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Sur la géométrie et la combinatoire du groupe T de Thompson

Ariadna Fossas

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GRENOBLE

THÈSE

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préparée au sein de l'Institut Fourier (UMR 5582) et du Département de
Matemàtica Aplicada IV et des écoles doctorales ED MSTII et ED UPC

Sur la géométrie et la combinatoire du groupe T de R. J. Thompson

Geometric and combinatorial aspects
of R. J. Thompson's group T

Thèse soutenue publiquement le **29 juin 2012**,
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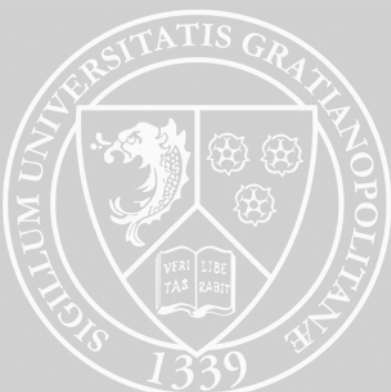
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Lance toi, comme si tu n'avais pas peur.

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Contents

Version française abrégée	9
1 Introduction	27
1.1 State of the art	28
1.2 Structure, motivations and perspectives	31
1.3 Definitions and statements of main results	34
2 Four definitions of Thompson's group T	43
2.1 Piecewise linear homeomorphisms of the circle	43
2.2 Equivalence classes of tree pair diagrams	46
2.3 $PPSL(2, \mathbb{Z})$ homeomorphisms of $\mathbb{R}P^1$	50
2.4 The asymptotic mapping class group of Σ	52
3 $PSL(2, \mathbb{Z})$ and F_2 as non-distorted subgroups of T	61
3.1 Tree pair diagram characterization of $PSL(2, \mathbb{Z})$	61
3.2 Non distortion of $PSL(2, \mathbb{Z})$ and F_2 on T	68
3.3 Piecewise linear characterization of $PSL(2, \mathbb{Z})$	70
4 Counting conjugacy classes of torsion elements on T	75
4.1 Dyadic (almost) triangulations of the disk	75
4.2 T as maps between dyadic polygons	78
4.3 On the number of conjugacy classes	79
5 T acts transitively on the infinite associahedron C	85
5.1 The asymptotic flip complex C^2	85
5.2 Stasheff's associahedra	88
5.3 Higher dimensional cells of C	91
5.4 T acts on the infinite associahedron	94
6 The group of automorphisms of C	99
6.1 Construction of the link complex $\mathcal{L}^2(\Delta_1)$	99
6.2 Extensions of link isomorphisms	101
6.3 The group of automorphisms of C	105

7	Adding a boundary to C^1	107
7.1	Geometric boundary at infinity of C^1	107
7.2	Points on the geometric boundary of C^1	113
7.3	Embedding C^1 into a functional space	122
8	Future work	127
	Bibliography	131

Version française abrégée

Cette thèse concerne la théorie géométrique des groupes et plus précisément le groupe T de Thompson. L'idée de la théorie géométrique des groupes est d'obtenir des informations sur les groupes en étudiant les propriétés des espaces sur lesquels ils agissent "joliment". Ce domaine est devenu extrêmement actif dans les années 80 grâce aux travaux de M. Gromov sur les groupes hyperboliques (voir, par exemple, [40, 39]).

Le but est d'étudier différents aspects du groupe T en comparant plusieurs points de vue et leurs interactions. Par exemple, nous trouvons un sous-groupe non distordu de T qui est isomorphe au groupe libre à deux générateurs en utilisant d'une part la version combinatoire de T en termes de classes d'équivalence de diagrammes d'arbres, et d'autre part l'approche de Thurston en termes d'homéomorphismes de la droite projective réelle qui sont " $PSL_2(\mathbb{Z})$ par morceaux". Par la suite, nous étudions l'action de T sur un complexe cellulaire C de dimension infinie qui peut être vu comme une généralisation de l'associaèdre de Stasheff au cas d'un polygone convexe ayant une infinité de côtés. En particulier, le 2-squelette de C avait été introduit par Funar et Kapoudjian et peut être plongé dans le complexe des pantalons d'une surface Σ de type infini.

Dans un premier temps, le complexe C nous permet de voir T comme un groupe d'automorphismes, ce qui n'est pas surprenant puisque T est simple (voir les travaux de Haglund et Paulin sur la simplicité des groupe d'automorphismes [41]). Malheureusement, le 1-squelette métrique de C n'est pas hyperbolique au sens de Gromov. Cependant, il peut être vu géométriquement en termes de (quasi-)triangulations dyadiques du disque de Poincaré. Cette interprétation géométrique nous permet de définir un bord à l'infini pour C^1 , par analogie avec la définition de Gromov [40, 39] du bord hyperbolique en termes de classes d'équivalence de rayons géodésiques. Cela nous permet ensuite de nous inspirer des méthodes de Thurston pour l'étude des homéomorphismes des surfaces pour plonger C^1 dans un espace fonctionnel compact et prendre l'adhérence de son image. Ces deux notions de bord admettent une prolongation naturelle de l'action de T . Nous avons pu construire explicitement un sous-ensemble du bord géométrique invariant par l'action de T . Un aspect positif est qu'on peut construire une application "naturelle" de ce sous-ensemble vers le bord fonctionnel qui est équivariante par l'action de T . En revanche, nous n'avons pas encore de description complète de tous les points du bord géométrique.

Cette introduction contient d'une part des rappels sur les objets connus, sans prétendre à l'exhaustion et en se focalisant sur les notions utilisées par la suite, et d'autre part des définitions des objets nouveaux permettant d'énoncer les résultats principaux.

Le groupe T de Thompson

L'objet d'étude principal de cette thèse est T , l'un des trois groupes de Thompson classiques $F < T < V$ introduits par Richard Thompson vers 1965 (voir [18], [3] et [69] pour des introductions). Ce sont des groupes infinis de présentation finie qui ont beaucoup de propriétés communes avec les groupes de type infini, ce qui est rare pour des groupes de type fini. Cela explique pourquoi les groupes de Thompson ont été utilisés comme contre-exemples à de nombreuses conjectures. Par exemple, T et V sont les premiers exemples de groupes simples infinis de présentation finie et ont été utilisés par McKenzie et Thompson [64] pour construire des nouveaux groupes de type fini dont le problème du mot n'est pas résoluble. Le groupe F n'est pas simple, seulement son commutateur $[F, F]$ l'est, mais il n'est pas finiment engendré. Il est important de mentionner que les groupes de Thompson sont en lien avec d'autres domaines mathématiques comme la logique [64, 11], la théorie de l'homotopie [12, 75, 13]), la théorie des représentations [51], l'algèbre [9, 12], la dynamique [7, 36, 56], la topologie [33, 32, 31]) et la cryptographie [71].

Le groupe T de Thompson est habituellement vu comme le sous-groupe de $\text{Homeo}^+(S^1)$ formé des homéomorphismes de $S^1 = [0, 1]/_{0 \sim 1}$ qui fixent l'ensemble des nombre dyadiques, sont linéaires par morceaux, différentiables sauf en un nombre fini de points dyadiques, et ont pour nombres dérivés des puissances de deux. Un des intérêts de ce point de vue est que T se comporte en tant que sous-groupe de $\text{Homeo}^+(S^1)$ comme un réseau dans un groupe de Lie (voir [36, 69]). Dans un sens général, T semble donc être un bon analogue discret du groupe $\text{Homeo}^+(S^1)$. Par exemple, les deux sont simples (voir [18] pour la simplicité de T et [35] pour celle de $\text{Homeo}^+(S^1)$). De plus, ces deux groupes ont pour groupe des automorphismes extérieurs $\mathbb{Z}/2\mathbb{Z}$ (voir [7, 35]).

La seconde version la plus utilisée est l'approche combinatoire (voir [18]) : T est vu comme le groupe des classes d'équivalence de diagrammes d'arbres. Un diagramme d'arbres est la donnée d'une paire d'arbres binaires finis enracinés (chaque sommet interne a exactement 2 descendants) ayant le même nombre de feuilles (sommets sans descendants), ainsi qu'une bijection entre les deux ensembles de feuilles préservant l'ordre cyclique défini par la lecture des feuilles de la gauche vers la droite. Une extension élémentaire d'un diagramme d'arbres est obtenue en remplaçant dans le premier arbre une feuille par deux branches et en faisant la même opération sur l'image de cette feuille dans le deuxième arbre. Une extension est obtenue par un nombre fini d'extensions élémentaires. Deux diagrammes d'arbres sont alors équivalents s'ils ont une extension commune. Dans chaque classe d'équivalence, il y a un unique représentant "minimal" dit réduit. Cette approche est très utile pour déduire des propriétés métriques. Par exemple, Burillo, Cleary, Stein et Taback ont établi dans [16] que le nombre de sommets internes du diagramme réduit donne une bonne estimation de la longueur dans un graphe de Cayley. La version combinatoire a aussi été utilisée par Burillo, Cleary, Rover, Stein et Taback pour étudier les plongements de sous-groupes, les commensurateurs et la distorsion dans les groupes de Thompson (voir [6, 14, 15, 16]).

Le troisième point de vue dont nous aurons besoin, dû à Thurston, relie T à $PSL_2(\mathbb{Z})$ (voir [61, 46, 69]). Le groupe T de Thompson est isomorphe au groupe $PPSL_2(\mathbb{Z})$ des homéomorphismes de la droite projective réelle préservant l'orientation, " $PSL_2(\mathbb{Z})$ par

morceaux”, ayant un nombre fini de points de non différentiabilité et tels que ces points de non différentiabilité soient rationnels (pour une idée de la preuve voir [46, 69]). Cette approche a été utilisée par Sergiescu dans [69] pour montrer comment plonger T dans le groupe des difféomorphismes du cercle.

Le quatrième et dernier point de vue sur T que nous considérerons ici est dû à Funar, Kapoudjian et Sergiescu [34]. Ils représentent le groupe T de Thompson (ainsi que les versions généralisées tressées de T et V) comme le groupe modulaire asymptotiquement rigide d’une surface connexe, non compacte, planaire ayant un nombre infini de composantes de bord. On peut voir ce groupe modulaire asymptotiquement rigide comme l’ensemble de classes d’homotopie d’homéomorphismes préservant une certaine triangulation de la surface en dehors d’un compact.

L’outil principal pour prouver l’équivalence entre la première définition (*version linéaire par morceaux*) et la deuxième (*version combinatoire*) est l’interprétation de l’arbre binaire (enraciné) infini en termes d’intervalles dyadiques standards (intervalles de la forme $[\frac{a}{2^n}, \frac{a+1}{2^n}]$ pour a et n entiers positifs ou nuls, voir [18]). Le lien entre la première définition et la troisième (*version “ $PSL_2(\mathbb{Z})$ par morceaux”*) est l’application “? de Minkowski”, qui envoie les nombres rationnels sur les nombres dyadiques et transforme les moyennes de Farey en moyennes arithmétiques (voir [1, 67]). L’équivalence entre la quatrième définition (groupe modulaire asymptotiquement rigide) et les trois autres repose sur le plongement de l’arbre régulier de valence 3 dans le disque et la triangulation duale de type Farey associée (voir [31]).

Funar, Kapoudjian et Sergiescu définissent les groupes modulaires asymptotiquement rigides de certaines surfaces connexes non-compactes ; ils s’en servent, entre autres, pour étendre la représentation de Bureau de B_∞ et pour construire des complexes cellulaires sur lesquels le groupe de Thompson agit par automorphismes (voir [31, 32, 33, 34, 51] et aussi les travaux de de Faria, Gardiner et Harvey [20] pour un point de vue conforme). Ces complexes peuvent être vus comme des généralisations des complexes de pantalons pour des surfaces de type infini. On peut aussi mentionner quelques autres complexes cellulaires sur lesquels T agit “joliment”. Par exemple, Brown [12], Brown et Geoghegan [13], et Stein [75] utilisent l’action de T sur des complexes de bases d’algèbres de Jonsson-Tarski [4] pour obtenir des propriétés de finitude et des propriétés homologiques. Par ailleurs, Farley construit des complexes cubiques $CAT(0)$ sur lesquels les groupes de Thompson agissent (voir [25, 27, 24, 26]).

Sur la théorie de Thurston des homéomorphismes de surfaces

Soit M une surface orientée hyperbolique de genre g avec b composantes de bord. Soit $\mathcal{S}(M)$ l’ensemble des classes d’isotopie des lacets simples essentiels et séparants de M . Considérons l’application symétrique

$$i : \mathcal{S}(M) \times \mathcal{S}(M) \longrightarrow \mathbb{Z}_{\geq 0},$$

où $i(\gamma, \gamma')$ est le minimum des nombres d’intersection géométriques entre les éléments de γ et ceux de γ' . Les résultats décrits ici ont été introduits par Thurston [76] et les détails ont été faits au Séminaire d’Orsay [28] et dans les travaux de Casson et Bleiler [19].

Soit $\mathbb{R}_{\geq 0}^{\mathcal{S}}$ l'ensemble des applications de \mathcal{S} vers l'ensemble des réels positifs ou nuls $\mathbb{R}_{\geq 0}$, muni de la topologie faible. Soit $P\mathbb{R}_{\geq 0}^{\mathcal{S}}$ l'espace projectif correspondant muni de la topologie quotient. Considérant les applications

$$\mathcal{S} \xrightarrow{i_*} \mathbb{R}_{\geq 0}^{\mathcal{S}} - \{0\} \xrightarrow{\pi} P\mathbb{R}_{\geq 0}^{\mathcal{S}},$$

où $i_*(\gamma')(\gamma) = i(\gamma, \gamma')$, Thurston a prouvé que $\pi \circ i_*$ est injective. Remarquons que la topologie induite sur \mathcal{S} par $\pi \circ i_*$ est non triviale. Il est alors naturel de considérer l'adhérence de $\pi \circ i_*(\mathcal{S})$ dans $P\mathbb{R}_{\geq 0}^{\mathcal{S}}$, dorénavant notée $\overline{\mathcal{S}}(M)$. Thurston a prouvé que :

1. $\overline{\mathcal{S}}(M)$ est topologiquement une sphère de dimension $6g + 2b - 7$, et
2. l'ensemble $\mathcal{S}(M)$ est dense dans $\overline{\mathcal{S}}(M)$.

Thurston a aussi défini une construction similaire pour l'ensemble de classes d'équivalence de feuilletages mesurés.

Soit H l'ensemble des métriques sur M de courbure constante $K = -1$ et telles que chaque composante de bord soit géodésique. Le groupe des difféomorphismes de M isotopes à l'identité, muni de la topologie \mathcal{C}^∞ , agit librement et continûment sur H . L'espace de Teichmüller $\mathcal{T}(M)$ de M peut être défini comme l'espace des orbites de cette action. Un résultat classique dû à Fricke et Klein est l'homéomorphisme suivant :

$$\mathcal{T}(M) \simeq \mathbb{R}^{6g+2b-6}.$$

Ainsi, en utilisant la construction de Thurston pour les feuilletages mesurés, l'espace $\overline{\mathcal{S}}(M)$ peut être vu comme le bord de $\mathcal{T}(M)$ et l'action du groupe de difféomorphismes de M peut être étendue à $\overline{\mathcal{S}}(M)$. De plus, le complexe des courbes peut être plongé dans $\overline{\mathcal{S}}(M)$ et le complexe des pantalons est quasi-isométrique à $\mathcal{T}(M)$, muni de la métrique de Weil-Peterson.

Les résultats précédents impliquent que $\mathcal{T}(M) \cup \overline{\mathcal{S}}(M)$ est topologiquement une boule fermée de dimension $6g + 2b - 6$. Alors, grâce au théorème du point fixe de Brouwer, chaque difféomorphisme φ de M fixe un point de $\mathcal{T}(M) \cup \overline{\mathcal{S}}(M)$. La nature de ce point fixe donne la classification de Thurston des difféomorphismes de surfaces suivante :

1. Soit φ est isotope à un difféomorphisme périodique.
2. Soit φ est isotope à un difféomorphisme réductible.
3. Soit φ est Pseudo-Anosov.

Un procédé analogue peut être mené dans le cas du tore 2-dimensionnel et cela redonne la classification des éléments de $PSL_2(\mathbb{Z})$ en elliptiques, paraboliques et hyperboliques.

Un autre objet important lors de l'étude des surfaces est le groupe modulaire de M , qui est le groupe des homéomorphismes de M modulo isotopie. Les groupes modulaires ont été abondamment étudiés au travers de leurs actions par automorphismes sur des complexes simpliciaux localement infinis : principalement le complexe des courbes, celui des pantalons, celui des arcs et celui de Hatcher-Thurston. Certains de ces complexes sont hyperboliques au sens de Gromov pour $n + b$ assez grand (voir [63]), et leurs groupes d'automorphismes sont reliés au groupe modulaire lui-même.

En particulier, Ivanov [50] puis Luo [58] ont prouvé que le groupe des automorphismes du complexe des courbes est le groupe modulaire étendu pour presque toutes les surfaces compactes, et Korkmaz [52] a étendu ce résultat aux sphères percées et aux tores percés. Des résultats analogues ont été prouvés pour le complexe des arcs par Ivanov [50] et Irmak et McCarthy [49], pour le complexe des pantalons par Margalit [60], pour le complexe de Hatcher-Thurston par Irmak et Korkmaz [48], pour le complexe des arcs et courbes par Korkmaz et Papadopoulos [53] et pour le complexe des courbes non-séparantes par Schmutz-Schaller [68] puis Irmak [47].

Structure de la thèse et résultats principaux

Le deuxième chapitre de cette thèse contient les quatre définitions du groupe T qui vont être utilisées par la suite, ainsi que quelques exemples et une idée de comment prouver l'équivalence entre elles.

$PSL(2, \mathbb{Z})$ et F_2 comme sous-groupes non-distordus de T

Au troisième chapitre on utilise les versions combinatoire et projective pour plonger quasi-isométriquement le groupe libre à deux générateurs dans T . Rappelons d'abord la définition la plus courante de T , ainsi que ses versions combinatoire et projective.

Le *groupe T de Thompson linéaire par morceaux* est le sous-groupe de $\text{Homeo}^+(S^1)$ formé des homéomorphismes de $S^1 = [0, 1]/_{0 \sim 1}$ qui fixent l'ensemble des nombres dyadiques, sont linéaires par morceaux, différentiables sauf en un nombre fini de points dyadiques, et ayant pour nombres dérivés des puissances de deux.

Un *diagramme d'arbres (cyclique)* est la donnée de (R, σ, S) , où R et S sont des arbres binaires finis (enracinés) avec le même nombre n de feuilles, et σ est une bijection entre l'ensemble de feuilles de R et celui de S qui préserve l'ordre cyclique défini par la lecture des feuilles de la gauche vers la droite. L'arbre binaire R est nommé *arbre de départ* et S est nommé *arbre d'arrivée*. Un sommet interne muni de ses deux descendants est nommé un *circonflexe*. Un circonflexe contenant deux feuilles est *exposé*. Un diagramme d'arbres est *réduit* si, pour tout circonflexe exposé de R , les images par σ de ses feuilles ne forment pas un circonflexe exposé de R .

Étant donné un diagramme d'arbres (R, σ, S) , sa *j -ième extension simple* s'obtient en rajoutant un circonflexe à la j -ième feuille de R commençant par la gauche, et en rajoutant un circonflexe à la feuille correspondante à $\sigma(j)$ dans S . De plus, on étend la bijection σ aux nouveaux ensembles de feuilles de telle façon que le descendant gauche (respectivement droit) de l'ancienne feuille j de R soit envoyé sur le descendant gauche (respectivement droit) de l'ancienne feuille $\sigma(j)$ de S . Un diagramme d'arbres obtenu à partir de (R, σ, S) par une succession finie d'extensions simples est une *extension* de (R, σ, S) .

Deux diagrammes d'arbres sont *équivalents* s'ils ont une extension en commun. Le *groupe T de Thompson combinatoire* est l'ensemble de classes d'équivalence de diagrammes d'arbres.

Le groupe T de Thompson projectif par morceaux (qu'on note $PPSL_2(\mathbb{Z})$) est le groupe des homéomorphismes de la droite projective réelle qui préservent l'orientation, sont $PSL_2(\mathbb{Z})$ par morceaux, et ont un nombre fini de points de non-différentiabilité, qui sont tous rationnels.

La version projective par morceaux de T permet de plonger "naturellement" $PSL_2(\mathbb{Z})$ dans T , et Sergiescu s'était posé la question de la caractérisation des éléments de ce sous-groupe.

Le groupe $PSL_2(\mathbb{Z})$ est isomorphe au produit libre $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. Ainsi, tout élément de $PSL_2(\mathbb{Z})$ admet une écriture sous forme normale par rapport au système générateur $\{a, b\}$, où $a^2 = 1$ et $b^3 = 1$ (voir [44]) :

$$a^{\epsilon_1} b^{\delta_1} a b^{\delta_2} a \dots a b^{\delta_k} a^{\epsilon_2},$$

où k est un entier positif ou nul, $\epsilon_1, \epsilon_2 \in \{0, 1\}$ et $\delta_1, \dots, \delta_k \in \{-1, 1\}$. Dorénavant on notera a et b les éléments du système générateur de $PSL_2(\mathbb{Z})$ ainsi que leurs images dans T par l'isomorphisme entre T et $PPSL_2(\mathbb{Z})$.

La réponse à la question de Sergiescu constitue une part importante du troisième chapitre et a été publiée dans [29], où la version combinatoire est utilisée pour donner une description explicite des éléments du plongement $PSL_2(\mathbb{Z}) \hookrightarrow T$. Avant de donner le théorème de caractérisation, on a besoin d'introduire une nouvelle définition : un arbre binaire (enraciné) fini est *fin* si tous ses circonflexes contiennent une feuille et un sommet interne, à l'exception d'un seul circonflexe, qui est exposé. On peut décrire les arbres fins à n feuilles en associant à chacun de leurs $n - 2$ sommets internes un poids $r_i \in \{-1, 1\}$ de la façon suivante : soient v_0, \dots, v_{n-3} ces sommets internes différents de la racine et v_{-1} la racine. Alors, pour $i \in \{1, \dots, n - 2\}$, $r_i = 1$ si v_i est le descendant gauche de v_{i-1} et $r_i = -1$ si v_i est le descendant droit de v_{i-1} .

Théorème A. Soit (R, σ, S) le diagramme d'arbres réduit de l'élément $t \in T$. Alors, $t \in \langle a, b \rangle \simeq PSL_2(\mathbb{Z})$ si et seulement si (R, σ, S) satisfait une des conditions suivantes :

1. le nombre de feuilles de R (et S) est au plus 4, ou
2. l'arbre R est fin à poids r_0, \dots, r_{k-1} , l'arbre S est fin à poids s_0, \dots, s_{k-1} , et ces poids vérifient les équations suivantes :

$$(i) \sum_{i=2}^{k-1} r_i s_{k+1-i} = 2 - k, \text{ et}$$

$$(ii) l + \sigma(1) + \epsilon(s_0) \equiv \frac{3 - s_1}{2} \pmod{k + 2},$$

où $l - 1$ est le cardinal de l'ensemble $\{i : r_i = -1, 0 \leq i \leq k - 1\}$, et

$$\epsilon(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x = -1. \end{cases}$$

La proposition suivante démontre l'une des implications du Théorème A. Pour l'autre inclusion il suffit de prouver que le nombre de solutions des équations (i) et (ii) coïncide avec le nombre de mots $w(a, b) = a^{\epsilon_1} b^{\delta_1} a b^{\delta_2} a \dots a b^{\delta_k} a^{\epsilon_2}$, avec $\epsilon_1, \epsilon_2 \in \{0, 1\}$ et $\delta_1, \dots, \delta_k \in \{-1, 1\}$:

Proposition. Soit $w(a, b)$ la forme normale d'un élément de $PSL_2(\mathbb{Z})$ par rapport au système générateur $\{a, b\}$, i.e.

$$w(a, b) = a^{\epsilon_1} b^{\delta_1} a b^{\delta_2} a \dots a b^{\delta_k} a^{\epsilon_2},$$

où $\epsilon_1, \epsilon_2 \in \{0, 1\}$ et $\delta_1, \dots, \delta_k \in \{-1, 1\}$. On suppose $k \geq 2$. Soit R l'arbre fin défini par les poids

$$r_0 = \epsilon^{-1}(\epsilon_1) \text{ et}$$

$$r_i = \delta_i, \forall 1 \leq i \leq k-1,$$

et soit S l'arbre fin défini par les poids

$$s_0 = \epsilon^{-1}(\epsilon_2) \text{ et}$$

$$s_i = -\delta_{k+1-i}, \forall 1 \leq i \leq k-1.$$

Soit σ la permutation cyclique définie par l'équation

$$\sigma(1) \equiv \frac{3 - s_1}{2} - \epsilon(s_0) - l \pmod{k+2},$$

où l et ϵ sont ceux du Théorème A.

Alors, (R, σ, S) est un diagramme réduit qui correspond à w . De plus, les poids r_0, \dots, r_{k-1} et s_0, \dots, s_{k-1} satisfont les équations (i) et (ii) du Théorème A.

À partir du Théorème A et la Proposition précédente, on peut estimer la longueur des éléments du sous-groupe par rapport aux système générateur $\{a, b\}$ en fonction du nombre de feuilles de leur diagramme réduit. Cette relation entre le nombre de circonflexes et la longueur des éléments de T avait déjà été montrée par Burillo, Cleary, Stein et Taback [16]. On en déduit donc le résultat suivant :

Proposition. Le groupe $PSL_2(\mathbb{Z})$ est un sous-groupe non-distordu de T .

Il y a plusieurs sous-groupes de $PSL_2(\mathbb{Z})$ qui sont isomorphes au groupe libre à deux générateurs. Un exemple est le sous-groupe dérivé $\langle abab^{-1}, ab^{-1}ab \rangle$ de $PSL_2(\mathbb{Z})$ (voir [21], page 26 ou [59] proposition III.12.3). De plus, ce sous-groupe est d'indice fini dans $PSL_2(\mathbb{Z})$. En utilisant le fait que les sous-groupes d'indice fini sont non-distordus, on obtient :

Corollaire. Le sous-groupe de T engendré par $\langle abab^{-1}, ab^{-1}ab \rangle$, qui est isomorphe au groupe libre engendré par deux éléments, est un sous-groupe non-distordu.

Il était déjà connu que le groupe libre pouvait être plongé dans T , cela étant la façon usuelle de prouver la non-moyennabilité de T . La preuve la plus connue utilise le lemme du ping-pong, ce qui complique l'étude des propriétés métriques de ce plongement. La partie intéressante du corollaire précédent est donc le fait que F_2 peut être plongé **quasi-isométriquement** dans T .

Sur le nombre de classes de conjugaison des éléments de torsion dans T

Au quatrième chapitre, on compte le nombre de classes de conjugaison (dans T) d'éléments d'ordre $n \in \mathbb{N}$: il est prouvé qu'il y en a exactement $\varphi(n)$, où φ est l'indicatrice d'Euler. Pour cela, on se sert de la version modulaire de T pour reformuler les diagrammes d'arbres en termes de polygones de Farey. Il est intéressant de remarquer que la base commune des deux versions est l'action de T sur l'ensemble des rationnels dyadiques, vue comme l'action de T sur le bord à l'infini de l'arbre binaire.

Soit D le disque unité de \mathbb{R}^2 . On suppose son bord S^1 paramétré par l'intervalle unité $[0, 1]$ proportionnellement à la longueur d'arc. On s'intéressera aux triangulations de D en un nombre infini de triangles ayant leurs sommets dans l'image des dyadiques par le paramétrage de S^1 . Bien que la nature de ces objets soit combinatoire, on peut les représenter par des triangulations du disque de Poincaré à l'image de la triangulation de Farey. Néanmoins, une représentation par triangulations du disque euclidien reste possible. Pour la commodité des dessins et la cohérence avec les versions *modulaire* et $PSL_2(\mathbb{Z})$ *par morceaux*, nous prendrons la représentation hyperbolique. Ainsi, on notera (p, q) (où $0 \leq p < q < 1$) l'arc géodésique dans le disque de Poincaré reliant les points $\gamma(p)$ et $\gamma(q)$ de S^1 , où γ est le paramétrage de S^1 par $[0, 1]$ décrit antérieurement.

Soit Δ_F l'équivalent de la triangulation de Farey, c'est à dire le sous-ensemble de l'ensemble des arcs géodésiques à extrémités dyadiques formé seulement des arcs reliant les extrémités des intervalles standards dyadiques :

$$\Delta_F = \left\{ \left(0, \frac{1}{2}\right) \right\} \cup \left\{ \left(\frac{m}{2^n}, \frac{m+1}{2^n}\right) : m, n \in \mathbb{Z}_{>0}, m \in \{0, \dots, 2^n - 1\}, n > 1 \right\}.$$

L'ensemble Δ_F est nommé l'*ensemble des arcs de Farey* et forme une triangulation de D au sens où $D \setminus \Delta_F$ est une réunion disjointe de triangles. Cette triangulation est nommée *triangulation dyadique de Farey* de D . Une *triangulation dyadique de type Farey* est une triangulation de D obtenue à partir de Δ_F en remplaçant un nombre fini d'arcs de Farey par d'autres arcs géodésiques d'extrémités dyadiques qui ne se croisent pas à l'intérieur de D .

Soit Δ une triangulation dyadique de D de type Farey et soient a_1, \dots, a_k des arcs de Δ . On dit que a_1, \dots, a_k définissent un Δ -polygone s'ils forment le bord d'un k -gone (idéal) $\mathcal{P}(a_1, \dots, a_k)$ dans D muni de la métrique hyperbolique (ou un k -gone convexe si l'on prend le modèle euclidien).

À partir de ces définitions, on obtient une nouvelle interprétation des éléments du groupe T :

Lemme. *Soit Δ une triangulation dyadique de D de type Farey, soient $\mathcal{P}(a_1, \dots, a_k)$ et $\mathcal{P}(a'_1, \dots, a'_k)$ deux Δ -polygones et σ une permutation cyclique de l'ensemble $\{1, \dots, k\}$. Alors, $(\Delta, \mathcal{P}(a_1, \dots, a_k), \mathcal{P}(a'_1, \dots, a'_k), \sigma)$ représente un unique élément $t \in T$ qui envoie le polygone $\mathcal{P}(a_1, \dots, a_k)$ sur le polygone $\mathcal{P}(a'_1, \dots, a'_k)$ de façon que l'arc a_i soit envoyé sur l'arc $a'_{\sigma(i)}$ et la triangulation Δ soit préservée en dehors de $\mathcal{P}(a_1, \dots, a_k)$. Cette bijection entre polygones préservant la triangulation à l'extérieur de ces polygones induit de façon canonique une bijection sur les dyadiques du bord qui coïncide avec t .*

On peut donc donner une caractérisation des éléments de torsion de T .

Proposition. *Soit t un élément du groupe T d'ordre n . Alors, il existe une triangulation dyadique Δ de type Farey, un Δ -polygone $\mathcal{P}(a_1, \dots, a_n)$ à n côtés et une permutation cyclique $\sigma \in S_n$ d'ordre n tels que $(\Delta, \mathcal{P}(a_1, \dots, a_n), \mathcal{P}(a_1, \dots, a_n), \sigma)$ représente t . De plus, tout élément de T admettant une représentation comme celle-ci est d'ordre n .*

Cela permet de démontrer le résultat suivant.

Théorème B. *Il y a exactement $\varphi(n)$ classes de conjugaison d'éléments d'ordre n dans T , où φ est l'indicatrice d'Euler.*

On s'est rendu compte *a posteriori* que ce résultat pouvait aussi être obtenu à partir de la caractérisation des éléments de torsion donné par Burillo, Cleary, Stein et Taback [16].

Lochak et Schneps [57] donnent une présentation finie de T dans un système générateur à deux éléments de torsion $\{\alpha, \beta\}$, où $\alpha^4 = 1$ et $\beta^3 = 1$. Il semble donc naturel de se demander si l'on peut trouver une preuve alternative du résultat de Brin [7] : $Out(T) = \mathbb{Z}/2\mathbb{Z}$, en utilisant le résultat sur le nombre de classes de conjugaison et le fait que T est le groupe d'automorphismes préservant l'orientation d'un certain complexe cellulaire C (voir [30]). Cette question reste ouverte.

T et l'associaèdre infini C

Au cinquième chapitre, on introduit l'associaèdre infini C et on montre que le groupe T de Thompson agit sur C par automorphismes. Le complexe cellulaire C peut être vu comme un analogue infini des associaèdres de Stasheff [73, 74, 37, 5, 55, 45, 72] et son squelette de dimension 1 devient ainsi le graphe des triangulations d'un polygone convexe à un nombre infini de côtés.

Avant de donner la définition de C , on introduit l'associaèdre de Stasheff d'un polygone convexe à n côtés. Il est, à notre avis, plus facile d'imaginer cet objet en tant que complexe dual du complexe des arcs du polygone. Soit n un entier strictement plus grand que 4 (pour les cas $n = 3$ et 4, la notion de complexe des arcs a aussi un sens, mais les complexes obtenus ne sont pas très intéressants) et soit P_n un polygone convexe à n côtés. Soit $V(P_n) = \{v_1, \dots, v_n\}$ l'ensemble des sommets de P_n , où $\{v_i, v_j\}$ ($i < j$) est un côté de P_n si et seulement si $j = i+1$ ou $i = 1, j = n$. Soit D_n l'ensemble des diagonales intérieures de P_n . Le *complexe des arcs* A_n de P_n est le complexe simplicial défini par les sous-ensembles de D_n contenant des diagonales qui ne se croisent pas deux à deux. La dimension de A_n est $n - 4$ et ses simplexes maximaux correspondent aux triangulations de P_n . De plus, A_n est un polytope convexe de dimension $(n - 3)$. L'*associaèdre de Stasheff's* K_n est le complexe dual de A_n .

Soit T_n l'ensemble des triangulations minimales de P_n , c'est-à-dire les triangulations obtenues à partir de $n - 3$ diagonales de P_n . L'ensemble des sommets $V(K_n)$ de K_n est T_n . La définition ensembliste du complexe cellulaire K_n est la suivante : pour $1 \leq k \leq n - 3$, un sous-ensemble K de T_n est une cellule de dimension k si K est exactement l'ensemble des triangulations de T_n ayant $n - 3 - k$ diagonales en commun. Il n'est pas difficile à voir que le sous-ensemble des parties de T_n qui définit K_n est fermé par intersection, et que l'intersection de deux cellules de dimension k est une cellule de dimension strictement

inférieure. Enfin, si on utilise le fait que le dual d'un polytope est un polytope de la même dimension, on obtient que l'associaèdre de Stasheff K_n est un polytope convexe de dimension $(n - 3)$.

Une *quasi-triangulation dyadique de type Farey* de D est l'objet $a\Delta$ obtenu à partir d'une triangulation dyadique de type Farey Δ si on enlève un nombre fini $\mathcal{D}(a\Delta)$ de cordes. L'*associaèdre infini* est le complexe cellulaire C qui a pour sommets les triangulations dyadiques de type Farey et où toute cellule de dimension $n \geq 1$ peut être vue comme une quasi-triangulation dyadique de type Farey telle que $\mathcal{D}(a\Delta) = n$. Soit $a\Delta$ une quasi-triangulation dyadique de type Farey avec $\mathcal{D}(a\Delta) = n > 0$, soient P_1, \dots, P_k les polygones non-triangulaires de $D - a\Delta$. Pour $i \in \{1, \dots, k\}$, on note $K(P_i)$ l'associaèdre de Stasheff d'un polygone convexe qui a le même nombre de côtés que P_i . Alors, on peut coller à l'ensemble de sommets $\{\Delta' \in C^0 : a\Delta \subset \Delta'\}$ une cellule de dimension n isomorphe au produit direct $K(P_1) \times \dots \times K(P_k)$, qui est topologiquement une boule fermée de dimension n .

L'associaèdre infini généralise un complexe cellulaire de dimension 2 introduit par Funar et Kapoudjian [31]. Leur complexe provient de la version modulaire du groupe T .

Soit T_3 l'arbre trivalent infini obtenu comme le graphe dual de la triangulation Δ_F de D . Soit Σ la surface topologique obtenue en recollant un voisinage tubulaire fermé sans auto-intersections autour de T_3 . Alors, Σ est planaire, orientable, non-compacte, contractile et elle a une infinité de composantes connexes de bord. La triangulation Δ_F munit Σ d'un pavage en hexagones, où chacun des bords des hexagones est formé en alternant bouts d'arcs de Farey et bouts de composantes de bord de Σ .

Un ensemble d'arcs de Farey est *séparant* si leur réunion est le bord d'une sous-surface compacte de Σ . Soit $\text{Homeo}^+(\Sigma)$ le groupe des homéomorphismes de Σ qui préservent l'orientation. Un élément $\varphi \in \text{Homeo}^+(\Sigma)$ est *asymptotiquement rigide* si il existe deux ensembles d'arcs séparants $\{a_1, \dots, a_k\}$, $\{a'_1, \dots, a'_k\}$ tels que :

1. pour tout $1 \leq i \leq k$, $\varphi(a_i \cap \Sigma) = a'_i \cap \Sigma$, et
2. le pavage par hexagones de Σ est préservé en dehors des deux sous-surfaces compactes bornées par $\{a_1, \dots, a_k\}$ et $\{a'_1, \dots, a'_k\}$.

Soit $\text{Homeo}_a(\Sigma)$ le groupe des homéomorphismes de Σ qui sont asymptotiquement rigides. Le *groupe modulaire asymptotique* de Σ est le quotient $\text{Homeo}_a^+(\Sigma)/\text{Homeo}_a^{0,+}$, où $\text{Homeo}_a^{0,+}$ est le groupe des homéomorphismes asymptotiquement rigides qui sont isotopes à l'identité.

On a déjà dit dans la première partie de cette introduction qu'étant donnée une surface connexe, orientable et compacte, on peut construire plusieurs complexes cellulaires localement infinis sur lesquels le groupe modulaire de la surface agit par automorphismes. Dans le cas des surfaces de type infini, il est difficile de définir des analogues du groupe modulaire qui soient de type fini. De plus, les analogues des complexes cellulaires définis trop naïvement n'ont pas les propriétés souhaitées pour de tels objets (par exemple, si on définit l'analogue naturel du complexe des courbes, son 1-squelette a diamètre 2). Dans cette direction, le 2-squelette du complexe cellulaire C est relié au complexe des pantalons de Σ par Funar et Kapoudjian [31].

Proposition. *Soit C l'associaèdre infini. Alors,*

1. C est connexe, simplement connexe et contractile.
2. Le réalisé du 1-squelette C^1 n'est pas hyperbolique au sens de Gromov.

La vision du groupe T en termes d'homéomorphismes linéaires par morceaux du cercle permet de voir l'action de T sur l'ensemble des rationnels dyadiques $\mathbb{Z}[1/2]$. Cette action peut être étendue à l'ensemble $\mathbb{Z}[1/2] \times \mathbb{Z}[1/2]$. Chaque arc dyadique de D est entièrement déterminé par ses extrémités dyadiques. Si on considère les (quasi-)triangulations dyadiques de type Farey comme sous-ensembles de $\mathbb{Z}[1/2] \times \mathbb{Z}[1/2]$, on peut prouver que l'image d'une (quasi-)triangulation dyadique de type Farey est bien une (quasi-)triangulation dyadique de type Farey. Cela nous permet de définir l'action de T sur C .

Proposition. *Le groupe T de Thompson agit par automorphismes sur l'associaèdre infini C . On a donc une application naturelle $\Psi : T \rightarrow \text{Aut}(C)$ qui est injective. De plus, cette action est transitive par rapport aux sommets et aux arêtes de C .*

Les méthodes utilisées dans la preuve peuvent être adaptées au sous-groupe des éléments de T qui fixent un point du cercle. Ces sous-groupes sont isomorphes au groupe de Thompson F . On obtient donc le résultat suivant :

Proposition. *Le groupe F de Thompson agit par automorphismes sur le complexe cellulaire C et l'application induite $\Psi : F \rightarrow \text{Aut}(C)$ est injective. L'action est transitive par rapport aux sommets.*

L'isomorphisme entre le groupe d'automorphismes de l'associaèdre infini C et $T \rtimes \mathbb{Z}/2\mathbb{Z}$ est établi au sixième chapitre (le cas particulier de la dimension 2 est un travail en commun avec M. Nguyen [30]) :

Théorème C. *Soit n un entier positif strictement plus grand que 1. Le groupe d'automorphismes du squelette de dimension n de C est isomorphe au produit semi-direct $T \rtimes \mathbb{Z}/2\mathbb{Z}$. En particulier, le groupe d'automorphismes de l'associaèdre infini C est isomorphe à $T \rtimes \mathbb{Z}/2\mathbb{Z}$.*

De plus, le co-noyau agit non trivialement par automorphisme extérieur. Ainsi, on obtient un plongement non-trivial de $\mathbb{Z}/2\mathbb{Z}$ dans $\text{Out}(T)$. Ce plongement est un isomorphisme d'après le résultat de Brin [7].

La preuve du Théorème C repose sur le fait que tout automorphisme de C est complètement déterminé par l'image de la boule de rayon 1 autour d'une triangulation dyadique de type Farey $\Delta \in C^0$ (on prend ici la métrique combinatoire). Étant donné un automorphisme φ de C , on trouve un élément $\Psi(t) \in \text{Aut}(C^2)$ tel que $t(\Delta_F) = \varphi(\Delta_F)$. Pour cela, on introduit un complexe auxiliaire qui codifie les 2-cellules pentagonales avec deux arêtes dans la boule de centre Δ et de rayon 1, ce qui nous permet de construire une suite exacte courte

$$1 \longrightarrow T \longrightarrow \text{Aut}(C^2) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1,$$

qui de plus est scindée.

Le complexe auxiliaire utilisé dans la preuve du théorème fait apparaître la notion d'*orientation* du complexe C . Ainsi, le groupe des automorphismes de C peut être séparé en deux classes : ceux qui préservent l'orientation et ceux qui la renversent. La première classe est un sous-groupe du groupe d'automorphismes de C et coïncide avec $\Psi(T)$. On peut donc interpréter géométriquement ce produit semi-direct $T \rtimes \mathbb{Z}/2\mathbb{Z}$ comme étant

le groupe modulaire étendu de Σ , ce qui fait du théorème C l'analogue du Théorème d'Ivanov sur les surfaces compactes.

Ce résultat qui relie T au groupe d'automorphismes d'un complexe cellulaire peut aussi être considéré d'un autre point de vue. En 1970, Tits [77] a prouvé que la plupart des sous-groupes des groupes d'automorphismes d'arbres engendrés par des éléments stabilisant des sommets sont simples. Ce résultat fut étendu par Haglund et Paulin [41] aux groupes d'automorphismes des complexes polyédraux négativement courbés, qui sont (généralement non-dénombrables) non-linéaires et virtuellement simples. Cela rend légitime la question de la réalisation des groupes simples T et V comme groupes d'automorphismes de complexes polyédraux. Farley [25, 27] donne une réponse partielle à cette question : il prouve que les trois groupes de Thompson classiques F , T et V agissent proprement et isométriquement sur des complexes CAT(0) cubiques. En particulier, ces groupes sont a-T-moyennables et satisfont la conjecture de Baum-Connes à coefficients arbitraires. Cependant, les groupes d'automorphismes de ces complexes sont bien plus gros que les groupes de Thompson. Notre associaèdre infini C fournit une réponse (presque) optimale puisque son groupe d'automorphismes est (presque) T . En revanche, C n'est pas hyperbolique au sens de Gromov et nous conjecturons qu'il n'est pas non plus CAT(0).

Un bord à l'infini pour C^1

Au septième et dernier chapitre, on définit un bord géométrique à l'infini pour le 1-squelette C^1 en considérant une classe particulière de rayons (qui contient les rayons géodésiques) et en associant à chacun de ces rayons un ensemble d'arcs dyadiques de D . N'ayant pas trouvé une topologie pour laquelle l'ensemble de classes d'équivalence de rayons géodésiques soit compact, on propose une compactification fonctionnelle à la Thurston de C^1 . On montre alors que cette compactification contient une partie du bord géométrique invariante par l'action de T , que nous décrivons explicitement.

Soit \mathcal{A} l'ensemble des arcs géodésiques de D d'extrémités dyadiques. On considère le 1-squelette C^1 muni de la métrique combinatoire d_{C^1} , c'est-à-dire que chaque arête de C^1 est de longueur 1. Un rayon $\rho : [0, \infty) \rightarrow C^1$ est *géodésique* dans C^1 si pour tous $s_1, s_2 \in [0, \infty)$, $d_{C^1}(\rho(s_1), \rho(s_2)) = |s_1 - s_2|$.

En utilisant un résultat de Sleator, Tarjan et Thurston [72] on obtient :

Lemme. *Soit ρ un rayon géodésique de C^1 , soit $(\Delta_n(\rho))_{n \in \mathbb{N}}$ la suite des sommets du rayon géodésique ρ . Soit a un arc dyadique. Alors, l'ensemble $s(a, \rho) = \{n \in \mathbb{N} : a \in \Delta_n(\rho)\}$ est formé d'entiers consécutifs (l'ensemble peut être éventuellement vide ou infini).*

Ce résultat est aussi valable pour des chemins géodésiques (finis), vu que tout chemin géodésique peut se prolonger en rayon géodésique. Remarquons aussi que cette propriété ne caractérise pas les rayons géodésiques, ce qui motive la définition suivante : un *rayon STT* dans C^1 est un rayon $\rho : [0, \infty) \rightarrow C^1$ tel que, pour tout arc dyadique a , l'ensemble $s(a, \rho)$ est formé d'entiers consécutifs (éventuellement vide ou infini). Soit ρ un rayon STT de C^1 . L'ensemble stable d'arcs $\mathcal{A}_{stab}(\rho)$ de ρ est

$$\mathcal{A}_{stab}(\rho) = \{a \in \mathcal{A} : s(a, \rho) \text{ est infini}\} \subset \mathcal{A}.$$

Soit ρ un rayon STT de C^1 et soit $(\Delta_n)_{n \in \mathbb{N}}$ la suite des triangulations dyadiques de type Farey qui apparaissent dans ρ (sommets du rayon). Rappelons que toute triangulation dyadique peut être vue comme un sous-ensemble de l'ensemble des arcs dyadiques. Alors,

$$\mathcal{A}_{stab}(\rho) = \liminf_{n \rightarrow \infty} \Delta_n.$$

De plus, grâce à la condition STT du rayon ρ , cette limite inférieure coïncide avec la limite supérieure

$$\mathcal{A}_{stab}(\rho) = \limsup_{n \rightarrow \infty} \Delta_n.$$

Ainsi, l'ensemble stable d'arcs est une limite au sens ensembliste du terme.

Lemme. Soient ρ et ρ' deux rayons STT différents de C^1 et soient $\mathcal{A}_{stab}(\rho)$ et $\mathcal{A}_{stab}(\rho')$ leurs ensembles stables d'arcs respectifs. On suppose que ρ et ρ' coïncident à partir d'un point, c'est à dire qu'il existe $s_0 \in [0, \infty)$ et $r \in \mathbb{R}$ tel que, pour tout $s \in [s_0, \infty)$, on ait $\rho'(s) = \rho(s + r)$. Alors, $\mathcal{A}_{stab}(\rho') = \mathcal{A}_{stab}(\rho)$.

Deux rayons STT ρ, ρ' de C^1 sont équivalents si $\mathcal{A}_{stab}(\rho) = \mathcal{A}_{stab}(\rho')$. Le bord géométrique à l'infini de C^1 est l'ensemble de classes d'équivalence de rayons STT de C^1 . De plus, on peut étendre l'action de T sur C^1 à son bord géométrique à l'infini.

Corollaire. Le groupe T de Thompson, et donc aussi le groupe F de Thompson, agissent sur le bord géométrique à l'infini de C^1 .

Les résultats suivants visent à décrire explicitement certains points du bord géométrique à l'infini de C^1 . Même si nous n'en avons pas une description totale, le sous-ensemble de points qu'on décrit est invariant par l'action de T .

Lemme. Soit ρ un rayon STT de C^1 . On suppose qu'il existe une composante connexe par arcs de $D \setminus \mathcal{A}_{stab}(\rho)$ ayant comme bord l'ensemble d'arcs dyadiques $\{a_1, \dots, a_k\} \subset \mathcal{A}_{stab}(\rho)$ ($k > 2$), c'est-à-dire une composante connexe par arcs de forme polygonale. Alors, $k = 3$ (le polygone est un triangle).

Proposition. Soit $\Delta' \subset \mathcal{A}$ une triangulation dyadique de D telle que $D \setminus (\Delta' \cap \Delta_F)$ est une réunion disjointe de polygones finis. Alors, il existe une infinité de rayons STT de C^1 qui ont pour ensemble stable d'arcs Δ' .

Une région étoilée de Farey est la donnée d'une paire $(P_\infty, \Delta(P_\infty))$, où :

1. P_∞ est une sous-région de D bornée par une ligne polygonale infinie d'arcs dyadiques consécutifs a_1, a_2, a_3, \dots qui commence à $\frac{m_1}{2^{n_1}}$ et telle que, pour un certain $n \in \mathbb{N}$ et pour tout $k \in \mathbb{N}$, les extrémités de l'arc a_{n+k} sont les dyadiques $\frac{2^k m_1 - 1}{2^{n_1+k}}$ et $\frac{m_1}{2^{n_1}}$,
2. $\Delta(P_\infty)$ est la triangulation dyadique de la région P_∞ dont tous les arcs contiennent l'extrémité de a_1 qui est différente de $\frac{m_1}{2^{n_1}}$.

On remarque que, étant donné un point de a_1 et un voisinage \mathcal{N} de ce point, l'intersection $\mathcal{N} \cap \Delta(P_\infty)$ est infinie. Si a_1 est un arc de Farey, on appelle $(P_\infty, \Delta(P_\infty))$ une région étoilée basée sur Farey.

Proposition. Soit $(P_\infty^{(1)}, \Delta(P_\infty^{(1)}), (P_\infty^{(2)}, \Delta(P_\infty^{(2)}), \dots$ une suite infinie de régions étoilées de Farey d'intérieurs deux à deux disjoints, et telle que toutes sauf un nombre fini sont des régions étoilées basées sur Farey. Soit Δ une triangulation dyadique de D

qui coïncide avec $\Delta(P_\infty)^{(i)}$ dans $P_\infty^{(i)}$ pour tout $i \in \mathbb{N}$, et de type Farey ailleurs. Alors, il existe une infinité de rayons STT ayant Δ pour ensemble stable d'arcs.

Proposition. Soit p un nombre dyadique de l'intervalle unité et soit $S(p)$ le sous-ensemble de \mathcal{A} contenant tous les arcs dyadiques qui ont une extrémité en p . Alors, il existe un rayon STT de C^1 dont l'ensemble stable d'arcs est $S(p)$.

Après avoir défini le bord géométrique de C^1 et décrit les différents types de points précédents, on aimerait plonger cet espace dans un espace compact. Pour cela, on s'inspire des méthodes de Thurston. Considérons l'espace fonctionnel suivant :

$$\mathcal{F} = \{f : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0} \cup \infty\}.$$

On remarque que $\mathcal{F} \simeq [0, \infty]^{\mathbb{N}}$ (\mathcal{A} est dénombrable) : \mathcal{F} est alors compact pour la topologie produit.

Soit $\mathcal{I} : C^1 \hookrightarrow \mathcal{F}$ l'application suivante :

1. Étant donnée une triangulation dyadique $\Delta \in C^0$ de type Farey, $\mathcal{I}(\Delta)$ est la fonction intersection géométrique avec Δ . Soient $a \in \mathcal{A}$ et $\Delta = \{a_i\}_{i \in \mathbb{N}} \subset \mathcal{A}$. Alors,

$$\mathcal{I}(\Delta)(a) = \sum_{i=1}^{\infty} i(a_i, a),$$

où

$$i(a, a') = \begin{cases} 1 & \text{si les arcs } a \text{ et } a' \text{ se croisent} \\ 0 & \text{sinon} \end{cases}$$

est le nombre d'intersection géométrique entre a et a' .

2. Étant donné un point p à l'intérieur d'une arête $a\Delta \in C^1$ entre deux triangulations dyadiques $\Delta_1, \Delta_2 \in C^0$ de type Farey, il existe une unique combinaison linéaire de Δ_1, Δ_2 telle que

$$p = s\Delta_1 + (1 - s)\Delta_2, \quad \text{avec } s \in (0, 1/2].$$

Alors, pour tout $a \in \mathcal{A}$ on définit

$$\mathcal{I}(p)(a) = s\mathcal{I}(\Delta_1)(a) + (1 - s)\mathcal{I}(\Delta_2)(a).$$

Proposition. L'application $\mathcal{I} : C^1 \hookrightarrow \mathcal{F}$ est injective.

Soit C_{Th}^1 l'adhérence de $\mathcal{I}(C^1)$ dans \mathcal{F} , qui est compacte pour la topologie induite (c'est un fermé dans un compact). On appelle C_{Th}^1 la compactification de Thurston de C^1 . On note $\partial_{\mathcal{F}}(C^1)$ l'ensemble $C_{Th}^1 \setminus \mathcal{I}(C^1)$, qu'on appelle bord de Thurston de C^1 . On peut voir tous les points du bord géométrique qu'on a décrits précédemment comme des points du bord de Thurston, à l'aide du résultat suivant :

Proposition. Soit $\rho : [0, \infty) \rightarrow C^1$ un rayon STT de C^1 tel que toute composante connexe par arcs de $D \setminus \mathcal{A}_{stab}(\rho)$ est soit un triangle, soit un arc avec une extrémité non-dyadique. Alors, la limite

$$\lim_{s \rightarrow \infty} \mathcal{I}(\rho(s))$$

existe et coïncide avec la fonction intersection géométrique avec $\mathcal{A}_{stab}(\rho)$, qui appartient donc au bord de Thurston $\partial_{\mathcal{F}}(C^1)$ de C^1 .

Qui plus est, on décrit précisément le comportement de la fonction limite ci-dessus suivant le type de point du bord considéré (“region étoilée de Farey”, “pavage de Farey” ou “étoile généralisée”).

Perspectives

Les perspectives de recherche issues de cette thèse se structurent le long de trois axes. Le premier concerne le bord géométrique de C^1 et ses possibles applications aux propriétés de nature “analytique” de T . On a déjà mentionné qu’à tout rayon de type STT (rayons géodésiques compris), on peut associer un objet géométrique ; plus précisément, un sous-ensemble de l’ensemble \mathcal{A} des arcs à extrémités dyadiques du cercle, qu’on nommera *ensemble stable*. Ainsi, deux rayons sont équivalents si leurs ensembles stables coïncident. L’action de T s’étend naturellement au bord géométrique. Les points du bord appartenant à $\partial_0 C$ correspondent aux sous-ensembles de \mathcal{A} tels que toute composante connexe par arcs de leur complémentaire est un triangle ou bien un arc. Cette partie $\partial_0 C$ du bord géométrique est invariante par l’action de T . Je voudrais d’abord déterminer tous les différents types de sous-ensembles de l’ensemble d’arcs dyadiques qui peuvent être atteints en tant qu’ensembles stables. Une fois le bord géométrique entièrement décrit, j’envisage de donner une classification des éléments de T inspirée de la classification de Thurston [76, 28] des difféomorphismes des surfaces, en utilisant l’action de T sur ce bord.

L’interprétation géométrique du 1-squelette C^1 en termes de triangulations dyadiques du disque et le fait que \mathcal{A} soit dénombrable permettent de plonger C^1 dans l’ensemble $\mathcal{F} = \{f : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}\}$, qui est compact lorsqu’on le munit de la topologie produit. Cela nous permet de définir une compactification “à la Thurston” de C^1 , c’est-à-dire en prenant l’adhérence de l’image du plongement $C^1 \hookrightarrow \mathcal{F}$. L’action de T s’étend naturellement à la compactification de Thurston de C^1 . Qui plus est, $\partial_0 C$ peut se plonger dans la compactification de Thurston de façon équivariante par rapport à l’action de T . Il est donc légitime de se demander si ce plongement s’étend à tout le bord géométrique et, si la réponse est affirmative, se demander si ce sous-ensemble est dense ou non. On peut s’intéresser aussi à la dimension de la compactification de Thurston de C^1 .

Une bonne compréhension de l’action de T sur la compactification de Thurston pourrait nous permettre d’étudier l’action de F sur ce même espace. Le groupe F est le plus petit, et probablement le plus connu, des trois groupes de Thompson classiques, et cela fait plus de 30 ans que la communauté mathématique se demande si ce groupe est moyennable ou non. Ainsi, l’étude des points fixes par l’action de F sur cet espace compact pourrait être une approche du problème de la moyennabilité de F . Cette construction de bord permettrait aussi de tenter d’adapter la preuve de l’exactitude des groupes hyperboliques aux groupes F et T (qui ne sont pas hyperboliques). Pour en finir avec l’aspect analytique, il serait par ailleurs légitime de se demander quelles autres définitions de bord à l’infini et quelles autres compactifications sont envisageables. Dans cette optique, il serait déjà intéressant d’arriver à les définir proprement et les comparer.

Le deuxième des axes concerne les *groupes de Thompson généralisés*. Les trois groupes classiques F , T et V de Thompson ont été généralisés par Higman, Brown, Stein, Röver

et d'autres. Par exemple, le groupe V de Thompson appartient à la famille de groupes $G_{n,r}$ introduite par Higman [43]; V correspond au cas particulier $n = 2$ et $r = 1$. Les trois groupes classiques de Thompson peuvent être inclus les uns dans les autres comme suit : $F \subset T \subset V$. De ce point de vue, Brown a introduit [12] des familles $F_{n,r} \subset T_{n,r} \subset V_{n,r}$, qui restent des groupes infinis, finiment présentés et simples. Stein a généralisé encore ces familles [75]. De plus, Greenberg et Sergiescu [38], Brin [8, 9], et Dehornoy [22] ont défini des versions tressées des différents groupes de Thompson généralisés.

Il semble donc naturel de se demander lesquelles des propriétés et définitions utilisés dans le cadre de ma thèse ont de bons analogues dans les groupes de Thompson généralisés et leurs tressages. Par exemple, existe-t-il une notion équivalente à celle de “ $PSL(2, \mathbb{Z})$ par morceaux” dans le cas des groupes de type T de Brown ? Et, si la réponse est affirmative, quels groupes peut-on plonger quasi-isométriquement dans ces groupes ? Ghys et Sergiescu ont montré que T peut se plonger dans le groupe des difféomorphismes du cercle de classe \mathcal{C}^∞ . Une caractérisation des groupes généralisés de type T ressemblant à celle de “ $PSL(2, \mathbb{Z})$ par morceaux” pourrait aider à trouver des plongements des groupes généralisés dans le groupe des difféomorphismes du cercle. Le lien entre la version combinatoire et la version “ $PSL(2, \mathbb{Z})$ par morceaux” s'est avéré utile dans l'étude des propriétés métriques des sous-groupes [29]. Serait-il possible d'utiliser ce lien pour aborder d'autres problèmes de plongement de sous-groupes ? Il est par ailleurs naturel de se demander si l'on peut adapter le résultat sur les classes de conjugaison dans ce cas.

De même, on pourrait s'intéresser aux différents complexes que l'on peut obtenir à partir de C en relaxant certaines propriétés de régularité. Par exemple, qu'obtient-on si on prend des pavages par carrés à la place de triangulations ? Cela permettrait des définitions de complexes généralisant l'associaèdre infini C , qui a pour groupe d'automorphismes une extension de T . Ainsi, l'étude de leurs groupes d'automorphismes donnerait lieu à de nouvelles généralisations des groupes de Thompson et on pourrait étudier leurs ressemblances avec les groupes de Thompson classiques et leurs différentes généralisations connues. Dans cette même direction, le passage à la dimension supérieure serait aussi intéressant. Par exemple, soit X un complexe simplicial de dimension 3 construit par récurrence à partir d'un tétraèdre X_0 en recollant, successivement, un tétraèdre à chaque face du bord de X_n . On considère le groupe des homéomorphismes de X qui préserve cette décomposition tétraédrale en dehors d'un compact. Est-il relié aux groupes de Thompson généralisés de type T ? Et, en cas de réponse affirmative, cela a-t-il un rapport avec les espaces hyperboliques de dimension supérieure et leurs groupes d'isométries ?

Le troisième et dernier des axes porte sur l'aspect *modulaire* du groupe T de Thompson. Cette version de Funar-Kapoudjian-Sergiescu a créé un lien fort entre le groupe T et les groupes modulaires. Ainsi, le résultat sur le groupe des automorphismes de C peut être interprété comme étant un analogue des résultats d'Ivanov décrivant les groupes modulaires comme groupes d'automorphismes de complexes de courbes. Il semble donc intéressant d'établir une carte de ressemblances et de différences entre le groupe de Thompson T et les groupes modulaires. Dans ce cadre, on peut s'intéresser à des problèmes de nature géométrique comme, par exemple, se demander si le groupe T est relativement hyperbolique, ou si tout plongement d'un réseau dans le groupe T a une image finie (voir les travaux de Navas [65]) .

On peut aussi utiliser l'approche modulaire pour des problèmes de nature combinatoire ou algébrique. Par exemple, T vérifie-t-il la propriété d'“(asymptotic) combability” ? T est-il automatique ? On pourrait se demander s'il existe $N \in \mathbb{N}$ tel que tout élément de T s'écrit comme produit d'au maximum N éléments de torsion. Ceci n'est pas le cas pour les groupes modulaires en genre plus grand que 2. Ou se demander si l'on peut trouver $g_1, \dots, g_k \in T$ (d'ordre infini) tels que tout élément s'écrit comme $g_1^{n_1} \dots g_k^{n_k}$. Ceci n'est pas non plus le cas pour les groupes modulaires en genre au moins deux, à cause de l'existence de beaucoup de quasi-morphismes non-triviaux.

Finalement, ce n'est pas impensable que la version modulaire permette de définir une sorte de théorie de Teichmüller pour T . Les premiers pas dans cette direction se trouvent dans les travaux de Greenberg [37], où l'on trouve un lien entre les groupes de Thompson de type F et certains espaces d'empilements finis de cercles (bracelets). Ce lien est repris par Martin [61, 62] et Sergiescu [69], notamment dans le cadre de la cohomologie. Laget [54] l'a aussi utilisé dans sa thèse pour introduire des nouvelles généralisations des groupes de Thompson. Ce point de vue se sert de l'action de $PSL(2, \mathbb{R})$ sur le disque de Poincaré pour faire agir F sur cet espace de bracelets. De plus, il faut mentionner que cette action et les différents espaces introduits sont en relation avec l'espace de Teichmüller universel défini par Penner [66] et font intervenir également des associaèdres. Il me semble intéressant de découvrir une façon d'adapter ces concepts aux groupes de Thompson de type T et de voir quel genre de liens apparaissent entre cette vision par bracelets et les triangulations du disque.

Chapter 1

Introduction

This PhD thesis is embedded in the field of geometric group theory and deals with Thompson's group T . Roughly speaking, in geometric group theory one obtains information of groups by studying the properties of the spaces where they act nicely. The field became extremely active in the decade of 1980 thanks to the work of M. Gromov on hyperbolic groups (see, for example, [40, 39]).

Our goal is to study different aspects of T by comparing multiple viewpoints of the same object and their interactions. For example, we are able to find a non distorted subgroup of T isomorphic to the free non abelian group of rank 2 by using both the combinatorial version of T in terms of equivalence classes of tree pair diagrams, and Thurston's approach of T in terms of piecewise $PSL_2(\mathbb{Z})$ homeomorphisms of the real projective line. Furthermore, we study the action of T on an infinite dimensional cellular complex C that can be seen as a generalisation of the associahedron for an infinitely sided convex polygon. In particular, the two dimensional skeleton of C was introduced by Funar and Kapoudjian, and can be embedded into the pants complex of a surface Σ of infinite type.

First, the complex C allow us to present T as an automorphism group which is not unexpected since T is simple (see Haglund and Paulin results in the simplicity of automorphism groups [41]). Unfortunately, the metric 1-skeleton of C is not Gromov hyperbolic. However, it can be geometrically represented in terms of dyadic (almost) triangulations of Poincaré's disk. This geometric interpretation yields to the definition of a boundary at infinity of C^1 , which is strongly inspired by Gromov's definition of the hyperbolic boundary as equivalence classes of geodesic paths. Finally, we adapt Thurston's methods on the study of homeomorphisms of surfaces to embed C^1 into a compact functional space and take the closure of its image. We have been able to construct explicitly a subset of the geometric boundary which is invariant by T . Moreover, we can "naturally" embed this subset into the Thurston's boundary in an equivariant manner. However, we do not have a complete understanding of the geometric boundary.

This introductory Chapter contains some explanations about the nature and the history of the objects treated later on, as well as a brief context for the results of this PhD thesis. The last section is much more technical; it contains the main results and all definitions needed to state them precisely.

1.1 State of the art

We do not pretend to summarise here all the existent results about the topics playing a role on this PhD thesis, but to present those that are relevant and will be strongly used further on this document.

About Thompson's group T

The main object of study of this PhD thesis is Thompson's group T , which is one of the three classical Thompson's groups F , T and V introduced by Richard Thompson in 1965 (see [18] and [3] for good introductions). These groups are finitely presented infinite groups and have a huge amount of common properties with infinitely generated groups, which is rare for groups of finite type. This explains why Thompson groups have been used to build counterexamples to numerous conjectures in group theory involving finitely presented infinite groups. For example, T and V are the first examples of finitely presented infinite simple groups and were used by McKenzie and Thompson in [64] to construct finitely presented groups with non solvable word problem. It is also worth to mention that Thompson's groups have remarkable connections with other mathematical domains as logic [64, 11], homotopy theory [12, 75, 13], representation theory [51], algebra [9, 12], dynamics [7, 36, 56], topology [33, 32, 31] and cryptography [71].

Thompson's group T is usually seen as the subgroup of $\text{Homeo}^+(S^1)$ corresponding to piecewise linear homeomorphisms of the circle $S^1 = [0, 1]/_{0 \sim 1}$ being differentiable except at finitely many dyadic rational numbers, having for derivatives only powers of 2, and fixing the set of dyadic rational numbers. From this point of view, the main interest of T is that its behaviour as a subgroup of $\text{Homeo}(S^1)$ has similarities with the behaviour of lattices in Lie groups (see [36] and [69]). In some general sense, this means that T seems to be a good discrete analogue of the group $\text{Homeo}(S^1)$. For example, both are simple groups (see [18] for a proof of the simplicity of T and [35] for the simplicity of homeomorphisms of the circle). Furthermore, $\text{Out}(T)$ is isomorphic to the cyclic group of order 2 (see [7]), which is the group of outer automorphisms of the circle (see [35]).

The second most used approach to Thompson group T is the combinatorial one (see [18]), where T is presented as the group of equivalent classes of tree pair diagrams. A tree pair diagram is a pair of finite rooted binary trees with the same number of leaves, with a cyclic numbering system pairing the leaves in the two trees. Two different tree pair diagrams are equivalent if they have a common expansion, and in every equivalence class of tree pair diagrams there is a unique reduced representative. This approach is very useful for deducing metric properties. For example, Burillo, Cleary, Stein and Taback established in [16] that the number of carets of the reduced tree pair diagram gives a good estimate of the length on the Cayley graph. The combinatorial version has also been used by Burillo, Cleary, Rover, Stein and Taback to study subgroup embeddings, commensurators and distortion in Thompson's groups [6, 14, 15, 16].

We still need to present two other viewpoints of T . The first one is due to Thurston and connects T with the projective special linear group $PSL_2(\mathbb{Z})$ (see, [61], [46] and [69]). Thompson's group T is isomorphic to the group $PPSL_2(\mathbb{Z})$ of orientation preserving

homeomorphisms of the real projective line $\mathbb{R}P^1$ which are piecewise $PSL_2(\mathbb{Z})$ and have a finite number of non differentiable points, all of them being rational numbers (for a sketch of the proof see [46] and [69]). This approach has been used by Sergiescu in [69] to explain that T can also be embedded in the group of diffeomorphisms of the circle of class \mathcal{C}^1 .

The other version we would like to explode in this PhD thesis is due to Funar, Kapoudjian and Sergiescu [34]. They present Thompson's group T (also some of the braided generalizations of T and V) as an asymptotically rigid mapping class group of a connected, non compact, planar surface with infinitely many boundary components. To give an idea of what does asymptotically rigid mapping class group mean, one can think about it as homotopy classes of homeomorphisms which preserve a given tessellation of the surface outside a compact subsurface and its image.

The main tool for proving the equivalence between the first and the second definitions is the infinite rooted binary tree and its interpretation as standard dyadic subintervals of $[0, 1]$ (see [18]). The tool for the equivalence between piecewise linear homeomorphisms of the circle and $PPSL_2(\mathbb{Z})$ is Minkowski's question mark function, which sends rational numbers to dyadic numbers and Farey medians to midpoints (see [1] and [67]). The equivalence between the asymptotically rigid mapping class group and the other definitions uses the embedding of the 3-regular infinite tree into the disc and the associated Farey type dual triangulation (see [31]).

Funar, Kapoudjian and Sergiescu use the asymptotically rigid mapping class group point of view to extend the Burau representation of the braid group B_∞ , and also to construct cellular complexes where Thompson group acts by automorphisms (see [31, 32, 33, 34, 51]). This complexes can be seen as generalizations of the pants complex for surfaces of infinite type. It is worth to mention a few other cellular complexes where T was proved to act nicely on. Brown [12], Brown and Geoghegan [13], and Stein [75] use the action of T in complexes of basis of Jonsson-Tarski algebras (see [4]) to obtain, respectively, finiteness properties and homological properties. Farley uses diagram groups to construct CAT(0) cubical complexes where Thompson's groups act (see [25, 27, 24, 26]).

Finally, we also should say that there are other interesting representations of Thompson's groups that will not be treated on this PhD thesis. For example, the one based on piecewise linear homeomorphisms of the real line (see [75] and [11]) together with the combinatorial associated version with infinite binary trees, and the viewpoint coming from diagram and picture groups (see [25]).

About Thurston's theory on surfaces

Let M be an oriented hyperbolic surface of genus g with b boundary components. Let $\mathcal{S}(M)$ be the set of isotopy classes of essential simple closed two-sided curves on M , and consider the symmetric map

$$i : \mathcal{S}(M) \times \mathcal{S}(M) \longrightarrow \mathbb{Z}_{\geq 0},$$

where $i(\gamma, \gamma')$ is the minimum of the geometric intersection numbers between an element of the class γ and an element of the class γ' . The results of this subsection were introduced

in [76], and the details can be found in [28] and [19].

Let $\mathbb{R}_{\geq 0}^{\mathcal{S}}$ denote the set of applications from \mathcal{S} to the positive real numbers $\mathbb{R}_{\geq 0}$, and consider the weak topology on it. Let $P\mathbb{R}_{\geq 0}^{\mathcal{S}}$ be the corresponding projective space with the quotient topology. Thus, we have

$$\mathcal{S} \xrightarrow{i_*} \mathbb{R}_{\geq 0}^{\mathcal{S}} - \{0\} \xrightarrow{\pi} P\mathbb{R}_{\geq 0}^{\mathcal{S}},$$

where $i_*(\gamma')(\gamma) = i(\gamma, \gamma')$. Thurston proved that the map $\pi \circ i_*$ is injective. Then, it is natural to consider the closure of $\pi \circ i_*(\mathcal{S})$ into $P\mathbb{R}_{\geq 0}^{\mathcal{S}}$, giving a non trivial topology to the set \mathcal{S} . This completion is denoted by $\overline{\mathcal{S}}(M)$, and Thurston proved that:

1. $\overline{\mathcal{S}}(M)$ is topologically a sphere of dimension $6g + 2b - 7$, and
2. The set $\mathcal{S}(M)$ is dense on $\overline{\mathcal{S}}(M)$.

Thurston also defined a similar construction for the set of equivalence classes of measured foliations.

Let H be the set of metrics on M of constant curvature $K = -1$ such that each boundary component is geodesic. The group of diffeomorphisms of M isotopic to the identity provided with the \mathcal{C}^∞ topology acts freely and continuously on H . The Teichmüller space $\mathcal{T}(M)$ of M can be defined as the space of orbits of this action. A classical result due to Fricke and Klein states that

$$\mathcal{T}(M) \simeq \mathbb{R}^{6g+2b-6}.$$

Thus, using Thurston's construction with measured foliations, the space $\overline{\mathcal{S}}(M)$ is a boundary of the Teichmüller space of M on which the action of the group of diffeomorphisms of M can be extended. Furthermore, the curve complex can be embedded into Thurston's boundary of the Teichmüller space of M , and the pants complex is quasi isometric to the Teichmüller space of M with the Weil-Peterson metric.

The previous results imply that $\mathcal{T}(M) \cup \overline{\mathcal{S}}(M)$ is a disk of dimension $6g + 2b - 6$. Hence, thanks to Brower's fixed point Theorem, every diffeomorphism φ of M fixes a point on $\mathcal{T}(M) \cup \overline{\mathcal{S}}(M)$. The nature of this fixed point gives the three different options of Thurston's classification Theorem on diffeomorphisms of surfaces:

1. Either φ is isotopic to a periodic diffeomorphism,
2. Or φ is isotopic to a reducible diffeomorphism,
3. Or φ is a Pseudo-Anosov diffeomorphism.

This construction can be adapted to the two dimensional torus. Then, Thurston's Classification Theorem gives the classification of elements in $PSL_2(\mathbb{Z})$ as elliptic, parabolic and hyperbolic.

Another important object on the study of surfaces is the mapping class group. Given a compact surface M of genus g with b boundary components, the mapping class group of M is the group of homeomorphisms of M modulo isotopies. Mapping class groups have been largely studied using their actions by automorphisms on locally infinite simplicial complexes; mainly the already cited curve complex and pants complex, the arc complex, and the Hatcher-Thurston complex. Some of these complexes are hyperbolic for $n + b$

large enough (see [63]), and their automorphism groups are related to the mapping class group itself.

Specifically, Ivanov [50] and subsequently Luo [58] proved that the automorphism group of the curve complex is the extended mapping class group for almost all compact surfaces, and Korkmaz extended this result to punctured spheres and punctured tori in [52]. Analogous results were proved for the arc complex by Ivanov [50] and Irmak and McCarthy [49], for the pants complex by Margalit [60], for the Hatcher-Thurston complex by Irmak and Korkmaz [48], for the arc and curve complex by Korkmaz and Papadopoulos [53], and for the complex of non-separating curves by Schmutz-Schaller [68] and subsequently by Irmak [47].

1.2 Structure, motivations and perspectives

The second Chapter of this PhD thesis is devoted to introduce four different viewpoints of Thompson’s group T that will be relevant to further chapters. We define and give examples of the classical approach as piecewise linear homeomorphisms of the circle, the combinatorial one on equivalence classes of tree pair diagrams, the version consisting on “piecewise $PSL_2(\mathbb{Z})$ ” homeomorphisms of the projective line, and the asymptotic mapping class group of a planar surface. A sketch of how to prove the equivalences between all of them is also included.

The third Chapter deals with the interaction between the “piecewise $PSL_2(\mathbb{Z})$ ” and the combinatorial approaches of T . Thanks to the $PPSL_2(\mathbb{Z})$ version of Thompson’s group T , it is natural to consider the group $PSL_2(\mathbb{Z})$ as a subgroup of T , and Sergiescu wondered how to characterise its elements. The answer to Sergiescu’s question is the main point of this chapter and can also be found in [29], where the combinatorial viewpoint of Thompson’s group T is used to explicitly describe the elements of the embedding $PSL_2(\mathbb{Z}) \hookrightarrow T$. It turns out that all the reduced tree pair diagrams representing this elements have exactly one leaf and one interior vertex for each level. In fact, we provide a bijection between tree pair diagrams of elements in $PSL_2(\mathbb{Z})$ and the corresponding normal forms on the classical generating system of $PSL_2(\mathbb{Z})$ consisting of an element of order 2 and another one of order 3 (see [44]).

This characterization of $PSL_2(\mathbb{Z})$ in terms of tree pair diagrams together with the results of Burillo, Cleary, Stein and Taback on the word metric of T (see [16]) allow us to prove that $PSL_2(\mathbb{Z})$ is a non distorted subgroup of Thompson’s group T . Although it was already known that T has subgroups isomorphic to the free non abelian group of rank 2 (this is the usual way to prove the non amenability of T and the classical proofs use the Ping-Pong lemma), it was not known if these subgroups were distorted. The piecewise projective approach together with our results give an easy example of a non distorted subgroup of T isomorphic to F_2 . There are several normal subgroups of $PSL_2(\mathbb{Z})$ isomorphic to the free non abelian group of rank two. One particular example is the commutator group of $PSL_2(\mathbb{Z})$ (see [21], page 26 or [59] proposition III.12.3). In addition, this subgroup has finite index on $PSL_2(\mathbb{Z})$. Using the fact that finite index subgroups are non distorted and our result that $PSL_2(\mathbb{Z})$ is non distorted as a subgroup

of Thompson’s group T , we have constructed a non distorted subgroup of Thompson’s group T isomorphic to the free non abelian group of rank 2.

Chapter four deals with the number of conjugacy classes of torsion elements in T . We prove that there are exactly $\varphi(n)$ conjugacy classes of elements of order n , where φ is the Euler function. For doing so, first the combinatorial and the asymptotic mapping class group approaches are mixed to reformulate the notion of tree pair diagram in terms of Farey polygons. Both representations are strongly based on the action of T on the set of dyadic rational numbers, coming from the action at infinity of the binary tree. Then, a more general geometric interpretation of elements of T inspired by Farey polygons is introduced. Thus, one can represent T in terms of dyadic triangulations of the disk of Farey type and maps between pairs of inscribed polygons. In this version of T , the elements of order n are given by two dyadic n -gons together with a cyclic numbering system pairing their sides. A posteriori, we realised that this result about the number of conjugacy classes of torsion elements can also be obtained from the characterisation of torsion elements of T given Burillo, Cleary, Stein and Taback in [16].

There is a result due to Lochak and Schneps [57] stating that T can be generated by two torsion elements α (of order 4) and β (of order 3). Thus, it is natural to wonder if one can find a direct proof of Brin’s theorem $Out(T) = \mathbb{Z}/2\mathbb{Z}$ [7] by using the result about the number of conjugacy classes and the fact that T is the group of orientation preserving isometries of a cellular complex C (see [30]). This remains an open question.

In the fifth Chapter we generalise to higher dimensions the 2-dimensional cellular complex C introduced by Funar and Kapoudjian [31]. We prove that T acts on it by automorphisms, and its action is transitive on vertices and edges. Furthermore, C is presented as an analogue to the associahedra of convex polygons [73, 74, 37, 5, 55, 45, 72] in the infinitely-sided case. For this, we use the concepts of dyadic triangulation of Farey type and dyadic almost triangulation of Farey type, from Chapter 4. This geometric viewpoint is used in Chapter 7 to define a geometric object for every “nice infinite path” on the 1-skeleton of C .

As we have aforementioned, given a connected (orientable) compact surface one can build various locally infinite simplicial complexes on which the mapping class group of the surface acts by automorphisms. When dealing with surfaces of infinite type, one issue is to find good analogues for mapping class groups. It is also hard to define complexes similar to those enumerated on the previous section. In this direction, the 2-dimensional skeleton C^2 of the cellular complex C introduced in Chapter 5 can be seen as a part of the pants complex of an infinite surface of genus zero Σ (see [31, 30]).

The automorphism group of the infinite associahedron C is proved to be isomorphic to $T \rtimes \mathbb{Z}/2\mathbb{Z}$ in Chapter six. The infinite associahedron C is thus a contractible cellular complex “optimal with respect to T ” in the sense that its automorphism group is “essentially” T itself. With M. Nguyen we proved [30] that the automorphism group of its 2-skeleton already is isomorphic to the extension of T by $\mathbb{Z}/2\mathbb{Z}$. When dealing with the 2-dimensional skeleton, this semi-direct product can be geometrically interpreted as the extended asymptotic mapping class group of Σ , obtaining in this way a complete analogue to Ivanov’s theorem for the surface Σ of infinite type. However, the 2-skeleton is not contractible, and the restriction to dimension 2 seems “artificial”, meaning that it does not

come from the group T itself but from the topological analogue to modular groups and pants complexes of Funar and Kapoudjian [31]. Thus, in our results the 2-dimensional complex C^2 is not presented in terms of dyadic triangulations (which is the key point to define a higher dimensional complex) but more as a complex coming from elementary moves (flips) and their relations.

This result about the automorphism group of the infinite associahedron can also be considered from a different viewpoint. In 1970, Tits [77] proved that most subgroups of automorphism groups of trees generated by vertex stabilizers are simple. This result was extended by Haglund and Paulin [41] to the case of automorphism groups of negatively curved polyhedral complexes which are (in general uncountable) non-linear and virtually simple. This leads to the legitimate question of realizing the simple groups T and V as automorphisms groups of suitable cell complexes. A partial answer to this question was given by Farley in [25] and [27], where he proved that Thompson's groups F , T and V act properly and isometrically on CAT(0) cubical complexes. In particular, these groups are a-T-menable and satisfy the Baum-Connes conjecture with arbitrary coefficients. Although the complex C is not Gromov hyperbolic, it gives a convenient answer to the question for the group T .

Finally, in the seventh Chapter we define a geometric boundary at infinity for the 1-skeleton of C by considering a class of infinite paths containing geodesics and mapping each one of them to a subset of the set of dyadic arcs. Furthermore, we try to adapt Thurston methods on the construction of a compactification of the 1-skeleton C^1 . Even though we are far from the complete understanding of this compactification, it is worth to mention that all points on the geometric boundary at infinity find to the date are naturally seen on the compactification.

To give a few hints on prospectives, it must be mentioned that the most famous open problem on Thompson's groups deals with the amenability (or non-amenability) of F . On this direction, the eventual understanding of the compact boundary at infinity of the cellular complex C could provide an extension of the action of T (and consequently also of F) to this border. One can then study the existence of fixed points by this action and, eventually, come up through this approach to the problem of the amenability. This viewpoint can also be used on the study of exactness of Thompson's groups F and T . On a different direction but still involving the cellular complex C , there seems to exist some hidden relation between Penner's Universal Teichmüller space [66] and C . It would be interesting to understand the nature of this interaction.

Another important fact is that the three classical Thompson groups have been included in bigger families of groups by Higman [43] and Brown [12], and then generalised by Stein [75], and Brin [8, 9], Brin and Guzmán [10] and others. Furthermore, there exist braided versions of all generalised Thompson groups initially defined by Greenberg and Sergiescu [38], and later by Dehornoy [22].

It is thus natural to ask if the geometric approach to Thompson's group T in terms of triangulations of the disk of Farey type and inscribed polygons can be adapted to all this different generalisations of classical Thompson's groups of type T . In addition, one can construct cellular complexes from regular tessellations of the disk and study their automorphism groups. By analogy with our result, this groups can be thought as

generalisations of Thompson's groups and one could wonder whether they coincide with already known generalisations or not.

1.3 Definitions and statements of main results

$PSL(2, \mathbb{Z})$ and F_2 as non-distorted subgroups of T

The *piecewise linear Thompson's group* T is the group of orientation-preserving piecewise linear homeomorphisms of the circle S^1 , thought as the unit interval with identified endpoints, that map the set $\mathbb{Z}[1/2] \cap [0, 1]$ of dyadic rational numbers to itself, are differentiable except at finitely many points, all non-differentiable points are dyadic rational numbers, and on intervals of differentiability, the derivatives are powers of 2.

A *tree pair diagram* is a triple (R, σ, S) , where R and S are finite rooted binary trees with the same number n of leaves, and σ is a bijection between the set of leaves of R and the set of leaves of S . The binary tree R is called the *source tree*, and S is called the *target tree*. An internal vertex together with its two downward descendants is called a *caret*. A caret is *exposed* if it contains two leaves of the tree. A tree pair diagram is *reduced* if, for all exposed carets of S , the images under σ of their leaves do not form an exposed caret of R . A tree pair diagram is *cyclic* if σ is a cyclic permutation. In this case, the i -th leaf of the source tree (considering the left-to-right order on its leaves) is sent to the $\sigma(i)$ -th leaf of the target tree.

Given a tree pair diagram (R, σ, S) , its j -th *simple expansion* is obtained by splitting the j -th leaf of R (adding the caret containing the leaf j and its descendants) and the $\sigma(j)$ -th leaf of S , and expanding the bijection σ by mapping the left (right) descendant of j on R to the left (right) descendant of $\sigma(j)$ on S . Every tree pair diagram obtained from (R, σ, S) by finitely many simple expansions is called an *expansion* of (R, σ, S) . Two tree pair diagrams are *equivalent* if they have a common expansion. The *combinatorial Thompson's group* T is the set of equivalence classes of tree pair diagrams.

The *piecewise projective Thompson's group* T ($PPSL_2(\mathbb{Z})$ for short) is the group of orientation preserving homeomorphisms of the real projective line $\mathbb{R}P^1$ which are piecewise $PSL_2(\mathbb{Z})$ and have a finite number of non differentiable points, all of them being rational numbers.

The projective special linear group $PSL_2(\mathbb{Z})$ is isomorphic to the free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. Then, every element of $PSL_2(\mathbb{Z})$ has a normal form in the generators a ($a^2 = 1$) and b ($b^3 = 1$) given by a word

$$a^{\epsilon_1} b^{\delta_1} a b^{\delta_2} a \dots a b^{\delta_k} a^{\epsilon_2},$$

where k is a non negative integer, $\epsilon_1, \epsilon_2 \in \{0, 1\}$ and $\delta_1, \dots, \delta_k \in \{-1, 1\}$. We will denote by a and b both the generators of $PSL_2(\mathbb{Z})$ as a group of matrices and their images in T under the isomorphism between $PPSL_2(\mathbb{Z})$ and Thompson's group T .

A finite rooted binary tree is *thin* if all its carets have one leaf and one internal vertex except for the last caret, which is exposed. It is easy to see that one can characterize thin trees with n leaves by associating a weight $r_i \in \{-1, 1\}$ to each one of its $n - 2$ internal

vertices. Let v_0, \dots, v_{n-3} be the internal vertices of a thin tree. Denote by v_{-1} its root. Then, $r_i = 1$ if v_i is the left descendant of v_{i-1} and $r_i = -1$ if v_i is the right descendant of v_{i-1} .

Theorem A. *Let (R, σ, S) be the reduced tree pair diagram of an element t of Thompson's group T . Then, t belongs to the subgroup $PSL_2(\mathbb{Z})$ if and only if (R, σ, S) satisfies one of the following conditions:*

1. *the number of leaves of R and S is less than 4; or*
2. *R is a thin tree with associated weights r_0, \dots, r_{k-1} , S is a thin tree with associated weights s_0, \dots, s_{k-1} , and the associated weights of R and S verify the equations:*

$$(i) \sum_{i=2}^{k-1} r_i s_{k+1-i} = 2 - k, \text{ and}$$

$$(ii) l + \sigma(1) + \epsilon(s_0) \equiv \frac{3 - s_1}{2} \pmod{k + 2},$$

where $l - 1$ is the cardinal of the set $\{i : r_i = -1, 0 \leq i \leq k - 1\}$, and

$$\epsilon(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x = -1. \end{cases}$$

The following Proposition proves the 'if' part of the main theorem. For the 'only if' part it suffices to show that the number of solutions to the Equations (i) and (ii) coincides with the number of words $w(a, b) = a^{\epsilon_1} b^{\delta_1} a b^{\delta_2} a \dots a b^{\delta_k} a^{\epsilon_2}$, with $\epsilon_1, \epsilon_2 \in \{0, 1\}$ and $\delta_1, \dots, \delta_k \in \{-1, 1\}$.

Proposition. *Let $w(a, b)$ be a word in normal form with respect to the generating system $\{a, b\}$ of $PSL_2(\mathbb{Z})$:*

$$w(a, b) = a^{\epsilon_1} b^{\delta_1} a b^{\delta_2} a \dots a b^{\delta_k} a^{\epsilon_2},$$

where $\epsilon_1, \epsilon_2 \in \{0, 1\}$ and $\delta_1, \dots, \delta_k \in \{-1, 1\}$. Assume that $k \geq 2$. Let R be the thin tree defined by the weights

$$r_0 = \epsilon^{-1}(\epsilon_1) \text{ and} \\ r_i = \delta_i, \forall 1 \leq i \leq k - 1,$$

and let S be the thin tree defined by the weights

$$s_0 = \epsilon^{-1}(\epsilon_2) \text{ and} \\ s_i = -\delta_{k+1-i}, \forall 1 \leq i \leq k - 1.$$

Let σ be the cyclic permutation defined by the equation

$$\sigma(1) \equiv \frac{3 - s_1}{2} - \epsilon(s_0) - l \pmod{k + 2},$$

where l and ϵ are those from Theorem A.

Then, the reduced tree pair diagram of w is (R, σ, S) . Furthermore, the weights r_0, \dots, r_{k-1} and s_0, \dots, s_{k-1} satisfy Equations (i) and (ii) of Theorem A.

Using Theorem A and the previous Proposition we can estimate the length of the elements in this subgroup. Furthermore, Burillo, Cleary, Stein and Taback gave an estimation of the length of elements in T in the number of carets in [16]. As a consequence of both results, we obtain:

Proposition. *The group $PSL_2(\mathbb{Z})$ is a non distorted subgroup of Thompson's group T .*

Corollary. *Let $\{a, b\}$ be the standard generating set of $PSL_2(\mathbb{Z})$ as a subgroup of Thompson's group T . Then, the subgroup $H = \langle abab^{-1}, ab^{-1}ab \rangle$ is a free non abelian group of rank 2 and it is non distorted in T .*

Counting conjugacy classes of torsion elements on T

Let D denote the closed disk of \mathbb{R}^2 of center $(0, 0)$ and perimeter 1. Suppose its boundary S^1 arc-parametrized by the unit interval $[0, 1]$. The set of segments joining two different dyadic boundary points of D

$$\Delta_F = \{(0, 1/2)\} \cup \left\{ \left(\frac{m}{2^n}, \frac{m+1}{2^n} \right) : m, n \in \mathbb{Z}_{>0}, m \in \{0, \dots, 2^n - 1\}, n > 1 \right\}$$

is called the set of *Farey arcs* and it is a triangulation of D in the sense that $D - \Delta_F$, on the interior of D , is a disjoint union of open triangles. The triangulation defined by Δ_F is called *Farey's dyadic triangulation* of D .

A *dyadic triangulation of Farey type* is a triangulation of the interior of D obtained from Δ_F by taking off a finite number of Farey arcs and replacing them for other non intersecting segments joining two different dyadic boundary points of D . Let Δ be a dyadic triangulation of D of Farey type, and let a_1, \dots, a_k be dyadic arcs of Δ . We say that a_1, \dots, a_k define a Δ -*polygon* if they bound a k -gon $\mathcal{P}(a_1, \dots, a_k)$ in D . A *dyadic almost triangulation of Farey type* of D is the object $a\Delta$ obtained from a dyadic triangulation of Farey type Δ by erasing a finite number $\mathcal{D}(a\Delta)$ of dyadic segments.

Lemma. *Let Δ be a dyadic triangulation of D of Farey type, let $\mathcal{P}(a_1, \dots, a_k)$ and $\mathcal{P}(a'_1, \dots, a'_k)$ be Δ polygons, and let σ be a cyclic permutation of the set $\{1, \dots, k\}$. Then, the quadruple $(\Delta, \mathcal{P}(a_1, \dots, a_k), \mathcal{P}(a'_1, \dots, a'_k), \sigma)$ represents a unique element t of Thompson's group T , meaning that the polygon $\mathcal{P}(a_1, \dots, a_k)$ is sent to the polygon $\mathcal{P}(a'_1, \dots, a'_k)$ by t . Moreover, t maps the arc a_i to the arc $a'_{\sigma(i)}$. Thus, the element t is defined on the dyadic numbers corresponding to the extremal points of a_1, \dots, a_k by this bijection of polygons, and the images of all other dyadic rational numbers are defined recursively by saying that the triangulation Δ is preserved outside $\mathcal{P}(a_1, \dots, a_k)$ and $\mathcal{P}(a'_1, \dots, a'_k)$.*

We derive from the previous result the following characterization of torsion elements:

Proposition. *Let t be an element of Thompson's group T of order n . Then, there exist a dyadic triangulation Δ of Farey type, a Δ polygon $\mathcal{P}(a_1, \dots, a_n)$, and a cyclic permutation $\sigma \in S_n$ such that $(\Delta, \mathcal{P}(a_1, \dots, a_n), \mathcal{P}(a_1, \dots, a_n), \sigma)$ is a representation of t , and σ has order n . Moreover, every element of T admitting a representation of this form has order n .*

This viewpoint allow us to count the number of conjugacy classes of torsion elements in T :

Theorem B. *There are exactly $\varphi(n)$ conjugacy classes of elements of T of order n , where φ is the Euler function.*

T and the infinite associahedron C

Let T_3 be the infinite (unrooted) trivalent tree obtained as the dual graph of the triangulation Δ_F of D , and let Σ be the topological surface obtained by gluing a closed neighbourhood without auto-intersections. The surface Σ is planar (thus oriented), non compact, contractible and its boundary has infinitely many connected components. In addition, Σ inherits a tessellation into hexagons from the triangulation Δ_F , where the boundaries of the hexagons alternate Farey arcs with boundary components of Σ .

A set of Farey arcs is *separating* if their union bounds a compact subsurface of Σ . Let $Homeo^+(\Sigma)$ denote the group of orientation-preserving homeomorphisms of Σ . An element $\varphi \in Homeo^+(\Sigma)$ is *asymptotically rigid* if there exist two sets of separating arcs $\{a_1, \dots, a_k\}$, $\{a'_1, \dots, a'_k\}$ such that:

1. for all $1 \leq i \leq k$, $\varphi(a_i \cap \Sigma) = a'_i \cap \Sigma$, and
2. the hexagonal tessellation of Σ is preserved outside the two compact subsurfaces bounded by $\{a_1, \dots, a_k\}$ and $\{a'_1, \dots, a'_k\}$.

Let $Homeo_a(\Sigma)$ denote the group of asymptotically rigid homeomorphisms of Σ . The asymptotic mapping class group of Σ is the quotient $Homeo_a(\Sigma)/Homeo_a^0$, where $Homeo_a^0$ is the group of asymptotically rigid homeomorphisms which are isotopic to the identity, and it is isomorphic to Thompson's group T .

Let n be an integer greater than 4 (for $n = 3, 4$ the notion still make sense, but one obtains degenerate cases) and let P_n denote a convex polygon with n sides. Let $V(P_n) = \{v_1, \dots, v_n\}$ be the set of its vertices, where $\{v_i, v_j\}$ ($i < j$) is a side of P_n if and only if $j = i + 1$ or $i = 1, j = n$. Let D_n be the set of internal diagonals of P_n . The *arc complex* A_n of P_n is the simplicial complex defined by the subsets of D_n which contain 2 by 2 non-crossing diagonals. The dimension of A_n is $n - 4$, and the maximal simplices correspond to triangulations of P_n . Furthermore, A_n is a convex $(n - 3)$ -dimensional polytope. *Stasheff's associahedra* K_n is the dual complex of A_n . Let T_n be the set of triangulations of P_n with exactly $n - 3$ internal diagonals. The set $V(K_n)$ of vertices of K_n is T_n . The set-theoretical definition of the cellular complex K_n is the following: for $1 \leq k \leq n - 3$, a subset K of T_n is a k -cell if it contains all triangulations in T_n which have $n - 3 - k$ internal diagonals in common. It is easy to see that the subset of parts of T_n defining K_n is closed by intersection, and that the intersection of two different k -cells is a cell of strictly lower dimension. Using the fact that the dual of a convex polytope is a convex polytope of the same dimension, one obtains that Stasheff's associahedra K_n is a convex $(n - 3)$ -dimensional polytope.

The *infinite associahedron* is the cellular complex C having the set of dyadic triangulations of Farey type as set of vertices C^0 and, for $n \geq 1$, the set of n -cells can be seen as the set of dyadic almost triangulations of Farey type satisfying $\mathcal{D}(a\Delta) = n$. For doing so, let $a\Delta$ be an almost triangulation of Farey type with $\mathcal{D}(a\Delta) = n > 0$. Let P_1, \dots, P_k be the non triangular polygons of $D - a\Delta$, and for $i \in \{1, \dots, k\}$ let $K(P_i)$ be

the associahedron of a convex polygon whose number of sides coincides with the number of sides of P_i . Then, one can attach to the sub-set $\{\Delta \in C^0 : a\Delta \subset \Delta\}$ of C^0 , an n -cell isomorphic to the direct product $K(P_1) \times \dots \times K(P_k)$, which is topologically a closed ball of dimension n .

Proposition. *The infinite associahedron C is connected, simply connected and contractible. Its 1-skeleton C^1 equipped with the combinatorial metric is not Gromov hyperbolic.*

The piecewise linear version of Thompson's group T shows that T acts on the set of rational dyadic numbers $\mathbb{Z}[1/2]$, thus it is easy to extend this action to the direct product $\mathbb{Z}[1/2] \times \mathbb{Z}[1/2]$. Moreover, every segment joining two dyadic points of the boundary of D is completely determined by its extremal points. This extended action of T can be used to define the action of T on C . The image by $t \in T$ of a dyadic (almost) triangulation of Farey type is still of Farey type, since t has only a finite number of linear pieces, and the derivatives are powers of two.

Proposition. *Thompson's group T acts on the infinite associahedron C by automorphisms. Thus, we have a natural map $\Psi : T \rightarrow \text{Aut}(C)$.*

Proposition. *Thompson's group T acts transitively on vertices and edges of the infinite associahedron C . Furthermore, the map $\Psi : T \rightarrow \text{Aut}(C)$ is injective.*

The methods used to prove the previous propositions can be easily adapted to Thompson's group F , which is the subgroup of elements of T fixing 0. One obtains the next result:

Proposition. *Thompson's group F acts on the cellular complex C by automorphisms. The action is transitive on vertices. Furthermore, the induced map $\Psi : F \rightarrow \text{Aut}(C)$ is injective.*

Theorem C. *The automorphism group of the n -th skeleton of the cellular complex C is isomorphic to the semi-direct product $T \rtimes \mathbb{Z}/2\mathbb{Z}$, for all $n \geq 2$. In particular, $\text{Aut}(C) \cong T \rtimes \mathbb{Z}/2\mathbb{Z}$.*

Moreover, the co-kernel act as a non trivial outer automorphism. Hence, we obtain a non trivial action of $\mathbb{Z}/2\mathbb{Z}$ on $\text{Out}(T)$, which is an isomorphism by Brin's theorem $\text{Out}(T) \simeq \mathbb{Z}/2\mathbb{Z}$.

The proof of Theorem C is based on the fact that every automorphism is determined by the image of the ball of radius one centred in some dyadic triangulation $\Delta \in C^0$ of D of Farey type (when considering the combinatorial metric). Then, given an automorphism φ of the complex C , we construct an element $t \in \Psi(T) \subset \text{Aut}(C^2)$ such that $t(\Delta_F) = \varphi(\Delta_F)$. For this, one introduces an auxiliary complex, which is related to the link of Δ in C^2 . Finally, one can construct a short exact sequence:

$$1 \longrightarrow T \longrightarrow \text{Aut}(C^2) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1,$$

and prove that it splits.

The link structure used to prove Theorem C admits a geometric interpretation leading to a reformulation of Theorem C. It turns out that one can classify the elements of $\text{Aut}(C^2)$ in two categories: those which preserve (in some natural sense) the orientation of the complex, and those which reverse the orientation. Then, it is natural to define the

extended mapping class group of Σ as $T \times \mathbb{Z}/2\mathbb{Z}$, which makes Theorem C a complete analogue to Ivanov's Theorem for compact surfaces.

Adding a boundary to C^1

A *dyadic arc* is a segment joining two dyadic numbers on the boundary of D . One denotes by \mathcal{A} the set of dyadic arcs.

An infinite ray $\rho : [0, \infty) \rightarrow C^1$ is *geodesic* if for all $s_1, s_2 \in [0, \infty)$, $d_{C^1}(\rho(s_1), \rho(s_2)) = |s_1 - s_2|$. Using a result of Sleator, Tarjan and Thurston one obtains:

Lemma. *Let ρ be a geodesic ray on C^1 , let $\{\Delta_n\}_{n \in \mathbb{N}}$ be the sequence of vertices of ρ , and let a be a dyadic arc. Then, $s(a, \rho) = \{s \in \mathbb{N} : a \in \Delta_n\}$ is a subset of consecutive integers (possibly empty or infinite).*

An *STT-ray* on C^1 is a ray $\rho : [0, \infty) \rightarrow C^1$ such that for every dyadic arc a , the set $s(a, \rho)$ is made of consecutive integers (possibly none or infinitely many). Let ρ be an infinite STT-ray of C^1 . The *stable arc set* of ρ is

$$\mathcal{A}_{stab}(\rho) := \{a \in \mathcal{A} : \exists s_a \in \mathbb{N} \mid s(a, \rho) \text{ is infinite}\} \subset \mathcal{A}.$$

Let ρ be an STT-ray on C^1 and let $\{\Delta_n\}_{n \in \mathbb{N}}$ be the ordered set of dyadic triangulations of Farey type appearing in ρ . Recall that every dyadic triangulation is seen as a set of dyadic arcs. Then,

$$A_{stab}(\rho) = \liminf_{n \rightarrow \infty} \Delta_n.$$

Furthermore, by the hypothesis STT on the ray ρ , this lower limit coincides with the upper limit

$$\mathcal{A}_{stab}(\rho) = \limsup_{n \rightarrow \infty} \Delta_n.$$

Thus, the stable arc set is a set-theoretical limit.

Lemma. *Let ρ and ρ' be two different STT-rays of C^1 , and let $\mathcal{A}_{stab}(\rho)$ and $\mathcal{A}_{stab}(\rho')$ be their respective stable arc sets. Suppose that ρ and ρ' coincide on their final segments (there exist $s_0 \in [0, \infty)$ and $r \in \mathbb{R}$ such that for all $s \in [s_0, \infty)$ the equation $\rho'(s) = \rho(s+r)$ is satisfied). Then, $\mathcal{A}_{stab}(\rho') = \mathcal{A}_{stab}(\rho)$.*

Lemma. *Let ρ be an STT-ray of C^1 . Suppose that there exists an arc-connected component of $D \setminus \mathcal{A}_{stab}(\rho)$ bounded by the dyadic arcs $\{a_1, \dots, a_k\} \subset \mathcal{A}_{stab}(\rho)$ ($k > 2$) which is a finite polygon P . Then, P is a triangle.*

Two STT-rays ρ, ρ' on C^1 are *equivalent* if $\mathcal{A}_{stab}(\rho) = \mathcal{A}_{stab}(\rho')$. The *geometric boundary at infinity* of C^1 is the set of equivalence classes of STT-rays on C^1 . Furthermore, the action of T on C^1 can be extended to its geometric boundary:

Corollary. *Thompson's group T (consequently also Thompson's group F) act on the geometric boundary at infinity of C^1 .*

Let Δ be a subset of \mathcal{A} . The set Δ is a *dyadic triangulation* of D if $D \setminus \Delta$ is a disjoint union of open triangles. Thus, given a dyadic arc $a \in \mathcal{A}$ we write $a \in \Delta$ if a belongs to the set Δ , and $a \notin \Delta$ if it does not.

Proposition. *Let Δ' be a dyadic triangulation of D such that $D \setminus (\Delta' \cap \Delta_F)$ is a disjoint union of polygons. Then, there exist infinitely many STT-rays with stable arc set equal to Δ' .*

A *Farey star region* is a pair $(P_\infty, \Delta(P_\infty))$, where:

1. P_∞ is a subregion of D bounded by an infinite polygonal line of consecutive dyadic arcs a_1, a_2, a_3, \dots starting at $\frac{m_1}{2^{n_1}}$ and such that, for some $n \in \mathbb{N}$, the arcs a_{n+k} with $k \in \mathbb{N}$ are equal to the Farey arcs joining $\frac{2^k m_1 - 1}{2^{n_1+k}}$ and $\frac{m_1}{2^{n_1}}$, and
2. $\Delta(P_\infty)$ is the dyadic triangulation of P_∞ given by the dyadic arcs joining the extremal point of the dyadic arc a_1 different from $\frac{m_1}{2^{n_1}}$, and an extremal point of a_l for some $l \geq 3$.

Note that, given an interior point a_1 and an open neighbourhood \mathcal{N} of it, $\mathcal{N} \cap \Delta(P_\infty)$ is infinite. If a_1 is a Farey arc, $(P_\infty, \Delta(P_\infty))$ is called a *Farey-based star region*.

Proposition. *Let $(P_\infty^{(1)}, \Delta(P_\infty^{(1)}), (P_\infty^{(2)}, \Delta(P_\infty^{(2)}), \dots$ be an infinite sequence of Farey star regions with 2 by 2 empty intersections of their interiors, and such that all but finitely many are Farey-based star regions. Let Δ be a dyadic triangulation of D coinciding with $\Delta(P_\infty^{(i)})$ on $P_\infty^{(i)}$ for every $i \in \mathbb{N}$, and of Farey type elsewhere. Then, there exist infinitely many STT-rays having Δ as stable arc set.*

Proposition. *Let p be a dyadic number of the unit interval and let $S(p)$ be the subset of \mathcal{A} containing all dyadic arcs having p as extremal point. Then, there exist an STT-ray ρ on C^1 with stable arc set equal to $S(p)$.*

Consider the following functional space:

$$\mathcal{F} = \{f : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0} \cup \infty\}.$$

Note that $\mathcal{F} \simeq [0, \infty]^{\mathbb{N}}$ (\mathcal{A} is countable). Thus, \mathcal{F} is compact with the product topology.

One defines $\mathcal{I} : C^1 \hookrightarrow \mathcal{F}$ as follows:

1. Given a dyadic triangulation $\Delta \in C^0$ of Farey type, $\mathcal{I}(\Delta)$ is the function *geometric intersection with Δ* . Let $a \in \mathcal{A}$ and $\Delta = \{a_i\}_{i \in \mathbb{N}} \subset \mathcal{A}$. Then,

$$\mathcal{I}(\Delta)(a) = \sum_{i=1}^{\infty} i(a_i, a),$$

where

$$i(a, a') = \begin{cases} 1 & \text{if the interiors of the dyadic arcs } a \text{ and } a' \text{ intersect} \\ 0 & \text{otherwise} \end{cases}$$

is the intersection number between a and a' .

2. Given a point p contained on the interior of an edge $a\Delta \in C^1$ joining two dyadic triangulations $\Delta_1, \Delta_2 \in C^0$, there is a unique convex linear combination of Δ_1, Δ_2

$$p = s\Delta_1 + (1-s)\Delta_2, \quad \text{with } s \in (0, 1/2]$$

associated to p . Then, for every $a \in \mathcal{A}$ one defines

$$\mathcal{I}(p)(a) = s\mathcal{I}(\Delta_1)(a) + (1-s)\mathcal{I}(\Delta_2)(a).$$

Proposition. *The map $\mathcal{I} : C^1 \hookrightarrow \mathcal{F}$ is injective.*

Let C_{Th}^1 be the closure of $\mathcal{I}(C^1)$ in \mathcal{F} , which is compact when considering the induced topology (it is a closed subspace of a topological compact space). The closure C_{Th}^1 will be called *Thurston's compactification of C^1* . We denote by $\partial_{\mathcal{F}}(C^1)$ the set $C_{Th}^1 \setminus \mathcal{I}(C^1)$ and we call it the *Thurston's boundary of C^1* .

The following result shows that the subset of the geometric boundary that we have been able to describe explicitly can be embedded into the Thurston's boundary of C^1 :

Proposition. *Let $\rho : [0, \infty) \rightarrow C^1$ be an STT-ray such that every arc-connected component of $D \setminus \mathcal{A}_{stab}(\rho)$ is either a triangle, either an arc with at least one of its extremal points being non-dyadic. Then, the limit*

$$\lim_{s \rightarrow \infty} \mathcal{I}(\rho(s))$$

exists and is equal to the function geometric intersection with $\mathcal{A}_{stab}(\rho)$, which belongs to Thurston's boundary $\partial_{\mathcal{F}}(C^1)$ of C^1 .

Furthermore, we describe precisely the behaviour of the limit function associated to the different types of explicit points on the geometric boundary.

Chapter 2

Four definitions of Thompson's group T

The group T is usually seen as a subgroup of the group of orientation-preserving piecewise linear homeomorphisms of the circle, thought as the unit interval $[0, 1]$ with identified endpoints, fixing the set of dyadic rational numbers $\mathbb{Z}[1/2] \cap [0, 1]$. In this chapter, we present four classical approaches to Thompson's group T , namely the piecewise linear approach, the combinatorial approach with tree pair diagrams, the piecewise projective approach due to Thurston, and the asymptotic mapping class group approach due to Funar, Kapoudjian and Sergiescu. Although there are other viewpoints of Thompson's group T , we restrict ourselves to the representations that will appear later on.

2.1 Piecewise linear homeomorphisms of the circle

Definition 1. *Thompson's group T* is the group of orientation-preserving piecewise linear homeomorphisms of the circle S^1 , thought as the unit interval $[0, 1]$ with identified endpoints, such that:

1. map the set $\mathbb{Z}[1/2] \cap [0, 1]$ of dyadic rational numbers to itself,
2. are differentiable except at finitely many points,
3. all non-differentiable points are dyadic rational numbers, and
4. on intervals of differentiability, the derivatives are powers of 2.

Example 1. Two non torsion elements of Thompson's group T :

$$A(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{4}, & \frac{1}{2} \leq x \leq \frac{3}{4} \\ 2x - 1, & \frac{3}{4} \leq x \leq 1. \end{cases} \quad B(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{8}, & \frac{3}{4} \leq x \leq \frac{7}{8} \\ 2x - 1, & \frac{7}{8} \leq x \leq 1. \end{cases}$$

Notation has been taken from [18]. Nowadays, $A(x)$ is usually found on the literature as X_0 , and $B(x)$ is usually denoted by X_1 .

Example 2. A torsion element of Thompson's group T :

$$C(x) = \begin{cases} \frac{x}{2} + \frac{3}{4}, & 0 \leq x \leq \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{4}, & \frac{3}{4} \leq x \leq 1. \end{cases}$$

$C(x)$ has order 3.

Proposition 2.1. ([18], Lemma 5.2) *Thompson's group T is generated by the elements $A(x)$, $B(x)$ and $C(x)$ of the previous examples.*

It follows from the definition that the elements of T are the maps

$$t(x) = \begin{cases} 2^{r_1}x + \frac{m_1}{2^{s_1}}, & \text{if } 0 \leq x \leq x_1 \\ 2^{r_2}x + \frac{m_2}{2^{s_2}}, & \text{if } x_1 \leq x \leq x_2 \\ \vdots \\ 2^{r_k}x + \frac{m_k}{2^{s_k}}, & \text{if } x_{k-1} \leq x \leq 1, \end{cases}$$

where $k \in \mathbb{N}$, for $i \in \{1, \dots, k\}$, r_i and s_i are integers, $m_i \in \mathbb{N}$, and x_i is a dyadic rational number. In this way, a natural partition of the unit interval on the x axis given by the non differentiable points of $t(x)$ is obtained:

$$0 = x_0 < x_1 < \dots < x_{k-1} < x_k = 1.$$

Note that $t(x)$ is affine on each interval $[x_i, x_{i+1}]$ of the partition. Note also that the images of the partition associated to $t(x)$ themselves form a partition of the unit interval as well, this time on the y axis, modulo a cyclic permutation ($t(0)$ might be different from 0):

$$0 = t(x_i) < t(x_{i+1}) < \dots < t(x_k) = t(x_0) < \dots < t(x_{i-1}) = 1.$$

Furthermore, all points of both partitions are dyadic rational numbers. Nevertheless, the piecewise linear homeomorphism of the circle obtained by gluing together two partitions of the unit intervals with dyadic rational points may not be an element of Thompson's group T . For example, the piecewise linear homeomorphism obtained by the partition $0 < 3/4 < 1$ on the x axis, and the partition $0 < 1/4 < 1$ on the y axis has slope $1/3$ on the first linear piece. The elements of Thompson's group T can be characterized using standard dyadic partitions of the unit interval.

Definition 2. A subinterval $[x_1, x_2]$ of the unit interval $[0, 1]$ is called a *standard dyadic interval* if it is of the form

$$\left[\frac{m}{2^n}, \frac{m+1}{2^n} \right]$$

for some positive integers m and n satisfying $0 \leq m \leq 2^n - 1$.

A partition of the unit interval given by

$$x_0 = 0 < x_1 < x_2 < \dots < x_{k-1} < x_k = 1$$

is a *standard dyadic partition* if, for all $i \in \{1, \dots, k-1\}$, the subinterval $[x_i, x_{i+1}]$ is a standard dyadic interval.

Proposition 2.2. ([18], Lemma 2.2) *Let t be an element of T . Then, there exists a standard dyadic partition of the unit interval*

$$0 = x_0 < x_1 < \dots < x_k = 1$$

such that:

1. t is affine on every subinterval of the partition, and
2. the induced partition on the y axis

$$0 = t(x_i) < t(x_{i+1}) < \dots < t(x_k) = t(x_0) < \dots < t(x_{i-1}) = 1$$

is also a standard dyadic partition of the unit interval.

Conversely, for each triple

$$(0 = x_0 < \dots < x_k = 1, 0 = y_0 < \dots < y_k = 1, \sigma),$$

where $x_0 < \dots < x_k$ and $y_0 < \dots < y_k$ are standard dyadic partitions of the unit interval and σ is a cyclic permutation, there exists a unique element t in T such that:

1. for all $i \in \{0, \dots, k\}$, $t(x_i) = y_{\sigma(i)}$, and
2. for all $i \in \{0, \dots, k-1\}$, t is affine on every subinterval $[x_i, x_{i+1}]$.

Proof. Consider the x axis partition associated to t , $0 = z_0 < z_1 < \dots < z_k = 1$. As $t \in T$, z_0, \dots, z_k are dyadic rational numbers and t is affine on each interval of the partition. Let $[z_i, z_{i+1}]$ be an interval of this partition and suppose that $t'(x) = 2^{-r}$, if $x \in [z_i, z_{i+1}]$. Let n be an integer such that $2^n z_i, 2^n z_{i+1}, 2^{n+r} t(z_i)$ and $2^{n+r} t(z_{i+1})$ are integers. Then,

$$z_i < z_i + \frac{1}{2^n} < z_i + \frac{2}{2^n} < z_i + \frac{3}{2^n} < \dots < z_{i+1}$$

is a standard dyadic partition of the interval $[z_i, z_{i+1}]$, and its images

$$t(z_i) < t(z_i) + \frac{1}{2^{n+r}} < t(z_i) + \frac{2}{2^{n+r}} < t(z_i) + \frac{3}{2^{n+r}} < \dots < t(z_{i+1})$$

form a standard dyadic partition of the interval $[t(z_i), t(z_{i+1})]$ as well.

In order to get the required partition it is enough to repeat the previous procedure for every interval of the x -axis partition associated to $t(x)$.

For the converse case, it suffices to join the pairs of points with dyadic rational coordinates $(x_i, \sigma(y_i))$ and $(x_{i+1}, \sigma(y_{i+1}))$ by straight lines. \square

Since any standard dyadic interval can be split in two standard dyadic intervals by taking the midpoint, one can not expect the partitions of Proposition 2.2 to be unique. However, there exists a standard dyadic partition satisfying Proposition 2.2 with a minimum number of standard dyadic intervals, which is called the *minimal standard dyadic partition*.

Remark 1. It can be proved that the minimal standard dyadic partition exists and every partition fulfilling Proposition 2.2 is a subpartition of this minimal standard dyadic partition.

To end this section, a presentation of Thompson's group T where both generators are torsion elements is introduced.

Proposition 2.3. (see [32, 57]) *Thompson's group T can be generated by two torsion elements, and the associated presentation is finite.*

$$T \simeq \langle \alpha, \beta \mid \alpha^4, \beta^3, [\beta\alpha\beta, \alpha^2\beta\alpha\beta\alpha^2], [\beta\alpha\beta, \alpha^2\beta^2\alpha^2\beta\alpha\beta\alpha^2\beta\alpha^2], (\beta\alpha)^5 \rangle.$$

2.2 Equivalence classes of tree pair diagrams

The set of standard dyadic subintervals of the unit interval inherits a combinatorial tree structure. Let R_2 be the rooted infinite binary tree (see Figure 2.1). Label the root by the unit interval $[0, 1]$, which is standard dyadic. Then, given an interior vertex labelled by the standard dyadic interval

$$\left[\frac{m}{2^n}, \frac{m+1}{2^n} \right],$$

its left descendant is labelled by the standard dyadic interval

$$\left[\frac{2m}{2^{n+1}}, \frac{2m+1}{2^{n+1}} \right],$$

which corresponds to its left half, and its right descendant is labelled by the standard dyadic interval

$$\left[\frac{2m+1}{2^{n+1}}, \frac{2m+2}{2^{n+1}} \right],$$

which corresponds to its right half. This way, every standard dyadic interval can be seen as a vertex of R_2 . Note that it is possible to provide the set of vertices of R_2 with a total order by starting for the root, and then listing the vertices on each level from left to right before and descending to the next level.

Remark 2. There is a bijection between the set of finite rooted binary subtrees of R_2 and the set of standard dyadic partitions of the unit interval by taking the labels on the leaves from left to right.

Definition 3. An internal vertex of a rooted binary tree together with its two descendants is called a *caret*, and a caret is *exposed* if it contains two leaves of the rooted binary tree.

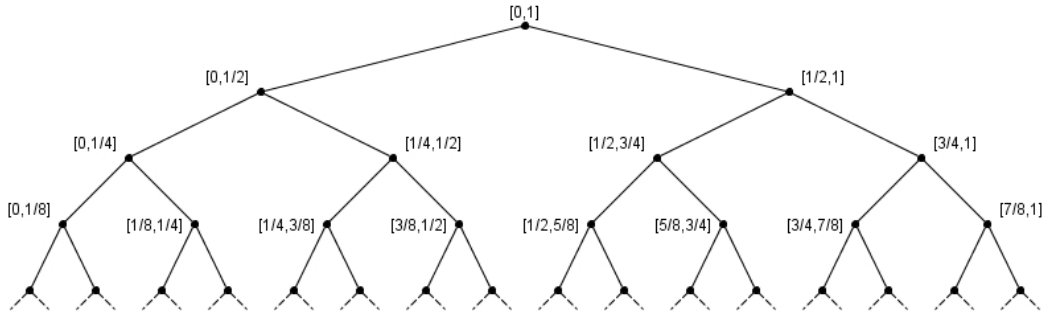
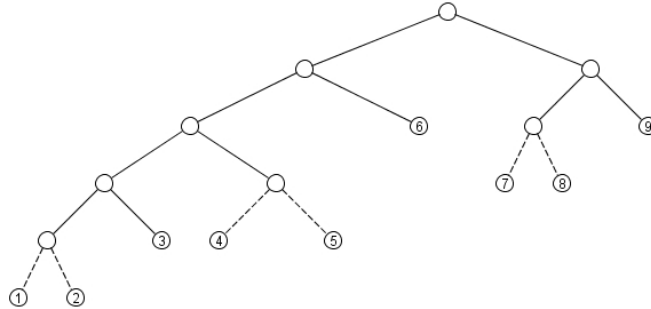


Figure 2.1: The infinite rooted binary tree R_2 .

Example 3. The following rooted binary tree corresponds to the standard dyadic partition

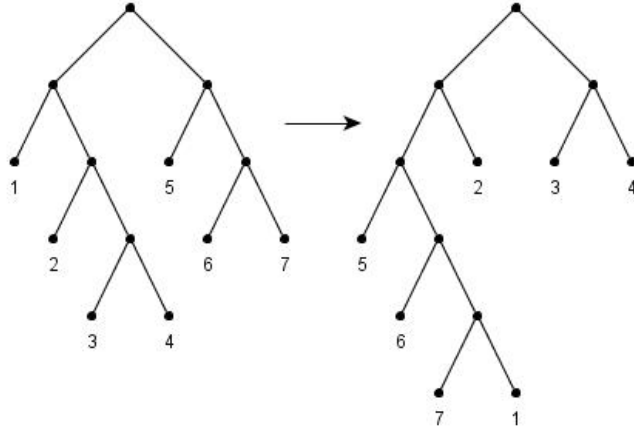
$$0 < \frac{1}{32} < \frac{1}{16} < \frac{1}{8} < \frac{3}{16} < \frac{1}{4} < \frac{1}{2} < \frac{5}{8} < \frac{3}{4} < 1$$

of the unit interval. The exposed carets are drawn in discontinuous lines.



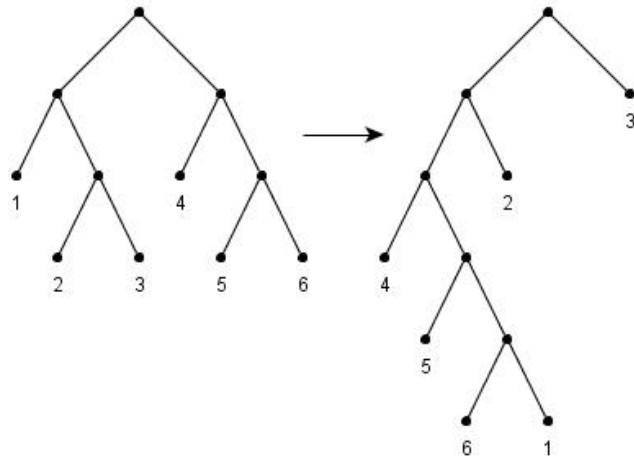
Definition 4. A *tree pair diagram* is a triple (R, σ, S) , where R and S are finite rooted binary trees with the same number n of leaves, and σ is a bijection between the set of leaves of R and the set of leaves of S . The binary tree R is called the *source tree*, and S is called the *target tree*. A tree pair diagram is *reduced* if, for all exposed carets of S , the images under σ of their leaves do not form an exposed caret of R . A tree pair diagram is *cyclic* if σ is a cyclic permutation. In this case, the i -th leaf of the source tree (considering the left-to-right order on its leaves) is sent to the $\sigma(i)$ -th leaf of the target tree.

Example 4. The cyclic tree pair diagram



is non reduced because there are exposed caret containing leaves 3 and 4 on both target and source trees.

Example 5. The cyclic tree pair diagram



is reduced. There are two exposed caret on the source tree containing, respectively, leaves 2-3 and 5-6. There is a single exposed caret on the target tree which contains leaves 6-1.

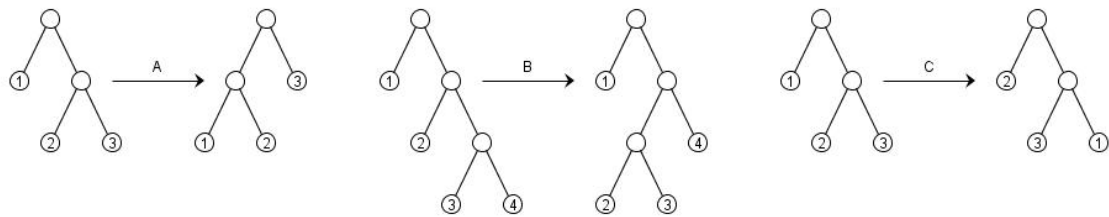
Remark 3. Given an element t of T and a standard dyadic partition $x_0 < \dots < x_k$ satisfying Proposition 2.2, a cyclic tree pair diagram (S, σ, R) representing t can be found. The source tree S is the rooted binary tree corresponding to the dyadic partition $x_0 < \dots < x_k$ (see Remark 2). Analogously, the target tree R is the one corresponding to the partition $t(x_i) < \dots < t(x_{i-1})$. Finally, the cyclic permutation σ is defined by $\sigma(1) = i$. As in the case of partitions, such a tree pair diagram is not unique.

Definition 5. Given a tree pair diagram (R, σ, S) , its j -th simple expansion is obtained by splitting the j -th leaf of R (i.e. adding the caret containing the leaf j and its descendants) and the $\sigma(j)$ -th leaf of S , and expanding the bijection σ by mapping the left (right) descendant of j on R to the left (right) descendant of $\sigma(j)$ on S . Every tree pair

diagram obtained from (R, σ, S) by finitely many simple expansions is called an *expansion* of (R, σ, S) . Two tree pair diagrams are *equivalent* if they have a common expansion.

Lemma 2.1. *Two equivalent cyclic tree pair diagrams represent the same element t of T . Furthermore, in each equivalence class of cyclic tree pair diagrams there is a unique reduced tree pair diagram, which corresponds to the minimal standard dyadic partition of t (see Remark 1).*

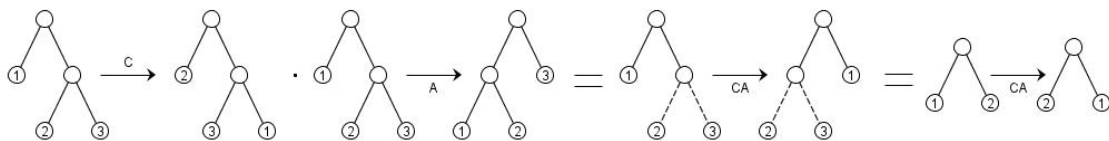
Example 6. The reduced tree pair diagrams of the generators A , B and C of Thompson group T .



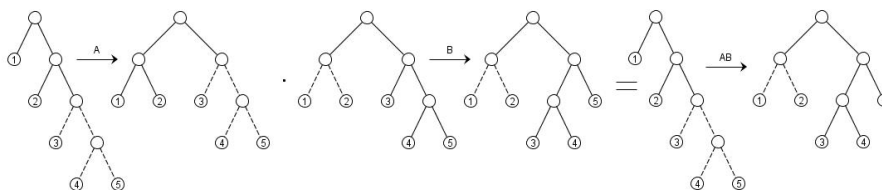
Definition 6. When elements of Thompson's group T are represented as equivalence classes of cyclic tree pair diagrams, the *multiplication* of (R, σ, S) by (R', σ', S') is described by the following procedure:

1. Let SR' be an ordered rooted binary tree which is a common expansion of R' and S . This element always exists.
2. Let $(RR', \sigma_{R'}, SR')$ and (SR', σ_S, SS') be the tree pair diagrams respectively equivalent to (R, σ, S) and (R', σ', S') .
3. Then, one sets $(R, \sigma, S) * (R', \sigma', S') = (RR', \sigma_{R'}\sigma_S, SS')$, and reduces the obtained tree pair diagram, if possible.

Example 7. The tree pair diagram multiplication CA , (denoted $A \circ C(x)$ when thought as piecewise linear homeomorphisms of the circle).



Example 8. The tree pair diagram multiplication AB .



2.3 $PPSL(2, \mathbb{Z})$ homeomorphisms of $\mathbb{R}P^1$

The projective special linear group $PSL_2(\mathbb{Z})$ is isomorphic to the free product

$$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$$

of the cyclic groups of order 2 and 3 (see [70], page 20 example 1.5.3). Thus, $PSL_2(\mathbb{Z})$ admit the presentation

$$\langle a, b | a^2 = b^3 = 1 \rangle,$$

where $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

Definition 7. The *piecewise projective Thompson's group* T ($PPSL_2(\mathbb{Z})$) is the group of orientation-preserving homeomorphisms of the real projective line $\mathbb{R}P^1$ which are piecewise $PSL_2(\mathbb{Z})$, fix the set of rational numbers, and have a finite number of non differentiable points, all of them rational.

In order to connect Thompson's group T with $PPSL_2(\mathbb{Z})$ we need the Minkowsky question mark function (see [67] or [78]).

Definition 8. Let $\bar{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ and let $\mathbb{Q}[\frac{1}{2}]$ denote the dyadic rational numbers of the unit interval. *Minkowski's question mark function*

$$? : \bar{\mathbb{Q}} \longrightarrow \mathbb{Q} \left[\frac{1}{2} \right]$$

is defined recursively.

The basic cases are

$$?(\infty) = \frac{1}{2}, \quad ?\left(\frac{0}{1}\right) = 0, \quad \text{and} \quad ?(-\infty) = \frac{1}{2},$$

where ∞ will be represented by the fraction $\frac{1}{0}$ and $-\infty$ by $\frac{-1}{0}$.

Then, for each pair $\left(\frac{p}{q}, \frac{r}{s}\right)$ of reduced fractions satisfying $|ps - qr| = 1$, Minkowski's question mark function of their Farey median

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}$$

is defined by the midpoint of the segment defined by their images

$$?\left(\frac{p}{q} \oplus \frac{r}{s}\right) = \frac{1}{2} \left(?\left(\frac{p}{q}\right) + ?\left(\frac{r}{s}\right) \right).$$

Minkowski's question mark function is clearly a bijection between $\bar{\mathbb{Q}}$ and $\mathbb{Q}[\frac{1}{2}]$. Thus, using both the density of $\bar{\mathbb{Q}}$ in the real projective line $\mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}$ and the density of $\mathbb{Q}[\frac{1}{2}]$ in $[0, 1]$, Minkowski's question mark function can be extended to

$$? : \mathbb{R}P^1 \longrightarrow [0, 1].$$

The pairs of reduced fractions $\left(\frac{p}{q}, \frac{r}{s}\right)$ satisfying $|ps - qr| = 1$ are called *consecutive Farey numbers*, and if $\frac{p}{q} < \frac{r}{s}$, the interval $\left[\frac{p}{q}, \frac{r}{s}\right]$ is called a *Farey interval*. Every rational number appears as the Farey median of a Farey interval.

See, [2] Section 5.4 or [42] Chapter 3 for more details on Farey fractions.

Lemma 2.2. *Let h be an element of $PPSL_2(\mathbb{Z})$ corresponding to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z})$ on the interval $[x_0, x_k]$ of the real projective line $\mathbb{R}P^1$. Let $x_0 < x_1 < \dots < x_k$ be a partition of $[x_0, x_k]$ such that every interval of the partition and every interval of the image partition $h(x_0) < \dots < h(x_k)$ is a Farey interval. Then, for all $0 \leq i < k$,*

$$h(x_i \oplus x_{i+1}) = h(x_i) \oplus h(x_{i+1}).$$

Proof. Let $x_i = \frac{p}{q}$ and $x_{i+1} = \frac{r}{s}$ be real rational numbers ($q \neq 0$ and $s \neq 0$). Then,

$$h(x_i) = \frac{ap + bq}{cp + dq} \quad \text{and} \quad h(x_{i+1}) = \frac{ar + bs}{cr + ds}.$$

Hence,

$$h\left(\frac{p}{q} \oplus \frac{r}{s}\right) = \frac{ap + ar + bq + bs}{cp + cr + dq + ds} = h\left(\frac{p}{q}\right) \oplus h\left(\frac{r}{s}\right).$$

If $x_{i+1} = \frac{1}{0}$, then $h(x_{i+1}) = \frac{a}{c}$ and

$$h\left(\frac{p}{q} \oplus \frac{1}{0}\right) = \frac{ap + a + bq}{cp + c + dq} = h\left(\frac{p}{q}\right) \oplus h\left(\frac{1}{0}\right).$$

Analogously if $x_i = \frac{-1}{0}$. □

The following result, due to Thurston and proved by Imbert and Sergiescu, identifies the two groups encountered above (see [46], Theorem 1.1).

Theorem 2.1. ([46], Theorem 1.1) *The group $PPSL_2(\mathbb{Z})$ is isomorphic to Thompson's group T .*

Proof. We claim that the homomorphism

$$Inn_{\gamma} : PPSL_2(\mathbb{Z}) \longrightarrow T$$

given by the conjugation

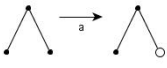
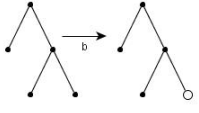
$$Inn_{\gamma}(g) = \gamma \circ g \circ \gamma^{-1}$$

is an isomorphism.

Consider the generating set $\{a, b\}$ of $PSL_2(\mathbb{Z})$ and calculate $Inn_{\gamma}(a)$ and $Inn_{\gamma}(b)$. By

Lemma 2.2, it suffices to find Farey partitions of $\bar{\mathbb{Q}}$ whose images are also Farey partitions. Then, applying Minkowski's question mark function we obtain standard dyadic partitions of the unit interval characterizing finite binary trees with the same number of leaves, thus elements of T .

The following table summarizes this procedure.

Generator	Farey partitions	Piecewise linear map	Tree pair diagram
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{matrix} [\frac{0}{1}, \frac{1}{0}] \mapsto [\frac{1}{0}, \frac{0}{1}] \\ [\frac{1}{0}, \frac{0}{1}] \mapsto [\frac{0}{1}, \frac{1}{0}] \end{matrix}$	$\begin{cases} x + \frac{1}{2}, & 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{2}, & \frac{1}{2} \leq x \leq 1. \end{cases}$	
$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$\begin{matrix} [\frac{0}{1}, \frac{1}{0}] \mapsto [\frac{-1}{1}, \frac{0}{1}] \\ [\frac{1}{0}, \frac{-1}{1}] \mapsto [\frac{0}{1}, \frac{1}{0}] \\ [\frac{-1}{1}, \frac{0}{1}] \mapsto [\frac{1}{0}, \frac{-1}{1}] \end{matrix}$	$\begin{cases} \frac{x}{2} + \frac{3}{4}, & 0 \leq x \leq \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{4}, & \frac{3}{4} \leq x \leq 1. \end{cases}$	

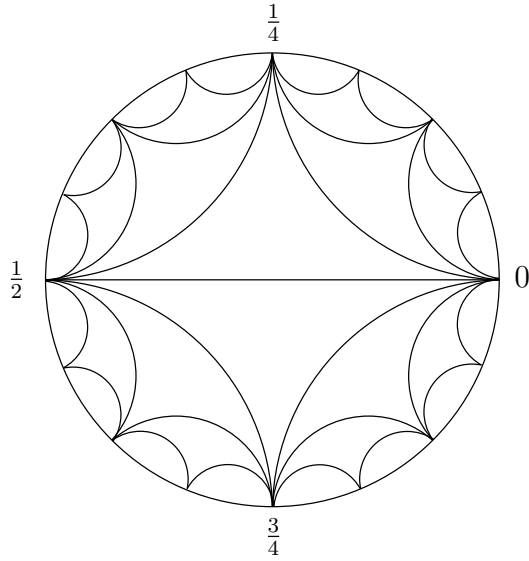
Thanks to this and Lemma 2.2, the map Inn_γ is well defined. Thus, Inn_γ is a homomorphism. The injectivity is also a consequence of Lemma 2.2. Let A , B and C be the three classical generators of Thompson's group T introduced before. Note that $Inn_\gamma(b) = C$ and $Inn_\gamma(a) = CA$. In order to prove that Inn_γ is surjective, we have to find an element d in $PPSL_2(\mathbb{Z})$ such that $Inn_\gamma(d) = B$. This element is

$$d(x) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } 0 \leq x \leq \infty, \\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, & \text{if } -\infty \leq x \leq -1, \\ \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}, & \text{if } -1 \leq x \leq -\frac{1}{2}, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{if } -\frac{1}{2} \leq x \leq 0. \end{cases}$$

□

2.4 The asymptotic mapping class group of Σ

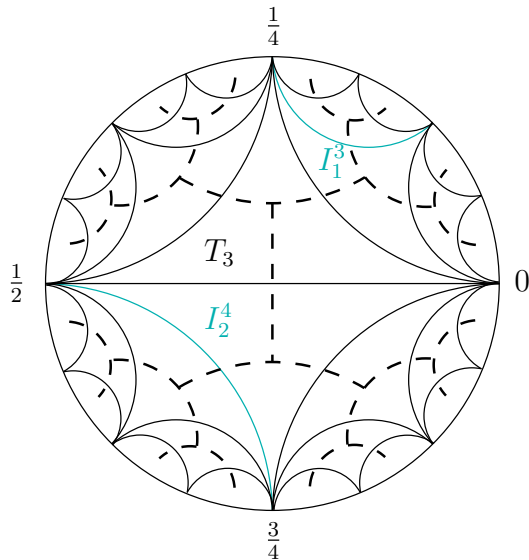
Let \mathbb{D}^2 be the hyperbolic disc and suppose that its boundary $\partial\mathbb{D}^2$ is parametrized by the unit interval (with identified endpoints). Let Δ_F denote the triangulation



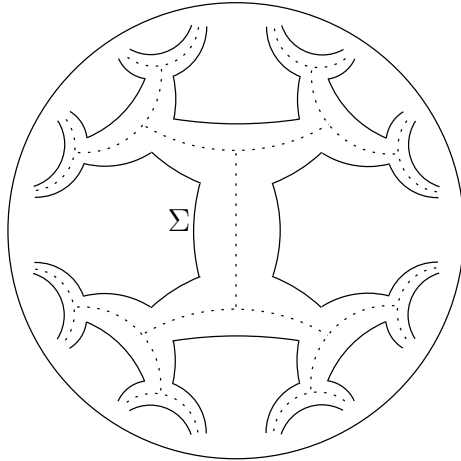
of \mathbb{D}^2 given by the family of bi-infinite geodesics

$$\Delta_F = \{I_m^n : n, m \in \mathbb{N}, 0 \leq m \leq 2^n - 1\},$$

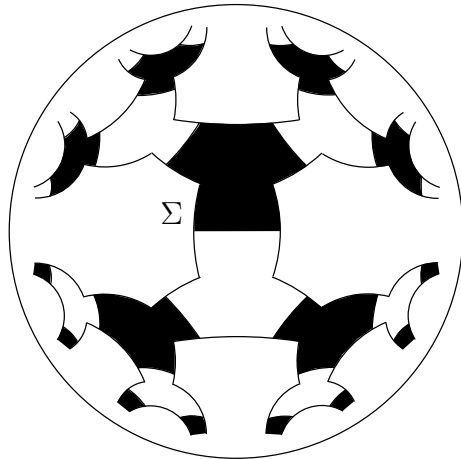
where the I_m^n denotes the bi-infinite geodesic of \mathbb{D}^2 joining the boundary points $\frac{m}{2^n}$ and $\frac{m+1}{2^n}$ of $\partial\mathbb{D}^2$ (in grey on the next figure two examples of I_m^n). Let T_3 be the dual graph of Δ_F , which is an infinite (unrooted) trivalent tree. The tree T_3 is realized in \mathbb{D}^2 by using geodesic segments obtained from the I_m^n (with $n \geq 1$ and m even) by rotations of angles $\pi/2^n$.



Let Σ be a closed δ -neighbourhood of T_3 , which is a planar surface (thus oriented), non compact, contractible and its boundary has infinitely many connected components.

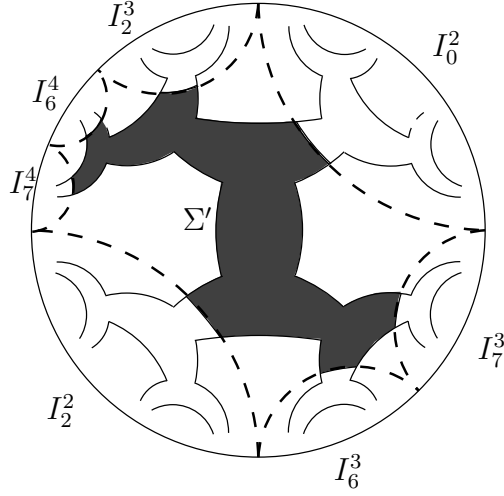


Note that δ can be selected small enough to avoid self-intersections of the boundary and such that each intersection $I_m^n \cap \Sigma$ is a single connected arc joining two boundary components of Σ . The triangulation Δ_F together with the boundary components of Σ define a tessellation *tes* of our surface into topological hexagons. Remark that the boundary of each hexagon alternates sides contained in different connected components of $\partial\Sigma$ with sides contained in different arcs of the triangulation Δ_F .



Definition 9. The set of arcs $\{I_{m_1}^{n_1}, \dots, I_{m_k}^{n_k}\}$ of Δ_F is *separating* if the union of its arcs bounds a compact subsurface of Σ containing $I_0^1 \cap \Sigma$.

Example 9. The subset $\{I_0^2, I_2^3, I_6^4, I_7^4, I_2^2, I_6^3, I_7^3\}$ of dyadic arcs of Δ_F plotted in discontinuous lines



is a set of separating arcs: they bound the filled compact subsurface Σ' of Σ .

Definition 10. Let $\text{Homeo}(\Sigma)$ denote the group of orientation-preserving homeomorphisms of Σ . An element $\varphi \in \text{Homeo}(\Sigma)$ is *asymptotically rigid* if there exists a set of separating arcs $\{I_{m_1}^{n_1}, \dots, I_{m_k}^{n_k}\}$ such that:

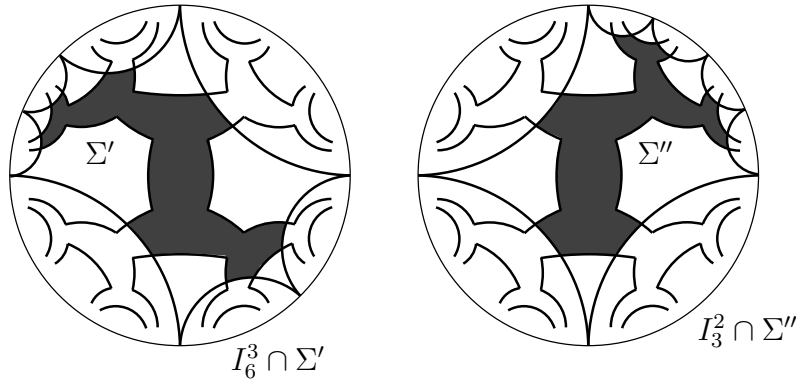
1. For all $1 \leq i \leq k$, there exists $I_{m'_i}^{n'_i}$ such that $\varphi(I_{m_i}^{n_i} \cap \Sigma) = I_{m'_i}^{n'_i} \cap \Sigma$, and
2. For all $1 \leq i \leq k$, for all $j \in \mathbb{N}$, and for all $0 \leq l < 2^j$ the arc $\varphi(I_{2^j m_i + l}^{n_i + j} \cap \Sigma)$ is equal to the arc $I_{2^j m'_i + l}^{n'_i + j} \cap \Sigma$.

Remark 4. The set of arcs

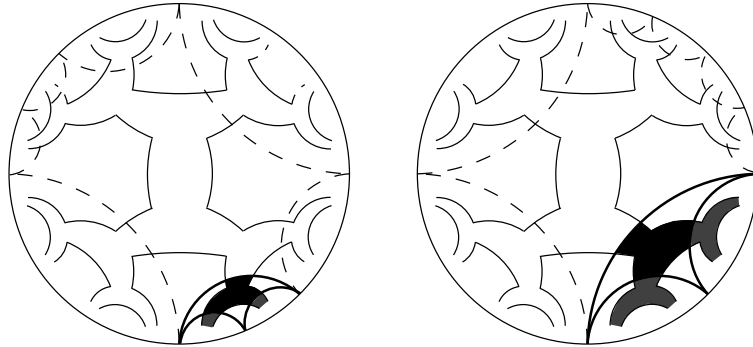
$$I_{m'_1}^{n'_1}, \dots, I_{m'_k}^{n'_k}$$

of the previous definition is a set of separating arcs.

Let φ be an asymptotically rigid homeomorphism. The first condition of the definition states that there exists a compact subsurface $\Sigma' \subset \Sigma$ whose boundary is composed by segments of $\partial\Sigma$ and arcs of Δ_F such that φ sends Σ' to another compact subsurface Σ'' of the same type (whose boundary is composed by segments of $\partial\Sigma$ and arcs of Δ_F). Furthermore, one has a cyclic matching between the boundary components of Σ' and those of Σ'' : in the plotted example the boundary component of Σ' corresponding to $I_6^3 \cap \Sigma$ is sent to the boundary component of Σ'' corresponding to $I_3^2 \cap \Sigma$.



The second condition deals with the behaviour of φ outside Σ' and Σ'' . Let c_1, \dots, c_k be the connected components of $T_3 \setminus (T_3 \cap \Sigma')$ and c'_1, \dots, c'_k be the connected components of $T_3 \setminus (T_3 \cap \Sigma'')$. Then the second condition says that each c_i is “naturally” sent to c'_i , meaning that the hexagonal tessellation of $\Sigma \setminus \Sigma'$ is sent to the hexagonal tessellation of $\Sigma \setminus \Sigma''$: in the plotted example the filled non-compact connected component of $\Sigma \setminus \Sigma'$ is sent to the filled non-compact connected component of $\Sigma \setminus \Sigma''$.



Definition 11. Let $\text{Homeo}_a(\Sigma)$ denote the group of asymptotically rigid homeomorphisms of Σ . The *asymptotic mapping class group* of Σ is the quotient

$$\text{Homeo}_a(\Sigma)/\text{Homeo}_a^0,$$

where Homeo_a^0 is the group of asymptotically rigid homeomorphisms which are isotopic to the identity.

Proposition 2.4. ([32], Section 1.3) *The asymptotic mapping class group of Σ is isomorphic to Thompson’s group T .*

We will give a sketch of a proof later on. First, we introduce a slightly different topological space X which allows us to simplify the situation. Both topological spaces Σ and X have the same asymptotic mapping class group.

Definition 12. The topological space X is obtained by glueing to the hyperbolic disk \mathbb{D}^2 the set of points

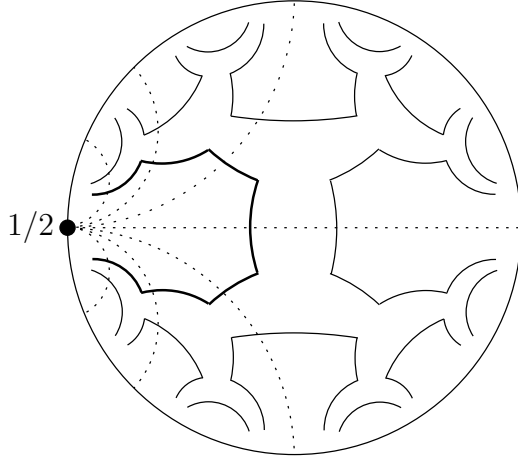
$$\mathbb{Z}[1/2] \cap \partial\mathbb{D}^2$$

of its boundary $\partial\mathbb{D}^2$ corresponding to the dyadic rational numbers.

Remark 5. The triangulation Δ_F of \mathbb{D}^2 is a triangulation of X having vertices at all dyadic rational numbers $\mathbb{Z}[1/2] \cap \partial\mathbb{D}^2$ of the boundary $\partial\mathbb{D}^2$.

Lemma 2.3. *Let $\partial_0\Sigma$ be a connected component of the boundary of Σ , which is a bi-infinite line on \mathbb{D}^2 . The closure of $\partial_0\Sigma$ in \mathbb{D}^2 is $\partial_0\Sigma \cup \{q\}$, where q is a dyadic rational number on $\partial\mathbb{D}^2$. Thus, there is a natural bijection between the set of dyadic rational numbers $\mathbb{Z}[1/2] \cap \partial\mathbb{D}^2$ and the set of connected components of the boundary $\partial\Sigma$.*

Proof. One can see that every connected component $\partial_0\Sigma$ of the boundary of Σ intersects all bi-infinite geodesics of type I_m^n having the same dyadic q as extremal point. In addition, as n tends to infinity, the arcs I_m^n (thought as curves on the euclidean disk) tend to $q \in \mathbb{D}^2$.



□

Corollary 2.1. *Let $\partial_0\Sigma$ be a connected component of the boundary of Σ , and let $I_{m_1}^{n_1}, I_{m_2}^{n_2}$ be two geodesics of Δ_F intersecting $\partial_0\Sigma$. Then, $I_{m_1}^{n_1}$ and $I_{m_2}^{n_2}$ have a common boundary point (dyadic).*

Corollary 2.2. *Let $\{I_{m_1}^{n_1}, \dots, I_{m_k}^{n_k}\}$ be a set of separating arcs and suppose they are cyclically ordered. Then, they are the sides of an ideal hyperbolic k -gon.*

Corollary 2.3. *One can deform continuously Σ to X by an homotopy*

$$H : \Sigma \times [0, 1] \longrightarrow \mathbb{D}^2 \cup \partial\mathbb{D}^2$$

such that:

1. For $t < 1$, $H(\cdot, t)$ is an homeomorphism.
2. For $t < 1$, $H(\Sigma \cap I_m^n, t) \subset I_m^n$.
3. $H(\cdot, 1)$ is an homeomorphism between the interior of Σ and \mathbb{D}^2 .
4. Let $\partial_0\Sigma$ be a connected component of the boundary of Σ and let q be the dyadic rational number on its adherence. Then, $H(\partial_0\Sigma, 1) = q$.

One can now define the notions of asymptotically rigid homeomorphism of X and asymptotic mapping class group of X .

Definition 13. An orientation preserving homeomorphism φ of X is *asymptotically rigid* if there exists a set of separating arcs $\{I_{m_1}^{n_1}, \dots, I_{m_k}^{n_k}\}$ such that:

1. For all $1 \leq i \leq k$, there exists $I_{m'_i}^{n'_i}$ such that $\varphi(I_{m_i}^{n_i}) = I_{m'_i}^{n'_i}$, and
2. For all $1 \leq i \leq k$, for all $j \in \mathbb{N}$, and for all $0 \leq l < 2^j$ the arc $\varphi(I_{2^j m_i + l}^{n_i + j})$ is equal to the arc $I_{2^j m'_i + l}^{n'_i + j}$.

Definition 14. The *asymptotic mapping class group* of X is the quotient

$$\text{Homeo}_a(X) / \text{Homeo}_a^0,$$

where Homeo_a^0 is the group of asymptotically rigid homeomorphisms which are isotopic to the identity.

Proposition 2.5. *The asymptotic mapping class groups of Σ and X are isomorphic.*

Proof. Let $\varphi \in \text{Homeo}_a(\Sigma)$. We define $\varphi' \in \text{Homeo}_a(X)$ as follows:

$$\varphi'(x) = \begin{cases} H(\varphi(y), 1), & \text{if } x \in \mathbb{D}^2 \\ H(\varphi(\partial_x \Sigma), 1), & \text{if } x \in \partial X, \end{cases}$$

where $y \in \Sigma$ such that $H(y, 1) = x$, and $\partial_x \Sigma$ is the connected component of $\partial \Sigma$ having x on its adherence and H is the homotopy from Lemma 2.3. Note that every isotopy of Σ can be extended to an isotopy of $\mathbb{D}^2 \cup \partial \mathbb{D}^2$. Thus, the application just defined induces an injective morphism of mapping class groups.

Reciprocally, let $[\varphi']$ be an element of the asymptotic mapping class group of X . Note that a representative $\varphi' \in \text{Homeo}_a(X)$ can be selected so that $\varphi'_\Sigma \in \text{Homeo}_a(\Sigma)$. This way an injective morphism from the asymptotic mapping class group of X to the asymptotic mapping class group of Σ is defined.

Verifying that the composition of both morphisms is the identity is straightforward. \square

A sketch of the proof of Proposition 2.4 follows:

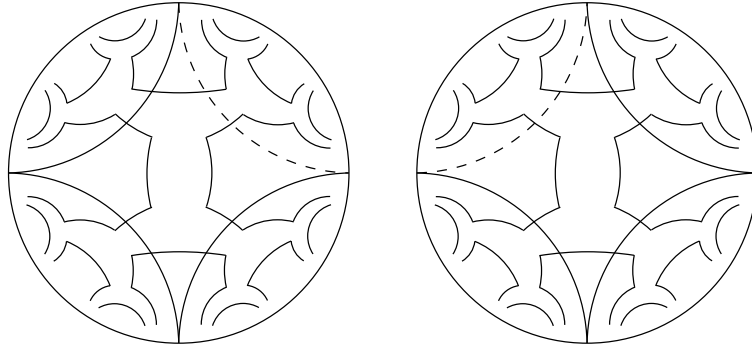
Let φ be a representative of an equivalence class of the asymptotic mapping class group of X . We claim that φ acts as an element of Thompson's group T on $\mathbb{Z}[1/2] \cap \partial \mathbb{D}^2$, which is the set of dyadic rational numbers, and that this action does not depend on the representative. To see this, let $\{I_{m_1}^{n_1}, \dots, I_{m_k}^{n_k}\}$ and $\{I_{m'_1}^{n'_1}, \dots, I_{m'_k}^{n'_k}\}$ be sets of separating arcs associated to φ , with $\varphi(I_{m_i}^{n_i}) = I_{m'_i}^{n'_i}$ for $1 \leq i \leq k$. Without loss of generality, it can be assumed that $m_1 = 0$, $m_k = 2^{n_k} - 1$ (I_m^1 is contained on the compact subsurface bounded by the arcs I_m^n) and that both sets are cyclically ordered. Let t be the unique map of the circle satisfying the following conditions:

1. For all $1 \leq i \leq k$, $t\left(\frac{m_i}{2^{n_i}}\right) = \frac{m'_i}{2^{n'_i}}$,
2. For all $1 \leq i \leq k$, $t\left(\frac{m_i + 1}{2^{n_i}}\right) = \frac{m'_i + 1}{2^{n'_i}}$, and
3. For all $x \in \left(\frac{m_i}{2^{n_i}}, \frac{m_i + 1}{2^{n_i}}\right)$, $t'(x) = 2^{n_i - m_i}$.

One can see that $t \in T$ using Corollary 2.2 and the fact that φ is orientation-preserving.

Reciprocally, given an element $t \in T$ and using Proposition 2.2 one can find two ideal k -gons inscribed in Δ_F defining an element φ in the asymptotic mapping class group of X which acts as t on $\mathbb{Z}[1/2] \cap \partial \mathbb{D}^2$. This ends the proof.

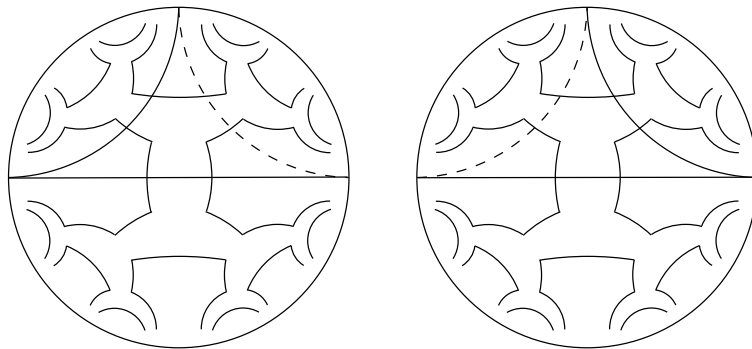
Example 10. The torsion generator α of Thompson's group T (see Proposition 2.3) can be represented as follows:



Thus, α sends $\{I_0^2, I_1^2, I_2^2, I_3^2\}$ respectively to $\{I_1^2, I_2^2, I_3^2, I_0^2\}$.

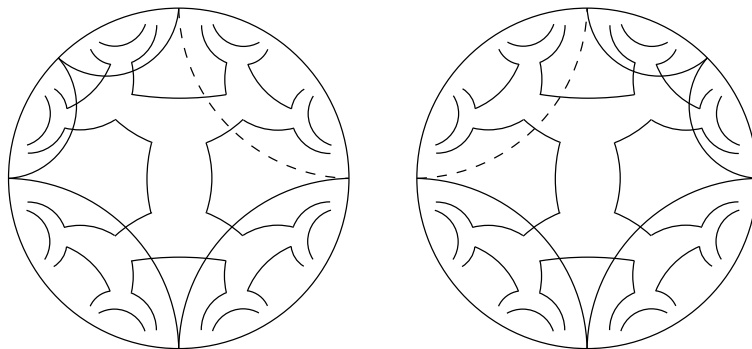
Remark 6. As in the case of tree pair diagrams, the polygons defining an element of T are not unique, although there is a minimal one which fulfils the conditions. Consider this minimal pair of polygons defining an element $t \in T$. Then, if we split a standard dyadic interval of the source polygon in two halves and we also split the standard dyadic interval corresponding to its image in two halves, the element defined by the new pair of polygons is the same. All possible pairs of polygons defining t are obtained through finitely many expanding operations.

Example 11. The torsion generator β of Thompson's group T (see Proposition 2.3) can be represented as follows:



Thus β sends $\{I_0^1, I_0^2, I_1^2\}$ respectively to $\{I_0^2, I_1^2, I_0^1\}$.

If I_0^1 have to be contained in the interior of both polygons defining β , at least the following representation is needed.



Chapter 3

$PSL(2, \mathbb{Z})$ and F_2 as non-distorted subgroups of T

The projective special linear group $PSL_2(\mathbb{Z})$ can be naturally seen as a subgroup of $PPSL(2, \mathbb{Z})$, which is isomorphic to Thompson's group T (see Theorem 2.1). In this chapter, the elements of $PSL(2, \mathbb{Z})$ are described in terms of tree pair diagrams. This characterization of $PSL_2(\mathbb{Z})$ together with the results of Burillo, Cleary, Stein and Taback in the word metric on T (see [16]) allow us to prove that $PSL_2(\mathbb{Z})$ is a non distorted subgroup of Thompson's group T . As a corollary, we obtain a non distorted subgroup of T isomorphic to the free non abelian group F_2 of rank two. Finally, we translate this results in terms of homeomorphisms of the circle. All the results of this chapter can be found in [29].

3.1 Tree pair diagram characterization of $PSL(2, \mathbb{Z})$

Using the $PPSL(2, \mathbb{Z})$ version of Thompson's group T , a copy of the projective special linear group $PSL_2(\mathbb{Z})$ can be embedded into T by considering the subgroup $Inn_?(PSL_2(\mathbb{Z}))$ (see Proof of Theorem 2.1). By abuse of notation, this subgroup will be simply referred $PSL_2(\mathbb{Z})$. The characters a and b will denote both generators of $PSL_2(\mathbb{Z})$ as a group of matrices and their images in T under the isomorphism between $PPSL_2(\mathbb{Z})$ and Thompson's group T .

Remark 7. Let A , B and C be the three classical generators of Thompson's group T (see Examples 1 and 2). It is easy to see that $a = CA$ and $b = C$. Hence, $PSL_2(\mathbb{Z}) \subset T$ is the subgroup generated by A and C .

Recall that the group $PSL_2(\mathbb{Z})$ is isomorphic to the free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, which means that it admits a generating set $\{a, b\}$, where $a^2 = 1$ and $b^3 = 1$ (see [70], page 20 example 1.5.3). Then, every element of $PSL_2(\mathbb{Z})$ has a normal form in the generators a and b given by a word of the form

$$a^{\epsilon_1} b^{\delta_1} a b^{\delta_2} a \dots a b^{\delta_k} a^{\epsilon_2},$$

where k is a non negative integer, $\epsilon_1, \epsilon_2 \in \{0, 1\}$ and $\delta_1, \dots, \delta_k \in \{-1, 1\}$. When studying the elements of $PSL_2(\mathbb{Z})$ in the language of tree pair diagrams, it turns out that all the

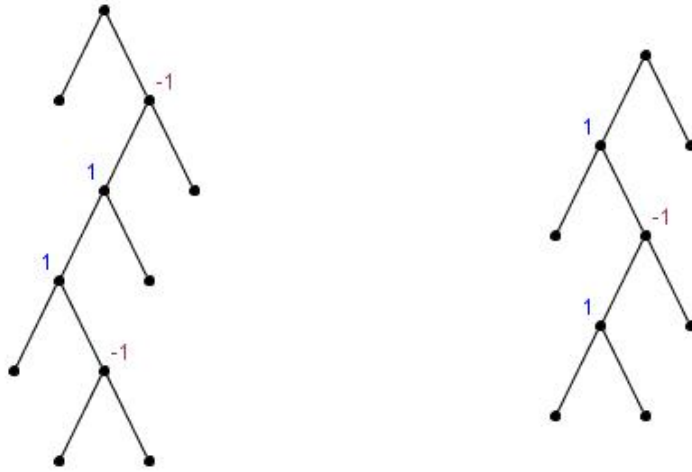
reduced tree pair diagrams representing elements of $PSL_2(\mathbb{Z})$ have exactly one leaf and one interior vertex for each level.

Definition 15. A finite rooted binary tree is *thin* if all its carets have one leaf and one interior vertex except for the last caret, which is exposed.

Remark 8. It is easy to see that one can characterize thin trees with n leaves by associating a weight $r_i \in \{-1, 1\}$ to each one of its $n - 2$ internal vertices. Let v_0, \dots, v_{n-3} be the internal vertices of a thin tree. Denote by v_{-1} its root. Then, $r_i = 1$ if v_i is the left descendant of v_{i-1} and $r_i = -1$ if v_i is the right descendant of v_{i-1} .

Remark 9. A thin tree with n leaves has exactly $n - 2$ associated weights and $n - 1$ carets. Furthermore, if l is the number of the left leaf of the exposed caret, then $l - 1$ is the cardinal of the set $\{i : r_i = -1, 0 \leq i \leq k - 1\}$.

Example 12. Two thin trees and their associated weights.



Theorem 3.1. Let (R, σ, S) be the reduced tree pair diagram of an element t of Thompson's group T . Then, t belongs to the subgroup $PSL_2(\mathbb{Z})$ if and only if (R, σ, S) satisfies one of the following conditions:

1. the number of leaves of R and S is less than 4; or
2. R is a thin tree with associated weights r_0, \dots, r_{k-1} , S is a thin tree with associated weights s_0, \dots, s_{k-1} , and the associated weights of R and S verify the equations:

$$\sum_{i=2}^{k-1} r_i s_{k+1-i} = 2 - k, \quad (3.1)$$

and

$$l + \sigma(1) + \epsilon(s_0) \equiv \frac{3 - s_1}{2} \pmod{k + 2}, \quad (3.2)$$

where $l - 1$ is the cardinal of the set $\{i : r_i = -1, 0 \leq i \leq k - 1\}$, and

$$\epsilon(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x = -1. \end{cases}$$

In particular, there is a relation between reduced words on the generators a and b of $PSL_2(\mathbb{Z})$ and reduced tree pair diagrams representing them as elements of Thompson's group T . The following result provide an explicit bijection between tree pair diagrams of elements in $PSL_2(\mathbb{Z})$ and the corresponding reduced words on the classical generators of $PSL_2(\mathbb{Z})$ of order 2 and 3 (see [44]). The inverse of the generator b is denoted by b^{-1} or \bar{b} .

Proposition 3.1. *Let $w(a, b)$ be a reduced word in normal form with respect to the generating system $\{a, b\}$ of $PSL_2(\mathbb{Z})$:*

$$w(a, b) = a^{\epsilon_1} b^{\delta_1} a b^{\delta_2} a \dots a b^{\delta_k} a^{\epsilon_2},$$

where $\epsilon_1, \epsilon_2 \in \{0, 1\}$ and $\delta_1, \dots, \delta_k \in \{-1, 1\}$. Assume that $k \geq 2$. Let R be the thin tree defined by the weights

$$\begin{aligned} r_0 &= \epsilon^{-1}(\epsilon_1) \text{ and} \\ r_i &= \delta_i, \forall 1 \leq i \leq k-1, \end{aligned}$$

and let S be the thin tree defined by the weights

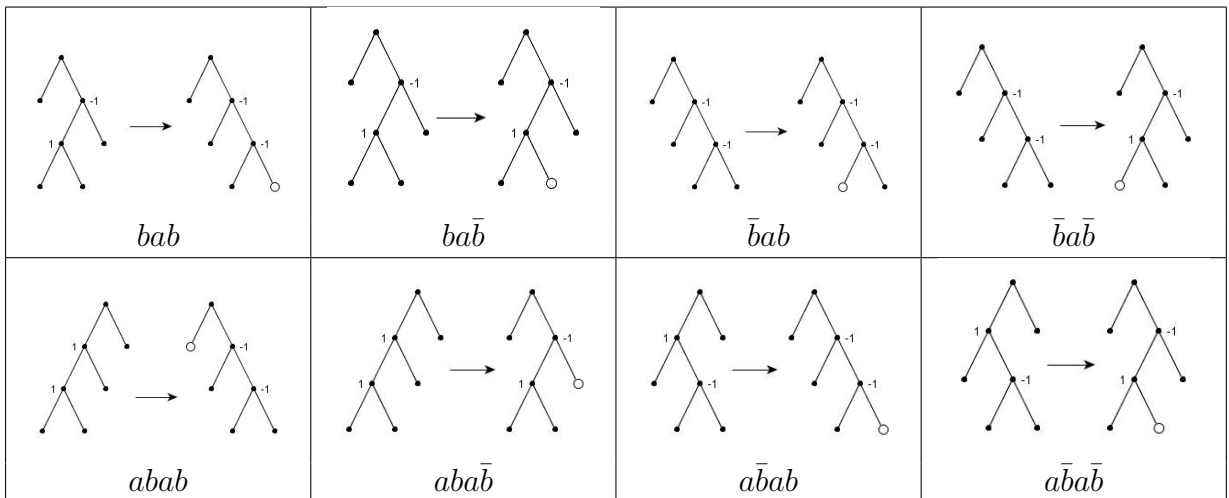
$$\begin{aligned} s_0 &= \epsilon^{-1}(\epsilon_2) \text{ and} \\ s_i &= -\delta_{k+1-i}, \forall 1 \leq i \leq k-1. \end{aligned}$$

Let σ be the cyclic permutation defined by the equation

$$\sigma(1) \equiv \frac{3 - s_1}{2} - \epsilon(s_0) - l \pmod{k+2},$$

where l and ϵ are those from Theorem 3.1. Then, (R, σ, S) is the reduced tree pair diagram for w . Furthermore, the weights r_0, \dots, r_{k-1} and s_0, \dots, s_{k-1} satisfy Equations (3.1) and (3.2) of Theorem 3.1.

Proof. We will proceed by induction. First, we consider the eight cases with $k = 2$ and $\epsilon_2 = 0$, so that the reduced word is $w = a^{\epsilon_1} b^{\delta_1} a b^{\delta_2}$.



The weights of their reduced tree pair diagrams verify the equations of the proposition

$$\begin{aligned} r_0 &= \epsilon^{-1}(\epsilon_1) \\ r_1 &= \delta_1 \\ s_0 &= -1 \\ s_1 &= -\delta_2, \end{aligned}$$

and

$$\sigma(1) \equiv \frac{3 - s_1}{2} - l \pmod{4}.$$

Now, let $w(a, b) = a^{\epsilon_1} b^{\delta_1} a b^{\delta_2} a \dots a b^{\delta_k} a^{\epsilon_2}$ be a reduced word verifying the induction hypothesis. We will see that all reduced words of type $w' = wy$, where $y \in \{a, b, \bar{b}\}$, also satisfy the equations of the proposition. There are several cases to be considered.

a.- Suppose that $\epsilon_2 = 0$. Then, $w' = wa$. By the induction hypothesis, wa has the reduced tree pair diagram of Figure 3.1.

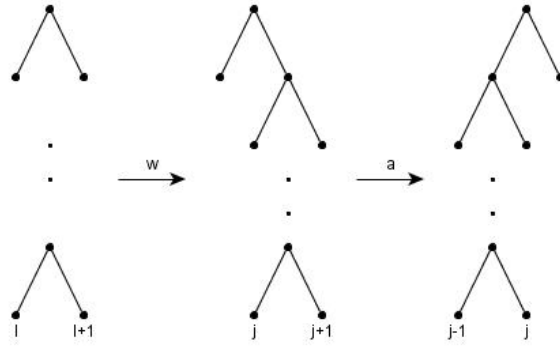


Figure 3.1: Tree pair diagrams multiplication for wa .

The tree pair diagram for wa in Figure 3.1 is reduced because the tree pair diagram for w in the same figure was reduced. As a consequence,

1. $r'_i = r_i$ for all $0 \leq i \leq k - 1$,
2. $s'_0 = 1$,
3. $s'_i = s_i$ for all $1 \leq i \leq k - 1$,
4. $l' = l$, and
5. $\sigma'(1) \equiv \sigma(1) - 1 \pmod{k + 2}$.

Since $w' = wa$, we have

1. $\epsilon'_1 = \epsilon_1$,
2. $\epsilon'_2 = 1$, and
3. $\delta'_i = \delta_i$ for all $1 \leq i \leq k$.

Thus, Equation (3.1) for w' follows from Equation (3.1) for w . Furthermore,

$$l' + \sigma'(1) + \epsilon(\epsilon'_2) = l + \sigma(1) - 1 + 1,$$

which is congruent to

$$\frac{3 - s'_1}{2}$$

by the induction hypothesis, thereby proving that Equation (3.2) is also verified.

b.- Suppose that $\epsilon_2 = 1$. Then, by the induction hypothesis w has the reduced tree pair diagram (R, σ, S) in Figure 3.2.

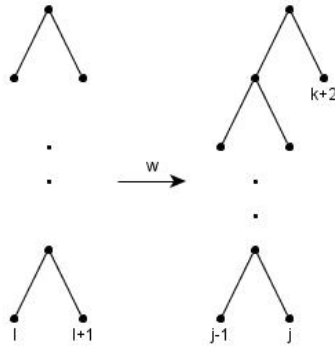


Figure 3.2: Reduced tree pair diagram for w .

Note that in order to apply b or \bar{b} to S we need to split the $(k + 2)$ -th vertex of S . That makes us split the p -th vertex of R as well, where p satisfies $\sigma(p) = k + 2$. Since

$$\sigma(p) = \sigma(1) + p - 1,$$

then

$$\sigma(1) + p - 1 = k + 2 \Rightarrow p \equiv k + 3 - \sigma(1) \equiv 1 - \sigma(1) \pmod{k + 2}.$$

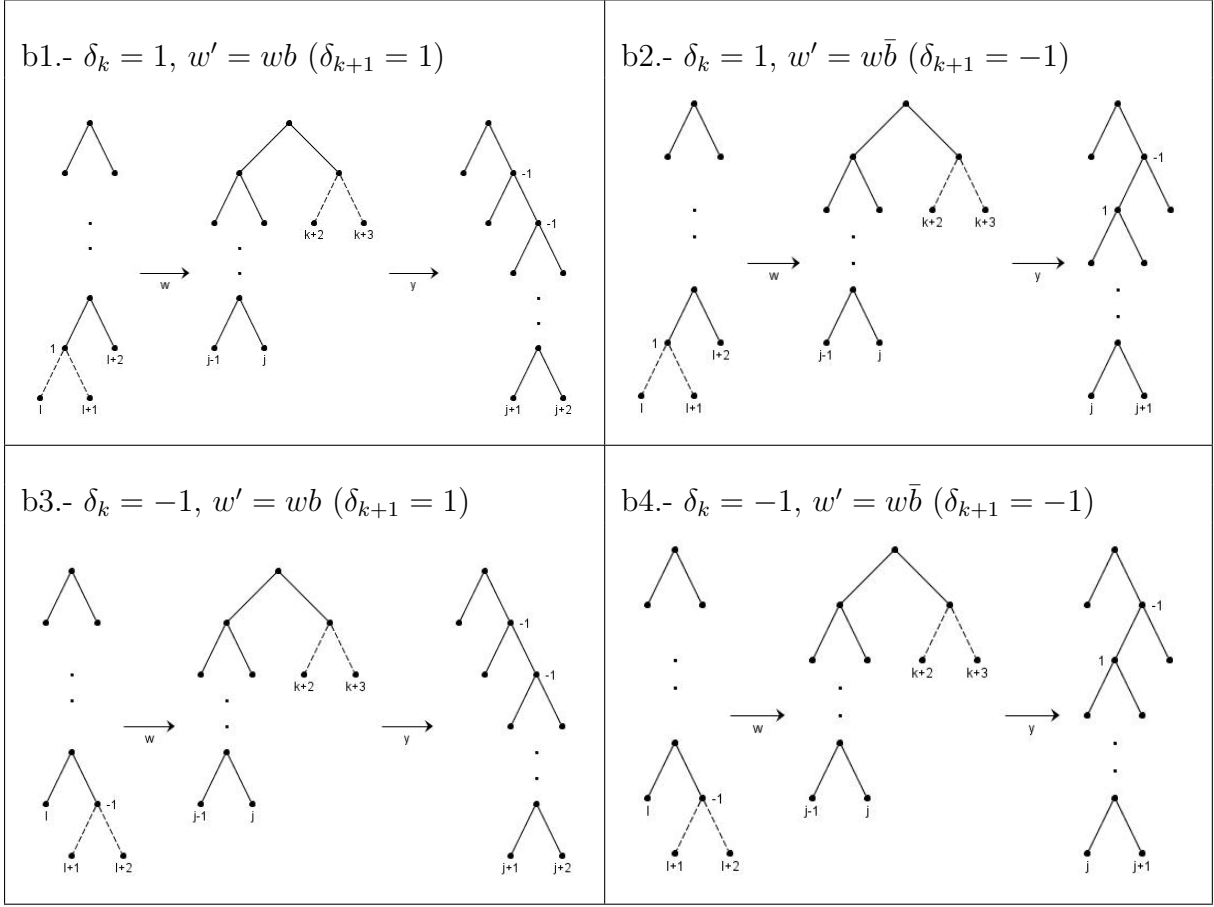
We want to see that $p \in \{l, l + 1\}$. By the induction hypothesis we have

$$\sigma(1) \equiv \begin{cases} 1 - l & \text{if } \delta_k = 1 \\ -l & \text{if } \delta_k = -1. \end{cases}$$

Thus,

$$p \equiv \begin{cases} l & \text{if } \delta_k = 1 \\ l + 1 & \text{if } \delta_k = -1. \end{cases}$$

Now, we must consider four subcases:



It is clear that the tree pair diagrams obtained for w' are reduced because the tree pair diagram for w was reduced. Note also that all four subcases fulfil the following relations:

- (r1) $k' = k + 1,$
- (r2) $r'_i = r_i, \text{ for } 0 \leq i \leq k - 1,$
- (r3) $r'_k = \delta_k,$
- (r4) $s'_0 = -1,$
- (r5) $s'_1 = -\delta_{k+1}, \text{ and}$
- (r6) $s'_i = s_{i-1}, \text{ for } 2 \leq i \leq k.$

As a consequence of (r1), (r2) and (r6) we obtain

$$\sum_{i=2}^{k'-1} r'_i s_{k'+1-i} = \sum_{i=2}^{k-1} r_i s_{k+1-i} + r'_k s'_2.$$

Thus, applying the induction hypothesis we get

$$\sum_{i=2}^{k'-1} r'_i s_{k'+1-i} = 2 - k - r'_k \delta_k.$$

Therefore, Equation (3.1) holds if and only if $r'_k \delta_k = 1$, and this follows from (r3).

Finally, we must verify that Equation (3.2) holds in our four subcases. Note that (r4) implies $\epsilon(s'_0) = 0$, so the left-hand-side of Equation (3.2) for w' is equal to $l' + \sigma'(1)$. The reduced tree pair diagrams show that

$$l' + \sigma'(1) = \begin{cases} l + \sigma(1) + 2, & \text{in case b1} \\ l + \sigma(1) + 1, & \text{in case b2} \\ l + \sigma(1) + 3, & \text{in case b3} \\ l + \sigma(1) + 2, & \text{in case b4.} \end{cases}$$

Moreover, since w verifies Equation (3.2) we obtain

$$l + \sigma(1) + 1 \equiv \begin{cases} 1, & \text{if } \delta_k = -1 \quad (\text{cases b1 and b2}) \\ 2, & \text{if } \delta_k = 1 \quad (\text{cases b3 and b4}) \end{cases} \pmod{k+2}.$$


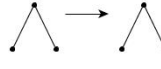

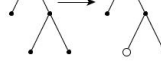
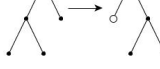
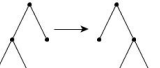
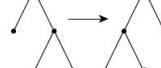
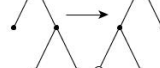
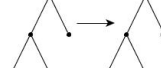
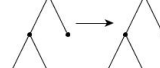
Furthermore, $1 \leq l \leq k+1$ and $1 \leq \sigma(1) \leq k+2$, so

$$l + \sigma(1) + 1 = \begin{cases} k+3, & \text{if } \delta_k = -1 \quad (\text{cases b1 and b2}) \\ k+4, & \text{if } \delta_k = 1 \quad (\text{cases b3 and b4}). \end{cases}$$

Hence, Equation (3.2) holds for w' . □

Now the proof of Theorem 3.1 follows naturally:

Proof. First, we list all reduced tree pair diagrams with less than 4 leaves and their reduced words in the generators a and b of $PSL_2(\mathbb{Z})$ (see the table below). Then, we focus on tree pair diagrams with thin trees and more than 3 leaves. The “if” part coincides with Proposition 3.1. The “only if” part will be proved by counting the number of elements fulfilling the two equations and showing this is exactly the number of elements that come from reduced words in a and b .

 1	 a	 b	 \bar{b}	 ab
 $a\bar{b}$	 ba	 $\bar{b}a$	 aba	 $a\bar{b}a$

Given a thin source tree, Equation (3.1) implies that we can only have four different thin target trees, namely those with $s_0 \in \{0, 1\}$ and $s_1 \in \{-1, 1\}$. Moreover, given a pair of compatible thin trees, l is determined by the source tree and all other terms of Equation (3.2) except $\sigma(1)$ are determined by the target tree. Then, there exists a unique permutation σ satisfying Equation (3.2). Therefore, given a thin tree R there exist exactly four reduced tree pair diagrams having R as the source tree and satisfying Equations (3.1) and (3.2), which is exactly the same number of reduced words in a, b whose associated source tree is R . □

Remark 10. From Theorem 3.1 and Proposition 3.1, we can read the reduced word in generators a, b of any element of $PSL_2(\mathbb{Z})$ directly from its reduced tree pair diagram.

3.2 Non distortion of $PSL(2, \mathbb{Z})$ and F_2 on T

Note that Theorem 3.1 and Proposition 3.1 give an estimate of the length of the elements in $PSL_2(\mathbb{Z})$ as a subgroup of Thompson's group T . It is thus natural to ask whether or not $PSL_2(\mathbb{Z})$ is distorted as a subgroup of Thompson's group T .

Definition 16. A finitely generated subgroup H of a finitely generated group G is *distorted* if there exists an infinite family $\{h_n\}_{n \in \mathbb{N}}$ of elements of H such that

1. $|h_n|_Z < |h_{n+1}|_Z$,
2. $\lim_{n \rightarrow \infty} |h_n|_Z = \infty$, and
3. the limit $\lim_{n \rightarrow \infty} \frac{|h_n|_Z}{|h_n|_Y}$ exists and is equal to zero,

where Z is a finite generating set for the group G which contains a finite generating set Y for the subgroup H , and $|\cdot|_Z$ and $|\cdot|_Y$ denote the word metrics on the generating sets Z and Y , respectively. Observe that $|\cdot|_Z \leq |\cdot|_Y$. See [40] for another definition of distortion.

Remark 11. The subgroup H is non distorted in G if and only if there are constants $K > 0$ and L such that

$$\frac{1}{K}|h|_Y - L \leq |h|_Z \leq K|h|_Y + L$$

holds for every $h \in H$.

Furthermore, Burillo, Cleary, Stein and Taback gave a general estimation of the length of elements in T by the number of carets of the reduced tree pair diagrams in [16]:

Theorem 3.2. ([16], theorem 5.1) *Let $N(w)$ be the number of carets of the reduced tree pair diagram representing $w \in T$, and let $|w|_{A,B,C}$ denote the word length of w in the classical generating set $\{A, B, C\}$. Then, there exists a positive constant K such that*

$$\frac{N(w)}{K} \leq |w|_{A,B,C} \leq KN(w).$$

As a consequence of Theorem 3.1, Proposition 3.1 and Theorem 3.2 we obtain:

Proposition 3.2. *The group $PSL_2(\mathbb{Z})$ is a non distorted subgroup of Thompson's group T .*

Proof. Let $w = a^{\epsilon_1} b^{\delta_1} a \dots a b^{\delta_k} a^{\epsilon_2}$ be a reduced word in the generators a and b of $PSL_2(\mathbb{Z})$, where $\delta_i \in \{-1, 1\}$ and $\epsilon_j \in \{0, 1\}$. We want to find constants $K' > 0$ and L (see Remark 11) such that

$$\frac{|w|_{a,b}}{K'} - L \leq |w|_{A,B,C,a,b} \leq K'|w|_{a,b} + L.$$

Let $N(w)$ be the number of carets of the reduced tree pair diagram representing $w \in T$. Recall that $a = CA$ and $b = C$. Thus,

$$\frac{|w|_{A,B,C}}{2} \leq |w|_{A,B,C,a,b} \leq |w|_{A,B,C}.$$

Then, thanks to Theorem 3.2 we obtain

$$\frac{N(w)}{2K} \leq |w|_{A,B,C,a,b} \leq 2KN(w). \quad (3.3)$$

Moreover,

$$|w|_{a,b} = |a^{\epsilon_1} b^{\delta_1} a \dots ab^{\delta_k} a^{\epsilon_2}|_{a,b} = 2k - 1 + \epsilon_1 + \epsilon_2.$$

From Proposition 3.1 and Theorem 3.1, the reduced tree pair diagram of w has exactly $k + 2$ leaves and $k + 1$ carets. Thus,

$$2N(w) - 3 \leq |w|_{a,b} \leq 2N(w) - 1,$$

which implies that

$$\frac{|w|_{a,b} + 1}{2} \leq N(w) \leq \frac{|w|_{a,b} + 3}{2}. \quad (3.4)$$

Therefore, from the Equations (3.3) and (3.4) we obtain the inequalities

$$\frac{|w|_{a,b} + 1}{4K} \leq |w|_{A,B,C,a,b} \leq K(|w|_{a,b} + 3),$$

which yield

$$\frac{|w|_{a,b}}{K'} - L \leq |w|_{A,B,C,a,b} \leq K'|w|_{a,b} + L,$$

where $K' = 4K$ and $L = 3K$. □

There are several normal subgroups of $PSL_2(\mathbb{Z})$ isomorphic to the free non abelian group F_2 of rank 2. One particular example is the commutator subgroup of $PSL_2(\mathbb{Z})$ (see [44]), which in addition is a finite index subgroup of $PSL_2(\mathbb{Z})$. Thus, knowing that finite index subgroups of non distorted subgroups are non distorted one obtains:

Corollary 3.1. *There exists a non distorted subgroup of Thompson's group T isomorphic to the free non abelian group F_2 of rank 2.*

Proof. Let H be the subgroup of $PSL_2(\mathbb{Z})$ generated by $g = ab\bar{a}$ and $h = \bar{a}b\bar{a}b$, which is isomorphic to the free non abelian group of rank 2 (see [21], page 26 or [59] proposition III.12.3). The subgroup H is the kernel of the morphism

$$\begin{array}{ccc} PSL_2(\mathbb{Z}) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \\ a & \longmapsto & (1, 0) \\ b & \longmapsto & (0, 1), \end{array}$$

so H is of finite index in $PSL_2(\mathbb{Z})$. Thus, H is non distorted in $PSL_2(\mathbb{Z})$ and using Proposition 3.2 H is non distorted in T . □

This is related to recent results of Calegari and Freedman [17], who analysed distorted cyclic subgroups of more general homeomorphisms groups.

3.3 Piecewise linear characterization of $PSL(2, \mathbb{Z})$

In this section we characterize the elements of $PSL_2(\mathbb{Z})$ as piecewise linear maps of the unit interval with identified endpoints. Note that a piecewise linear map of $[0, 1]$ with identified endpoints t can be given by the coordinates of its non differentiable points $(x_0, y_0), (x_1, y_1), \dots, (x_k, y_k)$. Furthermore, $x_0 = 0, x_k = 1$ and there exists $i \in \{0, \dots, k\}$ such that $y_i = 0$. Let $\Delta_j(x)$ denote the difference $x_j - x_{j-1}$ for $1 \leq j \leq k$, and $\Delta_j(y)$ denote $y_j - y_{j-1}$. Thus, the map t can be represented by the sequence

$$S(t) = \left\{ \frac{\Delta_1(y)}{\Delta_1(x)}, \frac{\Delta_2(y)}{\Delta_2(x)}, \dots, \frac{\Delta_i(y)}{\Delta_i(x)}, \frac{\circ\Delta_{i+1}(y)}{\Delta_{i+1}(x)}, \dots, \frac{\Delta_k(y)}{\Delta_k(x)} \right\},$$

since, for $0 \leq j \leq k$ we have

$$x_j = \sum_{l=1}^j \Delta_l \quad \text{and} \quad y_j = \sum_{l=i+1}^{i+1+j} \Delta_{l(\text{mod } k)}.$$

Note that the mark \circ on the sequence $S(t)$ denotes the cyclical permutation which prevents us to have $y_0 = 0$. We will denote by $\Delta_x(t)$ the sequence $\Delta_1(x), \Delta_2(x), \dots, \Delta_k(x)$, by $\Delta_y(t)$ the sequence $\Delta_1(y), \Delta_2(y), \dots, \Delta_k(y)$ and by $\Delta_y^\circ(t)$ the sequence $\Delta_{i+1}(y), \Delta_{i+2}(y), \dots, \Delta_k(y), \Delta_1(y), \dots, \Delta_i(y)$. Finally, we will denote by $\Delta(t)$ either $\Delta_x(t), \Delta_y(t)$ or $\Delta_y^\circ(t)$.

Remark 12. A piecewise linear map f is an element of Thompson group T if and only if for all $j \in \{1, \dots, k\}$, x_j, y_j are dyadic rational numbers and the fraction

$$\frac{\Delta_j(y)}{\Delta_j(x)}$$

is a power of two.

Definition 17. Let σ be a permutation of the set $\{1, \dots, k\}$. Then, σ is k -extremal if:

1. $\sigma(1) = 1$ or $\sigma(1) = k$;
2. for $2 \leq i \leq k-2$, $\sigma(i)$ is the maximum or the minimum of the set

$$\{1, \dots, k\} \setminus \bigcup_{j=1}^{i-1} \sigma(j);$$

3. $\sigma(k-1)$ is the minimum and $\sigma(k)$ is the maximum of the pair

$$\{1, \dots, k\} \setminus \bigcup_{j=1}^{k-2} \sigma(j).$$

Example 13. The permutation $\sigma = (1, 2, 3, 4, 5) \mapsto (5, 1, 2, 3, 4)$ is 5-extremal.

Definition 18. The sequence Δ is k -thin if there exists σ k -extremal such that:

(t1) $\Delta_{\sigma^{-1}(i)} = 2^{-i}$, for $1 \leq i \leq k-1$, and

(t2) $\Delta_{\sigma^{-1}(k)} = 2^{-k+1}$.

Lemma 3.1. *There exists a canonical bijection between k -thin sequences and thin trees with k leaves.*

Proof. Let Δ be a k -thin sequence. Note that $\Delta = \{\Delta^{(-1)}, 2^{-k+1}, 2^{-k+1}, \Delta^{(1)}\}$, where $\Delta^{(-1)}$ and $\Delta^{(1)}$ are subsequences of Δ . For $0 \leq i \leq k-3$, let us define $r_i = j$ if $\Delta_{\sigma^{-1}(i+1)} \in \Delta^{(j)}$. Then, the thin tree with weights r_0, \dots, r_{k-3} is associated to Δ . Finally, we remark that the number of thin trees with k leaves and the number of k -thin permutations are both equal to 2^{k-2} . \square

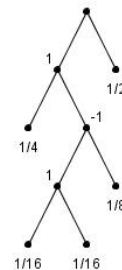
Example 14. A 5-thin sequence and its associated thin tree are shown below.

$$\Delta = \{2^{-2}, 2^{-4}, 2^{-4}, 2^{-3}, 2^{-1}\}.$$

$$\sigma = (1, 2, 3, 4, 5) \mapsto (5, 1, 4, 2, 3).$$

$$\Delta^{(-1)} = \{2^{-2}\} \Rightarrow r_1 = -1.$$

$$\Delta^{(1)} = \{2^{-3}, 2^{-1}\} \Rightarrow r_2 = r_0 = 1.$$



Definition 19. Let t be a piecewise linear map. The sequence $S(t)$ representing t is k -good if

(g1) the sequence $\Delta_x(t)$ is k -thin,

(g2) the sequence $\Delta_y^\circ(t)$ is k -thin, and

(g3) for $3 \leq j \leq k-2$, the fraction $\frac{2^{-k-1+j}}{2^{-j}}$ appears in $S(t)$.

Example 15. The sequence $\left\{ \frac{2^{-5}}{2^{-3}}, \frac{2^{-3}}{2^{-5}}, \frac{2^{-2}}{2^{-6}}, \frac{\circ 2^{-1}}{2^{-6}}, \frac{2^{-4}}{2^{-4}}, \frac{2^{-6}}{2^{-2}}, \frac{2^{-6}}{2^{-1}} \right\}$ is 7-good.

Theorem 3.3. *An element $t \in T$ belongs to the subgroup $PSL_2(\mathbb{Z}) < T$ if and only if the sequence $S(t)$ is k -good.*

Proof. For the “if” part, we first enumerate the 2-good and 3-good sequences $S(t)$ with their corresponding elements of $PSL_2(\mathbb{Z})$ in the following table.

$\frac{\circ 1}{1}, \frac{1}{1}$	$\frac{1}{1}, \frac{\circ 1}{1}$	$\frac{\circ 1}{2}, \frac{1}{1}, \frac{2}{1}$	$\frac{1}{2}, \frac{1}{1}, \frac{\circ 2}{1}$	$\frac{1}{2}, \frac{2}{1}, \frac{\circ 1}{1}$
1	a	$\bar{b}a$	\bar{b}	ba
$\frac{1}{2}, \frac{\circ 2}{1}, \frac{1}{1}$	$\frac{2}{1}, \frac{\circ 1}{1}, \frac{1}{2}$	$\frac{\circ 2}{1}, \frac{1}{1}, \frac{1}{2}$	$\frac{1}{1}, \frac{2}{1}, \frac{\circ 1}{2}$	$\frac{1}{1}, \frac{\circ 2}{1}, \frac{1}{2}$
b	aba	ab	$a\bar{b}a$	$a\bar{b}$

Now suppose that we are given a k -good sequence $S(t)$ with $k \geq 4$. Let r_0, \dots, r_{k-3} be the weights associated to $\Delta_x(t)$ and s_0, \dots, s_{k-3} be the weights associated to $\Delta_y^\circ(t)$ from Lemma 3.1. First, it will be proved that r_0, \dots, r_{k-3} and s_0, \dots, s_{k-3} satisfy Equation 3.1 of Theorem 3.1, by showing that $r_j = -s_{k-1-j}$ for $2 \leq j \leq k-3$.

From (g3) we know that, for $3 \leq j \leq k-2$, $S(t)$ contains the term

$$\frac{2^{-k+j-1}}{2^{-j}}.$$

Let $2 \leq i, j \leq k-3$. Thus, one of the following occurs:

1. $S(t) = \left\{ \frac{2^{-k+1}}{2^{-1}}, \frac{2^{-k+1}}{2^{-2}}, S^{(1)}, \frac{2^{-k+i-2}}{2^{-i-1}}, S^{(2)}, \frac{\circ}{2^{-k+1}}, \frac{\circ}{2^{-k+1}}, \frac{\circ}{2^{-m}}, S^{(3)}, \frac{2^{-k+j-2}}{2^{-j-1}}, S^{(4)} \right\}$
2. $S(t) = \left\{ S^{(1)}, \frac{2^{-k+i-2}}{2^{-i-1}}, S^{(2)}, \frac{\circ}{2^{-k+1}}, \frac{\circ}{2^{-k+1}}, \frac{\circ}{2^{-m}}, S^{(3)}, \frac{2^{-k+j-2}}{2^{-j-1}}, S^{(4)}, \frac{2^{-k+1}}{2^{-2}}, \frac{2^{-k+1}}{2^{-1}} \right\}$
3. $S(t) = \left\{ \frac{2^{-k+1}}{2^{-1-\delta}}, S^{(1)}, \frac{2^{-k+i-2}}{2^{-i-1}}, S^{(2)}, \frac{\circ}{2^{-k+1}}, \frac{\circ}{2^{-k+1}}, \frac{\circ}{2^{-m}}, S^{(3)}, \frac{2^{-k+j-2}}{2^{-j-1}}, S^{(4)}, \frac{2^{-k+1}}{2^{-2+\delta}} \right\}$

where $\delta \in \{0, 1\}$ and $S^{(q)}$ are subsequences, for $1 \leq q \leq 4$. Then, in each one of the above cases we have $r_i = -1 = -s_{k-1-i}$ and $r_j = 1 = -s_{k-1-j}$.

Now we find a cyclic permutation σ such that s_0, s_1 and $\sigma(1)$ fulfil Equation (3.2) of Theorem 3.1. Let p be the position of the mark on $\Delta_y(t)$. Define $\sigma(1)$ as the integer satisfying

$$\sigma(1) + p - 1 \equiv 1 \pmod{k}.$$

Let l be the position of the first 2^{-k+1} on $\Delta_x(t)$. Since $S(t)$ is k -good we have four possible cases:

1. $S(t) = \left\{ S^{(1)}, \frac{\circ 2^{-1}}{2^{-k+1}}, \frac{2^{-2}}{2^{-k+1}}, S^{(2)} \right\}$, so that $s_0 = s_1 = -1$ and $l \equiv p \pmod{k}$.
2. $S(t) = \left\{ S^{(1)}, \frac{2^{-1}}{2^{-k+1}}, \frac{\circ 2^{-2}}{2^{-k+1}}, S^{(2)} \right\}$, so that $s_0 = 1, s_1 = -1$ and $l \equiv p - 1 \pmod{k}$.
3. $S(t) = \left\{ S^{(1)}, \frac{2^{-2}}{2^{-k+1}}, \frac{\circ 2^{-1}}{2^{-k+1}}, S^{(2)} \right\}$, so that $s_0 = -1, s_1 = 1$ and $l \equiv p - 1 \pmod{k}$.
4. $S(t) = \left\{ S^{(1)}, \frac{2^{-2}}{2^{-k+1}}, \frac{2^{-1}}{2^{-k+1}} \frac{\circ}{2^{-m}}, S^{(2)} \right\}$, so that $s_0 = s_1 = 1$ and $l \equiv p - 2 \pmod{k}$.

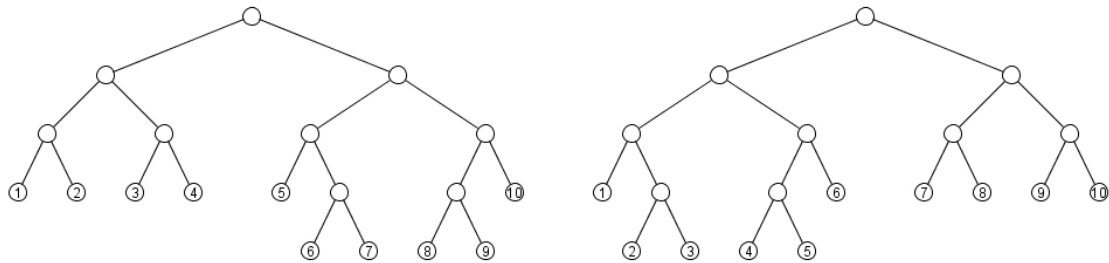
which corresponds exactly with Equation (3.2).

Finally, for the “only if” part, it suffices to remark that given a k -thin sequence $\Delta_x(t)$, there are exactly four k -good sequences $S(t)$. This coincides with the number of thin marked trees satisfying the equations of Theorem 3.1 that could be a target tree for a given thin source tree. \square

Remark 13. For all the elements of $PSL(2, \mathbb{Z})$, the number of linear pieces coincides with the number of leaves of its reduced tree pair diagram. This is not true in general. For example, consider the following element of T :

$$g(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{8} \\ \frac{x}{2} + \frac{1}{16}, & \frac{1}{8} \leq x \leq \frac{5}{8} \\ 2x - \frac{7}{8}, & \frac{5}{8} \leq x \leq \frac{7}{8} \\ x, & \frac{7}{8} \leq x \leq 1. \end{cases}$$

Its definition as a piecewise linear map has four pieces but its reduced tree pair diagram



has 10 leaves.

Chapter 4

Counting conjugacy classes of torsion elements on T

In this chapter we prove that there are exactly $\varphi(n)$ conjugacy classes of elements of order n in Thompson's group T , where φ is the Euler function. For doing so, we introduce a new viewpoint of T in terms of maps between dyadic polygons inscribed on dyadic triangulations of the unit disk. The first section is devoted to the notions of dyadic triangulation of the disk and dyadic almost triangulation of the disk, which will be strongly used from now on.

4.1 Dyadic (almost) triangulations of the disk

Let D be the closed disk in \mathbb{R}^2 of center $(0, 0)$ and perimeter 1. The boundary of D is denoted ∂D . Let $\gamma : [0, 1] \rightarrow \partial D$ be the arc-parametrization of ∂D with $\gamma(0) = (\frac{1}{2\pi}, 0)$.

We denote by $\mathbb{Z}[1/2]$ the set of dyadic rational numbers. Thus,

$$\mathbb{Z}[1/2] \cap [0, 1] = \left\{ \frac{m}{2^n} \in \mathbb{R} : m \in \mathbb{Z}_{\geq 0}, n \in \mathbb{N}, m \leq 2^n \right\}.$$

We denote by $\partial D[1/2]$ the image of $[0, 1] \cap \mathbb{Z}[1/2]$ by γ .

Definition 20. Let \mathcal{A} be the set of geodesic segments of D with extremal points in $\partial D[1/2]$. The elements of \mathcal{A} are called *dyadic arcs*.

Note that one can represent \mathcal{A} by

$$\mathcal{A} = \{(\gamma(x), \gamma(y)) : x, y \in \mathbb{Z}[1/2] \cap [0, 1], x < y\}.$$

Definition 21. Let Δ be a subset of \mathcal{A} . The set Δ is a *dyadic triangulation of D* if $D - \Delta$ is a disjoint union of open triangles. Thus, given a dyadic arc $a \in \mathcal{A}$ we write $a \in \Delta$ if a belongs to the set Δ , and $a \notin \Delta$ if it does not.

Example 16. Farey's triangulation of the Poincaré disk \mathbb{D}^2 can be adapted to the Euclidean disk D by considering dyadic arcs (Euclidean segments¹) instead of bi-infinite

1. For simplicity we will continue to draw hyperbolic geodesics.

hyperbolic geodesics. *Farey's dyadic triangulation* will also be denoted by $\Delta_F \subset \mathcal{A}$, and is defined as follows:

$$\begin{aligned} \Delta_F &= \{(\gamma(0), \gamma(\tfrac{1}{2}))\} \\ &\cup \{(\gamma(\tfrac{m}{2^n}), \gamma(\tfrac{m+1}{2^n})) : m, n \in \mathbb{Z}_{\geq 0}, m \in \{0, \dots, 2^n - 1\}, n > 1\}. \end{aligned}$$

Definition 22. A dyadic triangulation $\Delta \subset \mathcal{A}$ of D is *locally finite* if for every point x on the interior of D there exists an open neighbourhood $N(x)$ which intersects only a finite number of dyadic arcs of Δ .

Example 17. The set of dyadic arcs

$$\begin{aligned} \Delta &= \{(\gamma(\tfrac{1}{4}), \gamma(\tfrac{3}{4}))\} \\ &\cup \{(\gamma(\tfrac{m}{2^{2k+1}}), \gamma(\tfrac{m+1}{2^{2k+1}})) : k, m \in \mathbb{N}, 0 \leq m < 2^{2k+1}\} \\ &\cup \{(\gamma(\tfrac{2j+1}{2^{2k}}, \gamma(\tfrac{2j+2}{2^{2k}})) : j, k \in \mathbb{N}, 0 < 2j+1 < 2^{2k}\} \\ &\cup \{(\gamma(\tfrac{4j+1}{2^{2k+1}}, \gamma(\tfrac{2j+2}{2^{2k+1}})) : j, k \in \mathbb{N}, 0 < 4j+1 < 2^{2k+1}\} \end{aligned}$$

defines a dyadic triangulation of D which is locally finite on the interior of D .

Example 18. The set of dyadic arcs

$$\begin{aligned} \Delta &= \{(\gamma(0), \gamma(\tfrac{1}{2}))\} \\ &\cup \{(\gamma(\tfrac{m}{2^n}), \gamma(\tfrac{m+1}{2^n})) : m, n \in \mathbb{N}, 0 < m < 2^n\} \\ &\cup \{(\gamma(\tfrac{1}{2^k}), \gamma(\tfrac{1}{2})) : k \in \mathbb{N}\} \end{aligned}$$

defines a dyadic triangulation of D which is locally infinite on the interior of D on the interior points of the dyadic arc $(\gamma(0), \gamma(1/2))$, and locally finite elsewhere.

Definition 23. Let a_1, \dots, a_k be dyadic arcs of Δ_F . It is said that they define a *Farey polygon* if the arcs a_1, \dots, a_k bound a k -gon in D . This polygon is denoted by $\mathcal{P}(a_1, \dots, a_k)$. If the dyadic arc $(\gamma(0), \gamma(1/2)) \in \mathcal{A}$ is contained on the polygon $\mathcal{P}(a_1, \dots, a_k)$ (eventually on its boundary), $\mathcal{P}(a_1, \dots, a_k)$ is named a *central Farey polygon*.

Remark 14. Given a Farey polygon $\mathcal{P}(a_1, \dots, a_k)$, one can construct a minimal central Farey polygon $\mathcal{P}(a'_1, \dots, a'_l)$ containing $\mathcal{P}(a_1, \dots, a_k)$ by gluing to it as many adjacent triangles of $D - \mathcal{P}(a_1, \dots, a_k)$ as needed. Furthermore, it follows from Remark 1 that any other central Farey polygon containing $\mathcal{P}(a_1, \dots, a_k)$ contains $\mathcal{P}(a'_1, \dots, a'_l)$.

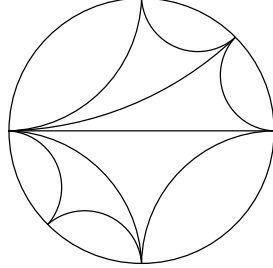
Definition 24. A dyadic triangulation Δ of D is of *Farey type* if the subset

$$n\mathcal{F}(\Delta) := \{a \in \Delta : a \notin \Delta_F\} \subset \Delta$$

is finite.

Remark 15. Given a dyadic triangulation Δ of Farey type, there exists a Farey polygon $\mathcal{P}(\Delta)$ containing all the arcs on $n\mathcal{F}(\Delta)$. Thus, Δ and Δ_F coincide outside $\mathcal{P}(\Delta)$. Furthermore, using Remark 14 there exists a Farey central polygon $\mathcal{P}_c(\Delta)$ containing all the arcs on $n\mathcal{F}(\Delta)$.

Example 19. A dyadic triangulation of Farey type:



In this case, $n\mathcal{F}(\Delta) = \{(\gamma(\frac{1}{8}), \gamma(\frac{1}{2}))\}$. One must imagine that outside the plotted polygon there are all dyadic arcs corresponding to Farey's dyadic triangulation Δ_F .

Definition 25. Let Δ be a dyadic triangulation of D of Farey type, and let a_1, \dots, a_k be dyadic arcs of Δ . We say that a_1, \dots, a_k define a Δ -polygon if they bound a k -gon $\mathcal{P}(a_1, \dots, a_k)$ in D . If the Farey central polygon $\mathcal{P}_c(\Delta)$ is contained on $\mathcal{P}(a_1, \dots, a_k)$ (eventually partially on its boundary), the Δ -polygon $\mathcal{P}(a_1, \dots, a_k)$ is *central*.

Remark 16. Given a Δ -polygon $\mathcal{P}(a_1, \dots, a_k)$, there exists a minimal central Δ -polygon $\mathcal{P}(a'_1, \dots, a'_l)$ containing $\mathcal{P}(a_1, \dots, a_k)$. Furthermore, any other central Δ -polygon containing $\mathcal{P}(a_1, \dots, a_k)$ contains $\mathcal{P}(a'_1, \dots, a'_l)$.

Before ending this section, we need to define the objects obtained from dyadic triangulations of D by taking off a finite set of dyadic arcs. Note that given Δ_1, Δ_2 two different dyadic triangulations of D of Farey type, their intersection $\Delta_1 \cap \Delta_2$ is a dyadic triangulation of D where there are finitely many missing dyadic arcs.

Definition 26. A subset $a\Delta$ of \mathcal{A} is a *dyadic almost triangulation of D* if all connected components of $D - a\Delta$ are polygons bounded by a finite number of dyadic arcs of $a\Delta$, and only a finite number of them are not triangles. Furthermore, if there are only a finite number of dyadic arcs in $a\Delta$ which do not belong to Δ_F , we say that $a\Delta$ is of *Farey type*.

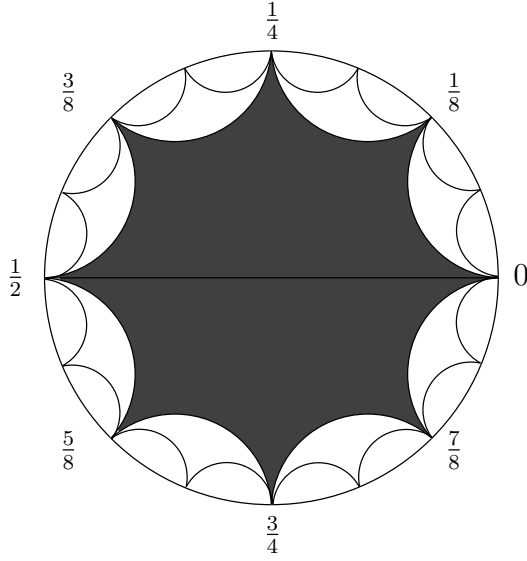
Definition 27. Let $a\Delta$ be a dyadic almost triangulation of D . We denote by $\mathcal{D}(a\Delta)$ the minimum number of dyadic arcs needed to transform $a\Delta$ into a triangulation. We denote by $\mathcal{P}(a\Delta)$ the minimal polygon whose boundary belongs to $a\Delta$ and which interior contains all non-triangular polygons of $D - a\Delta$. Finally, we denote by $\mathcal{S}(a\Delta)$ the number of sides of $\mathcal{P}(a\Delta)$.

Remark 17. Let $a\Delta$ be a dyadic almost triangulation of D . Then,

$$\mathcal{S}(a\Delta) \geq \mathcal{D}(a\Delta) + 3.$$

Example 20. The set of dyadic arcs

$$\begin{aligned} a\Delta &= \{(\gamma(0), \gamma(\frac{1}{2}))\} \\ &\cup \{(\gamma(\frac{m}{2^n}), \gamma(\frac{m+1}{2^n})) \mid m, n \in \mathbb{Z}_{>0}, m \in \{0, \dots, 2^n - 1\}, n > 2\} \end{aligned}$$



is a dyadic almost triangulation such that:

1. $\mathcal{D}(a\Delta) = 4$,
2. $\mathcal{P}(a\Delta)$ is the regular octagon with vertices $\{\gamma(m/8) : 0 \leq m \leq 7\}$,
3. $\mathcal{S}(a\Delta) = 8$, and
4. if we add the set of dyadic arcs

$$\left\{ \left(\gamma(0), \gamma\left(\frac{1}{4}\right) \right), \left(\gamma\left(\frac{1}{4}\right), \gamma\left(\frac{1}{2}\right) \right), \left(\gamma\left(\frac{1}{2}\right), \gamma\left(\frac{3}{4}\right) \right), \left(\gamma\left(\frac{3}{4}\right), \gamma(1) \right) \right\}$$

to $a\Delta$, we obtain Farey's dyadic triangulation Δ_F .

4.2 T as maps between dyadic polygons

Lemma 4.1. *The triple $(\mathcal{P}(a_1, \dots, a_k), \mathcal{P}(a'_1, \dots, a'_k), \sigma)$ represents a unique element t of Thompson's group T , where $\mathcal{P}(a_1, \dots, a_k)$ and $\mathcal{P}(a'_1, \dots, a'_k)$ are two central Farey polygons, the arcs a_1, \dots, a_k and a'_1, \dots, a'_k are cyclically ordered, and σ is a cyclic permutation of the set $\{1, \dots, k\}$, this meaning that the arc a_i is sent to the arc $a'_{\sigma(i)}$ by t . Furthermore, if we expand $\mathcal{P}(a_1, \dots, a_k)$ by adding the Farey triangle adjacent to a_i , we expand $\mathcal{P}(a'_1, \dots, a'_k)$ by adding the Farey triangle adjacent to $a'_{\sigma(i)}$, and we consider the cyclic permutation which is coherent with these expansions, the triple we obtain still represents $t \in T$. Thus, if two triples (P_1, Q_1, σ_1) and (P_2, Q_2, σ_2) have finite sequences of expansions leading to the same triple (P, Q, σ) , they represent the same element of T .*

Proof. Note that the dyadic arcs bounding a central Farey polygon which has the dyadic arc $(\gamma(0), \gamma(1/2))$ on its interior is a set of separating arcs. Reciprocally, every set of separating arcs bounds a central Farey polygon. Hence, there is a bijection between the set of standard dyadic partitions with at least 3 points and the set of central Farey polygons and, by Proposition 2.2, the triple $(\mathcal{P}(a_1, \dots, a_k), \mathcal{P}(a'_1, \dots, a'_k), \sigma)$ represents a unique element of Thompson's group T . \square

Note that Lemma 4.1 can be thought as a reformulation of the asymptotic mapping class group viewpoint of Thompson's group T . Thanks to Remark 14 we can generalise this representation to Farey polygons.

Lemma 4.2. *Let $\mathcal{P}(a_1, \dots, a_k)$ and $\mathcal{P}(a'_1, \dots, a'_k)$ be two Farey polygons and σ a cyclic permutation of the set $\{1, \dots, k\}$. Then, the triple*

$$(\mathcal{P}(a_1, \dots, a_k), \mathcal{P}(a'_1, \dots, a'_k), \sigma)$$

represents a unique element of Thompson's group T .

Proof. It follows from Remark 14 and Lemma 4.1. □

It is worth to mention that, even though the most used representations of Thompson's group T are based on the tree structure of standard dyadic intervals, T does not act on the rooted binary tree R_2 neither on its unrooted version T_3 , but on their boundary at infinity $\partial D[1/2]$. Somehow, what is actually preserved is the Farey type of dyadic (almost) triangulations. Therefore, there is no essential obstruction on considering a more general representation of T by maps between polygons inscribed on dyadic triangulations of Farey type. Our representation of elements of T can be related to Imbert's approach [46].

Lemma 4.3. *Let Δ be a dyadic triangulation of D of Farey type, let $\mathcal{P}(a_1, \dots, a_k)$ and $\mathcal{P}(a'_1, \dots, a'_k)$ be Δ polygons, and let σ be a cyclic permutation of the set $\{1, \dots, k\}$. Then, the quadruple*

$$(\Delta, \mathcal{P}(a_1, \dots, a_k), \mathcal{P}(a'_1, \dots, a'_k), \sigma)$$

represents a unique element of Thompson's group T .

Proof. It follows from Proposition 2.2 and Corollary 16. □

4.3 On the number of conjugacy classes

There is a result due to Burillo, Cleary, Stein and Taback (see Proposition 6.1 of [16]) concerning torsion elements of Thompson's group T , given in terms of tree pair diagrams. This result can be reformulated in the language of maps between dyadic polygons as follows:

Proposition 4.1. *([16], Proposition 6.1) Let t be an element of order n of Thompson's group T . Then, there exists*

$$(\Delta_F, \mathcal{P}(a_1, \dots, a_{kn}), \mathcal{P}(a_1, \dots, a_{kn}), \sigma)$$

a representation of t , where σ has order n . Furthermore, every element of T admitting a representation of this form has order n .

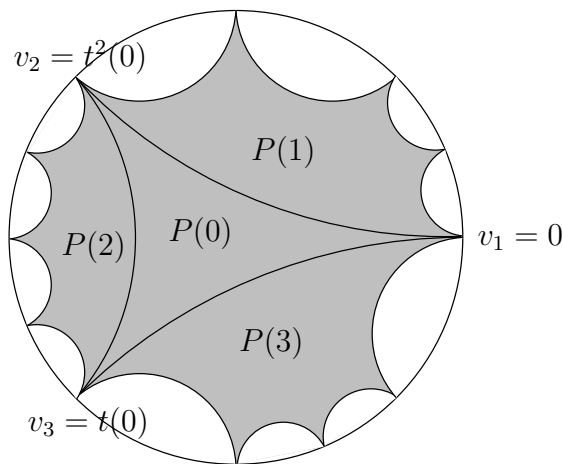
The generalization of the asymptotic mapping class group version of Thompson's group T to maps between dyadic triangulations of Farey type and inscribed dyadic polygons, allow us to characterise elements of order n by representations with n -gons.

Proposition 4.2. *Let t be an element of order n of Thompson's group T . Then, there exists a dyadic triangulation Δ of Farey type, an n -gon $\mathcal{P}(a_1, \dots, a_n)$ inscribed in Δ , and a cyclic permutation $\sigma \in S_n$ such that the triple*

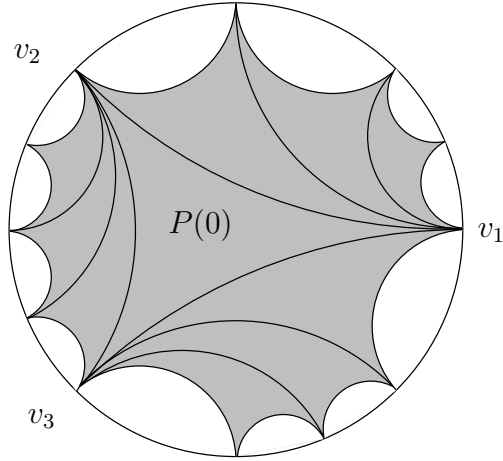
$$(\Delta, \mathcal{P}(a_1, \dots, a_n), \mathcal{P}(a_1, \dots, a_n), \sigma)$$

is a representation of t , and σ has order n . Furthermore, every element of T admitting a representation of this form has order n .

Proof. For the proof of the first part we will use Proposition 4.1. If $k = 1$ on the representation of Proposition 4.1, we are done. Otherwise, consider the polygon $P(0)$ with vertices $0, t(0), t^2(0), \dots, t^{n-1}(0)$. Note that $\mathcal{P}(a_1, \dots, a_{kn})$ (plotted filled in the figure) of Proposition 4.1 (which is a central Farey polygon) contains $P(0)$ on its interior.

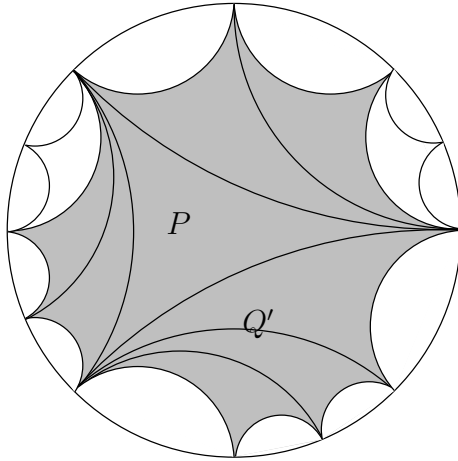


We denote by $0 = v_1, v_2, \dots, v_n$ the vertices of P_0 , previously cyclically ordered. Without loss of generality, we can suppose that a_1 is the first edge of $\mathcal{P}(a_1, \dots, a_{kn})$ in counter-clockwise order starting by $\gamma(0)$. In particular, $\gamma(0)$ is a vertex of a_1 . Now consider the polygon $P(1)$ defined as the connected component of $\mathcal{P}(a_1, \dots, a_{kn}) - P(0)$ containing a_1 on its boundary. Consider the star triangulation of $P(1)$ based on $\gamma(0)$ (all diagonals contain $\gamma(0)$ as extremal point). For $i \in \{2, \dots, n\}$, let $P(i)$ be the polygon defined as the connected component of $\mathcal{P}(a_1, \dots, a_{kn}) - P(0)$ containing v_i and not containing v_{i-1} on its boundary. Triangulate the polygons $P(2), \dots, P(n)$ by taking the successive images of $P(1)$ by t , meaning that $P(i)$ is triangulated by the star based on v_i . Choose a random triangulation of $P(0)$. Denote by Δ the dyadic triangulation of Farey type obtained this way inside $\mathcal{P}(a_1, \dots, a_{kn})$ and which coincide with Farey's triangulation Δ_F on $D - \mathcal{P}(a_1, \dots, a_{kn})$.



Thus, $(\Delta, P_0, P_0, \sigma')$ represents t by construction, where σ' is obtained by the restriction of σ of Proposition 4.1.

For the second statement, take $t \in T$ represented by the quadruple (Δ, P, P, σ) .



Let Q be a central Δ -polygon (plotted filled in the figure). Note that the connected components of $Q - P$ are Δ -polygons. Let Q' be a connected component of $Q - P$ with maximal number of sides. Thus, the successive images of Q' by t, t^2, \dots, t^n are adjacent to P and form a central Δ -polygon P' containing Q (is the exterior polygon in the figure, it contains Q' and two new adjacent triangles). The quadruple $(\Delta, P', P', \sigma')$ is a representation of t , where σ' is the canonical extension of σ . Finally, note that P' is a Farey central polygon. Thus, the quadruple $(\Delta_F, P', P', \sigma')$ also represents t and Proposition 4.1 can be applied to conclude. \square

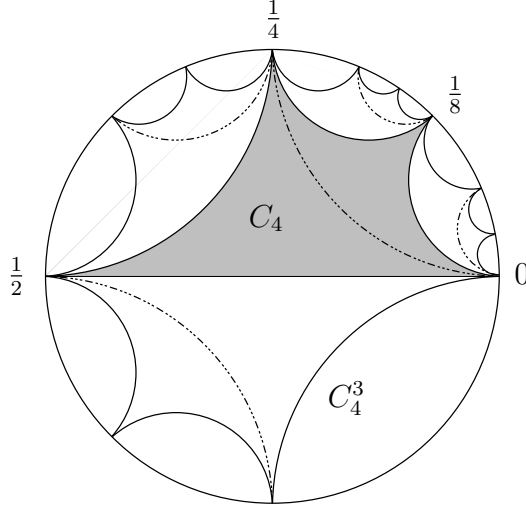
Theorem 4.1. *There are exactly $\varphi(n)$ conjugacy classes of elements of T of order n , where φ is the Euler function.*

Proof. Let t be an element of Thompson's group T of order n and let $(\delta, P(0), P(0), \sigma')$ be the representation of t obtained in Proposition 4.2 (we keep the same notations). Let $(\Delta_F, P_{kn}, P_{kn}, \sigma)$ be the representation of T obtained in Proposition 4.1. Consider the

central Farey polygon C_n given by the standard dyadic partition

$$0 < \frac{1}{2^{n-1}} < \frac{1}{2^{n-2}} < \dots < \frac{1}{2} < 1,$$

and let t_n be the element represented by $(\Delta_F, C_n, C_n, \sigma')$. We will construct an element $w_t \in T$ satisfying $t_n = w_t t w_t^{-1}$. Let C_n^k be the central Farey polygon



obtained from C_n by gluing to the first side $(\gamma(0), \gamma(\frac{1}{2^{n-1}}))$ the Farey polygon with $k+1$ vertices having as induced triangulation by Farey's triangulation the star based at $\gamma(0)$ and which is adjacent to C_n . Similarly, to each side $(\gamma(\frac{1}{2^{n-i}}, \gamma(\frac{1}{2^{n-i-1}}))$ we glue the Farey polygon with $k+1$ vertices which is adjacent to C_n and having as induced triangulation by Farey's triangulation the star based at $\gamma(\frac{1}{2^{n-i}}$. Thus, C_n^k is the Farey central polygon given by the following standard dyadic partition:

$$\begin{array}{cccccccc} 0 & < & \frac{1}{2^{n+k-2}} & < & \frac{1}{2^{n+k-3}} & < & \dots & < & \frac{1}{2^n} & < & \frac{1}{2^{n-1}} & = \\ \frac{1}{2^{n-1}} & < & \frac{1}{2^{n-1}} + \frac{1}{2^{n+k-2}} & < & \frac{1}{2^{n-1}} + \frac{1}{2^{n+k-3}} & < & \dots & < & \frac{1}{2^{n-1}} + \frac{1}{2^n} & < & \frac{1}{2^{n-2}} & = \\ \vdots & < & \vdots & < & \vdots & < & \vdots & < & \vdots & < & \vdots & = \\ \frac{1}{2} & < & \frac{1}{2} + \frac{1}{2^k} & < & \frac{1}{2} + \frac{1}{2^{k-1}} & < & \dots & < & \frac{3}{4} & < & 1. \end{array}$$

Hence, $w_t = (\Delta_F, C_n^k, P_{kn}, id)$ satisfies $t_n = w_t t w_t^{-1}$. Therefore there are at most $\varphi(n)$ conjugacy classes of elements of T of order n : at most one for every $\sigma \in S_n$ of order n .

To conclude, it suffices to prove that given σ_1 and σ_2 , two different cyclic permutations, then the elements $(\Delta_F, C_n, C_n, \sigma_1)$ and $(\Delta_F, C_n, C_n, \sigma_2)$ of T are not on the same conjugacy class. Note that given an element t of T of order n , the cyclic order of the points

$$x, t(x), t^2(x), \dots, t^{n-1}(x)$$

is independent of the choice of $x \in [0, 1]$. Note also that if $t = (\Delta_F, C_n, C_n, \sigma)$, the cyclic order on the orbits is given by σ . Let w be an element of T and take $x = w^{-1}(0)$. Then,

for every $k \in \{0, \dots, n-1\}$ we have:

$$(wtw^{-1})^k(x) = w(t^k(x)).$$

Remark that w preserves the cyclic order on $[0, 1]$ by the definition of T . Thus, conjugation preserves the cyclic order of the orbits, and there are exactly $\varphi(n)$ conjugacy classes on elements of order n in T . \square

Remark 18. Corollary 6.3 of [16] states that every torsion element is a conjugate of an element of the form (r_m, σ, r_m) , where r_m symbolises the binary rooted tree of m leaves and all right carets. Thus, we can give an alternative proof of Theorem 4.1 in the following way: if (r_m, σ, r_m) is a conjugate of a torsion element $t \in T$ of order n , then m must be a multiple of n ($m = kn$) and σ must be a product of k disjoint n -cycles. Hence, (r_m, σ, r_m) is a conjugate of (r_n, σ', r_n) . Finally, one uses the same argument used in the proof of Theorem 4.1 to prove that (r_n, σ', r_n) and (r_n, σ'', r_n) are conjugate if and only if $\sigma' = \sigma''$.

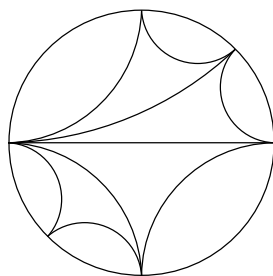
Chapter 5

T acts transitively on the infinite associahedron C

The infinite associahedron is a cellular complex C which contains all dyadic triangulations and almost triangulations of Farey type. In this chapter we prove that C is connected, simply connected, contractible and its 1-skeleton C^1 is non-hyperbolic. We also show that Thompson's group T acts on the infinite associahedron C by automorphisms. Furthermore, the action is transitive only on vertices and edges, and all stabilizers are finite except stabilizers of vertices, which are isomorphic to $PSL(2, \mathbb{Z})$.

5.1 The asymptotic flip complex C^2

Definition 28. The set of *vertices* of C

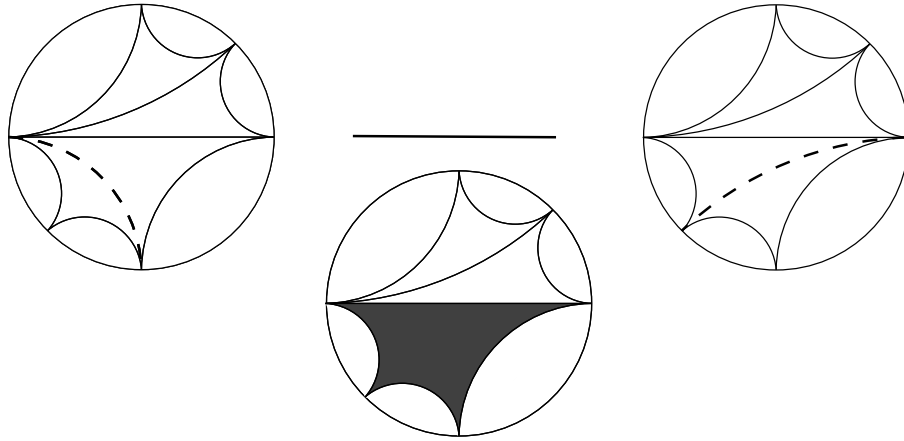


is the set of dyadic triangulations of D of Farey type

$$C^0 = \{\Delta \subset \mathcal{A} : D - \Delta = \sqcup \text{triangles}, |n\mathcal{F}(\Delta)| < \infty\}.$$

Remark 19. All dyadic triangulations $\Delta \in C^0$ of Farey type are locally finite triangulations on the interior of D .

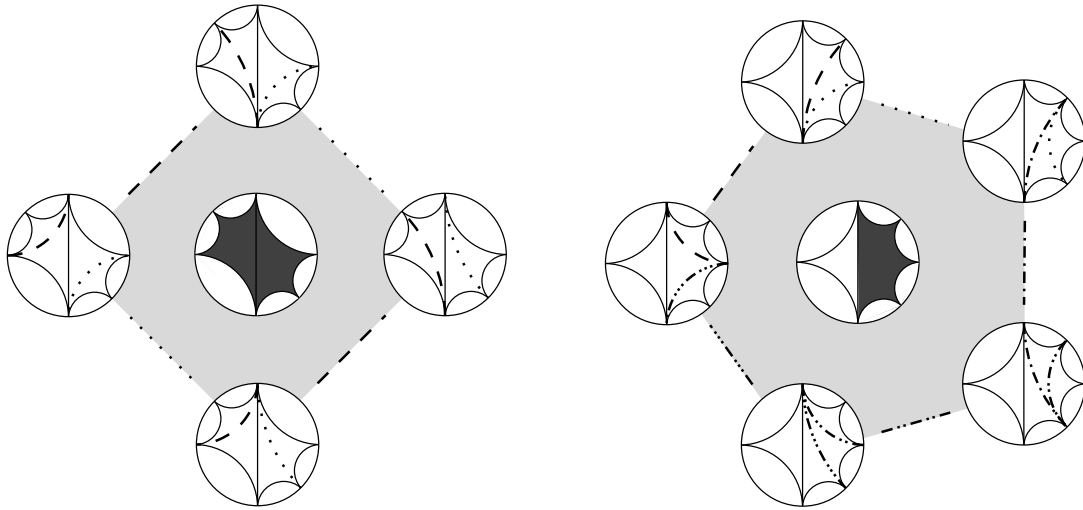
Definition 29. Two triangulations Δ_1 and Δ_2 are joined by an *edge* if their intersection $\Delta_1 \cap \Delta_2$ is a dyadic almost triangulation $a\Delta$ of D of Farey type with $\mathcal{D}(a\Delta) = 1$.



Thus, the set C^1 of edges of C^2 is the set of dyadic almost triangulations $a\Delta$ of D of Farey type such that $\mathcal{D}(a\Delta) = 1$. Given Δ_1 and a dyadic arc $a \in \Delta_1$, Δ_2 is obtained from Δ_1 by *flipping* the dyadic arc a if the edge joining Δ_1 and Δ_2 is the dyadic almost triangulation $a\Delta = \Delta_1 \setminus \{a\}$.

Remark 20. The cellular complexes C^1, C^2 are locally infinite.

Definition 30. There are two different kinds of *2-cells* in C^2 :



1. Squared 2-cells.

Each squared 2-cell can be thought as a dyadic almost triangulation $a\Delta$ of D of Farey type with $\mathcal{D}(a\Delta) = 2$ and $\mathcal{S}(a\Delta) \geq 6$. Thus, there are exactly two non-triangular connected components of $D - a\Delta$ and they are both quadrilaterals.

2. Pentagonal 2-cells.

Each pentagonal 2-cell can be thought as a dyadic almost triangulation $a\Delta$ of D of Farey type with $\mathcal{D}(a\Delta) = 2$ and $\mathcal{S}(a\Delta) = 5$. Thus, there is a single non-triangular connected component of $D - a\Delta$ and it is a pentagon.

The 2-dimensional cellular complex C^2 is called the *asymptotic flip complex*.

Remark 21. (Geometric interpretation of the 2-cells) Each 2-cell of the complex can be seen, geometrically, as the object obtained from a dyadic triangulation Δ of D of Farey

type by flipping consecutively two of its dyadic arcs $a_1, a_2 \in \Delta$. If there exists a triangle in $D - \Delta$ containing the two arcs a_1, a_2 on its boundary, then the 2-cell is pentagonal (the flips along a_1 and a_2 do not commute). Otherwise, the 2-cell is squared (the flips along a_1 and a_2 commute).

Lemma 5.1. *Given three different vertices $\Delta_1, \Delta_2, \Delta_3 \in C^0$ where both Δ_1 and Δ_3 are adjacent to Δ_2 , it exists a unique 2-cell on C^2 containing $\Delta_1, \Delta_2, \Delta_3$ on its boundary.*

Proof. It follows from Remark 21. □

Definition 31. A metric space X is *Gromov hyperbolic* if there exists $\delta > 0$ such that, for every geodesic triangle with vertices x, y, z on X , the side xy is contained in the δ -neighbourhood of the union of the two other sides, xz and yz .

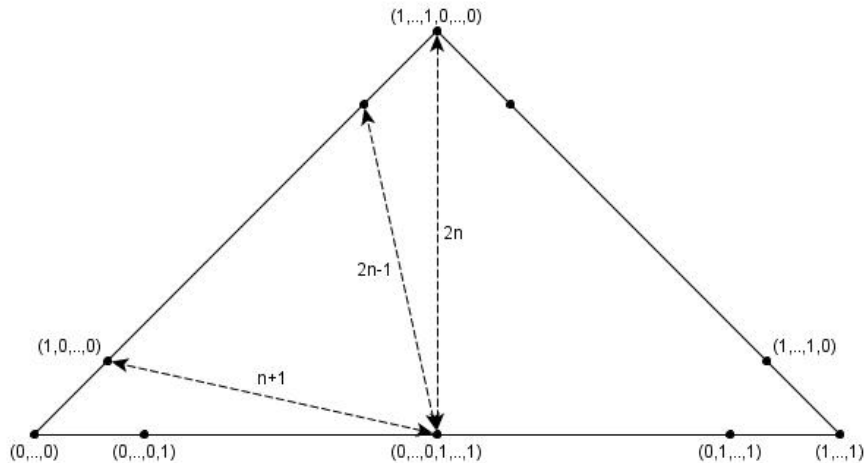
The 1-skeleton C^1 can be easily converted into a metric space parametrizing each edge by a segment of length 1 and taking the combinatorial distance.

Proposition 5.1. *The 1-skeleton C^1 of the cellular complex C^2 is not Gromov hyperbolic.*

The proof uses the following fact:

Remark 22. A flip changes a single arc of a given dyadic triangulation of D . Therefore, the distance between two vertices $\Delta_1, \Delta_2 \in C^0$ is bounded below by the number of arcs which belong to the dyadic triangulation Δ_1 and do not belong to Δ_2 .

Proof. Let n be a positive integer. First we construct a geodesic triangle $\Delta_1, \Delta_2, \Delta_3$ such that $d_C(\Delta_1, \Delta_2) = d_C(\Delta_2, \Delta_3) = n$, and $d_C(\Delta_1, \Delta_3) = 2n$. Furthermore, we give a point p in the geodesic path between Δ_1 and Δ_3 such that $d_C(p, x) \geq n$ for all points x in the geodesic path between Δ_2 and Δ_3 .



Let s_1, \dots, s_{2n} be $2n$ disjoint squares inscribed in Farey's dyadic triangulation Δ_F . Let $\Delta_1 = \Delta_F$, Δ_2 be the vertex obtained from Δ_F by flipping the diagonals of the squares s_1, \dots, s_n , and let Δ_3 be the vertex obtained from Δ_2 by flipping the diagonals of the squares s_{n+1}, \dots, s_{2n} . The geodesic path between Δ_1 and Δ_2 (respectively between Δ_2 and Δ_3) consists on flipping the diagonals of s_1, \dots, s_n in order (respectively s_{n+1}, \dots, s_{2n}).

The geodesic path between Δ_1 and Δ_3 consists on flipping first the diagonal of the square s_{2n} , then the diagonal of s_{2n-1} , and so on until the diagonal of s_1 , in decreasing order. The point p is the vertex obtained from Δ_F by flipping the diagonals of s_{n+1}, \dots, s_{2n} . All these paths are geodesics by Remark 22 and, for all points x in the geodesic path between Δ_1 and Δ_3 which passes through Δ_2 ,

$$d_C(p, x) = n + \min\{d_C(\Delta_1, x), d_C(\Delta_3, x)\}.$$

The points in the figure above correspond to triangulations obtained from Δ_F by flipping the diagonals of s_i if there is a one in the i -th coordinate. Discontinuous lines represent combinatorial distances in C^1 . \square

Remark 23. In some general sense, one can see the asymptotic flip complex C^2 embedded into a pants complex. For this, take another copy of the initial surface Σ , already tessellated in hexagons, and glue the two copies together along their boundary. Now, the arcs of Δ_F on the two copies are also glued and they become simple closed curves which decompose the doubled surface in infinitely many pairs of pants. Hence, the vertices of the cellular complex C^2 can be seen as decompositions in pairs of pants of the double surface, obtained from the initial decomposition by a finite number of elementary movements. Here, an elementary movement consists on changing a simple closed curve c by another simple closed curve c' , which does not intersect the other curves of the pants decomposition, and with minimal geometric intersection number $i([c], [c']) = 2$. In our case, we only consider the elementary movements that can be seen on the planar surface Σ . For the details of this construction see [31, 32].

From this viewpoint, the vertices of C^2 can also be seen as the isotopy classes relatively to the boundary of Σ of the tessellations of Σ into hexagons which differ from the distinguished tessellation only in a finite number of hexagons. We point out that the boundary is not pointwise fixed by the isotopies relatively to the boundary of Σ .

5.2 Stasheff's associahedra

It is worth to mention that the asymptotic flip complex is closely related to the graph of triangulations of a convex n -gon (see, for example, [5, 55, 45]) and hence, it is also related to the rotation distance in binary trees (see [72]). The graph of triangulations of a convex polygon corresponds to the 1-skeleton of Stasheff's associahedra (see [73, 74]), which can be seen as a combinatoric dual of the arc complex of a convex polygon. A relation between Stasheff's associahedra and Thompson's group F was first established by Greenberg [37].

Definition 32. The *graph of triangulations of a convex n -gon* ($n \geq 3$) is a finite graph whose vertices are the triangulations of the n -gon in $n - 2$ triangles, and where two vertices are joined by an edge if and only if all their diagonals coincide except one.

See [45] for a simple proof of the following result:

Proposition 5.2. *The graph of triangulations of a convex n -gon is connected.*

Before introducing Stasheff's associahedra, we will define the arc complex of a convex polygon. Let n be an integer greater than 4 (for $n = 3, 4$ the notion still make sense, but one obtains degenerate cases) and let P_n denote a convex polygon with n sides. Let $V(P_n) = \{v_1, \dots, v_n\}$ be the set of its vertices, where $\{v_i, v_j\}$ ($i < j$) is a side of P_n if and only if $j = i + 1$ or $i = 1, j = n$. Let D_n be the set of internal diagonals of P_n , i.e.

$$D_n = \{\{v_i, v_j\} : j > i + 1, \{v_i, v_j\} \neq \{v_1, v_n\}\}.$$

Definition 33. The *arc complex* A_n of P_n is the simplicial complex defined by the subsets of D_n which contain 2 by 2 non-crossing diagonals. The dimension of A_n is $n - 4$, and the maximal simplices correspond to triangulations of P_n .

Proposition 5.3. (Lee [55], see also [23]) *The arc complex A_n of P_n is a convex $(n - 3)$ -dimensional polytope.*

We introduce Stasheff's associahedra K_n (we take the notation from [37], but our K_n corresponds to Greenberg's K_{n-1}) as the dual complex of the arc complex A_n of P_n . Let T_n be the set of triangulations of P_n with exactly $n - 3$ internal diagonals. The set $V(K_n)$ of vertices of K_n is T_n :

$$V(K_n) = T_n.$$

The set-theoretical definition of the cellular complex K_n is the following: for $1 \leq k \leq n - 3$, a subset K of T_n is a k -cell if it contains all triangulations in T_n which have $n - 3 - k$ internal diagonals in common:

$$K \text{ is a } k\text{-cell} \Leftrightarrow \exists d_1, \dots, d_{n-3-k} \text{ diagonals of } P_n : K = \{s \in T_n : \{d_1, \dots, d_{n-3-k}\} \subset s\}.$$

It is easy to see that the subset of parts of T_n defining K_n is closed by intersection, and that the intersection of two different k -cells is a cell of strictly lower dimension.

It is not hard to prove that K_n is the dual complex of A_n by sending a k -cell K of K_n to the simplex K^* of A_n formed by the $n - 3 - k$ diagonals that all the triangulations in K have in common, which is of dimension $n - 3 - k - 1$. It follows from the definition that two cells $K \subset L$ of K_n are sent to $L^* \subset K^*$ on A_n . Thus, using the fact that the dual of a convex polytope is a convex polytope of the same dimension, one gets the following result:

Proposition 5.4. *Stasheff's associahedron K_n is a convex $(n - 3)$ -dimensional polytope.*

Corollary 5.1. *Stasheff's associahedra are contractible.*

One can also construct Stasheff's associahedra by induction (see [37] for a proof in terms of planar rooted trees with the same number of leaves). To translate Greenberg's construction into our viewpoint, one must consider tessellations of P_n by sub-polygons having vertices in $V(P_n)$. Consider one distinguished side of P_n , for example $\{v_1, v_2\}$. Then, each planar rooted tree with $n - 1$ leaves corresponds to a tessellation of P_n by taking the dual tree of the tessellation, rooted on the distinguished side.

In the language of tessellations of P_n , Greenberg's construction is the following one: let S_n be the set of tessellations of P_n with vertices in $V(P_n)$. Given $s \in S_n$, let $n(s)$

denote the set of sub-polygons of the tessellation s . For each $i \in n(s)$, let n_i be the number of sides of the corresponding sub-polygon minus 1. Denote by $|s|$ the integer

$$|s| = \sum_{i \in n(s)} (n_i - 2),$$

which corresponds to the minimal number of diagonals that one should add to the tessellation s to have a triangulation of P_n .

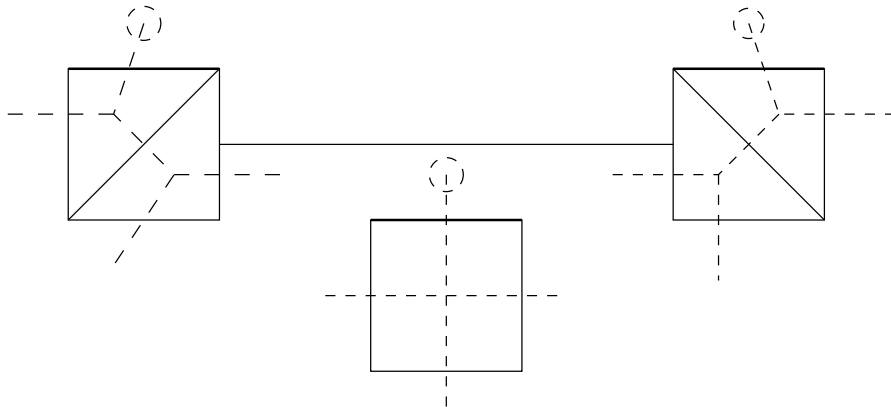
One can define a partial order on the set S_n : $s < r$ if and only if s is a tessellation of r , i.e. each polygon of r is either a polygon of s either the union of polygons of s . We will say that s is the *predecessor* of r if all polygons of r are polygons of s except one, which is the union of two polygons of s . This means that s is obtained from r by subdividing one of the non-triangular polygons of r in two sub-polygons. Note that if $s < r$, there exists an increasing sequence $s = s_0 < s_1 < \dots < s_k = r$, where each s_i is the predecessor of s_{i+1} .

The n -th associahedron K_n will be defined by induction. In particular, one associates to every $s \in S_n$ a closed polytope f_s of dimension $|s|$, and $K_n = f_{P_n}$, where P_n denotes the tessellation of P_n in a single polygon (itself). Furthermore,

$$\partial K_n = \bigcup_{s \in S_n - P_n} f_s,$$

and $f_s \subset f_r$ if $s < r$. The basic cases are the following:

1. K_3 is a point. There exists a unique way of tessellating a triangle.
2. K_4 is a segment:



For the construction of ∂K_n , suppose that:

- For all $j \in \{3, \dots, n-1\}$, and for all $s \in S_j$, f_s is defined.
- For all $s < r$ the inclusion $i_{rs} : f_s \rightarrow f_r$ is defined.
- For all $j \in \{3, \dots, n-1\}$,

$$(K_j, \partial K_j) \approx (\bar{B}^{j-3}, S^{j-4}).$$

Then, for $s \in S_n - \{P_n\}$ define

$$f_s \equiv \prod_{i \in n(s)} f_{P_{n_i}}.$$

If s is the predecessor of $r \neq P_n$, one must define $i_{rs} : f_s \rightarrow f_r$. Remark that in this case one have $|r| = |s| + 1$ and:

$$n(r) = n(s) - \{i_1, i_2\} \cup \{i_{1,2}\}.$$

Thus,

$$f_s \equiv f_{P_{n_{i_1}}} \times f_{P_{n_{i_2}}} \times \prod_{i \in n(s) - \{i_1, i_2\}} f_{P_{n_i}}$$

and

$$f_r \equiv f_{P_{n_{i_1, i_2}}} \times \prod_{i \in n(s) - \{i_1, i_2\}} f_{P_{n_i}}.$$

Hence, the inclusion i_{rs} can be defined as the identity with respect to the last direct factor. It is left to define

$$i_{rs|} : f_{P_{n_{i_1}}} \times f_{P_{n_{i_2}}} \rightarrow f_{P_{n_{i_1, i_2}}}.$$

Consider the sub-polygon P_k of r which is sub-divided in two polygons on the tessellation s , and let u be the tessellation of P_k given by the restriction of s to P_k . By definition we have

$$f_u \equiv f_{P_{n_{i_1}}} \times f_{P_{n_{i_2}}}.$$

By induction, the inclusion

$$i_{P_{n_{i_1, i_2}} u} : f_u \rightarrow f_{P_{n_{i_1, i_2}}}$$

is well defined. The composition of both gives the desired $i_{rs|}$.

In [73], Stasheff states that ∂K_n is isomorphic to S^{n-4} . Define $K_n = f_{P_n}$ to be the interior of this sphere together with the boundary, which corresponds to an $(n - 3)$ -cell.

5.3 Higher dimensional cells of C

Lemma 5.2. *Let $\Delta \in C^0$ be a dyadic triangulation of Farey type, and let a_1, \dots, a_k be dyadic arcs of Δ bounding a finite dyadic polygon P . Let p_1, \dots, p_k be the vertices of P and suppose they are cyclically ordered and that p_1 is the first point one finds following ∂D counter-clockwise and beginning at $\gamma(0)$. Let $\text{arc}(P)$ be the set of arcs (geometrically) contained on P ,*

$$\text{arc}(P) = \{(p_i, p_j) \in \mathcal{A} : 1 \leq i < j + 1 \leq k + 1, (p_i, p_j) \neq (p_1, p_k)\}.$$

Let $C^2(\Delta, P)$ be the subcomplex of the asymptotic flip complex C^2 induced by the set of dyadic triangulations of Farey type

$$\{\Delta_1 \in C^0 : \Delta_1 \setminus (\Delta_1 \cap \Delta) \subset \text{arc}(P)\}.$$

Then, $C^2(\Delta, P)$ is isomorphic to the 2-skeleton of Stasheff's associahedron K_k of a convex k -gon.

Proof. It follows from the definitions; one only needs to take into account the fact that P is a convex k -gon. \square

Corollary 5.2. *Let $\Delta_1, \Delta_2, \dots, \Delta_k$ be the vertices of a path ρ in C^1 . Then, ρ can be isometrically embedded into the graph of triangulations of a convex n -gon, where n is the number of sides of the minimal polygon P such that, for all $i \in \{1, \dots, k\}$, $D - \Delta_i$ defines the same triangulation outside P .*

By Lemma 5.2 and Corollary 5.2, the asymptotic flip complex C^2 can be naturally seen as a 2-dimensional complex associated to the triangulations graph of an infinite-sided convex polygon. We will use Stasheff's associahedra to define higher dimensional cells.

Let n be an integer greater than 2. By analogy with lower dimensional cells, the n -cells of the cellular complex C will be the dyadic almost triangulations $a\Delta$ of Farey type with $\mathcal{D}(a\Delta) = n$. We proceed by induction:

- 0-cells are already defined to be dyadic triangulations of D of Farey type.
- 1-cells correspond to dyadic almost triangulations $a\Delta$ of Farey type with $\mathcal{D}(a\Delta) = 1$. Remark that, by Lemma 5.2, given a dyadic almost triangulation $a\Delta$ with $\mathcal{D}(a\Delta) = 1$, the sub-complex $C^1(a\Delta)$ generated by $\{\Delta \in C^0 : a\Delta \subset \Delta\}$ is isomorphic to Stasheff's associahedron K_4 , which is a 1-simplex.
- 2-cells are already defined to be either squares or pentagons. Remark that, by Lemma 5.2, if $a\Delta$ is a dyadic almost triangulation with $\mathcal{D}(a\Delta) = 2$ and $\mathcal{S}(a\Delta) = 5$, the sub-complex $C^2(a\Delta)$ generated by $\{\Delta \in C^0 : a\Delta \subset \Delta\}$ is isomorphic to Stasheff's associahedron K_5 . For a squared 2-cell $a\Delta$, the sub-complex $C^2(a\Delta)$ generated by $\{\Delta \in C^0 : a\Delta \subset \Delta\}$ can be seen as the direct product of two Stasheff's associahedron K_4 in the following way: let P_1, P_2 be the non triangular polygons of $D - a\Delta$. For $i \in \{1, 2\}$ let $K(P_i)$ be the associahedron of a convex polygon whose number of sides coincides with the number of sides of P_i , in this case both are K_4 . Each $K(P_i)$ is topologically a closed ball of dimension 1 (an interval). Then,

$$C^2(a\Delta) \simeq K(P_1) \times K(P_2),$$

which is topologically a closed ball of dimension 2.

- Suppose that we have defined the n -th skeleton C^n , and suppose we have proved that for all dyadic almost triangulations $a\Delta$ of Farey type with $\mathcal{D}(a\Delta) = k \leq n$, the sub-complex $C^k(a\Delta)$ generated by $\{\Delta \in C^0 : a\Delta \subset \Delta\}$ is topologically a closed ball of dimension n . Let $a\Delta$ be an almost triangulation of Farey type with $\mathcal{D}(a\Delta) = n + 1$. Let P_1, \dots, P_k be the non triangular polygons of $D - a\Delta$, and for $i \in \{1, \dots, k\}$ let $K(P_i)$ be the associahedron of a convex polygon whose number of sides coincides with the number of sides of P_i . If $k > 1$, for all $i \in \{1, \dots, k\}$, $K(P_i)$ is topologically a closed ball of dimension $\mathcal{D}(P_i)$ by induction hypothesis. Furthermore, $C^n(a\Delta)$ is naturally isometric to the boundary of the direct product $K(P_1) \times \dots \times K(P_k)$, which is the boundary of a closed ball of dimension

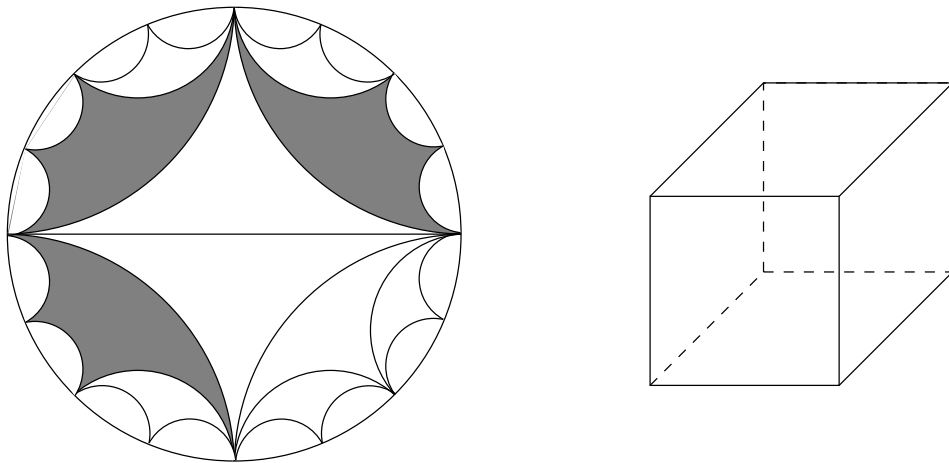
$$\mathcal{D}(P_1) + \dots + \mathcal{D}(P_k) = \mathcal{D}(a\Delta) = n + 1.$$

Thus, one can attach to $C^n(a\Delta)$ a closed ball of dimension $n + 1$. If $k = 1$, then $C^n(a\Delta)$ is isometric to the boundary of Stasheff's $(n + 1)$ -th associahedron, and one can attach a closed ball of dimension $n + 1$ to $C^n(a\Delta)$.

Definition 5.1. *The cellular complex C obtained by taking the limit when n tends to infinity of the cellular complexes C^n described before is called the infinite associahedron.*

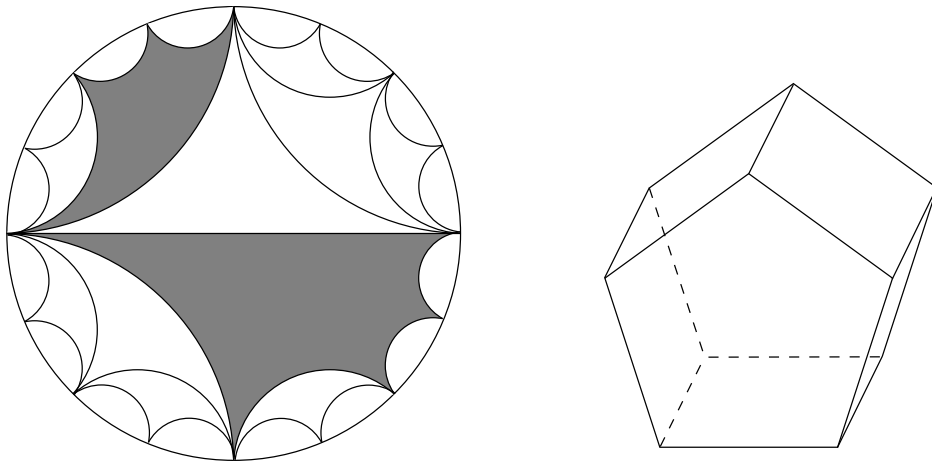
Example 21. The 3-cells of C correspond to the dyadic almost triangulations of Farey type with three missing arcs. There are three different kinds of 3-dimensional cells. The cell type depends only in whether or not the missing arcs are adjacent. Two missing arcs are adjacent if, once added to the almost triangulation to have a dyadic triangulation of D , they are on the boundary of the same triangle. Here we list the three different options and we draw both the almost dyadic triangulations corresponding to the 3-cells, and a geometric realisations of the cells.

1. Filled cubes.



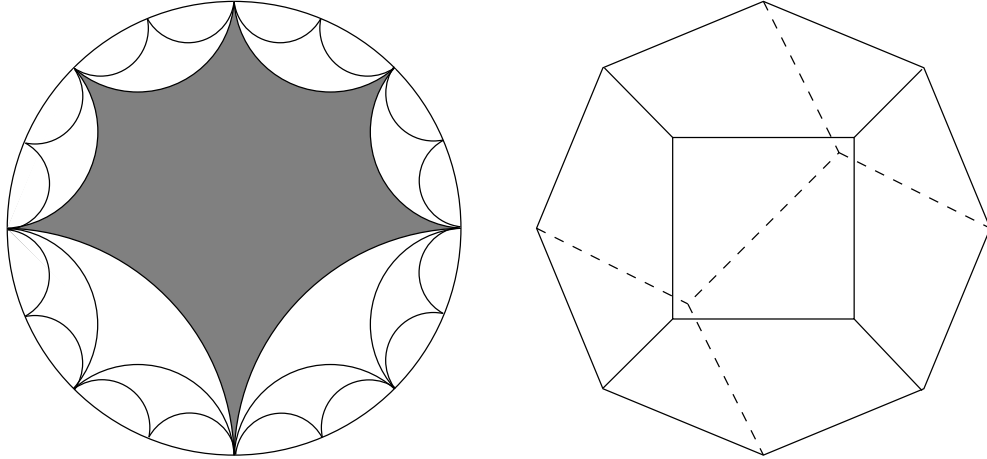
The tree arcs missing on $a\Delta$ are contained in squares with disjoint interiors.

2. Direct products pentagon cross interval.



Two of the three arcs missing on $a\Delta$ are the two diagonals of a pentagon, and the square containing the third one has disjoint interior with that pentagon.

3. The associahedron of a convex hexagon.



The three arcs missing on $a\Delta$ are the three diagonals of an hexagon. By Proposition 5.1 the complex of triangulations of a convex hexagon is a polytope of dimension 3, thus it can be topologically seen as a closed ball of dimension 3.

Proposition 5.5. *The infinite associahedron C is contractible.*

Proof. The connectivity of C follows from Proposition 5.2 and Corollary 5.2. The simple connectivity follows from Proposition 5.1 and Corollary 5.2. The contractibility follows from Whitehead's theorem stating that a CW-complex having all homotopy groups zero is contractible. One can see that C is a CW-complex because it has been constructed inductively by gluing balls of dimension n to the $(n-1)$ -skeleton for every $n \in \mathbb{N}$. To prove that all homotopy groups vanish, one can see that every compact subspace is contained in some (big enough) cell of associahedral type which contains the base-point (Farey's dyadic triangulation). Thus, the compact can be retracted to the base-point. \square

A cellular complex is connected if and only if its 1-skeleton is connected, and it is simply connected if and only if its 2-skeleton is simply connected. Thus, Proposition 5.5 is equivalent to the following result:

Proposition 5.6. *([30], Proposition 3.1) The 2-skeleton C^2 of the infinite associahedron is connected and simply connected.*

5.4 T acts on the infinite associahedron

Definition 34. Let $\Delta \in C^0$ be a dyadic triangulation of Farey type, and let t be an element of Thompson's group T . The dyadic triangulation $t \cdot \Delta$ is defined by

$$t \cdot \Delta = \left\{ \left(\gamma \left(\frac{p}{2^m} \right), \gamma \left(\frac{q}{2^n} \right) \right) \in \mathcal{A} : \left(\gamma \left(t^{-1} \left(\frac{p}{2^m} \right) \right), \gamma \left(t^{-1} \left(\frac{q}{2^n} \right) \right) \right) \in \Delta \right\}.$$

Lemma 5.3. *Let $\Delta \in C^0$ be a dyadic triangulation of Farey type, and let t be an element of T . Then, the dyadic triangulation $t \cdot \Delta$ is of Farey type.*

Proof. It is a direct consequence of the definitions of T and C . Note that it is possible to find a Farey central polygon P bounded by the dyadic arcs $\{a_1, \dots, a_k\}$, containing all dyadic arcs of $n\mathcal{F}(\Delta) = \{a \in \Delta : a \notin \Delta_F\}$ on its interior, such that the set of dyadic arcs $\{t(a_1), \dots, t(a_k)\}$ bounds another central Farey polygon $t(P)$. If a is the dyadic arc $(\gamma(x_1), \gamma(x_2))$ the dyadic arc $t(a)$ is the arc $(\gamma(t(x_1)), \gamma(t(x_2)))$. Thus, $n\mathcal{F}(t \cdot \Delta)$ is contained on the interior of $t(P)$, proving that $t \cdot \Delta \in C^0$. \square

Definition 35. Let $a\Delta$ be a dyadic almost triangulation of D of Farey type with $\mathcal{D}(a\Delta) = n > 0$. Let t be an element of T . The dyadic almost triangulation $t \cdot a\Delta$ is defined by

$$t \cdot a\Delta = \left\{ \left(\gamma \left(\frac{p}{2^m} \right), \gamma \left(\frac{q}{2^n} \right) \right) \in \mathcal{A} : \left(\gamma \left(t^{-1} \left(\frac{p}{2^m} \right) \right), \gamma \left(t^{-1} \left(\frac{q}{2^n} \right) \right) \right) \in a\Delta \right\}.$$

Lemma 5.4. *Let $a\Delta$ be an n -cell of C and let t be an element of T . The dyadic almost triangulation $t \cdot a\Delta$ is of Farey type and satisfies*

$$\mathcal{D}(t \cdot a\Delta) = \mathcal{D}(a\Delta) = n.$$

Proof. As in the case of Lemma 5.3, it is possible to find a Farey central polygon P bounded by the set of dyadic arcs $\{a_1, \dots, a_k\}$ containing all dyadic arcs of

$$n\mathcal{F}(a\Delta) = \{a \in a\Delta : a \notin \Delta_F\}$$

and all the non triangular polygons of $D - a\Delta$ on its interior, such that the set of dyadic arcs $\{t(a_1), \dots, t(a_k)\}$ bounds another central Farey polygon $t(P)$. Thus, $n\mathcal{F}(t \cdot a\Delta)$ is contained in the interior of $t(P)$, so $t \cdot a\Delta$ is an almost triangulation of Farey type. Furthermore, $\mathcal{D}(t \cdot a\Delta) = \mathcal{D}(a\Delta)$ and both $a\Delta$ and $t \cdot a\Delta$ have exactly the same almost triangular structure. Hence, $t \cdot a\Delta \in C^n$ and both n -cells $a\Delta$ and $t \cdot a\Delta$ are of the same type. \square

Proposition 5.7. *Thompson's group T acts on the infinite associahedron C by automorphisms. Thus, we have a natural map $\Psi : T \rightarrow \text{Aut}(C)$.*

Proof. It follows from Lemmas 5.3 and 5.4. \square

Proposition 5.8. *Thompson's group T acts transitively on vertices and edges of C . Furthermore, the map $\Psi : T \rightarrow \text{Aut}(C)$ is injective.*

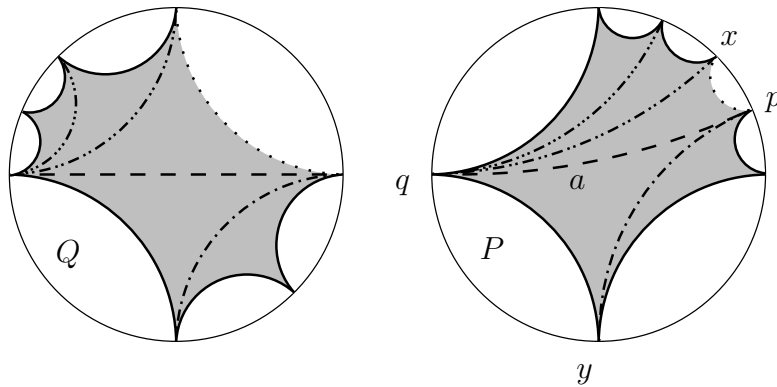
Proof. We prove the transitivity of the action in vertices by showing that given a dyadic triangulation of Farey type $\Delta \in C^0$, there exists an element $t \in T$ such that $f \cdot \Delta_F = \Delta$. Let P be a Farey central polygon containing all dyadic arcs of

$$n\mathcal{F}(\Delta) = \{a \in \Delta : a \notin \Delta_F\}$$

on its interior. Let a be a dyadic arc of Δ contained in the interior of P . Then,

$$a = (\gamma(p), \gamma(q))$$

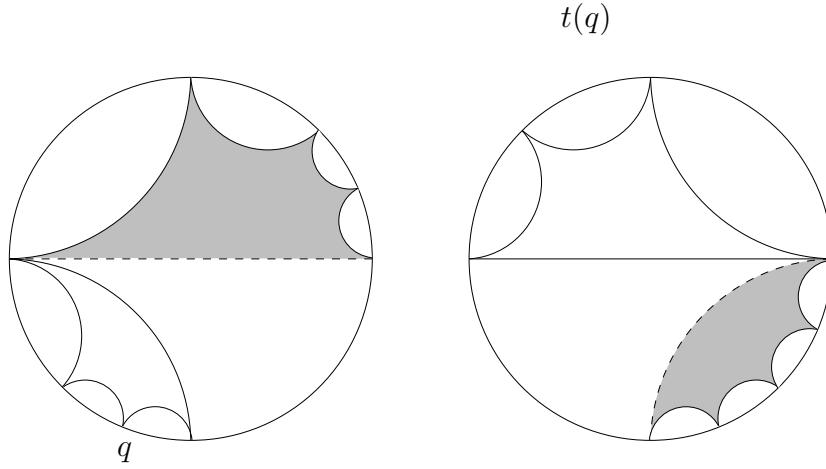
for some dyadic rational numbers p, q , $p < q$. There exists a unique triangle in P with vertices p, x, q , where x is a dyadic rational number $p < x < q$. We associate the triangle of Δ_F with vertices $\frac{1}{2}, 0, \frac{1}{4}$ to this triangle. Analogously, we associate to the triangle with vertices $p < q < y$ (or $y < p < q$) of P the triangle with vertices $\frac{3}{4}, 0, \frac{1}{2}$. Repeating this procedure finitely many times we construct a Farey central polygon Q triangulated by the restriction of Δ_F , whose sides are naturally associated to those of the polygon P . Then, the element $t \in T$ defined by $t(Q) = P$ satisfies our claim.



Analogously, it can be shown that every 1-cell can be sent by an element of T to the dyadic almost triangulation $a\Delta$ obtained from Farey's dyadic triangulation by erasing the horizontal edge (the geodesic joining $\gamma(0)$ and $\gamma(\frac{1}{2})$).

For the second assertion it is enough to prove that given an element $t \in T$ different from the identity, there exists a dyadic triangulation $\Delta \in C^0$ of D of Farey type such that $t \cdot \Delta \neq \Delta$. Let q be a dyadic rational number such that $t(q) \neq q$ (it exists because t is not the identity). Take a square \blacksquare inscribed in Farey's dyadic triangulation Δ_F such that:

1. q is a vertex of \blacksquare ,
2. the image of \blacksquare by t belongs to Δ_F , and
3. \blacksquare has empty intersection with its image by t .



In the figure, the two filled polygons represent an element of T matching them with the cyclic permutation defined by the discontinuous arcs. Note that \blacksquare exists because $t(q) \neq q$, the triangulation Δ_F have all dyadic rational number as vertices, and the dyadic rational numbers are dense in $[0, 1]$. Let Δ be the dyadic triangulation of Farey type obtained from Δ_F by flipping the diagonal of the square \blacksquare . Then, $t \cdot \Delta \neq \Delta$. \square

Remark that the action is not transitive on the n -cells for $n \geq 2$. On one hand, for a fixed $n \geq 2$, there are several kinds of n -cells and it is easy to see that the action of T fixes the type of the cell. But even among cells of the same type, the action is not transitive. To see this, we first define a combinatorial distance between inscribed polygons on a given triangulation.

Definition 5.2. *Let $a\Delta$ be a dyadic almost triangulation, and let P, Q be dyadic polygons inscribed in $a\Delta$. Then, the combinatorial distance $d_{a\Delta}(P, Q)$ in $a\Delta$ between P and Q is the minimum number of polygons of $a\Delta$ crossed by a geodesic joining an interior point of P and an interior point of Q .*

By definition of the action of T on C one gets the following result:

Lemma 5.5. *The action of T preserves the combinatorial distance between two inscribed polygons: let $a\Delta$ be a dyadic almost triangulation, let P, Q be two inscribed polygons of $a\Delta$ and let t be an element of Thompson's group T . Then,*

$$d_{a\Delta}(P, Q) = d_{t \cdot a\Delta}(t(P), t(Q)).$$

Remark that, given $n \in \mathbb{Z}_{\geq 0}$, one can define squared cells where the two inscribed squares of the dyadic almost triangulation defining the 2-cell are at combinatorial distance equal to n . Thus, one derives the non-transitivity of the action on cells of dimension at least 2. However, it is worth to mention that the proof of Proposition 5.8 can be adapted to n -cells isomorphic to Stasheff's associahedra K_n .

Corollary 5.3. *The action of T on C is not transitive for cells of dimension at least 2, except for those cells homeomorphic to Stasheff's associahedra.*

Note that the proof of Proposition 5.8 for the 0-skeleton C^0 can be adapted to Thompson's group F , which is the subgroup of elements of T fixing $\gamma(0)$, by taking $p = 0$. Thus, one obtains the following result:

Proposition 5.9. *Thompson's group F acts on C by automorphisms. The action on the 0-skeleton is transitive. Furthermore, the induced map $\Psi : F \rightarrow \text{Aut}(C)$ is injective.*

Remark that for F the action is no longer transitive in C^1 , because there is no way to send a square not having $\gamma(0)$ as a vertex to a square having $\gamma(0)$ as a vertex by an element of F .

Finally, we will end this chapter by some results about the stabilizers of the action of T on the infinite associahedron C .

Proposition 5.10. *Let $T \curvearrowright C$ be the action of T on C described before. Then,*

1. *If $\Delta \in C^0$ is a vertex of C , the stabilizer $\text{Stab}(\Delta)$ is isomorphic to $PSL(2, \mathbb{Z})$.*
2. *If $a\Delta$ is an n -cell for some $n \geq 1$, the stabilizer $\text{Stab}(\Delta)$ is finite.*

Proof. Suppose that the image of a dyadic (almost) triangulation of Farey type is fixed by the action of an element $t \in T$. Then, such element is completely determined by the image of an oriented dyadic arc of the dyadic (almost) triangulation. Thus, there are two different cases to consider:

- The dyadic (almost) triangulation is in fact a dyadic triangulation, in which case a given oriented dyadic arc can be sent to any other dyadic arc of the triangulation. By transitivity of $T \curvearrowright C$ on C^0 all vertex stabilizers are conjugated to $\text{Stab}(\Delta_F)$, and it is not difficult to see that

$$\text{Stab}(\Delta_F) \cong PSL(2, \mathbb{Z}).$$

- The dyadic almost triangulation $a\Delta$ have a finite number of non-triangular polygons, and a dyadic arc contained on the boundary of one of them must be sent to another dyadic arc bounding one non-triangular polygon. Thus, there are only finitely many $t \in T$ on the stabilizer of the cell.

□

Chapter 6

The group of automorphisms of C

We have seen that Thompson's group T acts on the asymptotic flip complex C^2 and on the infinite associahedron C by automorphisms. Thus, it is natural to ask whether or not the automorphism group of the those cellular complexes are related to T . In this chapter we show that both groups are the same by first constructing a short exact sequence:

$$1 \longrightarrow T \longrightarrow \text{Aut}(C^2) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1,$$

that splits. Thus, $\text{Aut}(C^2) \cong T \rtimes \mathbb{Z}/2\mathbb{Z}$. Furthermore, the co-kernel acts as a non trivial outer automorphism. Hence, we obtain a non trivial action of $\mathbb{Z}/2\mathbb{Z}$ on $\text{Out}(T)$. Thus, as Brin established in [7] that the group of outer automorphisms of T is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, we get an isomorphism between $\text{Out}(T)$ and $\mathbb{Z}/2\mathbb{Z}$. In addition, we show that each automorphism of C^2 can be extended to an automorphism of C , proving that both automorphism groups coincide.

6.1 Construction of the link complex $\mathcal{L}^2(\Delta_1)$

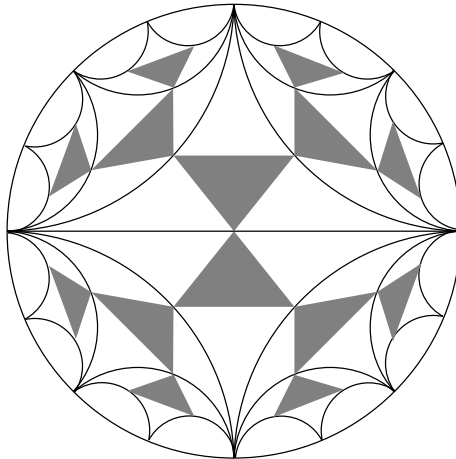
The goal of this chapter is to prove that both groups $\text{Aut}(C^2)$ and $\text{Aut}(C)$ are isomorphic to the semi-direct product $T \rtimes \mathbb{Z}/2\mathbb{Z}$. The proof of this result has some similarities to Ivanov's proof (see [50]) of the fact that the automorphism group of the arc complex of a compact surface is isomorphic to the extended mapping class group. In particular, he showed that each automorphism is completely determined by the image of a maximal simplex. In our case, each automorphism is determined by the image of the ball of radius one centred in some dyadic triangulation $\Delta \in C^0$ of D of Farey type. Then, given an automorphism φ of the complex C^2 , we construct an element $t \in \Psi(T) \subset \text{Aut}(C^2)$ such that $t(\Delta_F) = \varphi(\Delta_F)$. To this aim, we need to introduce an auxiliary complex, which is related to the link of Δ in C^2 .

Recall that in any dyadic triangulation Δ of D , the flips along two different dyadic arcs a_1, a_2 of Δ do not commute if and only if there is a triangle on $D - \Delta$ containing both dyadic arcs a_1, a_2 on its boundary (see Remark 21). We construct a sub-complex encoding this information.

Definition 36. Let $\Delta_1 \in C^0$ be a dyadic triangulation of Farey type. Let $\mathcal{L}^1(\Delta_1)$ be a graph whose set of vertices is the set of dyadic triangulations $\Delta_2 \in C^0$ of Farey type which are joined by an edge of C^1 to the dyadic triangulation Δ_1 . Two vertices $v = \Delta_2$ and $w = \Delta_3$ of $\mathcal{L}^1(\Delta_1)$ are connected by an edge in $\mathcal{L}^1(\Delta_1)$ if the 2-cell of C^2 which contains the path $\Delta_2, \Delta_1, \Delta_3$ is pentagonal. The *link* of Δ_1 is the 2-dimensional complex $\mathcal{L}^2(\Delta_1)$ obtained from $\mathcal{L}^1(\Delta_1)$ by gluing a 2-cell on each triangle of $\mathcal{L}^1(\Delta_1)$.

Remark 24. Given a dyadic triangulation of Farey type $\Delta_1 \in C^0$, there is a geometric interpretation of $\mathcal{L}^1(\Delta_1)$: we associate to every dyadic arc a of the dyadic triangulation Δ_1 a vertex which represents the dyadic triangulation $\Delta_2 \in C^0$ obtained from Δ_1 by flipping the arc a . Then, from the geometric interpretation of the 2-cells of C , which are exactly the 2-cells of C^2 , (see Remark 21) we see that two vertices of $\mathcal{L}^1(\Delta_1)$ associated to the arcs a_1, a_2 are joined by an edge if and only if there is a triangle in $D - \Delta_1$ having both a_1, a_2 on its boundary. Thus, $\mathcal{L}^1(\Delta_1)$ can be seen as a connected union of triangles, each of them inscribed on a triangle of $D - \Delta_1$, where the intersection between different triangles is either empty or a vertex, and where each vertex belongs to exactly two different triangles. Hence, the link $\mathcal{L}^2(\Delta_1)$ can be embedded in D and every triangle inherits an orientation.

Example 22. The link of Δ_F .



The link has been constructed only using the combinatorial structure of the 2-skeleton C^2 of the cellular complex C in the neighbourhood of a dyadic triangulation of Farey type Δ . Note also that the subcomplex C^2 is regular in the sense that all vertices have the same combinatorial structure in their neighbourhoods. Thus, the next result follows:

Lemma 6.1. *Let $\Delta_1 \in C^0$ be a dyadic triangulation of Farey type. Then, every automorphism $\phi \in \text{Aut}(C^2)$ induces an isomorphism $\phi_{*, \Delta_1} : \mathcal{L}^2(\Delta_1) \rightarrow \mathcal{L}^2(\Delta_2)$, where $\Delta_2 = \phi(\Delta_1)$.*

Using the transitivity of the action of T on C^0 (see Proposition 5.8) one obtains:

Corollary 6.1. *Let Δ_1 and Δ_2 be dyadic triangulations of Farey type. Then, their links $\mathcal{L}^2(\Delta_1)$ and $\mathcal{L}^2(\Delta_2)$ are isomorphic.*

Let $\Delta_1, \Delta_2 \in C^0$ be two dyadic triangulations of D of Farey type, and let

$$i : \mathcal{L}^2(\Delta_1) \longrightarrow \mathcal{L}^2(\Delta_2)$$

be an isomorphism between their links. Is there any automorphism $\varphi \in \text{Aut}(C^2)$ such that $\varphi_{*,\Delta_1} = i$? More specifically, which is the main obstruction to the existence of this automorphism?

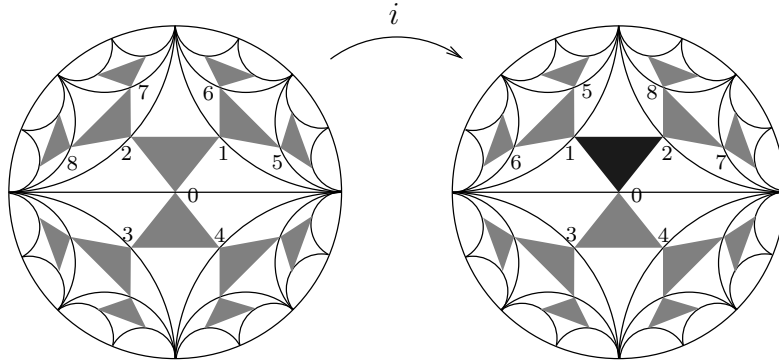
Let $\varphi \in \text{Aut}(C^2)$. Let $\Delta_2 \in C^0$ be the image of $\Delta_1 \in C^0$ by φ . Let (a, b, c) be three vertices of a triangle τ on the link $\mathcal{L}^2(\Delta_1)$, and suppose they are cyclically ordered according to the orientation of the triangle on D . The vertices $\varphi(a), \varphi(b), \varphi(c)$ are the vertices of a triangle $\phi(\tau)$ in the link $\mathcal{L}^2(\Delta_2)$. Now, there are two possible cases:

1. The orientation of $\varphi(\tau)$ on D induces the cyclic order $(\varphi(a), \varphi(b), \varphi(c))$ on its vertices. Then, we say that φ is *orientation preserving on τ* .
2. The orientation of $\varphi(\tau)$ on D induces the cyclic order $(\varphi(a), \varphi(c), \varphi(b))$ on its vertices. Then, we say that φ is *orientation reversing on τ* .

Definition 37. Let $\varphi \in \text{Aut}(C^2)$ and $\Delta_1 \in C^0$. We say that φ is Δ_1 -*orientation preserving (reversing)* if it is orientation preserving (reversing) on every triangle τ of $\mathcal{L}^2(\Delta_1)$.

Remark 25. Let t be an element of Thompson's group T . Then, the automorphism $\Psi(t) \in \text{Aut}(C^2)$ from Proposition 5.8 is Δ -orientation preserving for all dyadic triangulation $\Delta \in C^0$ of D of Farey type.

Example 23. A non-extensible link isomorphism.



6.2 Extensions of link isomorphisms

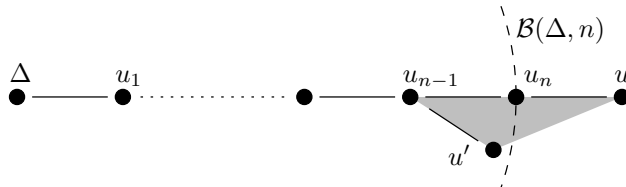
Lemma 6.2. Let $i : \mathcal{L}^2(\Delta_1) \rightarrow \mathcal{L}^2(\Delta_2)$ be an orientation preserving isomorphism of the links of two dyadic triangulations $\Delta_1, \Delta_2 \in C^0$ of Farey type. Then, there exists a unique automorphism $\varphi_i \in \text{Aut}(C^2)$ such that $\varphi_{*,\Delta_1} = i$. Furthermore, $\varphi_i \in \Psi(T)$.

Proof. We first prove the existence by constructing $\varphi \in \Psi(T)$ as follows: let $\{a'_1, \dots, a'_l\}$ be a set of dyadic arcs of Δ_2 bounding a dyadic polygon P such that all dyadic arcs of $n\mathcal{F}(\Delta_2)$ are contained in the interior of P . Let u'_1, \dots, u'_l be the vertices of $\mathcal{L}^1(\Delta_2)$ corresponding to the triangulations obtained from Δ_2 by flipping, respectively, the dyadic arcs a'_1, \dots, a'_l .

Let $\{a_1, \dots, a_k\}$ (where $k \geq l$) be a set of dyadic arcs of Δ_1 containing in the interior of the polygon they define the points $i^{-1}(u'_1), \dots, i^{-1}(u'_l)$. Let u_1, \dots, u_k be the vertices of $\mathcal{L}^1(\Delta_1)$ corresponding to the dyadic triangulations obtained from Δ_1 by flipping, respectively, the edges a_1, \dots, a_k . Then, $\{i(u_1), \dots, i(u_k)\}$ is a set of vertices corresponding to a set of dyadic arcs $\{b_1, \dots, b_k\}$ bounding a Farey central polygon inscribed on Δ_2 . Define φ by $\varphi(a_j) = b_j$ for $1 \leq j \leq k$. It is easy to see that $\varphi_{*,\Delta_1} = i$.

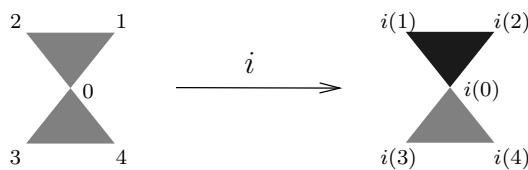
Uniqueness is based on Remarks 21 and 5.1. Let $\psi \in \text{Aut}(C^2)$ such that $\psi_{*,\Delta_1} = i$. Then, we prove that its action on the 0-skeleton C^0 must coincide with φ . We have:

1. $\psi(\Delta_1) = \Delta_2$.
2. For every element u on the unit sphere of C^1 centred in Δ_1 , $\psi(u)$ must be $i(u)$.
3. Suppose that ψ is defined in the ball $\mathcal{B}_{C^1}(\Delta_1, n)$ of C^1 of radius n centred in Δ_1 for $n \geq 2$, and let u be a dyadic triangulation of Farey type at distance $n + 1$ of Δ_1 . Consider a path $p = \Delta_1, u_1, \dots, u_{n-1}, u_n, u$ of length $n + 1$ joining Δ_1 and u . The vertices u_{n-1}, u_n, u define a unique 2-cell in C^2 (see Remark 5.1), and this 2-cell contains at least a fourth vertex $u' \in \mathcal{B}_{C^1}(\Delta_1, n)$.



Thus, $\psi(u) \in C^2$ must be the vertex of the unique 2-cell adjacent to $\psi(u_n)$, containing $\psi(u')$, $\psi(u_{n-1})$, $\psi(u_n)$ on its boundary, and different from $\psi(u_{n-1})$. □

The main obstruction to the extension of a link isomorphism to an automorphism of the asymptotic flip complex C^2 is sketched here:



where the dark triangle shows we have reversed the orientation inherited from D .

Lemma 6.3. *Let Δ_1 and Δ_2 be dyadic triangulations of C^0 and let $i : \mathcal{L}^2(\Delta_1) \rightarrow \mathcal{L}^2(\Delta_2)$ be an isomorphism of their links being orientation reversing on $\tau_1 = (u_0, u_1, u_2)$ and orientation preserving on $\tau_2 = (u_0, u_3, u_4)$. Then, it does not exist any $\varphi \in \text{Aut}(C^2)$ with $\varphi_{*,\Delta_1} = i$.*

Proof. Suppose there exists $\varphi \in \text{Aut}(C^2)$ such that $\varphi_{*,\Delta_1} = i$.

The dyadic triangulations u_1, Δ_1, u_0 define a pentagonal 2-cell of C^2 ; let u_5 be the dyadic triangulation at distance 1 of u_0 and different from Δ_2 on this cell. Analogously, let u_6 be the dyadic triangulation at distance 1 of u_0 , different from Δ_1 and lying on the

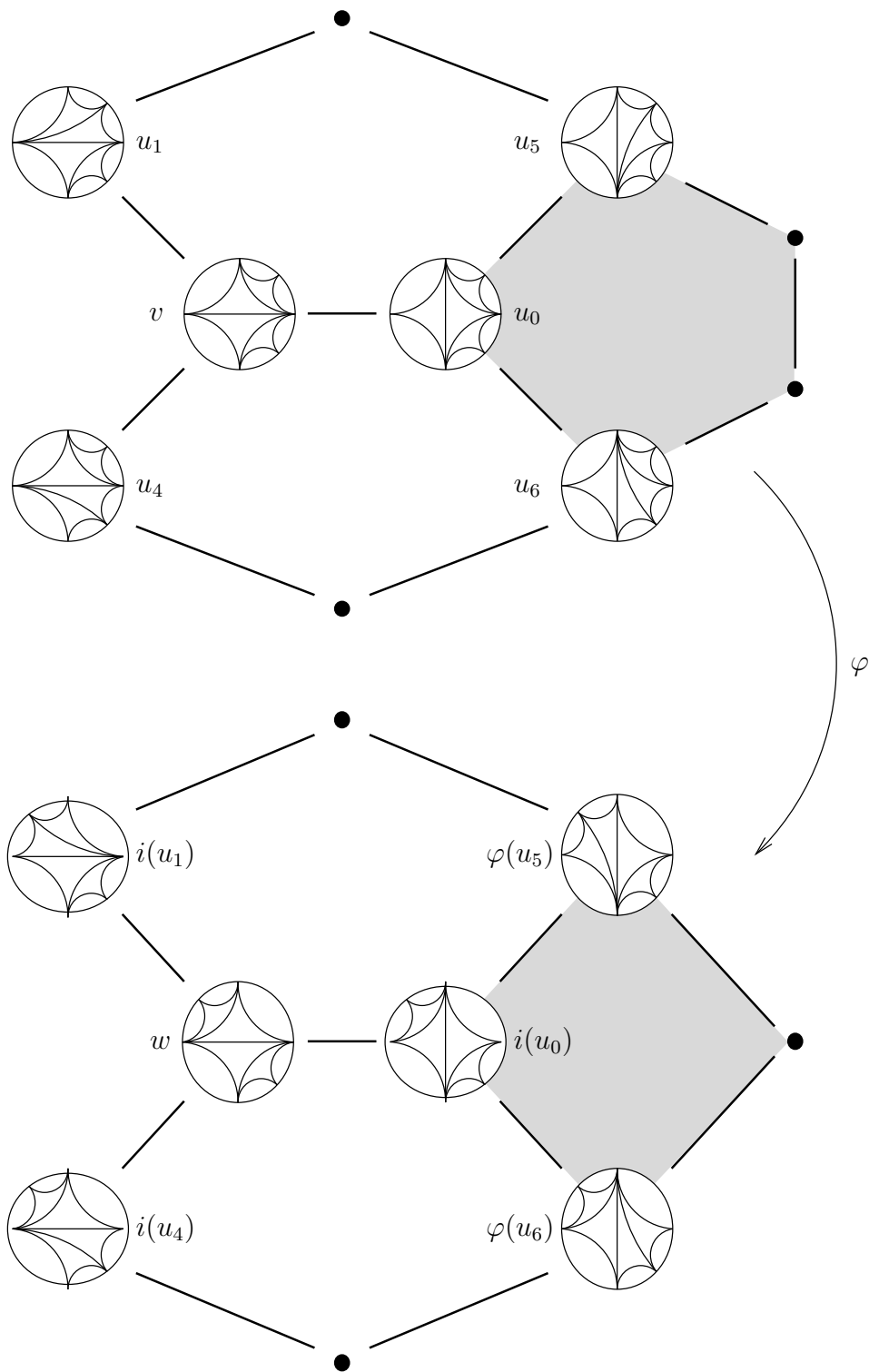
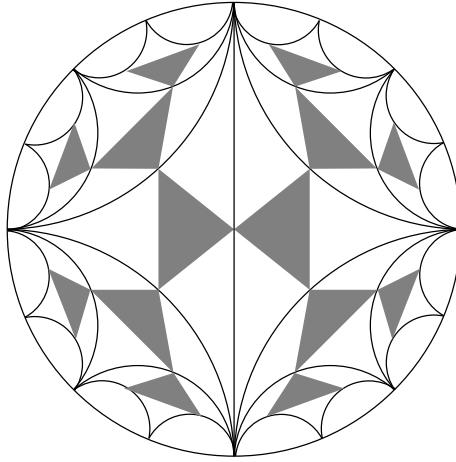


Figure 6.1: Example of Lemma 6.3.

pentagonal 2-cell defined by the path u_4, Δ_1, u_0 . The path u_5, u_0, u_6 defines a pentagonal 2-cell of C^2 .

Moreover, $i(u_1), i(\Delta_1), i(u_0)$ also define a pentagonal 2-cell of C^2 . Adapting the proof of uniqueness from Lemma 6.2, $\varphi(u_5)$ must be the vertex at distance 1 of $i(u_0)$ and different from Δ_2 in this pentagonal 2-cell. Furthermore, $\varphi(u_6)$ must be the vertex at distance 1 of $\varphi(u_0)$ different from Δ_2 and lying in the pentagonal 2-cell defined by the path $i(u_4), i(\Delta_1), i(u_0)$. But in this case, the path $\varphi(u_5), \varphi(u_0), \varphi(u_6)$ defines a squared 2-cell of C^2 . This contradicts the fact that $\varphi \in \text{Aut}(C^2)$. See Figure 6.1 (where $u = \Delta_1$ and $w = \Delta_2$) for a picture of the situation. \square

Example 24. The link of the dyadic triangulation Δ is



where Δ is obtained by flipping $(\gamma(0), \gamma(1/2))$ on Farey's dyadic triangulation Δ_F .

Remark 26. Let Δ_1 be a dyadic triangulation of Farey type and let Δ_2 be one of its neighbours in C^1 . Let Δ_2, u_1, u_2 and Δ_2, u_3, u_4 denote the triangles of $\mathcal{L}^2(\Delta_1)$ containing Δ_2 . Observe that the geometric representation of the link $\mathcal{L}^2(\Delta_2)$ of Δ_2 has the same vertices as $\mathcal{L}^2(\Delta_1)$ and only the above mentioned triangles change to Δ_2, u_1, u_4 , and to Δ_2, u_2, u_3 . See Examples 22 and 24.

This provides an alternative proof of Lemma 6.3:

Proof. Suppose there exists $\varphi \in \text{Aut}(C^2)$ such that $\varphi_{*,\Delta_1} = i$. Then φ_{*,u_0} should be an isomorphism between the link of u_0 and the link of $i(u_0)$. As a consequence of Remark 26, u_1 and u_4 are adjacent in the link of u_0 , but $i(u_1)$ and $i(u_4)$ are not adjacent in the link of $i(u_0)$. This contradicts the fact that φ_{*,u_0} is an isomorphism. \square

Lemma 6.4. Let $i_R : \mathcal{L}^2(\Delta_F) \rightarrow \mathcal{L}^2(\Delta_F)$ be the orientation reversing isomorphism obtained by the symmetry of axis $(\gamma(0), \gamma(1/2))$. Then, there exists a unique automorphism $\varphi_R \in \text{Aut}(C^2)$ such that $\varphi_{R,*,\Delta_F} = i_R$.

Proof. Uniqueness can be proved exactly as in Lemma 6.2. For the existence, consider the orientation reversing homeomorphism h of D consisting of the symmetry of axis $(\gamma(0), \gamma(1/2))$. Note that $h(\partial D[1/2]) = \partial D[1/2]$. Thus, given a dyadic triangulation Δ of Farey type, $h(\Delta)$ is a dyadic triangulation of D of Farey type. Similarly, given an almost

dyadic triangulation $a\Delta$ of Farey type, $h(a\Delta)$ is also an almost dyadic triangulation of Farey type and the n -cell associated to $h(a\Delta)$ is of the same type as the n -cell associated to $a\Delta$ (here n can be either 1 or 2). Hence, if we define φ_R to be the morphism of C^2 such that $\varphi_R(\Delta) = h(\Delta)$ for all $\Delta \in C^0$, and $\varphi_R(a\Delta) = h(a\Delta)$ for all $a\Delta \in C^n$, $\varphi_R \in \text{Aut}(C^2)$ and satisfies $\varphi_{R^*, \Delta_F} = i_R$ by construction. \square

6.3 The group of automorphisms of C

Definition 38. Let $\varphi \in \text{Aut}(C^2)$. It is *orientation preserving (reversing)* if it is Δ -orientation preserving (reversing) for all dyadic triangulations $\Delta \in C^0$.

Remark 27. As a consequence of Lemma 6.2, the subgroup of orientation preserving automorphisms of C^2 is isomorphic to Thompson's group T .

Remark 28. The automorphism φ_R from Lemma 6.4 is orientation reversing.

Proposition 6.1. *Let φ be an automorphism of C^2 . Then, φ is either orientation preserving or orientation reversing.*

Proof. As a consequence of Lemma 6.3, given an automorphism φ of C^2 , its restriction φ_{*, Δ_F} to the link $\mathcal{L}^2(\Delta_F)$ is either Δ_F -orientation preserving or Δ_F -orientation reversing. In the first case, $\varphi \in \Psi(T)$ by Lemma 6.2 and thus, it is orientation preserving by Remark 27. In the second case, consider the automorphism $\varphi' = \varphi \circ \varphi_R$, which is Δ_F -orientation preserving. By Lemma 6.2 and Remark 27 φ' is orientation preserving. Then, $\varphi = \varphi' \circ \varphi_R$ (note that $\varphi_R^{-1} = \varphi_R$), therefore φ is orientation reversing (Remark 28). \square

We are now ready to prove following theorem:

Theorem 6.1. *The automorphism group of the asymptotic flip complex C^2 is isomorphic to the semi-direct product $T \rtimes \mathbb{Z}/2\mathbb{Z}$.*

Proof. We want to construct the exact sequence

$$1 \longrightarrow T \longrightarrow \text{Aut}(C^2) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1,$$

and prove that it splits.

Let us denote by $\pi : \text{Aut}(C^2) \rightarrow \mathbb{Z}/2\mathbb{Z}$ the morphism defined as follows:

$$\varphi \mapsto \begin{cases} 0, & \text{if } \varphi \text{ is orientation preserving} \\ 1, & \text{if } \varphi \text{ is orientation reversing.} \end{cases}$$

Proposition 6.1 shows that the map π is a well defined morphism of groups. As a consequence of Lemmas 6.2 and 6.4, the map π is surjective and the isomorphism $\mathbb{Z}/2\mathbb{Z} \simeq \langle \varphi_R \rangle$ is a section of π .

Finally, $\ker(\pi) \simeq T$ follows from Lemma 6.2. \square

Therefore, we have an action of $\mathbb{Z}/2\mathbb{Z}$ into the group of automorphisms of T , where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts as the automorphism γ_R defined by $\gamma_R(t) = \varphi_R \circ t \circ \varphi_R$, for $t \in T$. In particular, the automorphism γ_R obtained in T is defined by $\gamma_R(\alpha) = \alpha^{-1}$ and

$\gamma_R(\beta) = \alpha^2\beta^{-1}\alpha^2$, where α and β are the torsion generators of T from Proposition 2.3. Thus, if we compose γ_R with the conjugation by α^2 we obtain the generator of the group of outer automorphisms of T sending α to α^{-1} and β to β^{-1} . This automorphism is the generator of $\text{Out}(T)$ given by Brin in [7].

Remark 29. (Geometric interpretation of Theorem 6.1.) By Proposition 6.1, all automorphisms of C^2 are either orientation preserving or orientation reversing. Thus, it seems natural to define the *asymptotic extended mapping class group* of Σ as

$$T \rtimes \mathbb{Z}/2\mathbb{Z}.$$

Using this notion, Theorem 6.1 turns out to be an exact analogue to Ivanov's theorem for the planar surface of infinite type Σ .

To end this chapter, let's go back to the automorphism of the infinite associahedron C . Recall that we have already proven that T is a subgroup of the automorphism group of the infinite associahedron C because T acts by automorphisms on C . Furthermore, as the asymptotic flip complex C^2 is the 2-dimensional skeleton of the complex C , the automorphism group of C must be a subgroup of $\text{Aut}(C^2)$. Thus, there are only two possibilities left to the group $\text{Aut}(C)$: either the orientation-reversing automorphism φ_R of C^2 can be extended to an automorphism of the complex C (in which case $\text{Aut}(C) \simeq \text{Aut}(C^2)$), either it does not and $\text{Aut}(C)$ is isomorphic to T .

Proposition 6.2. *The orientation-reversing automorphism φ_R of the asymptotic flip complex C^2 is the restriction of an automorphism of the infinite associahedron C to its 2-skeleton.*

The proof is analogous to the existence proof of Lemma 6.4.

Corollary 6.2. *The automorphism group of the infinite associahedron C is isomorphic to the semi-direct product $T \rtimes \mathbb{Z}/2\mathbb{Z}$.*

Chapter 7

Adding a boundary to C^1

This seventh chapter contains work in progress, so the results here are less conclusive than those from previous chapters. Once a “nice” topological space with a “good” action on it is established, it is natural to wonder about possible compactifications where the action could be extended. Here we define a geometric boundary at infinity to the metric 1-skeleton C^1 of the infinite associahedron. Furthermore, the action of T on C^1 can be extended to its geometric boundary. The methods used to present this boundary have flavours of Gromov’s definition of the boundary of an hyperbolic space. Thus, we first define a class of rays on C^1 (STT-rays) which includes all geodesic rays, and then we associate to every STT-ray a subset of the set of dyadic arcs of the disk. These subsets can somehow be seen as “generalised triangulations of D ”, and allow us to define an equivalence relation on the set of STT-rays.

Even though we do not have a result classifying explicitly all the points on the geometric boundary, we study significant examples of different behaviours (this explains why this chapter is more or less 10 pages longer than the others). In addition, the given examples suggest that one can adapt Thurston’s methods to embed C^1 into a compact functional space. Thus, we define the Thurston’s compactification of C^1 , and we prove that all the points from the geometric boundary that we have been able to describe precisely belong to Thurston’s compactification.

7.1 Geometric boundary at infinity of C^1

We have seen that the metric graph C^1 is not Gromov hyperbolic (see Proposition 5.1). Nevertheless, we would like to define a geometric boundary at infinity of C^1 inspired from the definition of the Gromov boundary by equivalence classes of infinite geodesic rays. Unfortunately, the study of geodesics on C^1 is not easy at all in the sense that, given an infinite ray, we do not know how to determine if it is geodesic. However, Sleator, Tarjan and Thurston [72] gave a property of geodesic rays on finite associahedra that can be generalised to define a larger family of infinite rays on C^1 containing all geodesic rays. Thus, we state that two infinite rays of C^1 satisfying the generalised property of Sleator Tarjan and Thurston are equivalent if they define the same stable arc set, which can be

seen geometrically.

Definition 39. An infinite ray

$$\rho : [0, \infty) \longrightarrow C^1$$

is *geodesic* on C^1 if, for all $s_1, s_2 \in [0, \infty)$,

$$d_{C^1}(\rho(s_1), \rho(s_2)) = |s_1 - s_2|,$$

where d_{C^1} denotes the combinatorial distance on C .

Lemma 7.1. (*Sleator-Tarjan-Thurston, see [72] Lemma 3*) *Let τ_1 and τ_2 be two different triangulations of a convex n -gon P .*

1. *If it is possible to flip one diagonal of τ_1 creating τ'_1 so that τ'_1 has one more diagonal in common with τ_2 , then there exists a geodesic path from τ_1 to τ_2 in which the first flip creates τ'_1 .*
2. *If τ_1 and τ_2 have a diagonal d in common, then a geodesic path from τ_1 to τ_2 will never flip d . In fact, any path between τ_1 and τ_2 that flips d is at least two flips longer than a geodesic path.*

Let $\rho : [0, k] \longrightarrow C^1$ be a finite path on C^1 between two dyadic triangulations of Farey type $\Delta_0, \Delta_k \in C^0$. For $i \in \{1, \dots, k\}$, let us define

$$\Delta_i = \rho(i).$$

Then, ρ can be naturally embedded into the n -th associahedron, where n is the number of missing diagonals

$$n = \mathcal{D}(\Delta_0 \cap \Delta_1 \cap \dots \cap \Delta_k)$$

of the dyadic almost triangulation $\Delta_0 \cap \Delta_1 \cap \dots \cap \Delta_k$ (see Proposition 5.2 and Corollary 5.2).

Definition 40. Let $a\Delta$ be a dyadic almost triangulation of D . We denote by $C(a\Delta)$ the subgraph of C^1 induced by the set of vertices

$$\{\Delta \in C^0 : a\Delta \subset \Delta\}.$$

Corollary 7.1. *Let $\Delta_1, \Delta_2 \in C^0$ be two different dyadic triangulations of D of Farey type. Note that $\Delta_1 \cap \Delta_2$ is a dyadic almost triangulation of Farey type. Then, each geodesic path of C^1 joining Δ_1 and Δ_2 is contained on $C(\Delta_1 \cap \Delta_2)$. Furthermore,*

$$\mathcal{D}(\Delta_1 \cap \Delta_2) \leq d_{C^1}(\Delta_1, \Delta_2),$$

where $d_{C^1}(\Delta_1, \Delta_2)$ is the combinatorial distance on C^1 , and $\mathcal{D}(\Delta_1 \cap \Delta_2)$ is the number of non-common diagonals between Δ_1 and Δ_2 .

Definition 41. Let $\rho : [0, \infty) \rightarrow C^1$ be a ray on C^1 and let $\{\Delta_i\}_{i \in \mathbb{N}}$ be the sequence of its vertices. Then, ρ is an *STT-ray* if, for each dyadic arc $a \in \mathcal{A}$, the subset

$$s(a, \rho) := \{i \in \mathbb{N} : a \in \Delta_i\}$$

is a (possibly empty or infinite) set of consecutive integers.

The following result is a direct consequence of Lemma 7.1:

Lemma 7.2. Let ρ be a geodesic ray on C^1 (or a finite geodesic path), let $\{\Delta_i\}_{i \in \mathbb{N}}$ be the sequence of its vertices, and let $a \in \mathcal{A}$ be a dyadic arc. Then,

$$s(a, \rho) := \{i \in \mathbb{N} : a \in \Delta_i\}$$

is a (possibly empty or infinite) set of consecutive integers. Equivalently, ρ is an *STT-ray*.

Remark 30. There exist STT-rays which are not geodesic. For example, any path of length 3 on the boundary of a pentagonal 2-cell.

Example 25. Let $\rho : [0, 3] \rightarrow C^1$ starting at Farey's dyadic triangulation and flipping, in the given order, the arcs $a_1 = (\gamma(0), \gamma(\frac{1}{4}))$, $a_2 = (\gamma(\frac{1}{4}), \gamma(\frac{1}{2}))$, and $a_3 = (\gamma(\frac{1}{8}), \gamma(\frac{1}{2}))$. Thus, $\rho(0) = \Delta_F$, $\rho(1)$ is the dyadic triangulation of Farey type

$$\Delta_F \cup \{a_3\} \setminus \{a_1\},$$

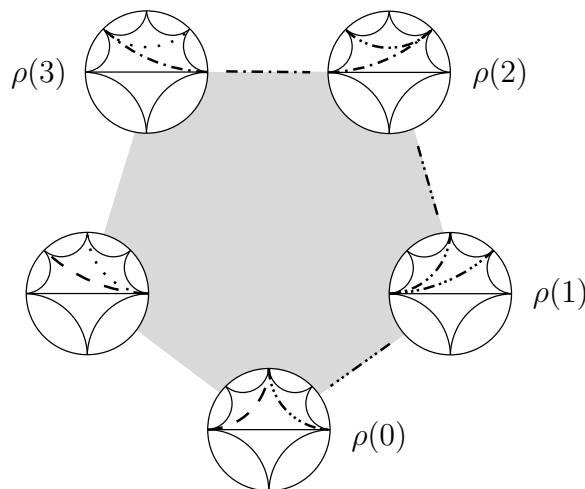
$\rho(2)$ is the dyadic triangulation of Farey type

$$\Delta_F \cup \{a_3, a_4\} \setminus \{a_1, a_2\},$$

where $a_4 = (\gamma(\frac{1}{8}), \gamma(\frac{3}{8}))$, and $\rho(3)$ is the dyadic triangulation of Farey type

$$\Delta_F \cup \{a_4, a_5\} \setminus \{a_1, a_2\},$$

where $a_5 = (\gamma(0), \gamma(\frac{3}{8}))$.



The path ρ is clearly non-geodesic:

$$d_{C^1}(\rho(0), \rho(3)) = 2 < |3 - 0| = 3.$$

From the figure it can be seen that

1. $s(a_1, \rho) = \{0\}$,
2. $s(a_2, \rho) = [0, 1]$,
3. $s(a_3, \rho) = [1, 2]$,
4. $s(a_4, \rho) = [2, 3]$, and
5. $s(a_5, \rho) = \{3\}$.

Furthermore,

1. For all $a \in \Delta_F \setminus \{a_1, a_2, a_3, a_4, a_5\}$, we have $s(a, \rho) = [0, 3]$, and
2. For all $a \notin \Delta_F \cup \{a_1, a_2, a_3, a_4, a_5\}$, we have $s(a, \rho) = \emptyset$,

proving that ρ is an STT-path.

Definition 42. Let ρ be an STT-ray of C^1 . The *stable arc set* of ρ is

$$\mathcal{A}_{stab}(\rho) := \{a \in \mathcal{A} : \exists s_a \in [0, \infty) \mid s(a, \rho) = [s_a, \infty)\} \subset \mathcal{A}.$$

Remark 31. Let ρ be an STT-ray on C^1 and let $\{\Delta_n\}_{n \in \mathbb{N}}$ be the sequence of dyadic triangulations of Farey type appearing in ρ (the vertices of the ray). Recall that every dyadic triangulation is seen as a set of dyadic arcs. Then,

$$\mathcal{A}_{stab}(\rho) = \liminf_{n \rightarrow \infty} \Delta_n.$$

Furthermore, by the hypothesis STT on the ray ρ , this lower limit coincides with the upper limit

$$\mathcal{A}_{stab}(\rho) = \limsup_{n \rightarrow \infty} \Delta_n.$$

Thus, the stable arc set is a set-theoretical limit.

Lemma 7.3. Let ρ and ρ' be two different STT-rays of C^1 , and let $\mathcal{A}_{stab}(\rho)$ and $\mathcal{A}_{stab}(\rho')$ be their respective stable arc sets. Suppose that ρ and ρ' coincide at their final segments (there exist $s_0 \in [0, \infty)$ and $r \in \mathbb{R}$ such that, for all $s \in [s_0, \infty)$, the equation $\rho'(s) = \rho(s + r)$ is satisfied). Then,

$$\mathcal{A}_{stab}(\rho') = \mathcal{A}_{stab}(\rho).$$

Proof. The lemma is directly obtained from the definition of the stable arc set. \square

Lemma 7.4. Let ρ be an STT-ray of C^1 . Suppose that there exists an arc-connected component of $D - \mathcal{A}_{stab}(\rho)$ bounded by a finite set of dyadic arcs

$$\{a_1, \dots, a_k\} \subset \mathcal{A}_{stab}(\rho) \quad (k > 2)$$

which is a finite polygon P . Then, P is a triangle.

Proof. Without loss of generality, suppose that $\rho(0)$ is a dyadic triangulation of D of Farey type. For $i \in \{1, \dots, k\}$ let n_i be the minimum of the interval $s(a_i, \rho)$,

$$n_i = \min \{s \in [0, \infty) : s \in s(a_i, \rho)\},$$

and let n_0 be the maximum of all n_i ,

$$n_0 = \max\{n_1, \dots, n_k\}.$$

Equivalently, n_0 is the minimum of the intersection of the intervals $s(a_i, \rho)$:

$$n_0 = \min \left\{ s \in [0, \infty) : s \in \bigcap_{i=1}^k s(a_i, \rho) \right\}.$$

Thus, given $i \in \{1, \dots, k\}$, and $n \in \mathbb{N}$ satisfying $n \geq n_0$, the dyadic arc a_i belongs to the dyadic triangulation of Farey type

$$\Delta_n = \rho(n).$$

Consider all the interior diagonals of the finite polygon P :

$$d_1, \dots, d_l \quad \left(l = \frac{k(k-3)}{2} \right).$$

Since none of the interior diagonals of P belongs to the stable arc set of ρ , the intervals

$$s(d_1, \rho), \dots, s(d_l, \rho)$$

are all finite. For $j \in \{1, \dots, l\}$ let m_j be the length of the interval $s(d_j, \rho)$. Let m_0 be

$$m_0 = \max\{m_1, \dots, m_l\} + 1.$$

Then,

$$\Delta_{n_0+m_0} = \rho(n_0 + m_0)$$

is a dyadic triangulation of D of Farey type containing the finite polygon P and none of its interior diagonals. Hence, P must be a triangle. \square

Definition 43. Two STT-rays ρ, ρ' on C^1 are *equivalent* if

$$\mathcal{A}_{stab}(\rho) = \mathcal{A}_{stab}(\rho').$$

The *geometric boundary at infinity* of C^1 is the set of equivalence classes of STT-rays on C^1 .

We would like to extend the action of Thompson's group T on the 1-skeleton C^1 to its geometric boundary at infinity. The main tool used to define the topological action of Thompson's group T on the cellular complex C is the fact that T acts on \mathcal{A} bijectively,

since \mathcal{A} can be embedded into $S^1 \times S^1$ and T is a subgroup of $\text{Homeo}(S^1)$ fixing the set of dyadic numbers. Let $t \in T$ and $a = (\gamma(p), \gamma(q)) \in \mathcal{A}$. Then,

$$t \cdot a = (\gamma \circ t(p), \gamma \circ t(q)) \in \mathcal{A}.$$

Since we have given a metric to C^1 , we would like to define the action of T on every point of the metric space C^1 . Note that the action is already well defined in every point of the 0-skeleton of C^1 by taking the topological action. Let t be an element of Thompson's group T and let

$$p = s\Delta_1 + (1-s)\Delta_2 \quad (s > 0)$$

be a point in the interior of an edge of C^1 . Then, we define the action of t on p by

$$t \cdot p = s(t \cdot \Delta_1) + (1-s)(t \cdot \Delta_2).$$

This is well defined thanks to Lemma 5.3. As a consequence of the previous definitions, T acts on the set of rays of C^1 . Let $\rho : [0, \infty) \rightarrow C^1$ be a ray on C^1 , and let $t \in T$. Then, t acts on ρ by

$$(t \cdot \rho)(s) = t \cdot \rho(s).$$

Lemma 7.5. *Let ρ be an STT-ray on C^1 , and let t be an element of Thompson's group T . Then, $t \cdot \rho$ is an STT-ray on C^1 .*

Proof. Let $a \in \mathcal{A}$. Since T acts on \mathcal{A} by bijections, there exist $a' \in \mathcal{A}$ such that $t \cdot a' = a$ ($a' = t^{-1} \cdot a$). Thus,

$$s(a, t \cdot \rho) = s(a', \rho),$$

which is an interval because ρ is an STT-ray. □

Corollary 7.2. *Thompson's group T acts on the set of STT-rays of C^1 . Furthermore, let $t \in T$ and let ρ, ρ' be two equivalent STT-rays. Then,*

$$\mathcal{A}_{stab}(t \cdot \rho) = \mathcal{A}_{stab}(t \cdot \rho').$$

Corollary 7.3. *Thompson's group T acts on the geometric boundary at infinity of C^1 .*

We can derive the following result from the previous Corollary using the inclusion of Thompson's group F on T :

Corollary 7.4. *Thompson's group F acts on the geometric boundary at infinity of C^1 .*

Given a Gromov hyperbolic space H , there is a definition of its Gromov boundary by taking equivalence classes of infinite geodesics. Then, every pair of geodesics staying at a bounded distance from each other define the same point at infinite. This is not true in our case.

Remark 32. *Non-hyperbolic behaviour of the geometric boundary at infinity.* There exist infinite STT-rays ρ and ρ' such that the inequality

$$d_{C^1}(\rho(s), \rho'(s + s')) \leq m$$

has a solution for $m \in \mathbb{N}$, for all $s \in [0, \infty)$, $\mathcal{A}_{stab}(\rho) \neq \mathcal{A}_{stab}(\rho')$.

Consider the geodesic ρ from the main Example in subsection 7.2, and let ρ' be the geodesic of the same kind which do not flip the arc $(\gamma(0), \gamma(\frac{1}{2}))$. The geodesics ρ and ρ' satisfy:

1. For all $s \in [0, \infty)$

$$d_{C^1}(\rho(s), \rho'(s)) \leq 2,$$

2. $(\gamma(\frac{1}{4}), \gamma(\frac{3}{4})) \in \mathcal{A}_{stab}(\rho)$, and
3. $(\gamma(\frac{1}{4}), \gamma(\frac{3}{4})) \notin \mathcal{A}_{stab}(\rho')$.

7.2 Points on the geometric boundary of C^1

It is useful introducing another representation of an STT-ray ρ before giving some examples of stable arc sets. Without loss of generality, suppose that ρ starts at a dyadic triangulation $\Delta_0 = \rho(0) \in C^0$ (see Property 2 from Lemma 7.3). The dyadic triangulation Δ_0 is a subset of \mathcal{A} , which is a countable set, thus it is countable. So we can represent Δ_0 as an ordered list of dyadic arcs

$$\Delta_0 = \{a_n^{(0)}\}_{n \in \mathbb{N}}.$$

There exists $m_1 \in \mathbb{N}$ such that the dyadic triangulation $\Delta_1 = \rho(1)$ is obtained from Δ_0 by flipping the dyadic arc a_{m_1} . Let $a_{m_1}^{(0)}$ be the dyadic arc satisfying

$$\begin{aligned} a_{m_1}^{(0)} &\in \Delta_0, \text{ and} \\ a_{m_1}^{(0)} &\notin \Delta_1. \end{aligned}$$

We can represent Δ_1 by the ordered set of dyadic arcs

$$\Delta_1 = \{a_n^{(1)}\}_{n \in \mathbb{N}},$$

with

$$a_j^{(1)} = \begin{cases} a_j^{(0)} & \text{if } j \neq m_1 \\ a_{m_1}^{(1)} & \text{if } j = m_1, \end{cases}$$

where $a_{m_1}^{(1)}$ is the dyadic arc obtained by flipping $a_{m_1}^{(0)}$ on Δ_0 . By repeating this process, we obtain a sequence of sequences of dyadic arcs associated to ρ .

Example 26. We have already seen that dyadic arcs of Farey's dyadic triangulation Δ_F are in bijection with vertices of the infinite rooted binary tree R_2 . Note that, due to the identification $0 \sim 1$ in ∂D , the root has no pre-image by this map, and the two vertices on the first level (which correspond to the intervals $[0, 1/2]$ and $[1/2, 1]$) represent the same geodesic arc in Farey's dyadic triangulation Δ_F (the horizontal edge). All other vertices are in one to one correspondence with dyadic arcs of Δ_F . Then, if one considers

the downward left to right order in the vertices of R_2 , Farey's dyadic triangulation can be thought as the following sequence of dyadic arcs:

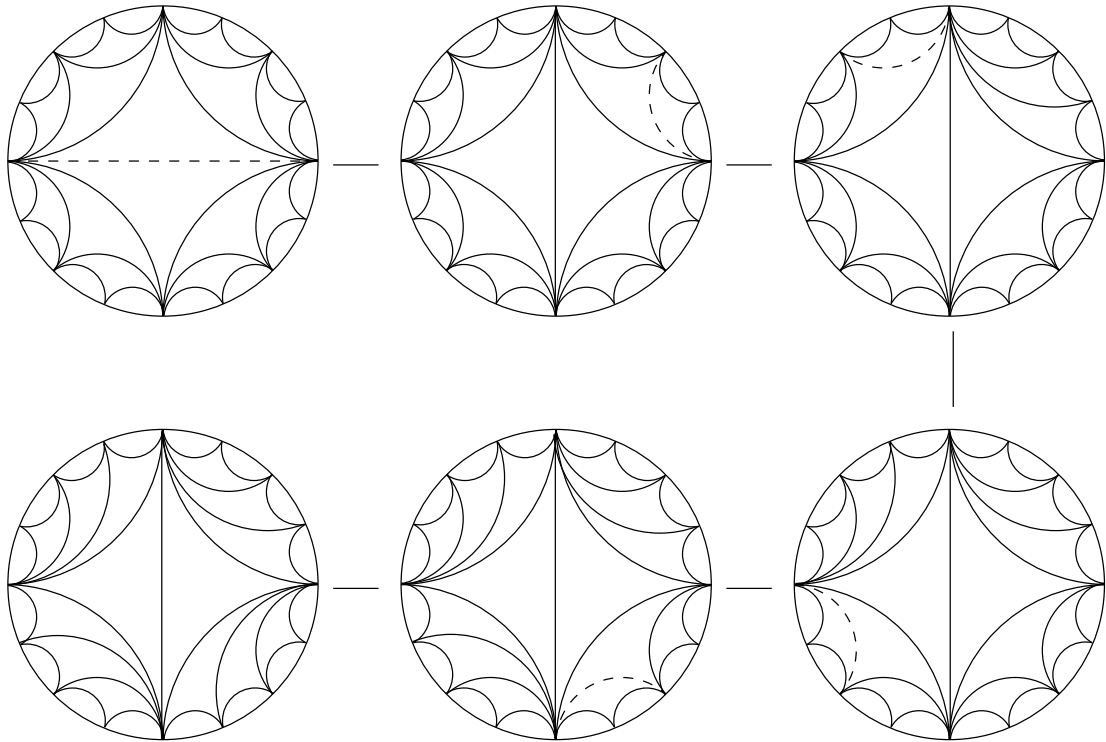
$$\Delta_F = \left\{ \left(\gamma(0), \gamma\left(\frac{1}{2}\right) \right), \left(\gamma(0), \gamma\left(\frac{1}{4}\right) \right), \left(\gamma\left(\frac{1}{4}\right), \gamma\left(\frac{1}{2}\right) \right), \right. \\ \left. \left(\gamma\left(\frac{1}{2}\right), \gamma\left(\frac{3}{4}\right) \right), \left(\gamma\left(\frac{3}{4}\right), \gamma(1) \right), \left(\gamma(0), \gamma\left(\frac{1}{8}\right) \right), \dots \right\}.$$

We present some examples of STT-rays with different types of stable arc sets. They are a representative set of all known examples, and they have Farey's dyadic triangulation Δ_F as basepoint (it has been proved in property 2 of Lemma 7.3 that this is not loss of generality).

Remark 33. In all the examples given on this section, every arc-connected component of $D - \mathcal{A}_{stab}(\rho)$ is either a triangle or a segment joining two boundary points of D , at least one of them being non-dyadic. Furthermore, when an arc-connected component of $D - \mathcal{A}_{stab}(\rho)$ is a segment, the stable arc set is locally infinite in it.

An infinite dimensional cube

Let $V = \{v_j\}_{j \in \mathbb{N}}$ be the subset of the set of vertices of R_2 which contains all even vertices of all even levels (the root is the level 0) of R_2 , and let a_{n_j} be the dyadic arc corresponding to v_j from Example 26. Consider the STT-ray ρ obtained by flipping first a_{n_1} , then a_{n_2} , then a_{n_3} , and so on. The initial segment of ρ is the following:



The distance between every pair of dyadic triangulations $\Delta_m = \rho(m)$, $\Delta_n = \rho(n)$ is exactly the number of non-common dyadic arcs $\mathcal{D}(\Delta_m \cap \Delta_n) = |m - n|$, thus ρ is a geodesic of C^1

(see Corollary 7.1). Let $a_i \in \Delta_F$ be a dyadic arc. The sequence associated to the dyadic arc a_i by ρ is

$$\{a_i^{(n)}\}_{n \in \mathbb{N}} = \begin{cases} \{a_{n_j}, \dots, a_{n_j}, b_{n_j}, b_{n_j}, \dots\}, & \text{if } v_{n_j} \in V \\ \{a_i, a_i, \dots\}, & \text{otherwise,} \end{cases}$$

where b_{n_j} is the dyadic arc obtained by flipping a_{n_j} on Δ_F . The stable arc set $\mathcal{A}_{stab}(\rho)$ is the dyadic and locally finite triangulation of D from Example 17

$$\begin{aligned} \mathcal{A}_{stab}(\rho) &= \left\{ \left(\gamma \left(\frac{1}{4} \right), \gamma \left(\frac{3}{4} \right) \right) \right\} \\ &\cup \left\{ \left(\gamma \left(\frac{m}{2^{2k+1}} \right), \gamma \left(\frac{m+1}{2^{2k+1}} \right) \right) : k, m \in \mathbb{N}, 0 \leq m < 2^{2k+1} \right\} \\ &\cup \left\{ \left(\gamma \left(\frac{2j+1}{2^{2k}} \right), \gamma \left(\frac{2j+2}{2^{2k}} \right) \right) : j, k \in \mathbb{N}, 0 < 2j+1 < 2^{2k} \right\} \\ &\cup \left\{ \left(\gamma \left(\frac{4j+1}{2^{2k+1}} \right), \gamma \left(\frac{2j+2}{2^{2k+1}} \right) \right) : j, k \in \mathbb{N}, 0 < 4j+1 < 2^{2k+1} \right\}. \end{aligned}$$

The flips associated to the set of dyadic arcs $\{a_{n_i} : i \in \mathbb{N}\}$ commute two by two. Roughly speaking, this means they define an infinite dimensional cube on the infinite associahedron C . Thus, every infinite geodesic ρ' on the 1-skeleton of this infinite cube fulfils

$$\mathcal{A}_{stab}(\rho') = \mathcal{A}_{stab}(\rho).$$

Proposition 7.1. *Let Δ' be a dyadic triangulation of D such that*

$$D \setminus (\Delta' \cap \Delta_F) = \bigcup_{n=1}^{\infty} P_n$$

is a disjoint union of polygons. Then, there exist infinitely many STT-rays ρ with stable arc set equal to Δ' .

Proof. For every $n \in \mathbb{N}$, let τ_n be the restriction of the dyadic triangulation Δ_F to the polygon P_n , and let τ'_n be the restriction of the dyadic triangulation Δ' to P_n . Let δ_n be a geodesic path on the associahedron of a convex k_n -gon joining τ and τ' . Let Δ be a dyadic triangulation of Farey type containing the boundary of P_n and such that the restriction of Δ to P_n is τ_n . Define $\rho_n(\Delta)$ as the geodesic path on C^1 fixing Δ outside P_n whose restriction to P_n coincides with δ_n . Then, given a bijection σ of \mathbb{N} , the ray with initial segment $\rho_{\sigma(1)}(\Delta_F)$ obtained by concatenation of the finite paths

$$\rho_{\sigma(2)}(\Delta_2), \rho_{\sigma(3)}(\Delta_3), \dots, \rho_{\sigma(n)}(\Delta_n), \rho_{\sigma(n+1)}(\Delta_{n+1}), \dots,$$

is an STT-ray satisfying

$$\mathcal{A}_{stab}(\rho) = \Delta',$$

where Δ_n is the dyadic triangulation of Farey type defined recursively by $\rho_{\sigma(n-1)}(\Delta_{n-1})$. \square

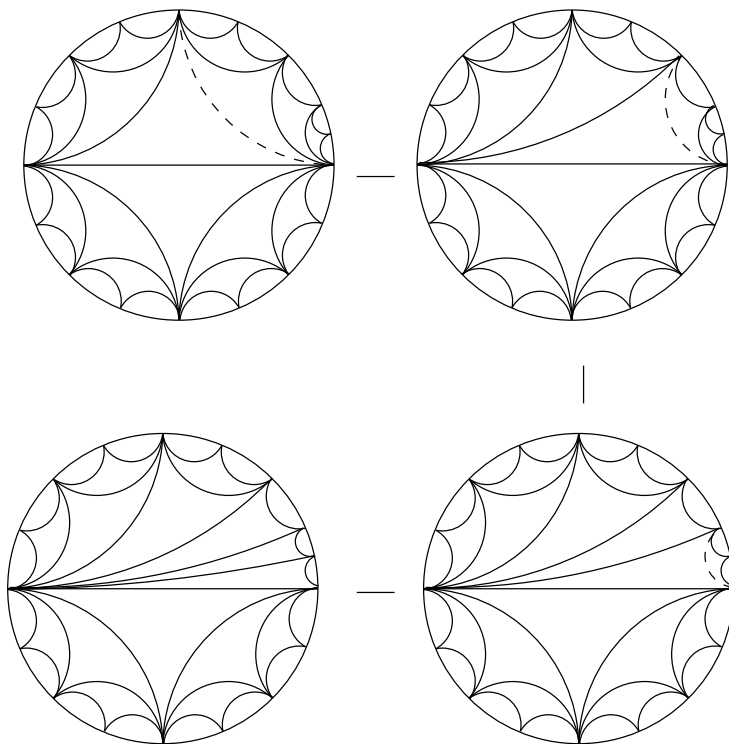
Remark 34. Let k_n denote the number of sides of the polygon P_n from Proposition 7.1. The sequence $\{k_n\}$ may not be bounded.

Definition 44. A dyadic triangulation of D satisfying the hypothesis of Proposition 7.1 is said to *contain a Farey tessellation* of D .

Question: does every locally finite dyadic triangulation of D contain a Farey tessellation of D ?

A geodesic ray following only pentagonal faces

Let $\rho \in \mathcal{G}$ be the geodesic ray obtained by flipping all dyadic arcs of the form $(\gamma(0), \gamma(p))$, with $p < \frac{1}{2}$, by decreasing euclidean length, *i.e.*, we first flip $(\gamma(0), \gamma(\frac{1}{4}))$, then we flip $(\gamma(0), \gamma(\frac{1}{8}))$, then $(\gamma(0), \gamma(\frac{1}{16}))$, and so on. The initial segment of ρ is the following:



The distance between every pair of dyadic triangulations of ρ is exactly the number of non-common dyadic arcs, hence ρ is a geodesic ray on C^1 . Let $k \in \mathbb{N}$, $k \geq 2$. The dyadic arc

$$a = \left(\gamma(0), \gamma\left(\frac{1}{2^k}\right) \right)$$

has as associated sequence

$$a^{(n)} = \begin{cases} a & \text{if } n < k, \\ \left(\gamma\left(\frac{1}{2^k}\right), \gamma\left(\frac{1}{2}\right) \right) & \text{if } n \geq k, \end{cases}$$

which is stationary and has only 2 different values. For the remaining dyadic arcs of Δ_F , their associated sequences are constant. The stable arc set $\mathcal{A}_{stab}(\rho)$ is the dyadic triangulation of D from Example 18

$$\begin{aligned}\mathcal{A}_{stab}(\rho) &= \left\{ \left(\gamma(0), \gamma\left(\frac{1}{2}\right) \right) \right\} \\ &\cup \left\{ \left(\gamma\left(\frac{m}{2^n}\right), \gamma\left(\frac{m+1}{2^n}\right) \right) : m, n \in \mathbb{N}, 0 < m < 2^n \right\} \\ &\cup \left\{ \left(\gamma\left(\frac{1}{2^k}\right), \gamma\left(\frac{1}{2}\right) \right) : k \in \mathbb{N} \right\},\end{aligned}$$

which is a dyadic triangulation of D , locally infinite in the interior of the dyadic arc $(\gamma(0), \gamma(1/2))$, and locally finite elsewhere.

Lemma 7.6. *Let a_1, \dots, a_k be two by two non-crossing dyadic arcs. Then, there exists a dyadic triangulation Δ of Farey type such that*

$$\{a_1, \dots, a_k\} \subset \Delta.$$

Proof. Consider the dyadic arcs a_1, \dots, a_k , and let

$$\gamma\left(\frac{m_1}{2^{n_1}}\right), \dots, \gamma\left(\frac{m_l}{2^{n_l}}\right)$$

be their extremal points. Suppose that, for all $i \in \{1, \dots, l\}$, m_i and 2^{n_i} are co-primes. Define

$$n = \max\{n_1, \dots, n_l\},$$

and let P_n be the polygon bounded by the set of dyadic arcs corresponding to the standard dyadic intervals of length $\frac{1}{2^n}$:

$$\left\{ \left(\gamma\left(\frac{0}{2^n}\right), \gamma\left(\frac{1}{2^n}\right) \right), \left(\gamma\left(\frac{1}{2^n}\right), \gamma\left(\frac{2}{2^n}\right) \right), \dots, \left(\gamma\left(\frac{2^n-1}{2^n}\right), \gamma\left(\frac{2^n}{2^n}\right) \right) \right\}.$$

The dyadic arcs a_1, \dots, a_l are two by two non-crossing interior diagonals of P_n , thus we can complete them to have a triangulation τ of P_n . Then, the dyadic triangulation Δ of Farey type coinciding with τ on P_n and with Δ_F elsewhere satisfies $\{a_1, \dots, a_k\} \subset \Delta$. \square

Definition 45. A *Farey star region* is a pair $(P_\infty, \Delta(P_\infty))$, where:

1. P_∞ is a subregion of D bounded by an infinite polygonal line of consecutive dyadic arcs a_1, a_2, a_3, \dots starting at $\gamma\left(\frac{m_1}{2^{n_1}}\right)$ and such that it exists $n \in \mathbb{N}$ satisfying

$$a_{n+k} = \left(\gamma\left(\frac{2^k m_1 - 1}{2^{n_1+k}}\right), \gamma\left(\frac{m_1}{2^{n_1}}\right) \right) \in \Delta_F,$$

for all $k \in \mathbb{N}$.

2. $\Delta(P_\infty)$ is the dyadic triangulation of P_∞ given by the dyadic arcs of the form

$$(\gamma(p), \gamma(q)),$$

where $\gamma(p)$ is the extremal point of the dyadic arc a_1 different from $\gamma(\frac{m_1}{2^{n_1}})$, and $\gamma(q)$ is an extremal point of a_l for some $l \geq 3$.

Note that $\Delta(P_\infty)$ is locally infinite on the points of a_1 . If the arc a_1 belongs to Farey's dyadic triangulation Δ_F we say that $(P_\infty, \Delta(P_\infty))$ is a *Farey-based star region*.

Next proposition follows from Lemma 7.6 and the fact that \mathbb{N}^k is countable, using the construction of the previous Example on every Farey star region:

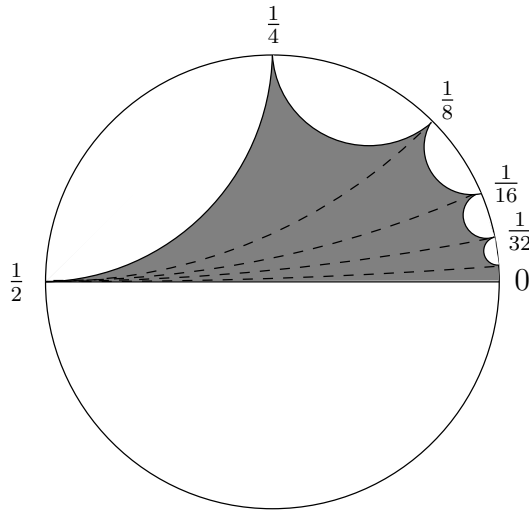
Proposition 7.2. *Let $(P_\infty^{(1)}, \Delta(P_\infty)^{(1)}), \dots, (P_\infty^{(k)}, \Delta(P_\infty)^{(k)})$ be Farey star regions with two by two empty intersections of their interiors, and let Δ be a dyadic triangulation of D coinciding with $\Delta(P_\infty)^{(i)}$ on $P_\infty^{(i)}$ for every $i \in \{1, \dots, k\}$ and of Farey type elsewhere. Then, there exist infinitely many STT-rays having Δ as stable arc set.*

Remark 35. If $(P_\infty^{(1)}, \Delta(P_\infty)^{(1)}), (P_\infty^{(2)}, \Delta(P_\infty)^{(2)})$ are Farey star regions locally infinite on a common dyadic arc a_1 and both of them approaching the same extremal point $\gamma(\frac{m_1}{2^{n_1}})$, then every dyadic triangulation satisfying Proposition 7.2 has only a finite number of dyadic arcs having $\gamma(\frac{m_1}{2^{n_1}})$ as extremal point.

With similar methods we obtain the following result:

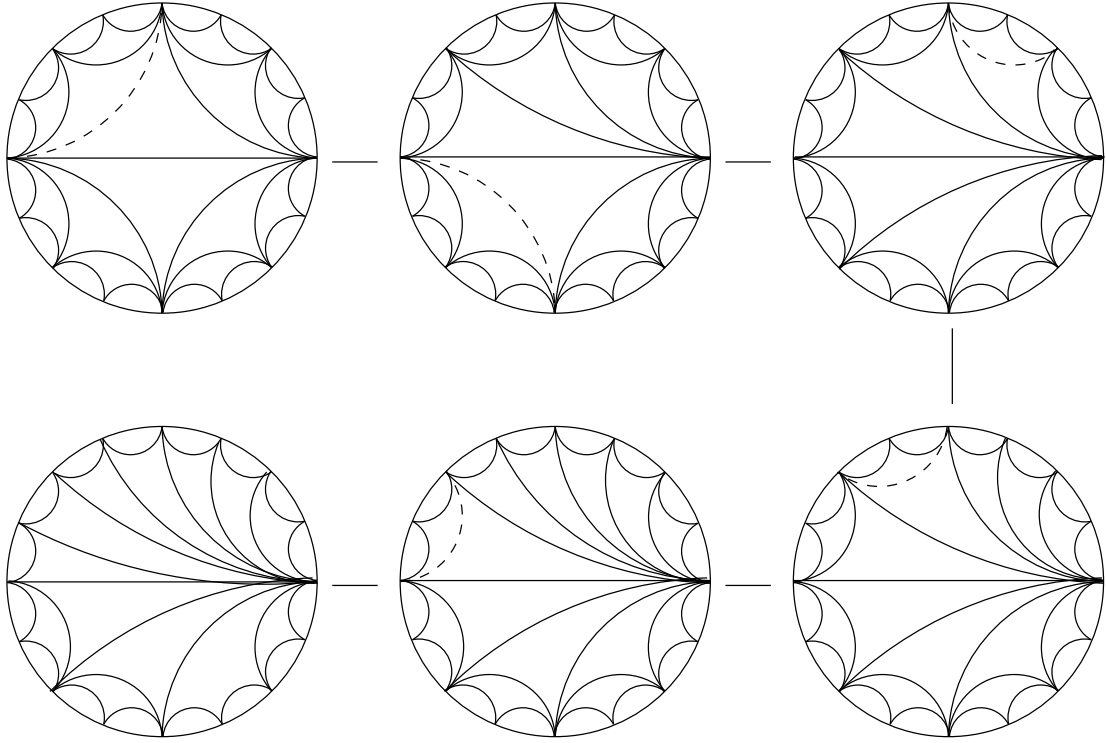
Proposition 7.3. *Let $(P_\infty^{(1)}, \Delta(P_\infty)^{(1)}), (P_\infty^{(2)}, \Delta(P_\infty)^{(2)}), \dots$ be an infinite sequence of Farey star regions with two by two empty intersections of their interiors, and such that all but finitely many are Farey-based star regions. Let Δ be a dyadic triangulation of D coinciding with $\Delta(P_\infty)^{(i)}$ on $P_\infty^{(i)}$ for every $i \in \mathbb{N}$, and of Farey type elsewhere. Then, there exist infinitely many STT-rays having Δ as stable arc set.*

Example 27. A Farey-based star region:



A completely locally infinite stable arc set

Let ρ be the infinite STT-ray obtained from the Farey's dyadic triangulation Δ_F by flipping all dyadic arcs not having $\gamma(0)$ as an extremal point in decreasing euclidean length. That is, one flips all dyadic arcs corresponding to the non-extremal vertices of the n -th level of the rooted binary tree R_2 , followed by the flips of all dyadic arcs corresponding to the non-extremal vertices of the $(n + 1)$ -th level of R_2 , etc. The initial segment of ρ is the following:



The STT-ray ρ is geodesic (see Corollary 7.1) because all pairs $m, n \in \mathbb{N}$ satisfy

$$\mathcal{D}(\rho(m) \cap \rho(n)) = |m - n|.$$

Furthermore, the flips corresponding to the arcs in bijection to the vertices of a fixed level of R_2 commute, so one can construct an infinite family of STT-rays having $\mathcal{A}_{stab}(\rho)$ as stable arc set. Thus, for $n \geq 2$ and $m \in \{1, \dots, 2^n - 2\}$, the sequence associated to

$$a = \left(\gamma \left(\frac{m}{2^n} \right), \gamma \left(\frac{m+1}{2^n} \right) \right)$$

is the 2-valued stationary sequence

$$a^n = \begin{cases} \left(\gamma \left(\frac{m}{2^n} \right), \gamma \left(\frac{m+1}{2^n} \right) \right) & \text{if } n \leq n_0 \\ \left(\gamma \left(\frac{2m+1}{2^{n+1}} \right), \gamma \left(\frac{2m+1}{2^{n+1}} \right) \right) & \text{if } n > n_0, \end{cases}$$

for some $n_0 \in \mathbb{N}$. The stable arc set of ρ is

$$\begin{aligned} \mathcal{A}_{stab}(\rho) = & \left\{ \left(\gamma(0), \gamma\left(\frac{m}{2^n}\right) \right) : 1 \leq m \leq 2^{n-1}, (m, 2^n) = 1 \right\} \\ & \cup \left\{ \left(\gamma\left(\frac{m}{2^n}\right), \gamma(1) \right) : 2^{n-1} + 1 \leq m \leq 2^n - 1, (m, 2^n) = 1 \right\}, \end{aligned}$$

which is locally infinite everywhere on the interior of D . Even though the set $\mathcal{A}_{stab}(\rho)$ is not a dyadic triangulation of D , it can be thought as an infinite analogue to the star triangulation of a convex polygon because every dyadic arc on \mathcal{A} has $\gamma(0)$ as an extremal point. The arc-connected components of $D - \mathcal{A}_{stab}(\rho)$ are all segments of the form $(\gamma(0), \gamma(z))$, for some $z \in [0, 1] \setminus \mathbb{Z}[1/2]$.

Either using similar methods either knowing that the dyadic translations are elements of Thompson's group T and considering the action of T on C , we obtain the following result:

Proposition 7.4. *Let p be a dyadic number of the unit interval and let $S(p)$ be the subset of \mathcal{A} containing all dyadic arcs of the form $(\gamma(p), \gamma(q))$ with $q \neq p$. Then, there exist an STT-ray ρ on C^1 with stable arc set equal to $S(p)$.*

An STT-ray with a non-stationary sequence

Consider the Farey's dyadic triangulation Δ_F with the already defined descending left-right order from Example 26. Let ρ be the STT-ray obtained by first flipping the dyadic arc

$$a_0 = \left(\gamma(0), \gamma\left(\frac{1}{2}\right) \right) \implies a_0^{(1)} = \left(\gamma\left(\frac{1}{4}\right), \gamma\left(\frac{3}{4}\right) \right),$$

then the dyadic arc

$$a_1 = \left(\gamma(0), \gamma\left(\frac{1}{4}\right) \right) \implies a_1^{(2)} = \left(\gamma\left(\frac{1}{8}\right), \gamma\left(\frac{3}{4}\right) \right),$$

then

$$a_2 = \left(\gamma\left(\frac{1}{2}\right), \gamma\left(\frac{3}{4}\right) \right) \implies a_2^{(3)} = \left(\gamma\left(\frac{1}{4}\right), \gamma\left(\frac{5}{8}\right) \right),$$

and going on with the following 3-step pattern:

1. One flips the dyadic arc

$$a_1^{(n)} \implies a_1^{(n+1)} \neq a_1^{(n)}.$$

Let $\Delta_{n+1} = \rho(n+1)$. The almost triangulation

$$a\Delta = \Delta_{n+1} \setminus \{a_1^{(n+1)}\}$$

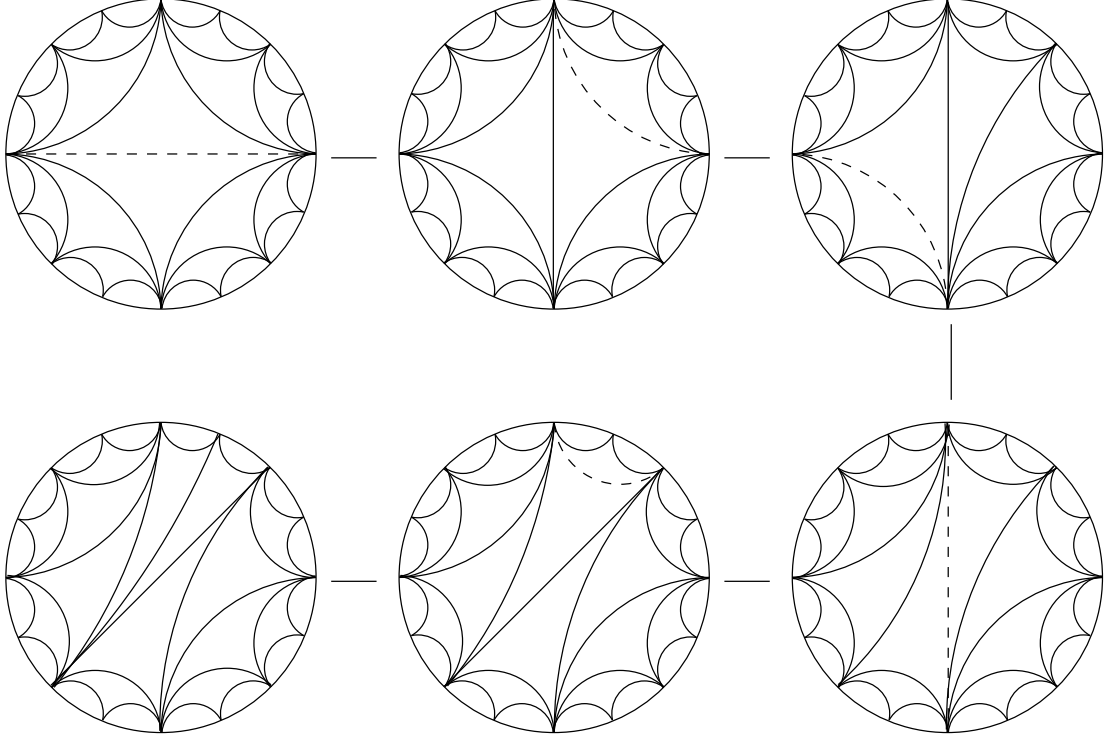
satisfies

$$\mathcal{D}(a\Delta) = 1.$$

Let $a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}$ be the dyadic arcs bounding the unique non triangular connected component of $a\Delta$. Suppose that a_1 is the first dyadic arc one reaches when travelling through ∂D counter-clockwise starting at $\gamma(0)$. Then, the dyadic arcs a_{i_1}, a_{i_3} belong to Farey's dyadic triangulation Δ_F , and a_{i_2}, a_{i_4} do not.

2. One flips a_{i_1} .
3. One flips a_{i_3} .

The initial segment of ρ is the following:



The sequence associated to a_0 (where only the non-repeated values are given) is defined by

$$a_0^{(n)} = \begin{cases} a_0 & \text{if } n = 1 \\ \left(\gamma \left(\sum_{i=2}^n \left(\frac{-1}{2} \right)^i \right), \gamma \left(\frac{1}{2} + \sum_{i=2}^n \left(\frac{-1}{2} \right)^i \right) \right) & \text{if } n \geq 2. \end{cases}$$

For the other dyadic arcs $a \in \Delta_F \setminus \{(\gamma(0), \gamma(\frac{1}{2}))\}$, their associated sequence is either constant either stationary with only two different values (as on Examples 7.2 and 7.2). The stable arc set $\mathcal{A}_{stab}(\rho)$ is not a dyadic triangulation of D because the non dyadic arc

$$(\gamma(1/6), \gamma(2/3))$$

is an arc-connected component of $D - \mathcal{A}_{stab}(\rho)$. But

$$\mathcal{A}_{stab}(\rho) \cup \left\{ \left(\gamma \left(\frac{1}{6} \right), \gamma \left(\frac{2}{3} \right) \right) \right\}$$

is a triangulation of D which is locally infinite in a neighbourhood of the non-dyadic arc $(\gamma(\frac{1}{6}), \gamma(\frac{2}{3}))$.

Conjecture: the STT-ray ρ is geodesic.

7.3 Embedding C^1 into a functional space

At the beginning of this chapter we defined a geometric boundary at infinity of the metric 1-skeleton C^1 , and we extended the action of Thompson's group T on it. Nevertheless, we do not know how to furnish the geometric boundary at infinity of C^1 with a topology so that it becomes compact. By definition, the points on the geometric boundary at infinity of C^1 have a geometric representation as stable arc sets. Furthermore, all known examples are, in some large sense, generalisations of dyadic triangulations of D (see Remark 33). It seems reasonable to try a functional compactification of C^1 inspired from Thurston's construction in his study of automorphisms of surfaces (see [76], [28] and [19]).

Consider the following functional space:

$$\mathcal{F} = \{f : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0} \cup \infty\}.$$

Note that $\mathcal{F} \simeq [0, \infty]^{\mathbb{N}}$, because \mathcal{A} is countable, and that \mathcal{F} is compact with the product topology.

We define $\mathcal{I} : C^1 \hookrightarrow \mathcal{F}$ as follows:

1. Given a dyadic triangulation $\Delta \in C^0$ of Farey type, we define $\mathcal{I}(\Delta)$ as the function *geometric intersection with Δ* . Let $a \in \mathcal{A}$ and $\Delta = \{a_i\}_{i \in \mathbb{N}} \subset \mathcal{A}$. Then,

$$\mathcal{I}(\Delta)(a) = \sum_{i=1}^{\infty} i(a_i, a),$$

where

$$i(a, a') = \begin{cases} 1 & \text{if } a \text{ and } a' \text{ intersect in one interior point} \\ 0 & \text{otherwise} \end{cases}$$

is the intersection number between a and a' .

2. Given a point p contained in the interior of an edge $a\Delta \in C^1$ joining two dyadic triangulations $\Delta_1, \Delta_2 \in C^0$, there is a unique convex linear combination of Δ_1, Δ_2

$$p = s\Delta_1 + (1 - s)\Delta_2, \quad \text{with } s \in (0, 1/2]$$

associated to p . Then, for every $a \in \mathcal{A}$ we define

$$\mathcal{I}(p)(a) = s\mathcal{I}(\Delta_1)(a) + (1 - s)\mathcal{I}(\Delta_2)(a).$$

Lemma 7.7. *Given $\Delta_1 \neq \Delta_2$ two different dyadic triangulations of D of Farey type, there exists a dyadic arc $a \in \mathcal{A}$ such that $\mathcal{I}(\Delta_1)(a) = 0$ and $\mathcal{I}(\Delta_2)(a) = 1$.*

Proof. It suffices to take $a \in \Delta_1$ and $a \notin (\Delta_1 \cap \Delta_2)$. □

Proposition 7.5. *The map $\mathcal{I} : C^1 \hookrightarrow \mathcal{F}$ is injective.*

Proof. The restriction of \mathcal{I} to the 0-skeleton C^0 is injective by Lemma 7.7. There are four different cases to consider:

1. Let $p_1 = \Delta \in C^0$ and p_2 be an interior point of an edge in C^1 :

$$p_2 = s\Delta_1 + (1-s)\Delta_2,$$

where $s \in (0, 1/2]$. Let $\Delta' \in \{\Delta_1, \Delta_2\} \setminus \Delta$, and let $a \in \mathcal{A}$ satisfying

$$\left. \begin{array}{l} \mathcal{I}(\Delta)(a) = 0, \\ \mathcal{I}(\Delta')(a) = 1. \end{array} \right\} \implies \left\{ \begin{array}{l} \mathcal{I}(p_2)(a) \geq s > 0, \\ \mathcal{I}(p_1)(a) = 0. \end{array} \right.$$

Such a exists by Lemma 7.7.

2. Let $s_1, s_2 \in (0, 1/2]$ and let p_1, p_2 be two different points of the same edge in C^1 :

$$\begin{aligned} p_1 &= s_1\Delta_1 + (1-s_1)\Delta_2, \\ p_2 &= s_2\Delta_1 + (1-s_2)\Delta_2 \quad (\text{or } p_2 = s_2\Delta_2 + (1-s_2)\Delta_1). \end{aligned}$$

Let $a \in \mathcal{A}$ be a dyadic arc satisfying:

$$\left. \begin{array}{l} \mathcal{I}(\Delta_1)(a) = 0, \\ \mathcal{I}(\Delta_2)(a) = 1. \end{array} \right\} \implies \left\{ \begin{array}{l} \mathcal{I}(p_1)(a) = s_1, \\ \mathcal{I}(p_2)(a) = s_2 \quad (\text{or } \mathcal{I}(p_2)(a) = 1-s_2). \end{array} \right.$$

Such a exists by Lemma 7.7. Then,

$$\mathcal{I}(p_1)(a) \neq \mathcal{I}(p_2)(a),$$

because $s_1 \neq s_2$ (or $s_1 \neq 1-s_2$) as a consequence of $p_1 \neq p_2$.

3. Let $s_1, s_2 \in (0, 1/2]$ and let p_1, p_2 be two points of two adjacent edges of C^1 :

$$\begin{aligned} p_1 &= s_1\Delta_1 + (1-s_1)\Delta_2 \quad (\text{or } p_1 = s_1\Delta_2 + (1-s_1)\Delta_1), \\ p_2 &= s_2\Delta_1 + (1-s_2)\Delta_3 \quad (\text{or } p_2 = s_2\Delta_3 + (1-s_2)\Delta_1). \end{aligned}$$

Then, $d_{C^1}(\Delta_2, \Delta_3) = 2$ and there exists a unique dyadic arc $a \in \mathcal{A}$ satisfying

$$\begin{aligned} \mathcal{I}(\Delta_2)(a) &= 0, \\ \mathcal{I}(\Delta_3)(a) &= 1, \text{ and} \\ \mathcal{I}(\Delta_1)(a) &= 1. \end{aligned}$$

Thus, $\mathcal{I}(p_1)(a) = s_1 < 1$ (or $\mathcal{I}(p_1)(a) = 1-s_1 < 1$) and $\mathcal{I}(p_2)(a) = 1$.

4. Let $s_1, s_2 \in (0, \frac{1}{2}]$ and p_1, p_2 be interior points of different and non-consecutive edges of C^1 :

$$\begin{aligned} p_1 &= s_1\Delta_1 + (1-s_1)\Delta_2, \\ p_2 &= s_2\Delta_3 + (1-s_2)\Delta_4, \end{aligned}$$

with $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ four different dyadic triangulations of D of Farey type. Since there are no closed paths of length 3 in C^1 , there exists $i \in \{1, 2\}$ such that

$$d_{C^1}(\Delta_i, \Delta_3) \geq 2.$$

Hence, there are at least two non common dyadic arcs between Δ_i and Δ_3 . Furthermore, there is a single non common dyadic arc between Δ_1 and Δ_2 . Thus, there exists a dyadic arc $a \in \mathcal{A}$ satisfying:

$$\begin{aligned}\mathcal{I}(\Delta_1)(a) &= 1, \\ \mathcal{I}(\Delta_2)(a) &= 1, \text{ and} \\ \mathcal{I}(\Delta_3)(a) &= 0.\end{aligned}$$

Then, $\mathcal{I}(p_1)(a) = 1$ and $\mathcal{I}(p_2)(a) \leq 1 - s_2 < 1$.

□

Definition 46. Let C_{Th}^1 be the closure of $\mathcal{I}(C^1)$ on \mathcal{F} , which is compact when considering the induced topology (it is a closed subspace of a topological compact space). The closure C_{Th}^1 will be called *Thurston's compactification of C^1* . We denote by $\partial_{\mathcal{F}}(C^1)$ the complement $C_{Th}^1 \setminus \mathcal{I}(C^1)$, and we call it *Thurston's boundary of C^1* .

The application \mathcal{I} allow us to slightly modify the definition of $s(a, \rho)$ given before: let $\rho : [0, \infty) \rightarrow C^1$ be a ray and let $a \in \mathcal{A}$ be a dyadic arc. We define

$$\tilde{s}(a, \rho) := \{s \in [0, \infty) : \mathcal{I}(\rho(s))(a) = 0\} \subset [0, \infty).$$

Then, ρ is an STT-ray on C^1 if, for every dyadic arc a , the subset $\tilde{s}(a, \rho)$ is a (possibly empty or infinite) interval. Furthermore, by Proposition 7.5 together with the condition for a ray of being STT we are tempted to say that one can embed STT-rays on Thurston's compactification in such a way that the limit of any ordered and non-bounded sequence of points exists and coincides with the function "geometric intersection with the stable arc set". However, by now we are only able to prove such a result on the subset of the geometric boundary that we have described before.

Proposition 7.6. *Let $\rho : [0, \infty) \rightarrow C^1$ be an infinite STT-ray such that every arc-connected component of the complement of the stable arc set on D*

$$D \setminus \mathcal{A}_{stab}(\rho)$$

is either a triangle, either an arc with at least one of its extremal points being non-dyadic. Then, the limit

$$\lim_{s \rightarrow \infty} \mathcal{I}(\rho(s))$$

exists and is equal to the function "geometric intersection with $\mathcal{A}_{stab}(\rho)$ ", which belongs to Thurston's boundary $\partial_{\mathcal{F}}(C^1)$ of C^1 .

Note that all the points in the geometric boundary of C^1 that we have described before satisfy Proposition 7.6 (see Remark 33).

Proof. Let $a \in \mathcal{A}$ be a dyadic arc. Without loss of generality, we suppose $\rho(0) \in C^0$.

1. Suppose that the dyadic arc a intersects a finite number k of arc-connected components of the complement of $D \setminus \mathcal{A}_{stab}(\rho)$. In this case, all k arc-connected components intersected by a are triangles. Thus, there is a finite set of dyadic arcs

$$\{a_1, \dots, a_{2k+1}\} \subset \mathcal{A}_{stab}(\rho)$$

bounding the k triangles, where $k - 1$ of them intersect a and the others bound a $k + 2$ -gon P . Furthermore,

$$\{a_1, \dots, a_{2k+1}\} \subset \rho(m)$$

for all

$$m \geq \min \left\{ s \in [0, \infty) : s \in \bigcap_{i=1}^{2k+1} \tilde{s}(a_i, \rho) \right\} = n_0.$$

Thus, for all $s \in [n_0, \infty)$ we have

$$\mathcal{I}(\rho(s))(a) = k - 1,$$

which is the geometric intersection between a and the stable arc set $\mathcal{A}_{stab}(\rho)$.

2. Otherwise, suppose that a intersects infinitely many arc-connected components of $D \setminus \mathcal{A}_{stab}(\rho)$. Let $\mathcal{A}(a) \subset \mathcal{A}_{stab}(\rho)$ be the set of dyadic arcs of the stable arc set intersecting a :

$$\mathcal{A}(a) = \{a' \in \mathcal{A}_{stab}(\rho) : i(a, a') = 1\} \subset \mathcal{A}_{stab}(\rho) \subset \mathcal{A}.$$

Since \mathcal{A} is countable, the subset $\mathcal{A}(a) \subset \mathcal{A}$ is also countable. We define an order on $\mathcal{A}(a) = \{a_1, a_2, \dots\}$ by

$$a' \prec a'' \Leftrightarrow \tilde{s}(a'', \rho) \subset \tilde{s}(a', \rho),$$

and let $m(a_i)$ denote

$$m_i = \min\{s \in [0, \infty) : s \in \tilde{s}(a_i, \rho)\} \in \mathbb{Z}_{\geq 0},$$

for every dyadic arc of $\mathcal{A}(a)$. Then, we have a strictly monotone sequence of integers:

$$\mathcal{I}(\rho(m_1))(a) < \mathcal{I}(\rho(m_2))(a) < \mathcal{I}(\rho(m_3))(a) < \dots$$

from where we obtain

$$\lim_{s \rightarrow \infty} \mathcal{I}(\rho(s))(a) = \infty.$$

□

To end this chapter, we analyse the behaviour of the different explicit examples of boundary points obtained from Proposition 7.6 as functions on \mathcal{A} . The following results are direct consequences of three facts. Namely:

1. Every dyadic arc contained in a locally finite region of a triangulation (in a large sense) of D intersects finitely many dyadic arcs of the triangulation.

2. By choosing a point $\gamma(p)$ on the boundary of D adjacent to an infinite, locally finite region of the triangulation, and given a positive integer z , one can find an arc which intersects exactly z dyadic arcs of the triangulation.
3. Given a (not necessarily dyadic) arc such that a triangulation (in a large sense) of D is locally infinite in each of their interior points, one can find infinitely many dyadic arcs crossing the given arc.

Proposition 7.7. *Let $\Delta \subset \mathcal{A}$ be a dyadic triangulation containing a Farey tessellation of D , and let ρ be an STT-path with stable arc set equal to Δ (it exists by Proposition 7.1). Then, for every dyadic arc $a \in \mathcal{A}$ we have*

$$\lim_{s \rightarrow \infty} \mathcal{I}(\rho(s))(a) < \infty.$$

Furthermore, for every positive integer $z \in \mathbb{Z}_{\geq 0}$ there are infinitely many dyadic arcs satisfying

$$\lim_{s \rightarrow \infty} \mathcal{I}(\rho(s))(a) = z.$$

Proposition 7.8. *Let $\Delta \subset \mathcal{A}$ be a dyadic triangulation of D , and let ρ be an STT-path with stable arc set equal to Δ , both satisfying Proposition 7.2 or Proposition 7.3. Then, for every generalized positive integer $z \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, there are infinitely many dyadic arcs satisfying*

$$\lim_{s \rightarrow \infty} \mathcal{I}(\rho(s))(a) = z.$$

Proposition 7.9. *Let $\Delta \subset \mathcal{A}$ be a dyadic star of D , and let ρ be an STT-path with stable arc set equal to Δ satisfying Proposition 7.4. Then, given a dyadic arc $a \in \mathcal{A}$*

- either $a \in \Delta$ and $\lim_{s \rightarrow \infty} \mathcal{I}(\rho(s))(a) = 0$,
- or $a \notin \Delta$ and $\lim_{s \rightarrow \infty} \mathcal{I}(\rho(s))(a) = \infty$.

Chapter 8

Future work

Future research prospects follow three main axes. The first one is concerned with the geometric compactification of C^1 and the possible applications to the study of analytic properties of T . Let $\partial_0 C$ denote the subset of the geometric boundary corresponding to equivalence classes of STT-rays having as stable arc set \mathcal{A}_{stab} a subset of \mathcal{A} such that each connected component of $D \setminus \mathcal{A}$ is either a triangle or an arc with at least one of the extremal points being non-dyadic. Recall that $\partial_0 C$ is T -invariant. The first step would be to determine all types of possible stable arc sets. Once the geometric boundary being completely determined, my goal would be to give a classification of the elements of T inspired by Thurston's classification of surface diffeomorphisms [76, 28], using the action of T on that geometric boundary.

The embedding of the 1-skeleton C^1 into the functional space $\mathcal{F} = \{f : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}\}$, which is compact for the product topology, permits the definition of the Thurston's compactification C^{Th} . Note that the action of T can be extended to C^{Th} by using the action on the dyadic rational numbers. Moreover, $\partial_0 C$ can be embedded in an equivariant manner into C^{Th} . This leads to the following questions: does the embedding extend to the whole geometric boundary? If yes, is the image dense? It would be also worth to examine the dimension of C^{Th} .

A good understanding of the action of T on the Thurston's compactification could be applied to the study the action of F , which is smaller. This latter group is probably the most known of the three classical Thompson groups, due to the still open question of its amenability. Thus, the study of fixed points under the action of F on this compact space could be an approach to that famous problem. Also, this boundary construction could enable an adaptation of the proof of the exactitude of hyperbolic groups to the groups T and F , which are not hyperbolic. A last idea concerning the analytic aspect is to investigate other types of compactifications, and to compare them to the preceding ones.

The second axis concerns the generalized Thompson groups. As we have already mentioned, the three classical groups, F , T et V have been generalized by Higman, Brown, Stein, Röver and others. Moreover, Greenberg and Sergiescu, Brin, and Dehornoy defined braided versions of those generalizations. It is thus natural to ask which of the definitions and properties introduced in this thesis have good analogues in generalized Thompson

groups and their braided versions. For example, does a notion analogous to “piecewise $PSL_2(\mathbb{Z})$ ” exist for Brown-Stein groups of type T and their generalisations? In the affirmative case, which groups can be quasi-isometrically embedded in those groups?

Ghys and Sergiescu showed that T can be embedded in the group of circle diffeomorphisms of class \mathcal{C}^∞ . Such an analogue of “piecewise $PSL_2(\mathbb{Z})$ ” could help to embed generalized Thompson groups of type T in the group of circle diffeomorphisms. Results of this kind together with Navas theorems about lattice embeddings on the group of diffeomorphisms of the circle [65] could be helpful to study the embeddings of lattices on groups of type T . Furthermore, the link between the “piecewise $PSL_2(\mathbb{Z})$ ” and the combinatorial version has been proved useful in the study of subgroups from a metric point of view. We wonder if it is possible to use this link for other problems on subgroup embeddings. It is also natural to ask whether the result on conjugacy classes extend to those cases.

We could also investigate different complexes that can be obtained from C by relaxing certain regularity properties. For instance, what happens if we replace triangulations by square tessellations? This could lead to generalizations of the infinite associahedron C , whose automorphisms group are finite extensions of groups of type T . The automorphisms groups of these generalized complexes could furnish interesting generalizations of Thompson groups, which could be compared to the known ones. Increase the dimension could also give interesting objects and results: for example, let X be a simplicial complex of dimension 3 constructed by induction from a tetrahedron X_0 by gluing, at each step n , a tetrahedron on each face of the boundary of X_n . Let us consider the group of homeomorphisms of X which preserve this tetrahedral tessellation outside a compact set. Is this group related to generalized Thompson groups of type T ? Is there a link with hyperbolic spaces of greater dimension and their isometry groups?

The third and last axis concerns the “modular” point of view on T , due to Funar-Kapoudjian-Sergiescu, which relates T to modular groups. The result on the automorphisms group of C can thus be interpreted as an analogue of Ivanov’s result describing modular groups as automorphisms groups of curve complexes. It could be interesting to study the resemblances and differences between T and the classical modular groups. For example, in the geometric setting, we could ask if each embedding of a lattice in T has a finite image or if T is relatively hyperbolic.

We could also use the modular approach for combinatorial or algebraical problems. For instance, is T asymptotically combable? Is T automatic? Does there exist a natural N such that every element of T can be written as a product of at most N torsion elements? This is not the case for modular groups in genus more than 2. Do there exist k non-torsion elements of T , g_1, g_2, \dots, g_k , such that for all $g \in T$, there exist n_1, n_2, \dots, n_k such that $g = g_1^{n_1} g_2^{n_2} \cdots g_k^{n_k}$? This is not the case for modular groups in genus at least 2, due to the presence of many non-trivial quasi-morphisms.

At last, it is possible that the modular version enables us to define a sort of Teichmüller theory for T . First steps in this direction were done by Greenberg, who exhibited a link between Thompson groups of type F and spaces of horocircles of the Poincaré disk, and made the first connection with associahedra. Moreover, we must mention that this action and the different spaces that have been introduced are somehow related to the universal Teichmüller space of Penner. It seems interesting to try to adapt these concepts to type

T Thompson groups and to see which relations could be established between Greenberg's point of view and the triangulations of the disk.

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Sur la géométrie et la combinatoire du groupe T de R.J. Thompson

Le groupe simple infini T de Thompson est généralement vu comme un sous-groupe du groupe des homéomorphismes dyadiques du cercle unité qui sont linéaires par morceaux et préservent l'orientation (T linéaire). Cependant, T peut aussi être vu comme le groupe des classes d'équivalence des paires équilibrées d'arbres binaires (T combinatoire), comme sous-groupe du groupe des homéomorphismes de la droite projective réelle qui préservent l'orientation et sont " $PSL(2, \mathbb{Z})$ par morceaux" (T projectif), et comme le groupe modulaire asymptotique de l'épaissi de l'arbre 3-régulier (T modulaire). On montre d'abord que la copie canonique de $PSL(2, \mathbb{Z})$ obtenue à partir de T projectif est un sous-groupe non distordu de T , en transportant ce sous-groupe pour en obtenir une caractérisation dans T combinatoire, ce qui permet d'estimer la longueur des mots de ses éléments. Comme corollaire, T a des sous-groupes non distordus isomorphes à F_2 . Un deuxième résultat utilise T modulaire pour prouver qu'il y a exactement $\varphi(n)$ classes de conjugaison d'éléments d'ordre n dans T , où φ est l'indicatrice d'Euler. On construit ensuite un complexe cellulaire C contractile et simplement connexe sur lequel le groupe T agit par automorphismes, et qui est minimal pour ces propriétés. Le groupe d'automorphismes de C est "essentiellement" T lui-même. Ce complexe cellulaire peut être vu comme une généralisation des associaèdres de Stasheff dans le cas d'un polygone convexe à une infinité de côtés. Dans la partie finale, on utilise l'interprétation géométrique du 1-squelette de C en termes de triangulations dyadiques du disque pour définir un bord géométrique à l'infini et une compactification dans un espace fonctionnel.

Geometric and combinatorial aspects of R.J. Thompson's group T

Thompson's infinite simple group T is usually seen as a subgroup of the group of dyadic, piecewise linear, orientation-preserving homeomorphisms of the unit circle (*linear T*). However, T can also be identified to a group of equivalence classes of balanced pairs of binary trees (*combinatorial T*), to a subgroup of piecewise $PSL(2, \mathbb{Z})$, orientation-preserving homeomorphisms of the projective real line (*projective T*), and to the asymptotic mapping class group of a fattened complete trivalent tree (*modular T*). First we prove that the canonical copy of $PSL(2, \mathbb{Z})$ obtained from *projective T* is a non-distorted subgroup of T . For this, one carries over this subgroup to obtain a characterization into *combinatorial T* , from which the word length of its elements can be estimated. As a corollary, T has non-distorted subgroups isomorphic to F_2 . The second result uses *modular T* to state that there are exactly $\varphi(n)$ conjugacy classes of elements of order n , where φ is the Euler function. The third result constructs a minimal simply-connected contractible cellular complex C on which the group T acts by automorphisms. The automorphism group of C is "essentially" T itself. The complex C can be seen as a generalization of Stasheff's associahedra for an infinitely sided convex polygon. On the final part, the geometric interpretation of the 1-skeleton of C in terms of dyadic triangulations of the disc is used to define a geometric boundary at infinity.