  $1 + 2\widehat{\varepsilon}_t \leq 1 + e^{-h_t/12}$ , we have for every  $x \in [\varepsilon, 1 - \varepsilon]$ ,

$$\begin{aligned} & \left\{ y \leq \tilde{\gamma}, H(\tilde{L}_1) > \sigma(\tilde{\gamma}t, \tilde{m}_1) \geq t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-) \right\} \\ \subset & \left\{ \sup_{z \in \mathcal{D}_1} \mathcal{L}(t(1-x), z) \leq y(1 + 2\widehat{\varepsilon}_t)t \leq \gamma t, H(\tilde{L}_1) > \sigma(\tilde{\gamma}t, \tilde{m}_1) \right\} \cup \mathcal{E}_\varepsilon^3 \end{aligned} \quad (2.5.42)$$

by definition of  $\tilde{\gamma}$ , where  $\mathcal{E}_\varepsilon^3$  is such that  $E(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}(\mathcal{E}_\varepsilon^3)) \leq (2 + C_+)e^{-(c_-)\varepsilon h_t}$ .

On the other hand, from the definition of  $\sigma(\cdot, \tilde{m}_1)$ , (2.5.23) and the definition of  $\tilde{\gamma}$ , we have for large  $t$  for every  $x \in [\varepsilon, 1 - \varepsilon]$ ,

$$\begin{aligned} & \left\{ y \leq \tilde{\gamma}, \sigma(\tilde{\gamma}t, \tilde{m}_1) > H(\tilde{L}_1) > t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-) \right\} \\ \subset & \left\{ \mathcal{L}(H(\tilde{L}_1), \tilde{m}_1) \leq \tilde{\gamma}t, \sigma(\tilde{\gamma}t, \tilde{m}_1) > H(\tilde{L}_1) > t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-) \right\} \\ \subset & \left\{ \sup_{z \in \mathcal{D}_1} \mathcal{L}(H(\tilde{L}_1), z) \leq \gamma t, \sigma(\tilde{\gamma}t, \tilde{m}_1) > H(\tilde{L}_1) > t(1-x) \right\} \cup \mathcal{E}_\varepsilon^4 \\ \subset & \left\{ \sup_{z \in \mathcal{D}_1} \mathcal{L}(t(1-x), z) \leq \gamma t, \sigma(\tilde{\gamma}t, \tilde{m}_1) > H(\tilde{L}_1) \right\} \cup \mathcal{E}_\varepsilon^4, \end{aligned} \quad (2.5.43)$$

where  $\mathcal{E}_\varepsilon^4$  is the event where (2.5.23) fails, it is such that  $E(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}(\mathcal{E}_\varepsilon^4)) \leq C_+e^{-(c_-)r_t}$ .

Combining (2.5.42) and (2.5.43) we get for large  $t$  for every  $x \in [\varepsilon, 1 - \varepsilon]$ , under  $\mathbb{P}_{\tilde{m}_1}^{W_\kappa}$ ,

$$\left\{ y \leq \tilde{\gamma}, H(\tilde{L}_1) > t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-) \right\} \subset \left\{ \sup_{z \in \mathcal{D}_1} \mathcal{L}(t(1-x), z) \leq \gamma t \right\} \cup \mathcal{E}_\varepsilon^5,$$

where  $\mathcal{E}_\varepsilon^5$  is such that  $E(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}(\mathcal{E}_\varepsilon^5)) \leq C_+e^{-(c_-)r_t} = C_+e^{-(c_-)C_0\phi(t)} = o(n_t^{-1})$  as  $t \rightarrow +\infty$  is we choose  $C_0$  large enough. Combining this with (2.5.28), (2.3.2) and Proposition 2.3.5, we obtain for large  $t$  for every  $x \in [\varepsilon, 1 - \varepsilon]$ ,

$$\begin{aligned} \tilde{f}_\gamma(x) & \geq \mathbb{P} \left( \frac{(1-x)}{R_1} \leq \gamma(1 - \varepsilon'_t), \mathbf{e}_1 S_1 R_1 > t(1-x)(1 + \varepsilon'_t) \right) - o(n_t^{-1}) \\ & = f_\gamma^-(x) - o(n_t^{-1}), \end{aligned}$$

where the constant  $c_2$  in the definition of  $\varepsilon'_t = e^{-c_2 h_t}$  has been decreased if necessary. This proves the lower bound for  $\tilde{f}_\gamma(x)$  and then finishes the proof of the lemma.  $\square$

**Proof of Lemma 2.5.4 :** Let  $0 < a < 1/4$ . We start with (2.5.2). By Proposition 2.3.5, the  $\mathcal{H}_i$ ,  $i \geq 1$  are i.i.d., so  $\bar{\mathcal{H}}_{k-1}$  and  $\bar{\mathcal{H}}_k - \bar{\mathcal{H}}_{k-1} = \mathcal{H}_k$  are independent for  $k \geq 1$ . Thus for  $t > 0$ ,

$$\begin{aligned} & \sum_{1 \leq k \leq n_t} \mathbb{P} [\bar{\mathcal{H}}_k > 1 - a/2, 1 - 2a < \bar{\mathcal{H}}_{k-1} \leq 1 - 3a/4] \\ & = \int_{1-2a}^{1-3a/4} d\mu_t(x) e^{\kappa\phi(t)} \mathbb{P} [\mathcal{H}_1 > 1 - x - a/2], \end{aligned} \quad (2.5.44)$$

where the measure  $\mu_t$  is defined by  $\int_0^x d\mu_t(y) := e^{-\kappa\phi(t)} \sum_{1 \leq k \leq n_t} \mathbb{P}[\bar{\mathcal{H}}_{k-1} \leq x]$ . We know that  $\mu_t$  converges vaguely as  $t \rightarrow +\infty$  to the measure  $\mu$  which has a density with respect to the Lebesgue measure equal to  $(\Gamma(\kappa)\mathcal{C}_\kappa)^{-1}x^{\kappa-1}\mathbf{1}_{x>0}$ , with  $\mathcal{C}_\kappa > 0$  (see Lemma 2.6.2). Also thanks to Lemma 2.4.1,  $e^{\kappa\phi(t)}P[\mathcal{H}_1/t > x]$  converges uniformly on every compact subset of  $(0, +\infty)$  to  $\mathcal{C}_\kappa x^{-\kappa}/\Gamma(1-\kappa)$ . Therefore,

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \sum_{1 \leq k \leq n_t} \mathbb{P}[\bar{\mathcal{H}}_k > 1 - a/2, 1 - 2a < \bar{\mathcal{H}}_{k-1} \leq 1 - 3a/4] \\ &= \frac{1}{\Gamma(\kappa)\Gamma(1-\kappa)} \int_{1-2a}^{1-3a/4} x^{\kappa-1}(1-x-a/2)^{-\kappa} dx \\ &\leq \text{const} \times a^{1-\kappa}. \end{aligned}$$

For (2.5.3), we apply (2.6.2) with  $r = \varepsilon \in (0, 1/2)$  and  $s = 1 - \varepsilon$ , which gives

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \mathbb{P}(\varepsilon t \leq H(m_{N_t}) \leq (1-\varepsilon)t) \\ &= 1 - \frac{\sin(\pi\kappa)}{\pi} \left( \int_0^\varepsilon x^{\kappa-1}(1-x)^{-\kappa} dx + \int_{1-\varepsilon}^1 x^{\kappa-1}(1-x)^{-\kappa} dx \right) \\ &\geq 1 - \frac{\sin(\pi\kappa)}{\pi} \left( \frac{(1-\varepsilon)^{-\kappa}}{\kappa} \varepsilon^\kappa + \frac{(1-\varepsilon)^{\kappa-1}}{1-\kappa} \varepsilon^{1-\kappa} \right), \end{aligned}$$

which implies the result.  $\square$

**Proof of Theorem 2.1.3 :** The proof of this theorem is a direct consequence of Propositions 2.5.1 and 2.1.4 and of Lemmata 2.4.4 and 2.4.5. Notice that the proof of the upper bound does not use the proof of the lower bound, but we use the upper bound for the proof of the lower bound. In particular from the upper bound of Theorem 2.1.3 (which makes use of the upper bound of Proposition 2.5.1 but not of its lower bound), we have  $\limsup_{t \rightarrow +\infty} \mathbb{P}(\mathcal{L}^*(t) < 2\tilde{w}_t) \leq \mathbb{P}(\mathcal{Y}_1^\natural(\mathcal{Y}_2^{-1}(1)^-) \leq \varepsilon)$  for any  $\varepsilon > 0$  as  $\lim_{t \rightarrow +\infty} \tilde{w}_t/t = 0$ . From this, as  $\mathcal{Y}_1^\natural(\mathcal{Y}_2^{-1}(1)^-)$  is positive, we obtain  $\lim_{t \rightarrow +\infty} \mathbb{P}(\mathcal{L}^*(t) < 2\tilde{w}_t) = 0$ , which proves assertion (2.5.18) at the beginning of the proof of the lower bound of Proposition 2.5.1.

Thanks to Proposition 2.5.1 and to the remark before this proposition, we only need to study the convergence of  $\mathcal{P}_1^\pm$  (the limit when  $t$  goes to infinity and then the limit when  $\varepsilon$  goes to 0). The latter can be written in term of functionals of  $(Y_1, Y_2)^t$  as follows. Let  $\mathbb{Y}_t := (Y_2^t)^{-1}(1 - 2\varepsilon)$ ; we have  $\mathcal{N}_t^{2\varepsilon} e^{-\kappa\phi(t)} = \mathbb{Y}_t$ , and

$$\begin{aligned} \mathcal{P}_1^\pm &= P \left[ (1 - Y_2^t(\mathbb{Y}_t^-)) \frac{Y_1^t(\mathbb{Y}_t) - Y_1^t(\mathbb{Y}_t^-)}{Y_2^t(\mathbb{Y}_t) - Y_2^t(\mathbb{Y}_t^-)} \leq \alpha_t^\pm, (Y_1^t)^\natural(\mathbb{Y}_t^-) \leq \alpha_t^\pm \right] \\ &= P \left[ (1 - \tilde{K}_{I,1-2\varepsilon}^-((Y_1, Y_2)^t)) \frac{K_{I,1-2\varepsilon}((Y_1, Y_2)^t) - K_{I,1-2\varepsilon}^-((Y_1, Y_2)^t)}{\tilde{K}_{I,1-2\varepsilon}((Y_1, Y_2)^t) - \tilde{K}_{I,1-2\varepsilon}^-((Y_1, Y_2)^t)} \leq \alpha_t^\pm, \right. \\ &\quad \left. J_{I,1-2\varepsilon}^-((Y_1, Y_2)^t) \leq \alpha_t^\pm \right], \end{aligned}$$

with the notation  $K_{I,a}, \tilde{K}_{I,a}, \dots$  introduced in (2.4.25) and before. The hypotheses of Lemma 4.5 are : finite number of large jumps on compact intervals, strictly increasing, starting at 0, and jumping over 1 without reaching it. These properties are naturally almost surely satisfied by a  $\kappa$ -stable subordinator so, almost surely, the paths of  $(\mathcal{Y}_1, \mathcal{Y}_2)$  satisfy the hypotheses of Lemma 2.4.5 (see e.g. [8] III.2 p. 75). Therefore they are points of continuity for  $J_{I,1-2\varepsilon}^-, K_{I,1-2\varepsilon}^-, \tilde{K}_{I,1-2\varepsilon}^-, \tilde{K}_{I,1-2\varepsilon}^-$  and  $\tilde{K}_{I,1-2\varepsilon}^-$ . Combining this continuity with Proposition 2.1.4, continuous mapping theorem, and replacing the functionals by their expressions, we obtain, when  $t$  goes to infinity, the convergence of  $\mathcal{P}_1^\pm$  to

$$P \left[ (1 - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1 - 2\varepsilon)^-)) \frac{\mathcal{Y}_1(\mathcal{Y}_2^{-1}(1 - 2\varepsilon)) - \mathcal{Y}_1(\mathcal{Y}_2^{-1}(1 - 2\varepsilon)^-)}{\mathcal{Y}_2(\mathcal{Y}_2^{-1}(1 - 2\varepsilon)) - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1 - 2\varepsilon)^-)} \leq \alpha, \right. \\ \left. \mathcal{Y}_1^\sharp(\mathcal{Y}_2^{-1}(1 - 2\varepsilon)^-) \leq \alpha \right].$$

Then, note that almost surely  $\mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)^-) < 1$  so we have a.s.  $\mathcal{Y}_2^{-1}(1 - 2\varepsilon) = \mathcal{Y}_2^{-1}(1)$  for all  $\varepsilon$  small enough. We deduce that the above expression converges to the repartition function of  $\max(\mathcal{I}_1, \mathcal{I}_2)$  (see (2.1.6) for definitions of  $\mathcal{I}_1$  and  $\mathcal{I}_2$ ) when  $\varepsilon$  goes to 0, and this yields Theorem 2.1.3.  $\square$

## 2.5.2 Favorite site (proof of Theorem 2.1.5)

$\hat{\text{E}}$

Thanks to Section 2.3, we know precisely the nature of the contribution of each  $h_t$ -valley to the local time. The difficulty in proving Theorem 2.1.3 was the need to consider only a part of the contribution of the last  $h_t$ -valley. The proofs of the first two points (2.1.9) and (2.1.10) of Theorem 2.1.5 are thus easier to obtain, since they do not require to "cut" the contribution of any valley. Let us prove the first point (2.1.9) (the second one, (2.1.10), is obtained similarly). We have, using (2.2.7),

$$\begin{aligned} & \mathbb{P}[\mathcal{L}^*(H(m_{N_t+1})) \leq \alpha t] \\ & \leq \mathbb{P}\left(\mathcal{L}^*(H(\tilde{L}_{N_t})) \leq \alpha t, \mathcal{Q}, \mathcal{V}_t\right) + \mathbb{P}(\overline{\mathcal{Q}}) + P(\overline{\mathcal{V}_t}) + \mathbb{P}(\overline{\mathcal{B}_3(n_t)}) \\ & \leq \mathbb{P}\left(\sup_{1 \leq j \leq N_t} \ell_j/t \leq (1 - \varepsilon_t)^{-1} \alpha, \mathcal{Q}, \mathcal{V}_t\right) + \mathbb{P}(\overline{\mathcal{Q}}) + o(1), \end{aligned}$$

where we fixed some  $\varepsilon > 0$  and  $\mathcal{Q} := \{\varepsilon t \leq H(m_{N_t}) \leq (1 - \varepsilon)t, 1 \leq N_t \leq n_t\}$  as after (2.5.4) (from there we see that  $\lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow +\infty} \mathbb{P}(\overline{\mathcal{Q}}) = 0$ ). In the last inequality we used Proposition 2.3.5, Lemma 2.2.1 and Lemma 2.3.2. To lighten notation, let

$\tilde{\alpha}_t := (1 - \varepsilon_t)^{-1}\alpha$ . We have

$$\begin{aligned} & \mathbb{P}\left(\sup_{1 \leq j \leq N_t} \ell_j/t \leq \tilde{\alpha}_t, \mathcal{Q}, \mathcal{V}_t\right) \\ & \leq \mathbb{P}\left(\sup_{1 \leq j \leq N_t} \ell_j/t \leq \tilde{\alpha}_t, \mathcal{B}_1(n_t), \mathcal{Q}, \mathcal{V}_t\right) + \mathbb{P}\left(\overline{\mathcal{B}_1(n_t)}\right) \\ & \leq \mathbb{P}\left(\sup_{1 \leq j \leq N_t} \ell_j/t \leq \tilde{\alpha}_t, \bar{\mathcal{H}}_{N_t} \geq 1 - \delta'_t, \bar{\mathcal{H}}_{N_t-1} \leq 1 - \varepsilon + \delta'_t, \mathcal{Q}\right) + o(1), \end{aligned}$$

with  $\delta'_t = 3\tilde{v}_t/t$  and where we used (2.3.1) together with Proposition 2.3.5. Partitioning on the values of  $N_t$  we get that the above is less than

$$\sum_{1 \leq k \leq n_t} \mathbb{P}\left(\sup_{1 \leq j \leq k} \ell_j/t \leq \tilde{\alpha}_t, \bar{\mathcal{H}}_k \geq 1 - \delta'_t, \bar{\mathcal{H}}_{k-1} \leq 1 - \varepsilon + \delta'_t, \mathcal{Q}\right) + o(1).$$

Since the sum  $\sum_1$  defined in the proof of the upper bound of Proposition 2.5.1 (see (2.5.15) and below) is smaller than  $s(\varepsilon, t)$  satisfying  $\lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow +\infty} s(\varepsilon, t) = 0$ , we can intersect the event on the above probability with  $\{k = \mathcal{N}_t^{2\varepsilon}\}$  and get

$$\mathbb{P}[\mathcal{L}^*(H(m_{N_t+1})) \leq \alpha t] \leq \mathbb{P}\left(\sup_{1 \leq j \leq \mathcal{N}_t^{2\varepsilon}} \ell_j/t \leq \tilde{\alpha}_t\right) + \mathbb{P}(\overline{\mathcal{Q}}) + s(\varepsilon, t) + o(1).$$

Then, as in the proof of Theorem 2.1.3 we have that  $(\mathcal{Y}_1, \mathcal{Y}_2)$  almost surely satisfies the hypothesis of Lemma 2.4.5, and is therefore almost surely a point of continuity for  $J_{I,12-\varepsilon}$  defined just above (2.4.25). From this continuity, Proposition 2.1.4 and the continuous mapping theorem we get

$$\sup_{1 \leq j \leq \mathcal{N}_t^{2\varepsilon}} \ell_j/t = J_{I,1-2\varepsilon}((Y_1, Y_2)^t) \xrightarrow[t \rightarrow +\infty]{\mathcal{L}} J_{I,1-2\varepsilon}(\mathcal{Y}_1, \mathcal{Y}_2) = \mathcal{Y}_1^\sharp(\mathcal{Y}_2^{-1}(1 - 2\varepsilon)).$$

Then, as in the proof of Theorem 2.1.3 we have almost surely  $\mathcal{Y}_2^{-1}(1 - 2\varepsilon) = \mathcal{Y}_2^{-1}(1)$  for all  $\varepsilon$  small enough so  $\mathcal{Y}_1^\sharp(\mathcal{Y}_2^{-1}(1 - 2\varepsilon))$  converges almost surely to  $\mathcal{Y}_1^\sharp(\mathcal{Y}_2^{-1}(1))$  when  $\varepsilon$  goes to 0. Thus, we get

$$\limsup_{t \rightarrow +\infty} \mathbb{P}[\mathcal{L}^*(H(m_{N_t+1})) \leq \alpha t] \leq \mathbb{P}\left(\mathcal{Y}_1^\sharp(\mathcal{Y}_2^{-1}(1)) \leq \alpha\right).$$

A lower bound is proved similarly, so we get the following, proving (2.1.9) :

$$\lim_{t \rightarrow +\infty} \mathbb{P}[\mathcal{L}^*(H(m_{N_t+1})) \leq \alpha t] = \mathbb{P}\left(\mathcal{Y}_1^\sharp(\mathcal{Y}_2^{-1}(1)) \leq \alpha\right).$$

To obtain the result (2.1.11) for the favorite site, we first argue that we essentially need to obtain the asymptotic behavior of  $N_t^*/N_t$ , where  $N_t^* := \min\{j \geq 1, \mathcal{L}(m_j, t) = \max_{1 \leq k \leq N_t} \mathcal{L}(m_k, t)\}$ . Indeed, define for any  $\varepsilon \in (0, 1/2)$ ,

$$\begin{aligned} \mathcal{K}_1 & := \{(1 - \varepsilon)m_{N_t} \leq X(t) \leq (1 + \varepsilon)m_{N_t}\}, \\ \mathcal{K}_2 & := \{(1 - \varepsilon)m_{N_t^*} \leq F_t^* \leq (1 + \varepsilon)m_{N_t^*}\}. \end{aligned}$$

## 2.5. SUPREMUM OF THE LOCAL TIME - AND OTHER FUNCTIONALS

Then, we have,  $\lim_{t \rightarrow +\infty} \mathbb{P}(\mathcal{K}_1) = 1$  by the localization result Theorem 2.1.2 combined with the fact that  $X(t)/t^\kappa$  converges in law under  $\mathbb{P}$  to a positive limit as  $t \rightarrow +\infty$  by [45].

Let us now justify that  $\lim_{t \rightarrow +\infty} \mathbb{P}(\mathcal{K}_2) = 1$ . According to (2.5.18) proved at the start of the proof of Theorem 2.1.3, to Lemma 2.3.4 and (2.3.3), we have

$$\mathbb{P} \left( \sup_{x \in \mathbb{R}} \mathcal{L}(t, x) \geq 2\tilde{w}_t, \mathcal{B}_4(n_t), N_t \leq n_t \right) \xrightarrow{t \rightarrow +\infty} 1.$$

Notice that on the event inside the above probability, for  $t$  large enough so that  $2\tilde{w}_t \geq te^{-2\phi(t)}$ , we have  $F_t^* \in \mathcal{D}_{N_t^*}$  (recall the definition of  $\mathcal{D}_j$  in (2.3.13)). Since  $\mathcal{D}_{N_t^*}$  is centered at  $m_{N_t^*}$  and its half-length is deterministic and equal to  $r_t = C_0\phi(t)$  we only need to justify that

$$\mathbb{P} \left( \varepsilon m_{N_t^*} \geq C_0\phi(t) \right) \xrightarrow{t \rightarrow +\infty} 1.$$

We have  $m_{N_t^*} \geq m_1$  and  $\mathbb{P}(m_1 \geq C_0\phi(t)/\varepsilon) \geq \mathbb{P}(\tilde{m}_1 \geq C_0\phi(t)/\varepsilon) - o(1)$  by Lemma 2.2.1. So using (2.6.13), we thus deduce that  $\lim_{t \rightarrow +\infty} \mathbb{P}(\mathcal{K}_2) = 1$ .

We can now write for  $x > 0$ ,

$$\mathbb{P} \left[ \frac{F_t^*}{X(t)} \leq x \right] = \mathbb{P} \left[ \frac{F_t^*}{X(t)} \leq x, \mathcal{K}_1, \mathcal{K}_2 \right] + v(\varepsilon, t) \leq \mathbb{P} \left[ \frac{m_{N_t^*}}{m_{N_t}} \leq x \frac{1+\varepsilon}{1-\varepsilon} \right] + v(\varepsilon, t).$$

where  $v(\varepsilon, t) \geq 0$ , satisfies  $\lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow +\infty} v(\varepsilon, t) = 0$ . Similarly, we have

$$\mathbb{P} \left[ \frac{F_t^*}{X(t)} \leq x \right] \geq \mathbb{P} \left[ \frac{m_{N_t^*}}{m_{N_t}} \leq x \frac{1-\varepsilon}{1+\varepsilon} \right] - v(\varepsilon, t).$$

Hence, we obtain

$$\mathbb{P} \left[ \frac{m_{N_t^*}}{m_{N_t}} \leq x \frac{1-\varepsilon}{1+\varepsilon} \right] - v(\varepsilon, t) \leq \mathbb{P} \left[ \frac{F_t^*}{X(t)} \leq x \right] \leq \mathbb{P} \left[ \frac{m_{N_t^*}}{m_{N_t}} \leq x \frac{1+\varepsilon}{1-\varepsilon} \right] + v(\varepsilon, t). \quad (2.5.45)$$

So, we observe that we only have to study the random variable  $\frac{m_{N_t^*}}{m_{N_t}}$ . For that we first remark that  $N_t^*$  and  $N_t$  diverge when  $t$  goes to infinity. Indeed by Lemma 2.6.1, the correct normalisation for the convergence in law of  $N_t$  is  $e^{\kappa\phi(t)}$ , so  $\mathbb{P}(N_t \geq e^{(1-\varepsilon)\kappa\phi(t)}) = 1 - o(1)$ . For  $N_t^*$ , we first notice that the previous result for  $N_t$  also gives for  $t$  large,  $\mathbb{P}(N_t \geq e^{(1-\varepsilon/2)\kappa\phi(t)}) = 1 - o(1)$ . Therefore

$$\mathbb{P} \left( N_t^* \leq e^{(1-\varepsilon)\kappa\phi(t)} \right) \leq \mathbb{P} \left( \max_{k \leq e^{(1-\varepsilon)\kappa\phi(t)}} \mathcal{L}(m_k, t) \geq \max_{k < e^{(1-\varepsilon/2)\kappa\phi(t)}} \mathcal{L}(m_k, t) \right) + o(1).$$

Now, since  $\mathcal{L}(m_k, t) = \mathcal{L}(\tilde{m}_k, H(\tilde{L}_k) \wedge (H(\tilde{m}_k) + H_{\tilde{m}_k \rightarrow \tilde{L}_k^-})) =: \hat{\ell}_k$  for  $k < N_t$  on  $\mathcal{V}_t \cap \{N_t \leq n_t\} \cap \mathcal{B}_2(n_t)$  which has probability  $1 - o(1)$  by Lemmas 2.2.1 and 2.3.1,

$$\begin{aligned} & \mathbb{P} \left( \max_{k \leq e^{(1-\varepsilon)\kappa\phi(t)}} \mathcal{L}(m_k, t) \geq \max_{k < e^{(1-\varepsilon/2)\kappa\phi(t)}} \mathcal{L}(m_k, t) \right) \\ & \leq \mathbb{P} \left( \max_{k \leq e^{(1-\varepsilon)\kappa\phi(t)}} \hat{\ell}_k \geq \max_{k < e^{(1-\varepsilon/2)\kappa\phi(t)}} \hat{\ell}_k \right) + o(1), \end{aligned}$$

with  $(\widehat{\ell}_k, k \leq e^{(1-\varepsilon/2)\kappa\phi(t)})$  i.i.d. random variables under  $\mathbb{P}$  by strong Markov property and the second part of Lemma 2.2.1, and with queue distributions given by (2.4.1) and Proposition 2.3.5.

It is then clear that for large  $t$ ,  $\mathbb{P}(\max_{k \leq e^{(1-\varepsilon)\kappa\phi(t)}} \widehat{\ell}_k \geq \max_{k \leq e^{(1-\varepsilon/2)\kappa\phi(t)}} \widehat{\ell}_k) = o(1)$ , and we therefore obtain that  $\mathbb{P}(N_t^* \geq e^{(1-\varepsilon)\kappa\phi(t)}) = 1 - o(1)$ .

Then, following the work of [37], we know that  $(m_i - m_{i-1}, i \geq 2)$  are i.i.d. random variables with a known Laplace transform (given by (2.19) in [37]), this allows to compute the first and fourth moments of  $\Delta m_1 := m_2 - m_1$  and therefore obtain after an elementary but tedious computation that for large  $t$ ,  $\mathbb{E}(\Delta m_1) \sim C_7 e^{\kappa h t}$  ( $C_7 > 0$ , see also (2.17) in [37]) and  $\mathbb{E}((\Delta m_1 - \mathbb{E}(\Delta m_1))^4) \sim C_8 e^{4\kappa h t}$  ( $C_8 > 0$ ), which yields as  $t \rightarrow +\infty$  and  $k \rightarrow +\infty$ ,

$$\mathbb{E}[(m_k/k - \mathbb{E}(\Delta m_1))^4] \sim C_8 e^{4\kappa h t} / k^2.$$

These facts allow us to write by a Markov inequality that

$$\begin{aligned} & \mathbb{P}[|m_{N_t} - \mathbb{E}(\Delta m_1)N_t| > \varepsilon \mathbb{E}(\Delta m_1)N_t] \\ & \leq \sum_{j \geq e^{(1-\varepsilon)\kappa\phi(t)}} \mathbb{P}[|m_j - \mathbb{E}(\Delta m_1)j| > \varepsilon \mathbb{E}(\Delta m_1)j] + o(1) \\ & \leq \sum_{j \geq e^{(1-\varepsilon)\kappa\phi(t)}} \frac{2C_8(C_7)^{-4}}{\varepsilon^4 j^2} + o(1) \\ & \leq C_+ \varepsilon^{-4} e^{-(1-\varepsilon)\kappa\phi(t)} + o(1). \end{aligned}$$

This yields that  $\{|m_{N_t} - \mathbb{E}(\Delta m_1)N_t| \leq \varepsilon \mathbb{E}(\Delta m_1)N_t\}$  as well as (with a similar computation)  $\{|m_{N_t^*} - \mathbb{E}(\Delta m_1)N_t^*| \leq \varepsilon \mathbb{E}(\Delta m_1)N_t^*\}$  are realized with a probability close to one.

Now including these events in the probability in (2.5.45), eventually enlarging  $v(\varepsilon, t)$  we get

$$\mathbb{P}\left[\frac{N_t^*}{N_t} \leq x \frac{(1-\varepsilon)^2}{(1+\varepsilon)^2}\right] - v(\varepsilon, t) \leq \mathbb{P}\left[\frac{F_t^*}{X(t)} \leq x\right] \leq \mathbb{P}\left[\frac{N_t^*}{N_t} \leq x \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}\right] + v(\varepsilon, t).$$

Notice that the random variables involved now ( $N_t^*$  and  $N_t$ ) only depend of what happens in the bottom of the  $h_t$ -valleys, and we have to deal with

$$\mathbb{P}\left[\frac{N_t^*}{N_t} \leq y\right] = \mathbb{P}[N_t^* = N_t] \mathbf{1}_{\{y=1\}} + \mathbb{P}\left[\frac{N_t^*}{N_t} \leq y, N_t^* < N_t\right] \mathbf{1}_{\{y \leq 1\}} + \mathbf{1}_{\{y > 1\}},$$

for any  $y > 0$ . We are now interested in the limit when  $t$  goes to infinity of the above two probabilities. We first use the same lines as for the proof of Section 5.1, that is to say we give a lower and an upper bound of this probability involving the i.i.d. sequences  $(\ell_j, j)$  and  $(\mathcal{H}_j, j)$ . In the same way we have obtained Proposition 2.5.1, we then have for any  $\varepsilon > 0$  and large  $t$ ,

$$\tilde{\mathcal{P}} - v(\varepsilon, t) \leq \mathbb{P}(N_t^* = N_t) \leq \tilde{\mathcal{P}} + v(\varepsilon, t)$$

with

$$\tilde{\mathcal{P}} := \mathbb{P} \left[ (1 - \bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon-1}}) \frac{\bar{\ell}_{\mathcal{N}_t^{2\varepsilon}} - \bar{\ell}_{\mathcal{N}_t^{2\varepsilon-1}}}{(\bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}} - \bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon-1}})} > \max_{1 \leq j \leq \mathcal{N}_t^{2\varepsilon-1}} \frac{\ell_j}{t} \right],$$

recall that  $\bar{\mathcal{H}}_k = Y_2(ke^{-\kappa\phi(t)}) = \frac{1}{t} \sum_{i=1}^k \mathcal{H}_i$ ,  $\bar{\ell}_k = Y_1(ke^{-\kappa\phi(t)}) = \frac{1}{t} \sum_{i=1}^k \ell_i$ ,  $\mathcal{N}_t^{2\varepsilon} := \inf\{m \geq 1, \bar{\mathcal{H}}_m > 1 - 2\varepsilon\}$ , and  $v$  is a positive function such that  $\lim_{t \rightarrow +\infty} v(\varepsilon, t) \leq \text{const} \times \varepsilon^{\kappa \wedge (1-\kappa)}$  with an eventually larger const than in Proposition 2.5.1. In the same way, for any  $y > 0$ ,  $\varepsilon > 0$  and  $t$  large enough,

$$\bar{\mathcal{P}}_1^- - v(\varepsilon, t) \leq \mathbb{P} \left[ \frac{N_t^*}{N_t} \leq y, N_t^* < N_t \right] \mathbb{1}_{y \leq 1} \leq \bar{\mathcal{P}}_1^+ + v(\varepsilon, t),$$

where

$$\begin{aligned} & \tilde{\mathcal{P}}_1^\pm \\ & := \mathbb{P} \left[ \mathcal{N}_t^* / \mathcal{N}_t^{2\varepsilon} \leq y \pm \varepsilon, (1 - \bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon-1}}) \frac{\bar{\ell}_{\mathcal{N}_t^{2\varepsilon}} - \bar{\ell}_{\mathcal{N}_t^{2\varepsilon-1}}}{(\bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}} - \bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon-1}})} \leq \max_{1 \leq j \leq \mathcal{N}_t^{2\varepsilon-1}} \frac{\ell_j}{t} \right] \mathbb{1}_{y \leq 1}, \end{aligned}$$

with  $\mathcal{N}_t^* := \min\{j \geq 1, \ell_j = \max_{k \leq \mathcal{N}_t^{2\varepsilon}} \ell_k\}$ . This together with Lemma 2.4.6 yields for large  $t$ ,

$$|\mathbb{P}[N_t^* = N_t] - \mathbb{P}[\mathcal{I}_1 < \mathcal{I}_2]| \leq \lim_{t \rightarrow +\infty} v(\varepsilon, t) + o(1)$$

and

$$\begin{aligned} & \left| \mathbb{P} \left[ \frac{N_t^* e^{-\kappa\phi(t)}}{N_t e^{-\kappa\phi(t)}} \leq y, N_t^* < N_t \right] - \mathbb{P} \left[ \frac{F^*(\mathcal{Y}_1, \mathcal{Y}_2)}{\mathcal{Y}_2^{-1}(1)} \leq y, \mathcal{I}_1 \geq \mathcal{I}_2 \right] \right| \\ & \leq \lim_{t \rightarrow +\infty} v(\varepsilon, t) + o(1), \end{aligned}$$

where  $F^*$  is defined at the beginning of Section 2.4.3. Replacing  $y$  by  $x \frac{(1-\varepsilon)^2}{(1+\varepsilon)^2}$  for the lower bound and by  $x \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}$  for the upper bound and taking the limit when  $t$  goes to infinity and then  $\varepsilon \rightarrow 0$  we obtain for  $0 < x < 1$ ,

$$\lim_{t \rightarrow +\infty} \mathbb{P} \left[ \frac{N_t^*}{N_t} \leq x \right] = \mathbb{P} \left[ \frac{F^*(\mathcal{Y}_1, \mathcal{Y}_2)}{\mathcal{Y}_2^{-1}(1)} \leq x, \mathcal{I}_1 \geq \mathcal{I}_2 \right].$$

To finish the proof of the last result of Theorem 2.1.5 we finally have to prove Lemma 2.5.5 below.

**Lemma 2.5.5.** *The random variable  $\frac{F^*(\mathcal{Y}_1, \mathcal{Y}_2)}{\mathcal{Y}_2^{-1}(1)}$  follows a uniform law  $U_{[0,1]}$  and is independent of the couple  $(\mathcal{I}_1, \mathcal{I}_2)$ .*

**Proof :**

For any  $s > 0$ , let  $\mathcal{G}_1(s) := \inf\{u \leq s, \mathcal{Y}_1(u) - \mathcal{Y}_1(u-) = \mathcal{Y}_1^\sharp(s)\}$ . The fact that for every  $s > 0$ ,  $\mathcal{G}_1(s)/s$  follows a uniform distribution is basic. Since the independence that we seek is specific we give some details.

The process of the jumps of  $(\mathcal{Y}_1, \mathcal{Y}_2)$  in  $[0, s]$  is a Poisson point process in  $[0, s] \times (\mathbb{R}_+)^2$  (the coordinate in  $[0, s]$  for the instant when the jump occurs and the other coordinate for the jump) with intensity measure  $\lambda \times \nu$  where  $\lambda$  is the Lebesgue measure on  $[0, s]$  and  $\nu$ , as defined in the introduction, is the Lévy measure of  $(\mathcal{Y}_1, \mathcal{Y}_2)$ . Let us give a particular construction of the process  $(\mathcal{Y}_1, \mathcal{Y}_2)$  on  $[0, s]$  :

Let  $(P_n)_{n \geq 1}$  be a countable partition of  $(\mathbb{R}_+)^2$  by Borelian sets such that  $\forall n \geq 1$ ,  $0 < \nu(P_n) < +\infty$ . For each  $n$  we define an *i.i.d.* sequence  $(S_k^n)_{k \geq 1}$  of random variables in  $(\mathbb{R}_+)^2$ , an *i.i.d.* sequence  $(U_k^n)_{k \geq 1}$  of random variables in  $[0, s]$  and a random variable  $T_n$  such that

- $\forall n \geq 1$ ,  $S_1^n \sim \nu(\cdot \cap P_n)/\nu(P_n)$ ,  $U_1^n \sim U_{[0, s]}$ ,  $T_n \sim \mathcal{P}(s\nu(P_n))$ ,
- For any  $n \geq 1$ , the variables  $(S_k^n)_{k \geq 1}$ ,  $(U_k^n)_{k \geq 1}$  and  $T_n$  are independent,
- The triplets  $((S_k^n)_{k \geq 1}, (U_k^n)_{k \geq 1}, T_n)_{n \geq 1}$  are independent,

where  $U$  stands for uniform and  $\mathcal{P}(\cdot)$  for Poisson distribution. We know that the random set

$$\mathcal{S}_n := \{(U_k^n, S_k^n), n \geq 1, 1 \leq k \leq T_n\}$$

is a Poisson point process in  $[0, s] \times (\mathbb{R}_+)^2$  with intensity measure  $\lambda \times \nu$ . Since  $(\mathcal{Y}_1, \mathcal{Y}_2)$  is pure jump, its restriction to  $[0, s]$  is equal in law to the process  $(\mathcal{Z}_1, \mathcal{Z}_2)$  defined by

$$\forall r \in [0, s], \quad (\mathcal{Z}_1, \mathcal{Z}_2)(r) = \sum_{n \geq 1, 1 \leq k \leq T_n} S_k^n \mathbb{1}_{U_k^n \leq r}.$$

In particular, with  $\pi_i(x_1, x_2) := x_i$  for  $(x_1, x_2) \in \mathbb{R}^2$ ,  $i \in \{1, 2\}$  and  $\mathcal{G}_1^{\mathcal{Z}}(s) := \inf\{u \leq s, \mathcal{Z}_1(u) - \mathcal{Z}_1(u-) = \mathcal{Z}_1^\sharp(s)\} \stackrel{\mathcal{L}}{=} \mathcal{G}_1(s)$ , we have

$$\begin{aligned} \mathcal{Z}_1^\sharp(s) &= \max\{\pi_1(S_k^n), n \geq 1, 1 \leq k \leq T_n\}, \\ \mathcal{G}_1^{\mathcal{Z}}(s) &= \inf\left\{U_k^n, n \geq 1, 1 \leq k \leq T_n, \pi_1(S_k^n) = \mathcal{Z}_1^\sharp(s)\right\}, \\ \mathcal{Z}_1(s) &= \sum_{n \geq 1, 1 \leq k \leq T_n} \pi_1(S_k^n), \quad \mathcal{Z}_2(s) = \sum_{n \geq 1, 1 \leq k \leq T_n} \pi_2(S_k^n). \end{aligned}$$

$\mathcal{G}_1^{\mathcal{Z}}(s)$  is the position of the highest jump of  $Z_1$  on  $[0, s]$  We thus have that  $\mathcal{G}_1(s)/s \stackrel{\mathcal{L}}{=} U_{[0, 1]}$  and it is independent from  $(\mathcal{Y}_1^\sharp(s), \mathcal{Y}_1(s), \mathcal{Y}_2(s))$  and from the sigma-field  $\sigma((\mathcal{Y}_1, \mathcal{Y}_2)(t+s) - (\mathcal{Y}_1, \mathcal{Y}_2)(s), t \geq 0)$ .

We now have to replace  $s$  by  $\mathcal{Y}_2^{-1}(1)$ . For that we can consider for example the dyadic approximations of  $\mathcal{Y}_2^{-1}(1)$ , that is,  $(t_n := \max\{k \in \mathbb{N}, \frac{k}{2^n} < \mathcal{Y}_2^{-1}(1)\}, n)$ . Then, partitioning on the values of  $t_n$ , using the independence we just proved and the fact that  $\mathcal{G}_1(s)/s$  follows a uniform distribution on  $[0, 1]$  we get that  $\mathcal{G}_1(t_n)/t_n$  follows a uniform distribution on  $[0, 1]$  and is independent from

$$((\mathcal{Y}_1^\sharp(t_n), \mathcal{Y}_2(t_n), \mathcal{Y}_1(t_n + 2^{-n}) - \mathcal{Y}_1(t_n), \mathcal{Y}_2(t_n + 2^{-n}) - \mathcal{Y}_2(t_n))). \quad (2.5.46)$$



We let  $n$  goes to infinity,  $t_n$  converges almost surely to  $\mathcal{Y}_2^{-1}(1)$  from below. As a consequence,  $\mathcal{G}_1(t_n)/t_n$  converges almost surely to  $\frac{F^*(\mathcal{Y}_1, \mathcal{Y}_2)}{\mathcal{Y}_2^{-1}(1)}$  while the quadruple in (2.5.46) converges almost surely to

$$\begin{aligned} & (\mathcal{Y}_1^\sharp(\mathcal{Y}_2^{-1}(1)-), \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)-), \\ & \quad \mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)-), \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)-)). \end{aligned}$$

As a consequence,  $\frac{F^*(\mathcal{Y}_1, \mathcal{Y}_2)}{\mathcal{Y}_2^{-1}(1)}$  follows a uniform distribution on  $[0, 1]$  and is independent from the above quadruple for which  $(\mathcal{I}_1, \mathcal{I}_2)$  is a measurable function, this yields the lemma.  $\square$

## 2.6 Results and additional arguments from the paper [3]

### 2.6.1 Some estimates on the diffusion $X$

The first lemma below gives the right normalisation in law of the number of  $h_t$ -valleys visited by  $X$  before time  $t$ .

**Lemma 2.6.1** (number of visited  $h_t$ -valleys). *Assume that  $0 < \kappa < 1$ . Then, under the annealed law  $\mathbb{P}$ ,  $N_t e^{-\kappa\phi(t)} \rightarrow_{t \rightarrow +\infty} \mathcal{N}$  in law. The law of  $\mathcal{N}$  is determined by its Laplace transform :*

$$\forall u > 0, \quad \mathbb{E}(e^{-u\mathcal{N}}) = \sum_{j=0}^{+\infty} \frac{1}{\Gamma(\kappa j + 1)} \left( \frac{-u}{\mathcal{C}_\kappa} \right)^j, \quad (2.6.1)$$

where  $\mathcal{C}_\kappa$  is a positive constant. Moreover  $\mathbb{P}(N_t > n_t) \leq e^{-\phi(t)}$ .

**Proof :** The convergence in distribution is exactly Proposition 1.6 of [3]. For the second fact we have  $\mathbb{P}(N_t \geq n_t) \leq \mathbb{P}(\tilde{N}_t \geq n_t) + \mathbb{P}(\tilde{\mathcal{V}}_t) \leq \mathbb{P}(\tilde{N}_t \geq n_t) + e^{[-\kappa/2 + o(1)]h_t}$  by Lemma 2.2.1, with  $\tilde{N}_t := \max\{j \geq 1, \tilde{m}_j \leq \sup_{s \leq t} X(s)\}$ . Then equation (5.3) in [3] gives  $\mathbb{P}(\tilde{N}_t \geq n_t) \leq \exp(-2\phi(t))$ , which yields the result.  $\square$

The lemma below deals with the renewal structure we speak about on the introduction, and the consequence on the hitting time  $H(m_{N_t})$  of the ultimate  $h_t$ -valley visited by  $X$  before time  $t$ .

**Lemma 2.6.2.** *Assume  $0 < \kappa < 1$  and  $0 < \delta < \inf\{2/27, \kappa^2/2\}$ . For  $t > 0$ , let  $\mu_t$  be the positive measure on  $\mathbb{R}_+$  such that*

$$\forall x \geq 0, \quad \mu_t([0, x]) := e^{-\kappa\phi(t)} \sum_{j=1}^{n_t} \mathbb{P}(\bar{\mathcal{H}}_j \leq x).$$

Recall that for any  $k$ ,  $\bar{\mathcal{H}}_k := \sum_{j=1}^k \mathcal{H}_j/t$ , and  $\mathcal{H}_1 = R_1 S_1 \mathbf{e}_1$  is defined in Proposition 2.3.5. Then,  $(\mu_t)_t$  converges vaguely as  $t \rightarrow +\infty$  to  $\mu$  defined by

$$d\mu(x) := (\mathcal{C}_\kappa \Gamma(\kappa))^{-1} x^{\kappa-1} \mathbf{1}_{(0,+\infty)}(x) dx,$$

with  $\mathcal{C}_\kappa$  is the same constant as in Lemma 2.6.1. For  $0 \leq r < s \leq 1$ ,

$$\lim_{t \rightarrow +\infty} \mathbb{P} \left( 1 - s \leq \frac{H(m_{N_t})}{t} \leq 1 - r \right) = \frac{\sin(\pi\kappa)}{\pi} \int_{1-s}^{1-r} x^{\kappa-1} (1-x)^{-\kappa} dx. \quad (2.6.2)$$

**Proof :** The first part of the above lemma is very close to Lemma 5.1 of [3], indeed Proposition 2.3.5 gives the proximity between the random variables  $(U_i, i \leq n_t)$  and the random variables  $(\mathcal{H}_i, i \leq n_t)$ , moreover an important preliminary result in [3] (Proposition 4.1) states that  $e^{\kappa\phi(t)}(1 - \mathbb{E}(e^{-\lambda U_1/t})) = \mathcal{C}_\kappa \lambda^\kappa + o(1)$  for large  $t$ . So we also know that

$$e^{\kappa\phi(t)}(1 - \mathbb{E}(e^{-\lambda \mathcal{H}_1/t})) = \mathcal{C}_\kappa \lambda^\kappa + o(1), \quad (2.6.3)$$

notice that this result could also be deduced from (2.4.2) with the help of a Tauberian theorem. Then by independence of the random variables  $\mathcal{H}_j$  and the fact that they are i.i.d., for any  $\lambda > 0$

$$\int_0^{+\infty} e^{-\lambda x} d\mu_t(x) = \frac{1}{e^{\kappa\phi(t)}} \sum_{j=1}^{n_t} \left( \mathbb{E} \left( e^{-\lambda \frac{\mathcal{H}_1}{t}} \right) \right)^j$$

By (2.6.3) as  $n_t e^{-\kappa\phi(t)} \rightarrow_{t \rightarrow +\infty} +\infty$ ,  $[\mathbb{E}(e^{-\lambda \mathcal{H}_1/t})]^{n_t+1} = o(1)$ . Hence, we get as  $t \rightarrow +\infty$ , again by 2.6.3

$$\begin{aligned} \int_0^{+\infty} e^{-\lambda x} d\mu_t(x) &= \frac{e^{-\kappa\phi(t)}(1 + o(1))}{1 - \mathbb{E}(e^{-\lambda \mathcal{H}_1/t})} + o(1) = \frac{1}{\mathcal{C}_\kappa \lambda^\kappa} + o(1) \\ &= \int_0^{+\infty} \frac{e^{-\lambda x} x^{\kappa-1}}{\mathcal{C}_\kappa \Gamma(\kappa)} dx + o(1), \end{aligned}$$

which gives the vague convergence of measure  $(\mu_t)_t$ . Also (2.6.2) is equation (1.2) of Corollary 1.5 in [3].  $\square$

In Lemma 2.6.3 below, we approximate  $\tilde{h}_j$ , the exit time of  $h_t$ -valley number  $j$  (if  $X$  leaves it on the right), by a product of 3 simpler random variables. To this aim, we recall that with the notation of Lemma 2.3.6 and of its proof, for each  $1 \leq j \leq n_t$ ,  $\tilde{R}_j = \int_{\tilde{\tau}_j^-(h_t/2)}^{\tilde{\tau}_j^+(h_t/2)} e^{-\tilde{V}^{(j)}(x)} dx$ , and  $A^j(u) = \int_{\tilde{m}_j}^u e^{\tilde{V}^{(j)}(x)} dx$ ,  $u \in \mathbb{R}$ . Moreover, for some independent Brownian motions  $B^j$ ,  $1 \leq j \leq n_t$ , independent of  $W_\kappa$ ,

$$\begin{aligned} \tilde{h}_j &= \int_{\tilde{L}_j^-}^{\tilde{L}_j^+} e^{-\tilde{V}^{(j)}(u)} \mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)] du, \\ \mathbf{e}_j &= \mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), 0] / A^j(\tilde{L}_j). \end{aligned}$$

**Lemma 2.6.3.** *Let  $0 < \varepsilon < \inf\{2/27, \kappa^2/2\}$ . For large  $t$ , we have for every  $1 \leq j \leq n_t$ ,*

$$\mathbb{P}\left(\left|\tilde{h}_j - A^j(\tilde{L}_j)\mathbf{e}_j\tilde{R}_j\right| > 2e^{-(1-3\varepsilon)h_t/6}A^j(\tilde{L}_j)\mathbf{e}_j\tilde{R}_j\right) \leq C_+e^{-(c_-)\varepsilon h_t}. \quad (2.6.4)$$

**Proof :** We first notice that  $(\tilde{h}_j, A^j(\tilde{L}_j), \mathbf{e}_j, \tilde{R}_j)$  is measurable with respect to the  $\sigma$ -field generated by  $(\tilde{V}^{(j)}(x + \tilde{L}_{j-1}^+), 0 \leq x \leq \tilde{L}_j^+ - \tilde{L}_{j-1}^+)$  and  $B^j$ , so, thanks to the second fact of Lemma 2.2.1, its law under  $\mathbb{P}$  does not depend on  $j$ . Thus, the left hand side of (2.6.4) does not depend on  $j$ . Hence we just have to prove (2.6.4) for  $j = 2$ .

This is actually already proved in [3], for which it is an important step. Indeed in this paper [3], our  $A^j$ ,  $\tilde{B}^2$  and  $\tilde{h}_2$  are denoted respectively by  $\tilde{A}_j$ ,  $B$  and  $\mathbf{U}$ , as defined in ([3], eq. (3.17) and (3.18)), and our  $\tilde{R}_2$  and  $\mathbf{e}_2$  by  $\tilde{\mathcal{I}}^-$  and  $\mathbf{e}_1$ , as defined in ([3], after eq. (4.17)). Hence our (2.6.4) for  $j = 2$  is exactly ([3], Lemma 4.7), which proves our lemma.

The proof of ([3], Lemma 4.7) is quite technical, however we can give a simple heuristic in order for the present paper to be more self-contained. The idea of the proof of ([3], Lemma 4.7) is that, loosely speaking, for  $u$  close to  $\tilde{m}_j$ , that is for  $u \in [\tilde{\tau}_j^-(h_t/2), \tilde{\tau}_j(h_t/2)]$ ,  $\mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)]$  is nearly  $\mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), 0] = A^j(\tilde{L}_j)\mathbf{e}_j$ , whereas for  $u$  far from  $\tilde{m}_j$ , that is for  $u \in [\tilde{L}_j^-, \tilde{L}_j]$  but  $u \notin [\tilde{\tau}_j^-(h_t/2), \tilde{\tau}_j(h_t/2)]$ ,  $e^{-\tilde{V}^{(j)}(x)}$  is "nearly" 0, with large probability. Finally, combining these heuristics gives  $\tilde{h}_j \approx A^j(\tilde{L}_j)\mathbf{e}_j\tilde{R}_j$ .  $\square$

The following lemma is used to prove Lemma 2.3.6 and uses the notation of this lemma, and where the independent r.v.  $G^+(h_t/2, h_t)$ ,  $F_1^+(h_t)$ ,  $F_2^-(h_t/2)$  and  $F_3^-(h_t/2)$  defined before Proposition 2.3.5.

**Lemma 2.6.4.** *Assume  $0 < \delta < \inf\{2/27, \kappa^2/2\}$ . For large  $t$ , possibly on an enlarged probability space, there exists  $R_2 \stackrel{\mathcal{L}}{=} F_2^-(h_t/2) + F_3^-(h_t/2)$  and  $S_2 \stackrel{\mathcal{L}}{=} F_1^+(h_t) + G^+(h_t/2, h_t)$ , such that  $R_2$ ,  $S_2$  and  $\mathbf{e}_2$  are independent and*

$$P\left(\left\{\left|\int_{\tilde{m}_2}^{\tilde{L}_2} e^{\tilde{V}^{(2)}(x)}dx - S_2\right| \leq e^{-(d_-)h_t}S_2, \tilde{R}_2 = R_2\right\}\right) \geq 1 - e^{-(D_-)h_t}, \quad (2.6.5)$$

where  $D_- > 0$ .

**Proof :** Due to ([3] Lemma 4.5) with its notation, we have  $\mathcal{I}_0^+ := \int_{m_2}^{\tau_2(h_t)} e^{V^{(2)}(x)}dx \stackrel{\mathcal{L}}{=} F^+(h_t)$ ,  $\mathcal{I}_2^+ := \int_{\tau_2(h_t)}^{L_2} e^{V^{(2)}(x)}dx \stackrel{\mathcal{L}}{=} G^+(h_t/2, h_t)$ ,  $\mathcal{I}_1^- := \int_{m_2}^{\tau_2(h_t/2)} e^{-V^{(2)}(x)}dx \stackrel{\mathcal{L}}{=} F^-(h_t/2)$  and finally  $\mathcal{I}_2^- := \int_{\tau_2^-(h_t/2)}^{m_2} e^{-V^{(2)}(x)}dx \stackrel{\mathcal{L}}{=} F^-(h_t/2)$  with  $L_2 := \inf\{x > \tau_2(h_t), V^{(2)}(x) = h_t/2\}$ . The problem is that  $\mathcal{I}_0^+$  is not independent of  $\mathcal{I}_1^-$ , so we would like to replace

it by some  $\mathcal{I}_1^+ \stackrel{\mathcal{L}}{=} \mathcal{I}_0^+$  of it with better independence properties. It is proved in ([3], at the top of page 32) that for large  $t$ , possibly in an enlarged probability space, there exists  $\mathcal{I}_1^+$  such that  $|\mathcal{I}_0^+ - \mathcal{I}_1^+| \leq e^{-(1-3\delta)h_t/2} \mathcal{I}_1^+$  with probability greater than  $1 - 4e^{-\kappa\delta h_t/2}$  and where  $\mathcal{I}_1^+ \stackrel{\mathcal{L}}{=} F^+(h_t)$  by ([3], eq. (4.35)).

Let  $S_2 := \mathcal{I}_1^+ + \mathcal{I}_2^+ \geq \mathcal{I}_1^+$ . Notice that on  $\mathcal{V}_t$ , by Remark 2.2.1,  $\tilde{R}_2 = \mathcal{I}_1^- + \mathcal{I}_2^- =: R_2$  and  $\int_{\tilde{m}_2}^{\tilde{L}_2} e^{\tilde{V}^{(2)}(x)} dx = \int_{m_2}^{L_2} e^{V^{(2)}(x)} dx = \mathcal{I}_0^+ + \mathcal{I}_2^+$ . The two previous inequalities give  $|\int_{\tilde{m}_2}^{\tilde{L}_2} e^{\tilde{V}^{(2)}(x)} dx - S_2| = |\mathcal{I}_0^+ - \mathcal{I}_1^+| \leq e^{-(1-3\delta)h_t/2} S_2$  and  $\tilde{R}_2 = R_2$  with probability at least  $1 - 5e^{-\kappa\delta h_t/2}$  thanks to Lemma 2.2.1. This proves (2.6.5).

Moreover, by ([3], Prop. 4.4 (i)),  $\mathcal{I}_1^+, \mathcal{I}_2^+, \mathcal{I}_1^-, \mathcal{I}_2^-$  and  $\mathbf{e}_2$  (which is denoted by  $\mathbf{e}_1$  in [3]) are independent. So,  $\mathbf{e}_2, S_2 = \mathcal{I}_1^+ + \mathcal{I}_2^+$  and  $R_2 = \mathcal{I}_1^- + \mathcal{I}_2^-$  are independent, and  $R_2 \stackrel{\mathcal{L}}{=} F_2^-(h_t/2) + F_3^-(h_t/2)$  and  $S_2 \stackrel{\mathcal{L}}{=} F_1^+(h_t) + G^+(h_t/2, h_t)$ .  $\square$

The last lemma of this section tells that with large probability, the diffusion  $X$  leaves every  $h_t$ -valley  $[\tilde{L}_j^-, \tilde{L}_j]$ ,  $1 \leq j \leq n_t$  from its right. Recall that  $B^j$  is defined after (2.3.19).

**Lemma 2.6.5.** *For large  $t$ , there exists  $c_- > 0$  such that*

$$\mathbb{P} \left[ \bigcap_{j=1}^{n_t} \left\{ \max_{u < \tilde{L}_j^-} \mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)] = 0 \right\} \right] \geq 1 - e^{-(c_-)h_t}. \quad (2.6.6)$$

**Proof :** (2.6.6) is essentially Lemma 3.2 in [3] :

Indeed, recall the definition of  $\mathcal{A}_j := \{\max_{u < \tilde{L}_j^-} \mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)] = 0\}$ , we have  $\bigcap_{j=1}^{n_t} \mathcal{A}_j = \bigcap_{j=1}^{n_t} \{H_j(\tilde{L}_j) < \{H_j(\tilde{L}_j^-)\}\}$ , with, for any  $\tilde{L}_j^- \leq x \leq \tilde{L}_j$ ,  $H_j(x) = \inf\{s > 0, B_j(s) = x\}$ , with  $B_j$  a Brownian motion. Therefore  $\mathbb{P}^{W_\kappa}(\mathcal{A}_j)$  is equal to the probability  $\mathbb{P}^{W_\kappa}(\bar{\mathcal{E}}_j)$  of Lemma 3.2 in [3]. It is proved in this lemma see (3.10) that for large  $t$ ,  $P(\mathcal{B} := \{\mathbb{P}^{W_\kappa}(\bar{\mathcal{E}}_j) \leq e^{-(\kappa/2)h_t}\}) \geq 1 - 3e^{-\kappa\delta h_t}$ , so we obtain (2.6.6) as  $\mathbb{P}(\bar{\mathcal{E}}_j) \leq E(\mathbb{P}^{W_\kappa}(\bar{\mathcal{E}}_j) \mathbf{1}_{\mathcal{B}}) + P(\bar{\mathcal{B}}) \leq e^{-c_- h_t}/n_t$ , for  $c_- > 0$  small enough.  $\square$

## 2.6.2 Some estimates on the potential $W_\kappa$ and its functionals

We start this section with the Laplace transform of the important functional  $\mathcal{R}_\kappa$  :

**Lemma 2.6.6.** *Recall that  $0 < \kappa < 1$ . For any  $\gamma > 0$ ,*

$$E(e^{-\gamma \mathcal{R}_\kappa}) = \left( \frac{(2\gamma)^{\kappa/2}}{\kappa \Gamma(\kappa) I_\kappa(2\sqrt{2\gamma})} \right)^2. \quad (2.6.7)$$

Moreover,  $\mathcal{R}_\kappa$  admits moments of any positive order.

**Proof :**  $\int_0^{+\infty} e^{-W_\kappa^\uparrow(u)} du$  is the limit in law under  $P$  of  $\int_0^{\tau^{W_\kappa^\uparrow}(x)} e^{-W_\kappa^\uparrow(u)} du$  as  $x \rightarrow +\infty$ . This limit is given by ([3], Lemma 4.2), which proves (2.6.7). Note that in ([3], Lemma 4.2),  $W_\kappa^\uparrow$  is denoted by  $R$ , and  $\int_0^{\tau^{W_\kappa^\uparrow}(x)} e^{-W_\kappa^\uparrow(u)} du$  is denoted respectively by  $F^-(x)$ . Moreover the Laplace transform of  $\mathcal{R}_\kappa$  is of class  $C^\infty$  on a neighborhood of 0 since  $x \mapsto x^\kappa/I_\kappa(x)$  is  $C^\infty$  on such a neighborhood (see e.g. [15] p. 638). Therefore  $\mathcal{R}_\kappa$  admits moments of any positive order.  $\square$

The following Lemma is a series of estimates concerning the different coordinates of valleys.

**Lemma 2.6.7.** *For  $t$  large enough, for every  $1 \leq i \leq n_t$ ,*

$$P(0 < M_0 < m_1) \leq C_+ h_t e^{-\kappa h_t}, \quad (2.6.8)$$

$$P(\tilde{\tau}_{i+1}^*(h_t) \neq \tilde{\tau}_{i+1}(h_t)) \leq C_+ h_t e^{-\kappa h_t}, \quad (2.6.9)$$

$$P\left(\inf_{[\tilde{\tau}_i^-(h_t^+), \tilde{\tau}_i^-(h_t)]} \tilde{V}^{(i)} < h_t/2\right) \leq e^{-\kappa h_t/8}, \quad (2.6.10)$$

$$P(\tilde{L}_i^+ - \tilde{L}_i^- \geq 40h_t^+/\kappa) \leq e^{-\kappa h_t/8}, \quad (2.6.11)$$

$$P(\tilde{\tau}_i(h) - \tilde{m}_i \geq 8h/\kappa) \leq C_+ e^{-\kappa h/(2\sqrt{2})}, \quad 0 \leq h \leq h_t, \quad (2.6.12)$$

$$P(\tilde{m}_1 \leq r) \leq e^r \exp\left((\kappa/2 - \sqrt{2 + \kappa^2/4})h_t^+\right) = o(1), \quad \forall r = o(h_t^+). \quad (2.6.13)$$

**Proof :** (2.6.8) follows from eq. (2.8) of [3]; (2.6.9) is eq. (3.41) of [3]. (2.6.10) and (2.6.11) are respectively eq. (2.34) and (2.32) of Lemma 2.7 of [3]. Moreover, (2.6.12) is eq. (2.22) of the same reference. For (2.6.13), we know from definitions in (2.2.3) that  $\tilde{m}_1 \geq \tilde{L}_1^\# = \tau^{W_\kappa}(-h_t^+)$ , where  $\tau^{W_\kappa}(-h_t^+)$  is the first positive time the drifted Brownian motion  $W_\kappa$  reaches  $-h_t$ . Using a Markov inequality together with (2.0.1) page 295 of [15] we obtain  $P(\tau^{W_\kappa}(-h_t^+) \leq r) = P(e^{-\tau^{W_\kappa}(-h_t^+)} \geq e^{-r}) \leq e^r e^{(\kappa/2 - \sqrt{2 + \kappa^2/4})h_t^+}$ , which is exactly (2.6.13).  $\square$

The lemma below deals with two functionals involving coordinates far from the bottom  $\tilde{m}_1$  of the first visited  $h_t$ -valley  $[\tilde{L}_1^-, \tilde{L}_1]$ .

**Lemma 2.6.8.** *There exists  $c_- > 0$  such that for any  $\varepsilon > 0$  and  $t$  large enough,*

$$P\left(\int_{\tilde{\tau}_1(h_t/2)}^{\tilde{L}_1} e^{-\tilde{V}^{(1)}(x)} dx \leq C_+ h_t^2 e^{-(1-\varepsilon)h_t/2}\right) \geq 1 - e^{-(c_-)\varepsilon h_t},$$

$$P\left(\int_{\tilde{L}_1^-}^{\tilde{\tau}_1^-(h_t/2)} e^{-\tilde{V}^{(1)}(x)} dx \leq C_+ h_t^2 e^{-(1-\varepsilon)h_t/2}\right) \geq 1 - e^{-(c_-)\varepsilon h_t}.$$

**Proof :** The proof is inspired from steps 1 and 2 of Lemma 4.7 of [3]. For the first integral, let

$$\mathcal{A}_1 := \left\{ \inf_{[\tilde{\tau}_1(h_t/2), \tilde{\tau}_1(h_t)]} \tilde{V}^{(1)} > (1 - \varepsilon)h_t/2 \right\}, \quad \mathcal{A}_2 := \left\{ \tilde{L}_1^+ - \tilde{L}_1^- \leq 40h_t^+/\kappa \right\}.$$

We have on  $\mathcal{A}_1 \cap \mathcal{A}_2$ ,

$$\int_{\tilde{\tau}_1(h_t/2)}^{\tilde{L}_1} e^{-\tilde{V}^{(1)}(u)} du \leq e^{-(1-\varepsilon)h_t/2} [\tilde{L}_1 - \tilde{\tau}_1(h_t/2)] \leq \frac{40h_t^+ h_t}{\kappa} e^{-(1-\varepsilon)h_t/2}. \quad (2.6.14)$$

Now, Fact 2.2.1, equation (2.7.3) with  $\alpha = 1/2$ ,  $\gamma = (1 - \varepsilon)/2$  and  $\omega = 1$ , and Lemma 2.2.1 give

$$P(\overline{\mathcal{A}}_1) \leq P[\inf_{[\tau_1(h_t/2), \tau_1(h_t)]} V^{(1)} \leq (1 - \varepsilon)h_t/2, \mathcal{V}_t] + P(\overline{\mathcal{V}}_t) \leq 3e^{-\kappa\varepsilon h_t/2}.$$

Moreover,  $P(\overline{\mathcal{A}}_2) \leq e^{-\kappa h_t/8} \leq e^{-\kappa\varepsilon h_t/2}$  by (2.6.11) since we can take  $\varepsilon < 1/4$ . The second inequality, can be proved similarly.  $\square$

**Lemma 2.6.9.** *Recall that for  $h > 0$ ,  $\beta_0(h) := E\left(\int_0^{\tau_1^*(h)} e^{W_\kappa(u)} du\right)$ , with  $\tau_1^*(h) := \inf\{u \geq 0, W_\kappa(u) - \inf_{[0,u]} W_\kappa \geq h\}$ . For large  $h$ ,*

$$\beta_0(h) \leq C_+ e^{(1-\kappa)h}. \quad (2.6.15)$$

**Proof :** (2.6.15) is ([3], eq. (3.38)), since in [3],  $\beta_0(h)$  is defined at the top of page 23 and  $\tau_1^*(h)$  in its Lemma 3.6.  $\square$

## 2.7 Appendix

### 2.7.1 Some estimates for Brownian motion, Bessel processes, $W_\kappa^\uparrow$ and their functionals

We provide in this section some known formulas for some processes that appear in our study. The first lemma is about Laplace transforms of the exponential functionals defined in (2.1.7) and (2.1.8). Its proof can be found in ([3], Lemma 4.2). Recall that  $C_+$  (respectively  $c_-$ ) is a positive constant that is as large (resp. small) as needed.

**Lemma 2.7.1.** *There exist  $C_9 > 0$ ,  $M > 0$  and  $\eta_1 \in (0, 1)$  such that  $\forall y > M, \forall \gamma \in (0, \eta_1]$ ,*

$$\left| E\left(e^{-\gamma F^+(y)/e^y}\right) - [1 - 2\gamma/(\kappa + 1)] \right| \leq C_9 \max(e^{-\kappa y}, \gamma^{3/2}), \quad (2.7.1)$$

$$\left| E\left(e^{-\gamma G^+(y/2, y)/e^y}\right) - [1 - \Gamma(1 - \kappa)(2\gamma)^\kappa/\Gamma(1 + \kappa)] \right| \leq C_9 \max(\gamma^\kappa e^{-\kappa y/2}, \gamma). \quad (2.7.2)$$

Moreover, there exists  $C_{10} > 0$  such that for all  $y > 0$ ,  $E(F^+(y)/e^y) \leq C_{10}$ .

Recall that  $W_\kappa^\uparrow$  is a  $(-\kappa/2)$ -drifted Brownian motion  $W_\kappa$  Doob-conditioned to stay positive (see above (2.1.7)). We have,

**Lemma 2.7.2.** *Let  $0 < \gamma < \alpha < \omega$ . For all  $h$  large enough, we have*

$$P^{\alpha h} \left( \tau^{W_\kappa^\uparrow}(\gamma h) < \tau^{W_\kappa^\uparrow}(\omega h) \right) \leq 2e^{-\kappa(\alpha-\gamma)h}, \quad (2.7.3)$$

$$P \left( \tau^{W_\kappa^\uparrow}(\omega h) - \tau^{W_\kappa^\uparrow}(\alpha h) \leq 1 \right) \leq 4e^{-[(\omega-\alpha)h]^2/3}, \quad (2.7.4)$$

$$P(\tau^{W_\kappa^\uparrow}(h) > 8h/\kappa) \leq C_+ e^{-\kappa h/(2\sqrt{2})}, \quad (2.7.5)$$

$$P(\tau^{W_\kappa^\uparrow}(h) \leq h) \leq C_+ e^{-(c_-)h}, \quad (2.7.6)$$

$$P(\tau^{W_\kappa^\uparrow}(\gamma h) \leq 1) \leq C_+ e^{-(c_-)[\gamma h]^2}, \quad (2.7.7)$$

where  $P^{\alpha h}$  denotes the law of  $W_\kappa^\uparrow$  starting from  $\alpha h$ . Moreover the first inequality is still true if  $\omega$  is a function of  $h$  such that  $\lim_{h \rightarrow +\infty} \omega(h) = +\infty$ .

**Proof :** The first 3 inequalities come from ([3], Lemma 2.6). The fact that, in (2.7.3),  $\omega$  can actually be taken as a function of  $h$  comes directly from eq. (2.31) of [3], which shows that the right hand side of (2.7.3) is equivalent to  $e^{-\kappa(\alpha-\gamma)h}$  as  $h \rightarrow +\infty$  if  $w = w(h) \rightarrow_{h \rightarrow +\infty} +\infty$ . (2.7.7) is a consequence of (2.7.4) with  $\omega = \gamma$  and  $\alpha = \gamma/2$ . We turn to (2.7.6). By ([3], eq. (2.7) and Fact 2.1, coming from [37]),  $E(e^{-\alpha\tau^{W_\kappa^\uparrow}(h)}) \sim_{h \rightarrow +\infty} \text{const.} e^{h(\kappa/2 - \sqrt{2\alpha + \kappa^2/4})}$ , in particular for  $\alpha = 1 - \kappa$ . Then a Markov inequality for  $P(e^{-\alpha\tau^{W_\kappa^\uparrow}(h)} > e^{-\alpha h})$  proves (2.7.6) since  $1 - \kappa/2 - \sqrt{2(1 - \kappa) + \kappa^2/4} < 0$ .  $\square$

We also need the following lemma, focusing only on some exponential functionals.

**Lemma 2.7.3.** *Recall that  $F^\pm$  and  $G^+$  are defined in (2.1.7) and (2.1.8). For all  $0 < \zeta \leq 1$  and  $0 < \varepsilon < 1$ , for  $h$  large enough,*

$$P[e^{(1-\varepsilon)\zeta h} \leq F^+(\zeta h) \leq e^{(1+\varepsilon)\zeta h}] \geq 1 - 4e^{-\kappa\varepsilon\zeta h/2}, \quad (2.7.8)$$

$$P[F^-(h) \geq e^{-\varepsilon h}] \geq 1 - e^{-(c_-)\varepsilon^2 h^2}, \quad (2.7.9)$$

$$P[G^+(\alpha h, h) \leq b(h)e^h] \geq 1 - C_+[b(h)]^{-\kappa}, \quad 0 < \alpha < 1, b(h) > 0. \quad (2.7.10)$$

**Proof :** By Markov inequality and the last line of Lemma 2.7.1,

$$P[F^+(\zeta h) > e^{(1+\varepsilon)\zeta h}] \leq C_{10} e^{-\varepsilon\zeta h} \leq e^{-\kappa\varepsilon\zeta h/2}$$

for large  $h$ . For the lower bound, we have by ([3], eq. (2.29)) for large  $h$ ,

$$P[F^+(\zeta h) \geq e^{(1-\varepsilon)\zeta h}] \geq 1 - 3e^{-\kappa\varepsilon\zeta h/2}.$$

These two inequalities prove (2.7.8). For (2.7.9), first  $F^-(h) \geq e^{-\varepsilon h} \tau^{W_\kappa^\dagger}(\varepsilon h)$ , and using (2.7.7),  $\tau^{W_\kappa^\dagger}(\varepsilon h) \geq 1$  with a probability larger than  $1 - e^{-(c_-)\varepsilon^2 h^2}$ , which proves (2.7.9). Finally, notice that in law  $G^+(\alpha h, h) \leq e^h \int_0^{+\infty} e^{W_\kappa(x)} dx = e^h A_\infty$ . By [32],  $2/A_\infty$  is a gamma variable of parameter  $(\kappa, 1)$ , and so has a density equal to  $e^{-x} x^{\kappa-1} \mathbf{1}_{\mathbb{R}_+}(x)/\Gamma(\kappa)$ , which leads to (2.7.10).  $\square$

The following lemma is exactly Lemma 4.3 in [3] which proof can be found in that paper.

**Lemma 2.7.4.** *Let  $(B(s), s \in \mathbb{R})$  be a standard two-sided Brownian motion. For every  $0 < \varepsilon < 1$ ,  $0 < \delta < 1$  and  $x > 0$ ,*

$$\mathbb{P}\left(\sup_{u \in [-\delta, \delta]} |\mathcal{L}_B(\tau^B(1), u) - \mathcal{L}_B(\tau^B(1), 0)| > \varepsilon \mathcal{L}_B(\tau^B(1), 0)\right) \leq C_+ \frac{\delta^{1/6}}{\varepsilon^{2/5}}, \quad (2.7.11)$$

$$\mathbb{P}\left(\sup_{u \in [0, 1]} \mathcal{L}_B(\tau^B(1), u) \geq x\right) \leq 4e^{-x/2}, \quad (2.7.12)$$

$$\mathbb{P}\left(\sup_{u \leq 0} \mathcal{L}_B(\tau^B(1), u) \geq x\right) \leq 4/x. \quad (2.7.13)$$

The next lemma says that with large probability, a 2-dimensional squared Bessel Process is bounded by some deterministic function. This lemma may be of independent interest.

**Lemma 2.7.5.** *Let  $(Q_2(u), u \geq 0)$  be a Bessel process of dimension 2, starting from 0, and two functions  $a(\cdot)$  and  $k(\cdot)$  from  $(0, +\infty)$  to  $(0, +\infty)$ , having limit  $+\infty$  on  $+\infty$ . We have for large  $t$ ,*

$$P\left(\forall u \in (0, k(t)], Q_2^2(u) \leq 2e[a(t) + 4 \log \log[ek(t)/u]]u\right) \geq 1 - C_+ \exp[-a(t)/2].$$

**Proof :** We consider for  $t > 0$  and  $i \in \mathbb{N}$ ,

$$\mathcal{A}_{1,i} := \left\{ \sup_{[k(t)/e^{i+1}, k(t)/e^i]} Q_2^2 \leq 2 \frac{k(t)}{e^i} [a(t) + 4 \log(i+1)] \right\}, \quad \mathcal{A}_2 := \bigcap_{i=0}^{\infty} \mathcal{A}_{1,i}.$$

We recall that there exist two standard independent Brownian motions  $(B_1(u), u \geq 0)$  and  $(B_2(u), u \geq 0)$  such that  $(Q_2^2(u), u \geq 0)$  is equal in law to  $(B_1^2(u) + B_2^2(u), u \geq 0)$ . So for  $i \in \mathbb{N}$ ,

$$\begin{aligned} P(\overline{\mathcal{A}}_{1,i}) &\leq 2P\left(\sup_{[k(t)/e^{i+1}, k(t)/e^i]} B_1^2 > k(t)e^{-i}[a(t) + 4 \log(i+1)]\right) \\ &\leq 4P\left(\sup_{[0, k(t)/e^i]} B_1 > \sqrt{k(t)e^{-i}[a(t) + 4 \log(i+1)]}\right) \\ &= 4P\left(|B_1(1)| > \sqrt{a(t) + 4 \log(i+1)}\right) \\ &\leq 8 \exp[-a(t)/2 - 2 \log(i+1)] \end{aligned}$$



for large  $t$  so that  $a(t) \geq 1$ , by scaling, and since  $B_1 \stackrel{\mathcal{L}}{=} -B_1$ ,  $\sup_{[0,1]} B_1 \stackrel{\mathcal{L}}{=} |B_1(1)|$  and  $P(B_1(1) \geq x) \leq e^{-x^2/2}$  for  $x \geq 1$ . Consequently for large  $t$ ,

$$P(\overline{\mathcal{A}}_2) \leq \sum_{i=0}^{\infty} P(\overline{\mathcal{A}}_{1,i}) \leq 8 \exp[-a(t)/2] \sum_{i=0}^{\infty} \frac{1}{(i+1)^2} = C_+ \exp[-a(t)/2]. \quad (2.7.14)$$

Now, let  $0 < u \leq k(t)$ . There exists  $i \in \mathbb{N}$  such that  $k(t)/e^{i+1} < u \leq k(t)/e^i$ . We have,  $e^i \leq k(t)/u$ , so  $e^{i+1} \leq ek(t)/u$  and then  $\log(i+1) \leq \log \log[ek(t)/u]$ . Consequently on  $\mathcal{A}_2$ ,

$$Q_2^2(u) \leq 2(k(t)/e^i)[a(t) + 4 \log(i+1)] \leq 2eu[a(t) + 4 \log \log[ek(t)/u]].$$

This, combined with (2.7.14), proves the lemma.  $\square$

We also need some estimates on the local time of  $B$  at a given coordinate  $y \in \mathbb{R}$  at the inverse of the local time of  $B$  at 0. Recall that  $\sigma_B(r, y) = \inf\{s > 0, \mathcal{L}_B(s, y) > r\}$  for  $r > 0, y \in \mathbb{R}$ . By the second Ray-Knight Theorem, the processes  $(\mathcal{L}_B(\sigma_B(r, 0), y), y \in \mathbb{R}_+)$  and  $(\mathcal{L}_B(\sigma_B(r, 0), -y), y \in \mathbb{R}_+)$  are two independent squared Bessel processes of dimension 0 starting at  $r$ . The following lemma is proved in ([67], Lemma 3.1; the results are stated for a Bessel process but are actually true for a *squared* Bessel process; see also [29], Lemma 2.3).

**Lemma 2.7.6.** *We denote by  $(Q_0(y), y \geq 0)$  the square of a 0-dimensional Bessel process starting at 1. Let  $M > 0, u > 0$  and  $v > 0$ . Then,*

$$P\left(\sup_{0 \leq y \leq v} |Q_0(y) - 1| \geq u\right) \leq 4 \frac{\sqrt{(1+u)v}}{u} \exp[-u^2/(8(1+u)v)], \quad (2.7.15)$$

$$P\left(\sup_{y \geq 0} Q_0(y) \geq M\right) = 1/M. \quad (2.7.16)$$

# Chapitre 3

## EXPONENTIAL FUNCTIONALS OF SPECTRALLY ONE-SIDED LÉVY PROCESSES CONDITIONED TO STAY POSITIVE

This work has been the object of an article [72] currently being reviewed.

### 3.1 Introduction

We consider a spectrally negative Lévy process  $V$  which is not the opposite of a subordinator. We denote its Laplace exponent by  $\Psi_V$  :

$$\forall t, \lambda \geq 0, \mathbb{E} [e^{\lambda V(t)}] = e^{t\Psi_V(\lambda)}.$$

In the case where  $V$  drifts to  $-\infty$ , it is well known that its Laplace exponent admits a non trivial zero that we denote here by  $\kappa$ ,  $\kappa := \inf\{\lambda > 0, \Psi_V(\lambda) = 0\}$ . If  $V$  does not drift to  $-\infty$ , then 0 is the only zero of  $\Psi_V$  so we put  $\kappa := 0$  in this case. We denote by  $(Q, \gamma, \nu)$  the generating triplet of  $V$  so  $\Psi_V$  can be expressed as

$$\Psi_V(\lambda) = \frac{Q}{2}\lambda^2 - \gamma\lambda + \int_{-\infty}^0 (e^{\lambda x} - 1 - \lambda x \mathbf{1}_{|x|<1})\nu(dx). \quad (3.1.1)$$

In the end of the paper, we also consider  $Z$ , a spectrally positive Lévy process drifting to  $+\infty$ .

We are interested in the basic exponential functionals of  $V$  and  $Z$  conditioned to stay positive,

$$I(V^\uparrow) := \int_0^{+\infty} e^{-V^\uparrow(t)} dt \quad \text{and} \quad I(Z^\uparrow) := \int_0^{+\infty} e^{-Z^\uparrow(t)} dt.$$

For both we study finiteness, exponential moments and the asymptotic tail at 0. For  $I(V^\uparrow)$ , we also get self-decomposability, more precise estimates on the asymptotic tail at 0 and a condition for smoothness of the density.

Our first motivation is to extend to spectrally one-sided Lévy processes conditioned to stay positive the general study of the exponential functionals of Lévy processes. Those functionals have been widely studied because of their importance in probability theory. For example they are fundamental to the study of diffusions in random environments and appear in many applications such as mathematical finance, see [12] for a survey on those functionals and their applications. For a general Lévy process, equivalent conditions for the finiteness of the exponential functional are given in [12], the asymptotic tail at  $+\infty$  of the functional is studied in [52], the absolute continuity is proved in [9] and properties of the density (such as regularity) are studied in [20] under some hypothesis on the jumps of the Lévy process and in [54]. In this paper we also obtain, as a by-product of our approach, some results on the exponential functionals of spectrally one-sided Lévy processes.

Our second motivation is the possibility to apply our results to the study of diffusions in a spectrally negative Lévy environment. Such processes, introduced by Brox [17] when the environment is given by a brownian motion have been specifically studied for the spectrally negative Lévy case by Singh [66]. In [4], they prove that the supremum of local time  $\mathcal{L}_X^*$  of a diffusion in a drifted brownian environment converges in law and they express the limit law in term of a subordinator and an exponential functional of the environment conditioned to stay positive. In order to generalize their result to a diffusion in a spectrally negative Lévy environment, knowledge on the exponential functionals involved is needed. These are precisely exponential functionals of the environment (which is spectrally negative) and its dual (which is spectrally positive) conditioned to stay positive.

Finally, we have hints that the almost sure asymptotic behavior of  $\mathcal{L}_X^*$ , for a diffusion in the spectrally negative Lévy environment  $V$ , is crucially linked to the right and left tails of the distribution of  $I(V^\uparrow)$ . This is why we study these tails here and give for the left tail a precise asymptotic estimate when it is possible, in particular, when  $\Psi_V(\lambda) \sim c\lambda^\alpha$ , for some constant  $c$  and  $\alpha \in ]1, 2]$ . For the right tail, we are only interested in the existence of some finite exponential moments. The application of the present work to diffusions in random environment is a work in preparation by the author [74], [73].

For  $A$  a process and  $S$  a borelian set, we denote

$$\tau(A, S) := \inf \{t \geq 0, A(t) \in S\}, \quad \mathcal{R}(A, S) := \sup \{t \geq 0, A(t) \in S\}.$$

We shall only write  $\tau(A, x)$  (respectively  $\mathcal{R}(A, x)$ ) instead of  $\tau(A, \{x\})$  (respectively  $\mathcal{R}(A, \{x\})$ ) and  $\tau(A, x+)$  instead of  $\tau(A, [x, +\infty[)$ . For example, since  $V^\uparrow$  has no positive jumps, we see that it reaches each positive level continuously :  $\forall x > 0, \tau(V^\uparrow, x+) = \tau(V^\uparrow, x)$  and since moreover  $V^\uparrow$  converges to  $+\infty$  we have  $\forall x > 0, \mathcal{R}(V^\uparrow, [0, x]) = \mathcal{R}(V^\uparrow, x)$ .

Also let  $\underline{A}(t) := \inf\{A(s), s \in [0, t]\}$  be the infimum process of  $A$ . If  $A$  is Markovian and  $x \in \mathbb{R}$  we denote  $A_x$  for the process  $A$  starting from  $x$ . For  $A_0$  we shall only write  $A$ . For any (possibly random) time  $T > 0$ , we write  $A^T$  for the process  $A$  shifted and centered at time  $T : \forall s \geq 0, A^T(s) := A(T + s) - A(T)$ .

We now recall some facts about  $V^\uparrow$ , that is,  $V$  conditioned to stay positive. For a spectrally negative Lévy process  $V$ , the Markov family  $(V_x^\uparrow, x \geq 0)$  may be defined as in [8], Section VII.3. For any  $x \geq 0$ , the process  $V_x^\uparrow$  must be seen as  $V$  conditioned to stay positive and starting from  $x$ . We denote  $V^\uparrow$  for the process  $V_0^\uparrow$ . It is known that  $V_x^\uparrow$  converges in the Skorokhod space to  $V^\uparrow$  when  $x$  goes to 0.

For  $X$  a positive random variable, we denote  $V_X^\uparrow$  for the process  $V$  conditioned to stay positive and starting from the random variable  $X$ . More rigorously,  $V_X^\uparrow$  is the Markov process that conditionally on  $\{X = x\}$  has law  $V_x^\uparrow$ .

For any positive  $x$ , we have from the Markov property and the absence of positive jumps that the process  $V^\uparrow$ , shifted at  $\tau(V^\uparrow, x)$ , its first passage time at  $x$ , is equal in law to  $V_x^\uparrow$ . In the case where  $V$  drifts to  $+\infty$ , it is known from [8], Section VII.3, that  $V_x^\uparrow$  has the same law as  $V_x$  conditioned in the usual sense to remain positive. This property, interesting for our study, is unfortunately not true when  $V$  oscillates or drifts to  $-\infty$  (in these cases we have to do the conditioning until an hitting time).

In the case where  $V$  drifts to  $-\infty$ , we define  $V^\sharp$  to be " $V$  conditioned to drift to  $+\infty$ ", as in [8], Section VII.1. The Laplace exponent  $\Psi_{V^\sharp}$  of  $V^\sharp$  satisfies  $\Psi_{V^\sharp} = \Psi_V(\kappa + \cdot)$  where  $\kappa$  is the non-trivial zero of  $\Psi_V$ . As a consequence  $\Psi'_{V^\sharp}(0) > 0$ , so  $V^\sharp$  drifts to infinity (this is deduced thanks to Corollary VII.2 in [8]) and it is also proven that  $V^\uparrow = (V^\sharp)^\uparrow$ .

In order to do our proofs in a systematic way, we often work with  $V^\sharp$  which is defined to be " $V$  conditioned to drift to  $+\infty$ " in the case where  $V$  drifts to  $-\infty$  and "only  $V$ " in the other cases (when  $V$  oscillates or drifts to  $+\infty$ ). As a consequence,  $V^\sharp$  always denotes a spectrally negative Lévy process that does not drifts to  $-\infty$  (it oscillates if  $V$  does and it drifts to  $+\infty$  if  $V$  drifts to  $+\infty$  or  $-\infty$ ). In any case we have that for all  $0 < x < y$ ,  $(V_x^\uparrow(t), 0 \leq t \leq \tau(V_x^\uparrow, y))$  is equal in law to  $(V_x^\sharp(t), 0 \leq t \leq \tau(V_x^\sharp, y))$  conditionally on  $\{\tau(V_x^\sharp, y) < \tau(V_x^\sharp, ]-\infty, 0])\}$ . Note that the same identity is true with  $V$  instead of  $V^\sharp$ , but the advantage of dealing with  $V^\sharp$  is that  $\tau(V_x^\sharp, y)$  is always finite (while  $\tau(V_x, y)$  can possibly be infinite when  $V$  drifts to  $-\infty$ ) which simplifies the argumentation.

Let  $W$  be the scale function of  $V$ , defined as in Section VII.2 of [8]. It satisfies

$$\forall 0 < x < y, \mathbb{P}(\tau(V_x, y) < \tau(V_x, ]-\infty, 0]) = W(x)/W(x + y).$$

According to Theorem VII.8 in [8], this function is continuous, increasing, and for any  $\lambda > \Phi_V(0)$ ,

$$\int_0^{+\infty} e^{-\lambda x} W(x) dx = \frac{1}{\Psi_V(\lambda)} < +\infty.$$

### 3.1.1 Results

In the special case of the exponential functional of a drifted brownian motion conditioned to stay positive, all the properties that are established here are already known and sometimes more explicitly. We discuss this case in the next subsection.

Our first result is the finiteness of  $I(V^\uparrow)$  and the fact that it admits exponential moments.

**Theorem 3.1.1.** *The random variable  $I(V^\uparrow)$  is almost surely finite, has finite expectation  $\mathbb{E}[I(V^\uparrow)]$  and*

$$\forall \lambda < 1/\mathbb{E}[I(V^\uparrow)], \quad \mathbb{E} \left[ e^{\lambda I(V^\uparrow)} \right] < +\infty. \quad (3.1.2)$$

Then, a fundamental point of our study is Proposition 3.3.2 which says that for any positive  $y$ ,  $I(V^\uparrow)$  satisfies the random affine equation

$$I(V^\uparrow) \stackrel{\mathcal{L}}{=} A^y + e^{-y} I(V^\uparrow), \quad (3.1.3)$$

where  $A^y$  is independent of the second term and will be specified later. We see that  $I(V^\uparrow)$  is a positive self-decomposable random variable and is therefore absolutely continuous and unimodal. It is well known (see for example expression (1.10) in [55]) that the exponential functional  $I(V)$  of a spectrally negative Lévy process  $V$  is also self-decomposable (as long as it is finite), it can be seen by splitting the trajectory at  $\tau(V, y)$ , the first passage time at  $y$ . Another consequence of (3.1.3) is that for any positive  $y$ ,  $I(V^\uparrow)$  can be written as the random series

$$I(V^\uparrow) \stackrel{\mathcal{L}}{=} \sum_{k \geq 0} e^{-ky} A_k^y,$$

where the random variables  $A_k^y$  are *iid* and have the same law as  $A^y$ . This decomposition is a very useful tool for the study of the random variable  $I(V^\uparrow)$  and is also the base of the proofs of the results we present below.

Our next results make a link between the asymptotic behavior of  $\Psi_V$  and the properties of  $I(V^\uparrow)$ .

**Theorem 3.1.2.** *Assume that there is  $\alpha > 1$  and a positive constant  $C$  such that for all  $\lambda$  large enough we have  $\Psi_V(\lambda) \leq C\lambda^\alpha$ . Then for all  $\delta \in ]0, 1[$  and  $x$  small enough we have*

$$\mathbb{P} (I(V^\uparrow) \leq x) \leq \exp \left( -\delta(\alpha - 1)/(Cx)^{1/(\alpha-1)} \right). \quad (3.1.4)$$

*Assume that there is  $\alpha > 1$  and a positive constant  $c$  such that for all  $\lambda$  large enough we have  $\Psi_V(\lambda) \geq c\lambda^\alpha$ . Then for all  $\delta > 1$  and  $x$  small enough we have*

$$\mathbb{P} (I(V^\uparrow) \leq x) \geq \exp \left( -\delta\alpha^{\alpha/(\alpha-1)}/(cx)^{1/(\alpha-1)} \right). \quad (3.1.5)$$

Let us now recall how is usually quantified the asymptotic behavior of  $\Psi_V$ . We define, as in [8], page 94,

$$\begin{aligned}\sigma &:= \sup \left\{ \alpha \geq 0, \lim_{\lambda \rightarrow +\infty} \lambda^{-\alpha} \Psi_V(\lambda) = \infty \right\}, \\ \beta &:= \inf \left\{ \alpha \geq 0, \lim_{\lambda \rightarrow +\infty} \lambda^{-\alpha} \Psi_V(\lambda) = 0 \right\}.\end{aligned}$$

Recall that  $\Psi_{V\#}(\cdot) = \Psi_V(\kappa + \cdot)$ , so  $\sigma$  and  $\beta$  are identical whether they are defined from  $\Psi_{V\#}$  or  $\Psi_V$ .

If  $\Psi_V$  has  $\alpha$ -regular variation for  $\alpha \in [1, 2]$  (for example if  $V$  is a drifted  $\alpha$ -stable Lévy process with no positive jumps), we have  $\sigma = \beta = \alpha$ . Recall that  $Q$  is the brownian component of  $V$ . It is well known that  $\Psi_V(\lambda)/\lambda^2$  converges to  $Q/2$  when  $\lambda$  goes to infinity so, when  $Q > 0$ ,  $\Psi_V$  has 2-regular variation, and when  $Q = 0$ ,

$$1 \leq \sigma \leq \beta \leq 2,$$

where  $1 \leq \sigma$  comes from the convexity of  $\Psi_V$ .

**Remark 3.1.3.** *When  $V$  has bounded variation, we know (see for example [8] Section I.1) that the brownian component of  $V$  is null, the Lévy measure  $\nu$  of  $V$  satisfies  $\int_{-1}^0 |x| \nu(dx) < +\infty$  and  $-\gamma - \int_{-1}^0 x \nu(dx)$ , the factor of  $\lambda$  in the expression of  $\Psi_V(\lambda)$ , is positive (otherwise  $V$  would be the opposite of a subordinator). It is thus easy to see that in this case  $\Psi_V(\lambda)/\lambda$  converges to  $-\gamma - \int_{-1}^0 x \nu(dx)$  when  $\lambda$  goes to infinity, so  $\sigma = \beta = 1$ . In the remaining, we sometimes assume that  $\sigma > 1$ , the reader should be aware that it excludes the case where  $V$  has bounded variation. However, this case is quite easy and shall be treated in the remarks.*

We are now ready to state our general results on the asymptotic tails at 0 of  $I(V^\uparrow)$  :

**Theorem 3.1.4.** *We have*

$$\forall \beta' > \beta, \lim_{x \rightarrow 0} x^{1/(\beta'-1)} \log \left( \mathbb{P} \left( I(V^\uparrow) \leq x \right) \right) = -\infty, \quad (3.1.6)$$

$$\text{if } \sigma > 1, \forall \sigma' \in ]1, \sigma[, \lim_{x \rightarrow 0} x^{1/(\sigma'-1)} \log \left( \mathbb{P} \left( I(V^\uparrow) \leq x \right) \right) = 0. \quad (3.1.7)$$

Theorem 3.1.4 gives for  $\mathbb{P}(I(V^\uparrow) \leq x)$  a lower bound involving  $\sigma$  and an upper bound involving  $\beta$ . In the case of  $\alpha$ -regular variation we can expect, under some extra hypothesis, to get a stronger result. We indeed have :

**Theorem 3.1.5.** *We assume that there is a positive constant  $C$  and  $\alpha \in ]1, 2]$  such that  $\Psi_V(\lambda) \sim_{\lambda \rightarrow +\infty} C \lambda^\alpha$ , then*

$$\log \left( \mathbb{P} \left( I(V^\uparrow) \leq x \right) \right) \underset{x \rightarrow 0}{\sim} -\frac{\alpha - 1}{(Cx)^{\frac{1}{\alpha-1}}}.$$

The above theorem is true in particular when, for some  $\alpha \in ]1, 2]$ ,  $V$  is an  $\alpha$ -stable spectrally negative Lévy process (with adjonction or not of a drift). In particular, it agrees exactly with the tail (3.1.11) given in the next subsection for the particular case of a drifted brownian motion.

**Remark 3.1.6.** *Since  $\Psi_V(\lambda)/\lambda^2$  has always a finite limit at  $+\infty$  we get, from Theorem 3.1.2, that there is always a positive constant  $K$  (depending on  $V$ ) such that for  $x$  small enough*

$$\mathbb{P}(I(V^\uparrow) \leq x) \leq e^{-K/x}.$$

**Remark 3.1.7.** *Note that Theorem 3.1.4 holds when  $\beta = 1$  and  $1/(\beta' - 1)$  can then equal any number in  $]0, +\infty[$ . When  $V$  has bounded variation, we even have a stronger result :  $\mathbb{P}(I(V^\uparrow) \leq x)$  is null for  $x$  small enough.*

**Remark 3.1.8.** *Recall that  $I(V^\uparrow)$  is unimodal. If 0 was a mode, then we would have  $\mathbb{P}(I(V^\uparrow) \leq x) \geq cx$  for some positive constant  $c$  and  $x$  small enough, which is incompatible with (3.1.6). As a consequence the density of  $I(V^\uparrow)$  is non-decreasing on a neighborhood of 0. This allows to remark that Theorems 3.1.2, 3.1.4, 3.1.5 and Remarks 3.1.6, 3.1.7 are true for the density of  $I(V^\uparrow)$  in place of the repartition function  $\mathbb{P}(I(V^\uparrow) \leq \cdot)$ .*

When  $V$  drifts to  $+\infty$  we prove Proposition 3.4.3 which says that the left tail of  $I(V^\uparrow)$  is the same as the left tail of  $I(V)$ . This implies that all the results we prove for the left tail of  $I(V^\uparrow)$  are true for the left tail of  $I(V)$  :

**Proposition 3.1.9.** *If  $V$  drifts to  $+\infty$ , then Theorems 3.1.2, 3.1.4, 3.1.5 and Remarks 3.1.6, 3.1.7 are true for  $I(V)$  in place of  $I(V^\uparrow)$ .*

Proposition 3.1.9 is an example of how the study of the exponential functional of the Lévy process conditioned to stay positive can be useful for the study of the exponential functional of the corresponding Lévy process.

We already mentioned that the law of  $I(V^\uparrow)$  is absolutely continuous but we do not know how smooth the density is in general. The following theorem provides a condition for smoothness :

**Theorem 3.1.10.** *If  $\sigma > 1$  and  $\beta$  are such that*

$$2\beta^2 - 3\sigma\beta + \sigma + \beta - 1 < 0, \tag{3.1.8}$$

*then the density of  $I(V^\uparrow)$  belongs to the Schwartz space. All its derivatives converge to 0 at  $+\infty$  and 0.*

This theorem admits the following corollary :

**Corollary 3.1.11.** *If  $V$  drifts to  $+\infty$ , is such that  $\sigma > 1$  and (3.1.8) is satisfied, then the density of  $I(V)$  is of class  $\mathcal{C}^\infty$  and all its derivatives converge to 0 at  $+\infty$  and 0.*

Here again, the study of the exponential functional of the Lévy process conditioned to stay positive implies results about the exponential functional of the corresponding Lévy process.

**Remark 3.1.12.** *If  $\Psi_V$  has  $\alpha$ -regular variation with  $\alpha > 1$ , then  $\sigma = \beta = \alpha$  so the condition (3.1.8) becomes  $-(\alpha - 1)^2 < 0$ , but this is always true for  $\alpha > 1$ , so Theorem 3.1.10 and Corollary 3.1.11 apply. In other words,  $\alpha$ -regular variation for the Laplace exponent of  $V$  implies smoothness of the density for  $I(V^\uparrow)$  (and  $I(V)$  if it is finite) when  $\alpha > 1$ .*

In the spectrally positive case, the finiteness of the exponential functional is quite easy to obtain, but our argument also yields the existence of some finite exponential moments. We can state the result as follows :

**Theorem 3.1.13.** *The random variable  $I(Z^\uparrow)$  is almost surely finite and admits some finite exponential moments.*

We also obtain a lower bound for the asymptotic tail at 0 of both  $I(Z)$  and  $I(Z^\uparrow)$ . This tail is heavier than the one given for  $I(V^\uparrow)$  and this comes from the positive jumps.

**Theorem 3.1.14.** *If  $Z$  has unbounded variation and non-zero Lévy measure then, there is a positive constant  $c$  such that*

$$e^{-c(\log(x))^2} \leq \mathbb{P}(I(Z) \leq x) \leq \mathbb{P}(I(Z^\uparrow) \leq x).$$

*The lower bound for  $\mathbb{P}(I(Z) \leq x)$  does not require the hypothesis of unbounded variation.*

**Remark 3.1.15.** *If the Lévy measure of  $Z$  is the zero measure then it is known, from the Lévy-Khintchine formula, that  $Z$  is a drifted brownian motion. The exact asymptotic tail at 0 of  $I(Z^\uparrow)$  is then given by Theorem 3.1.5 and it is thinner than the one provided by Theorem 3.1.14. The existence of jumps thus plays an important role for the asymptotic tail at 0 of the exponential functional and the proof of Theorem 3.1.14 indeed crucially relies on this hypothesis.*

The study of the spectrally positive case does not go as far as the study of the spectrally negative case. The reason for this is twofold. First, we do not have, in the spectrally positive case, a decomposition of the law of  $I(Z^\uparrow)$  as in (3.1.3), which deprives us of an important tool for the study. Secondly we do not need, in the applications, the results on the exponential functional to be as precise, in the spectrally positive case, as in the spectrally negative case. Indeed, in the study of a diffusion in a spectrally negative Lévy environment  $V$  drifting to  $-\infty$ , a random variable  $\mathcal{R}$  appears. Its law is the convolution of the laws of  $I(V^\uparrow)$  and  $I(\hat{V}^\uparrow)$ , where  $\hat{V} := -V$  is the dual process of  $V$  and is thus spectrally positive. The combination of the above theorems shows that for some things the behavior of  $I(V^\uparrow)$  is dominant in the study of  $\mathcal{R}$  when  $V$  has jumps. In particular, the asymptotic tail at 0 of  $\mathcal{R}$  is the same as the one of  $I(V^\uparrow)$ .



The rest of the paper is organized as follows. In Section 3.2 we prove some preliminary results on  $V^\uparrow$ . In Section 3.3 we prove Theorem 3.1.1 and establish Proposition 3.3.2 about the self-decomposability of  $I(V^\uparrow)$ . In Section 3.4 we prove Theorems 3.1.2, 3.1.4 and 3.1.5 by studying the asymptotic behavior of the Laplace transform of  $I(V^\uparrow)$ , and in the case where  $V$  drifts to  $+\infty$ , we establish a connection between the tails at 0 of the exponential functionals  $I(V^\uparrow)$  and  $I(V)$ . In Section 3.5 we prove Theorem 3.1.10 and Corollary 3.1.11 via a study of excursions. Section 3.6 is devoted to the spectrally positive case and the proofs of Theorems 3.1.13 and 3.1.14.

### 3.1.2 The example of drifted brownian motion conditioned to stay positive

The most simple case is the intersection of the spectrally positive and the spectrally negative case, that is, when  $V$  is a drifted brownian motion. All the results mentioned here are already known in this case. We define the  $\kappa$ -drifted brownian motion by  $W_\kappa(t) := W(t) - \frac{\kappa}{2}t$ . It is known that the two processes  $W_\kappa^\uparrow$  and  $W_{-\kappa}^\uparrow$  are equal in law. This follows, for example, from the expression of the generator of  $W_\kappa^\uparrow$ , or from the fact that for positive  $\kappa$ , the Laplace exponent of  $W_\kappa^\uparrow$  is equal to the Laplace exponent of  $W_{-\kappa}$ , so the processes conditioned to stay positive have the same law. We thus only consider positive  $\kappa$ .

It is known (see (4.6) in [3], see also Lemma 6.6 in [4]) that  $I(W_k^\uparrow)$  is almost surely finite and has Laplace transform

$$\mathbb{E}[e^{-\lambda I(W_k^\uparrow)}] = \frac{1}{2^\kappa \Gamma(1+\kappa)} \frac{(2\sqrt{2\lambda})^\kappa}{I_\kappa(2\sqrt{2\lambda})},$$

where  $I_\kappa$  is a modified Bessel function. This expression can also be written

$$\mathbb{E}[e^{-\lambda I(W_k^\uparrow)}] = \frac{1}{\Gamma(1+\kappa)} \frac{1}{\sum_{j=0}^{+\infty} \frac{(2\lambda)^j}{j! \Gamma(1+j+\kappa)}}, \quad (3.1.9)$$

and it is easy to see that it can be analytically extended in a neighborhood of 0, so the random variable  $I(W_k^\uparrow)$  admits some finite exponential moments.

An easy calculation on the asymptotic of this expression when  $\lambda$  goes to infinity yields

$$-\log \left( \mathbb{E}[e^{-\lambda I(W_k^\uparrow)}] \right) \underset{\lambda \rightarrow +\infty}{\sim} 2\sqrt{2\lambda}. \quad (3.1.10)$$

Combining (3.1.10) and De Bruijn's Theorem (see Theorem 4.12.9 in [14]) we get

$$-\log \left( \mathbb{P} \left( I(W_k^\uparrow) \leq x \right) \right) \underset{x \rightarrow 0}{\sim} \frac{2}{x}. \quad (3.1.11)$$

This estimate can be seen as a particular case of Theorem 3.1.5 (when it is applied in the case of a drifted brownian motion).

As the expression (3.1.9) extends to a neighborhood of 0, we get the expression of the characteristic function of  $I(W_k^\uparrow)$  which can be proved, using estimates on modified Bessel functions, to belong to the Schwartz space. Therefore, the density of  $I(W_k^\uparrow)$ , which is the Fourier transform of its characteristic function, belongs to the Schwartz space, but this is already included in Theorem 3.1.10.

## 3.2 Preliminary results on $V^\uparrow$ and finiteness of $I(V^\uparrow)$

### 3.2.1 Exponential functionals and excursions theory

We fix  $y > 0$ . In this subsection, we use excursions to prove that the integral of exponential  $V^\uparrow(\tau(V^\uparrow, y) + \cdot)$  or  $V^\sharp$  stopped at there last passage time at  $y$  and 0 respectively are equal in law to some subordinators stopped at independent exponential random variables.

It is easy to see that regularity of  $\{y\}$  for the markovian processes  $V_y^\sharp$  and  $V_y^\uparrow$  is equivalent to the regularity of  $\{0\}$  for  $V$  (or  $V^\sharp$ ) which in turn, according to Corollary VII.5 in [8], is equivalent to the fact that  $V$  has unbounded variation. The property of  $\{y\}$  being instantaneous for  $V_y^\sharp$  and  $V_y^\uparrow$  is equivalent to the same property of  $\{0\}$  for  $V$ , but this is a well known property of spectrally negative Lévy processes.  $\{y\}$  is thus always instantaneous for  $V_y^\sharp$  and  $V_y^\uparrow$  and the only alternative is whether it is regular or not, which corresponds to the fact that  $V$  has or not unbounded variation.

We apply excursions theory away from  $y$  (see [8]). Let us denote by  $L_y^\uparrow$  (respectively  $L_y^\sharp$ ) a local time at  $y$  of the process  $V_y^\uparrow$  (respectively  $V_y^\sharp$ ) and  $\eta_y^\uparrow$  (respectively  $\eta_y^\sharp$ ) the associated excursions measure. We denote  $\eta^\sharp$  for  $\eta_0^\sharp$ . The inverse of the local time  $L_y^{\uparrow,-1}$  (respectively  $L_y^{\sharp,-1}$ ) is a subordinator and  $V_y^\uparrow$  (respectively  $V_y^\sharp$ ) can be represented as a Poisson point process on the set of excursions, with intensity measure  $\eta_y^\uparrow$  (respectively  $\eta_y^\sharp$ ). Note that this is also true in the irregular case (when  $V$  has bounded variation) if the local time  $L_y^\uparrow$  (respectively  $L_y^\sharp$ ) is defined artificially as in [8], Section IV.5. In this case, the excursion measure is proportional to the law of the first excursion and in particular the total mass of the excursion measure is finite.

In the case where  $V$  drifts to  $+\infty$ , we also consider the excursions of  $V$  away from 0. Then,  $L$  denotes a local time at 0 of  $V$  and  $\eta$  the associated excursion measure.

Given  $\xi : [0, \zeta] \rightarrow \mathbb{R}$  an excursion away from  $y$ , we define  $\zeta(\xi)$  to be its life-time,  $H_y(\xi) := \sup_{[0, \zeta(\xi)]} \xi - y$  its height and  $G(\xi) := \int_0^{\zeta(\xi)} e^{-\xi(t)} dt$ .

For any  $h > 0$ , we consider  $IP_h$ ,  $FP_h$  and  $N$  three subsets that make a partition of the excursions of  $V^\sharp$  away from  $y$ . These three subsets are respectively : the set of excursions higher than  $h$  that stay positive, the set of excursions of height smaller

than  $h$  that stay positive, the set of excursions that reach  $] - \infty, 0]$  :

$$\begin{aligned} IP_h &:= \{\xi, \forall t \in [0, \zeta(\xi)], \xi(t) > 0, H_y(\xi) \geq h\}, \\ FP_h &:= \{\xi, \forall t \in [0, \zeta(\xi)], \xi(t) > 0, H_y(\xi) < h\}, \\ N &:= \{\xi, \tau(\xi, ] - \infty, 0]) < \zeta(\xi)\}. \end{aligned}$$

$N$  does not depend on  $h$ .  $IP_\infty$  and  $FP_\infty$  are defined as the monotone limits of the sets  $IP_h$  and  $FP_h$  :  $IP_\infty$  is the set of infinite excursions that stay positive and  $FP_\infty$  is the set of finite excursions that stay positive.  $\eta_y^\sharp(IP_\infty)$  and  $\eta_y^\sharp(N)$  are always finite whereas  $\eta_y^\sharp(FP_\infty)$  is infinite in the regular case (when  $V$  has unbounded variation). Also, note that  $\eta_y^\sharp(IP_\infty) = 0$  if  $V$  oscillates.

**Lemma 3.2.1.** *Let  $y$  be positive and let  $S$  be a pure jump subordinator with Lévy measure  $G\eta_y^\sharp(\cdot \cap FP_\infty)$ , the image measure of  $\eta_y^\sharp(\cdot \cap FP_\infty)$  by  $G$ . Let  $T$  be an exponential random variable with parameter  $\eta_y^\sharp(IP_\infty) + \eta_y^\sharp(N)$ , independent of  $S$ . We have*

$$\int_{\tau(V^\uparrow, y)}^{\mathcal{R}(V^\uparrow, y)} e^{-V^\uparrow(t)} dt \stackrel{\mathcal{L}}{=} S_T, \quad (3.2.1)$$

where  $\tau(\cdot, \cdot)$  and  $\mathcal{R}(\cdot, \cdot)$  are defined in the introduction.

*Démonstration.*  $V^\uparrow(\tau(V^\uparrow, y) + \cdot)$  has the same law as  $V_y^\uparrow$ , from the Markov property applied to  $V^\uparrow$  at time  $\tau(V^\uparrow, y)$  and the absence of positive jumps. As a consequence,  $\int_{\tau(V^\uparrow, y)}^{\mathcal{R}(V^\uparrow, y)} e^{-V^\uparrow(t)} dt$  is equal in law to  $\int_0^{\mathcal{R}(V_y^\uparrow, y)} e^{-V_y^\uparrow(t)} dt$  and we are left to prove the result for the latter.

Then, let us fix  $h > 0$ . As it is mentioned in the introduction,  $(V_y^\uparrow(t), 0 \leq t \leq \tau(V_y^\uparrow, y + h))$  is equal in law to  $(V_y^\sharp(t), 0 \leq t \leq \tau(V_y^\sharp, y + h))$  conditionally on  $\{\tau(V_y^\sharp, y + h) < \tau(V_y^\sharp, ] - \infty, 0])\}$ .

$(V_y^\sharp(t), 0 \leq t \leq \tau(V_y^\sharp, y + h))$  can be built from the Poisson point process on the set of excursions with intensity measure  $\eta_y^\sharp$ .  $(V_y^\uparrow(t), 0 \leq t \leq \tau(V_y^\uparrow, y + h))$  can be built from this same process, conditioned not to have jumps in  $N$  before its first jump in  $IP_h$ . In other words, we build  $(V_y^\uparrow(t), 0 \leq t \leq \tau(V_y^\uparrow, y + h))$  from the process of jumps in  $FP_h$  stopped at the exponential time (that has parameter  $\eta_y^\sharp(IP_h) + \eta_y^\sharp(N)$ ) at which occurs the first jump in  $IP_h \cup N$  and conditionally to the fact that this jump belongs to  $IP_h$ . Then, the process of jumps in  $FP_h$  and in  $IP_h \cup N$  are independent and by a property of Poisson point processes, the fact that the first jump in  $IP_h \cup N$  belongs to  $IP_h$  is independent of the time when this jump occurs. As a consequence,  $(V_y^\uparrow(t), 0 \leq t \leq \tau(V_y^\uparrow, y + h))$  is built from a Poisson point process with intensity measure  $\eta_y^\sharp(\cdot \cap FP_h)$ , until an independent exponential time,  $T_h$ , of parameter  $\eta_y^\sharp(IP_h) + \eta_y^\sharp(N)$  where we pick, independently, a jump following the law  $\eta_y^\sharp(\cdot \cap IP_h) / \eta_y^\sharp(IP_h)$  (and we only keep the part of this excursion that is before its hitting time of  $y + h$ ).

Let  $\mathcal{R}^h(V_y^\uparrow, y)$  be the last passage time of  $V_y^\uparrow$  at  $y$  before  $\tau(V_y^\uparrow, y + h)$  :

$$\mathcal{R}^h(V_y^\uparrow, y) := \sup\{t \in [0, \tau(V_y^\uparrow, y + h)], V_y^\uparrow(t) = y\}.$$

From above, if  $(p_s)_{s \geq 0}$  is a Poisson point process in  $FP_\infty$  with measure  $\eta_y^\sharp(\cdot \cap FP_\infty)$  and if  $T_h$  is an independent exponential random variable with parameter  $\eta_y^\sharp(IP_h) + \eta_y^\sharp(N)$ , then  $(V_y^\uparrow(t), 0 \leq t \leq \mathcal{R}^h(V_y^\uparrow, y))$  is built by putting aside the excursions of the process  $(p_s \mathbb{1}_{p_s \in FP_h}, 0 \leq s \leq T_h)$ . Since  $V_y^\uparrow$  converges almost surely to  $+\infty$ ,  $\mathcal{R}^h(V_y^\uparrow, y)$  converges almost surely to  $\mathcal{R}(V_y^\uparrow, y)$ , the last passage time at  $y$ , when  $h$  goes to infinity. On the other hand,  $IP_h \cup N$  decreases to  $IP_\infty \cup N$  when  $h$  goes to infinity. As a consequence  $T_h$  increases to an exponential random variable  $T$  with parameter  $\eta_y^\sharp(IP_\infty) + \eta_y^\sharp(N) > 0$ . Also,  $FP_h$  increases to  $FP_\infty$  when  $h$  goes to infinity. Then, identifying the limits when  $h$  goes to infinity of both  $(V_y^\uparrow(t), 0 \leq t \leq \mathcal{R}^h(V_y^\uparrow, y))$  and  $(p_s \mathbb{1}_{p_s \in FP_h}, 0 \leq s \leq T_h)$ , we get that  $(V_y^\uparrow(t), 0 \leq t \leq \mathcal{R}(V_y^\uparrow, y))$  is built by putting aside the excursions of the process  $(p_s, 0 \leq s \leq T)$ , where  $T$  is an exponential random variable with parameter  $\eta_y^\sharp(IP_\infty) + \eta_y^\sharp(N)$  and is independent from  $(p_s, s \geq 0)$ .

Now, remark that  $\int_0^{\mathcal{R}(V_y^\uparrow, y)} e^{-V_y^\uparrow(t)} dt$  is the sum of the images by  $G$  of the excursions of  $(V_y^\uparrow(t), 0 \leq t \leq \mathcal{R}(V_y^\uparrow, y))$  away from  $y$ . We thus have

$$\int_0^{\mathcal{R}(V_y^\uparrow, y)} e^{-V_y^\uparrow(t)} dt \stackrel{\mathcal{L}}{=} \sum_{0 < s < T} G(p_s). \quad (3.2.2)$$

By properties of Poisson point processes, the process in the right hand side,  $\sum_{0 < s < T} G(p_s)$ , is the sum of the jumps of a Poisson point process on  $\mathbb{R}_+$ , with intensity measure  $G\eta_y^\sharp(\cdot \cap FP_\infty)$ . Thus, from the Lévy-Itô decomposition, it has the same law as the subordinator  $S$ , which yields the result.  $\square$

**Remark 3.2.2.** *In the case where  $V$  has bounded variation, the total mass of  $\eta_y^\sharp$  is finite so  $S$  is only a compound Poisson process. In particular  $S_T$  can then be null with positive probability.*

$y > 0$  is still fixed (and arbitrary), let  $\mathcal{R}^y(V^\sharp, 0)$  be the last passage time of  $V^\sharp$  at 0 before  $\tau(V^\sharp, y)$  :

$$\mathcal{R}^y(V^\sharp, 0) := \sup\{t \in [0, \tau(V^\sharp, y)], V^\sharp(t) = 0\}.$$

In order to study the trajectory of  $V^\sharp$  before  $\mathcal{R}^y(V^\sharp, 0)$ , we now consider excursions away from 0. Let  $I_y$  and  $F_y$  denote respectively the subset of excursions higher than  $y$  and lower than  $y$  :

$$I_y := \{\xi, H_0(\xi) \geq y\}, \quad F_y := \{\xi, H_0(\xi) < y\}.$$

A similar proof as for Lemma 3.2.1 gives the following lemma.

**Lemma 3.2.3.** *Let  $S$  be a pure jump subordinator with Lévy measure  $G\eta^\sharp(\cdot \cap F_y)$ , the image measure of  $\eta^\sharp(\cdot \cap F_y)$  by  $G$ . Let  $T$  be an exponential random variable with parameter  $\eta^\sharp(I_y)$  which is independent of  $S$ . We have*

$$\int_0^{\mathcal{R}^y(V^\sharp, 0)} e^{-V^\sharp(t)} dt \stackrel{\mathcal{L}}{=} S_T. \quad (3.2.3)$$

In the case where  $V$  drifts to  $+\infty$  we need to study the trajectory before  $\mathcal{R}(V, 0)$ , the last passage time of  $V$  at 0. We still consider excursions away from 0. Let  $I$  and  $F$  denote respectively the subsets of infinite and finite excursions :

$$I := \{\xi, \zeta(\xi) = +\infty\}, \quad F := \{\xi, \zeta(\xi) < +\infty\}.$$

A similar proof as for Lemma 3.2.1 gives the following lemma.

**Lemma 3.2.4.** *We assume that  $V$  drifts to  $+\infty$ . Let  $S$  be a pure jump subordinator with Lévy measure  $G\eta(\cdot \cap F)$ , the image measure of  $\eta(\cdot \cap F)$  by  $G$ . Let  $T$  be an exponential random variable with parameter  $\eta(I)$  which is independent of  $S$ . We have*

$$\int_0^{\mathcal{R}(V, 0)} e^{-V(t)} dt \stackrel{\mathcal{L}}{=} S_T. \quad (3.2.4)$$

### 3.2.2 $V^\uparrow$ and $V^\sharp$ shifted at a last passage time

To obtain decomposition (3.1.3) of the law of  $I(V^\uparrow)$ , we split  $V^\uparrow$  at its last passage time at a point  $y$  and obtain two independent trajectories that we can identify.

**Lemma 3.2.5.** *(Corollary VII.19 of [8])*

*For any positive  $y$ , the two trajectories*

$$(V^\uparrow(t), 0 \leq t \leq \mathcal{R}(V^\uparrow, y)) \quad \text{and} \quad (V^\uparrow(t + \mathcal{R}(V^\uparrow, y)) - y, t \geq 0)$$

*are independent and the second is equal in law to  $V^\uparrow$ .*

**Lemma 3.2.6.** — *The two trajectories  $(V^\sharp(t), 0 \leq t \leq \mathcal{R}^y(V^\sharp, 0))$*

$$\text{and } (V^\sharp(t + \mathcal{R}^y(V^\sharp, 0)), 0 \leq t \leq \tau(V^\sharp, y) - \mathcal{R}^y(V^\sharp, 0))$$

*are independent and the second is equal in law to  $(V^\uparrow(t), 0 \leq t \leq \tau(V^\uparrow, y))$ .*

*As a consequence we have  $\tau(V^\uparrow, y) \stackrel{\mathcal{L}}{=} \tau(V^\sharp, y) - \mathcal{R}^y(V^\sharp, 0) \leq \tau(V^\sharp, y)$ .*

— *We assume that  $V$  drifts to  $+\infty$ . The two trajectories*

$$(V(t), 0 \leq t \leq \mathcal{R}(V, 0)) \quad \text{and} \quad (V(t + \mathcal{R}(V, 0)), t \geq 0)$$

*are independent and the second is equal in law to  $V^\uparrow$ .*

*Démonstration.* We fix  $y > 0$  and  $a \in ]0, y[$ . Let us denote by  $(e(s), s \geq 0)$  the excursions process of  $V^\sharp$  away from 0. Recall the notations  $I_y$  and  $F_y$ ,  $T_y := \inf\{s \geq 0, e(s) \in I_y\}$  is the time when occurs the first excursion higher than  $y$  and  $\xi_y$  is this excursion.

Decomposing  $V^\sharp$  as its excursions away from 0, we see that  $\mathcal{R}^y(V^\sharp, 0)$  is the instant when begins the first excursion higher than  $y$ , so

$$(V^\sharp(t + \mathcal{R}^y(V^\sharp, 0)), 0 \leq t \leq \tau(V^\sharp, y) - \mathcal{R}^y(V^\sharp, 0)) = (\xi_y(t), 0 \leq t \leq \tau(\xi_y, y)) \quad (3.2.5)$$

$(V^\sharp(t), 0 \leq t \leq \mathcal{R}^y(V^\sharp, 0))$  is thus a function of  $(e(s)\mathbb{1}_{e(s) \in F_y}, 0 \leq s \leq T_y)$  while  $(V^\sharp(t + \mathcal{R}^y(V^\sharp, 0)), 0 \leq t \leq \tau(V^\sharp, y) - \mathcal{R}^y(V^\sharp, 0))$  is a function of  $\xi_y$ . By properties of Poisson point processes,  $T_y$  is an exponential random variable independent of  $\xi_y$  and the process of finite excursions  $(e(s)\mathbb{1}_{e(s) \in F_y}, s \geq 0)$  is also independent of  $\xi_y$ . Therefore the objects  $(e(s)\mathbb{1}_{e(s) \in F_y}, 0 \leq s \leq T_y)$  and  $\xi_y$  are independent. From this independence we deduce that

$$(V^\sharp(t), 0 \leq t \leq \mathcal{R}^y(V^\sharp, 0)) \perp\!\!\!\perp (V^\sharp(t + \mathcal{R}^y(V^\sharp, 0)), 0 \leq t \leq \tau(V^\sharp, y) - \mathcal{R}^y(V^\sharp, 0)),$$

which is the required independence. It only remains to prove that the right hand side in (3.2.5) has the same law as  $(V^\uparrow(t), 0 \leq t \leq \tau(V^\uparrow, y))$ .

Using the Markov property at time  $\tau(\xi_a, a)$ , for an excursion  $\xi_a \in I_a$ , we have that  $\xi_a(\cdot + \tau(\xi_a, a))$  equals in law  $V_a^\sharp$  killed when it ever reaches 0.

Since  $I_y \subset I_a$  we can apply this to an excursion  $\xi_y \in I_y$  and get that  $(\xi_y(t + \tau(\xi_y, a)), 0 \leq t \leq \tau(\xi_y, y) - \tau(\xi_y, a))$  is equal in law to  $(V_a^\sharp(t), 0 \leq t \leq \tau(V_a^\sharp, y))$  conditioned to reach  $y$  before 0. Since  $V_a^\sharp$  has no positive jumps, reaching  $y$  before 0 is the same as reaching  $y$  before  $]-\infty, 0]$ .

As we mentioned in the introduction,  $(V_a^\sharp(t), 0 \leq t \leq \tau(V_a^\sharp, y))$  conditioned to reach  $y$  before  $]-\infty, 0]$  is equal in law to  $(V_a^\uparrow(t), 0 \leq t \leq \tau(V_a^\uparrow, y))$ . Putting all this together we get

$$(\xi_y(t + \tau(\xi_y, a)), 0 \leq t \leq \tau(\xi_y, y) - \tau(\xi_y, a)) \stackrel{\mathcal{L}}{=} (V_a^\uparrow(t), 0 \leq t \leq \tau(V_a^\uparrow, y)).$$

Since  $\tau(\xi_y, a)$  converges almost surely to 0 when  $a$  goes to 0 and  $V_a^\uparrow$  converges in law to  $V^\uparrow$  according to Proposition VII.14 in [8], we can let  $a$  go to 0 in both members and get

$$(\xi_y(t), 0 \leq t \leq \tau(\xi_y, y)) \stackrel{\mathcal{L}}{=} (V^\uparrow(t), 0 \leq t \leq \tau(V^\uparrow, y)). \quad (3.2.6)$$

As a consequence the right hand side in (3.2.5) has the same law as  $(V^\uparrow(t), 0 \leq t \leq \tau(V^\uparrow, y))$ , which concludes the proof of the first point of the lemma.

We now assume that  $V$  drifts to  $+\infty$  and prove the second point. For the independence, the arguments of the proof of the first point can be repeated, just replacing  $y$  by  $+\infty$  (we consider  $\xi_\infty$ , the infinite excursion away from 0, instead of the first excursion higher than  $y$ ). To prove that  $(V(t + \mathcal{R}(V, 0)), t \geq 0)$  is equal in law to  $V^\uparrow$ , it suffices to prove that  $\xi_\infty$  is equal in law to  $V^\uparrow$ . Let  $y$  be finite, we know from the proof of the first point that (3.2.6) is true for any excursion in  $I_y$ . Since  $\xi_\infty \in I_y$  we have

$$(\xi_\infty(t), 0 \leq t \leq \tau(\xi_\infty, y)) \stackrel{\mathcal{L}}{=} (V^\uparrow(t), 0 \leq t \leq \tau(V^\uparrow, y)).$$

Since  $y$  is arbitrary and  $\tau(V^\uparrow, y)$  converges almost surely to  $+\infty$  when  $y$  goes to  $+\infty$ , we get

$$(\xi_\infty(t), t \geq 0) \stackrel{\mathcal{L}}{=} (V^\uparrow(t), t \geq 0),$$

which gives the result. □

### 3.3 Finiteness, exponential moments, and self-decomposability

#### 3.3.1 Finiteness and exponential moments : Proof of Theorem 3.1.1

We are grateful to an anonymous referee for the following proof that is considerably simpler than the proof given by the author in the previous versions of this paper.

*Démonstration.* of Theorem 3.1.1

The idea of the proof is to provide finite upper bounds for the moments of  $I(V^\uparrow)$ . The first step is to prove that  $\mathbb{E}[I(V^\uparrow)] < +\infty$ . Using Fubini's Theorem and Corollary VII.16 of [8] we have

$$\begin{aligned} \mathbb{E}[I(V^\uparrow)] &= \int_0^{+\infty} \mathbb{E}[e^{-V^\uparrow(t)}] dt = \int_0^{+\infty} \int_0^{+\infty} e^{-y} \mathbb{P}(V^\uparrow(t) \in dy) dt \\ &= \int_0^{+\infty} \int_0^{+\infty} e^{-y} \frac{yW(y)}{t} \mathbb{P}(V(t) \in dy) dt. \end{aligned}$$

Now, using Corollary VII.3 of [8] we get

$$\mathbb{E}[I(V^\uparrow)] = \int_0^{+\infty} \int_0^{+\infty} e^{-y} W(y) \mathbb{P}(\tau(V, y) \in dt) dy = \int_0^{+\infty} e^{-y} W(y) \mathbb{P}(\tau(V, y) < +\infty) dy.$$

Since  $\mathbb{P}(\tau(V, y) < +\infty) = \mathbb{P}(\sup_{[0, +\infty[} V \geq y) = e^{-\kappa y}$  we obtain

$$\mathbb{E}[I(V^\uparrow)] = \int_0^{+\infty} e^{-(1+\kappa)y} W(y) dy < +\infty,$$

where the finiteness comes from the fact that  $\int_0^{+\infty} e^{-\lambda y} W(y) dy < +\infty$  for  $\lambda > \kappa$ . As a consequence, the exponential functional  $I(V^\uparrow)$  is almost surely finite and has finite expectation.

We now turn to the proof of the finiteness of the Laplace transform. We proceed by bounding the moments of the exponential functional. For any  $x \geq 0$  let us define  $h(x) := \mathbb{E}[I(V_x^\uparrow)]$ . For any  $x > 0$  we have

$$I(V^\uparrow) = \int_0^{+\infty} e^{-V^\uparrow(t)} dt \geq \int_{\tau(V, x)}^{+\infty} e^{-V^\uparrow(t)} dt \stackrel{\mathcal{L}}{=} \int_0^{+\infty} e^{-V_x^\uparrow(t)} dt = I(V_x^\uparrow),$$

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where, for the equality in law, we used the Markov property for  $V^\uparrow$  at time  $\tau(V, x)$ . As a consequence we have

$$\forall x > 0, h(x) = \mathbb{E}[I(V_x^\uparrow)] \leq \mathbb{E}[I(V^\uparrow)] < +\infty. \quad (3.3.1)$$

Now, note that for any  $k \geq 1$ ,

$$\begin{aligned} \mathbb{E} \left[ (I(V^\uparrow))^k \right] &= \mathbb{E} \left[ \int_0^{+\infty} \dots \int_0^{+\infty} e^{-V^\uparrow(t_1)} \times \dots \times e^{-V^\uparrow(t_k)} dt_1 \dots dt_k \right] \\ &= k! \mathbb{E} \left[ \int_{0 \leq t_1 < \dots < t_k} e^{-V^\uparrow(t_1)} \times \dots \times e^{-V^\uparrow(t_k)} dt_1 \dots dt_k \right], \end{aligned}$$

so that

$$\forall k \geq 1, \mathbb{E} \left[ (I(V^\uparrow))^k \right] / k! = \mathbb{E} \left[ \int_{0 \leq t_1 < \dots < t_k} e^{-V^\uparrow(t_1)} \times \dots \times e^{-V^\uparrow(t_k)} dt_1 \dots dt_k \right] \quad (3.3.2)$$

Let us prove by induction that for any  $k \geq 1$ ,

$$\mathbb{E} \left[ (I(V^\uparrow))^k \right] \leq k! (\mathbb{E}[I(V^\uparrow)])^k < +\infty \quad (3.3.3)$$

(3.3.3) is clearly true for  $k = 1$ . Let us assume that it is true for some arbitrary rank  $k$ . According to (3.3.2),  $\mathbb{E}[(I(V^\uparrow))^{k+1}]/(k+1)!$  equals

$$\begin{aligned} &\mathbb{E} \left[ \int_{0 \leq t_1 < \dots < t_{k+1}} e^{-V^\uparrow(t_1)} \times \dots \times e^{-V^\uparrow(t_k)} \times e^{-V^\uparrow(t_{k+1})} dt_1 \dots dt_k dt_{k+1} \right] \\ &= \mathbb{E} \left[ \int_{0 \leq t_1 < \dots < t_k} e^{-V^\uparrow(t_1)} \times \dots \times e^{-V^\uparrow(t_k)} \left( \int_{t_k}^{+\infty} e^{-V^\uparrow(s)} ds \right) dt_1 \dots dt_k \right] \\ &= \mathbb{E} \left[ \int_{0 \leq t_1 < \dots < t_k} e^{-V^\uparrow(t_1)} \times \dots \times e^{-V^\uparrow(t_k)} \mathbb{E} \left[ \int_{t_k}^{+\infty} e^{-V^\uparrow(s)} ds \mid \sigma(V^\uparrow(u), 0 \leq u \leq t_k) \right] dt_1 \dots dt_k \right]. \end{aligned}$$

From the Markov property at time  $t_k$ , the conditional expectation in the above expression equals  $h(V^\uparrow(t_k))$  which, according to (3.3.1), is almost surely less than  $\mathbb{E}[I(V^\uparrow)]$ . We thus get

$$\begin{aligned} \mathbb{E}[(I(V^\uparrow))^{k+1}]/(k+1)! &\leq \mathbb{E}[I(V^\uparrow)] \times \mathbb{E} \left[ \int_0^{0 \leq t_1 < \dots < t_k} e^{-V^\uparrow(t_1)} \times \dots \times e^{-V^\uparrow(t_k)} dt_1 \dots dt_k \right] \\ &= \mathbb{E}[I(V^\uparrow)] \times \mathbb{E} \left[ (I(V^\uparrow))^k \right] / k! \leq (\mathbb{E}[I(V^\uparrow)])^{k+1}, \end{aligned}$$

where we used (3.3.2) and the induction hypothesis. Thus the induction is proved. As a consequence for all  $\lambda \in ]0, 1/\mathbb{E}[I(V^\uparrow)][$  we have

$$\mathbb{E} \left[ e^{\lambda I(V^\uparrow)} \right] = \sum_{k \geq 0} \lambda^k \mathbb{E} \left[ (I(V^\uparrow))^k \right] / k! \leq \sum_{k \geq 0} (\lambda \mathbb{E}[I(V^\uparrow)])^k < +\infty.$$

The finiteness of the Laplace transform is obvious for  $\lambda \leq 0$ , so the result is proved.  $\square$



**Remark 3.3.1.** *If  $V$  does not oscillate, the finiteness of  $I(V^\uparrow)$  can be derived as a consequence of Lemma 3.2.6. Indeed, from the second statement of Lemma 3.2.6 applied to  $V^\sharp$ , we have*

$$I(V^\uparrow) \stackrel{\mathcal{L}}{=} \int_0^{+\infty} e^{-V^\sharp(t+\mathcal{R}(V^\sharp,0))} dt = \int_{\mathcal{R}(V^\sharp,0)}^{+\infty} e^{-V^\sharp(t)} dt \leq \int_0^{+\infty} e^{-V^\sharp(t)} dt = I(V^\sharp),$$

and  $V^\sharp$  drifts to  $+\infty$ , so Theorem 1 in [12] ensures that  $I(V^\sharp)$  is almost surely finite which yields the result.

### 3.3.2 Decomposition of the law of $I(V^\uparrow)$

In this subsection, we prove that the law of  $I(V^\uparrow)$  is solution of the random affine equation (3.1.3) and we give a decomposition of its non-trivial coefficient  $A^y$ . This is a key point of our analysis of the law of  $I(V^\uparrow)$ .

**Proposition 3.3.2.** *For any  $y > 0$ , the law of  $I(V^\uparrow)$  satisfies the random affine equation*

$$I(V^\uparrow) \stackrel{\mathcal{L}}{=} \int_0^{\tau(V^\uparrow,y)} e^{-V^\uparrow(t)} dt + S_T + e^{-y} I(\tilde{V}^\uparrow), \quad (3.3.4)$$

where the three terms of the right hand side are independent,  $S_T$  is as in Lemma 3.2.1, and  $\tilde{V}^\uparrow$  is an independent copy of  $V^\uparrow$ .

We define

$$A^y := \int_0^{\tau(V^\uparrow,y)} e^{-V^\uparrow(t)} dt + S_T$$

to lighten notations.

As a consequence,  $I(V^\uparrow)$  has the same law as the sum of a random series :

$$I(V^\uparrow) \stackrel{\mathcal{L}}{=} \sum_{k \geq 0} e^{-ky} A_k^y, \quad (3.3.5)$$

where the random variables  $A_k^y$  are iid and have the same law as  $A^y$ .

**Remark 3.3.3.** *The almost sure convergence of the random series in (3.3.5) is a consequence of the almost sure finiteness, given by Theorem 3.1.1, of the positive random variable  $I(V^\uparrow)$ . Also, it is a well known fact on random power series with iid coefficients that their radius of convergence is almost surely equal to a constant belonging to  $\{0,1\}$ . Since this constant, in the case of the power series in (3.3.5), has been proved to be greater than  $e^{-y}$ , we deduce that it equals 1.*

*Démonstration.* of Proposition 3.3.2

We fix  $y > 0$ . As  $V^\uparrow$  has no positive jumps and goes to infinity we have  $\tau(V^\uparrow, y) \leq \mathcal{R}(V^\uparrow, y) < +\infty$  and  $V^\uparrow(\tau(V^\uparrow, y)) = V^\uparrow(\mathcal{R}(V^\uparrow, y)) = y$ .

We write :

$$\begin{aligned}
 I(V^\uparrow) &= \int_0^{\mathcal{R}(V^\uparrow, y)} e^{-V^\uparrow(t)} dt + \int_{\mathcal{R}(V^\uparrow, y)}^{+\infty} e^{-V^\uparrow(t)} dt \\
 &= \int_0^{\mathcal{R}(V^\uparrow, y)} e^{-V^\uparrow(t)} dt + e^{-y} \int_0^{+\infty} e^{-(V^\uparrow(t+\mathcal{R}(V^\uparrow, y))-y)} dt \\
 &\stackrel{\mathcal{L}}{=} \int_0^{\mathcal{R}(V^\uparrow, y)} e^{-V^\uparrow(t)} dt + e^{-y} I(\tilde{V}^\uparrow),
 \end{aligned}$$

where we used Lemma 3.2.5 for the last equality in which  $\tilde{V}^\uparrow$  is an independent copy of  $V^\uparrow$ . We now decompose :

$$\int_0^{\mathcal{R}(V^\uparrow, y)} e^{-V^\uparrow(t)} dt = \int_0^{\tau(V^\uparrow, y)} e^{-V^\uparrow(t)} dt + \int_{\tau(V^\uparrow, y)}^{\mathcal{R}(V^\uparrow, y)} e^{-V^\uparrow(t)} dt \quad (3.3.6)$$

Since  $V^\uparrow(\tau(V^\uparrow, y)) = y$ , combining with the Markov property at time  $\tau(V^\uparrow, y)$ , the two terms in the right hand side of (3.3.6) are independent :

$$\int_0^{\tau(V^\uparrow, y)} e^{-V^\uparrow(t)} dt \perp\!\!\!\perp \int_{\tau(V^\uparrow, y)}^{\mathcal{R}(V^\uparrow, y)} e^{-V^\uparrow(t)} dt.$$

Now, thanks to Lemma 3.2.1, the second term has the same law as  $S_T$  with  $S_T$  as in the lemma. This achieves the proof.  $\square$

We have two remarks here :

**Remark 3.3.4.** *It is possible to prove Theorem 3.1.1 (the fact that  $I(V^\uparrow)$  is finite and admits some finite exponential moments) by invoking (3.3.5) and proving that each of the two terms composing  $A^y$  admit some finite exponential moments.*

**Remark 3.3.5.** *Let  $S$  and  $T$  be as in Proposition 3.3.2 and  $\epsilon \in ]0, \mathbb{E}[S_1][$  (where we do not bother with the fact that  $\mathbb{E}[S_1]$  is finite or not). Then we have*

$$\mathbb{P}(S_T \geq t) \geq \mathbb{P}(S_{t/\epsilon} \geq t) \times \mathbb{P}(T \geq t/\epsilon).$$

*The first factor in the right hand side converges to 1 thanks to the law of large numbers for Lévy processes (see for example Theorem 36.5 in [59]) and the second is equal to  $e^{-pt/\epsilon}$ , where  $p$  is the parameter of the exponential random variable  $T$ . Therefore, the Laplace transform of  $S_T$  is not finite everywhere, so neither is the Laplace transform of  $I(V^\uparrow)$  (because of (3.3.5)). This is why we can not say better than " $I(V^\uparrow)$  admits **some** finite exponential moments".*

### 3.4 Asymptotic tail at 0 : Proof of Theorems 3.1.2, 3.1.4 and 3.1.5

First, let us prove that Theorem 3.1.2 easily implies Theorem 3.1.4 and prove Remark 3.1.7.

*Démonstration.* of Theorem 3.1.4

Assume that Theorem 3.1.2 is proved.

We first prove (3.1.6). Let us fix  $\beta' > \beta$  and  $\epsilon > 0$ . From the definition of  $\beta$  we have that  $\Psi_V(\lambda) \leq \epsilon \lambda^{\beta'}$  for all  $\lambda$  large enough. Using the first point of Theorem 3.1.2 we deduce that

$$\limsup_{x \rightarrow 0} x^{1/(\beta'-1)} \log (\mathbb{P} (I(V^\uparrow) \leq x)) \leq -(\beta' - 1)/\epsilon^{1/(\beta'-1)}.$$

Since  $\epsilon$  can be chosen as small as we want we obtain (3.1.6).

We now assume that  $\sigma > 1$  and prove (3.1.7). Let us fix  $\sigma' \in ]1, \sigma[$  and  $M > 0$ . From the definition of  $\sigma$  we have that  $\Psi_V(\lambda) \geq M \lambda^{\sigma'}$  for all  $\lambda$  large enough. Using the second point of Theorem 3.1.2 we deduce that

$$0 \geq \liminf_{x \rightarrow 0} x^{1/(\sigma'-1)} \log (\mathbb{P} (I(V^\uparrow) \leq x)) \geq -\sigma'^{\sigma'/(\sigma'-1)}/M^{1/(\sigma'-1)}.$$

Since  $M$  can be chosen as large as we want we obtain (3.1.7). □

*Démonstration.* of Remark 3.1.7

Let us assume that  $V$  has bounded variation. As it can be seen from Remark 3.1.3, it is the difference of a positive drift  $dt$  and a pure jump subordinator  $S_t$  :  $\forall t > 0, V(t) = dt - S_t \leq dt$ . Let us fix  $y > 0$ , we have almost surely

$$\int_0^{\tau(V,y)} e^{-V(t)} dt \geq \int_0^{\tau(V,y)} e^{-dt} dt = \frac{1}{d} (1 - e^{-\tau(V,y)}). \quad (3.4.1)$$

Since  $V$  has bounded variation, we have  $\mathbb{P}(V(t) > 0, 0 \leq t \leq \tau(V, y)) > 0$  (see for example (47.1) in [59]) and we can see that  $(V^\uparrow(t), 0 \leq t \leq \tau(V^\uparrow, y))$  is equal in law to  $(V, 0 \leq t \leq \tau(V, y))$  conditioned in the usual sense to remain positive. Combining with (3.4.1), we see that almost surely

$$I(V^\uparrow) \geq \int_0^{\tau(V^\uparrow,y)} e^{-V^\uparrow(t)} dt \geq \frac{1}{d} (1 - e^{-\tau(V^\uparrow,y)}).$$

Then, since  $\tau(V^\uparrow, y)$  converges almost surely to  $+\infty$  when  $y$  goes to  $+\infty$ , we deduce that  $I(V^\uparrow)$  is more than the positive constant  $1/d$  almost surely. As a consequence  $\mathbb{P}(I(V^\uparrow) \leq x)$  is null for  $x \leq 1/d$ . □

In the next Subsection we prepare the proofs of Theorems 3.1.2 and 3.1.5.

### 3.4.1 Laplace transform of $I(V^\uparrow)$

In order to prove asymptotic estimates on  $\mathbb{P}(I(V^\uparrow) \leq x)$ , we first study the Laplace transform of  $I(V^\uparrow)$  via the decomposition given by Proposition 3.3.2. It is thus natural that we need first to study the Laplace transform of  $A^y$ .

First, let us define a notation.  $V^\sharp$  is a spectrally negative Lévy process, so, according to Theorem VII.1 in [8], the process  $\tau(V^\sharp, \cdot)$  is a subordinator which Laplace exponent  $\Phi_{V^\sharp}$  is defined for  $\lambda \geq 0$  by

$$\Phi_{V^\sharp}(\lambda) := -\log \left( \mathbb{E} \left[ e^{-\lambda \tau(V^\sharp, 1)} \right] \right),$$

and we have  $\Phi_{V^\sharp} = \Psi_{V^\sharp}^{-1}$ .

**Proposition 3.4.1.** *We fix  $y > 0$ . Let  $A^y$  be as in Proposition 3.3.2, then, for all  $\epsilon > 0$  and  $\lambda$  large enough we have*

$$(1 - \epsilon)y\Phi_{V^\sharp}(e^{-y}\lambda) \leq -\log \left( \mathbb{E} \left[ e^{-\lambda A^y} \right] \right) \leq (1 + \epsilon)y\Phi_{V^\sharp}(\lambda). \quad (3.4.2)$$

*Démonstration.* of Proposition 3.4.1

According to the definition of  $A^y$  in the Proposition 3.3.2,  $A^y$  can be decomposed as the sum of two independent random variables, one having the same law as  $\int_0^{\tau(V^\uparrow, y)} e^{-V^\uparrow(t)} dt$  and another having the same law as  $S_T$ , defined as in Lemma 3.2.1. Let  $\Phi_S$  be the Laplace exponent of the subordinator  $S$  :

$$\forall \lambda > 0, \Phi_S(\lambda) := -\log \left( \mathbb{E} \left[ e^{-\lambda S_1} \right] \right). \quad (3.4.3)$$

We can see that the Laplace transform of the random variable  $S_T$  is given by

$$\forall \lambda > 0, \mathbb{E} \left[ e^{-\lambda S_T} \right] = \frac{\eta_y^\sharp(IP_\infty) + \eta_y^\sharp(N)}{\eta_y^\sharp(IP_\infty) + \eta_y^\sharp(N) + \Phi_S(\lambda)}.$$

We thus have

$$\begin{aligned} -\log \left( \mathbb{E} \left[ e^{-\lambda A^y} \right] \right) &= -\log \left( \mathbb{E} \left[ \exp \left( -\lambda \int_0^{\tau(V^\uparrow, y)} e^{-V^\uparrow(t)} dt \right) \right] \right) - \log \left( \mathbb{E} \left[ e^{-\lambda S_T} \right] \right) \\ &= -\log \left( \mathbb{E} \left[ \exp \left( -\lambda \int_0^{\tau(V^\uparrow, y)} e^{-V^\uparrow(t)} dt \right) \right] \right) - \log \left( \frac{p}{p + \Phi_S(\lambda)} \right), \end{aligned} \quad (3.4.4)$$

where we denoted  $p$  for the constant  $\eta_y^\sharp(IP_\infty) + \eta_y^\sharp(N)$ . Using the fact that  $V^\uparrow$  is non-negative and the first point of Lemma 3.2.6 we have

$$\int_0^{\tau(V^\uparrow, y)} e^{-V^\uparrow(t)} dt \leq \tau(V^\uparrow, y) \stackrel{sto}{\leq} \tau(V^\sharp, y), \quad (3.4.5)$$

where  $\stackrel{sto}{\leq}$  denotes a stochastic inequality. As a consequence of (3.4.5) and of the definition of  $\Phi_{V^\#}$  we have

$$-\log \left( \mathbb{E} \left[ \exp \left( -\lambda \int_0^{\tau(V^\uparrow, y)} e^{-V^\uparrow(t)} dt \right) \right] \right) \leq -\log \left( \mathbb{E} \left[ e^{-\lambda \tau(V^\#, y)} \right] \right) = y \Phi_{V^\#}(\lambda).$$

Combining this inequality with (3.4.4) we obtain

$$-\log \left( \mathbb{E} \left[ e^{-\lambda A^y} \right] \right) \leq y \Phi_{V^\#}(\lambda) - \log \left( \frac{p}{p + \Phi_S(\lambda)} \right). \quad (3.4.6)$$

Using the first point of Lemma 3.2.6 and Lemma 3.2.3 we have

$$-\log \left( \mathbb{E} \left[ \exp \left( -\lambda \int_0^{\tau(V^\uparrow, y)} e^{-V^\uparrow(t)} dt \right) \right] \right) = -\log \left( \mathbb{E} \left[ e^{-\lambda \int_0^{\tau(V^\#, y)} e^{-V^\#(t)} dt} \right] / \mathbb{E} \left[ e^{-\lambda \tilde{S}_T} \right] \right),$$

where  $\tilde{S}_T$  is as  $S_T$  from Lemma 3.2.3. Here again, if  $\Phi_{\tilde{S}}$  denotes the Laplace exponent of the subordinator  $\tilde{S}$  as in (3.4.3) we have

$$\forall \lambda > 0, \mathbb{E} \left[ e^{-\lambda \tilde{S}_T} \right] = \frac{\eta^\#(I_y)}{\eta^\#(I_y) + \Phi_{\tilde{S}}(\lambda)}.$$

Moreover we have

$$\int_0^{\tau(V^\#, y)} e^{-V^\#(t)} dt \geq e^{-y} \tau(V^\#, y).$$

Putting together the above three expressions and the definition of  $\Phi_{V^\#}$  we obtain

$$-\log \left( \mathbb{E} \left[ \exp \left( -\lambda \int_0^{\tau(V^\uparrow, y)} e^{-V^\uparrow(t)} dt \right) \right] \right) \geq y \Phi_{V^\#}(e^{-y} \lambda) + \log \left( \frac{\eta^\#(I_y)}{\eta^\#(I_y) + \Phi_{\tilde{S}}(\lambda)} \right).$$

Combining the above inequality with (3.4.4) and the fact that the term  $-\log(p/(p + \Phi_S(\lambda)))$  is non-negative we get

$$-\log \left( \mathbb{E} \left[ e^{-\lambda A^y} \right] \right) \geq y \Phi_{V^\#}(e^{-y} \lambda) + \log \left( \frac{\eta^\#(I_y)}{\eta^\#(I_y) + \Phi_{\tilde{S}}(\lambda)} \right). \quad (3.4.7)$$

According to the Lévy-Khintchine formula for subordinators. The Laplace exponent  $\Phi_S$  can be written

$$\Phi_S(\lambda) = \gamma_S \lambda + \int_0^{+\infty} (1 - e^{-\lambda x}) \nu_S(dx),$$

so by dominated convergence, there exists a positive constant  $C_S$  such that for large  $\lambda$ ,  $\Phi_S(\lambda) \leq C_S \lambda$ . Similarly, there is a positive constant  $C_{\tilde{S}}$  such that for large  $\lambda$ ,  $\Phi_{\tilde{S}}(\lambda) \leq C_{\tilde{S}} \lambda$ . On the other hand, since  $\Psi_V(\lambda)/\lambda^2$  is bounded when  $\lambda$  goes to infinity, there is a positive constant  $c$  such that  $\Phi_{V^\#}(\lambda) \geq c\lambda^{1/2}$  for large  $\lambda$ . Combining all this with (3.4.6) and (3.4.7) we get (3.4.2) for any fixed  $\epsilon > 0$  and  $\lambda$  large enough.  $\square$

**Proposition 3.4.2.** *Assume that there is  $\alpha \geq 1$  and a positive constant  $C$  such that for all  $\lambda$  large enough we have  $\Psi_V(\lambda) \leq C\lambda^\alpha$ , then we have*

$$\liminf_{\lambda \rightarrow +\infty} -\log \left( \mathbb{E} \left[ e^{-\lambda I(V^\dagger)} \right] \right) / \lambda^{1/\alpha} \geq \alpha / C^{1/\alpha}. \quad (3.4.8)$$

*Assume that there is  $\alpha \geq 1$  and a positive constant  $c$  such that for all  $\lambda$  large enough we have  $\Psi_V(\lambda) \geq c\lambda^\alpha$ , then we have*

$$\limsup_{\lambda \rightarrow +\infty} -\log \left( \mathbb{E} \left[ e^{-\lambda I(V^\dagger)} \right] \right) / \lambda^{1/\alpha} \leq \alpha / c^{1/\alpha}. \quad (3.4.9)$$

*Démonstration.* Let us fix  $y > 0$  for which we apply the decomposition (3.3.5). Let us denote

$$\mathcal{M}(\lambda) := -\log \left( \mathbb{E} \left[ e^{-\lambda I(V^\dagger)} \right] \right) = \sum_{k \geq 0} -\log \left( \mathbb{E} \left[ e^{-\lambda e^{-ky} A^y} \right] \right),$$

where the second equality comes from (3.3.5) and from the fact that the sequence  $(A_k^y)_{k \geq 0}$  is *iid*. To establish the left tail of  $I(V^\dagger)$  we study the asymptotic behavior of  $\mathcal{M}(\lambda)$  and the latter, thanks to the above expression, is related to the asymptotic behavior of  $-\log(\mathbb{E}[e^{-\lambda A^y}])$ .

Unfortunately, Proposition 3.4.1 can not be applied simultaneously to all the terms of the sum defining  $\mathcal{M}(\lambda)$ . We separate this sum into three parts : a sum over a finite number of small indices for which we can apply (3.4.2) to each term, a sum over an infinite number of large indices that can be neglected, and a sum over the remaining indices (in finite number) that can be neglected.

We now prove the second point of the proposition. We assume that there is  $\alpha \geq 1$  and a positive constant  $c$  such that for all  $\lambda$  large enough we have  $\Psi_V(\lambda) \geq c\lambda^\alpha$ , and prove (3.4.9). Let us fix  $\delta > 1$ . Since  $\Psi_{V^\#}(\cdot) = \Psi_V(\kappa + \cdot)$  and  $\Phi_{V^\#} = \Psi_{V^\#}^{-1}$ , we have for all  $\lambda$  large enough that  $\Phi_{V^\#}(\lambda) \leq \delta \lambda^{1/\alpha} / c^{1/\alpha}$ .

According to Proposition 3.4.1 there exists  $\lambda_\delta > 1$  such that (3.4.2) is satisfied with  $\epsilon = \delta - 1$  for all  $\lambda \geq \lambda_\delta$ . By increasing  $\lambda_\delta$  if necessary, we can also assume that  $\Phi_{V^\#}(\lambda) \leq \delta \lambda^{1/\alpha} / c^{1/\alpha}$  for all  $\lambda \geq \lambda_\delta$ . Putting all this together we get

$$\lambda \geq \lambda_\delta \Rightarrow -\log \left( \mathbb{E} \left[ e^{-\lambda A^y} \right] \right) \leq \delta^2 y \lambda^{1/\alpha} / c^{1/\alpha}. \quad (3.4.10)$$

Also, let us choose  $M \in ]0, 1[$  small enough so that

$$\forall \lambda \in [0, M], \quad 0 \leq -\log \left( \mathbb{E} \left[ e^{-\lambda A^y} \right] \right) \leq 2\lambda \mathbb{E} [A^y].$$

For any  $\lambda > \lambda_\delta$ , we define  $n_1(\lambda) := \lfloor \log(\lambda/\lambda_\delta)/y \rfloor$  and  $n_2(\lambda) := \lfloor \log(\lambda/M)/y \rfloor$ . From the definition of  $\mathcal{M}(\lambda)$  we can write

$$\forall \lambda > \lambda_\delta, \quad \mathcal{M}(\lambda) = T_1(\lambda) + T_2(\lambda) + T_3(\lambda), \quad (3.4.11)$$

with

$$T_1(\lambda) := \sum_{k=0}^{n_1(\lambda)} -\log \left( \mathbb{E} \left[ e^{-\lambda e^{-ky} A^y} \right] \right), \quad T_2(\lambda) := \sum_{k=n_1(\lambda)+1}^{n_2(\lambda)} -\log \left( \mathbb{E} \left[ e^{-\lambda e^{-ky} A^y} \right] \right),$$

$$T_3(\lambda) := \sum_{k=n_2(\lambda)+1}^{+\infty} -\log \left( \mathbb{E} \left[ e^{-\lambda e^{-ky} A^y} \right] \right).$$

From the definition of  $n_1(\lambda)$ , (3.4.10) can be applied to each term of the sum defining  $T_1(\lambda)$ , we thus have

$$T_1(\lambda) \leq \delta^2 y (\lambda/c)^{1/\alpha} \sum_{k=0}^{n_1(\lambda)} e^{-ky/\alpha} = \delta^2 y (\lambda/c)^{1/\alpha} \frac{1 - e^{-y(n_1(\lambda)+1)/\alpha}}{1 - e^{-y/\alpha}}.$$

We get that for  $\lambda$  large enough

$$T_1(\lambda) \leq \delta^2 y (\lambda/c)^{1/\alpha} \frac{1}{1 - e^{-y/\alpha}}. \quad (3.4.12)$$

Using the monotony of the Laplace transform and the definitions of  $n_1(\lambda)$  and  $n_2(\lambda)$  we get

$$0 \leq T_2(\lambda) \leq -(n_2(\lambda) - n_1(\lambda)) \log \left( \mathbb{E} \left[ e^{-\lambda e^{-(n_1(\lambda)+1)y} A^y} \right] \right)$$

$$\leq -\frac{y + \log(\lambda_\delta/M)}{y} \log \left( \mathbb{E} \left[ e^{-\lambda_\delta A^y} \right] \right) < +\infty. \quad (3.4.13)$$

From the definitions of  $n_2(\lambda)$  and  $M$  we have

$$0 \leq T_3(\lambda) \leq 2\lambda \mathbb{E} [A^y] \sum_{k=n_2(\lambda)+1}^{+\infty} e^{-ky} = 2\lambda e^{-y(K_2(\lambda)+1)} \mathbb{E} [A^y] / (1 - e^{-y})$$

$$\leq 2M \mathbb{E} [A^y] / (1 - e^{-y}) < +\infty. \quad (3.4.14)$$

Putting (3.4.12), (3.4.13), and (3.4.14) into (3.4.11) we obtain that for  $\lambda$  large enough

$$\limsup_{\lambda \rightarrow +\infty} \mathcal{M}(\lambda) / \lambda^{1/\alpha} \leq \frac{\delta^2 y}{c^{1/\alpha} (1 - e^{-y/\alpha})}.$$

Since, from the definition  $\mathcal{M}(\lambda) = -\log(\mathbb{E}[e^{-\lambda I(V^\dagger)}])$  which does not depend on  $\delta$  nor on  $y$ , we can let  $\delta$  go to 1 and then  $y$  go to 0 in the above expression. We obtain

$$\limsup_{\lambda \rightarrow +\infty} -\log \left( \mathbb{E} \left[ e^{-\lambda I(V^\dagger)} \right] \right) / \lambda^{1/\alpha} \leq \alpha / c^{1/\alpha}.$$

For the first point of the proposition, we proceed exactly as for the second point, using the lower bound of (3.4.2) instead of the upper bound and noticing that  $n_1(\lambda)$  converges toward  $+\infty$  as  $\lambda$  goes to infinity, so that the factor  $(1 - e^{-y(n_1(\lambda)+1)/\alpha}) / (1 - e^{-y/\alpha})$  converges to  $1 / (1 - e^{-y/\alpha})$ .

□

### 3.4.2 Tail at 0 of $I(V^\dagger)$ : proof of Theorems 3.1.2 and 3.1.5

We now prove Theorems 3.1.2 and 3.1.5 by using Proposition 3.4.2 together with the deep link that exists between the left tail of a random variable and the asymptotic behavior of its Laplace transform.

*Démonstration.* of Theorem 3.1.2

We assume that there is  $\alpha > 1$  and a positive constant  $C$  such that for all  $\lambda$  large enough we have  $\Psi_V(\lambda) \leq C\lambda^\alpha$ . We fix  $\delta \in ]0, 1[$ . According to the first point of Proposition 3.4.2, there exists  $\lambda_\delta > 0$  such that for all  $\lambda > \lambda_\delta$  we have

$$\mathbb{E} \left[ e^{-\lambda I(V^\dagger)} \right] \leq \exp \left( -\delta^{1-1/\alpha} \alpha \lambda^{1/\alpha} / C^{1/\alpha} \right). \quad (3.4.15)$$

Let us fix  $x \in ]0, \delta^{(\alpha-1)/\alpha} C^{-1/\alpha} \lambda_\delta^{(1-\alpha)/\alpha}[$ . Using Markov inequality and (3.4.15) we get that for any  $\lambda > \lambda_\delta$ ,

$$\mathbb{P} \left( I(V^\dagger) \leq x \right) \leq e^{\lambda x} \mathbb{E} \left[ e^{-\lambda I(V^\dagger)} \right] \leq \exp \left( \lambda x - \delta^{1-1/\alpha} \alpha \lambda^{1/\alpha} / C^{1/\alpha} \right).$$

Since  $x \in ]0, \delta^{(\alpha-1)/\alpha} C^{-1/\alpha} \lambda_\delta^{(1-\alpha)/\alpha}[$  we have  $\delta C^{-1/(\alpha-1)} x^{-\alpha/(\alpha-1)} \geq \lambda_\delta$  so, in the above inequality, we can replace  $\lambda$  by  $\delta C^{-1/(\alpha-1)} x^{-\alpha/(\alpha-1)}$ . We obtain

$$\mathbb{P} \left( I(V^\dagger) \leq x \right) \leq \exp \left( -\delta(\alpha-1) / (Cx)^{1/(\alpha-1)} \right),$$

which is (3.1.4).

We now prove the second point. Assume that there is  $\alpha > 1$  and a positive constant  $c$  such that for all  $\lambda$  large enough we have  $\Psi_V(\lambda) \geq c\lambda^\alpha$ . We fix  $\delta > 1$  and  $r \in ]0, 1[$ . According to the second point of Proposition 3.4.2, there exists  $\lambda_\delta > 0$  such that for all  $\lambda > \lambda_\delta$  we have

$$\mathbb{E} \left[ e^{-\lambda I(V^\dagger)} \right] \geq \exp \left( -\delta^{r(\alpha-1)/\alpha} \alpha \lambda^{1/\alpha} / c^{1/\alpha} \right). \quad (3.4.16)$$

Let us fix  $x \in ]0, \delta^{(\alpha-1)/\alpha} \alpha c^{-1/\alpha} \lambda_\delta^{(1-\alpha)/\alpha}[$ . Using (3.4.16) we get that for any  $\lambda > \lambda_\delta$ ,

$$\begin{aligned} \exp \left( -\delta^{r(\alpha-1)/\alpha} \alpha \lambda^{1/\alpha} / c^{1/\alpha} \right) &\leq \mathbb{E} \left[ e^{-\lambda I(V^\dagger)} \right] \\ &= \mathbb{E} \left[ e^{-\lambda I(V^\dagger)} \mathbf{1}_{\{I(V^\dagger) \leq x\}} \right] + \mathbb{E} \left[ e^{-\lambda I(V^\dagger)} \mathbf{1}_{\{I(V^\dagger) > x\}} \right] \\ &\leq \mathbb{P} \left( I(V^\dagger) \leq x \right) + \exp \left( -\lambda x \right), \end{aligned}$$

so we get

$$\mathbb{P} \left( I(V^\dagger) \leq x \right) \geq \exp \left( -\delta^{r(\alpha-1)/\alpha} \alpha \lambda^{1/\alpha} / c^{1/\alpha} \right) - \exp \left( -\lambda x \right).$$

Since  $x \in ]0, \delta^{(\alpha-1)/\alpha} \alpha c^{-1/\alpha} \lambda_\delta^{(1-\alpha)/\alpha}[$  we have  $\delta \alpha^{\alpha/(\alpha-1)} c^{-1/(\alpha-1)} x^{-\alpha/(\alpha-1)} \geq \lambda_\delta$  so, in the above inequality, we can replace  $\lambda$  by  $\delta \alpha^{\alpha/(\alpha-1)} c^{-1/(\alpha-1)} x^{-\alpha/(\alpha-1)}$ . We obtain

$$\mathbb{P} \left( I(V^\dagger) \leq x \right) \geq \exp \left( -\delta^{(1+r(\alpha-1))/\alpha} \alpha^{\alpha/(\alpha-1)} / (cx)^{1/(\alpha-1)} \right) - \exp \left( -\delta \alpha^{\alpha/(\alpha-1)} / (cx)^{1/(\alpha-1)} \right).$$



Since  $\delta > \delta^{(1+r(\alpha-1))/\alpha}$ , the second term converges to 0 faster than the first one when 0 goes to 0 so we get, for  $x$  large enough,

$$\begin{aligned} \mathbb{P}(I(V^\uparrow) \leq x) &\geq \frac{1}{2} \exp(-\delta^{(1+r(\alpha-1))/\alpha} \alpha^{\alpha/(\alpha-1)} / (cx)^{1/(\alpha-1)}) \\ &\geq \exp(-\delta \alpha^{\alpha/(\alpha-1)} / (cx)^{1/(\alpha-1)}), \end{aligned}$$

where the last inequality is true for  $x$  small enough and comes from the fact that  $r \in ]0, 1[$ . This is precisely (3.1.5). □

*Démonstration.* of Theorem 3.1.5

We assume that there is a positive constant  $C$  and  $\alpha \in ]1, 2]$  such that  $\Psi_V(\lambda) \sim_{\lambda \rightarrow +\infty} C\lambda^\alpha$ . From Proposition 3.4.2 we deduce that

$$-\log \left( \mathbb{E} \left[ e^{-\lambda I(V^\uparrow)} \right] \right) \underset{\lambda \rightarrow +\infty}{\sim} \alpha \lambda^{1/\alpha} / C^{1/\alpha},$$

and the application of De Bruijn's Theorem (see Theorem 4.12.9 in [14]) yields the result. □

### 3.4.3 Connection between $I(V^\uparrow)$ and $I(V)$ : proof of Proposition 3.1.9

In this subsection, we assume that  $V$  drifts to  $+\infty$  (so that  $I(V) < +\infty$ ) and we prove a simple connection between the asymptotic tails at 0 of  $I(V^\uparrow)$  and  $I(V)$ .

**Proposition 3.4.3.** *If  $V$  drifts to  $+\infty$ , there is a positive constant  $c$  such that for all positive  $\epsilon$  and  $x$  small enough,*

$$\mathbb{P}(I(V) \leq x) \leq \mathbb{P}(I(V^\uparrow) \leq x) \leq \mathbb{P}(I(V) \leq (1 + \epsilon)x) / c\epsilon x.$$

As in the proof of Proposition 3.3.2, we decompose  $I(V)$  as the sum of two independent random variables, one having the same law as a subordinator stopped at an independent exponential time and the other having the same law as  $I(V^\uparrow)$ . We first need an easy lemma about the asymptotic tail at 0 of a subordinator stopped at an independent exponential time.

**Lemma 3.4.4.** *Let  $S$  be a subordinator and  $T$  an independent exponential random variable, there exists a positive constant  $c$  such that for all  $x$  small enough*

$$\mathbb{P}(S_T < x) \geq cx.$$

*Démonstration.* We prove in fact a stronger result : the function  $x \mapsto \mathbb{P}(S_T < x)$  is sub-additive, that is

$$\forall x, y \geq 0, \mathbb{P}(S_T < x + y) \leq \mathbb{P}(S_T < x) + \mathbb{P}(S_T < y) \tag{3.4.17}$$

and the lemma follows easily. Recall from the introduction the notation  $\tau(S, h+)$  for  $\tau(S, [h, +\infty[)$ . Let  $x, y > 0$  (the case when  $x = 0$  or  $y = 0$  is obvious), we have

$$\begin{aligned} \mathbb{P}(S_T < x + y) &= \mathbb{P}(S_T < x) + \mathbb{P}(T \geq \tau(S, x+), T < \tau(S, (x + y)+)) \\ &\leq \mathbb{P}(S_T < x) + \mathbb{P}(T \geq \tau(S, x+), T < \tau(S, [S_{\tau(S, x+)} + y, +\infty[)) \end{aligned}$$

because  $S_{\tau(S, x+)} \geq x$  almost surely,

$$= \mathbb{P}(S_T < x) + \mathbb{P}(T \geq \tau(S, x+)) \times \mathbb{P}(T < \tau(S, y+)),$$

from the characteristic property of the exponential distribution and the Markov property applied to  $S$  at time  $\tau(S, x+)$ . Since  $\mathbb{P}(T \geq \tau(S, x+)) \leq 1$  and  $\mathbb{P}(T < \tau(S, y+)) = \mathbb{P}(S_T < y)$  we obtain (3.4.17).  $\square$

We now prove the proposition

*Démonstration.* of Proposition 3.4.3.

We write :

$$\begin{aligned} I(V) &= \int_0^{\mathcal{R}(V,0)} e^{-V(t)} dt + \int_{\mathcal{R}(V,0)}^{+\infty} e^{-V(t)} dt \\ &= \int_0^{\mathcal{R}(V,0)} e^{-V(t)} dt + \int_0^{+\infty} e^{-(V(t+\mathcal{R}(V,0)))} dt \\ &\stackrel{\mathcal{L}}{=} \int_0^{\mathcal{R}(V,0)} e^{-V(t)} dt + I(V^\uparrow), \end{aligned} \tag{3.4.18}$$

where we used the second point of Lemma 3.2.6 for the last inequality in which the two terms  $\int_0^{\mathcal{R}(V,0)} e^{-V(t)} dt$  and  $I(V^\uparrow)$  are independent.

We thus have

$$\mathbb{P}(I(V) \leq x) \leq \mathbb{P}(I(V^\uparrow) \leq x). \tag{3.4.19}$$

According to Lemma 3.2.4, the term  $\int_0^{\mathcal{R}(V,0)} e^{-V(t)} dt$  has the same law as  $S_T$  where  $S$  is a pure jump subordinator with Lévy measure  $G\eta(\cdot \cap F)$ , the image measure by  $G$  of  $\eta(\cdot \cap F)$ , and  $T$  an independent exponential random variable with parameter  $\eta(I)$ .

For  $\epsilon > 0$  and  $x \geq 0$ , combining the equality in law of lemma 3.2.4 with (3.4.18) we obtain

$$\mathbb{P}(I(V) \leq (1 + \epsilon)x) \geq \mathbb{P}(I(V^\uparrow) \leq x) \times \mathbb{P}(S_T \leq \epsilon x) \geq c\epsilon x \mathbb{P}(I(V^\uparrow) \leq x),$$

for an appropriate constant  $c > 0$ , when  $x$  is small enough, according to lemma 3.4.4. Combining with (3.4.19) we get the result.  $\square$

Now, if  $V$  drifts to  $+\infty$ , Proposition 3.4.3 easily implies that the results of Theorems 3.1.2, 3.1.4, 3.1.5 and of Remarks 3.1.6, 3.1.7 are also true for  $I(V)$ , as long as they are for  $I(V^\uparrow)$ . Proposition 3.1.9 is thus proved.

## 3.5 Smoothness of the density : Proof of Theorem 3.1.10

According to Proposition 3.3.2,  $I(V^\uparrow)$  contains, as a convolution factor, the sum of infinitely many independent multiples of random variables having the same law as  $S_T$  ( $S_T$  being as in Lemma 3.2.1). We can thus use a condition on the Lévy measure of  $S$  to have the existence of the smooth density for  $I(V^\uparrow)$ . Actually, the condition that we check for  $S$  is the one of Proposition 28.3 in [59], which is a condition on the Lévy measure of a Lévy process for it to have a  $\mathcal{C}^\infty$  density with bounded derivatives.

As a jump of  $S$  is the image by the mapping  $G$  of an excursion of  $V^\sharp$ , we start by lemmas on the excursions of  $V^\sharp$ .

**Lemma 3.5.1.** *Assume  $\sigma > 1$  and choose  $\sigma'$  such that  $1 < \sigma' < \sigma$ . For all  $h$  small enough we have*

$$\eta^\sharp(\xi, \zeta(\xi) < +\infty, H_0(\xi) > h) \geq h^{-(\sigma'-1)}.$$

*Démonstration.* We consider excursions away from 0. Let  $M > 0$  be a fixed level and, for any  $h \in ]0, M[$ ,  $p_h$  denotes the probability that  $V^\sharp$  has no finite excursion of height in  $]h, M[$  before its first excursion higher than  $M$ . Since the set of finite excursions of height in  $]h, M[$  and the set of excursions higher than  $M$  are disjoint, we have, by a property of Poisson point processes

$$p_h = \frac{\eta^\sharp(\xi, H_0(\xi) > M)}{\eta^\sharp(\xi, H_0(\xi) > M) + \eta^\sharp(\xi, \zeta(\xi) < +\infty, H_0(\xi) \in ]h, M[)}, \quad (3.5.1)$$

so we only need to give an upper bound for  $p_h$ . Now, note that  $p_h$  is only the probability that  $\tau(V^\sharp, h)$  and  $\tau(V^\sharp, M)$  belong to the same excursion of  $V^\sharp$  away from 0, so

$$p_h = \mathbb{P}(\forall s \in ]\tau(V^\sharp, h), \tau(V^\sharp, M)[, V^\sharp(s) \neq 0) = \mathbb{P}(\forall s \in ]0, \tau(V_h^\sharp, M)[, V_h^\sharp(s) \neq 0),$$

where we used the Markov property at time  $\tau(V^\sharp, h)$ .

$$= \mathbb{P}(\tau(V_h^\sharp, M) < \tau(V_h^\sharp, ]-\infty, 0]) = W^\sharp(h)/W^\sharp(M+h) \underset{h \rightarrow 0}{\sim} W^\sharp(h)/W^\sharp(M),$$

where  $W^\sharp$  is the scale function of  $V^\sharp$ , and where the last equivalence comes from the continuity of the scale function (see the Introduction). Recall that  $W^\sharp$  has a Laplace transform given by the expression

$$\int_0^{+\infty} W^\sharp(x) e^{-\lambda x} dx = \frac{1}{\Psi_{V^\sharp}(\lambda)}, \quad \forall \lambda \geq 0. \quad (3.5.2)$$

Now, for any  $h > 0$ , by increases of  $W^\sharp$  we have

$$\begin{aligned} W^\sharp(h) &\leq h^{-1} \int_0^{2h} W^\sharp(x) dx \leq e^2 h^{-1} \int_0^{2h} W^\sharp(x) e^{-x/h} dx \\ &\leq e^2 h^{-1} \int_0^{+\infty} W^\sharp(x) e^{-x/h} dx \\ &= \frac{e^2}{h \Psi_{V^\sharp}(h^{-1})}, \end{aligned}$$

because of (3.5.2). From the definition of  $\sigma$  and the fact that  $\sigma' < \sigma$ , we know that  $\Psi_{V^\sharp}(h^{-1})/e^2 \geq h^{-\sigma'}$  provided  $h$  is small enough. We deduce that whenever  $h$  is small enough,

$$W^\sharp(h) \leq h^{\sigma'-1}.$$

We thus get for  $h$  small enough,

$$p_h \leq 2h^{\sigma'-1}/W^\sharp(M), \quad (3.5.3)$$

and taking the inverse in (3.5.1),

$$\eta^\sharp(\xi, \zeta(\xi) < +\infty, H_0(\xi) \in ]h, M[) \geq \eta^\sharp(\xi, H_0(\xi) > M) \times \left( \frac{1}{p_h} - 1 \right). \quad (3.5.4)$$

Since  $\eta^\sharp(\xi, \zeta(\xi) < +\infty, H_0(\xi) > h) \geq \eta^\sharp(\xi, \zeta(\xi) < +\infty, H_0(\xi) \in ]h, M[)$ , the combination of (3.5.3) and (3.5.4) yields the result. We got rid of the constants since the same result with the same constant is true for  $\sigma'$  increased a little bit.  $\square$

We now need the following lemma which states that for a spectrally negative Lévy process, the excursions of a given height can be split into two independents parts of which the law are known.

**Lemma 3.5.2.** *For any  $h > 0$ , assume that the process  $X$  follows the law  $\eta^\sharp(\cdot | H(\cdot) > h)$ . Then we have :*

- $(X(s), 0 \leq s \leq \tau(X, h)) \sim (V^\uparrow(s), 0 \leq s \leq \tau(V^\uparrow, h)),$
- $(X(s), \tau(X, h) \leq s \leq \tau(X(\cdot + \tau(X, h)), 0)) \sim (V_h^\sharp(s), 0 \leq s \leq \tau(V_h^\sharp, 0)),$
- $(X(s), 0 \leq s \leq \tau(X, h)) \perp\!\!\!\perp (X(s), \tau(X, h) \leq s \leq \tau(X(\cdot + \tau(X, h)), 0)).$

Note that the time  $\tau(V_h^\sharp, 0)$  may possibly be infinite, but this is unlikely when  $h$  is small.

*Démonstration.* of Lemma 3.5.2

The first point is a consequence of the first point of Lemma 3.2.6. The second and third points come from the Markov property.  $\square$

We can now prove the main lemma of this section, it will allow us to check the condition on the Lévy measure of  $S$ .

**Lemma 3.5.3.** *Assume  $\sigma > 1$  and choose  $\sigma'$  and  $\beta'$  such that  $1 < \sigma' < \sigma \leq \beta < \beta'$ . We also choose  $\epsilon > \frac{\beta'-1}{\sigma'-1} - 1$  and fix  $C > 0$  an arbitrary constant. Then, for  $r$  small enough we have*

$$\eta^\sharp(\xi, \zeta(\xi) \in ]x, r]) \geq x^{-(\sigma'-1)/\beta'}, \quad \forall x \in ]0, Cr^{1+\epsilon}].$$

*Démonstration.* Let us fix  $r > 0$  and  $x \in ]0, Cr^{1+\epsilon}]$ , then

$$\begin{aligned} \eta^\sharp(\xi, \zeta(\xi) \in ]x, r]) &\geq \eta^\sharp(\xi, \zeta(\xi) \in [x, r], H_0(\xi) > x^{1/\beta'}) \\ &= \eta^\sharp(\xi, \zeta(\xi) \in ]x, r] \mid H_0(\xi) > x^{1/\beta'}) \times \eta^\sharp(\xi, H_0(\xi) > x^{1/\beta'}), \end{aligned}$$

where  $\eta^\sharp(\cdot \mid H(\cdot) > x^{1/\beta'})$  is the measure of excursions conditioned to be higher than  $x^{1/\beta'}$ . The last quantity thus equals

$$\left[ \eta^\sharp(\xi, \zeta(\xi) > x \mid H_0(\xi) > x^{1/\beta'}) - \eta^\sharp(\xi, \zeta(\xi) > r \mid H_0(\xi) > x^{1/\beta'}) \right] \times \eta^\sharp(\xi, H_0(\xi) > x^{1/\beta'}). \quad (3.5.5)$$

We now study the three quantities appearing in (3.5.5) and show that this expression is of the same order as  $\eta^\sharp(\xi, H_0(\xi) > x^{1/\beta'})$  for which Lemma 3.5.1 provides a lower bound. We start by proving that  $\eta^\sharp(\xi, \zeta(\xi) > r \mid H_0(\xi) > x^{1/\beta'})$  converges to 0 uniformly in  $x \in ]0, Cr^{1+\epsilon}]$  when  $r$  goes to 0. First, Lemma 3.5.2 gives for all  $x$  in  $]0, Cr^{1+\epsilon}]$  that

$$\eta^\sharp(\xi, \zeta(\xi) > r \mid H_0(\xi) > x^{1/\beta'}) \leq \mathbb{P}\left(\tau(V_{x^{1/\beta'}}^\sharp, 0) > r/2\right) + \mathbb{P}\left(\tau(V^\uparrow, x^{1/\beta'}) > r/2\right). \quad (3.5.6)$$

The first thing is thus to prove that

$$\sup_{x \in ]0, Cr^{1+\epsilon}]} \mathbb{P}\left(\tau(V_{x^{1/\beta'}}^\sharp, 0) > r/2\right) \xrightarrow{r \rightarrow 0} 0. \quad (3.5.7)$$

Before reaching 0 (if it does) the process  $V_{x^{1/\beta'}}^\sharp$  reaches  $] -\infty, 0]$  (not necessarily at 0). We define  $\tau_{x^{1/\beta'}}^N := \tau(V_{x^{1/\beta'}}^\sharp, ] -\infty, 0])$ . For all  $x$  in  $]0, Cr^{1+\epsilon}]$  we have

$$\mathbb{P}\left(\tau(V_{x^{1/\beta'}}^\sharp, 0) > r/2\right) \leq \mathbb{P}\left(\tau_{x^{1/\beta'}}^N > r/4\right) + \mathbb{P}\left(\tau_{x^{1/\beta'}}^N \leq r/4, \tau(V_{x^{1/\beta'}}^\sharp(\cdot + \tau_{x^{1/\beta'}}^N), 0) > r/4\right). \quad (3.5.8)$$

Since  $(x \mapsto \tau_{x^{1/\beta'}}^N)$  is stochastically increasing in the variable  $x$  we have

$$\sup_{x \in ]0, Cr^{1+\epsilon}]} \mathbb{P}\left(\tau_{x^{1/\beta'}}^N > r/4\right) = \mathbb{P}\left(\tau_{C^{1/\beta'} r^{(1+\epsilon)/\beta'}}^N > r/4\right) = \mathbb{P}\left(\underline{V}^\sharp(r/4) > -C^{1/\beta'} r^{(1+\epsilon)/\beta'}\right).$$

We introduce  $T$ , an exponential random variable with parameter 1 independent of the process  $V$ , and a decreasing function  $q : ]0, +\infty[ \rightarrow ]0, +\infty[$  that converges to  $+\infty$  as  $r$  go to 0 and will be specified latter. We have

$$\begin{aligned} \mathbb{P}\left(\underline{V}^\sharp(r/4) > -C^{1/\beta'} r^{(1+\epsilon)/\beta'}\right) &\leq \mathbb{P}\left(\underline{V}^\sharp(T/q(r)) > -C^{1/\beta'} r^{(1+\epsilon)/\beta'}\right) + \mathbb{P}(T/q(r) > r/4) \\ &= \mathbb{P}\left(\exp\left(\frac{\underline{V}^\sharp(T/q(r))}{r^{(1+\epsilon)/\beta'}}\right) > \exp(-C^{1/\beta'})\right) + e^{-rq(r)/4} \\ &\leq e^{C^{1/\beta'}} \mathbb{E}\left[\exp\left(\frac{\underline{V}^\sharp(T/q(r))}{r^{(1+\epsilon)/\beta'}}\right)\right] + e^{-rq(r)/4}, \end{aligned}$$

from Markov inequality,

$$= \frac{1/\Phi_{V^\sharp}(q(r))r^{(1+\epsilon)/\beta'} - 1}{\Psi_{V^\sharp}(r^{-(1+\epsilon)/\beta'})/q(r) - 1} \times e^{C^{1/\beta'}} + e^{-rq(r)/4},$$

from the expression of the Laplace transform of the random variable  $\underline{V}^\sharp(T/q(r))$  that can be found page 192 of [8], and where  $\Phi_{V^\sharp} = \Psi_{V^\sharp}^{-1}$  as in Section 3.4.1. The last expression goes to 0 if these three conditions are satisfied :

$$\begin{aligned} q(r)r &\xrightarrow[r \rightarrow 0]{} +\infty, \\ \Phi_{V^\sharp}(q(r))r^{(1+\epsilon)/\beta'} &\xrightarrow[r \rightarrow 0]{} 0, \\ q(r)/\Psi_{V^\sharp}(r^{-(1+\epsilon)/\beta'})\Phi_{V^\sharp}(q(r))r^{(1+\epsilon)/\beta'} &\xrightarrow[r \rightarrow 0]{} 0. \end{aligned}$$

From the definition of  $\sigma$  and  $\beta$  and the fact that  $\sigma' < \sigma \leq \beta < \beta'$ , we have  $\Phi_{V^\sharp}(u) \leq u^{1/\sigma'}$ ,  $\Psi_{V^\sharp}(u) \geq u^{\sigma'}$  and  $\Phi_{V^\sharp}(u) \geq u^{1/\beta'}$ , provided  $u$  is large enough. The three conditions can thus be simplified and we only need to have :

$$\begin{aligned} q(r)r &\xrightarrow[r \rightarrow 0]{} +\infty, \\ (q(r))^{1/\sigma'} r^{(1+\epsilon)/\beta'} &\xrightarrow[r \rightarrow 0]{} 0, \\ (q(r))^{1-1/\beta'} r^{(\sigma'-1)(1+\epsilon)/\beta'} &\xrightarrow[r \rightarrow 0]{} 0. \end{aligned}$$

Elevating  $(q(r))$  to the right power so it makes its exponent disappear, these three conditions become

$$\begin{aligned} q(r)r &\xrightarrow[r \rightarrow 0]{} +\infty, \\ q(r)r^{\sigma'(1+\epsilon)/\beta'} &\xrightarrow[r \rightarrow 0]{} 0, \\ q(r)r^{(\sigma'-1)(1+\epsilon)/(\beta'-1)} &\xrightarrow[r \rightarrow 0]{} 0. \end{aligned}$$

Since  $\sigma' < \beta'$  we can check that  $\sigma'/\beta' > (\sigma' - 1)/(\beta' - 1)$ , so the third condition implies the second. We then only need to verify the first and the third condition,

but from the choice of  $\epsilon$ , we have that  $(\sigma' - 1)(1 + \epsilon)/(\beta' - 1) > 1$ , so  $q(r)$  can be chosen such that the first and the third conditions are satisfied. This yields

$$\sup_{x \in ]0, Cr^{1+\epsilon}] } \mathbb{P} \left( \tau_{x^{1/\beta'}}^N > r/4 \right) \xrightarrow[r \rightarrow 0]{} 0. \quad (3.5.9)$$

We now turn to the second term of (3.5.8). It is known that the jumps of  $V^\sharp$  is a Poisson process with intensity measure  $\nu^\sharp$ , the Lévy measure of  $V^\sharp$ . The probability that  $V^\sharp$  has a jump smaller than  $-r$  before time  $r/4$  is thus  $1 - e^{r\nu^\sharp(]-\infty, -r]) / 4} \sim r\nu^\sharp(]-\infty, -r]) / 4 := \gamma(r)$  and this goes to 0 for any Lévy measure. As a consequence, on  $\{\tau_{x^{1/\beta'}}^N \leq r/4\}$  we have  $V_{x^{1/\beta'}}^\sharp(\tau_{x^{1/\beta'}}^N) \in [-r, 0]$ , excepted on some event having probability less than  $\gamma(r)$ . We thus get

$$\sup_{x \in ]0, Cr^{1+\epsilon}] } \mathbb{P} \left( \tau_{x^{1/\beta'}}^N \leq r/4, \tau(V_{x^{1/\beta'}}^\sharp(\cdot + \tau_{x^{1/\beta'}}^N), 0) > r/4 \right) \leq \mathbb{P}(\tau(V^\sharp, r) > r/4) + \gamma(r),$$

and we now only need to show that  $\tau(V^\sharp, r)/r$  converges to 0 in probability. Now recall that we know from Theorem VII.1 in [8] that the Laplace transform of this random variable is given by

$$\mathbb{E} \left[ \exp \left( -\lambda \frac{\tau(V^\sharp, r)}{r} \right) \right] = e^{-r\Phi_{V^\sharp}(\lambda/r)}.$$

Since  $\Phi_{V^\sharp}(u) \leq u^{1/\sigma'}$  for  $u$  large enough, the last quantity converges to 1 when  $r$  goes to 0 so we indeed have the convergence to 0 in probability of  $\tau(V^\sharp, r)/r$  and as a consequence

$$\sup_{x \in ]0, Cr^{1+\epsilon}] } \mathbb{P} \left( \tau_{x^{1/\beta'}}^N \leq r/4, \tau(V_{x^{1/\beta'}}^\sharp(\cdot + \tau_{x^{1/\beta'}}^N), 0) > r/4 \right) \xrightarrow[r \rightarrow 0]{} 0.$$

Putting this, together with (3.5.9), in (3.5.8) we get (3.5.7). We now deal with the second term of (3.5.6), more precisely we prove that

$$\sup_{x \in ]0, Cr^{1+\epsilon}] } \mathbb{P} \left( \tau(V^\uparrow, x^{1/\beta'}) > r/2 \right) \xrightarrow[r \rightarrow 0]{} 0. \quad (3.5.10)$$

Because of the increases of the quantity  $\mathbb{P}(\tau(V^\uparrow, x^{1/\beta'}) > r/2)$  we can write

$$\begin{aligned} \sup_{x \in ]0, Cr^{1+\epsilon}] } \mathbb{P} \left( \tau(V^\uparrow, x^{1/\beta'}) > r/2 \right) &= \mathbb{P} \left( \tau(V^\uparrow, C^{1/\beta'} r^{(1+\epsilon)/\beta'}) > r/2 \right) \\ &\leq \mathbb{P} \left( \tau(V^\sharp, C^{1/\beta'} r^{(1+\epsilon)/\beta'}) > r/2 \right), \end{aligned}$$

according to the first point of Lemma 3.2.6 applied with  $y = C^{1/\beta'} r^{(1+\epsilon)/\beta'}$ .

Therefore, (3.5.10) will follow if we prove that the random variable  $\tau(V^\sharp, C^{1/\beta'} r^{(1+\epsilon)/\beta'})/r$  converge to 0 in probability as  $r$  goes to 0. Again, we see from Theorem VII.1 in [8] that the Laplace transform of this random variable is given by

$$\mathbb{E} \left[ \exp \left( -\lambda \frac{\tau(V^\sharp, C^{1/\beta'} r^{(1+\epsilon)/\beta'})}{r} \right) \right] = e^{-C^{1/\beta'} r^{(1+\epsilon)/\beta'} \Phi_{V^\sharp}(\lambda r^{-1})},$$

but, since  $\Phi_{V^\sharp}(u) \leq u^{1/\sigma'}$  provided  $\lambda$  is large enough, we have that, for small  $r$ ,

$$0 \leq r^{(1+\epsilon)/\beta'} \Phi_{V^\sharp}(\lambda r^{-1}) \leq r^{(1+\epsilon)/\beta' - 1/\sigma'} \lambda^{1/\sigma'}.$$

By the choice of  $\epsilon$ , we have  $1 + \epsilon > (\beta' - 1)(\sigma' - 1) > \beta'/\sigma'$ , so the last quantity converges to 0 as  $r$  goes to 0. This shows that the Laplace transform of  $\tau(V^\sharp, C^{1/\beta'} r^{(1+\epsilon)/\beta'})/r$  converges to 1 as  $r$  goes to 0 so we get the asserted convergence to 0 in probability as  $r$  goes to 0 and (3.5.10) follows.

Putting (3.5.7) and (3.5.10) in (3.5.6) we get

$$\sup_{x \in ]0, Cr^{1+\epsilon}] } \eta^\sharp \left( \xi, \zeta(\xi) > r \mid H_0(\xi) > x^{1/\beta'} \right) \xrightarrow[r \rightarrow 0]{} 0, \quad (3.5.11)$$

which means that among the excursions of heigh greater than  $x^{1/\beta'}$ , those of length greater than  $r$  are in negligible proportion, and it thus remains to show that those of length greater than  $x$  are in non-negligible proportion. We want to show that

$$\liminf_{r \rightarrow 0} \inf_{x \in ]0, Cr^{1+\epsilon}] } \eta^\sharp \left( \xi, \zeta(\xi) > x \mid H_0(\xi) > x^{1/\beta'} \right) > 0. \quad (3.5.12)$$

From the second point of Lemma 3.5.2 we have for all  $x$  in  $]0, Cr^{1+\epsilon}]$ ,

$$\eta^\sharp \left( \xi, \zeta(\xi) > x \mid H_0(\xi) > x^{1/\beta'} \right) \geq \mathbb{P} \left( \tau(V_{x^{1/\beta'}}^\sharp, 0) > x \right) \geq \mathbb{P} \left( \tau_{x^{1/\beta'}}^N > x \right). \quad (3.5.13)$$

As we did before, we introduce  $T$ , an exponential random variable with parameter 1 independent of the process  $V^\sharp$ , and a decreasing function  $q : ]0, +\infty[ \rightarrow ]0, +\infty[$  that converges to  $+\infty$  at 0 and will be specified latter.

For any  $x$  in  $]0, Cr^{1+\epsilon}]$  we have

$$\begin{aligned} \mathbb{P} \left( \tau_{x^{1/\beta'}}^N > x \right) &= \mathbb{P} \left( \underline{V}^\sharp(x) > -x^{1/\beta'} \right) \\ &\geq \mathbb{P} \left( \underline{V}^\sharp(T/q(x)) > -x^{1/\beta'} \right) - \mathbb{P} \left( T/q(x) < x \right) \\ &= \mathbb{P} \left( \frac{\underline{V}^\sharp(T/q(x))}{x^{1/\beta'}} > -1 \right) - 1 + e^{-xq(x)} \end{aligned}$$

We choose  $q(x) = 1/x$  and, because of (3.5.13), (3.5.12) will follow if we prove that the random variable  $\underline{V}^\sharp(T/q(x))/x^{1/\beta'}$  converges in probability to 0 as  $x$  goes to 0. According to [8] p 192, the Laplace transform of this random variable is given by

$$\mathbb{E} \left[ \exp \left( \lambda \frac{\underline{V}^\sharp(T/q(x))}{x^{1/\beta'}} \right) \right] = \frac{q(x) \left( \Phi_{V^\sharp}(q(x)) - \lambda/x^{1/\beta'} \right)}{\Phi_{V^\sharp}(q(x)) \left( q(x) - \Psi_{V^\sharp}(\lambda/x^{1/\beta'}) \right)}.$$

This last quantity converges to 1 when  $x$  goes to 0 if



- $\lambda/x^{1/\beta'}$  is negligible compared to  $\Phi_{V^\sharp}(q(x))$  when  $x$  goes to 0,
- $\Psi_{V^\sharp}(\lambda/x^{1/\beta'})$  is negligible compared to  $q(x)$  when  $x$  goes to 0.

These two conditions can be written

$$\begin{aligned} x^{1/\beta'} \Phi_{V^\sharp}(q(x)) &\xrightarrow[r \rightarrow 0]{} +\infty, \\ \Psi_{V^\sharp}(\lambda/x^{1/\beta'})/q(x) &\xrightarrow[r \rightarrow 0]{} 0. \end{aligned}$$

Now, because of the definition of  $\beta$ , because  $\beta' > \beta$ , and because  $q(x) = 1/x$ , it is easy to see that these two conditions are satisfied. This shows that the Laplace transform of the random variable  $V^\sharp(T/q(x))/x^{1/\beta'}$  converges to 1 as  $x$  goes to 0, so this random variable converges to 0 in probability and (3.5.12) follows.

For the factor  $\eta^\sharp(\xi, H_0(\xi) > x^{1/\beta'})$  of (3.5.5) we note that it is trivially more than  $\eta^\sharp(\xi, \zeta(\xi) < +\infty, H_0(\xi) > x^{1/\beta'})$  and we can use Lemma 3.5.1 which yields

$$\eta^\sharp\left(\xi, H_0(\xi) > x^{1/\beta'}\right) \geq x^{-(\sigma'-1)/\beta'}, \quad (3.5.14)$$

for  $x$  small enough. Putting (3.5.11), (3.5.12) and (3.5.14) in (3.5.5), we get the asserted result. Here again, we actually obtain the result up to a multiplicative constant, but since the result is still true if, for example, we increase  $\sigma'$  a little bit, we can get rid of the constant. □

We can now prove Theorem 3.1.10.

*Démonstration.* of Theorem 3.1.10

We make the assumption that (3.1.8) is satisfied so, by continuity of the left hand side of (3.1.8) in  $\sigma$  and  $\beta$ , we can choose  $\sigma'$  and  $\beta'$  to be as in Lemma 3.5.3, but close enough to respectively  $\sigma$  and  $\beta$  so that they also satisfy (3.1.8). We also choose  $\epsilon$  as in Lemma 3.5.3.

We fix  $y > 0$ . Let  $S$  is a pure jump subordinator with Lévy measure  $\mu(\cdot) := G\eta_y^\sharp(\cdot \cap FP_\infty)$ , the image measure of  $\eta_y^\sharp(\cdot \cap FP_\infty)$  by the mapping  $G$ . From the Lévy-Khintchine formula, the characteristic function of  $S(t)$  is

$$\mathbb{E} [e^{i\xi S(t)}] = e^{t\Phi_S(\xi)},$$

where

$$\forall \xi \in \mathbb{R}, \quad \Phi_S(\xi) := \int_0^{+\infty} (e^{i\xi x} - 1)\mu(dx),$$

so, taking the real part,

$$\begin{aligned} \forall \xi \in \mathbb{R}, \quad |\Re(\Phi_S(\xi))| &= \int_0^{+\infty} (1 - \cos(\xi x)) \mu(dx) \\ &\geq \int_0^{\pi/|\xi|} (1 - \cos(\xi x)) \mu(dx) \\ &\geq \frac{2\xi^2}{\pi^2} \int_0^{\pi/|\xi|} x^2 \mu(dx). \end{aligned} \quad (3.5.15)$$

We now prove that the measure  $\mu$  satisfies the hypothesis of Proposition 28.3 of [59]. We have for any  $r \in ]0, 1[$  :

$$\int_0^r x^2 \mu(dx) = 2 \int_0^r x \mu([x, r[) dx \geq 2 \int_0^{r^{1+\epsilon}} x \mu([x, r[) dx. \quad (3.5.16)$$

We thus need to minorate  $\mu([x, r[)$  for  $x \in ]0, r^{1+\epsilon}[$  :

$$\begin{aligned} \mu([x, r[) &= \eta_y^\sharp(\{\xi, G(\xi) \in [x, r[ \cap FP_\infty\}) \\ &= \eta_y^\sharp\left(\left\{\xi, \int_0^{\zeta(\xi)} e^{-\xi(t)} dt \in [x, r[ \right\} \cap FP_\infty\right), \end{aligned}$$

from the definition of  $G$ ,

$$\begin{aligned} &\geq \eta_y^\sharp(\{\xi, \zeta(\xi) \in [e^{2y}x, r[, \sup \xi \leq 2y\} \cap FP_\infty) \\ &\geq \eta_y^\sharp(\{\xi, \zeta(\xi) \in [e^{2y}x, r[ \}) - \eta_y^\sharp(\{\xi, \sup \xi > 2y\}) - \eta_y^\sharp(FP_\infty^c) \\ &= \eta^\sharp(\{\xi, \zeta(\xi) \in [e^{2y}x, r[ \}) - \eta_y^\sharp(\{\xi, \sup \xi > 2y\}) - \eta_y^\sharp(IP_\infty) - \eta_y^\sharp(N) \\ &= \eta^\sharp(\{\xi, \zeta(\xi) \in [e^{2y}x, r[ \}) - c, \end{aligned}$$

where we have put  $c := \eta_y^\sharp(\{\sup \xi > 2y\}) + \eta_y^\sharp(IP_\infty) + \eta_y^\sharp(N)$ . Note that  $c$  is well defined because the quantities  $\eta_y^\sharp(\{\sup \xi > 2y\})$ ,  $\eta_y^\sharp(IP_\infty)$  and  $\eta_y^\sharp(N)$  are finite.

We now apply Lemma 3.5.3 (taking  $C = e^{2y}$ ) and we deduce that for  $r$  small enough,

$$\forall x \in ]0, r^{1+\epsilon}[, \mu([x, r[) \geq e^{-2y(\sigma'-1)/\beta'} x^{-(\sigma'-1)/\beta'} - c.$$

Combining this with (3.5.16), we get that whenever  $r$  is small enough :

$$\begin{aligned} \int_0^r x^2 \mu(dx) &\geq 2e^{-2y(\sigma'-1)/\beta'} \int_0^{r^{1+\epsilon}} x^{1-(\sigma'-1)/\beta'} dx - 2c \int_0^{r^{1+\epsilon}} x dx \\ &\geq \frac{2e^{-2y(\sigma'-1)/\beta'}}{2 - (\sigma' - 1)/\beta'} r^{(1+\epsilon)(2-(\sigma'-1)/\beta')} - cr^{2(1+\epsilon)}. \end{aligned}$$

If

$$(1 + \epsilon)(2 - (\sigma' - 1)/(3\beta' - 1)) < 2, \quad (3.5.17)$$

then, choosing  $\delta \in ]0, 2 - (1 + \epsilon)(2 - (\sigma' - 1)/\beta')[$  and combining the above estimate with (3.5.15), we get that

$$|\Phi_S(\xi)| \geq |\mathcal{R}(\Phi_S(\xi))| \geq |\xi|^\delta, \quad (3.5.18)$$

whenever  $|\xi|$  is large enough. As the only assumption on  $\epsilon$  is that it is greater than  $(\beta' - 1)/(\sigma' - 1) - 1$ , we can choose  $\epsilon$  such that (3.5.17) is satisfied if and only if

$$\frac{\beta' - 1}{\sigma' - 1} (2 - (\sigma' - 1)/(3\beta' - 1)) < 2,$$

which is equivalent to the fact that  $\sigma'$  and  $\beta'$  satisfy (3.1.8). Therefore, we have proved that there exists  $\delta > 0$  such that (3.5.18) is true.

Let  $T$  be an exponential random variable with parameter  $p := \eta_y^\#(IP_\infty) + \eta_y^\#(N)$  which is independent of  $S$ , the Fourier transform of  $S_T$  is

$$\mathbb{E} [e^{i\xi S_T}] = \int_0^{+\infty} p e^{t\Phi_S(\xi)} e^{-pt} dt = \frac{p}{p - \Phi_S(\xi)} = \mathcal{O}_{|\xi| \rightarrow +\infty} (|\xi|^{-\delta}), \quad (3.5.19)$$

because of (3.5.18).

Proposition 3.3.2 gives the decomposition

$$I(V^\uparrow) = \sum_{k \geq 0} e^{-ky} B_k^y + \sum_{k \geq 0} e^{-ky} C_k^y$$

where all the random variables in the two series are mutually independent, and where each term  $B_k^y$  has the same law as  $S_T$ . Therefore, the characteristic function of  $I(V^\uparrow)$  is the product of a characteristic function bounded by 1 and of  $\prod_{k \geq 0} \mathbb{E}[e^{ie^{-ky}\xi S_T}]$  which, thanks to (3.5.19), goes to 0 faster than any negative power of  $|\xi|$ . This proves that the density of  $I(V^\uparrow)$  is of class  $\mathcal{C}^\infty$  and that all its derivatives converge to 0 when  $x$  goes to  $+\infty$ . The derivatives of the density of  $I(V^\uparrow)$  also converge to 0 when  $x$  goes to 0 since this density is of class  $\mathcal{C}^\infty$  and null on  $] -\infty, 0[$ .

To prove that  $\phi_{I(V^\uparrow)}$  actually belongs to the Schwartz space, we have to study a little more deeply the infinite product. Let us denote by  $\psi$  the characteristic function of  $\sum_{k \geq 0} e^{-ky} C_k^y$ . Then

$$\phi_{I(V^\uparrow)}(\xi) = \psi(\xi) \prod_{k \geq 0} \mathbb{E} [e^{ie^{-ky}\xi S_T}]. \quad (3.5.20)$$

The random variable  $S_T$  admits moments of any positive order because it is a convolution factor of  $I(V^\uparrow)$  which admits moments of any positive order thanks to Theorem 3.1.1. As a consequence the derivatives at any order of functions of the kind of  $(\xi \mapsto \mathbb{E}[e^{ie^{-ky}\xi S_T}])$ , for integers  $k$ , are defined and bounded. For any  $n \in \mathbb{N}$  and  $m > n$ , we can see by induction that the  $n^{\text{th}}$  derivative

$$P_{m,n} := \left( \xi \mapsto \prod_{k=0}^m \mathbb{E} [e^{ie^{-ky}\xi S_T}] \right)^{(n)}$$

is a finite sum of products. In each of these products, there are at least  $m - n$  factors of the form  $\mathbb{E}[e^{ie^{-ky}\xi S_T}]$  for some integers  $k$  and the other factors are derivatives at some orders of the functions  $(\xi \mapsto \mathbb{E}[e^{ie^{-ky}\xi S_T}])$  for some integers  $k$ . Therefore, from (3.5.19), we deduce that

$$P_{m,n}(\xi) = \mathcal{O}_{|\xi| \rightarrow +\infty} (|\xi|^{-(m-n)\delta}). \quad (3.5.21)$$

We decompose (3.5.20) into

$$\phi_{I(V^\uparrow)}(\xi) = \left( \prod_{k=0}^m \mathbb{E} [e^{ie^{-ky}\xi S_T}] \right) \times R_m(\xi),$$

where

$$R_m(\xi) := \psi(\xi) \prod_{k \geq m+1} \mathbb{E} \left[ e^{ie^{-ky}\xi S_T} \right].$$

From the Leibniz formula applied to the product, we have

$$\phi_{I(V^\uparrow)}^{(n)}(\xi) = \sum_{k=0}^n C_n^k P_{m,k}(\xi) R_m^{(n-k)}(\xi).$$

$R_m$  is the Fourier transform of a random variable that admits moments of any positive order (because it is a convolution factors of  $I(V^\uparrow)$ ), so its derivatives at any order are defined and bounded. From (3.5.21) we thus get that

$$\phi_{I(V^\uparrow)}^{(n)}(\xi) = \mathcal{O}_{|\xi| \rightarrow +\infty} (|\xi|^{-(m-n)\delta}).$$

As  $m$  is arbitrary,  $\phi_{I(V^\uparrow)}^{(n)}$  goes to 0 faster than any negative power of  $|\xi|$ . Therefore,  $\phi_{I(V^\uparrow)}$  belongs to the Schwartz space and so does the density of  $I(V^\uparrow)$ , since the Schwartz space is stable by Fourier transform. □

**Remark 3.5.4.** *The case where  $V$  has bounded variation is not contained in Theorem 3.1.10. Moreover, Remark 3.2.2 shows that the law of  $S_T$  ( $S_T$  being as in Lemma 3.2.1) has an atom at 0 if  $V$  has bounded variation, so there is no hope to generalize our proof of Theorem 3.1.10 to this case.*

We also prove Corollary 3.1.11.

*Démonstration.* of Corollary 3.1.11

Since  $V$  drifts to  $+\infty$  we have  $V^\# = V$  so the expression (3.4.18) in the proof of Proposition 3.4.3 tells us that  $I(V^\uparrow)$  is a convolution factor of  $I(V)$ . Now, under the assumptions of the corollary, Theorem 3.1.10 applies and we get the regularity of the density of  $I(V)$  thanks to the boundedness of the derivatives of the density of  $I(V^\uparrow)$  and the differentiation under the integral sign theorem. We get the convergence to 0 at  $+\infty$  of the derivatives of the density of  $I(V)$  thanks to the boundedness of the derivatives of the density of  $I(V^\uparrow)$  and the dominated convergence theorem. The convergence to 0 at 0 of the derivatives of the density of  $I(V)$  comes from the fact that this density is of class  $C^\infty$  and null on  $] -\infty, 0[$ . □

## 3.6 The spectrally positive case

We now make a brief study of the exponential functional of  $Z^\uparrow$  where  $Z$  is a spectrally positive Lévy process drifting to  $+\infty$ . If  $Z$  is a subordinator, then it stays positive and  $I(Z^\uparrow)$  is only  $I(Z)$  which is already known to be finite and have some finite exponential moments (see for example Theorem 2 in [12]), so Theorem 3.1.13 is already known in this case.

We thus assume that  $Z$  is not a subordinator. Since, in this case,  $-Z$  is spectrally negative and not the opposite of a subordinator (then, we denote by  $\kappa$  the non-trivial zero of  $\Psi_{-Z}$ ), it is regular for  $]0, +\infty[$  according to Theorem VII.1 in [8], so  $Z$  is regular for  $] -\infty, 0[$ . Moreover,  $Z$  drifts to  $+\infty$ . We can thus define the Markov family  $(Z_x^\uparrow, x \geq 0)$  as in [31], Chapter 8. It can be seen from there that the processes such defined are Markov, have infinite life-time (this is where we need the hypothesis that  $Z$  drifts to  $+\infty$ ) and that  $Z_0^\uparrow$ , that we denote by  $Z^\uparrow$ , is indeed well defined.

Here again, for any  $x \geq 0$ , the process  $Z_x^\uparrow$  must be seen as  $Z$  conditioned to stay positive and starting from  $x$ . Note that, since  $Z$  converges almost surely to infinity, for  $x > 0$ ,  $Z_x^\uparrow$  is only  $Z_x$  conditioned in the usual sense to remain positive.

### 3.6.1 Finiteness, exponential moments : Proof of Theorem 3.1.13

The idea is that adding a small term of negative drift to  $Z^\uparrow$  does not change its convergence to  $+\infty$ . It makes  $Z^\uparrow$  ultimately greater than a deterministic linear function for which the exponential functional is defined and deterministically bounded. The key point is thus to control the time taken by  $Z^\uparrow$  to become greater than the linear function once and for good. We start with the following lemma.

**Lemma 3.6.1.** *For any  $y > 0$ , there exists  $\epsilon > 0$  and positive constants  $c_1$  and  $c_2$  such that*

$$\forall s > 0, \mathbb{P}(\mathcal{R}(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot), ] - \infty, 0]) > s) \leq c_1 e^{-c_2 s}.$$

*Démonstration.* We fix  $y > 0$ . From Corollary VII.2 in [8], a spectrally negative Lévy process  $X$  drifts to  $-\infty$  if and only if  $\mathbb{E}[X(1)] < 0$ .  $Z$  is a spectrally positive Lévy process drifting to  $+\infty$  so taking the dual in the theorem we get  $\mathbb{E}[Z(1)] > 0$ . Now  $\mathbb{E}[(Z - \epsilon \cdot)(1)] = \mathbb{E}[Z(1)] - \epsilon$  which is positive for  $\epsilon$  chosen small enough. Still taking the dual in Corollary VII.2 in [8], this implies that  $Z - \epsilon \cdot$  is also a spectrally positive Lévy process that drifts to  $+\infty$  and which is not a subordinator. We have

$$\begin{aligned} \mathbb{P}(\mathcal{R}(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot), ] - \infty, 0]) > s) &= \mathbb{P}\left(\inf_{t \in [s, +\infty[} Z_y^\uparrow(t) - (y + \epsilon t) \leq 0\right) \\ &= \mathcal{C} \mathbb{P}\left(\inf_{t \in [s, +\infty[} Z_y(t) - (y + \epsilon t) \leq 0, \inf_{[0, +\infty[} Z_y > 0\right), \end{aligned}$$

where  $\mathcal{C} := 1/\mathbb{P}(\inf_{[0, +\infty[} Z_y > 0) = 1/(1 - e^{-\kappa y})$ . This comes from the fact that  $Z_y^\uparrow$  is only  $Z_y$  conditioned to stay positive in the usual sense. Now, noting that  $Z_y = y + Z$ ,

we bound the above quantity by

$$\begin{aligned}
 & \mathcal{C} \mathbb{P} \left( \inf_{t \in [s, +\infty[} Z(t) - \epsilon t \leq 0 \right) = \mathcal{C} \mathbb{P} \left( \sup_{t \in [s, +\infty[} -Z(t) + \epsilon t \geq 0 \right) \\
 & \leq \mathcal{C} \mathbb{P} \left( \sup_{t \in [s, +\infty[} -Z(t) + \epsilon t \geq 0, -Z(s) + \epsilon s \leq 0 \right) + \mathcal{C} \mathbb{P}(-Z(s) + \epsilon s \geq 0) \\
 & = \mathcal{C} \mathbb{P} \left( \sup_{t \in [0, +\infty[} -Z^s(t) + \epsilon t \geq -(-Z(s) + \epsilon s), -Z(s) + \epsilon s \leq 0 \right) + \mathcal{C} \mathbb{P}(-Z(s) + \epsilon s > 0).
 \end{aligned} \tag{3.6.1}$$

From the independence of the increments, the process  $-Z^s + \epsilon \cdot$  is equal in law to  $-Z + \epsilon \cdot$  and independent from  $-Z(s) + \epsilon s$ . From Corollary VII.2 in [8], the supremum over  $[0, +\infty[$  of the process  $-Z^s + \epsilon \cdot$  follows an exponential distribution with parameter  $\alpha$ , where  $\alpha$  is the non-trivial zero of  $\Psi_{-Z+\epsilon}$ . From this, combined with the independence from  $-Z(s) + \epsilon s$ , (3.6.1) becomes

$$\begin{aligned}
 & \mathcal{C} \mathbb{E} \left( e^{\alpha(-Z(s)+\epsilon s)} \mathbf{1}_{\{-Z(s)+\epsilon s \leq 0\}} \right) + \mathcal{C} \mathbb{P}(-Z(s) + \epsilon s > 0) \\
 & \leq \mathcal{C} \mathbb{E} \left( e^{\alpha(-Z(s)+\epsilon s)/2} \mathbf{1}_{\{-Z(s)+\epsilon s \leq 0\}} \right) + \mathcal{C} \mathbb{P} \left( e^{\alpha(-Z(s)+\epsilon s)/2} > 1 \right),
 \end{aligned}$$

from the decreases of negative exponential and composing by function  $x \mapsto \exp(\frac{\alpha}{2}x)$  in the probability of the second term,

$$\leq \mathcal{C} \mathbb{E} \left( e^{\alpha/2(-Z(s)+\epsilon s)} \right) + \mathcal{C} \mathbb{P} \left( e^{\alpha(-Z(s)+\epsilon s)/2} > 1 \right) \leq 2\mathcal{C} \mathbb{E} \left( e^{\alpha/2(-Z(s)+\epsilon s)} \right),$$

where we used Markov inequality in the second term,

$$= 2\mathcal{C} e^{s\Psi_{-Z+\epsilon}(\alpha/2)}.$$

As  $\Psi_{-Z+\epsilon}$  is negative on  $]0, \alpha[$ , we get the result with  $c_1 = 2\mathcal{C}$  and  $c_2 = -\Psi_{-Z+\epsilon}(\alpha/2)$ . □

We can now prove Theorem 3.1.13.

*Démonstration.* of Theorem 3.1.13

We fix  $y > 0$ . Let  $m_y$  be the point where the process  $Z_y^\uparrow$  reaches its infimum,  $m_y := \sup\{s \geq 0, Z_y^\uparrow(s-) \wedge Z_y^\uparrow(s) = \inf_{[0, +\infty[} Z_y^\uparrow\}$ . Note that from the absence of negative jumps the infimum is always reached at least at  $m_y -$  so  $Z_y^\uparrow(m_y -) = \inf_{[0, +\infty[} Z_y^\uparrow$ . In order to get  $Z^\uparrow$  from  $Z_y^\uparrow$ , we use the decomposition given by Theorem 24 in [31], that is :

- The two processes  $(Z_y^\uparrow(m_y + s) - Z_y^\uparrow(m_y -), s \geq 0)$  and  $(Z_y^\uparrow(s), 0 \leq s < m_y)$  are independent,

—  $(Z_y^\uparrow(m_y + s) - Z_y^\uparrow(m_y -), s \geq 0)$  is equal in law to  $Z^\uparrow$ .

Now,

$$\begin{aligned} I(Z_y^\uparrow) &= \int_0^{m_y} e^{-Z_y^\uparrow(u)} du + \int_{m_y}^{+\infty} e^{-Z_y^\uparrow(u)} du \\ &= \int_0^{m_y} e^{-Z_y^\uparrow(u)} du + e^{-Z_y^\uparrow(m_y -)} \int_0^{+\infty} e^{-(Z_y^\uparrow(m_y + u) - Z_y^\uparrow(m_y -))} du \\ &\geq e^{-y} \int_0^{+\infty} e^{-(Z_y^\uparrow(m_y + u) - Z_y^\uparrow(m_y -))} du, \end{aligned}$$

because almost surely  $Z_y^\uparrow(m_y -) \leq y$ ,

$$\stackrel{\mathcal{L}}{=} e^{-y} I(Z^\uparrow),$$

because of the above decomposition. We thus get

$$I(Z^\uparrow) \stackrel{sto}{\leq} e^y I(Z_y^\uparrow), \quad (3.6.2)$$

where  $\stackrel{sto}{\leq}$  denotes a stochastic inequality. As a consequence we only need to prove the result for  $I(Z_y^\uparrow)$ . We now choose  $\epsilon > 0$  as in Lemma 3.6.1. We have

$$\begin{aligned} 0 \leq I(Z_y^\uparrow) &= \int_0^{\mathcal{R}(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot), ] - \infty, 0])} e^{-Z_y^\uparrow(t)} dt + \int_{\mathcal{R}(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot), ] - \infty, 0])}^{+\infty} e^{-Z_y^\uparrow(t)} dt \\ &\leq \mathcal{R}(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot), ] - \infty, 0]) + \int_{\mathcal{R}(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot), ] - \infty, 0])}^{+\infty} e^{-Z_y^\uparrow(t)} dt, \end{aligned}$$

but for  $t \geq \mathcal{R}(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot), ] - \infty, 0])$  we have  $Z_y^\uparrow(t) \geq y + \epsilon t$ , so

$$\begin{aligned} 0 \leq I(Z_y^\uparrow) &\leq \mathcal{R}(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot), ] - \infty, 0]) + \int_{\mathcal{R}(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot), ] - \infty, 0])}^{+\infty} e^{-y - \epsilon t} dt \\ &\leq \mathcal{R}(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot), ] - \infty, 0]) + \int_0^{+\infty} e^{-y - \epsilon t} dt \\ &= \mathcal{R}(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot), ] - \infty, 0]) + e^{-y}/\epsilon. \end{aligned}$$

From Lemma 3.6.1, this is almost surely finite and admits some finite exponential moments. Thanks to (3.6.2), we have the same for  $I(Z^\uparrow)$ , which is the expected result.  $\square$

### 3.6.2 Tails at 0 of $I(Z^\uparrow)$ : Proof of Theorem 3.1.14

We need an analogous of Lemma 3.2.6 in order to compare, as we did in subsection 3.4.3, the exponential functionals  $I(Z^\uparrow)$  and  $I(Z)$ . We define  $m$ , the point where the process  $Z$  reaches its infimum :  $m := \sup\{s \geq 0, Z(s-) \wedge Z(s) = \inf_{[0, +\infty[} Z\}$ . Here again, from the absence of negative jumps, the infimum is always reached at least at  $m-$  so  $Z(m-) = \inf_{[0, +\infty[} Z$ .

**Lemma 3.6.2.** *If  $Z$  has unbounded variation, then  $Z(m+) - Z(m-)$  has the same law as  $Z^\uparrow$ .*

*Démonstration.*  $Z(m+) - Z(m-)$  is only the infinite excursion of the post-infimum process  $Z - \underline{Z}$ , so we only need to prove that this infinite excursion has the same law as  $Z^\uparrow$ , and for this we want to apply Proposition 4.7 of [33]. We already know that, because it is spectrally positive,  $Z$  is regular for  $] - \infty, 0[$ . Taking the dual of the process in Corollary VII.5 in [8], we get the regularity of  $\{0\}$  and  $]0, +\infty[$  for  $Z$ , thanks to the hypothesis of unbounded variation. The hypothesis of the proposition in [33] are thus fulfilled.

Let  $\mathcal{N}$  denote the excursion measure of the Markov process  $Z - \underline{Z}$  and  $(L^{-1}, U)$  the ladder process of  $Z : L^{-1}$  is the inverse of the local time at 0 of  $Z - \underline{Z}$  and for any positive  $t$ ,  $U(t) = \underline{Z}(L^{-1}(t))$ .

Recall from the introduction the notation  $\tau(A, h+)$  for  $\tau(A, [h, +\infty[)$ . We denote by  $\mathcal{U}$  the potential measure of  $U$  and, since  $-Z$  is spectrally negative, the formula page 191 in [8] applies and yields that  $\mathcal{U}(] - x, 0]) = (1 - e^{-\kappa x})/\kappa$  for any  $x \geq 0$ . Proposition 4.7 of [33] tells us that for any positive measurable function  $G$  defined on the space of càd-làg functions from  $[0, +\infty[$  to  $\mathbb{R}$  with finite lifetime, we have

$$\begin{aligned} \mathbb{E} \left[ G \left( (Z^\uparrow(s))_{0 \leq s \leq \tau(Z^\uparrow, h+)} \right) \right] &= \mathcal{N} \left( G \left( (\xi(s))_{0 \leq s \leq \tau(\xi, h+)} \right) \mathcal{U}(] - \xi(\tau(\xi, h+)), 0]) \mid \tau(\xi, h+) < \infty \right) \\ &\quad \times \mathcal{N} \left( \xi, \tau(\xi, h+) < \infty \right) \\ &= c_h \mathcal{N} \left( (1 - e^{-\kappa \xi(\tau(\xi, h+))}) G \left( (\xi(s))_{0 \leq s \leq \tau(\xi, h+)} \right) \mid \tau(\xi, h+) < \infty \right), \end{aligned} \quad (3.6.3)$$

replacing  $\mathcal{U}$  by its expression and where we set  $c_h := \mathcal{N}(\xi, \tau(\xi, h+) < \infty) / \kappa$ . Let  $\xi_\infty$  denote the infinite excursion of  $Z - \underline{Z}$ , then, for any positive measurable function  $F$ , we get that

$$\begin{aligned} \mathbb{E} \left[ F \left( (\xi_\infty(s))_{0 \leq s \leq \tau(\xi_\infty, h+)} \right) \right] &= \mathcal{N} \left( \mathbb{P} \left( \inf_{[0, +\infty[} Z_{\xi(\tau(\xi, h+))} > 0 \right) F \left( (\xi(s))_{0 \leq s \leq \tau(\xi, h+)} \right) \mid \tau(\xi, h+) < \infty \right) \\ &= \mathbb{E} \left[ \frac{\mathbb{P} \left( \inf_{[0, +\infty[} Z_{Z^\uparrow(\tau(Z^\uparrow, h+))} > 0 \right)}{c_h (1 - e^{-\kappa Z^\uparrow(\tau(Z^\uparrow, h+))})} F \left( (Z^\uparrow(s))_{0 \leq s \leq \tau(Z^\uparrow, h+)} \right) \right], \end{aligned} \quad (3.6.4)$$

where we used (3.6.3) with

$$G \left( (\xi(s))_{0 \leq s \leq \tau(\xi, h+)} \right) := \frac{\mathbb{P} \left( \inf_{[0, +\infty[} Z_{\xi(\tau(\xi, h+))} > 0 \right)}{c_h (1 - e^{-\kappa \xi(\tau(\xi, h+))})} \times F \left( (\xi_\infty(s))_{0 \leq s \leq \tau(\xi_\infty, h+)} \right).$$

It follows from (3.6.4) and from  $Z^\uparrow(\tau(Z^\uparrow, h+)) \geq h$  that,  $\mathbb{P}(\inf_{[0, +\infty[} Z_{\xi(\tau(\xi, h+))} > 0) / c_h (1 - e^{-\kappa \xi(\tau(\xi, h+))})$  is a bounded martingale with respect to the filtration  $\mathcal{F}_h := \sigma(Z^\uparrow(s), 0 \leq s \leq \tau(Z^\uparrow, h+))$  and that it converges almost surely to some constant. As a consequence, this quantity is almost surely equal to 1 for any positive  $h$ , hence,

$$\forall h > 0, \mathbb{E} \left[ F \left( (\xi_\infty(s))_{0 \leq s \leq \tau(\xi_\infty, h+)} \right) \right] = \mathbb{E} \left[ F \left( (Z^\uparrow(s))_{0 \leq s \leq \tau(Z^\uparrow, h+)} \right) \right],$$



so the infinite excursion of  $Z - \underline{Z}$  indeed has the same law as  $Z^\uparrow$ .

□

We can now prove Theorem 3.1.14.

*Démonstration.* of Theorem 3.1.14

We have

$$\begin{aligned} I(Z) &= \int_0^m e^{-Z(t)} dt + \int_m^{+\infty} e^{-Z(t)} dt \\ &= \int_0^m e^{-Z(t)} dt + e^{-Z(m-)} \int_m^{+\infty} e^{-(Z(m+t)-Z(m-))} dt \\ &\geq \int_m^{+\infty} e^{-(Z(m+t)-Z(m-))} dt, \end{aligned}$$

because almost surely  $Z(m-) \leq 0$ . Since  $Z$  has unbounded variation, we can use Lemma 3.6.2 which tells us that the last term is equal in law to  $I(Z^\uparrow)$ . We thus get

$$\mathbb{P}(I(Z) \leq x) \leq \mathbb{P}(I(Z^\uparrow) \leq x), \quad (3.6.5)$$

so we only need to prove the result for  $I(Z)$ . Obtaining (3.6.5) is the only thing for which we need the hypothesis of unbounded variation in this proof. The result that we now prove for  $I(Z)$  is thus true without this hypothesis.

Let  $(Q, \gamma, \nu)$  be the generating triplet of  $Z$  in the Lévy-Khintchine representation. Since  $\nu$  is non zero, there exist  $0 < \gamma_1 < \gamma_2 < +\infty$  such that  $\nu([\gamma_1, \gamma_2]) > 0$ . Then, for  $\eta \in ]0, 1[$  we define

$$\nu^{\eta,1} := \eta\nu(\cdot \cap [\gamma_1, \gamma_2]) \quad \text{and} \quad \nu^{\eta,2} := \nu - \eta\nu(\cdot \cap [\gamma_1, \gamma_2]),$$

and set  $Z^{\eta,1}$  and  $Z^{\eta,2}$  to be two independent Lévy processes which generating triplets are respectively  $(0, 0, \nu^{\eta,1})$  and  $(Q, \gamma, \nu^{\eta,2})$ . We have  $\nu = \nu^{\eta,1} + \nu^{\eta,2}$  so according to the Lévy-Khintchine formula,

$$Z \stackrel{\mathcal{L}}{=} Z^{\eta,1} + Z^{\eta,2}.$$

According to Corollary VII.2 in [8], a spectrally negative Lévy process  $X$  drifts to  $-\infty$  if and only if  $\mathbb{E}[X(1)] < 0$ .  $Z$  is a spectrally positive Lévy process drifting to  $+\infty$  so taking the dual in the theorem we get  $\mathbb{E}[Z(1)] > 0$ . Now since  $\mathbb{E}[Z(1)] = \mathbb{E}[Z^{\eta,1}(1)] + \mathbb{E}[Z^{\eta,2}(1)]$  and  $\mathbb{E}[Z^{\eta,1}(1)] < \gamma_2\eta\nu([\gamma_1, \gamma_2])$ , we have that  $\mathbb{E}[Z^{\eta,2}(1)]$  is positive for  $\eta$  small enough. Still taking the dual in Corollary VII.2 in [8], this implies that  $Z^{\eta,2}$  drifts to  $+\infty$  for  $\eta$  small enough. We thus choose such an  $\eta_0 \in ]0, 1[$  and denote by  $m_2$  the point at which  $Z^{\eta_0,2}$  reaches its minimum.

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$Z^{\eta_0,1}$  is a compound Poisson process, we define  $N$  the counting process of its jumps :  $N(t) := \#\{s \in [0, t], Z^{\eta_0,1}(s) - Z^{\eta_0,1}(s-) > 0\}$ .  $N$  is thus a standard Poisson process with parameter  $c_0 := \eta_0 \nu([\gamma_1, \gamma_2])$  and it is independent of  $Z^{\eta_0,2}$ . We have :

$$\begin{aligned} I(Z) &= \int_0^{+\infty} e^{-(Z^{\eta_0,1}(t)+Z^{\eta_0,2}(t))} dt \\ &\leq e^{-Z^{\eta_0,2}(m_2)} \int_0^{+\infty} e^{-Z^{\eta_0,1}(t)} dt \\ &\leq e^{-Z^{\eta_0,2}(m_2)} \int_0^{+\infty} e^{-\gamma_1 N(t)} dt \\ &= e^{-Z^{\eta_0,2}(m_2)} I(\gamma_1 N), \end{aligned}$$

so, from the independence between the two factors :

$$\mathbb{P}(I(\gamma_1 N) \leq x/2) \times \mathbb{P}(e^{-Z^{\eta_0,2}(m_2)} \leq 2) \leq \mathbb{P}(I(Z) \leq x). \quad (3.6.6)$$

We put  $c_1 := \mathbb{P}(e^{-Z^{\eta_0,2}(m_2)} \leq 2) > 0$ . Now by a property of standard Poisson processes, it is easy to see that  $I(\gamma_1 N)$  has the same law as

$$\frac{1}{c_0} \sum_{k=0}^{+\infty} e^{-\gamma_1 k} e_k,$$

where  $(e_k)_{k \in \mathbb{N}}$  is a sequence of *iid* exponential random variable with parameter 1. This allows us to compute the Laplace transform of  $I(\gamma_1 N)$  :

$$\forall \lambda \geq 0, \mathbb{E}[e^{-\lambda I(\gamma_1 N)}] = \prod_{k=0}^{+\infty} \frac{1}{1 + \frac{\lambda}{c_0} e^{-\gamma_1 k}}.$$

We put  $K(\lambda) := \min\{k \in \mathbb{N}, \lambda e^{-\gamma_1 k} \leq 1\}$  and taking the logarithm we get

$$\begin{aligned} \log(\mathbb{E}[e^{-\lambda I(\gamma_1 N)}]) &= - \sum_{k=0}^{+\infty} \log\left(1 + \frac{\lambda}{c_0} e^{-\gamma_1 k}\right) \\ &\geq -K(\lambda) \log\left(1 + \frac{\lambda}{c_0}\right) - \sum_{k \geq K(\lambda)} \log\left(1 + \frac{\lambda}{c_0} e^{-\gamma_1 k}\right) \\ &\geq -K(\lambda) \log\left(1 + \frac{\lambda}{c_0}\right) - \sum_{k \geq 0} \log\left(1 + \frac{1}{c_0} e^{-\gamma_1 k}\right). \end{aligned}$$

Now, since  $K(\lambda) \underset{\lambda \rightarrow +\infty}{\sim} \log(\lambda)/\gamma_1$ , we get that

$$\log(\mathbb{E}[e^{-\lambda I(\gamma_1 N)}]) \geq -2(\log(\lambda))^2/\gamma_1,$$

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for  $\lambda$  large enough. Now, reasoning as in the proof of (3.1.5) (where, from a lower bound on the Laplace transform of  $I(V^\uparrow)$ , we deduced a lower bound for its asymptotic tail at 0) we get

$$e^{-c_2(\log(x))^2} \leq \mathbb{P}(I(\gamma_1 N) \leq x),$$

for some positive constant  $c_2$ . Combining with (3.6.5) and (3.6.6), we get the sought result. □

# Chapitre 4

## PATH DECOMPOSITION OF A SPECTRALLY NEGATIVE LÉVY PROCESS, AND LOCAL TIME OF A DIFFUSION IN THIS ENVIRONMENT

This work has been the object of an article [74] that will be shortly submitted.

### 4.1 Introduction

Let  $V$  be a two-sided spectrally negative Lévy process which is not the opposite of a subordinator (in particular,  $V$  can not be a compound Poisson process), drifts to  $-\infty$  at  $+\infty$ , and such that  $V(0) = 0$ . We denote its Laplace exponent by  $\Psi_V$  :

$$\forall t, \lambda \geq 0, \mathbb{E} [e^{\lambda V(t)}] = e^{t\Psi_V(\lambda)}.$$

It is well-known, for such  $V$ , that  $\Psi_V$  admits a non trivial zero that we denote here by  $\kappa$ ,  $\kappa := \inf\{\lambda > 0, \Psi_V(\lambda) = 0\} > 0$ .

We are here interested in a diffusion in this potential  $V$ . Let us recall that such a diffusion  $(X(t), t \geq 0)$  in a random càd-làg potential  $V$  is defined informally by  $X(0) = 0$  and

$$dX(t) = d\beta(t) - \frac{1}{2}V'(X(t))dt,$$

where  $\beta$  is a Brownian motion independent from  $V$ . Rigorously,  $X$  is defined by its conditional generator given  $V$ ,

$$\frac{1}{2}e^{V(x)} \frac{d}{dx} \left( e^{-V(x)} \frac{d}{dx} \right).$$

The fact that  $V$  drifts to  $-\infty$  puts us in the case where the diffusion  $X$  is a.s. transient to the right. In [66], Singh makes the study of the asymptotic behavior of

$X$ . When  $0 < \kappa < 1$ , he proves in particular that

$$X(t)/t^\kappa \xrightarrow[t \rightarrow +\infty]{\mathcal{L}} \mathcal{C} (1/\mathcal{S}_\kappa)^\kappa,$$

where  $\mathcal{C}$  is some explicit positive constant depending on  $V$  and  $\mathcal{S}_\kappa$  follows a completely asymmetric  $\kappa$ -stable distribution. Putting this in relation with the results of Kawazu and Tanaka [45], we see that  $\kappa$  plays a similar role as the drift of the brownian environment, at least for the asymptotic behavior of the diffusion. In this paper, we prove that the same is true for the behavior of the local time of  $X$ . We denote by  $(\mathcal{L}_X(t, x), t > 0, x \in \mathbb{R})$  the version of the local time that is continus in time and càd-làg in space, and we define respectively the supremum of the local time and the favorite site until instant  $t$  as

$$\mathcal{L}_X^*(t) = \sup_{x \in \mathbb{R}} \mathcal{L}_X(t, x) \quad \text{and} \quad F^*(t) := \inf\{x \in \mathbb{R}, \mathcal{L}_X(t, x) \vee \mathcal{L}_X(t, x-) = \mathcal{L}_X^*(t)\}.$$

We study the convergence in distribution of  $\mathcal{L}_X^*(t)$  and  $F^*(t)$  when  $\kappa > 1$ , and of  $\mathcal{L}_X^*(t)$  when  $0 < \kappa < 1$ . When  $V$  is a drifted brownian motion, the case where  $0 < \kappa < 1$  has already been studied by Andreatti, Devulder and Véchambre in [4]. In this case, there is a useful renewal structure obtained from a valleys decomposition of the potential and the Markov property for the diffusion. The limit distribution they obtain involves a  $\kappa$ -stable subordinator and an exponential functional of the environment conditioned to stay positive. When  $0 < \kappa < 1$ , we extend their result to the diffusion in  $V$  and obtain a limit distribution in terms of the exponential functionals of  $V$  and its dual conditioned to stay positive :

$$I(V^\uparrow) := \int_0^{+\infty} e^{-V^\uparrow(t)} dt \quad \text{and} \quad I(\hat{V}^\uparrow) := \int_0^{+\infty} e^{-\hat{V}^\uparrow(t)} dt,$$

where  $\hat{V}$ , the dual of  $V$ , is equal in law to  $-V$ . It is proved in Theorems 1.1 and 1.13 of Véchambre [72] that these functionals are indeed finite and well-defined. The fact, proved in [72], that  $I(V^\uparrow)$  and  $I(\hat{V}^\uparrow)$  admit some finite exponential moments is of fundamental interest for our generalization of the results of [4].

The almost sure asymptotic behavior of the local time is studied, in the discrete transient case, by Gantert and Shi [41]. We believe that the present work will allow us, in the future, to link the asymptotic almost sure behavior of  $\mathcal{L}_X^*(t)$  with the left tail of  $I(V^\uparrow)$ . This tail is given in [72] and can be very different, when  $V$  is a general spectrally negative Lévy process, than when  $V$  is a drifted brownian motion. As a consequence we can expect, in the Lévy case, many possible behaviors for the almost sure asymptotic of  $\mathcal{L}_X^*(t)$ , and this is our main motivation to generalize, here, the study of [4] to the Lévy case.

When  $\kappa > 1$ , we adopt the point of view of [66] and link the local time to a generalized Ornstein-Uhlenbeck process. This approach also provides the convergence of the favorite site and can certainly be used to study the almost sure behavior of the supremum of the local time.

### 4.1.1 Main results

Our main results are the convergences in distribution for the supremum of the local time. Let us define the constants  $K$  and  $m$  similarly as in [66] :

$$K := \mathbb{E} \left[ \left( \int_0^{+\infty} e^{V(t)} dt \right)^{\kappa-1} \right] \quad \text{and} \quad m := \frac{-2}{\Psi_V(1)} > 0,$$

and for any  $\alpha, s > 0$ , let  $\mathcal{F}(\alpha, s)$  denote the Fréchet distribution with parameters  $\alpha$  and  $s$ , that is, the distribution with repartition function

$$\mathcal{F}(\alpha, s) ([0, t]) = e^{-(s/t)^\alpha}.$$

When  $\kappa > 1$ , the limit distribution of the supremum of the local time can be expressed as follows :

**Theorem 4.1.1.** *If  $\kappa > 1$ ,*

$$\mathcal{L}_X^*(t)/t^{1/\kappa} \xrightarrow[t \rightarrow +\infty]{\mathcal{L}} \mathcal{F}(\kappa, 2(\Gamma(\kappa)\kappa^2 K/m)^{1/\kappa}).$$

**Examples :** In some cases the parameters of the limit distribution are more explicit.

- Let  $W_\kappa$  be the  $\kappa$ -drifted brownian motion :  $W_\kappa(t) := W(t) - \frac{\kappa}{2}t$ . If we choose  $V = W_\kappa$  (for  $\kappa > 1$ ), then  $K = 2^{\kappa-1}/\Gamma(\kappa)$  (see Example 1.1 in [66]) and  $m = 4/(\kappa - 1)$ . The limit distribution of the supremum of the local time is therefore  $\mathcal{F}(\kappa, 4(\kappa^2(\kappa - 1)/8)^{1/\kappa})$ . This is precisely the second point of Theorem 1.6 of Devulder [28].
- If  $V$  is such that  $\kappa = 2$  then

$$K = \mathbb{E} \left[ \int_0^{+\infty} e^{V(t)} dt \right] = \int_0^{+\infty} \mathbb{E} [e^{V(t)}] dt = \int_0^{+\infty} e^{t\Psi_V(1)} dt = \frac{-1}{\Psi_V(1)} = \frac{m}{2},$$

and the limit distribution of  $\mathcal{L}_X^*(t)/t^{1/2}$  is therefore  $\mathcal{F}(2, 2\sqrt{2})$ .

We also prove that the distribution of the favorite site is asymptotically uniform :

**Theorem 4.1.2.** *If  $\kappa > 1$ ,*

$$mF^*(t)/t \xrightarrow[t \rightarrow +\infty]{\mathcal{L}} \mathcal{U},$$

where  $\mathcal{U}$  denotes the uniform distribution on  $[0, 1]$ .

When  $0 < \kappa < 1$ , we have to introduce some notations in order to express the limit distribution. Let  $G_1$  and  $G_2$  be two independent random variables with  $G_1 \stackrel{\mathcal{L}}{=} I(V^\uparrow)$  and  $G_2 \stackrel{\mathcal{L}}{=} I(\hat{V}^\uparrow)$ . We define  $\mathcal{R} := G_1 + G_2$ .  $\mathcal{R}$  is the analogue of  $\mathcal{R}_\kappa$  defined in [4] (if  $V$  if the  $\kappa$ -drifted brownian motion, then  $\mathcal{R} \stackrel{\mathcal{L}}{=} \mathcal{R}_\kappa$ , indeed, it is known that  $W_\kappa^\uparrow \stackrel{\mathcal{L}}{=} \hat{W}_\kappa^\uparrow$ , so  $\mathcal{R}$  is, as  $\mathcal{R}_\kappa$ , the sum of two independent copies of  $I(W_\kappa^\uparrow)$ ). Let also  $\mathcal{C}$  be the constant of Corollary 5 of Bertoin, Yor [12] applied to  $-V$  (which

gives the right tail of the exponential functional of the Lévy process  $-V$ ), the fact that this corollary can be applied is the object of Lemma 4.5.3 in Section 4.5. We put  $\mathcal{C}' := (\mathcal{C}/2) \int_0^{+\infty} u^\kappa e^{-u/2} du$ . Now, let  $\mathcal{Y}_1$  be the  $\kappa$ -stable subordinator with Laplace exponent  $\mathcal{C}'\Gamma(1-\kappa)\lambda^\kappa$  :

$$\forall t, \lambda \geq 0, \mathbb{E} [e^{-\lambda \mathcal{Y}_1(t)}] = e^{-t\mathcal{C}'\Gamma(1-\kappa)\lambda^\kappa}.$$

We now consider the pure jump Lévy process  $(\mathcal{Y}_1, \mathcal{Y}_2)$  where the component  $\mathcal{Y}_2$  is defined multiplying each jump of  $\mathcal{Y}_1$  by an independent copy of  $\mathcal{R}$ . We can also define  $(\mathcal{Y}_1, \mathcal{Y}_2)$  from its  $\kappa$ -stable Lévy measure  $\nu$  supported on  $]0, +\infty[ \times ]0, +\infty[$  and defined by

$$\forall x > 0, y > 0, \nu([x, +\infty[ \times ]y, +\infty[) = \frac{\mathcal{C}'}{y^\kappa} \mathbb{E} \left[ \mathcal{R}^\kappa \mathbf{1}_{\mathcal{R} \leq \frac{y}{x}} \right] + \frac{\mathcal{C}'}{x^\kappa} \mathbb{P} \left( \mathcal{R} > \frac{y}{x} \right).$$

It is easy to see that the first definition implies the second and also that the Lévy process  $(\mathcal{Y}_1, \mathcal{Y}_2)$  has Laplace transform

$$\forall t, \alpha, \beta \geq 0, \mathbb{E} [e^{-\alpha \mathcal{Y}_1(t) - \beta \mathcal{Y}_2(t)}] = e^{-t\mathcal{C}'\Gamma(1-\kappa)\mathbb{E}[(\alpha + \beta \mathcal{R})^\kappa]}.$$

As in [4], our limit distribution for the supremum of the local time is a function of  $(\mathcal{Y}_1, \mathcal{Y}_2)$ . For  $Z$  an increasing càd-làg process and  $s \geq 0$ , we put respectively  $Z(s-)$ ,  $Z^\natural(s)$  and  $Z^{-1}(s)$  for respectively the left-limit of  $Z$  at  $s$ , the largest jump of  $Z$  before  $s$  and the generalized inverse of  $Z$  at  $s$  :

$$Z(s-) = \lim_{r \nearrow s} Z(r), \quad Z^\natural(s) := \sup_{0 \leq r \leq s} (Z(r) - Z(r-)), \quad Z^{-1}(s) := \inf\{u \geq 0, Z(u) > s\},$$

where  $\inf \emptyset = +\infty$  by convention. We now define the couple of random variables  $(\mathcal{I}_1, \mathcal{I}_2)$  :

$$\mathcal{I}_1 := \mathcal{Y}_1^\natural(\mathcal{Y}_2^{-1}(1)-), \quad \mathcal{I}_2 := (1 - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)-)) \times \frac{\mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)-)}{\mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)-)},$$

and we have

**Theorem 4.1.3.** *If  $0 < \kappa < 1$ ,  $V$  has unbounded variations and  $V(1) \in L^p$  for some  $p > 1$  then*

$$\mathcal{L}_X^*(t)/t \xrightarrow[t \rightarrow +\infty]{\mathcal{L}} \mathcal{I} := \max(\mathcal{I}_1, \mathcal{I}_2).$$

This result is a generalization, for more general environments, of Theorem 1.3 of [4]. A key point in its proof is the fact that the contributions to the local time and to the time spent by the diffusion between the bottoms of two consecutive valleys are negligible, so the local maxima of the local time are localized at the bottom of the valleys, where most of the time is spent. This is where appears the pure jump subordinator  $(\mathcal{Y}_1, \mathcal{Y}_2)$  with two correlated components, one for the local time, and one for the time spent. Each jump of  $(\mathcal{Y}_1, \mathcal{Y}_2)$  represents the contribution of the bottom of a valley to these two quantities.

Our generalization of the results known in the brownian case also yields some other results such as the convergence of the supremum of the local time before and after the last valley (the first two points of Theorem 1.5 in [4]).

When  $0 < \kappa < 1$ , our study relies deeply on the decomposition of the potential into  $h$ -valleys and on the localization of the contributions to the local time and to the time spent near the  $h$ -minima. In fact, we have to make  $h$  going to infinity when  $t$  goes to infinity, this is why we briefly study the asymptotic of the  $h$ -valleys. The definition of the  $h$ -extrema and  $h$ -valleys are given in the next section but we already note the following result about the asymptotic of  $(m_1, m_2, \dots)$ , the sequence of  $h$ -minima.

**Theorem 4.1.4.** *When  $h$  goes to infinity, the renormalized random sequence  $e^{-\kappa h}(m_1, m_2, \dots)$  converges in distribution to the jumping times sequence of a standard Poisson process with parameter  $q$  (which depends explicitly on the law of  $V$ ). If  $V = W_\kappa$ , the  $\kappa$ -drifted brownian motion, then  $q = \kappa^2/2$ .*

Some of the estimates used to prove this theorem will also be useful to establish the negligibility of the local time between the bottoms of two consecutive valleys when  $0 < \kappa < 1$ . However, the main interest of this theorem is that it informs us on the typical distance between two consecutive minima, and this provides an heuristic explanation for why the method based on valleys fails when  $\kappa > 1$ .

As we can see in Theorem 4.1.4, the distance between two consecutive  $h$ -minima is of order  $e^{\kappa h}$ . In the case  $0 < \kappa < 1$ , what happens between the bottoms of the  $h$ -valleys can be neglected, so the main contributions to the local time and to the time spent by the diffusion are localized at the  $h$ -minima and are highly correlated, this explains the form of the limit distribution for the favorite site given in [4]. When  $\kappa > 1$ , this distance between two consecutive  $h$ -minima is so large that the time spent by the diffusion between the  $h$ -minima is no longer negligible compared to the time spent in the bottoms of the  $h$ -valleys, and there are extreme values taken by the local time between the  $h$ -valleys. In fact, it is impossible to use the valleys to localize the large values of the local time as we do when  $0 < \kappa < 1$ , this explains the asymptotic uniform distribution for the favorite site. Moreover, the case  $\kappa > 1$  is the case where  $X$  has positive speed. The local time at time  $t$  is then close to the local time at the hitting time of  $t/m$  and the latter can be expressed thanks to a generalized Ornstein-Uhlenbeck process. The supremum of the local time is therefore similar to the supremum of the heights of *iid* excursions of a Markov process. This explains the appearance of a Fréchet distribution in the limit distribution.

The rest of the paper is organized as follows. In section 4.2 we study the case where  $\kappa > 1$  and prove Theorems 4.1.1 and 4.1.2. In section 4.3 we recall the definitions of the  $h$ -extrema and  $h$ -valleys for  $V$  and establish some usual properties such as the independence of the consecutive slopes and the law of the bottoms of the valleys. We end this section by proving Theorem 4.1.4. In section 4.4 we assume  $0 < \kappa < 1$  and prove Theorem 4.1.3. In section 4.5 we prove some fundamental estimates on  $V$ ,  $V^\uparrow$  and  $\hat{V}^\uparrow$  and some technical results.



### 4.1.2 Facts and notations

We denote by  $(Q, \gamma, \nu)$  the generating triplet of  $V$  so  $\Psi_V$  can be expressed as

$$\Psi_V(\lambda) = \frac{Q}{2}\lambda^2 - \gamma\lambda + \int_{-\infty}^0 (e^{\lambda x} - 1 - \lambda x \mathbf{1}_{|x|<1})\nu(dx). \quad (4.1.1)$$

If  $V$  jumps at instant  $u$  we denote  $\Delta V(u) := V(u) - V(u-)$ , the jump of  $V$  at  $u$ . For  $r > 0$ , we define  $V^{<-r}$  to be the sum of the jumps of  $V$  that are less than  $-r$  :

$$\forall s \geq 0, V^{<-r}(s) := \sum_{0 \leq u \leq s} \Delta V(u) \mathbf{1}_{\{\Delta V(u) < -r\}}.$$

$V$  is the sum of the processes  $V - V^{<-r}$  and  $V^{<-r}$  that can be seen to be independent spectrally negative Lévy processes, thanks to the Lévy-Khintchine formula (4.1.1). According to Corollary VI.2 of Bertoin [8] we have  $\mathbb{E}[V(1)] < 0$  so for  $r$  chosen large enough we have  $\mathbb{E}[(V - V^{<-r})(1)] < 0$  which, thanks to Corollary VI.2 of [8], implies that  $V - V^{<-r}$  drifts to  $-\infty$ . In other words, removing the very large jumps of  $V$  does not change its convergence to  $-\infty$ .

For  $Y$  a process and  $S$  a borelian set, we denote

$$\tau(Y, S) := \inf \{t \geq 0, Y(t) \in S\}, \quad \mathcal{K}(Y, S) := \sup \{t \geq 0, Y(t) \in S\}.$$

We shall only write  $\tau(Y, x)$  instead of  $\tau(Y, \{x\})$  and  $\tau(Y, x+)$  instead of  $\tau(Y, [x, +\infty[)$ . Since  $V$  has no positive jumps we see that it reaches each positive level continuously :  $\forall x > 0, \tau(V, x+) = \tau(V, x)$ . Moreover, the law of the supremum of  $V$  is known, it is an exponential distribution with parameter  $\kappa$  (see Corollary V.II in [8]).

$\underline{Y}$  denotes the infimum process of  $Y$  :  $\forall t \geq 0, \underline{Y}(t) := \inf_{[0,t]} Y$ . The process  $V - \underline{V}$  is known as the process  $V$  reflected at its infimum, note that Proposition VI.1 of [8] tells that it is a càd-làg Markov process. The same holds for  $\hat{V} - \hat{\underline{V}}$ .

If  $Y$  is Markovian and  $x \in \mathbb{R}$  we denote  $Y_x$  for the process  $Y$  starting from  $x$ . For  $Y_0$  we shall only write  $Y$ . For any (possibly random) time  $T > 0$ , we write  $Y^T$  for the process  $Y$  shifted and centered at time  $T$  :  $\forall s \geq 0, Y^T(s) := Y(T+s) - Y(T)$ .

Let  $B$  be a brownian motion starting at 0 and independent from  $V$ . A diffusion in potential  $V$  can be defined via the formula

$$X(t) := A_V^{-1}(B(T_V^{-1}(t))), \quad (4.1.2)$$

where

$$A_V(x) := \int_0^x e^{V(u)} du \quad \text{and for } 0 \leq s \leq \tau \left( B, \int_0^{+\infty} e^{V(u)} du \right),$$

$$T_V(s) := \int_0^s e^{-2V(A_V^{-1}(B(u)))} du.$$

It is known that the local time of  $X$  at  $x$  until instant  $t$  has the following expression :

$$\mathcal{L}_X(t, x) = e^{-V(x)} \mathcal{L}_B(T_V^{-1}(t), A_V(x)). \quad (4.1.3)$$

For the hitting times of  $r \in \mathbb{R}$  by the diffusion  $X$  we shall use the frequent notation  $H(r)$  (instead of  $\tau(X, r)$ ). We denote by  $H_+(r)$  (respectively  $H_-(r)$ ) the total time spent by the diffusion in  $[0, +\infty[$  (respectively  $] - \infty, 0]$ ) before  $H(r)$ .

The quenched probability measure  $P^V$  is the probability measure conditionally on the potential  $V$ . When we deal with events relative to the diffusion  $X$ ,  $\mathbb{P}$  represents the annealed probability measure, it is defined as  $\mathbb{P} := \int P^\omega(\cdot) P(V \in d\omega)$ .

We often denote by  $d_{VT}$  the total variation distance between two probability distributions on the same space.

## 4.2 Supremum of the local time when $\kappa > 1$

We now treat the case  $\kappa > 1$ . Since some of the lemmas we state in this section are true in a general context we do not assume  $\kappa > 1$  yet. We thus have  $\kappa \in ]0, +\infty[$  unless mentioned otherwise. As we mentioned in the introduction, the valleys are of no use in this section so we have to study directly the expression of the local time. It is given by (4.1.3). According to [66], if  $m := -2/\psi_V(1)$ , then  $t$  is close to  $H(t/m)$  (when  $\kappa > 1$  and  $t$  is large). It is then convenient to look at the local time until the hitting times. It has a simpler expression :

$$\mathcal{L}_X(H(r), x) = e^{-V(x)} \mathcal{L}_B(\tau(B, A_V(r)), A_V(x)). \quad (4.2.1)$$

The supremum of the local time until instant  $H(r)$  can thus be written

$$\mathcal{L}_X^*(H(r)) = \max\{\mathcal{M}_1(r), \mathcal{M}_2(r)\}, \quad (4.2.2)$$

where

$$\begin{aligned} \mathcal{M}_1(r) &:= \sup_{x < 0} \mathcal{L}_X(H(r), x) = \sup_{x < 0} e^{-V(x)} \mathcal{L}_B(\tau(B, A_V(r)), A_V(x)) \\ &\leq \left( \sup_{x < 0} e^{-V(x)} \right) \times \left( \sup_{x < 0} \mathcal{L}_B(\tau(B, A_V(+\infty)), A_V(x)) \right) < +\infty \end{aligned} \quad (4.2.3)$$

and

$$\mathcal{M}_2(r) := \sup_{x \in [0, r]} \mathcal{L}_X(H(r), x) = \sup_{x \in [0, r]} e^{-V(x)} \mathcal{L}_B(\tau(B, A_V(r)), A_V(x)).$$

We also define the favorite positive site until time  $H(r)$  by

$$\begin{aligned} F_+^*(H(r)) &:= \operatorname{argmax}_{[0, r]} \mathcal{L}_X(H(r), \cdot) \\ &:= \inf \{x \in [0, r], \mathcal{L}_X(H(r), x) \vee \mathcal{L}_X(H(r), x-) = \mathcal{M}_2(r)\}. \end{aligned}$$

To lighten the notations, we often use the notation *argmax* in this section. Now, note that  $\mathcal{M}_2(r)$  is the same as  $J_2(r)$  of [66] with a supremum on  $[0, r]$  in place of an integral on this interval. In particular, if, similarly to [66], we define

$$Z(x) := e^{V(x)} R \left( \int_0^x e^{-V(y)} dy \right),$$

where  $R$  is a two-dimensional squared Bessel process independent from  $V$ , we get

$$\mathcal{M}_2(r) \stackrel{\mathcal{L}}{=} \sup_{x \in [0, r]} Z(x) \quad \text{and} \quad F_+^*(H(r)) \stackrel{\mathcal{L}}{=} r - \operatorname{argmax}_{[0, r]} Z. \quad (4.2.4)$$

We can therefore prove our results using some precise properties of  $Z$ . Recall from [66] the definition of  $L$ , the local time of  $Z$  for the position 1, of  $n$  the associated excursion measure, and of  $L^{-1}$  the right continus inverse of  $L$ . We denote by  $\xi$  a generic excursion.

### 4.2.1 The local time at hitting times

We now prove a lemma to justify rigorously that the local time of the diffusion at some instant can be approximated by the local time at an hitting time.

**Lemma 4.2.1.** *Let us denote by  $Q$  the positive constant denoted by  $n[\zeta]$  in [66], for  $r$  large enough we have*

$$\mathbb{P} \left( L^{-1}(r/Q - r^{3/4}) \leq r \leq L^{-1}(r/Q + r^{3/4}) \right) \geq 1 - r^{-1/4}. \quad (4.2.5)$$

Assume  $\kappa > 1$ . For any  $\alpha \in ]\max\{3/4, 1/\kappa\}, 1[$  their exists  $\epsilon > 0$  such that for  $r$  large enough we have

$$\mathbb{P} \left( H(r/m - r^\alpha) \leq r \leq H(r/m + r^\alpha) \right) \geq 1 - r^{-\epsilon}, \quad (4.2.6)$$

$$\mathbb{P} \left( \mathcal{L}_X^*(H(r/m - r^\alpha)) \leq \mathcal{L}_X^*(r) \leq \mathcal{L}_X^*(H(r/m + r^\alpha)) \right) \geq 1 - r^{-\epsilon}. \quad (4.2.7)$$

*Démonstration.* According to Lemma 5.1 in [66],  $\mathbb{E}[L^{-1}(1)] = Q$  and the subordinator  $L^{-1}$  admits some finite positive exponential moments, so, in particular, it has moments of the second order. Let us define the Lévy process  $U(t) := L^{-1}(t) - Qt$ .  $U$  has finite mean equal to 0 and moments of the second order, this implies that  $\mathbb{E}[|U(t)|^2] = t\mathbb{E}[|U(1)|^2]$ . Using Markov's inequality we get

$$\forall t, s > 0, \mathbb{P}(|U(t)| > s) \leq s^{-2} \mathbb{E}[|U(t)|^2] = s^{-2} t \mathbb{E}[|U(1)|^2]. \quad (4.2.8)$$

Then,

$$\begin{aligned} \mathbb{P} \left( r > L^{-1}(r/Q + r^{3/4}) \right) &\leq \mathbb{P} \left( |L^{-1}(r/Q + r^{3/4}) - (r + Qr^{3/4})| > Qr^{3/4} \right) \\ &= \mathbb{P} \left( |U(r/Q + r^{3/4})| > Qr^{3/4} \right) \leq (r/Q + r^{3/4}) \mathbb{E}[|U(1)|^2] / Q^2 r^{3/2}, \end{aligned}$$

where we used (4.2.8) with  $t = r/Q + r^{3/4}$  and  $s = Qr^{3/4}$ . We get a similar estimate for  $\mathbb{P}(r < L^{-1}(r/Q - r^{3/4}))$  so (4.2.5) follows. We now turn to (4.2.6). Until the end of this proof we assume  $\kappa > 1$ . We have

$$\begin{aligned} \mathbb{P}(H(r/m - r^\alpha) > r) &\leq \mathbb{P}(H_-(r/m - r^\alpha) + H_+(r/m - r^\alpha) - (r - mr^\alpha) > mr^\alpha) \\ &\leq \mathbb{P}(H_-(+\infty) > mr^\alpha/2) \\ &\quad + \mathbb{P}\left(\int_0^{r/m-r^\alpha} Z(x)dx - (r - mr^\alpha) > mr^\alpha/2\right), \end{aligned} \quad (4.2.9)$$

because, as we can see from [66],  $\int_0^t Z(x)dx$  has the same law as  $H_+(t)$  (denoted by  $J_2(t)$  there). The second term in the right hand side of (4.2.9) is less than

$$\begin{aligned} &\mathbb{P}\left(\int_0^{\tau(Z,1)} Z(x)dx > mr^\alpha/4\right) + \mathbb{P}\left(\int_{\tau(Z,1)}^{r/m-r^\alpha} Z(x)dx - (r - mr^\alpha) > mr^\alpha/4\right) \\ &\leq \mathbb{P}(\tau(Z,1) > mr^\alpha/4) + \mathbb{P}\left(\int_0^{r/m-r^\alpha} Z_1(x)dx - (r - mr^\alpha) > mr^\alpha/4\right), \end{aligned}$$

where we put  $Z_1 := Z(\tau(Z,1) + \cdot)$ . It is well defined since  $Z$  has no positive jumps and thus reaches  $[1, +\infty[$  continuously. Since  $Z$  is a Markov process,  $Z_1$  has indeed the same law as  $Z$  starting from 1. According to Proposition 4.3 of [66] :

$$\mathbb{P}(\tau(Z,1) > t) \leq e^{-ct}, \quad (4.2.10)$$

for some constant  $c > 0$  when  $t$  is large enough. We now put this in (4.2.9) where we also use Lemma 4.5.18 for the term  $\mathbb{P}(H_-(+\infty) > mr^\alpha/2)$ , we obtain for  $r$  large enough

$$\mathbb{P}(H(r/m - r^\alpha) > r) \leq \mathbb{P}\left(\left|\int_0^{r/m-r^\alpha} Z_1(x)dx - (r - mr^\alpha)\right| > mr^\alpha/4\right) + c_1 r^{-\frac{\alpha\kappa}{2+\kappa}}, \quad (4.2.11)$$

for some positive constant  $c_1$ . Let us define  $\tilde{U}(t) := \int_0^{L^{-1}(t)} Z_1(x)dx - mL^{-1}(t)$ . Note that, from Section 7 of [66] we can see that  $\tilde{U}$  is a Lévy process with finite mean equal to 0 and such that  $\mathbb{P}(\tilde{U}(1) > x) \sim_{x \rightarrow +\infty} cx^{-\kappa}$  for some positive constant  $c$ . As a consequence,  $\mathbb{E}[|\tilde{U}(1)|^\gamma] < +\infty$  for  $\gamma \in ]1/\alpha, \kappa[$ . We choose such a  $\gamma$  and use successively Markov's and Von Barh-Esseen's inequalities :

$$\forall t, s > 0, \mathbb{P}\left(|\tilde{U}(t)| > s\right) \leq s^{-\gamma} \mathbb{E}\left[|\tilde{U}(t)|^\gamma\right] \leq 2s^{-\gamma} \left( |t| \mathbb{E}\left[|\tilde{U}(1)|^\gamma\right] + \sup_{s \in [0,1]} \mathbb{E}\left[|\tilde{U}(s)|^\gamma\right] \right). \quad (4.2.12)$$

We now use (4.2.5) and get that for  $y$  large enough,  $y \leq L^{-1}(y/Q + y^{3/4})$  with probability greater than  $1 - y^{-1/4}$  so

$$\begin{aligned} \int_0^y Z_1(x)dx - my &\leq \int_0^{L^{-1}(y/Q + y^{3/4})} Z_1(x)dx - my \\ &= \tilde{U}(y/Q + y^{3/4}) + mU(y/Q + y^{3/4}) + mQy^{3/4}. \end{aligned}$$

Let  $C$  be an arbitrary positive constant. Applying (4.2.12) and (4.2.8) with  $t = y/Q + y^{3/4}$  and  $s = Cy^\alpha$  we get for  $y$  large enough

$$\mathbb{P}\left(\int_0^y Z_1(x)dx - my > Cy^\alpha\right) \leq c_2(y^{1-\alpha\gamma} + y^{1-2\alpha} + y^{-1/4}),$$

for some positive constant  $c_2$ , depending on  $C$ . We prove a similar inequality for  $my - \int_0^y Z_1(x)dx$  so we have actually, for  $c_3$  a positive constant (depending on  $C$ ) and  $y$  large enough,

$$\mathbb{P}\left(\left|\int_0^y Z_1(x)dx - my\right| > Cy^\alpha\right) \leq c_3(y^{1-\alpha\gamma} + y^{1-2\alpha} + y^{-1/4}). \quad (4.2.13)$$

Applying (4.2.13) with a good choice of  $C$ ,  $y = r/m - r^\alpha$  and putting into (4.2.11) we get

$$\mathbb{P}(H(r/m - r^\alpha) > r) \leq r^{-\epsilon}/2, \quad (4.2.14)$$

if  $\epsilon > 0$  is chosen to be less than  $\max\{\alpha\gamma - 1, 2\alpha - 1, \alpha\kappa/(2 + \kappa), 1/4\}$  and  $r$  is large enough. Then,

$$\begin{aligned} \mathbb{P}(H(r/m + r^\alpha) < r) &\leq \mathbb{P}(H_+(r/m + r^\alpha) < r) \leq \mathbb{P}\left(\int_{\tau(Z,1)}^{r/m+r^\alpha} Z(x)dx < r\right) \\ &\leq \mathbb{P}\left(\int_0^{r/m+r^\alpha/2} Z_1(x)dx < r\right) + \mathbb{P}(\tau(Z,1) > r^\alpha/2), \end{aligned}$$

so, using (4.2.10) and (4.2.13), we conclude the same way as for  $\mathbb{P}(H(r/m - r^\alpha) > r)$ : we get  $\mathbb{P}(H(r/m + r^\alpha) < r) \leq r^{-\epsilon}/2$  for  $r$  large enough, and combining with (4.2.14) we get (4.2.6).

Finally, (4.2.7) is only a consequence of (4.2.6) and of the increases of  $\mathcal{L}_X^*$ .  $\square$

We also need an almost sure version of (4.2.5) :

**Lemma 4.2.2.** *Almost surely, for all  $t$  large enough we have*

$$Qt - t^{3/4} \leq L^{-1}(t) \leq Qt + t^{3/4}.$$

*Démonstration.* Since  $L^{-1}(1)$  admits some finite exponential moments we have an iterated logarithme law.  $\square$

Now, we study the supremum of the process  $Z$  in terms of its excursion measure  $n$ .

**Lemma 4.2.3.** *There is  $\epsilon > 0$  and  $r_0 > 0$  such that for all  $r \geq r_0$  and  $h > 1$  we have*

$$e^{-(r/Q+r^{7/8})n(\sup \xi > h)} - r^{-\epsilon} \leq \mathbb{P} \left( \sup_{x \in [0, r]} Z(x) \leq h \right) \leq e^{-(r/Q-r^{7/8})n(\sup \xi > h)} + r^{-\epsilon}.$$

*Démonstration.* Recall the definition of  $Z_1$  in the proof of Lemma 4.2.1. For  $h > 1$  we have

$$\mathbb{P} \left( \sup_{x \in [0, r]} Z(x) \leq h \right) = \mathbb{P} \left( \sup_{x \in [\tau(Z, 1), r]} Z(x) \leq h \right) \geq \mathbb{P} \left( \sup_{x \in [0, r]} Z_1(x) \leq h \right), \quad (4.2.15)$$

because of the Markov property and the fact that the length of  $[\tau(Z, 1), r]$  is less than  $r$ . Let us choose  $\eta \in ]0, 3/4[$ . We have

$$\mathbb{P} \left( \sup_{x \in [\tau(Z, 1), r]} Z(x) \leq h \right) \leq \mathbb{P} \left( \sup_{x \in [0, r-r^\eta]} Z_1(x) \leq h \right) + \mathbb{P}(\tau(Z, 1) > r^\eta). \quad (4.2.16)$$

From (4.2.15), (4.2.16) and (4.2.10), we see that we only need to prove the lemma with  $Z_1$  instead of  $Z$  and  $r^{3/4}$  instead of  $r^{7/8}$ . We only prove the lower bound, since the proof of the upper bound is similar. For  $r$  large enough so that (4.2.5) is true we have

$$\mathbb{P} \left( \sup_{x \in [0, r]} Z_1(x) \leq h \right) \geq \mathbb{P} \left( \sup_{x \in [0, L^{-1}(r/Q+r^{3/4})]} Z_1(x) \leq h \right) - r^{-1/4},$$

and from the point of view of excursions, the probability in the right hand side is only the probability that no excursion higher than  $h$  occurs before  $L$  exceeds  $r/Q + r^{3/4}$ . It is known, from properties of Poisson point processes, that this probability equals  $e^{-(r/Q+r^{3/4})n(\sup \xi > h)}$  and the result follows.  $\square$

## 4.2.2 Proof of Theorems 4.1.1 and 4.1.2

In this subsection we assume that  $\kappa > 1$ .

*Démonstration.* of Theorem 4.1.1

According to (4.2.7), we only need to prove that

$$\frac{\mathcal{L}_X^*(H(r))}{(rm)^{1/\kappa}} \xrightarrow[r \rightarrow +\infty]{\mathcal{L}} \mathcal{F}(\kappa, 2(\Gamma(\kappa)\kappa^2 K/m)^{1/\kappa})$$

or, equivalently,

$$\frac{\mathcal{L}_X^*(H(r))}{r^{1/\kappa}} \xrightarrow[r \rightarrow +\infty]{\mathcal{L}} \mathcal{F}(\kappa, 2(\Gamma(\kappa)\kappa^2 K)^{1/\kappa}). \quad (4.2.17)$$

Now, recall (4.2.2). Combining it with (4.2.3) and (4.2.4), we are left to prove that

$$\forall a > 0, \mathbb{P} \left( \sup_{x \in [0, r]} Z(x) \leq ar^{1/\kappa} \right) \xrightarrow[r \rightarrow +\infty]{} e^{-2^\kappa \Gamma(\kappa) \kappa^2 K / a^\kappa}. \quad (4.2.18)$$

According to Proposition 5.1 of [66] we have

$$n(\sup \xi > h) \underset{h \rightarrow +\infty}{\sim} Q 2^\kappa \Gamma(\kappa) \kappa^2 K / h^\kappa. \quad (4.2.19)$$

Applying this with  $h = ar^{1/\kappa}$  and putting it into Lemma 4.2.3 we get (4.2.18) and the convergence of  $\mathcal{L}_X^*(t)/t^{1/\kappa}$  follows.  $\square$

*Démonstration.* of Theorem 4.1.2

We now study the asymptotic of the favorite site. We first prove that

$$F^*(H(r))/r \xrightarrow[r \rightarrow +\infty]{\mathcal{L}} \mathcal{U}. \quad (4.2.20)$$

$\mathcal{L}_X^*(H(r))$  converges in probability to  $+\infty$  (because of (4.2.17)). Therefore, the combination of (4.2.2), (4.2.3) and (4.2.4) tells us that (4.2.20) will follow if we prove that

$$\operatorname{argmax}_{[0, r]} Z/r \xrightarrow[r \rightarrow +\infty]{\mathcal{L}} \mathcal{U}. \quad (4.2.21)$$

Let us choose  $\gamma \in ]0, 1/8\kappa[$ . Similarly as in the proof of Lemma 4.2.3, we have

$$\mathbb{P} \left( \sup_{[0, L^{-1}(r/Q - r^{7/8})]} Z_1 > r^{1/\kappa - \gamma} \right) = 1 - e^{-(r/Q - r^{7/8})n(\sup \xi > r^{1/\kappa - \gamma})},$$

and using (4.2.19) with  $h = r^{1/\kappa - \gamma}$ , we get that this probability converges to 1 when  $r$  goes to infinity :

$$\mathbb{P} \left( \sup_{x \in [0, L^{-1}(r/Q - r^{7/8})]} Z_1(x) > r^{1/\kappa - \gamma} \right) \xrightarrow[r \rightarrow +\infty]{} 1. \quad (4.2.22)$$

Since  $L^{-1}(r/Q - r^{7/8})$  is a stopping time for  $Z_1$  and is such that  $Z_1(L^{-1}(r/Q - r^{7/8})) = 1$ , we have that

$$\mathbb{P} \left( \sup_{[L^{-1}(r/Q - r^{7/8}), L^{-1}(r/Q + r^{7/8})]} Z_1 < r^{1/\kappa - \gamma} \right) = \mathbb{P} \left( \sup_{[0, L^{-1}(2r^{7/8})]} Z_1 < r^{1/\kappa - \gamma} \right),$$

and we prove, similarly as for (4.2.22), that the probability in the right hand side converges to 1. We thus have

$$\mathbb{P} \left( \sup_{[L^{-1}(r/Q - r^{7/8}), L^{-1}(r/Q + r^{7/8})]} Z_1 < r^{1/\kappa - \gamma} \right) \xrightarrow{r \rightarrow +\infty} 1. \quad (4.2.23)$$

Putting (4.2.22) and (4.2.23) together, we get that, with a probability converging to 1,  $\operatorname{argmax}_{[0, \cdot]} Z_1$  stays constant on  $[L^{-1}(r/Q - r^{7/8}), L^{-1}(r/Q + r^{7/8})]$  :

$$\mathbb{P} \left( \forall x \in [L^{-1}(r/Q - r^{7/8}), L^{-1}(r/Q + r^{7/8})], \operatorname{argmax}_{[0, x]} Z_1 = \operatorname{argmax}_{[0, L^{-1}(r/Q)]} Z_1 \right) \xrightarrow{r \rightarrow +\infty} 1. \quad (4.2.24)$$

Now, note that, on  $\{\tau(Z, 1) < r\}$ , we have  $\operatorname{argmax}_{[0, r]} Z = \operatorname{argmax}_{[0, r - \tau(Z, 1)]} Z_1$ . Then, choose  $\eta \in ]0, 3/4[$ . According to (4.2.10),  $\tau(Z, 1) < r^\eta < r$  with a probability converging to 1 and, applying (4.2.5) we get

$$L^{-1}(r/Q - r^{7/8}) \leq L^{-1}((r - r^\eta)/Q - (r - r^\eta)^{3/4}) \leq r - r^\eta \leq r \leq L^{-1}(r/Q + r^{7/8}),$$

with a probability converging to 1. Combining with (4.2.24) we get

$$\mathbb{P} \left( \operatorname{argmax}_{[0, r]} Z = \operatorname{argmax}_{[0, L^{-1}(r/Q)]} Z_1 \right) \xrightarrow{r \rightarrow +\infty} 1. \quad (4.2.25)$$

Let us denote  $\hat{L} := L(\operatorname{argmax}_{[0, L^{-1}(r/Q)]} Z_1)$ . Considering the Poisson point process of excursions of  $Z_1$  associated with the local time  $L$ ,  $\hat{L}$  is the instant when occurs the highest excursion before the instant  $r/Q$ . It is a well-known property of Poisson point processes that it follows a uniform distribution on  $[0, r/Q]$  :

$$Q\hat{L}/r \stackrel{\mathcal{L}}{=} \mathcal{U}. \quad (4.2.26)$$

For the process  $Z_1$ , this excursion begins at  $L^{-1}(\hat{L}-)$  and ends at  $L^{-1}(\hat{L})$  so

$$L^{-1}(\hat{L}-) \leq \operatorname{argmax}_{[0, L^{-1}(r/Q)]} Z_1 \leq L^{-1}(\hat{L}). \quad (4.2.27)$$

Then, note that arbitrary high excursions of  $Z$  arise if we wait long enough. As a consequence,  $L^{-1}(\hat{L}-)$  converges almost surely to  $+\infty$  when  $r$  goes to infinity. We can thus apply Lemma 4.2.2 and get that almost surely, for  $r$  large enough :

$$Q\hat{L} - \hat{L}^{3/4} \leq L^{-1}(\hat{L}-) \leq \operatorname{argmax}_{[0, L^{-1}(r/Q)]} Z_1 \leq L^{-1}(\hat{L}) \leq Q\hat{L} + \hat{L}^{3/4}.$$

Combining with (4.2.26), we deduce that

$$\operatorname{argmax}_{[0, L^{-1}(r/Q)]} Z_1 / r \xrightarrow{t \rightarrow +\infty} \mathcal{U},$$

and putting it together with (4.2.25), we get (4.2.21) and (4.2.20) follows. We now have to prove the result for the favorite site until a deterministic time, instead of the



hitting time  $H(r)$ . For this, let us choose  $\alpha$  as in Lemma 4.2.1 and  $\gamma \in ]0, (1 - \alpha)/\kappa[$ . We prove that, with high probability, the favorite site remains constant between  $H(r/m - r^\alpha)$  and  $H(r/m + r^\alpha)$ , that is,

$$\mathbb{P}(\forall x \in [H(r/m - r^\alpha), H(r/m + r^\alpha)], F^*(x) = F^*(H(r/m))) \xrightarrow[r \rightarrow +\infty]{} 1. \quad (4.2.28)$$

First, note that, as a consequence of (4.2.20),

$$\mathbb{P}(F^*(H(r/m - r^\alpha)) \leq r/m - 2r^\alpha) \xrightarrow[r \rightarrow +\infty]{} 1, \quad (4.2.29)$$

and, as a consequence of (4.2.17), we have

$$\mathbb{P}(\mathcal{L}_X^*(H(r/m - r^\alpha)) > r^{1/\kappa - \gamma}) \xrightarrow[r \rightarrow +\infty]{} 1. \quad (4.2.30)$$

Then, since  $H(r/m - r^\alpha)$  is a stopping time for the diffusion  $X$  we have

$$\mathbb{P}\left(\inf_{[H(r/m - r^\alpha), +\infty[} X > r/m - 2r^\alpha\right) = \mathbb{P}\left(\inf_{[0, +\infty[} X > -r^\alpha\right) \xrightarrow[r \rightarrow +\infty]{} 1, \quad (4.2.31)$$

and using similarly the Markov property together with (4.2.17), we get

$$\begin{aligned} & \mathbb{P}\left(\sup_{[r/m - 2r^\alpha, r/m + r^\alpha]} \mathcal{L}_X(H(r/m + r^\alpha), \cdot) \leq r^{1/\kappa - \gamma}\right) \\ &= \mathbb{P}\left(\sup_{[0, 3r^\alpha]} \mathcal{L}_X(H(3r^\alpha), \cdot) \leq r^{1/\kappa - \gamma}\right) \xrightarrow[r \rightarrow +\infty]{} 1. \end{aligned} \quad (4.2.32)$$

The four estimates (4.2.29), (4.2.30), (4.2.31) and (4.2.32) tell us that with high probability when  $r$  is large, at time  $H(r/m - r^\alpha)$ , the supremum of the local time has been reached before  $r/m - 2r^\alpha$  and is larger than  $r^{1/\kappa - \gamma}$ , moreover, the diffusion will never reach back  $] -\infty, r/m - 2r^\alpha]$ , and at time  $H(r/m + r^\alpha)$ , the supremum of the local time on  $[r/m - 2r^\alpha, +\infty[$  is less than  $r^{1/\kappa - \gamma}$ . As a consequence, with a probability converging to 1, the favorite site does not move between  $H(r/m - r^\alpha)$  and  $H(r/m + r^\alpha)$ , this proves (4.2.28). Then, (4.2.28) together with (4.2.6) give

$$\mathbb{P}(F^*(r) = F^*(H(r/m))) \xrightarrow[r \rightarrow +\infty]{} 1,$$

and this together with (4.2.20) prove the sought convergence in distribution for the favorite site.  $\square$

### 4.3 Path decomposition of a spectrally negative Lévy process

#### 4.3.1 $h$ -extrema, $h$ -valleys and some processes conditioned to stay positive

Let us recall the notion of  $h$ -extrema which was first introduced by Neveu et al. [53], and studied in the case of drifted Brownian motion by Faggionato [37]. For  $h > 0$ , we say that  $x \in \mathbb{R}$  is an  $h$ -minimum for  $V$  if there exist  $u < x < v$  such that  $V(y) \wedge V(y-) \geq V(x) \wedge V(x-)$  for all  $y \in [u, v]$ ,  $V(u) \geq (V(x) \wedge V(x-)) + h$  and  $V(v-) \geq (V(x) \wedge V(x-)) + h$ . Moreover,  $x$  is an  $h$ -maximum for  $V$  if  $x$  is an  $h$ -minimum for  $-V$ , and  $x$  is an  $h$ -extremum for  $V$  if it is an  $h$ -maximum or an  $h$ -minimum for  $V$ .

Since  $V$  is not a compound Poisson process, it is known (see Proposition VI.4, in [8]) that it takes pairwise distinct values in its local extrema. Combining this with the fact that  $V$  has almost surely càd-làg paths and drifts to  $-\infty$  without being the opposite of a subordinator, we can check that the set of  $h$ -extrema is discrete, forms a sequence indexed by  $\mathbb{Z}$ , unbounded from below and above, and that the  $h$ -minima and  $h$ -maxima alternate.

We denote respectively by  $(m_i, i \in \mathbb{Z})$  and  $(M_i, i \in \mathbb{Z})$  the increasing sequences of  $h$ -minima and of  $h$ -maxima of  $V$ , such that  $m_0 \leq 0 < m_1$  and  $m_i < M_i < m_{i+1}$  for every  $i \in \mathbb{Z}$ .

As in [37], we define the *classical  $h$ -valleys* as the fragments of the trajectory of  $V$  between two  $h$ -maxima, translated at the  $h$ -minima between them : the  $i^{\text{th}}$  classical  $h$ -valley is the process

$$(V^{(i)}(x), M_{i-1} \leq x \leq M_i) \quad \text{where} \quad V^{(i)} := V(x) - V(m_i), \quad \forall x \in \mathbb{R}.$$

In order to state the law of these valleys, we need to recall some definitions about  $V$  and  $\hat{V}$  conditioned to stay positive but first, let us recall a useful fact.

**Fact 4.3.1.** *Let  $Y$  be a spectrally negative Lévy process which is not the opposite of a subordinator, then it is regular for  $]0, +\infty[$  and the regularity for  $] - \infty, 0[$  is equivalent with  $Y$  being of unbounded variations.*

*Démonstration.* The regularity for  $]0, +\infty[$  and the condition for the regularity for  $] - \infty, 0[$  are stated respectively in Theorem VII.1 and in Corollary VII.5 of [8].  $\square$

$V$  being spectrally negative, the Markov family  $(V_x^\uparrow, x \geq 0)$  may be defined as in [8], Section VII.3. For any  $x \geq 0$ , the process  $V_x^\uparrow$  must be seen as  $V$  conditioned to stay positive and starting from  $x$ . We denote  $V^\uparrow$  for the process  $V_0^\uparrow$ . It is known that  $V_x^\uparrow$  converges in the Skorokhod space to  $V^\uparrow$  when  $x$  goes to 0. Also, as well as  $V$ ,  $V^\uparrow$  has no positive jumps and reaches every positive level continuously.

According to Fact 4.3.1,  $V$  is regular for  $]0, +\infty[$ , so  $\hat{V}$  is for  $] -\infty, 0[$ . Moreover,  $\hat{V}$  drifts to  $+\infty$ . We can thus define the Markov family  $(\hat{V}_x^\uparrow, x \geq 0)$  as in Doney [31], Chapter 8. It can be seen from there that the processes thus defined are Markov and have infinite life-time. If moreover  $V$  has unbounded variations then  $\hat{V}$  is regular for  $]0, +\infty[$ , and from Theorem 24 of [31], we have that  $\hat{V}_0^\uparrow$ , that we shall denote by  $\hat{V}^\uparrow$ , is well defined.

Here again, for any  $x \geq 0$ , the process  $\hat{V}_x^\uparrow$  must be seen as  $\hat{V}$  conditioned to stay positive and starting from  $x$ . Note that, since  $\hat{V}$  converges almost surely to infinity, for  $x > 0$ ,  $\hat{V}_x^\uparrow$  is only  $\hat{V}_x$  conditioned in the usual sense to remain positive.

### 4.3.2 Law of the valleys

We now prove some facts about the law of the consecutive valleys near their bottom. In order to delimit the bottom of the valleys we define

$$\tau_i^-(h) := \sup\{x < m_i, V^{(i)}(x) \geq h\}, \quad \tau_i(h) := \inf\{x > m_i, V^{(i)}(x) = h\}.$$

For any  $i \in \mathbb{N}^*$ , let  $P_1^{(i)}$  be the truncated process  $(V^{(i)}(m_i - x), 0 \leq x \leq m_i - \tau_i^-(h))$ , and  $P_2^{(i)}$  the truncated process  $(V^{(i)}(m_i + x), 0 \leq x \leq \tau_i(h) - m_i)$ . We have,

**Proposition 4.3.2.** (*valleys decomposition*)

Assume  $V$  has unbounded variations. All the processes of the family  $(P_j^{(i)}, i \geq 1, j \in \{1, 2\})$ , are independent and :

- For all  $i \geq 2$ , the law of  $P_1^{(i)}$  is absolutely continuous with respect to the law of the process  $(\hat{V}^\uparrow(x))_{0 \leq x \leq \tau(\hat{V}^\uparrow, h+)}$  and has density  $c_h / (1 - e^{-\kappa \hat{V}^\uparrow(\tau(\hat{V}^\uparrow, h+) )})$  with respect to this law, where  $c_h$  is a constant depending on  $h$ .
- For all  $i \geq 1$ ,  $P_2^{(i)}$  is equal in law to  $(V^\uparrow(x))_{0 \leq x \leq \tau(V^\uparrow, h)}$  (this statement is true even if  $V$  has bounded variations).

Moreover, the density  $c_h / (1 - e^{-\kappa \hat{V}^\uparrow(\tau(\hat{V}^\uparrow, h+) )})$  is bounded by 2 for  $h$  deterministically large enough and it converges to 1 when  $h$  goes to infinity.

**Remark 4.3.3.**  $P_1^{(1)}$  may be a part of the so-called central slope, so its law is different.

*Démonstration.* We assume  $V$  has unbounded variations, recall from Fact 4.3.1 that this implies a regularity condition. In Lemma 4 of Cheliotis [21], the assertion that  $\underline{\lim}_{t \rightarrow +\infty} X_t = -\infty, \overline{\lim}_{t \rightarrow +\infty} X_t = +\infty$  can be dropped. Indeed, because of the regularity condition, the stopping times  $\underline{\tau}_k$  and  $\overline{\tau}_k$  are almost surely finite. This lemma is therefore true in our context. As a consequence, the proof of Lemma 1 of [21] also applies in our context. We thus get the fact that the slopes are independent and that (except the central slope), all descending (respectively ascending) slopes have the same law.

### 4.3. PATH DECOMPOSITION OF A SPECTRALLY NEGATIVE LÉVY PROCESS

For the law of the ascending slopes (not covering the origin) until their hitting time of  $h$ , we use the classical argument that can be found, for example, in the proof of Theorem 2 in [37] : the law of  $P_2^{(i)}$  is the law, before its hitting time of  $h$ , of the first excursion higher than  $h$  of  $V - \underline{V}$ , and using Proposition VII.15 of [8] (which does not require  $V$  to have unbounded variations), we prove that the latter is the law of  $(V^\uparrow(x))_{0 \leq x \leq \tau(V^\uparrow, h)}$ , that is,  $V^\uparrow$  killed when hitting  $h$ .

Then, by the time-reverse property the descending slopes (not covering the origin) have the same law as the ascending slopes of  $\hat{V}$  so, here again, we get that the law of  $P_1^{(i)}$  is the law, before its hitting time of  $[h, +\infty[$ , of the first excursion higher than  $h$  of  $\hat{V} - \hat{\underline{V}}$ . Unfortunately, since  $\hat{V}$  is not spectrally negative, we can no longer apply the previous argument to determine the law of the excursion. However, since  $] - \infty, 0[$  and  $]0, +\infty[$  are regular for  $\hat{V}$  (because they are for  $V$ ) and  $\hat{V}$  does not drift to  $-\infty$ , we can apply Proposition 4.7 of Duquesne [33] but we first need to introduce some notations.

Let  $(\hat{L}^{-1}, \hat{I})$  denote the ladder process of  $\hat{V}$ , of which the definition has been adapted to our setting :  $\hat{L}^{-1}$  is the inverse of a local time at 0, denoted by  $\hat{L}$ , of  $\hat{V} - \hat{\underline{V}}$ , and for any positive  $t$ ,  $\hat{I}(t) = -\hat{\underline{V}}(\hat{L}^{-1}(t))$ . We denote by  $\hat{\mathcal{N}}$  the excursion measure of  $\hat{V} - \hat{\underline{V}}$  associated with  $\hat{L}$ , and for  $\xi$  an excursion, let  $\zeta(\xi) := \inf\{s > 0, \xi(s) = 0\}$  denote its life-time.

We denote by  $\hat{\mathcal{I}}$  the potential measure of  $\hat{I}$ , and, since  $\hat{V}$  is spectrally positive, it can be seen from [8], Section VII.1, that  $\hat{\mathcal{I}}([0, x]) = (1 - e^{-\kappa x})/\kappa$  for any  $x \geq 0$ . Proposition 4.7 of [33] tells us that for any measurable function  $G$  defined on the space of càd-làg functions from  $[0, +\infty[$  to  $\mathbb{R}$  with possibly finite life-time, we have that  $\mathbb{E}[G((\hat{V}^\uparrow(s))_{0 \leq s \leq \tau(\hat{V}^\uparrow, h_+)})]$  equals

$$\begin{aligned} & \hat{\mathcal{N}} \left( G \left( (\xi(s))_{0 \leq s \leq \tau(\xi, h_+)} \right) \hat{\mathcal{I}}([0, \xi(\tau(\xi, h_+))] \mid \tau(\xi, h_+) < \zeta(\xi) \right) \times \hat{\mathcal{N}}(\xi, \tau(\xi, h_+) < \zeta(\xi)) \\ &= \hat{\mathcal{N}} \left( \frac{(1 - e^{-\kappa \xi(\tau(\xi, h_+))})}{c_h} G \left( (\xi(s))_{0 \leq s \leq \tau(\xi, h_+)} \right) \mid \tau(\xi, h_+) < \zeta(\xi) \right) \end{aligned}$$

where we have set  $c_h := \kappa/\hat{\mathcal{N}}(\xi, \tau(\xi, h_+) < \zeta(\xi))$ . So, for any measurable function  $F$ , we get that

$$\begin{aligned} & \hat{\mathcal{N}} \left( F \left( (\xi(s))_{0 \leq s \leq \tau(\xi, h_+)} \right) \mid \tau(\xi, h_+) < \zeta(\xi) \right) \\ &= \mathbb{E} \left[ \frac{c_h}{1 - e^{-\kappa \hat{V}^\uparrow(\tau(\hat{V}^\uparrow, h_+))}} F \left( (\hat{V}^\uparrow(s))_{0 \leq s \leq \tau(\hat{V}^\uparrow, h_+)} \right) \right]. \end{aligned}$$

That is, the law of the first excursion higher than  $h$ , and killed when reaching  $[h, +\infty[$ , of the process  $\hat{V} - \hat{\underline{V}}$  is absolutely continuous with respect to the law of the process  $(\hat{V}^\uparrow(s))_{0 \leq s \leq \tau(\hat{V}^\uparrow, h_+)}$  and has density  $c_h/(1 - e^{-\kappa \hat{V}^\uparrow(\tau(\hat{V}^\uparrow, h_+))})$  with respect to this law.

For the last assertion of the proposition, note that  $c_h = \kappa/\hat{\mathcal{N}}(\xi, \tau(\xi, h_+) < \zeta(\xi))$  increases, when  $h$  goes to infinity, to  $c_\infty := \kappa/\hat{\mathcal{N}}(\xi, \zeta(\xi) = \zeta(\xi))$ , where  $\hat{\mathcal{N}}(\xi, \zeta(\xi) =$

$+\infty$ ) is the measure of infinite excursions above 0 for  $\hat{V} - \underline{\hat{V}}$ , which is strictly positive, because  $\hat{V}$  almost surely converges to infinity. Also, we have almost surely that  $\hat{V}^\uparrow(\tau(\hat{V}^\uparrow, h+)) \geq h$ . Hence, for  $h$  deterministically large enough we have

$$c_h / (1 - e^{-\kappa \hat{V}^\uparrow(\tau(\hat{V}^\uparrow, h+))}) \leq 2c_\infty.$$

Then,

$$\mathbb{P} \left( c_h / (1 - e^{-\kappa \hat{V}^\uparrow(\tau(\hat{V}^\uparrow, h+))}) \xrightarrow{h \rightarrow +\infty} c_\infty \right) = 1,$$

so by the dominated convergence theorem we get  $c_\infty = 1$  and the last assertion of the proposition follows.  $\square$

### 4.3.3 Standard valleys

Since  $V$  drifts to  $-\infty$ , the descending phases of  $V$  between two  $h$ -minima are quite important and have to be taken in consideration for the study of the diffusion in  $V$ . Therefore, as in [4], we here define a new sequence  $(\tilde{m}_i)_{i \geq 1}$  of  $h$ -minima that are separated by descending phases of  $V$  and we then show that for a large number of indices, this sequence coincides with the sequence  $(m_i)_{i \geq 1}$  with an overwhelming probability. Our definition of the standard valleys are similar to the one given in [4] but we have to improve it to get a definition more adapted to our context. We first introduce some notations.

$\delta > 0$  is defined once and for all in the paper and can be chosen as small as we want. In this subsection  $h$  is a fixed positive number such that  $e^{(1-\delta)\kappa h} \geq h$ . We define  $\tilde{\tau}_0(h) = \tilde{L}_0 := 0$  and recursively for  $i \geq 1$ ,

$$\begin{aligned} \tilde{L}_i^\# &:= \inf \{x > \tilde{L}_{i-1}, V(x) \leq V(\tilde{L}_{i-1}) - e^{(1-\delta)\kappa h}\}, \\ \tilde{\tau}_i(h) &:= \inf \{x \geq \tilde{L}_i^\#, V(x) - \inf_{[\tilde{L}_i^\#, x]} V = h\}, \\ \tilde{m}_i &:= \inf \{x \geq \tilde{L}_i^\#, V(x) = \inf_{[\tilde{L}_i^\#, \tilde{\tau}_i(h)]} V\}, \\ \tilde{L}_i &:= \inf \{x > \tilde{\tau}_i(h), V(x) \leq h/2\}, \\ \tilde{\tau}_i^-(a) &:= \sup \{x < \tilde{m}_i, V(x) - V(\tilde{m}_i) \geq a\}, \quad \forall a \in [0, h], \\ \tilde{\tau}_i^+(a) &:= \inf \{x > \tilde{m}_i, V(x) - V(\tilde{m}_i) = a\}, \quad \forall a \in [0, h]. \end{aligned}$$

Note that all these random variables depend on  $h$ , even if this does not appear in the notations. We also introduce the equivalent of  $V^{(i)}$  for the  $\tilde{m}_i, i \in \mathbb{N}^*$  as follows :

$$\tilde{V}^{(i)}(x) := V(x) - V(\tilde{m}_i), \quad \forall x \in \mathbb{R}.$$

We call  $i^{\text{th}}$  *standard valley* the re-centered truncated potential  $(\tilde{V}^{(i)}(x), \tilde{L}_{i-1} \leq x \leq \tilde{L}_i)$ .

**Remark 4.3.4.** *The random times  $\tilde{L}_i^\#, \tilde{\tau}_i(h)$ , and  $\tilde{L}_i$  are stopping times. As a consequence, the sequence  $(\tilde{V}^{(i)}(x + \tilde{m}_i), \tilde{L}_{i-1} - \tilde{m}_i \leq x \leq \tilde{L}_i - \tilde{m}_i)_{i \geq 1}$  is iid.*

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Our definitions take in consideration the absence of positive jumps for  $V$ , in particular  $\tilde{\tau}_i(h) < +\infty$  and  $V(\tilde{\tau}_i(h)-) = V(\tilde{\tau}_i(h)) = V(\tilde{m}_i) + h$ . We can see that the  $\tilde{m}_i$ ,  $i \in \mathbb{N}^*$ , are  $h$ -minima. The next lemma, which is the analogue of Lemma 2.3 in Andreoletti, Devulder [3], shows that, with high probability, the sequence  $(\tilde{m}_i)_{i \geq 1}$  coincides with the sequence  $(m_i)_{i \geq 1}$  for indices  $i \leq n$  when  $n$  does not grow too fast with  $h$ .

**Lemma 4.3.5.** *For all  $n \geq 1$  and  $h$  large enough,*

$$\mathbb{P} \left( \mathcal{V}_{n,h} := \bigcap_{i=1}^n \{m_i = \tilde{m}_i\} \right) \geq 1 - ne^{-\delta\kappa h/3}.$$

*Démonstration.* As we said before,  $\{\tilde{m}_i, i \in \mathbb{N}^*\} \subset \{m_i, i \in \mathbb{N}^*\}$ . Hence on the complementary  $\mathcal{V}_{n,h}^c$  of  $\mathcal{V}_{n,h}$ , considering the smallest  $1 \leq i \leq n$  such that  $m_i \neq \tilde{m}_i$ , we have  $m_{i-1} = \tilde{m}_{i-1} < m_i < \tilde{m}_i$ .

For this  $i$ , we can not have  $m_i \in [\tilde{L}_i^\sharp, \tilde{m}_i[$ . Indeed, in this case,  $m_i - \tilde{L}_i^\sharp$  would be the starting point of an excursion higher than  $h$  for  $V^{\tilde{L}_i^\sharp} - \underline{V}^{\tilde{L}_i^\sharp}$ , but from the definitions,  $\tilde{m}_i - \tilde{L}_i^\sharp$  is the starting point of the first excursion higher than  $h$  for  $V^{\tilde{L}_i^\sharp} - \underline{V}^{\tilde{L}_i^\sharp}$ , which contradicts  $m_i < \tilde{m}_i$ . Hence,  $m_i \in ]\tilde{m}_{i-1}, \tilde{L}_i^\sharp[$ .

$\tilde{m}_{i-1}$  is an  $h$ -minimum and there cannot be any  $h$ -maximum belonging to  $[\tilde{m}_{i-1}, \tilde{\tau}_{i-1}(h)[$  so, as the  $h$  minima and the  $h$  maxima alternate, necessarily we have  $m_i > \tilde{\tau}_{i-1}(h)$ , so  $m_i \in ]\tilde{\tau}_{i-1}(h), \tilde{L}_i^\sharp[$ .

Since  $m_i$  is an  $h$ -minimum, there is  $v_i > m_i$  such that  $V(m_i) = \inf_{[m_i, v_i]} V$  and  $V(v_i) = V(m_i) + h$ . Since by definition of  $\tilde{L}_i^\sharp$  we have  $V(\tilde{L}_i^\sharp) = \inf_{[\tilde{\tau}_{i-1}(h), \tilde{L}_i^\sharp]} V$ , we cannot have  $\tilde{L}_i^\sharp \in ]m_i, v_i]$  so we must have

$$\tilde{\tau}_{i-1}(h) < m_i < v_i < \tilde{L}_i^\sharp.$$

For similar reasons we can neither have  $\tilde{L}_{i-1} \in ]m_i, v_i]$  so

$$\tilde{\tau}_{i-1}(h) < m_i < v_i < \tilde{L}_{i-1} \quad \text{or} \quad \tilde{L}_{i-1} \leq m_i < v_i < \tilde{L}_i^\sharp.$$

We have proved that

$$\mathcal{V}_{n,h}^c = \left( \bigcap_{i=1}^n \{m_i = \tilde{m}_i\} \right)^c \subset \bigcup_{i=1}^n (E_i^1 \cup E_i^2),$$

where

$$\begin{aligned} E_i^1 &:= \left\{ \tau \left( V^{\tilde{\tau}_{i-1}(h)} - \underline{V}^{\tilde{\tau}_{i-1}(h)}, h \right) < \tau \left( \underline{V}^{\tilde{\tau}_{i-1}(h)}, ] - \infty, -h/2 \right] \right\}, \\ E_i^2 &:= \left\{ \tau \left( V^{\tilde{L}_{i-1}} - \underline{V}^{\tilde{L}_{i-1}}, h \right) < \tau \left( \underline{V}^{\tilde{L}_{i-1}}, ] - \infty, -e^{(1-\delta)\kappa h} \right] \right\}. \end{aligned}$$

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Since  $\tilde{\tau}_{i-1}(h)$  and  $\tilde{L}_{i-1}$  are stopping times, we get for  $h$  large enough :

$$1 - \mathbb{P} \left( \bigcap_{i=1}^n \{m_i = \tilde{m}_i\} \right) \leq 2n\mathbb{P} \left( \tau(V - \underline{V}, h) < \tau(\underline{V}, ] - \infty, -e^{(1-\delta)\kappa h}) \right),$$

and the result follows for  $h$  large enough according to Lemma 4.5.5, applied with  $a = h$ ,  $b = e^{(1-\delta)\kappa h}$  and  $\eta = \delta/2$ . □

We now make use of the preceding lemma to precise the law of the bottoms of the standard valleys. First, for any  $i \in \mathbb{N}^*$ , let us define

$$\begin{aligned} \tilde{P}_1^{(i)} &:= (\tilde{V}^{(i)}(\tilde{m}_i - x), 0 \leq x \leq \tilde{m}_i - \tilde{\tau}_i^-(h)), \\ \tilde{P}_2^{(i)} &:= (\tilde{V}^{(i)}(\tilde{m}_i + x), 0 \leq x \leq \tilde{\tau}_i(h) - \tilde{m}_i), \\ \tilde{P}_3^{(i)} &:= (\tilde{V}^{(i)}(\tilde{\tau}_i(h) + x), 0 \leq x \leq \tilde{L}_i - \tilde{\tau}_i(h)). \end{aligned}$$

Note that for any index  $i$ ,  $m_i = \tilde{m}_i$  implies  $P_1^{(i)} = \tilde{P}_1^{(i)}$  and  $P_2^{(i)} = \tilde{P}_2^{(i)}$ .

**Proposition 4.3.6.** *Assume  $V$  has unbounded variations. All the processes of the family  $(\tilde{P}_j^{(i)}, i \geq 1, j \in \{1, 2, 3\})$ , are independent and for all  $i \geq 1$ ,*

$$\begin{aligned} d_{VT} \left( \tilde{P}_1^{(i)}, P_1^{(2)} \right) &\leq 2e^{-\delta\kappa h/3}, & \tilde{P}_2^{(i)} &\stackrel{\mathcal{L}}{=} (V^\uparrow(x))_{0 \leq x \leq \tau(V^\uparrow, h)}, \\ \tilde{P}_3^{(i)} &\stackrel{\mathcal{L}}{=} (h + V(x))_{0 \leq x \leq \tau(V, ] - \infty, -h/2)}. \end{aligned}$$

The statements about the laws of  $\tilde{P}_2^{(i)}$  and  $\tilde{P}_3^{(i)}$  do not require the hypothesis of unbounded variations. The statements about the laws of  $\tilde{P}_1^{(i)}$  is true for  $h$  large enough.

*Démonstration.* For the independence, we use Remark 4.3.4, so it only remains to prove that for any  $i \geq 1$ ,  $\tilde{P}_1^{(i)}$ ,  $\tilde{P}_2^{(i)}$  and  $\tilde{P}_3^{(i)}$  are independent.  $\tilde{P}_2^{(i)}$  is, after a stopping time, the first excursion of  $V - \underline{V}$  greater than  $h$ , considered up to its hitting time of  $h$ . It is therefore independent from the previous slopes and, using again Proposition VII.15 of [8] (which does not require  $V$  to have unbounded variations), we see that it has the same law as  $(V^\uparrow(x))_{0 \leq x \leq \tau(V^\uparrow, h)}$ . Also, the Markov property applied at time  $\tilde{\tau}_i(h)$  gives the asserted law of  $\tilde{P}_3^{(i)}$  and its independence from  $(\tilde{P}_1^{(i)}, \tilde{P}_2^{(i)})$ .

For the assertion on  $\tilde{P}_1^{(i)}$ , note that for  $h$  large enough,  $\mathbb{P}(\tilde{P}_1^{(2)} \neq P_1^{(2)}) \leq 2e^{-\delta\kappa h/3}$  according to Lemma 4.3.5 (applied with  $n := 2$ ) and that for any  $i \geq 1$ ,  $\tilde{P}_1^{(i)}$  is equal in law to  $\tilde{P}_1^{(2)}$  according to Remark 4.3.4. The assertion follows. □

We now consider the first ascend of  $h$  from the minimum, after  $\tilde{\tau}_i(h)$ ,

$$\begin{aligned} \tilde{\tau}_{i+1}^*(h) &:= \inf \left\{ u \geq \tilde{\tau}_i(h), V(u) - \inf_{[\tilde{\tau}_i(h), u]} V = h \right\}, \\ \tilde{m}_{i+1}^* &:= \inf \left\{ u \geq \tilde{\tau}_i(h), V(u) = \inf_{[\tilde{\tau}_i(h), \tilde{\tau}_{i+1}^*(h)]} V \right\}. \end{aligned}$$

We prove that these coincide with  $\tilde{\tau}_{i+1}(h)$  and  $\tilde{m}_{i+1}$  with a good probability.

**Lemma 4.3.7.** *There is a positive constant  $c$  such that for  $h$  large enough,*

$$\forall i \geq 1, \mathbb{P}(\tilde{\tau}_i^*(h) = \tilde{\tau}_i(h), \tilde{m}_i^* = \tilde{m}_i) \geq 1 - e^{-ch}.$$

*Démonstration.* Note that  $\tilde{m}_i^* = \tilde{m}_i$  whenever  $\tilde{\tau}_i^*(h) = \tilde{\tau}_i(h)$ . We thus only prove the latter. Fix  $i \geq 2$  and recall the definitions of  $\tilde{\tau}_i^*(h)$  and  $\tilde{\tau}_i(h)$ . We have  $V(\tilde{L}_i^\#) = \inf_{[\tilde{\tau}_{i-1}(h), \tilde{L}_i^\#]} V$  so, if  $\tilde{L}_i^\# < \tilde{\tau}_i^*(h)$ , we must have  $\tilde{\tau}_i^*(h) = \tilde{\tau}_i(h)$ . Therefore,

$$\{\tilde{\tau}_i^*(h) \neq \tilde{\tau}_i(h)\} \subset \left\{ \tilde{\tau}_i^*(h) \leq \tilde{L}_i^\# \right\} = \left\{ \tilde{\tau}_i^*(h) \leq \tilde{L}_{i-1} \right\} \cup \left\{ \tilde{L}_{i-1} < \tilde{\tau}_i^*(h) \leq \tilde{L}_i^\# \right\}. \quad (4.3.1)$$

Then, from the Markov property at time  $\tilde{\tau}_{i-1}(h)$  and the definitions of  $\tilde{L}_{i-1}$  and  $\tilde{\tau}_i^*(h)$  we get

$$\mathbb{P}\left(\tilde{\tau}_i^*(h) \leq \tilde{L}_{i-1}\right) = \mathbb{P}\left(\tau(V - \underline{V}, h) < \tau(\underline{V}, ] - \infty, -h/2])\right) \leq 2e^{-\kappa h/2}, \quad (4.3.2)$$

where the last inequality comes from Lemma 4.5.5 applied with  $a = h, b = h/2$  and  $\eta = 1/4$ . Then,  $V(\tilde{L}_{i-1}) = \inf_{[\tilde{\tau}_{i-1}(h), \tilde{L}_{i-1}]} V$  so, if  $\tilde{\tau}_i^*(h) > \tilde{L}_{i-1}$  we have  $\tilde{\tau}_i^*(h) - \tilde{L}_{i-1} = \tau(V^{\tilde{L}_{i-1} - \underline{V}^{\tilde{L}_{i-1}}}, h)$ . Combining this with the definition of  $\tilde{L}_i^\#$  and the Markov property at time  $\tilde{L}_{i-1}$  we get

$$\mathbb{P}\left(\tilde{L}_{i-1} < \tilde{\tau}_i^*(h) \leq \tilde{L}_i^\#\right) \leq \mathbb{P}\left(\tau(V - \underline{V}, h) < \tau(\underline{V}, ] - \infty, -e^{(1-\delta)\kappa h})\right) \leq e^{-\delta\kappa h/3}, \quad (4.3.3)$$

where the last inequality is true for  $h$  large enough, according to Lemma 4.5.5 applied with  $a = h, b = e^{(1-\delta)\kappa h}$  and  $\eta = \delta/2$ .

Putting together (4.3.1), (4.3.2) and (4.3.3) we get the result when  $i \geq 2$ . If  $i = 1$ , (4.3.1) and (4.3.3) are true but recall that  $\tilde{\tau}_0(h) = \tilde{L}_0 = 0$ , so  $\{\tilde{\tau}_1^*(h) \leq \tilde{L}_0\} = \emptyset$ . The result is therefore also true for  $i = 1$ . □

#### 4.3.4 Exponential functionals of the bottom of a standard valley

Let  $J(h)$  be the exponential functional of the bottom of the first standard valley :

$$J(h) := \int_{\tau_1^-(h/2)}^{\tau_1^+(h/2)} e^{-\hat{V}^{(1)}(u)} du.$$

We now make use of Propositions 4.3.2 and 4.3.6 and of the existence, proved in [72], of some finite exponential moments for  $I(V^\uparrow)$  and  $I(\hat{V}^\uparrow)$  to prove a very tight convergence of  $J(h)$  to  $\mathcal{R}$ . This result is crucial in Section 4.4 where we make  $\mathcal{R}$  appear in the limit distribution of the supremum of the local time.



**Proposition 4.3.8.** *Assume  $V$  has unbounded variations. The family of random variables  $(J(h))_{h>0}$  converges in distribution to  $\mathcal{R}$  and there exists a positive  $\lambda_0$  such that*

$$\forall \lambda < \lambda_0, \quad \mathbb{E} [e^{\lambda J(h)}] \xrightarrow{h \rightarrow +\infty} \mathbb{E} [e^{\lambda \mathcal{R}}], \quad (4.3.4)$$

and the above quantities are all finite. As a consequence, the moments of any positive order of  $J(h)$  converge to those of  $\mathcal{R}$  when  $h$  goes to infinity.

*Démonstration.* We consider a probability space on which are defined two independent processes  $Z_1$  and  $Z_2$  with  $Z_1 \stackrel{\mathcal{L}}{=} \hat{V}^\uparrow$  and  $Z_2 \stackrel{\mathcal{L}}{=} V^\uparrow$ , and we define

$$\begin{aligned} \tilde{I}(h) &:= \int_0^{\tau(Z_1, h/2+)} e^{-Z_1(x)} dx + \int_0^{\tau(Z_2, h/2)} e^{-Z_2(x)} dx, \quad \forall h > 0 \\ \tilde{\mathcal{R}} &:= \int_0^{+\infty} e^{-Z_1(x)} dx + \int_0^{+\infty} e^{-Z_2(x)} dx. \end{aligned}$$

We have trivially the equality in law  $\tilde{\mathcal{R}} \stackrel{\mathcal{L}}{=} \mathcal{R}$  and the almost sure increase of the family  $(\tilde{I}(h))_{h>0}$  to  $\tilde{\mathcal{R}}$ . Then, from the definitions of  $\tilde{P}_1^{(1)}$  and  $\tilde{P}_2^{(1)}$ ,

$$J(h) = \int_0^{\tau(\tilde{P}_1^{(1)}, h/2+)} e^{-\tilde{P}_1^{(1)}(u)} du + \int_0^{\tau(\tilde{P}_2^{(1)}, h/2)} e^{-\tilde{P}_2^{(1)}(u)} du.$$

According to Proposition 4.3.6, the two terms are independent and the second is equal in law to the second term of  $\tilde{I}(h)$ . It is easy to see that, having four random variables  $A, B, C_A$  and  $C_B$  where  $A$  and  $C_A$  (respectively  $B$  and  $C_B$ ) are defined on the same probability space and independent, and such that  $C_A$  and  $C_B$  have the same law, then we have this inequality for the total variation distance :

$$d_{VT}(C_A + A, C_B + B) \leq d_{VT}(A, B).$$

In our case, we thus have

$$d_{VT}(J(h), \tilde{I}(h)) \leq d_{VT} \left( \int_0^{\tau(\tilde{P}_1^{(1)}, h/2+)} e^{-\tilde{P}_1^{(1)}(u)} du, \int_0^{\tau(Z_1, h/2+)} e^{-Z_1(x)} dx \right). \quad (4.3.5)$$

Then, we see that if  $A$  and  $B$  are random variables (valued on a metric space  $E$ ), and  $f$  is a measurable mapping from  $E$  to  $\mathbb{R}$ , we have

$$d_{VT}(f(A), f(B)) \leq d_{VT}(A, B).$$

In our case, this yields

$$d_{VT} \left( \int_0^{\tau(\tilde{P}_1^{(1)}, h/2+)} e^{-\tilde{P}_1^{(1)}(u)} du, \int_0^{\tau(Z_1, h/2+)} e^{-Z_1(x)} dx \right) \leq d_{VT} \left( \tilde{P}_1^{(1)}, (Z_1(x))_{0 \leq x \leq \tau(Z_1, h+)} \right). \quad (4.3.6)$$

From the triangular inequality and Proposition 4.3.6, we have for  $h$  large enough,

$$d_{VT} \left( \tilde{P}_1^{(1)}, (Z_1(x))_{0 \leq x \leq \tau(Z_1, h+)} \right) \leq d_{VT} \left( \tilde{P}_1^{(1)}, P_1^{(2)} \right) + d_{VT} \left( P_1^{(2)}, (Z_1(x))_{0 \leq x \leq \tau(Z_1, h+)} \right) \leq 2e^{-\delta\kappa h/3} + d_{VT} \left( P_1^{(2)}, (Z_1(x))_{0 \leq x \leq \tau(Z_1, h+)} \right). \quad (4.3.7)$$

Now, using Proposition 4.3.2,  $P_1^{(2)}$  is absolutely continuous with respect to the law of the process  $(\hat{V}^\uparrow(s))_{0 \leq s \leq \tau(\hat{V}^\uparrow, h+)}$  and has density  $c_h / (1 - e^{-\kappa \hat{V}^\uparrow(\tau(\hat{V}^\uparrow, h+) )})$  with respect to this law. It is known that, if a random variable  $B$  has a density  $d_B$  with respect to a random variable  $A$  (both valued in a metric space), then, their total variation distance is expressed as follow :

$$d_{VT}(A, B) = \frac{1}{2} \int |1 - d_B| d\mathcal{L}_A.$$

That is, in our case,

$$d_{VT} \left( P_1^{(2)}, (Z_1(x))_{0 \leq x \leq \tau(Z_1, h+)} \right) = \frac{1}{2} \mathbb{E} \left[ \left| 1 - \frac{c_h}{1 - e^{-\kappa \hat{V}^\uparrow(\tau(\hat{V}^\uparrow, h+) )}} \right| \right]. \quad (4.3.8)$$

Now combining (4.3.5), (4.3.6), (4.3.7) and (4.3.8) we get for large  $h$ ,

$$d_{VT} \left( J(h), \tilde{I}(h) \right) \leq 2e^{-\delta\kappa h/3} + \frac{1}{2} \mathbb{E} \left[ \left| 1 - \frac{c_h}{1 - e^{-\kappa \hat{V}^\uparrow(\tau(\hat{V}^\uparrow, h+) )}} \right| \right],$$

and because of the last assertion in Proposition 4.3.2 and dominated convergence, the right-hand-side converges to 0 when  $h$  goes to infinity. From this and the convergence of  $(\tilde{I}(h))_{h>0}$  to  $\tilde{\mathcal{R}}$  we deduce that  $(J(h))_{h>0}$  converges in distribution to  $\tilde{\mathcal{R}}$  (and therefore to  $\mathcal{R}$ ). In order to prove (4.3.4), it only remain to prove a uniform integrability condition. In particular, (4.3.4) will follow if we prove the existence of a positive  $\lambda_0$  and a positive finite constant  $C$  such that

$$\forall h > 0, \quad \mathbb{E} \left[ e^{\lambda_0 \tilde{J}(h)} \right] \leq C. \quad (4.3.9)$$

Thanks to Theorems 1.1 and 1.13 of [72], the positive random variables  $I(V^\uparrow)$  and  $I(\hat{V}^\uparrow)$  admit some finite exponential moments. We can therefore choose a positive  $\lambda_0$  such that

$$\mathbb{E} \left[ e^{\lambda_0 I(V^\uparrow)} \right] < +\infty \quad \text{and} \quad \mathbb{E} \left[ e^{2\lambda_0 I(\hat{V}^\uparrow)} \right] < +\infty. \quad (4.3.10)$$

For any such choice of  $\lambda_0$  and  $h > 0$ ,

$$\begin{aligned} \mathbb{E} \left[ e^{\lambda_0 \tilde{J}(h)} \right] &= \mathbb{E} \left[ \exp \left( \lambda_0 \int_0^{\tau(\tilde{P}_1^{(1)}, h/2+)} e^{-\tilde{P}_1^{(1)}(u)} du \right) \right] \\ &\times \mathbb{E} \left[ \exp \left( \lambda_0 \int_0^{\tau(\tilde{P}_2^{(1)}, h/2)} e^{-\tilde{P}_2^{(1)}(u)} du \right) \right]. \end{aligned} \quad (4.3.11)$$

Then, according to Proposition 4.3.6,

$$\mathbb{E} \left[ \exp \left( \lambda_0 \int_0^{\tau(\tilde{P}_2^{(1)}, h/2)} e^{-\tilde{P}_2^{(1)}(u)} du \right) \right] = \mathbb{E} \left[ \exp \left( \lambda_0 \int_0^{\tau(V^\uparrow, h/2)} e^{-V^\uparrow(u)} du \right) \right] \leq \mathbb{E} \left[ e^{\lambda_0 I(V^\uparrow)} \right]. \quad (4.3.12)$$

From Remark 4.3.4 and the fact that  $(\tilde{m}_i)_{i \geq 1}$  is a subsequence of  $(m_i)_{i \geq 1}$  we have,

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( \lambda_0 \int_0^{\tau(\tilde{P}_1^{(1)}, h/2+)} e^{-\tilde{P}_1^{(1)}(u)} du \right) \right] \\ &= \mathbb{E} \left[ \exp \left( \lambda_0 \int_0^{\tau(\tilde{P}_1^{(2)}, h/2+)} e^{-\tilde{P}_1^{(2)}(u)} du \right) \right] \\ &= \mathbb{E} \left[ \exp \left( \lambda_0 \int_0^{\tau(P_1^{(2)}, h/2+)} e^{-P_1^{(2)}(u)} du \right) \mathbf{1}_{\tilde{P}_1^{(2)} = P_1^{(2)}} \right] \\ &+ \sum_{i>2} \mathbb{E} \left[ \exp \left( \lambda_0 \int_0^{\tau(P_1^{(i)}, h/2+)} e^{-P_1^{(i)}(u)} du \right) \mathbf{1}_{\tilde{P}_1^{(2)} = P_1^{(i)}} \right] \\ &\leq \mathbb{E} \left[ \exp \left( \lambda_0 \int_0^{\tau(P_1^{(2)}, h/2+)} e^{-P_1^{(2)}(u)} du \right) \right] \\ &+ \sum_{i>2} \sqrt{\mathbb{E} \left[ \exp \left( 2\lambda_0 \int_0^{\tau(P_1^{(i)}, h/2+)} e^{-P_1^{(i)}(u)} du \right) \right]} \sqrt{\mathbb{P} \left( \tilde{P}_1^{(2)} = P_1^{(i)} \right)}. \end{aligned}$$

Now, according to Proposition 4.3.2, the two expectations are (for large  $h$ ) less than respectively  $2\mathbb{E}[e^{\lambda_0 I(\hat{V}^\uparrow)}]$  and  $2\mathbb{E}[e^{2\lambda_0 I(\hat{V}^\uparrow)}]$ . Using the arguments of the proof of Lemma 4.3.5 we can prove that for  $h$  large enough  $\mathbb{P}(\tilde{P}_1^{(2)} = P_1^{(i)}) \leq e^{-\delta\kappa(i-2)h/3}$ . This proves that

$$\mathbb{E} \left[ \exp \left( \lambda_0 \int_0^{\tau(\tilde{P}_1^{(1)}, h/2+)} e^{-\tilde{P}_1^{(1)}(u)} du \right) \right] \leq 3\sqrt{\mathbb{E} \left[ e^{2\lambda_0 I(\hat{V}^\uparrow)} \right]}, \quad (4.3.13)$$

for  $h$  large enough. Now, combining (4.3.11), (4.3.12), (4.3.13) and (4.3.10), we get (4.3.9) so (4.3.4) is proved as well.  $\square$

### 4.3.5 Asymptotic of the $h$ -minima sequence

In this subsection, we are interested in the asymptotic distance between the  $h$ -minima when  $h$  goes to infinity. This leads to estimates that are useful to study the local time of the diffusion in  $V$  outside the bottoms of the valleys, and Theorem

4.1.4 is also proved in the end of the subsection. First, we define the first ascend of  $h$  for  $V - \underline{V}$  :

$$\tau^*(h) := \inf \{u \geq 0, (V - \underline{V})(u) = h\}, \quad m^*(h) := \inf \{u \geq 0, V(u) = \underline{V}(\tau^*(h))\}.$$

Note that, since  $\tilde{\tau}_0(h) = 0$ ,  $\tau^*(h)$  and  $m^*(h)$  coincide with respectively  $\tilde{\tau}_1^*(h)$  and  $\tilde{m}_1^*$ , defined in Subsection 4.3.3. We study  $m^*(h)$  by the mean of excursions theory. Let  $\mathcal{F}$  denote the space of excursions, that is, càd-làg functions from  $[0, +\infty[$  to  $\mathbb{R}$ , starting at zero and killed at the first positive instant when they reach 0. Note that this instant can possibly be infinite. For  $\xi \in \mathcal{F}$ , recall the notation  $\zeta(\xi) := \inf\{s > 0, \xi(s) = 0\}$  for the length of the excursion  $\xi$ . Also, let  $\mathcal{F}_{h,-}$  and  $\mathcal{F}_{h,+}$  denote respectively the set of excursions which height is strictly less than  $h$  and the set of excursions higher than  $h$  :

$$\mathcal{F}_{h,-} := \left\{ \xi \in \mathcal{F}, \sup_{[0, \zeta]} \xi < h \right\}, \quad \mathcal{F}_{h,+} := \left\{ \xi \in \mathcal{F}, \sup_{[0, \zeta]} \xi \geq h \right\}.$$

With the help of Fact 4.3.1, we see that  $\{0\}$  is instantaneous for  $V - \underline{V}$  (and it is regular if and only if  $V$  has unbounded variations), excursion theory above 0 can thus be applied for  $V - \underline{V}$  (see [8]). Let  $L$  be a local time at 0 of  $V - \underline{V}$ ,  $\mathcal{N}$  its associated excursion measure and  $L^{-1}$  its right continuous inverse. Then, the excursions above 0 of  $V - \underline{V}$  form a Poisson point process on  $\mathcal{F}$  with intensity measure  $\mathcal{N}$ . In the irregular case (when  $V$  has bounded variations) the local time  $L$  has to be defined artificially as in [8], Section IV.5. In this case, the excursion measure is proportional to the law of the first excursion and in particular the total mass of the excursion measure is finite.

Let us define  $S^{h,-}$  and  $S^{h,+}$  to be two independent pure jumps subordinators with Lévy measure respectively  $\zeta\mathcal{N}(\mathcal{F}_{h,-} \cap \cdot)$  and  $\zeta\mathcal{N}(\mathcal{F}_{h,+} \cap \cdot)$ , the image measures of respectively  $\mathcal{N}(\mathcal{F}_{h,-} \cap \cdot)$  and  $\mathcal{N}(\mathcal{F}_{h,+} \cap \cdot)$  by  $\zeta$ . Since  $\zeta\mathcal{N}(\mathcal{F}_{h,-} \cap \cdot) + \zeta\mathcal{N}(\mathcal{F}_{h,+} \cap \cdot) = \zeta\mathcal{N}$ , the Lévy-Khintchine representation yields that  $S := S^{h,-} + S^{h,+}$  is a pure jumps subordinator with Lévy measure  $\zeta\mathcal{N}$ , it is therefore equal in law to  $L^{-1}$ .

We also define  $T_h$  to be an exponential random variable with parameter  $\mathcal{N}(\mathcal{F}_{h,+})$ , independent from  $S^{h,-}$  and  $S^{h,+}$ . We can now express  $m^*(h)$  in term of these objects.

**Lemma 4.3.9.**

$$m^*(h) \stackrel{\mathcal{L}}{=} S^{h,-}(T_h), \quad \text{and} \quad \tau^*(h) - m^*(h) \stackrel{\mathcal{L}}{=} \tau(V^\uparrow, h).$$

*Démonstration.* Considering  $(e_t)_{t \geq 0}$ , the excursion process of  $V - \underline{V}$ , we have that  $L(\tau^*(h))$  is the instant when occurs the first jumps belonging to  $\mathcal{F}_{h,+}$ , and this jump corresponds to the excursions having  $m^*(h)$  as starting point. We can thus write

$$m^*(h) = \sum_{0 \leq t \leq L(\tau^*(h))} \zeta(e_t \mathbb{1}_{e_t \in \mathcal{F}_{h,-}}). \quad (4.3.14)$$

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Also,  $L(\tau^*(h))$  follows an exponential distribution with parameter  $\mathcal{N}(\mathcal{F}_{h,+})$  and is independent from the process  $(e_t \mathbb{1}_{e_t \in \mathcal{F}_{h,-}})_{t \geq 0}$ . By properties of Poisson point processes, the process in the right hand side of (4.3.14),  $\sum_{0 \leq t \leq \cdot} \zeta(e_t \mathbb{1}_{e_t \in \mathcal{F}_{h,-}})$ , is the sum of the jumps of a Poisson point process on  $\mathbb{R}_+$ , with intensity measure  $\zeta \mathcal{N}(\mathcal{F}_{h,-} \cap \cdot)$ . Thus, from the Lévy-Khintchine representation, it has the same law as the subordinator  $S^{h,-}$ , which yields the result for  $m^*(h)$ .

Then,  $(V(x) - V(m^*(h)), 0 \leq x \leq \tau^*(h) - m^*(h))$  is, considered up to its hitting time of  $h$ , the first excursion higher than  $h$  of  $V - \underline{V}$ . As we said in the proof of Propositions 4.3.2 and 4.3.6, the latter is equal in law to  $(V^\uparrow(x))_{0 \leq x \leq \tau(V^\uparrow, h)}$ . The result about  $\tau^*(h) - m^*(h)$  follows.  $\square$

We are now left to study  $S$  and  $\mathcal{N}(\mathcal{F}_{h,+})$ , the parameter of  $T_h$ . For  $S$ , we have the following lemma :

**Lemma 4.3.10.** *The random variable  $L^{-1}(1)$  (or equivalently  $S(1)$ ) admits some finite exponential moments.*

*Démonstration.* We fix an arbitrary  $t > 1$ .  $\mathcal{N}(\xi \in \mathcal{F}, \zeta(\xi) > t)$  is equal to

$$\mathcal{N}(\xi \in \mathcal{F}, \zeta(\xi) > t, \tau(\xi, 1) \leq 1) + \mathcal{N}(\xi \in \mathcal{F}, \zeta(\xi) > t, \tau(\xi, 1) > 1). \quad (4.3.15)$$

The first term is less than

$$\begin{aligned} & \mathcal{N}(\xi \in \mathcal{F}, \tau(\xi, 1) < +\infty, \inf \{s \geq 0, \xi(\tau(\xi, 1) + s) = 0\} > t - 1) \\ &= \mathcal{N}(\xi \in \mathcal{F}, \tau(\xi, 1) < +\infty) \times \mathbb{P}(\tau(V_1, ] - \infty, 0] > t - 1) \\ &\leq \mathcal{N}(\xi \in \mathcal{F}, \tau(\xi, 1) < +\infty) \times \mathbb{P}(V(t - 1) > -1) \\ &\leq \mathcal{N}(\xi \in \mathcal{F}, \tau(\xi, 1) < +\infty) \times e^\lambda \mathbb{E}[e^{\lambda V(t-1)}], \end{aligned}$$

where we used the Markov property in the excursions, chose some  $\lambda \in ]0, \kappa[$  and used Markov's inequality. We thus get that  $\mathcal{N}(\xi \in \mathcal{F}, \zeta(\xi) > t, \tau(\xi, 1) \leq 1)$  is less than

$$\mathcal{N}(\xi \in \mathcal{F}, \tau(\xi, 1) < +\infty) \times e^\lambda \times e^{(t-1)\Psi_V(\lambda)}, \quad (4.3.16)$$

and  $\Psi_V(\lambda) < 0$  since  $\lambda \in ]0, \kappa[$ . Then, the second term of (4.3.15) is less than

$$\int_0^1 \mathbb{P}(\tau(V_z, ] - \infty, 0] > t - 1) \times \mathcal{N}(\zeta(\xi) > 1, \xi(1) \in dz),$$

where we used the Markov property at time 1 in the excursions,

$$\begin{aligned} & \leq \int_0^1 \mathbb{P}(V(t - 1) > -z) \times \mathcal{N}(\zeta(\xi) > 1, \xi(1) \in dz) \\ & \leq \int_0^1 \mathbb{P}(V(t - 1) > -1) \times \mathcal{N}(\zeta(\xi) > 1, \xi(1) \in dz) \\ & \leq \mathcal{N}(\zeta(\xi) > 1) \times e^\lambda \mathbb{E}[e^{\lambda V(t-1)}], \end{aligned}$$

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for the same  $\lambda \in ]0, \kappa[$  and again we used Markov's inequality. We thus get that  $\mathcal{N}(\xi \in \mathcal{F}, \zeta(\xi) > t, \tau(\xi, 1) > 1)$  is less than

$$\mathcal{N}(\xi \in \mathcal{F}, \zeta(\xi) > 1) \times e^\lambda \times e^{(t-1)\Psi_V(\lambda)}. \quad (4.3.17)$$

Now putting (4.3.16) and (4.3.17) in (4.3.15), we get the existence of positive constants  $c_1, c_2$  such that

$$\forall t > 1, \mathcal{N}(\xi \in \mathcal{F}, \zeta(\xi) > t) \leq c_1 e^{-c_2 t}$$

which proves that for some  $c > 0$ ,

$$\int_1^{+\infty} e^{c\zeta(x)} \mathcal{N}(dx) < +\infty.$$

Using Theorem 25.3 in Sato [59], we deduce that  $L^{-1}(1)$  admits some finite exponential moments. □

The next lemma deals with the asymptotic behavior of  $\mathcal{N}(\mathcal{F}_{h,+})$ .

**Lemma 4.3.11.**

$$\mathcal{N}(\mathcal{F}_{h,+}) = e^{-\kappa h} \mathcal{N}(\mathcal{F}_{1,+}) \times (e^\kappa - \mathbb{E}[e^{\kappa V_1(\tau(V_1, ]-\infty, 0])}]) + \mathcal{O}(e^{-2\kappa h}).$$

*Démonstration.* We fix  $h > 1$ . From the strong Markov property applied at the hitting time of 1 in the excursions belonging to  $\mathcal{F}_{1,+}$  we get

$$\mathcal{N}(\mathcal{F}_{h,+})/\mathcal{N}(\mathcal{F}_{1,+}) = \mathbb{P}(\tau(V_1, h) < \tau(V_1, ]-\infty, 0]) =: p_h. \quad (4.3.18)$$

Then,

$$\begin{aligned} e^{-\kappa(h-1)} &= \mathbb{P}\left(\sup_{[0, +\infty[} V_1 > h\right) \\ &= p_h + \mathbb{P}\left(\tau(V_1, ]-\infty, 0]) < \tau(V_1, h), \sup_{[0, +\infty[} V_1(\tau(V_1, ]-\infty, 0]) + \cdot > h\right) \\ &= p_h + \mathbb{E}\left[e^{-\kappa(h-V_1(\tau(V_1, ]-\infty, 0]))} \mathbb{1}_{\tau(V_1, ]-\infty, 0]) < \tau(V_1, h)}\right], \end{aligned}$$

where we used the strong Markov property at the stopping time  $\tau(V_1, ]-\infty, 0])$ . We get

$$\begin{aligned} p_h &= e^{-\kappa h} \left( e^\kappa - \mathbb{E}\left[e^{\kappa V_1(\tau(V_1, ]-\infty, 0]))} \mathbb{1}_{\tau(V_1, ]-\infty, 0]) < \tau(V_1, h)}\right] \right) \\ &= e^{-\kappa h} \left( e^\kappa - \mathbb{E}\left[e^{\kappa V_1(\tau(V_1, ]-\infty, 0]))}\right] + \mathbb{E}\left[e^{\kappa V_1(\tau(V_1, ]-\infty, 0]))} \mathbb{1}_{\tau(V_1, ]-\infty, 0]) > \tau(V_1, h)}\right] \right). \end{aligned}$$

According to (4.3.18), it only remains to bound the last term. Since  $e^{\kappa V_1(\tau(V_1, ]-\infty, 0]))}$  is almost surely less than 1 we have

$$\mathbb{E}\left[e^{\kappa V_1(\tau(V_1, ]-\infty, 0]))} \mathbb{1}_{\tau(V_1, ]-\infty, 0]) > \tau(V_1, h)}\right] \leq \mathbb{P}(\tau(V_1, h) < +\infty) = e^{-\kappa(h-1)},$$

and the result follows. □

We can now get the asymptotic behavior of  $m^*(h)$ .

**Proposition 4.3.12.**

$$e^{-\kappa h} m^*(h) \xrightarrow[h \rightarrow +\infty]{\mathcal{L}} \mathcal{E}(q),$$

where  $\mathcal{E}(q)$  is the exponential distribution of parameter

$$q := \mathcal{N}(\mathcal{F}_{1,+}) \times (e^\kappa - \mathbb{E}[e^{\kappa V_1(\tau(V_1,]-\infty,0])}]) / \int_0^{+\infty} \zeta(x) \mathcal{N}(dx).$$

The parameter  $q$  here is the one appearing in Theorem 4.1.4. If  $V = W_\kappa$ , the  $\kappa$ -drifted brownian motion, the calculations can be made more explicit. Using (2.7) of [37], we can prove that the parameter  $q$  in the above proposition (which is the same  $q$  as in Theorem 4.1.4) equals  $\kappa^2/2$ .

*Démonstration.* of Proposition 4.3.12

Thanks to Lemma 4.3.9, we are reduced to study  $S^{h,-}(T_h)$ . We have

$$\frac{S^{h,-}(T_h)}{T_h} = \frac{S(T_h)}{T_h} - \frac{S^{h,+}(T_h)}{T_h} \quad (4.3.19)$$

According to Lemma 4.3.11,  $\mathbb{P}(T_h > M)$  converges to 1 when  $h$  goes to infinity, for any  $M > 0$ . According to Lemma 4.3.10,  $S(1)$  has finite expectation so we can use the law of large number for Lévy processes (see for example Theorem 36.5 in [59]) and deduce that  $S(t)/t$  converges almost surely (and therefore in distribution) to  $\mathbb{E}[S(1)]$ . We deduce that

$$S(T_h)/T_h \xrightarrow[h \rightarrow +\infty]{\mathcal{L}} \mathbb{E}[S(1)]. \quad (4.3.20)$$

We now deal with the second term of the right hand side of (4.3.19). From the independence between  $S^{h,+}$  and  $T_h$  we have

$$\mathbb{E} [S^{h,+}(T_h)/T_h] = \int_0^{+\infty} \mathbb{E} [S^{h,+}(u)/u] \mathcal{L}_{T_h}(du) = \mathbb{E} [S^{h,+}(1)].$$

Then,  $\zeta \mathcal{N}(\mathcal{F}_{h,+} \cap \cdot)$ , the Lévy measure of  $S^{h,+}$ , converges to 0 when  $h$  goes to infinity. Since  $S^{h,+}$  is pure jump, we deduce that  $S^{h,+}(1)$  converges to 0 in probability when  $h$  goes to infinity. Then,  $S^{h,+}(1) \leq S(1)$  and  $S(1)$  has finite expectation, so we deduce by dominated converges that  $\mathbb{E}[S^{h,+}(1)]$  converges to 0, we thus get

$$S^{h,+}(T_h)/T_h \xrightarrow[h \rightarrow +\infty]{\mathcal{L}} 0. \quad (4.3.21)$$

Combining (4.3.19), (4.3.20) and (4.3.21) we get

$$S^{h,-}(T_h)/T_h \xrightarrow[h \rightarrow +\infty]{\mathcal{L}} \mathbb{E}[S(1)]. \quad (4.3.22)$$

Recall also that  $\mathbb{E}[S(1)] = \mathbb{E}[L^{-1}(1)] = \int_0^{+\infty} x \zeta \mathcal{N}(dx)$ . Then,

$$e^{-\kappa h} S^{h,-}(T_h) = (e^{-\kappa h} / \mathcal{N}(\mathcal{F}_{h,+})) \times (\mathcal{N}(\mathcal{F}_{h,+}) \times T_h) \times (S^{h,-}(T_h) / T_h),$$

so combining Lemma 4.3.11, the definition of  $T_h$ , (4.3.22) and Slutsky Lemma, we get that  $e^{-\kappa h} S^{h,-}(T_h)$  converges to an exponential distribution with parameter  $q := \mathcal{N}(\mathcal{F}_{1,+}) \times (e^\kappa - \mathbb{E}[e^{\kappa V_1(\tau(V_1, ]-\infty, 0])}]) / \int_0^{+\infty} \zeta(x) \mathcal{N}(dx)$ . Thanks to Lemma 4.3.9 we get the result.  $\square$

We can now prove Theorem 4.1.4.

*Démonstration.* of Theorem 4.1.4

As we said in the proof of Proposition 4.3.2, Lemma 1 of [21] applies here, so the random variables  $(m_{i+1} - m_i)_{i \geq 1}$  are *iid*, and  $m_1$  is also independent from this sequence. To prove Theorem 4.1.4, we thus only need to prove that the random variables  $e^{-\kappa h} m_1$  and  $e^{-\kappa h} (m_2 - m_1)$  converge in distribution to an exponential distribution with parameter  $q$ . Then, note that according to Lemma 4.3.5 applied with  $n := 2$ , we only need to prove this convergence for  $e^{-\kappa h} \tilde{m}_1$  and  $e^{-\kappa h} (\tilde{m}_2 - \tilde{m}_1)$ . Now, according to subsection 4.3.3,

$$\tilde{m}_2 - \tilde{m}_1 = (\tilde{\tau}_1(h) - \tilde{m}_1) + (\tilde{L}_1 - \tilde{\tau}_1(h)) + (\tilde{L}_2^\sharp - \tilde{L}_1) + (\tilde{m}_2 - \tilde{L}_2^\sharp),$$

and according to Proposition 4.3.6 for the first term and the Markov property at times  $\tilde{\tau}_1(h)$ ,  $\tilde{L}_1$  and  $\tilde{L}_2^\sharp$  for the other terms, we get that the terms on the right hand side have respectively the same law as  $\tau(V^\uparrow, h)$ ,  $\tau(V, ]-\infty, -h/2])$ ,  $\tau(V, ]-\infty, -e^{(1-\delta)\kappa h})$ , and  $m^*(h)$ . Now, using Lemma 4.5.6 for the first term, and Lemma 4.5.4 for the second and third terms, we get that these terms, renormalized by  $e^{-\kappa h}$  converge to 0 when  $h$  goes to infinity. Proposition 4.3.12 gives the convergence for the last term renormalized by  $e^{-\kappa h}$ . Combining with Slutsky Lemma we get that  $e^{-\kappa h} (\tilde{m}_2 - \tilde{m}_1)$  converge in distribution to an exponential distribution with parameter  $q$ .

For  $\tilde{m}_1$  it is even simpler, we have  $\tilde{m}_1 = \tilde{L}_1^\sharp + (\tilde{m}_1 - \tilde{L}_1^\sharp)$ , so we can conclude the same way.  $\square$

## 4.4 Supremum of the local time when $0 < \kappa < 1$

We now generalize, in the context of the diffusion in  $V$ , the arguments of [4] to prove the convergence of the supremum of the local time when  $0 < \kappa < 1$ . First, let us recall some definitions of [3] and [4] for the study of the diffusion across the valleys.



The diffusions moves across valleys, and for these to be neither too small or too large with respect to the time scale, we have to make the size of the valleys grow with time  $t$ . We are thus interested in  $h_t$ -valleys where

$$h_t := \log(t) - \phi(t), \text{ for some function } \phi \text{ with } \log(\log(t)) \ll \phi(t) \ll \log(t).$$

We also define  $N_t$ , the indice of the largest  $h_t$ -minima visited by  $X$  until time  $t$ ,

$$N_t := \max \left\{ k \in \mathbb{N}, \sup_{0 \leq s \leq t} X(s) \geq m_k \right\}.$$

We need our estimates on the valleys to be true simultaneously for a large deterministic number of valleys that we will prove to be greater than  $N_t$  with good probability. This is why we define  $n_t := \lfloor e^{\kappa(1+\delta)\phi(t)} \rfloor$ .

In all this section, we assume that  $\delta$  is small enough so that  $(1 + 3\delta)\kappa < 1$ , and that the hypothesis of Theorem 4.1.3 are satisfied :  $V$  has unbounded variations and there exists  $p > 1$  such that  $V(1) \in L^p$ , so all the results of Sections 4.3 and 4.5 apply here.

#### 4.4.1 Proof of Theorem 4.1.3

As in [4], the idea is to approximate  $\mathcal{L}_X^*(t)/t$  by a functional of a sequence  $(e_i S_i^t, e_i S_i^t R_i^t)_{i \geq 1}$  that is defined later. Our approximation is similar to Proposition 5.1 of [4] and can be stated as follows :

**Proposition 4.4.1.** *Let us define the repartition functions*

$$\mathcal{P}_1^\pm(\alpha) := \mathbb{P} \left( \left( 1 - \frac{1}{t} \sum_{i=1}^{N_{t(1-\eta)}} e_i S_i^t R_i^t \right) \frac{e_{N_{t(1-\eta)}} S_{N_{t(1-\eta)}}^t}{e_{N_{t(1-\eta)}} S_{N_{t(1-\eta)}}^t R_{N_{t(1-\eta)}}^t} \leq \alpha_t^\pm, \max_{1 \leq i \leq N_{t(1-\eta)}-1} \frac{e_i S_i^t}{t} \leq \alpha_t^\pm \right),$$

where  $\alpha_t^\pm := \alpha(1 \pm (\log \log t)^{-1/2})$ . Let also  $v$  be a positive function such that  $\lim_{\eta \rightarrow 0} \lim_{t \rightarrow +\infty} v(\eta, t) = 0$ . For all  $t$  large enough we then have

$$\mathcal{P}_1^-(\alpha) - v(\eta, t) \leq \mathbb{P} \left( \sup_{x \in \mathbb{R}} \mathcal{L}_X(t, x)/t \leq \alpha \right) \leq \mathcal{P}_1^+(\alpha) + v(\eta, t).$$

The next step is to identify the objects in  $\mathcal{P}_1^\pm(\alpha)$  as continus functionals of something that converges to  $(\mathcal{Y}_1, \mathcal{Y}_2)$  (defined in the Introduction) when  $t$  goes to infinity. Let  $(D([0, +\infty[, \mathbb{R}^2), J_1)$  be the space of càd-làg functions taking values in  $\mathbb{R}^2$ , equipped with the  $J_1$ -Skorokhod topology. If, as in [4], we define  $(Y_1, Y_2)^t$  by

$$\forall s \geq 0, (Y_1, Y_2)^t(s) := \frac{1}{t} \sum_{j=1}^{\lfloor se^{\kappa\phi(t)} \rfloor} (e_j S_j^t, e_j S_j^t R_j^t),$$

then we have

**Proposition 4.4.2.**  $(Y_1, Y_2)^t$  converges in distribution to  $(\mathcal{Y}_1, \mathcal{Y}_2)$  in  $(D([0, +\infty[, \mathbb{R}^2), J_1)$ .

The objects in  $\mathcal{P}_1^\pm(\alpha)$  can be written in term of functionals of  $(Y_1, Y_2)^t$  : the functionals  $J_{I,a}^-, K_{I,a}, K_{I,a}^-, \tilde{K}_{I,a}$  and  $\tilde{K}_{I,a}^-$  (with  $a = 1 - \eta$ ) defined in Subsection 4.3 of [4]. Thanks to Lemma 4.5 there, we see that the  $\kappa$ -stable subordinator  $(\mathcal{Y}_1, \mathcal{Y}_2)$  is almost surely a point of continuity for these functionals so, by *continus mapping theorem* and Proposition 4.4.2, we get, when  $t$  goes to infinity, the convergence of  $\mathcal{P}_1^\pm(\alpha)$  to

$$\mathbb{P} \left( \max \left\{ \mathcal{Y}_1^{\sharp}(\mathcal{Y}_2^{-1}(1-\eta)-), \right. \right. \\ \left. \left. (1 - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1-\eta)-)) \frac{\mathcal{Y}_1(\mathcal{Y}_2^{-1}(1-\eta)) - \mathcal{Y}_1(\mathcal{Y}_2^{-1}(1-\eta)-)}{\mathcal{Y}_2(\mathcal{Y}_2^{-1}(1-\eta)) - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1-\eta)-)} \right\} \leq \alpha \right).$$

Then, we have almost surely  $\mathcal{Y}_2^{-1}(1-\eta) = \mathcal{Y}_2^{-1}(1)$  for all  $\eta$  small enough (since almost surely  $\mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)-) < 1$ ). As a consequence, when  $\eta$  goes to 0, the above expression converges to the repartition function of the random variable  $\mathcal{I}_1 \vee \mathcal{I}_2$  (defined in the Introduction) at  $\alpha$ .

As a consequence, Theorem 4.1.3 follows if we prove Propositions 4.4.1 and 4.4.2. For Proposition 4.4.1, we first need to show that the time spent by the diffusion and the local time are negligible outside the bottoms of the standard valleys, this is the object of the next two facts that are taken from [3] and [4] (however, the extension of these results to our context requires some precautions, this is why we give some details in the end of Section 4.5).

**Fact 4.4.3.** *There exists a positive constant  $C > 0$  such that for  $t$  large enough,*

$$\mathbb{P} \left( \mathcal{A}_t^1 := \bigcap_{j=1}^{n_t} \left\{ 0 \leq H(\tilde{m}_j) - \sum_{i=1}^{j-1} \left( H(\tilde{L}_i) - H(\tilde{m}_i) \right) \leq \frac{2t}{\log h_t} \right\} \right) \geq 1 - C e^{-\phi(t)} n_t \log h_t.$$

*The lower bound converges to 1 since  $(1 + 2\delta)\kappa < 1$ .*

Before stating the next fact, which proves that the supremum of the local time is negligible outside the "deep bottoms" of the standard valleys, we need to define what we mean exactly by "deep bottom". We define

$$\mathcal{D}_j := [\tilde{\tau}_j^-( (\phi(t))^2 ), \tilde{\tau}_j^+( (\phi(t))^2 )].$$

Note that the definition we give is different from the one in [4]. Indeed, the presence of negative jumps for the environment in our context implies to take some precautions and to adapt some definitions.

**Fact 4.4.4.** *Recall the definitions of  $\tau^*(h)$  and  $m^*(h)$  in Subsection 4.3.5. There are positive constants  $C_1, C_2, C_3$  such that for  $t$  large enough,*

$$\mathbb{P} \left( \sup_{x \in [0, m^*(h_t)]} \mathcal{L}_X(H(\tau^*(h_t)), x) > te^{(\kappa(1+3\delta)-1)\phi(t)} \right) \leq \frac{C_1}{n_t e^{\kappa\delta\phi(t)}}, \quad (4.4.1)$$

$$\mathbb{P} \left( \bigcap_{j=0}^{n_t-1} \left\{ \sup_{\mathbb{R}} \left( \mathcal{L}_X(H(\tilde{m}_{j+1}), x) - \mathcal{L}_X(H(\tilde{L}_j), x) \right) \leq te^{(\kappa(1+3\delta)-1)\phi(t)} \right\} \right) \geq 1 - \frac{C_2}{e^{\kappa\delta\phi(t)}}, \quad (4.4.2)$$

$$\mathbb{P} \left( \bigcap_{j=1}^{n_t} \left\{ \sup_{x \in [\tilde{L}_{j-1}, \tilde{L}_j] \cap \mathcal{D}_j^c} \left( \mathcal{L}_X(H(\tilde{L}_j), x) - \mathcal{L}_X(H(\tilde{m}_j), x) \right) \leq te^{-2\phi(t)}, \right\} \right) \geq 1 - \frac{C_3 n_t}{e^{2\phi(t)}}. \quad (4.4.3)$$

The next step in the proof of Proposition 4.4.1 is to show that the main contributions to the local time and to the time spent by the diffusion in the bottoms of the standard valleys can be approximated by the sequence  $(e_i S_i^t, e_i S_i^t R_i^t)_{i \geq 1}$  :

**Proposition 4.4.5.** *The random variables  $e_j, S_j^t, R_j^t$ ,  $j \geq 1$  are mutually independent and for any  $\epsilon \in ]0, \max(1/8, (1 - (1 + \delta)\kappa)/2)[$  there exists a positive constant  $c$  such that for  $t$  large enough,*

$$\mathbb{P} \left( \mathcal{A}_t^2 := \bigcap_{j=1}^{n_t} \left\{ (1 - e^{-ch_t/7}) e_j S_j^t \leq \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \leq (1 + e^{-ch_t/7}) e_j S_j^t \right\} \right) \geq 1 - e^{-ch_t},$$

$$\mathbb{P} \left( \mathcal{A}_t^3 := \bigcap_{j=1}^{n_t} \left\{ (1 - e^{-ch_t/7}) e_j S_j^t R_j^t \leq H(\tilde{L}_j) - H(\tilde{m}_j) \leq (1 + e^{-ch_t/7}) e_j S_j^t R_j^t \right\} \right) \geq 1 - e^{-ch_t}.$$

This proposition is proved in the following subsection. For the end of the proof of Proposition 4.4.1, the idea is simply to use the previous steps to translate " $\mathcal{L}_X^*(t)/t \leq \alpha$ " in term of events only involving the sequence  $(e_i S_i^t, e_i S_i^t R_i^t)_{i \geq 1}$ . We do this in Subsection 4.4.3, following the arguments of the proof of Proposition 5.1 of [4].

Finally, Proposition 4.4.2 is roughly speaking a Donsker Theorem for heavy tailed random variables, its proof relies on the study of the right tail of the random variables  $e_1 S_1^t$  and  $e_1 S_1^t R_1^t$  which is done in subsection 4.4.4.

#### 4.4.2 Proof of Proposition 4.4.5 and consequences

We now prove that  $(\mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j), H(\tilde{L}_j) - H(\tilde{m}_j))_{j \geq 1}$  can be approximated by an *iid* sequence. This generalizes Proposition 3.5 of [4] to our setting.

First, note that the proof of the first point of Lemma 3.6 in [4] can be repeated here. We only replace  $W_\kappa$  by  $V$ , consider the standard valleys to be defined in our sens, replace the random times  $\tilde{L}_j^-$  of [4] by  $\tilde{L}_{j-1}$ , apply Lemma 4.5.21 for the first

$n_t$  indices instead of Lemma 3.2 of [3], and use Remark 4.3.4 instead of Lemma 2.2 of [4]. We get

$$\mathbb{P} \left( \left( \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j), H(\tilde{L}_j) - H(\tilde{m}_j) \right)_{1 \leq j \leq n_t} = \left( \tilde{\mathcal{L}}_j, \tilde{h}_j \right)_{1 \leq j \leq n_t} \right) \geq 1 - n_t e^{-\kappa \delta h_t / 6}, \quad (4.4.4)$$

where the sequence  $(\tilde{\mathcal{L}}_j, \tilde{h}_j)_{j \geq 1}$  is *iid* and  $(\tilde{\mathcal{L}}_j, \tilde{h}_j)$  is equal to

$$A^j(\tilde{L}_j) \times \left( \mathcal{L}_{B^j}(\tau(B^j, 1), 0), \int_{\tilde{L}_{j-1}}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} \mathcal{L}_{B^j} \left( \tau(B^j, 1), A^j(u)/A^j(\tilde{L}_1) \right) du \right), \quad (4.4.5)$$

where  $A^j(u) := \int_{\tilde{m}_j}^u e^{\tilde{V}^{(j)}(x)} dx$  and  $B^j$  is a sequence of *iid* brownian motions starting at 0, and independent from  $V$ . Note that  $e_j := \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j)/A^j(\tilde{L}_j) = \mathcal{L}_{B^j}(\tau(B^j, 1), 0)$  follows an exponential distribution with parameter  $1/2$ . We are thus left to give approximations of  $\tilde{h}_j$  and  $A^j(\tilde{L}_j)$ . For this, we first prove lemmas to bound exponential functionals of the environment.

**Lemma 4.4.6.** *Choose  $\epsilon$  such that  $0 < \epsilon < \max(1/8, (1 - (1 + \delta)\kappa)/2)$ . Then,*

$$\forall j \geq 1, \mathbb{P} \left( \int_{\tilde{L}_{j-1}}^{\tilde{\tau}_j^-(h_t/2)} e^{-\tilde{V}^{(j)}(u)} du > e^{-\epsilon h_t} \right) \leq e^{-\epsilon h_t},$$

for  $t$  large enough and some positive constant  $c$  depending on  $\delta$  and  $\epsilon$ .

*Démonstration.* Fix  $j \geq 1$ . We have

$$\int_{\tilde{L}_{j-1}}^{\tilde{\tau}_j^-(h_t/2)} e^{-\tilde{V}^{(j)}(u)} du = \int_{\tilde{L}_{j-1}}^{\tilde{\tau}_j^-(h_t)} e^{-\tilde{V}^{(j)}(u)} du + \int_{\tilde{\tau}_j^-(h_t)}^{\tilde{\tau}_j^-(h_t/2)} e^{-\tilde{V}^{(j)}(u)} du. \quad (4.4.6)$$

The first term of the right hand side is less than

$$\left( \tilde{\tau}_j^-(h_t) - \tilde{L}_{j-1} \right) \times \sup_{[\tilde{L}_{j-1}, \tilde{\tau}_j^-(h_t)]} e^{-\tilde{V}^{(j)}}.$$

According to Lemma 4.5.17 for the first factor, and the definition of  $\tilde{L}_j^\sharp$  together with Lemma 4.5.16 applied with  $\alpha = 1, \eta = \epsilon$  for the second factor, there is a positive constant  $c_1$  (depending on  $\delta$  and  $\epsilon$ ) such that for  $t$  large enough,

$$\mathbb{P} \left( \int_{\tilde{L}_{j-1}}^{\tilde{\tau}_j^-(h_t)} e^{-\tilde{V}^{(j)}(u)} du > e^{-\epsilon h_t} / 2 > e^{((1+\delta)\kappa + \epsilon - 1)h_t} \right) \leq e^{-c_1 h_t}. \quad (4.4.7)$$

According to Propositions 4.3.6 and 4.3.2 we have,

$$\mathbb{P} \left( \int_{\tilde{\tau}_j^-(h_t)}^{\tilde{\tau}_j^-(h_t/2)} e^{-\tilde{V}^{(j)}(u)} du > e^{-\epsilon h_t} / 2 \right) \leq 2\mathbb{P} \left( \int_{\tau(\hat{V}^\uparrow, [h_t/2, +\infty])}^{\tau(\hat{V}^\uparrow, [h_t, +\infty])} e^{-\hat{V}^\uparrow(u)} du > e^{-\epsilon h_t} / 2 \right) + 2e^{-\delta \kappa h_t / 3},$$

and the integral on the right hand side is less than

$$\tau \left( \hat{V}^\uparrow, [h_t, +\infty[ \right) \times \sup_{[\tau(\hat{V}^\uparrow, [h_t/2, +\infty]), \tau(\hat{V}^\uparrow, [h_t, +\infty])]} e^{-\hat{V}^\uparrow}.$$

Then, according to Lemma 4.5.11 applied with  $y = h_t$ ,  $r = e^{h_t/8}$  and Lemma 4.5.9 applied with  $a = h_t/4$ ,  $b = h_t/2$ ,  $z = 0$ , there is a positive constant  $c_2$  such that for  $t$  large enough,

$$\mathbb{P} \left( \int_{\tilde{\tau}_j^-(h_t)}^{\tilde{\tau}_j^-(h_t/2)} e^{-\tilde{V}^{(j)}(u)} du > e^{-\epsilon h_t} / 2 > e^{(1/8-1/4)h_t} \right) \leq e^{-c_2 h_t}. \quad (4.4.8)$$

The combination of (4.4.6), (4.4.7) and (4.4.8) yields the result.  $\square$

**Lemma 4.4.7.** *Choose  $\epsilon$  such that  $0 < \epsilon < 1/8$ . There is a positive constant  $c$  such that for  $t$  large enough,*

$$\forall j \geq 1, \mathbb{P} \left( \int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} du > e^{-\epsilon h_t} \right) \leq e^{-c h_t}.$$

*Démonstration.* Fix  $j \geq 1$ . We have

$$\int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} du = \int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{\tau}_j^+(h_t)} e^{-\tilde{V}^{(j)}(u)} du + \int_{\tilde{\tau}_j^+(h_t)}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} du. \quad (4.4.9)$$

According to Proposition 4.3.6, the first term on the right hand side is equal in law to  $\int_{\tau(V^\uparrow, h_t/2)}^{\tau(V^\uparrow, h_t)} e^{-V^\uparrow(u)} du$  which is less than

$$\tau(V^\uparrow, h_t) \times \sup_{[\tau(V^\uparrow, h_t/2), \tau(V^\uparrow, h_t)]} e^{-V^\uparrow}.$$

According, for the first factor, to Lemma 4.5.6 applied with  $y = h_t$ ,  $r = e^{h_t/8}$  and, for the second factor, to Lemma 4.5.7 applied with  $a = h_t/4$ ,  $b = h_t/2$ , there is a positive constant  $c_1$  such that for  $t$  large enough,

$$\mathbb{P} \left( \int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{\tau}_j^+(h_t)} e^{-\tilde{V}^{(j)}(u)} du > e^{-\epsilon h_t} / 2 > e^{(1/8-1/4)h_t} \right) \leq e^{-c_1 h_t}. \quad (4.4.10)$$

Because of the definition of  $\tilde{L}_j$ , the second term in the right hand side of (4.4.9) is less than  $e^{-h_t/2}(\tilde{L}_j - \tilde{\tau}_j(h_t))$ . Because of Proposition 4.3.6,  $\tilde{L}_j - \tilde{\tau}_j(h_t)$  is equal in law to  $\tau(V, ] - \infty, -h_t/2])$ . Applying Lemma 4.5.4 with  $y = h_t/2$  and  $r = e^{h_t/4}$  we get, for  $t$  large enough,

$$\mathbb{P} \left( \int_{\tilde{\tau}_j(h_t)}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} du > e^{-\epsilon h_t} / 2 > e^{(1/4-1/2)h_t} \right) \leq e^{-h_t}. \quad (4.4.11)$$

The combination of (4.4.9), (4.4.10) and (4.4.11) yields the result.  $\square$

**Lemma 4.4.8.** *Choose  $\epsilon$  such that  $0 < \epsilon < 1/4$ . There is a positive constant  $c$  (depending on  $\epsilon$ ) such that for  $t$  large enough,*

$$\forall j \geq 1, \mathbb{P} \left( \sup_{u \in [\tilde{\tau}_j^-(h_t/2), \tilde{\tau}_j^+(h_t/2)]} \left| A^j(u) / A^j(\tilde{L}_j) \right| \leq e^{-(1-2\epsilon)h_t/2} \right) \geq 1 - e^{-ch_t}.$$

*Démonstration.* We have

$$\begin{aligned} A^j(\tilde{L}_j) &\geq \int_{\tilde{m}_j}^{\tilde{\tau}_j(h_t)} e^{\tilde{V}^{(j)}(u)} du \stackrel{\mathcal{L}}{=} \int_0^{\tau(V^\uparrow, h_t)} e^{V^\uparrow(u)} du \quad \text{and} \\ A^j(\tilde{\tau}_j^+(h_t/2)) &\stackrel{\mathcal{L}}{=} \int_0^{\tau(V^\uparrow, h_t/2)} e^{V^\uparrow(u)} du \end{aligned}$$

where we used Proposition 4.3.6 for the equalities in law. For  $A^j(\tilde{L}_j)$  we use (4.5.8) applied with  $h = h_t$ ,  $\eta = \epsilon/2$  and for  $A^j(\tilde{\tau}_j^+(h_t/2))$  we use (4.5.9) applied with  $h = h_t/2$ ,  $\eta = \epsilon$ . We get the existence of a positive constant  $c_1$  such that for  $t$  large enough,

$$\mathbb{P} \left( 0 \leq A^j(\tilde{\tau}_j^+(h_t/2)) / A^j(\tilde{L}_j) \leq e^{-(1-2\epsilon)h_t/2} \right) \geq 1 - e^{-c_1 h_t}. \quad (4.4.12)$$

Then,  $A^j(\tilde{\tau}_j^-(h_t/2)) = \int_{\tilde{m}_j}^{\tilde{\tau}_j^-(h_t/2)} e^{\tilde{V}^{(j)}(u)} du$  and according to Propositions 4.3.6 and 4.3.2 we have,

$$\begin{aligned} \mathbb{P} \left( A^j(\tilde{\tau}_j^-(h_t/2)) \leq -e^{h_t(1+\epsilon)/2} \right) &\leq 2\mathbb{P} \left( \int_0^{\tau(\hat{V}^\uparrow, h_t/2+)} e^{\hat{V}^\uparrow(u)} du \geq e^{h_t(1+\epsilon)/2} \right) + 2e^{-\delta\kappa h_t/3} \\ &\leq 2e^{-c_4 h_t} + 2e^{-\delta\kappa h_t/3}, \end{aligned}$$

where the last inequality is true for  $c_4$  a positive constant and  $t$  large enough. It comes from (4.5.14) applied with  $h = h_t/2$ ,  $\eta = \epsilon$ . Now combining with the lower bound for  $A^j(\tilde{L}_j)$  (given by (4.5.8) applied with  $h = h_t$ ,  $\eta = \epsilon/2$ ) we get

$$\mathbb{P} \left( -e^{-h_t(1-2\epsilon)/2} \leq A^j(\tilde{\tau}_j^-(h_t/2)) / A^j(\tilde{L}_j) \leq 0 \right) \geq 1 - e^{-c_5 h_t}, \quad (4.4.13)$$

for some positive constant  $c_5$ , and when  $t$  is large enough. The combination of (4.4.12) and (4.4.13) with the increase of  $A^j(\cdot)$  yields the result.  $\square$

We now approximate  $\tilde{h}_j$ .

**Lemma 4.4.9.** *Choose  $\epsilon$  such that  $0 < \epsilon < \max(1/8, (1 - (1 + \delta)\kappa)/2)$ . There is a positive constant  $c$  (depending on  $\delta$  and  $\epsilon$ ) such that for all  $t$  large enough we have*

$$\forall j \geq 1, \mathbb{P} \left( \left| \tilde{h}_j - A^j(\tilde{L}_j) R_j^t e_j \right| \leq e^{-\epsilon h_t/6} A^j(\tilde{L}_j) R_j^t e_j \right) \geq 1 - e^{-\epsilon h_t}, \quad (4.4.14)$$

where

$$R_j^t := \int_{\tilde{\tau}_j^-(h_t/2)}^{\tilde{\tau}_j^+(h_t/2)} e^{-\tilde{V}^{(j)}(u)} du.$$

*Démonstration.* Our proof has the same spirit as the one of Lemma 4.7 of [3] but relies on our estimates. We give the details. Here again,  $\tilde{h}_j/A^j(\tilde{L}_j)$  can be cut into three parts :

$$\begin{aligned} \tilde{h}_j/A^j(\tilde{L}_j) &= \int_{\tilde{L}_{j-1}}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} \mathcal{L}_{B^j} \left( \tau(B^j, 1), A^j(u)/A^j(\tilde{L}_j) \right) du \\ &= \int_{\tilde{L}_{j-1}}^{\tilde{\tau}_j^-(h_t/2)} + \int_{\tilde{\tau}_j^-(h_t/2)}^{\tilde{\tau}_j^+(h_t/2)} + \int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} \mathcal{L}_{B^j} \left( \tau(B^j, 1), A^j(u)/A^j(\tilde{L}_j) \right) du \\ &=: \mathcal{J}_0^j + \mathcal{J}_1^j + \mathcal{J}_2^j. \end{aligned} \quad (4.4.15)$$

We start by bounding  $\mathcal{J}_2^j$  and  $\mathcal{J}_0^j$  :

$$\mathcal{J}_2^j \leq \left( \sup_{[0,1]} \mathcal{L}_{B^j} \left( \tau(B^j, 1), \cdot \right) \right) \times \int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} du.$$

Thanks to estimate (7.12) of [4] in which we take  $x = e^{\epsilon h_t/2}$  and Lemma 4.4.7, there exists a positive constant  $c_1$  such that for  $t$  is large enough,

$$\mathbb{P} \left( \mathcal{J}_2^j < e^{-\epsilon h_t/2} \right) \geq 1 - e^{-c_1 h_t}. \quad (4.4.16)$$

For  $\mathcal{J}_0^j$  :

$$\mathcal{J}_0^j \leq \left( \sup_{]-\infty, 0]} \mathcal{L}_{B^j} \left( \tau(B^j, 1), \cdot \right) \right) \times \int_{\tilde{L}_{j-1}}^{\tilde{\tau}_j^-(h_t/2)} e^{-\tilde{V}^{(j)}(u)} du.$$

Thanks to estimate (7.13) of [4] in which we take  $x = e^{\epsilon h_t/2}$  and Lemma 4.4.6, there exists a positive constant  $c_2$  such that for  $t$  is large enough,

$$\mathbb{P} \left( \mathcal{J}_0^j < e^{-\epsilon h_t/2} \right) \geq 1 - e^{-c_2 h_t}. \quad (4.4.17)$$

For  $\mathcal{J}_1^j$  we apply Lemma 4.4.8 and estimate (7.11) of [4] (applied with  $\delta = e^{-(1-2\epsilon)h_t/2}$ ,  $\epsilon = e^{-(1-2\epsilon)h_t/6}$ ), we get

$$\left| \mathcal{J}_1^j - e_j \int_{\tilde{\tau}_j^-(h_t/2)}^{\tilde{\tau}_j^+(h_t/2)} e^{-\tilde{V}^{(j)}(u)} du \right| \leq e^{-(1-2\epsilon)h_t/6} e_j \int_{\tilde{\tau}_j^-(h_t/2)}^{\tilde{\tau}_j^+(h_t/2)} e^{-\tilde{V}^{(j)}(u)} du, \quad (4.4.18)$$

with probability greater than  $1 - e^{-c_6 h_t}$ , for some positive constant  $c_6$ , when  $t$  is large enough. Then,

$$\int_{\tilde{\tau}_j^-(h_t/2)}^{\tilde{\tau}_j^+(h_t/2)} e^{-\tilde{V}^{(j)}(u)} du \geq \int_{\tilde{\tau}_j^+(h_t/16)}^{\tilde{\tau}_j^+(h_t/8)} e^{-\tilde{V}^{(j)}(u)} du \stackrel{\mathcal{L}}{=} \int_{\tau(V^\uparrow, \epsilon h_t/16)}^{\tau(V^\uparrow, \epsilon h_t/8)} e^{-V^\uparrow(u)} du,$$

where we used Proposition 4.3.6 for the equality in law. We thus get

$$\mathbb{P} \left( \int_{\tilde{\tau}_j^-(h_t/2)}^{\tilde{\tau}_j^+(h_t/2)} e^{-\tilde{V}^{(j)}(u)} du \leq e^{-\epsilon h_t/8} \right) \leq \mathbb{P} \left( \tau(V^\uparrow, \epsilon h_t/8) - \tau(V^\uparrow, \epsilon h_t/16) \leq 1 \right) \leq e^{-c_7 h_t}, \quad (4.4.19)$$

when  $t$  is large enough, according to (4.5.7) applied with  $\alpha = \epsilon/8$ ,  $\omega = \epsilon/16$ , and where  $c_7$  is a positive constant.

We have  $\mathbb{P}(e_j \leq e^{-\epsilon h_t/8}) \underset{t \rightarrow +\infty}{\sim} e^{-\epsilon h_t/8}/2$ , so combining with (4.4.18) and (4.4.19) we get, for some positive constant  $c_8$  and  $t$  large enough,

$$\mathbb{P} \left( \mathcal{J}_1^j > e^{-\epsilon h_t/5} \right) \geq 1 - e^{-c_8 h_t}. \quad (4.4.20)$$

Combining (4.4.20) with (4.4.16) and (4.4.17) we get

$$\mathbb{P} \left( \mathcal{J}_0^j + \mathcal{J}_2^j \leq 2e^{-\epsilon h_t/5} \mathcal{J}_1^j \right) \geq 1 - e^{-c_9 h_t},$$

for some positive constant  $c_9$ , and when  $t$  is large enough. Combining with (4.4.15) and (4.4.18) we get the result.  $\square$

Now, we approximate  $A^j(\tilde{L}_j)$ .

**Lemma 4.4.10.**  $S_j^t := \int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{L}_j} e^{\tilde{V}^{(j)}(u)} du$  is independent from  $(e_j, R_j^t)$  and such that

$$\mathbb{P} \left( S_j^t \leq A^j(\tilde{L}_j) \leq (1 + e^{-h_t/7}) S_j^t \right) \geq 1 - e^{-c h_t},$$

for some positive constant  $c$ , when  $t$  is large enough.

*Démonstration.*  $S_j^t := \int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{L}_j} e^{\tilde{V}^{(j)}(u)} du$  so  $(S_j^t, R_j^t)$  only depends on  $V$  and is thus independent from  $e_j$  which only depends on  $B^j$ . The independence between  $S_j^t$  and  $R_j^t$  comes from Proposition 4.3.6 : the latter gives the independence between  $\tilde{P}_2^{(j)}$  and  $\tilde{P}_3^{(j)}$  and allows to apply the Markov property at time  $\tau(\tilde{P}_2^{(j)}, h_t/2)$ , since  $\tilde{P}_2^{(j)} \stackrel{\mathcal{L}}{=} (V^\uparrow(x))_{0 \leq x \leq \tau(V^\uparrow, h_t)}$ . The random variables  $e_j, S_j^t$  and  $R_j^t$  are therefore mutually independent. Then, from the definitions of  $S_j^t$  and  $A^j(\tilde{L}_j)$  we have

$$S_j^t \leq A^j(\tilde{L}_j) = \int_{\tilde{m}_j}^{\tilde{\tau}_j^+(h_t/2)} e^{\tilde{V}^{(j)}(u)} du + S_j^t. \quad (4.4.21)$$



Applying successively Proposition 4.3.6 and estimate (4.5.9) with  $h = h_t/2$ ,  $\eta = 1/3$ , we get

$$\mathbb{P} \left( \int_{\tilde{m}_j}^{\tilde{\tau}_j^+(h_t/2)} e^{\tilde{V}^{(j)}(u)} du \leq e^{4h_t/6} \right) \geq 1 - e^{-c_1 h_t}, \quad (4.4.22)$$

for some positive constant  $c_1$ , when  $t$  is large enough. Then,

$$S_j^t \geq \int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{\tau}_j(h_t)} e^{\tilde{V}^{(j)}(u)} du = \int_{\tilde{m}_j}^{\tilde{\tau}_j(h_t)} e^{\tilde{V}^{(j)}(u)} du - \int_{\tilde{m}_j}^{\tilde{\tau}_j^+(h_t/2)} e^{\tilde{V}^{(j)}(u)} du.$$

Applying Proposition 4.3.6 for each term, and, for the first term, estimate (4.5.8) with  $h = h_t$ ,  $\eta = 1/6$ , for the second term, (4.5.9) with  $h = h_t/2$ ,  $\eta = 1/3$ , we get

$$\mathbb{P} \left( S_j^t \geq e^{5h_t/6} - e^{4h_t/6} \right) \geq 1 - e^{-c_2 h_t}, \quad (4.4.23)$$

for some positive constant  $c_2$ , when  $t$  is large enough. Putting together (4.4.22) and (4.4.23) we get that

$$\mathbb{P} \left( \int_{\tilde{m}_j}^{\tilde{\tau}_j^+(h_t/2)} e^{\tilde{V}^{(j)}(u)} du / S_j^t \leq e^{-h_t/7} \right) \geq 1 - e^{-c_3 h_t},$$

for some positive constant  $c_3$ , when  $t$  is large enough. This, combined with (4.4.21), yields the result.  $\square$

Putting together (4.4.4), the *iid* character of the sequence  $(\tilde{\mathcal{L}}_j, \tilde{h}_j)_{1 \leq j \leq n_t}$ , Lemma 4.4.9 and Lemma 4.4.10, we obtain Proposition 4.4.5.

Thanks to this proposition, we can compare  $N_t$  with the overshoots of  $\sum_{i=1}^j e_i S_i^t R_i^t$ . For any  $a \geq 0$ , let us define

$$\mathcal{N}_a := \min \left\{ j \geq 0, \sum_{i=1}^j e_i S_i^t R_i^t > a \right\}.$$

**Lemma 4.4.11.** *Fix  $\epsilon$  as in Proposition 4.4.5 and  $\eta \in ]0, 1[$ . Assume that  $t$  is so large such that  $(1 - e^{-\epsilon h_t/7})^{-1} < (1 + \eta)$  and  $(1 + e^{-\epsilon h_t/7})^{-1} (1 - 2/\log(h_t)) \geq (1 - \eta)$ . Recall the events  $\mathcal{A}_t^1$  and  $\mathcal{A}_t^3$  introduced in respectively Fact 4.4.3 and Proposition 4.4.5. Then*

$$\mathcal{V}_{n_t, h_t} \cap \{N_t < n_t\} \cap \mathcal{A}_t^1 \cap \mathcal{A}_t^3 \subset \{\mathcal{N}_{(1-\eta)t} \leq N_t \leq \mathcal{N}_{(1+\eta)t}\}.$$

Even though it is used in the following subsection, an other interest of this Lemma is to prepare the further study of the almost sure behavior of  $\mathcal{L}_X^*(t)$ . Indeed, for the almost sure behavior, the contribution of the last valley can be sometimes omitted, sometimes totally included, so we are left to study some behavior of  $\sum_{i=1}^{N_t-1} e_i S_i^t R_i^t$  or  $\sum_{i=1}^{N_t} e_i S_i^t R_i^t$ , but the above lemma allows to replace  $N_t$  by some  $\mathcal{N}_a$ , which is more convenient since it only depends on the sequence  $(e_i S_i^t R_i^t, i \geq 1)$ .

*Démonstration.* Assume that the event  $\mathcal{V}_{n_t, h_t} \cap \{N_t < n_t\} \cap \mathcal{A}_t^1 \cap \mathcal{A}_t^3$  is realized. Then, for any  $k \leq n_t$  we have

$$\begin{aligned} N_t \geq k &\Rightarrow H(m_k) \leq t \Rightarrow H(\tilde{m}_k) \leq t \Rightarrow \sum_{i=1}^{k-1} \left( H(\tilde{L}_i) - H(\tilde{m}_i) \right) \leq t \\ &\Rightarrow \sum_{i=1}^{k-1} e_i S_i^t R_i^t \leq t(1 - e^{-\epsilon h_t/7})^{-1} \leq (1 + \eta)t \Rightarrow \mathcal{N}_{(1+\eta)t} \geq k, \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}_{(1-\eta)t} \geq k &\Rightarrow \sum_{i=1}^{k-1} e_i S_i^t R_i^t \leq (1 - \eta)t \Rightarrow \sum_{i=1}^{k-1} \left( H(\tilde{L}_i) - H(\tilde{m}_i) \right) \leq (1 - \eta)(1 + e^{-\epsilon h_t/7})t \\ &\Rightarrow H(\tilde{m}_k) \leq t \left[ (1 - \eta)(1 + e^{-\epsilon h_t/7}) + 2/\log(h_t) \right] \leq t \\ &\Rightarrow H(m_k) \leq t \Rightarrow N_t \geq k. \end{aligned}$$

We have thus proved that  $\mathcal{N}_{(1-\eta)t} \leq N_t \leq \mathcal{N}_{(1+\eta)t}$  is satisfied on  $\mathcal{V}_{n_t, h_t} \cap \{N_t < n_t\} \cap \mathcal{A}_t^1 \cap \mathcal{A}_t^3$ . □

### 4.4.3 Proof of Proposition 4.4.1

We now use the preceding results of this section to approach the repartition function of  $\mathcal{L}_X^*(t)/t$  by a repartition function involving the sequence  $(e_i S_i^t, e_i S_i^t R_i^t, i \geq 1)$ . We first state two facts which come from Lemmas 5.2 and 5.3 of [4].

**Fact 4.4.12.** *Fix  $\epsilon$  as in Proposition 4.4.5. Let us define the repartition functions, depending on  $t$ ,*

$$\begin{aligned} F_{\gamma, k}(x) &:= \mathbb{P} \left( \max_{1 \leq j \leq k-1} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \leq \gamma t, \sum_{j=1}^{k-1} (H(\tilde{L}_j) - H(\tilde{m}_j)) \leq xt \right), \\ F_{\gamma, k}^{\pm}(x) &:= \mathbb{P} \left( \max_{1 \leq j \leq k-1} e_j S_j^t \leq \gamma t(1 \pm 2e^{-\epsilon h_t/7}), \sum_{j=1}^{k-1} e_j S_j^t R_j^t \leq xt(1 \pm 2e^{-\epsilon h_t/7}) \right). \end{aligned}$$

*Then, there is a positive constant  $c$  such that for any  $2 \leq k \leq n_t$ ,  $0 < x \leq 1$  and  $\gamma > 0$  possibly depending on  $t$ ,*

$$F_{\gamma, k}^-(x) - e^{-ch_t} \leq F_{\gamma, k}(x) \leq F_{\gamma, k}^+(x) + e^{-ch_t},$$

*for all  $t$  large enough.*

*Démonstration.* This is a direct consequence of Proposition 4.4.5. □

Before stating the next fact we define  $X_{\tilde{m}_j} := X(\cdot + \tilde{m}_j)$  which is, according to the Markov property, a diffusion starting from  $\tilde{m}_j$ . We also define, for any  $r \in \mathbb{R}$ ,  $H_{X_{\tilde{m}_j}}(r)$  to be the hitting time of  $r$  by  $X_{\tilde{m}_j}$ .

**Fact 4.4.13.** *Fix  $\epsilon$  as in Proposition 4.4.5 and recall the definition of  $X_{\tilde{m}_1}$  in Subsection 4.5.6. We define the repartition functions, depending on  $t$ ,*

$$\begin{aligned} f_\gamma(x) &:= \mathbb{P} \left( \mathcal{L}_{X_{\tilde{m}_1}}(t(1-x), \tilde{m}_1) \leq \gamma t, H_{X_{\tilde{m}_1}}(\tilde{L}_1) > t(1-x), H_{X_{\tilde{m}_1}}(\tilde{L}_1) < H_{X_{\tilde{m}_1}}(\tilde{L}_0) \right), \\ \tilde{f}_\gamma(x) &:= \mathbb{P} \left( \sup_{y \in \mathcal{D}_1} \mathcal{L}_{X_{\tilde{m}_1}}(t(1-x), y) \leq \gamma t, H_{X_{\tilde{m}_1}}(\tilde{L}_1) > t(1-x), H_{X_{\tilde{m}_1}}(\tilde{L}_1) < H_{X_{\tilde{m}_1}}(\tilde{L}_0) \right), \\ f_\gamma^\pm(x) &:= \mathbb{P} \left( 1/R_1^t \leq \gamma(1 \pm 2e^{-cht/7})/(1-x), e_1 S_1^t R_1^t > t(1-x)(1 \mp 2e^{-cht/7}) \right). \end{aligned}$$

*Then, there is a positive constant  $c$  such that for any  $0 < x \leq 1$  and  $\gamma > 0$  possibly depending on  $t$ ,*

$$f_\gamma^-(x) - e^{-cht} \leq \tilde{f}_\gamma(x) \leq f_\gamma(x) \leq f_\gamma^+(x) + e^{-cht},$$

*for all  $t$  large enough.*

For the justification of this fact in our context, we give some details in the end of Section 4.5.

In order to generalize Lemma 5.4 of [4] to our context, let us recall the definition of the functionals  $\tilde{K}_{I,a}$  and  $\tilde{K}_{I,a}^-$  defined there :

$$\forall a > 0, \quad \tilde{K}_{I,a}(f_1, f_2) := f_2(f_2^{-1}(a)), \quad \tilde{K}_{I,a}^-(f_1, f_2) := f_2(f_2^{-1}(a)-).$$

Note that these functionals actually do not involve  $f_1$ . According to Lemma 4.5 of [4],  $(\mathcal{Y}_1, \mathcal{Y}_2)$  is almost surely a point of continuity for these functionals. Thanks to this we can prove :

**Lemma 4.4.14.**

$$\lim_{\eta \rightarrow 0} \limsup_{t \rightarrow +\infty} \left( s(\eta, t) := \sum_{k \leq n_t} \mathbb{P} \left( \frac{1}{t} \sum_{i=1}^k e_i S_i^t R_i^t > 1 - \eta/2, 1 - 2\eta < \frac{1}{t} \sum_{i=1}^{k-1} e_i S_i^t R_i^t \leq 1 - \eta \right) \right) = 0. \quad (4.4.24)$$

$$\lim_{\eta \rightarrow 0} \limsup_{t \rightarrow +\infty} (\tilde{s}(\eta, t) := 1 - \mathbb{P}(\eta t \leq H(m_{N_t}) \leq (1 - \eta)t)) = 0. \quad (4.4.25)$$

This result is less precise than Lemma 5.4 of [4] of which it is the analogue, but even in [4], they only need that the limits in  $t$  converge to 0 when  $\eta$  goes to 0. We can thus substitute our Lemma to theirs.

*Démonstration.* The events

$$\left\{ \frac{1}{t} \sum_{i=1}^k e_i S_i^t R_i^t > 1 - \eta/2, 1 - 2\eta < \frac{1}{t} \sum_{i=1}^{k-1} e_i S_i^t R_i^t \leq 1 - \eta \right\}$$

for  $k \in \{1, \dots, n_t\}$  are clearly disjoint so  $s(\eta, t)$  equals

$$\begin{aligned} & \mathbb{P} \left( \frac{1}{t} \sum_{i=1}^{\mathcal{N}_{(1-\eta)t}} e_i S_i^t R_i^t > 1 - \eta/2, 1 - 2\eta < \frac{1}{t} \sum_{i=1}^{\mathcal{N}_{(1-\eta)t-1}} e_i S_i^t R_i^t \leq 1 - \eta, \mathcal{N}_{(1-\eta)t} \leq n_t \right) \\ &= \mathbb{P} \left( Y_2^t(Y_2^{t,-1}(1-\eta)) > 1 - \eta/2, Y_2^t(Y_2^{t,-1}(1-\eta)-) > 1 - 2\eta, \mathcal{N}_{(1-\eta)t} \leq n_t \right) \\ &= \mathbb{P} \left( \tilde{K}_{I,1-\eta}^-(Y_2^t) > 1 - \eta/2, \tilde{K}_{I,1-\eta}^-(Y_2^t) > 1 - 2\eta, \mathcal{N}_{(1-\eta)t} \leq n_t \right). \end{aligned}$$

Now, according to Proposition 4.4.2 (to be proved in the following subsection),  $(Y_2^t, Y_2^t)$  converges to  $(\mathcal{Y}_2, \mathcal{Y}_2)$  for the convergence in distribution in  $D(\mathbb{R}_+, \mathbb{R}^2)$  with the  $J_1$  topology, and we have that for any fixed  $\eta \in ]0, 1/2[$ ,  $(\mathcal{Y}_1, \mathcal{Y}_2)$  is almost surely a point of continuity for  $\tilde{K}_{I,1-\eta}^-$  and  $\tilde{K}_{I,1-\eta}^-$ . We thus get

$$\limsup_{t \rightarrow +\infty} s(\eta, t) \leq \mathbb{P} \left( \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1-\eta)) > 1 - \eta/2, \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1-\eta)-) > 1 - 2\eta \right), \quad (4.4.26)$$

and we now study the limit when  $\eta$  goes to 0. Since  $\mathcal{Y}_2$  is a  $\kappa$ -stable subordinator, it is known that almost surely  $E_0 := 1 - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)-) > 0$  so  $\mathcal{Y}_2^{-1}(1-\eta) = \mathcal{Y}_2^{-1}(1)$  for all  $0 < \eta < E_0$  and  $\mathcal{Y}_2(\mathcal{Y}_2^{-1}(1-\eta)-) \leq 1 - 2\eta$  for all  $0 < \eta < E_0/2$ . This proves that almost surely, the event in the probability in (4.4.26) fails to happen for all  $\eta$  small enough, so by dominated convergence, this probability converges to 0 when  $\eta$  goes to 0. This proves (4.4.24).

Let us fix  $\epsilon$  as in Proposition 4.4.5, put  $\epsilon_t := e^{-\epsilon h_t/7}$  and choose  $t$  large enough so that  $2/\log h_t \leq \eta$ ,  $(1-\epsilon_t)^{-1} \leq 2$ ,  $(1-3\eta) \leq (1-2\eta)(1+\epsilon_t)^{-1}$ . Using successively the definitions of  $\mathcal{A}_t^1$  and  $\mathcal{A}_t^3$  and Lemma 4.4.11, we get that  $\mathbb{P}(\eta t \leq H(m_{N_t}) \leq (1-\eta)t)$  is greater than

$$\begin{aligned} & \mathbb{P}(\eta t \leq H(m_{N_t}) \leq (1-\eta)t, \mathcal{V}_{n_t, h_t}, N_t \leq n_t, \mathcal{A}_t^1 \cap \mathcal{A}_t^3) \\ & \geq \mathbb{P} \left( \eta t \leq \sum_{i=1}^{N_t-1} (H(\tilde{L}_i) - H(\tilde{m}_i)) \leq (1-2\eta)t \leq \left(1 - \eta - \frac{2}{\log h_t}\right) t, \mathcal{V}_{n_t, h_t}, \right. \\ & \quad \left. N_t \leq n_t, \mathcal{A}_t^1 \cap \mathcal{A}_t^3 \right) \\ & \geq \mathbb{P} \left( \eta(1-\epsilon_t)^{-1}t \leq 2\eta t \leq \sum_{i=1}^{N_t-1} e_i S_i^t R_i^t \leq (1-3\eta)t \leq (1-2\eta)(1+\epsilon_t)^{-1}t, \mathcal{V}_{n_t, h_t}, \right. \\ & \quad \left. N_t \leq n_t, \mathcal{A}_t^1 \cap \mathcal{A}_t^3 \right) \\ & \geq \mathbb{P} \left( 2\eta t \leq \sum_{i=1}^{\mathcal{N}_{(1-\eta)t-1}} e_i S_i^t R_i^t \leq \sum_{i=1}^{\mathcal{N}_{(1+\eta)t-1}} e_i S_i^t R_i^t \leq (1-3\eta)t, \mathcal{V}_{n_t, h_t}, N_t \leq n_t, \mathcal{A}_t^1 \cap \mathcal{A}_t^3 \right) \\ & = \mathbb{P} (2\eta \leq Y_2^t(Y_2^{t,-1}(1-\eta)-) \leq Y_2^t(Y_2^{t,-1}(1+\eta)-) \leq (1-3\eta), \mathcal{V}_{n_t, h_t}, N_t \leq n_t, \mathcal{A}_t^1 \cap \mathcal{A}_t^3) \\ & = \mathbb{P} \left( 2\eta \leq \tilde{K}_{I,1-\eta}^-(Y_2^t) \leq \tilde{K}_{I,1+\eta}^-(Y_2^t) \leq (1-3\eta), \mathcal{V}_{n_t, h_t}, N_t \leq n_t, \mathcal{A}_t^1 \cap \mathcal{A}_t^3 \right). \end{aligned}$$

According to Lemma 4.3.5, Fact 4.4.3, Proposition 4.4.5 and Lemma 4.4.16 of the following subsection, we have

$$\mathbb{P}(\mathcal{V}_{n_t, h_t} \cap \{N_t \leq n_t\} \cap \mathcal{A}_t^1 \cap \mathcal{A}_t^3) \xrightarrow{t \rightarrow +\infty} 1,$$

and again, from the convergence of  $(Y_2^t, Y_2^t)$  to  $(\mathcal{Y}_2, \mathcal{Y}_2)$  and the continuity of  $\tilde{K}_{I, 1-\eta}^-$ , we get

$$\limsup_{t \rightarrow +\infty} \tilde{s}(\eta, t) \leq 1 - \mathbb{P}(2\eta \leq \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1-\eta)-) \leq \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1+\eta)-) \leq (1-3\eta)). \quad (4.4.27)$$

Here again, we have almost surely  $E_0 := \min\{\mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)) - 1, 1 - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)-)\} > 0$  so  $\mathcal{Y}_2^{-1}(1+\eta) = \mathcal{Y}_2^{-1}(1)$  for all  $0 \leq \eta \leq E_0$  and  $\mathcal{Y}_2(\mathcal{Y}_2^{-1}(1+\eta)-) \leq (1-3\eta)$  for  $\eta \leq E_0/3$ . Also,  $\mathcal{Y}_2^{-1}(1-\eta) = \mathcal{Y}_2^{-1}(1)$  for all  $0 \leq \eta \leq E_0$  and  $\mathcal{Y}_2(\mathcal{Y}_2^{-1}(1-\eta)-) \geq 2\eta$  for  $0 \leq \eta \leq \min\{E_0, (1-E_0)/2\}$ . This proves that almost surely, the event in the probability in (4.4.27) happens for all  $\eta$  small enough, so by dominated convergence, this probability converges to 1 when  $\eta$  goes to 0. This proves (4.4.25).  $\square$

*Démonstration.* of Proposition 4.4.1 The proof of Proposition 5.1 of [4] can be identically repeated here. We only replace  $W_\kappa$  by  $V$ , consider the standard valleys to be defined in our sens, replace the random times  $\tilde{L}_j^-$  of [4] by  $\tilde{L}_{j-1}$ , and we use our estimates instead of theirs : Estimates (3.1), (3.2), (3.3), (5.2), (5.3), Lemmas 2.2, 3.2, 3.4, 4.1, 5.2, 5.3, Proposition 3.5 and the proof of Lemma 3.6 of [4] have to be replaced here by respectively Fact 4.4.3, the combination of Lemma 4.5.21 and Lemma 4.5.20, Lemma 4.4.16 of the following subsection, (4.4.24), (4.4.25), the combination of Lemma 4.3.5 and Remark 4.3.4, the second point of Fact 4.4.4, the third point of Fact 4.4.4, Lemma 4.4.15 of the following subsection, Fact 4.4.12, Fact 4.4.13, Proposition 4.4.5 and the discussion before Lemma 4.4.6.

The only difference is that, since our definition of  $\mathcal{D}_j$  is different, we do not need an analogue of (5.14) from [4] : it follows from our definition of  $\mathcal{D}_j$  that  $\mathcal{D}_j \subset [\tau_1^-(h_t/2), \tau_1(h_t/2)]$ , so we can use Lemma 4.4.8 to bound  $A^j(u)/A^j(\tilde{L}_j)$  on  $\mathcal{D}_j$ , which is the key to get the analogue of (5.17) from [4].  $\square$

#### 4.4.4 Proof of Proposition 4.4.2 and consequences

This proposition relies on :

**Lemma 4.4.15.** *Fix  $\eta \in ]0, 1/3[$  and recall the constant  $\mathcal{C}'$  defined in the Introduction. We have*

$$\lim_{t \rightarrow +\infty} \sup_{x \in [e^{-(1-2\eta)\phi(t)}, +\infty[} |x^\kappa e^{\kappa\phi(t)} \mathbb{P}(e_1 S_1^t / t > x) - \mathcal{C}'| = 0, \quad (4.4.28)$$

$$\lim_{t \rightarrow +\infty} \sup_{y \in [e^{-(1-3\eta)\phi(t)}, +\infty[} |y^\kappa e^{\kappa\phi(t)} \mathbb{P}(e_1 S_1^t R_1^t / t > y) - \mathcal{C}' \mathbb{E}[\mathcal{R}^\kappa]| = 0. \quad (4.4.29)$$

For any positive  $\alpha$ ,  $e^{\kappa\phi(t)}\mathbb{P}(e_1 S_1^t/t \geq x, e_1 S_1^t R_1^t/t \geq y)$  converges uniformly when  $t$  goes to infinity on  $[\alpha, +\infty[ \times [\alpha, +\infty[$  to  $\mathcal{C}' y^{-\kappa} \mathbb{E}[\mathcal{R}^\kappa \mathbf{1}_{\mathcal{R} \leq y/x}] + \mathcal{C}' x^{-\kappa} \mathbb{P}(\mathcal{R} > y/x)$ .

*Démonstration.* We start by proving a convergence analogue to (4.3) of [4] :

$$\lim_{t \rightarrow +\infty} \sup_{x \in [e^{-(1-\eta)\phi(t)}, +\infty[} |x^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1^t/t > x) - \mathcal{C}| = 0. \quad (4.4.30)$$

and we will deduce the first statement. First,

$$S_1^t = \int_{\tilde{\tau}_1^+(h_t/2)}^{\tilde{\tau}_1(h_t)} e^{\tilde{V}^{(1)}(u)} du + \int_{\tilde{\tau}_1(h_t)}^{\tilde{L}_1} e^{\tilde{V}^{(1)}(u)} du. \quad (4.4.31)$$

From Proposition 4.3.6 we get

$$\frac{1}{t} \int_{\tilde{\tau}_1(h_t)}^{\tilde{L}_1} e^{\tilde{V}^{(1)}(u)} du \stackrel{\mathcal{L}}{=} \frac{e^{h_t}}{t} \int_0^{\tau(V, ]-\infty, -h_t/2])} e^{V(y)} dy = e^{-\phi(t)} \int_0^{\tau(V, ]-\infty, -h_t/2])} e^{V(y)} dy, \quad (4.4.32)$$

and

$$\frac{1}{t} \int_{\tilde{\tau}_1(h_t/2)}^{\tilde{\tau}_1(h_t)} e^{\tilde{V}^{(1)}(u)} du \stackrel{\mathcal{L}}{=} \frac{1}{t} \int_{\tau(V^\uparrow, h_t/2)}^{\tau(V^\uparrow, h_t)} e^{V(y)} dy \leq e^{-\phi(t)} \tau(V^\uparrow, h_t). \quad (4.4.33)$$

By the second assertion of Lemma 4.5.3,

$$\mathbb{P}\left(\int_0^{+\infty} e^{V(y)} dy > u\right) \underset{u \rightarrow +\infty}{\sim} \mathcal{C} u^{-\kappa}. \quad (4.4.34)$$

We define  $\epsilon_t := e^{-\eta\phi(t)/2}$  and take  $x \in [e^{-(1-\eta)\phi(t)}, +\infty[$  (this implies  $x e^{\phi(t)} \geq e^{\eta\phi(t)}$  which converges to infinity). From (4.4.31), (4.4.32), and (4.4.33),  $x^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1^t/t > x)$  is less than

$$x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\tau(V^\uparrow, h_t) > x e^{\phi(t)} \epsilon_t) + x^\kappa e^{\kappa\phi(t)} \mathbb{P}\left(\int_0^{+\infty} e^{V(y)} dy > (1 - \epsilon_t) x e^{\phi(t)}\right).$$

From Lemma 4.5.6, the limit of the first term is 0, and from (4.4.34) the limit of the first second is  $\mathcal{C}$ . We get

$$\limsup_{t \rightarrow +\infty} \sup_{x \in [e^{-(1-\eta)\phi(t)}, +\infty[} x^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1^t/t > x) \leq \mathcal{C}. \quad (4.4.35)$$

From the Markov property applied at time  $\tau(V, ]-\infty, -h_t/2])$  we have

$$\begin{aligned} e^{-\phi(t)} \int_0^{+\infty} e^{V(y)} dy &\stackrel{\mathcal{L}}{=} e^{-\phi(t)} \int_0^{\tau(V, ]-\infty, -h_t/2])} e^{V(y)} dy \\ &\quad + e^{V(\tau(V, ]-\infty, -h_t/2]) - \phi(t)} \int_0^{+\infty} e^{\tilde{V}(y)} dy, \end{aligned}$$

where  $\tilde{V}$  is an independent copy of  $V$ . We now put  $\epsilon_t := e^{-h_t/4}$  and note that  $V(\tau(V, ] - \infty, -h_t/2]) < -h_t/2$ . Then,  $\mathbb{P}(e^{-\phi(t)} \int_0^{+\infty} e^{V(y)} dy > x(1 + \epsilon_t))$  is less than

$$\begin{aligned} & \mathbb{P} \left( e^{-\phi(t)} \int_0^{\tau(V, ] - \infty, -h_t/2])} e^{V(y)} dy > x \right) + \mathbb{P} \left( e^{-h_t/2 - \phi(t)} \int_0^{+\infty} e^{\tilde{V}(y)} dy > x\epsilon_t \right) \\ & \leq \mathbb{P}(S_1^t/t > x) + \mathbb{P} \left( e^{-h_t/2 - \phi(t)} \int_0^{+\infty} e^{\tilde{V}(y)} dy > x\epsilon_t \right), \end{aligned}$$

where we used (4.4.32) and (4.4.31) for the first term. Using (4.4.34) in the above inequality we get

$$\liminf_{t \rightarrow +\infty} \inf_{x \in [e^{-(1-\eta)\phi(t)}, +\infty[} x^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1^t/t > x) \geq \mathcal{C}.$$

Combining this with (4.4.35) we get 4.4.30.

Now that we have (4.4.30), the rest of the proof is exactly the same as the proof of Lemma 4.1 in [4] once they have proved (4.3). The argument crucially needs the fact that  $(R_1^t)_{t>1}$  converges in distribution to  $\mathcal{R}$  and is bounded in all  $L^p$  spaces. This is true from Proposition 4.3.8 applied with  $h = h_t$ , so the lemma is proved.  $\square$

In [4], the proof of Proposition 1.4 (that is, the convergence of  $(Y_1, Y_2)^t$  toward  $(\mathcal{Y}_1, \mathcal{Y}_2)$ ) relies only on their Lemma 4.1 from which is proved the tightness of the family  $(Y_1, Y_2)^t$  and the identification of the limit distribution. Using Lemma 4.4.15 instead of Lemma 4.1 of [4], the same proof can be repeated here and we get Proposition 4.4.2.

As an other consequence of Lemma 4.4.15, we can prove that no more than  $n_t$  valleys are visited until instant  $t$ . This is fundamental since most of the estimates we have are true not for all but for the first  $n_t$  valleys.

**Lemma 4.4.16.** *There is a positive constant  $c$  such that for all  $t$  large enough,*

$$\mathbb{P}(N_t \geq n_t) \leq e^{-cht}.$$

*Démonstration.* We have  $\{N_t \geq n_t\} = \{H(m_{n_t}) \leq t\}$  and  $\sum_{i=1}^{n_t-1} H(\tilde{L}_i) - H(\tilde{m}_i) \leq H(\tilde{m}_{n_t}) = H(m_{n_t})$  on  $\mathcal{V}_{n_t, h_t}$ . Let us fix  $\epsilon$  as in Proposition 4.4.5. Using the definition of  $\mathcal{A}_t^3$  there, we get that  $\mathbb{P}(\{N_t \geq n_t\} \cap \mathcal{V}_{n_t, h_t} \cap \mathcal{A}_t^3)$  is less than

$$\begin{aligned} & \mathbb{P} \left( \sum_{i=1}^{n_t-1} H(\tilde{L}_i) - H(\tilde{m}_i) \leq t, \mathcal{A}_t^3 \right) \leq \mathbb{P} \left( \sum_{i=1}^{n_t-1} e_i S_i^t R_i^t \leq t(1 - e^{-cht/7})^{-1} \right) \\ & \leq \mathbb{P} \left( \sup_{1 \leq i \leq n_t-1} e_i S_i^t R_i^t \leq t(1 - e^{-cht/7})^{-1} \right) \leq [1 - \mathbb{P}(e_1 S_1^t R_1^t / t > (1 - e^{-cht/7})^{-1})]^{n_t-1} \\ & \leq \exp \left[ -(n_t - 1) \mathbb{P}(e_1 S_1^t R_1^t / t > (1 - e^{-cht/7})^{-1}) \right], \end{aligned}$$

where the last inequality comes from  $\log(1-x) \leq -x$  for  $x \in [0, 1]$ . According to Lemma 4.4.15 and the definition of  $n_t$ , we have that

$$(n_t - 1)\mathbb{P}(e_1 S_1^t R_1^t / t > (1 - e^{-\epsilon h_t / 7})^{-1}) \underset{t \rightarrow +\infty}{\sim} C' \mathbb{E}[\mathcal{R}^\kappa] e^{\kappa \delta \phi(t)},$$

so  $\mathbb{P}(\{N_t \geq n_t\} \cap \mathcal{V}_{n_t, h_t} \cap \mathcal{A}_t^3) \leq e^{-h_t}$  for  $t$  large enough. Also,  $\mathbb{P}(\mathcal{V}_{n_t, h_t}^c) \leq n_t e^{-\delta \kappa h_t / 3}$  for  $t$  large enough according to Lemma 4.3.5 and  $\mathbb{P}(\mathcal{A}_t^{3,c})$  has an upper bound given by Proposition 4.4.5. The result follows.  $\square$

## 4.5 Some estimates on $V$ , $V^\uparrow$ , $\hat{V}^\uparrow$ and the diffusion in $V$

In this section, we prove some estimates for the processes  $V$ ,  $V^\uparrow$  and  $\hat{V}^\uparrow$ , especially about the hitting times and the exponential functionals of these processes. We also prove some facts used in Section 4.4. Even though Section 4.4 gives the main ideas, the estimates we prove here actually represent the biggest part of the work for the proof of Theorem 4.1.3. Some of them are rather classical but some others, especially those of Subsection 4.5.5, are new and technical. They are used in this paper in place of simpler estimates that are true in the case of a drifted brownian potential, but not for a general spectrally negative Lévy potential.

### 4.5.1 Estimates on $V$

**Lemma 4.5.1.** *Let  $a, b$  be positive numbers and define  $T := \inf\{x \geq 0, V(x) \notin [-a, b]\}$ , then*

$$(1 - e^{-\kappa a})e^{-\kappa b} \leq \mathbb{P}(V(T) = b) \leq e^{-\kappa b}.$$

*Démonstration.* For the upper bound,  $\mathbb{P}(V(T) = b) \leq \mathbb{P}(\sup_{[0, +\infty[} V \geq b) = e^{-\kappa b}$ .

For the lower bound, note that the process  $e^{\kappa V(\cdot \wedge T)}$  is a bounded martingale so, by the convergence theorem for martingales and the dominated convergence theorem we get

$$\begin{aligned} 1 &= \mathbb{E}[e^{\kappa V(T)}] = \mathbb{P}(V(T) \leq -a) \mathbb{E}[e^{\kappa V(T)} | V(T) \leq -a] + \mathbb{P}(V(T) = b) e^{\kappa b} \\ &\leq e^{-\kappa a} + \mathbb{P}(V(T) = b) e^{\kappa b}, \end{aligned}$$

which yields the result.  $\square$

We now study how  $V$  leaves an interval from below. More precisely, we control the moments of  $V(\tau(V, ] - \infty, -1])$ . This is where the assumption  $V(1) \in L^p$  for some  $p > 1$  becomes necessary for Theorem 4.1.3.



**Lemma 4.5.2.**

$$\forall p \geq 1, V(1) \in L^p \Rightarrow V(\tau(V, ] - \infty, -1]) \in L^p.$$

*Démonstration.* We use  $V^{<-r}$  as defined in Subsection 4.1.2, where  $r$  is chosen such that  $V - V^{<-r}$  drifts to  $-\infty$ . Let  $\kappa_r$  denote the non trivial zero of  $\Psi_{V-V^{<-r}}$ . We fix  $x \geq r$ . We have

$$\begin{aligned} \mathbb{P}(|1 + V(\tau(V, ] - \infty, -1])| > x) &\leq \mathbb{P}(\tau(\Delta V, ] - \infty, -x] \leq \tau(V, ] - \infty, -1]) \\ &\leq \mathbb{P}(V(\tau(\Delta V, ] - \infty, -x]) \geq -1) \\ &\leq \mathbb{P}((V - V^{<-r})(\tau(\Delta V, ] - \infty, -x]) \geq -1). \end{aligned}$$

Then, since  $x > r$ , we have  $\tau(\Delta V, ] - \infty, -x] = \tau(\Delta V^{<-r}, ] - \infty, -x])$  which is independent from  $(V - V^{<-r})$ , because  $(V - V^{<-r})$  and  $V^{<-r}$  are independent. We thus get that  $\mathbb{P}(|1 + V(\tau(V, ] - \infty, -1])| > x)$  is less than

$$\begin{aligned} \mathbb{P}((V - V^{<-r})(\tau(\Delta V^{<-r}, ] - \infty, -x]) \geq -1) &\leq e^{\kappa_r/2} \mathbb{E} \left[ e^{\kappa_r(V - V^{<-r})(\tau(\Delta V^{<-r}, ] - \infty, -x]) / 2} \right] \\ &= e^{\kappa_r/2} \mathbb{E} \left[ e^{\Psi_{V - V^{<-r}}(\kappa_r/2) \tau(\Delta V^{<-r}, ] - \infty, -x]} \right], \end{aligned}$$

where we used Markov's inequality. Then, note that  $\Psi_{V - V^{<-r}}(\kappa_r/2) < 0$  thanks to the definition of  $\kappa_r$ , and that  $\tau(\Delta V^{<-r}, ] - \infty, -x])$  follows an exponential distribution with parameter  $\nu(] - \infty, -x])$ . We thus get

$$\begin{aligned} \mathbb{P}(|1 + V(\tau(V, ] - \infty, -1])| > x) &\leq \frac{e^{\kappa_r/2}}{1 - \Psi_{V - V^{<-r}}(\kappa_r/2) / \nu(] - \infty, -x])} \\ &\leq C \nu(] - \infty, -x]), \end{aligned} \tag{4.5.1}$$

where we put  $C := -e^{\kappa_r/2} / \Psi_{V - V^{<-r}}(\kappa_r/2) > 0$ . We now choose  $p \geq 1$  and assume  $V(1) \in L^p$ . Theorem 25.3 in [59] implies that  $\int_{-\infty}^{-r} |x|^p \nu(dx) < +\infty$ , or equivalently  $\int_r^{+\infty} x^{p-1} \nu(] - \infty, -x]) dx < +\infty$ . Using (4.5.1) we deduce that

$$\int_{-\infty}^{-r} x^{p-1} \mathbb{P}(|1 + V(\tau(V, ] - \infty, -1])| > x) dx < +\infty,$$

so  $V(\tau(V, ] - \infty, -1]) \in L^p$ . □

The next lemma is fundamental. In Section 4.4, it allows us to compute precisely the right tails of the contributions to the local and to the time spent by the diffusion in the bottoms of the valleys. This allows to prove that the sum of these contributions converges to the  $\kappa$ -stable subordinator  $(\mathcal{Y}_1, \mathcal{Y}_2)$  from which is constructed the limit distribution in Theorem 4.1.3.

**Lemma 4.5.3.** — *There is a positive constant  $c$  such that for  $x$  small enough,*

$$\mathbb{P}\left(\int_0^{+\infty} e^{V(u)} du \leq x\right) \leq c\sqrt{x}.$$

—

$$\mathbb{P}\left(\int_0^{+\infty} e^{V(u)} du \geq x\right) \underset{x \rightarrow +\infty}{\sim} \mathcal{C}x^{-\kappa},$$

where as in the Introduction,  $\mathcal{C}$  is the constant in Corollary 5 of [12] applied to  $-V$ .

*Démonstration.* For the first assertion, our argument is very close to the one given at the beginning of the proof of Theorem 3 in [12], but to be into their setting we would have to assume the existence of finite exponential moments for  $-V$  which we do not. We thus give some details.

We use  $V^{<-1}$  as defined in Subsection 4.1.2. For  $x > 0$ , and  $\eta > 0$  that will be chosen later, the probability  $\mathbb{P}(\int_0^{+\infty} e^{V(u)} du \leq x)$  equals

$$\begin{aligned} & \mathbb{P}\left(\int_0^{+\infty} e^{V(u)} du \leq x, V^{<-1}(\eta) = 0\right) + \mathbb{P}\left(\int_0^{+\infty} e^{V(u)} du \leq x, V^{<-1}(\eta) \neq 0\right) \\ & \leq \mathbb{P}\left(\int_0^\eta e^{(V-V^{<-1})(u)} du \leq x, V^{<-1}(\eta) = 0\right) + \mathbb{P}(V^{<-1}(\eta) \neq 0) \\ & \leq \mathbb{P}\left(\int_0^\eta e^{(V-V^{<-1})(u)} du \leq x\right) + 1 - e^{-\eta\nu(]-\infty, -1])}, \end{aligned}$$

because  $V^{<-1}$  is a compound Poisson process that jumps at rate  $\nu(]-\infty, -1])$ . We get

$$\mathbb{P}\left(\int_0^{+\infty} e^{V(u)} du \leq x\right) \leq \mathbb{P}\left(\int_0^\eta e^{(V-V^{<-1})(u)} du \leq x\right) + \eta\nu(]-\infty, -1]). \quad (4.5.2)$$

$V - V^{<-1}$  is a Lévy process with bounded jumps so the arguments of [12] apply and yield

$$\begin{aligned} \mathbb{P}\left(\int_0^\eta e^{(V-V^{<-1})(u)} du \leq x\right) & \leq \mathbb{P}\left(\sup_{[0,\eta]} -(V - V^{<-1}) \geq \log(\eta/x)\right) \\ & \leq 2\mathbb{P}\left(-(V - V^{<-1})(\eta) \geq \log(\eta/x)\right) \\ & \leq \frac{2x}{\eta} \mathbb{E}\left[e^{-(V-V^{<-1})(\eta)}\right] \\ & = \frac{2x}{\eta} e^{\eta\Psi_{-(V-V^{<-1})}(1)}, \end{aligned}$$

where we used Markov's inequality. The crucial point is that the Laplace exponent at 1 :  $\Psi_{-(V-V^{<-1})}(1)$  is defined and finite since  $V - V^{<-1}$ , having bounded jumps, has

a Laplace transform defined on the whole complex plane (Theorem 25.3 in [59]). Now, combining with (4.5.2) and choosing  $\eta = \sqrt{x}$ , we get, for any  $x$  small enough so that  $e^{\sqrt{x}\Psi_{-(V-V^{<-1})}(1)} \leq 2$ ,

$$\mathbb{P}\left(\int_0^{+\infty} e^{V(u)} du \leq x\right) \leq (4 + \nu(\cdot - \infty, -1]) \sqrt{x},$$

which yields the first assertion.

We now prove the second assertion, it is only an application of Corollary 5 of [12] to  $-V$ . Since  $V$  is spectrally negative, we know that  $V$  has a Laplace transform which is defined on  $[0, +\infty[$  and the  $\theta$  in the corollary is the non-trivial zero of the Laplace exponent of  $V$ , that is  $\kappa$ . To check the Cramer's condition, we use the decomposition  $V = (V - V^{<-1}) + V^{<-1}$ . We have

$$\begin{aligned} \mathbb{E}[|V(x)| \exp(\kappa V(x))] &\leq \mathbb{E}[|(V - V^{<-1})(x)| \exp(\kappa(V - V^{<-1})(x) + \kappa V^{<-1}(x))] \\ &\quad + \mathbb{E}[|V^{<-1}(x)| \exp(\kappa(V - V^{<-1})(x) + \kappa V^{<-1}(x))] \\ &\leq \mathbb{E}[|(V - V^{<-1})(x)| \exp(\kappa(V - V^{<-1})(x))] \\ &\quad + M \mathbb{E}[\exp(\kappa(V - V^{<-1})(x))]. \end{aligned}$$

where we used the fact that  $V^{<-1}(x) \leq 0$  for the first term and, for the second term, the fact that, since  $V^{<-1}(x) \leq 0$ ,  $|V^{<-1}(x)| \exp(\kappa V^{<-1}(x))$  is deterministically bounded by  $M$ , the constant bounding  $(y \mapsto ye^{-\kappa y})$  on  $\mathbb{R}_+$ . Then, since  $V - V^{<-1}$  is a Lévy process with bounded jumps,  $(V - V^{<-1})(x)$  admits finite exponential moments of any positive and negative order (see Theorem 25.3 in [59]). As a consequence the above expression is finite which yields the Cramer's condition. Finally,  $V$  is indeed not arithmetic because it is spectrally negative and not the opposite of a subordinator. The hypothesis of Corollary 5 of [12] are thus satisfied for  $-V$  and we get the second assertion. □

**Lemma 4.5.4.** *There are two positive constants  $c_1, c_2$  such that*

$$\forall y, r > 0, \mathbb{P}(\tau(V, ] - \infty, -y]) > r) \leq e^{c_1 y - c_2 r}.$$

*Démonstration.* Let us choose  $c_1 \in ]0, \kappa[$  and define  $c_2 := -\Psi_V(c_1)$ .  $c_2$  is positive because of the definition of  $\kappa$ . We have

$$\mathbb{P}(\tau(V, ] - \infty, -y]) > r) \leq \mathbb{P}(V(r) > -y) \leq e^{c_1 y} \mathbb{E}[e^{c_1 V(r)}] = e^{c_1 y - c_2 r},$$

where we used Markov's inequality. □

**Lemma 4.5.5.** *Choose  $\eta \in ]0, 1[$ , then*

$$\forall a, b > 0, \mathbb{P}(\tau(V - \underline{V}, a) < \tau(\underline{V}, ] - \infty, -b]) \leq (b/\eta a + 1)e^{-\kappa(1-\eta)a}.$$

*Démonstration.* We first remark that

$$\{\tau(V - \underline{V}, a) < \tau(\underline{V}, ] - \infty, -\eta a)\} \subset \left\{ \sup_{[0, +\infty[} V > (1 - \eta)a \right\}$$

so

$$\mathbb{P}(\tau(V - \underline{V}, a) < \tau(\underline{V}, ] - \infty, -\eta a)) \leq e^{-\kappa(1-\eta)a}. \quad (4.5.3)$$

In order to establish a boundary with  $b$  instead of  $\eta a$ , we define the sequence of stopping times  $(T_i)_{i \geq 0}$  by  $T_0 := 0$  and

$$T_{i+1} := \min(\tau(V^{T_i} - \underline{V}^{T_i}, a), \tau(\underline{V}^{T_i}, ] - \infty, -\eta a)).$$

We have

$$\bigcap_{i=0}^{\lfloor b/\eta a + 1 \rfloor} \{\tau(V^{T_i} - \underline{V}^{T_i}, a) \geq \tau(\underline{V}^{T_i}, ] - \infty, -\eta a)\} \subset \{\tau(V - \underline{V}, a) \geq \tau(\underline{V}, ] - \infty, -b)\},$$

and by the Markov property applied at the stopping times  $T_i$ , the events in the intersection have all the same probability, so taking the complementary

$$\begin{aligned} \mathbb{P}(\tau(V - \underline{V}, a) < \tau(\underline{V}, ] - \infty, -b)) &\leq \lfloor b/\eta a + 1 \rfloor \mathbb{P}(\tau(V - \underline{V}, a) < \tau(\underline{V}, ] - \infty, -\eta a)) \\ &\leq (b/\eta a + 1)e^{-\kappa(1-\eta)a}. \end{aligned}$$

□

## 4.5.2 Estimates on $V^\uparrow$

We define  $V^\sharp$  to be " $V$  conditioned to drift to  $+\infty$ ", as in [8], page 193. The Laplace exponent  $\Psi_{V^\sharp}$  of  $V^\sharp$  satisfies  $\Psi_{V^\sharp} = \Psi_V(\kappa + \cdot)$ . As a consequence  $\Psi'_{V^\sharp}(0) > 0$ , so  $V^\sharp$  drift to infinity (because of Corollary VII.2 in [8]) and it is also proven that  $V^\uparrow = (V^\sharp)^\uparrow$ . Therefore, for  $x > 0$ ,  $V_x^\uparrow$  is only  $V_x^\sharp$  conditioned in the usual sense to remain positive, which is a useful property for our proofs.

Also, note that from the definition of the law of  $V^\sharp$  in [8], we have

$$\forall \lambda > -\kappa, \mathbb{E} \left[ e^{\lambda V^\sharp(1)} \right] = \mathbb{E} \left[ e^{(\lambda + \kappa)V(1)} \right] < +\infty, \quad (4.5.4)$$

so the spectrally negative Lévy process  $V^\sharp$  has its Laplace transform, as well as its Laplace exponent  $\Psi_{V^\sharp}$ , defined on the half-plane  $\{\mathcal{K}(z) > -\kappa\}$ .

**Lemma 4.5.6.** *There are two positive constants  $c_1, c_2$  such that*

$$\forall y, r > 0, \mathbb{P}(\tau(V^\uparrow, y) > r) \leq e^{c_1 y - c_2 r}.$$

*Démonstration.* Fix  $y$  and  $r > 0$ . From the first point of Lemma 2.6 of [72] we have

$$\mathbb{P}(\tau(V^\uparrow, y) > r) \leq \mathbb{P}(\tau(V^\sharp, y) > r). \quad (4.5.5)$$

Then,  $V^\sharp$  is a spectrally negative Lévy process, so, according to Theorem 1 page 189 in [8], the process  $\tau(V^\sharp, \cdot)$  is a subordinator which Laplace exponent  $\Phi_{V^\sharp}$  is defined for  $\lambda \geq 0$  by

$$\Phi_{V^\sharp}(\lambda) := -\log\left(\mathbb{E}\left[e^{-\lambda\tau(V^\sharp, 1)}\right]\right),$$

and we have  $\Phi_{V^\sharp} = \Psi_{V^\sharp}^{-1}$ .

From the discussion before the lemma, we know that  $V^\sharp$  has its Laplace transform, as well as its Laplace exponent  $\Psi_{V^\sharp}$ , defined in a neighborhood of 0. Then, since  $\Psi'_{V^\sharp}(0) > 0$ , the holomorphic local inversion theorem tells us that  $\Psi_{V^\sharp}^{-1}$ , that is  $\Phi_{V^\sharp}$ , extends in a neighborhood of 0.

Therefore, the subordinator  $\tau(V^\sharp, \cdot)$  has a Laplace transform defined in a neighborhood of 0. From Markov inequality we get, for a positive  $c_2$  in this neighborhood,

$$\mathbb{P}(\tau(V^\sharp, y) > r) \leq e^{-c_2 r} \mathbb{E}\left[e^{c_2 \tau(V^\sharp, y)}\right] = e^{-y\Phi_{V^\sharp}(-c_2)} e^{-c_2 r},$$

Note that  $c_1 := -\Phi_{V^\sharp}(-c_2)$  is positive. Combining with (4.5.5) we get the result.  $\square$

**Lemma 4.5.7.** *There are two positive constants  $c_1, c_2$  such that, for all  $1 < a < b$ , we have*

$$\mathbb{P}\left(\inf_{[0, +\infty[} V_b^\uparrow < a\right) \leq c_2 e^{-c_1(b-a)}.$$

*Démonstration.*  $V_b^\uparrow$  is only  $V_b^\sharp$  conditioned in the usual sense to stay positive, so

$$\begin{aligned} \mathbb{P}\left(\inf_{[0, +\infty[} V_b^\uparrow < a\right) &= \mathbb{P}\left(0 < \inf_{[0, +\infty[} V_b^\sharp < a\right) / \mathbb{P}\left(\inf_{[0, +\infty[} V_b^\sharp > 0\right) \\ &\leq \mathbb{P}\left(\inf_{[0, +\infty[} V^\sharp < a - b\right) / \mathbb{P}\left(\inf_{[0, +\infty[} V^\sharp > -b\right). \end{aligned}$$

Then, according to [8] page 192, the Laplace transform of  $\inf_{[0, +\infty[} V^\sharp$  is given by

$$\forall \lambda \geq 0, \mathbb{E}\left[e^{\lambda \inf_{[0, +\infty[} V^\sharp}\right] = \Psi'_{V^\sharp}(0) \frac{\lambda}{\Psi_{V^\sharp}(\lambda)}. \quad (4.5.6)$$

As we said in the beginning of this subsection,  $\Psi_{V^\sharp}$  extends analytically on a neighborhood of 0 and has a non null derivative at 0, so (4.5.6) extends too and  $\inf_{[0, +\infty[} V^\sharp$  admits a Laplace transform on a neighborhood of 0. Choosing  $c_1 > 0$  such that  $-c_1$  is in this neighborhood and  $c_2 := \mathbb{E}[e^{-c_1 \inf_{[0, +\infty[} V^\sharp}] / \mathbb{P}(\inf_{[0, +\infty[} V^\sharp > -1)$ , we get the result by Chernoff's inequality.  $\square$

**Lemma 4.5.8.** *There are two positive constants  $c_1$  and  $c_2$  such that for any  $0 < \alpha < \omega$ ,  $\eta \in ]0, 1[$  and all  $h$  large enough, we have*

$$P(\tau(V^\uparrow, \omega h) - \tau(V^\uparrow, \alpha h) \leq 1) \leq e^{-c_1(\omega - \alpha)h}, \quad (4.5.7)$$

$$\mathbb{P}\left(\int_0^{\tau(V^\uparrow, h)} e^{V^\uparrow(u)} du \geq e^{(1-\eta)h}\right) \geq 1 - e^{-c_2\eta h/2}, \quad (4.5.8)$$

$$\mathbb{P}\left(\int_0^{\tau(V^\uparrow, h)} e^{V^\uparrow(u)} du \leq e^{(1+\eta)h}\right) \geq 1 - e^{-h}. \quad (4.5.9)$$

*Démonstration.* From the Markov property at time  $\tau(V^\uparrow, \alpha h)$  and the fact that  $V_{\alpha h}^\uparrow$  is  $V_{\alpha h}^\sharp$  conditioned in the usual sense to stay positive we have that  $\mathbb{P}(\tau(V^\uparrow, \omega h) - \tau(V^\uparrow, \alpha h) \leq 1)$  equals

$$\begin{aligned} \mathbb{P}(\tau(V_{\alpha h}^\uparrow, \omega h) \leq 1) &= \mathbb{P}\left(\tau(V_{\alpha h}^\sharp, \omega h) \leq 1, \inf_{[0, +\infty[} V_{\alpha h}^\sharp > 0\right) / \mathbb{P}\left(\inf_{[0, +\infty[} V_{\alpha h}^\sharp > 0\right) \\ &= \mathbb{P}\left(\tau(V^\sharp, (\omega - \alpha)h) \leq 1, \inf_{[0, +\infty[} V^\sharp > -\alpha h\right) / \mathbb{P}\left(\inf_{[0, +\infty[} V^\sharp > -\alpha h\right) \\ &\leq 2\mathbb{P}(\tau(V^\sharp, (\omega - \alpha)h) \leq 1), \end{aligned}$$

for  $h$  large enough,

$$\leq 2e \times \mathbb{E}\left[e^{-\tau(V^\sharp, (\omega - \alpha)h)}\right] = 2e \times e^{-\psi_{V^\sharp}^{-1}(1)(\omega - \alpha)h},$$

where we used Chernoff's inequality, the fact that  $\tau(V^\sharp, \cdot)$  is a subordinator with Laplace exponent  $\psi_{V^\sharp}^{-1}$  (see Theorem VII.1 in [8]) and the fact that  $\psi_{V^\sharp}^{-1}(1) > 0$  (because  $\psi_{V^\sharp}(0) = 0$ ).

(4.5.8) is an easy consequence of Lemma 4.5.7 applied with  $a = (1 - \eta)h$ ,  $b = (1 - \eta/2)h$ , and (4.5.7) applied with  $\alpha = 1 - \eta/2$ ,  $\omega = 1$ .

Then, we have obviously,

$$\int_0^{\tau(V^\uparrow, h)} e^{V^\uparrow(u)} du \leq e^h \tau(V^\uparrow, h),$$

so

$$\mathbb{P}\left(\int_0^{\tau(V^\uparrow, h)} e^{V^\uparrow(u)} du > e^{(1+\eta)h}\right) \leq \mathbb{P}(\tau(V^\uparrow, h) > e^{\eta h}) \leq e^{-h},$$

for  $h$  large enough according to Lemma 4.5.6. This yields (4.5.9). □

### 4.5.3 Estimates on $\hat{V}^\uparrow$

The aim of this subsection is to get, for  $\hat{V}^\uparrow$ , estimates similar to those that we proved for  $V^\uparrow$  in the last subsection. We start with a generalization of Lemma 4.5.7.

**Lemma 4.5.9.** *For all  $z \geq 0$  and  $0 < a < b$  with  $b > z$ , we have,*

$$\mathbb{P} \left( \inf_{[\tau(\hat{V}_z^\uparrow), [b, +\infty[), +\infty[} \hat{V}_z^\uparrow < a \right) \leq e^{-\kappa(b-a)} / (1 - e^{-\kappa b}).$$

*Démonstration.* Let  $T := \tau(\hat{V}_z^\uparrow, [b, +\infty[)$ .  $T$  is a stopping time so, if  $U$  is a random variable having the same law as  $\hat{V}_z^\uparrow(T)$ , then  $\hat{V}_z^\uparrow(T + \cdot)$  is equal in law to  $\hat{V}_U^\uparrow$ , that is, the Markov process that conditionally on  $\{U = u\}$  has law  $\hat{V}_u^\uparrow$ . We thus have

$$\mathbb{P} \left( \inf_{[T, +\infty[} \hat{V}_z^\uparrow < a \right) = \mathbb{P} \left( \inf_{[0, +\infty[} \hat{V}_U^\uparrow < a \right) = \int_b^{+\infty} \mathbb{P} \left( \inf_{[0, +\infty[} \hat{V}_u^\uparrow < a \right) \mathcal{L}_U(du), \quad (4.5.10)$$

because almost surely,  $U \geq b$ . Now, since  $\hat{V}_u^\uparrow$  is only  $\hat{V}_u$  conditioned in the usual sens to remain positive, we have for any  $u \geq b$ ,

$$\begin{aligned} \mathbb{P} \left( \inf_{[0, +\infty[} \hat{V}_u^\uparrow < a \right) &= \mathbb{P} \left( 0 < \inf_{[0, +\infty[} \hat{V}_u < a \right) / \mathbb{P} \left( \inf_{[0, +\infty[} \hat{V}_u > 0 \right) \\ &\leq \mathbb{P} \left( \inf_{[0, +\infty[} \hat{V} < a - u \right) / \mathbb{P} \left( \inf_{[0, +\infty[} \hat{V} > -u \right) \\ &\leq \mathbb{P} \left( \inf_{[0, +\infty[} \hat{V} < a - b \right) / \mathbb{P} \left( \inf_{[0, +\infty[} \hat{V} > -b \right), \end{aligned}$$

where we used the fact that  $a - u \leq a - b$  for the numerator and the fact that  $u \geq b$  for the denominator. Since  $\hat{V}$  is the dual of  $V$ , the above estimate can be re-written

$$\mathbb{P} \left( \inf_{[0, +\infty[} \hat{V}_u^\uparrow < a \right) \leq \mathbb{P} \left( \sup_{[0, +\infty[} V > b - a \right) / \mathbb{P} \left( \sup_{[0, +\infty[} V < b \right) = e^{-\kappa(b-a)} / (1 - e^{-\kappa b}).$$

Putting into (4.5.10) we get the result. □

The proof of Lemma 4.5.6 relies on the fact that the hitting times of  $V^\uparrow$  are stochastically smaller than the hitting times of  $V^\sharp$ , for which we have precise estimates. Here, the procedure to link the hitting times of  $\hat{V}^\uparrow$  and  $\hat{V}$  is different. This is what we do now.

Let us fix  $x > 0$ . Let  $m_x$  be the point where the process  $\hat{V}_x^\uparrow$  reaches its infimum,  $m_x := \sup\{s \geq 0, \hat{V}_x^\uparrow(s-) \wedge \hat{V}_x^\uparrow(s) = \inf_{[0, +\infty[} \hat{V}_x^\uparrow\}$ . Note that from the absence of negative jumps, the infimum is always reached at least at  $m_x -$  so  $\hat{V}_y^\uparrow(m_x -) = \inf_{[0, +\infty[} \hat{V}_x^\uparrow$ . The next lemma is contained in Theorem 24 of [31].

**Lemma 4.5.10.** *Assume  $V$  has unbounded variations, then*

$$\left( \hat{V}_x^\uparrow(m_x + s) - \hat{V}_x^\uparrow(m_x -), s \geq 0 \right) \stackrel{\mathcal{L}}{=} \hat{V}^\uparrow.$$

We can thus obtain  $\hat{V}^\uparrow$  from  $\hat{V}_x^\uparrow$ , and  $\hat{V}_x^\uparrow$  is only  $\hat{V}_x$  conditioned in the usual sens to remain positive. This allows us to link  $\hat{V}^\uparrow$  and  $\hat{V}$ , so we are now able to prove our estimate :

**Lemma 4.5.11.** *Assume  $V$  has unbounded variations. There are three positive constants  $c_0, c_1$  and  $c_2$  such that*

$$\forall y > 1, r > 0, \mathbb{P} \left( \tau(\hat{V}^\uparrow, [y, +\infty[) > r \right) \leq c_0 (e^{-\kappa y} + e^{c_1 y - c_2 r}).$$

*Démonstration.* Let us fix  $r > 0, y > 1$  and choose  $x \in ]0, 1[$  (for example  $x := 1/2$ ). According to Lemma 4.5.10 we have

$$\begin{aligned} \mathbb{P} \left( \tau(\hat{V}^\uparrow, [y, +\infty[) > r \right) &= \mathbb{P} \left( \tau(\hat{V}_x^\uparrow(m_x + \cdot) - \hat{V}_x^\uparrow(m_x -), [y, +\infty[) > r \right) \\ &\leq \mathbb{P} \left( \tau(\hat{V}_x^\uparrow(m_x + \cdot), [y + x, +\infty[) > r \right), \end{aligned}$$

because  $\hat{V}_x^\uparrow(m_x -) \leq x$ . Now, if  $\hat{V}_x^\uparrow$  never reaches  $[0, x]$  after the instant  $T := \tau(\hat{V}_x^\uparrow, [y + x, +\infty[)$ , then the minimum  $m_x$  is reached before  $T$  so

$$\tau(\hat{V}_x^\uparrow(m_x + \cdot), [y + x, +\infty[) = \tau(\hat{V}_x^\uparrow, [y + x, +\infty[) - m_x = T - m_x \leq T.$$

We deduce that

$$\mathbb{P} \left( \tau(\hat{V}^\uparrow, [y, +\infty[) > r \right) \leq \mathbb{P}(T > r) + \mathbb{P} \left( \inf_{[T, +\infty[} \hat{V}_x^\uparrow < x \right). \quad (4.5.11)$$

We now bound the two terms of the right hand side. For the first term, since  $\hat{V}_x^\uparrow$  is only  $\hat{V}_x$  conditioned in the usual sens to remain positive, we have

$$\begin{aligned} \mathbb{P}(T > r) &= \mathbb{P} \left( \tau(\hat{V}_x, [y + x, +\infty[) > r, \inf_{[0, +\infty[} \hat{V}_x > 0 \right) / \mathbb{P} \left( \inf_{[0, +\infty[} \hat{V}_x > 0 \right) \\ &\leq \mathbb{P} \left( \tau(\hat{V}_x, [y + x, +\infty[) > r \right) / \mathbb{P} \left( \inf_{[0, +\infty[} \hat{V}_x > 0 \right) \\ &= \mathbb{P}(\tau(V, ] - \infty, -y]) > r) / (1 - e^{-\kappa x}). \end{aligned}$$

Combining with Lemma 4.5.4 we get

$$\mathbb{P}(T > r) \leq c_0 e^{c_1 y - c_2 r}, \quad (4.5.12)$$

where  $c_1$  and  $c_2$  are the constants in the lemma and  $c_0 := 1/(1 - e^{-\kappa x})$ . We now turn to the second term of (4.5.11). According to Lemma 4.5.9 applied with  $z = a = x$  and  $b = x + y$  we get

$$\mathbb{P} \left( \inf_{[T, +\infty[} \hat{V}_x^\uparrow < x \right) \leq e^{-\kappa y} / (1 - e^{-\kappa(x+y)}) \leq c_0 e^{-\kappa y}. \quad (4.5.13)$$

Now, combining (4.5.12) and (4.5.13) with (4.5.11) we get the result.  $\square$



We can now use our estimate on the hitting times to bound an exponential functional of  $\hat{V}^\uparrow$ .

**Lemma 4.5.12.** *Assume  $V$  has unbounded variations. There is a positive constant  $c$  such that for all  $h$  large enough, we have*

$$\mathbb{P} \left( \int_0^{\tau(\hat{V}^\uparrow, [h, +\infty[)} e^{\hat{V}^\uparrow(u)} du \leq e^{(1+\eta)h} \right) \geq 1 - e^{-ch}. \quad (4.5.14)$$

*Démonstration.* We choose  $c \in ]0, \kappa[$ . We have

$$\int_0^{\tau(\hat{V}^\uparrow, [h, +\infty[)} e^{\hat{V}^\uparrow(u)} du \leq e^h \tau(\hat{V}^\uparrow, [h, +\infty[),$$

so

$$\mathbb{P} \left( \int_0^{\tau(\hat{V}^\uparrow, [h, +\infty[)} e^{\hat{V}^\uparrow(u)} du > e^{(1+\eta)h} \right) \leq \mathbb{P} \left( \tau(\hat{V}^\uparrow, [h, +\infty[) > e^{\eta h} \right) \leq e^{-ch},$$

for any choice of  $c \in ]0, \kappa[$  and  $h$  large enough, according to Lemma 4.5.11. This yields (4.5.14).  $\square$

Note that we do not prove an analogue of (4.5.8) for  $\hat{V}^\uparrow$ , even if it would have been needed to repeat readily the arguments of [4] in our context. This is because the existence of possibly large negative jumps for  $V$  do not allow such an estimate to hold in general. Because of this, we have to take some precautions, and in particular, to prove some extra technical estimates in Subsection 4.5.5.

#### 4.5.4 Estimates on the first ascend of $h$ from the minimum

In order to bound the local time and the time spent by the diffusion between two valleys, we have to study the expectation of some functionals of  $V$  involving the first ascend of  $h$  from the minimum. This subsection uses the notations and estimates of Subsection 4.3.5.

**Lemma 4.5.13.** *There is a positive constant  $C$  such that for  $h$  large enough,*

$$\mathbb{E}[\tau^*(h)] \leq Ce^{\kappa h}.$$

*Démonstration.* We have  $\tau^*(h) = m^*(h) + (\tau^*(h) - m^*(h))$ , and using Lemma 4.3.9 we get

$$\mathbb{E}[\tau^*(h)] = \mathbb{E}[S^{h,-}(T_h)] + \mathbb{E}[\tau(V^\uparrow, h)]. \quad (4.5.15)$$

For the first term, note that  $\mathbb{E}[S^{h,-}(T_h)] = \mathbb{E}[S^{h,-}(1)] \times \mathbb{E}[T_h]$  since  $S^{h,-}$  is a subordinator and is independent from  $T_h$ . Also, recall that  $S^{h,-}(1) \leq S(1)$  where  $S(1)$

has finite expectation according to Lemma 4.3.10. Then,  $T_h$  follows an exponential distribution with parameter  $\mathcal{N}(\mathcal{F}_{h,+}) \sim ce^{-\kappa h}$ , according to Lemma 4.3.11, where  $c$  is the positive constant obtained in the lemma. For  $h$  large enough we thus get

$$\mathbb{E} [S^{h,-}(T_h)] \leq (2\mathbb{E}[S(1)]/c)e^{\kappa h}. \quad (4.5.16)$$

For the second term, we use Lemma 2.6 of [72] and the fact that, since  $V^\sharp$  is a spectrally negative Lévy process drifting to  $+\infty$ ,  $\tau(V^\sharp, \cdot)$  is a subordinator having finite expectation (see [8], Section VII.1). We obtain

$$\mathbb{E} [\tau(V^\uparrow, h)] \leq \mathbb{E} [\tau(V^\sharp, h)] = h\mathbb{E} [\tau(V^\sharp, 1)]. \quad (4.5.17)$$

Combining (4.5.15), (4.5.16) and (4.5.17) we get the result.  $\square$

**Lemma 4.5.14.** *There is a positive constant  $C$  such that*

$$\forall h > 0, \mathbb{E} \left[ \int_0^{\tau^*(h)} e^{V(u)} du \right] \leq Ce^{(1-\kappa)h}.$$

*Démonstration.* Since  $V$  is spectrally negative, single points are not essentially polar for  $V$  so, according to Theorem V.1 of [8], there is  $\mathcal{L}_V$ , a local time that satisfies the density of occupations formula for  $V$  : almost surely, for all measurable function  $f$  and  $t > 0$  we have  $\int_0^t f(V(s))ds = \int_{\mathbb{R}} f(x)\mathcal{L}_V(t, x)dx$ . Therefore, as in the proof of the majoration of  $\beta_0(h)$  in Lemma 3.6 of [3] we get :

$$\mathbb{E} \left[ \int_0^{\tau^*(h)} e^{V(u)} du \right] \leq \int_{-\infty}^h e^x \mathbb{E} [\mathcal{L}_V(+\infty, x)] dx,$$

and, according to the strong Markov property,

$$\mathbb{E} [\mathcal{L}_V(+\infty, x)] = \mathbb{P} (\tau(V, x) < +\infty) \times \mathbb{E} [\mathcal{L}_V(+\infty, 0)],$$

where

$$\mathbb{P} (\tau(V, x) < +\infty) \leq \mathbb{P} \left( \sup_{[0, +\infty[} V \geq x \right) = \mathbf{1}_{x \leq 0} + e^{-\kappa x} \mathbf{1}_{x > 0}.$$

We thus get

$$\mathbb{E} \left[ \int_0^{\tau^*(h)} e^{V(u)} du \right] \leq \frac{1}{1-\kappa} \mathbb{E} [\mathcal{L}_V(+\infty, 0)] \times (e^{(1-\kappa)h} - \kappa).$$

Then, considering the Poisson point process of excursions away from 0 associated with the local time  $\mathcal{L}_V(\cdot, 0)$ , we have that  $\mathcal{L}_V(+\infty, 0)$  is only the time when occurs the infinite excursion of  $V$ , and this follows an exponential distribution with parameter  $\eta_V(\xi, \zeta(\xi) = +\infty) > 0$ . As a consequence,  $\mathbb{E}[\mathcal{L}_V(+\infty, 0)] < +\infty$  and the result follows.  $\square$

### 4.5.5 Estimates on the valleys

We now use the previous results to prove some estimates on the standard valleys.

**Lemma 4.5.15.** *For all  $h$  large enough we have*

$$\forall j \geq 1, \mathbb{P} \left( \int_{\tilde{L}_{j-1}}^{\tilde{m}_j} e^{V^{(j)}(u)} du \geq e^{e^{(1-2\delta)\kappa h}} \right) \geq 1 - e^{-\kappa\delta h/4}.$$

*Démonstration.* Thanks to Remark 4.3.4 we only need to prove the result for  $j = 1$ . Recall the proof of Lemma 4.5.5 in which we set  $a = h$ ,  $b = e^{(1-\delta)\kappa h}$  and  $\eta = \delta/2$ . For  $h \geq 1$  let us define :

$$p(h) := \mathbb{P}(\min(\tau(V - \underline{V}, h), \tau(\underline{V}, ]-\infty, -\delta h/2])) \geq 1) = \mathbb{P}(T_1 \geq 1) \geq p(1) > 0.$$

Since, from the Markov property, the sequence  $(T_i - T_{i-1})_{i \geq 1}$  is *iid*, the probability that  $T_i - T_{i-1} < 1$  for all  $i \leq \lfloor e^{(1-2\delta)h} \rfloor$  is

$$(1 - p(h))^{\lfloor e^{(1-2\delta)\kappa h} \rfloor} \leq (1 - p(1))^{\lfloor e^{(1-2\delta)\kappa h} \rfloor} \leq e^{-h},$$

for  $h$  large enough. Then, recall that we proved that, with probability greater than  $1 - (b/\eta a + 1)e^{-\kappa(1-\eta)a}$  (which is more than  $1 - e^{-\kappa\delta h/3}$ , at least when  $h$  is large enough), the index  $i$  such that  $T_i = \tau(V - \underline{V}, h)$  is greater than  $2e^{(1-\delta)\kappa h}/\delta h$ .

It means that after the  $\lfloor e^{(1-2\delta)\kappa h} \rfloor$ th one,  $V$  will still make  $2e^{(1-\delta)\kappa h}/\delta h - \lfloor e^{(1-2\delta)\kappa h} \rfloor \geq e^{(1-2\delta)\kappa h}$  descents before  $\tau(V - \underline{V}, h)$  and so, before  $\tilde{m}_1$ . In  $[0, \tilde{m}_1]$ , there will therefore be an interval larger than 1 on which  $V^{(1)}$  is greater than  $e^{(1-2\delta)\kappa h}$ . We thus have

$$\int_{\tilde{L}_0}^{\tilde{m}_1} e^{V^{(1)}(u)} du \geq e^{e^{(1-2\delta)\kappa h}} \times 1$$

with probability greater than  $1 - e^{-h} - e^{-\kappa\delta h/3}$ , for  $h$  large enough, and the result follows. □

**Lemma 4.5.16.** *Fix  $0 < \eta < \alpha < 1$ . For  $h$  large enough we have*

$$\forall j \geq 1, \mathbb{P} \left( \tilde{L}_j^\sharp < \tilde{\tau}_j^-(\alpha h), \inf_{[\tilde{L}_j^\sharp, \tilde{\tau}_j^-(\alpha h)]} V^{(j)} > (\alpha - \eta) h \right) \geq 1 - e^{-\kappa\eta h/3}.$$

It would seem convenient to use the time-reverse property to prove this lemma. However this is not possible here so we have to show that  $V$  cannot get too close to its future minimum before time  $\tilde{\tau}_j^-(\alpha h)$ .

*Démonstration.* Thanks to Remark 4.3.4 we only need to prove the result for  $j = 1$ . Recall that  $\tilde{\tau}_1(h)$  is the first time after  $\tilde{L}_1^\sharp$  when  $V - \underline{V}$  reaches  $h$  and  $\tilde{m}_1$  is the associated minimum, so

$$\mathbb{P} \left( \tilde{L}_1^\sharp \geq \tilde{\tau}_1^-(\alpha h) \right) \leq \mathbb{P}(\tau(V - \underline{V}, h) < \tau(\underline{V}, ]-\infty, -h])) \leq 3e^{-\kappa h/2}, \quad (4.5.18)$$

where the last inequality comes from Lemma 4.5.5 applied with  $a = b = h$ ,  $\eta = 1/2$ . When  $\tilde{L}_1^\sharp < \tilde{\tau}_1^-(\alpha h)$ , let  $u_1$  be the unique point where  $V^{(1)}$  reaches its minimum on  $[\tilde{L}_1^\sharp, \tilde{\tau}_1^-(\alpha h)]$ . On the event  $\{\tilde{L}_1^\sharp < \tilde{\tau}_1^-(\alpha h), \inf_{[\tilde{L}_1^\sharp, \tilde{\tau}_1^-(\alpha h)]} V^{(1)} \leq (\alpha - \eta)h\}$  we have

$$V(\tilde{m}_1) \leq V(u_1) \leq V(\tilde{m}_1) + (\alpha - \eta)h \leq V(\tilde{\tau}_1^-(\alpha h)-) - \eta h. \quad (4.5.19)$$

It implies that  $V - \underline{V}$  reaches  $\eta h$  between  $u_1$  and  $\tilde{\tau}_1^-(\alpha h)$ , then  $V$  descends lower than its minimum level  $V(u_1)$  (because  $V(\tilde{m}_1) \in [V(u_1) - (\alpha - \eta)h, V(u_1)]$ ) and then,  $V - \underline{V}$  reaches  $h$  before  $V$  reaches  $]-\infty, V(u_1) - (\alpha - \eta)h[$ . We thus consider the times when  $V - \underline{V}$  reaches  $\eta h$  and separate them by the times when  $V$  gets lower than its previous minimum. We introduce the sequences of stopping times  $(S_j)_{j \geq 0}$  and  $(T_j)_{j \geq 1}$  where  $S_0 := \tilde{L}_1^\sharp$  and

$$T_j := \inf \{t \geq S_{j-1}, (V - \underline{V})(t) = \eta h\}, \quad S_j := \inf \{t \geq T_j, V(t) < \underline{V}(T_j)\}.$$

Note that for all  $j \geq 1$ ,  $\underline{V}(T_j) = V(T_j) - \eta h$  and  $V(S_j) = \underline{V}(S_j)$ . Combining with the Markov property, this implies that the sequences of truncated processes  $(V(t + S_{j-1}) - V(S_{j-1}), 0 \leq t \leq T_j - S_{j-1})_{j \geq 1}$  and  $(V(t + T_j) - V(T_j), 0 \leq t \leq S_j - T_j)_{j \geq 1}$  are both *iid*, and the two sequences are independent.

We see that  $\tilde{\tau}_1^+(\eta h) = T_{J_0}$  where we define  $J_0$  to be the first index  $j \geq 1$  for which  $\inf \{t \geq T_j, (V - \underline{V})(t) = h\} < S_j$  and  $\tilde{m}_1$  is the minimum of  $V$  before  $T_{J_0}$  ( $V(\tilde{m}_1) = \underline{V}(T_{J_0}) = V(T_{J_0}) - \eta h$ ). Moreover, we just saw that on the event  $\{\tilde{L}_1^\sharp < \tilde{\tau}_1^-(\alpha h), \inf_{[\tilde{L}_1^\sharp, \tilde{\tau}_1^-(\alpha h)]} V^{(1)} \leq (\alpha - \eta)h\}$ ,  $J_0 \geq 2$  and  $u_1$  is the minimum of  $V$  before  $T_K$  for some random  $K < J_0$  ( $V(u_1) = \underline{V}(T_K) = V(T_K) - \eta h$ ). Using (4.5.19), we get,

$$\underline{V}(T_{J_0-1}) \leq \underline{V}(T_K) = V(u_1) \leq V(\tilde{m}_1) + (\alpha - \eta)h = \underline{V}(T_{J_0}) + (\alpha - \eta)h = V(T_{J_0}) + (\alpha - 2\eta)h.$$

From the definition of  $S_j$  we have  $V(S_{J_0-1}) \leq \underline{V}(T_{J_0-1})$  so we get

$$\left\{ \tilde{L}_1^\sharp < \tilde{\tau}_1^-(\alpha h), \inf_{[\tilde{L}_1^\sharp, \tilde{\tau}_1^-(\alpha h)]} V^{(1)} \leq (\alpha - \eta)h \right\} \subset \{V(S_{J_0-1}) \leq V(T_{J_0}) + (\alpha - 2\eta)h\}. \quad (4.5.20)$$

For any  $k \geq 1$ , the event  $\{V(S_{k-1}) \leq V(T_k) + (\alpha - 2\eta)h\}$  only depends on  $(V(t + S_{k-1}) - V(S_{k-1}), 0 \leq t \leq T_k - S_{k-1})$  whereas the event  $\{J_0 = k\}$  only depends on the sequence  $(V(t + T_j) - V(T_j), 0 \leq t \leq S_j - T_j)_{j \geq 1}$ . Partitioning on the possible values for  $J_0$  and using the fact that the sequences  $(V(t + S_{j-1}) - V(S_{j-1}), 0 \leq t \leq T_j - S_{j-1})_{j \geq 1}$  and  $(V(t + T_j) - V(T_j), 0 \leq t \leq S_j - T_j)_{j \geq 1}$  are independent and both *iid*, we get

$$\begin{aligned} \mathbb{P}(V(S_{J_0-1}) \leq V(T_{J_0}) + (\alpha - 2\eta)h) &= \mathbb{P}(V(S_0) \leq V(T_1) + (\alpha - 2\eta)h) \\ &= \mathbb{P}(\tau(V - \underline{V}, \eta h) < \tau(V, ]-\infty, -(\alpha - \eta)h[)) \\ &\leq (1 + \alpha/\eta)e^{-\kappa\eta h/2}/2, \end{aligned}$$

where the inequality comes from Lemma 4.5.5 applied with  $a = \eta h$ ,  $b = (\alpha - \eta)h$ ,  $\eta = 1/2$ . Combining with (4.5.20) and (4.5.18) we get the result for  $h$  large enough.  $\square$

**Lemma 4.5.17.** *For all  $h$  large enough we have*

$$\forall j \geq 1, \mathbb{P} \left( \tilde{\tau}_j^-(h) - \tilde{L}_{j-1} > e^{(1+\delta)\kappa h} \right) \leq e^{-\delta\kappa h/2}.$$

*Démonstration.* Here again, we only need to prove the result for  $j = 1$ . We have

$$\tilde{\tau}_1^-(h) - \tilde{L}_0 \leq \tilde{\tau}_1(h) - \tilde{L}_0 = \left( \tilde{\tau}_1(h) - \tilde{L}_1^\sharp \right) + \left( \tilde{L}_1^\sharp - \tilde{L}_0 \right). \quad (4.5.21)$$

For the first term, according to the Markov property at  $\tilde{L}_1^\sharp$  and the definition of  $\tilde{\tau}_1(h)$ , we see that  $\tilde{\tau}_1(h) - \tilde{L}_1^\sharp$  has the same law as  $\tau^*(h)$ . We thus have  $\mathbb{P}(\tilde{\tau}_1(h) - \tilde{L}_1^\sharp > e^{(1+\delta)\kappa h}/2) = \mathbb{P}(\tau^*(h) > e^{(1+\delta)\kappa h}/2)$  and combining with Markov's inequality and Lemma 4.5.13, we get for  $h$  large enough,

$$\mathbb{P} \left( \tilde{\tau}_1(h) - \tilde{L}_1^\sharp > e^{(1+\delta)\kappa h}/2 \right) \leq 2C e^{-\delta\kappa h}, \quad (4.5.22)$$

where  $C$  is the constant in Lemma 4.5.13. For the second term, according to the definition of  $\tilde{L}_1^\sharp$  and Lemma 4.5.4 applied with  $y = e^{(1-\delta)\kappa h}$  and  $r = e^{(1+\delta)\kappa h}/2$  we have, for  $h$  large enough :

$$\mathbb{P} \left( \tilde{L}_1^\sharp - \tilde{L}_0 > \frac{e^{(1+\delta)\kappa h}}{2} \right) = \mathbb{P} \left( \tau(V, ] - \infty, -e^{(1-\delta)\kappa h}] > \frac{e^{(1+\delta)\kappa h}}{2} \right) \leq e^{-h}. \quad (4.5.23)$$

The combination of (4.5.21), (4.5.22) and (4.5.23) yields the result.  $\square$

## 4.5.6 Estimates on the diffusion in potential $V$

We now give an upper bound for the time spent by the diffusion and the local time in the negative half-line. Restricted to the drifted brownian case, our result is a little stronger than Lemma 3.5 of [3]. It is the only estimate that we need for both cases  $0 < \kappa < 1$  and  $\kappa > 1$ .

**Lemma 4.5.18.** *Recall the definition of  $H_-$  in Subsection 4.1.2. There is a positive constant  $C$  such that for  $r$  large enough,*

$$\mathbb{P}(H_-(+\infty) > r) \leq C r^{-\kappa/(2+\kappa)} \quad \text{and} \quad \mathbb{P} \left( \inf_{]-\infty, 0]} \mathcal{L}_X(+\infty, \cdot) > r \right) \leq 3r^{-\kappa/(2+\kappa)}.$$

*Démonstration.* From the definition of the local time and formula (4.2.1), we have

$$\begin{aligned} H_-(+\infty) &= \int_{-\infty}^0 \mathcal{L}_X(+\infty, x) dx = \int_{-\infty}^0 e^{-V(x)} \mathcal{L}_B(\tau(B, A_V(+\infty)), A_V(x)) dx \\ &= A_V(+\infty) \int_{-\infty}^0 e^{-V(x)} \mathcal{L}_{B'}(\tau(B', 1), A_V(x)/A_V(+\infty)) dx \end{aligned}$$

where  $B' := B((A_V(+\infty))^2)/A_V(+\infty)$ . By scale invariance, we see that conditionally to  $V$ ,  $B'$  is a brownian motion so

$$H_-(+\infty) \stackrel{\mathcal{L}}{=} A_V(+\infty) \int_{-\infty}^0 e^{-V(x)} \mathcal{L}_B(\tau(B, 1), A_V(x)/A_V(+\infty)) dx \quad (4.5.24)$$

$$\begin{aligned} &\leq \left( \sup_{y \leq 0} \mathcal{L}_B(\tau(B, 1), y) \right) \times A_V(+\infty) \times \int_{-\infty}^0 e^{-V(x)} dx \\ &= \left( \sup_{y \leq 0} \mathcal{L}_B(\tau(B, 1), y) \right) \times A_V(+\infty) \times \int_0^{+\infty} e^{\tilde{V}(x)} dx, \end{aligned} \quad (4.5.25)$$

where  $\tilde{V}(x) := -V(-x)$ . By time reversing, the process  $(\tilde{V}(x), x \geq 0)$  has the same law as  $(V(x), x \geq 0)$ . As a consequence, the right hand side of (4.5.25) features three factors, the last two of them being equal in law to  $\int_0^{+\infty} e^{V(u)} du$ . We thus have

$$\mathbb{P}(H_-(+\infty) > r) \leq \mathbb{P}\left(\sup_{y \leq 0} \mathcal{L}_B(\tau(B, 1), y) > r^{\kappa/(2+\kappa)}\right) + 2\mathbb{P}\left(\int_0^{+\infty} e^{V(u)} du > r^{1/(2+\kappa)}\right),$$

and combining with inequality (7.13) of [4] for the first term, and the second assertion of Lemma 4.5.3 for the second term, we get the result for  $r$  large enough. For the assertion about the local time, we only replace the integrals on  $] -\infty, 0]$  by a supremum (as well as the integral on  $[0, +\infty[$  for  $\tilde{V}$ ). Since  $\sup \tilde{V}$  follows an exponential distribution with parameter  $\kappa$ , the result follows.  $\square$

The next lemma provides a useful estimate to bound the time spent by the diffusion between two standard valleys. It generalizes a part of Lemma 3.6 of [3].

**Lemma 4.5.19.** *Recall the definition of  $H_+$  in Subsection 4.1.2. There exists a constant  $C > 0$  such that for  $h$  large enough,*

$$\mathbb{E}[H_+(\tau^*(h))] \leq C e^h. \quad (4.5.26)$$

*Démonstration.* The beginning of the proof is similar to the beginning of the proof of Lemma 3.6 in [3]. In fact (3.37) of [3] is still true in our setting, with  $V$  instead of the drifted brownian motion :

$$\mathbb{E}[H_+(\tau^*(h))] \leq 2\mathbb{E}[\tau^*(h)] \times \mathbb{E}\left[\int_0^{\tau^*(h)} e^{V(u)} du\right].$$

Combining this with Lemmas 4.5.13 and 4.5.14, we get the result.  $\square$

A fundamental point to have the renewal structure for the contributions to time and to local time is the fact that the diffusion never goes back to a previous valley. Let us define

$$X_{\tilde{L}_i} := X(H(\tilde{L}_i) + .)$$

which is, according to the Markov property, a diffusion in the environment  $V$  starting from  $\tilde{L}_i$ . We also denote by  $H_{X_{\tilde{L}_i}}(r)$  the hitting time of  $r$  by  $X_{\tilde{L}_i}$ . The following lemma proves that the diffusion does not go back :

**Lemma 4.5.20.** *There is a positive constant  $c$  such that for  $h$  large enough,*

$$\forall i \geq 1, \mathbb{P} \left( H_{X_{\tilde{L}_i}}(\tilde{\tau}_i(h)) < H_{X_{\tilde{L}_i}}(+\infty) \right) \leq e^{-ch}.$$

*Démonstration.* Let us fix  $i \geq 1$ . At fixed environment  $V$ ,  $P^V(H_{X_{\tilde{L}_i}}(\tilde{L}_i^*) < H_{X_{\tilde{L}_i}}(+\infty))$  equals

$$\begin{aligned} &= \frac{\int_{\tilde{L}_i}^{+\infty} e^{V(u)} du}{\int_{\tilde{\tau}_i(h)}^{+\infty} e^{V(u)} du} = e^{V(\tilde{L}_i) - V(\tilde{\tau}_i(h))} \frac{\int_0^{+\infty} e^{V(u+\tilde{L}_i) - V(\tilde{L}_i)} du}{\int_0^{+\infty} e^{V(u+\tilde{\tau}_i(h)) - V(\tilde{\tau}_i(h))} du} := e^{V(\tilde{L}_i) - V(\tilde{\tau}_i(h))} I_1 / I_2. \end{aligned} \quad (4.5.27)$$

Since  $\tilde{L}_i$  and  $\tilde{\tau}_i(h)$  are stopping times for  $V$ , both  $I_1$  and  $I_2$  have the same law as  $\int_0^{+\infty} e^{V(u)} du$  even though there are not independent. Then,

$$\mathbb{P}(I_1 / I_2 > e^{h/4}) \leq \mathbb{P}(I_1 > e^{h/8}) + \mathbb{P}(I_2 < e^{-h/8}) \leq 2\mathcal{C}e^{-\kappa h/8} + ce^{-h/16},$$

where the last inequality and the constants come from the two assertions of Lemma 4.5.3. It holds for  $h$  large enough. From the definition of  $\tilde{L}_i$  we have  $V(\tilde{L}_i) - V(\tilde{\tau}_i(h)) \leq -h/2$  so the last term of (4.5.27) is less than  $e^{-h/4}$  with probability greater than  $1 - 2\mathcal{C}e^{-\kappa h/8} + ce^{-h/16}$  and is bounded by 1 otherwise. Integrating (4.5.27) with respect to  $V$  we thus get

$$\mathbb{P} \left( H_{X_{\tilde{L}_i}}(\tilde{\tau}_i(h)) < H_{X_{\tilde{L}_i}}(+\infty) \right) \leq e^{-h/4} + 2\mathcal{C}e^{-\kappa h/8} + ce^{-h/16},$$

and the result follows.  $\square$

Recall the definitions of  $X_{\tilde{m}_j} := X(\cdot + \tilde{m}_j)$  and  $H_{X_{\tilde{m}_j}}(r)$ . The next lemma proves that the standard valleys are left from the right.

**Lemma 4.5.21.** *For  $h$  large enough,*

$$\forall i \geq 1, \mathbb{P} \left( H_{X_{\tilde{m}_i}}(\tilde{L}_{i-1}) < H_{X_{\tilde{m}_i}}(\tilde{L}_i) \right) \leq e^{-\kappa \delta h/6}.$$

This is where appears a difference with the brownian case. In Lemma 3.2 of [3], they have a similar result with, instead of  $\tilde{L}_{i-1}$ , an other random time denoted by  $\tilde{L}_i^-$ . In the context of a Lévy environment, the existence of jumps may allow one of the  $\tilde{m}_i - \tilde{L}_i^-$  to be quite small with a non negligible probability, which would allow some standard valleys to be left from  $\tilde{L}_i^-$ . In fact, we would need an analogue of (4.5.8) for  $\hat{V}^\uparrow$  to have the same result as in Lemma 3.2 of [3]. In order to still

have the standard valleys left from the right, we have to change our definition of "leave from the right" (replacing  $\tilde{L}_i^-$  by  $\tilde{L}_{i-1}$ ). A consequence of this is that the study of the descending parts of the standard  $h$ -valleys is more technical (since we have to consider a bigger part than in [3], and we do not know its law precisely) and requires Subsection 4.5.5. In particular, our proof that the standard valleys are left from the right requires the technical Lemma 4.5.15, but the idea of the result has no qualitative difference with the drifted brownian case.

*Démonstration.* of Lemma 4.5.21

Let us fix  $i \geq 1$ . At fixed environment  $V$ ,  $P^V(H_{X_{\tilde{m}_i}}(\tilde{L}_{i-1}) < H_{X_{\tilde{m}_i}}(\tilde{L}_i))$  equals

$$\begin{aligned} \frac{\int_{\tilde{m}_i}^{\tilde{L}_i} e^{\tilde{V}^{(i)}(u)} du}{\int_{\tilde{L}_{i-1}}^{\tilde{L}_i} e^{\tilde{V}^{(i)}(u)} du} &= \frac{1}{1 + \int_{\tilde{L}_{i-1}}^{\tilde{m}_i} e^{\tilde{V}^{(i)}(u)} du / \int_{\tilde{m}_i}^{\tilde{L}_i} e^{\tilde{V}^{(i)}(u)} du} \\ &\leq \max \left( 1, \int_{\tilde{m}_i}^{\tilde{L}_i} e^{\tilde{V}^{(i)}(u)} du / \int_{\tilde{L}_{i-1}}^{\tilde{m}_i} e^{\tilde{V}^{(i)}(u)} du \right). \end{aligned} \quad (4.5.28)$$

We first provide an upper bound for

$$\int_{\tilde{m}_i}^{\tilde{L}_i} e^{\tilde{V}^{(i)}(u)} du = \int_{\tilde{m}_i}^{\tilde{\tau}_i(h)} e^{\tilde{V}^{(i)}(u)} du + \int_{\tilde{\tau}_i(h)}^{\tilde{L}_i} e^{\tilde{V}^{(i)}(u)} du.$$

According to proposition 4.3.6, the terms of the right hand side have respectively the same law as  $\int_0^{\tau(V^\uparrow, h)} e^{V^\uparrow(u)} du$  and  $e^h \int_0^{\tau(V, ]-\infty, -h/2])} e^{V(u)} du \leq e^h \int_0^{+\infty} e^{V(u)} du$ . We thus have

$$\begin{aligned} \mathbb{P} \left( \int_{\tilde{m}_i}^{\tilde{L}_i} e^{\tilde{V}^{(i)}(u)} du > e^{(1+\delta)h} \right) &\leq \mathbb{P} \left( \int_0^{\tau(V^\uparrow, h)} e^{V^\uparrow(u)} du > e^{(1+\delta)h/2} \right) \\ &\quad + \mathbb{P} \left( \int_0^{+\infty} e^{V(u)} du > e^{\delta h/2} \right). \end{aligned} \quad (4.5.29)$$

For  $h$  large enough, the first term of the right hand side is bounded by  $e^{-h}$  because of (4.5.9) and the second is bounded by  $2\mathcal{C} \times 2^\kappa e^{-\kappa\delta h}$  because of the second assertion of Lemma 4.5.3. As a consequence, for  $h$  large enough,

$$\mathbb{P} \left( \int_{\tilde{m}_i}^{\tilde{L}_i} e^{\tilde{V}^{(i)}(u)} du > e^{(1+\delta)h} \right) \leq e^{-\kappa\delta h/3}. \quad (4.5.30)$$

A lower bound for  $\int_{\tilde{L}_{i-1}}^{\tilde{m}_i} e^{\tilde{V}^{(i)}(u)} du$  is given by Lemma 4.5.15. Combining with (4.5.30), we get that for  $h$  large enough

$$\mathbb{P} \left( \int_{\tilde{m}_i}^{\tilde{L}_i} e^{\tilde{V}^{(i)}(u)} du / \int_{\tilde{L}_{i-1}}^{\tilde{m}_i} e^{\tilde{V}^{(i)}(u)} du \geq e^{-h} \right) \leq e^{-\kappa\delta h/5},$$

so integrating (4.5.28) with respect to  $V$  we get the result for  $h$  large enough.  $\square$



### 4.5.7 Proof of some facts

We now prove Facts 4.4.3, 4.4.4 and 4.4.13. Even if they are taken from [3] and [4], their proof use estimates that are only true for the drifted brownian potential. This is why we have to adapt them to our context and to use our estimates instead of the original ones. We shall refer to [3] and [4] for the proofs and only precise what are the differences between their proofs and ours.

*Démonstration.* of Fact 4.4.3

This is Lemma 3.7 of [3]. For the proof, here are the differences :

$W_\kappa$  is, off course, replaced here by  $V$ . The standard valleys are to be considered as the ones defined in our sense. Also,  $\tilde{L}_i^*$  of [3] has to be replaced here by  $\tilde{\tau}_i(h_t)$  so Lemma 3.3 of [3] can be replaced here by Lemma 4.5.20 and  $\tilde{\tau}_i^*(h_t)$  is like we defined it in Subsection 4.3.3.

For the step 1, the event  $\mathcal{E}_2^{3.7}$  is replaced here by  $\cap_{i=1}^{n_t} \{\tilde{\tau}_i^*(h_t) = \tilde{\tau}_i(h_t)\}$  (note that this includes  $\{\tilde{\tau}_1^*(h_t) = \tilde{\tau}_1(h_t)\}$ ) and the negligibility of the complementary is proved by Lemma 4.3.7 (the constant in the exponential might be different from the one in [3] but it does not matter).

For the step 2, the event  $\mathcal{E}_1^{3.3}$  is replaced here by  $\cap_{i=1}^{n_t} \{H_{X_{\tilde{L}_i}}(\tilde{\tau}_i(h_t)) > H_{X_{\tilde{L}_i}}(\tilde{m}_{i+1})\}$  and the negligibility of the complementary is proved by Lemma 4.5.20. Then, estimate (3.34) of [3] (which is, in the proof of Lemma 3.7 there, referred to as "Lemma 3.6") has to be replaced here by Lemma 4.5.19. For  $H_+(\tilde{m}_1)$ , it is obviously less than  $H_+(\tilde{\tau}_1(h))$  which equals  $H_+(\tilde{\tau}_1^*(h))$  on  $\{\tilde{\tau}_1^*(h_t) = \tilde{\tau}_1(h_t)\}$  so, here, the expectation  $\mathbb{E}[H_+(\tilde{m}_1)\mathbf{1}_{\mathcal{E}_1^{3.3}}]$  is bounded by  $\mathbb{E}[H_+(\tau^*(h))]$ , just as the other terms, and in particular, there is no need, here, to bother with the event  $\mathcal{E}_3^{3.7}$  nor to specify we are on the event  $\mathcal{V}_t$  where the classical and standard valleys coincide.

Finally, to bound  $H_-(\tilde{m}_1)$ , we see from  $t/\log(h_t) > e^{ht}$  (which is true at least for large  $t$ ) and Lemma 4.5.18, that, for  $t$  large enough,

$$\mathbb{P}(H_-(\tilde{m}_1) > t/\log(h_t)) \leq \mathbb{P}(H_-(\tilde{m}_1) > e^{ht}) \leq \mathbb{P}(H_-(+\infty) > e^{ht}) \leq Ce^{-\kappa h_t/(2+\kappa)}.$$

□

*Démonstration.* of Fact 4.4.4

The first point is Lemma 3.3 of [4]. For the proof, here are the differences :

$W_\kappa$  is replaced here by  $V$  and we use the notations  $m^*(h_t)$  and  $\tau^*(h_t)$  instead of  $m_1^*(h_t)$  and  $\tau_1^*(h_t)$ . We change a little the definition of  $b(t)$ , that is,  $b(t) := 6R\phi(t)e^{\kappa h_t}/(1 - e^{-\kappa})$  where  $R := -\mathbb{E}[V(\tau(V, ] - \infty, -1))]$ . Note that  $R$  is finite according to Lemma 4.5.2 and the hypothesis assumed on  $V$ . For  $\mathbb{P}(\mathcal{A}_0^c)$  (where  $\mathcal{A}_0$  has the same definition as in the original proof of [4] with, off course,  $V$  instead of  $W_\kappa$ ), we get a similar upper bound for  $t$  large enough, thanks to the second assertion of Lemma 4.5.3. Because of the negative jumps, we have, before bounding  $\mathbb{P}(\mathcal{A}_2^c)$ , to

give a different definition for the stopping times  $f_i$  and to define a new event  $\mathcal{A}'_2$ . First,  $f_0 := 0$  and

$$\forall i \geq 1, f_i := \inf \{x \geq f_{i-1}, V(x) \leq V(f_{i-1}) - 1\}.$$

Let  $I(t) := \max\{i \in \mathbb{N}, f_{i-1} \leq m^*(h_t)\}$ , we now define  $\mathcal{A}'_2$  by

$$\mathcal{A}'_2 := \{I(t) \leq b(t)/2R\}.$$

We define, for  $i \geq 1$ ,  $E_i := \{\sup_{[f_{i-1}, f_i]} V - \underline{V} \geq h_t\}$ . Since  $\inf_{[0, f_{i-1}]} V = V(f_{i-1})$  we have

$$E_i = \left\{ \sup_{[0, f_i - f_{i-1}]} V^{f_{i-1}} - \underline{V}^{f_{i-1}} \geq h_t \right\},$$

and because of the Markov property applied at  $f_{i-1}$ , the events  $E_i$  are independent and have the same probability that we denote by  $p_t$ . Also,  $I(t)$  is the smallest  $i \geq 1$  for which  $E_i$  is realized, so  $I(t)$  follows a geometric distribution with parameter  $p_t$ . Then,

$$p_t := \mathbb{P}(E_1) \geq \mathbb{P}(V \text{ leaves } [-1, h_t] \text{ from above}) \geq (1 - e^{-\kappa})e^{-\kappa h_t},$$

where, for the last inequality, we applied Lemma 4.5.1 with  $a = 1$  and  $b = h_t$ . We deduce

$$\mathbb{P}(\mathcal{A}'_2) = (1 - p_t)^{\lfloor b(t)/2R \rfloor} = e^{\lfloor b(t)/2R \rfloor \log(1-p_t)} \leq e^{\lfloor b(t)/2R \rfloor \log(1-(1-e^{-\kappa})e^{-\kappa h_t})} \leq e^{-3\phi(t)}, \quad (4.5.31)$$

where the last inequality holds for  $t$  large enough. Then,

$$\{V(f_{\lfloor b(t)/2R \rfloor}) < -b(t)\} \subset \left\{ \sum_{i=1}^{\lfloor b(t)/2R \rfloor} (V(f_i) - V(f_{i-1}) + R) < -b(t)/2 \right\}.$$

The random variables  $V(f_i) - V(f_{i-1}) + R$  are *iid* having the same law as  $V(\tau(V, ]-\infty, -1]) + R$ , in particular they have 0 mean and belong to  $L^p$  (because of Lemma 4.5.2). We can thus apply successively Markov's and Von Barh-Esseen's inequalities :

$$\begin{aligned} \mathbb{P}(V(f_{\lfloor b(t)/2R \rfloor}) < -b(t)) &\leq \left(\frac{2}{b(t)}\right)^p \mathbb{E} \left[ \left| \sum_{i=1}^{\lfloor b(t)/2R \rfloor} V(f_i) - V(f_{i-1}) + R \right|^p \right] \\ &\leq 2 \left(\frac{2}{b(t)}\right)^p \times \left(\frac{b(t)}{2R}\right) \times \mathbb{E}[|V(\tau(V, ]-\infty, -1]) + R|^p] \\ &\leq e^{-(p-1)\kappa h_t}, \end{aligned} \quad (4.5.32)$$

where the last inequality comes from the definition of  $b(t)$  and holds for  $t$  large enough. Also,

$$\mathcal{A}'_2 \cap \{V(f_{\lfloor b(t)/2R \rfloor}) \geq -b(t)\} \subset \left\{ \inf_{[0, \tau^*(h_t)]} V \geq -b(t) \right\} = \mathcal{A}_2,$$

so combining with (4.5.31) and (4.5.32), the upper bound  $\mathbb{P}(\mathcal{A}'_2) \leq e^{-2\phi(t)}$  follows (for large  $t$ ).

#### 4.5. SOME ESTIMATES ON $V$ , $V^\uparrow$ , $\hat{V}^\uparrow$ AND THE DIFFUSION IN $V$

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Thanks to our definition of  $f_i$ , we still have  $e^{-V(x)} \leq e^{-V(f_i)+1}$  for  $f_i \leq x < f_{i+1}$  so (3.8) of [4] is true in our context.

Our upper bound for  $\mathbb{E}[\int_0^{\tau^*(h)} e^{V(u)} du]$  (denoted  $\beta_0(h)$  in [4]) is given by Lemma 4.5.14.

For  $\mathbb{P}(\mathcal{A}_4^c)$ , we use Lemma 4.3.9 and get

$$\mathbb{P}(\mathcal{A}_4^c) \leq \mathbb{P}(\tau(V^\uparrow, h_t) < 1) \leq \mathbb{P}(\tau(V^\uparrow, h_t) - \tau(V^\uparrow, h_t/2) < 1) \leq e^{-ch_t/2},$$

where the last inequality holds for some positive constant  $c$  and  $t$  large enough, according to (4.5.7) applied with  $\omega = 1$  and  $\alpha = 1/2$ .

We now justify the second point of Fact 4.4.4. We treat separately the case  $j = 0$ . Since  $H(\tilde{L}_0) = H(0) = 0$ , the term  $\mathcal{L}_X(H(\tilde{L}_j), x)$  can be omitted.  $\mathbb{P}(\sup_{\mathbb{R}} \mathcal{L}_X(H(\tilde{m}_1), x) > r_t)$  is less than

$$\mathbb{P}\left(\inf_{[-\infty, 0]} \mathcal{L}_X(+\infty, \cdot) > r_t\right) + \mathbb{P}\left(\sup_{[0, m^*(h_t)]} \mathcal{L}_X(H(\tau^*(h_t)), \cdot) > r_t\right) + \mathbb{P}(\tilde{m}_1 \neq m^*(h)(= \tilde{m}_1^*)).$$

Putting  $r_t := te^{(\kappa(1+3\delta)-1)\phi(t)}$  and applying Lemmas 4.5.18 and 4.3.7 together with the first point, we get for some constant  $C'_2$  and  $t$  large enough,

$$\mathbb{P}\left(\sup_{\mathbb{R}} \mathcal{L}_X(H(\tilde{m}_1), x) > te^{(\kappa(1+3\delta)-1)\phi(t)}\right) \leq \frac{C'_2}{n_t e^{\kappa\delta\phi(t)}}. \quad (4.5.33)$$

For  $j \geq 1$ , the proof has the same idea as the one of Lemma 3.2 in [4]. Thanks to Lemmas 4.5.20 and 4.3.7 we have, for some constant  $c$  and  $t$  large enough,

$$\mathbb{P}\left(\bigcap_{j=1}^{n_t-1} \{H_{X_j}(\tilde{m}_{j+1}) < H_{X_j}(\tilde{\tau}_j(h)), \tilde{\tau}_j^*(h) = \tilde{\tau}_j(h), \tilde{m}_j^* = \tilde{m}_j\}\right) \geq 1 - n_t e^{-ch_t}. \quad (4.5.34)$$

On this event we have that, for  $i \in \{1, \dots, n_t - 1\}$ , the chain of inequalities (3.11) of [4] is still true when  $\tilde{L}_i$ ,  $\tilde{m}_i^*$  and  $\tilde{\tau}_i^*(h)$  are defined in our sense (in Subsection 4.3.3) and  $\tilde{L}_i^*$  of [4] is replaced by  $\tilde{\tau}_i(h)$ . Moreover, the last term of this chain of inequality has the same law as  $\sup_{x \in [0, m^*(h_t)]} \mathcal{L}_X(H(\tau^*(h_t)), x)$ , so combining (4.5.34) and the first point we get

$$\mathbb{P}\left(\bigcap_{j=1}^{n_t-1} \left\{ \sup_{\mathbb{R}} \left( \mathcal{L}_X(H(\tilde{m}_{j+1}), x) - \mathcal{L}_X(H(\tilde{L}_j), x) \right) \leq te^{(\kappa(1+3\delta)-1)\phi(t)} \right\}\right) \geq 1 - \frac{C''_2(n_t - 1)}{n_t e^{\kappa\delta\phi(t)}}, \quad (4.5.35)$$

for some constant  $C''_2$  and  $t$  large enough. The combination of (4.5.33) and (4.5.35) is the sought result.

The third point of Fact 4.4.4 is Lemma 3.4 of [4]. For the proof, here are the differences :

$W_\kappa$  is replaced here by  $V$  and the standard valleys are to be considered as the ones defined in our sense (in Subsection 4.3.3). For convenience, we have  $r_t = (\phi(t))^2$  (instead of  $r_t = C_0\phi(t)$  in [4]). In this proof, we systematically replace  $\tilde{\tau}_j^-(h_t^+)$  of [4] by  $\tilde{L}_{j-1}$  (defined in our sense). For  $\mathcal{A}_2$ , we have that  $\mathbb{P}(\mathcal{A}_2^c)$  is less than

$$\begin{aligned} & \mathbb{P}\left(\int_{\tilde{m}_j}^{\tilde{\tau}_j^-(h_t)} e^{\tilde{V}^{(j)}(y)} dy > e^{h_t+2\phi(t)/\kappa}\right) + \mathbb{P}\left(\int_{\tilde{\tau}_j^-(h_t)}^{\tilde{L}_j} e^{\tilde{V}^{(j)}(y)} dy > e^{h_t+2\phi(t)/\kappa}\right) \\ = & \mathbb{P}\left(\int_0^{\tau(V^\uparrow, h_t)} e^{V^\uparrow(y)} dy > e^{h_t+2\phi(t)/\kappa}\right) + \mathbb{P}\left(e^{h_t} \int_0^{\tau(V, ]-\infty, -\frac{h_t}{2}])} e^{V(y)} dy > e^{h_t+2\phi(t)/\kappa}\right), \end{aligned}$$

where we used Proposition 4.3.6 for the laws of  $\tilde{P}_2^{(j)}$  and  $\tilde{P}_3^{(j)}$ ,

$$\begin{aligned} & \leq \mathbb{P}\left(\tau(V^\uparrow, h_t) > e^{2\phi(t)/\kappa}\right) + \mathbb{P}\left(\int_0^{+\infty} e^{V(y)} dy > e^{2\phi(t)/\kappa}\right), \\ & \leq e^{c_1 h_t - c_2 e^{2\phi(t)/\kappa}} + 2\mathcal{C}e^{-2\phi(t)}, \end{aligned}$$

for  $t$  large enough, according to Lemma 4.5.6 and the second assertion of Lemma 4.5.3 (the constants are the ones defined in there). For  $t$  large enough we get  $\mathbb{P}(\mathcal{A}_2^c) \leq ce^{-2\phi(t)}$ , where  $c$  is some positive constant.

The event  $\mathcal{A}_3$  of the original proof is not needed here thanks to our definition of  $\mathcal{D}_j$ , and we redefine

$$\mathcal{A}_4 := \left\{ \inf_{[\tilde{\tau}_j^+(\phi(t)^2), \tilde{\tau}_j^-(h_t)]} \tilde{V}^{(j)} \geq (\phi(t))^2/2 \right\}.$$

According to Proposition 4.3.6 and Lemma 4.5.7 applied with  $a = (\phi(t))^2/2$ ,  $b = (\phi(t))^2$ , we have  $\mathbb{P}(\mathcal{A}_4^c) \leq e^{-2\phi(t)}$  for  $t$  large enough. We also redefine

$$\mathcal{A}_5 := \left\{ \inf_{[\tilde{\tau}_j^-(h_t), \tilde{\tau}_j^-(\phi(t)^2)]} \tilde{V}^{(j)} \geq (\phi(t))^2/2 \right\} \text{ and } \mathcal{A}_6 := \left\{ \inf_{[\tilde{L}_{j-1}, \tilde{\tau}_j^-(h_t/2)]} V^{(j)} > h_t/4 \right\}.$$

Combining Proposition 4.3.6 and Lemma 4.5.9 applied with  $z = 0$ ,  $a = (\phi(t))^2/2$ ,  $b = (\phi(t))^2$ , we get for  $t$  large enough

$$\mathbb{P}(\mathcal{A}_5^c) \leq 2e^{-\delta\kappa h_t/3} + 2e^{-\kappa(\phi(t))^2/2}/(1 - e^{-\kappa(\phi(t))^2}) \leq e^{-2\phi(t)},$$

where the last inequality holds for  $t$  large enough. According to the definition of  $\tilde{L}_j^\sharp$  and Lemma 4.5.16 applied with  $\alpha = 1/2$ ,  $\eta = 1/4$ , we have  $\mathbb{P}(\mathcal{A}_6^c) \leq e^{-\kappa h_t/12}$  for large  $t$ .

On  $\cap_{i=4}^6 \mathcal{A}_i$ , we have  $V^{(j)}(x) \geq (\phi(t))^2/2$ ,  $\forall x \in [\tilde{L}_{j-1}, \tilde{L}_j] \cap \mathcal{D}_j^c$  so the conclusion follows as in the proof of Lemma 3.4 in [4] (but here we do not need to intersect with  $\mathcal{V}_t$  of [4]).  $\square$

*Démonstration.* of Fact 4.4.13

This is Lemma 5.3 of [4]. As in there, let  $\sigma_X(a, b) := \inf\{s \geq 0, \mathcal{L}_X(s, b) > a\}$  be the inverse of the local time of  $X$  at  $b$ . Here are the differences with the proof of Lemma 5.3 of [4] :

$W_\kappa$  is replaced here by  $V$  and the standard valleys are to be considered as the ones defined in our sense (in Subsection 4.3.3). In this proof, we systematically replace  $\tilde{L}_1^-$  of [4] by  $\tilde{L}_0 = 0$ , in particular, the domain of integration in the integral  $I$ , which represents the inverse of the local time, is  $[\tilde{L}_0, \tilde{L}_1]$ .

We get the analogue of (5.22) of [4] using Proposition 4.4.5. Our lower bound for  $\mathbb{P}(A^1(\tilde{\tau}_1^-(h_t/2)) \leq e^{h_t(1+\epsilon)/2})$  and  $\mathbb{P}(A^1(\tilde{\tau}_1(h_t/2)) \leq e^{h_t(1+\epsilon)/2})$  come from the proof of Lemma 4.4.8.

Recall also that, since the domain of integration in  $I$  is  $[\tilde{L}_0, \tilde{L}_1]$  we have in our case

$$I_2 \leq e^{\epsilon h_t/2} \left( \int_{\tilde{L}_0}^{\tilde{\tau}_1^-(h_t/2)} e^{-V^{(1)}(x)} dx + \int_{\tilde{\tau}_1(h_t/2)}^{\tilde{L}_1} e^{-V^{(1)}(x)} dx \right),$$

with probability larger than  $1 - 2e^{-\epsilon h_t/2}$  (note that we took  $\epsilon/2$  instead of  $\epsilon$  and  $h_t$  instead of  $\log t$ ). We bound the two integrals using Lemmas 4.4.6 and 4.4.7 and get, for  $t$  large enough :

$$\mathbb{P}(I_2 \leq 2e^{-\epsilon h_t/2}) \geq 1 - e^{-c_2 h_t}, \quad (4.5.36)$$

where  $c_2$  is a positive constant. Our lower bound for  $R_1^t$  comes from (4.4.19). Combing with (4.5.36) we get for some positive constant  $c_3$  and  $t$  large enough,

$$\mathbb{P}(I_2 \leq e^{-\epsilon h_t/4} R_1^t) \geq 1 - e^{-c_3 h_t}$$

so in place of (5.24) of [4] we have

$$\begin{aligned} \mathbb{P} \left( |I - \gamma t R_1^t| \geq e^{-\epsilon h_t/5} \gamma t R_1^t, H_{X_{\tilde{m}_1}}(\tilde{L}_1) > \sigma(\gamma t, \tilde{m}_1) \geq t(1-x), H_{X_{\tilde{m}_1}}(\tilde{L}_1) < H_{X_{\tilde{m}_1}}(\tilde{L}_0) \right) \\ \leq e^{-c_4 h_t} \end{aligned} \quad (4.5.37)$$

for all  $t$  large enough, and where  $c_4$  is a positive constant.

From our definition of  $e_1$  in Subsection 4.4.2 we still have

$$\sigma(\gamma t, \tilde{m}_1) > H(\tilde{L}_1) \Leftrightarrow \gamma t > A^1(\tilde{L}_1)e_1 \Leftrightarrow \gamma t R_1 > A^1(\tilde{L}_1)e_1 R_1.$$

(3.18) of [4] has to be replaced here by Lemma 4.4.10 and, as we already mentioned, (5.22) by Proposition 4.4.5. In place of (5.26) of [4] we thus have

$$\sigma(\gamma t, \tilde{m}_1) > H(\tilde{L}_1) \Rightarrow \gamma t R_1 > (1 - e^{-c_5 h_t}) H(\tilde{L}_1),$$

for some positive constant  $c_5$ , except on a event which probability is less than  $e^{-c_6 h_t}$  (for some positive constant  $c_6$ ). The fact that the constant are modified (with respect to the ones in the original proof) has no importance so we omit to mention it for the rest of the proof.

For  $\tilde{f}_\gamma(x)$ , note that we use our definition of  $\mathcal{D}_1$ . According to this definition we have  $\mathcal{D}_1 \subset [\tau_1^-(h_t/2), \tau_1(h_t/2)]$  for  $t$  large enough, so we can use the lower bounds for  $\mathbb{P}(A^1(\tilde{\tau}_1^-(h_t/2)) \leq e^{h_t(1+\epsilon)/2})$  and  $\mathbb{P}(A^1(\tilde{\tau}_1(h_t/2)) \leq e^{h_t(1+\epsilon)/2})$  (from the proof of Lemma 4.4.8) to bound  $\tilde{a}$  on  $\mathcal{D}_1$ .

(5.24) of [4] has, off course, to be replaced here by (4.5.37). Finally, (3.2) of [4] has to be replaced here by Lemma 4.5.21.

□



# Chapitre 5

## ALMOST SURE BEHAVIOR FOR THE LOCAL TIME OF A DIFFUSION IN A SPECTRALLY NEGATIVE LÉVY ENVIRONMENT

This work has been the object of an article [73] that will be shortly submitted.

### 5.1 Introduction

We study the almost sure asymptotic behavior of the supremum of the local time for a transient diffusion in a spectrally negative Lévy environment. Let  $V$  be a two-sided spectrally negative Lévy process which is not the opposite of a subordinator, drifts to  $-\infty$  at  $+\infty$ , and such that  $V(0) = 0$ . We denote its Laplace exponent by  $\Psi_V$  :

$$\forall t, \lambda \geq 0, \mathbb{E} [e^{\lambda V(t)}] = e^{t\Psi_V(\lambda)}.$$

It is well-known, for such  $V$ , that  $\Psi_V$  admits a non trivial zero that we denote here by  $\kappa$ ,  $\kappa := \inf\{\lambda > 0, \Psi_V(\lambda) = 0\} > 0$ .

We are here interested in a diffusion in this potential  $V$ . Such a diffusion  $(X(t), t \geq 0)$  is defined informally by  $X(0) = 0$  and

$$dX(t) = d\beta(t) - \frac{1}{2}V'(X(t))dt,$$

where  $\beta$  is a Brownian motion independent from  $V$ . Rigorously,  $X$  is defined by its conditional generator given  $V$ ,

$$\frac{1}{2}e^{V(x)} \frac{d}{dx} \left( e^{-V(x)} \frac{d}{dx} \right).$$



The fact that  $V$  drifts to  $-\infty$  puts us in the case where the diffusion  $X$  is a.s. transient to the right. The asymptotic behavior of this diffusion has been studied by Singh [66], he distinguishes three main possible behaviors depending on  $0 < \kappa < 1$ ,  $\kappa = 1$  or  $\kappa > 1$  (the case  $\kappa > 1$  being also divided into three subcases). We denote by  $(\mathcal{L}_X(t, x), t \geq 0, x \in \mathbb{R})$  the version of the local time that is continuous in time and càd-làg in space, and we define the supremum of the local time until instant  $t$  as

$$\mathcal{L}_X^*(t) = \sup_{x \in \mathbb{R}} \mathcal{L}_X(t, x).$$

Here, we study the almost sure asymptotic behavior of  $\mathcal{L}_X^*(t)$ . When the environment is a brownian motion with no drift, Shi [61] has studied this behavior and shown that

$$\mathbb{P}\text{-a.s. } \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t \log(\log(\log(t)))} \geq \frac{1}{32}, \quad (5.1.1)$$

where  $\mathbb{P}$  is the so-called annealed probability measure which definition is recalled in Subsection 5.1.3. In the same case, Andreatti and Diel [5] have proved more recently the convergence in distribution of  $\mathcal{L}_X^*(t)/t$ . Diel [29] has then continued the study by giving a finite upper bound for the limsup in (5.1.1) and doing the same study for the liminf :

$$\mathbb{P}\text{-a.s. } \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t \log(\log(\log(t)))} \leq \frac{e^2}{2} \quad \text{and} \quad \frac{j_0^2}{64} \leq \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t / \log(\log(\log(t)))} \leq \frac{e^2 \pi^2}{4},$$

where  $j_0$  is the smallest positive root of the Bessel function  $J_0$ .

In the case of a drifted-brownian environment, the almost sure behavior of  $\mathcal{L}_X^*(t)$  has been studied by Devulder [28] using annealed methods. He totally characterizes the almost sure behavior when  $\kappa > 1$ . In this case, for any positive non decreasing function  $a$  we have

$$\sum_{n=1}^{+\infty} \frac{1}{n(a(n))} \begin{cases} < +\infty \\ = +\infty \end{cases} \Leftrightarrow \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{(ta(t))^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \mathbb{P}\text{-a.s.}, \quad (5.1.2)$$

and

$$\mathbb{P}\text{-a.s. } \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{(t / \log(\log(t)))^{1/\kappa}} = 4(\kappa^2(\kappa - 1)/8)^{1/\kappa}. \quad (5.1.3)$$

When  $\kappa = 1$  he obtains

$$\mathbb{P}\text{-a.s. } \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t / \log(t) \log(\log(t))} \leq 1/2.$$

When  $0 < \kappa < 1$ , his method fails and only provides partial results for the almost sure behavior of the local time. More precisely he proves that the renormalisation for the limsup is greater than  $t$  :

$$\mathbb{P}\text{-a.s. } \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t} = +\infty, \quad (5.1.4)$$

and that the renormalization for the  $\liminf$  is at most  $t/\log(\log(t))$  and greater than  $t/(\log(t))^{1/\kappa}(\log(\log(t)))^{(2/\kappa)+\epsilon}$  for any  $\epsilon > 0$  :

$$\mathbb{P}\text{-a.s. } \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t/\log(\log(t))} \leq C(\kappa), \quad (5.1.5)$$

$$\forall \epsilon > 0, \mathbb{P}\text{-a.s. } \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t/(\log(t))^{1/\kappa}(\log(\log(t)))^{(2/\kappa)+\epsilon}} = +\infty, \quad (5.1.6)$$

where  $C(\kappa)$  is a positive non explicit constant.

For the discrete transient *Random Walk in Random Environment* (RWRE), the almost sure behavior of the supremum of the local time has been studied by Gantert and Shi [41], they obtain the behavior of the  $\limsup$  in the two subcases  $0 < \kappa \leq 1$  and  $\kappa > 1$ .

For the diffusion in the general potential  $V$ , the convergence in distribution of  $\mathcal{L}_X^*(t)/t$  has been studied by the author in [74] using two different methods : when  $0 < \kappa < 1$  a path decomposition of the environment that provides an interesting renewal structure to study the diffusion is used, this method is inspired from Andreatti and al. [4], [3] which are themselves inspired from the work of Enriquez and al. [34] in the discrete case. When  $\kappa > 1$  an equality in law between the local time and a generalized Ornstein-Uhlenbeck process that was introduced in [66] is used. The main contribution of this paper is in the case  $0 < \kappa < 1$ , pushing further the ideas of [74], we make a deep study of the renewal structure of the diffusion in order to establish the almost sure asymptotic behavior of  $\mathcal{L}_X^*(t)$ . We characterize this behavior when  $0 < \kappa < 1$  and  $\kappa > 1$ . In particular, the restriction of our results to the case of a drifted brownian potential with  $0 < \kappa < 1$  improves the results (5.1.4), (5.1.5) and (5.1.6) of [28] by giving the exact renormalizations and even the exact value of the constant for the  $\limsup$ , for the  $\liminf$  we get the exact renormalizations and an explicit upper bound of the constant.

### 5.1.1 Main results

We start with the case  $0 < \kappa < 1$ . In that case, the limit distribution of  $\mathcal{L}_X^*(t)/t$  given in [74] depends on exponential functionals of  $V$  and its dual conditioned to stay positive. They are defined as follow :

$$I(V^\uparrow) := \int_0^{+\infty} e^{-V^\uparrow(t)} dt \quad \text{and} \quad I(\hat{V}^\uparrow) := \int_0^{+\infty} e^{-\hat{V}^\uparrow(t)} dt,$$

where  $\hat{V}$ , the dual of  $V$ , is equal in law to  $-V$ . In Subsection 5.4.1 it is precised how  $V^\uparrow$  and  $\hat{V}^\uparrow$  are defined rigorously. These functionals are studied by the author in [72] where it is proved in Theorems 1.1 and 1.13 that they are indeed finite and well-defined. Let  $G_1$  and  $G_2$  be two independent random variables with  $G_1 \stackrel{\mathcal{L}}{=} I(V^\uparrow)$  and  $G_2 \stackrel{\mathcal{L}}{=} I(\hat{V}^\uparrow)$ . We define  $\mathcal{R} := G_1 + G_2$ .

To study the lim sup of  $\mathcal{L}_X^*(t)$ , we link the almost sur asymptotic behavior of  $\mathcal{L}_X^*(t)$  with the left tail of  $I(V^\uparrow)$  (or of  $\mathcal{R}$  according to the case). Before stating our results, let us recall what is known about the left tail of  $I(V^\uparrow)$ .

In [72], the left tail of  $I(V^\uparrow)$  is linked to the asymptotic behavior of  $\Psi_V$ . This asymptotic behavior is usually quantified thanks to two real numbers  $\sigma$  and  $\beta$  :

$$\begin{aligned} \sigma &:= \sup \left\{ \alpha \geq 0, \lim_{\lambda \rightarrow +\infty} \lambda^{-\alpha} \Psi_V(\lambda) = \infty \right\}, \\ \beta &:= \inf \left\{ \alpha \geq 0, \lim_{\lambda \rightarrow +\infty} \lambda^{-\alpha} \Psi_V(\lambda) = 0 \right\}. \end{aligned}$$

If  $\Psi_V$  has  $\alpha$ -regular variation for  $\alpha \in [1, 2]$  (for example if  $V$  is a drifted  $\alpha$ -stable Lévy process with no positive jumps), we have  $\sigma = \beta = \alpha$ . Note that when  $Q$ , the brownian component of  $V$ , is positive, then  $\Psi_V$  has 2-regular variation, and when  $Q = 0$ ,  $1 \leq \sigma \leq \beta \leq 2$ . The asymptotic behavior of  $\mathbb{P}(I(V^\uparrow) \leq x)$  as  $x$  goes to 0 is given by the following theorem from [72] :

**Theorem 5.1.1.** [Véchambre, [72]]

*There is a positive constant  $K_0$  (depending on  $V$ ) such that for  $x$  small enough*

$$\mathbb{P}(I(V^\uparrow) \leq x) \leq e^{-K_0/x}. \quad (5.1.7)$$

*More precisely we have*

$$\forall l < \frac{1}{\beta - 1}, \mathbb{P}(I(V^\uparrow) \leq x) \leq e^{-1/x^l}. \quad (5.1.8)$$

$$\text{If } \sigma > 1, \forall l > \frac{1}{\sigma - 1}, \mathbb{P}(I(V^\uparrow) \leq x) \geq e^{-1/x^l}. \quad (5.1.9)$$

*If there are two positive constants  $c < C$  and  $\alpha \in ]1, 2]$  such that  $c\lambda^\alpha \leq \Psi_V(\lambda) \leq C\lambda^\alpha$  for  $\lambda$  large enough, then for any  $\delta > 1$  we have, when  $x$  is small enough,*

$$\exp\left(-\frac{\delta\alpha^{\frac{\alpha}{\alpha-1}}}{(cx)^{\frac{1}{\alpha-1}}}\right) \leq \mathbb{P}(I(V^\uparrow) \leq x) \leq \exp\left(-\frac{\alpha - 1}{\delta(Cx)^{\frac{1}{\alpha-1}}}\right), \quad (5.1.10)$$

*If there is a positive constant  $C$  and  $\alpha \in ]1, 2]$  such that  $\Psi_V(\lambda) \sim_{\lambda \rightarrow +\infty} C\lambda^\alpha$ , then*

$$-\log(\mathbb{P}(I(V^\uparrow) \leq x)) \underset{x \rightarrow 0}{\sim} \frac{\alpha - 1}{(Cx)^{\frac{1}{\alpha-1}}}. \quad (5.1.11)$$

(5.1.7) is Remark 1.6 of [72], (5.1.8) and (5.1.9) are a reformulation of Theorem 1.4 of [72], and (5.1.10) comes from Theorem 1.2 of [72]. (5.1.11) is Theorem 1.5 of [72].

We can now state our results for the lim sup :

**Theorem 5.1.2.** — *Assume that  $0 < \kappa < 1$ ,  $V$  has unbounded variation,  $V(1) \in L^p$  for some  $p > 1$  and  $V$  possesses negative jumps.*

*If there is  $\gamma > 1$  and  $C > 0$  such that for  $x$  small enough*

$$\mathbb{P}(I(V^\uparrow) \leq x) \leq \exp\left(-\frac{C}{x^{\frac{1}{\gamma-1}}}\right), \quad (5.1.12)$$

*then we have*

$$\mathbb{P}\text{-a.s. } \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\gamma-1}} \leq C^{1-\gamma}. \quad (5.1.13)$$

*If there is  $\gamma > 1$  and  $C > 0$  such that for  $x$  small enough*

$$\mathbb{P}(I(V^\uparrow) \leq x) \geq \exp\left(-\frac{C}{x^{\frac{1}{\gamma-1}}}\right), \quad (5.1.14)$$

*then we have*

$$\mathbb{P}\text{-a.s. } \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\gamma-1}} \geq C^{1-\gamma}. \quad (5.1.15)$$

— *Assume now that  $V(t) = W_\kappa(t) := W(t) - \frac{\kappa}{2}t$  with  $0 < \kappa < 1$ , (i.e.  $V$  is the  $\kappa$ -drifted brownian motion), then the above implications ((5.1.12)  $\Rightarrow$  (5.1.13) and (5.1.14)  $\Rightarrow$  (5.1.15)) are still true with  $I(V^\uparrow)$  replaced by  $\mathcal{R}$ .*

In the above theorem we had to distinguish the case where  $V$  possesses negative jumps and the case where  $V$  is a drifted brownian motion. First, note that this is a true alternative : since  $V$  is spectrally negative, the case where  $V$  do not possess negative jumps is the case where  $V$  do not possess jumps at all and is therefore a drifted brownian motion. In this case we assumed for convenience that the gaussian component of  $V$  is normalized to 1. The difference between the two cases in the above proposition comes from the absence or presence of symmetry for the environment. When  $V$  possesses negative jumps,  $\hat{V}^\uparrow$  possesses positives jumps that might repulse it very fast from 0, this is why the left tail of  $I(V^\uparrow)$  is thinner than the left tail of  $I(\hat{V}^\uparrow)$ , as we can see comparing Theorems 1.4 and 1.14 of [72]. As a consequence, only the left tail of  $I(V^\uparrow)$  is relevant in the left tail of  $\mathcal{R}$ . When  $V$  is a drifted brownian motion, a symmetry appears :  $\hat{V}^\uparrow$  and  $V^\uparrow$  are equal in law,  $\mathcal{R}$  is then the sum of two independent random variables having the same law as  $I(V^\uparrow)$  and none of them can be neglected.

**Remark 5.1.3.** *It has to be noted that the lim sup above is  $\mathbb{P}$ -almost surely equal to a constant belonging to  $[0, +\infty]$  and that the inequalities (5.1.13) and (5.1.15) are inequalities relative to this value that the lim sup equals  $\mathbb{P}$ -almost surely. The same will be true in all the results below : all the lim sup and lim inf considered are  $\mathbb{P}$ -almost surely equal to a constant. This fact is justified in Subsection 5.4.4.*

Putting together Theorem 5.1.2 and what is known for the left tail of  $I(V^\uparrow)$  (Theorem 5.1.1), we can state precise results for the lim sup :

**Theorem 5.1.4.** *If  $0 < \kappa < 1$ ,  $V$  has unbounded variation and  $V(1) \in L^p$  for some  $p > 1$ , then we have*

$$\forall \beta' > \beta, \mathbb{P}\text{-a.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\beta'-1}} = 0, \quad (5.1.16)$$

and

$$\text{If } \sigma > 1, \forall \sigma' \in ]1, \sigma[, \mathbb{P}\text{-a.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\sigma'-1}} = +\infty. \quad (5.1.17)$$

If we make further hypothesis on the regularity of the variation of  $\Psi_V$  we can give the exact order of  $\mathcal{L}_X^*(t)$  :

**Theorem 5.1.5.** — *Assume that  $0 < \kappa < 1$ ,  $V$  has unbounded variation,  $V(1) \in L^p$  for some  $p > 1$  and  $V$  possesses negative jumps.*

*If there are two positive constants  $c < C$  and  $\alpha \in ]1, 2]$  such that  $c\lambda^\alpha \leq \Psi_V(\lambda) \leq C\lambda^\alpha$  for  $\lambda$  large enough, then we have*

$$\mathbb{P}\text{-a.s.} \frac{c}{\alpha^\alpha} \leq \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\alpha-1}} \leq \frac{C}{(\alpha-1)^{\alpha-1}}.$$

*If, more precisely, there is a positive constant  $C$  and  $\alpha \in ]1, 2]$  such that  $\Psi_V(\lambda) \sim C\lambda^\alpha$  for large  $\lambda$ , then we have*

$$\mathbb{P}\text{-a.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\alpha-1}} = \frac{C}{(\alpha-1)^{\alpha-1}}.$$

— *Assume now that  $V = W_\kappa$ , the  $\kappa$ -drifted brownian motion, with  $0 < \kappa < 1$ , then we have*

$$\mathbb{P}\text{-a.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))} = \frac{1}{8}.$$

**Remark 5.1.6.** *According to the combination of Theorem 5.1.2 and (5.1.7) we see that, if  $0 < \kappa < 1$ ,  $V$  has unbounded variation and  $V(1) \in L^p$  for some  $p > 1$ , then we always have*

$$\mathbb{P}\text{-a.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))} < +\infty.$$

*In other words,  $t(\log(\log(t)))$  is the maximal possible renormalisation for the lim sup.*

**Remark 5.1.7.** *As it was noticed by Shi [61] and Diel [29] for the recurrent case, we also notice a difference between the renormalization of the local time for the discrete transient RWRE with zero speed (given by Gantert and Shi [41]) and the renormalization of the local time that we give for the transient diffusion with zero speed. This difference can be explained as in the recurrent case : the valleys can potentially be much steeper in the continuous case with a potential having unbounded variation than in the discrete case, so the local maxima of the local time can potentially be higher in the first case.*

We see in the above two theorems that the renormalization of  $\mathcal{L}_X^*(t)$  for the limsup depends directly on the asymptotic behavior of  $\Psi_V$ . In particular, Theorem 5.1.5 says that for a drifted  $\alpha$ -stable environment (with no positive jumps and  $\alpha > 1$ ), the renormalization of  $\mathcal{L}_X^*(t)$  is  $t(\log(\log(t)))^{\alpha-1}$ . We see that we have much more possible behaviors with general spectrally negative Lévy environments, than with drifted brownian environments. Even if, for technical reasons, the above theorems do not apply when the environment  $V$  has bounded variation, we can conjecture that the behavior of  $\mathcal{L}_X^*(t)$  remains linked in the same way to the left tail of  $I(V^\uparrow)$  which is given by Remark 1.7 of [72]. This implies

**Conjecture 5.1.8.** *When  $V$  has bounded variation, we have*

$$\mathbb{P}\text{-a.s. } 0 < \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t} < +\infty.$$

If this conjecture is true, we would have, when the environment has bounded variation, the same renormalization as in the discrete transient case given by Theorem 1.1 of [41]. This would not be surprising since the discrete case gives rise to potentials of bounded variation. Moreover, if  $V$  has bounded variation then it is known to be the difference of a deterministic positive drift and a subordinator. The valleys can then not be steeper than the deterministic drift so, according to Remark 5.1.7, the expected renormalization of  $\mathcal{L}_X^*(t)$  has to be the same as in the discrete case.

For the lim inf, there is only one possible renormalization. Our result is as follows :

**Theorem 5.1.9.** *If  $0 < \kappa < 1$ ,  $V$  has unbounded variation and  $V(1) \in L^p$  for some  $p > 1$ , then we have*

$$\mathbb{P}\text{-a.s. } 0 < \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t/\log(\log(t))} \leq \frac{1 - \kappa}{\kappa(\mathbb{E}[I(V^\uparrow)] + \mathbb{E}[I(\hat{V}^\uparrow)])}. \quad (5.1.18)$$

Note that the expectations  $\mathbb{E}[I(V^\uparrow)]$  and  $\mathbb{E}[I(\hat{V}^\uparrow)]$  are finite and well defined since  $I(V^\uparrow)$  and  $I(\hat{V}^\uparrow)$  both admit some finite exponential moments according to Theorems 1.1 and 1.13 of [72].

**Example :** We consider  $W_\kappa$  the  $\kappa$ -drifted brownian motion ( $W_\kappa(t) := W(t) - \frac{\kappa}{2}t$ ), then, the expression of the Laplace transform of  $I(W_\kappa^\uparrow)$  is given by equation (1.9) of [72]. This expression allows to compute the moments of  $I(W_\kappa^\uparrow)$  and gives in particular  $\mathbb{E}[I(W_\kappa^\uparrow)] = 2/(1 + \kappa)$ . Moreover  $W_\kappa^\uparrow$  and  $\hat{W}_\kappa^\uparrow$  have the same law so  $\mathbb{E}[I(W_\kappa^\uparrow)] + \mathbb{E}[I(\hat{W}_\kappa^\uparrow)] = 4/(1 + \kappa)$ . If we choose, as an environment,  $V = W_\kappa$  (for  $0 < \kappa < 1$ ), then the above upper bound for the lim inf becomes  $(1 - \kappa^2)/4\kappa$ . Putting this in relation with the results of [28], we see that the application of Theorem 5.1.9 in the special case of a drifted brownian environment improves (5.1.6) and completes (5.1.5) by proving that this renormalization is exact and by providing an explicit upper bound.

The fact that we have many possible renormalizations for the lim sup, depending on the environment  $V$ , while only one for the lim inf, whatever is the environment  $V$ , might seem surprising, here is an heuristic explanation : In each valley the contribution to the time equals approximately the contribution to the local time multiplied by an exponential functional of the bottom of the valley (which is close to  $\mathcal{R}$ ). The lim sup concerns large values of the local time at a fixed time, it is reached when the contribution to the local time of some valley is large while the contribution to the time of the same valley has a fixed value, this happens when the exponential functional of the bottom of the valley has a small value. The link between the lim sup and the small values of an exponential functional is made rigorously in Theorem 5.1.2. The lim inf concerns small values of the local time at a fixed time, it is reached when the contributions to the local time of the valleys are small while the sum of their contributions to the time have a fixed value, this happens when the exponential functionals of the bottoms of some valleys are large. We see that the difference between lim sup and lim inf comes from the difference between the left and right tails of  $\mathcal{R}$ . The left tail is mainly the left tail of  $I(V^\uparrow)$  which depends on the asymptotic of  $\Psi_V$ , according to Theorem 5.1.1, and  $\Psi_V$  have many possible behaviors. On the other hand, the right tail is always exponential according to Theorems 1.1 and 1.13 and Remark 3.5 of [72]. This explains the difference of behaviors between the lim sup and the lim inf.

We now treat the case where  $\kappa > 1$ . In this case we use a different method and the results are different.

**Theorem 5.1.10.** *Let  $f$  be a positive non-increasing function. When  $\kappa > 1$ , we have*

$$\int_1^{+\infty} \frac{(f(t))^\kappa}{t} dt \begin{cases} < +\infty \\ = +\infty \end{cases} \Leftrightarrow \limsup_{t \rightarrow +\infty} \frac{f(t)\mathcal{L}_X^*(t)}{t^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \mathbb{P}\text{-a.s.}$$

The above result is the analogue of Theorem 1.2 of [41] for the continuous case. In the special case where  $V = W_\kappa$  (for  $\kappa > 1$ ), our result coincides with (5.1.2) (proved by Devulder in [28]). Indeed, since  $f$  is decreasing we have the easy equivalences

$$\int_1^{+\infty} \frac{(f(t))^\kappa}{t} dt < +\infty \Leftrightarrow \sum_{n=1}^{+\infty} \frac{(f(n))^\kappa}{n} < +\infty \Leftrightarrow \sum_{n=1}^{+\infty} (f(2^n))^\kappa < +\infty. \quad (5.1.19)$$

The first equivalence shows that our result agrees with (5.1.2) and the second equivalence allows to reformulate the integrability condition in a form that is convenient to prove Theorem 5.1.10.

Comparing Theorems 5.1.4, 5.1.5 and 5.1.10 we see that the renormalization for the lim sup is larger in the slow transient case than in the fast transient case : this is in accordance with intuition. However, it is surprising to see that the renormalization in the slow transient case is also greater than the renormalization in the recurrent case, given by Theorem 1.1 of [29]. Here is the heuristic explanation : In the recurrent case

the diffusion is trapped in the bottom of a large valley while in the slow transient case the diffusion gets successively trapped in the bottom of many valleys, these bottom being much more narrow. This explains that the large values of the local time have the tendency to be higher in the second case.

For the  $\liminf$ , we provide an explicite value. Let the constants  $K$  and  $m$  be defined similarly as in [66] :

$$K := \mathbb{E} \left[ \left( \int_0^{+\infty} e^{V(t)} dt \right)^{\kappa-1} \right] \quad \text{and} \quad m := \frac{-2}{\Psi_V(1)} > 0.$$

We have :

**Theorem 5.1.11.** *When  $\kappa > 1$ , we have*

$$\mathbb{P}\text{-a.s.} \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{(t/\log(\log(t)))^{1/\kappa}} = 2(\Gamma(\kappa)\kappa^2 K/m)^{1/\kappa}.$$

**Example :** If we choose  $V = W_\kappa$  (for  $\kappa > 1$ ), then  $K = 2^{\kappa-1}/\Gamma(\kappa)$  (see Example 1.1 in [66]) and  $m = 4/(\kappa - 1)$ . The above limit is then  $4(\kappa^2(\kappa - 1)/8)^{1/\kappa}$ . This coincides with (5.1.3) (proved by Devulder in [28]).

For the  $\liminf$  the behaviors of the different cases are in accordance with intuition : comparing Theorem 1.1 of [29] with Theorems 5.1.9 and 5.1.11 we see that the renormalization for the  $\liminf$  in the fast transient case is smaller than the renormalization in the slow transient case which is in turn smaller than the renormalization in the recurrent case.

## 5.1.2 Sketch of proofs and organisation of the paper

The rest of the paper is organized as follows.

*In section 5.2* we study the case  $0 < \kappa < 1$ . We first recall the decomposition of the environment into valleys from [74] and the behavior of the diffusion with respect to these valleys. In particular, we recall how the renewal structure of the diffusion allows to approximate the supremum of the local time and the time spent by the diffusion in the bottom of the valleys by an *iid* sequence of  $\mathbb{R}^2$ -valued random variables.

For the  $\limsup$ , we study the asymptotic of the distribution function  $\mathbb{P}(\mathcal{L}_X^*(t)/t \geq x_t)$ , where  $x_t$  is a suitably chosen quantity that goes to infinity with  $t$ . More precisely, in Proposition 5.2.11 we compare the asymptotic of this distribution function with the one of  $\mathbb{P}(Y_1^{\natural,t}(Y_2^{-1,t}(1)-) \geq x_t)$ , the distribution function of a functional of the above mentioned *iid* sequence, and in Proposition 5.2.15 we link the asymptotic behavior of  $\mathbb{P}(Y_1^{\natural,t}(Y_2^{-1,t}(1)-) \geq x_t)$  with the left tail of  $I(V^\uparrow)$  (or  $\mathcal{R}$  in the case of a drifted brownian potential). The synthesis of Propositions 5.2.11 and 5.2.15 allows to compare the distribution function  $\mathbb{P}(\mathcal{L}_X^*(t)/t \geq x_t)$  with the left tail of



$I(V^\uparrow)$  (or  $\mathcal{R}$  in the case of a drifted brownian potential) in Proposition 5.2.16. This proposition entails Theorem 5.1.2 by the mean of the Borel-Cantelli Lemma and the technical Lemma 5.2.10, which decomposes the trajectory of the diffusion into large independent parts in order to get the required independence to apply the Borel-Cantelli Lemma. The combination of Theorem 5.1.2 with what is known (and recalled in Theorem 5.1.1) for the left tail of  $I(V^\uparrow)$  easily yields Theorems 5.1.4 and 5.1.5 which solves the problem for the lim sup.

For the lim inf, we study the quantity  $\mathbb{P}(\mathcal{L}_X^*(t) \leq t/x_t)$ . In Proposition 5.2.17 it is compared with  $\mathbb{P}(Y_1^{\natural,t}(Y_2^{-1,t}(1)) \leq 1/x_t)$ , the distribution function of an other functional of the  $\mathbb{R}^2$ -valued *iid* sequence. In Lemmas 5.2.18 and 5.2.19 we study the Laplace transform of a random variable involved in this functional, this allows to give a lower and an upper bound for  $\mathbb{P}(Y_1^{\natural,t}(Y_2^{-1,t}(1)) \leq 1/x_t)$  in Proposition 5.2.20. The synthesis of Propositions 5.2.17 and 5.2.20 gives the asymptotic of  $\mathbb{P}(\mathcal{L}_X^*(t) \leq t/x_t)$  in Proposition 5.2.21. This Proposition entails Theorem 5.1.9 by the mean of the Borel-Cantelli Lemma and Lemma 5.2.10. This solves the problem for the lim inf.

In Section 5.3 we study the case  $\kappa > 1$ . In this case, the local time at  $t$  can be approximated by the local time at an hitting time and the latter has the same law as the generalized Ornstein-Uhlenbeck process introduced in [66]. Using what is known for the excursion measure of this process we prove Theorems 5.1.10 and 5.1.11.

In Section 5.4 we justify some facts about  $V$ ,  $V^\uparrow$  and the diffusion in  $V$  that are used along the paper.

### 5.1.3 Facts and notations

For  $Y$  a process and  $S$  a borelian set, we denote

$$\tau(Y, S) := \inf \{t \geq 0, Y(t) \in S\}, \quad \mathcal{K}(Y, S) := \sup \{t \geq 0, Y(t) \in S\}.$$

We shall only write  $\tau(Y, x)$  instead of  $\tau(Y, \{x\})$  and  $\tau(Y, x+)$  instead of  $\tau(Y, [x, +\infty[)$ . Since  $V$  has no positive jumps we see that each positive level is reached continuously (or not reached at all) :  $\forall x > 0, \tau(V, x+) = \tau(V, x)$  (which is possibly infinite). Moreover, the law of the supremum of  $V$  is known, it is an exponential distribution with parameter  $\kappa$  (see Corollary VII.2 in Bertoin [8]).

If  $Y$  is Markovian and  $x \in \mathbb{R}$  we denote  $Y_x$  for the process  $Y$  starting from  $x$ . For  $Y_0$  we shall only write  $Y$ . When it exists we denote by  $(\mathcal{L}_Y(t, x), t \geq 0, x \in \mathbb{R})$  the version of the local time that is continus in time and càd-làg in space and by  $(\sigma_Y(t, x), t \geq 0, x \in \mathbb{R})$  the inverse of the local time :  $\sigma_Y(t, x) := \inf\{s \geq 0, \mathcal{L}_Y(s, x) > t\}$ .

Let  $B$  be a brownian motion starting at 0 and independent from  $V$ . A diffusion in potential  $V$  can be defined via the formula :

$$X(t) := A_V^{-1}(B(T_V^{-1}(t))). \tag{5.1.20}$$

where

$$A_V(x) := \int_0^x e^{V(u)} du \quad \text{and for } 0 \leq s \leq \tau \left( B, \int_0^{+\infty} e^{V(u)} du \right),$$

$$T_V(s) := \int_0^s e^{-2V(A_V^{-1}(B(u)))} du.$$

It is known that the local time of  $X$  at  $x$  until instant  $t$  has the following expression :

$$\mathcal{L}_X(t, x) = e^{-V(x)} \mathcal{L}_B(T_V^{-1}(t), A_V(x)). \quad (5.1.21)$$

Recall the notation  $\mathcal{L}_X^*(t)$  for the supremum of the local time until time  $t$ . We also sometimes use the notation  $\mathcal{L}_X^{*,+}$  for the supremum of the local time on the positive half-line :  $\mathcal{L}_X^{*,+}(t) := \sup_{x \in [0, +\infty[} \mathcal{L}_X(t, x)$ .

For the hitting times of  $r \in \mathbb{R}$  by the diffusion  $X$  we shall use the frequent notation  $H(r)$  (instead of  $\tau(X, r)$ ).

$D(\mathbb{R}, \mathbb{R})$  is the space of càd-làg functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Let  $P$  be the probability measure on  $D(\mathbb{R}, \mathbb{R})$  inducing the law of  $V$ . For  $v \in D(\mathbb{R}, \mathbb{R})$ , the quenched probability measure  $P^v$  is the probability measure (associated with the diffusion  $X$ ) conditionally on  $\{V = v\}$ .  $\mathbb{P}$  represents the annealed probability measure, it is defined as  $\mathbb{P}(\cdot) := \int_{D(\mathbb{R}, \mathbb{R})} P^v(\cdot) P(dv)$ .  $X$  is a Markovian process under  $P^v$  but not under  $\mathbb{P}$ . Note that all the almost sure convergences stated in this Introduction are  $\mathbb{P}$ -almost sure convergences. For objects not related to the diffusion  $X$  we also use the natural notation  $\mathbb{P}$  for a probability.

If  $Z$  is a random variable, its law is denoted by  $\mathcal{L}(Z)$  and if  $A$  is an event of positive probability,  $\mathcal{L}(Z|A)$  denotes the law of  $Z$  conditionally to the event  $A$ .

For  $Z$  an increasing càd-làg process and  $s \geq 0$ , we put respectively  $Z(s-)$ ,  $Z^{\natural}(s)$  and  $Z^{-1}(s)$  for respectively the left-limit of  $Z$  at  $s$ , the largest jump of  $Z$  before  $s$  and the generalized inverse of  $Z$  at  $s$  :

$$Z(s-) = \lim_{r \nearrow s} Z(r), \quad Z^{\natural}(s) := \sup_{0 \leq r \leq s} (Z(r) - Z(r-)), \quad Z^{-1}(s) := \inf\{u \geq 0, Z(u) > s\}.$$

For two quantities  $a$  and  $b$  depending on a parameter,  $a \approx b$  means that  $\log(a) \sim \log(b)$  when the parameter converges (generally to 0 or infinity).

## 5.2 Almost sure behavior when $0 < \kappa < 1$

In all this section we assume the hypotheses of Theorems 5.1.4 and 5.1.9 :  $0 < \kappa < 1$ ,  $V$  has unbounded variation and there exists  $p > 1$  such that  $V(1) \in L^p$ . The hypothesis of unbounded variation is necessary to approximate the law of the left part of a valley by the law of  $\hat{V}^{\uparrow}$  and the hypothesis about moments for  $V(1)$  allows to neglect the local time outside the bottom of the valleys. For these reasons, many results of [74] (that are recalled in the next subsection) have been proved under these hypotheses.

### 5.2.1 Traps for the diffusion

We now recall some definitions about valleys and describe how the diffusion gets trapped into successive valleys. The facts and lemmas stated in this subsection are more or less classical and there are all proved or justified in Subsection 5.4.3 (except Fact 5.2.3 which is readily Lemma 3.5 of [74]).

We first recall the notion of  $h$ -extrema. For  $h > 0$ , we say that  $x \in \mathbb{R}$  is an  $h$ -minimum for  $V$  if there exist  $u < x < v$  such that  $V(y) \wedge V(y-) \geq V(x) \wedge V(x-)$  for all  $y \in [u, v]$ ,  $V(u) \geq (V(x) \wedge V(x-)) + h$  and  $V(v-) \geq (V(x) \wedge V(x-)) + h$ . Moreover,  $x$  is an  $h$ -maximum for  $V$  if  $x$  is an  $h$ -minimum for  $-V$ , and  $x$  is an  $h$ -extremum for  $V$  if it is an  $h$ -maximum or an  $h$ -minimum for  $V$ .

Since  $V$  is not a compound Poisson process, it is known (see Proposition VI.4, in [8]) that it takes pairwise distinct values in its local extrema. Combining this with the fact that  $V$  has almost surely càd-làg paths and drifts to  $-\infty$  without being the opposite of a subordinator, we can check that the set of  $h$ -extrema is discrete, forms a sequence indexed by  $\mathbb{Z}$ , unbounded from below and above, and that the  $h$ -minima and  $h$ -maxima alternate. Let  $\mathcal{V} \subset D(\mathbb{R}, \mathbb{R})$  be the set of the environments  $v$  that satisfy the above properties and that are such that

$$v(x) \xrightarrow{x \rightarrow +\infty} -\infty, \quad v(x) \xrightarrow{x \rightarrow -\infty} +\infty, \quad \int_0^{+\infty} e^{v(x)} dx < +\infty.$$

Note that the path of  $V$  belongs to  $\mathcal{V}$  with probability 1.

We denote respectively by  $(m_i, i \in \mathbb{Z})$  and  $(M_i, i \in \mathbb{Z})$  the increasing sequences of  $h$ -minima and of  $h$ -maxima of  $V$ , such that  $m_0 \leq 0 < m_1$  and  $m_i < M_i < m_{i+1}$  for every  $i \in \mathbb{Z}$ . An  $h$ -valleys is the fragments of the trajectory of  $V$  between two  $h$ -maxima.

The valleys are visited successively by the diffusions. For the size of the valleys to be well adapted with respect to the time scale, we have to make the size of the valleys grow with time  $t$ . We are thus interested in  $h_t$ -valleys where

$$h_t := \log(t) - \phi(t), \quad \text{with } \phi(t) := (\log(\log(t)))^\omega, \quad (5.2.1)$$

where  $\omega > 1$  will be chosen later in accordance with some other parameters. We also define  $N_t$ , the indice of the largest  $h_t$ -minima visited by  $X$  until time  $t$ ,

$$N_t := \max \left\{ k \in \mathbb{N}, \sup_{0 \leq s \leq t} X(s) \geq m_k \right\}.$$

We need deterministic bounds for the number of visited valleys. We define

$$n_t := \lfloor e^{\kappa(1+\delta)\phi(t)} \rfloor \quad \text{and} \quad \tilde{n}_t := e^{\rho\phi(t)},$$

where we fix  $\rho \in ]0, \kappa/(1+\kappa)[$  once and for all in all the paper. The following lemma says that with high probability,  $\tilde{n}_t \leq N_t \leq n_t$  :

**Lemma 5.2.1.** *There is a positive constant  $c$  such that for all  $t$  large enough,*

$$\mathbb{P}(N_t \geq n_t) \leq e^{-cht} \leq e^{-c\phi(t)}, \quad (5.2.2)$$

$$\mathbb{P}(N_t \leq \tilde{n}_t) \leq e^{-c\phi(t)}. \quad (5.2.3)$$

We recall the definition of the standard valleys given in [74]. Their interest is mainly the fact that they are defined via successive stopping time, which make them convenient to use in the calculations, and also the fact that they take in consideration the descending phases between two  $h_t$ -minima.

Let  $\delta > 0$ , small enough so that  $(1 + 3\delta)\kappa < 1$ , be defined once and for all in the paper. Assume  $t$  is large enough so that  $e^{(1-\delta)\kappa h_t} \geq h_t$ . We define  $\tilde{\tau}_0(h_t) = \tilde{L}_0 := 0$  and recursively for  $i \geq 1$ ,

$$\begin{aligned} \tilde{L}_i^\sharp &:= \inf\{x > \tilde{L}_{i-1}, V(x) \leq V(\tilde{L}_{i-1}) - e^{(1-\delta)\kappa h_t}\}, \\ \tilde{\tau}_i(h_t) &:= \inf\{x \geq \tilde{L}_i^\sharp, V(x) - \inf_{[\tilde{L}_i^\sharp, x]} V = h_t\}, \\ \tilde{m}_i &:= \inf\{x \geq \tilde{L}_i^\sharp, V(x) = \inf_{[\tilde{L}_i^\sharp, \tilde{\tau}_i(h_t)]} V\}, \\ \tilde{L}_i &:= \inf\{x > \tilde{\tau}_i(h_t), V(x) \leq h_t/2\}, \\ \tilde{\tau}_i^-(a) &:= \sup\{x < \tilde{m}_i, V(x) - V(\tilde{m}_i) \geq a\}, \quad \forall a \in [0, h_t], \\ \tilde{\tau}_i^+(a) &:= \inf\{x > \tilde{m}_i, V(x) - V(\tilde{m}_i) = a\}, \quad \forall a \in [0, h_t]. \end{aligned}$$

These random variables depend on  $h_t$  and therefore on  $t$ , even if this does not appear in the notations. We also define

$$\tilde{V}^{(i)}(x) := V(x) - V(\tilde{m}_i), \quad \forall x \in \mathbb{R}.$$

We call  $i^{\text{th}}$  *standard valley* the re-centered truncated potential  $(\tilde{V}^{(i)}(x), \tilde{L}_{i-1} \leq x \leq \tilde{L}_i)$ . The law of the bottom of these valleys is given in Fact 5.4.7 of Section 5.4.

Similarly as in [74] we define the *deep bottoms* of the  $j^{\text{th}}$  standard valleys to be the interval

$$\mathcal{D}_j := [\tilde{\tau}_j^-((\phi(t))^2), \tilde{\tau}_j^+((\phi(t))^2)].$$

**Remark 5.2.2.** *The random times  $\tilde{L}_i^\sharp$ ,  $\tilde{\tau}_i(h_t)$ , and  $\tilde{L}_i$  are stopping times. As a consequence, the sequence  $(\tilde{V}^{(i)}(x + \tilde{m}_i), \tilde{L}_{i-1} - \tilde{m}_i \leq x \leq \tilde{L}_i - \tilde{m}_i)_{i \geq 1}$  is iid.*

We also have that the sequence  $(\tilde{m}_i)_{i \geq 1}$  of the minima of the standard valleys coincides with the sequence  $(m_i)_{i \geq 1}$  with high probability for a large number of indices. Let

$$\mathcal{V}_t := \{v \in \mathcal{V}, \forall i \in \{1, \dots, n_t\}, m_i = \tilde{m}_i\}.$$

Note from the definition of  $\mathcal{V}$  that the sequences  $(\tilde{m}_i)_{i \geq 1}$  and  $(m_i)_{i \geq 1}$  are always defined for any  $v \in \mathcal{V}$  so in particular the event  $\mathcal{V}_t$  is well defined. We have :

**Fact 5.2.3.** *(Lemma 3.5 of [74])*

*There is a positive constant  $c$  such that for all  $t$  large enough,*

$$P(V \in \mathcal{V}_t) \geq 1 - e^{-cht}.$$

We define  $X_{\tilde{m}_j} := X(\cdot + \tilde{m}_j)$  which is, according to the Markov property, a diffusion in potential  $V$  starting from  $\tilde{m}_j$ . We also define, for any  $r \in \mathbb{R}$ ,  $H_{X_{\tilde{m}_j}}(r)$  to be the hitting time of  $r$  by  $X_{\tilde{m}_j}$ . When we deal with  $X_{\tilde{m}_j}$  we often need the notation  $A^j(x) = \int_{\tilde{m}_j}^x e^{\tilde{V}^{(j)}(s)} ds$ .

As in [4] and [74], we approximate the repartition function of the renormalized local time by repartition functions of functionals of the sequence  $(e_j S_j^t, e_j S_j^t R_j^t)_{j \geq 1}$  where

$$e_j := \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) / A^j(\tilde{L}_j), \quad S_j^t := \int_{\tilde{\tau}_j^-(ht/2)}^{\tilde{L}_j} e^{\tilde{V}^{(j)}(u)} du, \quad R_j^t := \int_{\tilde{\tau}_j^-(ht/2)}^{\tilde{\tau}_j^+(ht/2)} e^{-\tilde{V}^{(j)}(u)} du.$$

In [74], it is shown that  $e_j$  follows an exponential distribution with parameter  $1/2$  (since the distribution of  $e_j$  does not depend on  $t$  we omit the dependence in  $t$  for  $e_j$  in the notations) and that the random variables  $e_j, S_j^t, R_j^t$ ,  $j \geq 1$  are mutually independent. To simplify notations we define, as in [4] and [74], the process of the renormalized sum of the contributions :

$$\forall s \geq 0, (Y_1^t, Y_2^t)(s) := \frac{1}{t} \sum_{j=1}^{\lfloor se^{\kappa\phi(t)} \rfloor} (e_j S_j^t, e_j S_j^t R_j^t),$$

and the overshoots of  $\sum_{i=1}^j e_i S_i^t R_i^t$  : for any  $a \geq 0$ , let us define

$$\mathcal{N}_a := \min \left\{ j \geq 0, \sum_{i=1}^j e_i S_i^t R_i^t > a \right\}.$$

We have

**Fact 5.2.4.** (*Proposition 4.2 of [74]*)

$(Y_1^t, Y_2^t)$  converges in distribution in  $(D([0, +\infty[, \mathbb{R}^2), J_1)$  to a non-trivial bidimensional subordinator  $(\mathcal{Y}_1, \mathcal{Y}_2)$ .

Let us define some events that happen with high probability. They describe the behavior of the diffusion and provide effective approximations for the time and the local time, in the study of the case  $0 < \kappa < 1$ . On these events, the diffusion leaves the valleys from the right and never goes back to a previous valley, the local time and the time spent by the diffusion are negligible, compared with  $t$ , outside the bottom of the valleys, the supremum of the local time and the time spent by the diffusion in

the bottom of the valleys are approximated by the *iid* sequence  $(e_j S_j^t, e_j S_j^t R_j^t)_{j \geq 1}$  :

$$\begin{aligned} \mathcal{E}_t^1 &:= \bigcap_{j=1}^{n_t} \left\{ H_{X_{\tilde{m}_j}}(\tilde{L}_j) < H_{X_{\tilde{m}_j}}(\tilde{L}_{j-1}), H_{X_{\tilde{L}_j}}(+\infty) < H_{X_{\tilde{L}_j}}(\tilde{\tau}_j(h_t)) \right\}, \\ \mathcal{E}_t^2 &:= \bigcap_{j=0}^{n_t-1} \left\{ \sup_{y \in \mathbb{R}} \left( \mathcal{L}_X(H(\tilde{m}_{j+1}), y) - \mathcal{L}_X(H(\tilde{L}_j), y) \right) \leq t e^{(\kappa(1+3\delta)-1)\phi(t)} \right\}, \\ \mathcal{E}_t^3 &:= \bigcap_{j=1}^{n_t} \left\{ \sup_{y \in [\tilde{L}_{j-1}, \tilde{L}_j] \cap \overline{\mathcal{D}_j}} \left( \mathcal{L}_X(H(\tilde{L}_j), y) - \mathcal{L}_X(H(\tilde{m}_j), y) \right) \leq t e^{-2\phi(t)} \right\}, \\ \mathcal{E}_t^4 &:= \bigcap_{j=1}^{n_t} \left\{ \left| \sup_{y \in \mathcal{D}_j} \left( \mathcal{L}_X(H(\tilde{L}_j), y) - \mathcal{L}_X(H(\tilde{m}_j), y) \right) \right| \leq (1 + e^{-\tilde{c}h_t}) \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \right\}, \\ \mathcal{E}_t^5 &:= \bigcap_{j=1}^{n_t} \left\{ 0 \leq H(\tilde{m}_j) - \sum_{i=1}^{j-1} \left( H(\tilde{L}_i) - H(\tilde{m}_i) \right) \leq \frac{2t}{\log h_t} \right\}, \\ \mathcal{E}_t^6 &:= \bigcap_{j=1}^{n_t} \left\{ (1 - e^{-\tilde{c}h_t}) e_j S_j^t \leq \mathcal{L}_X(\tilde{m}_j, H(\tilde{L}_j)) \leq (1 + e^{-\tilde{c}h_t}) e_j S_j^t \right\}, \\ \mathcal{E}_t^7 &:= \bigcap_{j=1}^{n_t} \left\{ (1 - e^{-\tilde{c}h_t}) e_j S_j^t R_j^t \leq H(\tilde{L}_j) - H(\tilde{m}_j) \leq (1 + e^{-\tilde{c}h_t}) e_j S_j^t R_j^t \right\}. \end{aligned}$$

where  $\tilde{c}$  is a fixed positive constant that has been chosen small enough (the constraints for the choice of  $\tilde{c}$  will be precised in the proofs of Fact 5.2.5 and Fact 5.2.7). Note that the above events depend both on the environment  $V$  and the brownian motion driving the diffusion.

**Fact 5.2.5.** *There is a positive constant  $L$  such that for all  $t$  large enough,*

$$\mathbb{P}(\overline{\mathcal{E}_t^7}) \leq e^{-Lh_t}, \quad \sum_{i=1}^7 \mathbb{P}(\overline{\mathcal{E}_t^i}) \leq e^{-L\phi(t)}. \quad (5.2.4)$$

Fix  $\eta \in ]0, 1[$ . If  $t$  is so large such that  $(1 - e^{-\tilde{c}h_t})^{-1} < (1 + \eta)$  and  $(1 + e^{-\tilde{c}h_t})^{-1}(1 - 2/\log(h_t)) \geq (1 - \eta)$ , then

$$\{V \in \mathcal{V}_t\} \cap \{N_t < n_t\} \cap \mathcal{E}_t^5 \cap \mathcal{E}_t^7 \subset \{\mathcal{N}_{(1-\eta)t} \leq N_t \leq \mathcal{N}_{(1+\eta)t}\}. \quad (5.2.5)$$

Our proofs are based on the study of the asymptotic of the quantities  $\mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t)$  and  $\mathbb{P}(\mathcal{L}_X^*(t) \leq t/x_t)$  where  $x_t$  depends on  $t$  and goes to infinity with  $t$ . More precisely we define  $x_t$  such that

$$x_t \underset{t \rightarrow +\infty}{\sim} D(\log(\log(t)))^{\mu-1}, \quad (5.2.6)$$

where  $D > 0$  and  $\mu \in ]1, 2]$ . Precise choices of  $x_t$  will be made later, they will all satisfy (5.2.6) for some  $D > 0$  and  $\mu \in ]1, 2]$ .

Fix  $\epsilon$  small enough so that Fact 5.4.8 of Section 5.4 is satisfied. We define  $\mathcal{G}_t$  to be the set of "good environments" in the following sense :  $v \in \mathcal{V}_t$  belongs to  $\mathcal{G}_t$  if it satisfies the following conditions :

$$\forall j \in \{1, \dots, n_t\}, e^{-\epsilon h_t/4} \leq R_j^t = \int_{\tilde{\tau}_j^-(h_t/2)}^{\tilde{\tau}_j^+(h_t/2)} e^{-(v(u)-v(\tilde{m}_j))} du \leq e^{h_t/8}, \quad (5.2.7)$$

$$\forall j \in \{1, \dots, n_t\}, |A^j(\tilde{\tau}_j^-(h_t/2))| = \left| \int_{\tilde{m}_j}^{\tilde{\tau}_j^-(h_t/2)} e^{-(v(u)-v(\tilde{m}_j))} du \right| \leq e^{5h_t/8}, \quad (5.2.8)$$

$$\forall j \in \{1, \dots, n_t\}, A^j(\tilde{\tau}_j^+(h_t/2)) = \int_{\tilde{m}_j}^{\tilde{\tau}_j^+(h_t/2)} e^{-(v(u)-v(\tilde{m}_j))} du \leq e^{5h_t/8}, \quad (5.2.9)$$

$$\forall j \in \{1, \dots, n_t\}, \int_{\tilde{L}_{j-1}}^{\tilde{\tau}_j^-(h_t/2)} e^{-(v(u)-v(\tilde{m}_j))} du \leq e^{-\epsilon h_t}, \quad (5.2.10)$$

$$\forall j \in \{1, \dots, n_t\}, \int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{L}_j} e^{-(v(u)-v(\tilde{m}_j))} du \leq e^{-\epsilon h_t}, \quad (5.2.11)$$

$$P^v(\cup_{i=1}^7 \overline{\mathcal{E}_t^i}) \leq e^{-L\phi(t)/2}. \quad (5.2.12)$$

where  $P^v(\cdot)$  is defined in Subsection 5.1.3 and  $L$  is the constant defined in Fact 5.2.5. Note that, since  $\mathcal{G}_t \subset \mathcal{V}_t$ , we will often use the fact that  $\tilde{m}_j = m_j$  for  $v \in \mathcal{G}_t$  and  $j \leq n_t$ .

**Lemma 5.2.6.** *There is a positive constant  $c$  such that for all  $t$  large enough,*

$$P(V \in \mathcal{G}_t) \geq 1 - e^{-c\phi(t)}.$$

We need that, as in Lemma 5.3 of [4], an inequality for the local time in the bottom of a valley is related to an inequality for the corresponding random variable  $R_k^t$ . Since we deal with unlikely inequalities for the local time, we have to prove the negligibility of the event where such inequalities happen but not the corresponding inequalities for  $R_k^t$ . Recall the notations  $X_{\tilde{m}_j}$  and  $H_{X_{\tilde{m}_j}}(\cdot)$  defined above. For any fixed environment  $v \in \mathcal{G}_t$  and  $z \in [0, 1]$  we define

$$\begin{aligned} \mathcal{E}_t^8(v, k, z) &:= \left\{ (1-z) \frac{(1+e^{-\tilde{c}h_t})}{(1-e^{-\tilde{c}h_t})} < x_t R_k^t, \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \cdot) \geq tx_t, \right. \\ &\quad \left. H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\}, \\ \mathcal{E}_t^9(v, k, z) &:= \left\{ R_k^t/x_t < (1-e^{-\tilde{c}h_t})(1-z), \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \tilde{m}_k) \leq t/x_t, \right. \\ &\quad \left. H_{X_{\tilde{m}_k}}(\tilde{L}_k) \geq t(1-z), H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\}, \end{aligned}$$

where  $\tilde{c}$  is the same as in the definitions of  $\mathcal{E}_t^4$ ,  $\mathcal{E}_t^6$  and  $\mathcal{E}_t^7$ . Note that the inequality  $(1-z)(1+e^{-\tilde{c}h_t})/(1-e^{-\tilde{c}h_t}) < x_t R_k^t$  (resp.  $R_k^t/x_t < (1-e^{-\tilde{c}h_t})(1-z)$ ) only depends

on the environment  $v$ . We consider  $\mathcal{E}_t^8(v, k, z)$  (resp.  $\mathcal{E}_t^9(v, k, z)$ ) to equal  $\emptyset$  when  $v$  is such that this equality is not satisfied. In the rest of the paper, we use the same convention when we consider events, at fixed environment, that partially depend on the environment.

**Fact 5.2.7.** *There is a positive constant  $c$  such that for  $t$  large enough and any fixed environment  $v \in \mathcal{G}_t$  we have*

$$P^v \left( \bigcup_{k=1}^{n_t} \{N_t \geq k, H(\tilde{m}_k)/t \leq 1 - 4/\log(h_t)\} \cap \mathcal{E}_t^8(v, k, H(\tilde{m}_k)/t) \right) \leq e^{-c\phi(t)}, \quad (5.2.13)$$

$$P^v \left( \bigcup_{k=1}^{n_t} \mathcal{E}_t^8(v, k, 1 - 4/\log(h_t)) \right) \leq e^{-c\phi(t)}, \quad (5.2.14)$$

$$P^v \left( \bigcup_{k=1}^{n_t} \{N_t \geq k\} \cap \mathcal{E}_t^9(v, k, H(\tilde{m}_k)/t) \right) \leq e^{-c\phi(t)}. \quad (5.2.15)$$

Note that in the above fact,  $v$  is fixed in  $\mathcal{G}_t \subset \mathcal{V}_t$  so  $N_t \geq k$  implies  $H(\tilde{m}_k)/t \leq 1$ . The quantities in the above fact are thus well defined.

We know from Fact 5.2.5 that the contributions to the local time and to the time spent of the successive valleys are approximated by the *iid* sequence  $(e_j S_j^t, e_j S_j^t R_j^t)_{j \geq 1}$ . Since we deal with extreme values of the local time, we need to know the right tail of the distributions of  $e_j S_j^t$  and of  $e_j S_j^t R_j^t$ . We also need informations about the extreme values of  $R_j^t$ . All these are given by the next fact from [74] :

**Fact 5.2.8.** *Fix  $\eta \in ]0, 1/3[$  and let  $\mathcal{C}'$  be the constant in Lemma 4.15 of [74]. We have*

$$\lim_{t \rightarrow +\infty} \sup_{x \in [e^{-(1-2\eta)\phi(t)}, +\infty[} \left| x^\kappa e^{\kappa\phi(t)} \mathbb{P} \left( e_1 S_1^t / t > x \right) - \mathcal{C}' \right| = 0, \quad (5.2.16)$$

$$\lim_{t \rightarrow +\infty} \sup_{y \in [e^{-(1-3\eta)\phi(t)}, +\infty[} \left| y^\kappa e^{\kappa\phi(t)} \mathbb{P} \left( e_1 S_1^t R_1^t / t > y \right) - \mathcal{C}' \mathbb{E} [\mathcal{R}^\kappa] \right| = 0. \quad (5.2.17)$$

$(R_1^t)_{t>0}$  converges in distribution to  $\mathcal{R}$  and there exists a positive  $\lambda_0$  such that

$$\forall \lambda < \lambda_0, \quad \mathbb{E} \left[ e^{\lambda R_1^t} \right] \xrightarrow{t \rightarrow +\infty} \mathbb{E} \left[ e^{\lambda \mathcal{R}} \right], \quad (5.2.18)$$

where the above quantities are all finite. This entails the convergence of the moments of any positive order of  $R_1^t$  to those of  $\mathcal{R}$  when  $t$  goes to infinity.

Finally, let us state a general lemma about the diffusion  $X$  :



**Lemma 5.2.9.** *Let  $q$  be the constant in Theorem 1.4 of [74]. There is a positive constant  $c$  such that for all  $r$  and  $t$  large enough,*

$$\mathbb{P} \left( \sup_{[0,t]} X \geq 2t^\kappa e^{\kappa\delta(\log(\log(t)))^\omega} / q \right) \leq e^{-cht}, \quad (5.2.19)$$

$$\mathbb{P} \left( X(t) \leq t^\kappa e^{(\rho-\kappa)(\log(\log(t)))^\omega} / 2q \right) \leq e^{-c\phi(t)}, \quad (5.2.20)$$

$$\mathbb{P} \left( \inf_{[0,+\infty[} X \leq -r \right) \leq 3r^{-1}, \quad (5.2.21)$$

$$\mathbb{P} \left( \inf_{]-\infty,0]} \mathcal{L}_X(+\infty, \cdot) > r \right) \leq 3r^{-\kappa/(2+\kappa)}. \quad (5.2.22)$$

## 5.2.2 Decomposition of the diffusion into independent parts

An important point in our proofs is to give a decomposition of the trajectory of  $X$  that makes independence appear in order to apply the Borel-Cantelli Lemma. Let us fix  $a > 1$  and define the sequences  $t_n := e^{n^a}$ ,  $u_n := e^{\kappa(n^a - 2an^{a-1}/3)}$ ,  $v_n := e^{\kappa(n^a - an^{a-1}/3)}$ . Let  $X^n := X(H(v_n) + \cdot)$ , the diffusion shifted by the hitting time of  $v_n$  and  $T_n := \min\{t_n, \tau(X^n, u_n), \tau(X^n, u_{n+1})\}$ . Note that from the Markov property for  $X$  at time  $H(v_n)$  and the stationarity of the increments of  $V$ ,  $X^n - v_n$  is equal in law to  $X$  under the annealed probability  $\mathbb{P}$ . Let  $n_0$  be large enough so that  $u_n \leq v_n \leq u_{n+1}$  for all  $n \geq n_0$ . We define the events

$$\mathcal{C}_n := \{T_n = t_n\} \quad \text{and} \quad \mathcal{D}_n := \{H(v_n) < t_n/n\}.$$

The idea is that the sequence of processes  $(X^n(t), 0 \leq t \leq T_n)_{n \geq n_0}$  is independent so, intersecting a sequence  $(\mathcal{B}_n)_{n \geq n_0}$  of interesting events (where each event  $\mathcal{B}_n$  only depends on  $(X^n(t), 0 \leq t \leq t_n)$ ) with  $\mathcal{C}_n$  will result in  $(\mathcal{B}_n \cap \mathcal{C}_n)_{n \geq n_0}$ , a sequence of independent events. Since  $X^n$  is  $X$  shifted by  $H(v_n)$ , the event  $\mathcal{D}_n$  is useful to neglect this time shift when dealing with the renormalization of the local time. We will need the following lemma :

**Lemma 5.2.10.**

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{C}_n}) < +\infty, \quad (5.2.23)$$

and

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{D}_n}) < +\infty. \quad (5.2.24)$$

*Démonstration.* Here again, let  $q$  be the constant in Theorem 1.4 of [74]. First, notice from the definitions of  $t_n$ ,  $u_n$  and  $v_n$  that for all  $n$  large enough,

$$2t_n^\kappa e^{\kappa\delta(\log(\log(t_n)))^\omega} / q < u_{n+1} - v_n, \quad (5.2.25)$$

and

$$(t_n/n)^\kappa e^{(\rho-\kappa)(\log(\log(t_n/n)))^\omega} / 2q > v_n. \quad (5.2.26)$$

From the definition of  $\mathcal{C}_n$  and the Markov property applied to  $X$  at time  $H(v_n)$  we have

$$\begin{aligned} \mathbb{P}(\overline{\mathcal{C}_n}) &\leq \mathbb{P}(\tau(X^n, u_{n+1}) < t_n) + \mathbb{P}(\tau(X^n, u_n) < +\infty) \\ &\leq \mathbb{P}(\tau(X, u_{n+1} - v_n) < t_n) + \mathbb{P}(\tau(X, u_n - v_n) < +\infty) \\ &\leq \mathbb{P}\left(\sup_{[0, t_n]} X \geq u_{n+1} - v_n\right) + \mathbb{P}\left(\inf_{[0, +\infty[} X \leq u_n - v_n\right) \\ &\leq e^{-ct_n} + 3/(v_n - u_n) \end{aligned}$$

where  $c$  is the constant in Lemma 5.2.9. The last inequality is true for  $n$  large enough and comes, for the first term, from the combination of (5.2.25) and (5.2.19), and, for the second term, from (5.2.21). Recall that  $e^{-ht_n} \approx e^{-\log(t_n)} = e^{-n^a}$ . We thus deduce (5.2.23).

From the definition of  $\mathcal{D}_n$ , (5.2.26) and (5.2.20) we have for all  $n$  large enough,

$$\mathbb{P}(\overline{\mathcal{D}_n}) \leq \mathbb{P}(X(t_n/n) \leq v_n) \leq \mathbb{P}(X(t_n/n) \leq (t_n/n)^\kappa e^{(\rho-\kappa)(\log(\log(t_n/n)))^\omega} / 2q) \leq e^{-c\phi(t_n/n)},$$

where  $c$  is the constant in Lemma 5.2.9. Since  $e^{-c\phi(t_n/n)} = e^{-c(\log(\log(t_n/n)))^\omega} \approx e^{-ca^\omega(\log(n))^\omega}$  we deduce (5.2.24).  $\square$

### 5.2.3 The lim sup

We study the asymptotic of the quantity  $\mathbb{P}(\mathcal{L}_X^*(t)/t \geq x_t)$ . Recall that  $x_t$  is defined in (5.2.6) where  $D > 0$  and  $\mu \in ]1, 2]$  are fixed constants. In all this subsection the parameter  $\omega$  in (5.2.1) is fixed in  $]1, \mu[$ . We have :

**Proposition 5.2.11.** *There is a positive constant  $c$  such that for all  $a > 1$  and  $t$  large enough we have,*

$$\begin{aligned} \mathbb{P}\left(Y_1^{\natural, t}(Y_2^{-1, t}(1/a)-) \geq a x_t\right) - e^{-c\phi(t)} &\leq \mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t) \\ &\leq \mathbb{P}\left(Y_1^{\natural, t}(Y_2^{-1, t}(a)-) \geq x_t/a\right) \\ &\quad + \mathbb{P}\left(R_1^t \leq \frac{a}{x_t}\right) + e^{-c\phi(t)}. \end{aligned}$$

Note that the functional of  $(Y_1^t, Y_2^t)$  involved in this proposition is  $Y_1^{\natural, t}(Y_2^{-1, t}(\cdot)-)$  which represents the supremum of the local time before (and not including) the last valley. Even though, as we see in Proposition 4.1 of [74], the repartition function

of  $\mathcal{L}_X^*(t)/t$  involves a complex functional of  $(Y_1^t, Y_2^t)$  that represents the last valley (together with the functional  $Y_1^{h,t}(Y_2^{-1,t}(\cdot))$  that represents the previous valleys), Proposition 5.2.11 says that the right tail of this distribution function does not involve the last valley. Before proving this proposition we prove some lemmas.

**Lemma 5.2.12.** *There is a positive constant  $C$  such that for  $t$  large enough,*

$$\sum_{k=1}^{n_t} \mathbb{P}(H(\tilde{m}_k)/t \leq 1) \leq C e^{\kappa\phi(t)}.$$

*Démonstration.* Since for all  $k \geq 1$  we have almost surely  $H(\tilde{m}_k) \geq \sum_{j=1}^{k-1} H(\tilde{L}_j) - H(\tilde{m}_j)$ , we deduce that

$$\begin{aligned} \sum_{k=1}^{n_t} \mathbb{P}(H(\tilde{m}_k)/t \leq 1) &\leq \sum_{k=1}^{n_t} \mathbb{P}\left(\sum_{j=1}^{k-1} H(\tilde{L}_j) - H(\tilde{m}_j)/t \leq 1\right) \\ &\leq \sum_{k=1}^{n_t} \mathbb{P}\left(\sum_{j=1}^{k-1} (1 - e^{-\tilde{c}h_t}) e_j S_j^t R_j^t / t \leq 1\right) + n_t \mathbb{P}(\overline{\mathcal{E}}_t^7) \\ &\leq e^{(1-e^{-\tilde{c}h_t})^{-1}} \sum_{k=1}^{n_t} \mathbb{E}\left[e^{-\sum_{j=1}^{k-1} e_j S_j^t R_j^t / t}\right] + n_t e^{-Lh_t} \\ &= e^{(1-e^{-\tilde{c}h_t})^{-1}} \sum_{k=1}^{n_t} \left(\mathbb{E}\left[e^{-e_1 S_1^t R_1^t / t}\right]\right)^{k-1} + e^{(1+\delta)\phi(t) - Lh_t}, \end{aligned}$$

where we used the definition of  $\mathcal{E}_t^7$ , Markov's inequality, (5.2.4), the fact that the sequence  $(e_j S_j^t R_j^t)_{j \geq 1}$  is *iid* and the definition of  $n_t$ . For  $t$  large enough we thus get

$$\begin{aligned} \sum_{k=1}^{n_t} \mathbb{P}(H(\tilde{m}_k)/t \leq 1) &\leq e^2 \sum_{k=1}^{+\infty} \left(\mathbb{E}\left[e^{-e_1 S_1^t R_1^t / t}\right]\right)^{k-1} + e^{-Lh_t/2} \\ &= \frac{e^2}{1 - \mathbb{E}\left[e^{-e_1 S_1^t R_1^t / t}\right]} + e^{-Lh_t/2}. \end{aligned} \quad (5.2.27)$$

Then,

$$\begin{aligned} 1 - \mathbb{E}\left[e^{-e_1 S_1^t R_1^t / t}\right] &= \int_0^{+\infty} e^{-u} \mathbb{P}(e_1 S_1^t R_1^t / t > u) du \\ &\geq \int_{e^{-\phi(t)/2}}^{+\infty} e^{-u} \mathbb{P}(e_1 S_1^t R_1^t / t > u) du \\ &= e^{-\kappa\phi(t)} \int_{e^{-\phi(t)/2}}^{+\infty} u^{-\kappa} e^{-u} (u^\kappa e^{\kappa\phi(t)} \mathbb{P}(e_1 S_1^t R_1^t / t > u)) du. \end{aligned}$$

We can now use (5.2.17) with  $\eta = 1/6$  and get that for all  $t$  large enough

$$1 - \mathbb{E}\left[e^{-e_1 S_1^t R_1^t / t}\right] \geq \frac{\mathcal{C}' \mathbb{E}[\mathcal{R}^\kappa]}{2} e^{-\kappa\phi(t)} \int_{e^{-\phi(t)/2}}^{+\infty} u^{-\kappa} e^{-u} du \underset{t \rightarrow +\infty}{\sim} \frac{\mathcal{C}' \mathbb{E}[\mathcal{R}^\kappa]}{2} e^{-\kappa\phi(t)} \int_0^{+\infty} u^{-\kappa} e^{-u} du.$$

Putting into (5.2.27), we get the result for  $t$  large enough. □

We now link the asymptotic of  $\mathbb{P}(R_1^t \leq a/x_t)$  with the left tail of  $I(V^\uparrow)$ . We have to make a distinction between the case where  $V$  possesses negative jumps and the case where  $V$  possesses no negative jumps, that is,  $V$  is the  $\kappa$ -drifted brownian motion.  $R_1^t$  is an exponential functional of the bottom of the first valley. In the first case, due to the jumps, the left side of the bottom of the valley can be neglected, so only the right side counts. In the second case, both sides have the same law, so both have to be taken in consideration in the left tail of  $R_1^t$ .

**Lemma 5.2.13.** *Let  $z_t$  go to infinity with  $t$  satisfying  $(\log(z_t))^2 \ll h_t$ . Assume  $V$  possesses negative jumps. There is a positive constant  $c$  such that for any  $a > 1$  and  $t$  large enough,*

$$e^{-c(\log((1-1/a)/z_t))^2} \mathbb{P}(I(V^\uparrow) \leq 1/az_t) \leq \mathbb{P}(R_1^t \leq 1/z_t) \leq 2\mathbb{P}(I(V^\uparrow) \leq a/z_t). \quad (5.2.28)$$

If  $V := W_\kappa$ , the  $\kappa$ -drifted brownian motion then for  $t$  large enough,

$$\mathbb{P}(\mathcal{R} \leq 1/z_t) - 2e^{-\delta\kappa h_t/3} \leq \mathbb{P}(R_1^t \leq 1/z_t) \leq 2\mathbb{P}(\mathcal{R} \leq a/z_t) + 2e^{-\delta\kappa h_t/3}. \quad (5.2.29)$$

*Démonstration.* We first assume that  $V$  possesses negative jumps. Recall that  $R_1^t = \int_{\tilde{\tau}_1^-(h_t/2)}^{\tilde{\tau}_1^+(h_t/2)} e^{-\tilde{V}^{(1)}(u)} du$  so, using the equality in law between  $(\tilde{V}^{(i)}(\tilde{m}_i + x), 0 \leq x \leq \tilde{\tau}_i(h) - \tilde{m}_i)$  and  $(V^\uparrow(x), 0 \leq x \leq \tau(V^\uparrow, h))$  given by Fact 5.4.7, we get

$$R_1^t \geq \int_{\tilde{m}_1}^{\tilde{\tau}_1^+(h_t/2)} e^{-\tilde{V}^{(1)}(u)} du \stackrel{\mathcal{L}}{=} \int_0^{\tau(V^\uparrow, h_t/2)} e^{-V^\uparrow(u)} du,$$

so

$$\mathbb{P}(R_1^t \leq 1/z_t) \leq \mathbb{P}\left(\int_0^{\tau(V^\uparrow, h_t/2)} e^{-V^\uparrow(u)} du \leq 1/z_t\right). \quad (5.2.30)$$

According to Lemma 5.4.6, for some positive constant  $c$  and  $t$  large enough,  $(1 - e^{-ch_t}) \times \mathbb{P}\left(\int_0^{\tau(V^\uparrow, h_t/2)} e^{-V^\uparrow(u)} du \leq 1/z_t\right)$  is less than

$$\begin{aligned} \mathbb{P}\left(\int_0^{\tau(V^\uparrow, h_t/2)} e^{-V^\uparrow(u)} du \leq 1/z_t\right) &\times \mathbb{P}\left(\int_{\tau(V^\uparrow, h_t/2)}^{+\infty} e^{-V^\uparrow(u)} du \leq e^{-h_t/4}\right) \\ &\leq \mathbb{P}(I(V^\uparrow) \leq 1/z_t + e^{-h_t/4}). \end{aligned} \quad (5.2.31)$$

Combining with (5.2.30) we get that for  $t$  large enough,

$$\mathbb{P}(R_1^t \leq 1/z_t) \leq 2\mathbb{P}(I(V^\uparrow) \leq 1/z_t + e^{-h_t/4}) \leq 2\mathbb{P}(I(V^\uparrow) \leq a/z_t),$$

because  $e^{-h_t/4} \leq (a-1)/z_t$  for large  $t$  (thanks to the hypothesis  $(\log(z_t))^2 \ll h_t$ ). This is the asserted upper bound in (5.2.28).

On the other hand, using the independence between the left and right parts of the valleys (given by Fact 5.4.7), we get that  $\mathbb{P}(R_1^t \leq 1/z_t)$  is more than

$$\mathbb{P} \left( \int_{\tilde{m}_1}^{\tilde{\tau}_1^+(h_t/2)} e^{-\tilde{V}^{(1)}(u)} du \leq 1/az_t \right) \times \mathbb{P} \left( \int_{\tilde{\tau}_1^-(h_t/2)}^{\tilde{m}_1} e^{-\tilde{V}^{(1)}(u)} du \leq (1 - 1/a)/z_t \right). \quad (5.2.32)$$

From Fact 5.4.7, we get that the first factor equals

$$\mathbb{P} \left( \int_0^{\tau(V^\uparrow, h_t/2)} e^{-V^\uparrow(u)} du \leq 1/az_t \right) \geq \mathbb{P}(I(V^\uparrow) \leq 1/az_t),$$

while the second factor is more than

$$\begin{aligned} & \mathbb{E} \left[ \frac{c_{h_t}}{1 - e^{-\kappa \hat{V}^\uparrow(\tau(\hat{V}^\uparrow, h_t+))}}; \int_0^{\tau(\hat{V}^\uparrow, h_t/2+)} e^{-\hat{V}^\uparrow(u)} du \leq (1 - 1/a)/z_t \right] - 2e^{-\delta\kappa h_t/3} \\ & \geq c_{h_t} \mathbb{P} \left( \int_0^{+\infty} e^{-\hat{V}^\uparrow(u)} du \leq (1 - 1/a)/z_t \right) - 2e^{-\delta\kappa h_t/3}. \end{aligned}$$

Then,  $c_{h_t} \geq c_1$  when  $h_t \geq 1$  so, putting in (5.2.32), we get that for  $t$  large enough,  $\mathbb{P}(R_1^t \leq 1/z_t) \geq \mathbb{P}(I(V^\uparrow) \leq 1/az_t) \times (c_1 \mathbb{P}(I(\hat{V}^\uparrow) \leq (1 - 1/a)/z_t) - 2e^{-\delta\kappa h_t/3})$ .

According to Theorem 1.14 of [72], there is a positive constant  $c$  such that for  $t$  large enough,

$$\mathbb{P}(I(\hat{V}^\uparrow) \leq (1 - 1/a)/z_t) \geq e^{-c(\log((1-1/a)/z_t))^2}.$$

Thanks to the hypothesis  $(\log(z_t))^2 \ll h_t$  we deduce that, for  $c$  decreased a little, the second factor in the right hand side of (5.2.33) is more than  $e^{-c(\log((1-1/a)/z_t))^2}$ . This yields the lower bound in (5.2.28).

We now consider the case where  $V$  is the  $\kappa$ -drifted brownian motion  $W_\kappa$ . Let  $Z_1$  and  $Z_2$  be two independent versions of the process  $W_\kappa^\uparrow$ . Since  $W_\kappa$  has no jumps, the density of the process  $P^{(2)}$  in Fact 5.4.7 is almost surely constant so  $P^{(2)}$  is equal in law to  $(\hat{W}_\kappa^\uparrow(x), 0 \leq x \leq \tau(\hat{W}_\kappa^\uparrow, h_t+)) = (W_\kappa^\uparrow(x), 0 \leq x \leq \tau(W_\kappa^\uparrow, h_t))$  (the last equality comes from the fact that  $\hat{W}_\kappa^\uparrow = W_\kappa^\uparrow$  and  $W_\kappa$  is continus). Combining this with (5.4.5) and the equality in law between  $(\tilde{V}^{(i)}(\tilde{m}_i + x), 0 \leq x \leq \tilde{\tau}_i(h) - \tilde{m}_i)$  and  $(V^\uparrow(x), 0 \leq x \leq \tau(V^\uparrow, h))$  (both are from Fact 5.4.7), we get

$$\mathbb{P}(R_1^t \leq 1/z_t) \leq \mathbb{P} \left( \int_0^{\tau(Z_1, h_t/2)} e^{-Z_1(u)} du + \int_0^{\tau(Z_2, h_t/2)} e^{-Z_2(u)} du \leq 1/z_t \right) + 2e^{-\delta\kappa h_t/3}, \quad (5.2.34)$$

$$\mathbb{P}(R_1^t \leq 1/z_t) \geq \mathbb{P} \left( \int_0^{\tau(Z_1, h_t/2)} e^{-Z_1(u)} du + \int_0^{\tau(Z_2, h_t/2)} e^{-Z_2(u)} du \leq 1/z_t \right) - 2e^{-\delta\kappa h_t/3}. \quad (5.2.35)$$

Reasoning as in (5.2.31) we get for some positive constant  $c$  :

$$\begin{aligned} (1 - e^{-ch_t})^2 \mathbb{P} \left( \int_0^{\tau(Z_1, h_t/2)} e^{-Z_1(u)} du + \int_0^{\tau(Z_2, h_t/2)} e^{-Z_2(u)} du \leq 1/z_t \right) \\ \leq \mathbb{P} (I(Z_1) + I(Z_2) \leq 1/z_t + 2e^{-h_t/4}). \end{aligned}$$

Combining with (5.2.34) we get that for  $t$  large enough  $\mathbb{P}(R_1^t \leq 1/z_t)$  is less than

$$2\mathbb{P} (I(Z_1) + I(Z_2) \leq 1/z_t + 2e^{-h_t/4}) + 2e^{-\delta\kappa h_t/3} \leq 2\mathbb{P} (\mathcal{R} \leq a/z_t) + 2e^{-\delta\kappa h_t/3},$$

because  $2e^{-h_t/4} \leq (a-1)/z_t$  for large  $t$  and because, from the definitions of  $\mathcal{R}$ ,  $Z_1$  and  $Z_2$  we have  $\mathcal{R} \stackrel{\mathcal{L}}{=} I(Z_1) + I(Z_2)$ . The above inequality is the asserted upper bound in (5.2.29).

On the other hand, we have trivially

$$\mathbb{P} \left( \int_0^{\tau(Z_1, h_t/2)} e^{-Z_1(u)} du + \int_0^{\tau(Z_2, h_t/2)} e^{-Z_2(u)} du \leq 1/z_t \right) \geq \mathbb{P} (I(Z_1) + I(Z_2) \leq 1/z_t),$$

and combining with (5.2.35) we get that for  $t$  large enough,

$$\mathbb{P} (R_1^t \leq 1/z_t) \geq \mathbb{P} (I(Z_1) + I(Z_2) \leq 1/z_t) - 2e^{-\delta\kappa h_t/3} = \mathbb{P} (\mathcal{R} \leq 1/z_t) - 2e^{-\delta\kappa h_t/3}.$$

This yields the lower bound in (5.2.29). □

The next lemma studies the contribution of the last valley :

**Lemma 5.2.14.** *There is a positive constant  $c$  such that for all  $u > 1$  and  $t$  large enough,*

$$\mathbb{P} \left( \sup_{\mathcal{D}_{N_t}} (\mathcal{L}_X(t, \cdot) - \mathcal{L}_X(H(\tilde{m}_{N_t}), \cdot)) \geq tx_t \right) \leq \mathbb{P} \left( R_1^t \leq \frac{u}{x_t} \right) + e^{-c\phi(t)}.$$

*Démonstration.* We fix  $v \in \mathcal{G}_t$ , a realization of the environment. Let us define

$$\begin{aligned} \mathcal{E}_t(v, k, z) := \left\{ \sup_{y \in \mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), y) \geq tx_t, \right. \\ \left. H_{X_{\tilde{m}_k}}(\tilde{m}_{k+1}) \geq t(1-z), H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\}. \end{aligned}$$

We have

$$\begin{aligned} P^v \left( \sup_{\mathcal{D}_{N_t}} (\mathcal{L}_X(t, \cdot) - \mathcal{L}_X(H(\tilde{m}_{N_t}), \cdot)) \geq tx_t, N_t < n_t, \mathcal{E}_t^1 \right) \\ \leq \sum_{k=1}^{n_t} \int_0^1 P^v (\mathcal{E}_t(v, k, z), H(\tilde{m}_k)/t \in dz). \end{aligned} \tag{5.2.36}$$

The fact that the sum stops at  $n_t$  comes from  $N_t < n_t$  together with the fact that  $v \in \mathcal{G}_t \subset \mathcal{V}_t$ . From the definitions of  $\mathcal{E}_t(v, k, z)$ ,  $\mathcal{E}_t^8(v, k, z)$ ,  $\mathcal{E}_t^5$  and  $\mathcal{E}_t^7$  we have

$$\mathcal{E}_t(v, k, z) \subset \left\{ \frac{1-z}{R_k^t} \frac{(1+e^{-\tilde{c}h_t})}{(1-e^{-\tilde{c}h_t})} \geq x_t, (1+e^{-\tilde{c}h_t})e_k S_k^t R_k^t \geq t(1-z-2/\log(h_t)) \right\} \\ \cup \mathcal{E}_t^8(v, k, z) \cup \overline{\mathcal{E}_t^5} \cup \overline{\mathcal{E}_t^7}.$$

When  $z \leq 1 - 4/\log(h_t)$  we have, on the big event in the right hand side,

$$t(1-z)/2 \leq t(1-z-2/\log(h_t)) \leq (1+e^{-\tilde{c}h_t})e_k S_k^t R_k^t \\ \leq (1+e^{-\tilde{c}h_t})^2(1-e^{-\tilde{c}h_t})^{-1}(1-z)e_k S_k^t/x_t \\ \leq 2(1-z)e_k S_k^t/x_t,$$

for  $t$  large enough, and trivially  $(1-z)(1+e^{-\tilde{c}h_t})(1-e^{-\tilde{c}h_t})^{-1} \leq 1 + e^{-\tilde{c}h_t/2}$  for  $t$  large enough. As a consequence, for  $z \in [0, 1 - 4/\log(h_t)]$  and  $t$  large enough,

$$\mathcal{E}_t(v, k, z) \subset \left\{ R_k^t \leq (1+e^{-\tilde{c}h_t/2})/x_t, e_k S_k^t/t \geq x_t/4 \right\} \cup \mathcal{E}_t^8(v, k, z) \cup \overline{\mathcal{E}_t^5} \cup \overline{\mathcal{E}_t^7}.$$

Note also that the sum in (5.2.36) corresponds to disjoint events (so it is actually the probability of a union of events). We thus get that

$\sum_{k=1}^{n_t} \int_0^{1-4/\log(h_t)} P^v(\mathcal{E}_t(v, k, z), H(\tilde{m}_k)/t \in dz)$  is less than

$$\sum_{k=1}^{n_t} \int_0^{1-4/\log(h_t)} P^v \left( R_k^t \leq (1+e^{-\tilde{c}h_t/2})/x_t, e_k S_k^t/t \geq x_t/4, H(\tilde{m}_k)/t \in dz \right) \\ + P^v \left( \overline{\mathcal{E}_t^5} \cup \overline{\mathcal{E}_t^7} \cup \cup_{k=1}^{n_t} \{N_t \geq k, H(\tilde{m}_k)/t \leq 1 - 4/\log(h_t)\} \cap \mathcal{E}_t^8(v, k, H(\tilde{m}_k)/t) \right).$$

Now, recall that  $(S_k^t, R_k^t)$  only depends on  $v$  and that,  $v$  being fixed,  $e_k$  belongs to the  $\sigma$ -field  $\sigma(X(t), t \geq H(\tilde{m}_k))$ . In other words, it only depends on the diffusion after time  $H(\tilde{m}_k)$ . On the other hand,  $H(\tilde{m}_k)$  is measurable with respect to the  $\sigma$ -field  $\sigma(X(t), 0 \leq t \leq H(\tilde{m}_k))$ . From the Markov property applied to  $X$  at  $H(\tilde{m}_k)$ , we get that  $H(\tilde{m}_k)$  is independent from  $(e_k, S_k^t, R_k^t)$  so the above is less than

$$\sum_{k=1}^{n_t} P^v \left( R_k^t \leq (1+e^{-\tilde{c}h_t/2})/x_t, e_k S_k^t/t \geq x_t/4 \right) \times P^v \left( H(\tilde{m}_k)/t \leq 1 \right) \\ + P^v \left( \overline{\mathcal{E}_t^5} \cup \overline{\mathcal{E}_t^7} \cup \cup_{k=1}^{n_t} \{N_t \geq k, H(\tilde{m}_k)/t \leq 1 - 4/\log(h_t)\} \cap \mathcal{E}_t^8(v, k, H(\tilde{m}_k)/t) \right) \\ \leq \sum_{k=1}^{n_t} P^v \left( R_k^t \leq (1+e^{-\tilde{c}h_t/2})/x_t, e_k S_k^t/t \geq x_t/4 \right) \times P^v \left( H(\tilde{L}_{k-1})/t \leq 1 \right) + e^{-c\phi(t)},$$

where  $c$  is a positive constant and where we used the fact that  $\tilde{L}_{k-1} \leq \tilde{m}_k$  for the first term and the fact that  $v \in \mathcal{G}_t$  together with (5.2.12) and (5.2.13) for the second term. Now, note that the first factor in the above product only depends on

$(\tilde{v}^{(k)}(x), \tilde{L}_{k-1} \leq x \leq \tilde{L}_k)$  (that is, only on  $v$  shifted at time  $\tilde{L}_{k-1}$ ) while the second factor depends on  $v$  before time  $\tilde{L}_{k-1}$ . This time  $\tilde{L}_{k-1}$  is a stopping time for the Lévy process  $V$  of which  $v$  is a fixed possible path. As a consequence, when we integrate the above inequality with respect to  $v$  over  $D(\mathbb{R}, \mathbb{R}) = \mathcal{G}_t \cup \overline{\mathcal{G}_t}$  equipped with the probability measure  $P$ , we get that the two factors are independent so  $E[\sum_{k=1}^{n_t} \int_0^{1-4/\log(h_t)} P^V(\mathcal{E}_t(V, k, z), H(\tilde{m}_k)/t \in dz)]$ , is less than

$$\begin{aligned} & \sum_{k=1}^{n_t} \mathbb{P}(R_k^t \leq (1 + e^{-\tilde{c}h_t/2})/x_t, e_k S_k^t/t \geq x_t/4) \mathbb{P}(H(\tilde{L}_{k-1})/t \leq 1) \\ & + e^{-c\phi(t)} + P(V \notin \mathcal{G}_t) \\ & \leq \sum_{k=1}^{n_t} \mathbb{P}(R_k^t \leq (1 + e^{-\tilde{c}h_t/2})/x_t) \mathbb{P}(e_k S_k^t/t \geq x_t/4) \mathbb{P}(H(\tilde{L}_{k-1})/t \leq 1) + e^{-c\phi(t)} \end{aligned}$$

where we used the independence between  $R_k^t$  and  $e_k S_k^t$  and Lemma 5.2.6, and where the constant  $c$  has been suitably decreased. Since the sequence  $(e_k, S_k^t, R_k^t)_{k \geq 1}$  is *iid*, we deduce that the first part of the right-hand-side of (5.2.36),

$E[\sum_{k=1}^{n_t} \int_0^{1-4/\log(h_t)} P^V(\mathcal{E}_t(V, k, z), H(\tilde{m}_k)/t \in dz)]$ , is less than

$$\mathbb{P}\left(R_1^t \leq \frac{1 + e^{-\tilde{c}h_t/2}}{x_t}\right) \times \mathbb{P}(e_1 S_1^t/t \geq x_t/4) \sum_{k=1}^{n_t} \mathbb{P}(H(\tilde{L}_{k-1})/t \leq 1) + e^{-c\phi(t)}.$$

Using (5.2.16) and Lemma 5.2.12 to bound respectively the second and third factor of the second term, we get the existence of a positive constant  $C$  such that for  $t$  large enough,

$$E\left[\sum_{k=1}^{n_t} \int_0^{1-4/\log(h_t)} P^V(\mathcal{E}_t(V, k, z), H(\tilde{m}_k)/t \in dz)\right] \leq \frac{C}{x_t^\kappa} \mathbb{P}\left(R_1^t \leq \frac{1 + e^{-\tilde{c}h_t/2}}{x_t}\right) + e^{-c\phi(t)}. \quad (5.2.37)$$

It remains to study  $E[\sum_{k=1}^{n_t} \int_{1-4/\log(h_t)}^1 P^V(\mathcal{E}_t(V, k, z), H(\tilde{m}_k)/t \in dz)]$ . For  $v \in \mathcal{G}_t$  and  $z \in [1-4/\log(h_t), 1]$  we have, using the definitions of  $\mathcal{E}_t(v, k, z)$  and  $\mathcal{E}_t^8(v, k, z)$ :

$$\begin{aligned} \mathcal{E}_t(v, k, z) & \subset \left\{ \sup_{y \in \mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(4/\log(h_t), y) \geq tx_t, H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\} \\ & \subset \left\{ \frac{4}{\log(h_t)} \frac{(1 + e^{-\tilde{c}h_t})}{(1 - e^{-\tilde{c}h_t})} \geq x_t R_k^t \right\} \cup \mathcal{E}_t^8(v, k, 1 - 4/\log(h_t)) \\ & \subset \{8/x_t \log(h_t) \geq R_k^t\} \cup \mathcal{E}_t^8(v, k, 1 - 4/\log(h_t)), \end{aligned}$$

where, in the last inclusion, we used the fact that  $(1 + e^{-\tilde{c}h_t})/(1 - e^{-\tilde{c}h_t}) \leq 2$  for large  $t$ . Recall that the sum in (5.2.36) corresponds to disjoint events, we thus get



that  $\sum_{k=1}^{n_t} \int_{1-4/\log(h_t)}^1 P^v(\mathcal{E}_t(v, k, z), H(\tilde{m}_k)/t \in dz)$  is less than

$$\begin{aligned} & \sum_{k=1}^{n_t} P^v \left( R_k^t \leq 8/x_t \log(h_t), H(\tilde{m}_k)/t \in dz \right) + P^v(\cup_{k=1}^{n_t} \mathcal{E}_t^8(v, k, 1 - 4/\log(h_t))) \\ & \leq \sum_{k=1}^{n_t} \mathbb{1}_{R_k^t \leq 8/x_t \log(h_t)} \times P^v \left( H(\tilde{m}_k)/t \leq 1 \right) + P^v(\cup_{k=1}^{n_t} \mathcal{E}_t^8(v, k, 1 - 4/\log(h_t))) \\ & \leq \sum_{k=1}^{n_t} \mathbb{1}_{R_k^t \leq 8/x_t \log(h_t)} \times P^v \left( H(\tilde{L}_{k-1})/t \leq 1 \right) + e^{-c\phi(t)}, \end{aligned}$$

where, for the first inequality, we used the fact that  $R_k^t$  only depends on  $v$ , and for the second we used the fact that  $\tilde{L}_{k-1} \leq \tilde{m}_k$  for the first term and (5.2.14) for the second term. Here again, the first factor in the above product only depends on  $(\tilde{v}^{(k)}(x), \tilde{L}_{k-1} \leq x \leq \tilde{L}_k)$  (that is, only on  $v$  shifted at time  $\tilde{L}_{k-1}$ ) while the second factor depends on  $v$  before time  $\tilde{L}_{k-1}$ . As a consequence, when we integrate the above inequality with respect to  $v$  over  $D(\mathbb{R}, \mathbb{R}) = \mathcal{G}_t \cup \overline{\mathcal{G}_t}$  equipped with the probability measure  $P$ , we get that the two factors are independent so  $E[\sum_{k=1}^{n_t} \int_{1-4/\log(h_t)}^1 P^V(\mathcal{E}_t(V, k, z), H(\tilde{m}_k)/t \in dz)]$ , is less than

$$\begin{aligned} & \sum_{k=1}^{n_t} \mathbb{P} \left( R_k^t \leq 8/x_t \log(h_t) \right) \mathbb{P} \left( H(\tilde{L}_{k-1})/t \leq 1 \right) + e^{-c\phi(t)} + P(V \notin \mathcal{G}_t) \\ & \leq \mathbb{P} \left( R_1^t \leq 8/x_t \log(h_t) \right) \sum_{k=1}^{n_t} \mathbb{P} \left( H(\tilde{L}_{k-1})/t \leq 1 \right) + e^{-c\phi(t)}, \end{aligned}$$

where we used the fact that the sequence  $(R_k^t)_{k \geq 1}$  is *iid* and Lemma 5.2.6, and where the constant  $c$  has been suitably decreased. Note that from the definitions of  $x_t$  and  $h_t$  we have  $x_t \log(h_t) \sim K(\log(\log(t)))^\mu$ . Therefore, using Lemmas 5.2.13 (with  $z_t = x_t \log(h_t)/8$ ,  $a = 2$ ) and 5.2.12 to bound respectively the first and second factor we get that for  $t$  large enough the above is less than

$$C e^{\kappa\phi(t)} \mathbb{P} \left( I(V^\uparrow) \leq 1/K'(\log(\log(t)))^\mu \right) + C e^{\kappa\phi(t) - \delta\kappa h_t/3} + e^{-c\phi(t)},$$

for some positive constants  $C$  and  $K'$ . The term  $C e^{\kappa\phi(t) - \delta\kappa h_t/3}$  appears when  $V = W_\kappa$  (because of (5.2.29)) and it is not necessary otherwise, note that this term is ultimately less than  $e^{-c\phi(t)}$ . Combining with (5.1.7) we get

$$E \left[ \sum_{k=1}^{n_t} \int_{1-4/\log(h_t)}^1 P^V(\mathcal{E}_t(V, k, z), H(\tilde{m}_k)/t \in dz) \right] \leq C e^{\kappa\phi(t) - K''(\log(\log(t)))^\mu} + 2e^{-c\phi(t)},$$

where  $K''$  is a positive constant. Since we have chosen  $\omega \in ]1, \mu[$  in (5.2.1) we have  $(\log(\log(t)))^\mu \gg \phi(t)$  so for  $t$  large enough the above inequality yields

$$E \left[ \sum_{k=1}^{n_t} \int_{1-4/\log(h_t)}^1 P^V(\mathcal{E}_t(V, k, z), H(\tilde{m}_k)/t \in dz) \right] \leq e^{-c\phi(t)}, \quad (5.2.38)$$

where the constant  $c$  has been suitably decreased. Now putting (5.2.37) and (5.2.38) in (5.2.36) we get for some constant  $c$  and  $t$  large enough :

$$\begin{aligned} \mathbb{P} \left( \sup_{\mathcal{D}_{N_t}} (\mathcal{L}_X(t, \cdot) - \mathcal{L}_X(H(\tilde{m}_{N_t}), \cdot)) \geq tx_t \right) &\leq \mathbb{P} \left( R_1^t \leq \frac{u}{x_t} \right) + \mathbb{P}(N_t \geq n_t) \\ &\quad + \mathbb{P}(\overline{\mathcal{E}}_t^1) + e^{-c\phi(t)}. \end{aligned}$$

Using (5.2.2) and (5.2.4) we get the result for a suitably chosen constant  $c$  and  $t$  large enough.  $\square$

*Démonstration.* of Proposition 5.2.11

*Upper bound*

$$\begin{aligned} \mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t) &\leq \mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^2) \\ &\quad + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) + \mathbb{P}(\overline{\mathcal{E}}_t^1) + \mathbb{P}(\overline{\mathcal{E}}_t^2). \end{aligned}$$

The event  $\mathcal{E}_t^1$  ensures that for  $j \leq n_t$ ,  $\tilde{L}_{j-1}$  is no longer reached after  $H(\tilde{m}_j)$  and the event  $\mathcal{E}_t^2$  ensures that the local time does not grow too much between  $H(\tilde{L}_{j-1})$  and  $H(\tilde{m}_j)$ . The event  $\{V \in \mathcal{V}_t, N_t < n_t\}$  ensures that at time  $t$  the diffusion is trapped in one of the first  $n_t$  standard valleys. The first term of the right hand side is thus less than

$$\begin{aligned} \mathbb{P} \left( \sup_{[\tilde{L}_{N_t-1}, \tilde{L}_{N_t}]} (\mathcal{L}_X(t, \cdot) - \mathcal{L}_X(H(\tilde{m}_{N_t}), \cdot)) \vee \right. \\ \left. \sup_{1 \leq j \leq N_t-1} \sup_{[\tilde{L}_{j-1}, \tilde{L}_j]} (\mathcal{L}_X(H(\tilde{L}_j), \cdot) - \mathcal{L}_X(H(\tilde{m}_j), \cdot)) \geq t\tilde{x}_t^1, V \in \mathcal{V}_t, N_t < n_t \right), \end{aligned}$$

where  $\tilde{x}_t^1 := x_t - e^{(\kappa(1+3\delta)-1)\phi(t)} \sim x_t$ . Then, since  $\tilde{x}_t^1$  converge to  $+\infty$ , we have  $\tilde{x}_t^1 \geq e^{-2\phi(t)}$  for  $t$  large enough. Using the definition of  $\mathcal{E}_t^3$ , we get that for such large  $t$  the above is less than

$$\begin{aligned} \mathbb{P} \left( \sup_{\mathcal{D}_{N_t}} (\mathcal{L}_X(t, \cdot) - \mathcal{L}_X(H(\tilde{m}_{N_t}), \cdot)) \vee \sup_{1 \leq j \leq N_t-1} \sup_{\mathcal{D}_j} (\mathcal{L}_X(H(\tilde{L}_j), \cdot) - \mathcal{L}_X(H(\tilde{m}_j), \cdot)) \geq t\tilde{x}_t^1, \right. \\ \left. V \in \mathcal{V}_t, N_t < n_t \right) + \mathbb{P}(\overline{\mathcal{E}}_t^3). \end{aligned}$$

Putting all this together, we see that for  $t$  large enough,  $\mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t)$  is less than

$$\begin{aligned} \mathbb{P} \left( \sup_{1 \leq j \leq N_t-1} \sup_{\mathcal{D}_j} (\mathcal{L}_X(H(\tilde{L}_j), \cdot) - \mathcal{L}_X(H(\tilde{m}_j), \cdot)) \geq t\tilde{x}_t^1, V \in \mathcal{V}_t, N_t < n_t \right) \\ + \mathbb{P} \left( \sup_{\mathcal{D}_{N_t}} (\mathcal{L}_X(t, \cdot) - \mathcal{L}_X(H(\tilde{m}_{N_t}), \cdot)) \geq t\tilde{x}_t^1 \right) \\ + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) + \mathbb{P}(\overline{\mathcal{E}}_t^1) + \mathbb{P}(\overline{\mathcal{E}}_t^2) + \mathbb{P}(\overline{\mathcal{E}}_t^3). \end{aligned} \quad (5.2.39)$$

We now deal with the first term. Using the definition of  $\mathcal{E}_t^4$  we get that the first term of (5.2.39) is less than

$$\mathbb{P} \left( \sup_{1 \leq j \leq \tilde{N}_t-1} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \geq t\tilde{x}_t^2, V \in \mathcal{V}_t, N_t < n_t \right) + \mathbb{P}(\overline{\mathcal{E}}_t^4),$$

where  $\tilde{x}_t^2 := (1 + e^{-\tilde{c}h_t})^{-1} \tilde{x}_t^1 \sim x_t$ . Now using the definition of  $\mathcal{E}_t^6$ , the above is less than

$$\mathbb{P} \left( \sup_{1 \leq j \leq \tilde{N}_t-1} e_j S_j^t \geq \tilde{x}_t, V \in \mathcal{V}_t, N_t < n_t \right) + \mathbb{P}(\overline{\mathcal{E}}_t^4) + \mathbb{P}(\overline{\mathcal{E}}_t^6),$$

where  $\tilde{x}_t := (1 + e^{-\tilde{c}h_t})^{-1} \tilde{x}_t^2 \sim x_t$ . Using (5.2.5) with  $\eta = a - 1$ , the above is less than

$$\begin{aligned} & \mathbb{P} \left( \sup_{1 \leq j \leq \mathcal{N}_{at-1}} e_j S_j^t \geq \tilde{x}_t \right) + \mathbb{P}(\overline{\mathcal{E}}_t^4) + \mathbb{P}(\overline{\mathcal{E}}_t^6) + \mathbb{P}(\overline{\mathcal{E}}_t^5) + \mathbb{P}(\overline{\mathcal{E}}_t^7), \\ & = \mathbb{P} \left( Y_1^{\natural,t} (Y_2^{-1,t}(a)-) \geq \tilde{x}_t \right) + \mathbb{P}(\overline{\mathcal{E}}_t^4) + \mathbb{P}(\overline{\mathcal{E}}_t^6) + \mathbb{P}(\overline{\mathcal{E}}_t^5) + \mathbb{P}(\overline{\mathcal{E}}_t^7). \end{aligned}$$

Combining with (5.2.39) and the fact that  $\tilde{x}_t^1 \geq \tilde{x}_t$ , we get that  $\mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t)$  is less than

$$\begin{aligned} & \mathbb{P} \left( Y_1^{\natural,t} (Y_2^{-1,t}(a)-) \geq \tilde{x}_t \right) + \mathbb{P} \left( \sup_{\mathcal{D}_{N_t}} (\mathcal{L}_X(t, \cdot) - \mathcal{L}_X(H(\tilde{m}_{N_t}), \cdot)) \geq t\tilde{x}_t \right) + \mathbb{P}(V \notin \mathcal{V}_t) \\ & + \mathbb{P}(N_t \geq n_t) + \mathbb{P}(\overline{\mathcal{E}}_t^1) + \mathbb{P}(\overline{\mathcal{E}}_t^2) + \mathbb{P}(\overline{\mathcal{E}}_t^3) + \mathbb{P}(\overline{\mathcal{E}}_t^4) + \mathbb{P}(\overline{\mathcal{E}}_t^5) + \mathbb{P}(\overline{\mathcal{E}}_t^6) + \mathbb{P}(\overline{\mathcal{E}}_t^7). \end{aligned}$$

Applying Lemma 5.2.14 with  $u = \sqrt{a}$  (and  $x_t$  replaced by  $\tilde{x}_t$  which does not change anything since  $\tilde{x}_t$  also satisfies (5.2.6)), Fact 5.2.3, (5.2.2) and (5.2.4), we deduce the existence of a positive constant  $c$  such that for  $t$  large enough,

$$\mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t) \leq \mathbb{P} \left( Y_1^{\natural,t} (Y_2^{-1,t}(a)-) \geq \tilde{x}_t \right) + \mathbb{P} \left( R_1^t \leq \frac{\sqrt{a}}{\tilde{x}_t} \right) + e^{-c\phi(t)}.$$

Since  $\tilde{x}_t \sim x_t$  and  $a > 1$  we get the upper bound when  $t$  is large enough.

*Lower bound*

$$\begin{aligned} \mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t) & \geq \mathbb{P} \left( \sup_{1 \leq j \leq \tilde{N}_t-1} \mathcal{L}_X(t, \tilde{m}_j) \geq tx_t \right) \\ & \geq \mathbb{P} \left( \sup_{1 \leq j \leq \tilde{N}_t-1} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \geq tx_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1 \right) \end{aligned}$$

because, on  $\{V \in \mathcal{V}_t\} \cap \{N_t < n_t\} \cap \mathcal{E}_t^1$ ,  $\tilde{m}_j$  (for  $j < N_t$ ) is no longer reached between times  $H(\tilde{L}_j)$  and  $t$ ,

$$\geq \mathbb{P} \left( \sup_{1 \leq j \leq \tilde{N}_t-1} e_j S_j^t \geq t\hat{x}_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^6 \right)$$

because of the definition of  $\mathcal{E}_t^6$  and where  $\hat{x}_t := (1 - e^{-\tilde{c}h_t})^{-1} x_t \sim x_t$ ,

$$\geq \mathbb{P} \left( \sup_{1 \leq j \leq \mathcal{N}_t/a-1} e_j S_j^t \geq t\hat{x}_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^6, \mathcal{E}_t^5, \mathcal{E}_t^7 \right)$$

where we used (5.2.5) with  $\eta = 1 - a^{-1}$ ,

$$\begin{aligned} &\geq \mathbb{P} \left( Y_1^{\natural,t} (Y_2^{-1,t}(1/a)-) \geq \hat{x}_t \right) \\ &\quad - \left( \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) + \mathbb{P}(\overline{\mathcal{E}_t^1}) + \mathbb{P}(\overline{\mathcal{E}_t^5}) + \mathbb{P}(\overline{\mathcal{E}_t^6}) + \mathbb{P}(\overline{\mathcal{E}_t^7}) \right). \end{aligned}$$

Applying Fact 5.2.3, (5.2.2) and (5.2.4) we get the lower bound since  $\hat{x}_t \sim x_t$  and  $a > 1$ . □

**Proposition 5.2.15.** *Let  $y_t$  go to infinity with  $t$ . For any  $b > 0$  and  $u \in ]0, 1[$  there is a positive constant  $C$  (depending on  $b$  and  $u$ ) such that for all  $t$  large enough,*

$$C \mathbb{P} (R_1^t \leq ub/y_t) / y_t^\kappa \leq \mathbb{P} \left( Y_1^{\natural,t} (Y_2^{-1,t}(b)-) \geq y_t \right) \leq \mathbb{P} (R_1^t \leq b/y_t).$$

*Démonstration.* For any  $z > 0$ , let  $k^t(z)$  be the first index  $k \geq 1$  such that  $e_k S_k^t / t \geq z$ . We have

$$\begin{aligned} \left\{ Y_1^{\natural,t} (Y_2^{-1,t}(b)-) \geq y_t \right\} &= \left\{ k^t(y_t) < \mathcal{N}_{tb} \right\} = \left\{ \sum_{i=1}^{k^t(y_t)} e_i S_i^t R_i^t \leq tb \right\} \\ &\subset \left\{ e_{k^t(y_t)} S_{k^t(y_t)}^t R_{k^t(y_t)}^t \leq tb \right\}, \end{aligned} \quad (5.2.40)$$

and  $e_{k^t(y_t)} S_{k^t(y_t)}^t R_{k^t(y_t)}^t$  has the same law as  $e_1 S_1^t R_1^t$  conditionally on  $e_1 S_1^t \geq ty_t$  so, using (5.2.40) and the independence between  $e_1 S_1^t$  and  $R_1^t$  :

$$\mathbb{P} \left( Y_1^{\natural,t} (Y_2^{-1,t}(b)-) \geq y_t \right) \leq \mathbb{P} (e_1 S_1^t R_1^t \leq tb | e_1 S_1^t \geq ty_t) \leq \mathbb{P} (R_1^t \leq b/y_t),$$

which proves the upper bound. For the lower bound, we fix  $\eta < 1 - u$ . Note that according to (5.2.40),

$$\left\{ e_{k^t(y_t)} S_{k^t(y_t)}^t R_{k^t(y_t)}^t \leq tb(1 - \eta) \right\} \cap \left\{ \sum_{i=1}^{k^t(y_t)-1} e_i S_i^t R_i^t < tb\eta \right\} \subset \left\{ Y_1^{\natural,t} (Y_2^{-1,t}(b)-) \geq y_t \right\},$$

and both events on the left-hand-side are independent so

$$\mathbb{P} \left( Y_1^{\natural,t} (Y_2^{-1,t}(b)-) \geq y_t \right) \geq \mathbb{P} (e_1 S_1^t R_1^t \leq tb(1 - \eta) | e_1 S_1^t \geq ty_t) \times \mathbb{P} \left( \sum_{i=1}^{k^t(y_t)-1} e_i S_i^t R_i^t < tb\eta \right). \quad (5.2.41)$$

We first deal with the second factor. Let  $(Z_i)_{i \geq 1}$  be *iid* random variables such that  $\mathcal{L}(Z_1) = \mathcal{L}(e_1 S_1^t R_1^t | e_1 S_1^t \leq ty_t)$  and  $T$  be a geometric random variable with parameter  $\mathbb{P}(e_1 S_1^t \geq ty_t)$ , independent from the sequences  $(Z_i)_{i \geq 1}$  and  $(e_i S_i^t R_i^t)_{i \geq 1}$ . We then have

$$\mathbb{P} \left( \sum_{i=1}^{k^t(y_t)-1} e_i S_i^t R_i^t < tb\eta \right) = \mathbb{P} \left( \sum_{i=1}^{T-1} Z_i < tb\eta \right) \geq \mathbb{P} \left( \sum_{i=1}^{T-1} e_i S_i^t R_i^t < tb\eta \right),$$

because the random variable  $e_i S_i^t R_i^t$  is stochastically greater than the random variable  $Z_i$ . Then,

$$\mathbb{P} \left( \sum_{i=1}^{T-1} e_i S_i^t R_i^t < tb\eta \right) \geq \mathbb{P}(T \leq e^{\kappa\phi(t)}) \times \mathbb{P} \left( \sum_{1 \leq i \leq e^{\kappa\phi(t)}} e_i S_i^t R_i^t < tb\eta \right).$$

On the first hand we have

$$\mathbb{P} \left( \sum_{1 \leq i \leq e^{\kappa\phi(t)}} e_i S_i^t R_i^t < tb\eta \right) = \mathbb{P}(Y_2^t(1) < b\eta) \xrightarrow{t \rightarrow +\infty} \mathbb{P}(\mathcal{Y}_2(1) < b\eta) > 0,$$

where we used Fact 5.2.4 and where  $\mathcal{Y}_2$  is as in there, the second component of the limit process  $(\mathcal{Y}_1, \mathcal{Y}_2)$ . On the second hand

$$\mathbb{P}(T \leq e^{\kappa\phi(t)}) = 1 - (1 - \mathbb{P}(e_1 S_1^t \geq ty_t))^{[e^{\kappa\phi(t)}]} = 1 - e^{[e^{\kappa\phi(t)}] \ln(1 - \mathbb{P}(e_1 S_1^t \geq ty_t))}.$$

Using (5.2.16) we get  $\mathbb{P}(T \leq e^{\kappa\phi(t)}) \underset{t \rightarrow +\infty}{\sim} \mathcal{C}'/y_t^\kappa$ . Putting all this together, we get the existence of a positive constant  $c_1 > 0$  such that for  $t$  large enough,

$$\mathbb{P} \left( \sum_{i=1}^{k^t(y_t)-1} e_i S_i^t R_i^t < tb\eta \right) \geq c_1/y_t^\kappa. \quad (5.2.42)$$

We now study the first factor in the right hand side of (5.2.41). From the independence of the two factors  $e_1 S_1^t$  and  $R_1^t$  in  $e_1 S_1^t R_1^t$  we have

$$\mathbb{P}(e_1 S_1^t R_1^t \leq tb(1 - \eta) | e_1 S_1^t \geq ty_t) \geq \mathbb{P}(R_1^t \leq bu/y_t) \times \mathbb{P}(e_1 S_1^t \leq ty_t(1 - \eta)/u | e_1 S_1^t \geq ty_t)$$

and

$$\begin{aligned} \mathbb{P}(e_1 S_1^t \leq ty_t(1 - \eta)/u | e_1 S_1^t \geq ty_t) &= \frac{\mathbb{P}(ty_t \leq e_1 S_1^t \leq ty_t(1 - \eta)/u)}{\mathbb{P}(e_1 S_1^t \geq ty_t)} \\ &= 1 - \frac{\mathbb{P}(e_1 S_1^t > ty_t(1 - \eta)/u)}{\mathbb{P}(e_1 S_1^t \geq ty_t)} \\ &\xrightarrow{t \rightarrow +\infty} 1 - (u/(1 - \eta))^\kappa > 0, \end{aligned}$$

where the limit comes from (5.2.16), because  $y_t$  goes to infinity. We thus get the existence of a positive constant  $c_2 > 0$  such that for  $t$  large enough,

$$\mathbb{P}(e_1 S_1^t R_1^t \leq tb(1-\eta) | e_1 S_1^t \geq ty_t) \geq c_2 \mathbb{P}(R_1^t \leq bu/y_t). \quad (5.2.43)$$

Putting (5.2.42) and (5.2.43) in (5.2.41) we get the lower bound.  $\square$

Fix  $\theta > 1$ . We apply Proposition 5.2.11 (the upper bound with  $a = \theta^{1/3}$  and the lower bound with  $a = \theta^{1/4}$ ), Proposition 5.2.15 (the upper bound applied with  $b = \theta^{1/3}$ ,  $y_t = \theta^{-1/3}x_t$  and the lower bound with  $b = \theta^{-1/4}$ ,  $y_t = \theta^{1/4}x_t$ ,  $u = \theta^{-1/4}$ ) and Lemma 5.2.13 (the upper bounds of (5.2.28) and (5.2.29) applied with  $a = \theta^{1/3}$ ,  $z_t = \theta^{-2/3}x_t$ , and the lower bounds of these same expressions applied with  $a = \theta^{1/4}$ ,  $z_t = \theta^{3/4}x_t$ ). We get :

**Proposition 5.2.16.** *Fix  $\theta > 1$ . If  $V$  possesses negative jumps, there are positive constants  $\tilde{C}$  and  $c$  such that for  $t$  large enough,*

$$\frac{e^{-c(\log(x_t))^2}}{\tilde{C}x_t^\kappa} \mathbb{P}(I(V^\uparrow) \leq 1/\theta x_t) - e^{-c\phi(t)} \leq \mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t) \leq \tilde{C} \mathbb{P}(I(V^\uparrow) \leq \theta/x_t) + e^{-c\phi(t)}. \quad (5.2.44)$$

If  $V := W_\kappa$ , the  $\kappa$ -drifted brownian motion, there are positive constants  $C$  and  $c$  such that for  $t$  large enough,

$$\frac{1}{\tilde{C}x_t^\kappa} \mathbb{P}(\mathcal{R} \leq 1/\theta x_t) - e^{-c\phi(t)} \leq \mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t) \leq \tilde{C} \mathbb{P}(\mathcal{R} \leq \theta/x_t) + e^{-c\phi(t)}. \quad (5.2.45)$$

We can now link the asymptotic behavior of the local time with the left tail of  $I(V^\uparrow)$  :

*Démonstration.* of Theorem 5.1.2

First, we assume that  $V$  possesses negative jumps.

Let us assume that (5.1.12) is satisfied with some constants  $\gamma > 1$  and  $C > 0$ . We now prove (5.1.13). Let  $a > 1$  and define the events

$$\mathcal{A}_n := \left\{ \sup_{t \in [a^n, a^{n+1}]} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\gamma-1}} \geq a^3 C^{1-\gamma} \right\}.$$

We define  $x_t := C^{1-\gamma} a^2 (\log(\log(t/a)))^{\gamma-1}$ . Note that such a choice of  $x_t$  satisfies (5.2.6) with  $\mu = \gamma$  and  $D = C^{1-\gamma}$ . From the increase of  $\mathcal{L}_X^*(\cdot)$ , (5.2.44) (the upper

bound applied with  $t = a^{n+1}$ ,  $\theta = a$ ) and (5.1.12) we have, for  $n$  large enough,

$$\begin{aligned} \mathbb{P}(\mathcal{A}_n) &\leq \mathbb{P}(\mathcal{L}_X^*(a^{n+1}) \geq C^{1-\gamma} a^{n+3} (\log(\log(a^n)))^{\gamma-1}) \\ &\leq \tilde{C} \mathbb{P}(I(V^\dagger) \leq 1/a C^{1-\gamma} (\log(\log(a^n)))^{\gamma-1}) + e^{-c\phi(a^{n+1})} \\ &\leq \tilde{C} \exp\left(-a^{\frac{1}{\gamma-1}} \log(\log(a^n))\right) + e^{-c\phi(a^{n+1})} \\ &= \tilde{C} (\log(a))^{-a^{\frac{1}{\gamma-1}}} n^{-a^{\frac{1}{\gamma-1}}} + e^{-c\phi(a^{n+1})}. \end{aligned}$$

Since  $e^{-c\phi(a^{n+1})} = e^{-c(\log \log(a^{n+1}))^\omega} \leq n^{-2}$  for  $n$  large enough, the above is the general term of a converging series so, using the Borel-Cantelli lemma, we deduce that  $\mathbb{P}$ -almost surely,

$$\limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\gamma-1}} \leq a^3 C^{1-\gamma},$$

and letting  $a$  go to 1 we get (5.1.13).

Now, let us assume that (5.1.14) is satisfied with some constants  $\gamma > 1$  and  $C > 0$ . We now prove (5.1.15). Let  $a > 1$  and let  $t_n$ ,  $u_n$ ,  $v_n$  (defined from this  $a$ ) and  $X^n$  be as in Subsection 5.2.2 (recall that  $X^n - v_n$  is equal in law to  $X$  under the annealed probability  $\mathbb{P}$ ). We define

$$\mathcal{B}_n := \left\{ \frac{\mathcal{L}_{X^n}^*(t_n)}{t_n(\log(\log(t_n)))^{\gamma-1}} \geq \frac{C^{1-\gamma}}{a^{3(\gamma-1)}} \right\}.$$

We also define  $\mathcal{C}_n$  and  $\mathcal{D}_n$  to be as in Lemma 5.2.10 and  $\mathcal{E}_n := \mathcal{B}_n \cap \mathcal{C}_n$ . We define  $x_t := C^{1-\gamma} (\log(\log(t)))^{\gamma-1} / a^{3(\gamma-1)}$ . Note that such a choice of  $x_t$  satisfies (5.2.6) with  $\mu = \gamma$  and  $D = C^{1-\gamma} / a^{3(\gamma-1)}$ . According to (5.2.44) (the lower bound applied with  $t = t_n$ ,  $\theta = a^{\gamma-1}$ ), the definition of  $t_n$ , (5.1.14) and the fact that  $n$  is large we have, for some positive constants  $c_a$ ,  $K_a$  and  $n$  large enough,

$$\begin{aligned} \mathbb{P}(\mathcal{B}_n) &\geq K_a e^{-c_a(\log(\log(\log(t_n))))^2} \mathbb{P}(I(V^\dagger) \leq a^{2(\gamma-1)} / C^{1-\gamma} (\log(\log(t_n)))^{\gamma-1}) / (\log(\log(t_n)))^{\kappa(\gamma-1)} \\ &\quad - e^{-c\phi(t_n)} \\ &= K_a e^{-c_a(\log(a \log(n)))^2} \mathbb{P}(I(V^\dagger) \leq a^{\gamma-1} / C^{1-\gamma} (\log(n))^{\gamma-1}) / (a \log(n))^{\kappa(\gamma-1)} - e^{-c\phi(t_n)} \\ &\geq K_a e^{-c_a(\log(a \log(n)))^2} e^{-(\log(n))/a} / (a \log(n))^{\kappa(\gamma-1)} - e^{-c\phi(t_n)} \\ &= K_a \exp\left(-c_a (\log(a \log(n)))^2 - \log(n)/a + \kappa(\gamma-1) \log(a \log(n))\right) - e^{-c\phi(t_n)} \\ &\geq K_a \exp(-\log(n)) - e^{-c\phi(t_n)} \\ &= K_a n^{-1} - e^{-c\phi(t_n)}. \end{aligned}$$

Since  $e^{-c\phi(t_n)} = e^{-c(\log \log(t_n))^\omega} \leq n^{-2}$  for  $n$  large enough, we get

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{B}_n) = +\infty. \tag{5.2.46}$$

Then, the combination of (5.2.46) and (5.2.23) yields

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{E}_n) \geq \sum_{n \geq 1} \mathbb{P}(\mathcal{B}_n) - \sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{C}_n}) = +\infty. \quad (5.2.47)$$

Note that each event  $\mathcal{E}_n$  belongs to the  $\sigma$ -field  $\sigma(V(s) - V(u_n), u_n \leq s \leq u_{n+1}, X(t), H(v_n) \leq t \leq H(v_n) + T_n)$ , in other words, it only depends on the diffusion between times  $H(v_n)$  and  $H(v_n) + T_n$  and on the environment between positions  $u_n$  and  $u_{n+1}$ . From the Markov property and the independence of the increments of the environment, we get that the events  $(\mathcal{E}_n)_{n \geq 1}$  are independent. Combining this independence with (5.2.47) and the Borel-Cantelli Lemma we get that  $\mathbb{P}$ -almost surely, the event  $\mathcal{E}_n$  is realized infinitely many often. For  $n$  such that this event is realized we have

$$\frac{\mathcal{L}_X^*(H(v_n) + t_n)}{t_n(\log(\log(t_n)))^{\gamma-1}} \geq \frac{\mathcal{L}_{X^n}^*(t_n)}{t_n(\log(\log(t_n)))^{\gamma-1}} \geq \frac{C^{1-\gamma}}{a^{3(\gamma-1)}}. \quad (5.2.48)$$

According to (5.2.24) and the Borel-Cantelli Lemma we have  $\mathbb{P}$ -almost surely

$$H(v_n) + t_n \underset{n \rightarrow +\infty}{\sim} t_n,$$

so combining with (5.2.48) we deduce that  $\mathbb{P}$ -almost surely,

$$\limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\gamma-1}} \geq \frac{C^{1-\gamma}}{a^{3(\gamma-1)}},$$

and letting  $a$  go to 1 we get (5.1.15).

If  $V = W_\kappa$ , the  $\kappa$ -drifted brownian motion with  $0 < \kappa < 1$ , we proceed the same proof, only replacing  $I(V^\dagger)$  by  $\mathcal{R}$  and (5.2.44) by (5.2.45). We thus get that the same result is still true for  $V = W_\kappa$ , but with  $\mathcal{R}$  instead of  $I(V^\dagger)$ .  $\square$

The other theorems for the lim sup are now easy to prove.

*Démonstration.* of Theorems 5.1.4 and 5.1.5

In the case where  $V$  possesses negative jumps, Theorem 5.1.4 is a direct consequence of the combination of Theorem 5.1.2, (5.1.8) and (5.1.9). Similarly, the first point of Theorem 5.1.5 is obtained from the combination of Theorem 5.1.2 and (5.1.10). The second point of Theorem 5.1.5 is obtained from the combination of Theorem 5.1.2 and (5.1.11).

In the case where  $V = W_\kappa$ , the  $\kappa$ -drifted brownian motion with  $0 < \kappa < 1$ , we only need to prove the last point of Theorem 5.1.5, and this requires to determine exactly the left tail of  $\mathcal{R}$ . This variable is equal in law to the sum of two independent random variables having the same law as  $I(W_k^\dagger)$ . We thus have

$$-\log(\mathbb{E}[e^{-\lambda \mathcal{R}}]) = -\log\left(\left(\mathbb{E}[e^{-\lambda I(W_k^\dagger)}]\right)^2\right) = -2 \log\left(\mathbb{E}[e^{-\lambda I(W_k^\dagger)}]\right) \underset{\lambda \rightarrow +\infty}{\sim} 4\sqrt{2\lambda},$$



where the equivalent comes from (1.10) of [72]. Using this together with De Bruijn's Theorem (see Theorem 4.12.9 in [14]) we get

$$-\log(\mathbb{P}(\mathcal{R} \leq x)) \underset{x \rightarrow 0}{\sim} \frac{8}{x}.$$

The last point of Theorem 5.1.5 follows from this combined with Theorem 5.1.2.  $\square$

### 5.2.4 The lim inf

For the lim inf we study of the asymptotic of the quantity  $\mathbb{P}(\mathcal{L}_X^*(t)/t \leq 1/x_t)$ . Recall that  $x_t$  is defined in (5.2.6) where  $D > 0$  and  $\mu \in ]1, 2]$  are fixed constants. In all this subsection we take  $\omega := 2$  for the parameter in (5.2.1).

**Proposition 5.2.17.** *Recall the  $\lambda_0$  defined in Fact 5.2.8. There is a positive constant  $c$  such that for all  $a > 1$  and  $t$  large enough we have,*

$$\begin{aligned} \mathbb{P}\left(Y_1^{\natural,t}(Y_2^{-1,t}(a)) \leq 1/ax_t\right) - e^{-c\phi(t)} &\leq \mathbb{P}(\mathcal{L}_X^*(t) \leq t/x_t) \\ &\leq 2\mathbb{P}\left(Y_1^{\natural,t}(Y_2^{-1,t}(1/4)) \leq 2/x_t\right) \\ &\quad + e^{-\lambda_0 x_t/8} + e^{-c\phi(t)}. \end{aligned}$$

Here, the functional of  $(Y_1^t, Y_2^t)$  involved is  $Y_1^{\natural,t}(Y_2^{-1,t}(\cdot))$  which represents the supremum of the local time after leaving the last valley.

*Démonstration. Lower bound*

From the definition of  $N_t$ , we have  $H(\tilde{m}_{N_t+1}) \geq t$  on  $\{V \in \mathcal{V}_t, N_t < n_t\}$  so

$$\mathbb{P}(\mathcal{L}_X^*(t) \leq t/x_t) \geq \mathbb{P}(\mathcal{L}_X^*(H(\tilde{m}_{N_t+1})) \leq t/x_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^2).$$

The event  $\mathcal{E}_t^1$  ensures that for  $j \leq n_t$ ,  $\tilde{L}_{j-1}$  is no longer reached after  $H(\tilde{m}_j)$  and the event  $\mathcal{E}_t^2$  ensures that the local time does not grow too much between  $H(\tilde{L}_{j-1})$  and  $H(\tilde{m}_j)$ . The event  $\{V \in \mathcal{V}_t, N_t < n_t\}$  ensures that at time  $t$  the diffusion is trapped in one of the first  $n_t$  standard valleys. The right hand side is thus more than

$$\mathbb{P}\left(\sup_{1 \leq j \leq N_t} \sup_{[\tilde{L}_{j-1}, \tilde{L}_j]} \left(\mathcal{L}_X(H(\tilde{L}_j), \cdot) - \mathcal{L}_X(H(\tilde{m}_j), \cdot)\right) \leq t/\tilde{x}_t^1, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^2\right),$$

where  $\tilde{x}_t^1 := 1/((1/x_t) - e^{(\kappa(1+3\delta)-1)\phi(t)})$ . Then, since  $\tilde{x}_t^1 \sim x_t$ , we have  $1/\tilde{x}_t^1 \geq e^{-2\phi(t)}$  for  $t$  large enough. Using the definition of  $\mathcal{E}_t^3$ , we get that for such large  $t$  the above is more than

$$\mathbb{P}\left(\sup_{1 \leq j \leq N_t} \sup_{\mathcal{D}_j} \left(\mathcal{L}_X(H(\tilde{L}_j), \cdot) - \mathcal{L}_X(H(\tilde{m}_j), \cdot)\right) \leq t/\tilde{x}_t^1, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^2, \mathcal{E}_t^3\right).$$

From the definition of  $\mathcal{E}_t^4$  the above is more than

$$\mathbb{P} \left( \sup_{1 \leq j \leq N_t} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \leq t/\tilde{x}_t^2, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^2, \mathcal{E}_t^3, \mathcal{E}_t^4 \right),$$

where  $\tilde{x}_t^2 := (1 + e^{-\tilde{c}ht})\tilde{x}_t^1 \sim x_t$ . Now using the definition of  $\mathcal{E}_t^6$ , the above is more than

$$\mathbb{P} \left( \sup_{1 \leq j \leq N_t} e_j S_j^t \leq 1/\tilde{x}_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^2, \mathcal{E}_t^3, \mathcal{E}_t^4, \mathcal{E}_t^6 \right),$$

where  $\tilde{x}_t := (1 + e^{-\tilde{c}ht})\tilde{x}_t^2 \sim x_t$ . Let  $a > 1$ . Using (5.2.5) with  $\eta = a - 1$ , the above is more than

$$\begin{aligned} & \mathbb{P} \left( \sup_{1 \leq j \leq N_{at}} e_j S_j^t \leq 1/\tilde{x}_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^2, \mathcal{E}_t^3, \mathcal{E}_t^4, \mathcal{E}_t^5, \mathcal{E}_t^6, \mathcal{E}_t^7 \right) \\ & \geq \mathbb{P} \left( Y_1^{\ddagger, t} (Y_2^{-1, t}(a)) \leq 1/\tilde{x}_t \right) \\ & - \left( \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) + \mathbb{P}(\overline{\mathcal{E}}_t^1) + \mathbb{P}(\overline{\mathcal{E}}_t^2) + \mathbb{P}(\overline{\mathcal{E}}_t^3) + \mathbb{P}(\overline{\mathcal{E}}_t^4) + \mathbb{P}(\overline{\mathcal{E}}_t^5) + \mathbb{P}(\overline{\mathcal{E}}_t^6) + \mathbb{P}(\overline{\mathcal{E}}_t^7) \right), \end{aligned}$$

where we used the definition of  $(Y_1^t, Y_2^t)$ . Applying Fact 5.2.3, (5.2.2) and (5.2.4) we get the asserted lower bound for a suitably chosen constant  $c$  and  $t$  large enough, since  $\tilde{x}_t \sim x_t$  and  $a > 1$ .

*Upper bound*

$$\begin{aligned} \mathbb{P}(\mathcal{L}_X^*(t) \leq t/x_t) & \leq \mathbb{P} \left( \sup_{1 \leq j \leq N_t} \mathcal{L}_X(t, \tilde{m}_j) \leq t/x_t, V \in \mathcal{V}_t, N_t < n_t \right) \\ & + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) \\ & \leq \mathbb{P} \left( \mathcal{L}_X(t, \tilde{m}_{N_t}) \leq t/x_t, \sup_{1 \leq j \leq N_t-1} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \leq t/x_t, \right. \\ & \quad \left. V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1 \right) \\ & + \mathbb{P}(\overline{\mathcal{E}}_t^1) + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t), \end{aligned} \tag{5.2.49}$$

because, on  $\{V \in \mathcal{V}_t, N_t < n_t\} \cap \mathcal{E}_t^1$ ,  $\tilde{m}_j$  (for  $j < N_t$ ) is no longer reached between times  $H(\tilde{L}_j)$  and  $t$ . We fix  $v \in \mathcal{G}_t$ , a realization of the environment. Let us define

$$\begin{aligned} \mathcal{E}_t(v, k, z) & := \\ & \left\{ \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \tilde{m}_k) \leq t/x_t, H_{X_{\tilde{m}_k}}(\tilde{m}_{k+1}) \geq t(1-z), H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\}, \end{aligned}$$

and  $\nu_t(v, k, z) := P^v(\mathcal{E}_t(v, k, z))$ . The event  $\mathcal{E}_t(v, k, z)$  belongs to the  $\sigma$ -field  $\sigma(X(t), t \geq H(\tilde{m}_k))$ . In other words, it only depends on the diffusion after time  $H(\tilde{m}_k)$ . On the other hand,  $H(\tilde{m}_k)$  is measurable with respect to the  $\sigma$ -field  $\sigma(X(t), 0 \leq t \leq H(\tilde{m}_k))$ . From the Markov property applied to  $X$  at  $H(\tilde{m}_k)$ , we get that  $H(\tilde{m}_k)$  is independent from the event  $\mathcal{E}_t(v, k, z)$ . As a consequence,

$P^v(\mathcal{L}_X(t, \tilde{m}_{N_t}) \leq t/x_t, \sup_{1 \leq j \leq N_t-1} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \leq t/x_t, N_t < n_t, \mathcal{E}_t^1)$  is less than

$$\sum_{k=1}^{n_t} \int_0^1 \nu_t(v, k, z) \times P^v \left( \sup_{1 \leq j \leq k-1} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \leq t/x_t, H(\tilde{m}_k)/t \in dz \right). \quad (5.2.50)$$

The fact that the sum stops at  $n_t$  comes from  $N_t < n_t$  together with the fact that  $v \in \mathcal{G}_t \subset \mathcal{V}_t$ . From the definition of  $\mathcal{E}_t^9(v, k, z)$  we get

$$\mathcal{E}_t(v, k, z) \subset \left\{ \frac{1-z}{R_k^t} (1 - e^{-\tilde{c}h_t}) \leq \frac{1}{x_t}, H_{X_{\tilde{m}_k}}(\tilde{m}_{k+1}) \geq t(1-z) \right\} \cup \mathcal{E}_t^9(v, k, z)$$

and,  $v$  being fixed, the events in the above expression are independent from the  $\sigma$ -field  $\sigma(X(t), 0 \leq t \leq H(\tilde{m}_k))$ , so putting into (5.2.50), using the independence and the fact that the sum in (5.2.50) corresponds to disjoint events (so it is actually the probability of a union of events), we get that the first term in the right hand side of (5.2.49) is less than

$$\begin{aligned} & \mathbb{P} \left( \frac{1 - H(\tilde{m}_{N_t})/t}{R_{N_t}^t} (1 - e^{-\tilde{c}h_t}) \leq 1/x_t, \sup_{1 \leq j \leq N_t-1} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \leq t/x_t \right) \\ & + E \left[ \mathbf{1}_{V \in \mathcal{G}_t} P^V \left( \cup_{k=1}^{n_t} \{N_t \geq k\} \cap \mathcal{E}_t^9(V, k, H(\tilde{m}_k)/t) \right) \right] + P(V \notin \mathcal{G}_t) \\ & \leq \mathbb{P} \left( (1 - e^{-\tilde{c}h_t})^{-1} \frac{R_{N_t}^t}{x_t} + \frac{H(\tilde{m}_{N_t})}{t} \geq 1, \sup_{1 \leq j \leq N_t-1} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \leq t/x_t \right) + e^{-c\phi(t)}, \end{aligned}$$

where  $c$  is a positive constant and where we used (5.2.15) for the second term, Lemma 5.2.6 for the third term and the fact that  $t$  is large enough,

$$\begin{aligned} & \leq \mathbb{P} \left( (1 - e^{-\tilde{c}h_t})^{-1} \frac{R_{N_t}^t}{x_t} + \frac{(1 + e^{-\tilde{c}h_t})}{t} \sum_{j=1}^{N_t-1} e_j S_j^t R_j^t \geq 1 - 2/\log h_t, \right. \\ & \quad \left. \sup_{1 \leq j \leq N_t-1} e_j S_j^t \leq (1 - e^{-\tilde{c}h_t})^{-1} t/x_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^5, \mathcal{E}_t^6, \mathcal{E}_t^7 \right) \\ & + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) + \mathbb{P}(\overline{\mathcal{E}_t^5}) + \mathbb{P}(\overline{\mathcal{E}_t^6}) + \mathbb{P}(\overline{\mathcal{E}_t^7}) + e^{-c\phi(t)}, \end{aligned}$$

where we used the definitions of  $\mathcal{E}_t^5$ ,  $\mathcal{E}_t^6$  and  $\mathcal{E}_t^7$ ,

$$\begin{aligned} & \leq \mathbb{P} \left( \frac{R_{N_t}^t}{x_t} + \frac{1}{t} \sum_{j=1}^{N_t-1} e_j S_j^t R_j^t \geq 1/2, \sup_{1 \leq j < N_t-1} e_j S_j^t < 2t/x_t, V \in \mathcal{V}_t, N_t < n_t, \right. \\ & \quad \left. \mathcal{E}_t^5, \mathcal{E}_t^6, \mathcal{E}_t^7 \right) + e^{-c\phi(t)} \end{aligned}$$

where we used the fact that  $t$  is large enough, Fact 5.2.3, (5.2.2) and (5.2.4), and where the constant  $c$  has been suitably decreased,

$$\begin{aligned} & \leq \mathbb{P} \left( \frac{R_{N_t}^t}{x_t} + \frac{1}{t} \sum_{j=1}^{N_t-1} e_j S_j^t R_j^t \geq 1/2, \sup_{1 \leq j \leq N_t-1} e_j S_j^t < 2t/x_t, e_{N_t} S_{N_t}^t \geq 2t/x_t \right) \\ & + \mathbb{P} \left( \sup_{1 \leq j \leq N_t} e_j S_j^t < 2t/x_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^5, \mathcal{E}_t^6, \mathcal{E}_t^7 \right) + e^{-c\phi(t)}. \end{aligned}$$

On the event in the probability of the first term, we have  $N_t = k^t(2/x_t)$ . For the second term we use (5.2.5) with  $\eta = 3/4$ . The above is thus less than

$$\begin{aligned}
 & \mathbb{P} \left( \frac{R_{k^t(2/x_t)}^t}{x_t} + \frac{1}{t} \sum_{j=1}^{k^t(2/x_t)-1} e_j S_j^t R_j^t \geq 1/2 \right) + \mathbb{P} \left( \sup_{1 \leq j \leq N_{t/4}} e_j S_j^t < 2t/x_t \right) + e^{-c\phi(t)} \\
 & \leq \mathbb{P} \left( \frac{R_{k^t(2/x_t)}^t}{x_t} \geq 1/4 \right) + \mathbb{P} \left( \frac{1}{t} \sum_{j=1}^{k^t(2/x_t)-1} e_j S_j^t R_j^t \geq 1/4 \right) \\
 & + \mathbb{P} \left( \sup_{1 \leq j \leq N_{t/4}} e_j S_j^t < 2t/x_t \right) + e^{-c\phi(t)} \\
 & \leq \mathbb{P} (R_1^t \geq x_t/4) + 2\mathbb{P} (Y_1^{1,t} (Y_2^{-1,t}(1/4)) \leq 2/x_t) + e^{-c\phi(t)}
 \end{aligned}$$

where, for the last inequality, we used the fact that the sequence  $(R_j^t)_{j \geq 1}$  is *iid* and independent from the random index  $k^t(2/x_t)$ , (5.2.60) with  $b = 1/4$ ,  $y_t = x_t/2$ , and the definition of  $(Y_1^t, Y_2^t)$ . Then, note that according to (5.2.18) we have for all  $t$  large enough,

$$\mathbb{P} (R_1^t \geq x_t/4) \leq e^{-\lambda_0 x_t/8}.$$

Bounding the three terms in the right hand side of (5.2.49) thanks to the above, Fact 5.2.3, (5.2.2) and (5.2.4) we get the upper bound for a suitably chosen constant  $c$  and  $t$  large enough.  $\square$

We now study the functional involved in Proposition 5.2.17. For this we need two lemmas.

In the remaining part of this subsection we fix  $\eta \in ]0, 1/3[$ . Let  $y_t$  go to infinity with  $t$  satisfying  $\log(y_t) \ll \phi(t)$ . Let  $p_t := \mathbb{P}(e_1 S_1^t > t/y_t)$  and  $(\overline{\mathcal{H}}_i)_{i \geq 1}$  be *iid* random variables such that  $\overline{\mathcal{H}}_1$  has the same law as  $e_1 S_1^t R_1^t$  conditionally to  $\{e_1 S_1^t \leq t/y_t\}$ :  $\mathcal{L}(\overline{\mathcal{H}}_1) = \mathcal{L}(e_1 S_1^t R_1^t | e_1 S_1^t \leq t/y_t)$ . Since we have  $\log(y_t) \ll \phi(t)$ , (5.2.16) gives  $p_t \sim C' e^{-\kappa\phi(t)} y_t^\kappa$ . We have

**Lemma 5.2.18.**

$$1 - \mathbb{E} \left[ e^{-\lambda \overline{\mathcal{H}}_1/t} \right] \underset{t \rightarrow +\infty}{\sim} \lambda \frac{C' \kappa \mathbb{E}[\mathcal{R}]}{(1 - \kappa) e^{\kappa\phi(t)} y_t^{1-\kappa}}. \quad (5.2.51)$$

*Démonstration.* For any  $\lambda \geq 0$ ,  $\mathbb{E}[e^{-\lambda \overline{\mathcal{H}}_1/t}]$  equals

$$\begin{aligned}
 & \mathbb{E} \left[ e^{-\lambda e_1 S_1^t R_1^t/t} | e_1 S_1^t \leq t/y_t \right] = (1 - p_t)^{-1} \mathbb{E} \left[ e^{-\lambda e_1 S_1^t R_1^t/t} \mathbf{1}_{e_1 S_1^t \leq t/y_t} \right] \\
 & = (1 - p_t)^{-1} \int_0^{+\infty} \mathbb{E} \left[ e^{-\lambda u e_1 S_1^t/t} \mathbf{1}_{e_1 S_1^t \leq t/y_t} \right] \mathcal{L}(R_1^t)(du) \\
 & = (1 - p_t)^{-1} \int_0^{+\infty} \left( 1 - e^{-\lambda u/y_t} p_t - \lambda \int_0^{1/y_t} u e^{-\lambda x u} \mathbb{P}(e_1 S_1^t/t > x) dx \right) \mathcal{L}(R_1^t)(du),
 \end{aligned}$$

where we used iteration by parts,

$$= (1 - p_t)^{-1} \left( 1 - \mathbb{E} \left[ e^{-\lambda R_1^t / y_t} \right] p_t - \lambda \int_0^{1/y_t} \mathbb{E} \left[ R_1^t e^{-\lambda x R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > x) dx \right),$$

where we used Fubini's Theorem,

$$= (1 - p_t)^{-1} \left( 1 - p_t + p_t(1 - \mathbb{E} \left[ e^{-\lambda R_1^t / y_t} \right]) - \lambda \int_0^{1/y_t} \mathbb{E} \left[ R_1^t e^{-\lambda x R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > x) dx \right). \quad (5.2.52)$$

We now study the second and third term in (5.2.52). Using the fact that the difference between two points of a continuously differentiable function is the integral of its derivative, the last part of Fact 5.2.8 and the equivalent for  $p_t$  we get

$$p_t(1 - \mathbb{E} \left[ e^{-\lambda R_1^t / y_t} \right]) = p_t \frac{\lambda}{y_t} \int_0^1 \mathbb{E} \left[ R_1^t e^{-\lambda u R_1^t / y_t} \right] du \underset{t \rightarrow +\infty}{\sim} \lambda \frac{p_t \mathbb{E}[\mathcal{R}]}{y_t} \underset{t \rightarrow +\infty}{\sim} \lambda \frac{\mathcal{C}' \mathbb{E}[\mathcal{R}]}{e^{\kappa \phi(t)} y_t^{1-\kappa}}. \quad (5.2.53)$$

Then, from the last part of Fact 5.2.8 again,

$$\int_0^{1/y_t} \mathbb{E} \left[ R_1^t e^{-\lambda x R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > x) dx \underset{t \rightarrow +\infty}{\sim} \mathbb{E}[\mathcal{R}] \int_0^{1/y_t} \mathbb{P}(e_1 S_1^t / t > x) dx. \quad (5.2.54)$$

Recall that  $\eta \in ]0, 1/3[$ .  $\int_0^{1/y_t} \mathbb{P}(e_1 S_1^t / t > x) dx$  equals

$$\begin{aligned} & \int_0^{e^{-(1-2\eta)\phi(t)}} \mathbb{P}(e_1 S_1^t / t > x) dx + e^{-\kappa \phi(t)} \int_{e^{-(1-2\eta)\phi(t)}}^{1/y_t} x^{-\kappa} x^\kappa e^{\kappa \phi(t)} \mathbb{P}(e_1 S_1^t / t > x) dx \\ &= \int_0^{e^{-(1-2\eta)\phi(t)}} \mathbb{P}(e_1 S_1^t / t > x) dx + e^{-\kappa \phi(t)} \int_{e^{-(1-2\eta)\phi(t)}}^{1/y_t} x^{-\kappa} (x^\kappa e^{\kappa \phi(t)} \mathbb{P}(e_1 S_1^t / t > x) - \mathcal{C}') dx \\ &+ \mathcal{C}' e^{-\kappa \phi(t)} \int_0^{1/y_t} x^{-\kappa} dx - \mathcal{C}' e^{-\kappa \phi(t)} \int_0^{e^{-(1-2\eta)\phi(t)}} x^{-\kappa} dx. \end{aligned} \quad (5.2.55)$$

Now, the absolute values of the first and fourth terms of (5.2.55) are respectively less than  $e^{-(1-2\eta)\phi(t)}$  and  $\mathcal{C}' e^{-(2\eta\kappa + (1-2\eta)\phi(t))} / (1 - \kappa)$ . In particular, thanks to  $2\eta\kappa + (1 - 2\eta) > \kappa$  (which is trivial) and  $\log(y_t) \ll \phi(t)$ , both are negligible with respect to  $e^{-\kappa \phi(t)} / y_t^{1-\kappa}$ . From (5.2.16) we also have

$$e^{-\kappa \phi(t)} \int_{e^{-(1-2\eta)\phi(t)}}^{1/y_t} x^{-\kappa} (x^\kappa e^{\kappa \phi(t)} \mathbb{P}(e_1 S_1^t / t > x) - \mathcal{C}') dx = \mathcal{O}(e^{-\kappa \phi(t)} / y_t^{1-\kappa}),$$

and the third term of (5.2.55) equals  $\mathcal{C}' e^{-\kappa \phi(t)} / (1 - \kappa) y_t^{1-\kappa}$ . Combining with (5.2.54) we get

$$-\lambda \int_0^{1/y_t} \mathbb{E} \left[ R_1^t e^{-\lambda x R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > x) dx \underset{t \rightarrow +\infty}{\sim} -\lambda \frac{\mathcal{C}' \mathbb{E}[\mathcal{R}]}{(1 - \kappa) e^{\kappa \phi(t)} y_t^{1-\kappa}}. \quad (5.2.56)$$

Putting together (5.2.53) and (5.2.56) in (5.2.52) we obtain (5.2.51).  $\square$

**Lemma 5.2.19.** *Recall the  $\lambda_0$  defined in Fact 5.2.8. For any  $\lambda \in [0, \lambda_0[$  we have*

$$\left( \mathbb{E} \left[ e^{\lambda y_t \overline{\mathcal{H}}_1/t} \right] - 1 \right) / p_t \xrightarrow[t \rightarrow +\infty]{} 1 - \mathbb{E} \left[ e^{\lambda \mathcal{R}} \right] + \lambda \int_0^1 x^{-\kappa} \mathbb{E} \left[ \mathcal{R} e^{\lambda x \mathcal{R}} \right] dx, \quad (5.2.57)$$

and this limit is positive for all  $\lambda \in ]0, \lambda_0[$ .

*Démonstration.* For any  $\lambda \in [0, \lambda_0[$  we have  $\lambda y_t e_1 S_1^t / t \in [0, \lambda_0[$  on the event  $\{e_1 S_1^t / t \leq 1/y_t\}$ , so  $\mathbb{E}[e^{\lambda y_t \overline{\mathcal{H}}_1/t}]$  equals

$$\begin{aligned} & \mathbb{E} \left[ e^{\lambda y_t e_1 S_1^t R_1^t / t} \mathbf{1}_{e_1 S_1^t \leq t/y_t} \right] = (1 - p_t)^{-1} \mathbb{E} \left[ e^{\lambda y_t e_1 S_1^t R_1^t / t} \mathbf{1}_{e_1 S_1^t \leq t/y_t} \right] \\ & = (1 - p_t)^{-1} \int_0^{+\infty} \mathbb{E} \left[ e^{\lambda y_t u e_1 S_1^t / t} \mathbf{1}_{e_1 S_1^t \leq t/y_t} \right] \mathcal{L}(R_1^t)(du) \\ & = (1 - p_t)^{-1} \int_0^{+\infty} \left( 1 - e^{\lambda y_t u / y_t} p_t + \lambda y_t \int_0^{1/y_t} u e^{\lambda y_t x u} \mathbb{P}(e_1 S_1^t / t > x) dx \right) \mathcal{L}(R_1^t)(du) \\ & = (1 - p_t)^{-1} \int_0^{+\infty} \left( 1 - e^{\lambda u} p_t + \lambda \int_0^1 u e^{\lambda y u} \mathbb{P}(e_1 S_1^t / t > y/y_t) dy \right) \mathcal{L}(R_1^t)(du), \end{aligned}$$

where we used iteration by parts and made the change of variable  $y = y_t x$ ,

$$= (1 - p_t)^{-1} \left( 1 - \mathbb{E} \left[ e^{\lambda R_1^t} \right] p_t + \lambda \int_0^1 \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > y/y_t) dy \right)$$

where we used Fubini's Theorem,

$$= (1 - p_t)^{-1} \left( 1 - p_t + p_t (1 - \mathbb{E} \left[ e^{\lambda R_1^t} \right]) + \lambda \int_0^1 \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > y/y_t) dy \right). \quad (5.2.58)$$

According to (5.2.18) the second term is equivalent to  $p_t(1 - \mathbb{E}[e^{\lambda \mathcal{R}}])$ . We now study the third term in (5.2.58). Recall that  $\eta \in ]0, 1/3[$ . This term equals

$$\begin{aligned} & \lambda \int_0^{y_t e^{-(1-2\eta)\phi(t)}} \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > y/y_t) dy \\ & + \lambda y_t^\kappa e^{-\kappa\phi(t)} \int_{e^{-(1-2\eta)\phi(t)}}^1 y^{-\kappa} \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] (y/y_t)^\kappa e^{\kappa\phi(t)} \mathbb{P}(e_1 S_1^t / t > y/y_t) dy \\ & = \lambda \int_0^{y_t e^{-(1-2\eta)\phi(t)}} \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > y/y_t) dy \\ & + \lambda y_t^\kappa e^{-\kappa\phi(t)} \int_{y_t e^{-(1-2\eta)\phi(t)}}^1 y^{-\kappa} \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] \left( (y/y_t)^\kappa e^{\kappa\phi(t)} \mathbb{P}(e_1 S_1^t / t > y/y_t) - \mathcal{C}' \right) dy \\ & + \lambda \mathcal{C}' y_t^\kappa e^{-\kappa\phi(t)} \int_0^1 y^{-\kappa} \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] dy - \lambda \mathcal{C}' y_t^\kappa e^{-\kappa\phi(t)} \int_0^{y_t e^{-(1-2\eta)\phi(t)}} y^{-\kappa} \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] dy. \end{aligned} \quad (5.2.59)$$

Now, thanks to (5.2.18), the absolute values of the first and fourth terms of (5.2.59) are ultimately less than  $2\lambda\mathbb{E}[\mathcal{R}]y_t e^{-(1-2\eta)\phi(t)}$  and  $2\lambda\mathcal{C}'\mathbb{E}[\mathcal{R}]y_t e^{-(2\eta\kappa+(1-2\eta)\phi(t))/(1-\kappa)}$ . In particular, thanks to  $2\eta\kappa + (1 - 2\eta) > \kappa$  (which is trivial) and  $\log(y_t) \ll \phi(t)$ , both are negligible with respect to  $e^{-\kappa\phi(t)}y_t^\kappa$ . From (5.2.16) and (5.2.18) we also have

$$\begin{aligned} & \lambda y_t^\kappa e^{-\kappa\phi(t)} \int_{y_t e^{-(1-2\eta)\phi(t)}}^1 y^{-\kappa} \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] \left( (y/y_t)^\kappa e^{\kappa\phi(t)} \mathbb{P}(e_1 S_1^t/t > y/y_t) - \mathcal{C}' \right) dy \\ & = \mathcal{O}(e^{-\kappa\phi(t)} y_t^\kappa), \end{aligned}$$

and, thanks to (5.2.18), the third term of (5.2.59) is equivalent to  $\lambda\mathcal{C}'y_t^\kappa e^{-\kappa\phi(t)} \int_0^1 y^{-\kappa} \mathbb{E}[\mathcal{R}e^{\lambda y \mathcal{R}}] dy$ . We thus get

$$\begin{aligned} \lambda \int_0^1 \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] \mathbb{P}(e_1 S_1^t/t > y/y_t) dy & \underset{t \rightarrow +\infty}{\sim} \lambda\mathcal{C}'y_t^\kappa e^{-\kappa\phi(t)} \int_0^1 y^{-\kappa} \mathbb{E}[\mathcal{R}e^{\lambda y \mathcal{R}}] dy \\ & \underset{t \rightarrow +\infty}{\sim} \lambda p_t \int_0^1 y^{-\kappa} \mathbb{E}[\mathcal{R}e^{\lambda y \mathcal{R}}] dy. \end{aligned}$$

Putting into (5.2.58) we obtain (5.2.57).

We justify the positivity of the limit as follows : we see that the right hand side of (5.2.57) is equivalent to  $\lambda\kappa\mathbb{E}[\mathcal{R}]/(1-\kappa)$  when  $\lambda$  goes to 0. The limit in (5.2.57) is therefore positive for small  $\lambda$ . On the other hand,  $(\mathbb{E}[e^{\lambda y_t \overline{\mathcal{H}}_1/t}] - 1)/p_t$  increases with  $\lambda$  so the limit in (5.2.57) is non-decreasing on  $[0, \lambda_0[$ . We thus get the positivity of the limit for all  $\lambda \in ]0, \lambda_0[$ . □

We can now study the lower and upper bounds given by Proposition 5.2.17 :

**Proposition 5.2.20.** *Let  $y_t$  be chosen as before (that is,  $y_t \rightarrow +\infty$  and  $\log(y_t) \ll \phi(t)$ ). There is a positive constant  $L$  (not depending on the choice of  $y_t$ ), such that for any  $b > 0$ ,  $u > 1$  and  $t$  large enough we have*

$$e^{-ub(1-\kappa)y_t/\kappa\mathbb{E}[\mathcal{R}]} \leq \mathbb{P} \left( Y_1^{u,t} (Y_2^{-1,t}(b)) \leq 1/y_t \right) \leq e^{-Lby_t}.$$

*Démonstration. Lower bound*

Let us fix  $\alpha \in ]b(1-\kappa)/\kappa\mathbb{E}[\mathcal{R}], ub(1-\kappa)/\kappa\mathbb{E}[\mathcal{R}[$ . For any  $z > 0$ ,  $k^t(z)$  still denotes the first index  $k \geq 1$  such that  $e_k S_k^t/t \geq z$ . We have

$$\left\{ Y_1^{u,t} (Y_2^{-1,t}(b)) \leq 1/y_t \right\} = \left\{ k^t(1/y_t) > \mathcal{N}_{tb} \right\} = \left\{ \sum_{i=1}^{k^t(1/y_t)-1} e_i S_i^t R_i^t > tb \right\}. \quad (5.2.60)$$

Now, recall that  $(\overline{\mathcal{H}}_i)_{i \geq 1}$  are *iid* random variables such that  $\mathcal{L}(\overline{\mathcal{H}}_1) = \mathcal{L}(e_1 S_1^t R_1^t | e_1 S_1^t \leq t/y_t)$  and let  $T$  be a geometric random variable with parameter  $p_t = \mathbb{P}(e_1 S_1^t > t/y_t)$ ,

independent from the sequence  $(\overline{\mathcal{H}}_i)_{i \geq 1}$ . Recall that  $p_t \sim \mathcal{C}' e^{-\kappa\phi(t)} y_t^\kappa$ . We have

$$\mathbb{P} \left( \sum_{i=1}^{k^t(1/y_t)^{-1}} e_i S_i^t R_i^t > tb \right) = \mathbb{P} \left( \sum_{i=1}^{T-1} \overline{\mathcal{H}}_i > tb \right) \geq \mathbb{P}(T > \alpha h(t) + 1) \times \mathbb{P} \left( \sum_{i=1}^{\lfloor \alpha h(t) \rfloor} \overline{\mathcal{H}}_i > tb \right), \quad (5.2.61)$$

where we put  $h(t) := y_t/p_t \sim e^{\kappa\phi(t)} y_t^{1-\kappa}/\mathcal{C}'$ . We give a lower bound for the two factors in the left hand side of (5.2.61). We first study the Laplace transform of the normalized sum of the second factor to prove its convergence to a constant number. For any  $\lambda \geq 0$ , we have

$$\mathbb{E} \left[ e^{-\lambda \sum_{i=1}^{\lfloor \alpha h(t) \rfloor} \overline{\mathcal{H}}_i/t} \right] = \left( \mathbb{E} \left[ e^{-\lambda \overline{\mathcal{H}}_1/t} \right] \right)^{\lfloor \alpha h(t) \rfloor} = e^{\lfloor \alpha h(t) \rfloor \log(1 + \mathbb{E}[e^{-\lambda \overline{\mathcal{H}}_1/t} - 1])}.$$

According to Lemma 5.2.18, the exponent is equivalent to

$$-\lambda \mathcal{C}' \kappa \mathbb{E}[\mathcal{R}] \alpha h(t) / (1 - \kappa) e^{\kappa\phi(t)} y_t^{1-\kappa},$$

and since  $h(t) \sim e^{\kappa\phi(t)} y_t^{1-\kappa}/\mathcal{C}'$  we get

$$\mathbb{E} \left[ e^{-\lambda \sum_{i=1}^{\lfloor \alpha h(t) \rfloor} \overline{\mathcal{H}}_i/t} \right] \xrightarrow{t \rightarrow +\infty} e^{-\lambda \alpha \kappa \mathbb{E}[\mathcal{R}] / (1 - \kappa)},$$

so  $\sum_{i=1}^{\lfloor \alpha h(t) \rfloor} \overline{\mathcal{H}}_i/t$  converges in probability to  $\alpha \kappa \mathbb{E}[\mathcal{R}] / (1 - \kappa)$  which yields

$$\mathbb{P} \left( \sum_{i=1}^{\lfloor \alpha h(t) \rfloor} \overline{\mathcal{H}}_i > tb \right) \xrightarrow{t \rightarrow +\infty} 1, \quad (5.2.62)$$

since  $\alpha > b(1 - \kappa)/\kappa \mathbb{E}[\mathcal{R}]$ .

We now study the first factors in the left hand side of (5.2.61). Since  $T$  is geometric with parameter  $p_t$ , we have

$$\mathbb{P}(T > \alpha h(t) + 1) = (1 - p_t)^{\lfloor \alpha h(t) + 1 \rfloor} = e^{\lfloor \alpha h(t) + 1 \rfloor \log(1 - p_t)}.$$

Now, since  $h(t) \sim e^{\kappa\phi(t)} y_t^{1-\kappa}/\mathcal{C}'$  and  $p_t \sim \mathcal{C}' e^{-\kappa\phi(t)} y_t^\kappa$  we get

$$\log(\mathbb{P}(T > \alpha h(t) + 1)) \underset{t \rightarrow +\infty}{\sim} \alpha y_t. \quad (5.2.63)$$

Now, putting (5.2.62) and (5.2.63) into (5.2.61), and combining the latter with (5.2.60), we get the result for  $t$  large enough since  $\alpha < ub(1 - \kappa)/\kappa \mathbb{E}[\mathcal{R}]$ .

*Upper bound*



Recall (5.2.60) and the definitions of  $(\overline{\mathcal{H}}_i)_{i \geq 1}$ ,  $p_t$ ,  $T$  and  $h(t)$ . Let us fix  $\alpha > 0$  that will be chosen latter. We have

$$\mathbb{P} \left( \sum_{i=1}^{k^t(t/y_t)^{-1}} e_i S_i^t R_i^t > tb \right) = \mathbb{P} \left( \sum_{i=1}^{T-1} \overline{\mathcal{H}}_i > tb \right) \leq \mathbb{P}(T > \alpha h(t)) + \mathbb{P} \left( \sum_{i=1}^{[\alpha h(t)]} \overline{\mathcal{H}}_i > tb \right). \quad (5.2.64)$$

Let us choose  $\lambda \in ]0, \lambda_0[$  where  $\lambda_0$  defined in Fact 5.2.8. The second term equals

$$\begin{aligned} \mathbb{P} \left( \sum_{i=1}^{[\alpha h(t)]} y_t \overline{\mathcal{H}}_i / t > by_t \right) &\leq e^{-\lambda by_t} \mathbb{E} \left[ \exp \left( \lambda \sum_{i=1}^{[\alpha h(t)]} y_t \overline{\mathcal{H}}_i / t \right) \right] \\ &= e^{-\lambda by_t} (1 + \mathbb{E} [\exp(\lambda y_t \overline{\mathcal{H}}_i / t)] - 1)^{[\alpha h(t)]} \\ &= e^{-\lambda by_t + [\alpha h(t)] \log(1 + \mathbb{E}[\exp(\lambda y_t \overline{\mathcal{H}}_i / t)] - 1)}. \end{aligned}$$

According to Lemma 5.2.19, the fact that  $h(t) \sim e^{\kappa \phi(t)} y_t^{1-\kappa} / \mathcal{C}'$  and  $p_t \sim \mathcal{C}' e^{-\kappa \phi(t)} y_t^\kappa$  we have

$$[\alpha h(t)] \log(1 + \mathbb{E}[\exp(\lambda y_t \overline{\mathcal{H}}_i / t)] - 1) \underset{t \rightarrow +\infty}{\sim} \alpha y_t \left( 1 - \mathbb{E}[e^{\lambda \mathcal{R}}] + \lambda \int_0^1 x^{-\kappa} \mathbb{E}[\mathcal{R} e^{\lambda x \mathcal{R}}] dx \right).$$

Thanks to the positivity of the limit in Lemma 5.2.19 we can choose  $\alpha$  such that  $0 < \alpha < b\lambda/2(1 - \mathbb{E}[e^{\lambda \mathcal{R}}] + \lambda \int_0^1 x^{-\kappa} \mathbb{E}[\mathcal{R} e^{\lambda x \mathcal{R}}] dx)$ . We thus get for  $t$  large enough

$$\mathbb{P} \left( \sum_{i=1}^{[\alpha h(t)]} y_t \overline{\mathcal{H}}_i / t > by_t \right) \leq e^{-\lambda by_t / 2}. \quad (5.2.65)$$

Since  $T$  is geometric with parameter  $p_t$ , we have

$$\mathbb{P}(T > \alpha h(t)) = (1 - p_t)^{[\alpha h(t)]} = e^{[\alpha h(t)] \log(1 - p_t)} \underset{t \rightarrow +\infty}{\approx} e^{-\alpha y_t} \leq e^{-\alpha y_t / 2}, \quad (5.2.66)$$

where we used the equivalents for  $h(t)$  and  $p_t$  and where the last inequality holds for  $t$  large enough.

Now, putting (5.2.65) and (5.2.66) into (5.2.64), and combining the latter with (5.2.60), we get the result for  $t$  large enough.  $\square$

Fix  $\theta > 1$ . We apply Proposition 5.2.17 with  $a = \theta^{1/3}$  and Proposition 5.2.20 (the lower bound with  $b = \theta^{1/3}$ ,  $u = \theta^{1/3}$ ,  $y_t = \theta^{1/3} x_t$  and the upper bound with  $b = 1/4$ ,  $y_t = x_t/2$ ). We get :

**Proposition 5.2.21.** *Let  $\tilde{L} \in ]0, \min\{L/8, \lambda_0/8\}[$  where  $L$  is the positive constant defined in Proposition 5.2.20 and  $\lambda_0$  is defined in Fact 5.2.8. There is a positive constant  $c$  such that for any  $\theta > 1$  and  $t$  large enough we have*

$$e^{-\theta(1-\kappa)x_t/\kappa\mathbb{E}[\mathcal{R}]} - e^{-c\phi(t)} \leq \mathbb{P}(\mathcal{L}_X^*(t) \leq t/x_t) \leq e^{-\tilde{L}x_t} + e^{-c\phi(t)}.$$

We can now prove Theorem 5.1.9.

*Démonstration.* of Theorem 5.1.9

Recall that  $\tilde{L}$  is the positive constant defined in Proposition 5.2.21 and let  $x_t := 4 \log(\log(t))/2\tilde{L}$ . Note that such a choice of  $x_t$  satisfies (5.2.6) with  $\mu = 2$  and  $D = 4/2\tilde{L}$ . We define the events

$$\mathcal{A}_n := \left\{ \inf_{t \in [2^n, 2^{n+1}]} \frac{\mathcal{L}_X^*(t)}{t/\log(\log(t))} \leq \tilde{L}/4 \right\}.$$

From the increase of  $\mathcal{L}_X^*(\cdot)$  and Proposition 5.2.21 (the upper bound applied with  $t = 2^n$ ) we have for  $n$  large enough,

$$\begin{aligned} \mathbb{P}(\mathcal{A}_n) &\leq \mathbb{P}\left(\mathcal{L}_X^*(2^n) \leq 2^{n+1}\tilde{L}/4 \log(\log(2^n))\right) \leq \exp(-2 \log(\log(2^n))) + e^{-c\phi(2^n)} \\ &= (\log(2))^{-2} n^{-2} + e^{-c\phi(2^n)}. \end{aligned}$$

Since  $e^{-c\phi(2^n)} = e^{-c(\log \log(2^n))^2} \leq n^{-2}$  for  $n$  large enough, the above is the general term of a converging series so, using the Borel-Cantelli lemma we deduce that  $\mathbb{P}$ -almost surely,

$$\liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t/\log(\log(t))} \geq \tilde{L}/4,$$

so the lim inf is positive.

We now prove the upper bound for the lim inf. Let  $a > 1$  and let  $t_n, u_n, v_n$  (defined from this  $a$ ) and  $X^n$  be as in Subsection 5.2.2 (recall that  $X^n - v_n$  is equal in law to  $X$  under the annealed probability  $\mathbb{P}$ ). We define

$$\mathcal{B}_n := \left\{ \frac{\mathcal{L}_{X^n}^*(t_n)}{t_n/\log(\log(t_n))} \leq \frac{a^2(1-\kappa)}{\kappa \mathbb{E}[\mathcal{R}]} \right\}, \quad \mathcal{B}'_n := \left\{ \frac{\mathcal{L}_X^*(H(v_n))}{t_n/\log(\log(t_n))} \leq \frac{1}{n} \right\}.$$

We also define  $\mathcal{C}_n$  and  $\mathcal{D}_n$  to be as in Lemma 5.2.10 and  $\mathcal{E}_n := \mathcal{B}_n \cap \mathcal{C}_n$ . We define  $x_t := \kappa \mathbb{E}[\mathcal{R}] \log(\log(t))/a^2(1-\kappa)$ . Note that such a choice of  $x_t$  satisfies (5.2.6) with  $\mu = 2$  and  $D = \kappa \mathbb{E}[\mathcal{R}]/a^2(1-\kappa)$ . Recall the notation  $\mathcal{L}_X^{*,+}$  defined in Subsection 5.1.3. Let us choose  $\eta$  and  $C$  as in Lemma 5.3.2 of the next section and  $Q$  be as defined in the next section. According to (5.2.22) and Lemma 5.3.2 applied with  $u = t_n/n \log(\log(t_n)), v = v_n$  we get for all  $n$  large enough,

$$\begin{aligned} \mathbb{P}(\overline{\mathcal{B}'_n}) &\leq \mathbb{P}\left(\inf_{] -\infty, 0]} \mathcal{L}_X(+\infty, \cdot) > t_n/n \log(\log(t_n))\right) + \mathbb{P}\left(\frac{\mathcal{L}_X^{*,+}(H(v_n))}{t_n/\log(\log(t_n))} > \frac{1}{n}\right) \\ &\leq 3(t_n/n \log(\log(t_n)))^{-\kappa/(2+\kappa)} + C(v_n/Q + v_n^{7/8})(n \log(\log(t_n)))^\kappa / (t_n)^\kappa + v_n^{-\eta}. \end{aligned}$$

From the definition of  $t_n$ , the fact that  $n^\kappa v_n/t_n^\kappa = n^\kappa e^{-\kappa 2an^{a-1}/3}$  and  $\log(\log(t_n)) = a \log(n)$ , we get

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{B}'_n}) < +\infty. \tag{5.2.67}$$

According to Proposition 5.2.21 (the lower bound applied with  $t = t_n$ ,  $\theta = a$ ) and the definition of  $t_n$  we have

$$\mathbb{P}(\mathcal{B}_n) \geq e^{-\log(\log(t_n))/a} - e^{-c\phi(t_n)} = e^{-a\log(n)/a} - e^{-c\phi(t_n)} = n^{-1} - e^{-c\phi(t_n)}.$$

Since  $e^{-c\phi(t_n)} = e^{-c(\log \log(t_n))^2} \leq n^{-2}$  for  $n$  large enough, we get

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{B}_n) = +\infty. \quad (5.2.68)$$

Then, the combination of (5.2.68) and (5.2.23) yields

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{E}_n) \geq \sum_{n \geq 1} \mathbb{P}(\mathcal{B}_n) - \sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{C}_n}) = +\infty. \quad (5.2.69)$$

As in the proof of Theorem 5.1.2, we see that each event  $\mathcal{E}_n$  belongs to the  $\sigma$ -field  $\sigma(V(s) - V(u_n), u_n \leq s \leq u_{n+1}, X(t), H(v_n) \leq t \leq H(v_n) + T_n)$  so the events  $(\mathcal{E}_n)_{n \geq 1}$  are independent. Combining this independence with (5.2.69) and the Borel-Cantelli Lemma we get that  $\mathbb{P}$ -almost surely, the event  $\mathcal{E}_n$  is realized infinitely many often. Combining with (5.2.67) and the Borel-Cantelli Lemma we get that  $\mathbb{P}$ -almost surely, the event  $\mathcal{B}'_n \cap \mathcal{B}_n \cap \mathcal{C}_n$  is realized infinitely many often. For  $n$  such that this event is realized we have

$$\frac{\mathcal{L}_X^*(H(v_n) + t_n)}{t_n / \log(\log(t_n))} \leq \frac{\mathcal{L}_{X^n}^*(t_n)}{t_n / \log(\log(t_n))} + \frac{\mathcal{L}_X^*(H(v_n))}{t_n / \log(\log(t_n))} \leq \frac{a^2(1 - \kappa)}{\kappa \mathbb{E}[\mathcal{R}]} + \frac{1}{n}. \quad (5.2.70)$$

Recall that according to (5.2.24) and the Borel-Cantelli Lemma we have  $\mathbb{P}$ -almost surely

$$H(v_n) + t_n \underset{n \rightarrow +\infty}{\sim} t_n,$$

so combining with (5.2.70) we deduce that  $\mathbb{P}$ -almost surely,

$$\liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t / \log(\log(t))} \leq \frac{a^2(1 - \kappa)}{\kappa \mathbb{E}[\mathcal{R}]},$$

and letting  $a$  go to 1 we get the asserted upper bound for the lim inf. □

## 5.3 Almost sure behavior when $\kappa > 1$

In this section we prove Theorems 5.1.10 and 5.1.11. Let us first recall some facts and notations from [66] and [74]. Our proof is based on the study of the so-called generalized Ornstein-Uhlenbeck process defined by

$$Z(x) := e^{V(x)} R \left( \int_0^x e^{-V(y)} dy \right),$$

where  $R$  is a two-dimensional squared Bessel process independent from  $V$ . Let  $L$  be the local time of  $Z$  for the position 1,  $n$  the associated excursion measure, and  $L^{-1}$  the right continuous inverse of  $L$ . We denote by  $\xi$  a generic excursion. Let us denote by  $Q$  the positive constant denoted by  $n[\zeta]$  in [66]. Recall also the notations  $K$  and  $m$  defined in the Introduction. We have :

**Fact 5.3.1.** *There is  $\eta > 0$  and  $r_0 > 0$  such that for all  $r \geq r_0$  and  $h > 1$  we have*

$$e^{-(r/Q+r^{7/8})n(\sup \xi > h)} - r^{-\eta} \leq \mathbb{P} \left( \sup_{x \in [0,r]} Z(x) \leq h \right) \leq e^{-(r/Q-r^{7/8})n(\sup \xi > h)} + r^{-\eta}, \quad (5.3.1)$$

$$n(\sup \xi > h) \underset{h \rightarrow +\infty}{\sim} Q2^\kappa \Gamma(\kappa) \kappa^2 K/h^\kappa. \quad (5.3.2)$$

The first point is Lemma 2.3 of [74] while the second point is Proposition 5.1 of [66]. Note that Fact 5.3.1 is true for a general positive  $\kappa$  and not only for  $\kappa > 1$ .

Let us recall the link between  $Z$  and the local time until the hitting times. The local time at point  $x$  and within the hitting time  $H(r)$  is given by :

$$\mathcal{L}_X(H(r), x) = e^{-V(x)} \mathcal{L}_B(\tau(B, A_V(r)), A_V(x)). \quad (5.3.3)$$

$\mathcal{M}_1(r)$  and  $\mathcal{M}_2(r)$  denote respectively the supremum of the above expression for  $x \in ]-\infty, 0[$  and  $x \in [0, +\infty[$ . The supremum of the local time until instant  $H(r)$  can be written

$$\mathcal{L}_X^*(H(r)) = \max\{\mathcal{M}_1(r), \mathcal{M}_2(r)\}, \quad (5.3.4)$$

where

$$\mathcal{M}_1(r) \leq \mathcal{M}_1(+\infty) < +\infty \text{ and, as in [74], } \mathcal{M}_2(r) \stackrel{\mathcal{L}}{=} \sup_{x \in [0,r]} Z(x). \quad (5.3.5)$$

We can now study the behavior of the local time at a hitting time. This allows to prove the following useful lemma.

**Lemma 5.3.2.** *There exist  $\eta > 0$ , a positive constant  $C$ ,  $u_0 > 0$  and  $v_0 > 0$  such that*

$$\forall u \geq u_0, v \geq v_0, \mathbb{P}(\mathcal{L}_X^{*,+}(H(v)) > u) \leq C(v/Q + v^{7/8})u^{-\kappa} + v^{-\eta},$$

where  $\mathcal{L}_X^{*,+}$  is as defined in Subsection 5.1.3.

*Démonstration.* From the definition of  $\mathcal{M}_2$  and (5.3.5) we have

$$\mathbb{P}(\mathcal{L}_X^{*,+}(H(v)) \geq u) = \mathbb{P}(\mathcal{M}_2(v) > u) = \mathbb{P} \left( \sup_{[0,v]} Z > u \right). \quad (5.3.6)$$

Now let us choose  $\eta$  and  $v_0$  such that (5.3.1) is true for all  $r \geq v_0$  and  $h > 1$  with this  $\eta$ . We choose  $C > Q2^\kappa\Gamma(\kappa)\kappa^2K$  and  $u_0$  such that  $n(\sup \xi > u) \leq Cu^{-\kappa}$  for all  $u \geq u_0$ . Such a  $u_0$  exists thanks to (5.3.2). For  $u \geq u_0$  and  $v \geq v_0$  we have

$$\begin{aligned} \mathbb{P}\left(\sup_{[0,v]} Z > u\right) &\leq 1 - e^{-(v/Q+v^{7/8})\times n(\sup \xi > u)} + v^{-\eta} \\ &\leq (v/Q + v^{7/8}) \times n(\sup \xi > u) + v^{-\eta} \\ &\leq C(v/Q + v^{7/8})u^{-\kappa} + v^{-\eta}. \end{aligned}$$

Putting into (5.3.6) we get the result. □

**Remark 5.3.3.** *Neither Lemma 5.3.2 nor its proof require the hypothesis that  $\kappa > 1$ . The lemma is thus true whatever is the value of  $\kappa$ .*

We need to study the supremum of the local time until a deterministic time. The following fact from [74] says that we can replace a deterministic time by a hitting time when  $\kappa > 1$ . **We now assume  $\kappa > 1$  until the end of this section.** We have :

**Fact 5.3.4.** *For any  $\alpha \in ]\max\{3/4, 1/\kappa\}, 1[$  there exists  $\eta > 0$  such that for  $r$  large enough we have*

$$\mathbb{P}(H(r/m - r^\alpha) \leq r \leq H(r/m + r^\alpha)) \geq 1 - r^{-\eta}. \quad (5.3.7)$$

Let  $\alpha \in ]\max\{3/4, 1/\kappa\}, 1[$  be fixed until the end of this section and  $\eta > 0$  be small enough so that both (5.3.1) and (5.3.7) are satisfied (with this  $\alpha$ ).

### 5.3.1 The lim inf

In this subsection we prove Theorem 5.1.11. Let us define  $J := 2(\Gamma(\kappa)\kappa^2K/m)^{1/\kappa}$ , the expected lim inf. We begin to prove that

$$\liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{(t/\log(\log(t)))^{1/\kappa}} \geq J. \quad (5.3.8)$$

Let  $a > 1$  and define the events

$$\mathcal{A}_n := \left\{ \inf_{t \in [a^n, a^{n+1}]} \frac{\mathcal{L}_X^*(t)}{(t/\log(\log(t)))^{1/\kappa}} \leq \frac{J}{a^{3/\kappa}} \right\}.$$

From the increase of  $\mathcal{L}_X^*(\cdot)$ , (5.3.7), (5.3.4), (5.3.5), and (5.3.1), we have

$$\begin{aligned}
 \mathbb{P}(\mathcal{A}_n) &\leq \mathbb{P}(\mathcal{L}_X^*(a^n) \leq J(a^{n-2}/\log(\log(a^n)))^{1/\kappa}) \\
 &\leq \mathbb{P}(\mathcal{L}_X^*(H(a^n/m - a^{\alpha n})) \leq J(a^{n-2}/\log(\log(a^n)))^{1/\kappa}) + a^{-n\eta} \\
 &\leq \mathbb{P}(\mathcal{M}_2(a^n/m - a^{\alpha n}) \leq J(a^{n-2}/\log(\log(a^n)))^{1/\kappa}) + a^{-n\eta} \\
 &= \mathbb{P}\left(\sup_{[0, a^n/m - a^{\alpha n}]} Z \leq J(a^{n-2}/\log(\log(a^n)))^{1/\kappa}\right) + a^{-n\eta} \\
 &\leq e^{-\left(a^n/Qm - a^{\alpha n}/Q - (a^n/m - a^{\alpha n})^{7/8}\right) \times n \left(\sup \xi > J(a^{n-2}/\log(\log(a^n)))^{1/\kappa}\right)} + a^{-n\eta} \\
 &\quad + (a^n/m - a^{\alpha n})^{-\eta}. \tag{5.3.9}
 \end{aligned}$$

According to the equivalent given by (5.3.2), the exponent in the above expression is, for  $n$  large enough, less than  $-a \log(\log(a^n))$ , so for such large  $n$ ,

$$e^{-\left(a^n/Qm - a^{\alpha n}/Q - (a^n/m - a^{\alpha n})^{7/8}\right) \times n \left(\sup \xi > J(a^{n-2}/\log(\log(a^n)))^{1/\kappa}\right)} \leq (n \log(a))^{-a}.$$

The other two terms in the right hand side of (5.3.9) are also general terms of converging series so we obtain,

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{A}_n) < +\infty.$$

According to the Borel-Cantelli lemma we get

$$\liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{(t/\log(\log(t)))^{1/\kappa}} \geq J/a^{3/\kappa},$$

in which we can let  $a$  go to 1 which yields (5.3.8). We now prove that

$$\liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{(t/\log(\log(t)))^{1/\kappa}} \leq J. \tag{5.3.10}$$

Let us fix  $a > 0$ ,  $u_n := n^{2n}$ ,  $v_n := u_n/m + u_n^\alpha = n^{2n}/m + n^{2\alpha n}$  and  $X^n := X(H(2v_n) + \cdot)$ , the diffusion shifted by the hitting time of  $2v_n$ . Note that from the Markov property for  $X$  at time  $H(2v_n)$  and the stationarity of the increments of  $V$ ,  $X^n - 2v_n$  is equal in law to  $X$  under the annealed probability  $\mathbb{P}$ . We take  $n$  so large such that  $2v_n < v_{n+1}$  and define the events

$$\begin{aligned}
 \mathcal{B}_n &:= \left\{ \frac{\mathcal{L}_X^{*,+}(H(2v_n))}{(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}} \leq aJ \right\}, \\
 \mathcal{C}_n &:= \left\{ \frac{\mathcal{L}_{X^n}^*(\tau(X^n, v_{n+1}))}{(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}} \leq (1+a)J \right\}, \\
 \mathcal{D}_n &:= \{\tau(X^n, v_{n+1}) < \tau(X^n, v_n)\}, \\
 \mathcal{E}_n &:= \mathcal{C}_n \cap \mathcal{D}_n, \\
 \mathcal{F}_n &:= \{\mathcal{L}_X^{*,+}(u_n) \leq \mathcal{L}_X^{*,+}(H(v_n))\}.
 \end{aligned}$$

Recall that  $\eta >$  has been fixed so that (5.3.1) is satisfied. For this  $\eta$  and for  $C > Q2^\kappa\Gamma(\kappa)\kappa^2K$ , the inequality of Lemma 5.3.2 is true for  $u$  and  $v$  large enough. According to this lemma applied with  $u = aJ(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}$ ,  $v = 2v_n$  we get for all  $n$  large enough,

$$\mathbb{P}(\overline{\mathcal{B}}_n) \leq C (2v_n/Q + (2v_n)^{7/8}) \log(\log(u_{n+1}))/J^\kappa a^\kappa u_{n+1} + (2v_n)^{-\eta}.$$

Since, for  $n$  large enough,  $v_n/u_{n+1} \leq 1/mn^2$  and  $\log(\log(u_{n+1})) \sim \log(n)$  we can deduce that

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{B}}_n) < +\infty. \quad (5.3.11)$$

From the equality in law between  $X^n - 2v_n$  and  $X$  under  $\mathbb{P}$ , , (5.3.4), (5.3.5), (5.2.22), (5.3.1)

$$\begin{aligned} \mathbb{P}(\mathcal{C}_n) &= \mathbb{P}\left(\mathcal{L}_X^*(H(v_{n+1} - 2v_n)) \leq (1+a)J(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}\right) \\ &\geq \mathbb{P}\left(\mathcal{M}_2(H(v_{n+1} - 2v_n)) \leq (1+a)J(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}\right) \\ &\quad - \mathbb{P}\left(\inf_{]-\infty, 0]} \mathcal{L}_X(+\infty, \cdot) > (1+a)J(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}\right) \\ &\geq \mathbb{P}\left(\sup_{[0, v_{n+1}-2v_n]} Z \leq (1+a)J(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}\right) \\ &\quad - 3((1+a)J)^{-\kappa/(2+\kappa)} \times (u_{n+1}/\log(\log(u_{n+1})))^{-1/(2+\kappa)} \\ &\geq e^{-((v_{n+1}-2v_n)/Q+(v_{n+1}-2v_n)^{7/8}) \times n (\sup \xi > (1+a)J(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa})} \\ &\quad - (v_{n+1} - 2v_n)^{-\eta} - 3((1+a)J)^{-\kappa/(2+\kappa)} \times (u_{n+1}/\log(\log(u_{n+1})))^{-1/(2+\kappa)}. \end{aligned} \quad (5.3.12)$$

According to the equivalent given by (5.3.2) and the definitions of  $u_n$  and  $v_n$ , the exponent in the above expression is equivalent to  $(1+a)^{-\kappa} \log(\log(u_{n+1})) \sim (1+a)^{-\kappa} \log(n)$  so for  $n$  large enough,

$$e^{-((v_{n+1}-2v_n)/Q+(v_{n+1}-2v_n)^{7/8}) \times n (\sup \xi > (1+a)J(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa})} \geq \frac{1}{n},$$

and the remaining terms in the right hand side of (5.3.12) are the general terms of converging series. We thus get

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{C}_n) = +\infty. \quad (5.3.13)$$

$$\begin{aligned} \mathbb{P}(\overline{\mathcal{D}}_n) &= \mathbb{P}(\tau(X^n, v_{n+1}) > \tau(X^n, v_n)) \\ &\leq \mathbb{P}(\tau(X^n, v_n) < \tau(X^n, +\infty)) \\ &\leq \mathbb{P}(\tau(X, -v_n) < \tau(X, +\infty)) \\ &= \mathbb{P}\left(\inf_{[0, +\infty[} X < -v_n\right), \end{aligned}$$

where we used the equality in law between  $X^n - 2v_n$  and  $X$  under  $\mathbb{P}$ . Combining with (5.2.21) applied with  $r = v_n$  we get

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{D}_n}) < +\infty. \quad (5.3.14)$$

Then, the combination of (5.3.13) and (5.3.14) yields

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{E}_n) \geq \sum_{n \geq 1} \mathbb{P}(\mathcal{C}_n) - \sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{D}_n}) = +\infty. \quad (5.3.15)$$

According to the definitions of the sequences  $(u_n)_{n \geq 1}$  and  $(v_n)_{n \geq 1}$ , to (5.3.7), and to the increase of  $\mathcal{L}_X^{*,+}$ , we have, for  $n$  large enough,  $\mathbb{P}(\mathcal{F}_n) \leq u_n^{-\eta}$ , so

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{F}_n}) < +\infty. \quad (5.3.16)$$

Note that each event  $\mathcal{E}_n$  belongs to the  $\sigma$ -field  $\sigma(V(s) - V(v_n), v_n \leq s \leq v_{n+1}, X(t), H(2v_n) \leq t \leq \min(\tau(X^n, v_n), \tau(X^n, v_{n+1})))$ , in other words, it only depends on the diffusion between times  $H(2v_n)$  and  $\min(\tau(X^n, v_n), \tau(X^n, v_{n+1}))$  and on the environment between positions  $v_n$  and  $v_{n+1}$ . From the Markov property and the independence of the increments of the environment, we get that the events  $(\mathcal{E}_n)_{n \geq 1}$  are independent. Combining this independence with (5.3.15) and the Borel-Cantelli Lemma we get that  $\mathbb{P}$ -almost surely, the event  $\mathcal{E}_n$  is realized infinitely many often.

Combining (5.3.11), (5.3.16) and the Borel-Cantelli Lemma we get that  $\mathbb{P}$ -almost surely the event  $\mathcal{B}_n \cap \mathcal{F}_n$  is realized for all large  $n$ . We deduce that  $\mathbb{P}$ -almost surely, the event  $\mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n \cap \mathcal{F}_{n+1}$  is realized infinitely many often. Then, for  $n$  such that this event is realized we have,

$$\begin{aligned} \frac{\mathcal{L}_X^{*,+}(u_{n+1})}{(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}} &\leq \frac{\mathcal{L}_X^{*,+}(H(v_{n+1}))}{(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}} \\ &\leq \frac{\mathcal{L}_X^{*,+}(H(2v_n))}{(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}} + \frac{\mathcal{L}_{X^n}^*(\tau(X^n, v_{n+1}))}{(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}} \\ &\leq aJ + (1+a)J, \end{aligned}$$

so

$$\liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^{*,+}(t)}{(t/\log(\log(t)))^{1/\kappa}} \leq (1+2a)J.$$

Now, letting  $a$  go to 0 and combining with the finiteness of  $\sup_{]-\infty, 0[} \mathcal{L}_X(+\infty)$  (see (5.3.4) and (5.3.5)) we obtain (5.3.10) so Theorem 5.1.11 is proved.



### 5.3.2 The lim sup

In this subsection we prove Theorem 5.1.10. First, let us assume that  $\int_1^{+\infty} \frac{(f(t))^\kappa}{t} dt < +\infty$  and prove that

$$\limsup_{t \rightarrow +\infty} \frac{f(t)\mathcal{L}_X^*(t)}{t} = 0. \quad (5.3.17)$$

According to Remark 5.1.19 the condition  $\int_1^{+\infty} \frac{(f(t))^\kappa}{t} dt < +\infty$  is equivalent to

$$\sum_{n=1}^{+\infty} (f(2^n))^\kappa < +\infty. \quad (5.3.18)$$

Let us fix  $a > 0$  and define the events

$$\mathcal{A}_n := \left\{ \sup_{t \in [2^n, 2^{n+1}]} \frac{f(t)\mathcal{L}_X^{*,+}(t)}{t^{1/\kappa}} \geq a \right\}.$$

From the increase of  $\mathcal{L}_X^{*,+}(\cdot)$ , (5.3.7), (5.3.4), (5.3.5), and (5.3.1), we have

$$\begin{aligned} \mathbb{P}(\mathcal{A}_n) &\leq \mathbb{P}(\mathcal{L}_X^{*,+}(2^{n+1}) \geq 2^{n/\kappa} a / f(2^n)) \\ &\leq \mathbb{P}(\mathcal{L}_X^{*,+}(H(2^{n+1}/m + 2^{\alpha(n+1)})) \geq 2^{n/\kappa} a / f(2^n)) + 2^{-\eta(n+1)} \\ &= \mathbb{P}(\mathcal{M}_2(2^{n+1}/m + 2^{\alpha(n+1)}) \geq 2^{n/\kappa} a / f(2^n)) + 2^{-\eta(n+1)} \\ &= \mathbb{P}\left(\sup_{[0, 2^{n+1}/m + 2^{\alpha(n+1)}]} Z \geq 2^{n/\kappa} a / f(2^n)\right) + 2^{-\eta(n+1)} \\ &\leq 1 - e^{-\left(2^{n+1}/Qm + 2^{\alpha(n+1)}/Q + (2^{n+1}/m + 2^{\alpha(n+1)})^{7/8}\right) \times n(\sup \xi > 2^{n/\kappa} a / f(2^n))} \\ &\quad + 2^{-\eta(n+1)} + (2^{n+1}/m + 2^{\alpha(n+1)})^{-\eta} \\ &\leq \left(2^{n+1}/Qm + 2^{\alpha(n+1)}/Q + (2^{n+1}/m + 2^{\alpha(n+1)})^{7/8}\right) \times n(\sup \xi > 2^{n/\kappa} a / f(2^n)) \\ &\quad + 2^{-\eta(n+1)} + (2^{n+1}/m + 2^{\alpha(n+1)})^{-\eta}. \end{aligned}$$

According to (5.3.2), the first term in the right hand side is equivalent to

$$2^{1+\kappa} \Gamma(\kappa) \kappa^2 K(f(2^n)^\kappa / ma^\kappa)$$

which is the general term of a convergent series, according to (5.3.18). The two remaining terms are also the general terms of convergent series so we get

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{A}_n) < +\infty,$$

and applying the Borel-Cantelli lemma we deduce

$$\limsup_{t \rightarrow +\infty} \frac{f(t)\mathcal{L}_X^{*,+}(t)}{t} \leq a.$$

Now, letting  $a$  go to 0 and combining with the finiteness of  $\sup_{]-\infty, 0[} \mathcal{L}_X(+\infty)$  (see (5.3.4) and (5.3.5)) we obtain (5.3.17).

Let us now assume that  $\int_1^{+\infty} \frac{(f(t))^\kappa}{t} dt = +\infty$  and prove that

$$\limsup_{t \rightarrow +\infty} \frac{f(t) \mathcal{L}_X^*(t)}{t} = +\infty. \quad (5.3.19)$$

According to Remark 5.1.19 the condition  $\int_1^{+\infty} \frac{(f(t))^\kappa}{t} dt = +\infty$  is equivalent to

$$\sum_{n=1}^{+\infty} (f(2^n))^\kappa = +\infty. \quad (5.3.20)$$

Let  $M > 0$ ,  $u_n := 2^n/m - 2^{\alpha n}$  and  $X^n := X(H(\sqrt{2}u_n) + \cdot)$ , the diffusion shifted by the hitting time of  $\sqrt{2}u_n$ . Note that from the Markov property for  $X$  at time  $H(\sqrt{2}u_n)$  and the stationarity of the increments of  $V$ ,  $X^n - \sqrt{2}u_n$  is equal in law to  $X$  under the annealed probability  $\mathbb{P}$ . We take  $n$  so large such that  $\sqrt{2}u_n < u_{n+1}$  and define the events

$$\begin{aligned} \mathcal{C}_n &:= \left\{ \frac{f(2^{n+1}) \mathcal{L}_{X^n}^*(\tau(X^n, u_{n+1}))}{2^{(n+1)/\kappa}} \geq M \right\}, \\ \mathcal{D}_n &:= \{\tau(X^n, u_{n+1}) < \tau(X^n, u_n)\}, \\ \mathcal{E}_n &:= \mathcal{C}_n \cap \mathcal{D}_n, \\ \mathcal{F}_n &:= \{\mathcal{L}_X^*(H(u_n)) \leq \mathcal{L}_X^*(2^n)\}. \end{aligned}$$

From the equality in law between  $X^n - \sqrt{2}u_n$  and  $X$  under  $\mathbb{P}$ , (5.3.4), (5.3.5), and (5.3.1), we have

$$\begin{aligned} \mathbb{P}(\mathcal{C}_n) &= \mathbb{P}\left(\mathcal{L}_X(H(u_{n+1} - \sqrt{2}u_n)) \geq 2^{(n+1)/\kappa} M / f(2^{n+1})\right) \\ &\geq \mathbb{P}\left(\mathcal{L}_X^{*,+}(H(u_{n+1} - \sqrt{2}u_n)) \geq 2^{(n+1)/\kappa} M / f(2^{n+1})\right) \\ &= \mathbb{P}\left(\mathcal{M}_2(u_{n+1} - \sqrt{2}u_n) \geq 2^{(n+1)/\kappa} M / f(2^{n+1})\right) \\ &= \mathbb{P}\left(\sup_{[0, u_{n+1} - \sqrt{2}u_n]} Z \geq 2^{(n+1)/\kappa} M / f(2^{n+1})\right) \\ &\geq 1 - e^{-\left(\frac{u_{n+1} - \sqrt{2}u_n}{Q - (u_{n+1} - \sqrt{2}u_n)^{7/8}}\right) \times n \left(\sup \xi > 2^{(n+1)/\kappa} M / f(2^{n+1})\right)} \\ &\quad - (u_{n+1} - \sqrt{2}u_n)^{-\eta}. \end{aligned} \quad (5.3.21)$$

According to (5.3.2) and the definition of  $u_n$ , the exponent in the right hand side is equivalent to

$$(2 - \sqrt{2})2^{\kappa-1} \Gamma(\kappa) \kappa^2 K(f(2^{n+1}))^\kappa / m M^\kappa.$$

Since  $f$  converges to 0 at infinity, the above is also an equivalent for the term  $1 - e^{-\dots}$  in the right hand side of (5.3.21). Then, combining with (5.3.20) and the fact that the other term is the general term of a covering series we get

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{C}_n) = +\infty. \quad (5.3.22)$$

Reasoning as in the proof of (5.3.14) we can prove that

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{D}_n}) < +\infty, \quad (5.3.23)$$

so

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{E}_n) \geq \sum_{n \geq 1} \mathbb{P}(\mathcal{C}_n) - \sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{D}_n}) = +\infty. \quad (5.3.24)$$

According to (5.3.7), and to the increase of  $\mathcal{L}_X^*$ , we also prove that

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{F}_n}) < +\infty. \quad (5.3.25)$$

The events  $\mathcal{E}_n$  are independent since for each  $n$ ,  $\mathcal{E}_n$  belongs to the  $\sigma$ -field  $\sigma(V(s) - V(u_n), u_n \leq s \leq u_{n+1}, X(t), H(\sqrt{2}u_n) \leq t \leq H(\sqrt{2}u_n) + \min(\tau(X^n, u_n), \tau(X^{n+1}, u_{n+1}))$ ). Combining this independence with (5.3.24) and the Borel-Cantelli Lemma we get that  $\mathbb{P}$ -almost surely, the event  $\mathcal{E}_n$  is realized infinitely many often. Combining with (5.3.25) and the Borel-Cantelli Lemma we get that  $\mathbb{P}$ -almost surely, the event  $\mathcal{C}_n \cap \mathcal{D}_n \cap \mathcal{F}_{n+1}$  is realized infinitely many often. Then, for  $n$  such that this event is realized we have,

$$\begin{aligned} f(2^{n+1})\mathcal{L}_X^*(2^{n+1})/2^{(n+1)/\kappa} &\geq f(2^{n+1})\mathcal{L}_X^*(H(u_{n+1}))/2^{(n+1)/\kappa} \\ &\geq f(2^{n+1})\mathcal{L}_{X^n}^*(\tau(X^n, u_{n+1}))/2^{(n+1)/\kappa} \\ &\geq M, \end{aligned}$$

so

$$\limsup_{t \rightarrow +\infty} \frac{f(t)\mathcal{L}_X^*(t)}{t} \geq M.$$

Now, letting  $M$  go to infinity we get (5.3.19) so Theorem 5.1.10 is proved.

## 5.4 Some lemmas

In this section, we justify some technical facts and lemmas for  $V$ ,  $V^\uparrow$  and the diffusion in  $V$ . Some of them are known or can be easily obtained from results of [74], and we give some details for their justification when it is necessary. Some of these facts are new, like the approximation of the contributions of the valleys to the traveled distance by an *iid* sequence.

### 5.4.1 Properties of $V$ , $V^\uparrow$ and $\hat{V}^\uparrow$

**Lemma 5.4.1.** (Lemma 5.4 of [74])

There are two positive constants  $c_1, c_2$  such that

$$\forall y, r > 0, P(\tau(V, ] - \infty, -y]) > r) \leq e^{c_1 y - c_2 r}.$$

**Lemma 5.4.2.** *There are positive constants  $c_1, c_2$  such that,*

$$\forall t, a > 0, P \left( \sup_{[t, +\infty[} V > -a \right) \leq e^{c_1 a - c_2 t} + e^{-\kappa a}.$$

*Démonstration.* Let us choose  $\gamma \in ]0, \kappa[$ , we have

$$\begin{aligned} P \left( \sup_{[t, +\infty[} V > -a \right) &\leq P(V(t) > -2a) + P \left( V(t) \leq -2a, \sup_{[0, +\infty[} V(t + \cdot) - V(t) > a \right) \\ &\leq P(e^{\gamma V(t)} > e^{-2\gamma a}) + P \left( \sup_{[0, +\infty[} V(t + \cdot) - V(t) > a \right) \\ &= e^{2\gamma a} E[e^{\gamma V(t)}] + e^{-\kappa a}, \end{aligned}$$

where we used Markov's inequality for the first term and the Markov property at time  $t$  for the second term, together with the fact that the supremum of  $V$  on  $[0, +\infty[$  follows an exponential distribution with parameter  $\kappa$ . Since  $E[e^{\gamma V(t)}] = e^{t\Psi_V(\gamma)}$  and  $\Psi_V(\gamma) < 0$  (because  $0 < \gamma < \kappa$ ), we get the result with  $c_1 := 2\gamma$  and  $c_2 := -\Psi_V(\gamma)$ .  $\square$

**Lemma 5.4.3.** *(Lemma 5.3 of [74])*

*There is a positive constant  $\mathcal{C}$  such that*

$$\mathbb{P} \left( \int_0^{+\infty} e^{V(u)} du \geq x \right) \underset{x \rightarrow +\infty}{\sim} \mathcal{C} x^{-\kappa}.$$

We now state some Lemmas about  $V^\uparrow$ . First, we recall how  $V^\uparrow$  and  $\hat{V}^\uparrow$  are defined.

$V$  being spectrally negative, the Markov family  $(V_x^\uparrow, x \geq 0)$  may be defined as in [8], Section VII.3. For any  $x \geq 0$ , the process  $V_x^\uparrow$  must be seen as  $V$  conditioned to stay positive and starting from  $x$ . We denote  $V^\uparrow$  for the process  $V_0^\uparrow$ . It is known that  $V_x^\uparrow$  converges in the Skorokhod space to  $V^\uparrow$  when  $x$  goes to 0. Also, as well as  $V$ ,  $V^\uparrow$  has no positive jumps so it reaches every positive level continuously.

Since  $V$  is spectrally negative and not the opposite of a subordinator, it is regular for  $]0, +\infty[$  (see [8], Theorem VII.1), so  $\hat{V}$  is for  $] -\infty, 0[$ . Moreover,  $\hat{V}$  drifts to  $+\infty$ . We can thus define the Markov family  $(\hat{V}_x^\uparrow, x \geq 0)$  as in Doney [31], Chapter 8. It can be seen from there that the processes such defined are Markov and have infinite life-time. If moreover  $V$  has unbounded variation then  $\hat{V}$  is regular for  $]0, +\infty[$ , and from Theorem 24 of [31], we have that  $\hat{V}_0^\uparrow$ , that we denote by  $\hat{V}^\uparrow$ , is well defined.

Here again, for any  $x \geq 0$ , the process  $\hat{V}_x^\uparrow$  must be seen as  $\hat{V}$  conditioned to stay positive and starting from  $x$ . Note that, since  $\hat{V}$  converges almost surely to infinity, for  $x > 0$ ,  $\hat{V}_x^\uparrow$  is only  $\hat{V}_x$  conditioned in the usual sense to remain positive.

**Lemma 5.4.4.** (Lemma 5.7 of [74])

There are two positive constants  $c_1, c_2$  such that, for all  $1 < a < b$ , we have

$$\mathbb{P} \left( \inf_{[0, +\infty[} V_b^\uparrow < a \right) \leq c_2 e^{-c_1(b-a)}.$$

**Lemma 5.4.5.** There are positive constants  $c_3, c_4, c_5, c_6$  such that,

$$\forall 0 \leq x < y, r > 0, \mathbb{P}(\tau(V_x^\uparrow, y) > r) \leq e^{c_3 y - c_4 r}, \quad (5.4.1)$$

$$\forall z, r > 0, \mathbb{P}(\mathcal{K}(V_z^\uparrow, z) > r) \leq e^{2c_3 z - c_4 r} + c_6 e^{-c_5 z}. \quad (5.4.2)$$

*Démonstration.* Let us fix  $0 \leq x < y$ . From the Markov property applied at  $\tau(V^\uparrow, x)$ , the hitting time of  $x$  by  $V^\uparrow$ , we have

$$\mathbb{P}(\tau(V_x^\uparrow, y) > r) = \mathbb{P}(\tau(V^\uparrow(\tau(V^\uparrow, x) + \cdot), y) > r) \leq \mathbb{P}(\tau(V^\uparrow, y) > r),$$

so (5.4.1) follows from Lemma 5.6 of [74]. For the second point, we have

$$\{\tau(V_z^\uparrow, 2z) \leq r\} \cap \left\{ \inf_{[\tau(V_z^\uparrow, 2z), +\infty[} V_z^\uparrow > z \right\} \subset \{\mathcal{K}(V_z^\uparrow, z) \leq r\},$$

so taking the complementary,

$$\begin{aligned} \mathbb{P}(\mathcal{K}(V_z^\uparrow, z) > r) &\leq \mathbb{P}(\tau(V_z^\uparrow, 2z) > r) + \mathbb{P} \left( \inf_{[\tau(V_z^\uparrow, 2z), +\infty[} V_z^\uparrow \leq z \right) \\ &\leq e^{2c_3 z - c_4 r} + \mathbb{P} \left( \inf_{[0, +\infty[} V_{2z}^\uparrow \leq z \right), \end{aligned}$$

where, for the first term, we used (5.4.1) with  $x = z$ ,  $y = 2z$  and, for the second term, we used the Markov property at time  $\tau(V_z^\uparrow, 2z)$ . Combining with Lemma 5.4.4 applied with  $a = z$ ,  $b = 2z$ , we get (5.4.2).  $\square$

**Lemma 5.4.6.** There is a positive constant  $c$  such that for  $t$  large enough,

$$\mathbb{P} \left( \int_{\tau(V^\uparrow, h_t/2)}^{+\infty} e^{-V^\uparrow(x)} dx \geq e^{-h_t/4} \right) \leq e^{-c h_t}.$$

*Démonstration.*

$$\begin{aligned} \int_{\tau(V^\uparrow, h_t/2)}^{\mathcal{K}(V^\uparrow, h_t/2)} e^{-V^\uparrow(x)} dx &\leq (\mathcal{K}(V^\uparrow, h_t/2) - \tau(V^\uparrow, h_t/2)) \times \sup_{[\tau(V^\uparrow, h_t/2), \mathcal{K}(V^\uparrow, h_t/2)]} e^{-V^\uparrow} \\ &\stackrel{\mathcal{L}}{=} \mathcal{K}(V_{h_t/2}^\uparrow, h_t/2) \times \sup_{[0, \mathcal{K}(V_{h_t/2}^\uparrow, h_t/2)]} e^{-V_{h_t/2}^\uparrow}, \end{aligned}$$

where we used the Markov property at time  $\tau(V^\uparrow, h_t/2)$  for the equality in law. We thus get

$$\begin{aligned} \mathbb{P} \left( \int_{\mathcal{K}(V^\uparrow, h_t/2)} e^{-V^\uparrow(x)} dx \geq e^{-h_t/4/2} \right) &\leq \mathbb{P} \left( \mathcal{K}(V_{h_t/2}^\uparrow, h_t/2) \geq e^{h_t/8/2} \right) \\ &\quad + \mathbb{P} \left( \inf_{[0, +\infty[} V_{h_t/2}^\uparrow \leq 3h_t/8 \right) \\ &\leq e^{c_3 h_t - c_4 e^{h_t/8/2}} + c_6 e^{-c_5 h_t/2} + c_2 e^{-c_1 h_t/8}, \end{aligned} \quad (5.4.3)$$

where, for the first term, we applied (5.4.2) with  $z = h_t/2$ ,  $r = e^{h_t/8/2}$  and, for the second term, we applied Lemma 5.4.4 with  $a = 3h_t/8$ ,  $b = h_t/2$ .

Then, according to Corollary 19.VI of [8] we have

$$\int_{\mathcal{K}(V^\uparrow, h_t/2)} e^{-V^\uparrow(x)} dx \stackrel{\mathcal{L}}{=} e^{-h_t/2} \int_0^{+\infty} e^{-V^\uparrow(x)} dx = e^{-h_t/2} I(V^\uparrow),$$

and, according to Theorem 1.3 of [72],  $I(V^\uparrow)$  admits some finite exponential moments, so in particular it has finite expectation. We thus get

$$\mathbb{P} \left( \int_{\mathcal{K}(V^\uparrow, h_t/2)} e^{-V^\uparrow(x)} dx \geq e^{-h_t/4/2} \right) = \mathbb{P} (I(V^\uparrow) \geq e^{h_t/4/2}) \leq 2e^{-h_t/4} \mathbb{E} [I(V^\uparrow)]. \quad (5.4.4)$$

The result follows from the combination of (5.4.3) and (5.4.4).  $\square$

The next fact gives the law of the bottom of the valleys in terms of the laws of  $V^\uparrow$  and  $\hat{V}^\uparrow$ . It is a combination of Propositions 3.2 and 3.6 of [74].

**Fact 5.4.7.** *Assume  $V$  has unbounded variation. For all  $i \geq 1$  let*

$$\begin{aligned} P^{(i)} &:= (V^{(i)}(m_i - x), 0 \leq x \leq m_i - \tau_i^-(h_t)) \\ \tilde{P}^{(i)} &:= (\tilde{V}^{(i)}(\tilde{m}_i - x), 0 \leq x \leq \tilde{m}_i - \tilde{\tau}_i^-(h_t)). \end{aligned}$$

For all  $i \geq 1$  we have

$$d_{VT} \left( \tilde{P}^{(i)}, P^{(2)} \right) \leq 2e^{-\delta \kappa h_t/3} \quad (5.4.5)$$

where  $d_{VT}$  is the total variation distance. Moreover, the law of  $P^{(2)}$  is absolutely continuous with respect to the law of the process  $(\hat{V}^\uparrow(x), 0 \leq x \leq \tau(\hat{V}^\uparrow, h_t+))$  and has density  $c_{h_t}/(1 - e^{-\kappa \hat{V}^\uparrow(\tau(\hat{V}^\uparrow, h_t+))})$  with respect to this law, where  $c_{h_t}$  is a constant increasing with  $h_t$  and converging to 1 when  $t$  (and hence  $h_t$ ) goes to infinity.

For all  $i \geq 1$ , the two processes  $(\tilde{V}^{(i)}(\tilde{m}_i - x), 0 \leq x \leq \tilde{m}_i - \tilde{\tau}_i^-(h_t)) = \tilde{P}^{(i)}$  and  $(\tilde{V}^{(i)}(\tilde{m}_i + x), 0 \leq x \leq \tilde{\tau}_i(h_t) - \tilde{m}_i)$  are independent and the second is equal in law to  $(V^\uparrow(x), 0 \leq x \leq \tau(V^\uparrow, h_t))$ .

Let us now recall a fact from [74] :

**Fact 5.4.8.** *Assume that the hypotheses of Theorems 5.1.4 and 5.1.9 are satisfied. Fix  $\epsilon$  small enough. There is a positive constant  $c$  (depending on  $\epsilon$ ) such that for all  $t$  large enough*

$$\forall j \geq 1, P \left( \int_{\tilde{L}_{j-1}}^{\tilde{\tau}_j^-(h_t/2)} e^{-\tilde{V}^{(j)}(u)} du \leq e^{-\epsilon h_t} \right) \geq 1 - e^{-\epsilon h_t}, \quad (5.4.6)$$

$$\forall j \geq 1, P \left( \int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} du \leq e^{-\epsilon h_t} \right) \geq 1 - e^{-\epsilon h_t}, \quad (5.4.7)$$

$$\forall j \geq 1, P \left( \sup_{u \in [\tilde{\tau}_j^-(h_t/2), \tilde{\tau}_j^+(h_t/2)]} |A^j(u)/A^j(\tilde{L}_j)| \leq e^{-h_t/3} \right) \geq 1 - e^{-\epsilon h_t}, \quad (5.4.8)$$

$$\forall j \geq 1, P \left( \sup_{u \in [\tilde{\tau}_j^-(h_t/2), \tilde{\tau}_j^+(h_t/2)]} |A^j(u)| \leq e^{5h_t/8} \right) \geq 1 - e^{-\epsilon h_t}, \quad (5.4.9)$$

where  $A^j$  is defined in Subsection 5.2.1.

*Démonstration.* (5.4.6), (5.4.7) and (5.4.8) are respectively Lemma 4.6, Lemma 4.7 and Lemma 4.8 (applied with  $\epsilon = 1/6$ ) from [74]. For (5.4.9), note that

$$P \left( \sup_{u \in [\tilde{\tau}_j^-(h_t/2), \tilde{\tau}_j^+(h_t/2)]} |A^j(u)| \leq e^{5h_t/8} \right) \leq P \left( |A^j(\tilde{\tau}_j^-(h_t/2))| \vee |A^j(\tilde{\tau}_j^+(h_t/2))| \leq e^{5h_t/8} \right).$$

Then,  $|A^j(\tilde{\tau}_j^-(h_t/2))|$  and  $|A^j(\tilde{\tau}_j^+(h_t/2))|$  can be bounded as in the proof of Lemma 4.8 of [74] which yields (5.4.9).  $\square$

## 5.4.2 Contribution of the valleys to the traveled distance

We need to approximate the contribution to the distance traveled by the diffusion in each valley. We have :

**Proposition 5.4.9.** *On an enlarged probability space, there is an iid sequence  $(D_j^t)_{j \geq 1}$ , independent from the sequence  $(e_j, S_j^t, R_j^t)_{j \geq 1}$  and such that for  $t$  large enough,*

$$\forall j \geq 1, \mathbb{P} \left( \left| (\tilde{L}_j - \tilde{L}_{j-1}) - D_j^t \right| \leq e^{-c_1 h_t} D_j^t \right) \geq 1 - e^{-c_2 h_t} \text{ and } D_j^t \stackrel{\mathcal{L}}{\sim} \mathcal{E}(e^{-\kappa h_t} q),$$

where  $q$  is the constant in Theorem 1.4 of [74] and  $c_1, c_2$  are positive constants.

*Démonstration.* From the definitions of valleys we have

$$\tilde{L}_j - \tilde{L}_{j-1} = (\tilde{L}_j^\# - \tilde{L}_{j-1}) + (\tilde{m}_j - \tilde{L}_j^\#) + (\tilde{\tau}_j(h_t) - \tilde{m}_j) + (\tilde{L}_j - \tilde{\tau}_j(h_t)). \quad (5.4.10)$$

Recall the definition of  $m^*(h_t)$  from [74] :

$$\tau^*(h_t) := \inf \left\{ u \geq 0, V(u) - \inf_{[0,u]} V = h_t \right\}, \quad m^*(h_t) := \inf \left\{ u \geq 0, V(u) = \inf_{[0,\tau^*(h_t)]} V \right\}.$$

From the Markov property at the stopping times  $\tilde{L}_{j-1}$ ,  $\tilde{L}_j^\#$  and  $\tilde{\tau}_j(h_t)$ , and Fact 5.4.7 we get that the terms in the right hand side of (5.4.10) are respectively equal in law to  $\tau(V, ] - \infty, -e^{(1-\delta)\kappa h_t}]$ ,  $m^*(h_t)$ ,  $\tau(V^\uparrow, h_t)$  and  $\tau(V, ] - \infty, -h_t/2]$ . For the first and fourth term, applying Lemma 5.4.1 with  $y = e^{(1-\delta)\kappa h_t}$  and  $r = e^{(1-\delta/2)\kappa h_t}/2$  we get for  $t$  large enough,

$$\forall j \geq 1, \mathbb{P} \left( (\tilde{L}_j^\# - \tilde{L}_{j-1}) + (\tilde{L}_j - \tilde{\tau}_j(h_t)) > e^{(1-\delta/2)\kappa h_t} \right) \leq e^{-h_t}. \quad (5.4.11)$$

For the third term, applying (5.4.1) with  $x = 0$ ,  $y = h_t$  and  $r = e^{(1-\delta/2)\kappa h_t}$  we get for  $t$  large enough,

$$\forall j \geq 1, \mathbb{P} \left( (\tilde{\tau}_j(h_t) - \tilde{m}_j) > e^{(1-\delta/2)\kappa h_t} \right) \leq e^{-h_t}. \quad (5.4.12)$$

The main term is the second one, its law is given by Lemma 3.9 of [74]. Before studying it, let us recall some of the notations used in [74].

Let  $\mathcal{F}$  denote the space of excursions, that is, càd-làg functions from  $[0, +\infty[$  to  $\mathbb{R}$ , starting at zero and killed at the first positive instant when they reach 0 (this instant can possibly be infinite). For  $\xi \in \mathcal{F}$ , let us denote  $\zeta(\xi) := \inf\{s > 0, \xi(s) = 0\}$  for the length of the excursion  $\xi$ . For  $h > 0$ , let  $\mathcal{F}_{h,-}$  and  $\mathcal{F}_{h,+}$  denote respectively the set of excursions which height is strictly less than  $h$  and the set of excursions higher than  $h$  :

$$\mathcal{F}_{h,-} := \left\{ \xi \in \mathcal{F}, \sup_{[0,\zeta]} \xi < h \right\}, \quad \mathcal{F}_{h,+} := \left\{ \xi \in \mathcal{F}, \sup_{[0,\zeta]} \xi \geq h \right\}.$$

Let  $\mathcal{N}$  be the measure defined on  $\mathcal{F}$  as in Subsection 3.5 of [74]. Let  $S^{h,-}$  and  $S^{h,+}$  be two independent pure jumps subordinators with Lévy measures respectively  $\zeta\mathcal{N}(\mathcal{F}_{h,-} \cap \cdot)$  and  $\zeta\mathcal{N}(\mathcal{F}_{h,+} \cap \cdot)$ , the image measures of respectively  $\mathcal{N}(\mathcal{F}_{h,-} \cap \cdot)$  and  $\mathcal{N}(\mathcal{F}_{h,+} \cap \cdot)$  by  $\zeta$ . The sum of these measures equals  $\zeta\mathcal{N}$  so  $S := S^{h,-} + S^{h,+}$  is a pure jumps subordinator with Lévy measure  $\zeta\mathcal{N}$ . Let also  $T_h$  be an exponential random variable with parameter  $\mathcal{N}(\mathcal{F}_{h,+})$ , independent from  $S^{h,-}$  and  $S^{h,+}$ . According to Lemma 3.9 of [74] we have

$$\forall j \geq 1, \tilde{m}_j - \tilde{L}_j^\# \stackrel{\mathcal{L}}{=} S^{h_t,-}(T_{h_t}).$$



Recall that the sequence  $(\tilde{m}_j - \tilde{L}_j^\#)_{j \geq 1}$  is *iid* because of Remark 5.2.2. Therefore, on an enlarged probability space, there is an *iid* sequence  $(S_j^{h_t, -}, T_{h_t, j})_{j \geq 1}$  such that for all  $j \geq 1$ ,

$$S_j^{h_t, -} \perp\!\!\!\perp T_{h_t, j}, \quad S_j^{h_t, -} \stackrel{\mathcal{L}}{=} S^{h_t, -}, \quad T_{h_t, j} \stackrel{\mathcal{L}}{=} T_{h_t} \quad \text{and} \quad S_j^{h_t, -}(T_{h_t, j}) = \tilde{m}_j - \tilde{L}_j^\#. \quad (5.4.13)$$

Let  $d := \mathcal{N}(\mathcal{F}_{h_t, +})T_{h_t}$  and more generally  $d_j := \mathcal{N}(\mathcal{F}_{h_t, +})T_{h_t, j}$ . Then,  $d_j \stackrel{\mathcal{L}}{\sim} \mathcal{E}(1)$  and the sequence  $(d_j)_{j \geq 1}$  is *iid*. For large  $t$ , it is natural to approximate  $S^{h_t, -}(T_{h_t})$  by a multiple of  $d$ , for this we write

$$S^{h_t, -}(T_{h_t}) - e^{\kappa h_t} d/q = (S^{h_t, -}(T_{h_t}) - S(T_{h_t})) + (S(T_{h_t}) - \mathbb{E}[S(1)]T_{h_t}) + (\mathbb{E}[S(1)]T_{h_t} - e^{\kappa h_t} d/q). \quad (5.4.14)$$

For the first term, using the expression of  $S$  in terms of  $S^{h_t, +}$  and  $S^{h_t, -}$ , the independence between  $S^{h_t, +}$  and  $T_{h_t}$ , the definition of  $S^{h_t, +}$ , Cauchy-Schwarz's inequality and the definition of  $S$ , we have

$$\begin{aligned} \mathbb{E} [|S^{h_t, -}(T_{h_t}) - S(T_{h_t})|] &= \mathbb{E} [S^{h_t, +}(T_{h_t})] = \frac{1}{\mathcal{N}(\mathcal{F}_{h_t, +})} \mathbb{E} [S^{h_t, +}(1)] \\ &= \frac{1}{\mathcal{N}(\mathcal{F}_{h_t, +})} \int_{\mathcal{F}_{h_t, +}} \zeta(\xi) \mathcal{N}(d\xi) \leq \frac{1}{\sqrt{\mathcal{N}(\mathcal{F}_{h_t, +})}} \sqrt{\int \zeta^2(\xi) \mathcal{N}(d\xi)}, \\ &= \frac{1}{\sqrt{\mathcal{N}(\mathcal{F}_{h_t, +})}} \sqrt{\text{Var}(S(1))}. \end{aligned} \quad (5.4.15)$$

$\text{Var}(S(1))$  is indeed finite thanks to Lemma 3.10 of [74]. For the second term in the right hand side of (5.4.14), using Cauchy-Schwarz's inequality and the independence between  $S$  and  $T_{h_t}$ ,

$$\mathbb{E} [|S(T_{h_t}) - \mathbb{E}[S(1)]T_{h_t}|] \leq \sqrt{\mathbb{E} [(S(T_{h_t}) - \mathbb{E}[S(1)]T_{h_t})^2]} = \frac{1}{\sqrt{\mathcal{N}(\mathcal{F}_{h_t, +})}} \sqrt{\text{Var}(S(1))}. \quad (5.4.16)$$

For the third term in the right hand side of (5.4.14), using the definition of  $d$ ,

$$\mathbb{E}[S(1)]T_{h_t} - e^{\kappa h_t} d/q = (\mathbb{E}[S(1)]/\mathcal{N}(\mathcal{F}_{h_t, +}) - e^{\kappa h_t}/q)d.$$

Then, recall from Proposition 3.12 and Lemma 3.11 of [74] that

$$q := \mathcal{N}(\mathcal{F}_{1, +}) \times (e^\kappa - \mathbb{E}[e^{\kappa V_1(\tau(V_1, ]-\infty, 0])}]) / \mathbb{E}[S(1)],$$

and

$$\mathcal{N}(\mathcal{F}_{h_t, +}) = e^{-\kappa h_t} \mathcal{N}(\mathcal{F}_{1, +}) \times (e^\kappa - \mathbb{E}[e^{\kappa V_1(\tau(V_1, ]-\infty, 0])}]) + \mathcal{O}_{t \rightarrow +\infty}(e^{-2\kappa h_t}).$$

We thus get

$$\text{a.s. } \mathbb{E}[S(1)]T_{h_t} - e^{\kappa h_t} d/q = d \times \mathcal{O}_{t \rightarrow +\infty}(e^{-\kappa h_t}). \quad (5.4.17)$$

Let us define  $D^t := e^{\kappa h_t} d/q$  and more generally  $D_j^t := e^{\kappa h_t} d_j/q$ . We then have indeed  $D_j^t \stackrel{\mathcal{L}}{\sim} \mathcal{E}(e^{-\kappa h_t} q)$  and the fact that the sequence sequence  $(D_j^t)_{j \geq 1}$  is *iid.* (5.4.17) becomes

$$\text{a.s. } |\mathbb{E}[S(1)]T_{h_t} - D^t| \leq D^t \times C e^{-2\kappa h_t}, \quad (5.4.18)$$

where  $C$  is a positive constant. Using Markov's inequality, (5.4.15) and (5.4.16), and the asymptotic of  $\mathcal{N}(\mathcal{F}_{h_t, +})$  given by Lemma 3.11 of [74], we get the existence of a positive constant  $C$  such that for  $t$  large enough,

$$\mathbb{P}(|S^{h_t, -}(T_{h_t}) - S(T_{h_t})| + |S(T_{h_t}) - \mathbb{E}[S(1)]T_{h_t}| \geq e^{6\kappa h_t/10}) \leq C e^{-\kappa h_t/10}.$$

$$\mathbb{P}(D^t \leq e^{9\kappa h_t/10}) = \mathbb{P}(d \leq q e^{-\kappa h_t/10}) \leq q e^{-\kappa h_t/10}.$$

As a consequence there is a positive constant  $C$  such that for  $h_t$  large enough,

$$\mathbb{P}(|S^{h_t, -}(T_{h_t}) - S(T_{h_t})| + |S(T_{h_t}) - \mathbb{E}[S(1)]T_{h_t}| \geq e^{-3\kappa h_t/10} D^t) \leq C e^{-\kappa h_t/10}.$$

Combining with (5.4.18) and putting into (5.4.14) we get

$$\mathbb{P}(|S^{h_t, -}(T_{h_t}) - D^t| \geq e^{-2\kappa h_t/10} D^t) \leq C e^{-\kappa h_t/10}.$$

Combining with (5.4.13) we get

$$\forall j \geq 1, \mathbb{P}\left(|\tilde{m}_j - \tilde{L}_j^\# - D_j^t| \geq e^{-2\kappa h_t/10} D_j^t\right) \leq C e^{-\kappa h_t/10}. \quad (5.4.19)$$

Then, we have  $\mathbb{P}(D_j^t \leq e^{(1-\delta/2)\kappa h_t}) = \mathbb{P}(d_j \leq q e^{-\delta\kappa h_t/2}) \leq q e^{-\delta\kappa h_t/2}$  so putting into (5.4.11) and (5.4.12) we get for all  $j \geq 1$

$$\mathbb{P}\left((\tilde{L}_j^\# - \tilde{L}_{j-1}) + (\tilde{\tau}_j(h_t) - \tilde{m}_j) + (\tilde{L}_j - \tilde{\tau}_j(h_t)) > 2e^{-\delta\kappa h_t/2} D_j^t\right) \leq 2e^{-h_t} + q e^{-\delta\kappa h_t/2}. \quad (5.4.20)$$

Combining (5.4.19) and (5.4.20), and putting into (5.4.10) we get the result for  $t$  large enough.  $\square$

### 5.4.3 Proof of some facts and lemmas

This subsection is devoted to the justification of the facts stated in Subsection 5.2.1, which mainly come from [74]. As these results are included in Section 5.2, we prove them under the hypotheses of Theorems 5.1.4 and 5.1.9 :  $0 < \kappa < 1$ ,  $V$  has unbounded variation and there exists  $p > 1$  such that  $V(1) \in L^p$ .

In the facts and lemmas considered here, the value of the constant  $c$  is not important (as long as it is positive) so we allow it to decrease from line to line and it is implicit that the estimates are true for  $t$  large enough.

*Démonstration.* of Lemma 5.2.1

(5.2.2) is only Lemma 4.16 of [74]. For (5.2.3), note that  $\{N_t \leq \tilde{n}_t\} = \{H(m_{\tilde{n}_t+1}) > t\}$  and  $H(\tilde{m}_{\tilde{n}_t+1}) = H(m_{\tilde{n}_t+1})$  on  $\{V \in \mathcal{V}_t\}$  (since  $\tilde{n}_t < n_t$ ). Let  $t$  be large enough so that  $(1 - 2/\log(h_t))(1 + e^{-cht})^{-1} \geq 1/2$ . Using the definitions of  $\mathcal{E}_t^5$  and  $\mathcal{E}_t^7$  (which is possible here since  $\tilde{n}_t < n_t$ ) we get that  $\mathbb{P}(\{N_t \leq \tilde{n}_t\} \cap \{V \in \mathcal{V}_t\} \cap \mathcal{E}_t^5 \cap \mathcal{E}_t^7)$  is less than

$$\begin{aligned} \mathbb{P}(H(\tilde{m}_{\tilde{n}_t+1}) > t, \mathcal{E}_t^5, \mathcal{E}_t^7) &\leq \mathbb{P}\left(\sum_{i=1}^{\tilde{n}_t} e_i S_i^t R_i^t \geq t/2\right) \\ &\leq \mathbb{P}\left(\sup_{1 \leq i \leq \tilde{n}_t} e_i S_i^t R_i^t \geq t/2\tilde{n}_t\right) = 1 - [\mathbb{P}(e_1 S_1^t R_1^t / t < 1/2\tilde{n}_t)]^{\tilde{n}_t} \\ &= 1 - [1 - \mathbb{P}(e_1 S_1^t R_1^t / t \geq 1/2\tilde{n}_t)]^{\tilde{n}_t}. \end{aligned}$$

According to (5.2.17) applied with some  $\eta \in ]0, (1 - \rho)/3[$  we have that

$$\mathbb{P}(e_1 S_1^t R_1^t / t \geq 1/2\tilde{n}_t) \underset{t \rightarrow +\infty}{\sim} 2^\kappa \mathcal{C}' \mathbb{E}[\mathcal{R}^\kappa] \tilde{n}_t^\kappa e^{-\kappa\phi(t)} = 2^\kappa \mathcal{C}' \mathbb{E}[\mathcal{R}^\kappa] e^{\kappa(\rho-1)\phi(t)}.$$

Since  $\rho < 1$  the later converges to 0 so

$$\begin{aligned} [1 - \mathbb{P}(e_1 S_1^t R_1^t / t \geq 1/2\tilde{n}_t)]^{\tilde{n}_t} &\underset{t \rightarrow +\infty}{\approx} \exp(-2^\kappa \mathcal{C}' \mathbb{E}[\mathcal{R}^\kappa] \tilde{n}_t e^{\kappa(\rho-1)\phi(t)}) \\ &= \exp(-2^\kappa \mathcal{C}' \mathbb{E}[\mathcal{R}^\kappa] e^{(\rho+\kappa(\rho-1))\phi(t)}). \end{aligned}$$

Let  $\rho' := -(\rho + \kappa(\rho - 1))$ .  $\rho'$  is positive thanks to the hypothesis  $\rho \in ]0, \kappa/(1 + \kappa)[$  so  $e^{(\rho+\kappa(\rho-1))\phi(t)}$  converges to 0 and we deduce that for  $t$  large enough,

$$\mathbb{P}(\{N_t \leq \tilde{n}_t\} \cap \{V \in \mathcal{V}_t\} \cap \mathcal{E}_t^5 \cap \mathcal{E}_t^7) \leq 2\mathcal{C}' \mathbb{E}[\mathcal{R}^\kappa] e^{-\rho'\phi(t)}.$$

Combining with the bounds for  $\mathbb{P}(V \notin \mathcal{V}_t)$  and  $\mathbb{P}(\overline{\mathcal{E}}_t^5) + \mathbb{P}(\overline{\mathcal{E}}_t^7)$  given by respectively Fact 5.2.3 and (5.2.4) we get (5.2.3).  $\square$

Now, let us recall a fact about brownian local time :

**Fact 5.4.10.** *There is a positive constant  $c$  such that for all  $t$  large enough*

$$\mathbb{P}\left(\sup_{|y| \leq e^{-ht/3}} |\mathcal{L}_B(\tau(B, 1), y) - \mathcal{L}_B(\tau(B, 1), 0)| > e^{-ht/9} \mathcal{L}_B(\tau(B, 1), 0)\right) \leq e^{-cht}, \quad (5.4.21)$$

$$\mathbb{P}\left(\sup_{|y| \leq e^{-ht/8}} |\mathcal{L}_B(\sigma_B(1, 0), y) - 1| \geq e^{-ht/20}\right) \leq e^{-ht}, \quad (5.4.22)$$

$$\mathbb{P}\left(\sup_{y \in \mathbb{R}} \mathcal{L}_B(\sigma_B(1, 0), y) \geq u\right) \leq 2/u, . \quad (5.4.23)$$

*Démonstration.* (5.4.21) is (7.11) of [4] applied with  $\delta = e^{-h_t/3}$ ,  $\epsilon = e^{-h_t/9}$ . (5.4.22) comes from the second Ray-Knight Theorem combined with estimate (7.15) of [4] applied with  $u = e^{-h_t/20}$ ,  $v = e^{-h_t/8}$ . (5.4.23) comes from the second Ray-Knight Theorem combined with estimate (7.16) of [4]. □

*Démonstration.* of Fact 5.2.5

According to the combination of Lemmas 5.20 and 5.21 of [74], there is a positive constant  $c$  such that  $\mathbb{P}(\overline{\mathcal{E}}_t^1) \leq e^{-ch_t}$ . Since  $h_t$  is ultimately greater than  $\phi(t)$  we get  $\mathbb{P}(\overline{\mathcal{E}}_t^1) \leq e^{-c\phi(t)}$ .

$\mathbb{P}(\overline{\mathcal{E}}_t^2) \leq e^{-c\phi(t)}$  and  $\mathbb{P}(\overline{\mathcal{E}}_t^3) \leq e^{-c\phi(t)}$  come from Fact 4.4 of [74] (respectively the second and third point).

For  $\mathbb{P}(\overline{\mathcal{E}}_t^4)$ . Let us fix  $j \geq 1$  and use the Markov property at  $H(\tilde{m}_j)$  and the expression (5.3.3) of the local time within an hitting time. We get the existence of a Brownian motion  $(B(s), s \geq 0)$ , independent from  $\tilde{V}^{(j)}$ , such that

$$\forall y \in \mathcal{D}_j, \mathcal{L}_X(H(\tilde{L}_j), y) - \mathcal{L}_X(H(\tilde{m}_j), y) = e^{-\tilde{V}^{(j)}(y)} \mathcal{L}_B(\tau(B, A^j(\tilde{L}_j)), A^j(y)),$$

where  $A^j$  is defined in Subsection 5.2.1. Let  $\tilde{B} := B^j((A^j(\tilde{L}_j))^2)/A^j(\tilde{L}_j)$ . By the scaling property for brownian motion, we have that, conditionally to  $V$ ,  $\tilde{B}$  is a standard Brownian motion. We have

$$\forall y \in \mathcal{D}_j, \mathcal{L}_X(H(\tilde{L}_j), y) - \mathcal{L}_X(H(\tilde{m}_j), y) = e^{-\tilde{V}^{(j)}(y)} A^j(\tilde{L}_j) \mathcal{L}_{\tilde{B}}(\tau(\tilde{B}, 1), A^j(y)/A^j(\tilde{L}_j)).$$

From the definition of  $\mathcal{D}_j$  we have  $\mathcal{D}_j \subset [\tilde{\tau}_j^-(h_t/2), \tilde{\tau}_j^+(h_t/2)]$  so, thanks to (5.4.8) we have

$$\mathbb{P} \left( \sup_{y \in \mathcal{D}_j} \left| A^j(y)/A^j(\tilde{L}_j) \right| \leq e^{-h_t/3} \right) \geq 1 - e^{-ch_t}.$$

Combining with estimate (5.4.21) we get

$$\mathbb{P} \left( \sup_{y \in \mathcal{D}_j} \mathcal{L}_{\tilde{B}}(\tau(\tilde{B}, 1), A^j(y)/A^j(\tilde{L}_j)) > (1 + e^{-h_t/9}) \mathcal{L}_{\tilde{B}}(\tau(\tilde{B}, 1), 0) \right) \leq e^{-ch_t}$$

so with probability greater than  $1 - e^{-ch_t}$  we have

$$\begin{aligned} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) &\leq \sup_{y \in \mathcal{D}_j} \left( \mathcal{L}_X(H(\tilde{L}_j), y) - \mathcal{L}_X(H(\tilde{m}_j), y) \right) \\ &\leq (1 + e^{-h_t/9}) A^j(\tilde{L}_j) \mathcal{L}_{\tilde{B}}(\tau(\tilde{B}, 1), 0) \\ &= (1 + e^{-h_t/9}) \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j). \end{aligned}$$

Since  $\tilde{c} > 0$  has been chosen "small enough" in the Introduction, we can assume  $\tilde{c} \leq 1/9$  so  $\mathbb{P}(\overline{\mathcal{E}}_t^4) \leq e^{-ch_t} \leq e^{-c\phi(t)}$  follows from the above (and the fact that  $h_t$  is ultimately greater than  $\phi(t)$ ).

$\mathbb{P}(\overline{\mathcal{E}}_t^5) \leq e^{-c\phi(t)}$  comes from Fact 4.3 of [74].

$\tilde{c} > 0$  has been fixed "small enough" in the Introduction. We can assume that it was chosen so small such that Proposition 4.5 of [74] apply with  $\tilde{c}$  instead the constant  $\epsilon/7$  there.  $\mathbb{P}(\overline{\mathcal{E}}_t^6) \leq e^{-ch_t} \leq e^{-c\phi(t)}$  and  $\mathbb{P}(\overline{\mathcal{E}}_t^7) \leq e^{-ch_t} \leq e^{-c\phi(t)}$  thus come from Proposition 4.5 of [74] (and the fact that  $h_t$  is ultimately greater than  $\phi(t)$ ).

The second point of the fact is Lemma 4.11 of [74]. □

*Démonstration.* of Lemma 5.2.6

According to (5.2.18) and Chernoff inequality we have for any  $j \geq 1$

$$P(R_j^t \geq e^{h_t/8}) \leq e^{-\lambda_0 e^{h_t/8}/2}. \quad (5.4.24)$$

We have

$$\begin{aligned} P(R_j^t \leq e^{-\epsilon h_t/4}) &\leq P(R_j^t \leq e^{-h_t^{1/3}}) \leq 2P(I(V^\uparrow) \leq 2e^{-h_t^{1/3}}) + 2e^{-\delta\kappa h_t/3} \\ &\leq 2e^{-K_0 e^{h_t^{1/3}}/2} + 2e^{-\delta\kappa h_t/3}, \end{aligned} \quad (5.4.25)$$

where, for the last two inequalities, we used Lemma 5.2.13 (applied with  $z_t = e^{h_t^{1/3}}$ ,  $a = 2$ ) and (5.1.7), and where  $K_0$  is the constant in (5.1.7). The term  $2e^{-\delta\kappa h_t/3}$  is only necessary in the case  $V = W_\kappa$  (because of (5.2.29)). Combining (5.4.24) and (5.4.25) we get that (5.2.7) is satisfied with probability at least  $1 - e^{-ch_t}$ .

According to Fact 5.4.8 we have that (5.2.8), (5.2.9), (5.2.10) and (5.2.11) are satisfied with probability at least  $1 - e^{-ch_t}$ .

According to (5.2.4) we have  $\mathbb{P}(\cup_{i=1}^7 \overline{\mathcal{E}}_t^i) \leq e^{-L\phi(t)}$ . Then,

$$e^{-L\phi(t)/2} P\left(P^V(\cup_{i=1}^7 \overline{\mathcal{E}}_t^i) > e^{-L\phi(t)/2}\right) \leq E\left[P^V(\cup_{i=1}^7 \overline{\mathcal{E}}_t^i)\right] = \mathbb{P}\left(\cup_{i=1}^7 \overline{\mathcal{E}}_t^i\right) \leq e^{-L\phi(t)},$$

We thus deduce that (5.2.12) is satisfied with probability at least  $1 - e^{-L\phi(t)/2}$ . Combing all this with Fact 5.2.3, we get Lemma 5.2.6 for a suitably chosen constant  $c$ . □

*Démonstration.* of Fact 5.2.7

The ideas are very similar to the ones used for the proof of Lemma 5.3 of [4]. Since we are in the more general context of a Lévy potential and since we do not prove exactly the same thing (here, we work conditionally to the environment), we give the details.

Recall that  $\epsilon > 0$  has been fixed in the definition of  $\mathcal{G}_t$ .  $\tilde{c} > 0$  has been fixed "small enough" in the Introduction (and we already fixed some constraints about how small it must be in the proof of Fact 5.2.5). We can assume further that it was chosen so small such that  $\tilde{c} < \min(1/20, \epsilon/4)$ . Let us fix an environment  $v \in \mathcal{G}_t$ ,  $z \in [0, 1 - 4/\log(h_t)]$  and  $k \leq n_t$ . We put  $\tilde{x}_t := (1 + e^{-\tilde{c}h_t})^{-1}x_t$ ,  $Z := (1 - z)/(1 - e^{-\tilde{c}h_t})R_k^t$ .

To study  $\mathcal{E}_t^8(v, k, z)$ , we look at its intersections with the events  $\{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\}$  and  $\{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k)\}$ . We have

$$\begin{aligned} & \mathcal{E}_t^8(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\} \\ &= \left\{ tZ \leq t\tilde{x}_t, \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \cdot) \geq tx_t, \sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\}. \end{aligned} \quad (5.4.26)$$

On the above event,  $\sigma_{X_{\tilde{m}_k}}(tZ, \tilde{m}_k)$  is finite and the diffusion stays in  $[\tilde{L}_{k-1}, \tilde{L}_k]$  until this time.  $\sigma_{X_{\tilde{m}_k}}(tZ, \tilde{m}_k)$  is thus equal to

$$I := \int_{\tilde{L}_{k-1}}^{\tilde{L}_k} e^{-\tilde{v}^{(k)}(x)} \mathcal{L}_B(\sigma_B(tZ, 0), A^k(x)) dx, \quad (5.4.27)$$

where  $B$  is the brownian motion driving the diffusion  $X_{\tilde{m}_k}$  and  $A^k$  is defined in Subsection 5.2.1. Let  $\tilde{B} := B((tZ)^2)/tZ$  and  $\tilde{A}^k(\cdot) := A^k(\cdot)/tZ$ . From the scaling property  $\tilde{B}$  is still a brownian motion. We can write

$$I = tZ \int_{\tilde{L}_{k-1}}^{\tilde{L}_k} e^{-\tilde{v}^{(k)}(x)} \mathcal{L}_{\tilde{B}}(\sigma_{\tilde{B}}(1, 0), \tilde{A}^k(x)) dx. \quad (5.4.28)$$

Since  $v \in \mathcal{G}_t$  it satisfies (5.2.7). Recall also that  $t \geq e^{h_t}$ . For  $t$  large enough so that  $4e^{-h_t/8}/(1 - e^{-\tilde{c}h_t}) \log(h_t) \geq e^{-h_t/4}$  we thus have  $tZ \geq e^{3h_t/4}$  for any choice of  $z \in [0, 1 - 4/\log(h_t)]$ . Combining this with the fact that  $v$  satisfies (5.2.8) and (5.2.9) we get

$$\sup_{[\tilde{\tau}_k^-(h_t/2), \tilde{\tau}_k^+(h_t/2)]} \left| \tilde{A}^k(\cdot) \right| \leq (|A^k(\tilde{\tau}_k^-(h_t/2))| \vee A^k(\tilde{\tau}_k^+(h_t/2))) / tZ \leq e^{-h_t/8}.$$

Combining with (5.4.22) we get

$$P^v \left( \sup_{x \in [\tilde{\tau}_k^-(h_t/2), \tilde{\tau}_k^+(h_t/2)]} \left| \mathcal{L}_{\tilde{B}}(\sigma_{\tilde{B}}(1, 0), \tilde{A}^k(x)) - 1 \right| \geq e^{-h_t/20} \right) \leq e^{-h_t}, \quad (5.4.29)$$

and we deduce that

$$P^v \left( \left| \int_{\tilde{\tau}_k^-(h_t/2)}^{\tilde{\tau}_k^+(h_t/2)} e^{-\tilde{v}^{(k)}(x)} \mathcal{L}_{\tilde{B}}(\sigma_{\tilde{B}}(1, 0), \tilde{A}^k(x)) dx - R_k^t \right| \leq e^{-h_t/20} R_k^t \right) \geq 1 - e^{-h_t}. \quad (5.4.30)$$

Since  $v \in \mathcal{G}_t$  it satisfies (5.2.10) and (5.2.11). Combining with (5.4.23) (applied with

$u = e^{\epsilon h_t/2}$ ) and the lower bound for  $R_k^t$  in (5.2.7) we get

$$P^v \left( \int_{\tilde{\tau}_k^+(h_t/2)}^{\tilde{L}_k} e^{-\tilde{v}^{(k)}(x)} \mathcal{L}_{\tilde{B}}(\sigma_{\tilde{B}}(1, 0), A^k(x)) dx \leq e^{-\epsilon h_t/2} \leq e^{-\epsilon h_t/4} R_k^t \right) \geq 1 - e^{-\epsilon h_t}, \quad (5.4.31)$$

$$P^v \left( \int_{\tilde{L}_{k-1}}^{\tilde{\tau}_k^-(h_t/2)} e^{-\tilde{v}^{(k)}(x)} \mathcal{L}_{\tilde{B}}(\sigma_{\tilde{B}}(1, 0), A^k(x)) dx \leq e^{-\epsilon h_t/2} \leq e^{-\epsilon h_t/4} R_k^t \right) \geq 1 - e^{-\epsilon h_t}. \quad (5.4.32)$$

Now putting (5.4.30), (5.4.31) and (5.4.32) into (5.4.28) (and combining with the fact that  $\tilde{c} < \min(1/20, \epsilon/4)$ ) we get

$$P^v \left( \mathcal{E}_{neg}^1 := \{|I - tZR_k^t| > e^{-\tilde{c}h_t} tZR_k^t\} \right) \leq e^{-\epsilon h_t}.$$

Combining with the definition of  $Z$  we see that  $t(1 - z) \leq I$  on the complementary of  $\mathcal{E}_{neg}^2$ . Combining this with (5.4.26) we get that  $\mathcal{E}_t^8(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\}$  is included into

$$\left\{ tZ \leq t\tilde{x}_t, \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(\sigma_{X_{\tilde{m}_k}}(tZ, \tilde{m}_k), \cdot) \geq tx_t, \sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\} \cup \mathcal{E}_{neg}^1. \quad (5.4.33)$$

On the main event,  $\sigma_{X_{\tilde{m}_k}}(tZ, \tilde{m}_k)$  is finite. On this event,  $\mathcal{L}_{X_{\tilde{m}_k}}(\sigma_{X_{\tilde{m}_k}}(tZ, \tilde{m}_k), y)$  is thus equal to

$$L(y) := e^{-\tilde{v}^{(k)}(y)} \mathcal{L}_B(\sigma_B(tZ, 0), A^k(y)).$$

Here again, for  $\tilde{B} := B((tZ)^2)/tZ$  and  $\tilde{A}^k(\cdot) := A^k(\cdot)/tZ$ . We have

$$L(y) = tZ e^{-\tilde{v}^{(k)}(y)} \mathcal{L}_{\tilde{B}}(\sigma_{\tilde{B}}(1, 0), \tilde{A}^k(y)).$$

Since  $\mathcal{D}_k \subset [\tilde{\tau}_k^-(h_t/2), \tilde{\tau}_k^+(h_t/2)]$  we can apply (5.4.29) (together with  $\tilde{c} < 1/20$ ) and get

$$P^v \left( \mathcal{E}_{neg}^2 := \{\exists y \in \mathcal{D}_k, L(y) \geq (1 + e^{-\tilde{c}h_t})tZ\} \right) \leq e^{-\epsilon h_t}. \quad (5.4.34)$$

On the big event in (5.4.33) we thus have both  $(1 + e^{-\tilde{c}h_t})tZ > tx_t$  and  $tZ \leq t\tilde{x}_t$ , (expect possibly on the event  $\mathcal{E}_{neg}^2$ ). Since these two inequalities are not compatible we get

$$P^v \left( \mathcal{E}_t^8(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\} \right) \leq P^v \left( \mathcal{E}_{neg}^1 \right) + P^v \left( \mathcal{E}_{neg}^2 \right) \leq 2e^{-\epsilon h_t}. \quad (5.4.35)$$

We now study the case where  $\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k)$ . First, we have

$$\begin{aligned} & \mathcal{E}_t^8(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k)\} \\ &= \left\{ tZ \leq t\tilde{x}_t, \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \cdot) \geq tx_t, H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \wedge \sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) \right\} \\ &\subset \left\{ \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \cdot) \geq tx_t, \mathcal{L}_{X_{\tilde{m}_k}}(H_{X_{\tilde{m}_k}}(\tilde{L}_k), \tilde{m}_k) < t\tilde{x}_t \right\} \\ &\subset \left\{ \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(H_{X_{\tilde{m}_k}}(\tilde{L}_k), \cdot) \geq tx_t, \mathcal{L}_{X_{\tilde{m}_k}}(H_{X_{\tilde{m}_k}}(\tilde{L}_k), \tilde{m}_k) < t\tilde{x}_t \right\} \cup \overline{\mathcal{E}_t^1}. \end{aligned}$$

Indeed,  $\sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \cdot) \leq \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(H_{X_{\tilde{m}_k}}(\tilde{L}_k), \cdot)$  on  $\{t(1-z) \leq H_{X_{\tilde{m}_k}}(\tilde{L}_k)\}$  and  $\sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \cdot) = \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(H_{X_{\tilde{m}_k}}(\tilde{L}_k), \cdot)$  on  $\{H_{X_{\tilde{m}_k}}(\tilde{L}_k) < t(1-z)\} \cap \mathcal{E}_t^1$ . Since we are dealing with  $X_{\tilde{m}_k}$ , the diffusion shifted at time  $H(\tilde{m}_k)$ , we can see that the main event above is included in  $\overline{\mathcal{E}_t^4}$ . We thus get

$$\mathcal{E}_t^8(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k)\} \subset \overline{\mathcal{E}_t^1} \cup \overline{\mathcal{E}_t^4}. \quad (5.4.36)$$

(5.2.14) follows easily from the combination of (5.4.35) and (5.4.36) applied with  $z = 1 - 4/\log(h_t)$ . Then, the right hand sides of (5.4.35) and (5.4.36) do not depend on  $z$  (which is arbitrary in  $[0, 1 - 4/\log(h_t)]$ ) and  $H(\tilde{m}_k)/t$  is independent from  $X_{\tilde{m}_k}$ . We can thus replace  $z$  by  $H(\tilde{m}_k)/t$  in (5.4.35) and (5.4.36) (at least on  $\{N_t \geq k, H(\tilde{m}_k)/t \leq 1 - 4/\log(h_t)\}$ ). Using the combination of (5.4.35) and (5.4.36) to study the union of events in (5.2.13) we get

$$\begin{aligned} P^v \left( \bigcup_{k=1}^{n_t} \{N_t \geq k, H(\tilde{m}_k)/t \leq 1 - 4/\log(h_t)\} \cap \mathcal{E}_t^8(v, k, H(\tilde{m}_k)/t) \right) &\leq 2n_t e^{-cht} \\ &+ P^v \left( \overline{\mathcal{E}_t^1} \cup \overline{\mathcal{E}_t^4} \right). \end{aligned}$$

Since  $v \in \mathcal{G}_t$  it satisfies (5.2.12). We thus have  $P^v(\overline{\mathcal{E}_t^1} \cup \overline{\mathcal{E}_t^4}) \leq e^{-L\phi(t)/2}$  (where  $L$  is the constant defined in Fact 5.2.5) and from the definition of  $h_t$  and  $\phi(t)$  we have easily  $2n_t e^{-cht} \leq e^{-c\phi(t)}$  for large  $t$ . For  $t$  large enough we thus get (5.2.13).

We now prove (5.2.15). Let us fix an environment  $v \in \mathcal{G}_t$ ,  $z \in [0, 1]$  and  $k \leq n_t$ . Inversing the local time in the definition of  $\mathcal{E}_t^9(v, k, z)$  we get that  $\mathcal{E}_t^9(v, k, z)$  coincides with the event

$$\begin{aligned} & \left\{ R_k^t/x_t < (1 - e^{-\tilde{c}h_t})(1 - z), \sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) \wedge H_{X_{\tilde{m}_k}}(\tilde{L}_k) \geq t(1 - z), \right. \\ & \left. H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\}. \end{aligned} \quad (5.4.37)$$

We have to distinguish the cases  $H_{X_{\tilde{m}_k}}(\tilde{L}_k) > \sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k)$  and  $\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k)$ . On  $\mathcal{E}_t^9(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\}$ ,  $\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k)$  is finite and the diffusion stays in  $[\tilde{L}_{k-1}, \tilde{L}_k]$  until this time. On this event,  $\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k)$  is thus equal to

$$I' := \int_{\tilde{L}_{k-1}}^{\tilde{L}_k} e^{-\tilde{v}^{(k)}(x)} \mathcal{L}_B(\sigma_B(t/x_t, 0), A^k(x)) dx,$$



where, as in (5.4.27),  $B$  is the brownian motion driving the diffusion  $X_{\tilde{m}_k}$ . Now, let  $\tilde{B} := B((t/x_t)^2)/ (t/x_t)$  and  $\tilde{A}^k(\cdot) := A^k(\cdot)/(t/x_t)$ . From the scaling property  $\tilde{B}$  is still a brownian motion. We can write

$$I' = (t/x_t) \int_{\tilde{L}_{k-1}}^{\tilde{L}_k} e^{-\tilde{v}^{(k)}(x)} \mathcal{L}_{\tilde{B}}(\sigma_{\tilde{B}}(1, 0), \tilde{A}^k(x)) dx.$$

From  $t \geq e^{h_t}$ , the definition (5.2.6) of  $x_t$  and the definition (5.2.1) of  $h_t$  we see that  $t/x_t \geq e^{3h_t/4}$  for  $t$  large enough. Recall also that  $v \in \mathcal{G}_t$  implies that  $v$  satisfies (5.2.8) and (5.2.9). We thus get

$$\sup_{[\tilde{\tau}_k^-(h_t/2), \tilde{\tau}_k^+(h_t/2)]} |\tilde{A}^k(\cdot)| \leq (|A^k(\tilde{\tau}_k^-(h_t/2))| \vee A^k(\tilde{\tau}_k^+(h_t/2))) / (t/x_t) \leq e^{-h_t/8},$$

We can now proceed as in the proof of (5.2.13) to get analogues of (5.4.30), (5.4.31) and (5.4.32). We deduce that

$$P^v(\mathcal{E}_{neg}^3 := \{|I' - (t/x_t)R_k^t| > e^{-\tilde{c}h_t}(t/x_t)R_k^t\}) \leq e^{-ch_t}.$$

Now, note from (5.4.37) that  $\mathcal{E}_t^9(v, k, z) \subset \{(1 - e^{-\tilde{c}h_t})\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) \geq tR_k^t/x_t\}$ . Since  $\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) = I'$  on  $\mathcal{E}_t^9(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\}$  and  $(1 + e^{-\tilde{c}h_t})(1 - e^{-\tilde{c}h_t}) < 1$  we deduce that  $\mathcal{E}_t^9(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\} \subset \mathcal{E}_{neg}^3$ . Then,

$$P^v\left(\mathcal{E}_t^9(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\}\right) \leq e^{-ch_t}. \quad (5.4.38)$$

We now study the case where  $\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k)$ , following the ideas of the proof of Lemma 5.3 of [4]. Recall that  $B$  is the brownian motion driving the diffusion  $X_{\tilde{m}_k}$ . We have

$$\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k) \Leftrightarrow \sigma_B(t/x_t, 0) > \tau(B, A^k(\tilde{L}_k)) \Leftrightarrow t/x_t > \mathcal{L}_B\left(\tau(B, A^k(\tilde{L}_k)), 0\right)$$

Now, let  $\tilde{B} := B((A^k(\tilde{L}_k))^2)/A^k(\tilde{L}_k)$ . From the scaling property  $\tilde{B}$  is still a brownian motion.  $\mathcal{L}_B\left(\tau(B, A^k(\tilde{L}_k)), 0\right) = A^k(\tilde{L}_k)\mathcal{L}_{\tilde{B}}\left(\tau(\tilde{B}, 1), 0\right)$ . Note that from the definition of  $e_k$  given in Subsection 5.2.1 we have  $\mathcal{L}_{\tilde{B}}\left(\tau(\tilde{B}, 1), 0\right) = e_k$ . As a consequence,

$$\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k) \Leftrightarrow t/x_t > A^k(\tilde{L}_k)e_k \Leftrightarrow tR_k^t/x_t > A^k(\tilde{L}_k)e_k R_k^t.$$

According to the trivial inequality

$$A^k(\tilde{L}_j) = \int_{\tilde{m}_j}^{\tilde{L}_j} e^{-(v(u)-v(\tilde{m}_j))} du \geq \int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{L}_j} e^{v(u)-v(\tilde{m}_j)} du = S_j^t$$

and the definition of  $\mathcal{E}_t^7$  we deduce that

$$\left\{ \sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k) \right\} \subset \left\{ tR_k^t/x_t > (1 + e^{-\tilde{c}h_t})^{-1} H_{X_{\tilde{m}_k}}(\tilde{L}_k) \right\} \cup \overline{\mathcal{E}_t^7}$$

Now, note from (5.4.37) that  $\mathcal{E}_t^9(v, k, z) \subset \{(1 - e^{-\tilde{c}h_t})H_{X_{\tilde{m}_k}}(\tilde{L}_k) \geq tR_k^t/x_t\}$ . Since  $(1 + e^{-\tilde{c}h_t})(1 - e^{-\tilde{c}h_t}) < 1$ , the inequality  $tR_k^t/x_t > (1 + e^{-\tilde{c}h_t})^{-1}H_{X_{\tilde{m}_k}}(\tilde{L}_k)$  is in contradiction with the event  $\mathcal{E}_t^9(v, k, z)$ . We thus get

$$\mathcal{E}_t^9(v, k, z) \cap \left\{ \sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k) \right\} \subset \overline{\mathcal{E}_t^7} \quad (5.4.39)$$

Since the right hand sides of (5.4.38) and (5.4.39) do not depend on  $z$  (which is arbitrary in  $[0, 1]$ ) and  $H(\tilde{m}_k)/t$  is independent from  $X_{\tilde{m}_k}$ , we can replace  $z$  by  $H(\tilde{m}_k)/t$  in (5.4.38) and (5.4.39) (at least on  $\{N_t \geq k\}$ ). We can thus use the combination of (5.4.38) and (5.4.39) to study the union of events in (5.2.15). We get

$$P^v \left( \bigcup_{k=1}^{n_t} \{N_t \geq k\} \cap \mathcal{E}_t^9(v, k, H(\tilde{m}_k)/t) \right) \leq n_t e^{-ch_t} + P^v \left( \overline{\mathcal{E}_t^7} \right)$$

Since  $v \in \mathcal{G}_t$  it satisfies (5.2.12). We thus have  $P^v(\overline{\mathcal{E}_t^7}) \leq e^{-L\phi(t)/2}$  (where  $L$  is the constant defined in Fact 5.2.5) and from the definition of  $h_t$  and  $\phi(t)$  we have easily  $n_t e^{-ch_t} \leq e^{-c\phi(t)}$  for large  $t$ . For  $t$  large enough we thus get (5.2.15).  $\square$

*Démonstration.* of Fact 5.2.8

(5.2.16) and (5.2.17) are included into Lemma 4.15 of [74] while (5.2.18) comes from Proposition 3.8 of [74] applied with  $h = h_t$ .  $\square$

*Démonstration.* of Lemma 5.2.9

First, note that from (5.2.1) and the definition of  $n_t$  just after, we have  $2t^\kappa e^{\kappa\delta(\log(\log(t)))^\omega}/q = 2n_t e^{\kappa h_t}/q$ . From the definition of  $N_t$ , we know that on  $\{V \in \mathcal{V}_t\}$ , at time  $t$ ,  $\tilde{m}_{N_t+1}$  has never been reached by the diffusion and neither  $\tilde{L}_{N_t+1}$  (because  $\tilde{L}_{N_t+1} > \tilde{m}_{N_t+1}$ ), we thus have

$$\begin{aligned} \mathbb{P} \left( \sup_{[0,t]} X \geq 2n_t e^{\kappa h_t}/q \right) &\leq \mathbb{P} \left( \tilde{L}_{N_t+1} \geq 2n_t e^{\kappa h_t}/q \right) + \mathbb{P}(V \notin \mathcal{V}_t) \\ &\leq \mathbb{P} \left( \tilde{L}_{n_t} \geq 2n_t e^{\kappa h_t}/q \right) + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) \\ &\leq \mathbb{P} \left( (1 + e^{-c_1 h_t}) \sum_{j=1}^{n_t} D_j^t \geq 2n_t e^{\kappa h_t}/q \right) + e^{-ch_t}, \end{aligned}$$

where  $c$  is a positive constant,  $c_1$  has the same meaning as in Proposition 5.4.9 and  $t$  is large enough. For the last inequality, we used Proposition 5.4.9, Fact 5.2.3 and (5.2.2).

$$\leq e^{-(1+e^{-c_1 h_t})^{-1}n_t} \left( \mathbb{E} \left[ e^{qe^{-\kappa h_t} D_j^t/2} \right] \right)^{n_t} + e^{-ch_t} = e^{(\log(2)-(1+e^{-c_1 h_t})^{-1})n_t} + e^{-ch_t},$$

where we used Markov's inequality, the fact that the sequence  $(qe^{-\kappa h_t} D_j^t)$  is an *iid* sequence of exponential random variable with parameter 1 and the expression of the Laplace transform for the exponential distribution. Since  $\log(2) < 1$  and  $n_t = e^{\kappa(1+\delta)(\log(\log(t)))^\omega} \gg h_t$ , we get the first point for  $t$  large enough.

We now prove the second point. We first note that from (5.2.1) and the definition of  $\tilde{n}_t$  just after, we have  $t^\kappa e^{(\rho-\kappa)(\log(\log(t)))^\omega} / 2q = \tilde{n}_t e^{\kappa h_t} / 2q$ . On  $\{V \in \mathcal{V}_t\} \cap \{N_t < n_t\} \cap \mathcal{E}_t^1$  we have  $X(t) \geq \tilde{L}_{N_t-1}$  so

$$\begin{aligned} \mathbb{P}(X(t) \leq \tilde{n}_t e^{\kappa h_t} / 2q) &\leq \mathbb{P}(\tilde{L}_{N_t-1} \leq \tilde{n}_t e^{\kappa h_t} / 2q) + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) + \mathbb{P}(\overline{\mathcal{E}_t^1}) \\ &\leq \mathbb{P}(\tilde{L}_{\tilde{n}_t} \leq \tilde{n}_t e^{\kappa h_t} / 2q) + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \leq \tilde{n}_t) \\ &\quad + \mathbb{P}(N_t \geq n_t) + \mathbb{P}(\overline{\mathcal{E}_t^1}) \\ &\leq \mathbb{P}\left((1 - e^{-c_1 h_t}) \sum_{j=1}^{\tilde{n}_t} D_j^t \leq \tilde{n}_t e^{\kappa h_t} / 2q\right) + e^{-c\phi(t)} \end{aligned}$$

where  $c$  is a positive constant,  $c_1$  has the same meaning as in Proposition 5.4.9 and  $t$  is large enough. For the last inequality we used Proposition 5.4.9, Fact 5.2.3, Lemma 5.2.1, (5.2.4) and the fact that  $e^{-c h_t} \leq e^{-c\phi(t)}$  for large  $t$ .

$$\leq e^{(1-e^{-c_1 h_t})^{-1} \tilde{n}_t} \left(\mathbb{E}\left[e^{-2qe^{-\kappa h_t} D_j^t}\right]\right)^{\tilde{n}_t} + e^{-c\phi(t)} = e^{(-\log(3)+(1-e^{-c_1 h_t})^{-1})\tilde{n}_t} + e^{-c\phi(t)},$$

where we used Markov's inequality, the fact that the sequence  $(qe^{-\kappa h_t} D_j^t)_{j \geq 1}$  is an *iid* sequence of exponential random variable with parameter 1 and the expression for the Laplace transform of the exponential distribution. Since  $\log(3) > 1$  and  $\tilde{n}_t = e^{\rho(\log(\log(t)))^\omega} \gg h_t$ , we get the second point for  $t$  large enough.

We now prove (5.2.21). Note that for a fixed environment  $v \in \mathcal{V}$  we have

$$P^v \left( \inf_{[0, +\infty[} X \leq -r \right) = P^v (H(-r) < H(+\infty)) = \frac{\int_0^{+\infty} e^{v(x)} dx}{\int_{-r}^{+\infty} e^{v(x)} dx} \leq \frac{\int_0^{+\infty} e^{v(x)} dx}{\int_{-r}^{-r/2} e^{v(x)} dx} \quad (5.4.40)$$

and note that

$$\int_{-r}^{-r/2} e^{V(x)} dx = \int_{r/2}^r e^{V(-x)} dx \stackrel{\mathcal{L}}{=} \int_{r/2}^r e^{-V(x)} dx,$$

where the equality in law comes from the time-reverse property. Applying Lemma 5.4.2 with  $t = r/2$  and  $a = \sqrt{r}$  we get

$$P \left( \int_{r/2}^r e^{-V(x)} dx \leq \frac{r}{2} e^{\sqrt{r}} \right) \leq e^{c_1 \sqrt{r} - c_2 r/2} + e^{-\kappa \sqrt{r}}, \quad (5.4.41)$$

where  $c_1$  and  $c_2$  are the constants in the lemma. Then, applying Lemma 5.4.3 we get for  $r$  large enough,

$$P \left( \int_0^{+\infty} e^{V(x)} dx \geq e^{\sqrt{r}} \right) \leq e^{-\kappa\sqrt{r}/2}. \quad (5.4.42)$$

Putting (5.4.41) and (5.4.42) into (5.4.40) we get that with  $P$ -probability greater than  $1 - (e^{c_1\sqrt{r}-c_2r/2} + e^{-\kappa\sqrt{r}} + e^{-\kappa\sqrt{r}/2})$  :  $P^V(\inf_{[0,+\infty[} X \leq -r) \leq 2r^{-1}$  (and it is bounded by 1 when this estimates fails) so integrating on  $\mathcal{V}$  with respect to  $P$  we get

$$\mathbb{P} \left( \inf_{[0,+\infty[} X \leq -r \right) \leq 2r^{-1} + e^{c_1\sqrt{r}-c_2r/2} + e^{-\kappa\sqrt{r}} + e^{-\kappa\sqrt{r}/2}.$$

(5.2.21) follows for  $r$  large enough. Finally, (5.2.22) is included in Lemma 5.18 of [74]. □

#### 5.4.4 Almost sure constantness of lim sup and lim inf

We now use a classical argument involving Kolmogorov 0 – 1 law to justify the almost sure constantness of the lim sup stated in Remark 5.1.3. We here treat the case of the lim sup with the renormalisation  $t \log(\log(t))$ . The same argument can be used with the lim inf instead of the lim sup, or with any of the other renormalisations used in the paper.

We first fix  $v \in \mathcal{V}$ , a realization of the environment. For any  $n \in \mathbb{N}$ , the process  $X^n := X(\tau(X, n) + \cdot)$  is, according to the Markov property, a diffusion in the environment  $v(n + \cdot)$  and it is independent from  $(X(s), 0 \leq s \leq \tau(X, n))$ . We have

$$\limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t \log(\log(t))} = \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_{X^n}^*(t)}{(\tau(X, n) + t) \log(\log(\tau(X, n) + t))} = \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_{X^n}^*(t)}{t \log(\log(t))}, \quad (5.4.43)$$

where the first equality comes from the fact that  $\mathbb{P}$ -almost surely the favorite site  $F^*(t)$  goes to  $+\infty$ . Indeed, the diffusion  $\mathbb{P}$ -almost surely converges to  $+\infty$  and as we can see from the results of Subsection 5.1.1,  $\mathcal{L}_X^*(t)$  converges  $\mathbb{P}$ -almost surely to infinity, these two facts imply the convergence of  $F^*(t)$  to  $+\infty$ . As a consequence  $F^*(t)$  will become greater than  $n$  for  $t$  large enough which imply the first equality in (5.4.43). The second equality comes from the equivalence when  $t$  goes to infinity between  $(\tau(X, n) + t) \log(\log(\tau(X, n) + t))$  and  $t \log(\log(t))$ .

The lim sup in the right hand side of (5.4.43) belongs to the  $\sigma$ -field  $\sigma(X^n(t), t \geq 0)$ , it is thus independent from  $\sigma(X(s), 0 \leq s \leq \tau(X, n))$ . Since this is true for any  $n \in \mathbb{N}$  we get, according to Kolmogorov 0 – 1 law, that the lim sup is constant  $P^v$ -almost surely. In other words, the lim sup is only a deterministic function of the environment  $v$ , let us denote it by  $L(v)$ .

Let us fix  $v \in \mathcal{V}$  and  $n \in \mathbb{N}$ . Note that  $P^v(\inf X^n \geq n - 1) > 0$  so the lim sup still equals  $L(v)$  on  $\{\inf X^n \geq n - 1\}$ . As a consequence,  $L(v)$  is only a function of  $(v(x+n-1) - v(n-1), x \geq 0)$ . If we consider the space  $\mathcal{V}$  equipped with probability  $P$ , this implies that  $L(V)$  is independent from the  $\sigma$ -field  $\sigma(V(x), x \leq n - 1)$ . Since this is true for all  $n \in \mathbb{N}$  we deduce from Kolmogorov 0-1 law that  $L(V)$  is constant  $P$ -almost surely. This proves that the lim sup is in fact constant  $\mathbb{P}$ -almost surely. In other words,

$$\exists \lambda \in [0, +\infty] \text{ such that } \mathbb{P} - a.s. \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t \log(\log(t))} = \lambda.$$

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# Grégoire VÉCHAMBRE

## Fonctionnelles de processus de Lévy et diffusions en milieux aléatoires

Résumé :

Pour  $V$  un processus aléatoire càd-làg, on appelle diffusion dans le milieu aléatoire  $V$  la solution formelle de l'équation différentielle stochastique

$$dX_t = -\frac{1}{2}V'(X_t)dt + dB_t,$$

où  $B$  est un mouvement brownien indépendant de  $V$ . Le temps local au temps  $t$  et à la position  $x$  de la diffusion, noté  $\mathcal{L}_X(t, x)$ , donne une mesure de la quantité de temps passé par la diffusion au point  $x$ , avant l'instant  $t$ . Dans cette thèse nous considérons le cas où le milieu  $V$  est un processus de Lévy spectralement négatif convergeant presque sûrement vers  $-\infty$ , et nous nous intéressons au comportement asymptotique lorsque  $t$  tend vers l'infini de  $\mathcal{L}_X^*(t) := \sup_{\mathbb{R}} \mathcal{L}_X(t, \cdot)$ , le supremum du temps local de la diffusion, ainsi qu'à la localisation du point le plus visité par la diffusion. Nous déterminons notamment la convergence en loi et le comportement presque sûr du supremum du temps local. Cette étude révèle que le comportement asymptotique du supremum du temps local est fortement lié aux propriétés des fonctionnelles exponentielles des processus de Lévy conditionnés à rester positifs et cela nous amène à étudier ces dernières. Si  $V$  est un processus de Lévy,  $V^\uparrow$  désigne le processus  $V$  conditionné à rester positif. La fonctionnelle exponentielle de  $V^\uparrow$  est la variable aléatoire  $\int_0^{+\infty} e^{-V^\uparrow(t)} dt$ . Nous étudions en particulier sa finitude, son auto-décomposabilité, l'existence de moments exponentiels, sa queue en 0, l'existence et la régularité de sa densité.

Mots clés : Processus de diffusion, potentiel aléatoire, processus de Lévy, processus de renouvellement, temps local, processus de Lévy conditionné à rester positif, fonctionnelles exponentielles.

## Functionals of Lévy processes and diffusions in random media

Abstract :

For  $V$  a random càd-làg process, we call diffusion in the random medium  $V$  the formal solution of the stochastic differential equation

$$dX_t = -\frac{1}{2}V'(X_t)dt + dB_t,$$

where  $B$  is a brownian motion independent of  $V$ . The local time at time  $t$  and at the position  $x$  of the diffusion, denoted by  $\mathcal{L}_X(t, x)$ , gives a measure of the amount of time spent by the diffusion at point  $x$ , before instant  $t$ . In this thesis we consider the case where the medium  $V$  is a spectrally negative Lévy process converging almost surely toward  $-\infty$ , and we are interested in the asymptotic behavior, when  $t$  goes to infinity, of  $\mathcal{L}_X^*(t) := \sup_{\mathbb{R}} \mathcal{L}_X(t, \cdot)$ , the supremum of the local time of the diffusion. We are also interested in the localization of the point most visited by the diffusion. We notably establish the convergence in distribution and the almost sure behavior of the supremum of the local time. This study reveals that the asymptotic behavior of the supremum of the local time is deeply linked to the properties of the exponential functionals of Lévy processes conditioned to stay positive and this brings us to study them. If  $V$  is a Lévy process,  $V^\uparrow$  denotes the process  $V$  conditioned to stay positive. The exponential functional of  $V^\uparrow$  is the random variable  $\int_0^{+\infty} e^{-V^\uparrow(t)} dt$ . For this object, we study in particular finiteness, self-decomposability, existence of finite exponential moments, asymptotic tail at 0 and smoothness of the density.

Keywords : Diffusion, random potential, Lévy process, renewal process, local time, Lévy processes conditioned to stay positive, exponential functionals.