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Yang Lan

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## THÈSE DE DOCTORAT EN MATHÉMATIQUES

*Spécialité :*

Mathématiques fondamentales

**Yang LAN**

### **Dynamique asymptotique pour des équations de KdV généralisées $L^2$ critiques et surcritiques**

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# Résumé

Dans cette thèse, nous étudions la dynamique à temps long des solutions des équations de KdV généralisées (gKdV) critiques et surcritiques pour la masse.

La première partie de cette thèse est consacrée à la construction d'une dynamique explosive auto-similaire stable pour des équations de gKdV légèrement  $L^2$  surcritique dans l'espace d'énergie  $H^1$ . La preuve repose sur le profil auto-similaire construit par H. Koch. Nous donner une description précise de la formation des singularité près du temps d'explosion.

La deuxième partie est consacrée à la construction de solutions explosive aux équations de gKdV légèrement  $L^2$  surcritiques avec plusieurs points d'explosion. L'idée clé est d'envisager des solutions qui se comportent comme une somme de bulles découplée, chaque bulle se comportent comme un solution auto-similaire explosent en un seul point. Nous utilisons les argument topologique classique pour s'assurer que chaque bulle explose en même temps. Ici, nous avons besoin de données initiales plus grande régularité pour contrôler la solution entre les différents points d'explosion.

Enfin, dans la troisième partie, nous considérons les équations de gKdV  $L^2$  critiques avec une perturbation saturée. Dans ce cas, toute solution avec des données initiales dans  $H^1$  est toujours globale en le temps et bornée dans  $H^1$ . Nous donner une classification explicite de la dynamique près du solitons. Sous certaines hypothèses de décroissance, il n'y a que trois possibilités : (i) la solution converge asymptotiquement vers une onde solitaire ; (ii) la solution reste dans un petit voisinage de la famille modulée de l'état fondamental, en s'étalant par de temps infiniment grande (*Blow down*) ; (iii) la solution quitte tout petit voisinage de la famille modulée de solitons.

**Mots-clefs :** gKdV,  $L^2$ -critique et surcritique, explosion de auto-similaire, formation de singularités, points d'explosion multiple, perturbation saturée, dynamique près de soliton, formule de monotonie.



# Abstract

In this thesis, we deal with the long time dynamics for solutions of the mass critical and supercritical generalized KdV equations.

The first part of this work is devoted to construct a stable self-similar blow up dynamics for slightly  $L^2$  supercritical gKdV equations in the energy space  $H^1$ . The proof relies on the self-similar profile constructed by H. Koch. We will also give a specific description of the formation of singularity near the blow up time.

The second part is devoted to construct blow up solutions to the slightly  $L^2$  supercritical gKdV equations with multiple blow up points. The key idea is to consider solutions which behaves like a decoupled sum of bubbles. And each bubble behaves like a self-similar blow up solutions with a single blow up point. Then we can use a classic topological argument to ensure that each bubble blows up at the same time. Here, we require a higher regularity of the initial data to control the solution between the different blow up points.

Finally, in the third part, we consider the  $L^2$  critical gKdV equations with a saturated perturbation. In this case, any solution with initial data in  $H^1$  is always global in time and bounded in  $H^1$ . We will give a explicit classification of the flow near the ground states. Under some suitable decay assumptions, there are only three possibilities: (i) the solution converges asymptotically to a solitary wave; (ii) the solution is always in some small neighborhood of the modulated family of the ground state, but blows down at infinite time; (iii) the solution leaves any small neighborhood of the modulated family of the solitons.

**Keywords:** gKdV,  $L^2$ -critical and supercritical, self-similar blow-up, formation of singularity, multiple blow up points, saturated perturbation, dynamics near soliton, monotonicity formula



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# Organization of the thesis

As this thesis is composed of several articles written by the author, I decide to write this short paragraph to indicate origins of these chapters.

**Chapter 2** is based on the following work:

Y. Lan, *Stable self-similar blow-up dynamics for slightly  $L^2$ -supercritical generalized KdV equations*. *Comm. Math. Phys.* **345** (2016), no. 1, 223–269.

**Chapter 3** is based on the following work:

Y. Lan, *Blow-up solutions for  $L^2$ -supercritical gKdV equations with exactly  $k$  blow-up points*, arXiv:1602.08617, submitted.

**Chapter 4** is based on the following work:

Y. Lan, *On asymptotic dynamics for  $L^2$  critical generalized KdV equations with a saturated perturbation*, arXiv:1609.05146, submitted.



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# Chapitre 0

## Introduction (version française)

### 0.1 Vue d'ensemble la dynamique asymptotique des équations de KdV généralisées

#### 0.1.1 Préliminaire

Le objectif de cette thèse est d'étudier la dynamique asymptotique des *équations dispersives non-linéaires*.

L'équation dispersive non-linéaire, est une équation qui combine une équation linéaire avec un comportement dispersif (la solution se désintègre uniformément en temps) et une non linéarité appropriée. Typiquement, l'équation a la forme suivante :

$$u_t = Lu + N(u, \nabla u, \dots),$$

où  $N$  est le terme non linéaire et  $L$  est un opérateur linéaire anti-auto-adjoint donné par

$$\mathcal{F}(Lu)(\xi) = ip(\xi)\mathcal{F}u(\xi), \quad p(\xi) \in \mathbb{R}.$$

Ici  $\mathcal{F}$  est la transformée de Fourier et  $D_\xi^2 p(\xi) \neq 0$  pour tout  $\xi \neq 0$ . Des exemples typiques d'équations dispersives non-linéaires sont l'équation de Korteweg-de Vries, l'équation de Schrödinger non linéaire, l'équation de Benjamin-Ono, l'équation de KP-II etc.

Dans de nombreux cas, le problème de Cauchy locale est connu. Plus précisément, pour les données initiales qui se trouvent dans un espace de Banach approprié, on peut trouver une solution forte unique (locale en temps) à l'équation. De plus, l'application de flot est continue.

En notant  $(T_-, T_+)$  l'intervalle de temps maximal pour l'existence d'une solution. Un problème intéressant est de comprendre les comportements asymptotiques près de  $T_-$  et  $T_+$ . Les comportements asymptotiques typiques sont les suivants :

1. Scattering : la solution globale existe (i.e.  $T_- = -\infty$  or  $T_+ = +\infty$ ), et converge vers une solution de l'équation linéaire asymptotiquement ;



2. Explosion : la solution s'explode en temps fini (i.e.  $T_- > -\infty$  or  $T_+ < +\infty$ ), ou la solution existe globalement mais a comportements singuliers comme  $t \rightarrow \infty$ ;
3. Soliton : la solution globale existe et converge asymptotiquement vers une classe spéciale de solutions (le soi-disant *soliton*).

Dans le deuxième et le troisième cas, nous nous attendons aussi à donner une description spécifique de la dynamique asymptotique de la solution.

On considère l'un des plus simples cas d'équations dispersives, l'équation de Korteweg-de Vries généralisée (gKdV) :

$$\begin{cases} \partial_t u + (u_{xx} + u|u|^{p-1})_x = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0, x) = u_0(x), \end{cases} \quad (0.1.1)$$

avec  $p > 1$ .

L'équation d'origine de Korteweg-de Vries,

$$\partial_t u + (u_{xx} + u^2)_x = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (0.1.2)$$

apparaît dans la physique comme un modèle pour la propagation des ondes unidirectionnelles. La formulation exacte de cette équation vient de Korteweg et de Vries [40]. Cette équation ainsi que l'équation de Korteweg-de Vries modifiée :

$$\partial_t u + (u_{xx} + u^3)_x = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (0.1.3)$$

sont complètement intégrables, si on utilise la méthode de scattering inverse (Eckhaus & Schuur [19], Lax [47] et Miura [80]).

Les équations de KdV généralisées ainsi que les équations de Schrödinger non linéaires sont également considérées comme des modèles universels pour le système hamiltonien en dimension infinie. A partir de la structure hamiltonienne, on a les deux lois de conservation suivantes, i.e. la masse et l'énergie<sup>1</sup> :

$$M(u(t)) = \int u^2(t) = \int u_0^2 = M(u_0), \quad (0.1.4)$$

$$\begin{aligned} E(u(t)) &= \frac{1}{2} \int u_x^2(t) - \frac{1}{p+1} \int |u(t)|^{p+1} \\ &= \frac{1}{2} \int (\partial_x u_0)^2 - \frac{1}{p+1} \int |u_0|^{p+1} = E(u_0). \end{aligned} \quad (0.1.5)$$

De ces deux lois de conservation, l'espace de Sobolev  $H^1$  apparaît comme l'espace d'énergie, de sorte qu'il est l'espace naturel pour étudier les solutions de (0.1.1). La question générale est de comprendre le comportement des solutions de (0.1.1) avec les données initiales  $u_0 \in H^1$ .

L'existence de la solution locale est bien étudiée, Kato [33] et Ginibre, Tsutsumi [24] pour la théorie de  $H^s$  ( $s > \frac{3}{2}$ ); Kenig, Ponce et Vega [36] pour la théorie de  $L^2$  dans le

1. L'équation de KdV (0.1.2) a infiniment beaucoup de lois de conservation pour être complètement intégrable.

cas  $p = 5$  et la théorie de  $H^s$  pour  $p > 5$ ; Strunk [89] pour la théorie de  $B_{2,\infty}^s$  avec  $p > 5$ ; Bourgain [4] pour les cas périodiques.

Dans cette thèse, nous utilisons principalement l'existence locale suivante et les résultats d'unicité :

**Proposition 0.1.1** (Localement bien posées dans  $H^1$ , [36]). *Pour tout  $u_0 \in H^1$ , il existe une temps maximal  $T = T(\|u_0\|_{H^1}) > 0$  et une solution unique  $u(t) \in C([0, T], H^1)$  de (0.1.1), satisfaisant (0.1.4) et (0.1.5) dans  $[0, T)$ . De plus, nous avons le critère d'explosion suivant : soit  $T = +\infty$  soit  $T < +\infty$  et*

$$\lim_{t \rightarrow T} \|u(t)\|_{H^1} = +\infty.$$

**Définition 0.1.2.** Une *solution explosive* de (0.1.1) est une solution maximale  $u(t) \in C([0, T), H^1)$ , tel que

$$\lim_{t \rightarrow T} \|u(t)\|_{H^1} = +\infty. \quad (0.1.6)$$

Lorsque le temps maximal d'existence est fini (ou infini respectivement), on dit que la solution  $u(t)$  explose en temps fini (ou temps infini respectivement).

Le problème de Cauchy a une règle d'échelle standard. Pour tout  $\lambda > 0$ ,

$$u_\lambda(t, x) = \frac{1}{\lambda^{\frac{2}{p-1}}} u\left(\frac{t}{\lambda^3}, \frac{x}{\lambda}\right), \quad (0.1.7)$$

est une solution de (0.1.1). De plus, la norme  $\dot{H}^{\sigma_c}$  des données initiales est invariante sous cette échelle, où

$$\sigma_c = \frac{1}{2} - \frac{2}{p-1}.$$

- Si  $\sigma_c < 0$  (ou équivalent  $p < 5$ ), le problème de Cauchy (0.1.1) est appelée  $L^2$  sous-critique;
- Si  $\sigma_c = 0$  (ou  $p = 5$ ), le problème de Cauchy (0.1.1) est appelée  $L^2$  critique;
- Si  $\sigma_c > 0$  (ou  $p > 5$ ), le problème de Cauchy (0.1.1) est appelée  $L^2$  sur-critique.

## 0.1.2 Stabilité et instabilité des solitons

Il existe une classe spéciale de solutions à (0.1.1) appelée *onde solitaire*. Ils sont donnés par

$$u(t, x) = c^{\frac{1}{p-1}} \mathcal{Q}_p(\sqrt{c}(x - ct)), \quad (0.1.8)$$

ou

$$\mathcal{Q}_p'' - \mathcal{Q}_p + \mathcal{Q}_p |\mathcal{Q}_p|^{p-1} = 0, \quad \mathcal{Q}_p = \left( \frac{p+1}{2 \cosh^2((p-1)x/2)} \right)^{\frac{1}{p-1}}. \quad (0.1.9)$$

Ce  $\mathcal{Q}_p$  est aussi appelé *l'état fondamental*. De [91],  $\mathcal{Q}_p$  est liée à la meilleure constante de l'inégalité de Gagliardo-Nirenberg :

$$\int |f|^{p+1} \leq C_p \left( \int (\partial_x f)^2 \right)^{\frac{p-1}{4}} \left( \int f^2 \right)^{\frac{p+3}{4}}. \quad (0.1.10)$$

Plus précisément, la constante optimale pour cette inégalité est donnée par

$$C_p = J(Q_p) = \min_{f \in H^1, f \neq 0} J(f) = \frac{p+1}{2\|Q_p\|_{L^2}^{p-1}},$$

ou pour tout  $f \in H^1$ ,  $f \neq 0$ ,

$$J(f) = \frac{\|\partial_x f\|_{L^2}^{\frac{p-1}{2}} \|f\|_{L^2}^{\frac{p+3}{2}}}{\|f\|_{L^{p+1}}^{p+1}}.$$

Dans le cas critique  $p = 5$ , on note

$$Q = Q_5. \quad (0.1.11)$$

L'étude du flot près de ces solitons est cruciale pour la compréhension du comportement asymptotique des solutions de (0.1.1). Par conséquent, la stabilité et la stabilité asymptotique des solitons deviennent un problème important.

**Définition 0.1.3.** On dit que pour  $c > 0$  le soliton

$$c^{\frac{1}{p-1}} Q_p(\sqrt{c}(x - ct))$$

est stable dans  $H^1$ , si  $\forall \delta_0 > 0$ ,  $\exists \alpha_0 > 0$  tel que

$$\|u_0 - c^{\frac{1}{p-1}} Q_p(\sqrt{c} \cdot)\|_{H^1} \leq \alpha_0 \implies \forall t \geq 0, \exists x(t) / \|u(t) - c^{\frac{1}{p-1}} Q_p(\sqrt{c} \cdot - x(t))\|_{H^1} \leq \delta_0. \quad (0.1.12)$$

Nous disons que la famille de soliton

$$\{c^{\frac{1}{p-1}} Q_p(\sqrt{c}(x - x_0 - ct)) \mid c > 0, x_0 \in \mathbb{R}\}$$

est asymptotiquement stable dans  $H^1$ , si  $\exists \alpha_0 > 0$  tel que

$$\|u_0 - c^{\frac{1}{p-1}} Q_p(\sqrt{c} \cdot)\|_{H^1} \leq \alpha_0 \implies \exists c_\infty, x(t) / \|u(t, \cdot + x(t)) - c_\infty^{\frac{1}{p-1}} Q_p(\sqrt{c_\infty} \cdot)\|_{H^1} \xrightarrow[t \rightarrow +\infty]{} 0, \text{ dans } H^1. \quad (0.1.13)$$

Nous rappelons certains résultats connus pour la stabilité des solitons et la stabilité asymptotique de la famille des solitons :

- Dans le cas  $p = 2, 3, 4$ , il résulte de l'argument variationnel que les solitons sont stables dans  $H^1$  (Bona, Souganidis & Strauss [3], Cazenave & Lions [7], Merle & Vega [78] et Weinstein [93]).
- Dans le cas  $p = 2, 3, 4$ , la famille des solitons est asymptotiquement stable dans  $H^1$  (Martel & Merle [51] et Pego & Weinstein [84]).
- Dans le cas  $L^2$  critique (i.e.  $p = 5$ ), Martel & Merle [52] ont prouvé l'instabilité  $H^1$  des solitons.

- Dans le cas  $L^2$  surcritique (i.e.  $p > 5$ ), Bona, Souganidis & Strauss [3] prouvent l'instabilité des solitons dans  $H^1$  en utilisant l'argument de Grillakis, Shatah & Strauss [26].

Dans [60], Martel, Merle et Tsai ont prouvé la stabilité et la stabilité asymptotique de la somme de  $N$  solitons dans  $H^1$  pour le gKdV  $L^2$  sous-critique. Plus précisément, nous avons

**Théorème 0.1.1** (Stabilité et stabilité asymptotique de la somme de  $N$  solitons). *Soient  $p = 2, 3, 4$ ,  $0 < c_1^0 < \dots < c_N^0$ . Il existe  $\gamma_0, A_0, L_0, \alpha_0 > 0$  tels que  $\forall u_0 \in H^1(\mathbb{R})$  et supposons qu'il existe  $L > L_0$ ,  $\alpha < \alpha_0$ , et  $x_1^0 < \dots < x_N^0$ ,*

$$\left\| u_0 - \sum_{j=1}^N (c_j^0)^{\frac{1}{p-1}} \mathcal{Q}_p \left( \sqrt{c_j^0} \cdot -x_j^0 \right) \right\|_{H^1} \leq \alpha_0, \quad (0.1.14)$$

$$x_j^0 \geq x_{j-1}^0 + L, \text{ pour tout } j = 2, \dots, N. \quad (0.1.15)$$

Soit  $u(t)$  la solution correspondante de (0.1.1). Alors il existe  $x_1(t), \dots, x_N(t)$  tel que :

- *Stabilité de la somme de  $N$  solitons :*

$$\forall t \geq 0, \quad \left\| u(t) - \sum_{j=1}^N (c_j^0)^{\frac{1}{p-1}} \mathcal{Q}_p \left( \sqrt{c_j^0} \cdot -x_j(t) \right) \right\|_{H^1} \leq A_0(\alpha_0 + e^{-\gamma_0 L}). \quad (0.1.16)$$

- *Stabilité asymptotique de la somme de  $N$  solitons. Il existe  $0 < c_1^{+\infty} < \dots < c_N^{+\infty}$  avec*

$$|c_j^0 - c_j^{+\infty}| \leq A_0(\alpha_0 + e^{-\gamma_0 L}),$$

*tel que*

$$\left\| u(t) - \sum_{j=1}^N (c_j^{+\infty})^{\frac{1}{p-1}} \mathcal{Q}_p \left( \sqrt{c_j^{+\infty}} \cdot -x_j(t) \right) \right\|_{L^2(x > \frac{1}{10} c_1^{+\infty} t)} \rightarrow 0, \quad (0.1.17)$$

$$\dot{x}_j(t) \rightarrow c_j^{+\infty}, \quad (0.1.18)$$

*quand  $t \rightarrow +\infty$ .*

### 0.1.3 Dynamique explosive des équations de gKdV $L^2$ critique

En combinant les résultats de l'existence locale et des lois de conservation, on obtient : si  $p < 5$ , alors toute les solutions avec  $H^1$  données initiales sont globales en temps et bornées dans  $H^1$  ; Si  $p = 5$ , il est facile de voir que

$$E(u_0) = E(u(t)) \geq \frac{1}{2} \left( 1 - \left( \frac{\int u_0^2}{\int Q^2} \right)^2 \right) \int u_x^2(t).$$

Dans ce cas, si  $\|u_0\|_{L^2} < \|Q\|_{L^2}$  alors la solution pour le problème critique est également globale en temps et bornée dans  $H^1$ .

Alors que pour le problème critique ( $p = 5$ ) avec  $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$ , une explosion peut se produire. L'existence de singularité dans ce cas (en temps fini ou en temps infini) a été un problème ouvert.

Dans un contexte analogue, si l'on considère les équations de Schrödinger non linéaires focalisante  $L^2$  critiques (NLS) dans la dimension 1,

$$\begin{cases} i\partial_t u + \Delta u + u|u|^4 = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \quad (0.1.19)$$

Il existe des exemples simples de solutions de (0.1.19) qui explosent. En effet, l'équation (0.1.19) a les mêmes lois de conservation (i.e. (0.1.4) et (0.1.5)) et l'invariance de l'échelle (i.e. (0.1.7)) comme gKdV critique  $L^2$ .

Pour  $u_0 \in H^1$  avec énergie négative et décroissance rapide à l'infini (i.e.  $xu_0 \in L^2$ ), la solution associée explose en temps fini. En effet, nous avons l'identité dite de Virial

$$\frac{d}{dt} \int x^2 u^2(t, x) dx \leq CE_0,$$

Avec  $C > 0$ . Cette identité implique l'explosion en temps fini immédiatement.

D'autre part, pour le NLS  $L^2$  critique on a la transformation pseudo-conformale suivante. Si  $u(t, x)$  est une solution à (0.1.19), alors

$$\frac{1}{t^{1/2}} e^{\frac{ix^2}{4t}} \bar{u}\left(\frac{1}{t}, \frac{x}{t}\right)$$

est une solution.

Il existe une classe spéciale de solution, i.e. la solution périodique, donnée par

$$u(t, x) = c^{\frac{1}{4}} e^{ict} Q(\sqrt{cx}),$$

ou  $c > 0$ , et  $Q$  est l'état fondamental pour non-linéarité  $L^2$  critique donnée par (0.1.11). En appliquant la transformée pseudo-conformale à des solutions de ce type, on obtient une classe spéciale de solution

$$S(t, x) = \frac{c^{\frac{1}{4}}}{t^{1/2}} e^{-i(\frac{c}{t} + \frac{x^2}{4t})} Q\left(\frac{\sqrt{cx}}{t}\right),$$

qui explose à  $t = 0$ . Cette solution est aussi la solution explosive unique de (1.1.19) avec une masse minimale aux la symétrie du flux, [69].

En plus de la solution d'explosion avec une masse minimale, il existe également d'autres solutions explosives avec un taux de explosion conforme (voir [5]) :

$$\|u_x(t)\|_{L^2} \sim \frac{1}{T-t} \quad \text{quand } t \rightarrow T. \quad (0.1.20)$$

Tandis que pour la solution de (0.1.19) avec une énergie négative et une masse légèrement surcritique,

$$0 < \|u_0\|_{L^2} - \|Q\|_{L^2} \ll 1,$$

Merle et Raphaël démontrent l'existence et la stabilité d'une dynamique explosive avec un taux d'explosion de "log-log" :

$$\lim_{t \rightarrow T} \frac{\|\nabla u(t)\|_{L^2}}{\|\nabla Q\|_{L^2}} \sqrt{\frac{|\log |\log(T-t)||}{T-t}} = \frac{1}{\sqrt{2\pi}}. \quad (0.1.21)$$

Dans le cas de gKdV  $L^2$  critique, il n'y a pas de transformation pseudo-conforme non plus d'identité Virial qui puisse nous donner un exemple explicite de solutions explosives. L'existence d'une solution explosive et la description de la dynamique explosive ont été développées par Martel et Merle pour des solutions avec une masse légèrement surcritique et une énergie négative, i.e.

$$\|Q\|_{L^2} \leq \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*, \quad \alpha^* \ll 1 \text{ et } E(u_0) < 0. \quad (0.1.22)$$

Plus précisément, nous avons :

**Théorème 0.1.2** (Existence de solutions explosive, [70]). *Pour  $u_0 \in H^1$  avec (0.1.22), alors la solution associée de gKdV  $L^2$  critique explose en temps fini ou infini.*

Dans ce cas, nous n'avons plus d'informations sur le temps d'explosion et la dynamique explosive. En effet, dans [57, 59], Martel, Merle et Raphaël prouvent que l'explosion en temps fini et en temps infini sont tous possibles. De plus, la dynamique explosive peut être stable ou instable.

**Théorème 0.1.3** (Profil de explosion et borne inférieure de la taux d'explosion, [55]). *Pour donnée initiale  $u_0 \in H^1$  avec*

$$\|Q\|_{L^2} \leq \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*, \quad \alpha^* \ll 1,$$

*si la solution  $u(t)$  de (0.1.1) associée explose en temps fini ou en temps infini, alors il existe  $\lambda(t)$ ,  $x(t)$  tels que*

$$\lambda^{\frac{1}{2}}(t)u(t, \lambda(t) \cdot + x(t)) \rightharpoonup \pm Q, \text{ dans } H^1 - \text{faible, quand } t \rightarrow T. \quad (0.1.23)$$

*De plus, si  $T < +\infty$ , alors*

$$\lim_{t \rightarrow T} (T-t)^{1/3} \|u_x(t)\|_{L^2} = +\infty. \quad (0.1.24)$$

Ici,  $\lambda(t) \sim \|u_x(t)\|_{L^2}$  est le taux d'explosion de (0.1.23). il est impossible d'obtenir une convergence forte vers  $Q$  dans  $H^1$ , du fait de la conservation de la masse (0.1.4). Mais nous avons une convergence forte dans  $H_{\text{loc}}^1$ .

**Théorème 0.1.4** (Explosion en temps fini, [53]). *Soit  $u_0 \in H^1$  avec (0.1.22) et*

$$\int_{x > x_0} u_0^2(x) dx < \frac{\theta}{x_0^6}, \text{ pour tout } x_0 > 1, \quad (0.1.25)$$

*pour quelque constante universelle  $\theta > 0$ . Alors la solution  $u(t)$  de (0.1.1) associée explose en temps fini  $T < +\infty$ . De plus, nous avons la borne supérieure suivante de la vitesse d'explosion pour une séquence de temps  $t_n \rightarrow T$  :*

$$\|u_x(t_n)\|_{L^2} \leq \frac{C'}{T - t_n}. \quad (0.1.26)$$

Il existe un grand gap entre la borne inférieure donnée par le théorème 0.1.3 et la borne supérieure donnée par le théorème 0.1.4. Des exemples de solutions explosives avec différents taux de explosion sont construits dans [59].

**Théorème 0.1.5** (Existence globale pour une solution avec à masse critique et décroissance, [54]). *Pour donnée initiale avec à masse critique, i.e.  $\|u_0\|_{L^2} = \|Q\|_{L^2}$ , si*

$$\int_{x>x_0} u_0^2(x) dx < \frac{C}{x_0^3}, \text{ pour tout } x_0 > 1, \quad (0.1.27)$$

alors la solution  $u(t)$  de (0.1.1) associée est globale pour  $t \geq 0$ , et n'explose pas en temps infini.

Dans [57, 58, 59], Martel, Merle et Raphaël donnent une étude de la dynamique asymptotique près de l'état fondamental  $Q$ .

Plus précisément, considérez l'ensemble de données initial suivant

$$\mathcal{A}_{\alpha_0} = \left\{ u_0 \in H^1 \mid u_0 = Q + \varepsilon_0, \|\varepsilon_0\|_{H^1} < \alpha_0, \int_{y>0} y^{10} \varepsilon_0^2 < 1 \right\},$$

et le  $L^2$  tube suivant près de la famille des onde solitaires

$$\mathcal{T}_{\alpha^*} = \left\{ u_0 \in H^1 \mid \inf_{\lambda_0>0, x_0 \in \mathbb{R}} \left\| u_0 - \frac{1}{\lambda_0^{\frac{1}{2}}} Q \left( \frac{x-x_0}{\lambda_0} \right) \right\|_{L^2} < \alpha^* \right\}.$$

Alors nous avons :

**Théorème 0.1.6** (Solution explosive avec une masse minimale, [58]). *Il existe une solution  $S(t, x) \in C((0, +\infty), H^1)$  à (0.1.1) avec une masse minimale  $\|S(t)\|_{L^2} = \|Q\|_{L^2}$ , et*

$$\|S_x(t)\|_{L^2} \sim \frac{\|Q'\|_{L^2}}{t}, \quad \text{quand } t \rightarrow 0. \quad (0.1.28)$$

De plus, si une solution explosive  $u(t)$  dans  $H^1$  a une masse minimale  $\|u(t)\|_{L^2} = \|Q\|_{L^2}$ , alors  $u = S$ , aux symétries du flot près.

**Théorème 0.1.7** (Rigidité, [57]). *Pour  $0 < \alpha_0 \ll \alpha^* \ll 1$ , et  $u_0 \in \mathcal{A}_{\alpha_0}$ , soit  $u(t)$  la solution correspondante à (0.1.1), et  $0 < T \leq +\infty$  la temps maximal d'existence d'solution. Alors il n'y a que trois possibilités :*

**(Blow-up) :** *Soit la solution  $u(t)$  explose en temps fini  $0 < T < +\infty$  avec*

$$\|u(t)\|_{H^1} = \frac{\ell(u_0) + o(1)}{T-t}, \quad \ell(u_0) > 0.$$

De plus, pour tout  $t < T$ ,  $u(t) \in \mathcal{T}_{\alpha^*}$ .

**(Soliton) :** *Soit la solution est global, et pour tout  $t < T = +\infty$ ,  $u(t) \in \mathcal{T}_{\alpha^*}$ . De plus, il existe une constante  $\lambda_\infty > 0$  et une  $C^1$  fonction  $x(t)$  telle que*

$$\begin{aligned} \lambda_\infty^{\frac{1}{2}} u(t, \lambda_\infty \cdot + x(t)) &\rightarrow Q \text{ dans } H_{\text{loc}}^1, \text{ quand } t \rightarrow +\infty, \\ |\lambda_\infty - 1| &\lesssim \delta(\alpha_0), \quad x(t) \sim \frac{t}{\lambda_\infty^2}, \text{ quand } t \rightarrow +\infty. \end{aligned}$$

**(Exit) :** Soit pour un temps fini  $0 < t^* < T$ ,  $u(t^*) \notin \mathcal{T}_{\alpha^*}$ . Et il existe  $\lambda_u > 0$ ,  $x_u \in \mathbb{R}$ , tel que

$$\|\lambda_u^{1/2} u(t^*, \lambda_u \cdot + x_u) - \mathcal{S}(\tau^*, \cdot)\|_{L^2} \leq \delta(\alpha_0),$$

ou  $\delta(\alpha_0) \rightarrow 0$  quand  $\alpha_0 \rightarrow 0$ , et  $\tau^*$  dépend seulement de  $\alpha^*$ .

De plus, la scénarios (Blow-up) et (Exit) sont stables sous une petite perturbation dans  $\mathcal{A}_{\alpha_0}$ .

**Théorème 0.1.8** (Régimes explosive exotiques, [59]). Il existe des solutions avec taux de explosion non générique :

(i) Explosion en temps fini : pour tout  $\nu > \frac{11}{13}$ , il existe une solution  $u \in C((0, T_0], H^1)$  de (0.1.1) explose à  $t = 0$  avec

$$\|u(t)\|_{H^1} \sim t^{-\nu}, \quad \text{quand } t \rightarrow 0^+. \quad (0.1.29)$$

(ii) Explosion en temps infini :

1. Il existe une solution  $u \in C([0, +\infty), H^1)$  de (0.1.1) explose à  $+\infty$  avec

$$\|u(t)\|_{H^1} \sim e^t, \quad \text{quand } t \rightarrow +\infty. \quad (0.1.30)$$

2. Pour tout  $\nu > 0$ , il existe une solution  $u \in C([0, +\infty), H^1)$  de (0.1.1) explose à  $+\infty$  avec

$$\|u(t)\|_{H^1} \sim t^\nu, \quad \text{quand } t \rightarrow +\infty. \quad (0.1.31)$$

Dans [56], Martel, Merle, Nakanishi et Raphaël ont prouvé que les données initiales en  $\mathcal{A}_{\alpha_0}$  qui correspondent au régime de Soliton est une variété de seuil de codimension 1 entre (Blow up) et (Exit).

**Théorème 0.1.9** (Variété de seuil de codimension 1). Soit

$$\mathcal{A}_{\alpha_0}^\perp = \left\{ \varepsilon_0 \in H^1 \mid \|\varepsilon_0\|_{H^1} < \alpha_0, \int_{y>0} y^{10} \varepsilon_0^2 < 1, (\varepsilon_0, \mathcal{Q}) = 0 \right\}.$$

Alors, il existe  $\alpha_0 > 0$ ,  $\beta_0 > 0$ , et une  $C^1$  fonction  $A$  :

$$\mathcal{A}_{\alpha_0}^\perp \rightarrow (-\beta_0, \beta_0),$$

tel que pour tout  $\gamma_0 \in \mathcal{A}_{\alpha_0}^\perp$  et  $a \in (-\beta_0, \beta_0)$ , la solution de (0.1.1) correspondant à  $u_0 = (1+a)\mathcal{Q} + \gamma_0$  satisfait :

- (Soliton)** si  $a = A(\gamma_0)$ ;
- (Blow up)** si  $a > A(\gamma_0)$ ;
- (Exit)** si  $a < A(\gamma_0)$ .

En particulier, soit

$$\mathcal{Q} = \left\{ u_0 \in H^1 \mid \exists \lambda_0, x_0, \text{ tel que } u_0 = \lambda_0^{-\frac{1}{2}} \mathcal{Q}(\lambda_0^{-1}(x - x_0)) \right\}.$$

alors il existe un voisinage petit  $\mathcal{O}$  de  $\mathcal{Q}$  dans  $H^1 \cap L^2(y_+^{10} dy)$  et une  $C^1$ -variété  $\mathcal{M} \subset \mathcal{O}$  de codimension 1, telle que  $\mathcal{Q} \subset \mathcal{M}$  et pour tout  $u_0 \in \mathcal{O}$ , la solution associée de (0.1.1) est dans la régime de (Soliton) si et seulement si  $u_0 \in \mathcal{M}$ .



## 0.2 Principaux résultats de l'auteur

Nous avons vu quelques résultats de la dynamique asymptotique pour le gKdV  $L^2$  critique. Dans cette thèse, l'auteur a considéré le gKdV surcritique avec une non-linéarité légèrement supercritique et le gKdV critique avec une perturbation saturée. L'auteur a donné une description explicite de la dynamique explosive et la formation de singularité. Les résultats peuvent être divisés en trois parties :

La première partie concerne le gKdV légèrement surcritique. Sur la base de la construction du profil auto-similaire par Koch [38] et de la technique de gKdV critique [57], l'auteur a prouvé l'existence d'une dynamique stable avec un taux d'explosion auto-similaire dans l'espace d'énergie  $H^1$ . On a également donné une description spécifique de la formation de singularité.

La deuxième partie est consacrée à la construction d'une solution explosive à gKdV légèrement surcritique avec donnée initiale grande. L'idée clé est d'envisager des solutions qui se comportent comme une somme de bulles découplée. Et chaque bulle explose en même temps. Ceci est fait par un argument topologique standard. Par conséquent, ces solutions ne sont pas stables dans l'espace d'énergie  $H^1$ .

Enfin, dans la troisième partie, nous considérons le gKdV critique  $L^2$  avec une perturbation saturée. Dans ce cas, l'équation a encore une famille de solitons. L'auteur a donné une classification spécifique de la dynamique asymptotique près du soliton. Il n'y a que trois possibilités : (i) la solution converge asymptotiquement vers un soliton ; (ii) la solution reste dans un petit voisinage de la variété de solitons, en s'étalant par de temps infiniment grande (*Blow down*) ; (iii) solution quitte tout petit voisinage de la variété de solitons.

Les résultats mentionnés ci-dessus sont basés sur les travaux suivants de l'auteur :

- Y. Lan, *Stable self-similar blow-up dynamics for slightly  $L^2$ -supercritical generalized KdV equations*. Comm. Math. Phys. **345** (2016), 223–269.
- Y. Lan, *Blow-up solutions for  $L^2$ -supercritical gKdV equations with exactly  $k$  blow-up points*, arXiv :1602.08617, soumis.
- Y. Lan, *On asymptotic dynamics for  $L^2$  critical generalized KdV equations with a saturated perturbation*, arXiv :1609.05146, soumis.

### 0.2.1 Dynamique explosive auto-similaire stable pour la gKdV $L^2$ surcritique.

Dans cette partie, nous considérons le gKdV légèrement  $L^2$  surcritique, i.e. (0.1.1) avec

$$5 < p < 5 + \delta, \quad 0 < \delta \ll 1.$$

L'existence de solutions explosives dans le cas surcritique est un problème ouvert.

Considérons d'abord les équations de NLS  $L^2$  surcritiques focalisante :

$$\begin{cases} i\partial_t u + \Delta u + |u|^{p-1}u = 0, & (t, x) \in [0, T) \times \mathbb{R}^d, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}^d), \end{cases}$$

avec la non linéarité  $p > 1 + \frac{4}{d}$ .

De [29, 77, 87, 88], il existe des solutions radiales qui explosent sur une sphère asymptotiquement à la place d'un point d'explosion pour  $d \geq 2$ ,  $p = 5$ . De [27, 28, 94], il existe des solutions explosives avec la symétrie cylindrique pour  $d \geq 3$ ,  $p = 3$ . Et dans [75], Merle, Raphaël et Szeftel construisent une dynamique explosive auto-similaire stable pour une non linéarité légèrement  $L^2$ -surcritique, avec des données initiales non radiales en basse dimension (i.e.  $d \leq 5$ ).

Pour le cas légèrement surcritique de gKdV, la simulation numérique par Dix et McKinney [13] suggère qu'il existe des solutions explosives auto-similaires pour les équations des gKdV surcritiques<sup>2</sup>, i.e. une solution de la forme suivante :

$$u(t, x) = \frac{1}{\lambda^{\frac{2}{p-1}}(t)} Q_b \left( \frac{x}{\lambda(t)} \right), \quad \lambda(t) = \sqrt[3]{3b(T-t)},$$

avec  $b > 0$ . La solution exacte de ce type a été construite par Koch dans [38] pour des non-linéarités légèrement surcritiques. Mais une telle solution n'est pas dans l'espace Sobolev critique  $\dot{H}^{\sigma_c}$ , donc pas dans l'espace d'énergie  $H^1$ , ce qui rend difficile la stabilité de cette solution auto-similaire. Cependant, nous pouvons choisir une approximation appropriée du profil auto-similaire exact, ce qui conduit à un résultat similaire à celui des équations de Schrödinger légèrement surcritiques, i.e. [75].

Plus précisément, nous avons :

**Théorème 0.2.1** (Existence et stabilité d'une dynamique explosive auto-similaire, [46]). *Il existe un  $p^* > 5$  tel que pour tout  $p \in (5, p^*)$ , il existe un sous-ensemble ouvert non vide  $\mathcal{O}_p$  dans  $H^1$  tel que si  $u_0 \in \mathcal{O}_p$ , alors la solution correspondante de (0.1.1) explose en temps fini  $0 < T < +\infty$ , avec la dynamique suivante : il existe des paramètres géométriques  $(\lambda(t), x(t)) \in \mathbb{R}_+^* \times \mathbb{R}$  et un terme d'erreur  $\varepsilon(t)$  tel que :*

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} [Q_p + \varepsilon(t)] \left( \frac{x - x(t)}{\lambda(t)} \right) \quad (0.2.1)$$

avec

$$\|\varepsilon_y(t)\|_{L^2} \leq \sqrt{p-5}, \quad x(t) \rightarrow x(T), \quad \frac{\lambda^3(t)}{T-t} \sim p-5. \quad (0.2.2)$$

Théorème 0.2.1 est la première construction de solutions explosives des équations de gKdV surcritiques avec des données initiales en  $H^1$ . Il s'agit d'une dynamique explosive stable au lieu d'une seule solution explosive. Il est donc différent de la solution auto-similaire construite par Koch dans [38], bien que la construction dans cet article s'appuie profondément sur l'article de Koch.

<sup>2</sup>. Nous savons d'après le Théorème 0.1.3 qu'il n'existe pas de solution explosive auto-similaire pour l'équation de gKdV  $L^2$  critique.

La conclusion ici est presque la même que dans le cas de Schrödinger dans [75]. Mais nous avons besoin d'une stratégie totalement différente, en raison de la structure distincte de ces deux équations. En effet, notre stratégie ici est proche de celle de [57] pour gKdV critique, en raison de la condition légèrement surcritique sur la non-linéarité.

### Stratégie de la preuve :

Nous commençons par l'argument de modulation standard pour la solution proche de l'état fondamental, qui a été développé dans [53, 55, 70] pour gKdV  $L^2$  critique et [71, 72, 73, 74, 86] pour NLS  $L^2$  critique. Plus précisément, nous considérons la solution de la forme suivante :

$$u(t, x) = \frac{1}{\lambda^{\frac{2}{p-1}}(t)} [Q_{b(t)} + \varepsilon(t)] \left( \frac{x - x(t)}{\lambda(t)} \right),$$

ou  $Q_b \in H^1$  est une approximation appropriée du profil auto-similaire exact construit par Koch [38] et  $(b(t), \lambda(t), x(t))$  sont des paramètres géométriques satisfaisant certaines conditions d'orthogonalité appropriées pour  $\varepsilon$ .

Ensuite, en utilisant les conditions d'orthogonalité, on obtient l'estimation de modulation suivante pour les paramètres géométriques :

$$\begin{aligned} s &= \int_0^t \frac{1}{\lambda^3(\tau)} d\tau, \\ \left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| &\lesssim b_c^2 + \|\varepsilon\|_{L_{\text{loc}}^2}, \\ |b_s + 2(b - b_c)b_c| &\lesssim b_c^3 + b_c \|\varepsilon\|_{L_{\text{loc}}^2}, \end{aligned}$$

ou  $b_c > 0$  est une constante universelle qui ne dépend que de  $p$ .

On peut donc utiliser un argument de localisation de la loi de conservation d'énergie, la formule de monotonie développée dans [55, 57] et les estimations inhomogènes de Strichartz développées dans [23] pour obtenir un bon contrôle sur le terme d'erreur  $\varepsilon$  :

$$\int_{y > \kappa b_c^{-\frac{1}{20}}} \varepsilon_y^2 < b_c^{\frac{55}{7}}, \quad \|\varepsilon\|_{L_{\text{loc}}^2}^2 \leq b_c^{\frac{7}{2}}, \quad \|\varepsilon\|_{L^{5/2}} \leq b_c^{\frac{7}{15}},$$

pour une constante universelle  $\kappa \in (0, \frac{1}{2})$ .

Enfin, en combinant les estimations de modulation et le contrôle sur le terme d'erreur  $\varepsilon$ , on obtient :

$$0 < T < +\infty, \quad b(t) \sim b_c, \quad \lambda(t) \sim \sqrt[3]{3b_c(T-t)}, \quad x(t) \rightarrow x(T),$$

qui est le résultat du Théorème 0.2.1.

## 0.2.2 Solutions explosives avec plusieurs points d'explosion

Dans cette partie, nous considérons des solutions explosives avec plusieurs points d'explosion, où la motivation est de construire des solutions explosives grandes.

Il y a plusieurs exemples pour une telle construction :

- Merle [65] pour le NLS  $L^2$  critiques avec taux de explosion conforme ;
- Fan [20] et Planchon, Raphaël [85] pour les NLS  $L^2$  critiques avec taux de explosion de "log-log" ;
- Merle [68] et Merle, Zaag [79] pour les équations de chaleur avec taux d'explosion d'EDO.

Ces constructions correspondent au cas d'interaction faible, i.e. que l'interaction entre les bulles ne modifie pas le taux de explosion de chaque bulle. Il existe également quelques exemples de solution avec des bulles fortement en interaction :

- Martel, Raphaël [62] pour le NLS  $L^2$  critiques ;
- Cortázar, Del Pino, Musso [10] pour le équations de chaleur énergie-critique dans domaine ;
- Jendrej [30, 31] pour les équations de la ondes énergie-critique.

Pour le gKdV légèrement  $L^2$  surcritique, dans [46], l'auteur prouve l'existence d'une dynamique explosive stable où la singularité se concentre à un certain point fini (voir la section 1.2.1). Par conséquent, nous avons :

**Théorème 0.2.2** (Solutions avec plusieurs points d'explosion, [44]). *Il existe des constantes universelles  $p^* > 5$ ,  $c > 0$  telles que pour tout  $p \in (5, p^*)$ ,  $k \in \mathbb{N}^+$ , si  $2 \leq k \leq c|\log(p-5)|$ , alors pour tout  $k$  points distinct par paires :  $x_1, x_2, \dots, x_k \in \mathbb{R}$ , il existe une solution  $u(t)$  de (0.1.1), qui explose en temps fini  $T < +\infty$ . Et pour  $t$  près de  $T$ , il existe des paramètres d'échelle  $\lambda_j(t) \in \mathbb{R}^+$  et un terme d'erreur  $\tilde{u}(t, x)$  avec*

$$u(t, x) = \sum_{j=1}^k \frac{1}{\lambda_j^{\frac{2}{p-1}}(t)} Q_p \left( \frac{x - x_j}{\lambda_j(t)} \right) + \tilde{u}(t, x), \quad (0.2.3)$$

ou pour tout  $j = 1, \dots, k$ , aet  $t$  près de  $T$ ,

$$\frac{\lambda_j^3(t)}{T-t} \sim p-5, \quad \lambda_j(t)^{1-\sigma_c} \|\tilde{u}_x(t)\|_{L^2} \leq \delta(p), \quad (0.2.4)$$

pour une petite constante  $\delta(p)$  avec  $\lim_{p \rightarrow 5} \delta(p) = 0$ .

De plus, l'ensemble d'explosion de  $u(t)$  est exactement  $\{x_1, x_2, \dots, x_k\}$ .

De (0.2.3), on obtient de solutons explosives grandes pour (0.1.1) dans le sens que  $\|u_0\|_{L^{(p-1)/2}} \sim |\log(p-5)|$ , pour  $p$  assez proche de 5.

Pour  $k \geq 2$ , la solution explosive avec  $k$  bubbles est *n'est pas stable* dans  $H^1$ . En effet, nous avons besoin de certaines conditions spéciales (instables) sur les données initiales, qui peuvent être obtenues par un argument topologique standard. Un argument similaire est également utilisé dans [12, 20, 62, 64, 79] pour des solutions explosives avec plusieurs points d'explosion et [11, 41, 49, 60, 61] pour des solutions avec multi-solitons.

### Stratégie de la preuve :

De [46], il existe une dynamique explosive auto-similaire stable telle que la singularité se concentre à un certain point fini. Nous pouvons envisager des solutions qui peuvent être écrites comme la somme de  $k$  découplé solutions explosives self-similaires. Plus précisément, nous considérons des solutions de la forme suivante :

$$u(t, x) = \sum_{j=1}^k \frac{1}{\lambda_j^{\frac{2}{p-1}}(t)} Q_{b_j(t)} \left( \frac{x - x_j(t)}{\lambda_j(t)} \right) + \tilde{u}(t, x),$$

où  $Q_{b_j}$  est le profil approximatif auto-similaire défini dans [46], et les paramètres  $(b_j(t), \lambda_j(t), x_j(t))$  sont choisis pour ajuster certaines conditions d'orthogonalité appropriées pour le terme d'erreur  $\tilde{u}$ . L'idée cruciale ici est de construire une solution de cette forme telle que chaque bulle

$$\frac{1}{\lambda_j^{\frac{2}{p-1}}(t)} Q_{b_j(t)} \left( \frac{x - x_j(t)}{\lambda_j(t)} \right)$$

se comporte comme une solution explosive auto-similaire construite dans [46]. Nous savons de [46] que chaque bulle se concentre à un point fini. Nous nous attendons à ce que ces points (dits *emph* points d'explosion) soient disjoints et la distance entre ces points soit suffisamment grande. Dans ce cas, l'interaction entre les différentes bulles est très faible.

Ensuite, nous devons nous assurer que le temps d'explosion de chaque bulle est même. Cela nécessite une condition supplémentaire sur les données initiales, qui peut être obtenue par un argument topologique standard. Par conséquent, les solutions explosives avec plusieurs bulles ne sont pas stables dans  $H^1$ . Plus précisément, avec une condition appropriée sur les données initiales, on peut supposer que pour tout  $1 \leq i < j \leq k$  et tout  $t \in [0, T)$ ,

$$\frac{1}{2^{k+1}} < \frac{\lambda_i(t)}{\lambda_j(t)} < 2^{k+1}.$$

Ensuite, on peut suivre un argument similaire à [46] pour obtenir les estimations de modulation. Pour tout  $j = 1, \dots, k$ ,

$$\begin{aligned} \left| \frac{d\lambda}{ds_j} \frac{1}{\lambda_j} + b_j \right| + \left| \frac{dx_j}{ds_j} \frac{1}{\lambda_j} - 1 \right| &\lesssim b_c^2 + \|\varepsilon_j\|_{L_{\text{loc}}^2}, \\ \left| \frac{db_j}{ds_j} + 2(b_j - b_c)b_c \right| &\lesssim b_c^3 + b_c \|\varepsilon_j\|_{L_{\text{loc}}^2}, \end{aligned}$$

ou

$$s_j = \int_0^t \frac{1}{\lambda_j^3(\tau)} d\tau, \quad \varepsilon_j(t, y) = \lambda_j^{\frac{2}{p-1}}(t) \tilde{u}(t, \lambda_j(t)y + x_j(t)).$$

En utilisant la condition sur  $k$  (le nombre de points d'explosion), nous avons des estimations similaires sur le terme d'erreur. Pour tout  $j = 1, \dots, k$ ,

$$\int_{b_c^{-20} > y > \kappa b_c^{-\frac{1}{20}}} (\varepsilon_j)_y^2 < b_c^{\frac{55}{7}}, \quad \|\varepsilon_j\|_{L_{\text{loc}}^2}^2 \leq b_c^{\frac{7}{2}}, \quad \|\varepsilon_j\|_{L^{5/2}} \leq b_c^{\frac{7}{15}}.$$

Par conséquent, nous avons pour tous  $1 \leq i < j \leq k$ ,

$$\begin{aligned} 0 < T < +\infty, \quad b_j(t) \sim b_c, \quad \lambda_j(t) \sim \sqrt[3]{3b_c(T-t)}, \\ x_j(t) \rightarrow x_j(T), \quad x_i(T) \neq x_j(T). \end{aligned}$$

qui prouve l'existence de solutions explosives avec plusieurs points d'explosion.

Enfin, en utilisant un autre argument topologique (voir [68, Proposition 5.2]), on peut montrer que les points d'explosion peuvent être choisis arbitrairement, qui est exactement Théorème 0.2.2.

### 0.2.3 Dynamique asymptotique pour la gKdV $L^2$ -critique avec une perturbation saturée

Dans cette partie, nous considérons pour la gKdV  $L^2$  critique avec une perturbation saturée :

$$\begin{cases} \partial_t u + (u_{xx} + u^5 - \gamma u|u|^{q-1})_x = 0, & (t, x) \in [0, T] \times \mathbb{R}, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \quad (0.2.5)$$

avec  $q > 5$  et  $0 < \gamma \ll 1$ .

L'équation a deux lois de conservation, i.e. la masse et l'énergie :

$$\begin{aligned} M(u(t)) &= \int u(t)^2 = M_0, \\ E(u(t)) &= \frac{1}{2} \int u_x(t)^2 - \frac{1}{6} \int u(t)^6 + \frac{\gamma}{q+1} \int |u(t)|^{q+1} = E_0. \end{aligned}$$

Nous pouvons voir que la solution de (0.2.5) est toujours globale en temps et bornée dans  $H^1$ , à partir de la théorie locale [36] et des lois de conservation.

Cette équation a la règle de pseudo-échelle suivante : pour tout  $\lambda_0 > 0$ , si  $u(t, x)$  est une solution à (0.2.5), alors

$$u_{\lambda_0}(t, x) = \lambda_0^{-\frac{1}{2}} u(\lambda_0^{-3}t, \lambda_0^{-1}x), \quad (0.2.6)$$

est une solution à

$$\begin{cases} \partial_t v + (v_{xx} + v^5 - \lambda_0^{-m} \gamma v|v|^{q-1})_x = 0, & (t, x) \in [0, \lambda_0^{-3}T] \times \mathbb{R}, \\ v(0, x) = \lambda_0^{-\frac{1}{2}} u_0(\lambda_0^{-1}x) \in H^1(\mathbb{R}), \end{cases}$$

avec

$$m = \frac{q-5}{2} > 0. \quad (0.2.7)$$

La norme  $L^2$  est invariante par la transformation de (0.2.6).

Il existe encore des solutions d'onde solitaire données par

$$u(t, x) = \lambda_0^{-\frac{1}{2}} \tilde{Q}_{\lambda_0^{-m}\gamma}(\lambda_0^{-1}(x-x_0) - \lambda_0^{-3}(t-t_0)),$$

pour tout  $\lambda_0 > 0$ ,  $t_0 \in \mathbb{R}$ ,  $x_0 \in \mathbb{R}$  avec  $\lambda_0^{-m}\gamma \ll 1$ . Ici pour tout  $0 \leq \omega < \omega^* \ll 1$ ,  $\tilde{Q}_\omega$  est la solution radiale non négative unique avec décroissance exponentielle à l'EDO suivante :

$$\tilde{Q}_\omega'' - \tilde{Q}_\omega + \tilde{Q}_\omega^5 - \omega \tilde{Q}_\omega |\tilde{Q}_\omega|^{q-1} = 0.$$

Le but de cette partie est de classer la dynamique de (0.2.5) près de l'état fondamental, lorsque  $\gamma$  est suffisamment petite. L'idée principale est que le terme de défocalisation  $\gamma u |u|^{q-1}$  a un effet non linéaire plus faible que le terme de focalisation  $u^5$ . Ainsi, on peut s'attendre à ce que (0.2.5) ait un comportement de séparation similaire<sup>3</sup> comme (0.1.1), lorsque  $\gamma$  est petit.

Plus précisément, nous fixons une petite constante universelle  $\omega^* > 0$  (pour assurer l'existence de l'état fondamental  $\tilde{Q}_\omega$ ), et introduisons le suivant  $L^2$  tube autour de  $\tilde{Q}_\gamma$  :

$$\mathcal{T}_{\alpha^*, \gamma} = \left\{ u_0 \in H^1 \left| \inf_{\lambda_0 > 0, \lambda_0^{-m}\gamma < \omega^*, x_0 \in \mathbb{R}} \left\| u_0 - \frac{1}{\lambda_0^{\frac{1}{2}}} \tilde{Q}_{\lambda_0^{-m}\gamma} \left( \frac{x-x_0}{\lambda_0} \right) \right\|_{L^2} < \alpha^* \right. \right\},$$

et L'ensemble de données initiales

$$\mathcal{A}_{\alpha_0, \gamma} = \left\{ u_0 \in H^1 \left| u_0 = \tilde{Q}_\gamma + \varepsilon_0, \|\varepsilon_0\|_{H^1} < \alpha_0, \int_{y>0} y^{10} \varepsilon_0^2 < 1 \right. \right\},$$

pour certaines constantes universelles  $0 < \alpha_0 \ll \alpha^*$ . Ensuite nous avons :

**Théorème 0.2.3** (Dynamique dans  $\mathcal{A}_{\alpha_0, \gamma}$ , [45]). *Pour tout  $q > 5$ , il existe une constante  $0 < \alpha^*(q) \ll 1$ , tel que si  $0 < \gamma \ll \alpha_0 \ll \alpha^* < \alpha^*(q)$ , alors pour tout  $u_0 \in \mathcal{A}_{\alpha_0, \gamma}$ , La solution correspondante  $u(t)$  à (0.2.5) a un et un seul des comportements suivants :*

**-(Soliton) :** *Pour tout  $t \in [0, +\infty)$ ,  $u(t) \in \mathcal{T}_{\alpha^*, \gamma}$ . De plus, il existe une constante  $\lambda_\infty \in (0, +\infty)$  et une  $C^1$  fonction  $x(t)$  tel que*

$$\lambda_\infty^{\frac{1}{2}} u(t, \lambda_\infty \cdot + x(t)) \rightarrow \tilde{Q}_{\lambda_\infty^{-m}\gamma} \text{ dans } H_{\text{loc}}^1, \text{ quand } t \rightarrow +\infty; \quad (0.2.8)$$

$$x(t) \sim \frac{t}{\lambda_\infty^2}, \text{ quand } t \rightarrow +\infty. \quad (0.2.9)$$

**-(Blow down) :** *Pour tout  $t \in [0, +\infty)$ ,  $u(t) \in \mathcal{T}_{\alpha^*, \gamma}$ . De plus, il existe deux fonctions de classe  $C^1$ ,  $\lambda(t)$  et  $x(t)$ , telles que*

$$\lambda^{\frac{1}{2}}(t) u(t, \lambda(t) \cdot + x(t)) \rightarrow Q \text{ dans } H_{\text{loc}}^1, \text{ quand } t \rightarrow +\infty; \quad (0.2.10)$$

$$\lambda(t) \sim t^{\frac{2}{q+1}}, \quad x(t) \sim t^{\frac{q-3}{q+1}}, \text{ quand } t \rightarrow +\infty, \quad (0.2.11)$$

**-(Exit) :** *Il existe  $0 < t_\gamma^* < +\infty$  tel que  $u(t_\gamma^*) \notin \mathcal{T}_{\alpha^*, \gamma}$ .*

*Il existe des solutions associées à chaque régime. De plus, le régime (Soliton) et (Exit) sont stables sous les petites perturbations dans  $\mathcal{A}_{\alpha_0, \gamma}$ .*

3. Voir Théorème 0.1.7.

Le comportement de (blow down) ne semble pas apparaître dans le cas imperturbable de gKdV critique. Des exemples de solution avec ce type de comportement ont également été trouvés par Donninger, Krieger [16] pour les équations d'ondes énergie-critique. Alors que pour NLS masse-critique, le comportement de (blow down) peut être obtenu comme la transformation pseudo-conforme du régime de log-log<sup>4</sup>. Cependant, Théorème 0.2.3 est la première fois que ce type de comportement de (blow down) est obtenu dans le contexte d'une perturbation saturée. De plus, dans Théorème 0.2.3, nous pouvons démontrer (comme le Théorème 0.1.9) que le régime (blow down) est un seuil de codimension 1 entre deux régimes stable, ce qui est au contraste avec la cas de équations des Schrödinger non linéaire masse-critique, où le régime de (blow down) est stable.

Maintenant nous considérons le cas où  $\gamma \rightarrow 0$ . Comme nous l'avons mentionné précédemment, le terme de défocalisation  $\gamma u|u|^{q-1}$  a un effet non linéaire plus faible que le terme de focalisation  $u^5$ . On suppose donc que le régime de seuil (blow down) défini dans le Théorème 0.2.3 est une perturbation du régime (Soliton) défini dans le Théorème 0.1.7.

Plus précisément, nous avons :

**Théorème 0.2.4** (La cas de limite quand  $\gamma \rightarrow 0$ , [45]). *Fixons une non-linéarité  $q > 5$ , et choisissons  $0 < \alpha_0 \ll \alpha^* < \alpha^*(q)$  comme dans Théorème 0.2.3. Pour tout  $u_0 \in \mathcal{A}_{\alpha_0}$ , soit  $u(t)$  la solution correspondante de (0.1.1), et  $u_\gamma(t)$  la solution correspondante de (0.2.5). Ensuite nous avons :*

1. *Si  $u(t)$  appartient au régime (Blow up) défini dans le Théorème 0.1.7, alors il existe  $0 < \gamma(u_0, \alpha_0, \alpha^*, q) \ll \alpha_0$  tel que  $0 < \gamma < \gamma(u_0, \alpha_0, \alpha^*, q)$ , alors  $u_\gamma(t)$  appartient au régime (Soliton) défini dans le Théorème 0.2.3. De plus, il existe des constantes  $d_i = d_i(u_0, q) > 0$ ,  $i = 1, 2$ , tel que*

$$d_1 \gamma^{\frac{2}{q-1}} \leq \lambda_\infty \leq d_2 \gamma^{\frac{2}{q-1}}, \quad (0.2.12)$$

*ou  $\lambda_\infty$  est la constante définie dans (0.2.8).*

2. *Si  $u(t)$  appartient au régime (Exit) défini dans le Théorème 0.1.7, alors il existe  $0 < \gamma(u_0, \alpha_0, \alpha^*, q) \ll \alpha_0$  tel que si  $0 < \gamma < \gamma(u_0, \alpha_0, \alpha^*, q)$ , alors  $u_\gamma(t)$  appartient au régime (Exit) défini dans le Théorème 0.2.3.*

Théorème 0.2.3 et Théorème 0.2.4 donnent une description spécifique du comportement asymptotique de la solution à (0.2.5) avec des données initiales qui conduisent à une solution explosive pour l'équation non perturbée. Pour autant que nous sachions, c'est le premier résultat identique de ce type pour les équations dispersives non linéaires. On peut aussi s'attendre à des résultats similaires pour le NLS critique<sup>5</sup> ou les cas gKdV légèrement surcritiques. Mais ces problèmes sont encore complètement ouverts.

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4. Voir [77, (1.16)] par exemple.

5. Dans [48], Malkin prédit un comportement asymptotique similaire pour la solution au problème saturé du NLS critique.



### Stratégie de la preuve :

Nous suivons des arguments similaires à ceux de [57]. Nous considérons la solution de la forme suivante :

$$u(t, x) = \frac{1}{\lambda^{\frac{1}{2}}(t)} [Q_{b(t), \omega(t)} + \varepsilon(t)] \left( \frac{x - x(t)}{\lambda(t)} \right),$$

ou

$$\omega(t) = \frac{\gamma}{\lambda^m(t)},$$

et  $Q_{b, \omega}$  est une solution approximative à l'EDO suivante :

$$b \Lambda Q_{b, \omega} + (Q_{b, \omega}'' - Q_{b, \omega} + Q_{b, \omega}^5 - \omega Q_{b, \omega} |Q_{b, \omega}|^{q-1})' = 0.$$

On choisit aussi les paramètres  $\lambda(t), b(t), x(t)$  tel que certaines conditions d'orthogonalité sont satisfaites.

Les calculs formels montrent que les paramètres satisfont au système de EDO suivant :

$$\begin{cases} b_s + 2b^2 + c_0 \omega_s = 0, \\ \frac{\lambda_s}{\lambda} + b = \frac{x_s}{\lambda} - 1 = 0, \end{cases}$$

ou  $c_0 = c_0(q) > 0$  est une constante universelle et

$$s = \int_0^t \frac{1}{\lambda^3(\tau)} d\tau$$

est le temps redimensionné.

Il est facile de voir que le système de EDO a une quantité conservée (la loi de conservation d l'énergie de (0.2.5)) :

$$\frac{b(t) + mc_0/(m+2)\omega(t)}{\lambda^2(t)} \equiv L_0.$$

Si  $L_0 > 0$ , nous avons :

$$b(t) \rightarrow 0, \quad \lambda(t) \rightarrow \left( \frac{m\gamma c_0}{(m+2)L_0} \right)^{\frac{1}{m+2}}, \quad x(t) \sim \left( \frac{(m+2)L_0}{m\gamma c_0} \right)^{\frac{2}{m+2}} t,$$

quand  $t \rightarrow +\infty$ , qui correspond au le régime de (Soliton).

Si  $L_0 = 0$ , nous avons :

$$b(t) \sim -t^{-\frac{q-5}{q+1}}, \quad \lambda(t) \sim t^{\frac{2}{q+1}}, \quad x(t) \sim t^{\frac{q-3}{q+1}},$$

quand  $t \rightarrow +\infty$ , qui correspond au le régime de (Blow down).

Si  $L_0 < 0$ , nous avons :

$$b(t) \rightarrow -\infty, \quad \lambda(t) \rightarrow +\infty,$$

si  $t \rightarrow +\infty$ , qui correspond au le régime de (Exit).

Maintenant, nous considérons le plein ansatz. En utilisant un argument similaire à celui de [57], nous avons :

$$\left| \frac{\lambda_s}{\lambda} + b \right| \lesssim \|\varepsilon\|_{H_{\text{loc}}^1}^2 + |b|(\omega + |b|), \quad (0.2.13)$$

et pour tout  $0 \leq s_1 < s_2 < +\infty$ ,

$$\left| \left( \frac{\lambda + mc_0/(m+2\omega)}{\lambda^2} \right) \Big|_{s_1}^{s_2} \right| \lesssim \frac{\|\varepsilon\|_{H_{\text{loc}}^1}^2}{\lambda^2(s_1)} + \frac{b^2(s_1) + \omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2) + \omega^2(s_2)}{\lambda^2(s_2)}. \quad (0.2.14)$$

Nous avons aussi la formule de monotonie sur le terme d'erreur  $\varepsilon$ . Pour tout  $j = 1, 2$ , nous avons

$$\left( \frac{\mathcal{F}}{\lambda^{2(j-1)}} \right)_s + \mu \frac{\|\varepsilon\|_{H_{\text{loc}}^1}^2}{\lambda^{2(j-1)}} \lesssim \frac{b^2(b^2 + \omega^2)}{\lambda^{2(j-1)}},$$

ou  $\mathcal{F}$  est une fonctionnelle bien choisi. En raison de notre choix des conditions d'orthogonalité,  $\mathcal{F}$  est coercive :

$$\mathcal{F} \sim \|\varepsilon\|_{H_{\text{loc}}^1}^2.$$

Ensuite, nous pouvons voir la condition de séparation pour ces trois scénarios suivants<sup>6</sup> :

- **(Blow down)** : Pour tout  $t \in [0, +\infty)$ , nous avons

$$|b(t) + mc_0/(m+2)\omega(t)| < C^* \left( \|\varepsilon(t)\|_{H_{\text{loc}}^1}^2 + b^2(t) + \omega^2(t) \right). \quad (0.2.15)$$

- **(Exit)** : Il existe une  $t_1^* < +\infty$  tel que

$$b(t_1^*) + mc_0/(m+2)\omega(t_1^*) \leq -C^* \left( \|\varepsilon(t_1^*)\|_{H_{\text{loc}}^1}^2 + b^2(t_1^*) + \omega^2(t_1^*) \right). \quad (0.2.16)$$

- **(Soliton)** : Il existe une  $t_1^* < +\infty$  tel que

$$b(t_1^*) + mc_0/(m+2)\omega(t_1^*) \geq C^* \left( \|\varepsilon(t_1^*)\|_{H_{\text{loc}}^1}^2 + b^2(t_1^*) + \omega^2(t_1^*) \right). \quad (0.2.17)$$

Ici,  $C^*$  est une constante universelle grande.

Le comportement exact des paramètres peut être obtenu par l'intégrant l'EDO approximative des paramètres.

Enfin, il est facile de voir que Théorème 0.2.4 est une conséquence directe d'une théorie de  $H^1$  perturbation modifiée et des conditions de séparation (0.2.15)–(0.2.17).

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6. From (0.2.14) et un choix particulier de la constante universelle  $C^*$ , on peut voir que (0.2.16) et (0.2.17) ne peuvent pas se produire simultanément.



# Chapter 1

## Introduction (English version)

### 1.1 Overview of the asymptotic dynamics for generalized KdV equations

#### 1.1.1 Preliminary

The aim of this thesis is to study the asymptotic dynamics for some *nonlinear dispersive equations*.

By nonlinear dispersive equation we mean an equation which combines a linear equation with dispersive behavior (the solution decays uniformly in time) and a suitable nonlinearity. Typically, the equation is of the following form:

$$u_t = Lu + N(u, \nabla u, \dots),$$

where  $N$  is the nonlinear term and  $L$  is an anti-self-adjoint linear operator given by

$$\mathcal{F}(Lu)(\xi) = ip(\xi)\mathcal{F}u(\xi), \quad p(\xi) \in \mathbb{R}.$$

Here  $\mathcal{F}$  is the Fourier transform and  $D_\xi^2 p(\xi) \neq 0$  for all  $\xi \neq 0$ .

Typical examples for nonlinear dispersive equations are Korteweg-de Vries equation, nonlinear Schrödinger equation, Benjamin-Ono equation, KP-II equation and so on.

In many cases, the local wellposedness is known. More precisely, for initial data which is in a suitable Banach space, we can find a unique strong solution (local in time) to the equation. Moreover, the map from the initial data to the solution is continuous.

Denote by  $(T_-, T_+)$  the maximal interval of time where the solution is defined. An interesting problem is to understand the asymptotic behaviors near  $T_-$  and  $T_+$ . Typical asymptotic behaviors are the following:

1. Scattering: the solution exists globally (i.e.  $T_- = -\infty$  or  $T_+ = +\infty$ ), and converges to a linear solution;

2. Blow-up: the solution breaks down in finite time (i.e.  $T_- > -\infty$  or  $T_+ < +\infty$ ), or the solution exists globally but has some singular behaviors as  $t \rightarrow \infty$ ;
3. Soliton: the solution exists globally and converges asymptotically to some special class of solutions (the so called *soliton*).

In the second and the third case, we also expect to give a specific description of the asymptotic dynamics of the solution.

So let us consider some special class of nonlinear dispersive equations, i.e. the generalized focusing Korteweg-de Vries equations (gKdV):

$$\begin{cases} \partial_t u + (u_{xx} + u|u|^{p-1})_x = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0, x) = u_0(x), \end{cases} \quad (1.1.1)$$

with  $p > 1$ .

The original Korteweg-de Vries equation,

$$\partial_t u + (u_{xx} + u^2)_x = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (1.1.2)$$

arises in physics as a model for unidirectional waves motion. The exact formulation of this equation comes from Korteweg and de Vries [40]. This equation as well as the modified Korteweg-de Vries equation:

$$\partial_t u + (u_{xx} + u^3)_x = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (1.1.3)$$

have been studied well for being completely integrable using the method of inverse scattering (see Eckhaus & Schuur [19], Lax [47] and Miura [80]).

The generalized KdV equations as well as the nonlinear Schrödinger equations are also considered as universal models for Hamiltonian system in infinite dimension. From Hamiltonian structure, we have the following two conservation laws, i.e. mass and energy <sup>1</sup>:

$$M(u(t)) = \int u^2(t) = \int u_0^2 = M(u_0), \quad (1.1.4)$$

$$\begin{aligned} E(u(t)) &= \frac{1}{2} \int u_x^2(t) - \frac{1}{p+1} \int |u(t)|^{p+1} \\ &= \frac{1}{2} \int (\partial_x u_0)^2 - \frac{1}{p+1} \int |u_0|^{p+1} = E(u_0). \end{aligned} \quad (1.1.5)$$

From these two conservation laws, the Sobolev space  $H^1$  appears as the energy space, so that it is the natural space to study the solutions to (1.1.1). The general question is to understand the behavior of the solutions of (1.1.1) with initial data  $u_0 \in H^1$ .

Local existence in time for solution of (1.1.1) is well studied, Kato [33] and Ginibre, Tsutsumi [24] for the  $H^s$  theory ( $s > \frac{3}{2}$ ); Kenig, Ponce and Vega [36] for the  $L^2$  theory in the case  $p = 5$  and sharp  $H^s$  theory for  $p > 5$ ; Strunk [89] for the  $\dot{B}_{2,\infty}^s$  theory with  $p > 5$ ; Bourgain [4] for the periodic cases.

In this thesis, we mainly use the following local existence and uniqueness results:

- 
1. The KdV equation (1.1.2) has infinite many conservation laws for being completely integrable.

**Proposition 1.1.1** (Local wellposedness in  $H^1$ , [36]). *For all  $u_0 \in H^1$ , there exist a maximal lifespan  $T = T(\|u_0\|_{H^1}) > 0$  and a unique solution  $u(t) \in C([0, T], H^1)$  of (1.1.1), satisfying (1.1.4) and (1.1.5) on  $[0, T)$ . Moreover, we have the following blow-up criterion: either  $T = +\infty$  or  $T < +\infty$  and*

$$\lim_{t \rightarrow T} \|u(t)\|_{H^1} = +\infty.$$

**Definition 1.1.2.** A *blow-up solution* to (1.1.1) is a solution  $u(t) \in C([0, T], H^1)$ , with maximal lifespan  $0 < T \leq +\infty$ , such that

$$\lim_{t \rightarrow T} \|u(t)\|_{H^1} = +\infty. \quad (1.1.6)$$

When the maximal lifespan is finite (or infinite respectively), we say the solution  $u(t)$  blows up in finite time (or infinite time respectively).

The Cauchy problem (1.1.1) has a standard scaling rule. For all  $\lambda > 0$ ,

$$u_\lambda(t, x) = \frac{1}{\lambda^{\frac{2}{p-1}}} u\left(\frac{t}{\lambda^3}, \frac{x}{\lambda}\right), \quad (1.1.7)$$

is still a solution of (1.1.1). Moreover the  $\dot{H}^{\sigma_c}$  norm of the initial data is invariant under this scaling, where

$$\sigma_c = \frac{1}{2} - \frac{2}{p-1}.$$

- If  $\sigma_c < 0$  (or equivalently  $p < 5$ ), the Cauchy problem (1.1.1) is called  $L^2$  subcritical;
- If  $\sigma_c = 0$  (or  $p = 5$ ), the Cauchy problem (1.1.1) is called  $L^2$  critical;
- If  $\sigma_c > 0$  (or  $p > 5$ ), the Cauchy problem (1.1.1) is called  $L^2$  supercritical.

## 1.1.2 Stability and instability of the solitons

There is a special class of solutions to (1.1.1) called the *soliton solutions*. They are given by

$$u(t, x) = c^{\frac{1}{p-1}} Q_p(\sqrt{c}(x - ct)), \quad (1.1.8)$$

where

$$Q_p'' - Q_p + Q_p|Q_p|^{p-1} = 0, \quad Q_p = \left( \frac{p+1}{2 \cosh^2((p-1)x/2)} \right)^{\frac{1}{p-1}}. \quad (1.1.9)$$

This  $Q_p$  is also called the *ground state*. From [91],  $Q_p$  is related to the best constant of the Gagliardo-Nirenberg's inequality:

$$\int |f|^{p+1} \leq C_p \left( \int (\partial_x f)^2 \right)^{\frac{p-1}{4}} \left( \int f^2 \right)^{\frac{p+3}{4}}. \quad (1.1.10)$$

More precisely, the optimal constant for this inequality is given by

$$C_p = J(Q_p) = \min_{f \in H^1, f \neq 0} J(f) = \frac{p+1}{2 \|Q_p\|_{L^2}^{p-1}},$$

where for all  $f \in H^1$ ,  $f \neq 0$ ,

$$J(f) = \frac{\|\partial_x f\|_{L^2}^{\frac{p-1}{2}} \|f\|_{L^2}^{\frac{p+3}{2}}}{\|f\|_{L^{p+1}}^{p+1}}.$$

In the critical case  $p = 5$ , we denote by

$$Q = Q_5. \quad (1.1.11)$$

The study of the flow near these solitons is crucial for the understanding of the asymptotic behavior of the solutions of (1.1.1). Hence, the stability and asymptotic stability of the solitons become an important problem.

**Definition 1.1.3.** We say that for  $c > 0$  the soliton

$$c^{\frac{1}{p-1}} Q_p(\sqrt{c}(x - ct))$$

is stable in  $H^1$ , if  $\forall \delta_0 > 0$ ,  $\exists \alpha_0 > 0$  such that

$$\begin{aligned} \|u_0 - c^{\frac{1}{p-1}} Q_p(\sqrt{c} \cdot)\|_{H^1} \leq \alpha_0 \implies \\ \forall t \geq 0, \exists x(t) / \|u(t) - c^{\frac{1}{p-1}} Q_p(\sqrt{c} \cdot - x(t))\|_{H^1} \leq \delta_0. \end{aligned} \quad (1.1.12)$$

We say that the soliton family

$$\{c^{\frac{1}{p-1}} Q_p(\sqrt{c}(x - x_0 - ct)) \mid c > 0, x_0 \in \mathbb{R}\}$$

is asymptotically stable in  $H^1$ , if  $\exists \alpha_0 > 0$  such that

$$\begin{aligned} \|u_0 - c^{\frac{1}{p-1}} Q_p(\sqrt{c} \cdot)\|_{H^1} \leq \alpha_0 \implies \\ \exists c_\infty, x(t) / u(t, \cdot + x(t)) - c_\infty^{\frac{1}{p-1}} Q_p(\sqrt{c_\infty} \cdot) \xrightarrow[t \rightarrow +\infty]{} 0, \text{ in } H^1. \end{aligned} \quad (1.1.13)$$

We recall some known results for the stability of solitons and asymptotic stability of the family of solitons:

- In the case  $p = 2, 3, 4$ , it follows from variational argument that the solitons are stable in  $H^1$  (see Bona, Souganidis & Strauss [3], Cazenave & Lions [7], Merle & Vega [78] and Weinstein [93]).
- In the case  $p = 2, 3, 4$ , the family of solitons is asymptotically stable in  $H^1$  (see Martel & Merle [51] and Pego & Weinstein [84]).
- In the  $L^2$  critical case (i.e.  $p = 5$ ), Martel and Merle [52] proved the  $H^1$  instability of the solitons.

- In the  $L^2$  supercritical case (i.e.  $p > 5$ ), Bona, Souganidis and Strauss [3] proved the  $H^1$  instability of the solitons using the argument of Grillakis, Shatah and Strauss [26].

In [60], Martel, Merle and Tsai proved the stability and asymptotic stability of the sum of  $N$  solitons in  $H^1$  for subcritical gKdV. More precisely, we have

**Theorem 1.1.4** (Asymptotic stability of the sum of  $N$  solitons). *Let  $p = 2, 3, 4$ ,  $0 < c_1^0 < \dots < c_N^0$ . There exist  $\gamma_0, A_0, L_0, \alpha_0 > 0$  such that the following is true: Let  $u_0 \in H^1(\mathbb{R})$  and assume that there exist  $L > L_0$ ,  $\alpha < \alpha_0$ , and  $x_1^0 < \dots < x_N^0$ , such that*

$$\left\| u_0 - \sum_{j=1}^N (c_j^0)^{\frac{1}{p-1}} \mathcal{Q}_p \left( \sqrt{c_j^0} \cdot -x_j^0 \right) \right\|_{H^1} \leq \alpha, \quad (1.1.14)$$

$$x_j^0 \geq x_{j-1}^0 + L, \text{ for all } j = 2, \dots, N. \quad (1.1.15)$$

Let  $u(t)$  be the corresponding solution to (1.1.1). Then there exist  $x_1(t), \dots, x_N(t)$  such that:

- *Stability of the sum of  $N$  decoupled solitons:*

$$\forall t \geq 0, \quad \left\| u(t) - \sum_{j=1}^N (c_j^0)^{\frac{1}{p-1}} \mathcal{Q}_p \left( \sqrt{c_j^0} \cdot -x_j(t) \right) \right\|_{H^1} \leq A_0(\alpha_0 + e^{-\gamma_0 L}). \quad (1.1.16)$$

- *Asymptotic stability of the sum of  $N$  solitons. There exist  $0 < c_1^{+\infty} < \dots < c_N^{+\infty}$  with*

$$|c_j^0 - c_j^{+\infty}| \leq A_0(\alpha_0 + e^{-\gamma_0 L}),$$

such that

$$\left\| u(t) - \sum_{j=1}^N (c_j^{+\infty})^{\frac{1}{p-1}} \mathcal{Q}_p \left( \sqrt{c_j^{+\infty}} \cdot -x_j(t) \right) \right\|_{L^2(x > \frac{1}{10} c_1^{+\infty} t)} \rightarrow 0, \quad (1.1.17)$$

$$\dot{x}_j(t) \rightarrow c_j^{+\infty}, \quad (1.1.18)$$

as  $t \rightarrow +\infty$ .

### 1.1.3 Blow up dynamics for $L^2$ critical gKdV

Combining the results of local existence and conservation laws, we have: if  $p < 5$ , then every solution with  $H^1$  initial data is global in time and bounded in  $H^1$ ; if  $p = 5$ , it is easy to see that

$$E(u_0) = E(u(t)) \geq \frac{1}{2} \left( 1 - \left( \frac{\int u_0^2}{\int Q^2} \right)^2 \right) \int u_x^2(t).$$

Hence, in this case, if  $\|u_0\|_{L^2} < \|Q\|_{L^2}$  then the solution for the critical problem is also global in time and bounded in  $H^1$ .

While for critical problem ( $p = 5$ ) with  $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$ , blow-up may occur. The existence of singularity in this case (in either finite time or infinite time) has been a long standing open problem.



In an analogous context, if we consider the  $L^2$  critical focusing nonlinear Schrödinger equations (NLS) in dimension one,

$$\begin{cases} i\partial_t u + \Delta u + u|u|^4 = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \quad (1.1.19)$$

there are several examples of blow-up solutions to (1.1.19). Indeed, the Cauchy problem (1.1.19) has the same conservation laws (i.e. (1.1.4) and (1.1.5)) and scaling invariance (i.e. (1.1.7)) as the  $L^2$  critical gKdV.

For  $u_0 \in H^1$  with negative energy and fast decay at infinity (i.e.  $xu_0 \in L^2$ ), the corresponding solution will blow up in finite time. Indeed, we have the so-called Virial identity

$$\frac{d}{dt} \int x^2 u^2(t, x) dx \leq CE_0,$$

with  $C > 0$ . This identity implies finite time blow up immediately.

On the other hand, for the  $L^2$  critical NLS, we have the following pseudo-conformal transform. If  $u(t, x)$  is a solution to (1.1.19), then

$$\frac{1}{t^{1/2}} e^{i\frac{x^2}{4t}} \bar{u}\left(\frac{1}{t}, \frac{x}{t}\right)$$

is also a solution.

There is a special class of solution, i.e. the periodic solution, given by

$$u(t, x) = c^{\frac{1}{4}} e^{ict} Q(\sqrt{cx}),$$

where  $c > 0$ , and  $Q$  is the ground state for  $L^2$  critical nonlinearity given by (1.1.11). Applying the pseudo-conformal transform to solutions of this type, we obtain a special class of solution

$$S(t, x) = \frac{c^{\frac{1}{4}}}{t^{1/2}} e^{-i(\frac{c}{t} + \frac{x^2}{4t})} Q\left(\frac{\sqrt{cx}}{t}\right),$$

which blows up at  $t = 0$ . This solution is also the unique blow-up solution to (1.1.19) with minimal mass up to the symmetry of the flow, [69].

Besides the minimal mass blow-up solution, there also exist other blow-up solutions with conformal blow-up rate (see [5]):

$$\|u_x(t)\|_{L^2} \sim \frac{1}{T-t} \quad \text{as } t \rightarrow T. \quad (1.1.20)$$

While for solution to (1.1.19) with negative energy and slightly supercritical mass, i.e.

$$0 < \|u_0\|_{L^2} - \|Q\|_{L^2} \ll 1,$$

Merle and Raphaël [71, 72, 73, 74, 86] proved the existence and stability of a blow-up dynamics with log-log blow-up rate:

$$\lim_{t \rightarrow T} \frac{\|\nabla u(t)\|_{L^2}}{\|\nabla Q\|_{L^2}} \sqrt{\frac{|\log|\log(T-t)||}{T-t}} = \frac{1}{\sqrt{2\pi}}. \quad (1.1.21)$$

In the case of  $L^2$  critical gKdV, there is no pseudo-conformal transform or Virial identity which can give us explicit example of blow up solutions. The existence of blow-up solution and description of the blow-up dynamics have been developed by Martel and Merle for solutions with slightly supercritical mass and negative energy, i.e.

$$\|Q\|_{L^2} \leq \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*, \quad \alpha^* \ll 1 \text{ and } E(u_0) < 0. \quad (1.1.22)$$

More precisely, we have:

**Theorem 1.1.5** (Existence of blow-up solutions, [70]). *For  $u_0 \in H^1$  with (1.1.22), the corresponding solution to the critical gKdV will blow up in finite time or infinite time.*

In this case, we have no further information on the blow-up time and the the blow-up dynamics. Indeed, in [57, 59], Martel, Merle and Raphaël prove that blow-up in finite time and infinite time are both possible. Moreover, the blow-up dynamics may be stable or unstable.

**Theorem 1.1.6** (Blow-up profile and lower bound on blow-up rate, [55]). *For initial data  $u_0 \in H^1$  with*

$$\|Q\|_{L^2} \leq \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*, \quad \alpha^* \ll 1,$$

*if the corresponding solution  $u(t)$  blows up in finite time or infinite time, then there exist  $\lambda(t)$ ,  $x(t)$  such that*

$$\lambda^{\frac{1}{2}}(t)u(t, \lambda(t) \cdot +x(t)) \rightharpoonup \pm Q, \text{ weakly in } H^1, \text{ as } t \rightarrow T. \quad (1.1.23)$$

*Moreover, if  $T < +\infty$ , then*

$$\lim_{t \rightarrow T} (T-t)^{1/3} \|u_x(t)\|_{L^2} = +\infty. \quad (1.1.24)$$

Here,  $\lambda(t) \sim \|u_x(t)\|_{L^2}$  is the blow-up rate of (1.1.23). It is impossible to obtain strong convergence in  $H^1$ , due to the mass conservation law (1.1.4). But we have strong convergence in  $H^1_{\text{loc}}$ .

**Theorem 1.1.7** (Finite time blow-up dynamics, [53]). *Suppose the initial data  $u_0 \in H^1$  satisfies (1.1.22) and*

$$\int_{x>x_0} u_0^2(x) dx < \frac{\theta}{x_0^6}, \text{ for all } x_0 > 1, \quad (1.1.25)$$

*for some universal constant  $\theta > 0$ . Then the corresponding solution  $u(t)$  will blow up in finite time  $T < +\infty$ . Moreover, we have the following upper bound on the blow-up rate for a sequence  $t_n \rightarrow T$ :*

$$\|u_x(t_n)\|_{L^2} \leq \frac{C'}{T-t_n}. \quad (1.1.26)$$

There is a large gap between the lower bound given by Theorem 1.1.6 and the upper bound given by Theorem 1.1.7. Examples of blow up solutions with different blow up rates are constructed in [59].

**Theorem 1.1.8** (Global existence for solution with critical mass and decay, [54]). *For initial data with critical mass, i.e.  $\|u_0\|_{L^2} = \|Q\|_{L^2}$ , if we assume that*

$$\int_{x>x_0} u_0^2(x) dx < \frac{C}{x_0^3}, \text{ for all } x_0 > 1, \quad (1.1.27)$$

*then the corresponding solution is global for  $t \geq 0$ , and does not blow up in infinite time.*

In [57, 58, 59], Martel, Merle and Raphaël give an extensive study of the asymptotic dynamics near the ground state  $Q$ .

More precisely, consider the following initial data set

$$\mathcal{A}_{\alpha_0} = \left\{ u_0 \in H^1 \mid u_0 = Q + \varepsilon_0, \|\varepsilon_0\|_{H^1} < \alpha_0, \int_{y>0} y^{10} \varepsilon_0^2 < 1 \right\},$$

and the following  $L^2$  tube around the solitary wave family

$$\mathcal{T}_{\alpha^*} = \left\{ u_0 \in H^1 \mid \inf_{\lambda_0>0, x_0 \in \mathbb{R}} \left\| u_0 - \frac{1}{\lambda_0^{1/2}} Q \left( \frac{x-x_0}{\lambda_0} \right) \right\|_{L^2} < \alpha^* \right\}.$$

Then we have:

**Theorem 1.1.9** (Minimal mass blow up solution, [58]). *There exists a solution  $S(t, x) \in C((0, +\infty), H^1)$  to (1.1.1) with minimal mass  $\|S(t)\|_{L^2} = \|Q\|_{L^2}$ , and*

$$\|S_x(t)\|_{L^2} \sim \frac{\|Q'\|_{L^2}}{t}, \quad \text{as } t \rightarrow 0. \quad (1.1.28)$$

*Moreover, if an  $H^1$  blow up solution  $u(t)$  has minimal mass  $\|u(t)\|_{L^2} = \|Q\|_{L^2}$ , then  $u = S$  up to the symmetry of the critical gKdV (1.1.1).*

**Theorem 1.1.10** (Rigidity, [57]). *For  $0 < \alpha_0 \ll \alpha^* \ll 1$ , and  $u_0 \in \mathcal{A}_{\alpha_0}$ , let  $u(t)$  be the corresponding solution to (1.1.1), and  $0 < T \leq +\infty$  be the maximal lifetime. Then one of the following scenarios occurs:*

**(Blow-up):** *The solution  $u(t)$  blows up in finite time  $0 < T < +\infty$  with*

$$\|u(t)\|_{H^1} = \frac{\ell(u_0) + o(1)}{T-t}, \quad \ell(u_0) > 0.$$

*In addition, for all  $t < T$ ,  $u(t) \in \mathcal{T}_{\alpha^*}$ .*

**(Soliton):** *The solution is global, and for all  $t < T = +\infty$ ,  $u(t) \in \mathcal{T}_{\alpha^*}$ . In addition, there exist a constant  $\lambda_\infty > 0$  and a  $C^1$  function  $x(t)$  such that*

$$\begin{aligned} \lambda_\infty^{1/2} u(t, \lambda_\infty \cdot + x(t)) &\rightarrow Q \text{ in } H_{\text{loc}}^1, \text{ as } t \rightarrow +\infty, \\ |\lambda_\infty - 1| &\lesssim \delta(\alpha_0), \quad x(t) \sim \frac{t}{\lambda_\infty^2}, \text{ as } t \rightarrow +\infty. \end{aligned}$$

**(Exit):** *For some finite time  $0 < t^* < T$ ,  $u(t^*) \notin \mathcal{T}_{\alpha^*}$ . And there exist  $\lambda_u > 0$ ,  $x_u \in \mathbb{R}$ , such that*

$$\|\lambda_u^{1/2} u(t^*, \lambda_u \cdot + x_u) - S(\tau^*, \cdot)\|_{L^2} \leq \delta(\alpha_0),$$

*where  $\delta(\alpha_0) \rightarrow 0$  as  $\alpha_0 \rightarrow 0$ , and  $\tau^*$  depends only on  $\alpha^*$ .*

*Moreover, the scenarios (Blow-up) and (Exit) are stable by small perturbation in  $\mathcal{A}_{\alpha_0}$ .*

**Theorem 1.1.11** (Exotic blow-up regimes, [59]). *There exist solutions with non generic blow-up rate:*

(i) *Blow-up in finite time: for all  $\nu > \frac{11}{13}$ , there exist solution  $u \in C((0, T_0], H^1)$  of (1.1.1) blowing up at  $t = 0$  with*

$$\|u(t)\|_{H^1} \sim t^{-\nu}, \quad \text{as } t \rightarrow 0^+. \quad (1.1.29)$$

(ii) *Blow-up in infinite time:*

1. *There exist solution  $u \in C([0, +\infty), H^1)$  of (1.1.1) blowing up at  $+\infty$  with*

$$\|u(t)\|_{H^1} \sim e^t, \quad \text{as } t \rightarrow +\infty. \quad (1.1.30)$$

2. *For all  $\nu > 0$ , there exist solution  $u \in C([0, +\infty), H^1)$  of (1.1.1) blowing up at  $+\infty$  with*

$$\|u(t)\|_{H^1} \sim t^\nu, \quad \text{as } t \rightarrow +\infty. \quad (1.1.31)$$

In [56], Martel, Merle, Nakanishi and Raphaël proved that the initial data in  $\mathcal{A}_{\alpha_0}$  which corresponds to the (Soliton) regime is a codimension one threshold submanifold between (Blow up) and (Exit).

**Theorem 1.1.12** (Codimension one threshold manifold). *Let*

$$\mathcal{A}_{\alpha_0}^\perp = \left\{ \varepsilon_0 \in H^1 \mid \|\varepsilon_0\|_{H^1} < \alpha_0, \int_{y>0} y^{10} \varepsilon_0^2 < 1, (\varepsilon_0, \mathcal{Q}) = 0 \right\}.$$

*Then there exist  $\alpha_0 > 0$ ,  $\beta_0 > 0$ , and a  $C^1$  function  $A$ :*

$$\mathcal{A}_{\alpha_0}^\perp \rightarrow (-\beta_0, \beta_0),$$

*such that for all  $\gamma_0 \in \mathcal{A}_{\alpha_0}^\perp$  and  $a \in (-\beta_0, \beta_0)$ , the solution of (1.1.1) corresponding to  $u_0 = (1+a)\mathcal{Q} + \gamma_0$  satisfies:*

- (Soliton)** if  $a = A(\gamma_0)$ ;
- (Blow up)** if  $a > A(\gamma_0)$ ;
- (Exit)** if  $a < A(\gamma_0)$ .

*In particular, let*

$$\mathcal{Q} = \left\{ u_0 \in H^1 \mid \exists \lambda_0, x_0, \text{ such that } u_0 = \lambda_0^{-\frac{1}{2}} \mathcal{Q}(\lambda_0^{-1}(x - x_0)) \right\}.$$

*then there exists a small neighborhood  $\mathcal{O}$  of  $\mathcal{Q}$  in  $H^1 \cap L^2(y_+^{10} dy)$  and a codimension one  $C^1$  submanifold  $\mathcal{M}$  of  $\mathcal{O}$ , such that  $\mathcal{Q} \subset \mathcal{M}$  and for all  $u_0 \in \mathcal{O}$  the corresponding solution of (1.1.1) is in the (Soliton) regime if and only if  $u_0 \in \mathcal{M}$ .*

## 1.2 Main results by the author

We have reviewed some results of the asymptotic dynamics for the  $L^2$  critical gKdV. In this thesis, the author has considered the supercritical gKdV with a slightly supercritical

nonlinearity and critical gKdV with a saturated perturbation. The author has given an explicit description of the blow up dynamics and formation of singularity. The results can be split into three parts:

The first part deals with the slightly supercritical gKdV. Based on the construction of the exact self-similar profile by Koch [38] and the technique for critical gKdV [57], the author have proved the existence of a stable blow up dynamics with self-similar rate in the energy space  $H^1$ , and also have given a specific description of the formation of the singularity.

The second part is devoted to the construction of blow-up solution to slightly supercritical gKdV with large initial data. The basic idea is to construct solutions which are decoupled sums of bubbles. Each of them blows up at the same finite time. This is done by a standard topological argument. Hence, such solutions are not stable in the energy space  $H^1$ .

Finally, in the third part, we consider the  $L^2$  critical gKdV with a saturated perturbation. In this case, the equation still has a family of soliton solutions. The author have given a specific classification of the asymptotic dynamics near the soliton. There are only three possibilities: (i) the solution converges asymptotically to a soliton; (ii) the solution is close to the soliton manifold, but blows down in infinite time; (iii) the solution leaves any small neighborhood of the soliton manifold.

The results mentioned above are based on the following work by the author:

- Y. Lan, *Stable self-similar blow-up dynamics for slightly  $L^2$ -supercritical generalized KdV equations*. *Comm. Math. Phys.* **345** (2016), 223–269.
- Y. Lan, *Blow-up solutions for  $L^2$ -supercritical gKdV equations with exactly  $k$  blow-up points*, arXiv:1602.08617, submitted.
- Y. Lan, *On asymptotic dynamics for  $L^2$  critical generalized KdV equations with a saturated perturbation*, arXiv:1609.05146, submitted.

### 1.2.1 Stable self-similar blow up dynamics for $L^2$ supercritical gKdV

In this part, we consider the slightly  $L^2$  supercritical gKdV, i.e. (1.1.1) with

$$5 < p < 5 + \delta, \quad 0 < \delta \ll 1.$$

The existence of blow up solutions in the supercritical case has been a long standing open problem.

Let us first consider the focusing  $L^2$  supercritical NLS equations:

$$\begin{cases} i\partial_t u + \Delta u + |u|^{p-1}u = 0, & (t, x) \in [0, T) \times \mathbb{R}^d, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}^d), \end{cases}$$

with nonlinearity  $p > 1 + \frac{4}{d}$ .

From [29, 77, 87, 88], there exist radial solutions which blow up on an asymptotic blow-up sphere instead of a blow-up point for  $d \geq 2$ ,  $p = 5$ . From [27, 28, 94], there exist blow-up solutions with cylindrical symmetry for  $d \geq 3$ ,  $p = 3$ . And in [75], Merle, Raphaël and Szeftel construct a stable self-similar blow-up dynamics for slightly  $L^2$ -supercritical nonlinearity, with nonradial initial data in low dimensions (i.e.  $d \leq 5$ ).

For the slightly supercritical gKdV case, numerical simulation by Dix and McKinney [13] suggests that there are self-similar blow-up solutions to the supercritical gKdV equations<sup>2</sup>, i.e. a solution of the following form:

$$u(t, x) = \frac{1}{\lambda^{\frac{2}{p-1}}(t)} Q_b \left( \frac{x}{\lambda(t)} \right), \quad \lambda(t) = \sqrt[3]{3b(T-t)},$$

with  $b > 0$ . The exact solution of this type has been constructed by Koch in [38] for slightly supercritical nonlinearities. But such solution is not in the critical Sobolev space  $\dot{H}^{\sigma_c}$ , hence not in the energy space  $H^1$ , which makes it hard to consider the stability of this self-similar blow-up solution. However, we may choose a suitable approximation of the exact self-similar profile, which will lead to a similar result as the case of slightly supercritical Schrödinger equations, i.e. [75].

More precisely, we have:

**Theorem 1.2.1** (Existence and stability of a self-similar blow-up dynamics, [46]). *There exists a  $p^* > 5$  such that the following holds. For all  $p \in (5, p^*)$ , there exists a nonempty open subset  $\mathcal{O}_p$  in  $H^1$  such that if  $u_0 \in \mathcal{O}_p$ , then the corresponding solution to (1.1.1) blows up in finite time  $0 < T < +\infty$ , with the following dynamics: there exist geometrical parameters  $(\lambda(t), x(t)) \in \mathbb{R}_+^* \times \mathbb{R}$  and an error term  $\varepsilon(t)$  such that:*

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} [Q_p + \varepsilon(t)] \left( \frac{x - x(t)}{\lambda(t)} \right) \quad (1.2.1)$$

with

$$\|\varepsilon_y(t)\|_{L^2} \leq \sqrt{p-5}, \quad x(t) \rightarrow x(T), \quad \frac{\lambda^3(t)}{T-t} \sim p-5. \quad (1.2.2)$$

Theorem 1.2.1 is the first construction of blow-up solutions to the supercritical gKdV equations with initial data in  $H^1$ . This is a stable blow-up dynamics instead of a single blow-up solution. So it is different from the self-similar solution constructed by Koch in [38], though the construction in this paper relies deeply on Koch's work.

The conclusion here is almost the same as in the Schrödinger case in [75]. But we need a totally different strategy, due to the distinct structure of these two equations. Indeed, our strategy here is close to the one in [57] for critical gKdV, due to the slightly supercritical condition on the nonlinearity.

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2. We know from Theorem 1.1.6 that there are no self-similar blow-up solutions for the  $L^2$ -critical gKdV equation.

### Strategy of the proof:

We start with the standard modulation argument for solution near the ground state, which was developed in [53, 55, 70] for  $L^2$  critical gKdV and [71, 72, 73, 74, 86] for  $L^2$  critical NLS. More precisely, we consider solution of the following form:

$$u(t, x) = \frac{1}{\lambda^{\frac{2}{p-1}}(t)} [Q_{b(t)} + \varepsilon(t)] \left( \frac{x - x(t)}{\lambda(t)} \right),$$

where  $Q_b \in H^1$  is a suitable approximation to the exact self-similar profile constructed by Koch [38], and  $b(t), \lambda(t), x(t)$  are geometrical parameters satisfying some suitable orthogonality conditions for  $\varepsilon$ .

Next, using the orthogonality conditions, we obtain the following modulation estimate for the geometrical parameters:

$$\begin{aligned} s &= \int_0^t \frac{1}{\lambda^3(\tau)} d\tau, \\ \left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| &\lesssim b_c^2 + \|\varepsilon\|_{L_{\text{loc}}^2}, \\ |b_s + 2(b - b_c)b_c| &\lesssim b_c^3 + b_c \|\varepsilon\|_{L_{\text{loc}}^2}, \end{aligned}$$

where  $b_c > 0$  is a universal constant which depends only on  $p$ .

Then, we may use a localization argument of the energy conservation law, the monotonicity formula developed in [55, 57] and the inhomogeneous Strichartz estimates developed in [23] to get a good control on the error term  $\varepsilon$ :

$$\int_{y > \kappa b_c^{-\frac{1}{20}}} \varepsilon_y^2 < b_c^{\frac{55}{7}}, \quad \|\varepsilon\|_{L_{\text{loc}}^2}^2 \leq b_c^{\frac{7}{2}}, \quad \|\varepsilon\|_{L^{5/2}} \leq b_c^{\frac{7}{15}},$$

for some universal constant  $\kappa \in (0, \frac{1}{2})$ .

Finally, combining the modulation estimates and the control on the error term  $\varepsilon$ , we obtain:

$$0 < T < +\infty, \quad b(t) \sim b_c, \quad \lambda(t) \sim \sqrt[3]{3b_c(T-t)}, \quad x(t) \rightarrow x(T),$$

which is the result of Theorem 1.2.1.

## 1.2.2 Blow-up solutions with multiple blow up points

In this part, we consider blow up solutions with multiple blow up points, where the motivation is to construct large blow up solutions.

There are several examples for such construction:

- Merle [65] for  $L^2$  critical NLS with conformal blow up rate;
- Fan [20] and Planchon, Raphaël [85] for  $L^2$  critical NLS with log-log blow up rate;

- Merle [68] and Merle, Zaag [79] for nonlinear heat equation with ODE blow up rate.

These constructions correspond to the weak interaction case, i.e. the interaction between the bubbles does not change the blow up rate of each bubble. There are also some examples for solution with strongly interacting bubbles:

- Martel, Raphaël [62] for  $L^2$  critical NLS;
- Cortázar, Del Pino, Musso [10] for energy critical nonlinear heat equations in domain;
- Jendrej [30, 31] for focusing energy critical wave equations.

For the slightly  $L^2$  supercritical gKdV, in [46] the author proves the existence of a stable blow up dynamics where the singularity concentrates at some finite point (see Section (1.2.1)). Hence, we have:

**Theorem 1.2.2** (Solution with multiple blow up points, [44]). *There exist universal constants  $p^* > 5$ ,  $c > 0$  such that for all  $p \in (5, p^*)$ ,  $k \in \mathbb{N}^+$ , if  $2 \leq k \leq c|\log(p-5)|$ , then for all  $k$  pairwise distinct points  $x_1, x_2, \dots, x_k \in \mathbb{R}$ , there exists a solution  $u(t)$  of (1.1.1), which blows up in finite time  $T < +\infty$ . And for  $t$  close to  $T$ , there exist scaling parameters  $\lambda_j(t) \in \mathbb{R}^+$  and an error term  $\tilde{u}(t, x)$  with*

$$u(t, x) = \sum_{j=1}^k \frac{1}{\lambda_j^{\frac{2}{p-1}}(t)} Q_p \left( \frac{x - x_j}{\lambda_j(t)} \right) + \tilde{u}(t, x), \quad (1.2.3)$$

where for all  $j = 1, \dots, k$ , and  $t$  close to  $T$ ,

$$\frac{\lambda_j^3(t)}{T-t} \sim p-5, \quad \lambda_j(t)^{1-\sigma_c} \|\tilde{u}_x(t)\|_{L^2} \leq \delta(p), \quad (1.2.4)$$

for some small constant  $\delta(p)$  with  $\lim_{p \rightarrow 5} \delta(p) = 0$ .

Moreover, the blow-up set of  $u(t)$  is exactly  $\{x_1, x_2, \dots, x_k\}$ .

From (1.2.3), we obtain large blow up solutions for (1.1.1) in the sense that  $\|u_0\|_{L^{(p-1)/2}} \sim |\log(p-5)|$ , for  $p$  close enough to 5.

For  $k \geq 2$ , the blow up solution with  $k$  bubbles is *not stable* in  $H^1$ . Indeed, we require some special conditions (unstable) on the initial data, which can be obtained by a standard topological argument. Similar argument is also used in [12, 20, 62, 64, 79] for multiple bubble blow up solutions, and [11, 41, 49, 60, 61] for multi-soliton solutions.

### Strategy of the proof:

From [46], there exists a stable self-similar blow up dynamics such that the singularity concentrates at some finite point. We can consider solutions which can be written as the sum of  $k$  decoupled self-similar blow up solutions. More precisely, we consider solutions of the following form:

$$u(t, x) = \sum_{j=1}^k \frac{1}{\lambda_j^{\frac{2}{p-1}}(t)} Q_{b_j(t)} \left( \frac{x - x_j(t)}{\lambda_j(t)} \right) + \tilde{u}(t, x),$$



where  $Q_{b_j}$  is the approximate self-similar profile defined in [46], and the parameters  $b_j(t), \lambda_j(t), x_j(t)$  are chosen to adjust some suitable orthogonality conditions for the error term  $\tilde{u}$ . The crucial idea here is to construct solution of this form such that each bubble

$$\frac{1}{\lambda_j^{\frac{2}{p-1}}(t)} Q_{b_j(t)} \left( \frac{x - x_j(t)}{\lambda_j(t)} \right)$$

behaves like a self-similar blow-up solution constructed in [46]. We know from [46] that each bubble concentrates at some finite point. We expect these points (so called *blow-up points*) are disjoint and the distance between these points is large enough. In this case, the interaction between different bubbles are very small.

Next, we need to ensure that the blow up time of each bubble is the same. This requires an additional condition on the initial data, which can be obtained by a standard topological argument. Hence, the blow-up solutions with multiple bubble are not stable in  $H^1$ . More precisely, with some suitable condition on the initial data, we can assume that for all  $1 \leq i < j \leq k$ , and all  $t \in [0, T)$ ,

$$\frac{1}{2^{k+1}} < \frac{\lambda_i(t)}{\lambda_j(t)} < 2^{k+1}.$$

Then, we may follow similar argument in [46] to obtain the modulation estimates. For all  $j = 1, \dots, k$ ,

$$\begin{aligned} \left| \frac{d\lambda}{ds_j} \frac{1}{\lambda_j} + b_j \right| + \left| \frac{dx_j}{ds_j} \frac{1}{\lambda_j} - 1 \right| &\lesssim b_c^2 + \|\varepsilon_j\|_{L_{\text{loc}}^2}, \\ \left| \frac{db_j}{ds_j} + 2(b_j - b_c)b_c \right| &\lesssim b_c^3 + b_c \|\varepsilon_j\|_{L_{\text{loc}}^2}, \end{aligned}$$

where

$$s_j = \int_0^t \frac{1}{\lambda_j^3(\tau)} d\tau, \quad \varepsilon_j(t, y) = \lambda_j^{\frac{2}{p-1}}(t) \tilde{u}(t, \lambda_j(t)y + x_j(t)).$$

Using the condition on  $k$  (the number of the blow-up points), we have similar estimates on the error term. For all  $j = 1, \dots, k$ ,

$$\int_{b_c^{-20} > y > \kappa b_c^{-\frac{1}{20}}} (\varepsilon_j)_y^2 < b_c^{\frac{55}{7}}, \quad \|\varepsilon_j\|_{L_{\text{loc}}^2}^2 \leq b_c^{\frac{7}{2}}, \quad \|\varepsilon_j\|_{L^{5/2}} \leq b_c^{\frac{7}{15}}.$$

Hence, we have for all  $1 \leq i < j \leq k$ ,

$$\begin{aligned} 0 < T < +\infty, \quad b_j(t) &\sim b_c, \quad \lambda_j(t) \sim \sqrt[3]{3b_c(T-t)}, \\ x_j(t) &\rightarrow x_j(T), \quad x_i(T) \neq x_j(T). \end{aligned}$$

which proves the existence of blow-up solutions with multiple blow-u points.

Finally, using a second topological argument (see [68, Proposition 5.2]), we can show that the blow up points can be chosen arbitrarily, which is exactly Theorem 1.2.2.

### 1.2.3 Asymptotic dynamics for $L^2$ critical gKdV with a saturated perturbation

In this part, we consider the  $L^2$  critical gKdV with a saturated perturbation:

$$\begin{cases} \partial_t u + (u_{xx} + u^5 - \gamma u|u|^{q-1})_x = 0, & (t, x) \in [0, T) \times \mathbb{R}, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \quad (1.2.5)$$

with  $q > 5$  and  $0 < \gamma \ll 1$ .

The equation has two conservation laws, i.e. the mass and the energy:

$$\begin{aligned} M(u(t)) &= \int u(t)^2 = M_0, \\ E(u(t)) &= \frac{1}{2} \int u_x(t)^2 - \frac{1}{6} \int u(t)^6 + \frac{\gamma}{q+1} \int |u(t)|^{q+1} = E_0. \end{aligned}$$

We can see from local theory [36] and the conservation laws that the solution of (1.2.5) is always global in time and bounded in  $H^1$ .

This equation has the following pseudo-scaling rule: for all  $\lambda_0 > 0$ , if  $u(t, x)$  is a solution to (1.2.5), then

$$u_{\lambda_0}(t, x) = \lambda_0^{-\frac{1}{2}} u(\lambda_0^{-3}t, \lambda_0^{-1}x), \quad (1.2.6)$$

is a solution to

$$\begin{cases} \partial_t v + (v_{xx} + v^5 - \lambda_0^{-m} \gamma v|v|^{q-1})_x = 0, & (t, x) \in [0, \lambda_0^{-3}T) \times \mathbb{R}, \\ v(0, x) = \lambda_0^{-\frac{1}{2}} u_0(\lambda_0^{-1}x) \in H^1(\mathbb{R}), \end{cases}$$

with

$$m = \frac{q-5}{2} > 0. \quad (1.2.7)$$

The pseudo-scaling rule (1.2.6) leaves the  $L^2$  norm of the initial data invariant.

There still exists solitary wave solutions given by

$$u(t, x) = \lambda_0^{-\frac{1}{2}} \tilde{Q}_{\lambda_0^{-m}\gamma}(\lambda_0^{-1}(x-x_0) - \lambda_0^{-3}(t-t_0)),$$

for all  $\lambda_0 > 0$ ,  $t_0 \in \mathbb{R}$ ,  $x_0 \in \mathbb{R}$  with  $\lambda_0^{-m}\gamma \ll 1$ . Here for all  $0 \leq \omega < \omega^* \ll 1$ ,  $\tilde{Q}_\omega$  is the unique radial nonnegative solution with exponential decay to the following ODE:

$$\tilde{Q}_\omega'' - \tilde{Q}_\omega + \tilde{Q}_\omega^5 - \omega \tilde{Q}_\omega |\tilde{Q}_\omega|^{q-1} = 0.$$

The aim of this part is to classify the dynamics of (1.2.5) near the ground state, when  $\gamma$  is small enough. The main idea is that the defocusing term  $\gamma u|u|^{q-1}$  has weaker nonlinear effect than the focusing term  $u^5$ . So, we may expect that (1.2.5) has similar separation behavior<sup>3</sup> as (1.1.1), when  $\gamma$  is small.

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3. See Theorem 1.1.10.

More precisely, we fix a small universal constant  $\omega^* > 0$  (to ensure the existence of the ground state  $\tilde{Q}_\omega$ ), and introduce the following  $L^2$  tube around  $\tilde{Q}_\gamma$ :

$$\mathcal{T}_{\alpha^*, \gamma} = \left\{ u_0 \in H^1 \mid \inf_{\lambda_0 > 0, \lambda_0^{-m} \gamma < \omega^*, x_0 \in \mathbb{R}} \left\| u_0 - \frac{1}{\lambda_0^{\frac{1}{2}}} \tilde{Q}_{\lambda_0^{-m} \gamma} \left( \frac{x - x_0}{\lambda_0} \right) \right\|_{L^2} < \alpha^* \right\},$$

as well as the initial data set

$$\mathcal{A}_{\alpha_0, \gamma} = \left\{ u_0 \in H^1 \mid u_0 = \tilde{Q}_\gamma + \varepsilon_0, \|\varepsilon_0\|_{H^1} < \alpha_0, \int_{y > 0} y^{10} \varepsilon_0^2 < 1 \right\},$$

for some universal constants  $0 < \alpha_0 \ll \alpha^*$ . Then we have:

**Theorem 1.2.3** (Dynamics in  $\mathcal{A}_{\alpha_0, \gamma}$ , [45]). *For all  $q > 5$ , there exists a constant  $0 < \alpha^*(q) \ll 1$ , such that if  $0 < \gamma \ll \alpha_0 \ll \alpha^* < \alpha^*(q)$ , then for all  $u_0 \in \mathcal{A}_{\alpha_0, \gamma}$ , the corresponding solution  $u(t)$  to (1.2.5) has one and only one of the following behaviors:*

-(Soliton): *For all  $t \in [0, +\infty)$ ,  $u(t) \in \mathcal{T}_{\alpha^*, \gamma}$ . Moreover, there exist a constant  $\lambda_\infty \in (0, +\infty)$  and a  $C^1$  function  $x(t)$  such that*

$$\lambda_\infty^{\frac{1}{2}} u(t, \lambda_\infty \cdot + x(t)) \rightarrow \tilde{Q}_{\lambda_\infty^{-m} \gamma} \text{ in } H_{\text{loc}}^1, \text{ as } t \rightarrow +\infty; \quad (1.2.8)$$

$$x(t) \sim \frac{t}{\lambda_\infty^2}, \text{ as } t \rightarrow +\infty. \quad (1.2.9)$$

-(Blow down): *For all  $t \in [0, +\infty)$ ,  $u(t) \in \mathcal{T}_{\alpha^*, \gamma}$ . Moreover, there exist two  $C^1$  functions  $\lambda(t)$  and  $x(t)$ , such that*

$$\lambda^{\frac{1}{2}}(t) u(t, \lambda(t) \cdot + x(t)) \rightarrow Q \text{ in } H_{\text{loc}}^1, \text{ as } t \rightarrow +\infty; \quad (1.2.10)$$

$$\lambda(t) \sim t^{\frac{2}{q+1}}, \quad x(t) \sim t^{\frac{q-3}{q+1}}, \quad \text{as } t \rightarrow +\infty, \quad (1.2.11)$$

-(Exit): *There exists a  $0 < t_\gamma^* < +\infty$  such that  $u(t_\gamma^*) \notin \mathcal{T}_{\alpha^*, \gamma}$ .*

*There exist solutions associated to each regime. Moreover, the regime (Soliton) and (Exit) are stable under small perturbation in  $\mathcal{A}_{\alpha_0, \gamma}$ .*

The blow down behavior does not seem to appear in the unperturbed case. Examples of solution with such type of behavior was also found by Donninger, Krieger [16] for energy critical wave equations. While for mass critical NLS, the blow down behavior can be obtained as the pseudo-conformal transformation of the log-log regime<sup>4</sup>. However, Theorem 1.2.3 is the first time that this type of blow down behavior is obtained in the context of a saturated perturbation. Furthermore, in Theorem 1.2.3, we can similarly (as Theorem 1.1.12) prove that the (blow down) regime is a codimension one threshold between two stable ones, which is in contrast with the mass critical nonlinear Schrödinger case, where the blow down regime is stable.

Now we consider the case when  $\gamma \rightarrow 0$ . As we mentioned before, the defocusing term  $\gamma u|u|^{q-1}$  has weaker nonlinear effect than the focusing term  $u^5$ . So the threshold regime (blow down) defined in Theorem 1.2.3 is expected to be a perturbation of the (Soliton) regime defined in Theorem 1.1.10.

More precisely, we have:

---

4. See [77, (1.16)] for example.

**Theorem 1.2.4** (Limiting case as  $\gamma \rightarrow 0$ , [45]). *Let us fix a nonlinearity  $q > 5$ , and choose  $0 < \alpha_0 \ll \alpha^* < \alpha^*(q)$  as in Theorem 1.2.3. For all  $u_0 \in \mathcal{A}_{\alpha_0}$ , let  $u(t)$  be the corresponding solution of (1.1.1), and  $u_\gamma(t)$  be the corresponding solution of (1.2.5). Then we have:*

1. *If  $u(t)$  is in the (Blow up) regime defined in Theorem 1.1.10, then there exists  $0 < \gamma(u_0, \alpha_0, \alpha^*, q) \ll \alpha_0$  such that if  $0 < \gamma < \gamma(u_0, \alpha_0, \alpha^*, q)$ , then  $u_\gamma(t)$  is in the (Soliton) regime defined in Theorem 1.2.3. Moreover, there exist constants  $d_i = d_i(u_0, q) > 0$ ,  $i = 1, 2$ , such that*

$$d_1 \gamma^{\frac{2}{q-1}} \leq \lambda_\infty \leq d_2 \gamma^{\frac{2}{q-1}}, \quad (1.2.12)$$

where  $\lambda_\infty$  is the constant defined in (1.2.8).

2. *If  $u(t)$  is in the (Exit) regime defined in Theorem 1.1.10, then there exists  $0 < \gamma(u_0, \alpha_0, \alpha^*, q) \ll \alpha_0$  such that if  $0 < \gamma < \gamma(u_0, \alpha_0, \alpha^*, q)$ , then  $u_\gamma(t)$  is in the (Exit) regime defined in Theorem 1.2.3.*

Theorem 1.2.4 together with Theorem 1.2.3 gives a specific description of the asymptotic behavior of solution to (1.2.5) with initial data which leads to a blow-up solution for the unperturbed equation. As far as we know, this is the first identical result of this type for nonlinear dispersive equations. One may also expect similar results for the critical NLS<sup>5</sup> or the slightly supercritical gKdV cases. But these problems are still completely open.

### Strategy of the proof:

We follow similar arguments as in [57]. We consider solution of the following form:

$$u(t, x) = \frac{1}{\lambda^{\frac{1}{2}}(t)} [Q_{b(t), \omega(t)} + \varepsilon(t)] \left( \frac{x - x(t)}{\lambda(t)} \right),$$

where

$$\omega(t) = \frac{\gamma}{\lambda^m(t)},$$

and  $Q_{b, \omega}$  is an approximate solution to the following ODE:

$$b \Lambda Q_{b, \omega} + (Q_{b, \omega}'' - Q_{b, \omega} + Q_{b, \omega}^5 - \omega Q_{b, \omega} |Q_{b, \omega}|^{q-1})' = 0.$$

We also choose the parameters  $\lambda(t), b(t), x(t)$  such that some orthogonality conditions are satisfied.

Formal computations show that the parameters satisfy the following ODE system:

$$\begin{cases} b_s + 2b^2 + c_0 \omega_s = 0, \\ \frac{\lambda_s}{\lambda} + b = \frac{x_s}{\lambda} - 1 = 0, \end{cases}$$

---

5. In [48], Malkin predicted a similar asymptotic behavior for the solution to the saturated problem of critical NLS.

where  $c_0 = c_0(q) > 0$  is a universal constant and

$$s = \int_0^t \frac{1}{\lambda^3(\tau)} d\tau$$

is the rescaled time.

It is easy to see that the ODE system has a conserved quantity (the energy conservation law of (1.2.5)):

$$\frac{b(t) + mc_0/(m+2)\omega(t)}{\lambda^2(t)} \equiv L_0.$$

If  $L_0 > 0$ , we have:

$$b(t) \rightarrow 0, \quad \lambda(t) \rightarrow \left( \frac{m\gamma c_0}{(m+2)L_0} \right)^{\frac{1}{m+2}}, \quad x(t) \sim \left( \frac{(m+2)L_0}{m\gamma c_0} \right)^{\frac{2}{m+2}} t,$$

as  $t \rightarrow +\infty$ , which corresponds to the (Soliton) regime.

If  $L_0 = 0$ , then we have:

$$b(t) \sim -t^{-\frac{q-5}{q+1}}, \quad \lambda(t) \sim t^{\frac{2}{q+1}}, \quad x(t) \sim t^{\frac{q-3}{q+1}},$$

as  $t \rightarrow +\infty$ , which corresponds to the (Blow down) regime.

If  $L_0 < 0$ , we have:

$$b(t) \rightarrow -\infty, \quad \lambda(t) \rightarrow +\infty,$$

as  $t \rightarrow +\infty$ , which corresponds to the (Exit) regime.

Now we consider the full ansatz. Using similar argument as in [57], we have

$$\left| \frac{\lambda_s}{\lambda} + b \right| \lesssim \|\varepsilon\|_{H_{loc}^1}^2 + |b|(\omega + |b|), \quad (1.2.13)$$

and for all  $0 \leq s_1 < s_2 < +\infty$ ,

$$\left| \left( \frac{\lambda + mc_0/(m+2)\omega}{\lambda^2} \right) \Big|_{s_1}^{s_2} \right| \lesssim \frac{\|\varepsilon\|_{H_{loc}^1}^2}{\lambda^2(s_1)} + \frac{b^2(s_1) + \omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2) + \omega^2(s_2)}{\lambda^2(s_2)}. \quad (1.2.14)$$

We also have the monotonicity formula on the error term  $\varepsilon$ . For all  $j = 1, 2$ , we have

$$\left( \frac{\mathcal{F}}{\lambda^{2(j-1)}} \right)_s + \mu \frac{\|\varepsilon\|_{H_{loc}^1}^2}{\lambda^{2(j-1)}} \lesssim \frac{b^2(b^2 + \omega^2)}{\lambda^{2(j-1)}},$$

where  $\mathcal{F}$  is a well chosen functional. Due to our choice of the orthogonality conditions,  $\mathcal{F}$  has the following coercivity:

$$\mathcal{F} \sim \|\varepsilon\|_{H_{loc}^1}^2.$$

Then, we can see the separation condition for these three scenarios are the following<sup>6</sup>:

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6. From (1.2.14) and a special choice of the universal constant  $C^*$ , we can see that (1.2.16) and (1.2.17) cannot happen simultaneously.

- **(Blow down):** For all  $t \in [0, +\infty)$ , we have

$$|b(t) + mc_0/(m+2)\omega(t)| < C^* \left( \|\varepsilon(t)\|_{H_{\text{loc}}^1}^2 + b^2(t) + \omega^2(t) \right). \quad (1.2.15)$$

- **(Exit):** There exists a  $t_1^* < +\infty$  such that

$$b(t_1^*) + mc_0/(m+2)\omega(t_1^*) \leq -C^* \left( \|\varepsilon(t_1^*)\|_{H_{\text{loc}}^1}^2 + b^2(t_1^*) + \omega^2(t_1^*) \right). \quad (1.2.16)$$

- **(Soliton):** There exists a  $t_1^* < +\infty$  such that

$$b(t_1^*) + mc_0/(m+2)\omega(t_1^*) \geq C^* \left( \|\varepsilon(t_1^*)\|_{H_{\text{loc}}^1}^2 + b^2(t_1^*) + \omega^2(t_1^*) \right). \quad (1.2.17)$$

Here  $C^*$  is a large universal constant.

The exact behavior of the parameters can be obtained by reintegrating the approximate ODE of the parameters.

Finally, it is easy to see that Theorem 1.2.4 is a direct consequence of a modified  $H^1$  perturbation theory and the separation conditions (1.2.15)–(1.2.17).



# Chapter 2

## Stable self-similar blow-up dynamics for slightly mass supercritical gKdV equations

### 2.1 Introduction

#### 2.1.1 Setting of the problem

We consider the following gKdV equations:

$$\begin{cases} \partial_t u + (u_{xx} + u|u|^{p-1})_x = 0, & (t, x) \in [0, T) \times \mathbb{R}, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \quad (2.1.1)$$

with  $1 \leq p < +\infty$ .

From the result of C. E. Kenig, G. Ponce and L. Vega [36] and N. Strunk [89], (2.1.1) is locally well-posed in  $H^1$  and thus for all  $u_0 \in H^1$ , there exists a maximal lifetime  $0 < T \leq +\infty$  and a unique solution  $u(t, x) \in C([0, T), H^1(\mathbb{R}))$  to (2.1.1). Besides, we have the blow-up criterion: either  $T = +\infty$  or  $T < +\infty$  and  $\lim_{t \rightarrow T} \|u_x(t)\|_{L^2} = +\infty$ .

(2.1.1) admits 2 conservation laws, i.e. the mass and energy:

$$\begin{aligned} M(u(t)) &= \int |u(t, x)|^2 dx = M(u(0)), \\ E(u(t)) &= \frac{1}{2} \int |u(t, x)|^2 dx - \frac{1}{p+1} \int |u(t, x)|^{p+1} dx = E(u(0)). \end{aligned}$$

For all  $\lambda > 0$ ,  $u_\lambda(t, x) = \lambda^{\frac{2}{p-1}} u(\lambda^3 t, \lambda x)$  is also a solution which leaves the Sobolev space  $\dot{H}^{\sigma_c}$  invariant with the index:

$$\sigma_c = \frac{1}{2} - \frac{2}{p-1}. \quad (2.1.2)$$



We introduce the ground state  $Q_p$ , which is the unique radial nonnegative function with exponential decay at infinity to the following equation:

$$Q_p'' - Q_p + Q_p|Q_p|^{p-1} = 0. \quad (2.1.3)$$

$Q_p$  plays a distinguished role in the analysis. It provides a family of travelling wave solutions:

$$u(t, x) = \lambda^{\frac{2}{p-1}} Q_p(\lambda(x - \lambda^2 t - x_0)), \quad (\lambda, x_0) \in \mathbb{R}_+^* \times \mathbb{R}.$$

For  $p < 5$  or equivalently  $\sigma_c < 0$ , (2.1.1) is subcritical in  $L^2$ . The mass and energy conservation laws imply that the solution is always global and bounded in  $H^1$ . So a necessary condition for the occurrence of blow-up is  $p \geq 5$ . For  $p = 5$ , the blow up dynamics have been studied in a series of papers of Y. Martel, F. Merle and P. Raphaël in [50, 52, 53, 54, 55, 57, 58, 59, 70].

## 2.1.2 On the supercritical problem

Let us first consider the focusing  $L^2$  supercritical NLS equations:

$$\begin{cases} i\partial_t u + \Delta u + |u|^{p-1}u = 0, & (t, x) \in [0, T) \times \mathbb{R}^d, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}^d), \end{cases}$$

with nonlinearity  $p > 1 + \frac{4}{d}$ . From [77, 87, 88], for  $d \geq 2$ , there are radial solutions which blow up on an asymptotic blow-up sphere instead of a blow-up point. And in [75], F. Merle, P. Raphaël and J. Szeftel construct a stable self-similar blow-up dynamics for slightly  $L^2$ -supercritical nonlinearity, with nonradial initial data in low dimension (i.e.  $d \leq 5$ ).

Now let us return to the gKdV equations. In this paper we consider the slightly supercritical case:

$$5 < p < 5 + \varepsilon, \quad 0 < \varepsilon \ll 1.$$

The explicit description of blow-up dynamics for supercritical gKdV equations is mostly open. But numerical simulation of D. B. Dix and W. R. McKinney [13] suggests that there are self-similar blow-up solutions to supercritical gKdV equations<sup>1</sup>. We can expect a similar result to the slightly supercritical Schrödinger equations, i.e. [75]. More precisely, we expect a blow-up solution of the following form:

$$u(t, x) \sim \frac{1}{\lambda(t)^{\frac{2}{p-1}}} P\left(\frac{x}{\lambda(t)}\right), \quad \lambda(t) \sim \sqrt[3]{T-t}.$$

But here the delicate issue is that the profile  $P$  seems not to be provided by the ground state  $Q_p$ . If we explicitly let:

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q_b\left(\frac{x}{\lambda(t)}\right), \quad \lambda(t) = \sqrt[3]{3b(T-t)}, \quad b > 0.$$

---

1. We know from [55] that there are no self-similar blow-up solutions for the  $L^2$ -critical gKdV equation.

Then  $u$  solves (2.1.1) if and only if  $Q_b(y)$  solves the following ODE<sup>2</sup>:

$$b\Lambda Q_b + (Q_b'' - Q_b + Q_b|Q_b|^{p-1})' = 0. \quad (2.1.4)$$

The exact solutions of (2.1.4) have been exhibited by H. Koch [38], for the slightly supercritical nonlinearity  $5 < p < 5 + \varepsilon$ ,  $0 < \varepsilon \ll 1$ . It is related to an eigenvalue problem, i.e. for all  $5 < p < 5 + \varepsilon$ , there exists a unique  $b = b(p) > 0$ , such that a unique smooth solution  $Q_b$  to (2.1.4) with zero energy is found. Moreover  $Q_b$  belongs to  $\dot{H}^1 \cap L^{p+1}$ , but always misses the invariant Sobolev space  $\dot{H}^{\sigma_c}$  (hence  $Q_b \notin L^2$ ) due to a slowly decaying tail at the infinity:

$$Q_b(y) \sim \frac{1}{|y|^{\frac{1}{2}-\sigma_c}}.$$

This makes our analysis more complicated. Since if we expect some stability results for a Cauchy problem, we must work in some Cauchy space, i.e. a space in which a Cauchy theory holds. Here a natural Cauchy space for (2.1.1) is  $H^1$ , the *energy space* or  $\dot{H}^{\sigma_c}$  the *critical space*<sup>3</sup>, while  $Q_b$  is in neither of them. So we cannot use the profile  $Q_b$  directly.

Despite the slowly decaying tail of  $Q_b$ , we can choose a suitable cut-off of  $Q_b$  as an approximation, such that it is bounded in  $L^2$  with exponential decay on the right. We claim that the approximate self-similar profile generates a stable self-similar blow-up dynamics for the time dependent problems.

### 2.1.3 Statement of the result

**Theorem 2.1.1** (Existence and stability of a self-similar blow-up dynamics). *There exists a  $p^* > 5$  such that for all  $p \in (5, p^*)$ , there exist constants  $\delta(p) > 0$  and  $b^*(p) > 0$  with*

$$\lim_{p \rightarrow 5} \delta(p) = 0 \quad (2.1.5)$$

$$0 < c_0(p-5) \leq b^*(p) \leq C_0(p-5) \quad (2.1.6)$$

and a nonempty open subset  $\mathcal{O}_p$  in  $H^1$  such that the following holds. If  $u_0 \in \mathcal{O}_p$ , then the corresponding solution to (2.1.1) blows up in finite time  $0 < T < +\infty$ , with the following dynamics : there exist geometrical parameters  $(\lambda(t), x(t)) \in \mathbb{R}_+^* \times \mathbb{R}$  and an error term  $\varepsilon(t)$  such that:

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} [Q_p + \varepsilon(t)] \left( \frac{x - x(t)}{\lambda(t)} \right) \quad (2.1.7)$$

with

$$\|\varepsilon_y(t)\|_{L^2} \leq \delta(p). \quad (2.1.8)$$

Moreover, we have:

2. See the definition of “ $\Lambda$ ” in Section 1.4.

3. The Cauchy problem (2.1.1) is wellposed in  $H^1$  and  $\dot{H}^{\sigma_c}$ , but not in  $\dot{H}^1$ .

1. The blow-up point converges at the blow-up time:

$$x(t) \rightarrow x(T) \text{ as } t \rightarrow T, \quad (2.1.9)$$

2. The blow-up speed is self-similar:

$$\forall t \in [0, T), \quad (1 - \delta(p)) \sqrt[3]{3b^*(p)} \leq \frac{\lambda(t)}{\sqrt[3]{T-t}} \leq (1 + \delta(p)) \sqrt[3]{3b^*(p)}. \quad (2.1.10)$$

3. The following convergence holds:

$$\forall q \in [2, \frac{2}{1-2\sigma_c}), \quad u(t) \rightarrow u^* \text{ in } L^q \text{ as } t \rightarrow T. \quad (2.1.11)$$

4. The asymptotic profile  $u^*$  displays the following singular behavior:

$$(1 - \delta(p)) \int \mathcal{Q}_p^2 \leq \frac{1}{R^{2\sigma_c}} \int_{|x-x(T)| < R} |u^*|^2 \leq (1 + \delta(p)) \int \mathcal{Q}_p^2. \quad (2.1.12)$$

for  $R$  small enough. In particular, we have for all  $q \geq \frac{2}{1-2\sigma_c}$ :

$$u^* \notin L^q.$$

*Remark 2.1.2.* Here the meaning of  $q_c = \frac{2}{1-2\sigma_c}$  is given by the following Sobolev embedding:

$$\dot{H}^{\sigma_c} \hookrightarrow L^{q_c}.$$

That is, the asymptotic profile  $u^*$  is not in the critical space  $\dot{H}^{\sigma_c}$ , and the strong convergence (2.1.11) only exists in subcritical Lebesgue spaces.

*Remark 2.1.3.* It is easy to see from the  $L^2$  conservation law that  $\int |u^*|^2 = \int |u_0|^2$ .

*Remark 2.1.4.* The conclusion here is almost the same to the Schrödinger case in [75]. But we need a totally differential strategy, due to the different structure of these 2 equations. Indeed, our strategy here is very close to the one in [57] for critical gKdV. But there are some significant difference between critical equations and supercritical equations. For example the singular dynamics for gKdV is located around some point  $x(t)$ , which always goes to infinity in finite time for critical equation<sup>4</sup>. While in this supercritical case,  $x(t)$  converges to some finite point.

*Remark 2.1.5.* Theorem 2.1.1 is the first construction of blow-up solutions to the supercritical gKdV equations with initial data in  $H^1$ . This is a stable blow-up dynamics instead of a single blow-up solution. So it is not like the self-similar solution constructed by H. Koch in [38], though the construction in this paper relies deeply on H. Koch's work.

*Remark 2.1.6.* Stable self-similar blow-up dynamics are also obtained by Donninger [14] for equivariant wave maps; Donninger [15] for energy supercritical Yang-Mills equations; Collet-Raphaël-Szeftel [9] for energy supercritical heat equations.

4. This somehow explains why there is no self-similar blow-up solution for critical gKdV with initial data near soliton.

### 2.1.4 Notation

We first introduce the associated scaling generators:

$$\Lambda f = \frac{2}{p-1}f + yf'. \quad (2.1.13)$$

We denote the  $L^2$  scalar product by:

$$(f, g) = \int_{\mathbb{R}} f(x)g(x)dx \quad (2.1.14)$$

and observe the integration by parts:

$$(\Lambda f, g) = -(f, \Lambda g + 2\sigma_c g). \quad (2.1.15)$$

Then we let  $\mathcal{Q}_p$  be the ground state. For  $p = 5$ , we simply write  $\mathcal{Q}_p$  as  $\mathcal{Q}$ . We introduce the linearized operators at  $\mathcal{Q}_p$ :

$$Lf = -f'' + f - p\mathcal{Q}_p^{p-1}f. \quad (2.1.16)$$

A standard computation leads to:

$$L(\mathcal{Q}'_p) = 0, \quad L(\Lambda \mathcal{Q}_p) = -2\mathcal{Q}_p. \quad (2.1.17)$$

Finally, we denote by  $\delta(p)$  a small positive constant such that:

$$\lim_{p \rightarrow 5} \delta(p) = 0. \quad (2.1.18)$$

### 2.1.5 Strategy of the proof

We will give in this subsection a brief insight of the proof of Theorem 2.1.1. We will first use the self-similar solution constructed by H. Koch in [38], to derive a finite dimensional dynamics, which fully describe the blow-up regime. Since we are considering the slightly supercritical case, it is helpful to view this equation as a perturbation of the critical equation in some sense. So we can use some critical techniques in our analysis, though they may have a totally different meaning in the supercritical case.

#### 2.1.5.1 Derivation of the law

We look for a solution to (2.1.1) of the form:

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} V_{b(t)} \left( \frac{x - x(t)}{\lambda(t)} \right), \quad (2.1.19)$$

and introduce the rescaled time:

$$\frac{ds}{dt} = \frac{1}{\lambda(t)^3}.$$

Then  $u$  is a solution to (2.1.1) if and only if  $V_b$  solves the following equation:

$$b_s \frac{\partial V_b}{\partial b} - \frac{\lambda_s}{\lambda} \Lambda V_b + (V_b'' - V_b + V_b |V_b|^{p-1})' = \left( \frac{x_s}{\lambda} - 1 \right) V_b'. \quad (2.1.20)$$

Similar to the Schrödinger case, the self-similar blow-up regime of (2.1.1) corresponds to the following finite dimensional dynamics:

$$\frac{ds}{dt} = \frac{1}{\lambda^3}, \quad \frac{x_s}{\lambda} = 1, \quad \frac{\lambda_s}{\lambda} = -b, \quad b_s = 0, \quad (2.1.21)$$

which, after integrating, leads to finite time blow-up for  $b(0) > 0$  with:

$$\lambda(t) = c(u_0) \sqrt[3]{T-t}.$$

### 2.1.5.2 Decomposition of the flow and modulation equations (section 2 and section 3)

From the previous discussing we can see it is significant to find a solution  $Q_b$  to (2.1.4), which is done by H. Koch in [38]. For our analysis, it is better to work with the localized approximate self-similar profile<sup>5</sup>:

$$Q_b(y) = v(b, p, y) \chi_0(b_c y).$$

Then we can introduce the geometrical decomposition of the flow:

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} (Q_{b(t)} + \varepsilon) \left( t, \frac{x - x(t)}{\lambda(t)} \right),$$

where the 3 time dependent parameters are adjusted to ensure suitable orthogonality conditions<sup>6</sup> for  $\varepsilon$ . The modulation equations of the parameters are:

$$\begin{aligned} \frac{\lambda_s}{\lambda} + b &= O(b_c^{\frac{5}{2}} + \|\varepsilon\|_{H_{\text{loc}}^1}), \\ \frac{x_s}{\lambda} - 1 &= O(b_c^{\frac{5}{2}} + \|\varepsilon\|_{H_{\text{loc}}^1}), \\ b_s + c_p \tilde{b} b_c &= O(b_c^3 + b_c \|\varepsilon\|_{H_{\text{loc}}^1}). \end{aligned} \quad (2.1.22)$$

Our main task here is to control  $\|\varepsilon\|_{H_{\text{loc}}^1}$ , which is done by a bootstrap argument<sup>7</sup>. If such a control exists, we will see that (2.1.22) is just a small perturbation of the system (2.1.21), and has almost the same behavior<sup>8</sup>.

5. See detailed discussing in Section 2.2.

6. See (2.2.27).

7. See Proposition 2.2.9.

8. See detailed proof in Section 6.2.

### 2.1.5.3 Monotonicity formula (section 4 and section 5)

The key techniques in this paper are the monotonicity of energy and a dispersive control of  $\|\varepsilon\|_{H_{\text{loc}}^1}$ .

In supercritical case any critical or subcritical norm of the error term  $\varepsilon$  cannot be controlled, for example  $\|\varepsilon\|_{L^2}$  or even  $\int_{y>0} \varepsilon^2$ . This contrasts with the critical case, where  $\|\varepsilon\|_{L^2}$  is small. And for the same reason, we can no longer use the  $L^1$  control of  $\varepsilon$  as Martel, Merle and Raphaël do in the critical case in [57].

Fortunately, we can still control  $\|\varepsilon_y\|_{L^2}$ . Moreover, the monotonicity of the energy gives an even better control of the  $L^2$  norm of  $\varepsilon_y$  on the half-line  $[\kappa B, +\infty)$ . Together with Gagliardo-Nirenberg inequality, we can give a good control of the localized  $L^2$  norm of  $\varepsilon$  on the right.

Next, we construct a nonlinear functional:

$$\mathcal{F} \sim \int \left[ \varepsilon_y^2 \psi + \varepsilon^2 \zeta - \frac{2}{p+1} (|\varepsilon + Q_b|^{p+1} - Q_b^{p+1} - (p+1)\varepsilon Q_b^p) \psi \right],$$

for well chosen functions  $(\psi, \zeta)$ , which are exponentially decaying to the left and bounded on the right. A similar functional was introduced in [57] for the critical equations, but they have a totally different meaning. Here the key point in this case is that we cannot control  $\int_{y>0} \varepsilon^2$ . We need to find a different way to control  $\varepsilon$  on the right. However, if we choose  $\zeta$  such that it is compactly supported on the right, i.e.  $\text{supp } \zeta \subset (-\infty, 2B^2]$ , for some large constant  $B$ , then for  $y > 0$ , only localized  $L^2$  norm of  $\varepsilon$  appears in  $\mathcal{F}$ , which can be controlled by using the monotonicity of energy introduced before.

Moreover, from the choice of the orthogonality conditions, the leading order term of  $\mathcal{F}$  is coercive:

$$\mathcal{F} \sim \|\varepsilon\|_{H_{\text{loc}}^1}^2.$$

The most significant technique here is the *Lyapounov monotonicity*:

$$\frac{d\mathcal{F}}{ds} + \frac{1}{B} \|\varepsilon\|_{H_{\text{loc}}^1}^2 \lesssim b_c^{\frac{7}{2}}. \quad (2.1.23)$$

This formula shows that  $\|\varepsilon\|_{H_{\text{loc}}^1}$  (or equivalently  $\mathcal{F}$ ) is almost decreasing with respect to  $s \in [0, +\infty)$ . So it is controlled by a small constant (say,  $b_c^{3+8\nu}$ ) if we choose a good initial data.

### 2.1.5.4 End of the proof of Theorem 2.1.1

We will see that the monotonicity formula (2.1.23) and modulation equations have already led to the bootstrap bound on  $b$  and  $\|\varepsilon\|_{H_{\text{loc}}^1}$ . So we only need to prove the bound of  $\|\varepsilon\|_{L^{p_0}}$ . This is done by working on the original variable with the help of a refined Strichartz estimate<sup>9</sup>. Then we finish the bootstrap argument and the remaining part of Theorem 2.1.1 is followed by a standard procedure.

9. See Corollary 2.6.2.

## Acknowledgement

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## 2.2 Description of the blow-up set of initial data

This section is devoted to give a specific description of the open subset  $\mathcal{O}_p$  of the initial data, which leads to the self-similar blow-up dynamics in Theorem 1.1. The most important part here is to construct a suitable approximate self-similar profile.

### 2.2.1 Construction of the approximate self-similar profile

This part follows H. Koch's work [38]. To avoid misunderstanding, we use a different notation.

Let us consider a solution  $u(t, x)$  of the form:

$$u(t, x) = \frac{1}{(3(T-t))^{\frac{2}{3(p-1)}}} V\left(\frac{x}{(3(T-t))^{\frac{1}{3}}}\right).$$

Then by a standard computation,  $u(t, x)$  is a solution if and only if  $V(x)$  satisfies:

$$\Delta V + V''' + (V|V|^{p-1})' = 0. \quad (2.2.1)$$

For any constant  $b > 0$ , we introduce a change of variable:

$$x = b^{\frac{1}{3}}(y + b^{-1}), \quad v(y) = b^{\frac{2}{3(p-1)}} V(b^{\frac{1}{3}}(y + b^{-1})).$$

Then (2.2.1) is equivalent to (2.1.4), i.e.

$$b\Delta v + (v'' - v + v|v|^{p-1})' = 0. \quad (2.2.2)$$

The exact solution of (2.2.2) has been studied by H. Koch in [38]. Actually H. Koch gives a even larger range of solutions.

**Proposition 2.2.1** (H. Koch [38]). *There exist  $p^* > 5, b^* > 0$ , such that there exist 2 smooth maps:  $\gamma(b, p) : [0, b^*) \times [5, p^*) \rightarrow \mathbb{R}$ ,  $v(b, p, y) : [0, b^*) \times [5, p^*) \times \mathbb{R} \rightarrow \mathbb{R}$ , such that the following holds:*

1. *The self-similar equation:*

$$b((1 + \gamma(b, p))v + xv') + (v'' - v + v|v|^{p-1})' = 0, \quad (2.2.3)$$

$$(v(b, p, \cdot), \mathcal{Q}'_p(\cdot)) = 0, \quad v(b, p, y) > 0. \quad (2.2.4)$$

2. For all  $p \in [5, p^*)$ , there exists a unique  $b = b(p) \in [0, b^*)$  such that:

$$\gamma(b(p), p) = -1 + \frac{2}{p-1}, \quad b(5) = 0, \quad (2.2.5)$$

Moreover,

$$\left. \frac{db(p)}{dp} \right|_{p=5} = \frac{\|\mathcal{Q}\|_{L^2}^2}{\|\mathcal{Q}\|_{L^1}^2} > 0, \quad (2.2.6)$$

$$\left. \frac{\partial \gamma}{\partial b} \right|_{b=b(p)} = -\frac{\|\mathcal{Q}_p\|_{L^1}^2}{8\|\mathcal{Q}_p\|_{L^2}^2} + \mathcal{O}(|p-5|) < 0, \quad (2.2.7)$$

$$\frac{1}{2} \int |v_y(b(p), p, y)|^2 dy - \frac{1}{p+1} \int |v(b(p), p, y)|^{p+1} dy = 0. \quad (2.2.8)$$

3.  $v(b, p, \cdot) \in \dot{H}^1 \cap L^{p+1}$ ,  $v(b, p, \cdot) \notin L^2$  if  $b > 0$  and  $v(0, p, y) = \mathcal{Q}_p(y)$ . Moreover, let

$$w_p(b, y) = v(b, p, y) - \mathcal{Q}_p(y),$$

then for all  $k, n \in \mathbb{N}$  there holds:

$$|w_p(b, y)| \lesssim \begin{cases} e^{-\frac{1}{3b}(1+b^{-2/3}|1-by|)^{-1-\gamma}} & \text{if } y > b^{-1}, \\ b \exp\left(\frac{1}{3b}[(1-by)^{3/2}-1]\right) & \text{if } b^{-1} \geq y > 0, \\ b(1-by)^{-1-\gamma} & \text{if } y \leq 0, \end{cases} \quad (2.2.9)$$

$$|\partial_y^k \partial_b^n v| \lesssim \begin{cases} e^{-\frac{1}{3b}(1+b^{-2/3}|1-by|)^{-1-\gamma-k}} & \text{if } y > b^{-1}, \\ |\partial_y^k \partial_b^n (\text{Hi}_\gamma(b^{-2/3}(1-by))/\text{Hi}_\gamma(b^{-2/3}))| & \text{if } b^{-1} \geq y > 0, \\ |\partial_y^k \partial_b^n (b(1-by)^{-1-\gamma})| + e^y & \text{if } y \leq 0, \end{cases} \quad (2.2.10)$$

where

$$\text{Hi}_\gamma(x) = \frac{1}{\pi} \int_0^{+\infty} \sigma^\gamma e^{-\frac{1}{3}\sigma^2 + \sigma x} d\sigma.$$

*Remark 2.2.2.* (1) and (2) in Proposition 2.1 correspond to Theorem 3 in [38]. (2.2.9) corresponds to Proposition 12 in [38]. (2.2.10) corresponds to Proposition 15 in [38]<sup>10</sup>.

*Remark 2.2.3.* In [38], H. Koch gives the following asymptotic behavior of  $\text{Hi}_\gamma$ :

$$\text{Hi}_\gamma(x) = \left( \frac{1}{\sqrt{\pi}} |x|^{-\frac{1}{4} + \frac{\gamma}{2}} + \mathcal{O}(|x|^{-\frac{7}{4} + \frac{\gamma}{2}}) \right) e^{\frac{2}{3}x^{3/2}}, \quad \text{as } x \rightarrow +\infty.$$

together with the fact that  $\partial_x \text{Hi}_\gamma = \text{Hi}_{\gamma+1}$ , we have for  $b^{-1} \geq y > 0$ :

$$\begin{aligned} & \left| \partial_y^k \partial_b^n (\text{Hi}_\gamma(b^{-2/3}(1-by))/\text{Hi}_\gamma(b^{-2/3})) \right| \\ & \lesssim_{k,n} \exp\left(\frac{1}{3b}[(1-by)^{3/2}-1]\right) \leq e^{-\frac{y}{10}}. \end{aligned}$$

Hence (2.2.10) reads:

$$|\partial_y^k \partial_b^n v| \lesssim_{k,n} \begin{cases} e^{-\frac{1}{3b}(1+b^{-2/3}|1-by|)^{-1-\gamma-k}} & \text{if } y > b^{-1}, \\ e^{-y/10} & \text{if } b^{-1} \geq y > 0, \\ |\partial_y^k \partial_b^n (b(1-by)^{-1-\gamma})| + e^y & \text{if } y \leq 0, \end{cases} \quad (2.2.11)$$

<sup>10</sup> Let's mention that there is a slight problem in the original statement of this estimate in [38] (i.e. Proposition 15 in [38]). And (2.2.10) is the correct version.



Now we fix some  $p \in (5, p^*)$ , and denote

$$b_c = b(p) \sim p - 5, \quad \tilde{b} = b - b_c. \quad (2.2.12)$$

From now on, we will focus on the case  $|\tilde{b}| \ll b_c$ .

The exact self-similar solution  $v$  is not in  $L^2$ , which is not good for our analysis. We need to construct a suitable approximation of  $v$ . Fortunately, we observe that though  $v$  has a slowly decaying tail at infinity, it is with a small coefficient:

$$v(y) \sim \begin{cases} \frac{e^{-1/3b_c}}{|y|^{1+\gamma}} & \text{as } y \rightarrow +\infty, \\ \frac{b_c^{-\gamma}}{|y|^{1+\gamma}} & \text{as } y \rightarrow -\infty. \end{cases}$$

So it is reasonable to consider a suitable cut-off of  $v$ . Choose a smooth cut-off function  $\chi_0(y)$ , such that  $\chi_0(y) = 0$  if  $|y| > 2$ ,  $\chi_0(y) = 1$  if  $|y| < 1$ . Then we define the approximate self-similar profile  $Q_b(y)$  as:

$$Q_b(y) = v(b, p, y)\chi(y), \quad (2.2.13)$$

where  $\chi(y) = \chi_0(b_c y)$ . We have the following properties of the approximate self-similar profile:

**Lemma 2.2.4** (Properties of the localized profile). *Assume that  $b_c$  is small and  $|\tilde{b}| \ll b_c$ , then there holds:*

1. *Estimates on  $Q_b$ , for all  $k \in \mathbb{N}$ ,  $q \in [1, +\infty]$ :*

$$|\partial_y^k Q_b(y)| \lesssim_k e^{-\frac{y}{10}}, \quad \text{for } y \geq 0, \quad (2.2.14)$$

$$|\partial_y^k Q_b(y)| \lesssim_k e^y + b_c^{1+k} \mathbf{1}_{[-2b_c^{-1}, 0]}(y), \quad \text{for } y \leq 0, \quad (2.2.15)$$

$$\|Q_b - Q_p\|_{L^q} \lesssim b_c^{1-\frac{1}{q}}, \quad \| (Q_b - Q_p)_y \|_{L^2} \lesssim b_c. \quad (2.2.16)$$

Here  $\mathbf{1}_I$  is the characteristic function of any interval  $I$ .

2.  *$Q_b$  is an approximate solution to (2.1.4): Let*

$$-\Phi_b = b\Lambda Q_b + (Q_b'' - Q_b + Q_b|Q_b|^{p-1})', \quad (2.2.17)$$

then for  $k = 0, 1$ :

$$\partial_y^k \Phi_b = C_p \tilde{b} b_c \partial_y^k Q_b + O(|\tilde{b}|^2 \partial_y^k Q_b + b_c^2 \mathbf{1}_{[-2, -1]}(b_c y) + e^{-\frac{1}{10b_c}} \mathbf{1}_{[1, 2]}(b_c y)), \quad (2.2.18)$$

where  $C_p = \frac{d\gamma}{db}|_{b=b_c} < 0$ .

3. *Energy property of  $Q_b$ :*

$$|E(Q_b)| \lesssim b_c^3 + |\tilde{b}|. \quad (2.2.19)$$

4. *Properties of the first order term with respect to  $b$ : let  $P_b(y) = \frac{\partial Q_b}{\partial b}(y)$ , then*

$$|P_b(y)| \lesssim e^{-\frac{y}{10}} \mathbf{1}_{\{y>0\}}(y) + \mathbf{1}_{[-2b_c^{-1}, 0]}(y). \quad (2.2.20)$$

Furthermore, we have:

$$(P_b, Q_p) = \frac{1}{16} \left( \int Q_p \right)^2 + O(|p-5|) > 0. \quad (2.2.21)$$

*Remark 2.2.5.* The construction of  $Q_b$  is based on H. Koch's work, which seems to be much more complicated than the one Martel, Merle and Raphaël constructed in [57]. If we construct the self-similar profile in that way, all the above estimates will also hold, except the energy property (3). Indeed, for such profile (denoted by  $\widetilde{Q}_b$ ), we can only get  $E(\widetilde{Q}_b) \lesssim b_c + |\tilde{b}|$ , which is not enough for our analysis <sup>11</sup>.

*Proof.* (1) is a direct consequence of the asymptotic behavior of  $v$ , i.e. (2.2.9) and (2.2.11). For (2), a standard computation shows that:

$$\begin{aligned} -\Phi_b = & -C_p \tilde{b}(b_c + \tilde{b})Q_b - b(\gamma + 1 - \frac{2}{p-1} - C_p \tilde{b})Q_b + (byv\chi' + v\chi'''' \\ & + 3v'\chi'' + 3v''\chi' - v\chi' + pv'v^{p-1}(\chi^p - \chi) + p\chi'\chi^{p-1}v^p). \end{aligned}$$

Then (2) follows immediately from (2.2.9), (2.2.11) and the choice of  $\chi$ .

For (3), we note that  $E(v(b_c, p, \cdot)) = 0$ , and again from (2.2.9) we obtain:

$$|E(Q_b) - E(v(b_c, p, \cdot))| \lesssim |\tilde{b}| + b_c^3.$$

Finally we prove (4). First, (2.2.20) follows immediately from (2.2.11). For (2.2.21), we let  $P(y) = \frac{\partial v}{\partial b}|_{b=0}(y)$ . From (2.2.11) and continuity,

$$\begin{aligned} |P_b(y) - P(y)| &= \left| b \int_0^1 \frac{\partial^2 v}{\partial b^2}(tb, p, y)\chi(y)dt - P(y)(1 - \chi(y)) \right| \\ &\lesssim b_c |y| \mathbf{1}_{[-2b_c^{-1}, 0]}(y) + b_c \mathbf{1}_{[-2b_c^{-1}, 2b_c^{-1}]}(y) + \mathbf{1}_{\{|y| > 1/b_c\}}(y), \end{aligned}$$

which yields:

$$|(P_b, Q_p) - (P, Q_p)| \lesssim b_c = O(|p - 5|).$$

So we only need to show that:

$$(P, Q_p) = \frac{1}{16} \left( \int Q_p \right)^2 + O(|p - 5|) > 0. \quad (2.2.22)$$

We consider the Taylor's expansion of  $v$  with respect to  $b$  for  $b \rightarrow 0^+$  (here we ignore the assumption  $|\tilde{b}| \ll b_c$ ). And then keep track of the first order term of  $b$  in (2.2.3). Observe that  $\gamma(0, p) = \frac{2}{p-1} - 1 + O(|p - 5|)$ , so we obtain:

$$(LP)' = \Lambda Q_p + O(|p - 5|)Q_p.$$

Taking scalar product with  $\int_{-\infty}^y \Lambda Q_p$  yields

$$\frac{1}{2} \left( \int \Lambda Q_p \right)^2 + O(|p - 5|) = -(LP, \Lambda Q_p) = -(P, L(\Lambda Q_p)) = 2(P, Q_p).$$

Since

$$\int \Lambda Q_p = \left( \frac{2}{p-1} - 1 \right) \int Q_p = \left( -\frac{1}{2} + O(|p - 5|) \right) \int Q_p,$$

then (2.2.22) follows, which concludes the proof of the Lemma.  $\square$

<sup>11</sup>. See Remark 2.4.3 for details.

## 2.2.2 Description of the blow-up set of initial data

**Definition 2.2.6.** Fix a small universal constant  $\nu > 0$  (which will be chosen later). For  $p \in (5, p^*(\nu))$  with  $p^*(\nu)$  close enough to 5, we let  $\mathcal{O}_p$  be the set of initial data  $u_0 \in H^1$  of the form:

$$u_0(x) = \frac{1}{\lambda_0^{\frac{2}{p-1}}} (Q_{b_0} + \varepsilon_0) \left( \frac{x - x_0}{\lambda_0} \right)$$

with parameter  $(\lambda_0, x_0, b_0) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}_+^*$ , such that:

1.  $b_0$  is near  $b_c (= b(p) \sim \sigma_c \sim p - 5 > 0)$ :

$$|b_0 - b_c| < b_c^{\frac{7}{2}}; \quad (2.2.23)$$

2. Smallness of  $\varepsilon_0$  in  $H^1$ :

$$\int \varepsilon_0^2 + (\varepsilon_0)_y^2 < b_c^{30}; \quad (2.2.24)$$

3. Condition on the scaling parameter:

$$0 < \lambda_0 \leq 1. \quad (2.2.25)$$

*Remark 2.2.7.* It is easy to verify that  $\mathcal{O}_p$  is nonempty. We may choose suitable  $b_0, x_0, \lambda_0$ , and set  $\varepsilon_0 = 0$ .

## 2.2.3 Setting the bootstrap

Let  $u_0 \in \mathcal{O}_p$ , and  $u(t)$  be the corresponding solution to (2.1.1) with maximal time interval  $[0, T)$ ,  $0 < T \leq +\infty$ . By using the regularity  $u \in C([0, T), H^1)$  and a standard modulation theory<sup>12</sup> (up to some small perturbations), we can find a  $0 < T^* \leq T$ , such that for all  $t \in [0, T^*)$ ,  $u(t, x)$  admits a unique decomposition:

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} (Q_{b(t)} + \varepsilon(t)) \left( \frac{x - x(t)}{\lambda(t)} \right) \quad (2.2.26)$$

with geometrical parameters  $(\lambda(t), x(t), b(t)) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}_+^*$ , which are all  $C^1$  functions and the following orthogonality condition holds:

$$(\varepsilon(t), Q_p) = (\varepsilon(t), \Lambda Q_p) = (\varepsilon(t), y \Lambda Q_p) = 0. \quad (2.2.27)$$

Moreover, we may assume that:

$$|\tilde{b}(0)| = |b(0) - b_c| \leq b_c^2, \quad (2.2.28)$$

$$\int \varepsilon^2(0) + \varepsilon_y^2(0) < b_c^{20}, \quad (2.2.29)$$

$$0 < \lambda(0) \leq 2. \quad (2.2.30)$$

12. See Lemma 1 in [53] and Lemma 2.5 in [57].

Now we state the bootstrap argument. Denote

$$B = b_c^{-\frac{1}{20}} \quad (2.2.31)$$

and then choose a smooth function  $\varphi$  such that:

$$\varphi(y) = \begin{cases} e^y & \text{for } y < -1, \\ 1+y & \text{for } -\kappa < y < \kappa, \\ 3 & \text{for } y > 1, \end{cases} \quad (2.2.32)$$

$$\varphi'(y) \geq 0 \text{ for all } y \in \mathbb{R},$$

where  $0 < \kappa < 1$  is a small universal constant to be chosen later<sup>13</sup>. We let  $\varphi_B(y) = \varphi(\frac{y}{B})$ , and define the localized Sobolev norm of  $\varepsilon$ :

$$\mathcal{N}(t) = B \left( \int \varepsilon^2(t, y) \varphi_B'(y) dy + \int \varepsilon_y^2(t, y) \varphi_B'(y) dy \right). \quad (2.2.33)$$

By continuity, we may assume that on  $[0, T^*)$ , the following *a priori* bound holds:

$$|\tilde{b}(t)| \leq b_c^{\frac{3}{2} + \nu}, \quad (2.2.34)$$

$$\mathcal{N}(t) \leq b_c^{3+6\nu}, \quad (2.2.35)$$

$$\|\varepsilon(t)\|_{L^{p_0}} \leq b_c^{\frac{23}{50}}, \quad (2.2.36)$$

$$\|\varepsilon_y\|_{L^2} \leq b_c^{\frac{2}{3}}. \quad (2.2.37)$$

Here we choose

$$p_0 = \frac{5}{2}.$$

*Remark 2.2.8.* From bootstrap assumption (2.2.36), (2.2.37) and Gagliardo-Nirenberg inequality, we have for all  $q_0 \geq p_0$ ,

$$\|\varepsilon\|_{L^{q_0}} \lesssim \|\varepsilon\|_{L^{p_0}}^{\frac{p_0(q_0+2)}{q_0(p_0+2)}} \|\varepsilon_y\|_{L^2}^{\frac{2(q_0-p_0)}{q_0(p_0+2)}} \leq b_c^{\frac{149q_0-62}{270q_0}}.$$

In particular, for  $q_0 = p$  (note that  $p$  is slightly larger than 5) and  $q_0 = +\infty$ , we have:

$$\int |\varepsilon|^p \lesssim b_c^{\frac{5}{2}}, \quad \|\varepsilon\|_{L^\infty} \leq b_c^{\frac{149}{270}}, \quad (2.2.38)$$

Moreover, for all  $t \in [0, T^*)$ :

$$\int \varepsilon^2(t) e^{-\frac{|y|}{2}} \lesssim \mathcal{N}(t) + e^{-\kappa B/2} \|\varepsilon\|_{L^\infty}^2 \leq b_c^{20} + \mathcal{N}(t). \quad (2.2.39)$$

Our main claim is that the above regime is trapped:

---

13. See in Appendix A.

**Proposition 2.2.9.** *There holds for all  $t \in [0, T^*)$ ,*

$$|\tilde{b}(t)| \leq b_c^{\frac{3}{2}+2\nu}, \quad (2.2.40)$$

$$\mathcal{N}(t) \leq b_c^{3+8\nu}, \quad (2.2.41)$$

$$\|\varepsilon(t)\|_{L^{p_0}} \leq b_c^{\frac{13}{28}}, \quad (2.2.42)$$

$$\|\varepsilon_y\|_{L^2} \leq b_c^{\frac{3}{4}}. \quad (2.2.43)$$

and hence we may take  $T^* = T$ .

The next 3 sections are devoted to derive the dynamical controls of the geometrical parameters and monotonicity tools, which are the heart of the proof of the bootstrap bound in Proposition 2.2.9. Then Theorem 2.1.1 is just a simple consequence of Proposition 2.2.9, which will be shown in Section 6.

## 2.3 Modulation equations

In the framework of the geometrical decomposition (2.2.26), we introduce a new variable:

$$s = \int_0^t \frac{1}{\lambda^3(t')} dt', \quad y = \frac{x - x(t)}{\lambda(t)}. \quad (2.3.1)$$

Now we use  $(s, y)$  instead of the original variables  $(t, x)$ , and denote  $s^* = s(T^*)$ . Then we can claim the following properties:

**Proposition 2.3.1.** *The map  $s \in [0, s^*) \rightarrow (\lambda(s), x(s), b(s))$  is  $C^1$  and the following holds:*

1. *Equation of  $\varepsilon$ : for all  $s \in [0, s^*)$ ,*

$$\begin{aligned} \varepsilon_s - (L\varepsilon)_y + b\Lambda\varepsilon &= \left(\frac{\lambda_s}{\lambda} + b\right)(\Lambda Q_b + \Lambda\varepsilon) + \left(\frac{x_s}{\lambda} - 1\right)(Q_b + \varepsilon)_y \\ &\quad + \Phi_b - b_s P_b - (R_b(\varepsilon))_y - (R_{NL}(\varepsilon))_y, \end{aligned} \quad (2.3.2)$$

where

$$\Phi_b = -b\Lambda Q_b - (Q_b'' - Q_b + Q_b^p)', \quad (2.3.3)$$

$$R_b(\varepsilon) = p(Q_b^{p-1} - Q_b^{p-1})\varepsilon, \quad (2.3.4)$$

$$R_{NL}(\varepsilon) = (\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - p\varepsilon Q_b^{p-1} - Q_b^p. \quad (2.3.5)$$

2. *Modulation equation:*

$$\left|\frac{\lambda_s}{\lambda} + b_c\right| \lesssim b_c^{\frac{5}{2}} + \mathcal{N}^{\frac{1}{2}}, \quad (2.3.6)$$

$$\left|\frac{x_s}{\lambda} - 1\right| \lesssim b_c^{\frac{5}{2}} + \mathcal{N}^{\frac{1}{2}}, \quad (2.3.7)$$

$$|b_s + c_p \tilde{b} b_c| \lesssim b_c^3 + b_c \mathcal{N}^{\frac{1}{2}}, \quad (2.3.8)$$

where  $c_p$  is a positive constant with  $c_p = 2 + O(|p - 5|)$ .

*Proof.* The proof of (2.3.2) follows from a direct computation and the equation of  $u(t)$ . Now we prove (2.3.6)–(2.3.8). Let us differentiate the orthogonality condition  $(\varepsilon, \Lambda \mathcal{Q}_p) = (\varepsilon, y \Lambda \mathcal{Q}_p) = 0$  and use (2.2.38) to obtain:

$$\begin{aligned} & \left| \left( \frac{\lambda_s}{\lambda} + b \right) (\Lambda \mathcal{Q}_b, \Lambda \mathcal{Q}_p) \right| + \left| \left( \frac{x_s}{\lambda} - 1 \right) (\mathcal{Q}'_b, y \Lambda \mathcal{Q}_p) \right| \\ & \lesssim \left| \left( \frac{\lambda_s}{\lambda} + b \right) (\Lambda \mathcal{Q}_b, y \Lambda \mathcal{Q}_p) \right| + \left| \left( \frac{x_s}{\lambda} - 1 \right) (\mathcal{Q}'_b, \Lambda \mathcal{Q}_p) \right| + b_c |\tilde{b}| + |b_s| \\ & \quad + \int (\varepsilon^2 e^{-\frac{|y|}{2}} + |\varepsilon|^p) + b_c \left( \int \varepsilon^2 e^{-\frac{|y|}{2}} \right)^{\frac{1}{2}} + |(\varepsilon, L(\Lambda \mathcal{Q}_p)') + (\varepsilon, L(y \Lambda \mathcal{Q}_p)')| \\ & \lesssim \left| \left( \frac{\lambda_s}{\lambda} + b \right) (\Lambda \mathcal{Q}_b, y \Lambda \mathcal{Q}_p) \right| + \left| \left( \frac{x_s}{\lambda} - 1 \right) (\mathcal{Q}'_b, \Lambda \mathcal{Q}_p) \right| + b_c^{\frac{5}{2}} + |b_s| + \left( \int \varepsilon^2 e^{-\frac{|y|}{2}} \right)^{\frac{1}{2}}. \end{aligned}$$

From (2.2.9), we have for all  $y \in \mathbb{R}$ :

$$|\mathcal{Q}_b(y) - \mathcal{Q}_p(y)| \lesssim b_c,$$

which implies:

$$|(\Lambda \mathcal{Q}_b, \Lambda \mathcal{Q}_p) - (\Lambda \mathcal{Q}_p, \Lambda \mathcal{Q}_p)| \leq \|\mathcal{Q}_b - \mathcal{Q}_p\|_{L^\infty} \|\Lambda^* \Lambda \mathcal{Q}_p\|_{L^1} = O(b_c).$$

hence  $(\Lambda \mathcal{Q}_b, \Lambda \mathcal{Q}_p) = \|\Lambda \mathcal{Q}_p\|_{L^2}^2 + O(b_c)$ . Similarly, we have

$$(\mathcal{Q}'_b, y \Lambda \mathcal{Q}_p) = \|\Lambda \mathcal{Q}_p\|_{L^2}^2 + O(b_c), \quad (\Lambda \mathcal{Q}_b, y \Lambda \mathcal{Q}_p) = O(b_c), \quad (\mathcal{Q}'_b, \Lambda \mathcal{Q}_p) = O(b_c).$$

Combining these estimates with (2.2.39) we have:

$$\left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| \lesssim b_c^{\frac{5}{2}} + |b_s| + \mathcal{N}^{\frac{1}{2}}. \quad (2.3.9)$$

Now we differentiate the orthogonality condition  $(\varepsilon, \mathcal{Q}_p) = 0$ . A similar computation shows:

$$\begin{aligned} & |(P_b, \mathcal{Q}_p) b_s - (\Phi_b, \mathcal{Q}_p)| \\ & \lesssim O(b_c) \left( \left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| \right) + \int (\varepsilon^2 e^{-\frac{|y|}{2}} + |\varepsilon|^p) + b_c \left( \int \varepsilon^2 e^{-\frac{|y|}{2}} \right)^{\frac{1}{2}} \\ & = O(b_c) \left( b_c^{\frac{5}{2}} + \mathcal{N}^{\frac{1}{2}} + \left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| \right). \end{aligned} \quad (2.3.10)$$

Observe from (2.2.18) and (2.2.21):

$$\begin{aligned} & (\Lambda \mathcal{Q}_b, \mathcal{Q}_p) = O(b_c), \quad (\mathcal{Q}'_b, \mathcal{Q}_p) = O(b_c), \\ & (P_b, \mathcal{Q}_p) = \frac{1}{16} \|\mathcal{Q}_p\|_{L^1}^2 + O(|p-5|) > 0, \\ & (\Phi_b, \mathcal{Q}_p) = C_p b_c \tilde{b} (\mathcal{Q}_b, \mathcal{Q}_p) + O(|\tilde{b}|^2 + e^{-\frac{1}{2b_c}}) = -\tilde{c}_p \|\mathcal{Q}_p\|_{L^1}^2 b_c \tilde{b} + O(b_c^3), \end{aligned}$$

with  $\tilde{c}_p = \frac{1}{8} + O(|p-5|) > 0$ . First, from (2.3.10) we have:

$$|b_s| \lesssim b_c^{\frac{5}{2}} + O(b_c) \left( \mathcal{N}^{\frac{1}{2}} + \left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| \right). \quad (2.3.11)$$

Injecting (2.3.11) into (2.3.9), we obtain (2.3.6) and (2.3.7). Moreover (2.3.10) implies:

$$|b_s + c_p b_c \tilde{b}| = O(b_c) \left( b_c^2 + \mathcal{N}^{\frac{1}{2}} + \left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| \right), \quad (2.3.12)$$

where  $c_p = 2 + O(|p-5|)$ . Then (2.3.8) follows from (2.3.6), (2.3.7) and (2.3.12), which concludes the proof of the proposition.  $\square$

## 2.4 Monotonicity of the energy

This section is devoted to derive a control of the  $L^2$  norm of  $\varepsilon_y$  by the energy conservation law and monotonicity. We will first give a control of  $\|\varepsilon_y\|_{L^2}$  on the whole line, which proves the bootstrap bound (2.2.43). But furthermore, we will show that on the half line  $[\kappa B, +\infty)$ , there is a much better bound for the  $L^2$  norm of  $\varepsilon_y$ , which comes from the monotonicity of the localized energy<sup>14</sup>. Then by Gagliardo-Nirenberg inequality we can get a good control for the localized  $L^2$  norm of  $\varepsilon$ .

**Lemma 2.4.1.** *For all  $s \in [0, s^*)$ , the following estimates hold:*

$$\int \varepsilon_y^2(s) \lesssim b_c^{\frac{3}{2}+\nu}, \quad (2.4.1)$$

$$\int_{y>\kappa B} \varepsilon_y^2(s) \lesssim b_c^{\frac{55}{7}}. \quad (2.4.2)$$

*Remark 2.4.2.* (2.4.1) is the desired bootstrap bound (2.2.43).

*Proof of Lemma 4.1.* The first estimate (2.4.1) is a consequence of the energy conservation law. We write down the energy equality explicitly:

$$\begin{aligned} 2\lambda(s)^{2(1-\sigma_c)} E(u_0) &= 2E(Q_b) + \int \varepsilon_y(Q_b - Q_p)_y \\ &\quad + \int \varepsilon_y^2 - \int \varepsilon(Q_p)_{yy} - \frac{2}{p+1} \int ((Q_b + \varepsilon)^{p+1} - Q_b^{p+1}). \end{aligned} \quad (2.4.3)$$

From (2.2.35) and (2.3.6), we know for all  $s \in [0, s^*)$

$$-(1+\nu)b_c \leq \frac{\lambda_s}{\lambda} \leq -(1-\nu)b_c < 0. \quad (2.4.4)$$

---

14. See (2.4.14).

Therefore  $\lambda(s)$  is decreasing on  $[0, s^*)$ , then we have:

$$\begin{aligned}
\int \varepsilon_y^2 &\lesssim \lambda(s)^{2(1-\sigma_c)} |E(u_0)| + |\tilde{b}| + b_c^3 + \|(Q_b - Q_p)_y\|_{L^2}^2 \\
&\quad + \left( \int \varepsilon^2 e^{-\frac{|y|}{2}} \right)^{\frac{1}{2}} + \int (|\varepsilon|^p + Q_b^p) |\varepsilon| \\
&\lesssim b_c^{\frac{3}{2}+\nu} + \lambda(0)^{2(1-\sigma_c)} |E(u_0)| + \int |\varepsilon|^{p+1} + \int_{y>\kappa B} Q_b^p |\varepsilon| \\
&\quad + \int_{|y|\leq\kappa B} Q_b^p |\varepsilon| + \int_{y<-\kappa B} Q_b^p |\varepsilon| \\
&\lesssim b_c^{\frac{3}{2}+\nu} + \lambda(0)^{2(1-\sigma_c)} |E(u_0)| + b_c^3 + e^{-B} \left( \int_{y>\kappa B} \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} \\
&\quad + \left( \int_{|y|<\kappa B} |\varepsilon|^2 \right)^{\frac{1}{2}} + \left( \int |\varepsilon|^{p_0} \right)^{\frac{1}{p_0}} \left( \int_{y<-\kappa B} Q_b^{p_0} \right)^{\frac{1}{p_0}} \\
&\lesssim b_c^{\frac{3}{2}+\nu} + \lambda(0)^{2(1-\sigma_c)} |E(u_0)|.
\end{aligned}$$

Here we use the fact that  $|Q_b(y)| \lesssim b_c$ , if  $y < -\kappa B$ , and  $Q_b$  decays exponentially on the right.

So it remains to estimate  $\lambda(0)^{2(1-\sigma_c)} |E(u_0)|$ . We let  $s = 0$  in (2.4.3), from the assumption of the initial data, we have:

$$\lambda(0)^{2(1-\sigma_c)} |E(u_0)| \lesssim |E(Q_b(0))| + \|\varepsilon(0)\|_{H^1} \lesssim b_c^2 + |\tilde{b}(0)| \lesssim b_c^{\frac{3}{2}+\nu},$$

then (2.4.1) follows.

*Remark 2.4.3.* Here, the global  $L^2$  norm of  $\varepsilon_y$  is estimated in terms of  $E(Q_b)$ . If we choose the profile constructed in [57] instead of the one constructed in [38], we will only get  $\|\varepsilon_y\|_{L^2} \leq b_c^{\frac{1}{2}}$ . This leads to a bad control of the  $L^\infty$  norm of  $\varepsilon$ , i.e.  $\|\varepsilon\|_{L^\infty} \leq b_c^\alpha$ , where  $\alpha < \frac{1}{2}$ . This is not enough for our analysis<sup>15</sup>.

Now we prove (2.4.2). We use a bootstrap argument on  $[0, T^*)$ . We assume that for all  $t \in [0, T^*)$ , we have:

$$\int_{y>\kappa B} \varepsilon_y^2(t) \leq b_c^{\frac{15}{2}}. \quad (2.4.5)$$

Since this estimate is satisfied for  $t = 0$ , we only need to improve this estimate to:

$$\int_{y>\kappa B} \varepsilon_y^2(t) \lesssim b_c^{\frac{55}{7}} \quad \text{for } \forall t \in [0, T^*). \quad (2.4.6)$$

To do this we first choose a smooth function  $\theta$  such that:

$$\theta(y) = e^{-|y|} \text{ for } |y| > 1, \quad \theta(y) \geq \frac{1}{e} \text{ for } |y| < 1. \quad (2.4.7)$$

We then define

$$\Theta(y) = \frac{1}{K} \int_{-\infty}^y \theta(y') dy',$$

---

<sup>15</sup>. See details in (2.6.14).



where  $K = \int_{-\infty}^{+\infty} \theta(y') dy'$ .

Let  $t \in [0, T^*)$  be any fixed time. For all  $\tau \in [0, t]$ , we denote:

$$\tilde{x}(\tau) = \frac{1}{\sqrt{B}} \left( \frac{x - x(\tau)}{\lambda(\tau)} - \kappa B \right), \quad \tilde{y} = \frac{y - \kappa B}{\sqrt{B}},$$

$$\tilde{E}(\tau) = \int \left( \frac{1}{2} |u_x(\tau)|^2 - \frac{1}{p+1} |u(\tau)|^{p+1} \right) \Theta(\tilde{x}(\tau)) dx.$$

Observe that  $\Theta(\tilde{y}) \leq e^{-\frac{\kappa\sqrt{B}}{2}} \leq b_c^{20}$ , if  $y < \kappa B/2$ , so we have:

$$\begin{aligned} & \lambda(t)^{2(1-\sigma_c)} \tilde{E}(t) \\ &= \frac{1}{2} \int ((Q_b)_y + \varepsilon_y)^2 \Theta(\tilde{y}) dy - \frac{1}{p+1} \int |Q_b + \varepsilon|^{p+1} \Theta(\tilde{y}) dy \\ &\gtrsim \int_{y > \kappa B} \varepsilon_y^2(t) - \int_{y > \frac{\kappa B}{2}} (|(Q_b)_y|^2 + |Q_b|^{p+1}) - e^{-\frac{\kappa\sqrt{B}}{2}} \int_{y < \frac{\kappa B}{2}} (|(Q_b)_y|^2 + |Q_b|^{p+1}) \\ &\quad - \int_{y > \frac{\kappa B}{2}} |\varepsilon|^{p+1} - e^{-\frac{\kappa\sqrt{B}}{2}} \int_{y < \frac{\kappa B}{2}} |\varepsilon|^{p+1}. \end{aligned} \tag{2.4.8}$$

Next from (2.2.35), (2.2.36), (2.4.5) and localized Gagliardo-Nirenberg inequality, we know that (recall  $p_0 = \frac{5}{2}$ ):

$$\int_{y > \kappa B} |\varepsilon|^{p+1} \lesssim \left( \int |\varepsilon|^{p_0} \right)^{\frac{p+3}{p_0+2}} \left( \int_{y > \kappa B} \varepsilon_y^2 \right)^{\frac{p+1-p_0}{p_0+2}} \lesssim b_c^{\frac{173p-156}{90}} \leq b_c^{\frac{55}{7}}, \tag{2.4.9}$$

$$\|\varepsilon\|_{L^\infty(y > \kappa B)} \lesssim \left( \int |\varepsilon|^{p_0} \right)^{\frac{1}{p_0+2}} \left( \int_{y > \kappa B} \varepsilon_y^2 \right)^{\frac{1}{p_0+2}} \leq b_c^{\frac{173}{90}} \leq b_c^{\frac{3}{2}}. \tag{2.4.10}$$

On the other hand, by Sobolev embedding we can show:

$$\|\varepsilon\|_{L^\infty(|y| < \kappa B)} \lesssim \mathcal{N}^{\frac{1}{2}} \leq b_c^{\frac{3}{2}}, \tag{2.4.11}$$

hence

$$\int_{\frac{\kappa B}{2} < y < \kappa B} |\varepsilon|^{p+1} \leq \|\varepsilon\|_{L^\infty(|y| < \kappa B)}^{p-1} \left( \int_{|y| \leq \kappa B} \varepsilon^2 \right) \leq b_c^9. \tag{2.4.12}$$

Injecting (2.4.9) and (2.4.12) into (2.4.8) yields:

$$\int_{y > \kappa B} \varepsilon_y^2 \lesssim b_c^{\frac{55}{7}} + \lambda(t)^{2(1-\sigma_c)} \tilde{E}(t). \tag{2.4.13}$$

Therefore, it remains to estimate  $\tilde{E}(t)$ . We first use Kato's Localization identity for energy

to compute:

$$\begin{aligned}
\frac{d}{d\tau}\tilde{E}(\tau) &= -\frac{1}{2}\int(u_{xx}+u|u|^{p-1})^2g_x - \int u_{xx}^2g_x \\
&\quad + p\int u|u|^{p-2}u_x^2g_x + \frac{1}{2}\int u_x^2g_{xxx} \\
&\quad - \frac{x_t(\tau)}{\sqrt{B}\lambda(\tau)}\int\left(\frac{1}{2}|u_x(\tau)|^2 - \frac{1}{p+1}|u(\tau)|^{p+1}\right)\theta(\tilde{x}(\tau))dx \\
&\quad - \frac{\lambda_t(\tau)}{\sqrt{B}\lambda(\tau)}\int\left(\frac{1}{2}|u_x(\tau)|^2 - \frac{1}{p+1}|u(\tau)|^{p+1}\right)\left(\frac{x-x(\tau)}{\lambda(\tau)}\right)\theta(\tilde{x}(\tau))dx \\
&= I + II + III + IV,
\end{aligned}$$

where  $g(x, \tau) = \Theta(\tilde{x}(\tau))$ .

We claim that for some universal constant  $C > 0$ , there holds:

$$\frac{d}{d\tau}\tilde{E}(\tau) \leq \frac{Cb_c^9}{\lambda(\tau)^{3+2(1-\sigma_c)}} \quad (2.4.14)$$

First  $I \leq 0$ , since  $g$  is nondecreasing in  $x$ . We then deal with  $III$  and  $IV$ . From (2.3.6) and (2.3.7) we have:

$$x_t \sim \frac{1}{\lambda^2}, \quad \lambda_t \sim -\frac{b_c}{\lambda^2}.$$

For  $III$ , we use (2.4.10), (2.4.11) and the fact that  $|\theta(\tilde{y})| \leq e^{-\frac{\kappa\sqrt{B}}{2}}$ , if  $y \leq \kappa B/2$  to estimate:

$$\begin{aligned}
III &\leq -\frac{1}{4\sqrt{B}\lambda^3(\tau)}\int|u_x(\tau)|^2\theta(\tilde{x}(\tau)) + \frac{C}{\sqrt{B}\lambda(\tau)^{3+2(1-\sigma_c)}}\int|\varepsilon(\tau) + Q_{b(\tau)}|^{p+1}\theta(\tilde{y}) \\
&\leq -\frac{1}{4\sqrt{B}\lambda^3(\tau)}\int|u_x(\tau)|^2\theta(\tilde{x}(\tau)) \\
&\quad + \frac{1}{\sqrt{B}\lambda(\tau)^{3+2(1-\sigma_c)}}\left(\|\varepsilon\|_{L^\infty(y>\frac{\kappa B}{2})}^{p+1}\int_{y>\kappa B/2}\theta(\tilde{y})dy + e^{-\frac{\kappa\sqrt{B}}{2}}\int_{y<\kappa B/2}|\varepsilon|^{p+1}\right) \\
&\quad + \frac{1}{\sqrt{B}\lambda(\tau)^{3+2(1-\sigma_c)}}\left(e^{-\frac{\kappa\sqrt{B}}{2}}\int_{y<\kappa B/2}|Q_{b(\tau)}|^{p+1} + \int_{y>\kappa B/2}|Q_{b(\tau)}|^{p+1}\right) \\
&\leq -\frac{1}{4\sqrt{B}\lambda^3(\tau)}\int|u_x(\tau)|^2\theta(\tilde{x}(\tau)) + \frac{Cb_c^9}{\lambda(\tau)^{3+2(1-\sigma_c)}}.
\end{aligned} \quad (2.4.15)$$

For  $IV$ , similarly there holds:

$$\begin{aligned}
IV &\leq \frac{b_c}{\sqrt{B}\lambda^3(\tau)}\int|u_x(\tau)|^2\left|\frac{x-x(\tau)}{\lambda(\tau)}\right|\theta(\tilde{x}(\tau)) + \frac{Cb_c^9}{\lambda(\tau)^{3+2(1-\sigma_c)}} \\
&= \frac{b_c}{\sqrt{B}\lambda(\tau)^{3+2(1-\sigma_c)}}\int|y|\varepsilon_y^2(\tau)\theta(\tilde{y}) + \frac{Cb_c^9}{\lambda(\tau)^{3+2(1-\sigma_c)}}.
\end{aligned}$$

We then divide the integral  $\int |y|\varepsilon_y^2(\tau)\theta(\tilde{y})$  into 2 parts:  $\int_{|y-\kappa B|>B}$  and  $\int_{|y-\kappa B|\leq B}$ .

For the first part, we have  $|y\theta(\tilde{y})| \leq e^{-\frac{\kappa\sqrt{B}}{2}}$  on this region, hence:

$$\int_{|y-\kappa B|>B} |y|\varepsilon_y^2(\tau)\theta(\tilde{y}) \leq e^{-\frac{\kappa\sqrt{B}}{2}} \int \varepsilon_y^2(\tau) \leq Cb_c^9.$$

For another part, we have  $|yb_c| \ll 1$  on this region, hence:

$$\frac{b_c}{\sqrt{B}\lambda(\tau)^{3+2(1-\sigma_c)}} \int_{|y-\kappa B|\leq B} |y|\varepsilon_y^2(\tau)\theta(\tilde{y}) \leq \frac{1}{100\sqrt{B}\lambda^3(\tau)} \int \frac{1}{2}|u_x(\tau)|^2\theta(\tilde{x}(\tau)).$$

Collecting the above estimates, we obtain:

$$IV \leq \frac{1}{100\sqrt{B}\lambda^3(\tau)} \int |u_x(\tau)|^2\theta(\tilde{x}(\tau)) + \frac{Cb_c^9}{\lambda(\tau)^{3+2(1-\sigma_c)}}. \quad (2.4.16)$$

Finally, we estimate  $II$ :

$$\begin{aligned} II &\leq \frac{C}{\sqrt{B}\lambda(\tau)^{3+2(1-\sigma_c)}} \int |\varepsilon(\tau) + Q_{b(\tau)}|^{p-1} |\varepsilon_y(\tau) + (Q_{b(\tau)})_y|^2 \theta(\tilde{y}) \\ &\quad + \frac{C}{B^{\frac{3}{2}}\lambda^3(\tau)} \int |u_x(\tau)|^2 \theta''(\tilde{x}(\tau)) \\ &= II_1 + II_2. \end{aligned}$$

For the first term  $II_1$ , we divide the integral into 2 parts  $\int_{y<\kappa B/2}$  and  $\int_{y>\kappa B/2}$  as before, to obtain:

$$\begin{aligned} II_1 &\leq \frac{C}{\sqrt{B}\lambda(\tau)^{3+2(1-\sigma_c)}} \int \theta(\tilde{y}) \left( |\varepsilon|^{p-1} \varepsilon_y^2 + |\varepsilon|^{p-1} |Q'_b|^2 + |Q_b|^{p-1} (|\varepsilon_y|^2 + |Q'_b|^2) \right) \\ &\leq \frac{C}{\sqrt{B}\lambda(\tau)^{3+2(1-\sigma_c)}} \left( \|\varepsilon\|_{L^\infty(y>\frac{\kappa B}{2})}^{p-1} \int_{y>\frac{\kappa B}{2}} \varepsilon_y^2 + e^{-\frac{\kappa\sqrt{B}}{2}} \int_{y<\frac{\kappa B}{2}} (|(Q_b)_y|^2 + \varepsilon_y^2) \right. \\ &\quad \left. + \int_{y>\frac{\kappa B}{2}} |(Q_b)_y|^2 |Q_b|^{p-1} + \int_{y>\frac{\kappa B}{2}} e^{-\frac{|y|}{10}} (|\varepsilon|^{p-1} + \varepsilon_y^2) dy \right) \\ &\leq \frac{C}{\sqrt{B}\lambda(\tau)^{3+2(1-\sigma_c)}} \left( \|\varepsilon\|_{L^\infty(y>\frac{\kappa B}{2})}^{p-1} \int_{y>\frac{\kappa B}{2}} \varepsilon_y^2(\tau) + b_c^9 \right). \end{aligned}$$

Then from (2.4.10), (2.4.11) and the fact that:

$$\int_{y>\kappa B/2} \varepsilon_y^2(\tau) \leq \int_{\kappa B>y>\kappa B/2} \varepsilon_y^2(\tau) + \int_{y>\kappa B} \varepsilon_y^2(\tau) \leq b_c^3,$$

we obtain:

$$\|\varepsilon\|_{L^\infty(y>\kappa B/2)}^{p-1} \int_{y>\kappa B/2} \varepsilon_y^2(\tau) \lesssim b_c^{\frac{3(p-1)}{2}} \times b_c^3 \leq b_c^9,$$

hence

$$II_1 \leq \frac{Cb_c^9}{\lambda(\tau)^{3+2(1-\sigma_c)}}. \quad (2.4.17)$$

For the second term  $II_2$ , from the definition of  $\theta$ , we have  $|\theta''| \lesssim \theta$ , hence:

$$II_2 \leq \frac{1}{100\sqrt{B}\lambda^3(\tau)} \int |u_x(\tau)|^2 \theta(\tilde{x}(\tau)). \quad (2.4.18)$$

Collecting (2.4.15), (2.4.16), (2.4.17) and (2.4.18), we obtain (2.4.14).

Observe that for  $\beta > 3$  there holds:

$$\int_0^t \frac{1}{\lambda^\beta(\tau)} d\tau \leq -2 \int_0^t \frac{\lambda_t(\tau)}{b_c \lambda^{\beta-2}(\tau)} d\tau \leq \frac{2}{(\beta-3)b_c \lambda^{\beta-3}(t)}. \quad (2.4.19)$$

Integrating (2.4.14) from 0 to  $t$  yields:

$$\begin{aligned} \lambda(t)^{2(1-\sigma_c)} \tilde{E}(t) &\lesssim \lambda(t)^{2(1-\sigma_c)} \tilde{E}(0) + b_c^8 \lesssim \lambda(0)^{2(1-\sigma_c)} \tilde{E}(0) + b_c^8 \\ &\lesssim \int |\varepsilon_y(0) + (\mathcal{Q}_{b(0)})_y|^2 \theta(\tilde{y}) + b_c^8 \\ &\lesssim \int |(\mathcal{Q}_{b(0)})_y|^2 \theta(\tilde{y}) dy + \|\varepsilon_y(0)\|_{L^2}^2 + b_c^8 \\ &\lesssim b_c^8, \end{aligned} \quad (2.4.20)$$

where we use the assumption on the initial data, i.e. (2.2.29). Then (2.4.6) follows from (2.4.13) and (2.4.20), which completes the proof of Lemma 4.1.  $\square$

*Remark 2.4.4.* From (2.4.2) and Localized Gagliardo-Nirenberg inequality, we have the following  $L^\infty$  estimate of  $\varepsilon$ :

$$\|\varepsilon\|_{L^\infty(y > \kappa B)} \lesssim \left( \int |\varepsilon|^{p_0} \right)^{\frac{1}{p_0+2}} \left( \int_{y > \kappa B} \varepsilon_y^2 \right)^{\frac{1}{p_0+2}} \leq b_c^{\frac{1261}{630}} \leq b_c^2, \quad (2.4.21)$$

which is important in the derivation of the second monotonicity formula in the next section.

## 2.5 The second monotonicity formula

This section is devoted to derive a second monotonicity tool for  $\varepsilon$ , which is the key technique to our analysis. It is a Lyapunov functional based on a suitable localized Hamiltonian which is somehow similar to that of [57]. But here, due to the super-criticality, we cannot estimate the  $L^2$  norm of  $\varepsilon$  even on the half-line  $(1/b_c, +\infty)$ . We need to cut it off while this will generate some new terms to be controlled. But these new terms will be controlled by using the monotonicity of the energy introduced in the previous section.

**Pointwise monotonicity:** Recall from (2.32), the definition of  $\varphi$ . We let  $\psi, \eta$  be another 2 smooth functions such that:

$$\psi(y) = \begin{cases} e^y & \text{for } y < -1, \\ 1 & \text{for } y > -\kappa, \end{cases} \quad \psi' \geq 0, \quad (2.5.1)$$

$$\eta(y) = \begin{cases} 1 & \text{for } y < 1, \\ 0 & \text{for } y > 2, \end{cases} \quad \eta' \leq 0. \quad (2.5.2)$$

Here, we observe that  $\psi(-\kappa) = \varphi(-\kappa) + \kappa$ , and  $\psi(y) = \varphi(y)$  for all  $y < -1$ , so we may assume in addition:

$$\varphi(y) \leq \psi(y) \leq (1 + 3\kappa)\varphi(y), \text{ for all } y \leq -\kappa. \quad (2.5.3)$$

*Remark 2.5.1.* It is easy to check that for every  $\frac{1}{2} > \kappa > 0$ , such  $\psi$  and  $\varphi$  exist.

Now, recall  $B = b_c^{-\frac{1}{20}}$ . We let

$$\psi_B(y) = \psi\left(\frac{y}{B}\right), \quad \eta_B(y) = \eta\left(\frac{y}{B^2}\right), \quad \zeta_B(y) = \varphi_B \eta_B.$$

and then define the following Lyapunov functional for  $\varepsilon$ :

$$\mathcal{F} = \int \left[ \varepsilon_y^2 \psi_B + \varepsilon^2 \zeta_B - \frac{2}{p+1} (|\varepsilon + Q_b|^{p+1} - Q_b^{p+1} - (p+1)\varepsilon Q_b^p) \psi_B \right]. \quad (2.5.4)$$

Our main goal here is the following monotonicity formula of  $\mathcal{F}$ :

**Proposition 2.5.2** (The second monotonicity formula). *There exists a universal constant  $\mu > 0$  such that for all  $s \in [0, s^*)$ , the following holds:*

1. *Lyapunov control:*

$$\frac{d}{ds} \mathcal{F} + \mu \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B \lesssim b_c^{\frac{7}{2}}; \quad (2.5.5)$$

2. *Coercivity of  $\mathcal{F}$ :*

$$\mathcal{N} - b_c^{\frac{7}{2}} \lesssim \mathcal{F} \lesssim \mathcal{N} + b_c^{\frac{7}{2}}. \quad (2.5.6)$$

*Remark 2.5.3.* The proof of Proposition 2.5.2 is almost parallel to that of Proposition 3.1 in [57]. But since we have a control of the global  $L^2$  norm of  $\varepsilon$  (consequently the  $L^\infty$  norm of  $\varepsilon$ ), some part of the proof will be easier.

*Proof of Proposition 2.5.2.* We will prove (2.5.5) and (2.5.6) in several steps:

**Step 1** Algebraic computation of  $\mathcal{F}$  A direct computation shows:

$$\begin{aligned} \frac{d}{ds} \mathcal{F} &= 2 \int \psi_B(\varepsilon_y)_s \varepsilon_y + \varepsilon_s \left\{ \varepsilon \zeta_B - \psi_B [(\varepsilon + Q_b) |\varepsilon + Q_b|^{p-1} - Q_b^p] \right\} \\ &\quad - 2 \int \psi_B(Q_b)_s [(\varepsilon + Q_b) |\varepsilon + Q_b|^{p-1} - Q_b^p - p\varepsilon Q_b^{p-1}] \\ &= f_1 + f_2 + f_3, \end{aligned}$$

where

$$\begin{aligned} f_1 &= 2 \int \left( \varepsilon_s - \frac{\lambda_s}{\lambda} \Lambda \varepsilon \right) \left\{ -(\psi_B \varepsilon_y)_y + \varepsilon \zeta_B - \psi_B [(\varepsilon + Q_b) |\varepsilon + Q_b|^{p-1} - Q_b^p] \right\}, \\ f_2 &= 2 \frac{\lambda_s}{\lambda} \int \Lambda \varepsilon \left\{ -(\psi_B \varepsilon_y)_y + \varepsilon \zeta_B - \psi_B [(\varepsilon + Q_b) |\varepsilon + Q_b|^{p-1} - Q_b^p] \right\}, \\ f_3 &= -2 \int \psi_B(Q_b)_s [(\varepsilon + Q_b) |\varepsilon + Q_b|^{p-1} - Q_b^p - p\varepsilon Q_b^{p-1}]. \end{aligned}$$

We claim that the following estimates hold for some universal constant  $\mu_0 > 0$ :

$$f_1 \leq -\mu_0 \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C b \frac{7}{c}, \quad (2.5.7)$$

$$f_k \leq \frac{\mu_0}{10} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C b \frac{7}{c}, \quad \text{for } k = 2, 3. \quad (2.5.8)$$

It is obvious that (2.5.5) follows from (2.5.7) and (2.5.8).

In step 2 - step 5, we will prove (2.5.7) and (2.5.8). Observe that the definition of  $\varphi$ ,  $\psi$  and  $\zeta_B$  imply:

$$\text{for } \forall y \in (-\infty, \kappa], \quad |\varphi''''| + |\varphi''| + |\varphi| + |\psi''''| + |\psi'| + |\psi| \lesssim \varphi' \lesssim \varphi, \quad (2.5.9)$$

$$\zeta'_B = \begin{cases} 3\eta'_B & \text{for } y > B^2, \\ 0 & \text{for } B < y \leq B^2, \\ \varphi'_B & \text{for } y < B. \end{cases} \quad (2.5.10)$$

We will use these properties several times during the proof.

**Step 2** Control of  $f_1$ . We give the proof of (2.5.7) by using the equation (2.3.2) in the following form:

$$\begin{aligned} \varepsilon_s - \frac{\lambda_s}{\lambda} \Lambda \varepsilon &= (-\varepsilon_{yy} + \varepsilon - (\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} + Q_b^p)_y \\ &\quad + \left( \frac{\lambda_s}{\lambda} + b \right) \Lambda Q_b + \left( \frac{x_s}{\lambda} - 1 \right) (Q_b + \varepsilon)_y - b_s P_b + \Phi_b, \end{aligned} \quad (2.5.11)$$

where

$$\Phi_b = -b \Lambda Q_b - (Q_b'' - Q_b + Q_b^p)', \quad P_b = \frac{\partial Q_b}{\partial b}.$$

Injecting (2.5.11) into the definition of  $f_1$  yields:

$$f_1 = f_{1,1} + f_{1,2} + f_{1,3} + f_{1,4} + f_{1,5}$$

with

$$\begin{aligned} f_{1,1} &= 2 \int (-\varepsilon_{yy} + \varepsilon - (\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} + Q_b^p)_y \left\{ -(\psi_B \varepsilon_y)_y \right. \\ &\quad \left. + \varepsilon \zeta_B - \psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \right\}, \\ f_{1,2} &= 2 \left( \frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_b \left\{ -\psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] - (\psi_B \varepsilon_y)_y + \varepsilon \zeta_B \right\}, \\ f_{1,3} &= 2 \left( \frac{x_s}{\lambda} - 1 \right) \int (Q_b + \varepsilon)_y \left\{ -\psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \right. \\ &\quad \left. - (\psi_B \varepsilon_y)_y + \varepsilon \zeta_B \right\}, \\ f_{1,4} &= -2b_s \int P_b \left\{ -(\psi_B \varepsilon_y)_y + \varepsilon \zeta_B - \psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \right\}, \\ f_{1,5} &= 2 \int \Phi_b \left\{ -(\psi_B \varepsilon_y)_y + \varepsilon \zeta_B - \psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \right\}. \end{aligned}$$

Term  $f_{1,1}$ : Let us integrate by parts to obtain a more manageable formula <sup>16</sup>:

$$\begin{aligned} f_{1,1} = & 2 \int [-\varepsilon_{yy} + \varepsilon - (\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} + Q_b^p]_y (-\psi'_B \varepsilon_y + \varepsilon(\zeta_B - \psi_B)) \\ & + 2 \int [-\varepsilon_{yy} + \varepsilon - (\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} + Q_b^p]_y \\ & \quad \times [-\varepsilon_{yy} + \varepsilon - (\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} + Q_b^p] \psi_B. \end{aligned}$$

We compute these terms separately. First we integrate by parts to obtain:

$$\begin{aligned} 2 \int [-\varepsilon_{yy} + \varepsilon]_y [-\psi'_B \varepsilon_y + \varepsilon(\zeta_B - \psi_B)] = & -2 \left\{ \int \psi'_B \varepsilon_{yy}^2 \right. \\ & \left. + \int \varepsilon_y^2 \left( \frac{3}{2} \zeta'_B - \frac{1}{2} \psi'_B - \frac{1}{2} \psi_B''' \right) + \int \varepsilon^2 \left( \frac{1}{2} (\zeta'_B - \psi'_B) - \frac{1}{2} (\zeta_B - \psi_B)''' \right) \right\} \end{aligned}$$

and

$$\begin{aligned} & -2 \int [(Q_b + \varepsilon)|Q_b + \varepsilon|^{p-1} - Q_b^p]_y (\zeta_B - \psi_B) \varepsilon \\ = & -2 \int (\zeta_B - \psi_B) (Q_b)_y [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p - p\varepsilon Q_b^{p-1}] \\ & - \frac{2}{p+1} \int (\zeta_B - \psi_B)' [|Q_b + \varepsilon|^{p+1} - Q_b^{p+1} - (p+1)\varepsilon Q_b^p] \\ & + 2 \int (\zeta_B - \psi_B)' [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \varepsilon. \end{aligned}$$

Next by direct expansion:

$$\begin{aligned} \int [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p]_y \psi'_B \varepsilon_y = \\ p \int \psi'_B \varepsilon_y \left\{ (Q_b)_y [|Q_b + \varepsilon|^{p-1} - Q_b^{p-1}] + |Q_b + \varepsilon|^{p-1} \varepsilon_y \right\}. \end{aligned}$$

Finally,

$$\begin{aligned} & 2 \int [-\varepsilon_{yy} + \varepsilon - (\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} + Q_b^p]_y \\ & \quad \times [-\varepsilon_{yy} + \varepsilon - (\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} + Q_b^p] \psi_B \\ = & - \int \psi'_B [-\varepsilon_{yy} + \varepsilon - (\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} + Q_b^p]^2 \\ = & - \int \psi'_B \left\{ [-\varepsilon_{yy} + \varepsilon - (\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} + Q_b^p]^2 - [-\varepsilon_{yy} + \varepsilon]^2 \right\} \\ & - \int \psi'_B [-\varepsilon_{yy} + \varepsilon]^2 \\ = & - \int \psi'_B \left\{ [-\varepsilon_{yy} + \varepsilon - (\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} + Q_b^p]^2 - [-\varepsilon_{yy} + \varepsilon]^2 \right\} \\ & - \left[ \int \psi'_B (\varepsilon_{yy}^2 + 2\varepsilon_y^2) + \int \varepsilon^2 (\psi'_B - \psi_B''') \right]. \end{aligned}$$

16. See a similar computation in the proof of Proposition 3.1 in [57].

We collect all the above computations and obtain the following:

$$\begin{aligned}
f_{1,1} &= - \int [3\psi'_B \varepsilon_{yy}^2 + (3\zeta'_B + \psi'_B - \psi_B''') \varepsilon_y^2 + (\zeta'_B - \zeta_B''') \varepsilon^2] \\
&\quad - 2 \int \left[ \frac{|\varepsilon + Q_b|^{p+1} - Q_b^{p+1}}{p+1} - \varepsilon Q_b^p - \varepsilon((\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p) \right] (\zeta'_B - \psi'_B) \\
&\quad + 2 \int [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p - p\varepsilon Q_b^{p-1}] (Q_b)_y (\psi_B - \zeta_B) \\
&\quad + 2p \int \psi'_B \varepsilon_y \{ (Q_b)_y [|\varepsilon + Q_b|^{p-1} - Q_b^{p-1}] + |\varepsilon + Q_b|^{p-1} \varepsilon_y \} \\
&\quad - \int \psi'_B \left\{ [-\varepsilon_{yy} + \varepsilon - ((\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p)]^2 - [-\varepsilon_{yy} + \varepsilon]^2 \right\} \\
&= (f_{1,1})^< + (f_{1,1})^\sim + (f_{1,1})^>,
\end{aligned}$$

where  $(f_{1,1})^<, \sim, >$  correspond to the integration on  $y < -\kappa B$ ,  $|y| < \kappa B$  and  $y > \kappa B$ , respectively.

In the region  $y > \kappa B$ , we have  $\psi'_B = \psi_B''' \equiv 0$ . From (2.4.2), (2.4.9) and (2.4.21), we have:

$$\begin{aligned}
&\left| \int_{y>\kappa B} [3\psi'_B \varepsilon_{yy}^2 + (3\zeta'_B + \psi'_B - \psi_B''') \varepsilon_y^2 + (\zeta'_B - \zeta_B''') \varepsilon^2] \right| \\
&\lesssim \int_{y>\kappa B} \varepsilon_y^2 + \frac{1}{B} \int_{\kappa B < y < 2B^2} \varepsilon^2 \lesssim b_c^4 + B \|\varepsilon\|_{L^\infty(y>\kappa B)}^2 \\
&\lesssim B b_c^4 + b_c^4 \leq b_c^{\frac{7}{2}}.
\end{aligned}$$

Together with

$$\begin{aligned}
&\left| \int_{y>\kappa B} \left[ \frac{|\varepsilon + Q_b|^{p+1} - Q_b^{p+1}}{p+1} - \varepsilon Q_b^p - \varepsilon((\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p) \right] (\zeta'_B - \psi'_B) \right| \\
&\lesssim \int_{y>\kappa B} |\varepsilon|^{p+1} + |Q_b|^{p-1} \varepsilon^2 \lesssim \|\varepsilon\|_{L^{p_0}}^{\frac{p_0(p+3)}{p_0+2}} \|\varepsilon_y\|_{L^2(y>\kappa B)}^{\frac{2(p+1-p_0)}{p_0+2}} + e^{-\frac{\kappa B}{20}} \|\varepsilon\|_{L^\infty}^2 \\
&\lesssim b_c^{\frac{7}{2}}
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int_{y>\kappa B} [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p - p\varepsilon Q_b^{p-1}] (Q_b)_y (\psi_B - \zeta_B) \right| \\
&\lesssim e^{-\frac{\kappa B}{20}} \left( \int_{y>\kappa B} (\varepsilon^2 e^{-\frac{|y|}{2}} + |\varepsilon|^p) \right) \lesssim b_c^{\frac{7}{2}},
\end{aligned}$$

we obtain:

$$(f_{1,1})^> \lesssim b_c^{\frac{7}{2}}. \quad (2.5.12)$$



In the region  $|y| < \kappa B$ ,  $\zeta_B(y) = \varphi_B(y) = 1 + y/B$  and  $\psi_B(y) = 1$ . In particular,  $\zeta_B''' = \psi_B' = 0$ . We obtain:

$$\begin{aligned} (f_{1,1})^\sim &= -\frac{1}{B} \int_{|y| < \kappa B} \left\{ 3\varepsilon_y^3 + \varepsilon^2 + 2 \left[ \frac{|\varepsilon + Q_b|^{p+1} - Q_b^{p+1}}{p+1} \right. \right. \\ &\quad \left. \left. - \varepsilon Q_b^p - \varepsilon((\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p) \right] \right. \\ &\quad \left. + 2[(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p - p\varepsilon Q_b^{p-1}]y(Q_b)_y \right\} \\ &= -\frac{1}{B} \int_{|y| < \kappa B} \left\{ 3\varepsilon_y^2 + \varepsilon^2 - pQ_b^{p-1}\varepsilon^2 + p(p-1)yQ'_p Q_b^{p-2}\varepsilon^2 \right\} + R(\varepsilon), \end{aligned}$$

where

$$\begin{aligned} R(\varepsilon) &= -\frac{1}{B} \int_{|y| < \kappa B} \left\{ -p(Q_b^{p-1} - Q_p^{p-1})\varepsilon^2 + p(p-1)y((Q_b)_y Q_b^{p-2} - Q'_p Q_p^{p-2})\varepsilon^2 \right. \\ &\quad \left. + 2 \left( \frac{|\varepsilon + Q_b|^{p+1} - Q_b^{p+1}}{p+1} - \varepsilon Q_b^p - \frac{p}{2} Q_b^{p-1} \varepsilon^2 \right) \right. \\ &\quad \left. - 2\varepsilon \left( (\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p - p\varepsilon Q_b^{p-1} \right) \right. \\ &\quad \left. + 2 \left[ (\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p - p\varepsilon Q_b^{p-1} - \frac{p(p-1)}{2} \varepsilon^2 Q_b^{p-2} \right] y(Q_b)_y \right\}. \end{aligned}$$

We claim the following localized Virial estimate to obtain a coercivity result:

**Lemma 2.5.4** (Localized Virial estimate <sup>17</sup>). *There exists  $B_0 > 100$  and  $\mu_1 > 0$  such that if  $B > B_0$ , then:*

$$\begin{aligned} \int_{|y| < \kappa B} (3\varepsilon_y^2 + \varepsilon^2 - pQ_b^{p-1}\varepsilon^2 + p(p-1)yQ'_p Q_b^{p-2}\varepsilon^2) \\ \geq \mu_1 \int_{|y| < \kappa B} (\varepsilon_y^2 + \varepsilon^2) - \frac{1}{B} \int \varepsilon^2 e^{-\frac{|y|}{2}}. \end{aligned}$$

Since  $\int_{|y| > \kappa B} \varepsilon^2 e^{-\frac{|y|}{2}} \lesssim b_c^{10}$ , we have for some  $\mu_2 > 0$ :

$$\int_{|y| < \kappa B} (3\varepsilon_y^2 + \varepsilon^2 - pQ_b^{p-1}\varepsilon^2 + p(p-1)yQ'_p Q_b^{p-2}\varepsilon^2) \geq \mu_2 \int_{|y| < \kappa B} (\varepsilon_y^2 + \varepsilon^2) - b_c^{10}.$$

Using a similar strategy we have:

$$\begin{aligned} |R(\varepsilon)| &\lesssim \frac{1}{B} \left( b_c \int_{|y| < \kappa B} \varepsilon^2 + \int_{|y| < \kappa B} |\varepsilon|^3 + |\varepsilon|^{p+1} \right) \\ &\lesssim \frac{1}{B} \left( (b_c + \|\varepsilon\|_{L^\infty}) \int_{|y| < \kappa B} (\varepsilon_y^2 + \varepsilon^2) \right) \\ &\lesssim \frac{1}{1000} \int_{|y| < \kappa B} (\varepsilon_y^2 + \varepsilon^2) \varphi'_B. \end{aligned}$$

17. See proof in [57] (Lemma 3.4 & Lemma A.2).

Collecting the above estimates, we obtain for some  $\mu_3 > 0$ :

$$(f_{1,1})^\sim \leq -\mu_3 \int_{|y| < \kappa B} (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C b_c^{\frac{7}{2}}. \quad (2.5.13)$$

For the region  $y < -\kappa B$ , we have  $\zeta_B(y) = \varphi_B(y)$  and  $\psi_B \sim \varphi_B$ . Hence, we immediately have:

$$\begin{aligned} \int_{y < -\kappa B} \varepsilon^2 |\zeta_B''| &\lesssim \frac{1}{B^2} \int_{y < -\kappa B} \varepsilon^2 \varphi'_B \leq \frac{1}{100} \int_{y < -\kappa B} \varepsilon^2 \varphi'_B, \\ \int_{y < -\kappa B} \varepsilon_y^2 |\psi_B''| &\lesssim \frac{1}{B^2} \int_{y < -\kappa B} \varepsilon_y^2 \varphi'_B \leq \frac{1}{100} \int_{y < -\kappa B} \varepsilon_y^2 \varphi'_B. \end{aligned}$$

From Lemma 2.2.4, we know that for  $y < -\kappa B$ ,  $|Q_b(y)| \lesssim b_c$  and  $|Q'_b(y)| \lesssim b_c^2$ . Recall that we have  $\|\varepsilon\|_{L^\infty} \leq b_c^{\frac{1}{2}}$ , then we can estimate:

$$\begin{aligned} &\left| \int_{y < -\kappa B} \left[ \frac{|\varepsilon + Q_b|^{p+1} - Q_b^{p+1}}{p+1} - \varepsilon Q_b^p - \varepsilon((\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p) \right] (\zeta'_B - \psi'_B) \right| \\ &\lesssim \int_{y < -\kappa B} \left( |\varepsilon|^{p+1} + |Q_b|^{p-1} \varepsilon^2 \right) \varphi'_B \lesssim (b_c^{p-1} + \|\varepsilon\|_{L^\infty}^{p-1}) \int_{y < -\kappa B} \varepsilon^2 \varphi'_B \\ &\lesssim \frac{1}{100} \int_{y < -\kappa B} \varepsilon^2 \varphi'_B, \end{aligned}$$

$$\begin{aligned} &\left| \int_{y < -\kappa B} [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p - p\varepsilon Q_b^{p-1}] (Q_b)_y (\psi_B - \zeta_B) \right| \\ &\lesssim B \int_{y < -\kappa B} (\varepsilon^2 + |\varepsilon|^p) |(Q_b)_y| \varphi'_B \lesssim B b_c^2 (1 + \|\varepsilon\|_{L^\infty}^{p-2}) \int_{y < -\kappa B} \varepsilon^2 \varphi'_B \\ &\lesssim \frac{1}{100} \int_{y < -\kappa B} \varepsilon^2 \varphi'_B. \end{aligned}$$

Similarly, we have:

$$\begin{aligned} &\left| \int_{y < -\kappa B} \psi'_B \varepsilon_y \{ (Q_b)_y [ |Q_b + \varepsilon|^{p-1} - Q_b^{p-1} ] + |Q_b + \varepsilon|^{p-1} \varepsilon_y \} \right| \\ &\lesssim \int_{y < -\kappa B} \left( |\varepsilon_y \varepsilon (Q_b)_y Q_b^{p-2}| + |\varepsilon_y (Q_b)_y| |\varepsilon|^{p-1} + |\varepsilon_y^2 Q_b^{p-1}| + \varepsilon_y^2 |\varepsilon|^{p-1} \right) \varphi'_B \\ &\lesssim (b_c^{p-1} + \|\varepsilon\|_{L^\infty}^{p-1}) \int_{y < -\kappa B} (\varepsilon_y^2 + \varepsilon^2) \varphi'_B \\ &\lesssim \frac{1}{100} \int_{y < -\kappa B} (\varepsilon_y^2 + \varepsilon^2) \varphi'_B \end{aligned}$$

and

$$\begin{aligned}
& \left| \int_{y < -\kappa B} \psi'_B \left\{ [-\varepsilon_{yy} + \varepsilon - ((\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p)]^2 - [-\varepsilon_{yy} + \varepsilon]^2 \right\} \right| \\
& \lesssim \int_{y < -\kappa B} \left( |\varepsilon \varepsilon_{yy} Q_b^{p-1}| + |\varepsilon_{yy} Q_b^p| + |\varepsilon^2 Q_b^{p-1}| + |\varepsilon Q_b^p| + |\varepsilon|^{2p} + |\varepsilon_{yy}| |\varepsilon|^p \right) \psi'_B \\
& \lesssim (b_c^{p-1} + \|\varepsilon\|_{L^\infty}^{p-1}) \int_{y < -\kappa B} (\varepsilon_{yy}^2 + \varepsilon^2) \psi'_B + \frac{1}{100} \int_{y < -\kappa B} (\varepsilon_{yy}^2 + \varepsilon^2) \psi'_B \\
& \quad + 100 \int_{y < -\kappa B} Q_b^{2p} + \|\varepsilon\|_{L^\infty}^{2p-2} \int_{y < -\kappa B} \varepsilon^2 \psi'_B \\
& \lesssim \frac{1}{100} \int_{y < -\kappa B} (\varepsilon_{yy}^2 \psi'_B + \varepsilon^2 \varphi'_B) + b_c^{\frac{7}{2}}.
\end{aligned}$$

Therefore we obtain:

$$(f_{1,1})' \leq -\mu_4 \int_{y < -\kappa B} (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C b_c^{\frac{7}{2}} \quad (2.5.14)$$

for some  $\mu_4 > 0$ . From (2.5.12), (2.5.13), (2.5.14) and the following estimate:

$$\begin{aligned}
\int_{y > \kappa B} (\varepsilon_y^2 + \varepsilon^2) \varphi'_B & \lesssim \frac{1}{B} \int_{y > \kappa B} \varepsilon_y^2 + \frac{1}{B} \int_{\kappa B < y < 2B^2} \varepsilon^2 \\
& \lesssim b_c^4 + B \|\varepsilon\|_{L^\infty(y > \kappa B)}^2 \lesssim b_c^{\frac{7}{2}},
\end{aligned}$$

we obtain for some  $\mu_0 > 0$ ,

$$f_{1,1} \leq -\mu_0 \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C b_c^{\frac{7}{2}}. \quad (2.5.15)$$

Term  $f_{1,2}$ : We first rewrite  $f_{1,2}$ :

$$\begin{aligned}
f_{1,2} = 2 \left( \frac{\lambda_s}{\lambda} + b \right) \int (\Lambda(Q_b - Q_p)) \left\{ -(\psi_B \varepsilon_y)_y + \varepsilon \zeta_B \right. \\
\left. - \psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \right\} + \tilde{f}_{1,2},
\end{aligned}$$

where

$$\begin{aligned}
\tilde{f}_{1,2} & = 2 \left( \frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_p \left\{ -(\psi_B \varepsilon_y)_y + \varepsilon \zeta_B - \psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \right\} \\
& = 2 \left( \frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_p [ -(\psi_B)_y \varepsilon_y + (1 - \psi_B) \varepsilon_{yy} ] \\
& \quad + 2 \left( \frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_p \left\{ (1 - \psi_B) [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \right\} \\
& \quad - 2 \left( \frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_p [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p - p \varepsilon Q_b^{p-1}] + \tilde{\tilde{f}}_{1,2}, \\
\tilde{\tilde{f}}_{1,2} & = 2 \left( \frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_p (-\varepsilon_{yy} + \varepsilon \zeta_B - p Q_b^{p-1} \varepsilon) \\
& = 2 \left( \frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_p (L\varepsilon) - 2 \left( \frac{\lambda_s}{\lambda} + b \right) \int \varepsilon (1 - \zeta_B) \Lambda Q_p.
\end{aligned}$$

In conclusion, we have:

$$\begin{aligned}
f_{1,2} &= 2\left(\frac{\lambda_s}{\lambda} + b\right) \int \Lambda \mathcal{Q}_p(L\varepsilon) - 2\left(\frac{\lambda_s}{\lambda} + b\right) \int \varepsilon(1 - \zeta_B)\Lambda \mathcal{Q}_p \\
&\quad + 2\left(\frac{\lambda_s}{\lambda} + b\right) \int (\Lambda(\mathcal{Q}_b - \mathcal{Q}_p)) \left\{ -(\Psi_B \varepsilon_y)_y + \varepsilon \zeta_B \right. \\
&\quad \quad \left. - \Psi_B [(\varepsilon + \mathcal{Q}_b)|\varepsilon + \mathcal{Q}_b|^{p-1} - \mathcal{Q}_b^p] \right\} \\
&\quad + 2\left(\frac{\lambda_s}{\lambda} + b\right) \int \Lambda \mathcal{Q}_p [ -(\Psi_B)_y \varepsilon_y + (1 - \Psi_B) \varepsilon_{yy} ] \\
&\quad + 2\left(\frac{\lambda_s}{\lambda} + b\right) \int \Lambda \mathcal{Q}_p \left\{ (1 - \Psi_B) [(\varepsilon + \mathcal{Q}_b)|\varepsilon + \mathcal{Q}_b|^{p-1} - \mathcal{Q}_b^p] \right\} \\
&\quad - 2\left(\frac{\lambda_s}{\lambda} + b\right) \int \Lambda \mathcal{Q}_p [(\varepsilon + \mathcal{Q}_b)|\varepsilon + \mathcal{Q}_b|^{p-1} - \mathcal{Q}_b^p - p\varepsilon \mathcal{Q}_p^{p-1}].
\end{aligned}$$

We know from the orthogonality condition (2.2.27) that:

$$\int \Lambda \mathcal{Q}_p(L\varepsilon) = (\varepsilon, L\Lambda \mathcal{Q}_p) = -2(\varepsilon, \mathcal{Q}_p) = 0.$$

Again from the orthogonality condition  $(\varepsilon, y\Lambda \mathcal{Q}_p) = 0$ , we can estimate:

$$\begin{aligned}
\left| \int \Lambda \mathcal{Q}_p \varepsilon (1 - \zeta_B) \right| &= \left| \int \Lambda \mathcal{Q}_p \varepsilon \left( 1 - \zeta_B + \frac{y}{B} \right) \right| \\
&\lesssim e^{-\frac{\kappa B}{20}} \|\varepsilon\|_{L^\infty} \leq b_c^{\frac{7}{2}}.
\end{aligned}$$

For the next term, we first integrate by parts to remove all the derivatives on  $\varepsilon$ , then we divide the integral into 2 parts,  $\int_{y < \kappa B}$  and  $\int_{y > \kappa B}$ . For the first part we use Cauchy-Schwarz inequality, (2.2.9) and (2.2.11). While for the second part we use the fact that  $\mathcal{Q}_b$  decays exponentially on the right. So we have:

$$\begin{aligned}
&\left| \left( \frac{\lambda_s}{\lambda} + b \right) \int \Lambda(\mathcal{Q}_b - \mathcal{Q}_p) \left\{ -\Psi_B [(\varepsilon + \mathcal{Q}_b)|\varepsilon + \mathcal{Q}_b|^{p-1} - \mathcal{Q}_b^p] - (\Psi_B \varepsilon_y)_y + \varepsilon \zeta_B \right\} \right| \\
&= \left| \left( \frac{\lambda_s}{\lambda} + b \right) \int \left( (\Lambda \mathcal{Q}_b - \Lambda \mathcal{Q}_p) \left\{ -\Psi_B [(\varepsilon + \mathcal{Q}_b)|\varepsilon + \mathcal{Q}_b|^{p-1} - \mathcal{Q}_b^p] + \varepsilon \zeta_B \right\} \right. \right. \\
&\quad \left. \left. - [(\Lambda \mathcal{Q}_b - \Lambda \mathcal{Q}_p)' \Psi_B]' \varepsilon \right) \right| \\
&\lesssim \left( b_c^{\frac{5}{2}} + \mathcal{N}^{\frac{1}{2}} \right) \left( b_c \int_{y < \kappa B} \Psi_B (|\varepsilon| + |\varepsilon|^p) + \int_{y > \kappa B} e^{-\frac{|y|}{10}} (|\varepsilon| + |\varepsilon|^p) \right) \\
&\lesssim \left( b_c^{\frac{5}{2}} + \mathcal{N}^{\frac{1}{2}} \right) \left( b_c \left( \int_{y < \kappa B} \varepsilon^2 \Psi_B \right)^{\frac{1}{2}} \left( \int_{y < \kappa B} \Psi_B \right)^{\frac{1}{2}} + e^{-\frac{\kappa B}{20}} \|\varepsilon\|_{L^\infty} \right) \\
&\lesssim b_c B^{\frac{3}{2}} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + b_c^{\frac{7}{2}} B \left( \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B \right)^{\frac{1}{2}} + b_c^{\frac{7}{2}} \\
&\leq \frac{\mu_0}{1000} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C b_c^{\frac{7}{2}}.
\end{aligned}$$

For the following 2 terms, we first integrate by parts again to remove the derivatives on  $\varepsilon$ . Then we use the fact that  $\psi_B = 1$  on  $[-\kappa B, +\infty)$  and

$$|(\Lambda Q_p)''(y)| + |\Lambda Q_p(y)| \lesssim e^{-\frac{\kappa B}{20}} \phi_B'(y)$$

for  $y < -\kappa B$ , to obtain:

$$\begin{aligned} & \left| \left( \frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_p [ -(\psi_B)_y \varepsilon_y + (1 - \psi_B) \varepsilon_{yy} ] \right| \\ &= \left| \left( \frac{\lambda_s}{\lambda} + b \right) \int \left\{ [\Lambda Q_p(1 - \psi_B)]'' \varepsilon + [\Lambda Q_p(\psi_B)_y]' \varepsilon \right\} \right| \\ &\lesssim \left( b_c^{\frac{5}{2}} + \mathcal{N}^{\frac{1}{2}} \right) \left( \int_{y < -\kappa B} \varepsilon^2 \phi_B' \right)^{\frac{1}{2}} e^{-\frac{\kappa B}{30}} \\ &\leq \frac{\mu_0}{1000} \int (\varepsilon_y^2 + \varepsilon^2) \phi_B' + C b_c^{\frac{7}{2}} \end{aligned}$$

and

$$\begin{aligned} & \left| \left( \frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_p \left\{ (1 - \psi_B) [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \right\} \right| \\ &\lesssim \left( b_c^{\frac{5}{2}} + \mathcal{N}^{\frac{1}{2}} \right) \int_{y < -\kappa B} (|\varepsilon| + |\varepsilon|^p) e^{-\frac{\kappa B}{20}} \phi_B' \\ &\lesssim \left( b_c^{\frac{5}{2}} + \mathcal{N}^{\frac{1}{2}} \right) \left( \int_{y < -\kappa B} \varepsilon^2 \phi_B' \right)^{\frac{1}{2}} e^{-\frac{\kappa B}{20}} \\ &\leq \frac{\mu_0}{1000} \int (\varepsilon_y^2 + \varepsilon^2) \phi_B' + C b_c^{\frac{7}{2}}. \end{aligned}$$

Finally, by the same strategy we have :

$$\begin{aligned} & \left| \left( \frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_p [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p - p\varepsilon Q_b^{p-1}] \right| \\ &= \left| \left( \frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_p [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p - p\varepsilon Q_b^{p-1} + p\varepsilon(Q_b^{p-1} - Q_b^{p-1})] \right| \\ &\lesssim \left( b_c^{\frac{5}{2}} + \mathcal{N}^{\frac{1}{2}} \right) \left( \int_{y < \kappa B} \varepsilon^2 \phi_B' + b_c \mathcal{N}^{\frac{1}{2}} + e^{-\frac{\kappa B}{20}} \|\varepsilon\|_{L^\infty} \right) \\ &\leq \frac{\mu_0}{1000} \int (\varepsilon_y^2 + \varepsilon^2) \phi_B' + C b_c^{\frac{7}{2}}. \end{aligned}$$

The collection of the above estimates shows that:

$$|f_{1,2}| \leq \frac{\mu_0}{100} \int (\varepsilon_y^2 + \varepsilon^2) \phi_B' + C b_c^{\frac{7}{2}}. \quad (2.5.16)$$

Term  $f_{1,3}$ : We use the identity:

$$\begin{aligned} & \int \psi_B \left\{ (Q_b)_y [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p - p\varepsilon Q_b^{p-1}] \right. \\ & \quad \left. + \varepsilon_y [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \right\} \\ &= \frac{1}{p+1} \int \psi_B \partial_y [ |Q_b + \varepsilon|^{p+1} - Q_b^{p+1} - (p+1)\varepsilon Q_b^p ] \\ &= -\frac{1}{p+1} \int \psi'_B [ |Q_b + \varepsilon|^{p+1} - Q_b^{p+1} - (p+1)\varepsilon Q_b^p ] \end{aligned}$$

and a similar computation (as we do for term  $f_{1,2}$ ) to rewrite  $f_{1,3}$ :

$$\begin{aligned} f_{1,3} &= \frac{2}{p+1} \left( \frac{x_s}{\lambda} - 1 \right) \int \psi'_B [ |Q_b + \varepsilon|^{p+1} - Q_b^{p+1} - (p+1)\varepsilon Q_b^p ] \\ & \quad + 2 \left( \frac{x_s}{\lambda} - 1 \right) \int (Q_b - Q_p + \varepsilon)_y ( -\psi'_B \varepsilon_y - \psi_B \varepsilon_{yy} + \varepsilon \zeta_B ) \\ & \quad - 2p \left( \frac{x_s}{\lambda} - 1 \right) \int \varepsilon \psi_B [ Q_b^{p-1} (Q_b)_y - Q_p^{p-1} (Q_p)_y ] \\ & \quad + 2 \left( \frac{x_s}{\lambda} - 1 \right) \int Q'_p [ L\varepsilon - \psi'_B \varepsilon_y + (1 - \psi_B) \varepsilon_{yy} - \varepsilon(1 - \zeta_B) ]. \end{aligned}$$

For the first term, we use the bootstrap assumption  $\mathcal{N} \leq b_c^3$  to estimate:

$$\begin{aligned} & \left| \left( \frac{x_s}{\lambda} - 1 \right) \int \psi'_B [ |Q_b + \varepsilon|^{p+1} - Q_b^{p+1} - (p+1)\varepsilon Q_b^p ] \right| \\ & \lesssim \left( b_c^{\frac{5}{2}} + \mathcal{N}^{\frac{1}{2}} \right) \int \psi'_B ( |\varepsilon|^{p+1} + \varepsilon^2 Q_b^{p-1} ) \\ & \leq \frac{\mu_0}{1000} \int (\varepsilon_y^2 + \varepsilon^2) \phi'_B. \end{aligned}$$

For the second term, we first integrate by parts to remove the derivatives of  $\varepsilon$ , then we use Cauchy-Schwarz inequality, (2.2.9) and (2.2.11) to estimate  $\int_{y < \kappa B}$  and use (2.4.21) to

estimate  $\int_{y>\kappa B}$  as before:

$$\begin{aligned}
& \left| \left( \frac{x_s}{\lambda} - 1 \right) \int (Q_b - Q_p)_y (-\psi'_B \varepsilon_y - \psi_B \varepsilon_{yy} + \varepsilon \zeta_B) \right| \\
& \lesssim \left( b_c^{\frac{5}{2}} + \mathcal{N}^{\frac{1}{2}} \right) \left( b_c B \mathcal{N}^{\frac{1}{2}} + e^{-\frac{\kappa B}{20}} \|\varepsilon\|_{L^\infty} \right) \\
& \leq \frac{\mu_0}{1000} \int (\varepsilon_y^2 + \varepsilon^2) \phi'_B + C b_c^{\frac{7}{2}}, \\
& \left| \left( \frac{x_s}{\lambda} - 1 \right) \int \varepsilon_y (-\psi'_B \varepsilon_y - \psi_B \varepsilon_{yy} + \varepsilon \zeta_B) \right| \\
& \lesssim \left( b_c^{\frac{5}{2}} + \mathcal{N}^{\frac{1}{2}} \right) \left( \int (\varepsilon_y^2 + \varepsilon^2) \phi'_B + \frac{1}{B^2} \int_{B^2 < y < 2B^2} \varepsilon^2 \right) \\
& \leq \frac{\mu_0}{1000} \int (\varepsilon_y^2 + \varepsilon^2) \phi'_B + C b_c^{\frac{7}{2}}.
\end{aligned}$$

For the next term, we can estimate similarly by dividing the integral into 2 parts:

$$\begin{aligned}
& \left| \left( \frac{x_s}{\lambda} - 1 \right) \int \varepsilon \psi_B [Q_b^{p-1} (Q_b)_y - Q_p^{p-1} (Q_p)_y] \right| \\
& \lesssim \left( b_c^{\frac{5}{2}} + \mathcal{N}^{\frac{1}{2}} \right) \left( b_c B \mathcal{N}^{\frac{1}{2}} + e^{-\frac{\kappa B}{20}} \|\varepsilon\|_{L^\infty} \right) \\
& \leq \frac{\mu_0}{1000} \int (\varepsilon_y^2 + \varepsilon^2) \phi'_B + C b_c^{\frac{7}{2}}.
\end{aligned}$$

For the last term, we use the cancellation  $LQ'_p=0$  and the orthogonality condition  $(\varepsilon, yQ'_p) = (\varepsilon, \Lambda Q_p - \frac{2}{p-1} Q_p) = 0$  to estimate:

$$\begin{aligned}
& \left| \left( \frac{x_s}{\lambda} - 1 \right) \int Q'_p [L\varepsilon - \psi'_B \varepsilon_y + (1 - \psi_B) \varepsilon_{yy} - \varepsilon(1 - \zeta_B)] \right| \\
& = \left| \left( \frac{x_s}{\lambda} - 1 \right) \int Q'_p [L\varepsilon - \psi'_B \varepsilon_y + (1 - \psi_B) \varepsilon_{yy} - \varepsilon(1 + \frac{y}{B} - \zeta_B)] \right| \\
& \lesssim \left( b_c^{\frac{5}{2}} + \mathcal{N}^{\frac{1}{2}} \right) \left( e^{-\frac{\kappa B}{20}} \mathcal{N}^{\frac{1}{2}} + e^{-\frac{\kappa B}{20}} \|\varepsilon\|_{L^\infty} \right) \\
& \leq \frac{\mu_0}{1000} \int (\varepsilon_y^2 + \varepsilon^2) \phi'_B + C b_c^{\frac{7}{2}}.
\end{aligned}$$

In conclusion, we have:

$$|f_{1,3}| \leq \frac{\mu_0}{100} \int (\varepsilon_y^2 + \varepsilon^2) \phi'_B + C b_c^{\frac{7}{2}}. \quad (2.5.17)$$

Term  $f_{1,4}$ : Recall that

$$f_{1,4} = -2b_s \int P_b \left\{ -(\psi_B \varepsilon_y)_y + \varepsilon \zeta_B - \psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \right\}.$$

We estimate after integration by parts to remove the derivatives of  $\varepsilon$  and then divide the integral into 2 parts as before:

$$\begin{aligned}
& \left| \int P_b(-(\psi_B \varepsilon_y)_y + \varepsilon \zeta_B) \right| = \left| \int ((P_b)_y \varepsilon_y \psi_B + \varepsilon P_b \zeta_B) \right| \\
& \lesssim \left| \int_{y < \kappa B} (|\varepsilon P_b \zeta_B| + |\varepsilon_y (P_b)_y \psi_B|) + \int_{y > \kappa B} e^{-\frac{y}{8}} (|\varepsilon| + |\varepsilon_y|) \right| \\
& \lesssim B \left( \int_{y < \kappa B} (\varepsilon_y^2 + \varepsilon^2) \varphi'_B \right)^{\frac{1}{2}} + e^{-\frac{\kappa B}{20}} \left( \int_{y > \kappa B} \varepsilon_y^2 + \|\varepsilon\|_{L^\infty}^2 \right)^{\frac{1}{2}} \\
& \lesssim B \mathcal{N}^{\frac{1}{2}} + b_c^{\frac{7}{2}}.
\end{aligned}$$

For the nonlinear term, the same strategy shows:

$$\begin{aligned}
& \left| \int P_b \psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \right| \lesssim \int |P_b| \psi_B (Q_b^{p-1} |\varepsilon| + |\varepsilon|^p) \\
& \lesssim B \left( \int_{y < \kappa B} \varepsilon^2 \varphi'_B \right)^{\frac{1}{2}} + e^{-\frac{\kappa B}{20}} \|\varepsilon\|_{L^\infty} \\
& \lesssim B \mathcal{N}^{\frac{1}{2}} + b_c^{\frac{7}{2}}.
\end{aligned}$$

Recall from (2.3.8) we have:

$$|b_s| \lesssim b_c^{\frac{5}{2}} + b_c \mathcal{N}^{\frac{1}{2}}.$$

Then we obtain:

$$|f_{1,4}| \leq \frac{\mu_0}{100} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C b_c^{\frac{7}{2}}. \quad (2.5.18)$$

Term  $f_{1,5}$ : Recall from (2.2.18) we have for  $k = 0, 1$ :

$$|\partial_y^k \Phi_b| \lesssim b_c |\tilde{b}| |\partial_y^k Q_b| + b_c^2 \mathbf{1}_{[-2, -1]}(b_c y) + e^{-\frac{1}{10b_c}} \mathbf{1}_{[1, 2]}(b_c y)$$

So after integration by parts, we have:

$$\begin{aligned}
& \left| \int \Phi_b(-(\psi_B \varepsilon_y)_y + \varepsilon \zeta_B) \right| = \left| \int (\Phi_b)_y \psi_B \varepsilon_y + \int \Phi_b \varepsilon \zeta_B \right| \\
& \lesssim b_c^{\frac{5}{2}} \int (Q_b + |\partial_y Q_b|) (|\varepsilon_y \psi_B| + |\varepsilon \zeta_B|) + b_c^2 \int_{y \sim -b_c^{-1}} (|\varepsilon_y \psi_B| + |\varepsilon \zeta_B|) \\
& \quad + e^{-\frac{1}{10b_c}} \int_{y \sim b_c^{-1}} |\varepsilon_y \psi_B| + |\varepsilon \zeta_B| \\
& \lesssim b_c^{\frac{5}{2}} B \mathcal{N}^{\frac{1}{2}} + e^{-\frac{1}{2Bb_c}} (\|\varepsilon\|_{L^\infty} + \|\varepsilon_y\|_{L^2}) \\
& \leq \frac{\mu_0}{1000} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C b_c^{\frac{7}{2}}.
\end{aligned}$$

Here we use the fact that  $|\psi_B(y)| + |\zeta_B(y)| \lesssim e^{-\frac{1}{2Bb_c}} \lesssim b_c^{10}$ , for all  $y \in [-2b_c^{-1}, -b_c^{-1}]$ .



The nonlinear term can be similarly estimated as before:

$$\begin{aligned}
& \left| \int \Phi_b [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \Psi_B \right| \lesssim \int |\Phi_b| \Psi_B (|\varepsilon|^p + |Q_b^{p-1} \varepsilon|) \\
& \lesssim b_c^2 \int_{y \sim b_c^{-1}} (|\varepsilon Q_b^{p-1}| + |\varepsilon|^p) \Psi_B + e^{-\frac{1}{10b_c}} \int_{y \sim b_c^{-1}} (|\varepsilon Q_b^{p-1}| + |\varepsilon|^p) \Psi_B \\
& \quad + b_c^{\frac{5}{2}} \int Q_b (|\varepsilon Q_b^{p-1}| + |\varepsilon|^p) \Psi_B \\
& \leq \frac{\mu_0}{1000} \int (\varepsilon_y^2 + \varepsilon^2) \phi_B' + C b_c^{\frac{7}{2}}.
\end{aligned}$$

Thus we have shown that:

$$|f_{1,5}| \leq \frac{\mu_0}{100} \int (\varepsilon_y^2 + \varepsilon^2) \phi_B' + C b_c^{\frac{7}{2}}. \quad (2.5.19)$$

**Step 3** Control of  $f_2$ . Recall that:

$$f_2 = 2 \frac{\lambda_s}{\lambda} \int \Lambda \varepsilon \left\{ -(\Psi_B \varepsilon_y)_y + \varepsilon \zeta_B - \Psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \right\}.$$

We first claim the following identities:

$$\int \Lambda \varepsilon (\Psi_B \varepsilon_y)_y = -(1 - \sigma_c) \int \varepsilon_y^2 \Psi_B + \frac{1}{2} \int y \Psi_B' \varepsilon_y^2, \quad (2.5.20)$$

$$\int \Lambda \varepsilon (\varepsilon \zeta_B) = -\sigma_c \int \varepsilon^2 \zeta_B - \frac{1}{2} \int y \zeta_B' \varepsilon^2, \quad (2.5.21)$$

$$\begin{aligned}
& \int \Lambda \varepsilon \Psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \\
& = \frac{1}{p+1} \int \left( \frac{p+3}{p-1} \Psi_B - y \Psi_B' \right) [ |Q_b + \varepsilon|^{p+1} - Q_b^{p+1} - (p+1) \varepsilon Q_b^p ] \\
& \quad - \int \Psi_B \Lambda Q_b [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p - p \varepsilon Q_b^{p-1}].
\end{aligned} \quad (2.5.22)$$

We can see (2.5.20) and (2.5.21) are easily obtained by integrating by parts. While for (2.5.22), we have the following computation:

$$\begin{aligned}
& \int \Lambda (\varepsilon + Q_b) \Psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \\
& = \int \frac{2}{p-1} \Psi_B [|\varepsilon + Q_b|^{p+1} - Q_b^{p+1} - \varepsilon Q_b^p] \\
& \quad + \int y (\varepsilon + Q_b)' \Psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \\
& = \frac{2}{p-1} \int \Psi_B [|\varepsilon + Q_b|^{p+1} - Q_b^{p+1} - (p+1) \varepsilon Q_b^p] + p \int \Psi_B \varepsilon Q_b^{p-1} \left( \frac{2}{p-1} Q_b \right) + \Delta,
\end{aligned}$$

with

$$\Delta = \int y (\varepsilon + Q_b)' \Psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p].$$

Then we use the following identity:

$$\begin{aligned} & [|\varepsilon + Q_b|^{p+1} - Q_b^{p+1} - (p+1)\varepsilon Q_b^p]' \\ &= (p+1)(\varepsilon + Q_b)' [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] - p(p+1)\varepsilon Q_b' Q_b^{p-1} \end{aligned}$$

to compute:

$$\begin{aligned} \Delta &= \frac{1}{p+1} \int y \psi_B [|\varepsilon + Q_b|^{p+1} - Q_b^{p+1} - (p+1)\varepsilon Q_b^p]' - p \int y \psi_B \varepsilon Q_b' Q_b^{p-1} \\ &= -\frac{1}{p+1} \int (\psi_B - y \psi_B') [|\varepsilon + Q_b|^{p+1} - Q_b^{p+1} - (p+1)\varepsilon Q_b^p] \\ &\quad - p \int \psi_B \varepsilon Q_b^{p-1} (y Q_b'). \end{aligned}$$

Collecting all the above computation, we have:

$$\begin{aligned} & \int \Lambda(\varepsilon + Q_b) \psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \\ &= \frac{1}{p+1} \int \left( \frac{p+3}{p-1} \psi_B - y \psi_B' \right) [|\varepsilon + Q_b|^{p+1} - Q_b^{p+1} - (p+1)\varepsilon Q_b^p] \\ &\quad + p \int \psi_B \varepsilon Q_b^{p-1} (\Lambda Q_b), \end{aligned}$$

which is just (2.5.22).

Now we can use (2.5.20)–(2.5.22) to estimate  $f_{1,2}$ . Since

$$\frac{\lambda_s}{\lambda} \sim -b_c < 0,$$

we can drop the negative term to obtain:

$$2 \frac{\lambda_s}{\lambda} \int \Lambda \varepsilon (-\psi_B \varepsilon)_y \leq 0$$

and

$$\begin{aligned} 2 \frac{\lambda_s}{\lambda} \int \Lambda \varepsilon \zeta_B &\leq C \left( b_c^2 \int \varepsilon^2 \zeta_B + b_c \int_{0 < y < B} y \phi_B' \varepsilon^2 + b_c \int y \eta_B' \varepsilon^2 \right) \\ &\leq C \left( b_c B \int_{y < \kappa B} \varepsilon^2 \phi_B' + b_c^{\frac{1}{2}} \int_{\kappa B < y < 2B^2} \varepsilon^2 \right) \\ &\leq C \left( b_c^{\frac{1}{2}} \int_{y < \kappa B} \varepsilon^2 \phi_B' + \|\varepsilon\|_{L^\infty(y > \kappa B)}^2 \right) \\ &\leq \frac{\mu_0}{1000} \int (\varepsilon_y^2 + \varepsilon^2) \phi_B' + C b_c^{\frac{7}{2}}. \end{aligned}$$

For the nonlinear term we divide the integral into 3 parts:

$$\int \Lambda \varepsilon \psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] = m^< + m^\sim + m^>,$$

where  $m^<$ ,  $m^\sim$  and  $m^>$  correspond to the integration on  $y < -\kappa B$ ,  $|y| < \kappa B$  and  $y > \kappa B$  respectively. For  $y > \kappa B$ , we have:

$$|m^>| \lesssim \int_{y>\kappa B} (|\varepsilon|^{p+1} + \varepsilon^2 e^{-\frac{|y|}{2}}) \leq b_c^{\frac{7}{2}}.$$

Next for  $|y| < \kappa B$ , we can estimate:

$$|m^\sim| \lesssim \int_{|y|<\kappa B} |\varepsilon|^{p+1} + \varepsilon^2 \lesssim B \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B.$$

Finally, for  $y < -\kappa B$ , we have  $|Q_b| + |\Lambda Q_b| \lesssim b_c$  on this region. Together with  $\|\varepsilon\|_{L^\infty} \leq b_c^{\frac{1}{2}}$ , we obtain:

$$\begin{aligned} |m^<| &\lesssim \left( \|\varepsilon\|_{L^\infty}^{p+1} + b_c^{p-1} \|\varepsilon\|_{L^\infty}^2 + b_c \|\varepsilon\|_{L^\infty}^p \right) \int_{y<-\kappa B} (|\psi_B| + |y\psi'_B|) \\ &\lesssim B b_c^3 \leq b_c^{\frac{5}{2}}. \end{aligned}$$

Therefore, we obtain:

$$\left| \frac{\lambda_s}{\lambda} \int \Lambda \varepsilon \psi_B [(\varepsilon + Q_b)|\varepsilon + Q_b|^{p-1} - Q_b^p] \right| \lesssim \frac{\mu_0}{1000} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + b_c^{\frac{7}{2}},$$

hence

$$f_2 \leq \frac{\mu_0}{100} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C b_c^{\frac{7}{2}}. \quad (2.5.23)$$

**Step 4** Control of  $f_3$ . First from (3.8)

$$|(Q_b)_s| = |b_s P_b| \lesssim b_c^{\frac{5}{2}} |P_b|.$$

Recalling that  $P_b$  decays exponentially on the right, we have:

$$\begin{aligned} |f_3| &\lesssim b_c^{\frac{5}{2}} \left( \int_{y<\kappa B} \psi_B (|\varepsilon|^p + \varepsilon^2) + e^{-\frac{\kappa B}{20}} \|\varepsilon\|_{L^\infty}^2 \right) \\ &\leq \frac{\mu_0}{100} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C b_c^{\frac{7}{2}}. \end{aligned} \quad (2.5.24)$$

Collecting (2.5.15)–(2.5.24), we conclude the proof of (2.5.7) and (2.5.8).

**Step 5** Coercivity of  $\mathcal{F}$ . As before we divide the integral into 2 parts,  $\mathcal{F}^<$  and  $\mathcal{F}^>$ , which correspond to the integration on  $y < \kappa B$  and  $y > \kappa B$  respectively.

For the upper bound of  $\mathcal{F}$ , recall that  $B = b_c^{-\frac{1}{20}}$ , we have for  $y > \kappa B$ ,

$$\begin{aligned} |\mathcal{F}^>| &\lesssim \int_{y>\kappa B} (\varepsilon_y^2 + |\varepsilon|^{p+1} + \varepsilon^2 e^{-\frac{|y|}{2}}) + \int_{\kappa B < y < 2B^2} \varepsilon^2 \\ &\lesssim b_c^8 + B^2 \|\varepsilon\|_{L^\infty(y>\kappa B)}^2 \lesssim b_c^8 + b_c^{-\frac{1}{10}+4} \\ &\leq b_c^{\frac{7}{2}}. \end{aligned}$$

And for  $y < \kappa B$ , we have:

$$\begin{aligned} |\mathcal{F}^<| &\lesssim \int_{y < \kappa B} (\varepsilon_y^2 + \varepsilon^2 + |\varepsilon|^{p+1}) \psi_B \\ &\lesssim B \int_{y < \kappa B} (\varepsilon_y^2 + \varepsilon^2) \varphi'_B \leq \mathcal{N}. \end{aligned}$$

Then the upper bound follows.

For the lower bound, we rewrite  $\mathcal{F}$ :

$$\begin{aligned} \mathcal{F} &= \int (\varepsilon_y^2 \psi_B + \varepsilon^2 \zeta_B - p \psi_B Q_p^{p-1} \varepsilon^2) - p \int \psi_B (Q_b^{p-1} - Q_p^{p-1}) \varepsilon^2 \\ &\quad - \frac{2}{p+1} \int \psi_B \left[ |Q_b + \varepsilon|^{p+1} - Q_b^{p+1} - (p+1) \varepsilon Q_b^p - \frac{p(p+1)}{2} Q_b^{p-1} \varepsilon^2 \right]. \end{aligned}$$

First, we have:

$$\left| \int \psi_B (Q_b^{p-1} - Q_p^{p-1}) \varepsilon^2 \right| \lesssim b_c B \int_{y < \kappa B} \varepsilon^2 \varphi'_B + e^{-\frac{\kappa B}{20}} \|\varepsilon\|_{L^\infty}^2 \leq b_c^{\frac{1}{2}} \mathcal{N} + b_c^{\frac{7}{2}}.$$

For the nonlinear term, we use similar technique as before to estimate:

$$\begin{aligned} &\left| \int \psi_B \left[ |Q_b + \varepsilon|^{p+1} - Q_b^{p+1} - (p+1) \varepsilon Q_b^p - \frac{p(p+1)}{2} Q_b^{p-1} \varepsilon^2 \right] \right| \\ &\lesssim \int_{y < \kappa B} (|\varepsilon|^{p+1} + Q_b^{p-2} |\varepsilon|^3) \psi_B + \int_{y > \kappa B} |\varepsilon|^{p+1} + e^{-\frac{\kappa B}{20}} \|\varepsilon\|_{L^\infty}^3 \\ &\leq b_c^{\frac{1}{2}} \mathcal{N} + b_c^{\frac{7}{2}}. \end{aligned}$$

Finally, we claim there exists a constant  $0 < \kappa < 1$  independent of  $b$  (recall  $\kappa$  appears in the definition of the weight function  $\varphi$ ) such that the following holds for some universal constant  $v_1 > 0$ :

$$\int (\varepsilon_y^2 \psi_B + \varepsilon^2 \zeta_B - p \psi_B Q_p^{p-1} \varepsilon^2) \geq v_1 \mathcal{N} - \frac{1}{v_1} b_c^{\frac{7}{2}}, \quad (2.5.25)$$

Then the lower bound follows immediately. We leave the proof of (2.5.25) in Appendix A.

This concludes the proof of Proposition 2.5.2.  $\square$

## 2.6 Existence and stability of a self-similar dynamics

### 2.6.1 Closing the bootstrap

In this section, we will complete the proof of Proposition 2.2.9.

**Step 1** Dynamical trapping on  $b$ .

We first prove the dynamical trapping of  $b$ , i.e. (2.2.34). Suppose for some  $s_0 \in [0, s^*)$ , we have  $\tilde{b}(s_0) \geq b_c^{\frac{3}{2}+2\nu}$ . By the choice of the initial data, i.e. (2.28), we can find some  $s_1 \in [0, s_0)$  such that  $\tilde{b}(s_1) = b_c^{\frac{3}{2}+\frac{5}{2}\nu}$  and  $\tilde{b}(s) \geq b_c^{\frac{3}{2}+\frac{5}{2}\nu}$  for all  $s \in [s_1, s_0)$ , then  $\tilde{b}_s(s_1) \geq 0$ . From (2.2.35) and (2.3.8), we have:

$$\tilde{b}_s(s_1) \leq -c_p \tilde{b}(s_1) b_c + b_c^{\frac{5}{2}+3\nu} \leq -c_p b_c^{\frac{5}{2}+\frac{5\nu}{2}} + b_c^{\frac{5}{2}+3\nu} < 0, \quad (2.6.1)$$

if  $b_c$  is small enough (or equivalently  $p^*(\nu)$  is close enough to 5) such that  $b_c^\nu \ll 1$ . We get a contradiction. The opposite bound is similar.

**Step 2** Pointwise bound of the localized Sobolev norm of  $\varepsilon$ .

The bootstrap bound (2.2.41) is a consequence of the monotonicity formula which we proved in the last section. We argue again by contradiction and assume that there exists  $s_2 \in (0, s^*)$  s.t.  $\mathcal{N}(s_2) \geq b_c^{3+8\nu}$ . By continuity and the choice of initial data, i.e. (2.29), we can find  $s_3 \in (0, s_2)$  such that for all  $s \in [s_3, s_2]$ ,  $\mathcal{N}(s) \geq b_c^{3+10\nu}$ , and  $\mathcal{N}(s_3) = b_c^{3+10\nu}$ . Then we have for all  $s \in [s_3, s_2]$ :

$$\int (\varepsilon_y^2(s) + \varepsilon^2(s)) \varphi_B' \geq \frac{1}{B} b_c^{3+10\nu} = b_c^{3+\frac{1}{20}+10\nu} \gg b_c^{\frac{7}{2}},$$

provided that  $\nu$  is chosen small enough (say  $\nu = \frac{1}{1000}$ ). From (2.5.5), we know  $d\mathcal{F}/ds \leq 0$  on  $[s_3, s_2]$ , which yields  $\mathcal{F}(s_3) \geq \mathcal{F}(s_2)$ . Thus (2.5.6) leads to:

$$b_c^{3+8\nu} - b_c^{\frac{7}{2}} \leq \mathcal{N}(s_2) - b_c^{\frac{7}{2}} \lesssim \mathcal{F}(s_2) \leq \mathcal{F}(s_3) \lesssim \mathcal{N}(s_3) + b_c^{\frac{7}{2}} = b_c^{3+10\nu} + b_c^{\frac{7}{2}}.$$

This is a contradiction since  $b_c^\nu \ll 1$ . Therefore we conclude the proof of (2.2.41).

**Step 3**  $L^{p_0}$  control of  $\varepsilon$ .

For the  $L^{p_0}$  norm of  $\varepsilon$ , it is more convenient to work with the original variables. Consider the decomposition (see (2.2.26)):

$$u(t, x) = Q_S(t, x) + \tilde{u}(t, x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} (Q_{b(t)} + \varepsilon(t)) \left( \frac{x - x(t)}{\lambda(t)} \right).$$

By rescaling, it is sufficient to prove for all  $t \in [0, T^*)$ :

$$\|\tilde{u}(t)\|_{L^{p_0}} \leq \frac{b_c^{\frac{13}{28}}}{\lambda(t)^{\frac{2}{p-1} - \frac{1}{p_0}}}. \quad (2.6.2)$$

To prove this, we write down the equation of  $\tilde{u}$  and use a refined Strichartz estimate for the Airy equations. Indeed, the equation of  $\tilde{u}$  is:

$$\partial_t \tilde{u} + \tilde{u}_{xxx} = -\mathcal{E} - (f(\tilde{u}))_x$$

with

$$\mathcal{E} = \frac{1}{\lambda(t)^{3+\frac{2}{p-1}}} \left[ -\Phi_b + b_s P_b - \left( \frac{\lambda_s}{\lambda} + b \right) \Lambda Q_b - \left( \frac{x_s}{\lambda} - 1 \right) Q'_b \right] \left( t, \frac{x-x(t)}{\lambda(t)} \right),$$

$$f(\tilde{u}) = (Q_S + \tilde{u}) |Q_S + \tilde{u}|^{p-1} - Q_S |Q_S|^{p-1},$$

where  $\Phi_b$  is defined in (3.3).

Now we state the result of D. Foschi in [23] about the inhomogeneous Strichartz estimates:

**Proposition 2.6.1** (D. Foschi, Theorem 1.4 of [23]). *Consider a family of linear operators  $U(t): H \rightarrow L_X^2$ ,  $t \in \mathbb{R}$ , where  $H$  is a Hilbert space. Suppose the following properties of  $U(t)$  hold:*

1. For all  $t \in \mathbb{R}$ ,  $h \in H$ :

$$\|U(t)h\|_{L_X^2} \lesssim \|h\|_H.$$

2. There exists a constant  $\sigma > 0$ , such that for all  $f \in L_X^1 \cap L_X^2$  and  $t, s \in \mathbb{R}$ , there holds:

$$\|U(t)U(s)^*f\|_{L_X^\infty} \lesssim \frac{1}{|t-s|^\sigma} \|f\|_{L_X^1}.$$

We say a pair  $(q, r) \in [2, +\infty]^2$  is  $\sigma$ -acceptable if and only if they satisfy:

$$\frac{1}{q} < 2\sigma \left( \frac{1}{2} - \frac{1}{r} \right) \text{ or } (q, r) = (+\infty, 2).$$

Consider  $0 < \sigma < 1$  and 2  $\sigma$ -acceptable pairs:  $(q_i, r_i)$ ,  $i = 1, 2$ , such that the scaling rule is satisfied:

$$\frac{1}{q_1} + \frac{\sigma}{r_1} + \frac{1}{q_2} + \frac{\sigma}{r_2} = \sigma.$$

Then we have the following inhomogeneous Strichartz estimates:

$$\left\| \int_{s < t} U(t)U(s)^*F(s)ds \right\|_{L_t^{q_1}L_x^{r_1}} \lesssim \|F\|_{L_t^{q_2'}L_x^{r_2'}}. \quad (2.6.3)$$

Here, we can use Proposition 6.1 to derive a refined Strichartz estimate for the Airy equations with zero initial data. Let  $U(t) = \mathbf{1}_{[0, +\infty)}(t)e^{-t\partial_x^3}$ , then by the theory of oscillatory integral, we have <sup>18</sup>:

$$\|U(t)h\|_{L^2} \leq \|h\|_{L^2}, \quad \|U(t)h\|_{L^\infty} \lesssim \frac{1}{|t|^{\frac{1}{3}}} \|h\|_{L^1}, \quad \text{for } \forall t \neq 0.$$

Therefore, the following refined Strichartz estimates hold for Airy equations with zero initial data:

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18. See Page 13–15 in [39].

**Corollary 2.6.2** (Refined Strichartz estimates). *For all  $\frac{1}{3}$ -acceptable pairs  $(q_1, r_1)$  and  $(q_2, r_2)$ , if they satisfy:*

$$\frac{1}{q_1} + \frac{1}{3r_1} + \frac{1}{q_2} + \frac{1}{3r_2} = \frac{1}{3},$$

then there holds:

$$\left\| \int_0^t e^{-(t-s)\partial_x^3} (h(s, \cdot)) ds \right\|_{L_t^{q_1} L_x^{r_1}} \lesssim \|h\|_{L_t^{q_2'} L_x^{r_2'}}. \quad (2.6.4)$$

Now we fix  $\forall t \in [0, T^*)$ , and choose

$$(q_1, r_1) = (+\infty, p_0), \quad \frac{1}{r_2} = \frac{1}{p_0} - \delta, \quad \frac{1}{q_2} = \frac{p_0 - 2}{3p_0} + \frac{\delta}{3},$$

with  $\delta > 0$  to be chosen later. It is easy to check  $(q_i, r_i)$  satisfy the conditions in Corollary 6.2. Then we have the following estimate on  $[0, t]$ :

$$\begin{aligned} \|\tilde{u}\|_{L_{[0,t]}^\infty L_x^{p_0}} &\lesssim \|e^{-t\partial_x^3}(\tilde{u}(0))\|_{L_{[0,t]}^\infty L_x^{p_0}} + \|\mathcal{E}\|_{L_{[0,t]}^{q_2'} L_x^{r_2'}} + \|(f(\tilde{u}))_x\|_{L_{[0,t]}^{q_2'} L_x^{r_2'}} \\ &= I + II + III. \end{aligned} \quad (2.6.5)$$

We let  $\sigma_0 = \frac{1}{2} - \frac{1}{p_0} (= \frac{1}{10})$ , then by Sobolev embedding:

$$I \lesssim \|e^{-t\partial_x^3}(\tilde{u}(0))\|_{L_{[0,t]}^\infty \dot{H}^{\sigma_0}} = \frac{1}{\lambda(0)^{\frac{2}{p-1} - \frac{1}{p_0}}} \|\mathcal{E}(0)\|_{\dot{H}^{\sigma_0}} \leq \frac{b_c^{10}}{\lambda(t)^{\frac{2}{p-1} - \frac{1}{p_0}}}. \quad (2.6.6)$$

For  $II$ , from (2.2.18), (2.2.34), (2.2.35), (2.3.6), (2.3.7) and (2.3.8), there holds for all  $\tau \in [0, t]$ :

$$\begin{aligned} \|\mathcal{E}(\tau)\|_{L_x^{r_2'}} &= \frac{1}{\lambda(\tau)^{2 + \frac{2}{p-1} + \frac{1}{r_2}}} \left\| -\Phi_b + b_s P_b - \left(\frac{\lambda_s}{\lambda} + b\right) \Lambda Q_b - \left(\frac{x_s}{\lambda} - 1\right) Q_b' \right\|_{L_x^{r_2'}} \\ &\lesssim \frac{1}{\lambda(\tau)^{2 + \frac{2}{p-1} + \frac{1}{r_2}}} \left( \|\Phi_b\|_{L_x^{r_2'}} + b_c^{\frac{5}{2}} \|P_b\|_{L_x^{r_2'}} + \mathcal{N}^{\frac{1}{2}} + b_c^{\frac{5}{2}} \right) \\ &\lesssim \frac{b_c^{1 + \frac{1}{p_0} - \delta}}{\lambda(\tau)^{2 + \frac{2}{p-1} + \frac{1}{r_2}}}. \end{aligned}$$

From (2.4.19) we obtain:

$$II \lesssim \left( \int_0^t \left( \frac{b_c^{1 + \frac{1}{p_0} - \delta}}{\lambda(\tau)^{2 + \frac{2}{p-1} + \frac{1}{r_2}}} \right)^{q_2'} d\tau \right)^{\frac{1}{q_2'}} \lesssim \frac{b_c^{\frac{p_0+1}{3p_0} - \frac{2\delta}{3}}}{\lambda(t)^{\frac{2}{p-1} - \frac{1}{p_0}}} = \frac{b_c^{\frac{7}{15} - \frac{2\delta}{3}}}{\lambda(t)^{\frac{2}{p-1} - \frac{1}{p_0}}}. \quad (2.6.7)$$

Finally we deal with  $III$ . For all  $\tau \in [0, t]$ , there holds:

$$\begin{aligned} \|(f(\tilde{u}))_x\|_{L_x^{r_2'}} &= \frac{1}{\lambda(\tau)^{2 + \frac{2}{p-1} + \frac{1}{r_2}}} \left\| ((Q_b + \varepsilon)|Q_b + \varepsilon|^{p-1} - Q_b^p)_y \right\|_{L_x^{r_2'}} \\ &\lesssim \frac{1}{\lambda(\tau)^{2 + \frac{2}{p-1} + \frac{1}{r_2}}} \left( \|\varepsilon_y Q_b^{p-1}\|_{L_x^{r_2'}} + \|\varepsilon_y |\varepsilon|^{p-1}\|_{L_x^{r_2'}} \right. \\ &\quad \left. + \|\varepsilon(Q_b)_y |Q_b|^{p-2}\|_{L_x^{r_2'}} + \|(Q_b)_y |\varepsilon|^{p-1}\|_{L_x^{r_2'}} \right). \end{aligned} \quad (2.6.8)$$

We estimate these terms separately. First from (2.2.36), (2.2.37) and (2.2.38) we have:

$$\begin{aligned}\|\varepsilon(Q_b)_y|\varepsilon|^{p-2}\|_{L^{r'_2}} &\leq \|\varepsilon\|_{L^{r'_2(p-1)}}^{p-1} \leq b_c^{\frac{3}{2}}, \\ \|\varepsilon_y|\varepsilon|^{p-1}\|_{L^{r'_2}} &\leq \|\varepsilon_y\|_{L^2} \|\varepsilon\|_{L^{r(p-1)}}^{p-1} \leq b_c^{\frac{3}{2}},\end{aligned}$$

where

$$\frac{1}{r'_2} = \frac{1}{2} + \frac{1}{r}.$$

Next, by using the bootstrap bound (2.2.35), (2.2.37) and the decay property of  $Q_b$ , we have:

$$\begin{aligned}&\|\varepsilon_y Q_b^{p-1}\|_{L^{r'_2}} \\ &= \left( \int_{|y| < -\kappa B} |\varepsilon_y|^{r'_2} Q_b^{r'_2(p-1)} + \int_{|y| < \kappa B} |\varepsilon_y|^{r'_2} Q_b^{r'_2(p-1)} + \int_{|y| > \kappa B} |\varepsilon_y|^{r'_2} Q_b^{r'_2(p-1)} \right)^{\frac{1}{r'_2}} \\ &\lesssim \|\varepsilon_y\|_{L^2} \|Q_b\|_{L^{r(p-3)}}^{p-3} \|Q_b\|_{L^\infty(|y| > \kappa B)}^2 + \|\varepsilon_y\|_{L^2(|y| < \kappa B)} \|Q_b\|_{L^{r(p-1)}}^{p-1} \\ &\lesssim b_c^{\frac{3}{2}}.\end{aligned}$$

The same estimate holds for  $\|\varepsilon(Q_b)_y|Q_b|^{p-2}\|_{L^{r'_2}}$ .

Injecting all the above estimates into (2.6.8) yields:

$$\|(f(\tilde{u}))_x\|_{L^{r'_2}} \lesssim \frac{b_c^{\frac{3}{2}}}{\lambda(t)^{2+\frac{2}{p-1}+\frac{1}{r'_2}}}.$$

By a similar argument we have:

$$III \lesssim \frac{b_c^{\frac{1}{2}}}{\lambda(t)^{\frac{2}{p-1}-\frac{1}{p_0}}}. \quad (2.6.9)$$

Injecting (2.6.6), (2.6.7) and (2.6.9) into (2.6.5), we obtain (2.6.2), provided that  $\delta$  is small enough (since  $\frac{1}{2} > \frac{7}{15} > \frac{13}{28}$ ).

This concludes the proof of Proposition 2.2.9 (Recall we have proved (2.2.43) in Lemma 4.1).

## 2.6.2 Proof of Theorem 2.1.1

We are now in position to prove Theorem 2.1.1.

Pick a  $\nu > 0$  small enough and a  $p \in (5, p^*(\nu))$ . For all  $u_0 \in \mathcal{O}_p$ , we choose  $b^*(p) = b_c$  and denote  $u(t)$  the corresponding solution to the Cauchy problem (2.1.1) with maximal lifetime  $T$ . Proposition 2.2.9 implies that  $u(t)$  satisfies the geometrical decomposition introduced in Section 2 on  $[0, T)$ :

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} (Q_{b(t)} + \varepsilon(t)) \left( \frac{x - x(t)}{\lambda(t)} \right),$$



and the bounds in Proposition 2.2.9 hold on  $[0, T)$ . From (2.4.1), we have (2.1.7) and (2.1.8).

**Step 1** Finite time blow-up and self-similar rate.

From (2.3.6) we have:

$$\forall t \in [0, T), \quad (1 - \nu^2)b_c \leq -\lambda_t \lambda^2 \leq (1 + \nu^2)b_c. \quad (2.6.10)$$

Integrating it from 0 to  $t$  yields:

$$\forall t \in [0, T), \quad (1 - \nu^2)b_c t \leq \frac{1}{3} \lambda^3(0) \text{ and hence } T \leq \frac{\lambda^3(0)}{3b_c(1 - \nu^2)} < +\infty.$$

So the solution blows up in finite time. From  $H^1$  Cauchy theory we have:

$$\|u_x(t)\|_{L^2} \rightarrow +\infty \text{ as } t \rightarrow T,$$

which implies  $\lambda(t) \rightarrow 0$  as  $t \rightarrow T$ . We thus integrate (2.6.10) from  $t$  to  $T$  to obtain:

$$\forall t \in [0, T], \quad (1 - \nu^2)b_c(T - t) \leq \frac{\lambda^3(t)}{3} \leq (1 + \nu^2)b_c(T - t),$$

which implies (2.1.10).

**Step 2** Convergence of the blow-up point.

From (2.3.7) we have:

$$|x_t| = \frac{1}{\lambda^2} \left| \frac{x_s}{\lambda} \right| \leq \frac{1 + \nu^2}{\lambda^2}.$$

Thus from (2.1.10), we get:

$$\int_0^T |x_t| \leq \int_0^T \frac{1 + \nu^2}{((1 - \nu^2)b_c(T - t))^{\frac{2}{3}}} \leq (1 + \nu) \frac{\lambda(0)}{b_c} < +\infty,$$

and then (2.1.9) follows.

**Step 3** Strong convergence in  $L^q$ .

Fix a  $q \in [2, \frac{2}{1-2\sigma_c})$ , and let  $0 < \tau \ll T$  and  $0 < t < T - \tau$ , let  $u_\tau(t) = u(t + \tau)$  and  $v_\tau(t') = u_\tau(t') - u(t')$  for all  $t' \in [t, T - \tau)$ . Then  $v_\tau$  satisfies:

$$\partial_{t'} v_\tau + \partial_{xxx} v_\tau = (u|u|^{p-1} - u_\tau|u_\tau|^{p-1})_x.$$

Let  $\sigma_1 = \frac{1}{2} - \frac{1}{q}$ , and chose  $\tilde{q}$  and  $\tilde{r}$ , such that  $(+\infty, q)$  and  $(\tilde{q}, \tilde{r})$  satisfy the conditions in Corollary 2.6.2. Then we have:

$$\begin{aligned} \|(u|u|^{p-1})_x\|_{L_x^{\tilde{r}}} &= \frac{1}{\lambda^{2+\frac{1}{\tilde{q}}+\frac{1}{\tilde{r}}+\sigma_1-\sigma_c}} \left\| ((Q_b + \varepsilon)|Q_b + \varepsilon|^{p-1})_y \right\|_{L^{\tilde{r}}} \\ &\lesssim \frac{1}{\lambda^{2+\frac{1}{\tilde{q}}+\frac{1}{\tilde{r}}+\sigma_1-\sigma_c}} \left( (\|(Q_b)_y\|_{L^2} + \|\varepsilon_y\|_{L^2}) (\|Q_b\|_{L^{r_0}}^{p-1} + \|\varepsilon\|_{L^{r_0}}^{p-1}) \right) \\ &\lesssim \frac{1}{\lambda^{2+\frac{1}{\tilde{q}}+\frac{1}{\tilde{r}}+\sigma_1-\sigma_c}}, \end{aligned}$$

where

$$\frac{1}{\tilde{r}'} = \frac{1}{2} + \frac{p-1}{r_0}.$$

Since  $\sigma_1 < \sigma_c$  and  $\lambda(t) \sim \sqrt[3]{3b_c(T-t)}$ , we conclude:

$$\begin{aligned} & \left\| (|u|^{p-1} - |u_\tau|^{p-1})_x \right\|_{L_{[t, T-\tau]}^{\tilde{q}'} L_x^{\tilde{r}'}} \\ & \lesssim \left( \int_t^T \left( \frac{1}{\lambda(t')^{2+\frac{1}{q}+\frac{1}{\tilde{r}'}+\sigma_1-\sigma_c}} \right)^{\tilde{q}'} dt' \right)^{\frac{1}{\tilde{q}'}} \\ & \lesssim \frac{1}{b_c^2} (T-t)^{\frac{\sigma_c-\sigma_1}{3}} \rightarrow 0, \text{ as } t \rightarrow T, \text{ uniformly in } \tau. \end{aligned}$$

*Remark 2.6.3.* Here we can see the case  $q = q_c$  (i.e.  $\sigma_1 = \sigma_c$ ) will lead to a logarithm on the upper bound of the critical norm, therefore the strong convergence can't exist in the critical space.

Next from the refined Strichartz estimate (2.6.4) and Sobolev embedding we have:

$$\|v_\tau\|_{L_{[t, T-\tau]}^\infty L_x^q} \lesssim \|v_\tau(t)\|_{\dot{H}^{\sigma_1}} + \left( \int_t^T \left( \frac{1}{\lambda(t')^{2+\frac{1}{q}+\frac{1}{\tilde{r}'}+\sigma_1-\sigma_c}} \right)^{\tilde{q}'} dt' \right)^{\frac{1}{\tilde{q}'}}. \quad (2.6.11)$$

We claim (2.6.11) implies that  $u(t)$  is a Cauchy sequence in  $L^q$  as  $t \rightarrow T$ . Indeed, for all  $\varepsilon > 0$ , we can choose a  $t_\varepsilon$  close enough to  $T$ , such that:

$$\left( \int_{t_\varepsilon}^T \left( \frac{1}{\lambda(t')^{2+\frac{1}{q}+\frac{1}{\tilde{r}'}+\sigma_1-\sigma_c}} \right)^{\tilde{q}'} dt' \right)^{\frac{1}{\tilde{q}'}} \leq \frac{\varepsilon}{2C_0},$$

where  $C_0$  is the implicit constant in (2.6.11). From  $H^1$  Cauchy theory i.e.  $u(t) \in C([0, T], H^1)$ , there exists a  $\tau_0 = \tau_0(t_\varepsilon) \in (0, T - t_\varepsilon)$ , such that for all  $0 < \tau \leq \tau_0$ ,

$$\|v_\tau(t_\varepsilon)\|_{\dot{H}^{\sigma_1}} \leq \frac{\varepsilon}{2C_0}.$$

Choose a  $t_0 < T$  such that  $T - t_0 < \tau_0$ . Then for all  $t_1, t_2 \in (t_0, T)$ ,  $t_1 < t_2$ , let  $\tau = t_2 - t_1$ . From the above discussion, we have:

$$\|u(t_2) - u(t_1)\|_{L^q} = \|v_\tau(t_1)\|_{L^q} \leq \|v_\tau\|_{L_{[t_\varepsilon, T-\tau]}^\infty L_x^q} \leq \varepsilon,$$

which means  $u(t)$  is a Cauchy sequence in  $L^q$  as  $t \rightarrow T$ . Hence, we have proven (2.1.11).

#### Step 4 Singular behavior of the asymptotic profile.

Finally, we give the proof of (2.1.12). Let

$$A = b_c^{-\frac{21}{20}}, \quad R(\tau) = A\lambda(\tau) \text{ for all } \tau \in [t, T], \quad (2.6.12)$$

where  $t$  is a fixed time close enough to  $T$ . Then we choose a smooth cut-off function  $\chi$ , with  $\chi(y) = 0$  if  $|y| > 2$ ,  $\chi(y) = 1$  if  $|y| < 1$ . Denote

$$g(x) = \chi\left(\frac{x - x(T)}{R(t)}\right).$$

Then by Kato's localized identity for mass, we can estimate:

$$\begin{aligned} & \left| \frac{d}{d\tau} \int u^2(\tau) g \right| = \left| -3 \int u_x^2(\tau) g_x + \int u^2(\tau) g_{xxx} + \frac{2p}{p+1} \int |u(\tau)|^{p+1} g_x \right| \\ & \lesssim \frac{1}{R(t)} \left( \int |u_x(\tau)|^2 + |u(\tau)|^{p+1} \right) + \frac{1}{R(t)^3} \left| \int \chi''' \left( \frac{x - x(T)}{R(t)} \right) u^2(\tau) \right| \\ & \lesssim \frac{1}{R(t)} \frac{1}{\lambda(\tau)^{2-2\sigma_c}} \left( \int |(\varepsilon + Q_b)_y|^2 + |\varepsilon + Q_b|^{p+1} \right) + \frac{1}{R(t)^2} \|u(\tau)\|_{L^\infty}^2 \\ & \lesssim \frac{1}{R(t)} \frac{1}{\lambda(\tau)^{2-2\sigma_c}} + \frac{1}{R(t)^2} \frac{1}{\lambda(\tau)^{1-2\sigma_c}} (\|Q_b\|_{L^\infty}^2 + \|\varepsilon\|_{L^\infty}^2) \\ & \lesssim \frac{1}{R(t)} \frac{1}{\lambda(\tau)^{2-2\sigma_c}} + \frac{1}{R(t)^2} \frac{1}{\lambda(\tau)^{1-2\sigma_c}}. \end{aligned}$$

Since  $u(\tau)$  converges to  $u^*$  in  $L^2$  as  $\tau \rightarrow T$ , we can integrate the above inequality from  $t$  to  $T$  (with respect to  $\tau$ ) and use the fact that (which follows from (4.4)):

$$\text{for } \beta < 3, \quad \int_t^T \frac{d\tau}{\lambda(\tau)^\beta} \leq -2 \int_t^T \frac{\lambda_\tau(\tau)}{b_c \lambda(\tau)^{\beta-2}} d\tau = \frac{2\lambda(t)^{3-\beta}}{b_c(3-\beta)}$$

to obtain:

$$\begin{aligned} & \frac{1}{\lambda(t)^{2\sigma_c}} \left| \int \chi\left(\frac{x - x(T)}{R(t)}\right) |u^*|^2 - \int \chi\left(\frac{x - x(T)}{R(t)}\right) u^2(t) \right| \\ & \lesssim \frac{1}{A\lambda(t)^{1+2\sigma_c}} \int_t^T \frac{d\tau}{\lambda(\tau)^{2-2\sigma_c}} + \frac{1}{A^2\lambda(t)^{2+2\sigma_c}} \int_t^T \frac{d\tau}{\lambda(\tau)^{1-2\sigma_c}} \\ & \lesssim \frac{1}{b_c A} = b_c^{\frac{1}{20}}. \end{aligned} \tag{2.6.13}$$

On the other hand we have from the geometrical decomposition (2.2.26):

$$\begin{aligned} & \frac{1}{\lambda(t)^{2\sigma_c}} \int \chi\left(\frac{x - x(T)}{R(t)}\right) |u(t)|^2 \\ & = \int \chi \left[ \frac{1}{A} \left( y + \frac{x(t) - x(T)}{\lambda(t)} \right) \right] |Q_b + \varepsilon|^2 dy. \end{aligned} \tag{2.6.14}$$

From the properties of  $x(t)$  and  $\lambda(t)$ , we know that:

$$-\frac{x(t) - x(T)}{\lambda(t)} \sim \frac{1}{b_c} \ll A.$$

Together with Lemma 2.2.4 and (2.2.38) we have:

$$\int \chi \left[ \frac{1}{A} \left( y + \frac{x(t) - x(T)}{\lambda(t)} \right) \right] \varepsilon^2 \lesssim A \|\varepsilon\|_{L^\infty}^2 \leq A b_c^{\frac{149}{135}} \leq b_c^{\frac{1}{20}}, \quad (2.6.15)$$

$$\int \chi \left[ \frac{1}{A} \left( y + \frac{x(t) - x(T)}{\lambda(t)} \right) \right] |Q_b|^2 = (1 + \delta_0(p)) \int |Q_p|^2. \quad (2.6.16)$$

with  $\delta_0(p) \rightarrow 0$  as  $p \rightarrow 5$ . Injecting (2.6.14)–(2.6.16) into (2.6.13), yields:

$$\begin{aligned} \frac{1}{R(t)^{2\sigma_c}} \int \chi \left( \frac{x - x(T)}{R(t)} \right) |u^*|^2 &= \frac{1}{A^{2\sigma_c}} \int |Q_p|^2 (1 + \delta(p)) + O(b_c^{\frac{1}{40}}) \\ &= (1 + \delta(p)) \int |Q_p|^2. \end{aligned}$$

with  $\lim_{p \rightarrow 5} \delta(p) = 0$ . Let  $t \rightarrow T$ , i.e.  $R(t) \rightarrow 0$ , then (2.1.12) follows.

Finally, it is immediately seen from (2.1.12) that:

$$u^* \notin L^{1-\frac{2}{\sigma_c}},$$

which concludes the proof of Theorem 2.1.1.



# Chapter 3

## Blow up solutions for slightly mass supercritical gKdV equations with multiple blow up points

### 3.1 Introduction

#### 3.1.1 Setting of the problem

We consider the following gKdV equations:

$$\begin{cases} \partial_t u + (u_{xx} + u|u|^{p-1})_x = 0, & (t, x) \in [0, T) \times \mathbb{R}, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \quad (3.1.1)$$

with  $1 \leq p < +\infty$ .

From the result of C. E. Kenig, G. Ponce and L. Vega [36] and N. Strunk [89], (3.1.1) is locally well-posed in  $H^1$  and thus for all  $u_0 \in H^1$ , there exists a maximal lifetime  $0 < T \leq +\infty$  and a unique solution  $u(t, x) \in C([0, T), H^1(\mathbb{R}))$  to (3.1.1). Besides, we have the blow-up criterion: either  $T = +\infty$  or  $T < +\infty$  and  $\lim_{t \rightarrow T} \|u_x(t)\|_{L^2} = +\infty$ .

(3.1.1) admits two conservation laws, i.e. the mass and energy:

$$\begin{aligned} M(u(t)) &= \int |u(t, x)|^2 dx = M(u(0)), \\ E(u(t)) &= \frac{1}{2} \int |u(t, x)|^2 dx - \frac{1}{p+1} \int |u(t, x)|^{p+1} dx = E(u(0)). \end{aligned}$$

For all  $\lambda > 0$ ,  $u_\lambda(t, x) = \lambda^{\frac{2}{p-1}} u(\lambda^3 t, \lambda x)$  is also a solution. Moreover, the  $\dot{H}^{\sigma_c}$  norm of the initial data with the index:

$$\sigma_c = \frac{1}{2} - \frac{2}{p-1}, \quad (3.1.2)$$

is invariant under this scaling.

We introduce the ground state  $Q_p$ , which is the unique radial nonnegative function with exponential decay at infinity to the following equation:

$$Q_p'' - Q_p + Q_p|Q_p|^{p-1} = 0. \quad (3.1.3)$$

$Q_p$  plays a distinguished role in the analysis. It provides a family of travelling wave solutions:

$$u(t, x) = \lambda^{\frac{2}{p-1}} Q_p(\lambda(x - \lambda^2 t - x_0)), \quad (\lambda, x_0) \in \mathbb{R}_+^* \times \mathbb{R}.$$

For  $p < 5$  or equivalently  $\sigma_c < 0$ , (3.1.1) is called  $L^2$  subcritical. The mass and energy conservation laws imply that the solution is always global and bounded in  $H^1$ .

For  $p = 5$ , the solution is called  $L^2$  critical. From variation arguments [91], we know that if  $\|u_0\|_{L^2} < \|Q_5\|_{L^2}$ , then the solution to (3.1.1) is always global and bounded in  $H^1$ .

While for  $\|u_0\|_{L^2} \geq \|Q_5\|_{L^2}$ , blow up may occur. The blow up dynamics for solution with slightly supercritical mass:

$$\|Q_5\|_{L^2} < \|u_0\|_{L^2} < \|Q_5\|_{L^2} + \alpha^* \quad (3.1.4)$$

has been developed in a series paper of Martel and Merle [53, 54, 55, 70]. In particular, they obtain the existence of blow up solutions with negative energy, and the classification of the ground state  $Q_5$  as the unique global attractor for blow up solutions in  $H^1$ .

In [56, 57, 58, 59], Martel, Merle, Nakanishi and Raphaël, give a comprehensive study of the asymptotic dynamics near the ground state: classification of the flow near soliton; existence of the minimal mass blow up solutions; exotic blow up regime; codimension 1 threshold manifold for unstable regime.

### 3.1.2 On the supercritical problems

Let us consider the focusing  $L^2$  supercritical nonlinear Schrödinger equations (NLS)

$$\begin{cases} i\partial_t u + \Delta u + u|u|^{p-1} = 0, & (t, x) \in [0, T) \times \mathbb{R}^d, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}^d), \end{cases} \quad (\text{NLS})$$

with  $1 + \frac{4}{d} < p < +\infty$ . The blow up dynamics for supercritical NLS is mostly open. Only until recently, a few special examples are known. From [77, 87, 88], there exist blow-up solutions with log-log blow up rate for  $p = 5$  and  $d \geq 2$  with radial initial data. From [28, 94], there exist blow-up solutions with cylindrical symmetry blowing up at log-log blow up rate for  $p = 3$  and  $d \geq 3$ . In cite In [75], Merle, Raphaël and Szeftel construct a stable self-similar blow up dynamics for slightly supercritical nonlinearity in low dimensions (i.e.  $d \leq 5$ ).

For supercritical gKdV equations the existence of blow up solution in energy space  $H^1$  has been a long standing open problem. Numerical simulation [13] suggests the existence

of self-similar blow up solution in the slightly  $L^2$  supercritical case<sup>1</sup>, where a self similar solution is a solution of the following form:

$$u(t, x) \sim \frac{1}{[3b(T-t)]^{\frac{2}{3(p-1)}}} V_b \left( \frac{x}{\sqrt[3]{3b(T-t)}} \right),$$

where  $b > 0$ . Direct computation shows that  $V_b$  should be a solution to the following ODE<sup>2</sup>:

$$b\Lambda V_b + (V_b'' - V_b + V_b|V_b|^{p-1})' = 0. \quad (3.1.5)$$

The exact solution of (3.1.5) in slightly supercritical case has been constructed by H. Koch [38]. It is related to an eigenvalue problem, i.e. for all  $p > 5$  close enough to 5, there exists a unique  $b = b(p) > 0$  such that a unique solution  $V_b$  to (3.1.5) can be found. Hence, this  $V_b$  leads to a self-similar blow up solution to (3.1.1) directly.

But unfortunately, the exact solution  $V_b$  constructed in [38], has a slowly decaying tail:

$$V_b(y) \sim \frac{1}{|y|^{\frac{1}{2}-\sigma_c}}, \quad \text{as } |y| \rightarrow +\infty.$$

Thus,  $V_b$  belongs to  $L^{p+1} \cap \dot{H}^1$ , but always misses the critical Sobolev space  $\dot{H}^{\sigma_c}$  (hence  $L^2$ ), which makes it impossible to obtain a stability result for the exact self-similar blow up solution. Since, for typical Cauchy problem like (3.1.1), we can only expect a stability result in a Cauchy space, i.e. a space where local wellposedness holds. In our case, natural Cauchy spaces are the critical Sobolev space  $\dot{H}^{\sigma_c}$  and the energy space  $H^1$  from [36], while  $V_b$  is not in neither of them. Hence, we cannot use the profile  $V_b$  directly.

Despite the slowly decaying tail of  $V_b$ , we can choose a suitable cut-off of  $V_b$  as an approximation, such that it is bounded in  $L^2$  with exponential decay on the right. Based on this approximate self-similar profile, Lan [46] has construct a stable self-similar blow-up dynamics for slightly  $L^2$  supercritical gKdV:

**Theorem 3.1.1** (Existence and stability of a self-similar blow-up dynamics). *There exists a  $p^* > 5$  such that for all  $p \in (5, p^*)$ , there exist constants  $\delta(p) > 0$  and  $b^*(p) > 0$  with*

$$\lim_{p \rightarrow 5} \delta(p) = 0 \quad (3.1.6)$$

$$b^*(p) = \frac{4\pi^2}{\Gamma(1/4)^4} (p-5) + O(|p-5|^2), \quad \text{as } p \rightarrow 5 \quad (3.1.7)$$

and a nonempty open subset  $\mathcal{O}_p$  in  $H^1$  such that the following holds. If  $u_0 \in \mathcal{O}_p$ , then the corresponding solution to (3.1.1) blows up in finite time  $0 < T < +\infty$ , with the following dynamics: there exist geometrical parameters  $(\lambda(t), x(t)) \in \mathbb{R}_+^* \times \mathbb{R}$  and an error term  $\varepsilon(t)$  such that:

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} [\mathcal{Q}_p + \varepsilon(t)] \left( \frac{x - x(t)}{\lambda(t)} \right) \quad (3.1.8)$$

1. From [55], there is no self-similar blow up solutions for  $L^2$  critical gKdV with slightly supercritical mass.

2. See Section 1.6 for the definition of “ $\Lambda$ ”.



with

$$\|\varepsilon_y(t)\|_{L^2} \leq \delta(p). \quad (3.1.9)$$

Moreover, we have:

1. The blow-up point converges at the blow-up time:

$$x(t) \rightarrow x(T) \text{ as } t \rightarrow T, \quad (3.1.10)$$

2. The blow-up speed is self-similar:

$$\forall t \in [0, T), \quad (1 - \delta(p)) \sqrt[3]{3b^*(p)} \leq \frac{\lambda(t)}{\sqrt[3]{T-t}} \leq (1 + \delta(p)) \sqrt[3]{3b^*(p)}. \quad (3.1.11)$$

3. The following convergence holds:

$$\forall q \in [2, \frac{2}{1-2\sigma_c}), \quad u(t) \rightarrow u^* \text{ in } L^q \text{ as } t \rightarrow T. \quad (3.1.12)$$

4. The asymptotic profile  $u^*$  displays the following singular behavior:

$$(1 - \delta(p)) \int \mathcal{Q}_p^2 \leq \frac{1}{R^{2\sigma_c}} \int_{|x-x(T)| < R} |u^*|^2 \leq (1 + \delta(p)) \int \mathcal{Q}_p^2. \quad (3.1.13)$$

for  $R$  small enough. In particular, we have for all  $q \geq \frac{2}{1-2\sigma_c}$ :

$$u^* \notin L^q.$$

### 3.1.3 Blow up solution with multiple blow up points

The existence of large blow up solution is mostly open. One way to construct such solution is to consider blow up solutions with multiple blow up points. There are several examples:

- Merle [65] for  $L^2$  critical NLS with conformal blow up rate;
- Planchon, Raphaël [85] and Fan [20] for  $L^2$  critical NLS with log-log blow up rate;
- Merle [68] and Merle, Zaag [79] for nonlinear heat equation with ODE blow up rate.

Such constructions correspond to the weak interaction case, i.e. the interaction between the bubbles does not change the blow up rate of each bubble. There are also some examples for strongly interacting bubbles:

- Martel, Raphaël [62] for  $L^2$  critical NLS;
- Cortázar, Del Pino, Musso [10] for energy critical nonlinear heat equations in domain;
- Jendrej [31, 30] for focusing energy critical wave equations.

The goal of this paper is to construct blow up solutions for slightly supercritical gKdV with multiple bubbles, and each bubble concentrates at a finite point. First, we need to give the definition of the *blow up point* for solution to (3.1.1).

**Definition 3.1.2.** Let  $u(t)$  be a solution of (3.1.1), which blows up in finite time  $T$ . The *blow-up set* of  $u(t)$  is the set of all the points  $z$  such that:

$$\liminf_{R \rightarrow 0} \left( \liminf_{t \rightarrow T} \frac{1}{R^{2\sigma_c}} \int_{|x-z| < R} |u(t)|^2 \right) \geq \frac{1}{2} \int \mathcal{Q}_p^2.$$

*Remark 3.1.3.* From the definition, the blow-up set is “invariant” under the symmetry of the equation. More precisely, consider a solution  $u(t)$  of (3.1.1), which blows-up in finite time  $T$  with blow-up set  $B$ . Then for all  $\lambda_0 > 0$ ,  $x_0 \in \mathbb{R}$ ,

$$\bar{u}(t, x) = \frac{1}{\lambda_0^{\frac{2}{p-1}}} u\left(\frac{t}{\lambda_0^3}, \frac{x-x_0}{\lambda_0}\right),$$

is still a solution to (3.1.1), which blows up in finite time  $\bar{T} = \lambda_0^3 T$ . Moreover, its blow-up set is exactly:

$$\bar{B} = \{\lambda_0 x + x_0 | x \in B\}.$$

*Remark 3.1.4.* For all solution  $u(t)$  mentioned in Theorem 3.1.1, we can see from the proof of (3.1.13) in [46],

$$\begin{aligned} \limsup_{R \rightarrow 0} \lim_{t \rightarrow T} \frac{1}{R^{2\sigma_c}} \int_{|x-z_0| < R} |u(t)|^2 &\leq \delta(p), \quad \text{for all } z_0 \neq x(T), \\ \frac{1}{R^{2\sigma_c}} \int_{|x-x(T)| < R} |u(t)|^2 &\sim \int \mathcal{Q}_p^2, \quad \text{for } R \text{ small enough.} \end{aligned}$$

Therefore, the blow up set of  $u(t)$  is exactly  $\{x(T)\}$ .

### 3.1.4 Statement of the main theorem

**Theorem 3.1.5 (Main Theorem).** *There exist universal constants  $p^* > 5$ ,  $c > 0$  such that for all  $p \in (5, p^*)$ ,  $k \in \mathbb{N}^+$ , if  $2 \leq k \leq c |\log(p-5)|$ , then for all  $k$  pairwise distinct points  $x_1, x_2, \dots, x_k \in \mathbb{R}$ , there exists a solution  $u(t)$  of (3.1.1), which blows up in finite time  $T < +\infty$ . And for  $t$  close to  $T$ , there exist scaling parameters  $\lambda_j(t) \in \mathbb{R}^+$  and an error term  $\tilde{u}(t, x)$  with*

$$u(t, x) = \sum_{j=1}^k \frac{1}{\lambda_j^{\frac{2}{p-1}}(t)} \mathcal{Q}_p\left(\frac{x-x_j}{\lambda_j(t)}\right) + \tilde{u}(t, x), \quad (3.1.14)$$

where for all  $j = 1, \dots, k$ , and  $t$  close to  $T$ ,

$$\frac{\lambda_j^3(t)}{T-t} \sim p-5, \quad \|\tilde{u}_x(t)\|_{L^2} \leq \frac{\delta(p)}{\lambda_j(t)^{1-\sigma_c}}, \quad (3.1.15)$$

for some small constant  $\delta(p)$  with  $\lim_{p \rightarrow 5} \delta(p) = 0$ . Hence the blow-up rate is self-similar:

$$\|u_x(t)\|_{L^2} \sim \frac{k \|\mathcal{Q}'_p\|_{L^2}}{[(p-5)(T-t)]^{(1-\sigma_c)/3}}, \quad (3.1.16)$$

for  $t$  close to  $T$ .

Moreover, the blow-up set of  $u(t)$  is exactly  $\{x_1, x_2, \dots, x_k\}$ .

*Comments on Theorem 3.1.5:*

1. *Large blow up solutions.* For solutions constructed in Theorem 3.1.5, we know from the proof of Theorem 3.1.5 that  $\|u_0\|_{L^{qc}} \sim k\|\mathcal{Q}_p\|_{L^{qc}}$ . For  $p$  close enough to 5,  $c|\log(p-5)|$  is large, thus we prove the existence of blow up solutions with large initial data (i.e. the critical Lebesgue norm is comparable to  $|\log(p-5)|$ ) for slightly supercritical gKdV equations.

2. *Higher regularity for multiple bubble blow up solutions.* We will see in Section 3.1 that the solutions constructed in Theorem 3.1.5 are actually in  $H^2$ , when  $k \geq 2$ . This condition helps us to control the error term between the blow-up points<sup>3</sup>. But for solution with one bubble, Theorem 3.1.1 implies that  $u(t) \in H^2$  may not hold.

3. *Instability of multiple bubble blow up solutions.* For  $k \geq 2$ , the blow up solution with  $k$  bubbles is *not expected to be stable* in  $H^2$ . Indeed, we need some special conditions (unstable) on the initial data. These conditions are obtained by a standard topological argument. Similar argument is also used in [12, 20, 62, 68, 79] for multiple bubble blow up solutions, [11, 41, 49, 60, 61] for multi-soliton solutions.

4. *Blow up speed.* The blow up solution constructed in Theorem 3.1.5 corresponds to the weak interaction case, i.e. the blow up speed is still self-similar, same as the single bubble case. The existence of blow up solution to supercritical gKdV with blow up rate other than self-similar still remains open.

### 3.1.5 Outline of the proof

The main idea in this paper is to construct a solution which behaves like a decoupled sum of  $k$  self-similar blow-up solutions constructed in Theorem 3.1.1. To do this, we start with a nonempty open subset of initial data  $\mathcal{U}_{k,p} \subset H^2$ , consisting of  $H^2$  functions which can be written as a decoupled sum of bubbles. Then we establish the geometrical decomposition and the modulation estimates for the corresponding solutions just like what we do in [46]. Next we use a topological argument to show that there exists a nonempty subset  $\mathcal{O}_{k,p} \subset \mathcal{U}_{k,p}$ , such that the corresponding solution has exactly  $k$  blow-up points. Here for technical reasons, we have to assume that the distance between the blow-up points is large. Finally, by another topological argument and a standard argument of scaling, we can show that the blow-up points can be chosen arbitrarily. To be specific, we have the following steps:

#### 3.1.5.1 Geometrical decomposition and modulation estimate (Section 2)

We start with initial data which can be written as a decoupled sum of bubbles plus a small error term, i.e.

$$u_0(x) = \sum_{j=1}^k \frac{1}{\lambda_{j,0}^{\frac{2}{p-1}}} Q_{b_{j,0}} \left( \frac{x-x_{j,0}}{\lambda_{j,0}} \right) + \tilde{u}_0(x),$$

---

3. See the proof of (3.3.11).

where  $Q_b$  is the self-similar profile constructed in [46]. Moreover, we assume that

$$\min_{1 \leq i \neq j \leq k} |x_{i,0} - x_{j,0}| \geq b_c^{-100}, \quad (3.1.17)$$

where  $b_c$  is some universal constant with  $b_c \sim p - 5 > 0$ .

We then apply the standard argument of implicit function theory to establish the following geometrical decomposition on some time interval  $[0, T^*)$ :

$$u(t, x) = \sum_{i=1}^k \frac{1}{\lambda_i(t)^{\frac{2}{p-1}}} Q_{b_i(t)} \left( \frac{x - x_i(t)}{\lambda_i(t)} \right) + \tilde{u}(t, x), \quad (3.1.18)$$

with some orthogonality conditions on the error term  $\tilde{u}$ .

Following from similar argumetns as in [46] and [57], we can show the following modulation estimates hold:

$$\left| \frac{1}{\lambda_j} \frac{d\lambda_j}{ds_j} + b_j \right| + \left| \frac{1}{\lambda_j} \frac{dx_j}{ds_j} - 1 \right| \lesssim b_c^2 + \|\varepsilon_j\|_{H_{\text{loc}}^1}, \quad (3.1.19)$$

$$\left| \frac{db_j}{ds_j} + c_p(b_j - b_c)b_c \right| \lesssim b_c^{\frac{5}{2}} + b_c \|\varepsilon_j\|_{H_{\text{loc}}^1}, \quad (3.1.20)$$

$$s_j(t) = \int_0^t \frac{1}{\lambda_j^3(\tau)} d\tau, \quad \varepsilon_j(t, y) = \lambda_j^{\frac{2}{p-1}}(t) \tilde{u}(t, \lambda_j(t)y + x_j(t)),$$

for all  $1 \leq j \leq k$ , as long as the geometrical decomposition (3.1.18) holds.

### 3.1.5.2 Construction of the initial data set (Section 2)

To control the error term  $\varepsilon_j$ , we introduce some a priori estimates<sup>4</sup>. Suppose  $T^*$  is the maximal time for which these estimates hold. We expect that these estimates can be improved on  $[0, T^*)$  so that they actually hold on  $[0, T)$ , where  $T$  is the maximal lifespan. But this argument only works for initial data in an infinite subset  $\mathcal{O}_{k,p} \subset \mathcal{U}_{k,p}$ . Here  $\mathcal{O}_{k,p}$  contains  $H^2$  functions of the following form:

$$u_0 = \frac{1}{\lambda_{1,0}^{\frac{2}{p-1}}} Q_{b_{1,0}} \left( \frac{x - x_{1,0}}{\lambda_{1,0}} \right) + \sum_{j=2}^k \frac{1}{(\lambda_{j,0}^*)^{\frac{2}{p-1}}} Q_{b_{j,0}} \left( \frac{x - x_{j,0}}{\lambda_{j,0}^*} \right) + \tilde{u}_0(x),$$

with

$$\begin{aligned} \lambda_{k,0}^* &= F_k(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}) > 0, \\ \lambda_{k-1,0}^* &= F_{k-1}(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{k,0}^*) > 0, \\ &\vdots \\ \lambda_{2,0}^* &= F_2(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{3,0}^*, \dots, \lambda_{k,0}^*) > 0. \end{aligned}$$

4. See (3.2.34)–(3.2.40) for details.

Here  $\{F_j\}_{j=2}^k$  are some continuous functions constructed by a classic topological argument<sup>5</sup>.

Moreover, for all  $u_0 \in \mathcal{O}_{k,p}$ , we have for all  $1 \leq i, j \leq k, t \in [0, T^*)$

$$\lambda_i(t) \sim \lambda_j(t),$$

which ensures that the solution will blow up in finite time, and has exactly  $k$  blow-up points.

### 3.1.5.3 Estimates on $\varepsilon_j$ by using monotonicity tools (Section 3).

In Section 3, we will derive some crucial control of the error term  $\varepsilon_j$ , for all  $1 \leq j \leq k$ . More precisely, for all<sup>6</sup>  $s_j \in [0, s_j^*)$ ,

$$\int_{\kappa B < y < b_c^{-20}} (\varepsilon_j(s_j))_y^2 \lesssim b_c^{\frac{55}{7}}, \quad (3.1.21)$$

$$\frac{d}{ds_j} \mathcal{F}_j + \frac{\mu}{B} \|\varepsilon_j\|_{H_{loc}^1}^2 \lesssim b_c^{\frac{7}{2}}, \quad (3.1.22)$$

where  $\kappa, \mu > 0$  are some universal constants,  $B = b_c^{-1/20}$  is a large constant and

$$\mathcal{F}_j = \int \left[ (\varepsilon_j)_y^2 \psi_B + \varepsilon_j^2 \zeta_B - \frac{2}{p+1} (|\varepsilon_j + Q_{b_j}|^{p+1} - Q_{b_j}^{p+1} - (p+1)\varepsilon_j Q_{b_j}^p) \psi_B \right],$$

for some well chosen weight function  $(\psi_B, \zeta_B)$ .

The derivation of these estimates follows from almost the same strategy and computation as in [46, Section 4, Section 5], which is developed originally in [55] and [57]. The key observation is that the interaction of the bubbles:

$$\frac{1}{\lambda_j^{\frac{2}{p-1}}(t)} Q_{b_j(t)} \left( \frac{x - x_j(t)}{\lambda_j(t)} \right),$$

is extremely small due to the assumption of (3.1.17). For all  $j = 1, \dots, k$ , we may ignore the bubbles with an index other than  $j$ , due to the choice of the weigh function. Then the estimate of the error term is exactly the same to the single blow-up point case.

There are only two different things. One is that we need the  $H^2$  assumption to estimate  $\varepsilon_j$  on the interval between the blow-up points (clearly, there is no such interval in the single blow-up point case). The other one is that the error term  $\tilde{u}$  behaves like a sum of  $k$  error terms introduced in [46]. So if  $k$  is too large, we cannot obtain the smallness of any global norm<sup>7</sup> of  $\varepsilon_j$ . That's why we need to add a restriction on  $k$ .

5. See Lemma 3.2.11 for more details.

6. See Section 2.3 for the definition of  $s_j^*$ .

7. For example, (3.2.54) and (3.2.55).

### 3.1.5.4 End of the proof of Theorem 3.1.5 (Section 4 and Section 5).

Following from similar argument as in [46, Section 6.1], the modulation estimates (3.1.19), (3.1.20) and the estimates on the error term obtained in Section 3, we can see that for  $u_0 \in \mathcal{O}_{k,p}$ , the corresponding solution blows up in finite time  $T$ . We will also see that the translation parameters  $\{x_1(t), \dots, x_k(t)\}$  converge to  $k$  pairwise distinct points  $\{x_1(T), \dots, x_k(T)\}$  as  $t \rightarrow T$ . Moreover, the blow-up set is exactly  $\{x_1(T), \dots, x_k(T)\}$ .

Hence, we have already constructed solutions blow-up in finite time with exactly  $k$  blow-up points, where the distance between the blow-up points is large<sup>8</sup>. Then we can show that Theorem 3.1.5 follows from Proposition 3.2.16 by standard arguments.

Indeed for  $k$  given pairwise distinct points  $\{x_1, \dots, x_k\}$ , we first assume that the distance between them is large enough, i.e.

$$\min_{1 \leq i \neq j \leq k} |x_i - x_j| \geq b_c^{-120}.$$

Based on the following two facts

1. the blow-up points are continuously depend on the initial data in  $\mathcal{O}_{k,p}$  (due to the continuity of the functions  $F_j$ ,  $j = 2, \dots, k$ );
2. the blow-up points are not too far away from the the translation parameters, i.e.

$$\max_{1 \leq j \leq k} |x_j(0) - x_j(T)| \lesssim b_c^{-1}. \quad (3.1.23)$$

we can construct blow-up solutions whose blow-up set is exactly the set of these  $k$  points by a topological argument<sup>9</sup>(different from the one that is used to construct the set  $\mathcal{O}_{k,p}$ ).

While for arbitrarily given  $k$  pairwise distinct points, from Remark 3.1.3, we may use an argument of scaling to reduce to the case where the distance between the points is large. Thus, we conclude the proof of Theorem 3.1.5.

## 3.1.6 Notations

We first introduce the scaling generator:

$$\Lambda f = \frac{2}{p-1} f + y f'. \quad (3.1.24)$$

We denote the  $L^2$  scalar product by:

$$(f, g) = \int_{\mathbb{R}} f(x) g(x) dx \quad (3.1.25)$$

and observe the integration by parts:

$$(\Lambda f, g) = -(f, \Lambda g + 2\sigma_c g). \quad (3.1.26)$$

8. See (3.2.52) for more details.

9. See Lemma 3.5.1 for more details.

Then we let  $Q_p$  be the ground state. For  $p = 5$ , we simply write  $Q_p$  as  $Q$ . We introduce the linearized operators at  $Q_p$ :

$$Lf = -f'' + f - pQ_p^{p-1}f. \quad (3.1.27)$$

A standard computation leads to:

$$L(Q_p') = 0, \quad L(\Lambda Q_p) = -2Q_p. \quad (3.1.28)$$

We also denote by

$$v = \frac{1}{1000} > 0, \quad (3.1.29)$$

a small universal constant.

Next, we denote by  $A \lesssim B$  ( $A \gtrsim B$ ), if there exists a universal constant<sup>10</sup>  $C > 0$  such that

$$A \leq CB \quad (A \geq \frac{1}{C}B). \quad (3.1.30)$$

Finally, we denote by  $\delta(p)$  a small positive constant such that:

$$\lim_{p \rightarrow 5} \delta(p) = 0. \quad (3.1.31)$$

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## 3.2 Modulation estimate and topological argument

### 3.2.1 Self-similar profile

Let us first recall the properties of the self-similar profile  $Q_b$  constructed in [46]:

**Proposition 3.2.1** (H. Koch, [38]). *There exists  $p^* > 5$ ,  $b^* > 0$  and 2 smooth maps:  $\gamma(b, p) : [0, b^*) \times [5, p^*) \rightarrow \mathbb{R}$ ,  $v(b, p, y) : [0, b^*) \times [5, p^*) \times \mathbb{R} \rightarrow \mathbb{R}$ , such that the following holds:*

1. *The self-similar equation:*

$$b((1 + \gamma(b, p))v + xv') + (v'' - v + v|v|^{p-1})' = 0, \quad (3.2.1)$$

$$(v(b, p, \cdot), Q_p'(\cdot)) = 0, \quad v(b, p, y) > 0. \quad (3.2.2)$$

10. In this paper, “universal constant” means a constant which does not depend on  $p$  and  $k$ .

2. For all  $p \in [5, p^*)$ , there exists a unique  $b = b(p) \in [0, b^*)$  such that:

$$\gamma(b(p), p) = -1 + \frac{2}{p-1}, \quad b(5) = 0, \quad (3.2.3)$$

Moreover,

$$\left. \frac{db(p)}{dp} \right|_{p=5} = \frac{\|Q\|_{L^2}^2}{\|Q\|_{L^1}^2} = \frac{4\pi^2}{\Gamma(1/4)^4} > 0, \quad (3.2.4)$$

$$\left. \frac{\partial \gamma}{\partial b} \right|_{b=b(p)} = -\frac{\|Q_p\|_{L^1}^2}{8\|Q_p\|_{L^2}^2} + O(|p-5|) < 0, \quad (3.2.5)$$

$$\frac{1}{2} \int |v_y(b(p), p, y)|^2 dy - \frac{1}{p+1} \int |v(b(p), p, y)|^{p+1} dy = 0. \quad (3.2.6)$$

3.  $v(b, p, \cdot) \in \dot{H}^1 \cap L^{p+1}$ ,  $v(b, p, \cdot) \notin L^2$  if  $b > 0$  and  $v(0, p, y) = Q_p(y)$ . Moreover, let

$$w_p(b, y) = v(b, p, y) - Q_p(y),$$

then for all  $k, n \in \mathbb{N}$  there holds:

$$|w_p(b, y)| \lesssim \begin{cases} e^{-\frac{1}{3b}}(1+b^{-2/3}|1-by|)^{-1-\gamma} & \text{if } y > b^{-1}, \\ b \exp(\frac{1}{3b}[(1-by)^{3/2}-1]) & \text{if } b^{-1} \geq y > 0, \\ b(1-by)^{-1-\gamma} & \text{if } y \leq 0, \end{cases} \quad (3.2.7)$$

$$|\partial_y^k \partial_b^n v| \lesssim \begin{cases} e^{-\frac{1}{3b}}(1+b^{-2/3}|1-by|)^{-1-\gamma-k} & \text{if } y > b^{-1}, \\ e^{-\frac{y}{10}} & \text{if } 0 \leq y \leq b^{-1}, \\ |\partial_y^k \partial_b^n (b(1-by)^{-1-\gamma})| + e^y & \text{if } y \leq 0. \end{cases} \quad (3.2.8)$$

For  $p^* > p > 5$  fixed, we denote by  $b_c = b(p) \sim p-5$ . We choose a smooth cut-off function  $\chi$ , such that  $\chi(y) = 0$  if  $|y| > 2$ ,  $\chi(y) = 1$  if  $|y| < 1$ . We define the localized profile as follows:

$$Q_b(y) = v(b, p, y)\chi(b_c y).$$

Then  $Q_b$  has the following properties:

**Lemma 3.2.2** (Properties of the localized profile). *Assume that  $b_c$  is small and  $|\tilde{b}| \ll b_c$ , then there holds:*

1. Estimates on  $Q_b$ , for all  $\ell \in \mathbb{N}$ ,  $q \in [1, +\infty]$ :

$$|\partial_y^\ell Q_b(y)| \lesssim_\ell e^{-\frac{y}{10}}, \quad \text{for } y \geq 0, \quad (3.2.9)$$

$$|\partial_y^\ell Q_b(y)| \lesssim_\ell e^y + b_c^{1+k} \mathbf{1}_{[-2b_c^{-1}, 0]}(y), \quad \text{for } y \leq 0, \quad (3.2.10)$$

$$\|Q_b - Q_p\|_{L^q} \lesssim b_c^{1-\frac{1}{q}}, \quad \|(Q_b - Q_p)_y\|_{L^2} \lesssim b_c. \quad (3.2.11)$$

Here  $\mathbf{1}_I$  is the characteristic function of any interval  $I$ .



2.  $Q_b$  is an approximate self-similar profile: Let

$$-\Phi_b = b\Lambda Q_b + (Q_b'' - Q_b + Q_b|Q_b|^{p-1})', \quad (3.2.12)$$

then for  $\ell = 0, 1$ :

$$\partial_y^\ell \Phi_b = C_p \tilde{b} b_c \partial_y^\ell Q_b + O(|\tilde{b}|^2 \partial_y^\ell Q_b + b_c^2 \mathbf{1}_{[-2, -1]}(b_c y) + e^{-\frac{1}{10b_c}} \mathbf{1}_{[1, 2]}(b_c y)), \quad (3.2.13)$$

where  $C_p = \frac{dy}{db}|_{b=b_c} < 0$ .

3. Energy property of  $Q_b$ :

$$|E(Q_b)| \lesssim b_c^3 + |\tilde{b}|. \quad (3.2.14)$$

4. Properties of the first order term with respect to  $b$ : let  $P_b(y) = \frac{\partial Q_b}{\partial b}(y)$ , then

$$|P_b(y)| \lesssim e^{-\frac{y}{10}} \mathbf{1}_{\{y>0\}}(y) + \mathbf{1}_{[-2b_c^{-1}, 0]}(y). \quad (3.2.15)$$

Furthermore, we have:

$$(P_b, Q_p) = \frac{1}{16} \left( \int Q_p \right)^2 + O(|p-5|) > 0. \quad (3.2.16)$$

### 3.2.2 Geometrical decomposition

We first give definition of the open subset  $\mathcal{U}_{k,p}$  such that the corresponding solution has at least one blow-up point.

**Definition 3.2.3.** Let  $p^* > 5$  and close enough to 5, for all  $p \in (5, p^*)$  we define  $\mathcal{U}_{k,p}$  as the set of all  $u_0$  satisfying the following conditions:

1. Geometrical decomposition:

$$u_0(x) = \sum_{i=1}^k \frac{1}{\lambda_{i,0}^{\frac{2}{p-1}}} Q_{b_{i,0}} \left( \frac{x - x_{i,0}}{\lambda_{i,0}} \right) + \tilde{u}_0(x). \quad (3.2.17)$$

2.  $b_{j,0}$  is near  $b_c$ :

$$|b_{j,0} - b_c| \leq b_c^5. \quad (3.2.18)$$

3. Conditions on the scaling parameters:

$$0 < \lambda_{j,0} < 1, \quad (3.2.19)$$

$$\frac{1}{2^k} < \frac{\lambda_{i,0}}{\lambda_{j,0}} < 2^k, \quad \text{for all } i \neq j. \quad (3.2.20)$$

4. Distance between the blow-up points:

$$|x_{i,0} - x_{j,0}| > b_c^{-100}, \quad \text{for all } i \neq j. \quad (3.2.21)$$

5.  $H^2$  smallness on the error term:

$$\|\tilde{u}_0\|_{H^2} < b_c^{50} \quad (3.2.22)$$

*Remark 3.2.4.* It is easy to check that  $\mathcal{U}_{k,p}$  is nonempty and open.

*Remark 3.2.5.* We will see in the next section why the assumption  $\mathcal{U}_{k,p} \subset H^2$  is necessary for  $k \geq 2$ .

Now we can introduce the classical geometrical decomposition. From a standard inverse function argument<sup>11</sup>, we know that for all  $u_0 \in \mathcal{U}_{k,p}$ , there exist a  $t^* > 0$  and geometrical parameters  $\lambda_i(t)$ ,  $b_i(t)$ ,  $x_i(t)$ , such that the corresponding solution  $u(t)$  satisfies the following for all  $t \in [0, t^*)$ :

1. Geometrical decomposition:

$$u(t, x) = \sum_{i=1}^k \frac{1}{\lambda_i(t)^{\frac{2}{p-1}}} Q_{b_i(t)} \left( \frac{x - x_i(t)}{\lambda_i(t)} \right) + \tilde{u}(t, x). \quad (3.2.23)$$

2. Orthogonality condition:

$$(\varepsilon_j(t), Q_p) = (\varepsilon_j(t), \Lambda Q_p) = (\varepsilon_j(t), y \Lambda Q_p) = 0, \text{ for all } j = 1, \dots, k, \quad (3.2.24)$$

where  $Q_p$  is the ground state and

$$\varepsilon_j(t, y) = \lambda_j(t)^{\frac{2}{p-1}} \tilde{u}(t, \lambda_j(t)y + x_j(t)).$$

3. Estimates on the parameters at the initial time: for all  $i \neq j$ ,

$$0 < \lambda_j(0) < 2, \quad (3.2.25)$$

$$\frac{2\lambda_{i,0}}{3\lambda_{j,0}} < \frac{\lambda_i(0)}{\lambda_j(0)} < \frac{3\lambda_{i,0}}{2\lambda_{j,0}}, \quad (3.2.26)$$

$$|b_j(0) - b_c| \leq b_c^4, \quad (3.2.27)$$

$$|x_i(0) - x_j(0)| \geq b_c^{-80}, \quad (3.2.28)$$

$$\|\varepsilon_j(0)\|_{L^{(p-1)/2}} + \|(\varepsilon_j(0))_y\|_{H^1} \leq b_c^{30} \quad (3.2.29)$$

4. Continuity of the parameters: Consider  $u_{0,n} \in \mathcal{U}_{k,p}$ ,  $u_{0,n} \rightarrow u_0$  in  $H^1$ . Let  $u_n(t)$  be the solution of (3.1.1) with initial data  $u_{0,n}$  and  $\lambda_{j,n}(t)$ ,  $b_{j,n}(t)$ ,  $x_{j,n}(t)$ ,  $\tilde{u}_n(t, x)$  be the corresponding geometrical parameters and error terms. Suppose there exists a  $t_1^* > 0$  such that the geometrical decomposition for all  $u_n(t)$  and  $u(t)$  hold on  $[0, t_1^*)$ , then for all  $0 \leq t < t_1^*$ , we have:

$$\lim_{n \rightarrow +\infty} (\lambda_{j,n}(t), b_{j,n}(t), x_{j,n}(t), \tilde{u}_n(t)) = (\lambda_j(t), b_j(t), x_j(t), \tilde{u}(t)). \quad (3.2.30)$$

Next, we want to define the localized  $H^1$  norm of  $\varepsilon_j$ . We first denote:

$$B = b_c^{-\frac{1}{20}}, \quad (3.2.31)$$

11. See Lemma 1 in [53] and Lemma 2.5 in [57].

and choose a smooth weight function  $\varphi$  such that:

$$\varphi(y) = \begin{cases} e^y & \text{if } y < -1, \\ +y & \text{if } -\kappa < y < \kappa, \\ 3 & \text{if } y > 1. \end{cases} \quad (3.2.32)$$

$$\varphi' \geq 0 \text{ for all } y \in \mathbb{R},$$

where  $\kappa$  is a small universal positive constant which will be chosen later.

Then we define the following localized  $H^1$  norm:

$$\mathcal{N}_j(t) = \int ((\varepsilon_j)_y^2 + \varepsilon_j^2) \varphi'_B, \quad (3.2.33)$$

where  $\varphi_B(y) = \varphi(y/B)$ .

Let us consider the maximal time  $T^*$  such that the geometrical decomposition (3.2.23), orthogonality condition (3.2.24) and the following *a priori* estimates hold in  $[0, T^*)$ :

$$0 < \lambda_j(t) < 3, \quad (3.2.34)$$

$$\frac{1}{2^{k+2}} < \frac{\lambda_i(t)}{\lambda_j(t)} < 2^{k+2}, \quad (3.2.35)$$

$$|b_j(t) - b_c| \leq b_c^{\frac{3}{2}+\nu}, \quad (3.2.36)$$

$$|x_i(t) - x_j(t)| \geq b_c^{-70}, \text{ for all } i \neq j, \quad (3.2.37)$$

$$\|\varepsilon_j(t)\|_{L^{p_0}} \leq b_c^{\frac{23}{50}}, \quad (3.2.38)$$

$$\|(\varepsilon_j(t))_y\|_{L^2} \leq b_c^{\frac{2}{3}}, \quad (3.2.39)$$

$$\mathcal{N}_j(t) \leq b_c^{3+6\nu}, \quad (3.2.40)$$

where  $\nu > 0$  is a small universal constant to be chosen later, and

$$p_0 = \frac{5}{2}.$$

*Remark 3.2.6.* It is easy to see from (3.2.25)–(3.2.29) and the continuity of the flow that  $T^* > 0$ .

*Remark 3.2.7.* Our goal is to improve these estimates in  $[0, T^*)$ . Then from a standard bootstrap argument, these estimates actually hold on  $[0, T)$ , where  $T$  is the maximal life span. Indeed, following from similar argument as in [46], we can improve (3.2.34) and (3.2.36)–(3.2.40). But to improve the bound (3.2.35), we need to assume that  $u_0 \in \mathcal{O}_{k,p} \subset \mathcal{U}_{k,p}$ , where  $\mathcal{O}_{k,p}$  is an infinite subset of  $\mathcal{U}_{k,p}$ . This subset can be constructed by a topological argument.

*Remark 3.2.8.* From (3.2.38), (3.2.39) and Gagliardo-Nirenberg's inequality, we have for all  $q_0 \geq p_0$ ,  $t \in [0, T^*)$ ,

$$\|\varepsilon_j(t)\|_{L^{q_0}} \leq b_c^{\frac{149q_0-62}{270q_0}}. \quad (3.2.41)$$

### 3.2.3 Modulation estimate

In this subsection, we will prove the modulation estimates for the geometrical parameters on  $[0, T^*)$  by using the *a priori* estimates (3.2.34)–(3.2.40). We first introduce a rescaled coordinate  $(s_j, y)$  for all  $j = 1, \dots, k$ :

$$s_j = \int_0^t \frac{1}{\lambda_j(\tau)^3} d\tau, \quad y = \frac{x - x_j(t)}{\lambda_j(t)}.$$

Let

$$s_j^* = \int_0^{T^*} \frac{1}{\lambda_j(\tau)^3} d\tau.$$

Now we can state the modulation estimates:

**Proposition 3.2.9.** *For all  $j = 1, \dots, k$ , the following properties hold for all  $s_j \in [0, s_j^*)$ :*

1. *Equation of  $\varepsilon_j$ :*

$$\begin{aligned} \frac{d\varepsilon_j}{ds_j} &= (L\varepsilon_j)_y - b_j \Lambda \varepsilon_j + \left( \frac{1}{\lambda_j} \frac{d\lambda_j}{ds_j} + b_j \right) (\Lambda Q_{b_j} + \Lambda \varepsilon_j) + \left( \frac{1}{\lambda_j} \frac{dx_j}{ds_j} - 1 \right) (Q_{b_j} + \varepsilon_j)_y \\ &+ \Phi_{b_j} - \frac{db_j}{ds_j} P_{b_j} - (R_{b_j}(\varepsilon_j))_y - (R_{NL}(\varepsilon_j))_y \\ &+ \sum_{i=1, i \neq j}^k \left( \frac{\lambda_j}{\lambda_i} \right)^{\frac{3p-1}{p-1}} \left[ \Phi_{b_i} \left( \frac{\lambda_j \cdot + x_j - x_i}{\lambda_i} \right) + \frac{db_i}{ds_i} P_{b_i} \left( \frac{\lambda_j \cdot + x_j - x_i}{\lambda_i} \right) \right] \\ &+ \sum_{i=1, i \neq j}^k \left( \frac{\lambda_j}{\lambda_i} \right)^{\frac{3p-1}{p-1}} \left[ \left( \frac{1}{\lambda_i} \frac{d\lambda_i}{ds_i} + b_i \right) \Lambda Q_{b_i} + \left( \frac{1}{\lambda_i} \frac{dx_i}{ds_i} - 1 \right) (Q_{b_i})_y \right] \left( \frac{\lambda_j \cdot + x_j - x_i}{\lambda_i} \right) \\ &- p \sum_{i=1, i \neq j}^k \left( \frac{\lambda_j}{\lambda_i} \right)^{2+\frac{2}{p-1}} \left[ \varepsilon_j Q_{b_i}^{p-1} \left( \frac{\lambda_j \cdot + x_j - x_i}{\lambda_i} \right) \right]_y, \end{aligned} \tag{3.2.42}$$

where

$$\begin{aligned} \Phi_{b_i} &= -b_i \Lambda Q_{b_i} - (Q_{b_i}'' - Q_{b_i} + Q_{b_i}^p)', \\ R_{b_j}(\varepsilon_j) &= p(Q_{b_j}^{p-1} - Q_p^{p-1})\varepsilon_j, \\ R_{NL}(\varepsilon_j) &= \left[ \varepsilon_j + \sum_{i=1}^k \left( \frac{\lambda_j}{\lambda_i} \right)^{\frac{2}{p-1}} Q_{b_i} \left( \frac{\lambda_j \cdot + x_j - x_i}{\lambda_i} \right) \right]^p \\ &\quad - p \sum_{i=1}^k \left( \frac{\lambda_j}{\lambda_i} \right)^{2+\frac{2}{p-1}} \varepsilon_j Q_{b_i}^{p-1} \left( \frac{\lambda_j \cdot + x_j - x_i}{\lambda_i} \right) \\ &\quad - \sum_{i=1}^k \left( \frac{\lambda_j}{\lambda_i} \right)^{2+\frac{2}{p-1}} Q_{b_i}^p \left( \frac{\lambda_j \cdot + x_j - x_i}{\lambda_i} \right). \end{aligned}$$

2. Modulation estimates:

$$\left| \frac{1}{\lambda_j} \frac{d\lambda_j}{ds_j} + b_j \right| \lesssim b_c^2 + \mathcal{N}_j^{\frac{1}{2}}, \quad (3.2.43)$$

$$\left| \frac{1}{\lambda_j} \frac{dx_j}{ds_j} - 1 \right| \lesssim b_c^2 + \mathcal{N}_j^{\frac{1}{2}}, \quad (3.2.44)$$

$$\left| \frac{db_j}{ds_j} + c_p \tilde{b}_j b_c \right| \lesssim b_c^3 + b_c \mathcal{N}_j^{\frac{1}{2}}, \quad (3.2.45)$$

where  $c_p = 2 + O(|p-5|) > 0$ .

*Proof.* The proof of this proposition is almost the same as Proposition 3.1 in [46]. The only difference here is that we need to deal with some terms like

$$Q_{b_i} \left( \frac{\lambda_j \cdot + x_j - x_i}{\lambda_i} \right), P_{b_i} \left( \frac{\lambda_j \cdot + x_j - x_i}{\lambda_i} \right) \text{ or } \Phi_{b_i} \left( \frac{\lambda_j \cdot + x_j - x_i}{\lambda_i} \right), \quad (3.2.46)$$

for  $i \neq j$ .

We consider for example the following term

$$-p \sum_{i=1, i \neq j}^k \left( \frac{\lambda_j}{\lambda_i} \right)^{2 + \frac{2}{p-1}} \left[ \varepsilon_j Q_{b_i}^{p-1} \left( \frac{\lambda_j \cdot + x_j - x_i}{\lambda_i} \right) \right]_y. \quad (3.2.47)$$

Since  $Q_{b_i}$  is supported in  $[-2b_c^{-1}, 2b_c^{-1}]$ , if  $y$  belongs to the support of

$$Q_{b_i}^{p-1} \left( \frac{\lambda_j \cdot + x_j - x_i}{\lambda_i} \right),$$

then we have

$$\frac{x_i - x_j}{\lambda_j} - 2b_c^{-1} \frac{\lambda_i}{\lambda_j} < y < \frac{x_i - x_j}{\lambda_j} + 2b_c^{-1} \frac{\lambda_i}{\lambda_j}.$$

From (3.2.35) and (3.2.37), we know that if  $i \neq j$ ,

$$\left| \frac{x_i - x_j}{\lambda_j} \pm 2b_c^{-1} \frac{\lambda_i}{\lambda_j} \right| \geq b_c^{-70} - 2b_c^{-1} 2^{k+2} > b_c^{-10},$$

provided that  $10^k \leq b_c^{-\nu/4}$  for some small universal constant  $\nu > 0$ . Since

$$b_c \sim p - 5,$$

this is implied by the condition

$$k \leq c |\log(p-5)|,$$

if we choose

$$c = \frac{\nu}{8 \log 10} = \frac{1}{8000 \log 10} > 0.$$

Since we are considering a scalar product of (3.2.47) and some functions with exponential decay (i.e.  $Q_p, \Lambda Q_p, y \Lambda Q_p$ ), these terms can be controlled by

$$\sum_{i=1, i \neq j}^k 10^k \exp\left(-\frac{1}{10} b_c^{-10}\right) \leq b_c^{10},$$

provided that  $10^k \leq b_c^{-\nu/4}$ . Then we conclude the proof.  $\square$

### 3.2.4 First topological argument

In this subsection we will find a nonempty subset  $\mathcal{O}_{k,p} \subset \mathcal{U}_{k,p}$ , such that the corresponding scaling parameters (i.e.  $\lambda_j(t)$ ) are comparable to each other.

**Proposition 3.2.10.** *There exists a nonempty subset  $\mathcal{O}_{k,p} \subset \mathcal{U}_{k,p}$ , which contains infinite many elements, such that for all solution  $u(t)$  with initial data in  $\mathcal{O}_{k,p}$ , the corresponding scaling parameters  $\lambda_j(t)$  satisfy:*

$$\frac{1}{2^{k+1}} \leq \frac{\lambda_i(t)}{\lambda_j(t)} \leq 2^{k+1}, \text{ for all } t \in [0, T^*) \text{ and } 1 \leq i, j \leq k. \quad (3.2.48)$$

*Proof.* We first claim the following lemma:

**Lemma 3.2.11.** *For all  $t_0 \in [0, T^*)$ ,  $1 \leq i \neq j \leq k$ , if  $\lambda_i(t_0)/\lambda_j(t_0) \geq \frac{10}{9}$ , then for all  $t \in [t_0, T^*)$ , we have:*

$$\frac{\lambda_i(t)}{\lambda_j(t)} \geq \frac{10}{9}. \quad (3.2.49)$$

*Proof of Lemma 3.2.11.* The proof of (3.2.49) is a consequence of the a priori assumption (3.2.34)–(3.2.40) and the modulation estimates. Indeed, from (3.2.36), (3.2.40) and (3.2.43), we have:

$$-\frac{99}{100}b_c > (\lambda_j)_t \lambda_j^2 > -\frac{101}{100}b_c.$$

Then we can compute the derivative of  $\lambda_i/\lambda_j$  with respect to  $t$ :

$$\begin{aligned} \frac{d}{dt} \left( \frac{\lambda_i}{\lambda_j} \right) &= \frac{(\lambda_i)_t \lambda_j - (\lambda_j)_t \lambda_i}{\lambda_j^2} \\ &< \frac{1}{\lambda_j^2} \left( -\frac{99b_c \lambda_j}{100\lambda_i^2} + \frac{101b_c \lambda_i}{100\lambda_j^2} \right) \\ &= \frac{b_c}{\lambda_j \lambda_i^2} \left[ -\frac{99}{100} + \frac{101}{100} \left( \frac{\lambda_i}{\lambda_j} \right)^3 \right]. \end{aligned}$$

Similarly, we have:

$$\frac{d}{dt} \left( \frac{\lambda_i}{\lambda_j} \right) > \frac{b_c}{\lambda_j \lambda_i^2} \left[ -\frac{101}{100} + \frac{99}{100} \left( \frac{\lambda_i}{\lambda_j} \right)^3 \right].$$

The above two inequalities show that if for some time  $t_0 \in [0, T^*)$ , we have

$$\frac{\lambda_i(t_0)}{\lambda_j(t_0)} \geq \frac{10}{9},$$

then we have:

$$\left. \frac{d}{dt} \left( \frac{\lambda_i}{\lambda_j} \right) \right|_{t=t_0} > 0.$$

Hence, the lemma follows from a simple argument.  $\square$

For convenience we introduce the following notations:

1.  $\vec{\lambda}_0 = (\lambda_{1,0}, \dots, \lambda_{k,0})$ ,  $\vec{b}_0 = (b_{1,0}, \dots, b_{k,0})$ ,  $\vec{x}_0 = (x_{1,0}, \dots, x_{k,0})$ , and

$$F(\vec{\lambda}_0, \vec{b}_0, \vec{x}_0, \tilde{u}_0) = \sum_{i=1}^k \frac{1}{\lambda_{i,0}^{\frac{2}{p-1}}} Q_{b_{i,0}} \left( \frac{x - x_{i,0}}{\lambda_{i,0}} \right) + \tilde{u}_0(x),$$

2. Let  $\mathcal{C}$  be the set of  $(\vec{\lambda}_0, \vec{b}_0, \vec{x}_0, \tilde{u}_0)$  such that (3.2.18)–(3.2.22) hold (or equivalently  $F(\vec{\lambda}_0, \vec{b}_0, \vec{x}_0, \tilde{u}_0) \in \mathcal{U}_{k,p}$ ).
3. For  $\ell = 1, \dots, k$ , we let  $\mathcal{C}_\ell$  be the set of  $(\vec{\lambda}_0, \vec{b}_0, \vec{x}_0, \tilde{u}_0)$  such that (3.2.18)–(3.2.22) hold with (3.2.20) replaced by

$$\frac{1}{2^{k+1-\ell}} < \frac{\lambda_{i,0}}{\lambda_{j,0}} < 2^{k+1-\ell}.$$

Clearly, we have  $\mathcal{C}_k \subset \mathcal{C}_{k-1} \subset \dots \subset \mathcal{C}_1 = \mathcal{C}$ . Proposition 3.2.10 is a simple consequence of the following lemma

**Lemma 3.2.12.** *For all  $2 \leq \ell \leq k$  there exist continuous functions  $F_\ell$ :*

$$\begin{aligned} & \mathbb{R}_+^k \times \mathbb{R}^k \times H^2 \times \mathbb{R}_+^{k-\ell+1} \rightarrow \mathbb{R}_+, \\ & (\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{\ell+1,0}, \dots, \lambda_{k,0}) \mapsto F_\ell(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{\ell+1,0}, \dots, \lambda_{k,0}), \end{aligned}$$

such that for all  $2 \leq j \leq k$  and  $(\lambda_{1,0}, \lambda_{j+1,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0)$ , if there exist  $\lambda_{2,0}, \dots, \lambda_{j,0} > 0$  such that

$$(\lambda_{1,0}, \lambda_{2,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0) \in \mathcal{C}_j,$$

then the following holds:

1.  $(\lambda_{1,0}, \lambda_{2,0}^*, \dots, \lambda_{j,0}^*, \lambda_{j+1,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0) \in \mathcal{C}_1$ , where

$$\begin{aligned} \lambda_{j,0}^* &= F_j(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{j+1,0}, \dots, \lambda_{k,0}) > 0, \\ \lambda_{j-1,0}^* &= F_{j-1}(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{j,0}^*, \lambda_{j+1,0}, \dots, \lambda_{k,0}) > 0, \\ & \vdots \\ \lambda_{2,0}^* &= F_2(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{3,0}^*, \dots, \lambda_{j,0}^*, \lambda_{j+1,0}, \dots, \lambda_{k,0}) > 0. \end{aligned}$$

2. Let  $u(t)$  be the solution of (3.1.1) with initial data

$$u_0 = F(\lambda_{1,0}, \lambda_{2,0}^*, \dots, \lambda_{j,0}^*, \lambda_{j+1,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0),$$

and  $\{\lambda_i(t)\}_{i=1}^k$  be the corresponding scaling parameters, then for all  $t \in [0, T^*)$ ,  $1 \leq i_1, i_2 \leq j$ , we have

$$\frac{1}{2^{k+1}} \leq \frac{\lambda_{i_1}(t)}{\lambda_{i_2}(t)} \leq 2^{k+1}.$$

*Proof.* We will prove Lemma 3.2.12 by induction on  $j$ . We first prove Lemma 3.2.12 for  $j = 2$ .

Consider  $(\lambda_{1,0}, \lambda_{3,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \vec{u}_0)$  such that there exists  $\lambda_{2,0} > 0$  such that

$$(\lambda_{1,0}, \lambda_{2,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \vec{u}_0) \in \mathcal{C}_2.$$

We denote by  $S_2$ , the set of all  $\lambda_{2,0}$  such that  $(\vec{\lambda}_0, \vec{b}_0, \vec{x}_0, \vec{u}_0) \in \mathcal{C}_1$ . Clearly  $S_2$  is a nonempty interval. Next we define the following sets:

$$\begin{aligned} S_2^< &= \{\lambda_{2,0} \in S_2 \mid \text{The solution } u(t) \text{ with initial data } F(\vec{\lambda}_0, \vec{b}_0, \vec{x}_0, \vec{u}_0), \\ &\quad \text{satisfies } \lambda_2(t_0)/\lambda_1(t_0) < \frac{1}{2^{k+1}}, \text{ for some } t_0 \in [0, T^*).\}, \\ S_2^> &= \{\lambda_{2,0} \in S_2 \mid \text{The solution } u(t) \text{ with initial data } F(\vec{\lambda}_0, \vec{b}_0, \vec{x}_0, \vec{u}_0), \\ &\quad \text{satisfies } \lambda_2(t_0)/\lambda_1(t_0) > 2^{k+1}, \text{ for some } t_0 \in [0, T^*).\}. \end{aligned}$$

For these two sets, we have the following observations:

1.  $S_2^<$  and  $S_2^>$  are both contained in  $S_2$  and open. Here the openness comes from (3.2.30).
2.  $S_2^< \cap S_2^>$  is empty. This is a direct corollary of Lemma 3.2.11.
3.  $S_2^< \neq S_2$ ,  $S_2^> \neq S_2$ . Since if  $S_2^< = S_2$ , then from the definition of  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , we may always find some  $\lambda_{2,0} \in S_2$  such that

$$\frac{\lambda_{2,0}}{\lambda_{1,0}} > 2.$$

Then from (3.2.26), we have

$$\frac{\lambda_2(0)}{\lambda_1(0)} > \frac{2}{3} \times \frac{\lambda_{2,0}}{\lambda_{1,0}} > \frac{10}{9}.$$

then from Lemma 3.2.11, we have for all  $t \in [0, T^*)$ ,

$$\frac{\lambda_2(t)}{\lambda_1(t)} > \frac{10}{9},$$

which leads to a contradiction. Similarly, we have  $S_2^> \neq S_2$ .

Since  $S_2$  is a nonempty interval (i.e. connected), the above observations imply that  $S_2 / (S_2^< \cup S_2^>)$  is not empty. On the other hand, it is easy to check that if  $\delta > 0$  is small enough, then  $(\inf S_2 + \delta) \in S_2^<$ . So we have  $\inf S_2 / (S_2^< \cup S_2^>) \in S_2 / (S_2^< \cup S_2^>)$ . We then choose

$$F_2(\vec{b}_0, \vec{x}_0, \vec{u}_0, \lambda_{1,0}, \lambda_{3,0}, \dots, \lambda_{k,0}) = \inf S_2 / (S_2^< \cup S_2^>).$$

From Lemma 3.2.11 and (3.2.30), we know that  $F_2$  is continuous.

Next (if  $k \geq 3$ ), suppose for all  $2 \leq j \leq j_0 - 1$  ( $3 \leq j_0 \leq k$ ), Lemma 3.2.12 holds, i.e. there exists a continuous function  $F_2, \dots, F_j$ , such that for all

$$(\lambda_{1,0}, \lambda_{j+1,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \vec{u}_0),$$



if there exist  $\lambda_{2,0}, \dots, \lambda_{j,0} > 0$  such that

$$(\lambda_{1,0}, \lambda_{2,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0) \in \mathcal{C}_j,$$

then the following holds:

1.  $(\lambda_{1,0}, \lambda_{2,0}^*, \dots, \lambda_{j,0}^*, \lambda_{j+1,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0) \in \mathcal{C}_1$ , where

$$\lambda_{j,0}^* = F_j(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{j+1,0}, \dots, \lambda_{k,0}) > 0,$$

$$\lambda_{j-1,0}^* = F_{j-1}(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{j,0}^*, \lambda_{j+1,0}, \dots, \lambda_{k,0}) > 0,$$

$\vdots$

$$\lambda_{2,0}^* = F_2(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{3,0}^*, \dots, \lambda_{j,0}^*, \lambda_{j+1,0}, \dots, \lambda_{k,0}) > 0.$$

2. Let  $u(t)$  be the solution of (3.1.1) with initial data

$$u_0 = F(\lambda_{1,0}, \lambda_{2,0}^*, \dots, \lambda_{j,0}^*, \lambda_{j+1,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0),$$

and  $\{\lambda_i(t)\}_{i=1}^k$  be the corresponding scaling parameters, then for all  $t \in [0, T^*)$ ,  $1 \leq i_1, i_2 \leq j$ , we have

$$\frac{1}{2^{k+1}} \leq \frac{\lambda_{i_1}(t)}{\lambda_{i_2}(t)} \leq 2^{k+1}.$$

Now for  $j = j_0$ , we consider all  $(\lambda_{1,0}, \lambda_{j_0+1,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0)$ , such that there exist  $\lambda_{2,0}, \dots, \lambda_{j_0,0} > 0$ , such that

$$(\lambda_{1,0}, \lambda_{2,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0) \in \mathcal{C}_{j_0}.$$

We similarly denote by  $S_{j_0}$  the set of all  $\lambda_{j_0,0}$  such that there exist  $\lambda_{2,0}, \dots, \lambda_{j_0-1,0} > 0$  such that

$$(\lambda_{1,0}, \lambda_{2,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0) \in \mathcal{C}_{j_0-1}.$$

It is easy to see from the definition of  $\mathcal{C}_{j_0-1}$  and  $\mathcal{C}_{j_0}$ , that  $S_{j_0}$  is an interval and not empty. Moreover, from the induction hypothesis, for all  $\lambda_{j_0,0} \in S_{j_0}$ , we have:

$$(\lambda_{1,0}, \lambda_{2,0}^*, \dots, \lambda_{j_0-1,0}^*, \lambda_{j_0,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0) \in \mathcal{C}_1,$$

where

$$\lambda_{j_0-1,0}^* = F_{j_0-1}(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{j_0,0}, \dots, \lambda_{k,0}) > 0,$$

$$\lambda_{j_0-2,0}^* = F_{j_0-2}(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{j_0-1,0}^*, \lambda_{j_0,0}, \dots, \lambda_{k,0}) > 0,$$

$\vdots$

$$\lambda_{2,0}^* = F_2(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{3,0}^*, \dots, \lambda_{j_0-1,0}^*, \lambda_{j_0,0}, \dots, \lambda_{k,0}) > 0.$$

Next we define  $S_{j_0}^<$  be the set of all  $\lambda_{j_0,0} \in S_{j_0}$  such that the solution  $u(t)$  with initial data

$$u_0 = F(\lambda_{1,0}, \lambda_{2,0}^*, \dots, \lambda_{j_0-1,0}^*, \lambda_{j_0,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0),$$

satisfies  $\lambda_{j_0}(t_0)/\lambda_{i_0}(t_0) < 1/2^{k+1}$ , for some  $t_0 \in [0, T^*)$  and some  $i_0 \in \{1, \dots, j_0 - 1\}$ .

Similarly, we define  $S_{j_0}^>$  be the set of all  $\lambda_{j_0,0} \in S_{j_0}$  such that the solution  $u(t)$  with initial data

$$u_0 = F(\lambda_{1,0}, \lambda_{2,0}^*, \dots, \lambda_{j_0-1,0}^*, \lambda_{j_0,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0),$$

satisfies  $\lambda_{j_0}(t_0)/\lambda_{i_0}(t_0) > 2^{k+1}$ , for some  $t_0 \in [0, T^*)$  and some  $i_0 \in \{1, \dots, j_0 - 1\}$ .

We have the same observations:

1.  $S_{j_0}^<$  and  $S_{j_0}^>$  are both contained in  $S_{j_0}$  and open.
2.  $S_{j_0}^< \cap S_{j_0}^>$  is empty. Otherwise, there exist  $\lambda_{j_0,0} \in S_{j_0}$ ,  $i_1, i_2 \in \{1, \dots, j_0 - 1\}$   $t_0 \in [0, T^*)$  such that  $\lambda_{j_0}(t_0)/\lambda_{i_1}(t_0) > 2^{k+1}$ ,  $\lambda_{j_0}(t_0)/\lambda_{i_2}(t_0) < 1/2^{k+1}$ . Then we have  $\lambda_{i_1}(t_0)/\lambda_{i_2}(t_0) < 1/2^{2k+2}$ , which is a contradiction due to the choice of  $\lambda_{2,0}^*, \dots, \lambda_{j_0-1,0}^*$ .
3.  $S_{j_0}^< \neq S_{j_0}$ ,  $S_{j_0}^> \neq S_{j_0}$ . Suppose we have  $S_{j_0}^< = S_{j_0}$ . From our induction hypothesis, we know for all  $i_1, i_2 \in \{1, j_0 + 1, \dots, k\}$ ,

$$\frac{1}{2^{k+1-j_0}} < \frac{\lambda_{i_1,0}}{\lambda_{i_2,0}} < 2^{k+1-j_0}.$$

Choose  $\lambda_{j_0,0} > 0$ , such that

$$\lambda_{j_0,0} = (2^{k+2-j_0} - \delta)\lambda_{i_0,0},$$

where  $i_0 \in \{1, j_0 + 1, \dots, k\}$ ,  $\lambda_{i_0,0} = \min_{i \in \{1, j_0 + 1, \dots, k\}} \lambda_{i,0}$ , and  $\delta > 0$  is a small enough constant. Then for all  $i_1, i_2 \in \{1, j_0, \dots, k\}$ , we have

$$\frac{1}{2^{k+1-(j_0-1)}} < \frac{\lambda_{i_1,0}}{\lambda_{i_2,0}} < 2^{k+1-(j_0-1)}.$$

So there exist  $\lambda_{2,0}, \dots, \lambda_{j_0-1,0}$  such that

$$(\lambda_{1,0}, \lambda_{2,0}, \dots, \lambda_{j_0-1,0}, \lambda_{j_0,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0) \in \mathcal{C}_{j_0-1},$$

or equivalently  $\lambda_{j_0,0} \in S_{j_0}(= S_{j_0}^<)$ . From our induction hypothesis, we know that

$$(\lambda_{1,0}, \lambda_{2,0}^*, \dots, \lambda_{j_0-1,0}^*, \lambda_{j_0,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0) \in \mathcal{C}_1,$$

where

$$\begin{aligned} \lambda_{j_0-1,0}^* &= F_{j_0-1}(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{j_0,0}, \dots, \lambda_{k,0}) > 0, \\ \lambda_{j_0-2,0}^* &= F_{j_0-2}(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{j_0-1,0}^*, \lambda_{j_0,0}, \dots, \lambda_{k,0}) > 0, \\ &\vdots \\ \lambda_{2,0}^* &= F_2(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{3,0}^*, \dots, \lambda_{j_0-1,0}^*, \lambda_{j_0,0}, \dots, \lambda_{k,0}) > 0. \end{aligned}$$

But on the other hand, we have:

$$\frac{\lambda_{j_0}}{\lambda_{1,0}} = \frac{\lambda_{j_0,0}}{\lambda_{i_0,0}} \times \frac{\lambda_{i_0,0}}{\lambda_{1,0}} > (2^{k+1-(j_0-1)} - \delta)2^{-k-1+j_0} > \frac{9}{5},$$

if  $\delta$  is small enough. From (3.2.26), we know that

$$\frac{\lambda_{j_0}(0)}{\lambda_1(0)} > \frac{2}{3} \times \frac{\lambda_{j_0,0}}{\lambda_{1,0}} > \frac{6}{5} > \frac{10}{9},$$

where  $\{\lambda_\ell(t)\}_{\ell=1}^k$  are the scaling parameters of solution  $u(t)$  with initial data

$$u_0 = F(\lambda_{1,0}, \lambda_{2,0}^*, \dots, \lambda_{j_0-1,0}^*, \lambda_{j_0,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0).$$

By Lemma 3.2.11, we reach a contradiction. The proof of  $S_{j_0}^> \neq S_{j_0}$  is the same.

Therefore,  $S_{j_0}/(S_{j_0}^< \cup S_{j_0}^>)$  is not empty. Then we only need to choose

$$F_{j_0}(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{j_0+1,0}, \dots, \lambda_{k,0}) = \inf S_{j_0}/(S_{j_0}^< \cup S_{j_0}^>),$$

which ends the argument of the induction and concludes the proof of the Lemma.  $\square$

Now we turn back to the proof of Proposition 3.2.10. Let us consider  $\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}$  such that there exist  $\lambda_{2,0}, \dots, \lambda_{k,0} > 0$  with

$$(\lambda_{1,0}, \lambda_{2,0}, \dots, \lambda_{k,0}, \vec{b}_0, \vec{x}_0, \tilde{u}_0) \in \mathcal{C}_k.$$

Then we define  $\mathcal{O}_{k,p}$  be the set of all  $H^2$  functions in the following form:

$$u_0 = F(\lambda_{1,0}, \lambda_{2,0}^*, \dots, \lambda_{k,0}^*, \vec{b}_0, \vec{x}_0, \tilde{u}_0),$$

where

$$\begin{aligned} \lambda_{k,0}^* &= F_k(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}) > 0, \\ \lambda_{k-1,0}^* &= F_{k-1}(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{k,0}^*) > 0, \\ &\vdots \\ \lambda_{2,0}^* &= F_2(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{3,0}^*, \dots, \lambda_{k,0}^*) > 0 \end{aligned}$$

It is easy to see that  $\mathcal{O}_{k,p}$  contains infinite many elements.

On the other hand, the choice of  $\{\lambda_{j,0}^*\}_{j=2}^k$  implies that the scaling parameters of solution  $u(t)$  with initial data in  $\mathcal{O}_{k,p}$  satisfy:

$$\frac{1}{2^{k+1}} \leq \frac{\lambda_i(t)}{\lambda_j(t)} \leq 2^{k+1},$$

for all  $1 \leq i, j \leq k$ . This concludes the proof of Proposition 3.2.10.  $\square$

*Remark 3.2.13.* From the construction of the subset  $\mathcal{O}_{k,p}$ , if one can show that the functions  $F_j$  ( $j = 2, \dots, k$ ) are actually in  $C^1$ , then the subset  $\mathcal{O}_{k,p}$  has a codimension of  $k - 1$  in  $H^2$ . But this seems to be nontrivial.

*Remark 3.2.14.* From the proof of Lemma 3.2.12, the choice of  $\lambda_{j,0}^*$  ( $j = 2, \dots, k$ ) may not be unique<sup>12</sup>. Here in Lemma 3.2.12, we basically chose the “infimum” of all possible  $\lambda_{j,0}^*$ , which ensures that the functions  $F_j$  are all continuous. This argument is crucial to show that the blow-up points depend continuously on the initial data.

For this nonempty subset  $\mathcal{O}_{k,p}$ , the most important feature is that for  $u_0 \in \mathcal{O}_{k,p}$ , the estimates (3.2.34)–(3.2.40) can be improved on  $[0, T^*)$ . Hence from Remark 3.2.7, we have  $T^* = T$ . More precisely, we have:

**Proposition 3.2.15.** *If  $u_0 \in \mathcal{O}_{k,p}$ , then the following estimates hold on  $[0, T^*)$ :*

$$0 < \lambda_j(t) < 2, \quad (3.2.50)$$

$$\frac{1}{2^{k+1}} \leq \frac{\lambda_i(t)}{\lambda_j(t)} \leq 2^{k+1}, \quad (3.2.51)$$

$$|x_i(t) - x_j(t)| \geq b_c^{-75}, \text{ for all } i \neq j, \quad (3.2.52)$$

$$|b_j(t) - b_c| \leq b_c^{\frac{3}{2} + 2\nu}, \quad (3.2.53)$$

$$\|\varepsilon_j(t)\|_{L^{p_0}} \leq b_c^{\frac{13}{28}}, \quad (3.2.54)$$

$$\|(\varepsilon_j(t))_y\|_{L^2} \leq b_c^{\frac{3}{4}}, \quad (3.2.55)$$

$$\mathcal{N}_j(t) \leq b_c^{3+8\nu}, \quad (3.2.56)$$

From a standard bootstrap argument, we know that  $T^* = T$ . i.e. the estimates (3.2.50)–(3.2.56) hold for all  $t \in [0, T)$ . Then following from similar arguments as in [46, Section 6], we have:

**Proposition 3.2.16.** *For all  $u_0 \in \mathcal{O}_{k,p}$ , we have:*

1. *Finite time blow-up with self-similar rate: We have  $T < +\infty$ , and for all  $j = 1, \dots, k$  and  $t \in [0, T)$ :*

$$3(1 - \nu)b_c \leq \frac{\lambda_j^3(t)}{T - t} \leq 3(1 + \nu)b_c. \quad (3.2.57)$$

2. *The translation parameters converge to pairwise distinct points: For all  $j = 1, \dots, k$ ,*

$$x_j(t) \rightarrow x_j(T), \text{ as } t \rightarrow T, \quad (3.2.58)$$

$$|x_j(0) - x_j(T)| \lesssim \frac{1}{b_c}, \quad (3.2.59)$$

$$x_i(T) \neq x_j(T) \text{ for all } 1 \leq i \neq j \leq k. \quad (3.2.60)$$

12. The set  $S_{j_0}/(S_{j_0}^< \cup S_{j_0}^>)$  may contain more than one element.

3. Convergence in subcritical Lebesgue spaces: for all  $q \in [2, \frac{p-1}{2})$ ,

$$u(t) \rightarrow u^* \text{ in } L^q. \quad (3.2.61)$$

4. For  $R$  small enough, we have:

$$(1 - \delta(p)) \int \mathcal{Q}_p^2 \leq \frac{1}{R^{2\sigma_c}} \int_{|x-x_j(T)| < R} |u^*|^2 \leq (1 + \delta(p)) \int \mathcal{Q}_p^2, \quad (3.2.62)$$

$$\limsup_{R \rightarrow 0} \frac{1}{R^{2\sigma_c}} \int_{|x-z| < R} |u^*|^2 \leq \delta(p), \quad \text{for all } z \notin \{x_1(T), \dots, x_k(T)\}, \quad (3.2.63)$$

which implies that the blow-up set of  $u(t)$  is exactly  $\{x_1(T), \dots, x_k(T)\}$ .

5. The map from  $\mathcal{O}_{k,p}$  to  $\mathbb{R}^k$ :

$$u_0 \mapsto (x_1(T), \dots, x_k(T)) \quad (3.2.64)$$

is continuous under the topology of  $H^1$  and  $\mathbb{R}^k$ .

*Remark 3.2.17.* Proposition 3.2.16 implies that for all  $u_0 \in \mathcal{O}_{k,p}$ , the corresponding solution will blow up in finite time with self-similar rate, and has exactly  $k$  blow-up points.

### 3.3 Monotonicity tools and estimates on the error term

In this section, we will derive some crucial estimates on the error term  $\varepsilon_j$ , which imply the bootstrap bounds (3.2.55) and (3.2.56) immediately. Such estimates are similar to [46, Lemma 4.1, Proposition 5.2], and are the continuation of the monotonicity formula developed in [55] and [57].

#### 3.3.1 Monotonicity of the energy.

In this subsection we will give a control of  $\|(\varepsilon_j)_y\|_{L^2}$  and  $\|(\varepsilon_j)_y\|_{L^2(b_c^{-20} > y > \kappa B)}$ , which implies the bootstrap bound (3.2.55). These estimates provide a good control of the  $L^\infty$  norm of  $\varepsilon_j$  on the right.

**Proposition 3.3.1.** For all  $j = 1, \dots, k$  the following estimates hold for all  $s_j \in [0, s_j^*]$ :

$$\int (\varepsilon_j(s_j))_y^2 \lesssim b_c^{\frac{3}{2} + \frac{\nu}{2}}, \quad (3.3.1)$$

$$\int_{\kappa B < y < b_c^{-20}} (\varepsilon_j(s_j))_y^2 \lesssim b_c^{\frac{55}{7}}. \quad (3.3.2)$$

*Proof.* The proof of (3.3.1) is a consequence of the energy conservation law. Indeed, we have:

$$\begin{aligned} 2\lambda_j(s_j)^{2(1-\sigma_c)} E(u_0) &= 2E(\mathfrak{Q}_j) + \sum_{i=1}^k \left( \frac{\lambda_j(s_j)}{\lambda_i(s_j)} \right)^{2(1-\sigma_c)} \int (\varepsilon_i)_y(Q_{b_i})_y \\ &\quad + \int (\varepsilon_j)_y^2 - \frac{2}{p+1} \int ((\mathfrak{Q}_j + \varepsilon_j)^{p+1} - \mathfrak{Q}_j^{p+1}), \end{aligned} \quad (3.3.3)$$

where

$$\Omega_j(s_j, y) = \sum_{i=1}^k \left( \frac{\lambda_j(s_j)}{\lambda_i(s_j)} \right)^{\frac{2}{p-1}} Q_{b_i} \left( \frac{\lambda_j(s_j)y + x_j(s_j) - x_i(s_j)}{\lambda_i(s_j)} \right).$$

For the terms appear in the above summation, their supports are pairwise disjoint. So we have:

$$\begin{aligned} |E(\Omega_j(s_j, \cdot))| &= \left| \sum_{i=1}^k E \left( \left( \frac{\lambda_j(s_j)}{\lambda_i(s_j)} \right)^{\frac{2}{p-1}} Q_{b_i} \left( \frac{\lambda_j(s_j) \cdot + x_j(s_j) - x_i(s_j)}{\lambda_i(s_j)} \right) \right) \right| \\ &\leq \sum_{i=1}^k 6^{k+1} (|b_i - b_c| + b_c^3) \leq 10^k b_c^{\frac{3}{2} + \nu} \leq b_c^{\frac{3}{2} + \frac{\nu}{2}}. \end{aligned}$$

Here we use the fact that  $10^k \leq b_c^{-\nu/4}$ . The rest terms can be estimated similarly like what we do in [46]<sup>13</sup>, thus we conclude the proof of (3.3.1).

The proof of (3.3.2) is quite different from the single blow-up point case. We first choose 2 smooth functions  $\theta$  and  $\eta$ , such that  $\theta > 0$ ,  $\theta(y) = e^{-|y|}$  for  $|y| > 1$  and  $\eta(y) = 1$  for  $y < 1$ ,  $\eta(y) = e^{-y}$  for  $y > 2$ .

We introduce the following notations:

$$\begin{aligned} \Theta(y) &= \frac{\int_{-\infty}^y \theta(y') dy'}{\int_{-\infty}^{+\infty} \theta(y') dy'}, \quad \tilde{y} = \frac{y - \kappa B}{\sqrt{B}}, \\ \Psi_B(y) &= \Theta(\tilde{y}) \eta(b_c^{20} y). \end{aligned}$$

Then we assume that for all  $t \in [0, T^*)$ ,  $j \in \{1, \dots, k\}$ ,

$$\int_{y > \kappa B} [\varepsilon_j(t, y)]^2 \Psi_B(y) dy \leq b_c^{\frac{15}{2}}. \quad (3.3.4)$$

Since this estimate is satisfied for  $t = 0$ , so we only need to improve this estimate to:

$$\int_{y > \kappa B} [\varepsilon_j(t, y)]^2 \Psi_B(y) dy \lesssim b_c^{\frac{55}{7}}. \quad (3.3.5)$$

To do this, we fix  $t \in [0, T^*)$ . For  $\tau \in [0, t]$ , we introduce the localized energy:

$$\tilde{E}(\tau) = \int \left( \frac{1}{2} |u_x(\tau)|^2 - \frac{1}{p+1} |u(\tau)|^{p+1} \right) \Psi_B \left( \frac{x - x_j(\tau)}{\lambda_j(\tau)} \right) dx.$$

A direct computation shows:

$$\begin{aligned} &\lambda_j(t)^{2(1-\sigma_c)} \tilde{E}(t) \\ &= \frac{1}{2} \int ((Q_{b_j})_y + (\varepsilon_j)_y)^2 \Psi_B(y) dy - \frac{1}{p+1} \int |Q_{b_j} + \varepsilon_j|^{p+1} \Psi_B(y) dy + O(b_c^{20}) \\ &\gtrsim \int_{y > \kappa B} (\varepsilon_j)_y^2 \Psi_B - e^{-\frac{\kappa\sqrt{B}}{2}} \int_{y < \frac{\kappa B}{2}} (|(Q_{b_j})_y|^2 + |Q_{b_j}|^{p+1}) \\ &\quad - \int_{y > \frac{\kappa B}{2}} (|(Q_{b_j})_y|^2 + |Q_{b_j}|^{p+1}) - \int_{y > \frac{\kappa B}{2}} |\varepsilon|^{p+1} \Psi_B - e^{-\frac{\kappa\sqrt{B}}{2}} \int_{y < \frac{\kappa B}{2}} |\varepsilon|^{p+1} - b_c^{20}, \end{aligned} \quad (3.3.6)$$

13. See details in the first part of Section 4 in [46].

where we use the fact that  $\Psi_B(y) \leq e^{-\kappa\sqrt{B}/4}$  if  $y < \kappa B/2$ .

Next, we have:

$$\int_{y>\frac{\kappa B}{2}} |\varepsilon|^{p+1} \Psi_B \leq \left\| \varepsilon_j(t, \cdot)^{\frac{p_0}{2}+1} \eta(b_c^{20} \cdot)^{\frac{p_0+2}{2(p+1-p_0)}} \right\|_{L^\infty(y>\kappa B)}^{\frac{2(p+1-p_0)}{p_0+2}} \int \varepsilon_j^{p_0}.$$

For  $y > \kappa B$ , we have the following estimate:

$$\begin{aligned} & \left| |\varepsilon_j(t, y)|^{\frac{p_0}{2}+1} \eta(b_c^{20} y)^{\frac{p_0+2}{2(p+1-p_0)}} \right| \leq \left| |\varepsilon_j(t, y)|^{\frac{p_0}{2}+1} \eta(b_c^{20} y)^{\frac{1}{2}} \right| \\ & \leq \left| \int_y^{+\infty} (\varepsilon_j)_y |\varepsilon_j|^{\frac{p_0}{2}}(t, y') \eta(b_c^{20} y')^{\frac{1}{2}} dy' \right| + \left| b_c^{20} \int_y^{+\infty} |\varepsilon_j(t, y')|^{\frac{p_0}{2}+1} \frac{\eta'(b_c^{20} y')}{\eta(b_c^{20} y')^{\frac{1}{2}}} dy' \right| \\ & \lesssim \|\varepsilon_j\|_{L^{p_0}}^{\frac{p_0}{2}} \left( \int_{y'>\kappa B} (\varepsilon_j)_y^2 \eta(b_c^{20} y') dy' \right)^{\frac{1}{2}} + b_c^{10} \|\varepsilon_j\|_{L^{p_0+2}}^{\frac{p_0+2}{2}} \left( \int \frac{(\eta'(y'))^2}{\eta(y')} dy' \right)^{\frac{1}{2}} \\ & \lesssim b_c^{\frac{173}{40}}, \end{aligned}$$

where we use (3.2.38), (3.2.41), (3.3.4) and the basic fact that  $\eta(b_c^{20} y) \leq 2\Psi_B(y)$  for  $y > \kappa B$ . Combining the above 2 estimates, we have (recall that  $p_0 = \frac{5}{2}$  and  $p$  is slightly larger than 5):

$$\int_{y>\kappa B} |\varepsilon_j|^{p+1} \Psi_B \lesssim b_c^{\frac{173p-156}{90}} \leq b_c^{\frac{55}{7}} \quad (3.3.7)$$

On the other hand, from Sobolev embedding and (3.2.40), we have:

$$\int_{\kappa B/2 < y < \kappa B} |\varepsilon_j|^{p+1} \Psi_B \leq \|\varepsilon\|_{H^1(\kappa B/2 < y < \kappa B)}^{p+1} \leq b_c^9. \quad (3.3.8)$$

Injecting (3.3.7) and (3.3.8) into (3.3.6), we have:

$$\int_{\kappa B < y < b_c^{-20}} \varepsilon_j(t)^2 \lesssim b_c^{\frac{55}{7}} + \lambda_j(t)^{2(1-\sigma_c)} \tilde{E}(t)$$

Now it remains to estimate  $\lambda_j(t)^{2(1-\sigma_c)} \tilde{E}(t)$ . To do this, we first use the Kato's localized identities for the energy to compute the derivative of  $\tilde{E}(\tau)$ :

$$\begin{aligned} \frac{d}{d\tau} \tilde{E}(\tau) &= -\frac{1}{2} \int (u_{xx} + u|u|^{p-1})^2 g_x - \int u_{xx}^2 g_x \\ &+ p \int u|u|^{p-2} u_x^2 g_x + \frac{1}{2} \int u_x^2 g_{xxx} \\ &- \frac{x_l(\tau)}{\lambda(\tau)} \int \left( \frac{1}{2} |u_x(\tau)|^2 - \frac{1}{p+1} |u(\tau)|^{p+1} \right) \Psi'_B \left( \frac{x-x_j(\tau)}{\lambda_j(\tau)} \right) dx \\ &- \frac{\lambda_l(\tau)}{\lambda(\tau)} \int \left( \frac{1}{2} |u_x(\tau)|^2 - \frac{1}{p+1} |u(\tau)|^{p+1} \right) \left( \frac{x-x(\tau)}{\lambda(\tau)} \right) \Psi'_B \left( \frac{x-x_j(\tau)}{\lambda_j(\tau)} \right) dx \end{aligned} \quad (3.3.9)$$

where

$$g(x, \tau) = \Psi_B \left( \frac{x - x_j(\tau)}{\lambda_j(\tau)} \right).$$

We claim there exists a universal constant  $C$  such that for all  $\tau \in [0, t]$ ,

$$\frac{d}{d\tau} \tilde{E}(\tau) \leq \frac{Cb_c^{\frac{62}{7}}}{\lambda(\tau)^{3+2(1-\sigma_c)}} \quad (3.3.10)$$

We denote by

$$\begin{aligned} g_1(x, \tau) &= \frac{1}{\sqrt{B}\lambda_j(\tau)} \theta \left( \frac{x - x_j(\tau)}{\sqrt{B}\lambda_j(\tau)} - \kappa B \right) \eta \left( b_c^{20} \frac{x - x_j(\tau)}{\lambda_j(\tau)} \right), \\ g_2(x, \tau) &= \frac{b_c^{20}}{\lambda_j(\tau)} \Theta \left( \frac{x - x_j(\tau)}{\sqrt{B}\lambda_j(\tau)} - \kappa B \right) \eta' \left( b_c^{20} \frac{x - x_j(\tau)}{\lambda_j(\tau)} \right), \\ g_3(x, \tau) &= \frac{1}{(\sqrt{B}\lambda_j(\tau))^3} \theta'' \left( \frac{x - x_j(\tau)}{\sqrt{B}\lambda_j(\tau)} - \kappa B \right) \eta \left( b_c^{20} \frac{x - x_j(\tau)}{\lambda_j(\tau)} \right). \end{aligned}$$

Then we have

$$g = g_1 + g_2, \quad g_{xxx} - g_3 = O \left( \frac{b_c^{20}}{(\sqrt{B}\lambda_j(\tau))^3} \right).$$

So we can rewrite (3.3.9) as following:

$$\frac{d}{d\tau} \tilde{E}(\tau) = I + II + III + IV + V,$$

where

$$\begin{aligned} I &= -\frac{1}{2} \int (u_{xx} + u|u|^{p-1})^2 g_2 - \int u_{xx}^2 g_2, \\ II &= p \int u|u|^{p-2} u_x^2 g_2 + \frac{1}{2} \int u_x^2 (g_{xxx} - g_3), \\ III &= -x_t(\tau) \int \left( \frac{1}{2} |u_x(\tau)|^2 - \frac{1}{p+1} |u(\tau)|^{p+1} \right) g_2(x, \tau) dx, \\ IV &= -\lambda_t(\tau) \int \left( \frac{1}{2} |u_x(\tau)|^2 - \frac{1}{p+1} |u(\tau)|^{p+1} \right) \left( \frac{x - x(\tau)}{\lambda(\tau)} \right) g_2(x, \tau) dx, \\ V &= -\frac{1}{2} \int (u_{xx} + u|u|^{p-1})^2 g_1 - \int u_{xx}^2 g_1 \\ &\quad + p \int u|u|^{p-2} u_x^2 g_1 + \frac{1}{2} \int u_x^2 g_3 \\ &\quad - x_t(\tau) \int \left( \frac{1}{2} |u_x(\tau)|^2 - \frac{1}{p+1} |u(\tau)|^{p+1} \right) g_1(x, \tau) dx \\ &\quad - \lambda_t(\tau) \int \left( \frac{1}{2} |u_x(\tau)|^2 - \frac{1}{p+1} |u(\tau)|^{p+1} \right) \left( \frac{x - x(\tau)}{\lambda(\tau)} \right) g_1(x, \tau) dx. \end{aligned}$$



It is easy to estimate  $V$  by following the same argument as in [46, Section 4]. From the properties of  $\eta$  (i.e. exponential decay on the right), we know that on the support of  $g_2$  and  $g_{xxx} - g_3$ , the following term is negligible:

$$\sum_{i=1}^k \frac{1}{\lambda_i(\tau)^{\frac{2}{p-1}}} Q_{b_i(\tau)} \left( \frac{x - x_i(\tau)}{\lambda_i(\tau)} \right).$$

Together with (3.2.39), (3.2.40), (3.2.41) and (3.2.44) we have:

$$\begin{aligned} |II| &\lesssim \frac{b_c^{20}}{\lambda_j(\tau)^{3+2(1-\sigma_c)}} \left( b_c^{100} + \int (|\varepsilon_j|^{p-1} (\varepsilon_j)_y^2 + (\varepsilon_j)_y^2) \right) \leq \frac{Cb_c^{\frac{62}{7}}}{\lambda_j(\tau)^{3+2(1-\sigma_c)}}, \\ |III| &\lesssim \frac{b_c^{20} x_{s_j}}{\lambda_j(\tau)^{4+2(1-\sigma_c)}} \left[ b_c^{100} + \int (\varepsilon_j^2 + |\varepsilon_j|^{p+1}) \right] \leq \frac{Cb_c^{\frac{62}{7}}}{\lambda_j(\tau)^{3+2(1-\sigma_c)}}. \end{aligned}$$

While for  $IV$  we have  $g_2 \leq 0$ ,  $\lambda_\tau \leq 0$ , and on the support of  $g_2$ ,  $(x - x_j(\tau))/\lambda_j(\tau) \geq 0$ . So we have:

$$-\lambda_\tau(\tau) \int \left( \frac{1}{2} |u_x(\tau)|^2 \right) \left( \frac{x - x(\tau)}{\lambda(\tau)} \right) g_2(x, \tau) dx \leq 0.$$

Moreover, from (3.2.43) and the choice of  $\eta$  we have:

$$\begin{aligned} &\left| \lambda_\tau(\tau) \int \frac{1}{p+1} |u(\tau)|^{p+1} \left( \frac{x - x(\tau)}{\lambda(\tau)} \right) g_2(x, \tau) dx \right| \\ &\lesssim \frac{b_c}{\lambda_j(\tau)^{3+2(1-\sigma_c)}} \int |\varepsilon(\tau, y')|^{p+1} (b_c^{20} y') \eta'(b_c^{20} y') dy' \\ &\lesssim \frac{b_c}{\lambda_j(\tau)^{3+2(1-\sigma_c)}} \int_{y' > \kappa B} |\varepsilon(\tau, y')|^{p+1} \eta(b_c^{20} y')^{\frac{99}{100}} dy' \\ &\leq \frac{Cb_c^{\frac{62}{7}}}{\lambda_j(\tau)^{3+2(1-\sigma_c)}}, \end{aligned}$$

where the last inequality follows from the same argument which is used to estimate (3.3.7). Thus we obtain:

$$IV \leq \frac{Cb_c^{\frac{62}{7}}}{\lambda_j(\tau)^{3+2(1-\sigma_c)}}.$$

Finally, we deal with  $I$ . First of all, we have

$$I \lesssim \frac{b_c^{20}}{\lambda_j(\tau)} \left( \int (u_{xx}^2 + |u|^{2p}) \right) \lesssim \frac{b_c^{20}}{\lambda_j(\tau)} \int u_{xx}^2 + \frac{b_c^{10}}{\lambda_j(\tau)^{3+2(1-\sigma_c)}}, \quad (3.3.11)$$

where we use the fact that

$$\begin{aligned} \int |u|^{2p} &\lesssim \frac{1}{\lambda_j(\tau)^{2+2(1-\sigma_c)}} \int \left( |\varepsilon_j(\tau)|^{2p} + \sum_{i=1}^k \left( \frac{\lambda_j(\tau)}{\lambda_i(\tau)} \right)^{\frac{3p+1}{p-1}} |Q_{b_i(\tau)}|^{2p} \right) \\ &\lesssim \frac{10^k}{\lambda_j(\tau)^{2+2(1-\sigma_c)}} \leq \frac{b_c^{-\nu/4}}{\lambda_j(\tau)^{2+2(1-\sigma_c)}}. \end{aligned}$$

While for  $\int u_{xx}^2$ , we can use pseudo-conservation law to estimate. Precisely, we have the following estimate for all  $\tau_0 \in [0, \tau]$ :

$$\frac{d}{d\tau_0} E_2(\tau_0) \lesssim \int u_x^3 |u|^{2p-3}(\tau_0) + \int u_x^5 |u|^{p-4}(\tau_0), \quad (3.3.12)$$

where

$$E_2(\tau_0) = \int u_{xx}^2(\tau_0) - \frac{5p}{3} \int u_x^2 |u|^{p-1}(\tau_0).$$

It is easy to prove (3.3.12) by integrating by parts.

Now we assume the following *a priori* estimate for all  $\tau_0 \in [0, \tau]$ :

$$\int u_{xx}(\tau_0)^2 \leq \frac{b_c^{-8}}{\lambda_j(\tau_0)^{2+2(1-\sigma_c)}}. \quad (3.3.13)$$

Then Sobolev embedding implies that:

$$\|u_x(\tau_0)\|_{L^\infty} \leq \|u_x(\tau_0)\|_{L^2}^{\frac{1}{2}} \|u_{xx}(\tau_0)\|_{L^2}^{\frac{1}{2}} \lesssim \frac{b_c^{-2-\nu}}{\lambda_j(\tau_0)^{\frac{1}{2}+2(1-\sigma_c)}},$$

where we use the fact that

$$\int u_x^2(\tau_0) \lesssim \frac{b_c^{-\nu/4}}{\lambda_j(\tau_0)^{2(1-\sigma_c)}}.$$

From (3.3.12) and (3.3.13) we obtain

$$\begin{aligned} \frac{d}{d\tau_0} E_2(\tau_0) &\lesssim \|u_x\|_{L^\infty} \|u\|_{L^\infty}^{2p-3} \int u_x^2 + \|u_x\|_{L^\infty}^3 \|u\|_{L^\infty}^{p-4} \int u_x^2 \\ &\lesssim \frac{b_c^{-6-10\nu}}{\lambda_j(\tau_0)^{5+2(1-\sigma_c)}}. \end{aligned}$$

Note that for  $\beta > 3$ ,

$$\int_0^{\tau_0} \frac{1}{\lambda_j(\tau')^\beta} d\tau' \leq \frac{2}{b_c(\beta-3)\lambda_j(\tau_0)^{\beta-3}}.$$

We have:

$$E_2(\tau_0) \lesssim E_2(0) + \frac{b_c^{-7-10\nu}}{\lambda_j(\tau_0)^{2+2(1-\sigma_c)}}.$$

The conditions on the initial data lead to:

$$E_2(0) \lesssim \frac{b_c^{-10\nu}}{\lambda_j(0)^{2+2(1-\sigma_c)}} \leq \frac{b_c^{-10\nu}}{\lambda_j(\tau_0)^{2+2(1-\sigma_c)}}.$$

Together with

$$\int u_x^2 |u|^{p-1}(\tau_0) \lesssim \frac{b_c^{-10\nu}}{\lambda_j(\tau_0)^{2+2(1-\sigma_c)}},$$

we have for all  $\tau_0 \in [0, \tau]$ ,

$$\int u_{xx}(\tau_0)^2 \leq \frac{b_c^{-7-10\nu}}{\lambda_j(\tau_0)^{2+2(1-\sigma_c)}}.$$

From a standard bootstrap argument (if  $\nu$  is small enough), we have shown that

$$\int u_{xx}(\tau)^2 \leq \frac{b_c^{-7-10\nu}}{\lambda_j(\tau)^{2+2(1-\sigma_c)}}.$$

Injecting this into (3.3.11) we get

$$I \leq \frac{b_c^{10}}{\lambda_j(\tau)^{3+2(1-\sigma_c)}},$$

which concludes the proof of (3.3.10), hence the proof of (3.3.5) and (3.3.2).  $\square$

The main goal of this subsection is to control the  $L^\infty$  norm of  $\varepsilon_j$  on the right, i.e.

**Corollary 3.3.2.** *The following  $L^\infty$  control hold for all  $t \in [0, T^*)$ :*

$$\|\varepsilon_j(t)\|_{L^\infty(b_c^{-10} > y > \kappa B)} \lesssim b_c^2. \quad (3.3.14)$$

*Proof.* Let  $\eta_0$  be a smooth function with  $\eta_0(y) = 1$  for  $y < 1$ ,  $\eta_0(y) = 0$  for  $y > 2$ . Let  $f(y) = \varepsilon_j(y)\eta_0(b_c^{10}y)$ . Applying the localized Gagliardo-Nirenberg inequality to  $f$ , we have

$$\begin{aligned} \|f\|_{L^\infty(y > \kappa B)} &\leq \|f\|_{L^{p_0}}^{\frac{p_0}{p_0+2}} \|f_y\|_{L^2(y > \kappa B)}^{\frac{2}{p_0+2}} \\ &\lesssim \|\varepsilon_j\|_{L^{p_0}}^{\frac{p_0}{p_0+2}} \left( \|(\varepsilon_j)_y\|_{L^2(b_c^{20} > y > \kappa B)}^{\frac{2}{p_0+2}} + \left( b_c^5 \|\varepsilon_j\|_{L^\infty} \right)^{\frac{2}{p_0+2}} \right) \\ &\lesssim b_c^2. \end{aligned}$$

$\square$

### 3.3.2 Monotonicity formula

In this subsection, we will derive a monotonicity formula for the localized Sobolev norm of  $\varepsilon_j$ , which will imply the bootstrap bound (3.2.56) immediately and is important in the derivation of the asymptotic dynamics of the flow. This formula here is almost the same to the one in [46]. Such monotonicity tools were introduced originally in [55] and [57] for critical gKdV.

Recall from (2.32), the definition of  $\varphi$ . We let  $\psi, \eta_0$  be 2 other smooth functions such that:

$$\psi(y) = \begin{cases} e^y & \text{for } y < -1, \\ 1 & \text{for } y > -\kappa, \end{cases} \quad \psi' \geq 0, \quad (3.3.15)$$

$$\eta_0(y) = \begin{cases} 1 & \text{for } y < 1, \\ 0 & \text{for } y > 2, \end{cases} \quad \eta_0' \leq 0. \quad (3.3.16)$$

Here, we observe that  $\psi(-\kappa) = \varphi(-\kappa) + \kappa$ , and  $\psi(y) = \varphi(y)$  for all  $y < -1$ , so we may assume in addition:

$$\varphi(y) \leq \psi(y) \leq (1 + 3\kappa)\varphi(y), \text{ for all } y \leq -\kappa. \quad (3.3.17)$$

*Remark 3.3.3.* It is easy to check that for every  $\frac{1}{2} > \kappa > 0$ , such  $\psi$  and  $\varphi$  exist.

Now, recall  $B = b_c^{-\frac{1}{20}}$ . We let

$$\psi_B(y) = \psi\left(\frac{y}{B}\right)\eta_0(b_c^{10}y), \quad \zeta_B(y) = \varphi_B\eta_0\left(\frac{y}{B^2}\right).$$

and then define the following *Lyapunov* functional for  $\varepsilon_j$ :

$$\mathcal{F}_j = \int \left[ (\varepsilon_j)_y^2 \psi_B + (\varepsilon_j)^2 \zeta_B - \frac{2}{p+1} (|\varepsilon_j + Q_{b_j}|^{p+1} - Q_{b_j}^{p+1} - (p+1)\varepsilon_j Q_{b_j}^p) \psi_B \right]. \quad (3.3.18)$$

Our main goal here is the following monotonicity formula for  $\mathcal{F}_j$ :

**Proposition 3.3.4** (The second monotonicity formula). *There exists a universal constant  $\mu > 0$  and  $0 < \kappa < \frac{1}{2}$ , such that for all  $j = 1, \dots, k$ ,  $s_j \in [0, s_j^*)$ , the following holds:*

1. *Lyapunov control:*

$$\frac{d}{ds_j} \mathcal{F}_j + \mu \int ((\varepsilon_j)_y^2 + (\varepsilon_j)^2) \varphi_B' \lesssim b_c^{\frac{7}{2}}; \quad (3.3.19)$$

2. *Coercivity of  $\mathcal{F}$ : there exists a universal constant  $\kappa > 0$  such that*

$$\mathcal{N}_j - b_c^{\frac{7}{2}} \lesssim \mathcal{F}_j \lesssim \mathcal{N}_j + b_c^{\frac{7}{2}}. \quad (3.3.20)$$

*Remark 3.3.5.* The bootstrap bound (3.2.56) follows immediately from (3.3.19), (3.3.20) and Grownwell's inequality.

*Proof.* The proof of Proposition 3.3.4 is exactly the same to the one in [46, Section 5]. The idea is that the error term  $\varepsilon_j$  has been “localized” to the support of  $Q_{b_j}$ , due to our choice of the weight functions. Then the estimate is exactly the same to the single blow-up point case.

The only difference here is that we add a cut-off on the right of  $\psi_B$ . This will lead to some additional terms on the right hand side of (3.3.19). But if we check the proof of [46, Proposition 5.1], we will see these additional terms can always be bounded by

$$-b_c^{10} \int (\varepsilon_j)_y^2(s_j, y') \eta_0'(b_c^{10}y') dy',$$

and hence bounded by  $b_c^{10}$ . □

## 3.4 Existence of blow-up solutions with exactly $k$ blow-up points

This section is devoted to prove Proposition 3.2.15 and Proposition 3.2.16. Hence for all  $u_0 \in \mathcal{O}_{k,p}$ , the corresponding solution has exactly  $k$  blow-up points.

### 3.4.1 Closing the bootstrap

In this subsection we will finish the bootstrap argument and finally prove Proposition 3.2.15.

*Proof of Proposition 3.2.15.* The bounds (3.2.50)–(3.2.53) are consequences of the modulation estimates (3.2.43)–(3.2.45). Indeed, (3.2.50) follows from the fact that  $0 < \lambda_j(0) < 2$  and that  $\lambda_j(t)$  is decreasing. (3.2.51) is just the definition of  $\mathcal{O}_{k,p}$ . For (3.2.52), we have

$$\begin{aligned} |x_i(t) - x_j(t)| &\geq |x_i(0) - x_j(0)| - \int_0^t |(x_i)_t| + |(x_j)_t| \\ &\geq b_c^{-80} - 2 \int_0^t \left( \frac{1}{\lambda_i^2} + \frac{1}{\lambda_j^2} \right) \\ &\geq b_c^{-80} + \frac{1}{b_c} \int_0^t (\lambda_i + \lambda_j)_t \geq b_c^{-75}. \end{aligned}$$

While for (3.2.53), suppose for some  $s_{j,0} \in (0, s_j^*)$ , we have  $b(s_{j,0}) > b_c + b_c^{\frac{3}{2}+2\nu}$ . By the choice of the initial data, i.e. (3.2.27), we can find some  $s_{j,1} \in [0, s_{j,0})$  such that  $b(s_{j,1}) = b_c + b_c^{\frac{3}{2}+\frac{5}{2}\nu}$  and  $b(s_j) \geq b_c + b_c^{\frac{3}{2}+\frac{5}{2}\nu}$  for all  $s_j \in [s_{j,1}, s_{j,0})$ . Then  $b_{s_j}(s_{j,1}) \geq 0$ . From (3.2.40) and (3.2.45), we have:

$$b_{s_j}(s_{j,1}) \leq -c_p(b(s_{j,1}) - b_c)b_c + b_c^{\frac{5}{2}+3\nu} = -c_p b_c^{\frac{5}{2}+\frac{5\nu}{2}} + b_c^{\frac{5}{2}+3\nu} < 0, \quad (3.4.1)$$

if  $b_c$  is small enough such that  $b_c^\nu \ll 1$ . We get a contradiction. The opposite bound is similar.

The proof of (3.2.54)–(3.2.56) is parallel to the one for the single blow-up point case in [46], using the similar monotonicity tools developed in the previous section.  $\square$

### 3.4.2 Proof of Proposition 3.2.16

First we prove the finite time blow-up and self-similar result, i.e. (3.2.57). From Proposition 3.2.15 and (3.2.43), we know for all  $t \in [0, T)$ ,  $j \in \{1, \dots, k\}$ ,

$$(1 - \nu)b_c \leq -(\lambda_j)_t \lambda_j^2 \leq (1 + \nu)b_c. \quad (3.4.2)$$

Integrating (3.4.2) from 0 to  $t$  we know that for all  $j \in \{1, \dots, k\}$ ,

$$\forall t \in [0, T), \quad (1 - \nu)b_c t \leq \frac{1}{3} \lambda_j^3(0) \text{ and hence } T \leq \frac{\lambda_j^3(0)}{3b_c(1 - \nu)} < +\infty. \quad (3.4.3)$$

So the solution blows up in finite time. Then from local Cauchy theory, we know  $\lim_{t \rightarrow T} \|u_x(t)\|_{L^2} = +\infty$ . But we know from the geometrical decomposition:

$$\|u_x(t)\|_{L^2} \sim \sum_{i=1}^k \frac{1}{\lambda_i(t)^{1-\sigma_c}}.$$

Combining with (3.2.51), we have for all  $j \in \{1, \dots, k\}$ ,

$$\lim_{t \rightarrow T} \lambda_j(t) = 0.$$

We then integrate (3.4.2) from  $t$  to  $T$  to obtain:

$$\forall t \in [0, T), \quad (1 - \nu)b_c(T - t) \leq \frac{\lambda_j^3(t)}{3} \leq (1 + \nu)b_c(T - t),$$

which implies (3.2.57).

While for (3.2.59), from (3.2.44), (3.2.25) and (3.4.3), we have for all  $j = 1, \dots, k$ ,

$$|x_j(0) - x_j(T)| \lesssim \int_0^{+\infty} |(x_j)_{s_j}| ds_j \lesssim \int_0^T \frac{dt}{\lambda_j(t)} \lesssim \int_0^T \frac{dt}{\sqrt[3]{b_c(T-t)}} \lesssim \frac{T^{2/3}}{b_c^{1/3}} \lesssim \frac{1}{b_c}.$$

The proof of (3.2.58)–(3.2.63) is exactly the same to the one in [46, Section 6.2].

Finally, for (3.2.64), from (3.2.30) and Lemma 3.2.12, we only need to show that the blow-up time is continuous with respect to the initial data in  $\mathcal{O}_{k,p}$ . More precisely, we have:

**Lemma 3.4.1.** *Consider  $\{u_{0,n}\}_{n=1}^{+\infty} \subset \mathcal{O}_{k,p}$ ,  $u_0 \in \mathcal{O}_{k,p}$ , such that  $u_{0,n}$  converges to  $u_0$  in  $H^1$  as  $n \rightarrow +\infty$ . Let  $u_n(t)$ ,  $u(t)$  be the corresponding solutions to (3.1.1), and  $T_n$ ,  $T$  be the corresponding blow-up times, then we have*

$$\lim_{n \rightarrow +\infty} T_n = T.$$

*Proof.* First of all, from a classical argument of continuity with respect to the initial data (i.e. the perturbation theory<sup>14</sup>), we obtain:

$$\liminf_{n \rightarrow +\infty} T_n \geq T. \quad (3.4.4)$$

On the other hand, for all  $\delta > 0$ , there exists  $n(\delta) > 0$ , such that if  $n > n(\delta)$ ,  $u_n(T - \delta)$  exists. Integrating (3.4.2) from  $T - \delta$  to  $T_n$ , we have

$$T_n < T - \delta + \frac{\lambda_{j,n}^3(T - \delta)}{3(1 - \nu)b_c},$$

where  $\lambda_{j,n}(t)$  is the  $j$ -th scaling parameter of the solution  $u_n(t)$ . Let  $n \rightarrow +\infty$ , we will obtain:

$$\limsup_{n \rightarrow +\infty} T_n \leq T - \delta + \frac{\lambda_j^3(T - \delta)}{3(1 - \nu)b_c} \leq T + 2\delta.$$

Then let  $\delta \rightarrow 0$ , we have  $\limsup_{n \rightarrow +\infty} T_n \leq T$ , which concludes the proof of the lemma.  $\square$

Therefore, we finish the proof of (3.2.64) and hence the proof of Proposition 3.2.16.

14. See for example Lemma 2.4 in [58].

### 3.5 Proof of Theorem 3.1.5 by Brouwer's theorem

In this section, we will prove Theorem 3.1.5. Actually, Proposition 3.2.16 has already given the existence of solutions with exactly  $k$  blow-up points. And here we will use another topological argument to show that the blow-up points can be chosen arbitrarily.

Given any  $k$  points  $\{x_1, \dots, x_k\}$ , we want to find a solution whose blow-up set is exactly  $\{x_1, \dots, x_k\}$ .

**Step 1.** First, we show that if

$$|x_i - x_j| \geq b_c^{-120}, \text{ for all } i \neq j, \quad (3.5.1)$$

then there exists a solution  $u(t)$  satisfying (3.1.14) and (3.1.15), whose blow-up set is exactly  $\{x_1, \dots, x_k\}$ .

For  $j = 1, \dots, k$ , we let  $I_j = [x_j - b_c^{-3}, x_j + b_c^{-3}]$ . Then for all  $x_{i,0} \in I_i$ , we have

$$|x_{i_1,0} - x_{i_2,0}| \geq b_c^{-100}, \text{ for all } i_1 \neq i_2.$$

Next, we choose suitable  $\lambda_{1,0}, b_{1,0}, \dots, b_{k,0} > 0$  and  $\tilde{u}_0 \in H^2$ , such that for all

$$(x_{1,0}, \dots, x_{k,0}) \in I_1 \times \dots \times I_k,$$

there exist  $(\lambda_{2,0}, \dots, \lambda_{k,0})$  such that conditions (3.2.18)–(3.2.22) is satisfied for<sup>15</sup>

$$v_0(x) = F(\vec{\lambda}_0, \vec{b}_0, \vec{x}_0, \tilde{u}_0) = \sum_{i=1}^k \frac{1}{\lambda_{i,0}^{\frac{2}{p-1}}} Q_{b_{i,0}} \left( \frac{x - x_{i,0}}{\lambda_{i,0}} \right) + \tilde{u}_0(x).$$

Then, we can consider the solution  $u(t)$  with initial data

$$u_0 = F(\lambda_{1,0}, \lambda_{2,0}^*, \dots, \lambda_{k,0}^*, \vec{b}_0, \vec{x}_0, \tilde{u}_0),$$

where<sup>16</sup>

$$\lambda_{k,0}^* = F_k(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}) > 0,$$

$$\lambda_{i,0}^* = F_i(\vec{b}_0, \vec{x}_0, \tilde{u}_0, \lambda_{1,0}, \lambda_{i+1,0}^*, \dots, \lambda_{k,0}^*) > 0, \text{ for } i = 2, \dots, k-1.$$

Then from Proposition 3.2.16, it is easy to check that  $u(t)$  satisfies (3.1.14) and (3.1.15), and the blow-up set of  $u(t)$  is  $\{x_1(T), \dots, x_k(T)\}$ . Therefore we have constructed a map  $M$  from  $D = I_1 \times \dots \times I_k$  to  $\mathbb{R}^k$ , i.e.

$$M(x_{1,0}, \dots, x_{k,0}) = (x_1(T), \dots, x_k(T)).$$

From (3.2.64) and the fact that  $F_j$  is continuous for all  $j = 2, \dots, k$ , it is easy to see that the map  $M$  is continuous.

15. We mention here that this is possible due to the assumption (3.5.1).

16. Here,  $F_j$ ,  $j = 2, \dots, k$  are the continuous functions defined in Lemma 3.2.12.

Now we claim there exists a  $(x_{1,0}, \dots, x_{k,0}) \in D$  such that

$$M(x_{1,0}, \dots, x_{k,0}) = (x_1, \dots, x_k). \quad (3.5.2)$$

From the construction of geometrical decomposition (i.e. the argument of implicit function theorem), we have

$$|x_{i,0} - x_i(0)| \ll 1,$$

for all  $i = 1, \dots, k$ . Together with (3.2.59), we have for all  $i \in \{1, \dots, k\}$ ,

$$|x_{i,0} - x_i(T)| \leq b_c^{-2}. \quad (3.5.3)$$

We then introduce the following topological lemma, which is a corollary of the Brouwer's fixed point theorem, [6].

**Lemma 3.5.1.** *Let  $f$  be a continuous map from  $\mathbb{R}^k$  to  $\mathbb{R}^k$ , and  $T_r = [-r, r]^k \subset \mathbb{R}^k$  be a cube centered at 0, for some  $r > 0$ . Suppose we have for all  $y \in \partial T_r$ ,*

$$|f(y) - y| < r, \quad (3.5.4)$$

*then there exists a  $y_0 \in T_r$  such that  $f(y_0) = 0$ .*

*Proof of Lemma 3.5.1.* Suppose for all  $y \in T_r$ ,  $f(y) \neq 0$ . Then we can define a map  $g$  from  $T_r$  to  $\partial T_r$  as following:

$$g(y) = \partial T_r \cap \{tf(y) | t \geq 0\}.$$

It is easy to check that  $g$  is well-defined and continuous. The assumption (3.5.4) ensures that for all  $y \in \partial T_r$ , and  $t \in [0, 1]$ , we have

$$tg(y) + (1-t)y \neq 0,$$

which implies that  $g|_{\partial T_r}$  is homotopic to  $\text{Id}_{\partial T_r}$ . Indeed we can consider the following map:

$$\begin{aligned} G: [0, 1] \times \partial T_r &\rightarrow \partial T_r, \\ (t, x) &\mapsto \partial T_r \cap \{s[tg(y) + (1-t)y] | s \geq 0\}. \end{aligned}$$

It is easy to check that  $G$  is well-defined and continuous. Moreover, we have  $G(0, y) = y$ ,  $G(1, y) = g(y)$ , for all  $y \in \partial T_r$ . Thus  $g|_{\partial T_r}$  is homotopic to  $\text{Id}_{\partial T_r}$ .

Then the homeomorphism of the homology groups induced by  $g$  (i.e.  $g_*: H_*(T_r) \rightarrow H_*(\partial T_r)$ ) is surjective. But this is a contradiction, since  $H_{k-1}(T_r) = 0$ ,  $H_{k-1}(\partial T_r) = \mathbb{Z}$ . Therefore, we conclude the proof of Lemma 3.5.1.  $\square$

Now we apply Lemma 3.5.1 to  $f = M$ , and  $T_r = D$  with  $r = b_c^{-3}$ . From (3.5.3), we can see that condition (3.5.4) is satisfied. Then we obtain (3.5.2), which concludes the proof of Theorem 3.1.5 under the assumption of (3.5.1).

**Step 2.** Now for arbitrarily  $k$  pairwise distinct points  $\{x_1, \dots, x_k\}$ , choose  $\underline{\lambda} > 0$ , such that

$$\min_{1 \leq i \neq j \leq k} |\underline{\lambda}x_i - \underline{\lambda}x_j| \geq b_c^{-120}.$$



Now from the above arguments, there exists a solution  $v(t)$  blowing up in finite time  $T_v < +\infty$ , whose blow-up set is  $\{\underline{\lambda}x_1, \dots, \underline{\lambda}x_k\}$ . Moreover, for  $t$  close to  $T_v$ , there exist  $\lambda_{j,v}(t)$  and  $\tilde{v}(t, x)$  such that

$$v(t, x) = \sum_{j=1}^k \frac{1}{\lambda_{j,v}^{\frac{2}{p-1}}(t)} \mathcal{Q}_p \left( \frac{x - \underline{\lambda}x_j}{\lambda_{j,v}(t)} \right) + \tilde{v}(t, x),$$

$$\frac{\lambda_{j,v}^3(t)}{T_v - t} \sim p - 5, \quad \lambda_{j,v}(t)^{1-\sigma_c} \|\tilde{v}_x(t)\|_{L^2} \leq \delta(p).$$

Then we let

$$u(t, x) = \underline{\lambda}^{\frac{2}{p-1}} v(\underline{\lambda}^3 t, \underline{\lambda}x).$$

It is easy to see from Remark 3.1.3 that  $u(t)$  is a solution to (3.1.1) blowing up in finite time  $T_u = \underline{\lambda}^{-3} T_v < +\infty$ . And its blow-up set is exactly  $\{x_1, \dots, x_k\}$ . Moreover,  $u(t)$  satisfies (3.1.14) and (3.1.15) with

$$\lambda_j(t) = \frac{\lambda_{j,v}(\underline{\lambda}^3 t)}{\underline{\lambda}}, \quad \tilde{u}(t, x) = \underline{\lambda}^{\frac{2}{p-1}} \tilde{v}(\underline{\lambda}^3 t, \underline{\lambda}x), \quad \text{for all } t \text{ close to } T_u = \underline{\lambda}^{-3} T_v.$$

Therefore, we conclude the proof of Theorem 3.1.5.

# Chapter 4

## On asymptotic dynamics for mass critical gKdV equation with a saturated perturbation

### 4.1 Introduction

#### 4.1.1 Setting of the problem

Let us consider the following Cauchy problem:

$$\begin{cases} \partial_t u + (u_{xx} + u^5 - \gamma u|u|^{q-1})_x = 0, & (t, x) \in [0, T) \times \mathbb{R}, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \quad (\text{gKdV}_\gamma)$$

with  $q > 5$  and  $0 < \gamma \ll 1$ .

The equation has two conservation laws, i.e. the mass and the energy:

$$\begin{aligned} M(u(t)) &= \int u(t)^2 = M_0, \\ E(u(t)) &= \frac{1}{2} \int u_x(t)^2 - \frac{1}{6} \int u(t)^6 + \frac{\gamma}{q+1} \int |u(t)|^{q+1} = E_0. \end{aligned}$$

We can see that the solution of  $(\text{gKdV}_\gamma)$  is always global in time and bounded in  $H^1$ . First of all,  $(\text{gKdV}_\gamma)$  is locally wellposed in  $H^1$  due to [33, 36], i.e. for any  $u_0 \in H^1$ , there exists a unique strong solution in  $C([0, T), H^1)$ , with either  $T = +\infty$  or  $T < +\infty$  and  $\lim_{t \rightarrow T} \|u_x(t)\|_{L^2} = +\infty$ . Since  $\gamma > 0$ ,  $q > 5$ , the mass and energy conservation laws ensure that for all  $t \in [0, T)$ ,

$$\|u_x(t)\|_{L^2}^2 \lesssim |E_0| + \gamma^{-\frac{4}{q-5}} M_0 < +\infty,$$

so  $T = +\infty$  and  $u(t)$  is always bounded in  $H^1$ .

This equation does not have a standard scaling rule, but has the following pseudo-scaling rule: for all  $\lambda_0 > 0$ , if  $u(t, x)$  is a solution to  $(\mathbf{gKdV}_\gamma)$ , then

$$u_{\lambda_0}(t, x) = \lambda_0^{-\frac{1}{2}} u(\lambda_0^{-3}t, \lambda_0^{-1}x), \quad (4.1.1)$$

is a solution to

$$\begin{cases} \partial_t v + (v_{xx} + v^5 - \lambda_0^{-m} \gamma v |v|^{q-1})_x = 0, & (t, x) \in [0, \lambda_0^{-3}T) \times \mathbb{R}, \\ v(0, x) = \lambda_0^{-\frac{1}{2}} u_0(\lambda_0^{-1}x) \in H^1(\mathbb{R}), \end{cases}$$

with

$$m = \frac{q-5}{2} > 0. \quad (4.1.2)$$

The pseudo-scaling rule (4.1.1) leaves the  $L^2$  norm of the initial data invariant.

There is a special class of solutions. We first introduce the ground state  $Q_\omega$  for  $0 \leq \omega < \omega^* \ll 1$ , which is the unique radial nonnegative solution with exponential decay to the following ODE<sup>1</sup>:

$$Q_\omega'' - Q_\omega + Q_\omega^5 - \omega Q_\omega |Q_\omega|^{q-1} = 0.$$

Then for all  $\lambda_0 > 0$ ,  $t_0 \in \mathbb{R}$ ,  $x_0 \in \mathbb{R}$  with  $\lambda_0^{-m} \gamma < \omega^*$ , the following is a solution to  $(\mathbf{gKdV}_\gamma)$ :

$$u(t, x) = \lambda_0^{-\frac{1}{2}} Q_{\lambda_0^{-m} \gamma}(\lambda_0^{-1}(x - x_0) - \lambda_0^{-3}(t - t_0)).$$

A solution of this type is called a *solitary wave* solution.

## 4.1.2 On the critical problem with saturated perturbation

The saturated perturbation was first introduced for the nonlinear Schrödinger (NLS):

$$i\partial_t u + \Delta u + g(|u|^2)u = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^d. \quad (\text{NLS})$$

In many applications, the leading order approximation of the nonlinearity,  $g(s)$ , is the power nonlinearity, i.e.  $g(s) = \pm s^\sigma$ . For example,  $g(s) = s$ , leads to the focusing cubic NLS equation, which appears in many contexts.

But such approximation may lead to nonphysical predictions. For example, from [21, 73, 75, 90], for NLS with critical or supercritical focusing nonlinearities (i.e.  $g(s) = s^\sigma$  with  $\sigma d \geq 2$ ), blow up may occur. However, this contradicts with the experiments in the optical settings [32], which shows that there is no ‘‘singularity’’ and the solution always remains bounded.

One way to correct this model is to replace the power nonlinearities by saturated nonlinearities. A typical example<sup>2</sup> is  $g(s) = s^\sigma - \gamma s^{\sigma+\delta}$ , with  $\delta > 0$ ,  $\gamma > 0$ . Similar as  $(\mathbf{gKdV}_\gamma)$ , in this case any  $H^1$  solution to (NLS) is global in time and bounded in  $H^1$ .

1. The existence of such  $Q_\omega$  was proved in [2, Section 6], but, in this paper we will give an alternative proof for the existence.

2. See [25, 63] for other kind of saturated perturbations.

On the other hand, the saturated perturbation is also related to the problem of continuation after blow up time. This kind of problems arising in physics is poorly understood even at a formal level. One approach is to consider the solution  $u_\varepsilon(t)$  to the following critical NLS with saturated perturbation:

$$\begin{cases} i\partial_t u + \Delta u + |u|^{\frac{4}{d}} u - \varepsilon |u|^q u = 0, & (t, x) \in [0, T) \times \mathbb{R}^d, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}^d), \end{cases}$$

where  $4/d < q < 4/(d-2)$ . Suppose the solution  $u(t)$  to the unperturbed NLS (i.e.  $\varepsilon = 0$ ) with initial data  $u_0$ , blows up in finite time  $T < +\infty$ . Then, it is easy to see that for all  $\varepsilon > 0$ ,  $u_\varepsilon(t)$  exists globally in time and for all  $t < T$ ,

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon(t) = u(t), \text{ in } H^1.$$

Now, we may consider the following limit:

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon(t), \quad t > T,$$

to see whether the limiting function exists and in what sense it satisfies the critical NLS. Such construction for blow up solutions using the Virial identity was given by Merle [66]. Alternative way to construct the approximate solution  $u_\varepsilon(t)$  can also be found in [64, 67, 76]. But, this only remains for very special cases. General constructions of this type are mostly open. In all cases, the asymptotic behavior of the approximate solution  $u_\varepsilon(t)$  is crucial in the analysis.

Therefore, the asymptotic dynamics of dispersive equations with a saturated perturbation becomes a natural question.

### 4.1.3 Results for $L^2$ critical gKdV equations

Let us recall some results for the following  $L^2$  critical gKdV equations:

$$\begin{cases} \partial_t u + (u_{xx} + u^5)_x = 0, & (t, x) \in [0, T) \times \mathbb{R}, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}). \end{cases} \quad (\text{gKdV})$$

This equation is  $L^2$  critical, since for all  $\lambda > 0$ ,

$$u_\lambda(t, x) = \lambda^{-\frac{1}{2}} u(\lambda^{-3}t, \lambda^{-1}x),$$

is still a solution to (gKdV) and  $\|u_\lambda\|_{L^2} = \|u\|_{L^2}$ .

There is a special class of solutions i.e. the solitary waves, which is given by

$$u(t, x) = \frac{1}{\lambda_0^{1/2}} Q\left(\frac{x - x_0 - \lambda_0^{-2}(t - t_0)}{\lambda_0}\right),$$

with

$$Q(x) = \left( \frac{3}{\cosh^2(2x)} \right)^{\frac{1}{4}}, \quad Q'' - Q + Q^5 = 0.$$

The function  $Q$  is called the ground state.

From variational arguments [91], we know that if  $\|u_0\|_{L^2} < \|Q\|_{L^2}$ , then the solution to (gKdV) is global in time and bounded in  $H^1$ , while for  $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$ , blow up may occur. The blow up dynamics for solution with slightly supercritical mass:

$$\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^* \quad (4.1.3)$$

has been developed in a series paper of Martel and Merle [53, 54, 55, 70]. In particular, they prove the existence of blow up solutions with negative energy, and give a specific description of the blow up dynamics and the formation of singularity.

In [57, 58, 59], Martel, Merle and Raphaël give a exclusive study of the asymptotic dynamics near the ground state  $Q$ .

More precisely, consider the following initial data set

$$\mathcal{A}_{\alpha_0} = \left\{ u_0 \in H^1 \mid u_0 = Q + \varepsilon_0, \|\varepsilon_0\|_{H^1} < \alpha_0, \int_{y>0} y^{10} \varepsilon_0^2 < 1 \right\},$$

and the following  $L^2$  tube around the solitary wave family

$$\mathcal{T}_{\alpha^*} = \left\{ u_0 \in H^1 \mid \inf_{\lambda_0 > 0, x_0 \in \mathbb{R}} \left\| u_0 - \frac{1}{\lambda_0^{\frac{1}{2}}} Q \left( \frac{x - x_0}{\lambda_0} \right) \right\|_{L^2} < \alpha^* \right\}.$$

Then we have:

**Theorem 4.1.1.** *For  $0 < \alpha_0 \ll \alpha^* \ll 1$ , and  $u_0 \in \mathcal{A}_{\alpha_0}$ , let  $u(t)$  be the corresponding solution to (gKdV), and  $0 < T \leq +\infty$  be the maximal lifetime. Then one of the following scenarios occurs:*

(Blow up): *The solution  $u(t)$  blows up in finite time  $0 < T < +\infty$  with*

$$\|u(t)\|_{H^1} = \frac{\ell(u_0) + o(1)}{T - t}, \quad \ell(u_0) > 0.$$

*In addition, for all  $t < T$ ,  $u(t) \in \mathcal{A}_{\alpha_0}$ .*

(Soliton): *The solution is global, and for all  $t < T = +\infty$ ,  $u(t) \in \mathcal{A}_{\alpha_0}$ . In addition, there exist a constant  $\lambda_\infty > 0$  and a  $C^1$  function  $x(t)$  such that*

$$\begin{aligned} \lambda_\infty^{\frac{1}{2}} u(t, \lambda_\infty \cdot + x(t)) &\rightarrow Q \text{ in } H_{\text{loc}}^1, \text{ as } t \rightarrow +\infty, \\ |\lambda_\infty - 1| &\lesssim \delta(\alpha_0), \quad x(t) \sim \frac{t}{\lambda_\infty^2}, \text{ as } t \rightarrow +\infty. \end{aligned}$$

(Exit): *For some finite time  $0 < t^* < T$ ,  $u(t^*) \notin \mathcal{T}_{\alpha^*}$ .*

*Moreover, the scenarios (Blow up) and (Exit) are stable by small perturbation in  $\mathcal{A}_{\alpha_0}$ .*

In [56], Martel, Merle, Nakanishi and Raphaël proved that the initial data in  $\mathcal{A}_{\alpha_0}$  which corresponds to the (Soliton) regime is a codimension one threshold submanifold between (Blow up) and (Exit).

**Theorem 4.1.2.** *Let*

$$\mathcal{A}_{\alpha_0}^{\perp} = \left\{ \varepsilon_0 \in H^1 \mid \|\varepsilon_0\|_{H^1} < \alpha_0, \int_{y>0} y^{10} \varepsilon_0^2 < 1, (\varepsilon_0, Q) = 0 \right\}.$$

*Then there exist  $\alpha_0 > 0$ ,  $\beta_0 > 0$ , and a  $C^1$  function  $A$ :*

$$\mathcal{A}_{\alpha_0}^{\perp} \rightarrow (-\beta_0, \beta_0),$$

*such that for all  $\gamma_0 \in \mathcal{A}_{\alpha_0}^{\perp}$  and  $a \in [-\beta_0, \beta_0]$ , the solution of (gKdV) corresponding to  $u_0 = (1+a)Q + \gamma_0$  satisfies:*

- (Soliton) if  $a = A(\gamma_0)$ ;
- (Blow up) if  $a > A(\gamma_0)$ ;
- (Exit) if  $a < A(\gamma_0)$ .

*In particular, let*

$$\mathcal{Q} = \left\{ u_0 \in H^1 \mid \exists \lambda_0, x_0, \text{ such that } u_0 = \lambda_0^{-\frac{1}{2}} Q(\lambda_0^{-1}(x - x_0)) \right\}.$$

*then there exists a small neighborhood  $\mathcal{O}$  of  $\mathcal{Q}$  in  $H^1 \cap L^2(y_+^{10} dy)$  and a codimension one  $C^1$  submanifold  $\mathcal{M}$  of  $\mathcal{O}$ , such that  $\mathcal{Q} \subset \mathcal{M}$  and for all  $u_0 \in \mathcal{O}$  the corresponding solution of (gKdV) is in the (Soliton) regime if and only if  $u_0 \in \mathcal{M}$ .*

#### 4.1.4 Statement of the main result

The aim of this paper is to classify the dynamics of (gKdV $_{\gamma}$ ) near the ground state  $Q$  for (gKdV), when  $\gamma$  is small enough. The main idea is that the defocusing term  $\gamma u|u|^{q-1}$  has weaker nonlinear effect than the focusing term  $u^5$ . So, we may expect that (gKdV $_{\gamma}$ ) has similar separation behavior as (gKdV), when  $\gamma$  is small.

More precisely, we fix a small universal constant  $\omega^* > 0$  (to ensure the existence of the ground state  $Q_{\omega}$ ), and then introduce the following  $L^2$  tube around  $Q_{\gamma}$ :

$$\mathcal{T}_{\alpha^*, \gamma} = \left\{ u_0 \in H^1 \mid \inf_{\lambda_0 > 0, \lambda_0^{-m} \gamma < \omega^*, x_0 \in \mathbb{R}} \left\| u_0 - \frac{1}{\lambda_0^{\frac{1}{2}}} Q_{\lambda_0^{-m} \gamma} \left( \frac{x - x_0}{\lambda_0} \right) \right\|_{L^2} < \alpha^* \right\}.$$

Then we have:

**Theorem 4.1.3** (Dynamics in  $\mathcal{A}_{\alpha_0}$ ). *For all  $q > 5$ , there exists a constant  $0 < \alpha^*(q) \ll 1$ , such that if  $0 < \gamma \ll \alpha_0 \ll \alpha^* < \alpha^*(q)$ , then for all  $u_0 \in \mathcal{A}_{\alpha_0}$ , the corresponding solution  $u(t)$  to (gKdV $_{\gamma}$ ) has one and only one of the following behaviors:*

- (Soliton): *For all  $t \in [0, +\infty)$ ,  $u(t) \in \mathcal{T}_{\alpha^*, \gamma}$ . Moreover, there exist a constant  $\lambda_{\infty} \in (0, +\infty)$  and a  $C^1$  function  $x(t)$  such that*

$$\lambda_{\infty}^{\frac{1}{2}} u(t, \lambda_{\infty} \cdot + x(t)) \rightarrow Q_{\lambda_{\infty}^{-m} \gamma} \text{ in } H_{\text{loc}}^1, \text{ as } t \rightarrow +\infty; \quad (4.1.4)$$

$$x(t) \sim \frac{t}{\lambda_{\infty}^2}, \text{ as } t \rightarrow +\infty. \quad (4.1.5)$$

-(Blow down): For all  $t \in [0, +\infty)$ ,  $u(t) \in \mathcal{T}_{\alpha^*, \gamma}$ . Moreover, there exist two  $C^1$  functions  $\lambda(t)$  and  $x(t)$ , such that

$$\lambda^{\frac{1}{2}}(t)u(t, \lambda(t) \cdot +x(t)) \rightarrow Q \text{ in } H_{\text{loc}}^1, \text{ as } t \rightarrow +\infty; \quad (4.1.6)$$

$$\lambda(t) \sim t^{\frac{2}{q+1}}, \quad x(t) \sim t^{\frac{q-3}{q+1}}, \quad \text{as } t \rightarrow +\infty, \quad (4.1.7)$$

-(Exit): There exists a  $0 < t_\gamma^* < +\infty$  such that  $u(t_\gamma^*) \notin \mathcal{T}_{\alpha^*, \gamma}$ .

There exist solutions associated to each regime. Moreover, the regime (Soliton) and (Exit) are stable under small perturbation in  $\mathcal{A}_{\alpha_0}$ .

*Comments on Theorem 4.1.3:*

1. *Classification of the flow near ground state.* Theorem 4.1.3 gives a detailed description of the flow near the ground state  $Q_\gamma$  of (gKdV $_\gamma$ ). This kind of problems has attracted considerable attention especially for the dispersive equations. For example, Nakanishi and Schlag [81, 82, 83] for Klein-Gordon and mass supercritical nonlinear Schrödinger equations; Merle-Raphaël [22, 71, 72, 73, 74, 86] and Merle-Raphaël-Szeftel [76] for mass critical nonlinear Schrödinger equations; Martel-Merle-Raphaël [57, 58] for  $L^2$  critical gKdV equations; Kenig-Merle [34] and Duyckaerts-Merle [18] for energy-critical nonlinear Schrödinger equations; Kenig-Merle [35], Duyckaerts-Merle [17] and Krieger-Nakanishi-Schlag [42, 43] for energy-critical wave equations; Collot-Merle-Raphaël [8] for energy critical nonlinear heat equations. Note that the fact that the regime (blow down) near the ground state is a codimension one threshold submanifold of initial data in  $\mathcal{A}_{\alpha_0}$  could be proved similarly as in [56].

2. *Asymptotic stability of solitons for (gKdV $_\gamma$ ).* Since the (Soliton) regime is open, Theorem 4.1.3 also implies the asymptotic stability of the soliton  $Q_\gamma$  for (gKdV $_\gamma$ ) under some suitable decay assumption. Recall that from [52], the soliton  $Q$  for the unperturbed critical gKdV equation is not stable in  $H^1$ .

3. *Blow down behaviors.* Theorem 4.1.3 shows that a saturated perturbation may lead to some chaotic behaviors (i.e. the blow down behaviors), which does not seem to appear in the unperturbed case. Examples for solution with a blow down behavior was also found by Donninger, Krieger [16] for energy critical wave equations. While for mass critical NLS, the blow down behavior can be obtained as the pseudo-conformal transformation of the log-log regime<sup>3</sup>. However, Theorem 4.1.3 is the first time that this type of blow down behavior is obtained in the context of a saturated perturbation. Furthermore, in Theorem 4.1.3, the (blow down) regime is a codimension one threshold between two stable ones, which is in contrast with the mass critical nonlinear Schrödinger case, where the blow down regime is stable.

Now we consider the case when  $\gamma \rightarrow 0$ . As we mentioned before, the defocusing term  $\gamma u|u|^{q-1}$  has weaker nonlinear effect than the focusing term  $u^5$ . So the results in Theorem 4.1.3 are expected to be a perturbation of the one in Theorem 4.1.1.

More precisely, we have:

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3. See [76, (1.16)] for example.

**Theorem 4.1.4.** *Let us fix a nonlinearity  $q > 5$ , and choose  $0 < \alpha_0 \ll \alpha^* < \alpha^*(q)$  as in Theorem 4.1.3. For all  $u_0 \in \mathcal{A}_{\alpha_0}$ , let  $u(t)$  be the corresponding solution of (gKdV), and  $u_\gamma(t)$  be the corresponding solution of (gKdV $_\gamma$ ). Then we have:*

1. *If  $u(t)$  is in the (Blow up) regime defined in Theorem 4.1.1, then there exists  $0 < \gamma(u_0, \alpha_0, \alpha^*, q) \ll \alpha_0$  such that if  $0 < \gamma < \gamma(u_0, \alpha_0, \alpha^*, q)$ , then  $u_\gamma(t)$  is in the (Soliton) regime defined in Theorem 4.1.3. Moreover, there exist constants  $d_i = d_i(u_0, q) > 0$ ,  $i = 1, 2$ , such that*

$$d_1 \gamma^{\frac{2}{q-1}} \leq \lambda_\infty \leq d_2 \gamma^{\frac{2}{q-1}}, \quad (4.1.8)$$

where  $\lambda_\infty$  is the constant defined in (4.1.4).

2. *If  $u(t)$  is in the (Exit) regime defined in Theorem 4.1.1, then there exists  $0 < \gamma(u_0, \alpha_0, \alpha^*, q) \ll \alpha_0$  such that if  $0 < \gamma < \gamma(u_0, \alpha_0, \alpha^*, q)$ , then  $u_\gamma(t)$  is in the (Exit) regime defined in Theorem 4.1.3.*

*Remark 4.1.5.* We can see from Theorem 4.1.4 that (gKdV $_\gamma$ ) is a perturbation of (gKdV) as  $\gamma \rightarrow 0$ : the (Soliton) regime of (gKdV $_\gamma$ ) “converges” to the (Blow up) regime of (gKdV), and the (Exit) regime “converges” to the (Exit) regime of (gKdV).

*Remark 4.1.6.* Theorem 4.1.4 is the first result of this type for nonlinear dispersive equations. One may also expect similar results for the critical NLS<sup>4</sup> or the slightly supercritical gKdV cases. But they are still completely open.

## 4.1.5 Notation

For  $0 \leq \omega < \omega^* \ll 1$ , we let  $Q_\omega$  be the unique nonnegative radial solution with exponential decay to the following ODE:

$$Q_\omega'' - Q_\omega + Q_\omega^5 - \omega Q_\omega |Q_\omega|^{q-1} = 0. \quad (4.1.9)$$

For simplicity, we denote by  $Q = Q_0$ . Recall that we have:

$$Q(x) = \left( \frac{3}{\cosh^2(2x)} \right)^{\frac{1}{4}}.$$

We also introduce the linearized operator at  $Q_\omega$ :

$$L_\omega f = -f'' + f - 5Q_\omega^4 f + q\omega |Q_\omega|^{q-1} f.$$

Similarly, we denote by  $L = L_0$ .

Next, we introduce the scaling operator:

$$\Lambda f = \frac{1}{2} f + y f'.$$

---

4. In [48], Malkin predicted a similar asymptotic behavior for the solution to the saturated problem of critical NLS.



Then, for a given small constant  $\alpha^*$ , we denote by  $\delta(\alpha^*)$  a generic small constant with

$$\lim_{\alpha^* \rightarrow 0} \delta(\alpha^*) = 0.$$

Finally, we denote the  $L^2$  scalar product by

$$(f, g) = \int f(x)g(x) dx$$

## 4.1.6 Outline of the proof

### 4.1.6.1 Decomposition of the flow

We are searching for solutions of the following form:

$$u(t, x) \sim \frac{1}{\lambda(t)^{\frac{1}{2}}} Q_{b(t), \omega(t)} \left( \frac{x - x(t)}{\lambda(t)} \right),$$

$$\omega = \frac{\gamma}{\lambda^m}, \quad \frac{ds}{dt} = \frac{1}{\lambda^3}, \quad \frac{\lambda_s}{\lambda} = -b, \quad \frac{x_s}{\lambda} = 1,$$

which lead to the modified self-similar equation:

$$b\Lambda Q_{b, \omega} + (Q_{b, \omega}'' - Q_{b, \omega} + Q_{b, \omega}^5 - \omega Q_{b, \omega} |Q_{b, \omega}|^{q-1})' = 0. \quad (4.1.10)$$

Formal computations show that  $b$  and  $\omega$  must satisfy the following condition:

$$b_s + 2b^2 + c_0 \omega_s = 0,$$

where  $c_0 = c_0(q) > 0$  is a universal constant.

Combining all the above, we get the following formal finite dimensional system:

$$\begin{cases} \frac{ds}{dt} = \frac{1}{\lambda^3}, \quad \frac{\lambda_s}{\lambda} = -b, \quad \frac{x_s}{\lambda} = 1, \\ b_s + 2b^2 + c_0 \omega_s = 0, \quad \omega = \frac{\gamma}{\lambda^m}. \end{cases} \quad (4.1.11)$$

By standard computations, it is easy to see that (4.1.11) has the following behavior. Let

$$L_0 = \frac{b(0)}{\lambda^2(0)} + \frac{mc_0\gamma}{(m+2)\lambda^{m+2}(0)}.$$

We have:

1. If  $L_0 > 0$ , then

$$b(t) \rightarrow 0, \quad \lambda(t) \rightarrow \left( \frac{m\gamma c_0}{(m+2)L_0} \right)^{\frac{1}{m+2}}, \quad x(t) \sim \left( \frac{(m+2)L_0}{m\gamma c_0} \right)^{\frac{2}{m+2}} t,$$

as  $t \rightarrow +\infty$ , which corresponds to the (Soliton) regime.

2. If  $L_0 = 0$ , then

$$b(t) \rightarrow 0, \quad \lambda(t) \rightarrow +\infty, \quad x(t) \rightarrow +\infty,$$

as  $t \rightarrow +\infty$ , which corresponds to the (Blow down) regime.

3. If  $L_0 < 0$ , then

$$b(t) \rightarrow -\infty, \quad \lambda(t) \rightarrow +\infty,$$

as  $t \rightarrow +\infty$ , which corresponds to the (Exit) regime.

### 4.1.6.2 Modulation theory

Our first step is to find a solution to (4.1.10). But for our analysis, it is enough to consider a suitable approximation<sup>5</sup>:

$$Q_{b,\omega}(y) = Q_\omega(y) + b\chi(|b|^\beta y)P_\omega(y).$$

As long as the solution remains in  $\mathcal{T}_{\alpha^*,\gamma}$ , we can introduce the following geometrical decomposition:

$$u(t) = \frac{1}{\lambda(t)^{\frac{1}{2}}} \left[ Q_{b(t),\omega(t)} + \varepsilon(t) \right] \left( \frac{x-x(t)}{\lambda(t)} \right)$$

with  $\omega(t) = \gamma/\lambda(t)^m$ , and the error term satisfies some orthogonality conditions. Then the equation of the parameters are roughly speaking of the following form:

$$\frac{\lambda_s}{\lambda} + b = \frac{dJ_1}{ds} + O(\|\varepsilon\|_{H_{\text{loc}}^1}^2), \quad b_s + 2b^2 + c_0\omega_s = \frac{dJ_2}{ds} + O(\|\varepsilon\|_{H_{\text{loc}}^1}^2),$$

with

$$|J_i| \lesssim \|\varepsilon\|_{H_{\text{loc}}^1} + \int_{y>0} |\varepsilon|.$$

Therefore, a  $L^1$  control on the right is needed, otherwise  $J_i$  will perturb the formal system (4.1.11).

### 4.1.6.3 Monotonicity Formula

Our next step is to derive a control for  $\|\varepsilon\|_{H_{\text{loc}}^1}$ . Similar to [57], we introduce the following nonlinear functional:

$$\mathcal{F} \sim \int (\psi\varepsilon_y^2 + \varphi\varepsilon^2 - 5\varepsilon^2 Q_{b,\omega}\psi + q\omega\varepsilon^2 |Q_{b,\omega}|^{q-1}\psi),$$

for some well-chosen weight functions  $(\psi, \varphi)$ , which decay exponentially to the left, and grow polynomially on the right. We will see from the choice of the orthogonality condition that the leading quadratic term of  $\mathcal{F}$  is coercive:

$$\mathcal{F} \gtrsim \|\varepsilon\|_{H_{\text{loc}}^1}^2.$$

Most importantly, we have the following monotonicity formula:

$$\frac{d}{ds} \left( \frac{\mathcal{F}}{\lambda^{2j}} \right) + \frac{\|\varepsilon\|_{H_{\text{loc}}^1}^2}{\lambda^{2j}} \lesssim \frac{\omega^2 b^2 + b^4}{\lambda^{2j}},$$

for  $j = 0, 1$ . This formula is crucial in all three cases.

---

5. See Section 2.1 for more details.

#### 4.1.6.4 Rigidity

The selection of the dynamics depends on:

1. For all  $t$ ,

$$\left| b(t) + \frac{mc_0}{m+2} \omega(t) \right| \lesssim \|\varepsilon(t)\|_{H_{\text{loc}}^1}^2 + b^2(t) + \omega^2(t).$$

2. For some  $t_1^* < T = +\infty$ ,

$$b(t_1^*) + \frac{mc_0}{m+2} \omega(t_1^*) \gg \|\varepsilon(t)\|_{H_{\text{loc}}^1}^2 + b^2(t) + \omega^2(t).$$

3. For some  $t_1^* < T = +\infty$ ,

$$-b(t_1^*) - \frac{mc_0}{m+2} \omega(t_1^*) \gg \|\varepsilon(t)\|_{H_{\text{loc}}^1}^2 + b^2(t) + \omega^2(t).$$

We will see that in the first case we have for all  $t$ ,

$$|b(t)| \sim \omega(t) \gg \|\varepsilon(t)\|_{H_{\text{loc}}^1}^2,$$

And in the second case we have

$$\omega(t) \gg |b(t)| \gg \|\varepsilon(t)\|_{H_{\text{loc}}^1}^2,$$

for  $t > t_2^* \geq t_1^*$  as long as  $u(t)$  remains in  $\mathcal{T}_{\alpha^*, \gamma}$ . While in the third case, we have

$$-b(t) \gg \omega(t) \gg \|\varepsilon(t)\|_{H_{\text{loc}}^1}^2,$$

for  $t > t_1^*$  as long as  $u(t)$  remains in  $\mathcal{T}_{\alpha^*, \gamma}$ . Then reintegrating the modulation equations, we will see that these three cases correspond to the (Blow down), (Soliton) and (Exit) regimes respectively.

Moreover, the condition on  $b(t_1^*)$  and  $\omega(t_1^*)$  which determines the (Soliton) and (Exit) regimes is an open condition to the initial data due to the continuity of the flow. On the other hand, it is easy to construct solutions, which belongs to the (Soliton) and (Exit) regime respectively. Since, the initial data set  $\mathcal{A}_{\alpha_0}$  is connected, we can see that there exist solutions corresponding to the (Blow down) regime.

#### 4.1.6.5 Proof of Theorem 4.1.4

The proof of Theorem 4.1.4 is based on the fact that the separation condition for (gKdV $_{\gamma}$ ) is close to the separation condition for (gKdV), when  $\gamma \rightarrow 0$ . Then Theorem 4.1.4 follows immediately from a modified  $H^1$  perturbation theory<sup>6</sup>.

### Acknowledgement

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6. See [37, Theorem 3.1] for the standard  $L^2$  perturbation theory.

## 4.2 Nonlinear profile and decomposition of the flow

In this section we will introduce the nonlinear profile and the geometrical decomposition similar to the one in [57], which turns out to lead to the desired rigidity dynamics.

### 4.2.1 Structure of the linearized operator $L_\omega$

Denote by  $\mathcal{Y}$  the set of smooth function  $f$  such that for all  $k \in \mathbb{N}$ , there exist  $r_k > 0$ ,  $C_k > 0$ , with

$$|\partial_y^k f(y)| \leq C_k (1 + |y|)^{r_k} e^{-|y|}.$$

Let us first recall some results about the linearized operator  $L$ .

**Lemma 4.2.1** (Properties of  $L$ , [52], [57], [92]). *The self-adjoint operator  $L$  in  $L^2$  has the following properties:*

1. *Eigenfunction:  $LQ^3 = -8Q^3$ ,  $LQ' = 0$ ,  $\ker L = \{aQ' | a \in \mathbb{R}\}$ .*
2. *Scaling:  $L(\Lambda Q) = -2Q$ .*
3. *For any function  $f \in L^2$  orthogonal to  $Q'$ , there exist a unique  $g \in H^2$  such that  $Lg = f$  with  $(g, Q') = 0$ . Moreover, if  $f$  is even (or respectively odd), then  $g$  is even (or respectively odd).*
4. *If  $f \in L^2$  such that  $Lf \in \mathcal{Y}$ , then  $f \in \mathcal{Y}$ .*
5. *Coercivity: For all  $f \in H^1$ , if  $(f, Q^3) = (f, Q') = 0$ , then  $(Lf, f) \geq (f, f)$ . Moreover, there exists a  $\kappa_0 > 0$  such that for all  $f \in H^1$ ,*

$$(Lf, f) \geq \kappa_0 \|f\|_{H^1}^2 - \frac{1}{\kappa_0} \left[ (f, Q)^2 + (f, \Lambda Q)^2 + (f, y\Lambda Q)^2 \right].$$

**Proposition 4.2.2** (Nonlocalized profiles, [57]). *There exist a unique function  $P$  with  $P' \in \mathcal{Y}$ , such that:*

$$(LP)' = \Lambda Q, \quad \lim_{y \rightarrow -\infty} P(y) = \frac{1}{2} \int Q, \quad |P(y)| \lesssim e^{-\frac{y}{2}} \text{ for } y > 0, \quad (4.2.1)$$

$$(P, Q) = \frac{1}{16} \left( \int Q \right)^2, \quad (P, Q') = 0. \quad (4.2.2)$$

Now for the ground state  $Q_\omega$  and the linearized operator  $L_\omega$ , we have the following properties:

**Lemma 4.2.3.** *For  $0 < \omega < \omega^* \ll 1$ , we have:*

1. *Null space:  $\ker L_\omega = \{aQ'_\omega | a \in \mathbb{R}\}$ .*
2. *Pseudo-scaling rule:  $L_\omega(\Lambda Q_\omega) = -2Q_\omega + \frac{q-5}{2}\omega Q_\omega^q$ .*
3. *For any function  $f \in L^2$  orthogonal to  $Q'_\omega$ , there exist a unique  $g \in H^2$  such that  $L_\omega g = f$  with  $(g, Q'_\omega) = 0$ . Moreover, if  $f$  is even (or respectively odd), then  $g$  is even (or respectively odd).*

4. If  $f \in L^2$  such that  $L_\omega f \in \mathcal{Y}$ , then  $f \in \mathcal{Y}$ .
5. Let  $Z_\omega = \frac{\partial Q_\omega}{\partial \omega}$ , then  $Z_\omega \in \mathcal{Y}$ , and  $L_\omega Z_\omega = -Q_\omega |Q_\omega|^{q-1}$ .
6. Coercivity: There exists a  $\kappa_0 > 0$  such that for all  $f \in H^1$ ,

$$(L_\omega f, f) \geq \kappa_0 \|f\|_{H^1}^2 - \frac{1}{\kappa_0} \left[ (f, Q_\omega)^2 + (f, \Lambda Q_\omega)^2 + (f, y \Lambda Q_\omega)^2 \right].$$

*Proof.* (1) follows from the arguments in [92]. (2) follows from direct computation. (3) is a direct corollary of (1). While for (4), from standard elliptic theory, we know that  $f$  is smooth and bounded. So we have  $Lf \in \mathcal{Y}$ , from Lemma 4.2.1, we have  $f \in \mathcal{Y}$ .

Now we turn to the proof of (5). Differentiating the equation (4.1.9), we obtain  $L_\omega Z_\omega = -Q_\omega |Q_\omega|^{q-1}$ . Since  $Q_\omega |Q_\omega|^{q-1} \in \mathcal{Y}$ , if we can show that  $Z_\omega \in L^2$ , then we have  $Z_\omega \in \mathcal{Y}$ . To do this, we introduce the following map:

$$F : H_e^2 \times \mathbb{R} \mapsto L_e^2, \quad (u, \omega) \mapsto -u'' + u - u^5 + \omega u |u|^{q-1},$$

where  $H_e^2$  (respectively  $L_e^2$ ) is the Banach space consists of all  $H^2$  (respectively  $L^2$ ) functions which are even. Since  $H^2(\mathbb{R})$  is continuously embedded into  $L^\infty(\mathbb{R})$ , the map  $F$  is well-defined.

We claim that there exists a small  $\omega^* > 0$ , such that if  $0 \leq \omega < \omega^*$ , then there exist a unique  $u(\omega) \in H_e^2$ , such that  $F(u(\omega), \omega) = 0$ . Since we have  $F(Q, 0) = 0$ , from implicit function theory, we only remains to show that the Fréchet derivative with respect to  $u$ , i.e.  $\frac{\partial F}{\partial u} \Big|_{(Q,0)} \in \mathcal{L}(H_e^2, L_e^2)$  is invertible and continuous. But it is easy to see that

$$\frac{\partial F}{\partial u} \Big|_{(Q,0)} = L,$$

which is invertible and continuous due to (3) of Lemma 4.2.1. Hence, we obtain the existence of such  $u(\omega)$ . Moreover, since  $F$  is continuously differentiable with respect to both  $u$  and  $\omega$ . So we have  $u(\omega)$  is continuously differentiable with respect to  $\omega$ . In particular, we have  $\frac{\partial u}{\partial \omega} \in H_e^2$ . But from the uniqueness of  $u(\omega)$ , we must have  $u(\omega) = Q_\omega$ . As a consequence, we have  $Z_\omega = \frac{\partial Q_\omega}{\partial \omega} = \frac{\partial u}{\partial \omega} \in H_e^2$ , which concludes the proof of (5).

Finally, (6) follows immediately from a some perturbation of (5) of Lemma 4.2.1. We then finish the proof of Lemma 4.2.3. □

**Proposition 4.2.4.** For  $0 < \omega < \omega^* \ll 1$ , there exist a smooth function  $P_\omega$  with  $P'_\omega \in \mathcal{Y}$ , such that:

$$(L_\omega P_\omega)' = \Lambda Q_\omega, \quad \lim_{y \rightarrow -\infty} P_\omega(y) = \frac{1}{2} \int Q_\omega, \quad (4.2.3)$$

$$(P_\omega, Q'_\omega) = 0, \quad (P_\omega, Q_\omega) = \frac{1}{16} \left( \int Q \right)^2 + F(\omega), \quad (4.2.4)$$

where  $F$  is a  $C^1$  function with  $F(0) = 0$ . Moreover there exist constants  $C_0, C_1, \dots$ , independent of  $\omega$ , such that

$$|P_\omega(y)| + \left| \frac{\partial P_\omega}{\partial \omega}(y) \right| \leq C_0 e^{-\frac{y}{2}}, \text{ for all } y > 0, \quad (4.2.5)$$

$$|P_\omega(y)| + \left| \frac{\partial P_\omega}{\partial \omega}(y) \right| \leq C_0, \text{ for all } y \in \mathbb{R}, \quad (4.2.6)$$

$$|\partial_y^k P_\omega(y)| \leq C_k e^{-\frac{|y|}{2}} \text{ for all } k \in \mathbb{N}_+, y \in \mathbb{R}. \quad (4.2.7)$$

*Proof.* The proof of Proposition 4.2.4 is almost parallel to Proposition 4.2.2. We look for solution of the form  $P_\omega = \tilde{P}_\omega - \int_y^{+\infty} \Lambda Q_\omega$ . The function  $y \rightarrow \int_y^{+\infty} \Lambda Q_\omega$  is bounded and decays exponentially as  $y \rightarrow +\infty$ . Then,  $P_\omega$  solves (4.2.3) if and only if  $\tilde{P}_\omega$  solves

$$(L_\omega \tilde{P}_\omega)' = \Lambda Q_\omega + \left( L_\omega \int_y^{+\infty} \Lambda Q_\omega \right)' = R'_\omega,$$

where

$$R_\omega = (\Lambda Q_\omega)' - 5Q_\omega^4 \int_y^{+\infty} \Lambda Q_\omega + q\omega |Q_\omega|^{q-1} \int_y^{+\infty} \Lambda Q_\omega.$$

Note that  $R_\omega \in \mathcal{Y}$ . Since  $(\Lambda Q_\omega, Q_\omega) = 0$  and  $L_\omega Q'_\omega = 0$ , we have  $(R_\omega, Q'_\omega) = -(R'_\omega, Q_\omega) = 0$ . Then from Lemma 4.2.3, there exists a unique  $\tilde{P}_\omega \in \mathcal{Y}$ , orthogonal to  $Q'_\omega$ , such that  $L_\omega \tilde{P}_\omega = R_\omega$ . Then  $P_\omega = \tilde{P}_\omega - \int_y^{+\infty} \Lambda Q_\omega$  satisfies (4.2.3) with  $(P_\omega, Q'_\omega) = 0$  and  $\lim_{y \rightarrow -\infty} P_\omega(y) = \frac{1}{2} \int Q_\omega$ . Moreover, we have

$$\begin{aligned} 2 \int P_\omega Q_\omega &= - \int (L_\omega P_\omega) \Lambda Q_\omega + O(\omega) = \int \Lambda Q_\omega \int_y^{+\infty} \Lambda Q_\omega + O(\omega) \\ &= \frac{1}{2} \left( \int \Lambda Q_\omega \right)^2 + O(\omega) = \frac{1}{8} \left( \int Q \right)^2 + O(\omega). \end{aligned}$$

Let

$$F(\omega) = (P_\omega, Q_\omega) - \frac{1}{16} \left( \int Q \right)^2,$$

then  $F(0) = 0$ .

Next we claim that  $\frac{\partial \tilde{P}_\omega}{\partial \omega} \in \mathcal{Y}$ . Let us differentiate the equation  $L_\omega \tilde{P}_\omega = R_\omega$  to get

$$L_\omega \left( \frac{\partial \tilde{P}_\omega}{\partial \omega} \right) = \frac{\partial R_\omega}{\partial \omega} - 20Z_\omega Q_\omega^3 \tilde{P}_\omega + q(q-1)\omega Z_\omega Q_\omega |Q_\omega|^{q-3} \tilde{P}_\omega + q|Q_\omega|^{q-1} \tilde{P}_\omega. \quad (4.2.8)$$

Since  $Z_\omega \in \mathcal{Y}$ , it is easy to check that  $\frac{\partial R_\omega}{\partial \omega} \in \mathcal{Y}$ . So Lemma 4.2.3 implies that  $\frac{\partial \tilde{P}_\omega}{\partial \omega} \in \mathcal{Y}$ .

Now it only remains to prove (4.2.5)–(4.2.7). But from [2, Section 6], there exist constants  $M_0, M_1, \dots$ , independent of  $\omega$ , such that for all  $k \in \mathbb{N}$ ,  $y \in \mathbb{R}$ ,

$$|\partial_y^k Q_\omega(y)| \leq M_k e^{-\frac{2|y|}{3}}.$$

Together with (4.2.8) and the construction of  $P_\omega$ , we obtain (4.2.5)–(4.2.7). It is easy to see that (4.2.5)–(4.2.7) also implies that  $F \in C^1$ . Then we conclude the proof of Lemma 4.2.3.  $\square$

Now, we proceed to a simple localization of the profile to avoid the nontrivial tail on the left. Let  $\chi$  be a smooth function with  $0 \leq \chi \leq 1$ ,  $\chi' \geq 0$ ,  $\chi(y) = 1$  if  $y > -1$ ,  $\chi(y) = 0$  if  $y < -2$ . We fix a

$$\beta = \frac{3}{4}. \quad (4.2.9)$$

And define the localized profile:

$$\chi_b(y) = \chi(|b|^\beta y), \quad Q_{b,\omega}(y) = Q_\omega + b\chi_b(y)P_\omega(y). \quad (4.2.10)$$

**Lemma 4.2.5** (Localized Profiles). *For  $|b| < b^* \ll 1$ ,  $0 < \omega < \omega^* \ll 1$ , there holds:*

1. *Estimates on  $Q_b$ : For all  $y \in \mathbb{R}$ ,  $k \in \mathbb{N}$ ,*

$$|Q_{b,\omega}(y)| \lesssim e^{-|y|} + |b|(\mathbf{1}_{[-2,0]}(|b|^\beta y) + e^{-\frac{|y|}{2}}), \quad (4.2.11)$$

$$|\partial_y^k Q_{b,\omega}(y)| \lesssim e^{-|y|} + |b|e^{-\frac{|y|}{2}} + |b|^{1+k\beta} \mathbf{1}_{[-2,-1]}(|b|^\beta y), \quad (4.2.12)$$

where  $\mathbf{1}_I$  denotes the characteristic function of the interval  $I$ .

2. *Equation of  $Q_{b,\omega}$ : Let*

$$-\Psi_{b,\omega} = b\Lambda Q_{b,\omega} + (Q_{b,\omega}'' - Q_{b,\omega} + Q_{b,\omega}^5 - \omega Q_{b,\omega}|Q_{b,\omega}|^{q-1})'. \quad (4.2.13)$$

Then, for all  $y \in \mathbb{R}$ ,

$$\begin{aligned} -\Psi_{b,\omega} &= b^2 \left( (10Q_\omega^3 P_\omega^2)_y + \Lambda P_\omega \right) - \frac{1}{2} b^2 (1 - \chi_b) P_\omega \\ &\quad + O\left( |b|^{1+\beta} \mathbf{1}_{[-2,-1]}(|b|^\beta y) + b^2 (\omega + |b|) e^{-\frac{|y|}{2}} \right). \end{aligned} \quad (4.2.14)$$

Moreover, we have

$$|\partial_y \Psi_{b,\omega}(y)| \lesssim |b|^{1+2\beta} \mathbf{1}_{[-2,-1]}(|b|^\beta y) + b^2 e^{-\frac{|y|}{2}}. \quad (4.2.15)$$

3. *Mass and energy properties of  $Q_{b,\omega}$ :*

$$\left| \int Q_{b,\omega}^2 - \left( \int Q_\omega^2 + 2b \int P_\omega Q_\omega \right) \right| \lesssim |b|^{2-\beta}, \quad (4.2.16)$$

$$|E(Q_{b,\omega})| \lesssim |b| + \omega. \quad (4.2.17)$$

*Proof.* The proof of (1) follows immediately from the definition of  $Q_{b,\omega}$  and Proposition 4.2.4. For (2), let us expand  $Q_{b,\omega} = Q_\omega + b\chi_b P_\omega$  in the expression of  $\Psi_{b,\omega}$ , using the fact that

$$Q_\omega'' - Q_\omega + Q_\omega^5 - \omega Q_\omega |Q_\omega|^{q-1} = 0, \quad (L_\omega P_\omega)' = \Lambda Q_\omega,$$

we have:

$$\begin{aligned} -\Psi_{b,\omega} &= b(1 - \chi_b)\Lambda Q_\omega \\ &\quad + b(\chi_b''' P_\omega + 3\chi_b'' P_\omega' + 2\chi_b' P_\omega'' - \chi_b' P_\omega + 5\chi_b' Q_\omega P_\omega - q\omega \chi_b' |Q_\omega|^{q-1} P_\omega) \\ &\quad + b^2 \left( (10Q_\omega^3 \chi_b^2 P_\omega^2)_y + P_\omega \Lambda \chi_b + \chi_b y P_\omega' \right) \\ &\quad + b^3 (10Q_\omega^2 \chi_b^3 P_\omega^3)_y + b^4 (5Q_\omega \chi_b^4 P_\omega^4)_y + b^5 (\chi_b^5 P_\omega^5)_y \\ &\quad - \omega \left( (Q_\omega + b\chi_b P_\omega) |Q_\omega + b\chi_b P_\omega|^{q-1} - Q_\omega |Q_\omega|^{q-1} - qb\chi_b P_\omega |Q_\omega|^{q-1} \right)_y. \end{aligned}$$

We keep track of all terms up to  $b^2$ . Then (4.2.14) and (4.2.15) follow from the construction of the profile  $Q_{b,\omega}$ .

Finally, for (3), we have

$$\int \chi_b^2 P_\omega^2 \lesssim |b|^{-\beta}.$$

Then (4.2.16) follows from

$$\int Q_{b,\omega}^2 = \int Q_\omega^2 + 2b \int \chi_b P_\omega Q_\omega + b^2 \int \chi_b^2 P_\omega^2.$$

While for (4.2.17), since  $E(Q_\omega) = O(\omega)$ , we have:

$$|E(Q_{b,\omega})| \lesssim |b| + |E(Q_\omega)| \lesssim |b| + \omega,$$

which concludes the proof of Lemma 4.2.5.  $\square$

## 4.2.2 Geometrical decomposition and modulation estimates

In this paper we consider  $H^1$  solution to (gKdV $_\gamma$ ) a priori in the modulates tube  $\mathcal{T}_{\alpha^*,\gamma}$  of functions near the soliton manifold. More precisely, we have

**Lemma 4.2.6.** *Assume that there exist  $(\lambda_1(t), x_1(t)) \in ((\gamma/\omega^*)^{1/m}, +\infty) \times \mathbb{R}$  and  $\varepsilon_1(t)$  such that for all  $t \in [0, t_0)$ , the solution  $u(t)$  to (gKdV $_\gamma$ ) satisfies*

$$u(t, x) = \frac{1}{\lambda_1^{\frac{1}{2}}(t)} [Q_{\omega_1(t)} + \varepsilon_1(t)] \left( \frac{x - x_1(t)}{\lambda_1(t)} \right), \quad (4.2.18)$$

with,  $\forall t \in [0, t_0)$ ,

$$\omega_1(t) + \|\varepsilon_1(t)\|_{L^2} \leq \kappa \ll 1, \quad (4.2.19)$$

where

$$\omega_1(t) = \frac{\gamma}{\lambda_1^m(t)}.$$

Then we have

1. *There exist continuous functions  $(\lambda(t), x(t), b(t)) \in (0, +\infty) \times \mathbb{R}^2$ , such that for all  $t \in [0, t_0)$ ,*

$$\varepsilon(t, y) = \lambda^{\frac{1}{2}}(t) u(t, \lambda(t)y + x(t)) - Q_{b(t), \omega(t)} \quad (4.2.20)$$

*satisfies the orthogonality conditions:*

$$(\varepsilon(t), Q_{\omega(t)}) = (\varepsilon(t), \Lambda Q_{\omega(t)}) = (\varepsilon(t), y \Lambda Q_{\omega(t)}) = 0, \quad (4.2.21)$$

where

$$\omega(t) = \frac{\gamma}{\lambda^m(t)}.$$

Moreover,

$$\omega(t) + \|\varepsilon(t)\|_{L^2} + |b(t)| + \left| 1 - \frac{\lambda_1(t)}{\lambda(t)} \right| \lesssim \delta(\kappa), \quad (4.2.22)$$

$$\|\varepsilon(0)\|_{H^1} \lesssim \delta(\|\varepsilon_1(0)\|_{H^1}). \quad (4.2.23)$$



2. The parameters and error term depend continuously on the initial data. Considering a family of solutions  $u_n(t)$ , with  $u_{0,n} \in \mathcal{A}_{\alpha_0}$ , and  $u_{0,n} \rightarrow u_0$  in  $H^1$ , as  $n \rightarrow +\infty$ . Let  $(\lambda_n(t), b_n(t), x_n(t), \varepsilon_n(t))$  be the corresponding geometrical parameters and error terms of  $u_n(t)$ . Suppose the geometrical decomposition of  $u_n(t)$  and  $u(t)$  hold on  $[0, T_0]$  for some  $T_0 > 0$ . Then for all  $t \in [0, T_0]$ , we have:

$$(\lambda_n(t), b_n(t), x_n(t), \varepsilon_n(t)) \xrightarrow{\mathbb{R}^3 \times H^1} (\lambda(t), b(t), x(t), \varepsilon(t)), \quad (4.2.24)$$

as  $n \rightarrow +\infty$ .

*Remark 4.2.7.* Lemma 4.2.6 is a standard consequence of the implicit function theory. We refer to [53, 55, 57] for detailed proof.

*Remark 4.2.8.* The smallness of  $\omega(t)$  ensures that  $\mathcal{Q}_{\omega(t)}$  and  $\mathcal{Q}_{b(t), \omega(t)}$  are both well defined.

### 4.2.3 Modulation Equation

In the frame work of Lemma 4.2.6, we introduce the rescaled variables  $(s, y)$

$$y = \frac{x - x(t)}{\lambda(t)}, \quad s = \int_0^t \frac{1}{\lambda^3(\tau)} d\tau. \quad (4.2.25)$$

Then, we have the following properties:

**Proposition 4.2.9.** Assume for all  $t \in [0, t_0)$ ,

$$\omega(t) + \|\varepsilon(t)\|_{L^2} + \int \varepsilon_y^2 e^{-\frac{3|y|}{2(q-2)}} dy \leq \kappa \quad (4.2.26)$$

for some small universal constant  $\kappa > 0$ . Then the functions  $(\lambda(s), x(s), b(s))$  are all  $C^1$  and the following holds

1. Equation of  $\varepsilon$ : For all  $s \in [0, s_0)$ ,

$$\begin{aligned} \varepsilon_s - (L\omega\varepsilon)_y + b\Lambda\varepsilon &= \left(\frac{\lambda_s}{\lambda} + b\right)(\Lambda\mathcal{Q}_{b,\omega} + \Lambda\varepsilon) + \left(\frac{x_s}{\lambda} - 1\right)(\mathcal{Q}_{b,\omega} + \varepsilon)_y \\ &\quad - b_s \frac{\partial \mathcal{Q}_{b,\omega}}{\partial b} - \omega_s \frac{\partial \mathcal{Q}_{b,\omega}}{\partial \omega} + \Psi_{b,\omega} - (R_b(\varepsilon))_y - (R_{NL}(\varepsilon))_y, \end{aligned} \quad (4.2.27)$$

where

$$\Psi_{b,\omega} = -b\Lambda\mathcal{Q}_{b,\omega} - (\mathcal{Q}_{b,\omega}'' - \mathcal{Q}_{b,\omega} + \mathcal{Q}_{b,\omega}^5 - \omega\mathcal{Q}_{b,\omega}|\mathcal{Q}_{b,\omega}|^{q-1})', \quad (4.2.28)$$

$$R_b(\varepsilon) = 5(\mathcal{Q}_{b,\omega}^4 - \mathcal{Q}_{\omega}^4)\varepsilon - q\omega(|\mathcal{Q}_{b,\omega}|^{q-1} - |\mathcal{Q}_{\omega}|^{q-1})\varepsilon, \quad (4.2.29)$$

$$\begin{aligned} R_{NL}(\varepsilon) &= (\varepsilon + \mathcal{Q}_{b,\omega})^5 - 5\mathcal{Q}_{b,\omega}^4\varepsilon - \mathcal{Q}_{b,\omega}^5 - \omega[(\varepsilon + \mathcal{Q}_{b,\omega})|\varepsilon + \mathcal{Q}_{b,\omega}|^{q-1} \\ &\quad - q\varepsilon|\mathcal{Q}_{b,\omega}|^{q-1} - \mathcal{Q}_{b,\omega}|\mathcal{Q}_{b,\omega}|^{q-1}]. \end{aligned} \quad (4.2.30)$$

2. *Estimate induced by the conservation laws: for  $s \in [0, s_0)$ , there holds:*

$$\|\varepsilon\|_{L^2} \lesssim |b|^{\frac{1}{4}} + \omega^{\frac{1}{2}} + \left| \int u_0^2 - \int Q^2 \right|^{\frac{1}{2}}, \quad (4.2.31)$$

$$\frac{\|\varepsilon_y\|_{L^2}^2}{\lambda^2} \lesssim \frac{1}{\lambda^2} \left( \omega + |b| + \int \varepsilon^2 e^{-\frac{|y|}{10}} \right) + \gamma \frac{\|\varepsilon_y\|_{L^2}^{m+2}}{\lambda^{m+2}} + |E_0|. \quad (4.2.32)$$

3.  *$H^1$  modulation equation: for all  $s \in [0, s_0)$ ,*

$$\left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| \lesssim \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} + |b|(\omega + |b|), \quad (4.2.33)$$

$$|b_s| + |\omega_s| \lesssim (\omega + |b|) \left[ \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} + |b| \right] + \int \varepsilon^2 e^{-\frac{|y|}{10}}. \quad (4.2.34)$$

4.  *$L^1$  control on the right: Assume the uniformly  $L^1$  control on the right:  $\forall t \in [0, t_0)$*

$$\int_{y>0} |\varepsilon(t)| \lesssim \delta(\kappa). \quad (4.2.35)$$

*then the quantities  $J_1$  and  $J_2$  below are well-defined. Moreover, we have*

(a) *Law of  $\lambda$ : let*

$$\rho_1(y) = \frac{4}{(\int Q)^2} \int_{-\infty}^y \Lambda Q, \quad J_1(s) = (\varepsilon(s), \rho_1), \quad (4.2.36)$$

*where  $Q$  is the ground state for (gKdV). Then we have:*

$$\begin{aligned} & \left| \frac{\lambda_s}{\lambda} + b - 2 \left( (J_1)_s + \frac{1}{2} \frac{\lambda_s}{\lambda} J_1 \right) \right| \\ & \lesssim (\omega + |b|) \left[ \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} + |b| \right] + \int \varepsilon^2 e^{-\frac{|y|}{10}}. \end{aligned} \quad (4.2.37)$$

(b) *Law of  $b$ : let*

$$\begin{aligned} \rho_2 &= \frac{16}{(\int Q)^2} \left( \frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2} \Lambda Q + P - \frac{1}{2} \int Q \right) - 8\rho_1, \\ J_2(s) &= (\varepsilon(s), \rho_2), \end{aligned} \quad (4.2.38)$$

*where  $P$  was introduced in Proposition 4.2.2. Then we have*

$$\begin{aligned} & \left| b_s + 2b^2 + \omega_s G'(\omega) + b \left( (J_2)_s + \frac{1}{2} \frac{\lambda_s}{\lambda} J_2 \right) \right| \\ & \lesssim \int \varepsilon^2 e^{-\frac{|y|}{10}} + (\omega + |b|) b^2, \end{aligned} \quad (4.2.39)$$

*where  $G \in C^2$  with  $G(0) = 0$ ,  $G'(0) = c_0 > 0$ , for some universal constant  $c_0$ .*

(c) Law of  $\frac{b}{\lambda^2}$ : let

$$\rho = 4\rho_1 + \rho_2 \in \mathcal{Y}, \quad J(s) = (\varepsilon(s), \rho), \quad (4.2.40)$$

then we have:

$$\begin{aligned} & \left| \frac{d}{ds} \left( \frac{b}{\lambda^2} \right) + \frac{b}{\lambda^2} \left( J_s + \frac{1}{2} \frac{\lambda_s}{\lambda} J \right) + \frac{\omega_s G'(\omega)}{\lambda^2} \right| \\ & \lesssim \frac{1}{\lambda^2} \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} + (\omega + |b|) b^2 \right). \end{aligned} \quad (4.2.41)$$

*Remark 4.2.10.* The proof of Proposition 4.2.9 follows almost the same procedure as [57, Lemma 2.7]. It is important that there is no a priori assumption on the upper bound of  $\lambda(t)$ . This fact ensures that Proposition 4.2.9 can be used in all the 3 regimes <sup>7</sup>.

*Proof. Proof of (1):* Equation (4.2.27) follows by direct computation from the equation of  $u(t)$ .

*Proof of (2):* We write down the mass conservation law:

$$\int \mathcal{Q}_{b,\omega}^2 - \int \mathcal{Q}^2 + \int \varepsilon^2 + 2(\varepsilon, \mathcal{Q}_{b,\omega}) = \int u_0^2 - \int \mathcal{Q}^2. \quad (4.2.42)$$

From (4.2.16) and the orthogonality condition (4.2.21), we have

$$\int \varepsilon^2 \lesssim |b| + \omega + |b|^{1-\beta} \|\varepsilon\|_{L^2} + \left| \int u_0^2 - \int \mathcal{Q}^2 \right|.$$

then (4.2.31) follows from  $\beta = \frac{3}{4}$ .

Similarly, we use the energy conservation law and (4.2.17) to obtain:

$$\begin{aligned} 2\lambda^2 E_0 &= 2E(\mathcal{Q}_{b,\omega}) - 2 \int \varepsilon(\mathcal{Q}_{b,\omega})_{yy} + \int \varepsilon_y^2 - \frac{1}{3} \int [(\mathcal{Q}_{b,\omega} + \varepsilon)^6 - \mathcal{Q}_{b,\omega}^6] \\ &\quad + \frac{2\omega}{q+1} \int [|\mathcal{Q}_{b,\omega} + \varepsilon|^{q+1} - |\mathcal{Q}_{b,\omega}|^{q+1}] \\ &= O(|b| + \omega) + \int \varepsilon_y^2 - 2 \int \varepsilon [(\mathcal{Q}_{b,\omega} - \mathcal{Q}_\omega)_{yy} + (\mathcal{Q}_{b,\omega}^5 - \mathcal{Q}_\omega^5) \\ &\quad + \omega(\mathcal{Q}_{b,\omega} |\mathcal{Q}_{b,\omega}|^{q-1} - \mathcal{Q}_\omega |\mathcal{Q}_\omega|^{q-1})] - \frac{1}{3} \int [(\mathcal{Q}_{b,\omega} + \varepsilon)^6 - \mathcal{Q}_{b,\omega}^6 - 6\varepsilon \mathcal{Q}_{b,\omega}^5] \\ &\quad + \frac{2\omega}{q+1} \int [|\mathcal{Q}_{b,\omega} + \varepsilon|^{q+1} - |\mathcal{Q}_{b,\omega}|^{q+1} - (q+1)\varepsilon \mathcal{Q}_{b,\omega} |\mathcal{Q}_{b,\omega}|^{q-1}] \end{aligned}$$

We estimate all terms in the above identity. By the definition of  $\mathcal{Q}_{b,\omega}$ , we have:

$$\begin{aligned} & \left| \int \varepsilon [(\mathcal{Q}_{b,\omega} - \mathcal{Q}_\omega)_{yy} + (\mathcal{Q}_{b,\omega}^5 - \mathcal{Q}_\omega^5) + \omega(\mathcal{Q}_{b,\omega} |\mathcal{Q}_{b,\omega}|^{q-1} - \mathcal{Q}_\omega |\mathcal{Q}_\omega|^{q-1})] \right| \\ & \lesssim |b| \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} + |b|^{1+2\beta} \int_{-2|b|^{-\beta} \leq y \leq 0} |\varepsilon| \\ & \lesssim |b| + \int \varepsilon^2 e^{-\frac{|y|}{10}}. \end{aligned}$$

<sup>7</sup> We will see in Section 4 that we can't expect any (finite) upper bound on the scaling parameter  $\lambda(t)$  in both (Blow down) and (Exit) case.

For the nonlinear term, we use Gagliardo-Nirenberg's inequality to estimate:

$$\begin{aligned} \left| \int [(Q_{b,\omega} + \varepsilon)^6 - Q_{b,\omega}^6 - 6\varepsilon Q_{b,\omega}^5] \right| &\lesssim \int \varepsilon^2 Q_{\omega}^4 + \int \varepsilon^6 + |b| \int \varepsilon^2 \\ &\lesssim \int \varepsilon^2 e^{-\frac{|y|}{10}} + |b| + \|\varepsilon\|_{L^2}^4 \|\varepsilon_y\|_{L^2}^2, \end{aligned}$$

and

$$\begin{aligned} &\left| \omega \int [ |Q_{b,\omega} + \varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1} - (q+1)\varepsilon Q_{b,\omega} |Q_{b,\omega}|^{q-1} ] \right| \\ &\lesssim \omega \left( |b| + \int \varepsilon^2 e^{-\frac{|y|}{10}} + \int |\varepsilon|^{q+1} \right) \\ &\lesssim |b| + \int \varepsilon^2 e^{-\frac{|y|}{10}} + \frac{\gamma}{\lambda^m} \|\varepsilon\|_{L^2}^{\frac{q+3}{2}} \|\varepsilon_y\|_{L^2}^{m+2} \\ &\lesssim |b| + \int \varepsilon^2 e^{-\frac{|y|}{10}} + \gamma \frac{\|\varepsilon_y\|_{L^2}^{m+2}}{\lambda^m}. \end{aligned}$$

Collecting all the estimates above, we obtain (4.2.32).

*Proof of (3):* Let us differentiate the orthogonality conditions

$$(\varepsilon(t), \Lambda Q_{\omega(t)}) = (\varepsilon(t), y \Lambda Q_{\omega(t)}) = 0.$$

Note that

$$\frac{d}{ds}(\varepsilon, \Lambda Q_{\omega}) = (\varepsilon_s, \Lambda Q_{\omega}) + \omega_s(\varepsilon, \Lambda Z_{\omega}),$$

where  $Z_{\omega} = \partial Q_{\omega} / \partial \omega \in \mathcal{Y}$ . So we have:

$$\begin{aligned} &\left| \left( \frac{\lambda_s}{\lambda} + b \right) - \frac{(\varepsilon, L_{\omega}(\Lambda Q_{\omega}))'}{\|\Lambda Q_{\omega}\|_{L^2}^2} \right| + \left| \left( \frac{x_s}{\lambda} - 1 \right) - \frac{(\varepsilon, L_{\omega}(y \Lambda Q_{\omega}))'}{\|\Lambda Q_{\omega}\|_{L^2}^2} \right| \\ &\lesssim \left( \left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| + |b| \right) \times \left( \omega + |b| + \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} \right) \\ &\quad + |b_s| + |\omega_s| + \int \varepsilon^2 e^{-\frac{|y|}{10}} + \int \varepsilon^5 e^{-\frac{9|y|}{10}} + \int |\varepsilon|^q e^{-\frac{9|y|}{10}}. \end{aligned}$$

For the nonlinear term, we use Sobolev embedding and the a priori smallness (4.2.26):

$$\|\varepsilon e^{-\frac{|y|}{4}}\|_{L^{\infty}}^2 \leq \|\varepsilon e^{-\frac{3|y|}{4(q-2)}}\|_{L^{\infty}}^2 \lesssim \int (\partial_y \varepsilon^2 + \varepsilon^2) e^{-\frac{3|y|}{4(q-2)}} \ll 1,$$

to estimate

$$\int \varepsilon^5 e^{-\frac{9|y|}{10}} + \int |\varepsilon|^q e^{-\frac{9|y|}{10}} \lesssim \left( \|\varepsilon e^{-\frac{|y|}{4}}\|_{L^{\infty}}^3 + \|\varepsilon e^{-\frac{3|y|}{4(q-2)}}\|_{L^{\infty}}^{q-2} \right) \int \varepsilon^2 e^{-\frac{|y|}{10}}. \quad (4.2.43)$$

Here we use the basic fact that  $q > 5$ .

For  $\omega_s$ , we have

$$\omega_s = -m\omega \frac{\lambda_s}{\lambda} = m\omega b - m\omega \left( \frac{\lambda_s}{\lambda} + b \right) \quad (4.2.44)$$

The above estimates imply that

$$\left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| \lesssim (\omega + |b|)|b| + |b_s| + \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}}, \quad (4.2.45)$$

and

$$\begin{aligned} & \left| \left( \frac{\lambda_s}{\lambda} + b \right) - \frac{(\varepsilon, L_\omega(\Lambda Q_\omega)')}{\|\Lambda Q_\omega\|_{L^2}^2} \right| + \left| \left( \frac{x_s}{\lambda} - 1 \right) - \frac{(\varepsilon, L_\omega(y\Lambda Q_\omega)')}{\|\Lambda Q_\omega\|_{L^2}^2} \right| \\ & \lesssim (\omega + |b|) \left[ \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} + |b| \right] + \int \varepsilon^2 e^{-\frac{|y|}{10}}. \end{aligned} \quad (4.2.46)$$

Next, let us differentiate the relation  $(\varepsilon, Q_\omega) = 0$ . Then we use the following facts:  $L_\omega Q'_\omega = 0$ ,  $(Q_\omega, \Lambda Q_\omega) = (Q_\omega, Q'_\omega) = 0$ ,  $(\varepsilon, \Lambda Q_\omega) = 0$ , the nondegeneracy  $(P_\omega, Q_\omega) > 0$ , and (4.2.14), (4.2.15), (4.2.44) to obtain:

$$\begin{aligned} |b_s| & \lesssim \left( \omega + |b| + \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} \right) \times \left( \left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| \right) \\ & \quad + (\omega + |b|) \left[ \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} + |b| \right] + \int \varepsilon^2 e^{-\frac{|y|}{10}}. \end{aligned} \quad (4.2.47)$$

Combining (4.2.44), (4.2.45) and (4.2.47), we get (4.2.33) and (4.2.34).

*Proof of (4):* Firstly, we claim the following sharp equation:

$$\begin{aligned} & b_s + 2b^2 + \omega_s G'(\omega) - \frac{16b}{(\int Q)^2} \left[ \frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2} (\varepsilon, L(\Lambda Q)') + 20(\varepsilon, P Q^3 Q') \right] \\ & = O\left( b^2(\omega + |b|) + \int \varepsilon^2 e^{-\frac{|y|}{10}} \right). \end{aligned} \quad (4.2.48)$$

To prove this, we take the scalar product of (4.2.27) with  $Q_\omega$ . We keep track of all terms up to  $b^2$ .

First, from (4.2.14), we have

$$\begin{aligned} (\Psi_{b,\omega}, Q_\omega) & = -b^2((10P_\omega^2 Q_\omega^3)_y + \Lambda P_\omega, Q_\omega) + O(b^2(|b| + \omega)) \\ & = -b^2((10P^2 Q^3)_y + \Lambda P, Q) + O(b^2(|b| + \omega)) \\ & = -\frac{b^2}{8} \|Q\|_{L^1}^2 + O(b^2(|b| + \omega)), \end{aligned} \quad (4.2.49)$$

where for the last step use the following computation:

$$\begin{aligned} (\Lambda P, Q) & = -(P, \Lambda Q) = -(P, (LP)') = (P, (P'' - P + 5Q^4 P)') \\ & = (P, P''' - P') + 10 \int Q^3 Q' P^2, \end{aligned}$$

and from Proposition 4.2.2, we obtain:

$$((10P^2 Q^3)_y + \Lambda P, Q) = \frac{1}{2} \lim_{y \rightarrow -\infty} P^2 = \frac{1}{8} \|Q\|_{L^1}^2.$$

Next, from Proposition 4.2.4, we have:

$$\begin{aligned} (b_s \frac{\partial Q_{b,\omega}}{\partial b}, Q_\omega) &= b_s((\chi_b + \beta y \chi'_b) P_\omega, Q_\omega) = b_s(P_\omega, Q_\omega) + O(b^{10}) \\ &= \frac{b_s}{16} \|Q\|_{L^1}^2 + F(\omega) b_s + O(b^{10}), \end{aligned} \quad (4.2.50)$$

where  $F$  is the  $C^1$  function introduced in Proposition 4.2.4. From Lemma 4.2.3, we have

$$\begin{aligned} (Z_\omega, Q_\omega) &= -\frac{1}{2}(L_\omega Z_\omega, \Lambda Q_\omega) + O(\omega) = \frac{1}{2} \int (\Lambda Q_\omega) Q_\omega |Q_\omega|^{q-1} + O(\omega) \\ &= \frac{q-1}{4(q+1)} \int |Q_\omega|^{q+1} + O(\omega) > 0. \end{aligned}$$

Then from (4.2.34), we have:

$$\begin{aligned} (\omega_s \frac{\partial Q_{b,\omega}}{\partial \omega}, Q_\omega) &= \omega_s \frac{1}{2} \frac{\partial \|Q_\omega\|_{L^2}^2}{\partial \omega} + O(|b\omega_s|) \\ &= \omega_s \tilde{G}'(\omega) + O\left(b^2(\omega + |b|) + \int \varepsilon^2 e^{-\frac{|y|}{10}}\right), \end{aligned} \quad (4.2.51)$$

with  $\tilde{G}(\omega) = \frac{1}{2}(\|Q_\omega\|_{L^2}^2 - \|Q\|_{L^2}^2)$ . It is easy to check  $\tilde{G}(0) = 0$ ,  $\tilde{G} \in C^1$ , and

$$\tilde{G}'(0) = (Z_\omega, Q_\omega) \Big|_{\omega=0} = \frac{q-1}{4(q+1)} \int |Q|^{q+1} > 0.$$

Next, from Proposition 4.2.4 we have:

$$|(Q'_{b,\omega} + \varepsilon_y, Q_\omega)| \lesssim \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} + |(Q'_\omega, Q_\omega)| + |(P'_\omega, Q_\omega)| + b^{10},$$

which together with (4.2.33) implies that

$$\left| \left( \frac{x_s}{\lambda} - 1 \right) (Q'_{b,\omega} + \varepsilon_y, Q_\omega) \right| \lesssim b^2(\omega + |b|) + \int \varepsilon^2 e^{-\frac{|y|}{10}}. \quad (4.2.52)$$

For the small linear term, we have:

$$\begin{aligned} \int R_b(\varepsilon) Q'_\omega &= 20b \int P_\omega Q_\omega^3 Q'_\omega \varepsilon + |b|(\omega + |b|) O\left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} \\ &= 20b \int P Q^3 Q' \varepsilon + |b|(\omega + |b|) O\left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}}. \end{aligned} \quad (4.2.53)$$

Since the nonlinear term can be estimated with the help of (4.2.43), we then have:

$$\begin{aligned} b_s + \frac{2b^2 + \omega_s \tilde{G}'(\omega)}{1 + H(\omega)} - \frac{16}{(1 + H(\omega))(fQ)^2} \left[ (\Lambda Q_{b,\omega}, Q_\omega) \left( \frac{\lambda_s}{\lambda} + b \right) + 20b(\varepsilon, P Q^3 Q') \right] \\ = O\left( b^2(\omega + |b|) + \int \varepsilon^2 e^{-\frac{|y|}{10}} \right), \end{aligned}$$

where

$$H(\omega) = \frac{16}{(\int Q)^2} F(\omega).$$

From (4.2.46) we have

$$\left| \left( \frac{\lambda_s}{\lambda} + b \right) - \frac{(\varepsilon, L(\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2} \right| \lesssim (\omega + |b|) \left[ \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} + |b| \right] + \int \varepsilon^2 e^{-\frac{|y|}{10}}.$$

Moreover, we have

$$|(\Lambda Q_{b,\omega}, Q_\omega) - b(\Lambda P, Q)| \lesssim b^{10} + |b(\Lambda P, Q) - b(\Lambda P_\omega, Q_\omega)| \lesssim |b|(\omega + |b|).$$

We then conclude that

$$\begin{aligned} b_s + \frac{2b^2 + \omega_s \tilde{G}'(\omega)}{1 + H(\omega)} - \frac{16b}{(1 + H(\omega))(\int Q)^2} \left[ \frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2} (\varepsilon, L(\Lambda Q)') + 20(\varepsilon, PQ^3Q') \right] \\ = O\left( b^2(\omega + |b|) + \int \varepsilon^2 e^{-\frac{|y|}{10}} \right). \end{aligned} \quad (4.2.54)$$

Finally, since  $H \in C^1$ ,  $H(0) = 0$ , it is to check that the following function

$$G(\omega) = \int_0^\omega \frac{\tilde{G}'(x)}{1 + H(x)} dx$$

satisfies  $G \in C^2$ ,  $G(0) = 0$ ,  $G'(0) = c_0 > 0$ . Then, (4.2.54) implies (4.2.48) immediately.

Now, we turn to the proof of (4.2.37), (4.2.39) and (4.2.41). For all  $f \in \mathcal{Y}$ , independent of  $s$ ,  $(\varepsilon, \int_{-\infty}^y f)$  is well defined due to (4.2.35). Moreover, we have:

$$\begin{aligned} \frac{d}{ds} \left( \varepsilon, \int_{-\infty}^y f \right) &= -(\varepsilon, L_\omega f) + \left( \frac{\lambda_s}{\lambda} + b \right) \left( \Lambda Q_{b,\omega}, \int_{-\infty}^y f \right) + \frac{\lambda_s}{\lambda} \left( \Lambda \varepsilon, \int_{-\infty}^y f \right) \\ &\quad - \left( \frac{x_s}{\lambda} - 1 \right) (Q_{b,\omega} + \varepsilon, f) - \left( b_s \frac{\partial Q_{b,\omega}}{\partial b} + \omega_s \frac{\partial Q_{b,\omega}}{\partial \omega}, \int_{-\infty}^y f \right) \\ &\quad + \left( \Psi_{b,\omega}, \int_{-\infty}^y f \right) + (R_b(\varepsilon) + R_{NL}(\varepsilon), f). \end{aligned}$$

Using (4.2.33), (4.2.34), (4.2.43) and Proposition 4.2.4, we have:

$$\begin{aligned} \frac{d}{ds} \left( \varepsilon, \int_{-\infty}^y f \right) &= -(\varepsilon, Lf) + \left( \frac{\lambda_s}{\lambda} + b \right) \left( \Lambda Q, \int_{-\infty}^y f \right) + \left( \frac{x_s}{\lambda} - 1 \right) (f, Q) \\ &\quad - \frac{1}{2} \frac{\lambda_s}{\lambda} \left( \varepsilon, \int_{-\infty}^y f \right) + O\left( (|b| + \omega) \left[ \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} \right] \right) \\ &\quad + O((|b| + \omega)|b|) + O\left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right). \end{aligned} \quad (4.2.55)$$

-Proof of (4.2.37): We apply (4.2.55) to  $f = \Lambda Q$ , using the following fact

$$L\Lambda Q = -2Q, \quad \left( \Lambda Q, \int_{-\infty}^y \Lambda Q \right) = \frac{1}{8} \left( \int Q \right)^2, \quad \left( Q', \int_{-\infty}^y \Lambda Q \right) = 0,$$

to obtain:

$$2(J_1)_s = \frac{16(\varepsilon, Q)}{(fQ)^2} + \left(\frac{\lambda_s}{\lambda} + b\right) - \frac{\lambda_s}{\lambda} J_1 \\ + O\left((|b| + \omega) \left[ \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} + |b| \right] + \int \varepsilon^2 e^{-\frac{|y|}{10}}\right).$$

Then (4.2.37) follows immediately from the orthogonality condition (4.2.21).

-Proof of (4.2.39): We apply (4.2.55) to  $f = \rho'_2$ . Then from Lemma 4.2.1 and Proposition 4.2.2, we have:

$$(\Lambda Q, \rho_2) = \frac{16}{(fQ)^2} \left( \frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}} \Lambda Q + P - \frac{1}{2} \int Q, \Lambda Q \right) - \frac{32}{(fQ)^2} \left( \Lambda Q, \int_{-\infty}^y \Lambda Q \right) \\ = \frac{16}{(fQ)^2} [(\Lambda P, Q) + (\Lambda Q, P)] + \frac{4\|Q\|_{L^1}^2}{(fQ)^2} - \frac{16}{(fQ)^2} \left( \int \Lambda Q \right)^2 = 0, \\ (\rho'_2, Q) = \frac{16}{(fQ)^2} \left( \frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}} (\Lambda Q)' + P', Q \right) - 8(\rho'_1, Q).$$

Next, from

$$L(P') = (LP')' + 20Q'Q^3P = \Lambda Q + 20Q'Q^3P,$$

and the orthogonality condition  $(\varepsilon, \Lambda Q_\omega) = 0$ , we have

$$(\varepsilon, L\rho'_2) = \frac{16}{(fQ)^2} \left( \varepsilon, L \left[ \frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}} (\Lambda Q)' + P' \right] \right) - 8(\varepsilon, L\rho'_1) \\ = \frac{16}{(fQ)^2} \left[ \frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2} (\varepsilon, L(\Lambda Q)') + 20(\varepsilon, PQ^3Q') \right] + O(\omega) \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}},$$

Injecting all the above estimates into (4.2.55) with  $f = \rho'_2$ , we obtain:

$$(J_2)_s = - \frac{16}{(fQ)^2} \left[ \frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2} (\varepsilon, L(\Lambda Q)') + 20(\varepsilon, PQ^3Q') \right] - \frac{1}{2} \frac{\lambda_s}{\lambda} J_2 \\ + O\left((|b| + \omega) \left[ \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} + |b| \right] + \int \varepsilon^2 e^{-\frac{|y|}{10}}\right). \quad (4.2.56)$$

Then (4.2.39) follows from (4.2.48) and (4.2.56).

-Proof of (4.2.41): From (4.2.37) and (4.2.41),

$$\frac{d}{ds} \left( \frac{b}{\lambda^2} \right) = \frac{b_s + 2b^2}{\lambda^2} - \frac{2b}{\lambda^2} \left( \frac{\lambda_s}{\lambda} + b \right) \\ = - \frac{b}{\lambda^2} \left[ (J_2)_s + \frac{1}{2} \frac{\lambda_s}{\lambda} J_2 \right] - \frac{2b}{\lambda^2} \left[ 2(J_1)_s + \frac{\lambda_s}{\lambda} J_1 \right] - \frac{\omega_s G'(\omega)}{\lambda^2} \\ + O\left( \frac{1}{\lambda^2} \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} + (\omega + |b|)b^2 \right) \right) \\ = - \frac{b}{\lambda^2} \left[ J_s + \frac{1}{2} \frac{\lambda_s}{\lambda} J \right] - \frac{\omega_s G'(\omega)}{\lambda^2} + O\left( \frac{1}{\lambda^2} \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} + (\omega + |b|)b^2 \right) \right)$$



which is exactly (4.2.41).

Finally, it is easy to check that  $\lim_{|y| \rightarrow +\infty} \rho(y) = 0$ , which implies that  $\rho \in \mathcal{Y}$ .  $\square$

## 4.3 Monotonicity formula

In this section, we will introduce the monotonicity tools developed in [55] and [57]. This is the key technical argument of the analysis for solution near the soliton.

### 4.3.1 Pointwise monotonicity

Let  $(\varphi_i)_{i=1,2}, \psi \in C^\infty(\mathbb{R})$  be such that:

$$\varphi_i(y) = \begin{cases} e^y, & \text{for } y < -1, \\ 1 + y, & \text{for } -\frac{1}{2} < y < \frac{1}{2}, \\ y^i, & \text{for } y > 2, \end{cases} \quad \varphi'(y) > 0, \text{ for all } y \in \mathbb{R}, \quad (4.3.1)$$

$$\psi(y) = \begin{cases} e^{2y}, & \text{for } y < -1, \\ 1, & \text{for } y > -\frac{1}{2}, \end{cases} \quad \psi'(y) \geq 0, \text{ for all } y \in \mathbb{R}. \quad (4.3.2)$$

Let  $B > 100$  be a large universal constant to be chosen later. We then define the following weight function:

$$\psi_B(y) = \psi\left(\frac{y}{B}\right), \quad \varphi_{i,B}(y) = \varphi\left(\frac{y}{B}\right), \quad (4.3.3)$$

and the following weighted Sobolev norm of  $\varepsilon$ :

$$\mathcal{N}_i(s) = \int \left( \varepsilon_y^2(s, y) \psi_B(y) + \varepsilon^2(s, y) \varphi_{i,B}(y) \right) dy, \quad i = 1, 2, \quad (4.3.4)$$

$$\mathcal{N}_{i,\text{loc}}(s) = \int \varepsilon^2(s, y) \varphi'_{i,B}(y) dy, \quad i = 1, 2. \quad (4.3.5)$$

Then we have the following monotonicity:

**Proposition 4.3.1** (Monotonicity formula). *There exist universal constants  $\mu > 0$ ,  $B = B(q) > 100$  and  $0 < \kappa \ll 1$ , such that the following holds. Let  $u(t)$  be a solution of  $(\text{gKdV}_\gamma)$  satisfying (4.2.19) on  $[0, t_0]$ , and hence the geometrical decomposition (4.2.20) holds on  $[0, t_0]$ . Let  $s_0 = s(t_0)$ , and assume the following a priori bounds hold for all  $s \in [0, s_0]$ :*

(H1) *Scaling invariant bounds:*

$$\omega(s) + |b(s)| + \mathcal{N}_2(s) + \|\varepsilon(s)\|_{L^2} + \omega(s) \|\varepsilon_y(s)\|_{L^2}^m \leq \kappa; \quad (4.3.6)$$

(H2) *Bounds related to  $H^1$  scaling:*

$$\frac{\omega(s) + |b(s)| + \mathcal{N}_2(s)}{\lambda^2(s)} \leq \kappa; \quad (4.3.7)$$

(H3)  $L^2$  weighted bound on the right:

$$\int_{y>0} y^{10} \varepsilon^2(s, y) dy \leq 50 \left( 1 + \frac{1}{\lambda^{10}(s)} \right). \quad (4.3.8)$$

We define the Lyapounov functionals for  $(i, j) \in \{1, 2\}^2$  as following:

$$\begin{aligned} \mathcal{F}_{i,j} = & \int \left( \varepsilon_y^2 \psi_B + (1 + \mathcal{J}_{i,j}) \varepsilon^2 \varphi_{i,B} - \frac{1}{3} \psi_B [(Q_{b,\omega} + \varepsilon)^6 - Q_{b,\omega}^6 - 6\varepsilon Q_{b,\omega}^5] \right. \\ & \left. + \frac{2\omega}{q+1} [ |Q_{b,\omega} + \varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1} - (q+1)\varepsilon Q_{b,\omega} |Q_{b,\omega}|^{q-1} ] \psi_B \right), \end{aligned} \quad (4.3.9)$$

with <sup>8</sup>

$$\mathcal{J}_{i,j} = (1 - J_1)^{-4(j-1)-2i} - 1. \quad (4.3.10)$$

Then the following estimates hold on  $[0, s_0]$ :

1. *Scaling invariant Lyapounov control: for  $i = 1, 2$ ,*

$$\frac{d\mathcal{F}_{i,1}}{ds} + \mu \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} \lesssim_B b^2 (\omega^2 + b^2). \quad (4.3.11)$$

2.  *$H^1$  scaling Lyapounov control: for  $i = 1, 2$ ,*

$$\frac{d}{ds} \left( \frac{\mathcal{F}_{i,2}}{\lambda^2} \right) + \frac{\mu}{\lambda^2} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} \lesssim_B \frac{b^2 (\omega^2 + b^2)}{\lambda^2}. \quad (4.3.12)$$

3. *Coercivity and pointwise bounds: there holds for all  $(i, j) \in \{1, 2\}^2$ ,*

$$\mathcal{N}_i \lesssim \mathcal{F}_{i,j} \lesssim \mathcal{N}_i, \quad (4.3.13)$$

$$|J_i| + |\mathcal{J}_{i,j}| \lesssim \mathcal{N}_2^{\frac{1}{2}}. \quad (4.3.14)$$

*Remark 4.3.2.* The proof of Proposition 4.3.1 is almost the same to [57, Proposition 3.1]. The only difference here is the additional terms involving  $\omega$ .

*Remark 4.3.3.* Similar as Proposition 4.2.9, we do not assume any a priori control on the upper bound of  $\lambda(s)$  so that the monotonicity formula can be used in all the 3 cases.

*Remark 4.3.4.* As mentioned in [57, Proposition 3.1], the weight function  $\psi$  decays faster than  $\varphi_i$  on the left. As a result,  $\mathcal{N}_2$  and  $\mathcal{F}_{i,j}$  do not control  $\int \varepsilon_y^2 \varphi'_{i,B}$  (See [57, Remark 3.5] for more details).

*Proof of Proposition 4.3.1.* The proof of (4.3.13) and (4.3.14) is exactly the same as [57, Proposition 3.1]. We only need to prove (4.3.11) and (4.3.12). To do this, we compute directly to obtain that for all  $(i, j) \in \{1, 2\}^2$ ,

$$\lambda^{2(j-1)} \left( \frac{\mathcal{F}_{i,j}}{\lambda^{2(j-1)}} \right)_s = f_1 + f_2 + f_3 + f_4 + f_5, \quad (4.3.15)$$

8. Recall that  $J_1$  was defined in (4.2.36).

where

$$\begin{aligned}
f_1 &= 2 \int \left( \varepsilon_s - \frac{\lambda_s}{\lambda} \Lambda \varepsilon \right) \left( -(\Psi_B \varepsilon)_y + \varepsilon \varphi_{i,B} - \Psi_B \Delta_{b,\omega}(\varepsilon) \right), \\
f_2 &= 2 \int \left( \varepsilon_s - \frac{\lambda_s}{\lambda} \Lambda \varepsilon \right) \varepsilon \mathcal{J}_{i,j} \varphi_{i,B}, \\
f_3 &= 2 \frac{\lambda_s}{\lambda} \int \Lambda \varepsilon \left( -(\Psi_B \varepsilon_y)_y + (1 + \mathcal{J}_{i,j}) \varepsilon \varphi_{i,B} - \Psi_B \Delta_{b,\omega}(\varepsilon) \right) \\
&\quad + (\mathcal{J}_{i,j})_s \int \varphi_{i,B} \varepsilon^2 - 2(j-1) \frac{\lambda_s}{\lambda} \mathcal{F}_{i,j}, \\
f_4 &= -2 \int \Psi_B(Q_{b,\omega})_s \left[ \Delta_{b,\omega} - 5\varepsilon Q_{b,\omega}^4 + q\omega \varepsilon |Q_{b,\omega}|^{q-1} \right], \\
f_5 &= \frac{2\omega_s}{q+1} \int \left[ |Q_{b,\omega} + \varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1} - (q+1)\varepsilon Q_{b,\omega} |Q_{b,\omega}|^{q-1} \right] \Psi_B, \\
\Delta_{b,\omega}(\varepsilon) &= (Q_{b,\omega} + \varepsilon)^5 - Q_{b,\omega}^5 - \omega(Q_{b,\omega} + \varepsilon) |Q_{b,\omega} + \varepsilon|^{q-1} + \omega Q_{b,\omega} |Q_{b,\omega}|^{q-1}.
\end{aligned}$$

Our goal is to show that for some  $\mu_0 > 0$ ,

$$\frac{d}{ds} f_1 \leq -\mu_0 \int \left( (\varepsilon^2 + \varepsilon^2) \varphi'_{i,B} + \varepsilon_{yy}^2 \Psi'_B \right) + Cb^2(\omega^2 + b^2), \quad (4.3.16)$$

$$\left| \frac{d}{ds} f_k \right| \leq \frac{\mu_0}{10} \int \left( (\varepsilon^2 + \varepsilon^2) \varphi'_{i,B} + \varepsilon_{yy}^2 \Psi'_B \right) + Cb^2(\omega^2 + b^2), \text{ for } k = 2, 3, 4, 5. \quad (4.3.17)$$

The following properties will be used several times in this paper<sup>9</sup>:

$$|\varphi_i'''(y)| + |\varphi_i''(y)| + |\psi'''(y)| + |y\psi'(y)| + |\psi(y)| \lesssim \varphi'_i \lesssim \varphi_i, \text{ for all } y \in \mathbb{R}, \quad (4.3.18)$$

$$e^{|y|} (|\psi(y)| + |\psi'(y)|) \lesssim \varphi'_i \sim \varphi_i, \text{ for all } y \in (-\infty, \frac{1}{2}], \quad (4.3.19)$$

$$\mathcal{N}_{1,\text{loc}} \lesssim \mathcal{N}_{2,\text{loc}} \lesssim \mathcal{N}_1 \lesssim \mathcal{N}_2, \quad \int \varepsilon^2 \varphi_{1,B} dy \lesssim \mathcal{N}_{2,\text{loc}}, \quad (4.3.20)$$

$$\int_{y>0} y^2 \varepsilon^2(s) \lesssim \left( 1 + \frac{1}{\lambda^{\frac{10}{9}}(s)} \right) \mathcal{N}_{2,\text{loc}}^{\frac{8}{9}}(s). \quad (4.3.21)$$

**Control of  $f_1$ .** First, we rewrite  $f_1$  using the equation of  $\varepsilon$  in the following form:

$$\begin{aligned}
\varepsilon_s - \frac{\lambda_s}{\lambda} \Lambda \varepsilon &= \left( -\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon) \right)_y + \left( \frac{\lambda_s}{\lambda} + b \right) \Lambda Q_{b,\omega} \\
&\quad + \left( \frac{x_s}{\lambda} - 1 \right) (Q_{b,\omega} + \varepsilon)_y - b_s \frac{\partial Q_{b,\omega}}{\partial b} - \omega_s \frac{\partial Q_{b,\omega}}{\partial \omega} + \Psi_{b,\omega},
\end{aligned} \quad (4.3.22)$$

where  $-\Psi_{b,\omega} = b\Lambda Q_{b,\omega} + (Q_{b,\omega}'' - Q_{b,\omega} + Q_{b,\omega}^5 - \omega Q_{b,\omega} |Q_{b,\omega}|^{q-1})_y$ . This yields:

$$f_1 = f_{1,1} + f_{1,2} + f_{1,3} + f_{1,4} + f_{1,5},$$

9. See [57, Section 3] for more details.

with

$$\begin{aligned}
f_{1,1} &= 2 \int (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))_y (-\psi_B \varepsilon_y)_y + \varepsilon \varphi_{i,B} - \psi_B \Delta_{b,\omega}(\varepsilon), \\
f_{1,2} &= 2 \left( \frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_{b,\omega} (-\psi_B \varepsilon_y)_y + \varepsilon \varphi_{i,B} - \psi_B \Delta_{b,\omega}(\varepsilon), \\
f_{1,3} &= 2 \left( \frac{x_s}{\lambda} - 1 \right) \int (Q_{b,\omega} + \varepsilon)_y (-\psi_B \varepsilon_y)_y + \varepsilon \varphi_{i,B} - \psi_B \Delta_{b,\omega}(\varepsilon), \\
f_{1,4} &= -2 \int \left( b_s \frac{\partial Q_{b,\omega}}{\partial b} + \omega_s \frac{\partial Q_{b,\omega}}{\partial \omega} \right) (-\psi_B \varepsilon_y)_y + \varepsilon \varphi_{i,B} - \psi_B \Delta_{b,\omega}(\varepsilon), \\
f_{1,5} &= 2 \int \Psi_{b,\omega} (-\psi_B \varepsilon_y)_y + \varepsilon \varphi_{i,B} - \psi_B \Delta_{b,\omega}(\varepsilon).
\end{aligned}$$

For the term  $f_{1,1}$ , we integrate by parts to obtain a more manageable formula:

$$\begin{aligned}
f_{1,1} &= 2 \int (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))_y (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon)) \psi_B \\
&\quad + 2 \int (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))_y (-\psi'_B \varepsilon_y + \varepsilon(\varphi_B - \psi_B)).
\end{aligned}$$

We compute these terms separately. First, we have

$$\begin{aligned}
&2 \int (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))_y (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon)) \psi_B \\
&= - \int \psi'_B (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))^2 \\
&= - \int \psi'_B ([-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon)]^2 - (-\varepsilon_{yy} + \varepsilon)^2) - \int \psi'_B (-\varepsilon_{yy} + \varepsilon)^2 \\
&= - \left[ \int \psi'_B (\varepsilon_{yy}^2 + 2\varepsilon_y^2) + \varepsilon^2 (\psi'_B - \psi'''_B) \right] \\
&\quad - \int \psi'_B ([-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon)]^2 - (-\varepsilon_{yy} + \varepsilon)^2).
\end{aligned}$$

Next, we integrate by parts to obtain

$$\begin{aligned}
&-2 \int (\Delta_{b,\omega}(\varepsilon))_y (\varphi_{i,B} - \psi_B) \varepsilon \\
&= -\frac{1}{3} \int (\varphi_{i,B} - \psi_B)' ([(Q_{b,\omega} + \varepsilon)^6 - Q_{b,\omega}^6 - 6\varepsilon Q_{b,\omega}^5] - 6\varepsilon [(Q_{b,\omega} + \varepsilon)^5 - Q_{b,\omega}^5]) \\
&\quad -2 \int (\varphi_{i,B} - \psi_B) (Q_{b,\omega})_y [(Q_{b,\omega} + \varepsilon)^5 - Q_{b,\omega}^5 - 5\varepsilon Q_{b,\omega}^4] \\
&\quad + \frac{2\omega}{q+1} \int (\varphi_{i,B} - \psi_B)' \left( [|Q_{b,\omega} + \varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1} - (q+1)\varepsilon Q_{b,\omega} |Q_{b,\omega}|^{q-1}] \right. \\
&\quad \quad \left. - (q+1)\varepsilon [(Q_{b,\omega} + \varepsilon) |Q_{b,\omega} + \varepsilon|^{q-1} - Q_{b,\omega} |Q_{b,\omega}|^{q-1}] \right) \\
&\quad + 2\omega \int (\varphi_{i,B} - \psi_B) (Q_{b,\omega})_y \left[ (Q_{b,\omega} + \varepsilon) |Q_{b,\omega} + \varepsilon|^{q-1} \right. \\
&\quad \quad \left. - Q_{b,\omega} |Q_{b,\omega}|^{q-1} - q\varepsilon |Q_{b,\omega}|^{q-1} \right],
\end{aligned}$$

and

$$\begin{aligned} & 2 \int (-\varepsilon_{yy} + \varepsilon)_y (-\psi'_B \varepsilon_y + \varepsilon(\varphi_{i,B} - \psi_B)) \\ &= -2 \left[ \int \psi'_B \varepsilon_{yy}^2 + \int \varepsilon_y^2 \left( \frac{3}{2} \varphi'_{i,B} - \frac{1}{2} \psi'_B - \frac{1}{2} \psi_B''' \right) \right. \\ & \quad \left. + \int \varepsilon^2 \left( \frac{1}{2} (\varphi_B - \psi_B)' - \frac{1}{2} (\varphi_B - \psi_B)''' \right) \right]. \end{aligned}$$

Finally, by direct expansion, we have

$$\begin{aligned} & \int (\Delta_{b,\omega}(\varepsilon))_y \psi'_B \varepsilon_y \\ &= 5 \int \psi'_B \varepsilon_y \left( (Q_{b,\omega})_y [(Q_{b,\omega} + \varepsilon)^4 - Q_{b,\omega}^4] + \varepsilon_y (Q_{b,\omega} + \varepsilon)^4 \right) \\ & \quad - q\omega \int \psi'_B \varepsilon_y \left( (Q_{b,\omega})_y [|Q_{b,\omega} + \varepsilon|^{q-1} - |Q_{b,\omega}|^{q-1}] + \varepsilon_y |Q_{b,\omega} + \varepsilon|^{q-1} \right). \end{aligned}$$

Collecting all the estimates above, we have

$$f_{1,1} = I + II,$$

where

$$\begin{aligned} I &= - \int [3\psi'_B \varepsilon_{yy}^2 + (3\varphi'_{i,B} + \psi'_B - \psi_B''') \varepsilon_y^2 + (\varphi'_{i,B} - \varphi_B''') \varepsilon^2] \\ & \quad - 2 \int \left[ \frac{(Q_{b,\omega} + \varepsilon)^6 - Q_{b,\omega}^6 - 6\varepsilon Q_{b,\omega}^5}{6} - \varepsilon [(Q_{b,\omega} + \varepsilon)^5 - Q_{b,\omega}^5] \right] (\varphi'_{i,B} - \psi'_B) \\ & \quad + 2 \int [(Q_{b,\omega} + \varepsilon)^5 - Q_{b,\omega}^5 - 5\varepsilon Q_{b,\omega}^4] (Q_{b,\omega})_y (\psi_B - \varphi_{i,B}) \\ & \quad + 10 \int \psi'_B \varepsilon_y \left( (Q_{b,\omega})_y [(Q_{b,\omega} + \varepsilon)^4 - Q_{b,\omega}^4] + \varepsilon_y (Q_{b,\omega} + \varepsilon)^4 \right) \\ & \quad - \int \psi'_B ([-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon)]^2 - (-\varepsilon_{yy} + \varepsilon)^2) \\ &= I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned}$$

and

$$\begin{aligned} II &= 2\omega \int \left[ \frac{|Q_{b,\omega} + \varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1} - (q+1)\varepsilon Q_{b,\omega} |Q_{b,\omega}|^{q-1}}{q+1} \right. \\ & \quad \left. - \varepsilon [(Q_{b,\omega} + \varepsilon) |Q_{b,\omega} + \varepsilon|^{q-1} - Q_{b,\omega} |Q_{b,\omega}|^{q-1}] \right] (\varphi'_{i,B} - \psi'_B) \\ & \quad - 2\omega \int [(Q_{b,\omega} + \varepsilon) |Q_{b,\omega} + \varepsilon|^{q-1} - Q_{b,\omega} |Q_{b,\omega}|^{q-1} \\ & \quad - q\varepsilon |Q_{b,\omega}|^{q-1}] (Q_{b,\omega})_y (\psi_B - \varphi_{i,B}) \\ & \quad - 2q\omega \int \psi'_B \varepsilon_y \left( (Q_{b,\omega})_y [|Q_{b,\omega} + \varepsilon|^{q-1} - |Q_{b,\omega}|^{q-1}] + \varepsilon_y |Q_{b,\omega} + \varepsilon|^{q-1} \right). \end{aligned}$$

For  $I_k$ ,  $k = 1, 2, 3, 4$ , we can use the same strategy as in [57, Proposition 3.1] to obtain:

$$\sum_{k=1}^4 I_k \leq -\mu_1 \int \left( \varepsilon_{yy}^2 \psi'_B + \varepsilon_y^2 \phi'_{i,B} + \varepsilon^2 \phi'_{i,B} \right) + Cb^4, \quad (4.3.23)$$

for some universal constant  $\mu_1 > 0$ .

While for  $I_5$ , we have:

$$\begin{aligned} |I_5| &\lesssim \int \psi'_B (|\varepsilon_{yy}| + |\varepsilon| + |\varepsilon|^5 + \omega|\varepsilon|^q) (|\varepsilon|^5 + \omega|\varepsilon|^q + |Q_{b,\omega}\varepsilon|) \\ &\leq \frac{\mu_1}{100} \int (\varepsilon_{yy}^2 + \varepsilon^2) \psi'_B + C(\mu_1) \left( \int Q_{b,\omega}^2 \varepsilon^2 \psi'_B + \int \varepsilon^{10} \psi'_B + \omega^2 \int |\varepsilon|^{2q} \psi'_B \right). \end{aligned} \quad (4.3.24)$$

Using the following weighted Sobolev bound introduced in [70, Lemma 6] and [57, Proposition 3.1]:

$$\begin{aligned} \left\| \varepsilon^2 \sqrt{\phi'_{i,B}} \right\|_{L^\infty}^2 &\lesssim \|\varepsilon\|_{L^2}^2 \left( \int \varepsilon_y^2 \phi'_{i,B} + \int \varepsilon^2 \frac{(\phi'_{i,B})^2}{\phi'_B} \right) \\ &\lesssim \delta(\kappa) \int (\varepsilon_y^2 + \varepsilon^2) \phi'_{i,B}, \end{aligned} \quad (4.3.25)$$

and the hypothesis (H1), we have:

$$\int Q_{b,\omega}^2 \varepsilon^2 \psi'_B \lesssim \|Q_{b,\omega}\|_{L^\infty(y < -\frac{b}{2})}^2 \int \varepsilon^2 \phi'_{i,B} \leq \frac{\mu_1}{500} \int (\varepsilon_{yy}^2 + \varepsilon^2) \psi'_B, \quad (4.3.26)$$

$$\int \varepsilon^{10} \psi'_B \lesssim \|\varepsilon^2 (\psi'_B)^{\frac{1}{4}}\|_{L^\infty}^4 \int \varepsilon^2 \lesssim \delta(\kappa) \int (\varepsilon_y^2 + \varepsilon^2) \phi'_{i,B} + \left( \int \varepsilon_y^2 (\psi'_B)^{\frac{1}{2}} \right)^2, \quad (4.3.27)$$

and

$$\begin{aligned} \omega^2 \int |\varepsilon|^{2q} \psi'_B &\lesssim \omega^2 \|\varepsilon^2 (\psi'_B)^{\frac{1}{4}}\|_{L^\infty}^4 \int |\varepsilon|^{2q-8} \lesssim \omega^2 \|\varepsilon^2 (\psi'_B)^{\frac{1}{4}}\|_{L^\infty}^4 \|\varepsilon\|_{L^2}^{q-3} \|\varepsilon_y\|_{L^2}^{q-5} \\ &\lesssim \delta(\kappa) \int (\varepsilon_y^2 + \varepsilon^2) \phi'_{i,B} + \left( \int \varepsilon_y^2 (\psi'_B)^{\frac{1}{2}} \right)^2, \end{aligned} \quad (4.3.28)$$

where we use the fact that  $\omega \|\varepsilon_y\|_{L^2}^m \leq \kappa$  for the last inequality.

From  $((\psi')^{\frac{1}{2}})'' \lesssim \phi'_i$  and (H1), we have

$$\begin{aligned} \left( \int \varepsilon_y^2 (\psi'_B)^{\frac{1}{2}} \right)^2 &= \left( - \int \varepsilon \varepsilon_{yy} (\psi'_B)^{\frac{1}{2}} + \frac{1}{2} \int \varepsilon^2 ((\psi'_B)^{\frac{1}{2}})'' \right)^2 \\ &\lesssim \int \varepsilon^2 \int \varepsilon_{yy}^2 \psi'_B + \left( \int \varepsilon^2 \phi'_{i,B} \right)^2 \\ &\lesssim \delta(\kappa) \int (\varepsilon_{yy}^2 \psi'_B + \varepsilon^2 \phi'_{i,B}). \end{aligned} \quad (4.3.29)$$

Injecting (4.3.26)–(4.3.29) into (4.3.24), we have:

$$|I_5| \lesssim \frac{\mu_1}{50} \left( \int \varepsilon_{yy}^2 \psi'_B + \int (\varepsilon^2 + \varepsilon_y^2) \phi'_{i,B} \right). \quad (4.3.30)$$

Now, we turn to the estimate of  $II$ . We write  $II$  in the following form:

$$II = II^< + II^>,$$

where  $II^< >$  correspond to the integration on  $y < -\frac{B}{2}$  and  $y > -\frac{B}{2}$  respectively.

For  $II^<$ , using the fact that  $\psi'_B \sim (\varphi'_{i,B})^2$  for  $y < -\frac{B}{2}$ , we have:

$$\begin{aligned} |II^<| &\lesssim \omega \left( \int_{y < -\frac{B}{2}} (|\varepsilon|^{q+1} + |Q_{b,\omega}|^{q-1} \varepsilon^2) \varphi'_{i,B} + \int_{y < -\frac{B}{2}} |Q'_{b,\omega}| (|\varepsilon|^q + \varepsilon^2) \varphi_{i,B} \right) \\ &\quad + \omega \int_{y < -\frac{B}{2}} \psi'_B |\varepsilon_y| \left( |\varepsilon|^{q-1} + |Q_{b,\omega}|^{q-2} |\varepsilon| + |\varepsilon_y| |\varepsilon|^{q-1} + |\varepsilon_y| |Q_{b,\omega}|^{q-1} \right) \\ &\leq C(\mu_1) \omega \left( \int \varphi'_{i,B} (|\varepsilon|^{q+1} + |\varepsilon|^q) + \int \psi'_B (\varepsilon_y^2 |\varepsilon|^{q-1} + |\varepsilon_y| |\varepsilon|^{q-1}) \right) \\ &\quad + \frac{\mu_1}{500} \int_{y < -\frac{B}{2}} (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B}. \end{aligned}$$

We use (H1)–(H3) and Gagliardo-Nirenberg's inequality to estimate these terms separately. First, we have

$$\begin{aligned} \omega \int |\varepsilon|^{q+1} \varphi'_{i,B} &\lesssim \omega \|\varepsilon^2 (\varphi'_{i,B})^{1/2}\|_{L^\infty}^2 \int |\varepsilon|^{q-3} \\ &\lesssim \omega \left( \int \varepsilon^2 \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B} \right) \left( \|\varepsilon_y\|_{L^2}^{\frac{q-5}{2}} \|\varepsilon\|_{L^2}^{\frac{q-1}{2}} \right) \\ &\lesssim (\omega \|\varepsilon_y\|_{L^2}^m) \left( \int \varepsilon^2 \right)^{\frac{q+3}{4}} \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B} \\ &\lesssim \delta(\kappa) \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B}, \end{aligned}$$

and

$$\begin{aligned} \omega \int |\varepsilon|^q \varphi'_{i,B} &\lesssim \omega \|\varepsilon^2 (\varphi'_{i,B})^{1/2}\|_{L^\infty}^{\frac{3}{2}} \|\varepsilon\|_{L^4}^{\frac{1}{2}} (\varphi'_{i,B})^{\frac{1}{4}} \|\varepsilon\|_{L^4}^{\frac{2q-7}{2}} \| \varepsilon \|_{L^{4/3}} \\ &\lesssim \omega \left( \int \varepsilon^2 \right)^{\frac{3}{4}} \left( \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B} \right) \|\varepsilon\|_{L^2}^{\frac{q-2}{2}} \|\varepsilon_y\|_{L^2}^{\frac{q-5}{2}} \\ &\lesssim \delta(\kappa) \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B}. \end{aligned}$$

From  $\psi' \lesssim (\varphi'_i)^2$  and (4.3.29), we also have:

$$\begin{aligned} \omega \int \psi'_B \varepsilon_y^2 |\varepsilon|^{q-1} &\lesssim \omega \|\varepsilon^2 (\psi'_B)^{1/4}\|_{L^\infty}^2 \|\varepsilon\|_{L^\infty}^{q-5} \int \varepsilon_y^2 (\psi'_B)^{1/2} \\ &\lesssim (\omega \|\varepsilon_y\|_{L^2}^m) \|\varepsilon\|_{L^2}^{m+2} \left( \int (\varepsilon^2 + \varepsilon_y^2) (\psi'_B)^{1/2} \right) \int \varepsilon_y^2 (\psi'_B)^{1/2} \\ &\lesssim \delta(\kappa) \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B} + \left( \int \varepsilon_y^2 (\psi'_B)^{1/2} \right)^2 \\ &\lesssim \delta(\kappa) \left( \int \varepsilon_{yy}^2 \psi'_B + \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B} \right), \end{aligned}$$

and

$$\begin{aligned}
\omega \int \psi'_B |\varepsilon_y| |\varepsilon|^{q-1} &\lesssim \omega \|\varepsilon^2 (\psi'_B)^{1/4}\|_{L^\infty}^{\frac{3}{2}} \|\varepsilon_y (\psi'_B)^{1/4}\|_{L^2} \|\varepsilon|^{q-4}\|_{L^2} \\
&\lesssim (\omega \|\varepsilon_y\|_{L^2}^m) \|\varepsilon\|_{L^2}^{m+2} \left( \int (\varepsilon^2 + \varepsilon_y^2) (\psi'_B)^{1/2} \right) \left( \int \varepsilon_y^2 (\psi'_B)^{1/2} \right)^{\frac{1}{2}} \\
&\lesssim \delta(\kappa) \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B} + \left( \int \varepsilon_y^2 (\psi'_B)^{1/2} \right)^{\frac{3}{2}} \\
&\leq \frac{\mu_1}{1000} \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B} + C(\mu_1) \left( \int \varepsilon_y^2 (\psi'_B)^{1/2} \right)^2 \\
&\leq \frac{\mu_1}{500} \left( \int \varepsilon_{yy}^2 \psi'_B + \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B} \right).
\end{aligned}$$

In conclusion, we have

$$|II^<| \leq \frac{\mu_1}{50} \left( \int \varepsilon_{yy}^2 \psi'_B + \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B} \right). \quad (4.3.31)$$

For  $II^>$ , we know that  $\psi'_B \equiv 0$  for  $y > -\frac{B}{2}$ . Using Sobolev Embedding, we have

$$\|\varepsilon\|_{L^\infty(y > -\frac{B}{2})}^2 \lesssim \|\varepsilon\|_{L^2(y > -\frac{B}{2})} \|\varepsilon_y\|_{L^2(y > -\frac{B}{2})} \leq \mathcal{N}_2 \leq 1.$$

Thus, we have

$$|II^>| \lesssim \omega \left( \int \varepsilon^2 \varphi'_{i,B} + \int (Q_{b,\omega})_y \varepsilon^2 \varphi_{i,B} \right) \lesssim_B \delta(\kappa) \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B}. \quad (4.3.32)$$

Combining (4.3.23), (4.3.30), (4.3.31) and (4.3.32), we have

$$f_{1,1} \leq -\mu_0 \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B} + Cb^4, \quad (4.3.33)$$

for some universal constant  $\mu_0 > 0$ .

Now, let us deal with  $f_{1,2}$ , it is easy to see that

$$f_{1,2} = \tilde{I} + \tilde{II},$$

where

$$\begin{aligned}
\tilde{I} &= 2 \left( \frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_{b,\omega} \left( -(\psi_B \varepsilon_y)_y + \varepsilon \varphi_{i,B} - \psi_B [(Q_{b,\omega} + \varepsilon)^5 - Q_{b,\omega}^5] \right), \\
\tilde{II} &= 2\omega \left( \frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_{b,\omega} \psi_B \left( (Q_{b,\omega} + \varepsilon) |Q_{b,\omega} + \varepsilon|^{q-1} - Q_{b,\omega} |Q_{b,\omega}^{q-1}| \right).
\end{aligned}$$

The term  $\tilde{I}$  can be estimated by the same argument as in [57, Propostion 3.1]. Thus, we have<sup>10</sup>

$$|\tilde{I}| \leq \frac{\mu_0}{500} \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B} + Cb^2(\omega^2 + b^2).$$

10. There is a slightly difference between the estimate of  $(\lambda_s/\lambda + b)$  in this paper and in [57, Lemma 2.7], so there is an additional term i.e.  $\omega^2 b^2$  on the right hand side of this inequality.



While for  $\tilde{H}$ , we have

$$|\tilde{H}| \lesssim \omega \left| \frac{\lambda_s}{\lambda} + b \right| \left( B^{\frac{1}{2}} \mathcal{N}_{i,\text{loc}}^{\frac{1}{2}} + \int |\varepsilon|^q \Psi_B \right).$$

Using (4.2.33) and the strategy for  $f_{1,1}$ , we have

$$|\tilde{H}| \leq \frac{\mu_0}{500} \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B} + Cb^2(\omega^2 + b^2).$$

Similar argument can be applied to  $f_{1,k}$ ,  $k = 3, 4, 5$ . Together with (4.3.33), we conclude the proof of (4.3.16).

**Control of  $f_2$ .** For  $f_2$ , we integrate by parts, using (4.3.22) to get

$$\begin{aligned} f_2 = 2\mathcal{J}_{i,j} \int \varepsilon \varphi_{i,B} & \left[ (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))_y + \left( \frac{\lambda_s}{\lambda} + b \right) \Lambda Q_{b,\omega} \right. \\ & \left. + \left( \frac{x_s}{\lambda} - 1 \right) (Q_{b,\omega} + \varepsilon)_y - b_s \frac{\partial Q_{b,\omega}}{\partial b} - \omega_s \frac{\partial Q_{b,\omega}}{\partial \omega} + \Psi_{b,\omega} \right]. \end{aligned}$$

We integrate by parts, estimating all terms like we did for  $f_1$ . Together with

$$|\mathcal{J}_{i,j}| \lesssim |J_1| \lesssim \mathcal{N}_2^{\frac{1}{2}} \lesssim \delta(\kappa),$$

we have

$$|f_2| \lesssim \delta(\kappa) \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B} + b^2(\omega^2 + b^2). \quad (4.3.34)$$

**Control of  $f_3$ .** Recall that

$$\begin{aligned} f_3 = 2 \frac{\lambda_s}{\lambda} \int \Lambda \varepsilon & \left( -(\Psi_B \varepsilon_y)_y + (1 + \mathcal{J}_{i,j}) \varepsilon \varphi_{i,B} - \Psi_B \Delta_{b,\omega}(\varepsilon) \right) \\ & + (\mathcal{J}_{i,j})_s \int \varphi_{i,B} \varepsilon^2 - 2(j-1) \frac{\lambda_s}{\lambda} \mathcal{F}_{i,j}. \end{aligned}$$

Integrating by parts<sup>11</sup>, we have

$$f_3 = I + II,$$

where

$$\begin{aligned} I = & \frac{\lambda_s}{\lambda} \int [(2 - 2(j-1)) \Psi_B - y \Psi'_B] \varepsilon_y^2 \\ & - \frac{1}{3} \frac{\lambda_s}{\lambda} \int [(2 - 2(j-1)) \Psi_B - y \Psi'_B] [(Q_{b,\omega} + \varepsilon)^6 - Q_{b,\omega}^6 - 6\varepsilon Q_{b,\omega}^5] \\ & + 2 \frac{\lambda_s}{\lambda} \int \Psi_B \Lambda Q_b [(Q_{b,\omega} + \varepsilon)^5 - Q_{b,\omega}^5 - 5\varepsilon Q_{b,\omega}^4] \\ & + (\mathcal{J}_{i,j})_s \int \varepsilon^2 \varphi_{i,B} - \frac{\lambda_s}{\lambda} (1 + \mathcal{J}_{i,j}) \int y \varphi'_{i,B} \varepsilon^2 - 2(j-1) \frac{\lambda_s}{\lambda} (1 + \mathcal{J}_{i,j}) \int \varepsilon^2 \varphi_{i,B}, \end{aligned}$$

11. See [57, Proposition 3.1, Step 5] and [46, (5.22)] for more details.

and

$$\begin{aligned} II = & \frac{2\omega}{q+1} \frac{\lambda_s}{\lambda} \int \left[ \left( \frac{q+3}{q-1} - 2(j-1) \right) \psi_B - y\psi'_B \right] \times \left[ |Q_{b,\omega} + \varepsilon|^{q+1} \right. \\ & \left. - |Q_{b,\omega}|^{q+1} - (q+1)\varepsilon Q_{b,\omega} |Q_{b,\omega}|^{q-1} \right] \\ & - 2\omega \frac{\lambda_s}{\lambda} \int \psi_B \Lambda Q_b \left[ (Q_{b,\omega} + \varepsilon) |Q_{b,\omega} + \varepsilon|^{q-1} - Q_{b,\omega} |Q_{b,\omega}|^{q-1} - q\varepsilon |Q_{b,\omega}|^{q-1} \right]. \end{aligned}$$

Similarly, we can use the same strategy as in [57, Proposition 3.1] to estimate  $I$ . So we have:

$$|I| \lesssim \delta(\kappa) \int (\varepsilon^2 + \varepsilon_y^2) \phi'_{i,B} + b^2(\omega^2 + b^2).$$

While for  $II$ , from

$$\psi_B + \left| \left( \frac{q+3}{q-1} - 2(j-1) \right) \psi_B - y\psi'_B \right| \lesssim_B \phi'_{i,B},$$

we have

$$|II| \lesssim \omega \left| \frac{\lambda_s}{\lambda} \right| \int (|\varepsilon|^{q+1} + |\varepsilon|^q + \varepsilon^2) \phi'_{i,B}.$$

Using  $|\lambda_s/\lambda| \lesssim \delta(\kappa)$  and the strategy for  $f_{1,1}$ , we have:

$$|II| \lesssim \delta(\kappa) \int (\varepsilon^2 + \varepsilon_y^2) \phi'_{i,B}.$$

In conclusion, we have

$$|f_3| \lesssim \delta(\kappa) \int (\varepsilon^2 + \varepsilon_y^2) \phi'_{i,B} + b^2(\omega^2 + b^2). \quad (4.3.35)$$

**Control of  $f_4$ .** From (4.2.4) and (4.3.12), we have

$$|(Q_{b,\omega})_s| \lesssim |b_s| \left| \frac{\partial Q_{b,\omega}}{\partial b} \right| + |\omega_s| \left| \frac{\partial Q_{b,\omega}}{\partial \omega} \right| \leq (\omega + |b|)(|b| + \mathcal{N}_{i,\text{loc}}^{\frac{1}{2}}) \lesssim \delta(\kappa).$$

Using the Sobolev bounds (4.3.25) and the strategy for  $f_{1,1}$ , we have

$$\begin{aligned} |f_4| & \lesssim \delta(\kappa) \left( \int (|\varepsilon|^q + |\varepsilon|^5 + \varepsilon^2) \phi'_{i,B} \right) \\ & \lesssim \delta(\kappa) \left( \int \varepsilon_{yy}^2 \psi'_B + \int (\varepsilon^2 + \varepsilon_y^2) \phi'_{i,B} \right). \end{aligned} \quad (4.3.36)$$

**Control of  $f_5$ .** From (4.2.33) we know that

$$|\omega_s| = m\omega \left| \frac{\lambda_s}{\lambda} \right| \lesssim \delta(\kappa).$$

Thus, the Sobolev bounds (4.3.25) and the strategy for  $f_{1,1}$ , we have

$$\begin{aligned} |f_5| &\lesssim \delta(\kappa) \int (|\varepsilon|^{q+1} + \varepsilon^2) \varphi'_{i,B} \\ &\lesssim \delta(\kappa) \left( \int \varepsilon_{yy}^2 \psi'_B + \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{i,B} \right). \end{aligned} \quad (4.3.37)$$

Combining (4.3.34)–(4.3.37), we conclude the proof of (4.3.17), hence the proof of Proposition 4.3.1.  $\square$

### 4.3.2 Dynamical control of the tail on the right

In order to close the bootstrap bound (H3), we need the dynamical control of the  $L^2$  tail on the right introduced in [57]. More precisely, we choose a smooth function

$$\varphi_{10}(y) = \begin{cases} 0, & \text{for } y < 0, \\ y^{10}, & \text{for } y > 1, \end{cases} \quad \varphi'_{10} \geq 0.$$

Then we have

**Lemma 4.3.5** (Dynamical control of the tail on the right, [57]). *Under the assumption of Proposition 4.3.1, there holds:*

$$\frac{1}{\lambda^{10}} \frac{d}{ds} \left( \lambda^{10} \int \varphi_{10} \varepsilon^2 \right) \lesssim \mathcal{N}_{1,\text{loc}} + b^2. \quad (4.3.38)$$

*Proof.* The proof of Lemma 4.3.5 is exactly the same as [57, Lemma 3.7]. We omit the proof here.  $\square$

## 4.4 Rigidity of the dynamics in $\mathcal{A}_{\alpha_0}$ and proof of Theorem 4.1.3

In this section we will classify the behavior of any solution with initial data in  $\mathcal{A}_{\alpha_0}$ , which directly implies Theorem 4.1.3. To begin, we define

$$t^* = \sup\{0 < t < +\infty \mid \text{for all } t' \in [0, t], u(t') \in \mathcal{T}_{\alpha^*, \gamma}\}. \quad (4.4.1)$$

Assume  $0 < \gamma \ll \alpha_0 \ll \alpha^* \ll 1$ , then the condition on the initial data, i.e.  $u_0 \in \mathcal{A}_{\alpha_0}$  implies  $t^* > 0$ .

Next, by Lemma 4.2.6,  $u(t)$  admits the following geometrical decomposition on  $[0, t^*]$ :

$$u(t, x) = \frac{1}{\lambda^{\frac{1}{2}}(t)} [Q_{b(t), \omega(t)} + \varepsilon(t)] \left( \frac{x - x(t)}{\lambda(t)} \right).$$

The condition  $u_0 \in \mathcal{A}_{\alpha_0}$  implies:

$$\omega(0) + \|\varepsilon(0)\|_{H^1} + \omega(0) \|\varepsilon_y(0)\|_{L^2}^m + |b(0)| + |1 - \lambda(0)| \lesssim \delta(\alpha_0), \quad (4.4.2)$$

$$\int_{y>0} y^{10} \varepsilon^2(0) dy \leq 2. \quad (4.4.3)$$

Using Hölder's inequality, we have:

$$\mathcal{N}_2(0) \lesssim \delta(\alpha_0). \quad (4.4.4)$$

Then let us fix a  $0 < \kappa \ll 1$  as in Proposition 4.2.9 and 4.3.1, and define

$$t^{**} = \sup\{0 < t < t^* \mid \text{(H1), (H2) and (H3) hold for all } t' \in [0, t]\}. \quad (4.4.5)$$

Note that from (4.4.2)–(4.4.4), we have  $t^{**} > 0$ . Let  $s^* = s(t^*)$ ,  $s^{**} = s(t^{**})$ ,  $s_1^* = s(t_1^*)$ .

#### 4.4.1 Consequence of the monotonicity formula

We derive some crucial estimates from the monotonicity formula introduced in Section 3.

**Lemma 4.4.1.** *We have the following:*

1. *Almost monotonicity of the localized Sobolev norm. There exists a universal constant  $K_0 > 1$ , such that for  $i = 1, 2$  and  $0 \leq s_1 < s_2 \leq s^{**}$ , there holds:*

$$\begin{aligned} & \mathcal{N}_i(s_2) + \int_{s_1}^{s_2} \int (\varepsilon_y^2(s, y) + \varepsilon^2(s, y)) \varphi'_{i,B}(y) dy ds \\ & \leq K_0 \left[ \mathcal{N}_i(s_1) + \sup_{s \in [s_1, s_2]} |b(s)|^3 + \sup_{s \in [s_1, s_2]} \omega^3(s) \right], \end{aligned} \quad (4.4.6)$$

$$\begin{aligned} & \frac{\mathcal{N}_i(s_2)}{\lambda^2(s_2)} + \int_{s_1}^{s_2} \frac{1}{\lambda^2(s)} \left[ \left( \int (\varepsilon_y^2 + \varepsilon^2)(s) \varphi'_{i,B} \right) + b^2(s) (|b(s)| + \omega(s)) \right] ds \\ & \leq K_0 \left( \frac{\mathcal{N}_i(s_1)}{\lambda^2(s_1)} + \left[ \frac{b^2(s_1) + \omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2) + \omega^2(s_2)}{\lambda^2(s_2)} \right] \right). \end{aligned} \quad (4.4.7)$$

2. *Control of  $b$  and  $\omega$ . For all  $0 \leq s_1 < s_2 \leq s^{**}$ , there holds:*

$$\omega(s_2) + \int_{s_1}^{s_2} b^2(s) ds \lesssim \mathcal{N}_1(s_1) + \omega(s_1) + \sup_{s \in [s_1, s_2]} |b(s)|, \quad (4.4.8)$$

3. *Control of  $\frac{b}{\lambda^2}$ . Let  $c_1 = \frac{m}{m+2} G'(0) > 0$ , where  $G$  is the  $C^2$  function introduced in (4.2.39). Then there exists a universal constant  $K_1 > 1$  such that for all  $0 \leq s_1 < s_2 \leq s^{**}$ , there holds:*

$$\begin{aligned} & \left| \frac{b(s_2) + c_1 \omega(s_2)}{\lambda^2(s_2)} - \frac{b(s_1) + c_1 \omega(s_1)}{\lambda^2(s_1)} \right| \\ & \leq K_1 \left( \frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_1) + \omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2) + \omega^2(s_2)}{\lambda^2(s_2)} \right). \end{aligned} \quad (4.4.9)$$

4. *Refined control of  $\lambda$ .* Let  $\lambda_0(s) = \lambda(s)(1 - J_1(s))^2$ . Then there exists a universal constant  $K_2 > 1$  such that for all  $s \in [0, s^{**}]$ ,

$$\left| \frac{(\lambda_0)_s}{\lambda_0} + b \right| \leq K_2 \left[ \mathcal{N}_1 + (|b| + \omega)(\mathcal{N}_2^{\frac{1}{2}} + |b|) \right]. \quad (4.4.10)$$

*Proof.* Proof of (4.4.6) and (4.4.8). From (4.2.48), we have:

$$\frac{d}{ds} G(\omega) + b^2 \leq -b_s + C\mathcal{N}_{1,\text{loc}}.$$

Integrating from  $s_1$  to  $s_2$ , we have

$$\begin{aligned} G(\omega(s_2)) + \int_{s_1}^{s_2} b^2 &\lesssim \int_{s_1}^{s_2} \mathcal{N}_{1,\text{loc}} + G(\omega(s_1)) + |b(s_2) - b(s_1)| \\ &\lesssim \int_{s_1}^{s_2} \mathcal{N}_{1,\text{loc}} + G(\omega(s_1)) + \sup_{s \in [s_1, s_2]} |b(s)| \end{aligned}$$

Since  $G(\omega) \sim \omega$ , we then obtain (4.4.8).

Next, from the monotonicity formula (4.3.11) and (4.3.13) we obtain:

$$\begin{aligned} \mathcal{N}_i(s_2) + \int_{s_1}^{s_2} \int (\varepsilon_y^2(s, y) + \varepsilon^2(s, y)) \varphi'_{i,B}(y) dy ds \\ \lesssim \mathcal{N}_i(s_1) + \left[ \sup_{s \in [s_1, s_2]} b^2(s) + \sup_{s \in [s_1, s_2]} \omega^2(s) \right] \int_{s_1}^{s_2} b^2, \end{aligned} \quad (4.4.11)$$

Combining (4.4.8) and (4.4.11), we obtain (4.4.6).

*Proof of (4.4.7).* First, from (4.2.48) and (4.2.34), we have

$$\begin{aligned} 2 \int_{s_1}^{s_2} \frac{|b|^3}{\lambda^2} &\leq \int_{s_1}^{s_2} \left[ - \frac{|b|b_s - \omega_s G'(\omega)|b| + C\mathcal{N}_{1,\text{loc}} + \delta(\kappa)|b|^3}{\lambda^2} \right] \\ &\leq - \frac{1}{2} \frac{b|b|}{\lambda^2} \Big|_{s_1}^{s_2} + O \left( \int_{s_1}^{s_2} \frac{\mathcal{N}_{1,\text{loc}} + \omega b^2}{\lambda^2} \right) + \delta(\kappa) \int_{s_1}^{s_2} \frac{|b|^3}{\lambda^2}. \end{aligned} \quad (4.4.12)$$

Recall that  $\omega = \frac{\gamma}{\lambda^m}$ . Then from (4.2.33) we have:

$$\begin{aligned} \int_{s_1}^{s_2} \frac{\omega b^2}{\lambda^2} &= - \int_{s_1}^{s_2} \frac{\lambda_s}{\lambda} \frac{\omega b}{\lambda^2} + \int_{s_1}^{s_2} \frac{\omega b}{\lambda^2} \left( \frac{\lambda_s}{\lambda} + b \right) \\ &\leq \frac{1}{m+2} \int_{s_1}^{s_2} \left( \frac{\omega}{\lambda^2} \right)_s b + \delta(\kappa) \int_{s_1}^{s_2} \frac{\omega b^2}{\lambda^2} + O \left( \int_{s_1}^{s_2} \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2} \right) \\ &= \frac{1}{m+2} \int_{s_1}^{s_2} \frac{\omega}{\lambda^2} (-b_s) + \delta(\kappa) \int_{s_1}^{s_2} \frac{\omega b^2}{\lambda^2} \\ &\quad + O \left( \int_{s_1}^{s_2} \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2} + \frac{b^2(s_1) + \omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2) + \omega^2(s_2)}{\lambda^2(s_2)} \right). \end{aligned} \quad (4.4.13)$$

From (4.2.48) and (4.2.34), we have:

$$\begin{aligned} \int_{s_1}^{s_2} \frac{\omega}{\lambda^2} (-b_s) &\leq \int_{s_1}^{s_2} \frac{\omega}{\lambda^2} \left[ \left(2 + \frac{m}{10}\right) b^2 + \omega_s G'(\omega) + C(m) \mathcal{N}_{1,\text{loc}} \right] \\ &\leq \left(2 + \frac{m}{10}\right) \int_{s_1}^{s_2} \frac{\omega b^2}{\lambda^2} + \int_{s_1}^{s_2} \frac{\omega_s \omega G'(\omega)}{(\gamma/\omega)^{2/m}} \\ &\quad + O\left( \int_{s_1}^{s_2} \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2} + \frac{b^2(s_1) + \omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2) + \omega^2(s_2)}{\lambda^2(s_2)} \right). \end{aligned} \quad (4.4.14)$$

From (4.2.33) and (4.2.34) again, we have:

$$\left| \int_{s_1}^{s_2} \frac{\omega_s \omega G'(\omega)}{(\gamma/\omega)^{2/m}} \right| = \frac{|M(\omega(s_2)) - M(\omega(s_1))|}{\gamma^{2/m}} \lesssim \frac{\omega^2(s_1)}{\lambda^2(s_1)} + \frac{\omega^2(s_2)}{\lambda^2(s_2)}, \quad (4.4.15)$$

where

$$M(\omega) = \int_0^\omega x^{1+2/m} G'(x) dx \sim \omega^{2+2/m}.$$

Therefore, combining (4.4.13)–(4.4.15), we have

$$\begin{aligned} \int_{s_1}^{s_2} \frac{\omega b^2}{\lambda^2} &\leq \left( \frac{2+m/10}{m+2} + \delta(\kappa) \right) \int_{s_1}^{s_2} \frac{\omega b^2}{\lambda^2} \\ &\quad + O\left( \int_{s_1}^{s_2} \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2} + \left[ \frac{b^2(s_1) + \omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2) + \omega^2(s_2)}{\lambda^2(s_2)} \right] \right). \end{aligned} \quad (4.4.16)$$

Taking  $\kappa > 0$  small enough, from (4.4.12) and (4.4.16) we have

$$\int_{s_1}^{s_2} \frac{b^2(\omega + |b|)}{\lambda^2} \lesssim \int_{s_1}^{s_2} \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2} + \left[ \frac{b^2(s_1) + \omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2) + \omega^2(s_2)}{\lambda^2(s_2)} \right], \quad (4.4.17)$$

Now, integrating the monotonicity formula (4.3.12), we have:

$$\begin{aligned} &\frac{\mathcal{N}_i(s_2)}{\lambda^2(s_2)} + \int_{s_1}^{s_2} \frac{1}{\lambda^2(s)} \left[ \left( \int (\varepsilon_y^2 + \varepsilon^2)(s) \varphi'_{i,B} \right) \right] ds \\ &\lesssim \frac{\mathcal{N}_i(s_1)}{\lambda^2(s_1)} + \delta(\kappa) \int_{s_1}^{s_2} \frac{b^2(s)(\omega(s) + |b(s)|)}{\lambda^2(s)} ds, \end{aligned}$$

which implies (4.4.7) immediately.

*Proof of (4.4.9).* The proof of (4.4.9) based on integrating the equation of  $\frac{b}{\lambda^2}$ , i.e. (4.2.41). More precisely, from (4.2.33), (4.2.41) and the fact that  $|J| \lesssim \mathcal{N}_{1,\text{loc}}^{\frac{1}{2}}$  (recall that  $J$  given by (4.2.40) is a well localized  $L^2$  scalar product), we have:

$$\begin{aligned} &\left| \left( \frac{b}{\lambda^2} e^J \right)_s + \frac{\omega_s G'(\omega)}{\lambda^2} e^J \right| = \left| \left( \frac{b}{\lambda^2} \right)_s + \frac{b}{\lambda^2} J_s + \frac{\omega_s G'(\omega)}{\lambda^2} \right| e^J \\ &\lesssim \left| \frac{\lambda_s}{\lambda} \frac{b}{\lambda^2} J \right| + O\left( \frac{1}{\lambda^2} \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} + (\omega + |b|) b^2 \right) \right) \\ &\lesssim \frac{b^2}{\lambda^2} |J| + O\left( \frac{1}{\lambda^2} \left( \int \varepsilon^2 e^{-\frac{|y|}{10}} + (\omega + |b|) b^2 \right) \right) \\ &\lesssim O\left( \frac{1}{\lambda^2} (\mathcal{N}_{1,\text{loc}} + (\omega + |b|) b^2) \right). \end{aligned}$$

We integrate this estimate in time using (4.4.7) to get

$$\left| \left[ \frac{b}{\lambda^2} e^J \right]_{s_1}^{s_2} + \int_{s_1}^{s_2} \frac{\omega_s G'(\omega)}{\lambda^2} e^J \right| \lesssim \frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)} + \left[ \frac{b^2(s_1) + \omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2) + \omega^2(s_2)}{\lambda^2(s_2)} \right]. \quad (4.4.18)$$

Note that  $|e^J - 1| \leq 2|J| \lesssim \mathcal{N}_{1,\text{loc}}^{\frac{1}{2}}$ . Together with (4.4.7), we have

$$\begin{aligned} \left[ \frac{b}{\lambda^2} e^J \right]_{s_1}^{s_2} &= \frac{b}{\lambda^2} \Big|_{s_1}^{s_2} + \left| \left[ \frac{b}{\lambda^2} \mathcal{N}_{1,\text{loc}}^{\frac{1}{2}} \right]_{s_1}^{s_2} \right| \\ &= \frac{b}{\lambda^2} \Big|_{s_1}^{s_2} + O\left( \frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_1) + \omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2) + \omega^2(s_2)}{\lambda^2(s_2)} \right). \end{aligned} \quad (4.4.19)$$

Next, from (4.2.34), (4.4.7) and  $|J| \lesssim \mathcal{N}_{1,\text{loc}}^{\frac{1}{2}}$ , we have

$$\begin{aligned} \left| \int_{s_1}^{s_2} \frac{\omega_s G'(\omega)}{\lambda^2} (e^J - 1) \right| &\lesssim \int_{s_1}^{s_2} \frac{(b^2 + \omega^2)b^2 + \mathcal{N}_{1,\text{loc}}}{\lambda^2} \\ &\lesssim \frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_1) + \omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2) + \omega^2(s_2)}{\lambda^2(s_2)}. \end{aligned} \quad (4.4.20)$$

Finally, recall  $\omega = \gamma/\lambda^m$ , so we have:

$$\int_{s_1}^{s_2} \frac{\omega_s G'(\omega)}{\lambda^2} = \int_{s_1}^{s_2} \frac{\omega_s G'(\omega)}{(\gamma/\omega)^{2/m}} = \frac{\Sigma(\omega)}{\lambda^2} \Big|_{s_1}^{s_2},$$

where

$$\Sigma(\omega) := \frac{1}{\omega^{2/m}} \int_0^\omega x^{2/m} G'(x) dx.$$

Recall that  $G$  is the  $C^2$  function introduced in (4.2.39). We then have  $\Sigma \in C^2$  and  $c_1 = \Sigma'(0) = \frac{m}{m+2} G'(0) > 0$ . Hence, we have

$$\int_{s_1}^{s_2} \frac{\omega_s G'(\omega)}{\lambda^2} = \frac{c_1 \omega}{\lambda^2} \Big|_{s_1}^{s_2} + O\left( \frac{\omega^2(s_1)}{\lambda^2(s_1)} + \frac{\omega^2(s_2)}{\lambda^2(s_2)} \right), \quad (4.4.21)$$

Combining (4.4.18)–(4.4.21), we conclude the proof of (4.4.9).

*Proof of (4.4.10).* From (4.3.14), we have

$$\left| \frac{\lambda}{\lambda_0} - 1 \right| \lesssim |J_1| \lesssim \mathcal{N}_2^{\frac{1}{2}} \lesssim \delta(\kappa),$$

thus we obtain from (4.2.37):

$$\begin{aligned} \left| \frac{(\lambda_0)_s}{\lambda_0} + b \right| &= \left| \frac{1}{1 - J_1} \left[ (1 - J_1) \frac{\lambda_s}{\lambda} + b - 2(J_1)_s \right] - \frac{J_1}{1 - J_1} b \right| \\ &\lesssim \int \varepsilon^2 e^{-\frac{|y|}{10}} + (|b| + \omega) (\mathcal{N}_2^{\frac{1}{2}} + |b|). \end{aligned}$$

This concludes the proof of (4.4.10), hence the proof of Lemma 4.4.1.  $\square$

### 4.4.2 Rigidity dynamics in $\mathcal{A}_{\alpha_0}$

In this part, we will give a specific classification for the asymptotic behavior of solution with initial data in  $\mathcal{A}_{\alpha_0}$ .

We first introduce the separation time  $t_1^*$ :

$$\begin{aligned} t_1^* &= 0, \text{ if } |b(0) + c_1 \omega(0)| \geq C^* (\mathcal{N}_1(0) + b^2(0) + \omega^2(0)), \\ t_1^* &= \sup \left\{ 0 < t < t^* \mid \text{for all } t' \in [0, t], \right. \\ &\quad \left. |b(t') + c_1 \omega(t')| \leq C^* (\mathcal{N}_1(t') + b^2(t') + \omega^2(t')) \right\}, \text{ otherwise,} \end{aligned} \quad (4.4.22)$$

where <sup>12</sup>

$$C^* = 100(K_1 + K_0 K_2) > 0. \quad (4.4.23)$$

Then we have:

**Proposition 4.4.2** (Rigidity Dynamics). *There exist universal constants  $0 < \gamma \ll \alpha_0 \ll \alpha^* \ll 1$  and  $C^* > 1$  such that the following holds. Let  $u_0 \in \mathcal{A}_{\alpha_0}$ , and  $u(t)$  be the corresponding solution to  $(\text{gKdV}_\gamma)$ . Then we have:*

(1) *The following trichotomy holds:*

- **(Blow down):** *If  $t_1^* = t^*$ , then  $t_1^* = t^* = T = +\infty$  with,*

$$|b(t)| + \mathcal{N}_2(t) \rightarrow 0, \quad \text{as } t \rightarrow +\infty, \quad (4.4.24)$$

$$\lambda(t) \sim t^{\frac{2}{q+1}}, \quad x(t) \sim t^{\frac{q-3}{q+1}}, \quad \text{as } t \rightarrow +\infty. \quad (4.4.25)$$

- **(Exit):** *If  $t_1^* < t^*$  with*

$$b(t_1^*) + c_1 \omega(t_1^*) \leq -C^* (\mathcal{N}_1(t_1^*) + b^2(t_1^*) + \omega^2(t_1^*)),$$

*then  $t^* < T = +\infty$ . In particular,*

$$\inf_{\lambda_0 > 0, \lambda_0^{-m} \gamma < \omega^*, x_0 \in \mathbb{R}} \left\| u(t^*) - \frac{1}{\lambda_0^{\frac{1}{2}}} \mathcal{Q}_{\lambda_0^{-m} \gamma} \left( \frac{x - x_0}{\lambda_0} \right) \right\|_{L^2} = \alpha^*. \quad (4.4.26)$$

*Moreover, we have*

$$b(t^*) \leq -C(\alpha^*) < 0, \quad \lambda(t^*) \geq \frac{C(\alpha^*)}{\delta(\alpha_0)} \gg 1. \quad (4.4.27)$$

- **(Soliton):** *If  $t_1^* < t^*$  with*

$$b(t_1^*) + c_1 \omega(t_1^*) \geq C^* (\mathcal{N}_1(t_1^*) + b^2(t_1^*) + \omega^2(t_1^*)),$$

*then  $t^* = T = +\infty$ . Moreover, we have:*

$$\mathcal{N}_2(t) + |b(t)| \rightarrow 0, \quad \text{as } t \rightarrow +\infty, \quad (4.4.28)$$

$$\lambda(t) = \lambda_\infty (1 + o(1)), \quad x(t) = \frac{t}{\lambda_\infty^2} (1 + o(1)), \quad \text{as } t \rightarrow +\infty, \quad (4.4.29)$$

*for some  $\lambda_\infty \in (0, +\infty)$ .*

---

12. Recall that  $K_0, K_1, K_2$  and  $c_1$  were introduced in Lemma 4.4.1.



(2) All of the three scenarios introduced in (1) are known to occur. Moreover, the initial data sets which lead to the (Soliton) and (Exit) case are open in  $\mathcal{A}_{\alpha_0}$  (under the topology of  $H^1 \cap L^2(y_+^{10} dy)$ ).

*Remark 4.4.3.* It is easy to see Proposition 4.4.2 implies Theorem 4.1.3 immediately.

*Remark 4.4.4.* The constant  $C^*$  chosen here is not sharp. We can replace it by some slightly different ones.

*Proof of Proposition 4.4.2:*

### 1. The blow down case.

Assume that  $t_1^* = t^*$ , i.e. for all  $t \in [0, t^*]$ ,

$$|b(t) + c_1 \omega(t)| \leq C^* (\mathcal{N}_1(t) + b^2(t) + \omega^2(t)). \quad (4.4.30)$$

We claim that  $t^{**} = t^*$ , i.e. the bootstrap assumptions (H1), (H2) and (H3) hold on  $[0, t^*]$ .

Indeed, we claim that for all  $s \in [0, s^{**})$ ,

$$\omega(s) + |b(s)| + \|\varepsilon(s)\|_{L^2} + \mathcal{N}_2(s) \lesssim \delta(\alpha_0), \quad (4.4.31)$$

$$\lambda(s) \geq \frac{4}{5}, \quad (4.4.32)$$

$$\int_{y>0} y^{10} \varepsilon^2(s) dy \leq 5. \quad (4.4.33)$$

Then choosing  $\alpha^*$ ,  $\alpha_0$ ,  $\gamma$  such that  $0 < \gamma \ll \alpha_0 \ll \alpha^* \ll \kappa$ , we can see that (4.4.31)–(4.4.33) imply  $t^{**} = t^*$  immediately.

First, from (4.4.30) we have: for all  $s \in [0, s^{**})$ ,

$$b(s) \leq 4C^* \mathcal{N}_1(s) - |b(s)|, \quad (4.4.34)$$

$$|b(s)| \lesssim \mathcal{N}_1(s) + \omega(s), \quad (4.4.35)$$

$$\omega(s) \lesssim \mathcal{N}_1(s) + |b(s)|. \quad (4.4.36)$$

Then we apply (4.4.34) and (4.4.36) to (4.4.10) to obtain:

$$\begin{aligned} \frac{(\lambda_0)_s}{\lambda_0} &\geq -b - \mathcal{N}_1 - C(\omega + |b|)(\mathcal{N}_2^{\frac{1}{2}} + |b|) \\ &\geq -5C^* \mathcal{N}_1 + |b| - \delta(\kappa)|b| \\ &\gtrsim -\mathcal{N}_1. \end{aligned}$$

Integrating this from  $s_1$  to  $s_2$  for some  $0 \leq s_1 < s_2 \leq s^{**}$ , and using (4.4.6), we have:

$$\lambda(s_2) \geq \frac{9}{10} \lambda(s_1). \quad (4.4.37)$$

In particular, we know from (4.4.2) that for all  $s \in [0, s^{**})$

$$\lambda(s) \geq \frac{9}{10} \lambda(0) \geq \frac{4}{5}. \quad (4.4.38)$$

By our choice of  $\gamma$ , we have

$$\omega(s) = \frac{\gamma}{\lambda^m(s)} \leq 2^m \gamma \lesssim \delta(\alpha_0). \quad (4.4.39)$$

Next, from (4.4.4), (4.4.6) and (4.4.35), we have for all  $s \in [0, s^{**})$

$$\mathcal{N}_2(s) \lesssim \mathcal{N}_2(0) + \sup_{s' \in [0, s]} \mathcal{N}_2^3(s') + \sup_{s' \in [0, s]} \omega^3(s'),$$

which together with (4.4.35) implies that

$$|b(s)| + \mathcal{N}_2(s) \lesssim \delta(\alpha_0),$$

for all  $s \in [0, s^{**})$ . Then from (4.2.31) and the condition on the initial data, we obtain

$$\|\varepsilon(s)\|_{L^2} \lesssim \delta(\alpha_0). \quad (4.4.40)$$

From (4.2.32) and the condition on the initial data, we have

$$\frac{\|\varepsilon_y(s)\|_{L^2}^2}{\lambda^2(s)} \lesssim \delta(\alpha_0) + \frac{\|\varepsilon_y(s)\|_{L^2}^{m+2}}{\lambda^{m+2}(s)}.$$

Since  $\|\varepsilon_y(0)\|_{L^2} \lesssim \delta(\alpha_0)$ ,  $\lambda(0) \sim 1$ , from a standard bootstrap argument, we have:

$$\frac{\|\varepsilon_y(s)\|_{L^2}^2}{\lambda^2(s)} \lesssim \delta(\alpha_0).$$

Thus, we have

$$\omega(s) \|\varepsilon_y(s)\|_{L^2}^m \lesssim \gamma \frac{\|\varepsilon_y(s)\|_{L^2}^m}{\lambda^m(s)} \lesssim \delta(\alpha_0). \quad (4.4.41)$$

Finally, let us integrate (4.3.38) from 0 to  $s$ , using (4.4.3), (4.4.6), (4.4.8), (4.4.37) and (4.4.38) to obtain

$$\begin{aligned} \int \varphi_{10} \varepsilon^2(s) dy &\leq \frac{\lambda^{10}(0)}{\lambda^{10}(s)} \int \varphi_{10} \varepsilon^2(0) dy + C \int_0^s \frac{\lambda^{10}(s')}{\lambda^{10}(s)} (\mathcal{N}_{1,\text{loc}}(s') + b^2(s')) ds' \\ &\leq 3 + C \int_0^s (\mathcal{N}_{1,\text{loc}}(s') + b^2(s')) ds' \leq 3 + \delta(\kappa) < 5. \end{aligned}$$

We therefore conclude the proof of (4.4.31)–(4.4.33), and obtain  $t^{**} = t^*$ . Since  $0 < \alpha_0 \ll \alpha^*$ , the estimate (4.4.31) implies that  $t^{**} = t^* = T = +\infty$ .

Now we claim that  $\lambda(t) \rightarrow +\infty$  as  $t \rightarrow +\infty$ . Let

$$S = \int_0^{+\infty} \frac{1}{\lambda^3(\tau)} d\tau \in (0, +\infty].$$

From (4.2.34), (4.4.6), (4.4.8) and (4.4.36) we have:

$$\begin{aligned}\int_0^{+\infty} |\omega_t| dt &= \int_0^S |\omega_s| ds \lesssim \int_0^S (\mathcal{N}_{2,\text{loc}}(s) + b^2(s)) ds < +\infty, \\ \int_0^{+\infty} \frac{\gamma^2}{\lambda^{3+2m}(t)} dt &= \int_0^S \omega^2(s) ds \lesssim \int_0^S (\mathcal{N}_{2,\text{loc}}(s) + b^2(s)) ds < +\infty.\end{aligned}$$

This leads to  $\lambda(t) \rightarrow +\infty$  as  $t \rightarrow +\infty$ , or equivalently  $\lim_{t \rightarrow +\infty} \omega(t) = 0$ .

Next, we claim that  $S = +\infty$ . Otherwise,  $b(s), \omega(s) \in L^1([0, S])$ . Applying this to (4.4.10), we obtain:

$$\frac{(\lambda_0)_s}{\lambda_0} \in L^1([0, S]).$$

But since  $\lambda_0(s) \rightarrow +\infty$  as  $s \rightarrow S$ , we have:

$$\left| \int_0^{S-\delta_0} \frac{(\lambda_0)_s(s')}{\lambda_0} ds' \right| = \left| \log \left( \frac{\lambda_0(S-\delta_0)}{\lambda_0(0)} \right) \right| \rightarrow +\infty,$$

as  $\delta_0 \rightarrow 0$ , which leads to a contradiction.

Now we can prove (4.4.24) and (4.4.25). To do this, we claim that for all  $s \in [0, +\infty)$ ,

$$\lambda^m(s) \mathcal{N}_2(s) + \int_0^s \lambda^m(s') (\varepsilon^2(s') + \varepsilon_y^2(s')) \varphi'_{2,B} ds' \lesssim 1. \quad (4.4.42)$$

From (4.3.11) we have:

$$\frac{1}{\lambda^m} \frac{d}{ds} \left( \lambda^m \mathcal{F}_{2,1} \right) \leq -\mu \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{2,B} + O(b^4 + \omega^2 b^2) - m \frac{\lambda_s}{\lambda} \mathcal{F}_{2,1}. \quad (4.4.43)$$

From (4.2.33), (4.3.13), (4.3.21) and (4.4.38), we have

$$\begin{aligned}\left| \frac{\lambda_s}{\lambda} \mathcal{F}_{2,1} \right| &\lesssim (|b| + \mathcal{N}_{1,\text{loc}}^{\frac{1}{2}}) \left[ \left( 1 + \frac{1}{\lambda^{\frac{10}{9}}(s)} \right) \mathcal{N}_{2,\text{loc}}^{\frac{8}{9}} + \int \varepsilon_y^2 \psi_B \right] \\ &\lesssim b^2 + \delta(\kappa) \int (\varepsilon^2 + \varepsilon_y^2) \varphi'_{2,B}.\end{aligned}$$

Injecting this into (4.4.43) and integrating from 0 to  $s$ , using (4.4.35) and (4.4.36), we have,

$$\begin{aligned}\lambda^m(s) \mathcal{N}_2(s) + \int_0^s \lambda^m(s') (\varepsilon^2(s') + \varepsilon_y^2(s')) \varphi'_{2,B} ds' \\ \lesssim \int_0^s \lambda^m(s') \omega^4(s') ds' + \delta(\kappa) \int_0^s \lambda^m(s') \mathcal{N}_1(s') ds' \\ \lesssim \gamma \int_0^s \omega^3(s') ds' + \delta(\kappa) \int_0^s \lambda^m(s') \mathcal{N}_1(s') ds' \\ \lesssim \gamma \int_0^s b^2(s') ds' + \delta(\kappa) \int_0^s \lambda^m(s') \mathcal{N}_1(s') ds'.\end{aligned}$$

Together with (4.4.8), we obtain (4.4.42).

Since  $\lambda(s) \rightarrow +\infty$  as  $s \rightarrow +\infty$ , we have

$$\mathcal{N}_2(s) \lesssim \lambda^{-m}(s) \rightarrow 0 \quad \text{as } s \rightarrow +\infty. \quad (4.4.44)$$

Now, using (4.4.10), (4.4.30) and (4.4.35), we have

$$\begin{aligned} \left| -\frac{(\lambda_0)_s}{\lambda_0} + c_1 \omega \right| &\lesssim \left| \frac{(\lambda_0)_s}{\lambda_0} + b \right| + |b + c_1 \omega| \\ &\lesssim \mathcal{N}_1 + b^2 + \omega^2 + (|b| + \omega)(\mathcal{N}_2^{\frac{1}{2}} + |b|) \\ &\lesssim \mathcal{N}_1 + \delta(\kappa)\omega. \end{aligned}$$

Multiplying the above inequality by  $\lambda_0^m$  and integrating from 0 to  $s$ , we obtain

$$-C \int_0^s \lambda_0^m \mathcal{N}_1 + \frac{1}{2} c_1 \gamma s \leq \int_0^s (\lambda_0)_s \lambda_0^{m-1} \leq C \int_0^s \lambda_0^m \mathcal{N}_1 + 2c_1 \gamma s.$$

From (4.4.42) and  $|1 - \lambda/\lambda_0| \lesssim \delta(\kappa)$ , we obtain

$$\lambda^m(s) \sim s, \quad \text{as } s \rightarrow +\infty.$$

We then have,

$$t(s) = \int_0^s \lambda^3(s') ds' \sim s^{\frac{m+3}{m}} = s^{\frac{q+1}{q-5}}, \quad \text{as } s \rightarrow +\infty,$$

which implies

$$\lambda(t) \sim t^{\frac{2}{q+1}}, \quad \text{as } t \rightarrow +\infty.$$

Next, from (4.4.30) and (4.4.35), we have

$$b(t) \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

Finally, integrating (4.2.33), we obtain:

$$x(t) \sim t^{\frac{q-3}{q+1}}, \quad \text{as } t \rightarrow +\infty,$$

which concludes the proof of (4.4.24) and (4.4.25).

## 2. The Exit case.

Assume  $t_1^* < t^*$  with

$$b(t_1^*) + c_1 \omega(t_1^*) \leq -C^* (\mathcal{N}_1(t_1^*) + b^2(t_1^*) + \omega^2(t_1^*)). \quad (4.4.45)$$

First of all, following the same procedure as in the (Blow down) case, we have for all  $s \in [0, s_1^*]$ ,

$$\omega(s) + |b(s)| + \|\varepsilon(s)\|_{L^2} + \omega(s) \|\varepsilon_y(s)\|_{L^2}^m + \mathcal{N}_2(s) \lesssim \delta(\alpha_0), \quad (4.4.46)$$

$$\lambda(s) \geq \frac{4}{5}, \quad (4.4.47)$$

$$\int_{y>0} y^{10} \varepsilon^2(s) dy \leq 5. \quad (4.4.48)$$

In particular, we have  $t_1^* < t^{**} \leq t^*$ . Now, we claim  $t^{**} = t^* < T = +\infty$ .

To prove this, we use a standard bootstrap argument by improving (H1), (H2) and (H3) on  $[t_1^*, t^{**}]$ . Let

$$\ell^* = \frac{b(t_1^*) + c_1 \omega(t_1^*)}{\lambda^2(t_1^*)} < 0.$$

It is easy to see that  $|\ell^*| \lesssim \delta(\alpha_0)$ . Now we observe from (4.4.9) that for all  $s \in [s_1^*, s^{**}]$ ,

$$2\ell^* - C^* \frac{b^2(s) + \omega^2(s)}{\lambda^2(s)} \leq \frac{b(s) + c_1 \omega(s)}{\lambda^2(s)} \leq \frac{\ell^*}{2} + C^* \frac{b^2(s) + \omega^2(s)}{\lambda^2(s)},$$

which implies

$$-b(s) \gtrsim \omega(s) > 0, \quad (4.4.49)$$

$$3\ell^* - C \frac{\omega(s)}{\lambda^2(s)} \leq \frac{b(s)}{\lambda^2(s)} \leq \frac{\ell^*}{3} < 0. \quad (4.4.50)$$

We then observe from (4.4.10) and (4.4.49) that,

$$\frac{(\lambda_0)_s}{\lambda_0} \gtrsim -\mathcal{N}_{1,\text{loc}},$$

which after integration, yields the almost monotonicity:

$$\forall s_1^* \leq s_1 < s_2 \leq s^{**}, \quad \lambda(s_2) \geq \frac{9}{10} \lambda(s_1) \geq \frac{1}{2}. \quad (4.4.51)$$

So we obtain for all  $s \in [s_1^*, s^{**}]$ ,

$$\omega(s) + \frac{\omega(s)}{\lambda^2(s)} \lesssim \gamma \lesssim \delta(\alpha_0).$$

Together with (4.4.7) and (4.4.50), we have for all  $s \in [s_1^*, s^{**}]$ ,

$$\frac{|b(s)| + \mathcal{N}_2(s)}{\lambda^2(s)} \lesssim \delta(\alpha_0),$$

which improves (H2) if we choose  $\alpha_0 \ll \kappa$ . Next, Using the same strategy as in the (Blow down) case, we have for all  $s \in [s_1^*, s^{**}]$ ,

$$\int \varphi_{10} \varepsilon^2(s) dy \leq 7.$$

Then, (H3) is improved. We now only remains to improve (H1). Since for all  $t \in [t_1^*, t^*)$ ,  $u(t) \in \mathcal{T}_{\alpha^*, \gamma}$ . Following the argument in Lemma 4.2.6, we have for all  $t \in [0, t^*)$ ,  $|b(t)| \lesssim \delta(\alpha^*)$ . By (4.2.31), (4.4.6), and (4.4.49), we have for all  $s \in [s_1^*, s^{**}]$ ,

$$\omega(s) + \|\varepsilon(s)\|_{L^2} + \mathcal{N}_2(s) \lesssim \delta(\alpha^*).$$

Now, following from the same argument as we did for (4.4.41), we have:

$$\omega(s) \|\varepsilon_y(s)\|_{L^2}^m \lesssim \delta(\alpha_0).$$

Then (H1) is improved, due to our choice of the universal constant, i.e.  $\alpha^* \ll \kappa$ .

In conclusion, we have proved  $t^{**} = t^*$ .

*Proof of  $t^* < T = +\infty$ .* We claim that (Exit) occurs in finite time  $t^* < +\infty$ . Dividing (4.4.10) by  $\lambda_0^2$ , and use (4.4.49) to estimate on  $[t_1^*, t^*)$

$$-\frac{\ell^*}{3} - C \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2} \leq (\lambda_0)_t \leq -3\ell^* + C \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2}.$$

Integrating from  $t_1^*$  to  $t$ , we get

$$\frac{|\ell^*|(t-t_1^*)}{3} - C_1 \int_{t_1^*}^t \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2} \leq \lambda_0(t) - \lambda_0(t_1^*) \leq 3|\ell^*|(t-t_1^*) + C_2 \int_{t_1^*}^t \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2}.$$

From (4.4.51), we have

$$\int_{t_1^*}^t \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2} = \int_{s_1^*}^s \lambda \mathcal{N}_{1,\text{loc}} \lesssim \lambda(s) \int_{s_1^*}^s \mathcal{N}_{1,\text{loc}} \lesssim \delta(\kappa) \lambda(t).$$

Thus, for all  $t \in [t_1^*, t^*)$ ,

$$\frac{1}{4} (|\ell^*|(t-t_1^*) + \lambda_0(t_1^*)) \leq \lambda(t) \leq 4(|\ell^*|(t-t_1^*) + \lambda_0(t_1^*)).$$

Next, from (4.4.49), we have for all  $t \in [t_1^*, t^*)$ ,

$$-100|\ell^*|(|\ell^*|(t-t_1^*) + \lambda_0(t_1^*))^2 \leq b(t) \leq -\frac{|\ell^*|}{100}(|\ell^*|(t-t_1^*) + \lambda_0(t_1^*))^2.$$

If  $t^* = T = +\infty$ , then the above estimate leads to  $b(t) \rightarrow -\infty$  as  $t \rightarrow +\infty$ , which contradicts with the fact that  $|b(t)| \lesssim \delta(\alpha^*)$  for all  $t \in [t_1^*, t^*)$ . Thus, we have  $t^* < T = +\infty$ .

Finally, since  $0 < t^* < +\infty$ , by the definition of  $t^*$ , we must have  $-b(t^*) \geq C(\alpha^*) > 0$ . While from (4.4.49), we have

$$\lambda^2(t^*) \geq \frac{1}{2} \frac{|b(t^*)|}{|\ell^*|} \gtrsim \frac{C(\alpha^*)}{\delta(\alpha_0)} \gg 1,$$

which concludes the proof of (4.4.26) and (4.4.27).

### 3. The Soliton case.

Assume  $t_1^* < t^*$  with

$$b(t_1^*) + c_1 \omega(t_1^*) \geq C^* (\mathcal{N}_1(t_1^*) + b^2(t_1^*) + \omega^2(t_1^*)) > 0. \quad (4.4.52)$$

Similar to the (Exit) case, we have for all  $s \in [0, s_1^*]$ ,

$$\omega(s) + |b(s)| + \|\varepsilon(s)\|_{L^2} + \omega(s) \|\varepsilon_y(s)\|_{L^2}^m + \mathcal{N}_2(s) \lesssim \delta(\alpha_0), \quad (4.4.53)$$

$$\lambda(s) \geq \frac{4}{5}, \quad (4.4.54)$$

$$\int_{y>0} y^{10} \varepsilon^2(s) dy \leq 5. \quad (4.4.55)$$

But here, we can't directly prove that  $t^{**} = t^*$  as we did in the (Exit) case. The main difficulty is that we lack some control on the upper bound of  $\lambda(t_1^*)$ , which makes it hard to improve the bootstrap assumption (H2) and (H3). However, we will see that the bootstrap assumption (H2) and (H3) is related to the scaling symmetry of the problem. If we use the pseudo-scaling rule (4.1.1) on  $[t_1^*, t^*]$  to rescale  $\lambda(t_1^*)$  to 1, then we can get the desired result. Roughly speaking, on  $[t_1^*, t^*]$ , the bootstrap assumption (H2) and (H3) should be replaced by some other suitable assumptions (H2)' and (H3)'.

More precisely, we introduce the following change of coordinates. For all  $t \in [t_1^*, t^*]$ , let

$$\bar{t} = \frac{t - t_1^*}{\lambda^3(t_1^*)}, \quad \bar{x} = \frac{x - x(t_1^*)}{\lambda(t_1^*)}, \quad \bar{\gamma} = \frac{\gamma}{\lambda^m(t_1^*)}, \quad \bar{t}^* = \frac{t^* - t_1^*}{\lambda^3(t_1^*)} \quad (4.4.56)$$

$$\bar{u}(\bar{t}, \bar{x}) = \lambda^{\frac{1}{2}} u(\lambda^3(t_1^*)\bar{t} + t_1^*, \lambda(t_1^*)\bar{x} + x(t_1^*)). \quad (4.4.57)$$

Then, from the pseudo-scaling rule (4.1.1),  $\bar{u}(\bar{t}, \bar{x})$  is a solution to the following Cauchy problem:

$$\begin{cases} \partial_{\bar{t}} \bar{u} + (\bar{u}_{\bar{x}\bar{x}} + \bar{u}^5 - \bar{\gamma} \bar{u} |\bar{u}|^{q-1})_{\bar{x}} = 0, & (\bar{t}, \bar{x}) \in [0, \bar{t}^*] \times \mathbb{R}, \\ \bar{u}(0, \bar{x}) = \mathcal{Q}_{b(t_1^*), \omega(t_1^*)}(\bar{x}) + \varepsilon(t_1^*, \bar{x}) \in H^1(\mathbb{R}). \end{cases} \quad (4.4.58)$$

Moreover, for all  $\bar{t} \in [0, \bar{t}^*]$  we define:

$$\bar{\varepsilon}(\bar{t}, y) = \varepsilon(\lambda^3(t_1^*)\bar{t} + t_1^*, y), \quad \bar{\lambda}(\bar{t}) = \frac{\lambda(\lambda^3(t_1^*)\bar{t} + t_1^*)}{\lambda(t_1^*)}, \quad \bar{\omega}(\bar{t}) = \frac{\bar{\gamma}}{\bar{\lambda}^m(\bar{t})}, \quad (4.4.59)$$

$$\bar{b}(\bar{t}) = b(\lambda^3(t_1^*)\bar{t} + t_1^*), \quad \bar{x}(\bar{t}) = \frac{x(\lambda^3(t_1^*)\bar{t} + t_1^*) - x(t_1^*)}{\lambda(t_1^*)}. \quad (4.4.60)$$

Then, from (4.2.20), it is easy to check that

$$\bar{u}(\bar{t}, \bar{x}) = \frac{1}{\bar{\lambda}^{\frac{1}{2}}(\bar{t})} \left[ \mathcal{Q}_{\bar{b}(\bar{t}), \bar{\omega}(\bar{t})} + \bar{\varepsilon}(\bar{t}) \right] \left( \frac{\bar{x} - \bar{x}(\bar{t})}{\bar{\lambda}(\bar{t})} \right),$$

with:

$$(\bar{\varepsilon}(\bar{s}), \mathcal{Q}_{\bar{\omega}(\bar{s})}) = (\bar{\varepsilon}(\bar{s}), \Lambda \mathcal{Q}_{\bar{\omega}(\bar{s})}) = (\bar{\varepsilon}(\bar{s}), \bar{\gamma} \Lambda \mathcal{Q}_{\bar{\omega}(\bar{s})}) = 0.$$

where  $(\bar{s}, \bar{y})$  are the scaling invariant variables:

$$\bar{s} = \int_0^{\bar{t}} \frac{1}{\bar{\lambda}^3(\tau)} d\tau, \quad \bar{y} = \frac{\bar{x} - \bar{x}(\bar{t})}{\bar{\lambda}(\bar{t})},$$

We then introduce the weighted Sobolev norms:

$$\bar{\mathcal{N}}_i(\bar{s}) = \int \left( \bar{\varepsilon}_{\bar{y}}^2(\bar{s}, \bar{y}) \psi_B(\bar{y}) + \bar{\varepsilon}^2(\bar{s}, \bar{y}) \varphi_{i,B}(\bar{y}) \right) d\bar{y}, \quad (4.4.61)$$

$$\bar{\mathcal{N}}_{i,\text{loc}}(\bar{s}) = \int \bar{\varepsilon}^2(\bar{s}, \bar{y}) \varphi'_{i,B}(\bar{y}) d\bar{y}, \quad (4.4.62)$$

where  $\varphi_{i,B}$  and  $\psi_B$  are the weight functions introduced in Section 3.

From now on, for all  $\bar{t} \in [0, \bar{t}^*)$ , we let  $t = \lambda^3(t_1^*)\bar{t} + t_1^*$ . In this setting, we have  $\bar{s}(\bar{t}) = s(t) - s_1^*$ . Since the pseudo-scaling rule (4.1.1) is  $L^2$  invariant, we have

$$\bar{u}(\bar{t}) \in \mathcal{T}_{\alpha^*, \bar{\gamma}} \iff u(t) \in \mathcal{T}_{\alpha^*, \gamma},$$

which yields

$$\bar{t}^* = \sup\{0 < \bar{t} < +\infty \mid \text{for all } t' \in [0, \bar{t}], \bar{u}(t') \in \mathcal{T}_{\alpha^*, \bar{\gamma}}\}.$$

Next, let  $\kappa > 0$  be the universal constant introduced in Proposition 4.2.9, Proposition 4.3.1 and Lemma 4.4.1. We then define the following bootstrap assumptions for the rescaled solution  $\bar{u}(\bar{t}, \bar{x})$ . For all  $\bar{s} \in [0, \bar{s}(\bar{t})]$ :

(H1)' Scaling invariant bounds:

$$\bar{\omega}(\bar{s}) + |\bar{b}(\bar{s})| + \bar{\mathcal{N}}_2(\bar{s}) + \|\bar{\varepsilon}(\bar{s})\|_{L^2} + \bar{\omega}(\bar{s}) \|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m \leq \kappa; \quad (4.4.63)$$

(H2)' Bounds related to  $H^1$  scaling:

$$\frac{\bar{\omega}(\bar{s}) + |\bar{b}(\bar{s})| + \bar{\mathcal{N}}_2(\bar{s})}{\bar{\lambda}^2(\bar{s})} \leq \kappa; \quad (4.4.64)$$

(H3)'  $L^2$  weighted bound on the right:

$$\int_{\bar{y} > 0} \bar{y}^{10} \bar{\varepsilon}^2(\bar{s}, \bar{y}) d\bar{y} \leq 50 \left(1 + \frac{1}{\bar{\lambda}^{10}(\bar{s})}\right). \quad (4.4.65)$$

We define  $\bar{t}^{**}$  as following:

$$\bar{t}^{**} = \sup\{0 < \bar{t} < \bar{t}^* \mid \text{(H1)', (H2)' and (H3)' hold for all } t' \in [0, \bar{t}]\}. \quad (4.4.66)$$

Our goal here is to prove that  $\bar{t}^{**} = \bar{t}^* = +\infty$ , which gives us the desired asymptotic behaviors<sup>13</sup>. Let  $\bar{s}^* = \bar{s}(\bar{t}^*)$ ,  $\bar{s}^{**} = \bar{s}(\bar{t}^{**})$ . Since

$$\bar{\lambda}(0) = 1, \bar{x}(0) = 0, \bar{b}(0) = b(t_1^*), \bar{\omega}(0) = \omega(t_1^*), \bar{\varepsilon}(0, \bar{y}) = \varepsilon(t_1^*, \bar{y}), \bar{\gamma} \lesssim \gamma, \quad (4.4.67)$$

we know from (4.4.53)–(4.4.55), that  $\bar{s}^{**} > 0$ .

On the other hand, on  $[0, \bar{s}^{**})$ , all conditions of Proposition 4.2.9, Proposition 4.3.1, Lemma 4.3.5 and Lemma 4.4.1 are satisfied for  $\bar{u}(\bar{t}, \bar{x})$ . Repeating the same procedure, we have

**Lemma 4.4.5** (Estimates for the rescaled solution). *For all  $\bar{s} \in [0, \bar{s}^{**})$  or equivalently  $s \in [s_1^*, s_1^* + \bar{s}^{**})$ , all estimates of Proposition 4.2.9, Proposition 4.3.1, Lemma 4.3.5 and Lemma 4.4.1 hold with*

$$(t, x, u, \gamma, \lambda(t), b(t), x(t), \omega(t), \varepsilon(t), s, y)$$

replaced by

$$(\bar{t}, \bar{x}, \bar{u}, \bar{\gamma}, \bar{\lambda}(\bar{t}), \bar{b}(\bar{t}), \bar{x}(\bar{t}), \bar{\omega}(\bar{t}), \bar{\varepsilon}(\bar{t}), \bar{s}, \bar{y}).$$

13. Since  $\lambda(t_1^*) \gtrsim 1$ , we know that (H1) is equivalent to (H1)', (H2) is weaker than (H2)', while (H3) is stronger than (H3)'. It is hard to determine whether  $t^{**} = \lambda^3(t_1^*)\bar{t}^{**} + t_1^*$  holds.



*Remark 4.4.6.* For simplicity, we skip the statement of these similar estimates for  $\bar{u}$ . We also refer to the equation number of the corresponding inequality for  $u(t)$ , when we need to use these estimates for  $\bar{u}(\bar{t})$ .

Now we are in a position to prove  $\bar{t}^{**} = \bar{t}^* = +\infty$ . We first define

$$\bar{t}_2^* = \begin{cases} 0, & \text{if } |\bar{b}(0)| \leq \frac{1}{100}c_1\bar{\omega}(0), \\ \sup\{0 < \bar{t} < \bar{t}^* \mid \text{for all } t' \in [0, \bar{t}], |\bar{b}(t')| \geq \frac{1}{100}c_1\bar{\omega}(t')\}, & \text{otherwise.} \end{cases}$$

Our first observation is that  $\bar{t}_2^* < \bar{t}^*$ . Otherwise, since  $\bar{t}_2^* = \bar{t}^* \geq \bar{t}^{**} > 0$ , we have for all  $\bar{t} \in [0, \bar{t}^{**})$ ,  $\bar{b}(\bar{t}) \neq 0$ .

If  $\bar{b}(0) > 0$ , we claim that  $\bar{t}^{**} = \bar{t}_2^* = \bar{t}^* = +\infty$ . To prove this, we need to improve (H1)', (H2)' and (H3)' on  $[0, \bar{t}^{**}]$ . Indeed, from the definition of  $\bar{t}_2^*$ , we have

$$0 < \bar{\omega}(\bar{t}) \lesssim \bar{b}(\bar{t}). \quad (4.4.68)$$

for all  $\bar{t} \in [0, \bar{t}^{**})$ . Applying this to (4.4.10), we have:

$$\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0} \leq -\bar{b} + O(\bar{\mathcal{N}}_{2,\text{loc}}) + \delta(\kappa)|\bar{b}|.$$

Integrating this from 0 to  $\bar{t}$  using (4.4.6) and the fact that  $\bar{\lambda}(0) = 1$ , we obtain the almost monotonicity:

$$\forall 0 \leq \bar{s}_1 < \bar{s}_2 \leq \bar{s}^{**}, \quad \bar{\lambda}(\bar{s}_2) \leq \frac{10}{9}\bar{\lambda}(\bar{s}_1) \leq \frac{5}{4}. \quad (4.4.69)$$

On the other hand, we learn from (4.4.9), (4.4.52) and (4.4.67), that for all  $\bar{s} \in [0, \bar{s}^{**})$ ,

$$\frac{99}{100}\bar{\ell}^* - K_1 \frac{\bar{b}^2(\bar{s}) + \bar{\omega}^2(\bar{s})}{\bar{\lambda}^2(\bar{s})} \leq \frac{\bar{b}(\bar{s}) + c_1\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \leq \frac{101}{100}\bar{\ell}^* + K_1 \frac{\bar{b}^2(\bar{s}) + \bar{\omega}^2(\bar{s})}{\bar{\lambda}^2(\bar{s})}, \quad (4.4.70)$$

where

$$0 < \bar{\ell}^* = \frac{\bar{b}(0) + c_1\bar{\omega}(0)}{\bar{\lambda}^2(0)} = b(t_1^*) + c_1\omega(t_1^*) \lesssim \delta(\alpha_0).$$

Together with (4.4.68), we have for all  $\bar{s} \in [0, \bar{s}^{**})$ ,

$$\frac{\bar{b}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \sim \bar{\ell}^* \lesssim \delta(\alpha_0), \quad \frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \lesssim \bar{\ell}^* \lesssim \delta(\alpha_0). \quad (4.4.71)$$

Then from (4.4.69), (4.4.6) and (4.4.7), we have for all  $\bar{s} \in [0, \bar{s}^{**})$ ,

$$\frac{\bar{\mathcal{N}}_2(\bar{s})}{\bar{\lambda}^2(\bar{s})} \lesssim \delta(\alpha_0), \quad \bar{\mathcal{N}}_2(\bar{s}) + \bar{\omega}(\bar{s}) + |\bar{b}(\bar{s})| \lesssim \bar{\lambda}^2(\bar{s})\bar{\ell}^* + \delta(\alpha_0) \leq \delta(\alpha_0), \quad (4.4.72)$$

Then, from (4.2.31), (4.4.53) and following fact

$$\bar{u}(0, \bar{x}) = \mathcal{Q}_{b(t_1^*), \omega(t_1^*)}(\bar{x}) + \varepsilon(t_1^*, \bar{x}),$$

we know that

$$\begin{aligned} \|\bar{\varepsilon}(\bar{s})\|_{L^2} &\lesssim \delta(\alpha_0) + \left| \int \bar{u}^2(0) - \int \mathcal{Q}^2 \right|^{\frac{1}{2}} \\ &\lesssim \delta(\alpha_0) + \|\varepsilon(t_1^*)\|_{L^2} + |b(t_1^*)|^{\frac{1}{2}} + \omega^{\frac{1}{2}}(t_1^*) \\ &\lesssim \delta(\alpha_0). \end{aligned} \quad (4.4.73)$$

Now, from (4.2.32) and (4.4.72), we have:

$$\bar{\omega}(\bar{s}) \|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m = \bar{\gamma} \frac{\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m}{\bar{\lambda}^m(\bar{s})} \lesssim \delta(\alpha_0) + \left( \bar{\gamma} \frac{\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m}{\bar{\lambda}^m(\bar{s})} \right)^{\frac{m+2}{2}} + |\bar{\gamma}^{\frac{2}{m}} \bar{E}(\bar{u}(0))|^{\frac{m}{2}},$$

where  $\bar{E}(\bar{u}(0))$  is the energy of the Cauchy problem (4.4.58), i.e.

$$\bar{E}(\bar{u}(0)) = \frac{1}{2} \int \bar{u}_{\bar{x}}^2(0) - \frac{1}{6} \int \bar{u}^6(0) + \frac{\bar{\gamma}}{q+1} \int |\bar{u}(0)|^{q+1},$$

Since

$$\bar{u}(0, \bar{x}) = \lambda^{\frac{1}{2}}(t_1^*) u(t_1^*, \lambda(t_1^*) \bar{x} + x(t_1^*)),$$

from the energy conservation law of (gKdV $_{\gamma}$ ) and the condition on the initial data, we have

$$|\bar{\gamma}^{\frac{2}{m}} \bar{E}(\bar{u}(0))| = \left| \gamma^{\frac{2}{m}} \frac{\bar{E}(\bar{u}(0))}{\lambda^2(t_1^*)} \right| = |\gamma^{\frac{2}{m}} E(u(t_1^*))| = |\gamma^{\frac{2}{m}} E_0| \lesssim \delta(\alpha_0).$$

Thus, for all  $\bar{s} \in [0, \bar{s}^{**})$ , we have

$$\bar{\omega}(\bar{s}) \|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m \lesssim \delta(\alpha_0) + (\bar{\omega}(\bar{s}) \|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m)^{1+\frac{m}{2}}.$$

From (4.4.53) and (4.4.67), we have

$$\bar{\omega}(0) \|\bar{\varepsilon}_{\bar{y}}(0)\|_{L^2}^m = \omega(s_1^*) \|\varepsilon_y(s_1^*)\|_{L^2}^m \lesssim \delta(\alpha_0),$$

then a standard bootstrap argument leads to:

$$\bar{\omega}(\bar{s}) \|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m \lesssim \delta(\alpha_0), \quad (4.4.74)$$

for all  $\bar{s} \in [0, \bar{s}^{**})$ .

Finally, integrating (4.3.38), using (4.4.6) and (4.4.69) we obtain:

$$\begin{aligned} \int \varphi_{10}(\bar{y}) \bar{\varepsilon}^2(\bar{s}, \bar{y}) d\bar{y} &\leq \frac{\bar{\lambda}^{10}(0)}{\bar{\lambda}^{10}(\bar{s})} \int \varphi_{10}(\bar{y}) \bar{\varepsilon}^2(0, \bar{y}) d\bar{y} + \frac{C}{\bar{\lambda}^{10}(\bar{s})} \int_0^{\bar{s}} \bar{\lambda}^{10} (\bar{\mathcal{N}}_{1,\text{loc}} + \bar{b}^2) \\ &\leq \frac{1}{\bar{\lambda}^{10}(\bar{s})} \left[ 5 + C \bar{\lambda}^{10}(0) \int_0^{\bar{s}} (\bar{\mathcal{N}}_{1,\text{loc}} + \bar{b}^2) \right] \leq \frac{5 + \delta(\kappa)}{\bar{\lambda}^{10}(\bar{s})}. \end{aligned} \quad (4.4.75)$$

Combining (4.4.71)–(4.4.75), we conclude that  $\bar{t}^{**} = \bar{t}^*$ . Since all  $H^1$  solution of (4.4.58) is global in time, we must have  $\bar{t}^{**} = \bar{t}^* = +\infty$ , provided that  $\alpha_0 \ll \alpha^*$ . Now we inject (4.4.71) into (4.4.10) to obtain:

$$\frac{\bar{\ell}^*}{3} - C \frac{\bar{\mathcal{N}}_{1,\text{loc}}}{\bar{\lambda}^2} \leq -(\bar{\lambda}_0)_{\bar{t}} \leq 3\bar{\ell}^* + C \frac{\bar{\mathcal{N}}_{1,\text{loc}}}{\bar{\lambda}^2}$$

Integrating in time, we have for all  $\bar{t} \in [0, +\infty)$

$$0 < \bar{\lambda}_0(\bar{t}) \leq \bar{\lambda}(0) - \frac{\bar{\ell}^* \bar{t}}{3} + C \int_0^{\bar{t}} \frac{\bar{\mathcal{N}}_{1,\text{loc}}}{\bar{\lambda}^2}.$$

From (4.4.69) and (4.4.6) we have

$$\int_0^{\bar{t}} \frac{\bar{\mathcal{N}}_{1,\text{loc}}}{\bar{\lambda}^2} = \int_0^{\bar{s}} \bar{\lambda}(\tau) \bar{\mathcal{N}}_{1,\text{loc}}(\tau) d\tau \lesssim \int_0^{\bar{s}} \bar{\mathcal{N}}_{1,\text{loc}}(\tau) d\tau \lesssim \delta(\kappa),$$

which implies that the solution blows up in finite time. This is a contradiction.

Now we consider the other case  $\bar{b}(0) < 0$ . We claim again that  $\bar{t}_2^* = \bar{t}^{**} = \bar{t}^* = +\infty$ . It is also done by improving the 3 bootstrap assumptions. First, we know from (4.4.9), (4.4.52) and (4.4.67) that (4.4.70) still holds in this case. And the definition of  $\bar{t}_2^*$  implies that

$$0 < \bar{\ell}^* \lesssim -\frac{\bar{b}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \sim \frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})}. \quad (4.4.76)$$

Then we apply the fact that  $0 < \bar{\omega} \lesssim -\bar{b}$  to (4.4.10) to obtain:

$$\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0} \geq -\frac{1}{2}\bar{b} - O(\bar{\mathcal{N}}_{2,\text{loc}}).$$

Integrating in time we have:

$$\forall 0 \leq \bar{s}_1 < \bar{s}_2 \leq \bar{s}^{**}, \quad \bar{\lambda}(\bar{s}_2) \geq \frac{9}{10} \bar{\lambda}(\bar{s}_1) \geq \frac{4}{5}, \quad (4.4.77)$$

which yields for all  $\bar{s} \in [0, \bar{s}^{**})$ ,

$$\bar{\omega}(\bar{s}) + \frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \lesssim \bar{\gamma} \lesssim \delta(\alpha_0). \quad (4.4.78)$$

From (4.4.76), (4.4.6) and (4.4.7), we get

$$\bar{\mathcal{N}}_2(\bar{s}) + |\bar{b}(\bar{s})| + \frac{\bar{\mathcal{N}}_2(\bar{s}) + |\bar{b}(\bar{s})|}{\bar{\lambda}^2(\bar{s})} \lesssim \delta(\alpha_0) \quad (4.4.79)$$

Using the same argument as we did for (4.4.73)–(4.4.75), we have

$$\|\bar{\varepsilon}(\bar{s})\|_{L^2} \lesssim \delta(\alpha_0), \quad \bar{\omega}(\bar{s}) \|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m \lesssim \delta(\alpha_0), \quad \int \varphi_{10} \bar{\varepsilon}^2(\bar{s}) d\bar{y} \leq 7. \quad (4.4.80)$$

Combining (4.4.78)–(4.4.80), we conclude that  $\bar{t}^{**} = \bar{t}^* = +\infty$ . But from (4.4.76), we have

$$-\bar{b} \sim \bar{\omega}(\bar{s}) \gtrsim \bar{\gamma}^{\frac{2}{m+2}} (\bar{\ell}^*)^{\frac{m}{m+2}} > 0. \quad (4.4.81)$$

On the other hand, from (4.4.8), we have

$$\int_0^{\bar{s}^{**}} \bar{b}^2(s') ds' \lesssim 1.$$

The above 2 estimates imply that

$$\bar{s}^{**} = \int_0^{+\infty} \frac{1}{\bar{\lambda}^3(\tau)} d\tau < +\infty.$$

which leads to  $\bar{\lambda}(\bar{t}_n) \rightarrow +\infty$  as  $n \rightarrow +\infty$ , for some sequence  $\bar{t}_n \rightarrow +\infty$ , or equivalently  $\lim_{n \rightarrow +\infty} \bar{\omega}(\bar{t}_n) = 0$ . This contradicts with (4.4.81).

In conclusion, we have proved that  $\bar{t}_2^* < \bar{t}^*$  with

$$|\bar{b}(\bar{t}_2^*)| \leq \frac{1}{100} c_1 \bar{\omega}(\bar{t}_2^*).$$

Let  $\bar{s}_2^* = \bar{s}(\bar{t}_2^*)$ . Repeating the same procedure as before, we have for all  $\bar{s} \in [0, \bar{s}_2^*]$ ,

$$\bar{\omega}(\bar{s}) + |\bar{b}(\bar{s})| + \|\bar{\varepsilon}(\bar{s})\|_{L^2} + \bar{\omega}(\bar{s}) \|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m + \bar{\mathcal{N}}_2(\bar{s}) \lesssim \delta(\alpha_0), \quad (4.4.82)$$

$$\frac{\bar{\omega}(\bar{s}) + |\bar{b}(\bar{s})| + \bar{\mathcal{N}}_2(\bar{s})}{\bar{\lambda}^2(\bar{s})} \lesssim \delta(\alpha_0), \quad (4.4.83)$$

$$\int_{\bar{y}>0} \bar{y}^{10} \bar{\varepsilon}^2(\bar{s}) d\bar{y} \leq 7 \left( 1 + \frac{1}{\bar{\lambda}^{10}(\bar{s})} \right). \quad (4.4.84)$$

In particular, we have  $\bar{t}_2^* < \bar{t}^{**} \leq \bar{t}^*$ . Similarly, we need to improve the 3 bootstrap assumptions on  $[\bar{t}_2^*, \bar{t}^{**})$  to obtain  $\bar{t}^{**} = \bar{t}^* = +\infty$ .

First, it is easy to see that (4.4.70) holds on  $[\bar{s}_2^*, \bar{s}^{**})$ . So the definition of  $\bar{s}_2^*$  yields <sup>14</sup>

$$\frac{19}{20} \bar{\ell}^* \leq \frac{c_1 \bar{\omega}(\bar{s}_2^*)}{\bar{\lambda}^2(\bar{s}_2^*)} \leq \frac{21}{20} \bar{\ell}^*, \quad (4.4.85)$$

which implies

$$\frac{9}{10} \left( \frac{\bar{\ell}^*}{c_1 \bar{\gamma}} \right)^{\frac{2}{m+2}} \leq \frac{1}{\bar{\lambda}^2(\bar{s}_2^*)} \leq \frac{11}{10} \left( \frac{\bar{\ell}^*}{c_1 \bar{\gamma}} \right)^{\frac{2}{m+2}}. \quad (4.4.86)$$

Next, we let

$$C_1 = \frac{99}{100} c_1 < c_1, \quad C_2 = \frac{101}{100} c_1 > c_1,$$

Then, we learn from (4.4.70) that for all  $\bar{s} \in [\bar{s}_2^*, \bar{s}^{**})$ ,

$$\begin{aligned} \frac{99}{100} \bar{\ell}^* &\leq \frac{\bar{b}(\bar{s}) + C_2 \bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} - \frac{c_1}{100} \frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} + O\left(\frac{\bar{b}^2(\bar{s}) + \bar{\omega}^2(\bar{s})}{\bar{\lambda}^2(\bar{s})}\right) \\ &\leq \frac{\bar{b}(\bar{s}) + C_2 \bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} - \frac{c_1}{100} \frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} + \delta(\kappa) \left( \left| \frac{\bar{b}(\bar{s}) + C_2 \bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \right| + \left| \frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \right| \right), \end{aligned}$$

which implies <sup>15</sup>

$$\frac{49}{50} \bar{\ell}^* \leq \frac{\bar{b}(\bar{s}) + C_2 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} - \frac{c_1}{200} \frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}, \quad (4.4.87)$$

14. Recall that  $c_1 = G'(0) > 0$ , where  $G$  is the  $C^2$  function introduced in (4.2.39).

15. Here we use the fact that  $|1 - (\bar{\lambda}/\bar{\lambda}_0)| \lesssim |\bar{J}_1| \lesssim \delta(\kappa)$ .

where

$$\bar{\omega}_0(\bar{s}) = \frac{\bar{\gamma}}{\bar{\lambda}_0^m(\bar{s})}.$$

Injecting (4.4.10) into (4.4.87), using (4.4.7) and the fact that <sup>16</sup>

$$\frac{\bar{b}(\bar{s}) + C_2 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} > 0,$$

we have

$$\begin{aligned} \frac{49}{50} \bar{\ell}^* &\leq \frac{101}{100} \left( -\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0^3} + \frac{C_2 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right) - \frac{1}{100} \left( \frac{\bar{b}(\bar{s}) + C_2 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right) - \frac{c_1}{200} \frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \\ &\quad + \frac{101K_2}{100} \frac{\bar{\mathcal{N}}_1(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} + \delta(\kappa) \left( \left| \frac{\bar{b}(\bar{s}) + C_2 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right| + \left| \frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right| \right), \\ &\leq \frac{101}{100} \left( -\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0^3} + \frac{C_2 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right) - \frac{1}{100} \left( \frac{\bar{b}(\bar{s}) + C_2 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right) - \frac{c_1}{300} \frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \\ &\quad + \frac{101K_0K_2}{100} \frac{(\bar{\mathcal{N}}_1(0) + \bar{b}^2(0) + \bar{\omega}^2(0))}{\bar{\lambda}_0^2(0)} + \delta(\kappa) \left( \left| \frac{\bar{b}(\bar{s}) + C_2 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right| + \left| \frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right| \right), \\ &\leq \frac{101}{100} \left( -\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0^3} + \frac{C_2 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right) + \frac{51K_0K_2}{50} \frac{(\bar{\mathcal{N}}_1(0) + \bar{b}^2(0) + \bar{\omega}^2(0))}{\bar{\lambda}_0^2(0)}. \end{aligned} \quad (4.4.88)$$

From (4.4.52) and (4.4.67), we have

$$\bar{\ell}^* = \frac{\bar{b}(0) + c_1 \bar{\omega}(0)}{\bar{\lambda}^2(0)} \geq 100(K_1 + K_0K_2) \frac{(\bar{\mathcal{N}}_1(0) + \bar{b}^2(0) + \bar{\omega}^2(0))}{\bar{\lambda}^2(0)}.$$

So (4.4.88) implies that for all  $\bar{s} \in [\bar{s}_2^*, \bar{s}^{**})$ ,

$$\frac{1}{2} \left( \frac{1}{\bar{\lambda}_0^2} \right)_{\bar{s}} + C_2 \bar{\gamma} \left( \frac{1}{\bar{\lambda}_0^2} \right)^{1+\frac{m}{2}} \geq \frac{9}{10} \bar{\ell}^*. \quad (4.4.89)$$

Similar to (4.4.87), we have

$$\frac{51}{50} \bar{\ell}^* \geq \frac{\bar{b}(\bar{s}) + C_1 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} + \frac{c_1}{200} \frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} - \delta(\kappa) \left| \frac{\bar{b}(\bar{s}) + C_2 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right|, \quad (4.4.90)$$

which leads to

$$\begin{aligned} \frac{51}{50} \bar{\ell}^* &\geq \frac{99}{100} \left( -\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0^3} + \frac{C_1 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right) + \frac{1}{100} \left( \frac{\bar{b}(\bar{s}) + C_1 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right) + \frac{c_1}{200} \frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \\ &\quad + \frac{99K_2}{100} \frac{\bar{\mathcal{N}}_1(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} - \delta(\kappa) \left( \left| \frac{\bar{b}(\bar{s}) + C_2 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right| + \left| \frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right| \right), \end{aligned}$$

16. This is a direct consequence of (4.4.87).

and

$$\begin{aligned} \frac{51}{50} \bar{\ell}^* &\geq \frac{101}{100} \left( -\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0^3} + \frac{C_1 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right) - \frac{1}{100} \left( \frac{\bar{b}(\bar{s}) + C_1 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right) + \frac{c_1}{200} \frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \\ &\quad + \frac{101K_2}{100} \frac{\bar{N}_1(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} - \delta(\kappa) \left( \left| \frac{\bar{b}(\bar{s}) + C_2 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right| + \left| \frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \right| \right). \end{aligned}$$

Using the same strategy as (4.4.88), and discussing the sign of  $\frac{\bar{b}(\bar{s}) + C_1 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}$ , we have:

$$\frac{1}{2} \left( \frac{1}{\bar{\lambda}_0^2} \right)_{\bar{s}} + C_1 \bar{\gamma} \left( \frac{1}{\bar{\lambda}_0^2} \right)^{1+\frac{m}{2}} \leq \frac{11}{10} \bar{\ell}^*. \quad (4.4.91)$$

Then we need following basic lemma:

**Lemma 4.4.7.** *Let  $F: [0, x_0) \rightarrow (0, +\infty)$  be a  $C^1$  function. Let  $\nu > 0$ ,  $L > 0$  be 2 positive constants. Then we have:*

1. *If for all  $x \in [0, x_0)$*

$$F_x + F^{1+\nu} \geq L,$$

*then for all  $x \in [0, x_0)$ ,*

$$F(x) \geq \min(F(0), L^{\frac{1}{1+\nu}}).$$

2. *If for all  $x \in [0, x_0)$*

$$F_x + F^{1+\nu} \leq L,$$

*then for all  $x \in [0, x_0)$ ,*

$$F(x) \leq \max(F(0), L^{\frac{1}{1+\nu}}).$$

It is easy to prove Lemma 4.4.7 by standard ODE theory. Now we apply Lemma 4.4.7 to (4.4.89) and (4.4.91) on  $[\bar{s}_2^*, \bar{s}^{**})$ , using (4.4.86) to obtain

$$\frac{90}{101} \left( \frac{\bar{\ell}^*}{c_1 \bar{\gamma}} \right)^{\frac{2}{m+2}} \leq \frac{1}{\bar{\lambda}^2(\bar{s})} \leq \frac{10}{9} \left( \frac{\bar{\ell}^*}{c_1 \bar{\gamma}} \right)^{\frac{2}{m+2}}, \quad (4.4.92)$$

for all  $\bar{s} \in [\bar{s}_2^*, \bar{s}^{**})$ . This also implies that for all  $\bar{s} \in [\bar{s}_2^*, \bar{s}^{**})$ ,

$$\bar{\omega}(\bar{s}) \sim \bar{\gamma}^{\frac{2}{m+2}} (\bar{\ell}^*)^{\frac{m}{m+2}} \lesssim \delta(\alpha_0), \quad \frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \sim \bar{\ell}^* \lesssim \delta(\alpha_0). \quad (4.4.93)$$

From (4.4.87) and (4.4.90), we have

$$\frac{\bar{b}(\bar{s}) + C_2 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \geq \frac{49}{50} \bar{\ell}^*, \quad \frac{\bar{b}(\bar{s}) + C_1 \bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \leq 2 \bar{\ell}^*,$$

together with (4.4.93), we have

$$\left| \frac{\bar{b}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \right| \lesssim \bar{\ell}^* \lesssim \delta(\alpha_0), \quad |\bar{b}(\bar{s})| \lesssim \bar{\gamma}^{\frac{2}{m+2}} (\bar{\ell}^*)^{\frac{m}{m+2}} \lesssim \delta(\alpha_0). \quad (4.4.94)$$

Again, from the mass conservation law (4.2.31), energy conservation law (4.2.32) and the almost monotonicity (4.4.6), (4.4.7), we have for all  $\bar{s} \in [\bar{s}_2^*, \bar{s}^{**}]$ :

$$\|\bar{\varepsilon}(\bar{s})\|_{L^2} + \bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m + \bar{\mathcal{N}}_2(\bar{s}) + \frac{\bar{\mathcal{N}}_2(\bar{s})}{\bar{\lambda}^2(\bar{s})} \lesssim \delta(\alpha_0). \quad (4.4.95)$$

Finally, we learn from (4.4.92), that for all  $\bar{s}_2^* \leq \bar{s}_1 < \bar{s}_2 \leq \bar{s}^{**}$ ,

$$\frac{1}{4} < \left(\frac{81}{101}\right)^5 \leq \left(\frac{\bar{\lambda}(\bar{s}_1)}{\bar{\lambda}(\bar{s}_2)}\right)^{10} \leq \left(\frac{101}{81}\right)^5 < 4.$$

Then for all  $\bar{s} \in [\bar{s}_2^*, \bar{s}^{**}]$ , we integrate (4.3.38) from  $\bar{s}_2^*$  to  $\bar{s}$  to obtain:

$$\begin{aligned} \int \varphi_{10}(\bar{y}) \bar{\varepsilon}^2(\bar{s}, \bar{y}) d\bar{y} &\leq \frac{\bar{\lambda}^{10}(\bar{s}_2^*)}{\bar{\lambda}^{10}(\bar{s})} \int \varphi_{10}(\bar{y}) \bar{\varepsilon}^2(\bar{s}_2^*, \bar{y}) d\bar{y} + \frac{C}{\bar{\lambda}^{10}(\bar{s})} \int_{\bar{s}_2^*}^{\bar{s}} \bar{\lambda}^{10}(\bar{\mathcal{N}}_{1,\text{loc}} + \bar{b}^2) \\ &\leq \frac{\bar{\lambda}^{10}(\bar{s}_2^*)}{\bar{\lambda}^{10}(\bar{s})} \times 7 \left(1 + \frac{1}{\bar{\lambda}^{10}(\bar{s}_2^*)}\right) + 4C \int_{\bar{s}_2^*}^{\bar{s}} (\bar{\mathcal{N}}_{1,\text{loc}} + \bar{b}^2) \\ &\leq 28 \left(1 + \frac{1}{\bar{\lambda}^{10}(\bar{s})}\right) + \delta(\kappa) < 30 \left(1 + \frac{1}{\bar{\lambda}^{10}(\bar{s})}\right). \end{aligned} \quad (4.4.96)$$

Combining (4.4.93)–(4.4.96), we have improved (H1)', (H2)' and (H3)', hence  $\bar{t}^{**} = \bar{t}^* = +\infty$ . This also implies that  $t^* = +\infty$ . Now it is sufficient to prove the following

$$|\bar{b}(\bar{t})| + \bar{\mathcal{N}}_2(\bar{t}) \rightarrow 0, \quad \bar{\lambda}(\bar{t}) \rightarrow \bar{\lambda}_\infty \in (0, +\infty),$$

as  $\bar{t} \rightarrow +\infty$ . First of all, from (4.4.92), we know that

$$\bar{s}^{**} = \bar{s}^* = \int_0^{+\infty} \frac{1}{\bar{\lambda}^3(\tau)} d\tau = +\infty.$$

Then we claim that  $\bar{b}_s \bar{b} \in L^1((0, +\infty))$ . Indeed, from (4.2.48), we have

$$|\bar{b} \bar{b}_s + \bar{\omega}_s G'(\bar{\omega}) \bar{b}| \lesssim \bar{b}^2 + \int \bar{\varepsilon}^2 e^{-\frac{|\bar{y}|}{10}} \in L^1((0, +\infty)).$$

From (4.2.33), we have:

$$\bar{\omega}_s G'(\bar{\omega}) \bar{b} = m \bar{\omega} G'(\bar{\omega}) \bar{b}^2 + O\left(\bar{\omega} \left| \bar{b} \left( \frac{\bar{\lambda}_{\bar{s}}}{\bar{\lambda}} + \bar{b} \right) \right| \right) = O\left(\bar{b}^2 + \int \bar{\varepsilon}^2 e^{-\frac{|\bar{y}|}{10}}\right).$$

The above 2 estimates imply that

$$\int_0^{+\infty} |\bar{b}_s \bar{b}(s')| ds' = \int_0^{+\infty} \frac{1}{2} |(\bar{b}^2)_s| < +\infty.$$

Together with

$$\int_0^{+\infty} \bar{b}^2(s) ds < +\infty,$$

we conclude that  $\bar{b}(\bar{t}) \rightarrow 0$ , as  $\bar{t} \rightarrow +\infty$ . Next, We use (4.2.48) again to obtain:

$$|\bar{b}_{\bar{s}} + \bar{\omega}_{\bar{s}} G'(\bar{\omega})| \lesssim \bar{b}^2 + \int \bar{\varepsilon}^2 e^{-\frac{|\bar{y}|}{10}} \in L^1((0, +\infty)).$$

Thus, we have

$$\int_0^{+\infty} |(\bar{b} + G(\bar{\omega}))_{\bar{s}}(s')| ds' < +\infty.$$

We then know that  $b(\bar{t}) + G(\bar{\omega}(\bar{t}))$  has a limit as  $\bar{t} \rightarrow +\infty$ . Since  $\lim_{\bar{t} \rightarrow +\infty} \bar{b}(\bar{t}) = 0$ , we obtain that  $G(\bar{\omega}(\bar{t}))$  has a limit as  $\bar{t} \rightarrow +\infty$ . On the other hand, we have  $G'(0) > 0$ ,  $\bar{\omega}(\bar{t}) \ll 1$ , so there exists a constant  $\bar{\omega}_\infty > 0$ , such that

$$\lim_{\bar{t} \rightarrow +\infty} \bar{\omega}(\bar{t}) = \bar{\omega}_\infty \sim \bar{\gamma}^{\frac{2}{m+2}} (\bar{\ell}^*)^{\frac{m}{m+2}},$$

or equivalently

$$\lim_{\bar{t} \rightarrow +\infty} \bar{\lambda}(\bar{t}) = \bar{\lambda}_\infty \sim \left( \frac{c_1 \bar{\gamma}}{\bar{\ell}^*} \right)^{\frac{1}{m+2}}.$$

Let

$$\ell^* = \frac{b(t_1^*) + c_1 \omega(t_1^*)}{\lambda^2(t_1^*)} > 0.$$

Recall that

$$\bar{\gamma} = \frac{\gamma}{\lambda^m(t_1^*)}, \quad \bar{\ell}^* = b(t_1^*) + c_1 \omega(t_1^*), \quad \bar{\lambda}(\bar{t}) = \frac{\lambda(\lambda^3(t_1^*)\bar{t} + t_1^*)}{\lambda(t_1^*)}.$$

We obtain

$$\lim_{t \rightarrow +\infty} \lambda(t) = \lambda_\infty \sim \left( \frac{c_1 \gamma}{\ell^*} \right)^{\frac{1}{m+2}}. \quad (4.4.97)$$

Next, the inequality (4.4.6) implies the existence of a sequence  $\bar{s}_n$  such that

$$\bar{\mathcal{N}}_1(\bar{s}_n) \lesssim \int (\bar{\varepsilon}^2(\bar{s}_n) + \bar{\varepsilon}_{\bar{y}}^2(\bar{s}_n)) \varphi'_{2,B} \rightarrow 0, \text{ as } n \rightarrow +\infty,$$

where  $\lim_{n \rightarrow +\infty} \bar{s}_n = +\infty$ . Using the monotonicity (4.4.11), we have

$$\bar{\mathcal{N}}_1(\bar{s}) \rightarrow 0, \text{ as } \bar{s} \rightarrow +\infty.$$

Together with (4.3.21) and (4.4.92), we obtain

$$\bar{\mathcal{N}}_2(\bar{t}) \rightarrow 0, \text{ as } \bar{t} \rightarrow +\infty,$$

which implies that

$$\mathcal{N}_2(t) \rightarrow 0, \text{ as } t \rightarrow +\infty,$$

Finally, from (4.2.33), we have:

$$\lambda^2(t)x_t(t) \sim 1, \text{ as } t \rightarrow +\infty,$$

which after integration implies

$$x(t) \sim \frac{t}{\lambda_\infty^2}, \text{ as } t \rightarrow +\infty.$$

We then conclude the proof of (4.4.28) and (4.4.29), hence the proof of the first part of Proposition 4.4.2.



#### 4. Nonemptiness and stability.

Now we give the proof of the second part of Proposition 4.4.2.

First, we show that the (Soliton) and (Exit) regimes are stable under small perturbation in  $\mathcal{A}_{\alpha_0}$ . From (4.2.24), we know that the parameters depend continuously on the initial data, which implies that the cases (Exit) and (Soliton) are both open in  $\mathcal{A}_{\alpha_0}$ , since the separation condition is an open condition of initial data in  $\mathcal{A}_{\alpha_0}$ .

Indeed, for all  $u_0 \in \mathcal{A}_{\alpha_0}$ , if the corresponding solution  $u(t)$  to (gKdV $_{\gamma}$ ) belongs to the (Soliton) regime, we let  $t_1^*$  be the separation time introduced in Proposition 4.4.2. For all  $\tilde{u}_0 \in \mathcal{A}_{\alpha_0}$ , close enough to  $u_0$ , we let  $\tilde{u}(t)$  be the corresponding solution to (gKdV $_{\gamma}$ ), and  $\tilde{b}(t)$ ,  $\tilde{x}(t)$ ,  $\tilde{\lambda}(t)$ ,  $\tilde{\varepsilon}(t)$  be the corresponding geometrical parameters and error term. Then from local theory, we have  $\sup_{t \in [0, t_1^*]} \|u(t) - \tilde{u}(t)\|_{H^1} \ll 1$ , which together with (4.2.24), leads to

$$\tilde{b}(t_1^*) + c_1 \tilde{\omega}(t_1^*) \geq \frac{999}{1000} C^* (\tilde{\mathcal{N}}_1(t_1^*) + \tilde{b}^2(t_1^*) + \tilde{\omega}^2(t_1^*)).$$

So  $\tilde{u}(t)$  must belong to the (Soliton) regime. This implies the openness of (Soliton) regime. The openness of the (Exit) regime follows from the same argument.

Next, we claim that there exists initial data in  $\mathcal{A}_{\alpha_0}$  such that the corresponding solution to (gKdV $_{\gamma}$ ) belongs to the (Soliton) and (Exit) regimes respectively. First, it is easy to check that the traveling wave solution

$$u(t, x) = \mathcal{Q}_{\gamma}(x - t)$$

belongs to the (Soliton) regime. On the other hand, from (4.2.42), we can see, in both the (Soliton) and (Blow down) cases, we have

$$\|u_0\|_{L^2} \geq \|\mathcal{Q}\|_{L^2}.$$

Hence, for initial data  $u_0 \in \mathcal{A}_{\alpha_0}$  with <sup>17</sup>  $\|u_0\|_{L^2} < \|\mathcal{Q}\|_{L^2}$ , the corresponding solution must belong to the (Exit) regime.

Finally, since the sets of initial data which leads to the (Soliton) and (Exit) regime are both open and nonempty in  $\mathcal{A}_{\alpha_0}$ . Together with the fact that  $\mathcal{A}_{\alpha_0}$  is connected, we conclude that there exists  $u_0 \in \mathcal{A}_{\alpha_0}$ , such that the corresponding solution to (gKdV $_{\gamma}$ ) belongs to the (Blow down) regime.  $\square$

### 4.5 Proof of Theorem 4.1.4

In this part we will use the local Cauchy theory of generalized KdV equations developed in KPv to prove Theorem 4.1.4.

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17. Since we assume that  $\gamma \ll \alpha_0$ , such  $u_0$  exists in  $\mathcal{A}_{\alpha_0}$ .

### 4.5.1 $H^1$ perturbation theory

First of all, let us introduce the following linear estimates proved by Kenig, Ponce and Vega in [36].

**Lemma 4.5.1** (Linear estimates, [36]). *The following linear estimates hold:*

1. For all  $u_0 \in H^1$ ,

$$\left\| \frac{\partial}{\partial x} W(t)u_0 \right\|_{L_x^\infty L_t^2(\mathbb{R})} + \|W(t)u_0\|_{L_x^5 L_t^{10}(\mathbb{R})} \lesssim \|u_0\|_{L^2}, \quad (4.5.1)$$

$$\left\| D_x^{\alpha_q} D_t^{\beta_q} W(t)u_0 \right\|_{L_x^p L_t^r(I)} \lesssim \|D_x^{s_q} u_0\|_{L^2}, \quad (4.5.2)$$

where  $q > 5$  is the power of the defocusing nonlinear term of (gKdV $_\gamma$ ), and

$$\begin{aligned} W(t)f &= e^{-t\partial_x^3} f, \quad s_q = \frac{1}{2} - \frac{2}{(q-1)}, \\ \alpha_q &= \frac{1}{10} - \frac{2}{5(q-1)}, \quad \beta_q = \frac{3}{10} - \frac{6}{5(q-1)}, \\ \frac{1}{p} &= \frac{2}{5(q-1)} + \frac{1}{10}, \quad \frac{1}{r} = \frac{3}{10} - \frac{4}{5(q-1)}. \end{aligned}$$

2. For all well localized  $g$ , we have:

$$\sup_{t \in I} \left\| \frac{\partial}{\partial x} \int_0^t W(t-t')g(\cdot, t') dt' \right\|_{L_x^1} \lesssim \|g\|_{L_x^1 L_t^2(I)}, \quad (4.5.3)$$

$$\left\| \frac{\partial^2}{\partial x^2} \int_0^t W(t-t')g(\cdot, t') dt' \right\|_{L_x^\infty L_t^2(I)} \lesssim \|g\|_{L_x^1 L_t^2(I)}, \quad (4.5.4)$$

$$\left\| \int_0^t W(t-t')g(\cdot, t') dt' \right\|_{L_x^5 L_t^{10}(I)} \lesssim \|g\|_{L_x^{5/4} L_t^{10/9}(I)}, \quad (4.5.5)$$

$$\left\| D_x^{\alpha_q} D_t^{\beta_q} \int_0^t W(t-t')g(\cdot, t') dt' \right\|_{L_x^p L_t^r(I)} \lesssim \|g\|_{L_x^{p'} L_t^{r'}(I)}, \quad (4.5.6)$$

$$\|g\|_{L_x^{5(q-1)/4} L_t^{5(q-1)/2}} \lesssim \|D_x^{\alpha_q} D_t^{\beta_q} g\|_{L_x^p L_t^r}, \quad (4.5.7)$$

where

$$1 = \frac{1}{p} + \frac{1}{p'} = \frac{1}{r} + \frac{1}{r'}.$$

*Proof.* See Theorem 3.5, Corollary 3.8, Lemma 3.14, Lemma 3.15 and Corollary 3.16 in [36] for the proof of (1) and (2).  $\square$

Now we define the following norms:

$$\begin{aligned}\eta_I^1(w) &= \|w\|_{L_x^5 L_t^{10}(I)}, \quad \eta_I^2(w) = \|w_x\|_{L_x^\infty L_t^2(I)}, \quad \eta_I^3(w) = \|D_x^{\alpha_q} D_t^{\beta_q} w\|_{L_x^p L_t^r(I)}, \\ \Omega_I(w) &= \max_{j=1,2} [\eta_I^j(w) + \eta_I^j(w_x)] + \eta_I^3(w), \\ \Delta_I(h) &= \|h\|_{L_x^1 L_t^2(I)} + \|h_x\|_{L_x^{5/4} L_t^{10/9}(I)} + \|h_x\|_{L_x^1 L_t^2(I)} + \|h_{xx}\|_{L_x^{5/4} L_t^{10/9}(I)} \\ &\quad + \|h_x\|_{L_x^{p'} L_t^{r'}(I)},\end{aligned}$$

for all interval  $I \subset \mathbb{R}$ .

Then we have the following modified  $H^1$  perturbation theory:

**Proposition 4.5.2** (Modified long time  $H^1$  perturbation theory). *Let  $I$  be an interval containing 0, and  $\tilde{u}$  be an  $H^1$  solution to*

$$\begin{cases} \partial_t \tilde{u} + (\partial_{xx} \tilde{u} + \tilde{u}^5 - \gamma \tilde{u} |\tilde{u}|^{q-1})_x = e_x, & (t, x) \in I \times \mathbb{R}, \\ \tilde{u}(0, x) = \tilde{u}_0 \in H^1. \end{cases} \quad (4.5.8)$$

Suppose we have

$$\sup_{t \in I} \|\tilde{u}(t)\|_{H^1} + \Omega_I(\tilde{u}) \leq M,$$

for some  $M > 0$  independent of  $\gamma$ . Let  $u_0 \in H^1$  be such that

$$\|u_0 - \tilde{u}_0\|_{H^1} + \Delta_I(e) \leq \varepsilon,$$

for some small  $0 < \varepsilon < \varepsilon_0(M)$ . Then the solution of  $(\mathbf{gKdV}_\gamma)$  with initial data  $u_0$  satisfies:

$$\sup_{t \in I} \|u - \tilde{u}\|_{H^1} + \Omega_I(u - \tilde{u}) \leq C(M)\varepsilon. \quad (4.5.9)$$

*Remark 4.5.3.* The perturbation theory still holds true if we replace  $H^1$  by  $H^s$ , with  $s \geq \frac{1}{2} - \frac{2}{q-1} > 0$ .

*Proof of Proposition 4.5.2.* Without loss of generality, we assume that  $I = [0, T_0]$  for some  $T_0 > 0$ .

We first claim the following short time perturbation theory.

**Lemma 4.5.4** (Short time perturbation theory). *Under the same notation of Proposition 4.5.2, if we assume in addition that  $\Omega_I(\tilde{u}) \leq \varepsilon_0$ , for some small  $0 < \varepsilon_0 = \varepsilon_1(M) \ll 1$ . Then there exists a constant  $C_0(M)$  which depends only on  $M$  such that if  $0 < \varepsilon < \varepsilon_0 = \varepsilon_1(M)$ , then*

$$\sup_{t \in I} \|u - \tilde{u}\|_{H^1} + \Omega_I(u - \tilde{u}) \leq C_0(M)\varepsilon. \quad (4.5.10)$$

We leave the proof of Lemma 4.5.4 in Appendix B.

Now we turn to the proof of Proposition 4.5.2. Let  $\varepsilon_0 = \varepsilon_1(2M) > 0$  as in Lemma 4.5.4. We then choose  $0 = t_0 < t_1 < \dots < t_N = T_0$  (recall that we assume  $I = [0, T_0]$ ), such that for all  $j = 1, \dots, N$ ,

$$\Omega_{[t_{j-1}, t_j]}(\tilde{u}) \leq \varepsilon_0.$$

From a standard argument, we know that  $N = N(M, \varepsilon_0) = N(M) > 0$ . We use Lemma 4.5.4 on each interval  $[t_{j-1}, t_j]$  to obtain:

$$\sup_{t \in [t_{j-1}, t_j]} \|u(t) - \tilde{u}(t)\|_{H^1} + \Omega_{[t_{j-1}, t_j]}(\tilde{u}) \leq C_0(M) \max(\varepsilon, \|u(t_{j-1}) - \tilde{u}(t_{j-1})\|_{H^1}).$$

Arguing by induction, using  $\|u(0) - \tilde{u}(0)\|_{H^1} \leq \varepsilon$ , we have for all  $j = 1, \dots, N$ ,

$$\sup_{t \in [t_{j-1}, t_j]} \|u(t) - \tilde{u}(t)\|_{H^1} + \Omega_{[t_{j-1}, t_j]}(\tilde{u}) \leq C(j, M)\varepsilon.$$

Summarizing these estimates, we have:

$$\begin{aligned} \sup_{t \in I} \|u - \tilde{u}\|_{H^1} + \Omega_I(\tilde{u}) &\leq \sum_{j=1}^N \sup_{t \in [t_{j-1}, t_j]} \|u(t) - \tilde{u}(t)\|_{H^1} + \Omega_{[t_{j-1}, t_j]}(\tilde{u}) \\ &\leq \sum_{j=1}^N C(j, M)\varepsilon = C(M)\varepsilon, \end{aligned}$$

which concludes the proof of Proposition 4.5.2.  $\square$

## 4.5.2 End of the proof of Theorem 4.1.4

Now for  $0 < \gamma \ll \alpha_0 \ll \alpha^* \ll 1$ , we choose a  $u_0 \in \mathcal{A}_{\alpha_0/2} \subset \mathcal{A}_{\alpha_0}$ , such that the corresponding solution  $u(t)$  to (gKdV) belongs to the (Blow up) regime with blow up time  $T < +\infty$ . Let  $u_\gamma(t)$  be the corresponding solution to (gKdV $_\gamma$ ). From [57, Section 4.4], we know that there exists a  $0 < T_1^* < T < +\infty$ , geometrical parameters  $(\lambda(t), b(t), x(t))$  and an error term  $\varepsilon(t)$  such that the following geometrical decomposition holds on  $[0, T_1^*]$ :

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{1}{2}}} [Q_{b(t)} + \varepsilon(t)] \left( \frac{x - x(t)}{\lambda(t)} \right), \quad (4.5.11)$$

with

$$(\varepsilon, Q) = (\varepsilon, \Lambda Q) = (\varepsilon, y\Lambda Q) = 0. \quad (4.5.12)$$

Moreover, we have for all  $t \in [0, T_1^*]$ ,

$$\mathcal{N}_2(t) + \|\varepsilon(t)\|_{L^2} + |b(t)| + |1 - \lambda(t)| \lesssim \delta(\alpha_0), \quad (4.5.13)$$

$$\int_{y>0} y^{10} \varepsilon^2(t, y) dy \leq 5. \quad (4.5.14)$$

and

$$b(T_1^*) \geq 2C^* \mathcal{N}_1(T_1^*), \quad (4.5.15)$$

where  $C^*$  is the universal constant<sup>18</sup> introduced in Section 4.2. One may easily check that  $C^*$  defined by (4.4.23) is independent of  $\gamma$ .

Next, we claim that there exists a constant  $C(u_0, q) > 1$  which depends only on  $u_0$  and  $q$ , such that

$$\sup_{t \in [0, T_1^*]} \|u(t)\|_{H^1} + \Omega_{[0, T_1^*]}(u) + \Delta_{[0, T_1^*]}(u|u|^{q-1}) \leq C(u_0, q) < +\infty. \quad (4.5.16)$$

Indeed, from [36, Corollary 2.11] (taking  $s = 1$ ), we have

$$\eta_{[0, T_1^*]}^1(u) + \eta_{[0, T_1^*]}^1(u_x) + \eta_{[0, T_1^*]}^2(u) + \eta_{[0, T_1^*]}^2(u_x) \leq C(u_0, q) < +\infty.$$

Then, from Duhamel's principle, we have:

$$u(t) = W(t)u_0 + \int_0^t \left( W(t-t')\partial_x(u^5) \right) dt'.$$

Together with (4.5.2), (4.5.6) and the Gagliardo-Nirenberg's inequality introduced in [1, Theorem 2.44], we have

$$\begin{aligned} \eta_{[0, T_1^*]}^3(u) &\lesssim \|u_x u^4\|_{L_x^{p'} L_t^{r'}} + \|u_0\|_{H^1} \lesssim \|u\|_{L_x^5 L_t^{10}}^4 \|u_x\|_{L_x^{p_0} L_t^{r_0}} + \|u_0\|_{H^1} \\ &\lesssim \|u\|_{L_x^5 L_t^{10}}^4 \|D_x^{s_q} u\|_{L_x^5 L_t^{10}}^{1-s_q} \|D_x^{s_q} u_x\|_{L_x^\infty L_t^2}^{s_q} + \|u_0\|_{H^1} \\ &\lesssim \|u\|_{L_x^5 L_t^{10}}^4 \left( \|u\|_{L_x^5 L_t^{10}}^{1-s_q} \|u_x\|_{L_x^5 L_t^{10}}^{s_q} \right)^{1-s_q} \left( \|u_x\|_{L_x^\infty L_t^2}^{1-s_q} \|u_{xx}\|_{L_x^\infty L_t^2}^{s_q} \right)^{s_q} \\ &\quad + \|u_0\|_{H^1} \\ &\lesssim \left( \eta_{[0, T_1^*]}^1(u) + \eta_{[0, T_1^*]}^1(u_x) + \eta_{[0, T_1^*]}^2(u) + \eta_{[0, T_1^*]}^2(u_x) \right)^5 + \|u_0\|_{H^1}, \end{aligned}$$

where

$$\frac{1}{p_0} = \frac{1}{10} - \frac{2}{5(q-1)}, \quad \frac{1}{r_0} = \frac{3}{10} + \frac{4}{5(q-1)}.$$

This implies  $\Omega_{[0, T_1^*]}(u) \leq C(u_0, q) < +\infty$ .

Next, using the arguments in [36, Section 6], we obtain

$$\Delta_{[0, T_1^*]}(u|u|^{q-1}) \lesssim (\Omega_{[0, T_1^*]}(u))^q \leq C(u_0, q)$$

which yields (4.5.16).

Then we apply Proposition 4.5.2 to  $u(t)$  and  $u_\gamma(t)$ , with  $e = \gamma u|u|^{q-1}$ . Note that from (4.5.16), we have

$$\Delta_{[0, T_1^*]}(e) < \gamma C(u_0, q) \leq \gamma^{\frac{1}{2}} \ll \varepsilon_0(C(u_0, q)),$$

provided that  $0 < \gamma < \gamma(u_0, \alpha_0, \alpha^*, q) \ll 1$ . Then Proposition 4.5.2 implies that for all  $t \in [0, T_1^*]$ , we have

$$\|u(t) - u_\gamma(t)\|_{H^1} \lesssim \gamma^{\frac{1}{2}}. \quad (4.5.17)$$

18. The constant  $C^*$  chosen here might be different from the one in [57, (4.23)]. But we can always replace  $C^*$  (both constants in this paper and in [57]) by some larger universal constant.

Combining with (4.5.11)–(4.5.14), we know that for all  $t \in [0, T_1^*]$ ,  $u_\gamma(t) \in \mathcal{T}_{\alpha_0, \gamma}$ . This allows us to apply Lemma 4.2.6 to  $u_\gamma(t)$  on  $[0, T_1^*]$ , i.e. there exist geometrical parameters  $(b_\gamma(t), \lambda_\gamma(t), x_\gamma(t))$  and an error term  $\varepsilon_\gamma(t)$ , such that

$$u_\gamma(t, x) = \frac{1}{\lambda_\gamma(t)^{\frac{1}{2}}} [\mathcal{Q}_{b_\gamma(t), \omega_\gamma(t)} + \varepsilon_\gamma(t)] \left( \frac{x - x_\gamma(t)}{\lambda_\gamma(t)} \right),$$

with

$$\omega_\gamma(t) = \frac{\gamma}{\lambda_\gamma^m(t)}.$$

Moreover, the orthogonality conditions (4.2.21) hold.

Now, from Lemma 4.2.6 and (4.5.17) we obtain that for all  $t \in [0, T_1^*]$ ,

$$\left| 1 - \frac{\lambda(t)}{\lambda_\gamma(t)} \right| + |b_\gamma(t) - b(t)| + |x_\gamma(t) - x(t)| + \|\varepsilon_\gamma(t) - \varepsilon(t)\|_{H^1} \lesssim \delta(\gamma). \quad (4.5.18)$$

Together with (4.5.13)–(4.5.15), we have the following:

1. For all  $t \in [0, T_1^*]$ , (4.4.53)–(4.4.55) hold for  $u_\gamma(t)$ .
2. At the time  $t = T_1^*$ , there holds:

$$b_\gamma(T_1^*) + c_1 \omega_\gamma(T_1^*) \geq C^* (\mathcal{N}_{1, \gamma}(T_1^*) + b_\gamma^2(T_1^*) + \omega_\gamma^2(T_1^*)),$$

where

$$\mathcal{N}_{i, \gamma}(t) = \int (\varepsilon_\gamma)_y^2 \psi_B + \varepsilon_\gamma^2 \phi_{i, B}.$$

By the argument in Section 4, we know that  $u_\gamma(t)$  belongs to the (Soliton) regime introduced in Theorem 4.1.3. Moreover, we also obtain (4.1.8) from (4.4.97). This concludes the proof of the first part of Theorem 4.1.4.

The second part of Theorem 4.1.4 follows from exactly the same procedure. Thus, we complete the proof of Theorem 4.1.4.



# Appendix A

## Coercivity of the Lyapunov functional

The coercivity result of  $\mathcal{F}$  i.e. (2.5.25), follows from the following lemma<sup>1</sup>:

**Lemma A.0.1** (Coercivity of  $L$ ). *There exists a constant  $\kappa_0 > 0$  such that for all  $f \in H^1$ , there holds:*

$$(Lf, f) \geq \kappa_0 \|f\|_{H^1}^2 - \frac{1}{\kappa_0} \left[ (f, \mathcal{Q}_p)^2 + (f, \Lambda \mathcal{Q}_p)^2 + (f, y \Lambda \mathcal{Q}_p)^2 \right]. \quad (\text{A.0.1})$$

Now we can prove (2.5.25) by using Lemma A.1, orthogonality condition (2.2.27) and a localization argument:

Choose a smooth function  $\eta_0$  such that  $\eta_0(y) = 1$ , if  $y < \kappa$ ,  $\eta_0(y) = e^{-y}$  if  $y > 1$  and  $\eta_0'(y) \leq 0$  for all  $y$ . Let

$$\Psi_B(y) = \psi_B(y) \eta_0\left(\frac{y}{B}\right).$$

Then we apply (A.0.1) for  $f = \varepsilon \sqrt{\Psi_B}$ . We compute every term in (A.0.1) separately: First, from (2.5.3) and the definition of  $\psi$  and  $\varphi$  we have for all  $y \leq \kappa B$ ,

$$\psi_B(y) \leq (1 + 3\kappa) \varphi_B(y).$$

By the same strategy as in Section 5, we obtain:

$$\begin{aligned} (Lf, f) &= \int \varepsilon_y^2 \Psi_B + \varepsilon^2 \Psi_B - p \Psi_B \mathcal{Q}_p^{p-1} \varepsilon^2 + \int \varepsilon^2 \frac{(\Psi_B)_y^2}{4\Psi_B} - \frac{1}{2} \int \varepsilon^2 (\Psi_B)_{yy} \\ &\leq \int_{y \leq \kappa B} (\varepsilon_y^2 + \varepsilon^2 - p \mathcal{Q}_p^{p-1} \varepsilon^2) \Psi_B + O\left(\frac{1}{B}\right) \int_{y < -\kappa B} (\varepsilon_y^2 + \varepsilon^2) \Psi_B \\ &\quad + C \int_{y > \kappa B} (\varepsilon_y^2 + \varepsilon^2 e^{-\frac{y}{B}}) \\ &\leq \int_{y \leq \kappa B} (\varepsilon_y^2 \psi_B + \varepsilon^2 \varphi_B - p \psi_B \mathcal{Q}_p^{p-1} \varepsilon^2) + C\kappa \int_{y \leq \kappa B} \varepsilon^2 \varphi_B \\ &\quad + O\left(\frac{1}{B}\right) \int_{y < -\kappa B} (\varepsilon_y^2 + \varepsilon^2) \varphi_B + Cb_c^{\frac{7}{2}} \\ &\leq \int (\varepsilon_y^2 \psi_B + \varepsilon^2 \varphi_B - p \psi_B \mathcal{Q}_p^{p-1} \varepsilon^2) + Cb_c^{\frac{7}{2}} + C(\kappa B + 1) \int_{y < \kappa B} (\varepsilon_y^2 + \varepsilon^2) \varphi_B', \end{aligned}$$

1. See for example, Lemma 2.1 in [57].



with some constant  $C > 0$  independent of  $\kappa$  and  $B$ .

Next, a direct computation shows:

$$\begin{aligned} \kappa_0 \|f\|_{H^1}^2 &\geq \kappa_0 \int_{y \leq \kappa B} (\varepsilon_y^2 \Psi_B + \varepsilon^2 \Psi_B) - C \int \varepsilon^2 \frac{(\Psi_B)_y^2}{4\Psi_B} \\ &\geq \frac{1}{C} \int_{y \leq \kappa B} (\varepsilon_y^2 + \varepsilon^2) \Psi_B - C b_c^{\frac{7}{2}}. \end{aligned}$$

Then, from the orthogonality condition (2.2.27) we have:

$$|(f, \mathcal{Q}_p)| \lesssim \int_{|y| > \kappa B} |\varepsilon| e^{-|y|} \lesssim e^{-\frac{\kappa B}{2}} \|\varepsilon\|_{L^\infty} \lesssim b_c^{10}.$$

The same estimates hold for  $(f, \Lambda \mathcal{Q}_p)$  and  $(f, y \Lambda \mathcal{Q}_p)$ . Injecting all the above estimates into (A.0.1), we have:

$$\begin{aligned} & B \int_{y \leq \kappa B} (\varepsilon_y^2 + \varepsilon^2) \varphi'_B \\ & \leq C \int (\varepsilon_y^2 \Psi_B + \varepsilon^2 \varphi_B - p \Psi_B \mathcal{Q}_p^{p-1} \varepsilon^2) + C(\kappa B + 1) \int_{y < \kappa B} (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C b_c^{\frac{7}{2}} \quad (\text{A.0.2}) \\ & \leq C \int (\varepsilon_y^2 \Psi_B + \varepsilon^2 \varphi_B - p \Psi_B \mathcal{Q}_p^{p-1} \varepsilon^2) + \frac{B}{2} \int_{y \leq \kappa B} (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C b_c^{\frac{7}{2}}, \end{aligned}$$

provided that  $\kappa$  is small enough (We can take  $\kappa$  such that it is independent of  $b$ ). Then (A.0.2) implies (2.5.25) immediately.

# Appendix B

## Proof of Lemma 4.5.4

In this section, we give the proof of the modified short time perturbation theory, i.e. Lemma 4.5.4.

First, we let  $v(t, x) = u(t, x) - \tilde{u}(t, x)$ ,  $S(t) = \Omega_{[0,t]}(v)$ . We claim the following estimate holds true for all  $t \in I$ :

$$S(t) \lesssim_M \varepsilon + S(t)(S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}). \quad (\text{B.0.1})$$

Since  $S(0) = 0$  and  $\Omega_I(\tilde{u}) \leq \varepsilon_0$ , we know that Lemma 4.5.4 follows from a standard bootstrap argument. Now it only remains to prove (B.0.1).

First, by Duhamel's principle, we have

$$\begin{aligned} v(t) &= W(t)(\tilde{u}_0 - u_0) + \int_0^t \left( W(t-t') \partial_x [\tilde{u}^5 - \gamma \tilde{u} |\tilde{u}|^{q-1} \right. \\ &\quad \left. - (\tilde{u} + v)^5 + \gamma(\tilde{u} + v) |\tilde{u} + v|^{q-1} - e] \right) dt' \\ &= v_L(t) + v_N(t). \end{aligned}$$

For the linear part  $v_L$ , from Lemma 5.1, we have:

$$\Omega_{[0,t]}(v_L) + \sup_{t' \in [0,t]} \|v_L\|_{H^1} \lesssim \|\tilde{u}_0 - u_0\|_{H^1} \lesssim \varepsilon. \quad (\text{B.0.2})$$

Now, for the nonlinear part  $v_N$ , we use Lemma 4.5.1 to estimate:

$$\begin{aligned} \eta_{[0,t]}^1(v_N) &\lesssim \|e_x\|_{L_x^{5/4} L_t^{10/9}([0,t])} + \|(v + \tilde{u})^4 (v + \tilde{u})_x - \tilde{u}^4 \tilde{u}_x\|_{L_x^{5/4} L_t^{10/9}([0,t])} \\ &\quad + \||v + \tilde{u}|^{q-1} (v + \tilde{u})_x - |\tilde{u}|^{q-1} \tilde{u}_x\|_{L_x^{5/4} L_t^{10/9}([0,t])}. \end{aligned}$$

By Hölder's inequality, we have:

$$\begin{aligned}
& \| (v + \tilde{u})^4 (v + \tilde{u})_x - \tilde{u}^4 \tilde{u}_x \|_{L_x^{5/4} L_t^{10/9}([0,t])} \\
& \lesssim \| ((v + \tilde{u})^4 - \tilde{u}^4) \tilde{u}_x \|_{L_x^{5/4} L_t^{10/9}} + \| (v + \tilde{u})^4 v_x \|_{L_x^{5/4} L_t^{10/9}} \\
& \lesssim (\| \tilde{u} \|_{L_x^5 L_t^{10}}^3 + \| v \|_{L_x^5 L_t^{10}}^3) \| v \|_{L_x^5 L_t^{10}} \| \tilde{u}_x \|_{L_x^\infty L_t^2} \\
& \quad + \| v \|_{L_x^5 L_t^{10}}^4 (\| v_x \|_{L_x^\infty L_t^2} + \| \tilde{u}_x \|_{L_x^\infty L_t^2}) \\
& \lesssim S(t) (S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}),
\end{aligned}$$

and

$$\begin{aligned}
& \| |v + \tilde{u}|^{q-1} (v + \tilde{u})_x - |\tilde{u}|^{q-1} \tilde{u}_x \|_{L_x^{5/4} L_t^{10/9}([0,t])} \\
& \lesssim \| (|v + \tilde{u}|^{q-1} - |\tilde{u}|^{q-1}) \tilde{u}_x \|_{L_x^{5/4} L_t^{10/9}} + \| |v + \tilde{u}|^{q-1} v_x \|_{L_x^{5/4} L_t^{10/9}} \\
& \lesssim (\| \tilde{u} \|_{L_x^{5(q-1)/4} L_t^{5(q-1)/2}}^{q-2} + \| v \|_{L_x^{5(q-1)/4} L_t^{5(q-1)/2}}^{q-2}) \| v \|_{L_x^{5(q-1)/4} L_t^{5(q-1)/2}} \| \tilde{u}_x \|_{L_x^\infty L_t^2} \\
& \quad + \| v \|_{L_x^{5(q-1)/4} L_t^{5(q-1)/2}}^{q-1} (\| v_x \|_{L_x^\infty L_t^2} + \| \tilde{u}_x \|_{L_x^\infty L_t^2}) \\
& \lesssim (\| D_x^{\alpha_q} D_t^{\beta_q} \tilde{u} \|_{L_x^p L_t^r}^{q-2} + \| D_x^{\alpha_q} D_t^{\beta_q} v \|_{L_x^p L_t^r}^{q-2}) \| D_x^{\alpha_q} D_t^{\beta_q} v \|_{L_x^p L_t^r} \| \tilde{u}_x \|_{L_x^\infty L_t^2} \\
& \quad + \| D_x^{\alpha_q} D_t^{\beta_q} v \|_{L_x^p L_t^r}^{q-1} (\| v_x \|_{L_x^\infty L_t^2} + \| \tilde{u}_x \|_{L_x^\infty L_t^2}) \\
& \lesssim S(t) (S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}),
\end{aligned}$$

where we use (4.5.7) for the last but two inequality. The above two estimates imply that

$$\eta_{[0,t]}^1(v_N) \lesssim S(t) (S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}) + \varepsilon. \quad (\text{B.0.3})$$

Similarly, we have

$$\begin{aligned}
\eta_{[0,t]}^1(\partial_x v_N) & \lesssim \| e_{xx} \|_{L_x^{5/4} L_t^{10/9}([0,t])} + \| ((v + \tilde{u})^5 - \tilde{u}^5)_{xx} \|_{L_x^{5/4} L_t^{10/9}([0,t])} \\
& \quad + \| ((v + \tilde{u})|v + \tilde{u}|^{q-1} - \tilde{u}|\tilde{u}|^{q-1})_{xx} \|_{L_x^{5/4} L_t^{10/9}([0,t])}.
\end{aligned}$$

By Hölder's inequality again, we have:

$$\begin{aligned}
& \| ((v + \tilde{u})^5 - \tilde{u}^5)_{xx} \|_{L_x^{5/4} L_t^{10/9}([0,t])} \\
& \lesssim \| (v + \tilde{u})^4 v_{xx} \|_{L_x^{5/4} L_t^{10/9}} + \| (v + \tilde{u})^3 (v_x + 2\tilde{u}_x) v_x \|_{L_x^{5/4} L_t^{10/9}} \\
& \lesssim (\| \tilde{u} \|_{L_x^5 L_t^{10}}^4 + \| v \|_{L_x^5 L_t^{10}}^4) \| v_{xx} \|_{L_x^\infty L_t^2} \\
& \quad + \| v \|_{L_x^5 L_t^{10}}^3 \| v_x \|_{L_x^\infty L_t^2} (\| v_x \|_{L_x^5 L_t^{10}} + \| \tilde{u}_x \|_{L_x^5 L_t^{10}}) \\
& \lesssim S(t) (S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}),
\end{aligned}$$

and

$$\begin{aligned}
& \left\| \left( (v + \tilde{u})|v + \tilde{u}|^{q-1} - \tilde{u}|\tilde{u}|^{q-1} \right)_{xx} \right\|_{L_x^{5/4} L_t^{10/9}([0,t])} \\
& \lesssim \left\| |v + \tilde{u}|^{q-1} v_{xx} \right\|_{L_x^{5/4} L_t^{10/9}} + \left\| |v + \tilde{u}|^{q-2} (v_x + 2\tilde{u}_x) v_x \right\|_{L_x^{5/4} L_t^{10/9}} \\
& \lesssim \left( \|\tilde{u}\|_{L_x^{5(q-1)/4} L_t^{5(q-1)/2}}^{q-1} + \|v\|_{L_x^{5(q-1)/4} L_t^{5(q-1)/2}}^{q-1} \right) \|v_{xx}\|_{L_x^\infty L_t^2} \\
& \quad + \|v\|_{L_x^{5(q-1)/4} L_t^{5(q-1)/2}}^{q-2} \|v_x\|_{L_x^\infty L_t^2} \left( \|v_x + 2\tilde{u}_x\|_{L_x^{5(q-1)/4} L_t^{5(q-1)/2}} \right) \\
& \lesssim S(t) (S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}).
\end{aligned}$$

Collecting these estimates, we have:

$$\eta_{[0,t]}^1(\partial_x v_N) \lesssim S(t) (S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}) + \varepsilon. \quad (\text{B.0.4})$$

Next, using similar strategy, we have:

$$\begin{aligned}
\eta_{[0,t]}^2(v_N) & \lesssim \|e\|_{L_x^1 L_t^2([0,t])} + \|(v + \tilde{u})^5 - \tilde{u}^5\|_{L_x^1 L_t^2([0,t])} \\
& \quad + \left\| \left( (v + \tilde{u})|v + \tilde{u}|^{q-1} - \tilde{u}|\tilde{u}|^{q-1} \right) \right\|_{L_x^1 L_t^2([0,t])} \\
& \lesssim \varepsilon + \left( \|\tilde{u}\|_{L_x^5 L_t^{10}}^4 + \|v\|_{L_x^5 L_t^{10}}^4 \right) \|v\|_{L_x^5 L_t^{10}} \\
& \quad + \left( \|v\|_{L_x^{5(q-1)/4} L_t^{5(q-1)/2}}^{q-1} + \|\tilde{u}\|_{L_x^{5(q-1)/4} L_t^{5(q-1)/2}}^{q-1} \right) \|v\|_{L_x^5 L_t^{10}} \\
& \lesssim S(t) (S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}) + \varepsilon,
\end{aligned} \quad (\text{B.0.5})$$

and

$$\begin{aligned}
\eta_{[0,t]}^2(\partial_x v_N) & \lesssim \|e_x\|_{L_x^1 L_t^2([0,t])} + \left\| \left( (v + \tilde{u})^5 - \tilde{u}^5 \right)_x \right\|_{L_x^1 L_t^2([0,t])} \\
& \quad + \left\| \left( (v + \tilde{u})|v + \tilde{u}|^{q-1} - \tilde{u}|\tilde{u}|^{q-1} \right)_x \right\|_{L_x^1 L_t^2([0,t])} \\
& \lesssim \varepsilon + \left\| \left( (v + \tilde{u})^4 - \tilde{u}^4 \right) \tilde{u}_x \right\|_{L_x^1 L_t^2([0,t])} + \left\| (v + \tilde{u})^4 v_x \right\|_{L_x^1 L_t^2([0,t])} \\
& \quad + \left\| \left( |v + \tilde{u}|^{q-1} - |\tilde{u}|^{q-1} \right) \tilde{u}_x \right\|_{L_x^1 L_t^2([0,t])} + \left\| |v + \tilde{u}|^{q-1} v_x \right\|_{L_x^1 L_t^2([0,t])} \\
& \lesssim S(t) (S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}) + \varepsilon.
\end{aligned} \quad (\text{B.0.6})$$

Finally, we need to estimate  $\eta_{[0,t]}^3(v_N)$ . From Lemma 5.1, we have:

$$\begin{aligned}
\eta_{[0,t]}^3(v_N) & \lesssim \|e_x\|_{L_x^{p'} L_t^{r'}([0,t])} + \left\| \left( (v + \tilde{u})^5 - \tilde{u}^5 \right)_x \right\|_{L_x^{p'} L_t^{r'}([0,t])} \\
& \quad + \left\| \left( (v + \tilde{u})|v + \tilde{u}|^{q-1} - \tilde{u}|\tilde{u}|^{q-1} \right)_x \right\|_{L_x^{p'} L_t^{r'}([0,t])} \\
& \lesssim \varepsilon + \left\| \left( (v + \tilde{u})^4 - \tilde{u}^4 \right) \tilde{u}_x \right\|_{L_x^{p'} L_t^{r'}} + \left\| (v + \tilde{u})^4 v_x \right\|_{L_x^{p'} L_t^{r'}} \\
& \quad + \left\| \left( |v + \tilde{u}|^{q-1} - |\tilde{u}|^{q-1} \right) \tilde{u}_x \right\|_{L_x^{p'} L_t^{r'}} + \left\| |v + \tilde{u}|^{q-1} v_x \right\|_{L_x^{p'} L_t^{r'}}.
\end{aligned}$$

By similar technique we use for (B.0.6), we have:

$$\begin{aligned}
& \left\| \left( (v + \tilde{u})^4 - \tilde{u}^4 \right) \tilde{u}_x \right\|_{L_x^{p'} L_t^{r'}} + \left\| (v + \tilde{u})^4 v_x \right\|_{L_x^{p'} L_t^{r'}} \\
& \lesssim \left\| (v + \tilde{u})^4 - \tilde{u}^4 \right\|_{L_x^{5/4} L_t^{5/2}} \| \tilde{u}_x \|_{L_x^{p_0} L_t^{r_0}} + \left\| (v + \tilde{u})^4 \right\|_{L_x^{5/4} L_t^{5/2}} \| v_x \|_{L_x^{p_0} L_t^{r_0}} \\
& \lesssim \| v_x \|_{L_x^{p_0} L_t^{r_0}} (S(t)^4 + \Omega_{[0,t]}(\tilde{u})^4) + \| \tilde{u}_x \|_{L_x^{p_0} L_t^{r_0}} S(t) (S(t)^3 + \Omega_{[0,t]}(\tilde{u})^3),
\end{aligned}$$

and

$$\begin{aligned} & \left\| (|v + \tilde{u}|^{q-1} - |\tilde{u}|^{q-1}) \tilde{u}_x \right\|_{L_x^{p'} L_t^{r'}} + \left\| |v + \tilde{u}|^{q-1} v_x \right\|_{L_x^{p'} L_t^{r'}} \\ & \lesssim \|v_x\|_{L_x^{p_0} L_t^{r_0}} (S(t)^{q-1} + \Omega_{[0,t]}(\tilde{u})^{q-1}) + \|\tilde{u}_x\|_{L_x^{p_0} L_t^{r_0}} S(t) (S(t)^{q-2} + \Omega_{[0,t]}(\tilde{u})^{q-2}), \end{aligned}$$

where

$$\frac{1}{p_0} = \frac{1}{10} - \frac{2}{5(q-1)}, \quad \frac{1}{r_0} = \frac{3}{10} + \frac{4}{5(q-1)}.$$

By the Gagliardo-Nirenberg's inequality introduced in [1, Theorem 2.44], we have:

$$\begin{aligned} \|v_x\|_{L_x^{p_0} L_t^{r_0}} & \lesssim \|D_x^{s_q} v\|_{L_x^5 L_t^{10}}^{1-s_q} \|D_x^{s_q} v_x\|_{L_x^\infty L_t^2}^{s_q} \\ & \lesssim \left( \|v\|_{L_x^5 L_t^{10}}^{1-s_q} \|v_x\|_{L_x^5 L_t^{10}}^{s_q} \right)^{1-s_q} \left( \|v_x\|_{L_x^\infty L_t^2}^{1-s_q} \|v_{xx}\|_{L_x^\infty L_t^2}^{s_q} \right)^{s_q} \\ & \lesssim S(t), \end{aligned}$$

Similarly, we have:

$$\|\tilde{u}_x\|_{L_x^{p_0} L_t^{r_0}} \lesssim \Omega_{[0,t]}(\tilde{u}),$$

hence

$$\eta_{[0,t]}^3(v_N) \lesssim S(t) (S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}) + \varepsilon. \quad (\text{B.0.7})$$

Combining (B.0.2)–(B.0.7), we conclude the proof of (B.0.1), hence the proof of Lemma 4.5.4.

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**Titre :** Dynamique asymptotique pour des équations de KdV généralisées  $L^2$  critiques et surcritiques.

**Mots-clés :** gKdV, critique et surcritique pour la masse, formation de singularités, dynamique près de soliton, formule de monotonie.

**Résumé :** Dans cette thèse, nous étudions la dynamique à temps long des solutions des équations de KdV généralisées (gKdV) critiques et surcritiques pour la masse.

La première partie de cette thèse est consacrée à la construction d'une dynamique explosive auto-similaire stable pour des équations de gKdV légèrement  $L^2$  surcritique dans l'espace d'énergie  $H^1$ . La preuve repose sur le profil auto-similaire construit par H. Koch. Nous donner une description précise de la formation de singularité près du temps d'explosion.

La deuxième partie est consacrée à la construction de solutions explosive aux équations de gKdV légèrement  $L^2$  surcritiques avec plusieurs points d'explosion. L'idée clé est d'envisager des solutions qui se comportent comme une somme de bulles découpée, chaque bulle se comportent comme un solution auto-similaire explosent en un seul point. Nous utilisons les argument topologique

classique pour s'assurer que chaque bulle explose en même temps. Ici, nous avons besoin de données initiales plus grande régularité pour contrôler la solution entre les différents points d'explosion.

Enfin, dans la troisième partie, nous considérons les équations de gKdV  $L^2$  critiques avec une perturbation saturée. Dans ce cas, toute solution avec des données initiales dans  $H^1$  est toujours globale en le temps et bornée dans  $H^1$ . Nous donner une classification explicite de la dynamique près du solitons. Sous certaines hypothèses de décroissance, il n'y a que trois possibilités : (i) la solution converge asymptotiquement vers une onde solitaire; (ii) la solution reste dans un petit voisinage de la famille modulée de l'état fondamental, en s'étalant par de temps infiniment grande (*Blow down*); (iii) la solution quitte tout petit voisinage de la famille modulée de solitons.

**Title:** Asymptotic dynamics for  $L^2$  critical and supercritical generalized KdV equations.

**Keywords:** gKdV, mass critical and supercritical, formation of singularity, dynamics near ground state, monotonicity formula.

**Abstract:** In this thesis, we deal with the long time dynamics for solutions of the mass critical and supercritical generalized KdV equations.

The first part of this work is devoted to construct a stable self-similar blow up dynamics for slightly  $L^2$  supercritical gKdV equations in the energy space  $H^1$ . The proof relies on the self-similar profile constructed by H. Koch. We will also give a specific description of the formation of singularity near the blow up time.

The second part is devoted to construct blow up solutions to the slightly  $L^2$  supercritical gKdV equations with multiple blow up points. The key idea is to consider solutions which behaves like a decoupled sum of bubbles. And each bubble behaves like a self-similar blow up solutions with a single blow up point. Then we can use a classic topological argument to ensure that each bubble

blows up at the same time. Here, we require a higher regularity of the initial data to control the solution between the different blow up points.

Finally, in the third part, we consider the  $L^2$  critical gKdV equations with a saturated perturbation. In this case, any solution with initial data in  $H^1$  is always global in time and bounded in  $H^1$ . We will give an explicit classification of the flow near the ground states. Under some suitable decay assumptions, there are only three possibilities: (i) the solution converges asymptotically to a solitary wave; (ii) the solution is always in some small neighborhood of the modulated family of the ground state, but blows down at infinite time; (iii) the solution leaves any small neighborhood of the modulated family of the ground state.