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Laboratoire d'Analyse et de Mathématiques Appliquées

Thèse

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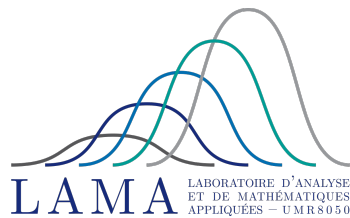
Xiaochuan Yang

Étude dimensionnelle de la régularité de processus de diffusion à sauts

Spécialité : Mathématiques

Soutenue le 1er juillet 2016 devant un jury composé de :

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Thèse effectuée au sein du **Laboratoire d'Analyse et de Mathématiques Appliquées**
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Résumé

Dans cette thèse, on étudie diverses propriétés dimensionnelles de la régularité de processus de diffusions à sauts, solutions d'une classe d'équations différentielles stochastiques à sauts. En particulier, on décrit la fluctuation de la régularité höldérienne de ces processus et celle de la dimension locale de la mesure d'occupation qui leur est associée en calculant leur spectre multifractal. La dimension de Hausdorff de l'image et du graphe de ces processus sont aussi calculées.

Dans le dernier chapitre, on utilise une nouvelle notion de dimension dite "de grande échelle" pour décrire l'asymptote à l'infini du temps de séjour d'un mouvement brownien en dimension 1 sous une frontière glissante.

Mots-clé:

Equations différentielles stochastiques à sauts, Processus de Markov, Régularité locale, Multifractals, Image et graphe, Dimension de Hausdorff, Dimension de Hausdorff macroscopique, Processus ponctuel de Poisson.

Dimension properties of the regularity of jump diffusion processes

Abstract

In this dissertation, we study various dimension properties of the regularity of jump diffusion processes, solution of a class of stochastic differential equations with jumps. In particular, we describe the fluctuation of the Hölder regularity of these processes and that of the local dimensions of the associated occupation measure by computing their multifractal spectra. The Hausdorff dimension of the range and the graph of these processes are also calculated.

In the last chapter, we use a new notion of “large scale” dimension in order to describe the asymptotics of the sojourn set of a Brownian motion under moving boundaries.

Keywords:

Stochastic differential equations with jumps, Markov processes, Local regularity, Multifractal analysis, Range and graph, Hausdorff dimension, Macroscopic Hausdorff dimension, Poisson point process.

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Introduction

Dans la théorie des probabilités, les processus aléatoires les plus naturels sont partout irréguliers. Par exemple, le mouvement brownien est continu mais nulle part dérivable. De même, les processus de Lévy sautent en général sur un ensemble dense de points; nous verrons d'autres exemples de processus irréguliers par la suite. Il est donc naturel d'étudier la régularité locale des processus stochastiques. Un outil pertinent pour décrire la fluctuation de régularité des trajectoires est l'analyse multifractale. C'est l'approche que nous allons développer dans la majeure partie de cette thèse.

Beaucoup d'auteurs se sont intéressés à la régularité fine du mouvement brownien. Par autosimilarité, la croissance locale du mouvement brownien se comporte en moyenne comme une fonction racine carrée. Lévy [82] a établi le module de continuité global exact du brownien qui est effectivement la fonction racine carrée avec une correction logarithmique. Cela établit une borne supérieure pour la régularité locale. En contrepartie, Paley, Wiener et Zygmund [96] ont montré que la régularité höldérienne locale ne peut être supérieure à $1/2$. La situation change totalement pour les processus de Lévy car la régularité höldérienne dépend du point considéré. En effet, un subordinateur est différentiable en Lebesgue presque tout point grâce à la monotonie des trajectoires, pourtant il saute sur un ensemble dense de points. En réalité, il y a un continuum de valeurs possibles pour l'exposant de Hölder des trajectoires d'un processus de Lévy. L'analyse multifractale permet de décrire l'ensemble des comportements locaux possibles d'une fonction (en particulier un processus de Lévy), et de quantifier la taille (en terme de dimension de Hausdorff) des ensembles de points ayant un comportement local donné. Mais paradoxalement, ce type d'étude n'a été mené que pour des classes très restreintes de processus, desquelles font partie le mouvement brownien [95, 98], les processus de Lévy [61] et les temps locaux [87].

Il est naturel de s'intéresser à ces questions d'analyse multifractale pour des classes plus générales de processus, notamment les processus de Markov à sauts. Ces derniers sont l'objet central de cette thèse. Nous allons étudier une classe importante de processus de Markov à sauts, appelée diffusion à sauts, qui est définie trajectoriellement par une équation différentielle stochastique guidée par un mouvement brownien et une mesure aléatoire de Poisson. Tout comme le processus de diffusion continue (EDS guidé par un brownien seul) généralise le mouvement brownien, le processus de diffusion à sauts généralise les processus de Lévy. Ces processus ont eu de nombreuses applications en finance (évaluation d'option, modèle de Merton [2, 35]), en physique (dynamique de Langevin, diffusion des neutrons dans une liquide [33, 79]), en biologie (coalescence, évolution d'une population [26, 77]). *Les deux premiers chapitres de cette thèse sont consacrés à l'analyse multifractale de diffusion à sauts.*

Il y a bien d'autres caractéristiques qui décrivent la rugosité d'un processus.

Une mesure pertinente de la rugosité d'un processus est fournie par la dimension de Hausdorff de son image et son graphe. Ce type d'étude a été initié par Taylor [111], qui a prouvé que l'image du mouvement brownien d -dimensionnel admet $\min\{d, 2\}$ pour dimension de Hausdorff, presque

surement. Depuis, beaucoup de travaux ont été menés pour déterminer la dimension de l'image d'un ensemble quelconque par différents types de processus stochastiques : les processus de Lévy [54, 102], le mouvement brownien fractionnaire [110, 116], ainsi que leurs extensions multi-paramètres [4, 115]. En pratique, ces quantités ont l'intérêt de pouvoir être estimées rapidement par ordinateur, donc elles font partie des estimateurs importants pour la statistique des données financières [18, 86, 94]. *Dans le Chapitre 3, nous allons étudier la dimension de l'image et du graphe de la diffusion à sauts de type stable ("stable-like"), introduite par Bass [15]. La dimension de l'image d'un ensemble quelconque par ces processus est aussi calculée, voir la fin du Chapitre 4.*

Une autre façon de décrire la régularité d'un processus consiste à étudier la mesure d'occupation qui lui est naturellement associée.

La mesure d'occupation d'un processus X à valeurs dans \mathbb{R}^d , définie pour tout borélien $A \subset \mathbb{R}^d$ par

$$\mu_t(A) = \int_0^t \mathbf{1}_A(X_s) ds \quad (1)$$

décrit le temps passé par X dans l'ensemble A avant l'instant t . C'est l'objet "dual" du processus X dans le sens suivant : plus la mesure est régulière, plus le processus est irrégulier. L'existence de temps locaux (densité d'occupation par rapport à la mesure de Lebesgue) pour le mouvement brownien uni-dimensionnel a été établie par Lévy. En dimension supérieure, la densité d'occupation brownienne n'existe pas. Dans ce cas, l'outil approprié pour décrire la régularité locale est la dimension locale de μ en tout point x de son support, définie par

$$\lim_{r \rightarrow 0} \frac{\ln(\mu(B(x, r)))}{\ln r}.$$

Perkins et Taylor [99] ont prouvé que le mouvement brownien d -dimensionnel ($d \geq 2$) admet 2 pour dimension locale en tout point de son support. La situation est à nouveau complètement différente pour un processus de Lévy. En effet, Hu et Taylor [55] ont découvert que les points typiques de la mesure d'occupation d'un subordonneur α -stable admettent α pour dimension locale, mais l'ensemble des points dont la dimension locale supérieure (remplacer \lim par \limsup dans la définition) vaut $h \in [\alpha, 2\alpha]$ a une dimension de Hausdorff non triviale. Signalons également que la mesure d'occupation admet l'image de X pour support. Ainsi, l'étude de la dimension locale de μ est utile pour calculer la dimension de l'image de X . *Dans le Chapitre 4, nous allons effectuer l'analyse multifractale de la mesure d'occupation de diffusions à sauts de type stable.*

Pour décrire des phénomènes d'échelle des sous-ensembles de l'espace discret (comme \mathbb{Z}^d), Barlow et Taylor [8, 9] ont introduit dans la fin des années 80 une nouvelle dimension fractale qu'ils ont appelé "dimension de Hausdorff macroscopique". L'idée est que cette dimension ne dépend que du comportement asymptotique à l'infini des ensembles.

Il existe également une "dimension de masse" qui est utilisée par exemple en théorie des nombres et en physique statistique. Des exemples classiques d'ensembles étudiés comprennent la com-

posante connexe infinie d'un processus de percolation, l'image d'une marche aléatoire. C'est l'analogie de la dimension de Minkowski à grande échelle. Intuitivement, si $E \subset \mathbb{N}$, la valeur de cette dimension de masse est le réel positif α tel que

$$\#\{k \in E : k \leq n\} \sim n^\alpha$$

pour tout n suffisamment grand.

Signalons que Barlow et Taylor [8] ont construit des exemples pour lesquels la dimension de Hausdorff macroscopique et la dimension de masse sont différentes.

Très récemment, Xiao et Zheng [118] ont étudié l'image d'une marche aléatoire dans l'environnement aléatoire via cette dimension. Georgiou *et al.* [53] ont établi des résultats sur l'image d'une marche aléatoire transiente quelconque, résolvant une question de Barlow et Taylor [9]. La dimension de Hausdorff macroscopique est utile également dans l'étude des sous-ensembles de \mathbb{R}^d : par exemple, Khoshnevisan, Kim et Xiao [71] ont quantifié via cette dimension les hauts pics spatiaux dans un voisinage de l'infini de la solution d'une équation de la chaleur stochastique avec un bruit additif (ou multiplicatif).

Le dernier chapitre est consacré à l'étude du temps de séjour d'un mouvement brownien en dimension 1 sous une frontière glissante. Soit $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ et

$$E(\varphi) = \{t \geq 0 : |B_t| \leq \varphi(t)\}.$$

Uchiyama [113] a considéré l'existence de densité supérieure (à l'infini) par rapport à la mesure de Lebesgue de ces ensembles $E(\varphi)$. Il a démontré des identités remarquables sous l'hypothèse que la frontière glissante φ croît vers l'infini comme une fonction racine carrée avec une correction logarithmique. On va s'intéresser au temps de séjour du brownien sous des frontières qui croissent beaucoup plus lentement que la fonction racine carrée et on calcule la dimension de Hausdorff macroscopique et la dimension de masse de ces ensembles de séjour. Cela permet de quantifier la fluctuation et la récurrence des trajectoires browniennes.

0.1 Notations, définitions et historique

Soit $(X_t)_{t \geq 0}$ un processus stochastique à valeurs dans \mathbb{R}^d défini sur un espace probabilisé $(\Omega, \mathcal{F}, \mathbb{P})$. L'étude systématique de la continuité des trajectoires a été initiée par Doob [37, 38]. Il a été compris ensuite, à travers de travaux de Lévy [81, 82], Doob [39, 40], Kinney [72], Kolmogorov-Chentsov [32], que la plupart des processus (par exemple, martingales, processus de Feller et beaucoup de processus gaussiens) ont des trajectoires continues à droite avec limites à gauche (càdlàg). Notons cependant l'exemple d'un processus dont les trajectoires sont presque sûrement nulle part bornées introduit par Maejima [84] et récemment considéré par Balança [6].

En 1828, le botaniste écossais Brown a observé le mouvement erratique d'un pollen en suspension dans un fluide. C'est l'origine du premier processus stochastique en temps continu : le mouvement brownien. En 1900, Bachelier se sert du mouvement brownien dans la théorie de la spéculation [5]. Plus tard en 1905, Einstein [47] l'a utilisé pour la théorie cinétique des molécules. Cependant, c'est Wiener [114] en 1923 qui a fondé l'étude mathématique du mouvement brownien. Son approche est de construire une mesure de probabilité qui porte maintenant son nom sur l'espace des fonctions continues.

La nulle part dérivabilité du brownien peut être précisée par une notion de régularité locale entre la continuité et la dérivabilité. La notion naturelle est l'exposant de Hölder ponctuel que l'on rappelle maintenant.

Définition 1. Soit $t_0 \in \mathbb{R}^+$ et $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ localement bornée. On dit que f appartient à $C^h(t_0)$ s'il existe des constantes $C, \delta > 0$, un polynôme P de degré au plus $\lfloor h \rfloor$, tels que pour tout $t \in B(t_0, \delta)$,

$$|f(t) - P(t - t_0)| \leq |t - t_0|^h.$$

On définit l'exposant de Hölder de f en t_0 par

$$H_f(t_0) = \sup\{h \geq 0 : f \in C^h(t_0)\}.$$

Remarque 1. Le polynôme P joue le rôle du développement de Taylor pour les fonctions lisses. Lorsque $H_f(t_0) < 1$, ce polynôme est nécessairement la constante $f(t_0)$.

Il s'avère que le mouvement brownien n'est pas sensible à l'exposant de Hölder ponctuel. En effet, [82, 96], avec probabilité 1, pour tout $t \in [0, 1]$, $H_B(t) = 1/2$, voir cependant le travail de Orey et Taylor [95] et celui de Perkins [98] qui ont cherché la fluctuation logarithmique de la régularité locale du brownien. Mentionnons également la méthode d'analyse harmonique qui consiste à décomposer X sur une base de Schauder ou sur une base d'ondelettes plus générales pour établir des résultats fins sur la régularité locale [4, 19, 30]. On peut aussi regarder un interview de Kahane [64] pour plus d'historique sur le brownien.

Les processus à sauts occupent une place incontournable dans l'ensemble des processus stochastiques. Le premier exemple est le processus de Poisson simple qui est utilisé par Lundberg en 1903 pour modéliser l'arrivée successive des accidents. On rencontre tout de suite des processus à sauts plus compliqués si on étudie des fonctionnelles très simples de processus continus. Par

exemple, le processus de temps de passage d'un brownien

$$\forall a \geq 0, \quad T_a = \inf\{t \geq 0 : B_t > a\}$$

est un processus à sauts. De plus, T fait partie d'une classe très importante de processus à sauts : les processus à accroissements indépendants et stationnaires (PAIS), appelés les processus de Lévy en l'honneur du probabiliste français Paul Lévy qui a eu une contribution fondatrice dans ce domaine. En 1934, le travail révolutionnaire de Lévy [81] a caractérisé les processus à accroissements indépendants en terme d'une diffusion et une composante de saut. Itô [57] a montré en 1942 que la composante de saut peut s'écrire comme une intégrale de Poisson, qui donne la formulation finale de la décomposition de Lévy-Itô énoncée ci-dessous. La représentation de la fonction caractéristique des PAIS est donnée par Lévy lui-même [82], puis Khintchine [66]. On pourra consulter [24, 25, 104] pour des références sur le sujet.

Soit π une mesure sur \mathbb{R}^d . Le processus ponctuel de Poisson d'intensité π est un processus ponctuel $\mathcal{P} = (\Delta_t, t \geq 0)$ à valeurs dans \mathbb{R}^d tel que pour tout $E \subset \mathbb{R}^d$ avec $\pi(E) < +\infty$, le processus de comptage $t \mapsto \#\{s \leq [0, t] : \Delta_s \in E\}$ est un processus de Poisson simple d'intensité $\pi(E)$. La mesure aléatoire de Poisson N engendrée par \mathcal{P} est la mesure discrète

$$N(dt, dx) = \sum_{t: \Delta_t \in \mathbb{R}^d \setminus \{0\}} \delta_{(t, \Delta_t)}.$$

La mesure de Poisson compensée est la mesure signée qui s'écrit

$$\tilde{N}(ds, dx) = N(ds, dx) - ds \otimes \pi(dx).$$

Théorème 1. (Lévy [81], Itô [57]) Soit X un processus de Lévy à valeurs dans \mathbb{R}^d . Alors il existe une matrice $d \times d$ définie positive A , un vecteur $b \in \mathbb{R}^d$, et une mesure π sur \mathbb{R}^d vérifiant

$$\int 1 \wedge |x|^2 \pi(dx) < +\infty$$

tels que

$$X_t = AB_t + bt + \int_0^t \int_{|x| \leq 1} x \tilde{N}(ds, dx) + \int_0^t \int_{|x| > 1} x N(ds, dx).$$

où B est un mouvement brownien d -dimensionnel, N est une mesure aléatoire de Poisson compensée d'intensité $\pi(dx)$ indépendante de B .

Cette décomposition est extrêmement utile pour l'étude de la régularité des processus de Lévy. En effet, Jaffard [61] a fait une analyse fine de la configuration du temps et de la taille des sauts, qui a conduit à la détermination de l'exposant de Hölder $H_X(t)$ pour tout t . En particulier, il a montré que l'exposant de Hölder ponctuel d'un processus de Lévy général dépend du point con-

sidéré et que la fonction $t \mapsto H_X(t)$ est partout discontinue. On voit tout de suite que l'analyse multifractale est le cadre adéquat pour donner des informations organisées de la régularité locale.

Nous allons faire l'analyse multifractale d'une classe de processus plus générale, englobant les processus de Lévy, qu'on appelle les diffusions à sauts.

Bien que l'idée d'une équation différentielle stochastique (EDS) apparaisse déjà dans les travaux de Langevin [79] et Bernstein [21], la théorie rigoureuse et systématique d'EDS est fondée grâce à des travaux du probabiliste japonais Kiyoshi Itô [58, 59]. L'EDS guidée par un processus de Lévy apparaît déjà dans [59].

On appelle diffusion à sauts la solution d'une EDS à sauts définie comme suit. Soit $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ un espace probabilisé filtré qui vérifie les conditions usuelles. Soient B un mouvement brownien standard d -dimensionnel et \mathcal{P} un processus ponctuel de Poisson sur \mathbb{R}^d définis tous les deux sur cet espace probabilisé. Notons N la mesure aléatoire de Poisson engendrée par \mathcal{P} et \tilde{N} la mesure compensée. Considérons l'équation suivante (dans \mathbb{R}^d)

$$X_t = X_0 + \int_0^t \sigma(X_{s-}) dB_s + \int_0^t b(X_s) ds + \int_0^t \int_{|z| \leq 1} G(X_{s-}, z) \tilde{N}(ds, dz). \quad (2)$$

La solution existe s'il y a un processus X adapté à (\mathcal{F}_t) qui vérifie (2). Il y a unicité trajectorielle si deux solutions quelconques X et X' sont indistinguables, c'est-à-dire que $\mathbb{P}(\forall t \geq 0, X_t = X'_t) = 1$. Les conditions classiques sur les coefficients σ, b, G pour que l'unique solution trajectorielle existe se trouvent dans [59]. La recherche de conditions optimales pour avoir unicité trajectorielle reste un domaine très actif dans l'étude des EDS, voir [16, 50, 51, 83]. *L'équation (2) sera l'objet d'étude central dans cette thèse. On s'attend à obtenir l'exposant de Hölder encore plus chaotique que celui des processus de Lévy, vu que la dynamique de diffusion à sauts est plus compliquée, moins explicite.*

Pour décrire la taille des ensembles de singularités d'un processus X , c'est-à-dire les ensembles de niveau de l'exposant de Hölder

$$E_X(h) = \{t \geq 0 : H_X(t) = h\},$$

on a besoin de la notion de dimension de Hausdorff. En effet, comme l'a montré Jaffard, les ensembles de singularités pour un processus de Lévy sont en général denses dans n'importe quel intervalle. La dimension de Minkowski vaut toujours la même valeur pour tous les ensembles de

singularité, donc ne permet pas de distinguer les différentes régularités. On pourrait également considérer la dimension de packing, mais on va travailler avec la dimension de Hausdorff.

Définition 2. Soient $E \subset \mathbb{R}^d$ et $0 \leq s \leq d$. La mesure de Hausdorff s -dimensionnelle de E est définie par

$$\mathcal{H}^s(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(E) = \lim_{\delta \rightarrow 0} \inf \left\{ \sum_{i=1}^{+\infty} |Q_i|^s : E \subset \bigcup_{i=1}^{+\infty} Q_i \text{ et } |Q_i| \leq \delta \right\}.$$

L'application $s \mapsto \mathcal{H}^s(E)$ étant décroissante, on définit la dimension de Hausdorff de E par

$$\dim_{\mathcal{H}} E = \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = +\infty\}.$$

La dimension de Hausdorff permet d'analyser finement des ensembles "compliqués" (comme par exemple un ensemble de Cantor) dans l'espace euclidien ou un espace métrique plus général. Elle est la notion centrale dans une branche récente de mathématiques : la géométrie fractale. C'est le mathématicien franco-américain Benoit Mandelbrot qui a inventé la terminologie "fractal" pour décrire mathématiquement beaucoup de phénomènes qui exhibent d'une façon ou d'une autre des propriétés d'auto-similarité ou de rugosité, voir [48, 85].

L'image, le graphe et la ligne de niveau d'un processus stochastique donnent des classes très intéressantes de fractals aléatoires. Prenons l'image comme un exemple. Le tout premier travail sur ce sujet est celui de Taylor [111], qui a utilisé la méthode de la théorie potentielle pour calculer la dimension de l'image du brownien. On a attendu ensuite des années [27, 54, 63, 88, 101] avant que Pruitt [102] établisse la formule de dimension de l'image d'un processus de Lévy général en terme de sa mesure potentielle. Récemment, Khoshnevisan, Xiao et Zhong [69, 70] ont développé la théorie potentielle des processus multiparamétrés (les processus de Lévy additifs) et l'argument de co-dimension pour écrire la dimension de l'image d'un processus de Lévy général en terme de son exposant caractéristique. Comme déjà dit, pour étudier l'image de X , sa mesure d'occupation μ joue un rôle primordial car cette dernière est la mesure naturelle portée par l'image. Le lien entre la dimension de l'image et la mesure d'occupation est résumé dans le lemme suivant, appelé le principe de distribution de masse.

Lemme 1. Soit μ une mesure finie positive portée par un ensemble $E \subset \mathbb{R}^d$. Supposons que pour μ -presque tout x ,

$$\liminf_{r \rightarrow 0} \frac{\ln(\mu(B(x, r)))}{\ln r} \geq s.$$

Alors on a $\dim_{\mathcal{H}} E \geq s$.

On introduit maintenant le spectre multifractal, connu aussi sous le nom du spectre de singularités.

Définition 3. *Le spectre multifractal d'une fonction X est l'application*

$$D_X : h \mapsto \dim_{\mathcal{H}} E_X(h)$$

pour tout $h \geq 0$ où $E_X(h)$ est l'ensemble iso-höldérien $E_X(h) = \{t : H_X(t) = h\}$. Par convention, $\dim_{\mathcal{H}} \emptyset = -\infty$.

En pratique, l'exposant de Hölder $H_X(\cdot)$ peut être une fonction très erratique (c'est le cas des processus de Lévy), le calcul de $H_X(\cdot)$ est donc totalement instable numériquement. On désire plutôt dans ce cas obtenir des informations moins précises, plus qualitatives : quelles sont les valeurs de h l'exposant $H_X(\cdot)$ peut prendre, et quelle est la dimension de Hausdorff de $E_X(h)$? L'analyse multifractale cherche à répondre ces questions. La régularité locale de diffusion à sauts est encore plus erratique que celle des processus de Lévy, donc l'analyse multifractale reste une machinerie satisfaisante pour décrire sa régularité locale.

Avec le même esprit que pour des processus stochastiques, l'analyse multifractale permet de décrire la régularité locale de mesures aléatoires. La notion de régularité pour des mesures que nous allons utiliser est la suivante.

Définition 4. *Soit μ une mesure positive sur \mathbb{R}^d et $x \in \mathbb{R}^d$. La dimension locale supérieure de μ en x est*

$$\overline{\dim}(\mu, x) = \limsup_{r \rightarrow 0} \frac{\ln(\mu(B(x, r)))}{\ln r}$$

On définit de façon similaire la dimension locale inférieure de μ en x

$$\underline{\dim}(\mu, x) = \liminf_{r \rightarrow 0} \frac{\ln(\mu(B(x, r)))}{\ln r}$$

Lorsque la dimension locale supérieure et inférieure de μ en x coïncident et sont égales à h , on dit que la dimension locale de μ en x vaut h , noté par $\dim(\mu, x) = h$.

L'accent est mis sur la mesure d'occupation de diffusion à sauts dans cette thèse.

Définition 5. *Le spectre multifractal supérieur de la mesure d'occupation μ d'un processus X est l'application*

$$\overline{d}_\mu(\cdot) : h \mapsto \dim_{\mathcal{H}} \{x \in \text{Supp}(\mu) : \overline{\dim}(\mu, x) = h\}.$$

On définit de façon similaire le spectre multifractal inférieur $\underline{d}_\mu(\cdot)$.

On termine cette section introductive avec un résumé de la théorie de dimensions à grande échelle récemment développée par Barlow et Taylor [8, 9]. Des modèles discrets en physique statistique

sont couramment gouvernés par une loi de puissance. L'intention initiale de Barlow et Taylor est de définir une notion de fractals dans un espace discret, comme par exemple \mathbb{Z}^d , à l'aide d'une nouvelle notion de dimension adaptée, appelée dimension de Hausdorff macroscopique. Ils déterminent la dimension de Hausdorff macroscopique de l'image d'une marche aléatoire transiente en terme de la fonction de Green associée [9]. Cette dimension de grande échelle a été étendue dans [9] aux ensembles quelconques d'un espace continu comme \mathbb{R}^d . L'idée est de trouver une notion de taille qui ignore la structure locale des ensembles (qui est naturelle si on s'intéresse à des ensembles dans \mathbb{Z}^d), qui en même temps quantifie le comportement asymptotique à l'infini des ensembles.

Pour définir la dimension de Hausdorff macroscopique, on a besoin de quelques notations. On définit des cubes dans \mathbb{R}^d

$$Q(x, r) = \{u \in \mathbb{R}^d : x_i \leq u_i < x_i + r\}.$$

Notons $s(Q) = r > 0$ la longueur du côté d'un cube Q et introduisons les couronnes

$$\mathcal{S}_n = \{x \in \mathbb{R}^d : |x| \in [2^{n-1}, 2^n]\}$$

On considère une sorte de capacité d'ensembles restreinte à toutes les grandes couronnes. Pour tout $E \subset \mathbb{R}^d$, $n \in \mathbb{N}$ et $\rho \geq 0$, on introduit la quantité

$$\nu_\rho^n(E) = \inf \left\{ \sum_{i=1}^m \left(\frac{s(Q_i)}{2^n} \right)^\rho : E \cap \mathcal{S}_n \subset \bigcup_{i=1}^m Q_i \text{ avec } s(Q_i) \geq 1 \text{ et } Q_i \subset \mathcal{S}_n \right\}.$$

Les quantités $\nu_\rho^n(E)$ ressemblent à la mesure de Hausdorff d'un ensemble mais les recouvrements sont composés des cubes de taille grande (supérieure à 1). Ceci est fait exprès pour que la structure locale des ensembles n'ait pas d'influences sur cette dimension de grande échelle. En plus, la contribution de chaque cube est calculée en tenant compte de la couronne repère \mathcal{S}_n .

On peut maintenant définir la dimension de Hausdorff macroscopique et les dimensions de masse.

Définition 6. Soit $E \subset \mathbb{R}^d$. La dimension de Hausdorff macroscopique de E est définie par

$$\text{Dim}_H E = \inf \left\{ \rho \geq 0 : \sum_{n \geq 0} \nu_\rho^n(E) < +\infty \right\}. \quad (3)$$

Les dimensions de masse supérieure et inférieure de E sont

$$\begin{aligned} \text{Dim}_{UM} E &= \limsup_{n \rightarrow +\infty} \frac{\log(|E \cap [0, n]|)}{\log n}, \\ \text{Dim}_{LM} E &= \liminf_{n \rightarrow +\infty} \frac{\log(|E \cap [0, n]|)}{\log n}. \end{aligned}$$

Il est clair [8] que pour tout $E \subset \mathbb{R}^d$,

$$\text{Dim}_H E \leq \text{Dim}_{LM} E \leq \text{Dim}_{UM} E.$$

Afin d'obtenir une borne inférieure de $\text{Dim}_H E$, un analogue du principe de distribution de masse est utile. L'enjeu est de trouver la bonne mesure portée par l'ensemble E .

Lemme 2. *Soit $E \subset \mathcal{S}_n$. Soit μ une mesure finie sur \mathbb{R}^d portée par E . Supposons qu'il existe des constantes finies positives C and ρ , telles que pour tout cube $Q(x, r) \subset \mathcal{S}_n$ avec $r \geq 1$, on a*

$$\mu(Q(x, r)) \leq Cr^\rho.$$

Alors

$$\nu_\rho^n(E) \geq C^{-1}2^{-n\rho}\mu(\mathcal{S}_n).$$

Par la suite, on va présenter les travaux de thèse en utilisant toutes les notions mentionnées dans cette section.

0.2 Présentation des travaux de thèse

Rappelons que l'étude principale de cette thèse porte sur la solution de l'EDS (2), appelée un processus de diffusion à sauts.

Multifractalité de diffusion à sauts

Diverses propriétés de régularité des processus de Lévy ont été étudiées par beaucoup d'auteurs. Par exemple, Bertoin a considéré la différentiabilité et le taux de croissance local [22, 23]. En 1999, Stéphane Jaffard [61] a établi la nature multifractale des processus de Lévy. Son approche originale a suscité beaucoup de travaux sur la multifractalité des processus à sauts [6, 10, 12, 43, 44].

Soit X un processus de Lévy à valeurs dans \mathbb{R}^d , avec triplet (A, b, π) dans sa décomposition de Lévy-Itô. Pour simplifier l'énoncé, on suppose que $A = 0$. Le cas général peut se traiter facilement une fois que le cas sans diffusion ($A = 0$) est fait. Soit β l'indice supérieur de Blumenthal-

Gettoor [28] de X , c'est-à-dire,

$$\beta = \inf \left\{ \gamma \geq 0 : \int_{|x| \leq 1} |x|^\gamma \pi(dx) < +\infty \right\}.$$

Le théorème de Jaffard établit que le spectre multifractal de X s'écrit presque sûrement,

$$D_X(h) = \begin{cases} \beta h & \text{si } h \in [0, 1/\beta], \\ -\infty & \text{sinon.} \end{cases}$$

Le but dans le premier travail de ma thèse est d'étudier la structure multifractale des processus de diffusion à sauts.

Expliquons pourquoi les processus de diffusion à sauts sont plus compliqués que les processus de Lévy. Pour fixer des idées, on considère le cas uni-dimensionnel et on prend $\sigma = b = 0$ et $G(x, z) = \text{Sign}(z)|z|^{1/\beta(x)}$ avec $\beta(\cdot)$ une fonction régulière à valeurs dans $[\varepsilon, 2 - \varepsilon]$. Par convention, on écrit $z^{1/\beta(\cdot)} = \text{Sign}(z)|z|^{1/\beta(\cdot)}$. La mesure de Poisson N admet $\pi(dz) = dz/z^2$ pour intensité. L'équation (2) devient

$$X_t = X_0 + \int_0^t \int_{-1}^1 z^{1/\beta(X_{s-})} \tilde{N}(ds, dz). \quad (4)$$

Lorsque β est constante, X est un processus de Lévy. En appliquant la formule d'Itô et un changement de variable, on observe que le générateur de ce processus de Markov est

$$\mathcal{L}f(x) = \int_{-1}^1 [f(x+u) - f(x) - uf'(x)] \frac{\beta(x) du}{|u|^{1+\beta(x)}}.$$

Ainsi, la dynamique markovienne de X est non homogène en espace. En particulier, l'évolution du futur proche de X à chaque instant dépend de la position actuelle de X . Ce n'est pas le cas pour les processus de Lévy. On perd la stationnarité des accroissements.

On considère toujours l'équation (4). Dans un petit voisinage d'un temps de continuité de X (disons t_0), le processus d'indice $t \mapsto \beta(X_t)$ est à peu près une constante, heuristiquement. Donc X se comporte localement comme

$$t > t_0, \quad \hat{X}_t = X_{t_0} + \int_{t_0}^t \int_{-1}^1 z^{1/\beta(X_{t_0})} \tilde{N}(ds, dz)$$

qui est un processus de Lévy.

La difficulté est la suivante : il y a une infinité non dénombrable de valeurs β (aléatoires) prises par la fonction $t \mapsto \beta(X_t)$, et surtout, dans un intervalle, il peut y avoir un "grand" saut, tel que le processus d'indice $t \mapsto \beta(X_t)$ varie brutalement.

Pour s'affranchir de la difficulté, on a introduit une technique de découpage par tranche ("slicing")

qui consiste à découper le processus selon la valeur du processus d'indice. Précisément, on écrit pour tout grand entier m

$$X_t = X_0 + \sum_{k=0}^{m-1} X_t^m \text{ où } X_t^m = \int_0^t \int_{-1}^1 z^{1/\beta(X_{s-})} \mathbf{1}_{\beta(X_{s-}) \in [\frac{2k}{m}, \frac{2k+2}{m}[} \tilde{N}(ds, dz).$$

Plus la valeur de m est grande, plus chaque tranche X^m dans la somme se rapproche intuitivement d'un processus de Lévy. L'estimation simultanée des accroissements de ces processus "tranche" rend possible une estimation pour les accroissements de X . L'exposant de Hölder s'en suit une fois que les accroissements sont bien contrôlés.

Cette technique est une des contributions importantes dans cette thèse. Elle est originale et peut servir dans l'étude des autres caractéristiques de diffusion à sauts, comme par exemple la p -variation de ces processus (voir Chapitre 3). Nous pensons qu'elle serait utile afin d'étudier la régularité des processus de Markov (à sauts) plus généraux.

Le calcul de dimension des ensembles de singularités nécessite des technicités en appliquant un théorème de la théorie géométrique de la mesure développé par Barral et Seuret [11].

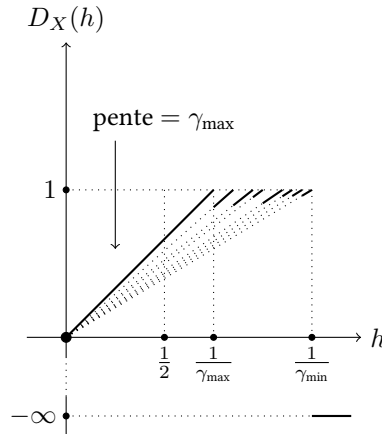
Dans le Chapitre 1, on montre la multifractalité de diffusion à sauts uni-dimensionnelle sous certaines hypothèses pour les coefficients σ, b, G . Dans le Chapitre 2, on étend le résultat aux dimensions supérieures, et on relâche les hypothèses pour les coefficients et la mesure d'intensité π afin d'introduire de l'anisotropie dans le modèle. Pour simplifier la présentation, on n'énonce le résultat sur le spectre que pour l'équation (4).

Théorème. *Posons*

$$\begin{aligned} \gamma(h) &:= \sup \left\{ \beta(X_s) : \beta(X_s) \leq 1/h \right\}, \\ \gamma_{\max} &:= \sup \{ \beta(X_s) : s \geq 0 \} \\ \gamma_{\min} &:= \inf \left\{ \beta(X_s) : s \geq 0 \right\}. \\ J &:= \{t : X_t \neq X_{t-}\}. \end{aligned}$$

Avec probabilité 1, le spectre multifractal de X est

$$D_X(h) = \begin{cases} h \cdot \gamma(h) & \text{si } h < 1/\gamma_{\min} \text{ et } h \notin (\beta(X(J)))^{-1}, \\ -\infty & \text{si } h > 1/\gamma_{\min}. \end{cases}$$



Le spectre dépend de la trajectoire de X , qui reflète la richesse de la régularité locale de diffusion à sauts. On renvoie le lecteur aux deux premiers chapitres pour des résultats concernant des diffusions à sauts plus générales. En particulier, des EDS guidées par une grande classe de processus de Lévy sont considérées.

Dimension de Hausdorff de l'image et du graphe

Depuis les années 80, il y a un intérêt particulier pour des processus de Feller dont le générateur peut s'écrire comme un opérateur pseudo-différentiel [76]. L'approche probabiliste (problème de martingale) permet de découvrir des exemples très intéressants d'opérateurs auxquels on peut associer un processus de Feller. Richard Bass [15] a introduit en 1988 une classe de processus de Feller qu'il appelle processus de type stable ("stable-like processes"). L'opérateur associé à ces processus s'écrit pour une classe de fonctions appropriées

$$\mathcal{L}^B f(x) = \int_{\mathbb{R}} [f(x+u) - f(u) - \mathbf{1}_{|u| \leq 1} u f'(x)] \frac{C_{\beta(x)} du}{|u|^{1+\beta(x)}}$$

où β est une fonction avec certaine régularité telle que son image est incluse dans $[\varepsilon_0, 2 - \varepsilon_0]$. La constante $C_{\beta(x)}$ est telle que l'on a

$$\mathcal{L}^B f(x) = \mathcal{F}^{-1} \left(|\xi|^{\beta(x)} \mathcal{F} f(\xi) \right) (x)$$

où \mathcal{F} est la transformée de Fourier, \mathcal{F}^{-1} son inverse. Donc, cette classe de processus correspond à l'opérateur pseudo-différentiel avec symbole d'ordre variable $(\xi, x) \rightarrow |\xi|^{\beta(x)}$. Considérons

une variante de cet opérateur

$$\mathcal{L}f(x) = \int_{-1}^1 [f(x+u) - f(u) - uf'(x)] \frac{\beta(x)du}{|u|^{1+\beta(x)}}.$$

On observe que ces processus admettent une représentation poissonnienne qui est le processus de diffusion à sauts de type stable (4).

Schilling [105] et Schilling, Knopova et Wang [73] ont trouvé des bornes déterministes pour la dimension de l'image de ces processus. Leur approche est intéressante mais ne conduit pas à une formule exacte pour la dimension de l'image.

Le but du deuxième travail de ma thèse est de déterminer la dimension exacte de l'image et du graphe de diffusion à sauts de type stable au sens de Bass.

La non homogénéité en espace de la dynamique markovienne est toujours la difficulté majeure dans cette étude. L'approche analytique (analyse de Fourier, calcul fonctionnel) permet de déduire l'estimation uniforme en espace du noyau de "chaleur" pour l'EDP parabolique associée à l'opérateur \mathcal{L} . De façon équivalente, elle donne une estimation uniforme en espace pour la probabilité de transition. Cette estimation n'est pas suffisamment fine pour déduire une formule exacte pour la dimension de l'image.

L'idée pour s'affranchir de cette difficulté est de bien localiser notre étude là où la dimension de l'image est presque atteinte. Pour ce faire, on exploite la nature EDS et l'unicité trajectorielle du processus de diffusion à sauts (4) pour construire soigneusement un couplage des processus, solutions d'EDS qui coïncident avec (4) en temps court.

Expliquons comment construire le couplage pour obtenir une borne inférieure pour la dimension de l'image. Posons pour tout $a \in (0, 2)$,

$$X_t^a = X_0 + \int_0^t \int_{|z| \leq 1} z^{\frac{1}{\beta(X_{s-}^a) \vee a}} \tilde{N}(ds, dz)$$

Lorsque $a < \beta(X_0)$, l'unicité trajectorielle montre que X et X^a coïncident sur un intervalle aléatoire non trivial (disons $[0, \delta]$), donc $\dim_{\mathcal{H}} X[0, 1]$ est nécessairement supérieure à $\dim_{\mathcal{H}} X^a[0, \delta]$. Or, Knopova *et al.* [73] ont montré que la dimension de l'image de X^a est bornée inférieurement par la constante a . La construction simultanée des X^a et X implique $\dim_{\mathcal{H}} X[0, 1] \geq \beta(X_0)$ quitte à faire tendre a vers $\beta(X_0)$.

Pour montrer que la borne inférieure obtenue est aussi une borne supérieure, la technique de "couplage en tranche" introduite dans le premier travail s'applique à nouveau. Nous l'utilisons pour montrer que la p -variation de X est finie pour un p approprié. On conclut avec un lemme de McKean [88] qui éclaire le lien entre la p -variation et la borne supérieure de dimension de l'image d'une fonction càdlàg.

Signalons que l'approche développée en ci-dessus s'applique aussi au calcul de la dimension du

graphe de diffusion à sauts de type stable.

Terminons cette partie par nos résultats en dimension quelconque. L'opérateur des processus de diffusion à sauts de type stable, noté toujours par X , dans \mathbb{R}^d est

$$\mathcal{L}\varphi(x) = \int_{\mathbb{R}^d} [\varphi(x+u) - \varphi(x) - \mathbf{1}_{|u|\leq 1} u \cdot \nabla\varphi(x)] \frac{\beta(x)du}{|u|^{d+\beta(x)}}$$

où β est lipschitzienne dont l'image est incluse dans un compact de $(0, 2)$.

Théorème. 1. Avec probabilité 1,

$$\dim_{\mathcal{H}}(X[0, 1]) = d \wedge \sup_{s \in [0, 1]} \beta(X_s).$$

2. Deux cas se présentent lorsqu'on considère la dimension du graphe de X .

(a) Si $d \geq 2$, on a presque sûrement

$$\dim_{\mathcal{H}}(\text{Gr}_{[0, 1]}(X)) = 1 \vee \sup_{t \in [0, 1]} \beta(X_t).$$

(b) Si $d = 1$, on a presque sûrement

$$\dim_{\mathcal{H}}(\text{Gr}_{[0, 1]}(X)) = 1 \vee \left(2 - \frac{1}{\sup_{t \in [0, 1]} \beta(X_t)} \right).$$

Ces résultats montrent que, contrairement aux processus de Lévy, les processus de Markov à sauts d'ordre variable (le symbole de l'opérateur-générateur est d'ordre variable) ont typiquement une dimension de l'image (et du graphe) qui est aléatoire.

Multifractalité de la mesure d'occupation de diffusion à sauts de type stable

En 1997, Hu et Taylor [55] ont fait l'analyse multifractale de la mesure d'occupation d'un subordonneur α -stable (processus de Lévy stable croissant).

Soit X^α un subordonneur α -stable avec $\alpha \in]0, 1[$ et $\mu^\alpha = \mu_1^\alpha$ sa mesure d'occupation sur l'intervalle du temps $[0, 1]$ définie dans (1). Hu et Taylor ont montré que pour tout $h \in [\alpha, 2\alpha]$,

presque surement,

$$\bar{d}_{\mu^\alpha}(h) = \begin{cases} \alpha(\frac{2\alpha}{h} - 1) & \text{si } h \in [\alpha, 2\alpha], \\ -\infty & \text{sinon.} \end{cases}$$

Le but du troisième travail de cette thèse est d'étudier la multifractalité de la mesure d'occupation du processus de diffusion à sauts de type stable *croissant*, c'est-à-dire la solution de

$$X_t = \int_0^t \int_0^1 z^{1/\beta(X_{s-})} N(ds, dz).$$

On considère sa mesure d'occupation μ sur l'intervalle du temps $[0, 1]$. Pour calculer la dimension des ensembles de singularité de μ , on doit construire une famille d'ensembles de Cantor sur lesquels la dimension locale de la mesure d'occupation vaut exactement h , simultanément pour tout h . L'approche de Hu et Taylor ne s'applique pas dans notre cas pour au moins trois raisons :

1. dans [55], le quantificateur "p.s." placé après " $\forall h$ " indique que l'événement de pleine probabilité dépend de l'indice h qui prend ses valeurs sur un intervalle non trivial ($[\alpha, 2\alpha]$). Nous souhaitons obtenir le résultat presque surement pour toutes les valeurs de h en même temps, autrement dit, intervertir " $\forall h$ " et "p.s."
2. dans [55], au lieu de travailler directement sur l'ensemble "iso"-höldérien, ils considèrent l'ensemble des points où la dimension locale supérieure de la mesure d'occupation est plus grande que h

$$\{x : \overline{\dim}(\mu, x) \geq h\}.$$

Ils en déduisent le résultat après avoir estimé la fonction de jauge pour la mesure de Hausdorff de ces ensembles. Dans notre cas, la non homogénéité de la dynamique en espace nous impose de travailler directement sur l'ensemble iso-höldérien

$$\{x : \overline{\dim}(\mu, x) = h\}.$$

3. dans [55], la preuve pour calculer le spectre est réalisée en deux étapes. Dans un premier temps, Hu et Taylor ont construit des ensembles de temps t pour lesquels la dimension locale supérieure de μ^α en X_t^α est supérieure ou égale à une valeur donnée (dans $[\alpha, 2\alpha]$). Ensuite, en utilisant le théorème classique de dimension uniforme de l'image pour les processus stables (ce résultat établit que p.s. pour tout ensemble dans $[0, 1]$, la dimension de son image par un processus stable vaut la dimension de l'ensemble considéré multipliée par l'indice de stabilité, voir [24]), ils ont automatiquement le spectre en espace. Pour la diffusion à sauts de type stable, l'étape deux ne se fait pas automatiquement. On doit

établir un résultat de dimension uniforme des ensembles par le processus de diffusion à sauts de type stable.

Les deux premiers points concernent la construction d'ensembles de Cantor. L'idée clé est de les construire de telle façon que l'on capte la hétérogénéité en temps de diffusion à sauts de type stable. Pour ce faire, on identifie des configurations de "zero saut" et "double sauts" dans le processus ponctuel de Poisson et on les utilise avec un poids différent selon l'échelle du temps, l'échelle de l'espace et l'endroit où on se trouve. Des estimations fines pour le processus ponctuel de Poisson sont nécessaires. Cette construction bien localisée et hétérogène pourrait servir pour étudier la mesure d'occupation des autres processus de Markov à sauts.

Ici on ne présente que le résultat pour le spectre multifractal en espace. Pour plus de résultats (comme par exemple sur le spectre en temps, le spectre associé à la dimension locale inférieure, le théorème de dimension uniforme de l'image), voir Chapitre 4.

Théorème. *Posons*

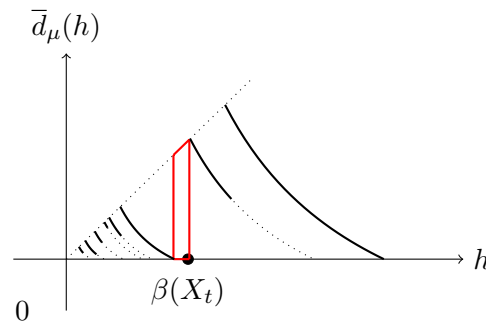
$$\mathcal{E} = \{\beta(X_t) : X_{t-} \neq X_t \text{ et } \beta(X_t) \geq 2\beta(X_{t-})\} \cup \{2\beta(X_{t-}) : X_{t-} \neq X_t \text{ et } \beta(X_t) \geq 2\beta(X_{t-})\}$$

et

$$g_\alpha(h) := \begin{cases} \alpha \left(\frac{2\alpha}{h} - 1 \right) & \text{si } h \in [\alpha, 2\alpha), \\ -\infty & \text{sinon.} \end{cases}$$

Le spectre multifractal de la mesure d'occupation de diffusion à sauts de type stable croissant s'écrit presque sûrement pour tout $h \notin \mathcal{E}$,

$$\bar{d}_\mu(h) = \sup \{g_\alpha(h) : \alpha \in \{\beta(X_t) : t \in \mathbb{R}^+\}\}.$$



Observons que $\mathcal{E} = \emptyset$ si, par exemple, l'image de $\beta(\cdot)$ est incluse dans $[1/2, 1 - \varepsilon_0]$. On renvoie le lecteur au Chapitre 4 pour le spectre en les points exceptionnels $h \in \mathcal{E}$.

Temps de séjour d'un brownien sous des frontières glissantes

Je me suis intéressé récemment aux propriétés de séjour du brownien, qui sont bien utiles pour des applications en finance et en physique. Théoriquement, l'ensemble de séjour du brownien par rapport à une frontière glissante procure beaucoup d'informations sur les propriétés trajectorielles du brownien. Plus précisément, soit $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ une fonction croissante, considérons les ensembles

$$E^o(\varphi) = \{t \geq 0 : |B_t| \geq \varphi(t)\}, \quad (5)$$

$$E^i(\varphi) = \{t \geq 0 : |B_t| \leq \varphi(t)\}, \quad (6)$$

où le premier est appelé le séjour brownien à l'extérieur de la frontière $\varphi(\cdot)$ et le second à l'intérieur de la frontière. Signalons deux types d'étude particulièrement intéressants.

- Propriétés géométriques de $E^i(\varphi)$ et $E^o(\varphi)$ autour de $t_0 = 0$. L'ensemble $E^o(\varphi)$ procure une description de l'irrégularité du brownien. En effet, si 0 est un point d'accumulation de $E^o(\varphi)$, alors le module de continuité local du mouvement brownien ne peut pas dépasser φ . En contrepartie, $E^i(\varphi)$ correspond à un comportement plus régulier.
- Propriétés asymptotiques de ces ensembles à l'infini. Cela correspond au comportement à long terme du brownien. Par autosimilarité, la croissance du brownien est en moyenne une fonction racine carrée, qui sert donc de fonction référence pour décrire trajectoriellement la croissance du brownien. Concrètement,

1. l'ensemble $E^o(\varphi)$, lorsque φ croît plus vite que la racine carrée, concerne les hauts pics du brownien. Par exemple, la loi de logarithme itéré de Strassen [109] implique que la densité supérieure à l'infini de $E^o(\varphi_a)$ avec $\varphi_a(t) = a(2t \ln \ln(t))^{1/2}$ est non triviale, pour tout $a \in [0, 1)$ et la densité supérieure de $E^o(\varphi_1)$ est zéro. Signalons que Khoshnevisan, Kim et Xiao [71] ont prouvé que l'ensemble des hauts pics du brownien dans le cas critique (c'est-à-dire l'ensemble $E^o(\varphi_1)$) a pour une dimension de Hausdorff macroscopique pleine.
2. l'ensemble $E^i(\varphi)$, lorsque φ croît moins vite que la racine, concerne la croissance plus petite que normale du brownien. Uchiyama [113] a établi des bornes pour la densité supérieure à l'infini de $E^i(\varphi)$ où $\varphi(t) = \sqrt{t}/h(t)$ avec h prise dans une classe de fonctions avec croissance au plus logarithmique.

Le but du dernier travail de ma thèse est de déterminer la dimension macroscopique du séjour brownien à l'intérieur d'une frontière avec croissance beaucoup moins importante que la fonction racine carrée. Par beaucoup moins importante, on parle des ensembles

$$\forall \gamma \in [0, 1/2], \quad E(\gamma) = \{t \geq 0 : |B| \leq t^\gamma\}.$$

Les méthodes pour calculer la dimension macroscopique sont comparables à celles utilisées pour calculer la dimension de Hausdorff classique.

- Pour majorer la dimension de Hausdorff macroscopique, on doit trouver un recouvrement économique.
- Pour la minoration, l'enjeu est de bien choisir une mesure portée par l'ensemble considéré qui vérifie la propriété d'échelle énoncée dans le Lemme 2.

Présentons nos résultats sur la dimension de Hausdorff macroscopique et la dimension de masse supérieure pour les ensembles $E(\gamma)$.

Théorème. *Presque surement,*

$$\begin{aligned} \text{Dim}_{\text{H}}E(\gamma) &= \begin{cases} \frac{1}{2} & \text{si } \gamma \in [0, 1/2), \\ 1 & \text{si } \gamma = 1/2. \end{cases} \\ \text{Dim}_{\text{UM}}E(\gamma) &= \frac{1}{2} + \gamma \quad \text{pour tout } \gamma \in [0, 1/2]. \end{aligned}$$

Il est remarquable que la dimension de Hausdorff macroscopique des $E(\gamma)$ est une constante pour tout $\gamma \in [0, 1/2)$ et qu'il y a un saut en $1/2$. Un ingrédient important dans la preuve est l'estimation de "hitting probability" qui consiste à estimer la probabilité que le mouvement brownien touche une frontière glissante sur un intervalle quelconque. Il est intéressant de voir que l'ensemble de séjour du brownien montre en quelque sorte la nature multifractale à grande échelle (en termes de dimension de masse) du mouvement brownien. Ce théorème donne aussi une classe d'ensembles naturels pour lesquels la dimension de Hausdorff macroscopique et la dimension de masse diffèrent.

Pour la majoration (Dim_{H}), la toute première question concerne l'interprétation du "recouvrement économique" dans le cadre de dimension macroscopique du séjour brownien. Le meilleur recouvrement pour nos ensembles $E(\gamma)$ n'est ni le plus grand (\mathcal{S}_n pour chaque couronne \mathcal{S}_n) ni le plus petit (des intervalles de longueur un dans chaque \mathcal{S}_n). Plaçons nous en un point où le brownien vaut 0. Supposons qu'il s'agit d'un point t_0 dans la couronne \mathcal{S}_n . Il est connu que le temps de sortie pour un brownien issu de zéro

$$\sigma_a = \inf\{t \geq 0 : |B_t| \geq a\}$$

vérifie $\mathbb{E}[\sigma_a] = a^2$. Ainsi, pour sortir la frontière $2^{n\gamma}$, le temps nécessaire est en moyenne $2^{2n\gamma}$. En prenant le recouvrement de $E(\gamma) \cap \mathcal{S}_n$ par des intervalles de longueur $2^{2n\gamma}$, on a trouvé la borne supérieure optimale pour la dimension de Hausdorff macroscopique de $E(\gamma)$, pour tout $\gamma \in [0, 1/2)$.

Pour la minoration (Dim_H), on applique le fait que l'ensemble des zéros du brownien

$$\mathcal{Z} = \{t : B_t = 0\}$$

(qui est inclus dans tous les $E(\gamma)$) coïncide avec l'image d'un subordonneur 1/2-stable σ

$$\mathcal{R} = \sigma[0, +\infty[,$$

il suffit de minorer $\text{Dim}_H \mathcal{R}$ en considérant la mesure d'occupation de σ et de vérifier la propriété d'échelle dans le Lemme 2 pour cette mesure. Signalons que les propriétés microscopiques (ou locales) de \mathcal{Z} sont connues grâce au travail de Taylor-Wendel [112] et celui de Perkins [97] qui ont considéré la mesure de Hausdorff classique de \mathcal{Z} en utilisant les temps locaux du brownien. En particulier, $\dim_{\mathcal{H}} \mathcal{Z} = 1/2$. Nous montrons que la dimension de Hausdorff macroscopique de \mathcal{Z} vaut également 1/2. Ceci doit être comparé avec un résultat de Khoshnevisan [67] qui prouve que l'ensemble des temps de visite à 0 d'une marche aléatoire ξ de variance finie dans \mathbb{Z}^1

$$\{n \in \mathbb{N} : \xi_n = 0\}$$

admet 1/2 pour dimension de Hausdorff macroscopique.

Le calcul de $\text{Dim}_{\text{UM}} E(\gamma)$ est basé à la fois sur une estimation d'un certain type de "hitting probability", et une suite d'échelles du temps bien choisie pour pouvoir observer le comportement désiré. Le premier entraîne la borne supérieure et le deuxième la borne inférieure.

0.3 Perspectives

Il y a encore beaucoup de questions intéressantes que j'ai envie de traiter, en voici quelques-unes.

- Depuis le travail fondateur de Lamperti [78], beaucoup de propriétés trajectoires des processus de Markov positifs auto-similaires (pssMp) ont été découvertes via la transformée (trajectoirelle) de Lamperti qui établit un lien entre un sspMp quelconque et un certain processus de Lévy sous-jacent. Mon objectif est de comprendre le rôle de la transformée de Lamperti pour la régularité locale des processus de Lévy afin d'effectuer l'analyse multifractale des pssMp.
- La mesure d'occupation considérée dans le Chapitre 4 est singulière par rapport à la mesure de Lebesgue. Bass [14] a montré l'existence de la densité d'occupation de diffusion à sauts de type stable sous l'hypothèse que l'image de $\beta(\cdot)$ est incluse dans $[1 + \varepsilon, 2 - \varepsilon]$. Une étude sur la régularité locale de cette densité d'occupation va améliorer la compréhension de diffusion à sauts de type stable.

- La dimension macroscopique semble un outil prometteur pour étudier plusieurs types de fluctuation des processus stochastiques. je compte travailler sur ce sujet.

Introduction (English)

In the theory of probability, the most natural stochastic processes are everywhere irregular. For example, Brownian motion is continuous but nowhere differentiable ; Lévy processes jump in general on a dense set of times; we are going to see other examples of irregular processes in the sequel. So it is natural to study the regularity of stochastic processes. A relevant tool to describe the fluctuation of regularity of the sample paths is multifractal analysis. This is the approach that we are going to develop in the majority of this thesis.

Many authors are interested in the fine regularity results of Brownian motion. By self-similarity, the local growth rate of Brownian motion scales on average like a square root function. Lévy [82] has established the exact modulus of continuity of Brownian motion which is indeed the square root function with a logarithmic order correction. This establishes an upper bound for the local regularity of Brownian motion. On the other hand, Paley, Wiener et Zygmund [96] have shown that the local Hölder regularity can not be larger than $1/2$. The situation is completely different for Lévy processes since the Hölder regularity depends on the point under consideration. Indeed, a subordinator is differentiable for Lebesgue almost every point by monotonicity of the sample paths, however it jumps on a dense set of points. In fact, there is a continuum of possible values for the Hölder exponent of a general Lévy process. Multifractal analysis permits to describe all the possible local behaviors of a function (in particular a Lévy process), and to quantify the size (in terms of the Hausdorff dimension) of the set of points having a given local behavior. Paradoxically, this kind of study has been investigated only for several restricted classes of processes, e.g. Brownian motion [95, 98], Lévy processes [61] and local times [87].

It is natural to investigate these questions of multifractal analysis for more general classes of processes, especially Markov processes with jumps. We are going to study an important class of Markov processes with jumps, called jump diffusions, which are defined by a stochastic differential equation driven by a Brownian motion and a Poisson point process. Jump diffusion processes correspond to Lévy processes in the same way as the continuous diffusion corresponds to Brownian motion. These processes have many applications in finance (option pricing, Merton's model [2, 35]), in physics (Langevin's dynamics, neutron scattering in a liquid [33, 79]), in biologie (coalescence, evolution of a population [26, 77]). *In the first two chapters of this thesis, we concentrate on the multifractal analysis of jump diffusion processes.*

There is also other characteristics which can describe the roughness of a process.

A relevant measurement of the roughness of a process is provided by the Hausdorff dimension of its range and its graph. This type of study is initiated by Taylor [111] who proved that the range of d -dimensional Brownian motion is $\min\{d, 2\}$ almost surely. Since then, many results have been obtained on the fractal dimensions for the range of different types of stochastic processes : Lévy processes [54, 102], fractional Brownian motion [110, 116], and their multi-parameter extensions [4, 115]. *In Chapter 3, we are going to study the Hausdorff dimension of the range and the graph of stable-like jump diffusions, introduced by Bass [15]. The dimension of the image of an*

arbitrary set by these processes is also studied, see the end of Chapter 4.

The occupation measure of a process provides another way to describe the local regularity of the process.

Let X be a process taking values in \mathbb{R}^d . The occupation measure of X is defined for any measurable set $A \subset \mathbb{R}^d$ by

$$\mu_t(A) = \int_0^t \mathbf{1}_A(X_s) ds \quad (7)$$

This measure describes the time spent by X in the set A before time t . It is the “dual” object of the process X in the sense that, more regular is the occupation measure, more irregular is the process. The existence of local times (occupation density with respect to the Lebesgue measure) for the one-dimensional Brownian motion is established by Lévy. In higher dimensions, the occupation density of the Brownian motion does not exist. In such cases, the appropriate tool to describe the local regularity is the local dimension of μ at each point x in its support, defined by

$$\lim_{r \rightarrow 0} \frac{\ln(\mu(B(x, r)))}{\ln r}.$$

Perkins et Taylor [99] have proved that the local dimension of d -dimensional Brownian occupation measure ($d \geq 2$) is 2 at every point in its support. The situation is again completely different for Lévy processes. Indeed, Hu et Taylor [55] have shown that the typical points of the α -stable occupation measure have local dimension α , but the set of times with upper local dimension (replace \lim by \limsup in the definition) h has non-trivial Hausdorff dimension for all $h \in [\alpha, 2\alpha]$. Remark that the occupation measure is supported by the range of X . Hence, the study of local dimensions of μ is useful to calculate various fractal dimensions for the range of X . *In the chapter 4, we are going to perform multifractal analysis of the occupation measure of stable-like jump diffusions.*

In order to describe the complexity of sets in discrete space such as \mathbb{Z}^d , Barlow et Taylor [8, 9] have introduced in the late 80’s a new notion of fractal dimension, called “macroscopic Hausdorff dimension”. The idea is that this dimension depends only on some type of asymptotic behavior at infinity of the sets.

There exists a “mass dimension” which is used for example in number theory and physical statistics. Examples of studied sets include the connected component of a percolation process, the range of a transient random walk. This is the counterpart of Minkowski dimensions on a large scale. Intuitively, if $E \subset \mathbb{N}$, the value of this mass dimension is the positive real α such that

$$\#\{k \in E : k \leq n\} \sim n^\alpha$$

for every large n .

Remark that Barlow et Taylor [8] have constructed examples of sets which have different macro-

scopic Hausdorff dimension and mass dimension.

Very recently, Xiao et Zheng [118] have studied the range of a random walk in random environment via these dimensions. Georgiou *et al.* [53] have established results on the range of an arbitrary transient random walk, resolving a question of Barlow et Taylor [9]. The macroscopic Hausdorff dimension is also useful in the study of asymptotics (at infinity) of sets in \mathbb{R}^d : Khoshnevisan, Kim and Xiao [71] have quantified by this dimension the spatial high peaks (near infinity) for the solution of stochastic heat equation with additive or multiplicative noise.

The last chapter is devoted to the study of sojourn times of Brownian motion in dimension one inside moving boundaries. Let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and

$$E(\varphi) = \{t \geq 0 : |B_t| \leq \varphi(t)\}.$$

Uchiyama [113] have considered the existence of upper density (at infinity) with respect to the Lebesgue measure of these sets. He has found remarkable identities for the upper density of $E(\varphi)$ under the hypotheses that the moving boundary φ grows like a square root function with a logarithmic order correction. We are going to study the Brownian sojourn inside moving boundaries with growth which is much smaller than the square root function, and we compute the macroscopic Hausdorff dimension and the mass dimension of these sojourn sets. This allows to quantify the fluctuation and the recurrence of Brownian sample paths on a large scale.

Notations, definitions and history

Let $(X_t)_{t \geq 0}$ be a stochastic process taking values in \mathbb{R}^d , defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The systematic study of the continuity of the sample paths of X is initiated by Doob [37, 38]. It is soon understood via the works of Lévy [81, 82], Doob [39, 40], Kinney [72], Kolmogorov-Chentsov [32], that most of processes (e.g. martingales, Feller processes, many Gaussian processes) have right-continuous with left limit (càdlàg) sample paths. However, there are examples of processes with nowhere bounded sample paths studied by Maejima [84] and recently by Balança [6].

In 1828, Scottish botanist Brown observed the erratic motion of a pollen suspended in a fluid. It is the origin of the first stochastic process in continuous time : the Brownian motion. In 1900, Bachelier applied Brownian motion to the theory of speculation [5]. Later in 1905, Einstein [47] used it in the kinetic theory of molecules. However, the mathematical investigation of Brownian motion has been founded upon the seminal work of Wiener [114] in 1923. His approach is to construct a probability measure which is named after him on the space of continuous functions.

The nowhere differentiability of Brownian motion can be described more precisely by a notion of local regularity between continuity and differentiability. The natural choice is the pointwise Hölder exponent that we recall now.

Definition 1. Let $t_0 \in \mathbb{R}^+$ and $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ locally bounded. We say that f belongs to $C^h(t_0)$ if there exist constants $C, \delta > 0$, a polynomial P with order less than h such that for all $t \in B(t_0, \delta)$,

$$|f(t) - P(t - t_0)| \leq |t - t_0|^h.$$

We define the Hölder exponent of f at t_0 by

$$H_f(t_0) = \sup\{h \geq 0 : f \in C^h(t_0)\}.$$

Remark 1. The polynomial plays the role of the Taylor expansion for smooth functions. When $H_f(t_0) < 1$, this polynomial is necessarily the constant $f(t_0)$.

It turns out that the Brownian motion is not sensible to the pointwise Hölder exponent. Indeed, with probability 1, for all $t \in [0, 1]$, $H_B(t) = 1/2$, see however the paper by Orey and Taylor [95] and the one by Perkins [98] which concern the logarithmic fluctuation of the local regularity of Brownian motion. One can see an interview of Kahane [64] for more history about Brownian motion. We mention as well harmonic analytical methods in the study of local regularity, which consist in decompose a process X into a Schauder basis or more general wavelets, see [4, 19, 30].

Jump processes occupy an important place among all the stochastic processes. The first example is the simple Poisson process which is used by Lundberg in 1903 in order to model the successive arrival of accidents. One encounters immediately more complicated jump processes if one studies simple functionals of continuous processes. For example, the process of first passage time of a Brownian motion

$$\forall a \geq 0, T_a = \inf\{t \geq 0 : B_t > a\}$$

is a jump process. Furthermore, T belongs to a very important class of jump processes : processes with independent and stationary increments, also called Lévy processes in honor of French probabilist Paul Lévy which has made fundamental contributions in this domain. In 1934, the seminal work of Lévy [81] characterized the processes with independent and stationary increment in terms of a diffusion and a component of jumps. Itô [57] showed in 1942 that the component of jumps can be written as a Poisson integral, which gave rise to the final formulation of the Lévy-Itô decomposition stated below. The representation of the characteristic exponent of Lévy processes is given by Lévy himself [82], and Khintchine [66]. One can consult [24, 25, 104] for more references on this subject.

Let π be a measure on \mathbb{R}^d . The Poisson point process of intensity π is a point process $\mathcal{P} = (\Delta_t, t \geq 0)$ with value in \mathbb{R}^d such that for each $E \subset \mathbb{R}^d$ with $\pi(E) < +\infty$, the counting process $t \mapsto \#\{s \leq [0, t] : \Delta_s \in E\}$ is a simple Poisson process of intensity $\pi(E)$. The random Poisson measure N generated by \mathcal{P} is the discrete measure

$$N(dt, dx) = \sum_{t: \Delta_t \in \mathbb{R}^d \setminus \{0\}} \delta_{(t, \Delta_t)}.$$

The compensated Poisson measure is the signed measure

$$\tilde{N}(ds, dx) = N(ds, dx) - ds \otimes \pi(dx).$$

Theorem 1. (Lévy [81], Itô [57]) *Let X be a Lévy process taking values in \mathbb{R}^d . Then there exist a $d \times d$ positive definite matrix A , a vector $b \in \mathbb{R}^d$, and a measure π on \mathbb{R}^d satisfying*

$$\int 1 \wedge |x|^2 \pi(dx) < +\infty$$

such that

$$X_t = AB_t + bt + \int_0^t \int_{|x| \leq 1} x \tilde{N}(ds, dx) + \int_0^t \int_{|x| > 1} x N(ds, dx).$$

where B is a d -dimensional Brownian motion, N is a Poisson random measure of intensity $\pi(dx)$ independent of B .

This decomposition is extremely useful in the study of local regularity of Lévy processes. Indeed, Jaffard [61] performed a fine analysis of the configuration of the jumps, which allowed to determine the Hölder exponent $H_X(t)$ for all t . In particular, he showed that the pointwise Hölder exponent of a Lévy process depends on the point under consideration and the mapping $t \mapsto H_X(t)$ is everywhere discontinuous. Thus, multifractal analysis is a relevant framework to give organized information of the local regularity.

We are going to perform multifractal analysis for a more general class of processes, including Lévy processes, that we call jump diffusion processes (or jump diffusions for short).

Although the idea of stochastic differential equations (SDE) appeared already in the works of Langevin [79] and Bernstein [21], the rigorous and systematic study of SDE is founded by the works of Japanese probabilist Kiyoshi Itô [58, 59]. SDE driven by a Lévy process appeared already in [59].

We call *jump diffusion processes* the solution of a SDE with jumps defined as follows. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a probability space that satisfies the usual conditions. Let B be a standard Brownian motion

in \mathbb{R}^d and \mathcal{P} be a Poisson point process in \mathbb{R}^d defined both on this probability space. Let N be the Poisson random measure generated by \mathcal{P} and \tilde{N} the compensated measure. Consider the following equation (in \mathbb{R}^d)

$$X_t = X_0 + \int_0^t \sigma(X_{s-}) dB_s + \int_0^t b(X_s) ds + \int_0^t \int_{|z| \leq 1} G(X_{s-}, z) \tilde{N}(ds, dz). \quad (8)$$

The solution exists if there is a process X adapted to (\mathcal{F}_t) which verifies (8). The solution is pathwise unique if two solutions X and X' are indistinguishable, i.e. $\mathbb{P}(\forall t \geq 0, X_t = X'_t) = 1$. Classical conditions on the coefficients σ, b, G for the existence of a unique pathwise solution are presented in [59]. It is still an active research domain to pursue the optimal conditions for the pathwise uniqueness, see [16, 50, 51, 83]. *The equation (8) is the central object of study in this thesis. Its solution is expected to have more irregular behaviors than Lévy processes, for its Markovian dynamic is more complicated, less explicit.*

To describe the size of the sets of singularities of a process X , i.e. the level sets of the Hölder exponent

$$E_X(h) = \{t \geq 0 : H_X(t) = h\},$$

we need the notion of Hausdorff dimension. Indeed, as is shown by Jaffard [61], the sets of singularities for a Lévy process are in general dense in an arbitrary non trivial interval. The Minkowski dimension has the same value for all the sets of singularities, thus does not allow to distinguish different local behaviors. One can also consider the packing dimensions, but we work rather with the Hausdorff dimension.

Definition 2. *Let $E \subset \mathbb{R}^d$ and $0 \leq s \leq d$. The s -dimensional Hausdorff measure of E is defined by*

$$\mathcal{H}^s(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(E) = \lim_{\delta \rightarrow 0} \inf \left\{ \sum_{i=1}^{+\infty} |Q_i|^s : E \subset \bigcup_{i=1}^{+\infty} Q_i \text{ et } |Q_i| \leq \delta \right\}.$$

The mapping $s \mapsto \mathcal{H}^s(E)$ being decreasing, one defines the Hausdorff dimension of E by

$$\dim_{\mathcal{H}} E = \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = +\infty\}.$$

The Hausdorff dimension allows to analyse “complicated” sets in a Euclidean space or a more general metric space. It is the central notion in a recent branch of mathematics : the fractal geometry. It was Benoit Mandelbrot who invented the terminology “fractal” to describe mathematically many phenomena which exhibit in one way or another the property of selfsimilarity or roughness, see [48, 85].

The range, the graph and the level sets of a stochastic process provide some classes of interesting random fractals. Take the range as an example. The first work on this subject is the one by Taylor [111], who used potential theoretical methods to compute the Hausdorff dimension of the range of Brownian motion. One had to wait several years [27, 54, 63, 88, 101] until Pruitt [102] established the dimension formula for the range of a general Lévy process. Recently, Khoshnevisan, Xiao et Zhong [69, 70] developed the potential theory of some multiparameter process (the so-called additive Lévy process) and used it to give dimension formulas for the range of a Lévy process in terms of its characteristic exponent. As is said, in order to study the range of X , its occupation measure μ plays a crucial role since the latter is the natural measure supported by the range of X . The link between the dimension of the range and the occupation measure is summarized in the following lemma, called the mass distribution principle.

Lemma 1. *Let μ be a finite positive measure supported by a set $E \subset \mathbb{R}^d$. Assume that for μ -almost every x ,*

$$\liminf_{r \rightarrow 0} \frac{\ln(\mu(B(x, r)))}{\ln r} \geq s.$$

Then one has $\dim_{\mathcal{H}} E \geq s$.

We introduce now the multifractal spectrum, known also as the spectrum of singularities.

Definition 3. *The multifractal spectrum of a function X is the mapping*

$$D_X : h \mapsto \dim_{\mathcal{H}} E_X(h)$$

for all $h \geq 0$ where $E_X(h)$ is the iso-Hölder set $E_X(h) = \{t : H_X(t) = h\}$. By convention, $\dim_{\mathcal{H}} \emptyset = -\infty$.

In practice, the Hölder exponent $H_X(\cdot)$ might be a very erratic function (it is the case of Lévy processes), the computation of $H_X(\cdot)$ is thus very instable, numerically. In such cases, one desires rather to obtain less precise, more qualitative informations : which are the values of h the exponent mapping $H_X(\cdot)$ can take, and which is the Hausdorff dimension of $E_X(h)$ for all the possible values of h ? The multifractal analysis intends to answer these questions. The local regularity of jump diffusion processes is even more erratic than that of Lévy processes, so the multifractal analysis remains a promising framework to describe their local regularity.

In the same spirit as for processes, the multifractal analysis allows to describe the local regularity of random measures. The notion of local regularity for measures that we are going to use is the following.

Definition 4. Let μ be a measure on \mathbb{R}^d and $x \in \mathbb{R}^d$. The upper local dimension of μ at x is

$$\overline{\dim}(\mu, x) = \limsup_{r \rightarrow 0} \frac{\ln(\mu(B(x, r)))}{\ln r}$$

One defines similarly the lower local dimension of μ at x

$$\underline{\dim}(\mu, x) = \liminf_{r \rightarrow 0} \frac{\ln(\mu(B(x, r)))}{\ln r}$$

When the limit exists, one calls the limit local dimension of μ at x , denoted by $\dim(\mu, x)$.

The measure considered in this thesis is the occupation measure of jump diffusion processes.

Definition 5. The upper multifractal spectrum of the occupation measure μ of a process X is the mapping

$$\overline{d}_\mu(\cdot) : h \mapsto \dim_{\mathcal{H}}\{x \in \text{Supp}(\mu) : \overline{\dim}(\mu, x) = h\}.$$

One defines in a similar way the lower multifractal spectrum $\underline{d}_\mu(\cdot)$.

Let us end this introductory section with a short review of the theory of large scale dimensions recently developed by Barlow and Taylor [8, 9]. Discrete models in statistical physics are often governed by a power law. The initial intention of Barlow and Taylor is to define a notion of fractals in a discrete space, such as \mathbb{Z}^d , with a new notion of adapted dimension, called the macroscopic Hausdorff dimension. They determined the macroscopic Hausdorff dimensions of the range of a transient random walk in terms of the associated Green function [9]. This large scale dimension can be extended [9] to continuous space such as \mathbb{R}^d . The idea is to find a notion of size which ignores the local structure of the sets (this is natural if one considers the subsets of \mathbb{Z}^d), which at the same time quantifies the asymptotic behavior at infinity of the sets.

In order to define the macroscopic Hausdorff dimension, one needs some notations. Define cubes in \mathbb{R}^d

$$Q(x, r) = \{u \in \mathbb{R}^d : x_i \leq u_i < x_i + r\}.$$

Let $s(Q) = r > 0$ be the side length of a cube Q . Introduce the annuli

$$\mathcal{S}_n = \{x \in \mathbb{R}^d : |x| \in [2^{n-1}, 2^n]\}.$$

Let us consider a sort of capacity of sets restricted to all the large annuli. For every $E \subset \mathbb{R}^d$, $n \in \mathbb{N}$ and $\rho \geq 0$, one introduces the quantity

$$\nu_\rho^n(E) = \inf \left\{ \sum_{i=1}^m \left(\frac{s(Q_i)}{2^n} \right)^\rho : E \cap \mathcal{S}_n \subset \bigcup_{i=1}^m Q_i \text{ with } s(Q_i) \geq 1 \text{ and } Q_i \subset \mathcal{S}_n \right\}.$$

The quantity $\nu_\rho^n(E)$ looks like the Hausdorff measure of a set but the covering are composed of large cubes (with side larger than 1). This is done on purpose such that the local structure of the sets does not have influence on their large scale dimension. Further, the contribution of each cube is computed by taking into account the landmark annulus \mathcal{S}_n .

We can now define the macroscopic Hausdorff dimension and the mass dimensions.

Definition 6. Let $E \subset \mathbb{R}^d$. The macroscopic Hausdorff dimension of E is defined by

$$\text{Dim}_H E = \inf \left\{ \rho \geq 0 : \sum_{n \geq 0} \nu_\rho^n(E) < +\infty \right\}. \quad (9)$$

The upper and lower mass dimensions of E are

$$\begin{aligned} \text{Dim}_{UM} E &= \limsup_{n \rightarrow +\infty} \frac{\log(|E \cap [0, n]|)}{\log n}, \\ \text{Dim}_{LM} E &= \liminf_{n \rightarrow +\infty} \frac{\log(|E \cap [0, n]|)}{\log n}. \end{aligned}$$

It is clear [8] that for all $E \subset \mathbb{R}^d$,

$$\text{Dim}_H E \leq \text{Dim}_{LM} E \leq \text{Dim}_{UM} E.$$

In order to obtain a lower bound for $\text{Dim}_H E$, a large scale analog of the mass distribution principle is useful. The “challenge” is to find a good measure supported by the set E .

Lemma 2. Let $E \subset \mathcal{S}_n$. Let μ be a measure on \mathbb{R}^d supported by E . Assume that there exist positive constants C and ρ , such that for every cube $Q(x, r) \subset \mathcal{S}_n$ with $r \geq 1$, one has

$$\mu(Q(x, r)) \leq Cr^\rho.$$

Then

$$\nu_\rho^n(E) \geq C^{-1} 2^{-n\rho} \mu(\mathcal{S}_n).$$

In the sequel, we are going to present main results obtained in this thesis by applying all the notions introduced in this section.

Presentation of the thesis

Recall that the main object of study in this thesis is the solution of the SDE (8), called a jump diffusion process.

Multifractality of jump diffusion processes

Various properties of the regularity of Lévy processes have been studied by many authors. For example, Bertoin considered the differentiability and the local growth rate [22, 23]. In 1999, Stéphane Jaffard [61] established the multifractal nature of Lévy processes. His approach is original and evokes many works on the multifractality of jump processes [6, 10, 12, 43, 44].

Let X be a Lévy process in \mathbb{R}^d , with triplet (A, b, π) in its Lévy-Itô decomposition. To simplify the presentaiton, one supposes that $A = 0$. The general case can be treated simply once the the case without diffusion ($A = 0$) is done. Let β be the upper index of Blumenthal-Gettoor [28] of X , i.e.

$$\beta = \inf \left\{ \gamma \geq 0 : \int_{|x| \leq 1} |x|^\gamma \pi(dx) < +\infty \right\}.$$

Jaffard's theorem establishes that the multifractal spectrum of X is almost surely

$$D_X(h) = \begin{cases} \beta h & \text{if } h \in [0, 1/\beta], \\ -\infty & \text{otherwise.} \end{cases}$$

The purpose in the first work of this thesis is to study the multifractal structure of jump diffusion processes.

Let us explain why jump diffusion processes are more complicated than Lévy processes. To fix the ideas, one considers the one-dimensional case and takes $\sigma = b = 0$ and $G(x, z) = z^{1/\beta(x)}$ with $\beta(\cdot)$ a lisse function taking values in $[\varepsilon, 2-\varepsilon]$. The Poisson measure N has intensity $\pi(dz) = dz/z^2$. The equation (8) becomes

$$X_t = X_0 + \int_0^t \int_{-1}^1 z^{1/\beta(X_{s-})} \tilde{N}(ds, dz). \quad (10)$$

When β is constant, X is a Lévy process. By applying Itô's formula and a variable change, one observes that the generator of this Markov process is

$$\mathcal{L}f(x) = \int_{-1}^1 [f(x+u) - f(x) - uf'(x)] \frac{\beta(x) du}{u^{1+\beta(x)}}.$$

So, the Markovian dynamic of X is no longer homogeneous in space. In particular, the evolution of the nearby future of X at each instance depends on the current position of X . This is in sharp contrast with Lévy processes. The process X has no longer stationary increments. However, the following observation is important. In a small neighborhood of a continuous time of X (say, t_0),

the index process $t \mapsto \beta(X_t)$ is nearly a constant, heuristically. So X behaves locally like

$$t > t_0, \quad \widehat{X}_t = X_{t_0} + \int_{t_0}^t \int_{-1}^1 z^{1/\beta(X_{t_0})} \widetilde{N}(ds, dz)$$

which is a Lévy process.

The difficulty is the following : there are non-countable infinite (random) values β taken by the function $t \mapsto \beta(X_t)$, and in an interval, there might be a “large” jump, such that the index process $t \mapsto \beta(X_t)$ varies dramatically.

To overcome this difficulty, we introduce a slicing technique which consists in slicing the process according to the value of the index process. More precisely, we writes for all large integer m

$$X_t = X_0 + \sum_{k=0}^{m-1} X_t^{m,k} \text{ où } X_t^{m,k} = \int_0^t \int_{-1}^1 z^{1/\beta(X_{s-})} \mathbf{1}_{\beta(X_{s-}) \in [\frac{2k}{m}, \frac{2k+2}{m}]} \widetilde{N}(ds, dz).$$

The larger is the value of m , the closer to a Lévy process is each $X^{m,k}$. The simultaneous estimates for the increments of these sliced processes makes possible an estimate for the increments of X . The Hölder exponent follows once the increments of X are well controlled.

This technique is one of the important contribution of this thesis. It is original and can be useful to study other characteristics of jump diffusions, e.g. the p -variation of these processes (see Chapter 3). We believe that it may be useful to study the local regularity of more general Markov processes with jumps.

The computation of the multifractal spectrum needs further technicalities while applying a theorem in geometric measure theory developed by Barral and Seuret [11].

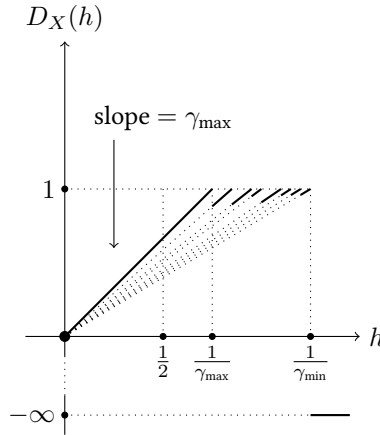
In Chapter 1, we show the multifractality of one dimensional jump diffusion processes with several conditions on the coefficients σ, b, G . In Chapter 2, we extend the results to higher dimensions, and we weaken the conditions on the coefficients and the intensity measure π to order to introduce anisotropy in the model. To simplify the presentation, we only state the result on the spectrum for the SDE (10).

Theorem. *Set*

$$\begin{aligned} \gamma(h) &:= \sup \left\{ \beta(X_s) : \beta(X_s) \leq 1/h \right\}, \\ \gamma_{\max} &:= \sup \{ \beta(X_s) : s \geq 0 \}, \\ \gamma_{\min} &:= \inf \left\{ \beta(X_s) : s \geq 0 \right\}, \\ J &:= \{ t : X_t \neq X_{t-} \}. \end{aligned}$$

With probability 1, the multifractal spectrum of X is

$$D_X(h) = \begin{cases} h \cdot \gamma(h) & \text{if } h < 1/\gamma_{\min} \text{ and } h \notin (\beta(X(J)))^{-1}, \\ -\infty & \text{if } h > 1/\gamma_{\min}. \end{cases}$$



The spectrum depends on the trajectory of X , which reflects the richness of the local regularity of jump diffusions. We refer the readers to Chapter 1-2 for more results concerning more general jump diffusions. In particular, SDEs driven by a large class of Lévy processes are considered.

Hausdorff dimension of the range and the graph

Since the 80's, there has been a particular interest to study Feller processes whose generator can be written as a pseudo-differential operator [76]. The probabilistic approach (martingale problem) allows to discover very interesting operators to which one can associate a Feller process. Richard Bass [15] introduced in 1988 a class of Feller processes that he called “stable-like processes”. The operator corresponding to these processes is written

$$\mathcal{L}^B f(x) = \int [f(x+u) - f(u) - \mathbf{1}_{|u| \leq 1} u f'(x)] \frac{C_{\beta(\cdot)} du}{|u|^{1+\beta(x)}}$$

where β is a function having certain regularity such that its range is included $[\varepsilon_0, 2 - \varepsilon_0]$, and the constant $C_{\beta(\cdot)}$ is such that one has

$$\mathcal{L}^B f(x) = \mathcal{F}^{-1} \left(|\xi|^{\beta(x)} \mathcal{F} f(\xi) \right) (x)$$

with \mathcal{F} the Fourier transform, \mathcal{F}^{-1} its inverse. Here $(\xi, x) \rightarrow |\xi|^{\beta(x)}$ is called the symbol of the operator \mathcal{L}^B . Consider a variant of this operator

$$\mathcal{L}f(x) = \int_{-1}^1 [f(x+h) - f(x) - hf'(x)] \frac{\beta(x) du}{|u|^{1+\beta(x)}}.$$

One observes that the related processes admit a SDE representation which is the jump diffusion process (10).

Schilling [105] and Schilling, Knopova and Wang [73] have found deterministic bounds for the dimension of the range. Their approach is interesting but does not give the exact formula for the dimension of the range.

The purpose of the second work of this thesis is to determine the exact dimension of the range and the graph of jump diffusion processes in the sense of Bass.

The non-homogeneity in space of the Markovian dynamic is always the major difficulty in this study. The analytic approach (Fourier analysis, functional calculus) allows to obtain uniform in space estimates of the heat kernel of the parabolic PDE associated with the operator \mathcal{L} . Equivalently, one can get uniform in space estimate for the transition probability. These estimates are not sufficient to deduce exact dimension formula of the range.

The idea to overcome this difficulty is to localize our study to where the dimension of the range is almost attained. To this end, we explore the SDE nature and pathwise uniqueness of the SDE (10) to construct a coupling of processes, solutions of SDE which coincide with (10) in short time.

Let us explain how to construct such a coupling which gives a lower bound for the dimension of the range. Set for all $a \in (0, 2)$,

$$X_t^a = X_0 + \int_0^t \int_{-1}^1 z^{\frac{1}{\beta(X_{s-}^a) \vee a}} \tilde{N}(ds, dz).$$

When $a < \beta(X_0)$, the pathwise uniqueness implies that X and X^a coincide in a non trivial random interval (say $[0, \delta]$), thus $\dim_{\mathcal{H}} X[0, 1]$ is necessarily larger than $\dim_{\mathcal{H}} X^a[0, \delta]$. But Knopova *et al.* [73] established that the dimension of range of X^a is larger than the constant a . The simultaneous construction of X^a and X implies $\dim_{\mathcal{H}} X[0, 1] \geq \beta(X_0)$ by letting a tend to $\beta(X_0)$.

To show that the obtained lower bound is actually optimal, the slicing technique introduced in the first work is applied to show that the p -variation of X is finite for an appropriate p . One concludes with a lemma by McKean [88] who makes explicit the link between the p -variation and the upper bound for the dimension of the range of a càdlàg function.

We mention that the approach developed above can also be applied to the computation of the dimension of the graph of stable-like jump diffusions.

Let us end this part by our result in arbitrary dimension. The operator of stable-like jump diffu-

sions, still denoted by X , in \mathbb{R}^d is

$$\mathcal{L}\varphi(x) = \int_{\mathbb{R}^d} [\varphi(x+u) - \varphi(x) - \mathbf{1}_{|u|\leq 1} u \cdot \nabla\varphi(x)] \frac{\beta(x)du}{|u|^{d+\beta(x)}},$$

where β is a Lipschitz continuous function with range in some compact set of $(0, 2)$.

Theorem. 1. *With probability 1,*

$$\dim_{\mathcal{H}}(X[0, 1]) = d \wedge \sup_{s \in [0, 1]} \beta(X_s).$$

2. *Two cases may occur when one considers the dimension of the graph of X .*

(a) *If $d \geq 2$, one has a.s.*

$$\dim_{\mathcal{H}}(\text{Gr}_{[0, 1]}(X)) = 1 \vee \sup_{t \in [0, 1]} \beta(X_t).$$

(b) *If $d = 1$, one has a.s.*

$$\dim_{\mathcal{H}}(\text{Gr}_{[0, 1]}(X)) = 1 \vee \left(2 - \frac{1}{\sup_{t \in [0, 1]} \beta(X_t)} \right).$$

These results indicate that, in contrast to Lévy processes, the range of Markov processes with variable-order generator (i.e. the symbol of the operator-generator is of variable order) has typically a random dimension.

Multifractality of the occupation measure of stable-like jump diffusions

In 1997, Hu et Taylor [55] performed multifractal analysis of the occupation measure of an α -stable subordinator (an increasing Lévy process).

Let X^α be an α -stable subordinator with $\alpha \in]0, 1[$ and $\mu^\alpha = \mu_1^\alpha$ be its occupation measure up to time 1 defined in (7). Hu and Taylor proved that for all $h \in [\alpha, 2\alpha]$, a.s.

$$\bar{d}_{\mu^\alpha}(h) = \begin{cases} \alpha(\frac{2\alpha}{h} - 1) & \text{if } h \in [\alpha, 2\alpha], \\ -\infty & \text{otherwise.} \end{cases}$$

The purpose of the third work in this thesis is to study the multifractal nature of the occupation measure of *increasing* stable-like jump diffusions, i.e. solution to

$$X_t = \int_0^t \int_0^1 z^{1/\beta(X_{s-})} N(ds, dz).$$

One considers μ its occupation measure up to time 1. To compute the dimension of the set of singularities of μ , one has to construct a family of Cantor sets, each point of which has upper local dimension h , simultaneously for all h . The approach of Hu and Taylor can not be applied to our cases for at least three reasons:

1. In [55], the quantifier “a.s.” placed after “ $\forall h$ ” indicates that the full probability events depend on the index h which takes its values in a continuum (the non trivial interval $[\alpha, 2\alpha]$). We intend to obtain almost sure result for all h at the same time, in other words, interchange “ $\forall h$ ” and “a.s.”.
2. in [55], instead of work directly with the “iso”-Hölder sets, they consider the set of points with upper local dimension larger than h

$$\{x : \overline{\dim}(\mu, x) \geq h\}.$$

They estimate then the Hausdorff measure of these sets, which implies dimension estimate of the “iso”-Hölder sets. In our case, the non-homogeneity of the dynamic in space imposes us to work directly with the “iso”-Hölder sets

$$\{x : \overline{\dim}(\mu, x) = h\}.$$

3. in [55], the proof for the computation of the spectrum is realized in two steps. First, Hu et Taylor construct the set of times t , each of which satisfies $\overline{\dim}(\mu, X_t^\alpha) \geq h$ with $h \in [\alpha, 2\alpha]$. Second, by applying the classic uniform dimension result for the range of stable processes (this result states that a.s. for all set in $[0, 1]$, the dimension of its range by stable processes is the stability parameter multiplied by the dimension of the considered set, see [24]), they have automatically the space spectrum. For stable-like jump diffusions, the second step is not automatic, one needs a similar uniform dimension result.

The two first points concern the construction of Cantor sets. The key idea is to construct in such a way that one captures the heterogeneity in time of stable-like jump diffusions. To this end, we identify the “zero jump” and “double jump” configurations in the Poisson point process and use them with different weight according to the time scale, space scale and the position of the process. Fine estimates for the Poisson point process are necessary. This well-localized and

heterogeneous construction might be useful to study the occupation measure of other Markov processes with jumps.

Here we only present the result on the multifractal spectrum in space. For more results, e.g. time spectrum, spectrum related to the lower local dimension, uniform dimension theorem, see Chapter 4.

Theorem. *Set*

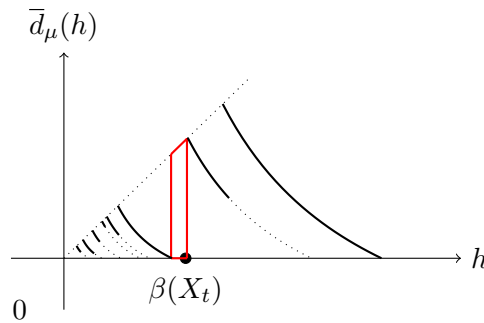
$$\mathcal{E} = \{\beta(X_t) : X_{t-} \neq X_t \text{ et } \beta(X_t) \geq 2\beta(X_{t-})\} \cup \{2\beta(X_{t-}) : X_{t-} \neq X_t \text{ et } \beta(X_t) \geq 2\beta(X_{t-})\}$$

and

$$g_\alpha(h) := \begin{cases} \alpha \left(\frac{2\alpha}{h} - 1 \right) & \text{if } h \in [\alpha, 2\alpha), \\ -\infty & \text{otherwise.} \end{cases}$$

The multifractal spectrum of the occupation measure of increasing stable-like jump diffusionss is a.s. for all $h \notin \mathcal{E}$,

$$\bar{d}_\mu(h) = \sup \{g_\alpha(h) : \alpha \in \{\beta(X_t) : t \in \mathbb{R}^+\}\}.$$



Observe that $\mathcal{E} = \emptyset$ if, for instance, the range of $\beta(\cdot)$ is included in $[1/2, 1 - \varepsilon_0]$. We refer the readers to Chapter 4 for the spectrum at each exceptional value $h \in \mathcal{E}$.

Sojourn of Brownian motion inside moving boundaries

I am recently interested in sojourn properties of the Brownian motion, which is very important for applications in finance and in physics. Theoretically, the set of Brownian sojourn with respect

to a moving boundary provides much information on the Brownian paths. More precisely, let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an increasing function, consider the sets

$$\begin{aligned} E^o(\varphi) &= \{t \geq 0 : |B_t| \geq \varphi(t)\}, \\ E^i(\varphi) &= \{t \geq 0 : |B_t| \leq \varphi(t)\}, \end{aligned}$$

where the first is called the Brownian sojourn outside a moving boundary, the second inside a boundary. Let us mention two types of study with particular interests.

- Geometric properties of $E^i(\varphi)$ and $E^o(\varphi)$ around $t_0 = 0$. The set $E^o(\varphi)$ provides a description of the irregularity of Brownian motion. Indeed, if 0 is a point of accumulation of $E^o(\varphi)$, then the local modulus of continuity of Brownian motion can not be larger than φ . On the other side, $E^i(\varphi)$ corresponds to a more regular behavior.

- Asymptotic properties of these sets at infinity. This corresponds to the long time behavior of Brownian motion. By self-similarity, the growth rate of the Brownian motion is on average a square root function, which is considered as a reference function for typical Brownian growth rate. Concretely,

1. the set $E^o(\varphi)$, when φ grows quicker than the square root, concerns the high peaks of the Brownian motion. For example, the iterated logarithm law of Strassen [109] implies that the upper density at infinity of $E^o(\varphi_a)$ with $\varphi_a(t) = a(2t \ln \ln(t))^{1/2}$ are non trivial, for all $a \in [0, 1)$. Let us mention that Khoshnevisan, Kim et Xiao [71] have proved recently that the set of Brownian high peaks in the critical case, i.e. the set $E^o(\varphi_1)$, has macroscopic Hausdorff dimension one, even if it has zero upper density.
2. the set $E^i(\varphi)$, when φ grows slower than the square root, concerns the smaller than normal growth rate of the Brownian motion. Uchiyama [113] established bounds for the upper density at infinity of $E^i(\varphi)$ where $\varphi(t) = \sqrt{t}/h(t)$ with h taken in a class of functions with at most logarithmic growth rate.

The purpose of the last work in this thesis is to determine the macroscopic dimension of the Brownian sojourn inside moving boundaries with growth rate much smaller than the square root function. By much smaller, we mean the sets

$$\forall \gamma \in [0, 1/2], \quad E(\gamma) = \{t \geq 0 : |B| \leq t^\gamma\}.$$

The methods of computation of the macroscopic Hausdorff dimension are comparable to those used to compute the classic Hausdorff dimension.

- To bound from above the macroscopic Hausdorff dimension, one needs to find an economic covering.

- To bound from below the dimension, one might choose a measure supported by the involved set, and verifies the scaling condition for this measure stated in Lemma 2.

Let us present our results on the macroscopic Hausdorff dimension and the upper mass dimension for the sets $E(\gamma)$.

Theorem. *With probability one,*

$$\begin{aligned} \text{Dim}_{\text{H}}E(\gamma) &= \begin{cases} \frac{1}{2} & \text{if } \gamma \in [0, 1/2), \\ 1 & \text{if } \gamma = 1/2. \end{cases} \\ \text{Dim}_{\text{UM}}E(\gamma) &= \frac{1}{2} + \gamma \quad \text{for all } \gamma \in [0, 1/2]. \end{aligned}$$

It is remarkable that the macroscopic Hausdorff dimension of $E(\gamma)$ is a constant for all $\gamma \in [0, 1/2)$ and there is a jump at $1/2$. An important ingredient in the proof is the estimate of the “hitting probability” which consists in estimating the probability that the Brownian motion hits the region inside a moving boundary during any fixed time interval. It is interesting to see that Brownian sojourn sets, measured by upper mass dimension, form somehow a large scale multifractal.

To derive the upper bound, the first problem concerns the interpretation of an economic covering in the framework of the macroscopic dimension for the Brownian sojourn. The best covering for our sets $E(\gamma)$ is neither the largest (\mathcal{S}_n for each annulus \mathcal{S}_n), nor the smallest (unit interval inside each \mathcal{S}_n). Let us localize ourselves to a Brownian zero point. Imagine that it is a point in the annulus \mathcal{S}_n . It is known that the exit time for a Brownian motion starting from zero

$$\sigma_a = \inf\{t \geq 0 : |B_t| \geq a\}$$

satisfies $\mathbb{E}[\sigma_a] = a^2$. Hence, to escape from the boundary $2^{n\gamma}$, the Brownian motion needs in average $2^{2n\gamma}$ unit of times. By taking non overlapping intervals of length $2^{2n\gamma}$ as a covering of $E(\gamma) \cap \mathcal{S}_n$, one can find the optimal upper bound for the macroscopic Hausdorff dimension of $E(\gamma)$, for all $\gamma \in [0, 1/2)$.

To deduce a lower bound, one applies the fact that the zero set of the Brownian motion

$$\mathcal{Z} = \{t : B_t = 0\}$$

(which is included in all the $E(\gamma)$) coincides with the range of a $1/2$ -stable subordinator σ

$$\mathcal{R} = \sigma[0, +\infty[.$$

Thus, it suffices to bound from below $\text{Dim}_{\text{H}}\mathcal{R}$ by considering the occupation measure of σ and verifies the scaling property in Lemma 2 for this measure. Let us mention that the microscopic

(or local) properties of \mathcal{Z} is well known thanks to the work of Taylor-Wendel [112] and that of Perkins [97] who considered the classical Hausdorff measure of \mathcal{Z} by using Brownian local times. In particular, $\dim_{\mathcal{H}} \mathcal{Z} = 1/2$. We show that the macroscopic Hausdorff dimension of \mathcal{Z} is $1/2$ as well. This should be compared with a result of Khoshnevisan [67] who proved that the set of passage time to zero of a random walk ξ with finite variance in \mathbb{Z}

$$\{n \in \mathbb{N} : \xi_n = 0\}$$

has macroscopic Hausdorff dimension $1/2$.

The computation of $\text{Dim}_{\text{UM}} E(\gamma)$ is based on an estimate of certain type of hitting probability, and a sequence of well chosen time scales in order to observe the desired sojourn behavior for the Brownian motion. The first implies the upper bound, and the second the lower bound.

Perspectives

There are many interesting questions that I want to investigate, let us list some of them.

- Since the seminal work by Lamperti [78], many path properties of positive self-similar Markov processes (pssMp) have been discovered by the (pathwise) Lamperti transform which establishes a link between a pssMp and certain underlying Lévy process. My purpose is to understand the role of the Lamperti transform to the local regularity of Lévy processes in order to perform multifractal analysis of pssMp.
- The occupation measure considered in Chapter 4 is singular with respect to the Lebesgue measure. Bass [14] showed the existence of the occupation density of stable-like jump diffusions under the condition that the range of $\beta(\cdot)$ is included in $[1 + \varepsilon, 2 - \varepsilon]$. It will be of great interest to study the local regularity of this occupation density.
- The macroscopic dimension seems like a promising tool to study various type of fluctuation of stochastic processes. I intend to work on this subject.

Chapitre 1

Multifractalité de diffusion à sauts

On étudie la régularité locale et la nature multifractale des trajectoires d'une classe de diffusion à sauts, solutions d'une classe d'EDS à sauts. Cette classe contient les EDS guidées par un processus de Lévy et les processus de type stable au sens de Bass. On montre que \mathcal{M} a des propriétés multifractales remarquables, et que son spectre multifractal est non seulement aléatoire, mais aussi dépend du temps. On explique, notamment en calculant ses processus tangents, pourquoi les diffusion à sauts comportent localement comme un processus de Lévy. De nouvelles techniques sont introduites pour étudier ces processus.

Ce chapitre est basé sur l'article [120].

1.1 Introduction

Multifractal properties are now identified as important features of sample paths of stochastic processes. The variation of the regularity of random measures and processes has been observed considerably since mid-70's, e.g. fast and slow points of (fractional) Brownian motion [68, 95, 98], Mandelbrot's cascades [65], (multifractal time changed) Lévy processes [6, 10, 43, 44, 61], superprocesses [92, 100], among many other examples. Multifractal analysis turns out to be a relevant approach to provide organized information about the distribution of singularities and to describe the roughness of the object under consideration.

The regularity exponent we consider is the pointwise Hölder exponent. Let us recall some relevant definitions in this context.

Definition 1.1. *Let $f \in L_{loc}^\infty(\mathbb{R})$, $x_0 \in \mathbb{R}$, $h \in \mathbb{R}^+ \setminus \mathbb{N}^*$. The function f belongs to $C^h(x_0)$ if there exist two positive constants $C, M > 0$, a polynomial P with degree less than h , such that when $|x - x_0| < M$,*

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^h.$$

The pointwise Hölder exponent of f at x_0 is

$$H_f(x_0) = \sup\{h \geq 0 : f \in C^h(x_0)\}.$$

We aim at describing the distribution of the singularities of a function, via the computation of its multifractal spectrum. $\dim_{\mathcal{H}}$ stands for the Hausdorff dimension and by convention $\dim_{\mathcal{H}} \emptyset = -\infty$, see [48] for more on dimensions.

Definition 1.2. *Let $f \in L_{loc}^\infty(\mathbb{R})$. For $h \geq 0$, the iso-Hölder set is*

$$E_f(h) = \{x \in \mathbb{R} : H_f(x) = h\}$$

and the multifractal spectrum of f is the mapping $D_f : \mathbb{R}^+ \rightarrow [0, 1] \cup \{-\infty\}$ defined by

$$h \longmapsto D_f(h) = \dim_{\mathcal{H}} E_f(h).$$

We also define, for any open set $A \subset \mathbb{R}^+$, the local spectrum of f on A as

$$D_f(A, h) = \dim_{\mathcal{H}}(A \cap E_f(h)). \tag{1.1}$$

The aforementioned stochastic processes have *homogeneous* multifractal spectra, meaning that there is no dependency on the region where the spectra are computed: $D_f(\mathbb{R}^+, h) = D_f(A, h)$, for all open sets $A \in \mathbb{R}^+$. Recently, Barral, Fournier, Jaffard and Seuret [12] investigated a specific example which has multifractal characteristics that change as time passes. So, we also focus in the pointwise multifractal spectrum at a given point.

Definition 1.3. Let $f \in L_{loc}^\infty(\mathbb{R})$, $t_0 \in \mathbb{R}^+$, and let $I_n(t_0) = [t_0 - 1/n, t_0 + 1/n]$ for every $n \geq 1$. The pointwise multifractal spectrum of f at t_0 is the mapping defined by

$$\forall h \geq 0, \quad D_f(t_0, h) = \lim_{n \rightarrow +\infty} D_f(I_n, h).$$

The local spectrum $D_f(A, h)$ on any open set A can be completely recovered from the pointwise spectrum $D_f(t_0, h)$, for $t_0 \in A$, as stated by next Proposition. Hence the pointwise multifractal spectrum results are finer than the multifractal spectrum results on an interval.

Proposition 1.1 (Proposition 2, [13]). Let $f \in L_{loc}^\infty(\mathbb{R})$. Then for any open interval $I = (a, b) \subset \mathbb{R}_+$, for any $h \geq 0$, we have $D_f(I, h) = \sup_{t \in I} D_f(t, h)$. Consequently, the mapping $t \mapsto D_f(t, h)$ is upper semi-continuous.

The multifractal analysis of a Lévy process was performed by Jaffard [61].

Theorem 1.1 ([61]). Let $(\mathcal{L}_t)_{t \geq 0}$ be a Lévy process of upper index $\beta \in (0, 2)$, with a non-zero Brownian component. Almost surely, at every $t > 0$, the sample path of \mathcal{L} has the (deterministic) pointwise spectrum

$$D_{\mathcal{L}}(t_0, h) = D_{\mathcal{L}}(h) = \begin{cases} \beta h & \text{if } h \in [0, 1/2), \\ 1 & \text{if } h = 1/2, \\ -\infty & \text{if } h > 1/2. \end{cases}$$

In particular, Lévy processes are homogeneously multifractal.

Examples of stochastic process with varying pointwise spectrum were given in [6, 12, 42]. Here we deal with a general class of Markov processes, \mathcal{M} , defined below by Equation (1.2), called the jump diffusions. The motivation for the study of these processes is twofold. First, real life data, particularly in finance, exhibit often jump activities and their characteristics change as time passes [1, 35]. A method widely used for model selection consist in computing global quantities of the signal, which are closely related to the multifractal spectrum (the so-called multifractal formalism). Second, Çinlar and Jacod [34] showed that any one-dimensional Markov semimartingale has, up to a time change, a SDE representation as (1.2). This large class of processes includes the stable-like processes constructed by Bass [15] (see also [93]) and stable Lévy-driven SDE recently studied by Fu-Li [51], Li-Mytnik [83], Fournier [50]. Our main theorems establish the multifractal properties of these processes.

We work in the probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions. The process \mathcal{M} is defined as the solution to the one-dimensional stochastic differential equation with jumps:

$$\mathcal{M}_t = \int_0^t \sigma(\mathcal{M}_{s-}) dB_s + \int_0^t b(\mathcal{M}_s) ds + \int_0^t \int_{C(0,1)} G(\mathcal{M}_{s-}, z) \tilde{N}(ds dz), \quad (1.2)$$

where:

- $\forall 0 \leq a < b$, $C(a, b)$ is the annulus $C(a, b) = [-b, -a] \cup [a, b] \setminus \{0\}$,
- $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ are Lipschitz continuous functions,
- either $|\sigma|$ is bounded below away from 0, or $\sigma \equiv 0$,
- B is a $(\mathfrak{F}_t)_{t \geq 0}$ -standard Brownian motion,
- \tilde{N} is a compensated Poisson random measure of intensity $dt \otimes \pi(dz)$, with $\pi(dz) = \mathbf{1}_{C(0,1)}(z)dz/z^2$.

The condition we impose on the function G is that it belongs to the following class of admissible functions.

Definition 1.4. *The set \mathcal{G} is the set of those functions $G : \mathbb{R} \times C(0, 1) \rightarrow [-1, 1]$ satisfying :*

1. *Symmetry* : $\forall (x, y) \in \mathbb{R}^2, \forall z \in C(0, 1)$,

$$G(x, z) = \text{sign}(z)|G(x, |z|)| \text{ and } G(x, z)G(y, z) > 0.$$

2. *Asymptotically stable-like* : $\forall x \in \mathbb{R}$,

$$\liminf_{z \rightarrow 0} \frac{\log |G(x, z)|}{\log |z|} \text{ exists and denoted by } \frac{1}{\tilde{\beta}(x)}.$$

3. *Lipschitz condition* : there exists $C > 0$ such that $\forall (x, y) \in \mathbb{R}^2, \forall z \in C(0, 1)$,

$$\left| \frac{\log |G(y, z)| - \log |G(x, z)|}{\log |z|} \right| \leq C|x - y|.$$

4. *Boundedness* : $\overline{\text{Range } \tilde{\beta}} \subset (0, 2)$ and for all $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for every $z \in C(0, r_\varepsilon)$ and every x ,

$$|G(x, z)| \leq |z|^{\frac{1}{\tilde{\beta}(x)+\varepsilon}}.$$

The reader should keep in mind that when $G \in \mathcal{G}$, one has intuitively

$$G(x, z) \text{ "}\sim\text{" } |z|^{1/\tilde{\beta}(x)} \text{ as } z \rightarrow 0,$$

for some function $\tilde{\beta}$ which ranges in $[\varepsilon, 2 - \varepsilon]$ for some $\varepsilon > 0$. This class includes stable-like processes and the solutions to the Lévy stable-driven SDE given by

$$\mathcal{M}_t = \int_0^t \int_{C(0,1)} g(\mathcal{M}_{s-}) dL_s^\alpha$$

with L^α an α -stable Lévy process and g a reasonable function (see Section 1.2.1 for details).

We introduce the following notation. For the rest of the paper, we set

$$t \in \mathbb{R}^+ \mapsto \beta(t) = \tilde{\beta}(\mathcal{M}(t)).$$

The quantity $\beta(t)$ is key: it shall be understood as *the local Blumenthal-Gettoor index of \mathcal{M} at time t* , and governs the local behavior of \mathcal{M} at t .

We state now the multifractal properties of \mathcal{M} . When the diffusion part does not vanish, the pointwise multifractal spectrum of \mathcal{M} takes a simple form, which is the main result of this paper.

Theorem 1.2. *Assume that $G \in \mathcal{G}$ and $\sigma \neq 0$. Then, with probability one, for every $t \in \mathbb{R}^+$, the pointwise multifractal spectrum of \mathcal{M} at t is*

$$D_{\mathcal{M}}(t, h) = \begin{cases} h \cdot (\beta(t) \vee \beta(t-)) & \text{if } h < 1/2, \\ 1 & \text{if } h = 1/2, \\ -\infty & \text{if } h > 1/2. \end{cases}$$

In particular, if t is a continuous time for \mathcal{M} , the formula reduces to $D_{\mathcal{M}}(t, h) = h \cdot \beta(t)$ when $h < 1/2$.

We prove this result in Section 1.7. The case when the diffusion part vanishes will also be entirely treated in Theorems 1.5 and 1.6 of Section 1.7. It is more complicated to state, since many cases must be distinguished according to various relationships between t , \mathcal{M}_t and β . The presence of the Brownian component eliminates these difficulties.

Observe that the pointwise spectrum is linear up to the exponent $h = 1/2$. Recalling Theorem 1.1, Corollary 1.1 implies that the multifractal spectrum of \mathcal{M} looks like that of a Lévy process, except that the slope of the linear part of the spectrum is random and depends on the interval on which we compute the spectrum. This remarkable property reflects the fact that the “local Blumenthal-Gettoor” index of a jump diffusion \mathcal{M} depends on time.

From the pointwise spectrum of \mathcal{M} we deduce its local spectrum.

Corollary 1.1. *Assume that $G \in \mathcal{G}$, $\sigma \neq 0$. For $I = (a, b) \subset \mathbb{R}^+$, let*

$$\gamma_I := \sup \{ \beta(s) : s \in I \}.$$

With probability one, the local multifractal spectrum of \mathcal{M} on I is

$$D_{\mathcal{M}}(I, h) = \begin{cases} h \cdot \gamma_I & \text{if } h < 1/2, \\ 1 & \text{if } h = 1/2, \\ -\infty & \text{if } h > 1/2. \end{cases}$$

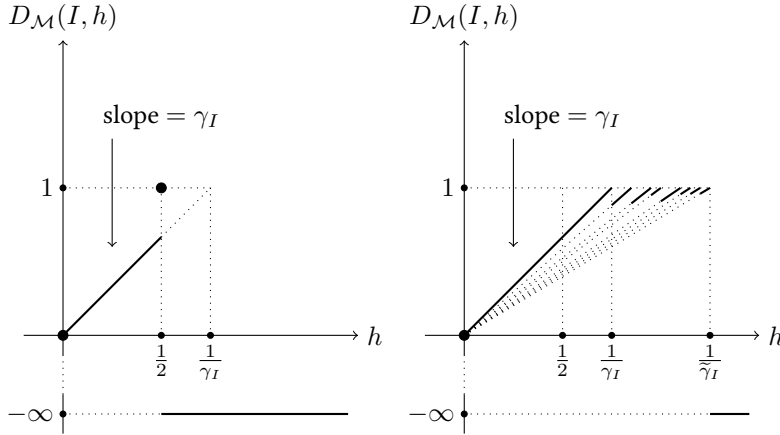


Figure 1.1: Local multifractal spectrum $D_{\mathcal{M}}(I, \cdot)$ of \mathcal{M} in the interval I when $\sigma \neq 0$ (left) and $\sigma \equiv 0$ (right). The right figure is a representation, since there is a countable number of small affine parts. When $\sigma \neq 0$, the diffusion "hides" the complicated right part of $D_{\mathcal{M}}(I, \cdot)$.

We can also give the statement for the local multifractal spectrum when the diffusion vanishes (this is a corollary of Theorems 1.5 and 1.6).

Corollary 1.2. *Assume that $G \in \mathcal{G}$, $\sigma \equiv 0$ and (\mathcal{H}) holds (see Theorem 1.3 below for the definition of (\mathcal{H})). Let $I = (a, b) \subset \mathbb{R}^+$ and*

$$\begin{aligned}\gamma_I(h) &:= \sup \left\{ \beta(s) : s \in I, \beta(s) \leq 1/h \right\}, \\ \tilde{\gamma}_I &:= \inf \left\{ \beta(s) : s \in I \right\}.\end{aligned}$$

With probability one, the local multifractal spectrum of \mathcal{M} on I is

$$D_{\mathcal{M}}(I, h) = \begin{cases} h \cdot \gamma_I(h) & \text{if } h < 1/\tilde{\gamma}_I \text{ and } h \notin (\beta(J))^{-1}, \\ -\infty & \text{if } h > 1/\tilde{\gamma}_I. \end{cases}$$

Both corollaries are consequences of Proposition 1.1. The difference between the corollaries follows from the fact that the diffusion has regularity 1/2 at every point, so the complicated part of the multifractal spectrum ($h > 1/2$) in Corollary 1.2 disappears (see Figure ??).

Observe that we do not give the value of the spectrum on the countable set $(\beta(J))^{-1}$. This is due to the occurrence of various delicate situations depending on the trajectory of \mathcal{M} , which are described in Section 1.7.

In order to compute $D_{\mathcal{M}}$, we have to investigate the pointwise exponent of the process \mathcal{M} at every point. To state our main result in this direction, let us introduce the Poisson point process \mathcal{P} associated with the process \mathcal{M} and the notion of the approximation rate by \mathcal{P} .

The Poisson random measure N can be derived from a Lévy process \mathcal{L} with characteristic triplet

$(0, 0, \pi(dz))$, meaning that there are no Brownian component and no drift. For all $s, t \in \mathbb{R}^+$ and every Borel set $A \in \mathcal{B}(C(0, 1))$, denote $N([s, t], A) = \#\{u \in [s, t] : \Delta\mathcal{L}_u \in A\}$. Then, almost surely, N is a Poisson random measure with intensity $dt \otimes \pi(dz)$. Let

$$\mathcal{P} = (T_n, Z_n)_{n \geq 0}. \quad (1.3)$$

denote the sequence of jump times and jump sizes of \mathcal{L} . Then almost surely,

$$N(dt, dz) = \sum_{n \geq 1} \delta_{(T_n, Z_n)}(dt, dz).$$

We can assume that $(|Z_n|)_{n \in \mathbb{N}}$ forms a decreasing sequence by rearrangement. Let $J := \{t \in \mathbb{R}^+ : \Delta\mathcal{M}_t \neq 0\}$ be the set of locations of the jumps, where $\Delta\mathcal{M}_t := \mathcal{M}_t - \mathcal{M}_{t-}$. Using the property of Poisson integral,

$$J = \{T_n : n \in \mathbb{N}\}$$

and for every $n \in \mathbb{N}$,

$$\Delta\mathcal{M}_{T_n} = G(\mathcal{M}_{T_n-}, Z_n).$$

See for instance Section 2.3 of [3] for details.

The approximation rate δ_t by \mathcal{P} describes how close to the jump points T_n a point t is. Intuitively, the larger δ_t is, the closer to large jumps t is.

Definition 1.5. *The approximation rate of $t \in \mathbb{R}^+$ by \mathcal{P} is defined by*

$$\delta_t := \sup\{\delta \geq 1 : |T_n - t| \leq (Z_n)^\delta \text{ for infinitely many } n\}.$$

This random approximation rate plays a key role when investigating the pointwise regularity of \mathcal{M} , as stated by the following result.

Theorem 1.3. *Let $G \in \mathcal{G}$ be an admissible function.*

1. *Assume that $\sigma \neq 0$. Then almost surely*

$$\forall t \notin J, \quad H_{\mathcal{M}}(t) = \frac{1}{\delta_t \beta(t)} \wedge \frac{1}{2}.$$

2. *Assume that $\sigma \equiv 0$ and that the following assumption (\mathcal{H}) holds:*

- (a) *either $b \in C^\infty(\mathbb{R})$ and $\overline{\text{Range } \tilde{\beta}} \subset [1, 2)$.*
- (b) *or $b \in C^\infty(\mathbb{R})$ and $x \mapsto \tilde{b}(x) := \int_0^1 G(x, z) \frac{dz}{z^2} \in C^\infty(\mathbb{R})$.*

Then, almost surely, $\forall t \notin J, H_{\mathcal{M}}(t) = \frac{1}{\delta_t \beta(t)}$.

This theorem is proved in Section 1.5 and 1.6.

Observe that the terms in the right hand side of (1.2) yield a semimartingale decomposition of the process \mathcal{M} , see [31]. To simplify notations, we write

$$\mathcal{M}_t = \mathcal{X}_t + \mathcal{Y}_t + \mathcal{Z}_t,$$

where:

- $\mathcal{X}_t = \int_0^t \sigma(\mathcal{M}_{s-}) dB_s$ is the diffusion term.
- $\mathcal{Y}_t = \int_0^t b(\mathcal{M}_s) ds$ is the drift term.
- $\mathcal{Z}_t = \int_0^t \int_{C(0,1)} G(\mathcal{M}_{s-}, z) \tilde{N}(dsdz)$ is the jump term.

Let us make additional remarks on our main results:

- The drift part of a jump diffusion is not as simple as that of a Lévy process, which is linear hence belongs to $C^\infty(\mathbb{R})$. The regularity of the drift depends on that of \mathcal{M} which varies along time. However, this has no consequence on the statement of Theorem 1.3 because we are able to prove that the drift is always more regular than \mathcal{M} (see Section 1.5).
- The sum of the diffusion \mathcal{X}_t and the jump term \mathcal{Z}_t has, almost surely, everywhere a pointwise exponent less than $1/2$. Indeed, \mathcal{X} has an exponent everywhere equal to $1/2$ (see Proposition 1.5). When the pointwise exponent of \mathcal{Z}_t is not equal to $1/2$, we use the fact $H_{\mathcal{X}+\mathcal{Z}}(t) \geq H_{\mathcal{X}}(t) \wedge H_{\mathcal{Z}}(t)$ with equality if $H_{\mathcal{X}}(t) \neq H_{\mathcal{Z}}(t)$. When $H_{\mathcal{X}}(t) = H_{\mathcal{Z}}(t)$, equality is not true in general. But it does hold in our context, because the irregularity generated by jump discontinuities cannot be compensated by a continuous term (see Section 1.6 for details).
- One key argument in our proof consists in constructing simultaneously with \mathcal{M} a family of martingales $(\mathcal{P}^j)_{j \geq 1}$, whose local index does not vary much, such that $\mathcal{Z} = \sum_j \mathcal{P}^j$ plus a process with only large jumps whose regularity can be controlled. The increments of the \mathcal{P}^j are easier to control, and using this decomposition we are able to estimate the values of the increments of \mathcal{M} on all dyadic intervals, see Proposition 1.8.

This paper is organized as follows. In next Section, first properties of the process \mathcal{M} are given. In Section 3, we determine the pointwise exponent of the diffusion term. In Section 4, we deal with the pointwise regularity of the jump term. We prove Theorem 1.3 in Sections 1.5-1.6 and we compute the local spectrum of \mathcal{M} in different situations (Theorems 1.2, 1.5 and 1.6) in Section 7.

We show the existence of tangent processes of the process \mathcal{M} in some special cases in Section 8. Appendices 1 and 2 contain auxiliary results.

Let us mention some extensions. The symmetry assumption in \mathcal{G} can certainly be relaxed. One can also replace the regular intensity measure dz/z^2 by singular ones, under the condition that the associated Poisson point process satisfies some good covering properties. Multidimensional versions of the main theorems can be obtained with extra technicalities. Other dimensional properties of stochastic processes, such as dimensions of the range, of the graph of \mathcal{M} , are important mathematical properties with application in physics for modeling purposes, and are investigated in [121]. More generally, the present article opens the way to a systematic multifractal study of other classes of variable-order Markov processes (in the sense that the symbol of the generator of the process is of variable order, see [31]).

Let us end this introduction with the following open question. Certain classes of Markov processes having a SDE representation are not covered by our main theorem due to the presence of the variable jump rate. This is the case of continuous state branching processes [51] and of positive self-similar Markov processes [41]. It would be very interesting to determine their multifractal structure.

1.2 Properties of \mathcal{M}

We comment the assumptions on \mathcal{M} , then show the existence of a unique pathwise solution to (1.2), and that the jump part of the process is a L^2 -martingale. We also compute its generator.

1.2.1 Comments

Let us comment the assumptions made on the parameters of the problem, especially on the set \mathcal{G} given by Definition 2.3.

The symmetry (part 1.) facilitates some computations and statement of the results. Looked at from a generator point of view (see Proposition 4.13 and the discussion below it), Part 2. is a natural extension of the assumption that \mathcal{M} is a stable-like process. Parts 3. and 4. in Definition 2.3 ensure the existence of \mathcal{M} , and enable us to perform multifractal analysis.

The following specific choice of $G \in \mathcal{G}$ shows that the class \mathcal{G} is general enough to include interesting examples. We consider the pure jump equations, i.e. $\sigma = b \equiv 0$. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be smooth and bounded below from 0, $\tilde{\beta} : \mathbb{R} \rightarrow \mathbb{R}$ be smooth with its range included in some compact subset of $(0, 2)$. We take

$$G(x, z) = g(x)z^{\frac{1}{\tilde{\beta}(x)}}.$$

If $\tilde{\beta} \equiv \alpha \in (0, 2)$, \mathcal{M} is the solution to the Lévy stable-driven SDE

$$\mathcal{M}_t = \int_0^t \int_{C(0,1)} g(\mathcal{M}_{s-}) dL_s^\alpha \quad (1.4)$$

with L^α an α -stable Lévy process. Indeed, we remark that the image measure of dz/z^2 by the mapping $z \mapsto z^{1/\alpha}$ is $const \cdot dz/z^{1+\alpha}$. Our main theorem establishes that (1.4) has the same multifractal nature as an α -stable Lévy process, under the condition that g is also bounded from above. By a classical localization argument (see for instance Proposition 2.4 in [51]), one can easily extend the result to unbounded coefficient g . If g is constant, then \mathcal{M} is a stable-like process in the sense of Bass [15]. We clarify this point in Section 1.2.3.

It is worth mentioning that the intensity measure can be any stable Lévy measure, i.e. $|z|^{-1-\alpha} dz$ with $\alpha \in (0, 2)$ (in this case, the definition of the set \mathcal{G} must be adapted). With a suitable change of measure for the Poisson integral, it suffices to consider our specific Lévy measure $\pi(dz) = dz/z^2$.

1.2.2 Basic properties

Proposition 1.2. *The SDE (1.2) has a unique pathwise solution.*

By the classical Picard iteration procedure (see e.g. [3] Theorem 6.2.3), it is enough to check that there are $K_0, K_1 < +\infty$ such that:

(i) Growth condition : $\forall x \in \mathbb{R}$,

$$|\sigma(x)|^2 + |b(x)|^2 + \int_{C(0,1)} |G(x, z)|^2 \pi(dz) \leq K_0(1 + x^2).$$

(ii) Lipschitz condition : $\forall (x, y) \in \mathbb{R}^2$,

$$|\sigma(x) - \sigma(y)|^2 + |b(x) - b(y)|^2 + \int_{C(0,1)} |G(x, z) - G(y, z)|^2 \pi(dz) \leq K_1|x - y|^2.$$

We check both conditions for every $G \in \mathcal{G}$ in Appendix 1.

Proposition 1.3. *The jump term \mathcal{Z} is a martingale and there is a strictly positive sequence $(\alpha_n)_{n \geq 1}$ with $\alpha_n \rightarrow_{n \rightarrow +\infty} 0$ such that a.s.,*

$$\mathcal{Z}_t = \lim_{n \rightarrow +\infty} \int_0^t \int_{C(\alpha_n, 1)} G(\mathcal{M}_{s-}, z) \tilde{N}(dsdz),$$

where the convergence is uniform on $[0, 1]$.

Proof. By a classical argument using uniform convergence on every compact set in probability

of a sequence of L^2 -martingale, it is enough to show the following (see e.g. Theorem 4.2.3 of Applebaum [3])

$$\forall t > 0, \quad \int_0^t \int_{C(0,1)} \mathbb{E}[G(\mathcal{M}_{s-}, z)^2] ds \pi(dz) < +\infty.$$

Observe that the fourth property of the functions in \mathcal{G} gives that, for any small $\varepsilon_0 > 0$, there exists $z_0 \in (0, 1]$ such that

$$\forall x, \forall z \in C(0, z_0), \quad |G(x, z)| \leq |z|^{\frac{1}{2} + \varepsilon_0}. \quad (1.5)$$

Using (1.5) and the uniform boundedness of G , one obtains that for all $t > 0$,

$$\begin{aligned} & \int_0^t \int_{C(0,1)} \mathbb{E}[G(\mathcal{M}_{s-}, z)^2] ds \pi(dz) \\ &= \int_0^t \int_{C(0, z_0)} \mathbb{E}[G(\mathcal{M}_{s-}, z)^2] ds \pi(dz) + \int_0^t \int_{C(z_0, 1)} \mathbb{E}[G(\mathcal{M}_{s-}, z)^2] ds \pi(dz) \\ &\leq \int_0^t \int_{C(0, z_0)} (|z|^{\frac{1}{2} + \varepsilon_0})^2 ds \frac{dz}{z^2} + t \int_{C(z_0, 1)} \frac{dz}{z^2} \leq t \left(\frac{z_0^{2\varepsilon_0}}{\varepsilon_0} + \frac{2}{z_0} - 2 \right) < +\infty. \end{aligned}$$

□

Proposition 1.4. *The generator of the process \mathcal{M} is*

$$\begin{aligned} \mathfrak{L}f(x) &:= b(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) \\ &+ \int_{C(0,1)} [f(x + G(x, z)) - f(x) - G(x, z)f'(x)] \frac{dz}{z^2}. \end{aligned}$$

Proof. Let us denote by $t \mapsto \langle X \rangle_t$ the quadratic variation process of X . We compute the generator through Itô's formula (for semimartingales). For every twice continuously differentiable function f with compact support,

$$\begin{aligned} & f(\mathcal{M}_t) - f(0) \\ &= \int_0^t f'(\mathcal{M}_{s-})\sigma(\mathcal{M}_{s-}) dB_s + \int_0^t f'(\mathcal{M}_s)b(\mathcal{M}_s) ds + \frac{1}{2} \int_0^t f''(\mathcal{M}_s)\sigma^2(\mathcal{M}_s) d\langle B \rangle_s \\ &+ \int_0^t \int_{C(0,1)} [f(\mathcal{M}_{s-} + G(\mathcal{M}_{s-}, z)) - f(\mathcal{M}_{s-})] \tilde{N}(dsdz) \\ &+ \int_0^t \int_{C(0,1)} [f(\mathcal{M}_{s-} + G(\mathcal{M}_{s-}, z)) - f(\mathcal{M}_{s-}) - G(\mathcal{M}_{s-}, z)f'(\mathcal{M}_{s-})] \frac{dz}{z^2} ds. \end{aligned}$$

Taking expectation yields $\mathbb{E}[f(\mathcal{M}_t)] = f(0) + \int_0^t \mathbb{E}[\mathfrak{L}f(\mathcal{M}_s)] ds$, so that the generator of \mathcal{M} is indeed \mathfrak{L} . □

1.2.3 Relation with stable-like processes

We clarify the relation between our processes in the case $\sigma = b \equiv 0$ and the symmetric stable-like processes.

The infinitesimal generator governs the short time behavior of the semi-group of a Markov process, hence characterizes the Markov process under consideration. A famous class of Markov processes consists of symmetric α -stable processes, whose generator is

$$\mathfrak{L}^\alpha f(x) = \int_{[-1,1]} [f(x+u) - f(x) - uf'(x)] |u|^{-1-\alpha} du$$

with $\alpha \in (0, 2)$ and this measure controls the density of the jumps. Note that we integrate on $[-1, 1]$ instead of on \mathbb{R} , because the number of large jumps, which is a.s. finite, does not influence the sample path properties outside these points and we do not want to worry about the integrability of the process. One natural way to enrich the family of the α -stable processes consists in adding a dependency of the jump measure on the location. Let

$$\mathfrak{L}^{\tilde{\beta}(\cdot)} f(x) = \int_{[-1,1]} [f(x+u) - f(x) - uf'(x)] \tilde{\beta}(x) |u|^{-1-\tilde{\beta}(x)} du.$$

Assume that $\overline{\text{Range } \tilde{\beta}} \subset (0, 2)$. The corresponding processes were first constructed by Bass [15] by solving a martingale problem, and was called stable-like processes. There exist other constructions by using pseudo-differential operator or Hille-Yosida-Ray theorem, see [31] and references therein.

Let $u = \text{sign}(z)|z|^{1/\tilde{\beta}(x)}$. This change of variable yields

$$\mathfrak{L}^{\tilde{\beta}(\cdot)} f(x) = \int_{[-1,1]} \left[f\left(x + \text{sign}(z)|z|^{1/\tilde{\beta}(x)}\right) - f(x) - \text{sign}(z)|z|^{1/\tilde{\beta}(x)} f'(x) \right] \frac{dz}{z^2}$$

The class of jump-diffusions \mathcal{M} has a jump generator behaving asymptotically like $\mathfrak{L}^{\tilde{\beta}(\cdot)}$. In particular, take a Lipschitz continuous function $\tilde{\beta}$ and

$$G(x, z) = G_0(x, z) = \text{sign}(z)|z|^{1/\tilde{\beta}(x)},$$

satisfying $\overline{\text{Range } \tilde{\beta}} \subset (0, 2)$. Then $G \in \mathcal{G}$ and we recover $\mathfrak{L}^{\tilde{\beta}(\cdot)}$.

1.3 Pointwise regularity of the diffusion term

Proposition 1.5. *With probability one, for every $t \geq 0$, $H_{\mathcal{X}}(t) = \frac{1}{2}$.*

By Dambis-Dubins-Swartz theorem, \mathcal{X} (which is a local martingale) can be written as a Brownian motion subordinated in time where the subordinated process is a bi-Lipschitz continuous

function. The Hölder regularity of Brownian motion can thus be inherited by the martingale in question. This is somewhat classical, we will include a complete proof in Appendix 2.

Remark 1.1. *The condition that σ stays away from 0 cannot be dropped. Indeed, when $\sigma(\mathcal{M}_t) = 0$, the process \mathcal{X} may gain more regularity at t and the computation of $H_{\mathcal{M}}(t)$ involves the regularity of $\sigma(\mathcal{M}_\cdot)$ at time t .*

1.4 Pointwise regularity of the jump term

In the rest of the paper, we restrict our study to the time interval $[0, 1]$, the extension to \mathbb{R}^+ is straightforward.

Following Jaffard [61], we introduce a family of limsup sets on which we can control the growth of the jump part \mathcal{Z} . For every $\delta \geq 1$, let

$$A_\delta = \limsup_{n \rightarrow +\infty} B(T_n, |Z_n|^\delta),$$

where $\mathcal{P} = (T_n, Z_n)_{n \geq 1}$ is the point process (1.3) generating the Poisson measure N in (1.2). We first prove a covering property for the system \mathcal{P} .

Proposition 1.6. *With probability one, $[0, 1] \subset A_1$.*

Proof. The Poisson random measure N has intensity $ds \pi(dz)$ where $\pi(dz) = dz/z^2$. Using Shepp's theorem [107] (and a reformulated version by Bertoin [22]), it suffices to prove that

$$S = \int_0^1 \exp\left(2 \int_t^1 \pi((u, 1)) du\right) dt = +\infty.$$

But $\pi((u, 1)) = u^{-1} - 1$, so that $S = \int_0^1 e^{2(t-1-\log t)} dt = +\infty$. \square

It is clear from Proposition 1.6 that a.s., the approximation rate δ_t by the system of points \mathcal{P} (see Definition 1.5) is well-defined, always greater than 1, and random because it depends on $(T_n, Z_n)_{n \geq 1}$. Another consequence of this proposition is that the set of jumps is dense in $[0, 1]$. These considerations lead to the upper bound for the pointwise exponent of the jump term.

Proposition 1.7. *With probability one, $\forall t \in [0, 1]$, $H_{\mathcal{Z}}(t) \leq \frac{1}{\beta(t)\delta_t}$.*

This proposition is based on two lemmas. The first is observed by Jaffard [60] which sheds light on the importance of the jump times.

Lemma 1.1 ([60]). *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a càdlàg function discontinuous on a dense set of points, and let $t \in \mathbb{R}$. Let $(t_n)_{n \geq 1}$ be a real sequence converging to t such that, at each t_n , $|f(t_n) - f(t_n^-)| = z_n > 0$. Then*

$$H_f(t) \leq \liminf_{n \rightarrow +\infty} \frac{\log z_n}{\log |t_n - t|}.$$

The second lemma establishes a first link between the pointwise regularity and the approximation rate.

Lemma 1.2. *For all $\delta \geq 1$, almost surely*

$$\forall t \in A_\delta, \quad H_{\mathcal{Z}}(t) \leq \frac{1}{\beta(t)\delta}. \quad (1.6)$$

Proof. Recall that almost surely the set of jumps J is

$$J = \{t \in [0, 1] : \Delta \mathcal{M}_t \neq 0\} = \{T_n : n \in \mathbb{N}_*\}$$

and that $\forall n \in \mathbb{N}_*$, $\Delta \mathcal{Z}_{T_n} = G(\mathcal{M}_{T_n-}, Z_n)$. Consider $t \in A_\delta \setminus J$. Necessarily, t is a continuous time of \mathcal{M} and there is an infinite number of n such that

$$t \in B(T_n, |Z_n|^\delta). \quad (1.7)$$

Applying Lemma 1.1 to the process \mathcal{Z} and the jumps satisfying (1.7), one has

$$\begin{aligned} H_{\mathcal{Z}}(t) &\leq \liminf_{n \rightarrow +\infty} \frac{\log |G(\mathcal{M}_{T_n-}, Z_n)|}{\log |T_n - t|} \leq \liminf_{n \rightarrow +\infty} \frac{\log |G(\mathcal{M}_{T_n-}, Z_n)|}{\delta \log |Z_n|} \\ &\leq \liminf_{n \rightarrow +\infty} \frac{-|\log |G(\mathcal{M}_{T_n-}, Z_n)| - \log |G(\mathcal{M}_t, Z_n)||}{\delta \log |Z_n|} \\ &\quad + \liminf_{n \rightarrow +\infty} \frac{\log |G(\mathcal{M}_t, Z_n)|}{\delta \log |Z_n|} \\ &\leq \frac{1}{\beta(t)\delta} + \frac{C}{\delta} \liminf_{n \rightarrow +\infty} |\mathcal{M}_{T_n-} - \mathcal{M}_t| = \frac{1}{\beta(t)\delta}. \end{aligned}$$

where we used (1.7) for the second inequality, the Lipschitz condition of G for the last inequality and the continuity of \mathcal{M} at time t for the last equality.

For all $t \in J$, $H_{\mathcal{Z}}(t) = 0$, which completes the proof. \square

Proof of Proposition 1.7 : It follows from Lemma 1.2 that a.s., for all rational number $\delta \geq 1$, (1.6) holds. Using the monotonicity of $\delta \mapsto A_\delta$ and the density of rational numbers in $[0, 1]$, we deduce that (1.6) holds for all $\delta \geq 1$, a.s..

If $\delta_t < +\infty$, then $t \in A_{\delta_t - \varepsilon}$, for every $\varepsilon > 0$. Hence, $H_{\mathcal{Z}}(t) \leq \frac{1}{\beta(t)(\delta_t - \varepsilon)}$ as a consequence of Lemma 1.2. Letting ε tend to 0, we obtain the result.

If $\delta_t = +\infty$, then $t \in \bigcap_{\delta \geq 1} A_\delta$, meaning that $t \in B(T_n, |Z_n|^\delta)$ for infinitely many integers n , for all $\delta \geq 1$. We deduce by Lemma 1.2 that $H_{\mathcal{Z}}(t) \leq \frac{1}{\beta(t)\delta}$, for all $\delta \geq 1$, thus $H_{\mathcal{Z}}(t) = 0$, which completes the proof. \square

Recall that we want to prove that the upper bound that we obtain in Proposition 1.7 is in fact optimal. Let us first make a useful remark which determines the configuration of the jumps

around a point t . Let $t \notin A_\delta \cup J$. Then there exists a random integer n_0 , such that

$$\forall n \geq n_0, \quad |T_n - t| \geq |Z_n|^\delta. \quad (1.8)$$

Let $s > t$ sufficiently close to t such that $[t, s]$ does not contain those T_n which violate (1.8). It is possible because the cardinality of such T_n is finite. For each s , there exists a unique integer m (which depends on s) such that

$$2^{-m-1} \leq |s - t| < 2^{-m}.$$

Assume that $T_n \in [t, s]$, then $2^{-m} > |t - s| \geq |T_n - s| \geq |Z_n|^\delta$, so that $|Z_n| \leq 2^{-m/\delta}$. This means that in an interval of length 2^{-m} with one extreme point in the complementary of $A_\delta \cup J$, there is no jump whose corresponding Poisson jump size is larger than $2^{-m/\delta}$. Therefore, to consider the increment of our process near such time t , we can split the increment of the compensated Poisson integral $\mathcal{Z}_t - \mathcal{Z}_s$ into two parts:

$$\int_s^t \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{u-}, z) \tilde{N}(dudz) + \int_s^t \int_{C(2^{-m/\delta}, 1)} G(\mathcal{M}_{u-}, z) \tilde{N}(dudz). \quad (1.9)$$

The last remark motivates Proposition 1.8. Before stating it, we define the random quantities

$$\beta_{s,t}^m = \left(\sup_{u \in [s,t]} \beta(u) + \frac{2}{m} \right) \quad \text{and} \quad \hat{\beta}_{s,t}^m = \left(\sup_{u \in [s,t] \pm 2^{-m}} \beta(u) + \frac{2}{m} \right),$$

where

$$[s, t] = \begin{cases} [s, t] & \text{if } s < t, \\ [t, s] & \text{otherwise,} \end{cases} \quad [s, t] \pm 2^{-m} = \begin{cases} [s - 2^{-m}, t + 2^{-m}] & \text{if } s < t, \\ [t - 2^{-m}, s + 2^{-m}] & \text{otherwise.} \end{cases}$$

Proposition 1.8. *There exists a finite positive constant C such that for every $\delta > 1$, $\varepsilon > 0$, and every integer $m \geq m_0$ with m_0 depending only on ε et G ,*

$$\mathbb{P} \left(\sup_{\substack{|s-t| \leq 2^{-m} \\ s, t \in [0,1]}} \left| 2^{\frac{m}{\delta(\hat{\beta}_{s,t}^m + \varepsilon)}} \int_s^t \int_{C(0, 2^{-\frac{m}{\delta}})} G(\mathcal{M}_{u-}, z) \tilde{N}(dudz) \right| \geq 6m^2 \right) \leq Ce^{-m}.$$

Balança [6] proved a similar result for Lévy processes. Here, the idea is to exploit the martingale nature of our process and some “freezing” procedure for the local upper index process $t \mapsto \beta(t)$. Intuitively, in the neighborhood of a continuity point t , since $\beta(t)$ is also continuous at t , one may say that our process behaves locally like a Lévy process with Blumenthal-Gettoor’s index $\beta(t)$. A good way to make explicit this intuition is to cut the index process in the spirit of Lebesgue integral. Roughly speaking, we decompose the first term of (1.9) as a sum of m processes \mathcal{P}^j ,

whose local index takes value in $[2j/m, 2(j+1)/m)$. When m becomes large, the local behavior of these processes is comparable with that of some Lévy process, in probability.

Proposition 1.8 brings information about the uniform increment estimate of \mathcal{Z} . In contrast with the Lévy case, it is remarkable that the exponent depends on two parameters: the approximation rate δ_t and $\beta(t)$, both random and correlated with \mathcal{M} . This observation complicates in the proof.

The proof is decomposed into several lemmas. The following lemma gives an increment estimate in the first dyadic interval with “frozen” index.

Lemma 1.3. *There exists a finite positive constant C such that for each $\delta > 1$, $\varepsilon > 0$, $m \geq m_0$ (depending only on ε and G), and $j \in \{0, \dots, m-1\}$,*

$$\mathbb{P} \left(\sup_{t \leq 2^{-m}} \left| 2^{\frac{m}{\delta(2j+2+m\varepsilon)/m}} \int_0^t \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{s-}, z) \mathbf{1}_{\beta(s-) \in [\frac{2j}{m}, \frac{2j+2}{m})} \tilde{N}(dsdz) \right| \geq 2m \right) \leq Ce^{-2m}.$$

Proof. Let

$$\begin{aligned} H_j(\mathcal{M}_{s-}, z) &:= 2^{\frac{m}{\delta(2j+2+m\varepsilon)/m}} G(\mathcal{M}_{s-}, z) \mathbf{1}_{\beta(s-) \in [\frac{2j}{m}, \frac{2j+2}{m})} \mathbf{1}_{C(0, 2^{-m/\delta})}(z), \\ \mathcal{P}_t^j &:= \int_0^t \int_{C(0,1)} H_j(\mathcal{M}_{s-}, z) \tilde{N}(dsdz). \end{aligned}$$

For every $t \leq 2^{-m}$, one has

$$\begin{aligned} & \int_0^t \int_{C(0,1)} \mathbb{E}[H_j(\mathcal{M}_{s-}, z)^2] \frac{dz}{z^2} ds \\ &= \int_0^t \mathbb{E} \left[2^{\frac{2m}{\delta(2j+2+m\varepsilon)/m}} \mathbf{1}_{\beta(s-) \in [\frac{2j}{m}, \frac{2j+2}{m})} \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{s-}, z)^2 \frac{dz}{z^2} \right] ds \\ &\leq 2 \int_0^t \mathbb{E} \left[2^{\frac{2m}{\delta(2j+2+m\varepsilon)/m}} \mathbf{1}_{\beta(s-) \in [\frac{2j}{m}, \frac{2j+2}{m})} \int_0^{2^{-\frac{m}{\delta}}} z^{\frac{2}{\beta(s-)+\varepsilon/2}-2} dz \right] ds \\ &\leq C \int_0^t \mathbb{E} \left[2^{\frac{2m}{\delta(2j+2+m\varepsilon)/m}} \mathbf{1}_{\beta(s-) \in [\frac{2j}{m}, \frac{2j+2}{m})} 2^{-\frac{m}{\delta} \left(\frac{2}{\beta(s-)+\varepsilon/2} - 1 \right)} \right] ds \\ &= C \int_0^t \mathbb{E} \left[2^{\frac{m}{\delta} \left(\frac{2}{(2j+2)/m+\varepsilon} - \frac{2}{\beta(s-)+\varepsilon/2} + 1 \right)} \mathbf{1}_{\beta(s-) \in [\frac{2j}{m}, \frac{2j+2}{m})} \right] ds, \end{aligned}$$

where we used the property of class \mathcal{G} for the first inequality. Hence,

$$\int_0^t \int_{C(0,1)} \mathbb{E}[H_j(\mathcal{M}_{s-}, z)^2] \frac{dz}{z^2} ds \leq C \int_0^t \mathbb{E} \left[2^{\frac{m}{\delta}} \right] du = Ct2^{m/\delta} \leq C.$$

Hence, $(t \mapsto \mathcal{P}_t^j)_{t \leq 2^{-m}}$ is a martingale by Proposition 1.3. One deduces by convexity and Jensen's

inequality that $t \mapsto e^{\mathcal{P}_t^j}$ and $t \mapsto e^{-\mathcal{P}_t^j}$ are positive submartingales. By Doob's maximal inequality for positive martingales,

$$\begin{aligned} \mathbb{P} \left(\sup_{t \leq 2^{-m}} |\mathcal{P}_t^j| \geq 2m \right) &\leq \mathbb{P} \left(\sup_{t \leq 2^{-m}} e^{\mathcal{P}_t^j} \leq e^{2m} \right) + \mathbb{P} \left(\sup_{t \leq 2^{-m}} e^{-\mathcal{P}_t^j} \leq e^{2m} \right) \\ &\leq e^{-2m} \left(\mathbb{E} \left[e^{\mathcal{P}_{2^{-m}}^j} \right] + \mathbb{E} \left[e^{-\mathcal{P}_{2^{-m}}^j} \right] \right). \end{aligned}$$

We now compute $\mathbb{E}[e^{\mathcal{P}_{2^{-m}}^j}]$ and $\mathbb{E}[e^{-\mathcal{P}_{2^{-m}}^j}]$. If these expectations are finite and independent of the value of m , then the proof is done. We only study the positive submartingales $e^{\mathcal{P}_t^j}$ ($e^{-\mathcal{P}_t^j}$ can be studied similarly). By Itô's Formula for compensated Poisson integral,

$$\begin{aligned} e^{\mathcal{P}_t^j} &= 1 + \int_0^t \int_{C(0,1)} e^{\mathcal{P}_{s-}^j} \left(e^{H_j(\mathcal{M}_{s-}, z)} - 1 \right) \tilde{N}(dsdz) \\ &\quad + \int_0^t \int_{C(0,1)} e^{\mathcal{P}_{s-}^j} \left(e^{H_j(\mathcal{M}_{s-}, z)} - 1 - H_j(\mathcal{M}_{s-}, z) \right) \frac{dz}{z^2} ds. \end{aligned} \quad (1.10)$$

Note that for all $s \in [0, 1]$, using the property of $G \in \mathcal{G}$,

$$|H_j(\mathcal{M}_{s-}, z)| \leq 2^{\frac{m}{\delta(2j+2+m\varepsilon)/m}} |z|^{\frac{1}{\beta(s-)+\varepsilon/2}} \mathbf{1}_{|z| \leq 2^{-m/\delta}} \mathbf{1}_{\beta(s-) \in [\frac{2j}{m}, \frac{2j+2}{m}]} \leq 1.$$

Then, since $|e^u - 1 - u| \leq |u|^2$ for $|u| \leq 1$, taking expectation in (1.10) yields

$$\begin{aligned} &\mathbb{E}[e^{\mathcal{P}_t^j}] \\ &= 1 + \mathbb{E} \left[\int_0^t \int_{C(0,1)} e^{\mathcal{P}_{s-}^j} \left(e^{H_j(\mathcal{M}_{s-}, z)} - 1 - H_j(\mathcal{M}_{s-}, z) \right) \frac{dz}{z^2} ds \right] \\ &\leq 1 + \mathbb{E} \left[\int_0^t \int_{C(0,1)} e^{\mathcal{P}_{s-}^j} H_j(\mathcal{M}_{s-}, z)^2 \frac{dz}{z^2} ds \right] \\ &= 1 + \mathbb{E} \left[\int_0^t \int_{C(0, 2^{-m/\delta})} e^{\mathcal{P}_{s-}^j} 2^{\frac{2m}{\delta(2j+2+m\varepsilon)/m}} \mathbf{1}_{\beta(s-) \in [\frac{2j}{m}, \frac{2j+2}{m}]} G(\mathcal{M}_{s-}, z)^2 \frac{dz}{z^2} ds \right] \\ &\leq 1 + 2\mathbb{E} \left[\int_0^t e^{\mathcal{P}_{s-}^j} 2^{\frac{2m}{\delta(2j+2+m\varepsilon)/m}} \mathbf{1}_{\beta(s-) \in [\frac{2j}{m}, \frac{2j+2}{m}]} \int_0^{2^{-m/\delta}} z^{\frac{2}{\beta(s-)+\varepsilon/2}-2} dz ds \right], \end{aligned}$$

where we used the definition of the class \mathcal{G} . We deduce that

$$\begin{aligned} \mathbb{E}[e^{\mathcal{P}_t^j}] &\leq 1 + C\mathbb{E} \left[\int_0^t e^{\mathcal{P}_{s-}^j} 2^{\frac{2m}{\delta(2j+2+m\varepsilon)/m}} \mathbf{1}_{\beta(s-) \in [\frac{2j}{m}, \frac{2j+2}{m}]} 2^{-\frac{m}{\delta}(\frac{2}{\beta(s-)+\varepsilon/2}-1)} ds \right] \\ &\leq 1 + C\mathbb{E} \left[\int_0^t e^{\mathcal{P}_{s-}^j} 2^{\frac{m}{\delta}(\frac{2}{(2j+2)/m+\varepsilon} - \frac{2}{\beta(s-)+\varepsilon/2} + 1)} \mathbf{1}_{\beta(s-) \in [\frac{2j}{m}, \frac{2j+2}{m}]} ds \right] \\ &\leq 1 + C\mathbb{E} \left[\int_0^t e^{\mathcal{P}_{s-}^j} 2^{m/\delta} ds \right] = 1 + C \int_0^t \mathbb{E}[e^{\mathcal{P}_s^j}] 2^{m/\delta} ds. \end{aligned}$$

By Gronwall's inequality applied to $s \mapsto \mathbb{E}[e^{\mathcal{P}_s^j}]$, one concludes that

$$\mathbb{E}[e^{\mathcal{P}_{2^{-m}}^j}] \leq e^{\int_0^{2^{-m}} C 2^{m/\delta} ds} \leq e^{C 2^{m/\delta} 2^{-m}} \leq e^C.$$

□

Now we can consider the whole jump process.

Lemma 1.4. *There exists $0 < C < +\infty$ such that for each $\delta > 1$, $\varepsilon > 0$, $m \geq m_0$ (depending only on ε and G), and $i \in \{0, \dots, 2^m - 1\}$, one has*

$$\begin{aligned} & \mathbb{P} \left(\sup_{t \leq 2^{-m}} \left| \int_{i 2^{-m}}^{i 2^{-m} + t} \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{s-}, z) \tilde{N}(dsdz) \right| \geq 2m^2 2^{-\frac{m}{\delta(\beta_{i 2^{-m}, (i+1) 2^{-m} + \varepsilon}^m)}} \right) \\ & \leq C m e^{-2m}. \end{aligned}$$

Proof. It suffices to show this inequality for the first dyadic interval, namely $i = 0$. For every j , let us introduce the event

$$A_j = \left\{ \sup_{t \leq 2^{-m}} \left| \int_0^t \int_{C(0, 2^{m/\delta})} G(\mathcal{M}_{s-}, z) \mathbf{1}_{\beta(s-) \in [\frac{2j}{m}, \frac{2j+2}{m}]} \tilde{N}(dsdz) \right| \geq 2m^2 2^{-\frac{m}{\delta(\beta_{0, 2^{-m} + \varepsilon}^m)}} \right\}.$$

We have

$$A_j = \left(A_j \cap \left\{ \sup_{s \leq 2^{-m}} \beta(s-) < \frac{2j}{m} \right\} \right) \cup \left(A_j \cap \left\{ \sup_{s \leq 2^{-m}} \beta(s-) \geq \frac{2j}{m} \right\} \right).$$

Under the event $\left\{ \sup_{s \leq 2^{-m}} \beta(s-) < \frac{2j}{m} \right\}$, the compensated Poisson integral in A_j is zero, thus the first event is a null set. One has

$$\begin{aligned} \mathbb{P}(A_j) &= \mathbb{P} \left(A_j \cap \left\{ \sup_{s \leq 2^{-m}} \beta(s-) \geq \frac{2j}{m} \right\} \right) \\ &= \mathbb{P} \left(A_j \cap \left\{ \sup_{s \leq 2^{-m}} \beta(s-) + \frac{2}{m} + \varepsilon \geq \frac{2j+2}{m} + \varepsilon \right\} \right) \\ &\leq \mathbb{P} \left(\sup_{t \leq 2^{-m}} \left| \int_0^t \int_{C(0, 2^{m/\delta})} G(\mathcal{M}_{s-}, z) \mathbf{1}_{\beta(s-) \in [\frac{2j}{m}, \frac{2j+2}{m}]} \tilde{N}(dsdz) \right| \geq 2m^2 2^{-\frac{m}{\delta(2j+2+m\varepsilon)/m}} \right). \end{aligned}$$

Using Lemma 1.3, one concludes that $\mathbb{P}(A_j) \leq C e^{-2m}$.

In order to get the increment estimate with “moving” index, we note that

$$\left\{ \sup_{t \leq 2^{-m}} \left| \int_0^t \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{s-}, z) \tilde{N}(dsdz) \right| \geq 2m^2 2^{-\frac{m}{\delta(\beta_{0, 2^{-m} + \varepsilon}^m)}} \right\} \subset \bigcup_{j=1}^m A_j.$$

One deduces that

$$\mathbb{P} \left(\sup_{t \leq 2^{-m}} \left| \int_0^t \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{s-}, z) \tilde{N}(dsdz) \right| \geq 2m^2 2^{-\frac{m}{\delta(\beta_{0, 2^{-m} + \varepsilon}^m)}} \right) \leq Cme^{-2m},$$

which completes the proof of Lemma 1.4. \square

We prove Proposition 1.8, using a classical discretization procedure.

Proof of Proposition 1.8 : We discretize the first term of (1.9) in the time domain. Let $s, t \in [0, 1]$ such that $|s - t| \leq 2^{-m}$. There exists $i \in \{0, \dots, 2^m - 1\}$ such that $[s, t] \subset [(i-1)/2^m, (i+1)/2^m]$, thus

$$\begin{aligned} & 2^{\frac{m}{\delta(\hat{\beta}_{s,t}^m + \varepsilon)}} \left| \int_s^t \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{s-}, z) \tilde{N}(dudz) \right| \\ & \leq 2^{\frac{m}{\delta(\beta_{i2^{-m}, (i+1)2^{-m} + \varepsilon}^m)}} \sup_{t \leq 2^{-m}} \left| \int_{i2^{-m}}^{i2^{-m}+t} \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{s-}, z) \tilde{N}(dsdz) \right| \\ & + 2 \cdot 2^{\frac{m}{\delta(\beta_{(i-1)2^{-m}, i2^{-m} + \varepsilon}^m)}} \sup_{t \leq 2^{-m}} \left| \int_{(i-1)2^{-m}}^{(i-1)2^{-m}+t} \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{s-}, z) \tilde{N}(dsdz) \right| \end{aligned}$$

Therefore,

$$\begin{aligned} & \left\{ \sup_{\substack{|s-t| \leq 2^{-m} \\ s, t \in [0, 1]}} 2^{\frac{m}{\delta(\hat{\beta}_{s,t}^m + \varepsilon)}} \left| \int_s^t \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{s-}, z) \tilde{N}(dudz) \right| \geq 6m^2 \right\} \subset \\ & \bigcup_{i=0}^{2^m-1} \left\{ 2^{\frac{m}{\delta(\beta_{i2^{-m}, (i+1)2^{-m} + \varepsilon}^m)}} \sup_{t \leq 2^{-m}} \left| \int_{i2^{-m}}^{i2^{-m}+t} \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{s-}, z) \tilde{N}(dsdz) \right| \geq 2m^2 \right\} \end{aligned}$$

so that by Lemma 1.4, one sees that for every m large enough,

$$\begin{aligned} & \mathbb{P} \left(\sup_{\substack{|s-t| \leq 2^{-m} \\ s, t \in [0, 1]}} 2^{\frac{m}{\delta(\hat{\beta}_{s,t}^m + \varepsilon)}} \left| \int_s^t \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{s-}, z) \tilde{N}(dudz) \right| \geq 6m^2 \right) \\ & \leq 2^m Cme^{-2m} \leq Ce^{-m}. \end{aligned}$$

\square

1.5 Proof of Theorem 1.3 (ii) jumps without diffusion

Recall that we assume that (\mathcal{H}) holds, and in this case, one has

$$\mathcal{M}_t = \int_0^t b(\mathcal{M}_u) du + \int_0^t \int_{C(0,1)} G(\mathcal{M}_{u-}, z) \tilde{N}(dudz) = \mathcal{Y}_t + \mathcal{Z}_t.$$

Case (a) of (\mathcal{H}) : $b \in C^\infty(\mathbb{R})$ and $\overline{\text{Range } \tilde{\beta}} \subset [1, 2)$. Under the conditions on b , the drift term \mathcal{Y} does not influence the pointwise regularity of \mathcal{M} . This follows from the next Lemma.

Lemma 1.5. *Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ and $F(\cdot) = \int_0^\cdot f(y) dy$.*

(i) *Let $g \in C^\infty(\mathbb{R})$ and $x \in \mathbb{R}$, then $H_{g \circ f}(x) \geq H_f(x)$.*

(ii) *$\forall x \in \mathbb{R}, H_F(x) \geq H_f(x) + 1$.*

It is a simple exercise to check the lemma. Hence, for every t , one has $H_{\mathcal{Y}}(t) \geq H_{b(\mathcal{M}_\cdot)}(t) + 1 \geq H_{\mathcal{M}}(t) + 1$, which yields $H_{\mathcal{M}}(t) = H_{\mathcal{Z}}(t)$. Therefore, it is enough to prove that a.s. for every $t \in [0, 1] \setminus J$, $H_{\mathcal{Z}}(t) = \frac{1}{\delta_t \beta(t)}$.

The upper bound is obtained by Proposition 1.7, so it remains us to get the lower bound, which will be deduced from the following property:

$$\forall \delta > 1, \forall \varepsilon > 0, \text{ almost surely, } \forall t \notin J \cup A_\delta, H_{\mathcal{Z}}(t) \geq \frac{1}{\delta(\beta(t) + \varepsilon)}. \quad (1.11)$$

Indeed, assume that (1.11) holds true. This implies that, almost surely, for all rational pair $\varepsilon > 0$ and $\delta > 1$, one has $H_{\mathcal{Z}}(t) \geq \frac{1}{\delta(\beta(t) + \varepsilon)}$ for all points $t \notin J \cup A_\delta$. The monotonicity of the mapping $\delta \mapsto A_\delta$ yields that if $\delta' > \delta$, $(t \notin A_\delta) \Rightarrow (t \notin A_{\delta'})$. One deduces that almost surely, for every $\delta > 1$ real, for every rationals $\delta' > \delta$ and $\varepsilon > 0$, if $t \notin J \cup A_\delta$, the exponent $H_{\mathcal{Z}}(t)$ satisfies $H_{\mathcal{Z}}(t) \geq \frac{1}{\delta'(\beta(t) + \varepsilon)}$. Using the density of the rational numbers in \mathbb{R} and taking ε arbitrarily small in \mathbb{Q} yields that almost surely,

$$\forall \delta > 1, \forall t \notin J \cup A_\delta, H_{\mathcal{Z}}(t) \geq \frac{1}{\delta \beta(t)}.$$

We deduce the lower bound for $H_{\mathcal{Z}}(t)$. Let $\varepsilon' > 0$. If $t \notin J$ and $\delta_t < +\infty$, one has $t \notin A_{\delta_t + \varepsilon'}$ by the definition of the approximation rate δ_t . hence $H_{\mathcal{Z}}(t) \geq \frac{1}{(\delta_t + \varepsilon')\beta(t)}$. If $t \notin J$ but $\delta_t = +\infty$, then $\frac{1}{(\delta_t + \varepsilon')\beta(t)} = 0$, the desired inequality is trivial. Letting $\varepsilon' \rightarrow 0$ yields that a.s., $\forall t \notin J$, $H_{\mathcal{Z}}(t) \geq \frac{1}{\delta_t \beta(t)}$.

Now we prove (1.11). Applying Proposition 1.8 and Borel-Cantelli lemma, one sees that $\forall \varepsilon > 0$, $\forall \delta > 1$, almost surely, $\exists m(\omega), \forall m \geq m(\omega)$,

$$\sup_{|s-t| \leq 2^{-m}, s, t \in [0,1]} \left| 2^{\frac{m}{\delta(\tilde{\beta}_{s,t}^m + \varepsilon)}} \int_s^t \int_{C(0,2^{-m/\delta})} G(\mathcal{M}_{u-}, z) \tilde{N}(dudz) \right| \leq 6m^2. \quad (1.12)$$

For every $t \notin J \cup A_\delta$, pick a point s close to t such that $|s - t| < 2^{-m(\omega)}$. Hence

$$2^{-m-1} \leq |s - t| < 2^{-m}. \quad (1.13)$$

for some unique $m \geq m(\omega)$, and

$$\left| 2^{\frac{m}{\delta(\widehat{\beta}_{s,t}^m + \varepsilon)}} \int_s^t \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{u-}, z) \widetilde{N}(dudz) \right| \leq 6m^2$$

where we used (1.12). Therefore, by (1.13), one has

$$\begin{aligned} \left| \int_t^s \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{u-}, z) \widetilde{N}(dudz) \right| &\leq 6m^2 2^{-\frac{m}{\delta(\widehat{\beta}_{s,t}^m + \varepsilon)}} \\ &\leq 6|s - t|^{\frac{1}{\delta(\beta(t) + 2\varepsilon)}} \left(\log \frac{1}{|s - t|} \right)^2 \end{aligned} \quad (1.14)$$

where we used the continuity of $\beta := \widetilde{\beta} \circ \mathcal{M}$ at t (because $\widetilde{\beta}$ is Lipschitz and \mathcal{M} is continuous at t) and the fact that $\widehat{\beta}_{s,t}^m \leq \beta(t) + \varepsilon$ for large m .

Recalling (1.9), in the interval $[s, t]$ with $t \notin J \cup A_\delta$ and $|s - t| \sim 2^{-m}$, there is no time whose corresponding jump size is larger than $2^{-m/\delta}$. Hence,

$$\int_s^t \int_{C(2^{-m/\delta}, 1)} G(\mathcal{M}_{u-}, z) \widetilde{N}(dudz) = \int_s^t \int_{C(2^{-m/\delta}, 1)} G(\mathcal{M}_{u-}, z) \frac{dz}{z^2} du. \quad (1.15)$$

By the fact that $G \in \mathcal{G}$, for every $\varepsilon > 0$, there exists $z(\varepsilon) > 0$, such that for all $z \in C(0, z(\varepsilon))$ and $x \in [0, 1]$, one has $|G(x, z)| \leq |z|^{\frac{1}{\beta(x) + \varepsilon}}$. Hence

$$\begin{aligned} \left| \int_s^t \int_{C(2^{-m/\delta}, z(\varepsilon))} G(\mathcal{M}_{u-}, z) \frac{dz}{z^2} du \right| &\leq 2 \left| \int_s^t \int_{2^{-m/\delta}}^{z(\varepsilon)} z^{\frac{1}{\beta(u-) + \varepsilon/2}} \frac{dz}{z^2} du \right| \\ &\leq 2 \left| \int_s^t \int_{2^{-m/\delta}}^{z(\varepsilon)} z^{\frac{1}{\beta(t) + \varepsilon}} \frac{dz}{z^2} du \right| \\ &\leq C|s - t| 2^{-\frac{m}{\delta} \left(\frac{1}{\beta(t) + \varepsilon} - 1 \right)} \\ &= C|s - t|^{1 - \frac{1}{\delta} + \frac{1}{\delta(\beta(t) + \varepsilon)}} \end{aligned} \quad (1.16)$$

and

$$\left| \int_s^t \int_{C(z(\varepsilon), 1)} G(\mathcal{M}_{u-}, z) \frac{dz}{z^2} du \right| \leq \left| \int_s^t \int_{C(z(\varepsilon), 1)} \frac{dz}{z^2} du \right| \leq Cz(\varepsilon)^{-1} |s - t| \quad (1.17)$$

by the uniform boundedness of G . Combining the estimates (1.14)-(1.17),

$$\begin{aligned}
& |\mathcal{Z}_t - \mathcal{Z}_s| \\
& \leq \left| \int_s^t \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{u-}, z) \tilde{N}(dudz) \right| + \left| \int_s^t \int_{C(2^{-m/\delta}, 1)} G(\mathcal{M}_{u-}, z) \tilde{N}(dudz) \right| \\
& \leq \left| \int_s^t \int_{C(0, 2^{-m/\delta})} G(\mathcal{M}_{u-}, z) \tilde{N}(dudz) \right| + \left| \int_s^t \int_{C(2^{-m/\delta}, z(\varepsilon))} G(\mathcal{M}_{u-}, z) \frac{dz}{z^2} du \right| \\
& \quad + \left| \int_s^t \int_{C(z(\varepsilon), 1)} G(\mathcal{M}_{u-}, z) \frac{dz}{z^2} du \right| \\
& \leq 6|s - t|^{\frac{1}{\delta(\beta(t)+2\varepsilon)}} \left(\log \frac{1}{|s - t|} \right)^2 + C|s - t|^{1 - \frac{1}{\delta} + \frac{1}{\delta(\beta(t)+\varepsilon)}} + Cz(\varepsilon)^{-1}|s - t| \\
& \leq 8|s - t|^{\frac{1}{\delta(\beta(t)+2\varepsilon)}} \left(\log \frac{1}{|s - t|} \right)^2
\end{aligned}$$

for s sufficiently close to t , where we used the fact that $1 > \frac{1}{\delta(\beta(t)+\varepsilon)}$ and $1 - 1/\delta > 0$. The desired lower bound is obtained.

Case (b) of (\mathcal{H}) : $b \in C^\infty$ and $x \mapsto \tilde{b}(x) := \int_0^1 G(x, z) dz / z^2 \in C^\infty$, whenever the integral is well defined.

If t is such that $\beta(t) \geq 1$, we follow the exact same lines of Case (a) to obtain the exponent.

If t is such that $\beta(t) < 1$, by continuity, for every $u \in [t - \varepsilon, t + \varepsilon]$ with ε sufficiently small, $\beta(u) < 1$. Then we write

$$\begin{aligned}
\mathcal{M}_u - \mathcal{M}_{t-\varepsilon} &= \mathcal{Y}_u - \mathcal{Y}_{t-\varepsilon} + \mathcal{Z}_u - \mathcal{Z}_{t-\varepsilon} \\
&= \int_{t-\varepsilon}^u b(\mathcal{M}_s) ds + \int_{t-\varepsilon}^u \int_{C(0,1)} G(\mathcal{M}_{s-}, z) \tilde{N}(dsdz) \\
&= \int_{t-\varepsilon}^u \bar{b}(\mathcal{M}_s) ds + \int_{t-\varepsilon}^u \int_{C(0,1)} G(\mathcal{M}_{s-}, z) N(ds dz) \\
&:= \bar{\mathcal{Y}}_u - \bar{\mathcal{Z}}_u
\end{aligned}$$

where $\bar{b}(x) = b(x) - \tilde{b}(x)$. Using that $\beta(t) < 1$, one easily checks that both integrals in the last line are well defined for every $u \in [t - \varepsilon, t + \varepsilon]$. Under the conditions we impose on b and \tilde{b} , it follows that $h_{\bar{\mathcal{Y}}}(t) \geq H_{\mathcal{M}}(t) + 1$, always due to Lemma 1.5. Therefore, $H_{\mathcal{M}}(t) = H_{\bar{\mathcal{Z}}}(t)$.

Notice that the upper bound for $H_{\mathcal{Z}}(t)$ is also an upper bound for $H_{\bar{\mathcal{Z}}}(t)$ since Lemma 1.1 depends only on the system of Poisson points \mathcal{P} . We only need to obtain the lower bound for $H_{\bar{\mathcal{Z}}}(t)$. By the same arguments as those developed in case (a), it suffices to prove (1.11) for $\bar{\mathcal{Z}}$. But for s close

to t , if m is the integer given by (1.13) (since $s \in [t - \varepsilon, t + \varepsilon]$ for m large), one has

$$\begin{aligned}
|\bar{\mathcal{Z}}_t - \bar{\mathcal{Z}}_s| &= \left| \int_s^t \int_{C(0,1)} G(\mathcal{M}_{u-}, z) N(dudz) \right| \\
&= \left| \int_s^t \int_{C(0,2^{-m/\delta})} G(\mathcal{M}_{u-}, z) N(dudz) \right| \\
&\leq \left| \int_s^t \int_{C(0,2^{-m/\delta})} G(\mathcal{M}_{u-}, z) \tilde{N}(dudz) \right| \\
&\quad + \left| \int_s^t \int_{C(0,2^{-m/\delta})} G(\mathcal{M}_{u-}, z) \frac{dz}{z^2} du \right| \\
&:= I_1 + I_2.
\end{aligned}$$

We estimate I_1 by (1.14). For I_2 , we apply (1.13) to get

$$\begin{aligned}
I_2 &\leq 2 \left| \int_s^t \int_0^{2^{-m/\delta}} z^{\frac{1}{\beta(t)+\varepsilon\wedge(1-\beta(t))/2}} \frac{dz}{z^2} du \right| \leq C \left| \int_s^t 2^{-\frac{m}{\delta}} \left(\frac{1}{\beta(t)+\varepsilon\wedge(1-\beta(t))/2} - 1 \right) du \right| \\
&\leq C |s - t|^{1+\frac{1}{\delta}} \left(\frac{1}{\beta(t)+\varepsilon\wedge(1-\beta(t))/2} - 1 \right) \leq C |s - t|^{1-\frac{1}{\delta} + \frac{1}{\delta(\beta(t)+\varepsilon)}},
\end{aligned}$$

so that

$$\begin{aligned}
|\bar{\mathcal{Z}}_t - \bar{\mathcal{Z}}_s| &\leq 6 |s - t|^{\frac{1}{\delta(\beta(t)+2\varepsilon)}} \left(\log \frac{1}{|s - t|} \right)^2 + C |s - t|^{1-\frac{1}{\delta} + \frac{1}{\delta(\beta(t)+\varepsilon)}} \\
&\leq 7 |s - t|^{\frac{1}{\delta(\beta(t)+2\varepsilon)}} \left(\log \frac{1}{|s - t|} \right)^2,
\end{aligned} \tag{1.18}$$

which yields (1.11). From this we deduce the desired lower bound.

1.6 Proof of Theorem 1.3 (i) jump with a non-zero diffusion

Along the proof, whenever $\beta(t) < 1$, we focus on a small interval $[t - \varepsilon, t + \varepsilon]$ and study $\bar{\mathcal{Y}}$ and $\bar{\mathcal{Z}}$, as we did in the previous section.

Recall that we want to show that almost surely, $\forall t \notin J$, $H_{\mathcal{M}}(t) = \frac{1}{\delta_t \beta(t)} \wedge \frac{1}{2}$. It is enough to show that $\forall \delta > 1, \forall \varepsilon > 0$, almost surely,

$$\begin{aligned}
\forall t \notin J \cup A_\delta, \quad \beta(t) \geq 1 &\Rightarrow H_{\mathcal{Z}}(t) \geq \frac{1}{\delta(\beta(t)+\varepsilon)}, \\
\beta(t) < 1 &\Rightarrow H_{\bar{\mathcal{Z}}}(t) \geq \frac{1}{\delta(\beta(t)+\varepsilon)}.
\end{aligned} \tag{1.19}$$

Indeed, assume that (1.19) holds true.

Then, for every $\varepsilon > 0$, almost surely, $\forall \delta > 1$ rational, $\forall t \notin J \cup A_\delta$, one has

$$\begin{aligned} \beta(t) \geq 1 &\Rightarrow H_{\mathcal{Z}}(t) \geq \frac{1}{\delta(\beta(t) + \varepsilon)} \\ \text{and } \beta(t) < 1 &\Rightarrow H_{\overline{\mathcal{Z}}}(t) \geq \frac{1}{\delta(\beta(t) + \varepsilon)}. \end{aligned}$$

By the same argument as the one used in the proof of Theorem 1.3 (ii), one can remove the (rational) restriction on δ and remove ε . Hence, almost surely, $\forall \delta > 1$, $\forall t \notin J \cup A_\delta$, one has

$$\begin{aligned} \beta(t) \geq 1 &\Rightarrow H_{\mathcal{Z}}(t) \geq \frac{1}{\delta\beta(t)} \\ \text{and } \beta(t) < 1 &\Rightarrow H_{\overline{\mathcal{Z}}}(t) \geq \frac{1}{\delta\beta(t)}. \end{aligned}$$

Then by the definition of δ_t and Proposition 1.7 (upper bound for the exponent of the jump part), one deduces that $\forall \varepsilon' > 0$, almost surely one has

$$\forall t \notin \mathcal{J}, \quad \begin{aligned} \beta(t) \geq 1 &\Rightarrow \frac{1}{(\delta_t + \varepsilon')\beta(t)} \leq H_{\mathcal{Z}}(t) \leq \frac{1}{\delta_t\beta(t)}, \\ \beta(t) < 1 &\Rightarrow \frac{1}{(\delta_t + \varepsilon')\beta(t)} \leq H_{\overline{\mathcal{Z}}}(t) \leq \frac{1}{\delta_t\beta(t)}. \end{aligned}$$

Letting $\varepsilon' \rightarrow 0$ yields almost surely

$$\forall t \notin \mathcal{J}, \quad \begin{aligned} \beta(t) \geq 1 &\Rightarrow H_{\mathcal{Z}}(t) = \frac{1}{\delta_t\beta(t)}, \\ \beta(t) < 1 &\Rightarrow H_{\overline{\mathcal{Z}}}(t) = \frac{1}{\delta_t\beta(t)}. \end{aligned}$$

But almost surely, for every t , $H_{\mathcal{X}+\mathcal{Y}}(t) = H_{\mathcal{X}+\overline{\mathcal{Y}}} = \frac{1}{2}$ (use Proposition 1.5 and the facts that $H_{\mathcal{Y}}(t) \geq 1$, $H_{\overline{\mathcal{Y}}}(t) \geq 1$).

Consider now a time $t \notin J$ where the process is continuous. Four cases must be distinguished.

- If $\beta(t) \geq 1$ and $\frac{1}{2} \neq \frac{1}{\delta_t\beta(t)}$, then $H_{\mathcal{M}}(t) = H_{\mathcal{X}+\mathcal{Y}+\mathcal{Z}}(t) = \frac{1}{2} \wedge \frac{1}{\delta_t\beta(t)}$.
- If $\beta(t) \geq 1$ and $\frac{1}{2} = \frac{1}{\delta_t\beta(t)}$, then by Lemma 1.1 for \mathcal{M} ,

$$\frac{1}{2} \wedge \frac{1}{\delta_t\beta(t)} \geq H_{\mathcal{M}}(t) = H_{\mathcal{X}+\mathcal{Y}+\mathcal{Z}}(t) \geq H_{\mathcal{X}+\mathcal{Y}}(t) \wedge H_{\mathcal{Z}}(t) = \frac{1}{2} \wedge \frac{1}{\delta_t\beta(t)}.$$

- If $\beta(t) < 1$ and $\frac{1}{2} \neq \frac{1}{\delta_t\beta(t)}$, then $H_{\mathcal{M}}(t) = H_{\mathcal{X}+\overline{\mathcal{Y}}+\overline{\mathcal{Z}}}(t) = \frac{1}{2} \wedge \frac{1}{\delta_t\beta(t)}$.
- If $\beta(t) < 1$ and $\frac{1}{2} = \frac{1}{\delta_t\beta(t)}$, then

$$\frac{1}{2} \wedge \frac{1}{\delta_t\beta(t)} \geq H_{\mathcal{M}}(t) = H_{\mathcal{X}+\overline{\mathcal{Y}}+\overline{\mathcal{Z}}}(t) \geq H_{\mathcal{X}+\overline{\mathcal{Y}}}(t) \wedge H_{\overline{\mathcal{Z}}}(t) = \frac{1}{2} \wedge \frac{1}{\delta_t\beta(t)}.$$

In all cases, if t is not a jump time, $H_{\mathcal{M}}(t) = \frac{1}{2} \wedge \frac{1}{\delta_t\beta(t)}$.

Finally, we prove (1.19). Let $\delta > 1$ and $\varepsilon > 0$, and consider $t \notin J \cup A_\delta$.

- If $\beta(t) \geq 1$, we use (1.14)-(1.17) to get, for s sufficiently close to t ,

$$|\mathcal{Z}_t - \mathcal{Z}_s| \leq C|s - t|^{\frac{1}{\delta(\beta(t)+2\varepsilon)}} \left(\log \frac{1}{|s - t|} \right)^2.$$

- If $\beta(t) < 1$, using (1.18) for s close to t , one gets

$$|\bar{\mathcal{Z}}_t - \bar{\mathcal{Z}}_s| \leq 7|s - t|^{\frac{1}{\delta(\beta(t)+2\varepsilon)}} \left(\log \frac{1}{|s - t|} \right)^2.$$

1.7 Computation of the pointwise multifractal spectrum

In this section, we compute the pointwise spectrum of \mathcal{M} in all possible settings, i.e. Theorem 1.2 for jumps with diffusion and Theorems 1.5, 1.6 for jumps without diffusion. The main tool comes from geometric measure theory, the so-called ubiquity theorem, which consists in determining the Hausdorff dimension of some limsup sets. This theory finds its origin in Diophantine approximation and the localized version developed by Barral and Seuret [11] is very useful in studying random objects with varying spectra.

Theorem 1.4 ([11]). *Consider a Poisson point process S with intensity $dt \otimes \mathbf{1}_{z \in C(0,1)} dz / z^2$. Let $I = (a, b) \subset [0, 1]$ and $f : I \rightarrow [1, +\infty)$ be continuous at every $t \in I \setminus \mathfrak{C}$, for some countable set $\mathfrak{C} \subset [0, 1]$. Consider*

$$S(I, f) = \{t \in I : \delta_t \geq f(t)\} \quad \text{and} \quad \tilde{S}(I, f) = \{t \in I : \delta_t = f(t)\}.$$

We have almost surely for each $I = (a, b) \subset [0, 1]$ and each f as described above,

$$\dim_{\mathcal{H}} S(I, f) = \dim_{\mathcal{H}} \tilde{S}(I, f) = \sup\{1/f(t) : t \in I \setminus \mathfrak{C}\}.$$

1.7.1 Proof of Theorem 1.2: Pointwise spectrum of \mathcal{M} when $\sigma \neq 0$

When the diffusion term does exist, the computation of the pointwise spectrum is easier to state. Fix one point $t \in \mathbb{R}^+$. Let

$$I_t^n := \left[t - \frac{1}{n}, t + \frac{1}{n} \right] \cap \mathbb{R}^+.$$

- If $h > 1/2$, then $D_{\mathcal{M}}(t, h) = -\infty$ by item 1. of Theorem 1.3.

• If $h < 1/2$, then

$$\begin{aligned} E_{\mathcal{M}}(h) \cap I_t^n &= \left\{ s \in I_t^n : h = \frac{1}{\delta_s \beta(s)} \wedge \frac{1}{2} \right\} \\ &= \left\{ s \in I_t^n : h = \frac{1}{\delta_s \beta(s)} \right\} = \left\{ s \in I_t^n : \delta_s = \frac{1}{h\beta(s)} \right\}. \end{aligned}$$

But $\overline{\text{Range } \beta} \subset (0, 2)$ so that $\frac{1}{h\beta(s)} > 1, \forall s \in I_t^n$. This yields that $\dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_t^n = \sup \{h\beta(s) : s \in I_t^n\}$ by Theorem 1.4. Hence

$$D_{\mathcal{M}}(t, h) = \lim_{n \rightarrow +\infty} \dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_t^n = h \cdot (\beta(t) \vee \beta(t-)).$$

• If $h = 1/2$. For every $h' < 1/2$, let $\tilde{E}_{h'} = \left\{ s \in \mathbb{R}^+ : \delta_s \geq \frac{1}{h'\beta(s)} \right\}$. Clearly, for every $h' < 1/2$, $E_{\mathcal{M}}(h') \subset \tilde{E}_{h'}$ and Theorem 1.4 gives

$$\dim_{\mathcal{H}} E_{\mathcal{M}}(h') = \dim_{\mathcal{H}} \tilde{E}_{h'}, \text{ for all } h' < 1/2. \quad (1.20)$$

Now decompose

$$\begin{aligned} I_t^n &= \left(\bigcup_{h' < 1/2} (E_{\mathcal{M}}(h') \cap I_t^n) \right) \cup (E_{\mathcal{M}}(1/2) \cap I_t^n) \\ &\subset \left(\bigcup_{h' < 1/2} (\tilde{E}_{h'} \cap I_t^n) \right) \cup (E_{\mathcal{M}}(1/2) \cap I_t^n). \end{aligned}$$

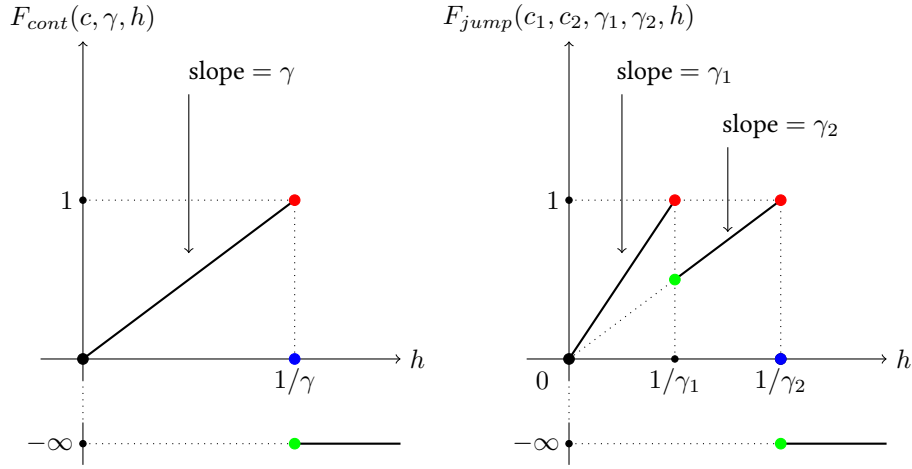
Hence

$$\begin{aligned} 1 = \dim_{\mathcal{H}}(I_t^n) &\leq \left(\dim_{\mathcal{H}} \bigcup_{h' < 1/2} (\tilde{E}_{h'} \cap I_t^n) \right) \vee (\dim_{\mathcal{H}} E_{\mathcal{M}}(1/2) \cap I_t^n) \\ &= \left(\lim_{h' \uparrow 1/2} \dim_{\mathcal{H}} (\tilde{E}_{h'} \cap I_t^n) \right) \vee (\dim_{\mathcal{H}} E_{\mathcal{M}}(1/2) \cap I_t^n) \\ &= \left(\frac{1}{2} \sup \{ \beta(s) : s \in I_t^n \} \right) \vee (\dim_{\mathcal{H}} E_{\mathcal{M}}(1/2) \cap I_t^n), \end{aligned}$$

where we used the monotonicity of the sets $(\tilde{E}_{h'})_{h' < 1/2}$ and (1.20). Since $\overline{\text{Range } \beta} \subset (0, 2)$, $\dim_{\mathcal{H}} E_{\mathcal{M}}(1/2) \cap I_t^n = 1$, which yields $D_{\mathcal{M}}(t, 1/2) = 1$.

1.7.2 Definitions and statement of the results when $\sigma \equiv 0$

When the diffusion does not vanish, it eliminates all the problems we encounter at some specific points. To deal with all possible situations, we need to introduce some notations. For $t \in J$, we


 Figure 1.2: Functions F_{cont} and F_{jump}

define

$$I_{t+}^n := \begin{cases} (t, t + \frac{1}{n}] & \text{if } \beta(t) > \beta(t-), \\ [t - \frac{1}{n}, t) & \text{otherwise.} \end{cases}$$

For all $t \in J$, we need the mapping $\beta^{t+}(s) : I_{t+}^n \cup \{t\} \rightarrow \mathbb{R}$ defined by

$$\beta^{t+}(s) = \begin{cases} \beta(s) & \text{if } s \in I_{t+}^n, \\ \lim_{I_{t-}^n \ni u \rightarrow t} \beta(u) & \text{if } s = t. \end{cases}$$

The map β^{t+} coincides with β except at t . Similarly, we set

$$I_{t-}^n := \left[t - \frac{1}{n}, t + \frac{1}{n} \right] \setminus I_{t+}^n \setminus \{t\}, \quad \beta^{t-}(s) = \begin{cases} \beta(s) & \text{if } s \in I_{t-}^n, \\ \lim_{I_{t+}^n \ni u \rightarrow t} \beta(u) & \text{if } s = t. \end{cases}$$

Throughout this section, we write $t \in LM(F)$ to mean that t is a strict local minimum for a mapping F .

Finally, we introduce two functions F_{cont} and F_{jump} (see Figure 1.2) which correspond to different cases of the pointwise spectra.

- For a time t where the process is continuous, we will use

$$F_{cont}(c, \gamma, h) = \begin{cases} \gamma h & \text{if } h \in [0, 1/\gamma), \\ c & \text{if } h = 1/\gamma, \\ -\infty & \text{otherwise.} \end{cases}$$

There will be only three possible values for c (1, 0 and $-\infty$).

- If t is a jump time for the process, we will use the function F_{jump}

$$F_{jump}(c_1, c_2, \gamma_1, \gamma_2, h) = \begin{cases} \gamma_1 \cdot h & \text{if } h \in [0, 1/\gamma_1), \\ c_1 & \text{if } h = 1/\gamma_1, \\ \gamma_2 \cdot h & \text{if } h \in [1/\gamma_1, 1/\gamma_2), \\ c_2 & \text{if } h = 1/\gamma_2, \\ -\infty & \text{otherwise,} \end{cases}$$

when $\gamma_1 > \gamma_2$. There will be three possible values for c_2 (1, 0 and $-\infty$) and two for c_1 (1 and $h \cdot \gamma_2$).

The several cases in the theorems below correspond to assigning a precise value to the discontinuous points of the pointwise spectrum, and various scenarii may occur, depending on the fact that t is or not a local minimum for the functions β^{t+} and β^{t-} . The reader shall keep in mind the following heuristics:

if \mathcal{M} is continuous at t , its pointwise spectrum looks like F_{cont} ,
if t is a jump time, the pointwise spectrum looks like F_{jump} .

Theorem 1.5. *Assume that $\sigma \equiv 0$ and (\mathcal{H}) holds (see Theorem 1.3). Then, with probability one,*

1. *for every $t \notin J$, the pointwise spectrum of \mathcal{M} at time t is given by*

$$D_{\mathcal{M}}(t, h) = \begin{cases} F_{cont}(1, \beta(t), h) & \text{if } t \notin LM(\beta), \\ F_{cont}(0, \beta(t), h) & \text{if } t \in LM(\beta) \text{ and } \delta_t = 1, \\ F_{cont}(-\infty, \beta(t), h) & \text{if } t \in LM(\beta) \text{ and } \delta_t \neq 1. \end{cases}$$

2. *Assume that $t \in J$ and that $t \notin LM(\beta^{t-}) \cup LM(\beta^{t+})$. Define for $t \in J$*

$$\beta_m(t) = \beta(t) \wedge \beta(t-) \quad \text{and} \quad \beta_M(t) = \beta(t) \vee \beta(t-). \quad (1.21)$$

We have

$$D_{\mathcal{M}}(t, h) = F_{jump}(1, 1, \beta_M(t), \beta_m(t)).$$

This theorem covers the most frequent cases, i.e. when t is a continuous time or t is a jump time and not a strict local minimum for β^{t+} and β^{t-} .

Next theorem covers all the "annoying" cases, i.e. when t is a jump time and is a minimum for at least one of the two functions β^{t+} and β^{t-} . Observe that this concerns at most a countable number of times.

Theorem 1.6. *Assume that $\sigma \equiv 0$ and (\mathcal{H}) holds. Almost surely:*

1. *If $t \notin LM(\beta^{t+})$, then*

$$D_{\mathcal{M}}(t, h) = \begin{cases} F_{jump}(1, 0, \beta_M(t), \beta_m(t), h) & \text{if } t \in LM(\beta^{t-}), \Delta\beta(t) > 0 \text{ and } \delta_t = 1, \\ F_{jump}(1, -\infty, \beta_M(t), \beta_m(t), h) & \text{if } t \in LM(\beta^{t-}), \Delta\beta(t) < 0 \text{ or } \delta_t \neq 1. \end{cases}$$

2. *If $t \in LM(\beta^{t+})$, then*

$$D_{\mathcal{M}}(t, h) = \begin{cases} F_{jump}(h \cdot \beta_m(t), 1, \beta_M(t), \beta_m(t), h) & \text{if } t \notin LM(\beta^{t-}), \\ F_{jump}(h \cdot \beta_m(t), 0, \beta_M(t), \beta_m(t), h) & \text{if } t \in LM(\beta^{t-}), \Delta\beta(t) > 0, \delta_t = 1. \\ F_{jump}(h \cdot \beta_m(t), -\infty, \beta_M(t), \beta_m(t), h) & \text{if } t \in LM(\beta^{t-}), \Delta\beta(t) < 0 \text{ or } \delta_t \neq 1. \end{cases}$$

When t is a jump time, the behaviors of \mathcal{M} on the right hand-side and on the left hand-side of t may differ a lot. So the pointwise spectrum reflects the superposition of two local behaviors, which explains the formulas above. Though not easy to read, these formulas are simple consequences of these complications that may arise as very special cases.

1.7.3 First part of the proof of Theorems 1.5 and 1.6: the linear parts

We start with an easy lemma.

Lemma 1.6. *Assume that $\sigma \equiv 0$ and (\mathcal{H}) holds. Almost surely, for Lebesgue-almost every $t \in [0, 1]$, $h_{\mathcal{M}}(t) = 1/\beta(t)$.*

Proof. Using Theorem 1.3, one sees that for $I = (a, b) \subset [0, 1]$,

$$\dim_{\mathcal{H}}\{t \in I : H_{\mathcal{M}}(t) = \kappa/\beta(t)\} = \dim_{\mathcal{H}}\{t \in I : \delta_t = 1/\kappa\}.$$

Let $\kappa \in (0, 1)$. We apply Theorem 1.4 to the Poisson system \mathcal{P} and the mapping $f(t) \equiv 1/\kappa$: this yields directly $\dim_{\mathcal{H}}\{t \in I : H_{\mathcal{M}}(t) = \kappa/\beta(t)\} = \kappa$. Still by Theorem 1.4, one has

$$\dim_{\mathcal{H}}\{t \in I : h_{\mathcal{M}}(t) \leq \kappa/\beta(t)\} = \kappa. \quad (1.22)$$

Next, let us decompose the interval I as

$$I = \left\{t \in I : H_{\mathcal{M}}(t) = 1/\beta(t)\right\} \cup \left(\bigcup_{n \geq 1} S_n\right) \quad (1.23)$$

where $S_n := \{t \in I : H_{\mathcal{M}}(t) \leq (1 - 1/n)/\beta(t)\}$. For every $n \geq 1$, the Lebesgue measure of S_n is zero since it has Hausdorff dimension strictly less than 1, by (1.22). We deduce by (1.23) that for Lebesgue-a.e. $t \in I$, $H_{\mathcal{M}}(t) = 1/\beta(t)$. Since this holds for any interval $(a, b) \subset [0, 1]$, the conclusion follows. \square

We only prove the result for $t \in J$. If t is a continuous time for \mathcal{M} , the pointwise spectrum at t is obtained directly since in this case $\beta(t) = \beta_m(t) = \beta_M(t)$. We treat separately three linear parts of F_{jump} .

• If $h < \frac{1}{\beta_M(t)}$, there exists $\varepsilon > 0$ such that $h < \frac{1}{\beta_M(t) + \varepsilon}$. But $\forall s \in I_t^n$, $\beta(s) < \beta_M(t) + \varepsilon/2$ by the càdlàg property of the sample paths, which implies

$$\frac{1}{h\beta(s)} > \frac{\beta_M(t) + \varepsilon}{\beta_M(t) + \varepsilon/2} > 1$$

for all $s \in I_t^n$ with n large enough. Theorem 1.4 implies that

$$\begin{aligned} \dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_t^n &= \dim_{\mathcal{H}} \left\{ s \in I_t^n : \delta_s = \frac{1}{h\beta(s)} \right\} \\ &= \sup \{ h\beta(s) : s \in I_t^n \} = h \cdot \sup \{ \beta(s) : s \in I_t^n \}. \end{aligned}$$

for large n , which yields $D_{\mathcal{M}}(t, h) = \lim_{n \rightarrow +\infty} \sup \{ h\beta(s) : s \in I_t^n \} = h \cdot (\beta_M(t))$.

• If $h \in \left(\frac{1}{\beta_M(t)}, \frac{1}{\beta_m(t)} \right)$, there exists $\varepsilon > 0$ small enough so that h is in the interval $\left(\frac{1}{\beta_M(t) - \varepsilon}, \frac{1}{\beta_m(t) + \varepsilon} \right)$. Let us consider separately I_{t+}^n and I_{t-}^n . For all $s \in I_{t+}^n$,

$$\frac{1}{h\beta(s)} \leq \frac{\beta_M(t) - \varepsilon}{\beta_M(t) - \varepsilon/2} < 1$$

by the càdlàg property of the sample paths, for n large enough. Hence $E_{\mathcal{M}}(h) \cap I_{t+}^n = \left\{ s \in I_{t+}^n : \delta_s = \frac{1}{h\beta(s)} \right\} = \emptyset$, because $\delta_s \geq 1$ almost surely. For all $s \in I_{t-}^n$,

$$\frac{1}{h\beta(s)} > \frac{\beta_m(t) + \varepsilon}{\beta_m(t) + \varepsilon/2} > 1$$

which implies by Theorem 1.4 that

$$\dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_t^n = \dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_{t-}^n = h \cdot \sup \{ \beta(s) : s \in I_{t-}^n \}$$

for large n . Using the càdlàg property of the sample paths, one concludes that $D_{\mathcal{M}}(t, h) = \lim_{n \rightarrow +\infty} \sup \{ h\beta(s) : s \in I_{t-}^n \} = h \cdot (\beta_m(t))$.

• If $h > \frac{1}{\beta_m(t)}$, one can choose $\varepsilon > 0$ small enough so that $h > \frac{1}{\beta_m(t) - \varepsilon}$. But $\forall s \in I_t^n$,

$\beta(s) > \beta_m(t) - \varepsilon$ for large n . So,

$$h > \frac{1}{\beta(s)} \geq \frac{1}{\delta_s \beta(s)} = H_{\mathcal{M}}(s)$$

yields $E_{\mathcal{M}}(h) \cap I_t^n = \emptyset$. Hence, $D_{\mathcal{M}}(t, h) = \lim_{n \rightarrow +\infty} \dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_t^n = -\infty$.

1.7.4 Second part of the proof of Theorems 1.5 and 1.6: the points of discontinuities of F_{cont} and F_{jump}

There are two possible discontinuities for F_{jump} , which are $\frac{1}{\beta_M(t)}$ and $\frac{1}{\beta_m(t)}$.

• If $h = \frac{1}{\beta_M(t)}$, we distinguish between two cases.

Case 1 : $t \in LM(\beta^{t+})$. Then $\forall s \in I_{t+}^n$, $\frac{1}{h\beta(s)} = \frac{\beta_M(t)}{\beta(s)} < 1$, which implies $E_{\mathcal{M}}(h) \cap I_{t+}^n = \emptyset$. Notice that

$$E_{\mathcal{M}}(h) \cap I_{t-}^n = \left\{ s \in I_{t-}^n : \delta_s = \frac{\beta_M(t)}{\beta(s)} \right\}.$$

For every $s \in I_{t-}^n$, one has

$$\frac{\beta_M(t)}{\beta(s)} \geq \frac{\beta_M(t)}{\beta_m(t) + |\Delta\beta_t|/2} > 1.$$

This ensures that $\dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_{t-}^n = \sup \left\{ \frac{\beta(s)}{\beta_M(t)} : s \in I_{t-}^n \right\}$, still by Theorem 1.4. Therefore,

$$\dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_t^n = \dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap (I_{t-}^n \cup \{t\}) = \dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_{t-}^n,$$

which yields $D_{\mathcal{M}}(t, h) = \lim_{n \rightarrow +\infty} \dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_{t-}^n = \frac{\beta_m(t)}{\beta_M(t)} = h \cdot \beta_m(t)$.

Case 2 : $t \notin LM(\beta^{t+})$. In this case, either t is not a local minimum for β^{t+} , or β^{t+} is locally constant near t . If t is not a local minimum for β^{t+} , one can extract a monotone sequence $\{s_k\} \subset I_{t+}^n$ tending to t such that

$$\beta(s_k) < \beta_M(t). \quad (1.24)$$

Since β is càdlàg and the cardinality of J is at most countable, we can choose s_k as continuous points for β . Let us first compute the pointwise spectrum of \mathcal{M} on points s_k and deduce the result by a regularity restriction of the pointwise spectrum.

Fix $k \geq 1$ and let p be large enough. For every $s \in I_{s_k}^p$, one has $\frac{\beta_M(t)}{\beta(s)} > 1$ by (1.24). Further, Theorem 1.4 ensures that

$$\dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_{s_k}^p = \sup \{ h\beta(s) : s \in I_{s_k}^p \},$$

which yields that $D_{\mathcal{M}}(s_k, h) = h\beta(s_k)$, for every integer $k \geq 1$. Hence

$$1 \geq D_{\mathcal{M}}(t, h) = \limsup_{s \rightarrow t} D_{\mathcal{M}}(s, h) \geq \limsup_{k \rightarrow +\infty} D_{\mathcal{M}}(s_k, h) = h\beta_M(t) = 1$$

where we used Proposition 1.1. If β^{t+} is locally constant near t , then $E_{\mathcal{M}}(h) \cap I_{t+}^n = \{s \in I_{t+}^n : \delta_s = \frac{1}{h\beta(s)} = 1\}$ for n large. Applying Lemma 1.6, one deduces that $\text{Leb}(E_{\mathcal{M}}(h) \cap I_{t+}^n) = \text{Leb}(I_{t+}^n)$ where Leb denotes the Lebesgue measure, and so $\dim_{\mathcal{H}}(E_{\mathcal{M}}(h) \cap I_{t+}^n) = 1$. One concludes that

$$1 \geq D_{\mathcal{M}}(t, h) = \lim_{n \rightarrow +\infty} \dim_{\mathcal{H}}(E_{\mathcal{M}}(h) \cap I_t^n) \geq \lim_{n \rightarrow +\infty} \dim_{\mathcal{H}}(E_{\mathcal{M}}(h) \cap I_{t+}^n) = 1.$$

• If $h = \frac{1}{\beta_m(t)}$: As before, we distinguish two cases.

Case 1 : $t \in LM(\beta^{t-})$. Then $\forall s \in I_{t-}^n$, one has $\frac{\beta_m(t)}{\beta(s)} < 1$, which implies that $E_{\mathcal{M}}(h) \cap I_{t-}^n = \emptyset$. Notice that $E_{\mathcal{M}}(h) \cap I_{t+}^n = \left\{s \in I_{t+}^n : \delta_s = \frac{\beta_m(t)}{\beta(s)}\right\}$ and that $\forall s \in I_{t+}^n$,

$$\frac{\beta_m(t)}{\beta(s)} < \frac{\beta_m(t)}{\beta_M(t) - |\Delta\beta_t|/2} < 1.$$

Hence $E_{\mathcal{M}}(h) \cap I_{t+}^n = \emptyset$, for large n . But

$$E_{\mathcal{M}}(h) \cap \{t\} = \begin{cases} \{t\} & \text{if } \beta(t-) > \beta(t) \text{ and } \delta_t = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence,

$$\dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_t^n = \dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap \{t\} = \begin{cases} 0 & \text{if } \beta(t-) > \beta(t) \text{ and } \delta_t = 1, \\ -\infty & \text{otherwise,} \end{cases}$$

for large n , which yields

$$D_{\mathcal{M}}(t, h) = \begin{cases} 0 & \text{if } \beta(t-) > \beta(t) \text{ and } \delta_t = 1, \\ -\infty & \text{otherwise.} \end{cases}$$

Case 2 : $t \notin LM(\beta^{t-})$. Then, either t is not a local minimum for β^{t-} , or β^{t-} is locally constant near t . If t is not a local minimum for β^{t-} . By a similar argument as in the second case of the last situation, we can prove that $d(s_k, h) = h\beta(s_k)$ where $\{s_k\} \subset I_{t-}^n \setminus \mathcal{J}$ is a strictly monotone sequence tending to t satisfying $\beta(s_k) < \beta_m(t)$. Therefore,

$$1 \geq D_{\mathcal{M}}(t, h) = \limsup_{s \rightarrow t} D_{\mathcal{M}}(s, h) \geq \limsup_{k \rightarrow +\infty} D_{\mathcal{M}}(s_k, h) = h(\beta_m(t)) = 1.$$

If β^{t-} is locally constant near t , then $E_{\mathcal{M}}(h) \cap I_{t-}^n = \{s \in I_{t-}^n : \delta_s = \frac{\beta_m(t)}{\beta(s)} = 1\}$ for n large. Still by Lemma 1.6, one has $\text{Leb}(E_{\mathcal{M}}(h) \cap I_{t-}^n) = \text{Leb}(I_{t-}^n)$, which yields $\dim_{\mathcal{H}}(E_{\mathcal{M}}(h) \cap I_{t-}^n) = 1$. Hence, $1 \geq D_{\mathcal{M}}(t, h) \geq \lim_{n \rightarrow +\infty} \dim_{\mathcal{H}}(E_{\mathcal{M}}(h) \cap I_{t-}^n) = 1$.

1.8 Existence of tangent processes

In order to describe the local structure of stochastic processes which are often rough (not differentiable), several authors consider the tangent processes associated with them, see for instance [49]. Precisely, given a stochastic process X and t_0 a fixed time, one wonders if there exist two sequences $(\alpha_n)_{n \geq 1}$, $(r_n)_{n \geq 1}$ decreasing to zero such that the sequence of process $(r_n(X_{t_0 + \alpha_n t} - X_{t_0}))_{t \geq 0}$ converges in law to some limit process $(Y_t)_{t \geq 0}$, and call it, if exists, a tangent process. One observes in Theorem 1.2 and Theorem 1.5 that the pointwise spectrum of the process \mathcal{M} looks like (but not exactly) the spectrum of some Lévy process. Then natural questions concern the connections between the pointwise spectrum of the process at t_0 and its tangent process at this point. In the stable-like case, we show the existence of tangent processes of \mathcal{M} , which are some stable Lévy processes. Their spectra coincide with the pointwise spectra of \mathcal{M} at time t except for one value of h . Here, the scaling (r_n, α_n) must be carefully chosen and plays an important role.

Throughout this section, the Skorohod space of càdlàg functions on $[0, 1]$ is endowed with the uniform convergence topology. We consider the function $G_0(x, z) = \text{sign}(z)|z|^{1/\tilde{\beta}(x)}$ with $\tilde{\beta}$ Lipschitz continuous and $\text{Range } \tilde{\beta} \subset (0, 2)$, and the pure jump diffusion

$$\mathcal{M}_t = \int_0^t \int_{C(0,1)} G_0(\mathcal{M}_{s-}, z) \tilde{N}(dsdz).$$

Proposition 1.9. *Let $t_0 \geq 0$ be fixed, conditionally on \mathcal{F}_{t_0} , the family of processes $\left(\frac{\mathcal{M}_{t_0 + \alpha t} - \mathcal{M}_{t_0}}{\alpha^{1/\tilde{\beta}(t_0)}}\right)_{t \in [0,1]}$ converges in law to a stable Lévy process with Lévy measure $\beta(t_0)u^{-1-\beta(t_0)} du$, when $\alpha \rightarrow 0$.*

The next lemma gives some moment estimate for \mathcal{M} near 0. The second point was proved in [12], we still prove it for the sake of completeness. Let us introduce the stopping times for every $\eta > 0$

$$\tau_\eta := \inf\{t > 0 : \beta(t) > \tilde{\beta}(0) + \eta\}.$$

Lemma 1.7. *Let $\eta > 0$ be small.*

(i) *If $\tilde{\beta}(0) \geq 1$, for every $\gamma \in (\tilde{\beta}(0) + \eta, 2)$, there exists a constant c_γ such that $\forall \alpha > 0$,*

$$\mathbb{E}[|\mathcal{M}_{\alpha \wedge \tau_\eta}|^\gamma] \leq c_\gamma \alpha.$$

(ii) *If $\tilde{\beta}(0) < 1$, for every $\gamma \in (\tilde{\beta}(0) + \eta, 1 \wedge 2\tilde{\beta}(0))$, the same moment inequality holds.*

Proof. (i) Since \mathcal{M} is a martingale, by Burkholder-Davis-Gundy inequality and the symmetry of G_0 in z , we have

$$\begin{aligned} \mathbb{E}[|\mathcal{M}_{\alpha \wedge \tau_\eta}|^\gamma] &\leq \mathbb{E}\left[\sup_{0 \leq t \leq \alpha \wedge \tau_\eta} |\mathcal{M}_t|^\gamma\right] \\ &\leq c_\gamma \mathbb{E}\left[\left|\int_0^{\alpha \wedge \tau_\eta} \int_{C(0,1)} |G_0(\mathcal{M}_{s-}, z)|^2 N(ds dz)\right|^{\gamma/2}\right] \\ &\leq c_\gamma \mathbb{E}\left[\int_0^{\alpha \wedge \tau_\eta} \int_0^1 |G_0(\mathcal{M}_{s-}, z)|^\gamma N(ds dz)\right] \\ &= c_\gamma \mathbb{E}\left[\int_0^{\alpha \wedge \tau_\eta} \int_0^1 |G_0(\mathcal{M}_{s-}, z)|^\gamma dz/z^2 ds\right] \end{aligned}$$

For every $s \in [0, \tau_\eta)$, one has

$$\int_0^1 |G_0(\mathcal{M}_{s-}, z)|^\gamma dz/z^2 = \int_0^1 |z|^{\gamma/\beta(s-)} dz/z^2 \leq \int_0^1 |z|^{\gamma/(\tilde{\beta}(0)+\eta)} dz/z^2 < +\infty,$$

where we used that $\gamma > \tilde{\beta}(0) + \eta$. Hence,

$$\mathbb{E}[|\mathcal{M}_{\alpha \wedge \tau_\eta}|^\gamma] \leq c_\gamma \mathbb{E}\left[\int_0^{\alpha \wedge \tau_\eta} ds\right] \leq c_\gamma \alpha.$$

(ii) For every $s \in [0, \tau_\eta)$ with η small enough, it makes sense to separate the compensated Poisson measure, i.e. the difference of the Poisson measure and its intensity. Using $(a+b)^\gamma \leq c_\gamma(a^\gamma + b^\gamma)$ for all $(a, b) \in \mathbb{R}_+^2$, subadditivity and symmetry, we have

$$\begin{aligned} \mathbb{E}[|\mathcal{M}_{\alpha \wedge \tau_\eta}|^\gamma] &\leq c_\gamma \mathbb{E}\left[\left|\int_0^{\alpha \wedge \tau_\eta} \int_{C(0,1)} |G_0(\mathcal{M}_{s-}, z)| N(ds dz)\right|^\gamma\right] \\ &\quad + c_\gamma \mathbb{E}\left[\left|\int_0^{\alpha \wedge \tau_\eta} \int_{C(0,1)} |G_0(\mathcal{M}_{s-}, z)| dz/z^2 ds\right|^\gamma\right] \\ &\leq c_\gamma \mathbb{E}\left[\int_0^{\alpha \wedge \tau_\eta} \int_0^1 |G_0(\mathcal{M}_{s-}, z)|^\gamma dz/z^2 ds\right]. \end{aligned}$$

Repeating the arguments of the first point yields the result. \square

Lemma 1.8. *Let x_0 be fixed. For all $\gamma > \tilde{\beta}(x_0)$, there exist strictly positive constants C_γ and δ such that for all $x \in B(x_0, \delta)$*

$$\int_{C(0,1)} |G_0(x, z) - G_0(x_0, z)|^\gamma \pi(dz) \leq C_\gamma |x - x_0|^\gamma.$$

It is easy to check Lemma 1.8. Now we prove Proposition 1.9, using the self-similarity of the limit process and last two lemmas.

Proof. By the Markov property, it is enough to prove the proposition for $t_0 = 0$. Let us introduce

$$\mathcal{L}_t = \int_0^t \int_{C(0,1)} G_0(0, z) \tilde{N}(dsdz), \quad \mathcal{S}_t = \int_0^t \int_{C(0,+\infty)} G_0(0, z) \tilde{N}(dsdz).$$

Note that \mathcal{L} and \mathcal{S} are pure jump Lévy processes whose Lévy measure are $\tilde{\beta}(0)|z|^{-\tilde{\beta}(0)-1}\mathbf{1}_{C(0,1)}dz$ and $\tilde{\beta}(0)|z|^{-\tilde{\beta}(0)-1}dz$, respectively. As is well known, \mathcal{S} is $1/\tilde{\beta}(0)$ self-similar, meaning that for every $\alpha > 0$, one has

$$\left(\alpha^{-1/\tilde{\beta}(0)} \mathcal{S}_{\alpha t} \right)_{t \in [0,1]} = (\mathcal{S}_t)_{t \in [0,1]}$$

in law, see for instance Chapter 3 of Sato [104]. Observe that $\forall \delta > 0$,

$$\mathbb{P} \left(\sup_{0 \leq t \leq 1} \left| \alpha^{-1/\tilde{\beta}(0)} (\mathcal{L}_{\alpha t} - \mathcal{S}_{\alpha t}) \right| \leq \delta \right) \geq \mathbb{P} (N([0, \alpha], D(1, +\infty)) = 0) = e^{-\alpha} \rightarrow_{\alpha \downarrow 0} 1.$$

This computation yields that

$$\alpha^{-1/\tilde{\beta}(0)} \sup_{t \in [0,1]} |\mathcal{L}_{\alpha t} - \mathcal{S}_{\alpha t}| \rightarrow 0$$

in probability, when $\alpha \rightarrow 0$. Recall that the self-similarity of \mathcal{S} ensures that $(\alpha^{-1/\tilde{\beta}(0)} \mathcal{S}_{\alpha t})_{t \in [0,1]}$ converges (equals) in law to $(\mathcal{S}_t)_{t \in [0,1]}$, thus the process $(\alpha^{-1/\tilde{\beta}(0)} \mathcal{L}_{\alpha t})_{t \in [0,1]}$ converges in law to $(\mathcal{S}_t)_{t \in [0,1]}$. To conclude, it remains to prove the following

$$\alpha^{-1/\tilde{\beta}(0)} \Delta_\alpha \rightarrow 0 \text{ in probability,}$$

where $\Delta_\alpha := \sup_{0 \leq t \leq \alpha} |\mathcal{M}_t - \mathcal{L}_t|$. There are two cases.

Case 1 : $\tilde{\beta}(0) \geq 1$. Applying the Burkholder–Davis–Gundy inequality and a subadditivity property, one has, for every $\gamma \in (\tilde{\beta}(0) + \eta, 2)$,

$$\begin{aligned} \mathbb{E}[|\Delta_{\alpha \wedge \tau_\eta}|^\gamma] &\leq c_\gamma \mathbb{E} \left[\int_0^{\alpha \wedge \tau_\eta} \int_{C(0,1)} |G_0(\mathcal{M}_{s-}, z) - G_0(0, z)|^\gamma dz/z^2 ds \right] \\ &\leq c_\gamma \mathbb{E} \left[\int_0^{\alpha \wedge \tau_\eta} |\mathcal{M}_s|^\gamma ds \right] \leq c_\gamma \int_0^\alpha \mathbb{E}[|\mathcal{M}_{s \wedge \tau_\eta}|^\gamma] ds \leq c_\gamma \alpha^2, \end{aligned}$$

where we used Lemma 1.8 and Lemma 1.7. Hence, for every $\delta > 0$, one has

$$\mathbb{P} \left(\alpha^{-1/\tilde{\beta}(0)} \Delta_\alpha \geq \delta \right) \leq \mathbb{P}(\tau_\eta \leq \alpha) + \mathbb{P} \left(\alpha^{-1/\tilde{\beta}(0)} \Delta_{\alpha \wedge \tau_\eta} \geq \delta \right), \quad (1.25)$$

where $\lim_{\alpha \downarrow 0+} \mathbb{P}(\tau_\eta \leq \alpha) = \mathbb{P}(\tau_\eta = 0) = 0$ and

$$\mathbb{P} \left(\alpha^{-1/\tilde{\beta}(0)} \Delta_{\alpha \wedge \tau_\eta} \geq \delta \right) \leq \delta^{-\gamma} \alpha^{-\gamma/\tilde{\beta}(0)} \mathbb{E}[|\Delta_{\alpha \wedge \tau_\eta}|^\gamma] \leq c_{\delta, \gamma} \alpha^{2-\gamma/\tilde{\beta}(0)} \rightarrow 0, \quad (1.26)$$

since $2\tilde{\beta}(0) \geq 2 > \gamma > \tilde{\beta}(0) + \eta$.

Case 2 : $\tilde{\beta}(0) < 1$. As in Lemma 1.7, for every $s \in [0, \tau_\eta)$ with η small enough, it makes sense to separate the compensated Poisson measure. By subadditivity, for every $\gamma \in (\tilde{\beta}(0) + \eta, 1 \wedge 2\tilde{\beta}(0))$,

$$\begin{aligned} \mathbb{E}[|\Delta_{\alpha \wedge \tau_\eta}|^\gamma] &\leq c_\gamma \mathbb{E} \left[\left| \int_0^{\alpha \wedge \tau_\eta} \int_{C(0,1)} |G_0(\mathcal{M}_{s-}, z) - G_0(0, z)| N(ds dz) \right|^\gamma \right] \\ &\quad + c_\gamma \mathbb{E} \left[\left| \int_0^{\alpha \wedge \tau_\eta} \int_{C(0,1)} |G_0(\mathcal{M}_{s-}, z) - G_0(0, z)| dz/z^2 ds \right|^\gamma \right] \\ &\leq c_\gamma \mathbb{E} \left[\int_0^{\alpha \wedge \tau_\eta} \int_{C(0,1)} |G_0(\mathcal{M}_{s-}, z) - G_0(0, z)|^\gamma dz/z^2 ds \right] \\ &\leq c_\gamma \mathbb{E} \left[\int_0^{\alpha \wedge \tau_\eta} |\mathcal{M}_s|^\gamma ds \right] \leq c_\gamma \int_0^\alpha \mathbb{E}[|\mathcal{M}_{s \wedge \tau_\eta}|^\gamma] ds \leq c_\gamma \alpha^2, \end{aligned}$$

where we used again Lemma 1.8 and Lemma 1.7. Repeating the computations (1.25), (1.26) and using $\gamma \in (\tilde{\beta}(0) + \eta, 1 \wedge 2\tilde{\beta}(0))$ yield the result. \square

Appendix 1

Proof of Proposition 1.2 : By the Lipschitz assumption on σ and b , it suffices to show that $G \in \mathcal{G}$ satisfies those two conditions.

(i) We check the growth condition. We divide the integral into two parts, use (1.5) and the uniform boundedness of G to get

$$\begin{aligned} \int_{C(0,1)} G(x, z)^2 \frac{dz}{z^2} &= \int_{C(0, z_0)} G(x, z)^2 \frac{dz}{z^2} + \int_{C(z_0, 1)} G(x, z)^2 \frac{dz}{z^2}. \\ &\leq \int_{C(0, z_0)} |z|^{1+2\varepsilon_0} \frac{dz}{z^2} + \int_{C(z_0, 1)} \frac{dz}{z^2} \\ &= \frac{2z_0^{2\varepsilon_0}}{2\varepsilon_0} + 2 \left(\frac{1}{z_0} - 1 \right) := K_0 \leq K_0(1 + x^2). \end{aligned}$$

(ii) Let us verify the Lipschitz condition. Let $(x, y) \in \mathbb{R}^2$ with $x \neq y$. Assume without loss of generality that $\log |G(x, z)| > \log |G(y, z)|$. By symmetry,

$$\begin{aligned} \int_{C(0,1)} |G(x, z) - G(y, z)|^2 \frac{dz}{z^2} &= 2 \int_0^1 |G(x, z)|^2 \left(1 - e^{\log |G(y, z)| - \log |G(x, z)|} \right)^2 \frac{dz}{z^2} \\ &\leq 2 \int_0^1 |G(x, z)|^2 (\log |G(x, z)| - \log |G(y, z)|)^2 \frac{dz}{z^2} \\ &\leq C|x - y|^2 \int_0^1 |G(x, z)|^2 (\log z)^2 \frac{dz}{z^2}, \end{aligned}$$

where we used the inequality $1 - e^{-u} \leq u$ for all $u > 0$ and the Lipschitz condition on $G \in \mathcal{G}$. To conclude, it remains to show that this integral is finite and independent of the value of x . Indeed, still by (1.5) and the uniform boundedness of G ,

$$\begin{aligned} \int_0^1 G(x, z)^2 (\log z)^2 \frac{dz}{z^2} &= \int_0^{z_0} G(y, z)^2 (\log z)^2 \frac{dz}{z^2} + \int_{z_0}^1 G(y, z)^2 (\log z)^2 \frac{dz}{z^2} \\ &\leq \int_0^{z_0} |z|^{1+2\varepsilon_0-2} (\log z)^2 dz + \int_{z_0}^1 (\log z)^2 \frac{dz}{z^2} \\ &\leq c_1^2 \left(\int_0^{z_0} z^{\varepsilon_0-1} dz + \int_{z_0}^1 z^{-2-\varepsilon_0} dz \right) \\ &\leq c_1^2 \frac{z_0^{\varepsilon_0}}{\varepsilon_0} + \frac{c_1^2}{1+\varepsilon_0} \left(\frac{1}{(z_0)^{1+\varepsilon_0}} - 1 \right) := K_1/C < +\infty, \end{aligned}$$

□

Appendix 2

Recall the martingale representation theorem.

Theorem 1.7 (Dambis-Dubins-Swartz, Th. 5.1.6 [103]). *Let M be a $(\mathcal{F}_t, \mathbb{P})$ -continuous local martingale such that $M_0 = 0$ and $\langle M \rangle_{+\infty} = +\infty$. Let*

$$T_t = \inf\{s \geq 0 : \langle M \rangle_s > t\},$$

then $B_t = M_{T_t}$ is a (\mathcal{F}_{T_t}) -Brownian motion and a.s. $\forall t \in \mathbb{R}^+$, $M_t = B_{\langle M \rangle_t}$.

Proof of Proposition 1.5: Recall that $\mathcal{X}_t = \int_0^t \sigma(\mathcal{M}_s) dB_s$ is a local martingale starting from 0. The quadratic variation process of \mathcal{X}

$$\langle \mathcal{X} \rangle_t = \int_0^t \sigma(\mathcal{M}_s)^2 ds,$$

satisfies $\langle \mathcal{X} \rangle_\infty = \infty$ almost surely, since σ stays away from 0 by assumption. Applying Theorem of Dambis-Dubins-Swartz to \mathcal{X} , one can find a standard Brownian motion \tilde{B} on $(\mathcal{F}, \mathbb{P})$ such that a.s. $\forall t$, $\mathcal{X}_t = \tilde{B}_{\langle \mathcal{X} \rangle_t}$.

First computation yields a.s. for every $t \in \mathbb{R}^+$, $\forall r > 0$, for all $u \in B(t, r)$,

$$c|u - t| \leq |\langle \mathcal{X} \rangle_u - \langle \mathcal{X} \rangle_t| \leq \left| \int_t^u C(1 + |\mathcal{M}_s|)^2 ds \right| \leq C|u - t|, \quad (1.27)$$

where we used that σ stays away from 0 to find the constants $c, C \in \mathbb{R}_*^+$.

By the properties of Lévy's modulus (Theorem 1.2.7 of [103]), for every $\varepsilon > 0$, a.s. for every t ,

for u sufficiently close to t , one has by (1.27)

$$|\mathcal{X}_u - \mathcal{X}_t| = |\tilde{B}_{\langle \mathcal{X} \rangle_u} - \tilde{B}_{\langle \mathcal{X} \rangle_t}| \leq C' |\langle \mathcal{X} \rangle_u - \langle \mathcal{X} \rangle_t|^{\frac{1}{2} - \varepsilon} \leq C' |u - t|^{\frac{1}{2} - \varepsilon}.$$

Hence, almost surely, $\forall t$, $H_{\mathcal{X}}(t) \geq \frac{1}{2} - \varepsilon$.

On the other hand, Dvoretzky [46] proved that, for a standard Brownian motion B , there exists a constant $K > 0$, such that almost surely

$$\forall t, \quad \limsup_{h \rightarrow 0^+} \frac{|B_{t+h} - B_t|}{h^{1/2}} \geq K.$$

Applying Dvoretzky's Theorem to our Brownian motion \tilde{B} , we get that almost surely for every $t \geq 0$, there exists a positive sequence $(h_n)_{n \geq 1}$ converging to zero such that

$$|\tilde{B}_{\langle \mathcal{X} \rangle_{t+h_n}} - \tilde{B}_{\langle \mathcal{X} \rangle_t}| \geq K |h_n|^{1/2}. \quad (1.28)$$

As $t \mapsto \langle \mathcal{X} \rangle_t$ is a strictly increasing (always by the assumption that σ stays away from 0) continuous function, there exists a sequence $(u_n)_{n \geq 1}$ such that $\langle \mathcal{X} \rangle_t + h_n = \langle \mathcal{X} \rangle_{u_n}$. By the first inequality of (1.27),

$$|h_n| \geq c |u_n - t|. \quad (1.29)$$

It follows from (1.28) and (1.29) that

$$|\mathcal{X}_{u_n} - \mathcal{X}_t| = |\tilde{B}_{\langle \mathcal{X} \rangle_{u_n}} - \tilde{B}_{\langle \mathcal{X} \rangle_t}| = |\tilde{B}_{\langle \mathcal{X} \rangle_{t+h_n}} - \tilde{B}_{\langle \mathcal{X} \rangle_t}| \geq Kc |u - t|^{1/2},$$

This yields a.s. $\forall t$, $H_{\mathcal{X}}(t) \leq 1/2$, and letting ε tend to 0 gives the result. \square

Chapitre 2

Extension : dimension supérieure et anisotropie

Dans ce chapitre, on continue l'étude de la régularité locale et la multifractalité des diffusion à sauts. On étend les résultats dans le premier chapitre en dimension supérieure, et on introduit l'anisotropie qui porte sur les coefficients d'EDS et la mesure d'intensité de processus ponctuel de Poisson qui apparaissent dans la construction d'EDS.

In this chapter, we seek to extend the result in [120] in a general setting. In particular, we are interested in asymmetric, higher dimensional construction. We give two ways to add anisotropy to the model. The first is achieved by enlarging the class of admissible functions for the coefficient G . Another way is to allow more general intensity measures for the Poisson point process driving the SDE. The study of tangent processes is also refined.

2.1 Anisotropic coefficient in \mathbb{R}^d

2.1.1 Construction

In this section, we construct the process in higher dimension with anisotropic coefficient. We consider the Markov process which is solution to the following d -dimensional jumping SDE

$$\mathcal{M}_t = \int_0^t \int_{\mathbb{S}^{d-1}} \int_0^1 G(\mathcal{M}_{s-}, \theta, r) \tilde{N}(ds, d\theta, dr). \quad (2.1)$$

Here, N is a Poisson random measure with intensity $\lambda \otimes \alpha \otimes \pi_0$ where λ is Lebesgue measure on \mathbb{R} , α is the uniform probability measure on \mathbb{S}^{d-1} and π_0 is a Lévy measure on \mathbb{R} with density $r^{-2}dr$. The anisotropy comes from the assumptions on G , which is given by the following definition. The idea is that the jump direction of \mathcal{M} is determined by the random Poisson angle θ , the jump time and jump size are determined by both the Poisson system and the function G , which exhibits different asymptotic behavior in different direction θ . Note $C(a, b)$ the annulus in \mathbb{R}^d with radius $0 \leq a < b$.

Definition 2.1. *The set \mathcal{G}^d is the set of those functions $G : \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, 1] \rightarrow C(0, 1)$ satisfying*

1. For every $x \in \mathbb{R}^d, \theta \in \mathbb{S}^{d-1}$,

$$\langle G(x, \theta, r), \theta \rangle = |G(x, \theta, r)|.$$

2. Asymptotically stable-like :

$$\frac{\log |G(x, \theta, r)|}{\log r} \text{ converge uniformly to } \frac{1}{\tilde{\beta}(x, \theta)} \text{ as } r \rightarrow 0.$$

3. Lipschitz condition : there exists $C > 0$ such that for all $(x, y) \in \mathbb{R}^{2d}$, for every $(\theta, r) \in \mathbb{S}^{d-1} \times (0, 1]$,

$$\left| \frac{\log |G(y, \theta, r)| - \log |G(x, \theta, r)|}{\log r} \right| \leq C|x - y|.$$

4. Boundedness : $\overline{\text{Range } \tilde{\beta}} \subset (0, 2)$,

5. Continuity : for all $x \in \mathbb{R}^d, \theta \mapsto \tilde{\beta}(x, \theta)$ is continuous.

Clearly, this is a natural extension of the class \mathcal{G} in dimension d . We need the technical assumption (5), in order to obtain pointwise regularity of \mathcal{M} . Another remark is that the uniform in (2) can be weakened if we add some "symmetry" on the function G , for example, this is the case if we assume that for some large $j^* \in \mathbb{N}$,

$$G(x, \theta, r) = \sum_{k=0}^{2^{(d-1)j^*} - 1} \mathbf{1}_{\Sigma_{j^*,k}}(\theta) G(x, \theta_{j^*,k}, r)$$

where $\theta_{j^*,k}$ is the "leftmost" point of the dyadic "cube" $\Sigma_{j^*,k} \subset \mathbb{S}^{d-1}$.

We give two simple but important examples in the class \mathcal{G}^d .

Example 2.1. • Take $G(x, \theta, r) = \sigma(x)r^{1/\beta}\theta$ with $\beta \in (0, 2)$ and σ Lipschitz continuous bounded from below and above, then (2.1) becomes the following SDE driven by an isotropic β -stable Lévy process $(L_t)_{t \geq 0}$ in \mathbb{R}^d

$$X_t = \int_0^t \sigma(X_{s-}) dL_s$$

- Given $\gamma : \mathbb{R}^d \times \mathbb{S}^{d-1} \mapsto [\varepsilon, 2 - \varepsilon]$ which is Lipschitz continuous in the first variable and continuous in the second. Let $G(x, \theta, r) = r^{1/\gamma(x,\theta)}\theta$, we get the anisotropic stable-like process.

We wrap up main properties of the solution to this SDE in the following proposition.

Proposition 2.1. For every $G \in \mathcal{G}^d$, there exists a unique pathwise solution for (2.1) which is a càdlàg strong Markov process whose generator writes for every $\varphi \in C_b^2(\mathbb{R}^d)$,

$$\mathfrak{L}\varphi(x) = \int_{\mathbb{S}^{d-1}} \int_0^1 (\varphi(x + G(x, \theta, r)) - \varphi(x) - G(x, \theta, r) \cdot \nabla \varphi(x)) \frac{dr}{r^2} \alpha(d\theta).$$

Moreover, \mathcal{M} is a L^2 -martingale with value in \mathbb{R}^d .

Proof. 1. As usual, to prove the pathwise uniqueness and existence, we check the growth condition and the Lipschitz condition. Note $\tilde{\beta}^*(x) := \sup_{\theta \in \mathbb{S}^{d-1}} \tilde{\beta}(x, \theta)$. Let $\varepsilon > 0$ be small, using the fourth item of \mathcal{G} ,

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} \int_0^{r(\varepsilon)} |G(x, \theta, r)|^2 \frac{dr}{r^2} \alpha(d\theta) &\leq \int_{\mathbb{S}^{d-1}} \int_0^{r(\varepsilon)} r^{\frac{2}{\tilde{\beta}^*(x)+\varepsilon}} \frac{dr}{r^2} \alpha(d\theta) \\ &\leq c\alpha(\mathbb{S}^{d-1})r(\varepsilon)^{\frac{2}{\tilde{\beta}^*(x)+\varepsilon}-1} \end{aligned}$$

and

$$\int_{\mathbb{S}^{d-1}} \int_{r(\varepsilon)}^1 |G(x, \theta, r)|^2 \frac{dr}{r^2} \alpha(d\theta) \leq c\alpha(\mathbb{S}^{d-1})r(\varepsilon)^{-1}$$

by uniform boundedness of G . Thus the growth condition is satisfied

$$\int_{\mathbb{S}^{d-1}} \int_0^1 |G(x, \theta, r)|^2 \frac{dr}{r^2} \alpha(d\theta) \leq c \leq c(1 + |x|^2).$$

Now we check Lipschitz condition. Without loss of generality, we assume $|G(x, \theta, r)| > |G(y, \theta, r)|$. Using the property of functions in \mathcal{G}^d ,

$$\begin{aligned} & \int_{\mathbb{S}^{d-1}} \int_0^1 |G(x, \theta, r) - G(y, \theta, r)|^2 \frac{dr}{r^2} \alpha(d\theta) \\ &= \int_{\mathbb{S}^{d-1}} \int_0^1 \langle G(x, \theta, r) - G(y, \theta, r), \theta \rangle^2 \frac{dr}{r^2} \alpha(d\theta) \\ &= \int_{\mathbb{S}^{d-1}} \int_0^1 (|G(x, \theta, r)| - |G(y, \theta, r)|)^2 \frac{dr}{r^2} \alpha(d\theta) \\ &\leq \int_{\mathbb{S}^{d-1}} \int_0^1 |G(x, \theta, r)|^2 \left(1 - e^{\log |G(y, \theta, r)| - \log |G(x, \theta, r)|}\right)^2 \frac{dr}{r^2} \alpha(d\theta) \\ &\leq \int_{\mathbb{S}^{d-1}} \int_0^1 |G(x, \theta, r)|^2 (\log |G(y, \theta, r)| - \log |G(x, \theta, r)|)^2 \frac{dr}{r^2} \alpha(d\theta) \\ &\leq c|x - y|^2 \int_{\mathbb{S}^{d-1}} \int_0^1 |G(x, \theta, r)|^2 (\log \frac{1}{r})^2 \frac{dr}{r^2} \alpha(d\theta). \end{aligned}$$

where $\log(1/r)^2/r^2$ is bounded by $cr^{-2+\varepsilon}$ for any $\varepsilon > 0$. The last integral thus can be bounded from above, independent of x . The Lipschitz condition is proved.

2. The SDE structure (2.1) together with Itô's formula for compensated Poisson integral yields that, for every φ bounded twice continuously differentiable functions in \mathbb{R}^d

$$\begin{aligned} \varphi(\mathcal{M}_t) &= \varphi(0) + \int_0^t \int_{\mathbb{S}^{d-1}} \int_0^1 (\varphi(\mathcal{M}_{s-} + G(\mathcal{M}_{s-}, \theta, r)) - \varphi(\mathcal{M}_{s-})) \tilde{N}(ds, d\theta, dr) \\ &\quad + \int_0^t \int_{\mathbb{S}^{d-1}} \int_0^1 (\varphi(\mathcal{M}_{s-} + G(\mathcal{M}_{s-}, \theta, r)) - \varphi(\mathcal{M}_{s-}) - G(\mathcal{M}_{s-}, \theta, r) \cdot \nabla \varphi(\mathcal{M}_{s-})) \frac{dr}{r^2} \alpha(d\theta) ds \end{aligned}$$

Taking expectation, we get

$$\mathbb{E}[\varphi(\mathcal{M}_t)] = \varphi(0) + \int_0^t \mathbb{E}[\mathcal{L}\varphi(\mathcal{M}_s)] ds$$

which yields the result.

3. To prove that \mathcal{M} is a L^2 martingale, it suffices to check that for all t ,

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{S}^{d-1}} \int_0^1 |G(\mathcal{M}_{s-}, \theta, r)|^2 \frac{dr}{r^2} \alpha(d\theta) ds \right] < +\infty.$$

which is obvious due to the boundedness assumption on G .

□

2.1.2 Decomposition of the Poisson system in $\mathbb{R}^+ \times \mathbb{R}^d$

As in one dimension, the Poisson measure can be constructed from a Lévy process with characteristic triplet $(0, 0, \alpha \otimes \pi_0)$. Under the polar parametrization, we write the jump time and jump size of the corresponding Lévy process as a sequence of triplet (T_n, Θ_n, R_n) , where $T_n \in [0, 1]$ denotes the jump time, $\Theta_n \in \mathbb{S}^{d-1}$ denotes the direction which is a vector on the hyper sphere, $R_n \in (0, 1]$ denotes the jump size. To simplify notations, we will focus on $d = 2$, in which case $\mathbb{S}^1 = [0, 2\pi)$.

We aim at distinguishing different behavior in different direction. So we decompose the intensity measure for all $j \in \mathbb{N}$

$$\lambda \otimes \alpha \otimes \pi_0 = \sum_{k=0}^{2^j-1} \lambda \otimes \alpha_{j,k} \otimes \pi_0$$

where

$$\alpha_{j,k}(d\theta) = \mathbf{1}_{I_{j,k}}(\theta)\alpha(d\theta) \quad \text{with} \quad I_{j,k} = 2\pi \cdot [k2^{-j}, (k+1)2^{-j})$$

We decompose the Poisson system as follows : for every (j, k) , we note $(T_n^{j,k}, \Theta_t^{j,k}, R_n^{j,k})$ those Poisson point such that the angle is in the dyadic interval $I_{j,k}$. We introduce the limsup sets and the approximating rate as before

$$A_\delta^{j,k} = \limsup_{n \rightarrow +\infty} B(T_n^{j,k}, |R_n^{j,k}|^\delta)$$

$$\delta_t^{j,k} = \sup\{\delta \geq 1 : t \in A_\delta^{j,k}\}.$$

The idea is that we select those points, which are approximated infinitely often by the Poisson point lying in some prescribed angle with some rate of approximation, to form the sets $A_\delta^{j,k}$.

We write the related decomposition for the process \mathcal{M}

$$\mathcal{M} = \sum_{k=0}^{2^j-1} \mathcal{M}^{j,k}$$

where

$$\mathcal{M}_t^{j,k} = \int_0^t \int_{I_{j,k}} \int_0^1 G(\mathcal{M}_{s-}, \theta, r) \tilde{N}(ds, d\theta, dr).$$

2.1.3 Regularity

We first estimate the regularity of decomposed processes $\mathcal{M}^{j,k}$. Note

$$\beta^{j,k}(t) = \sup_{\theta \in I_{j,k}} \tilde{\beta}(\mathcal{M}_t, \theta) \quad \text{and} \quad \beta_{j,k}(t) = \inf_{\theta \in I_{j,k}} \tilde{\beta}(\mathcal{M}_t, \theta).$$

We have the following lemma.

Lemma 2.1. *Given $G \in \mathcal{G}^d$. With probability 1, for every $t \in [0, 1] \setminus J$,*

$$\frac{1}{\delta_t^{j,k} \beta_{j,k}(t)} \leq H_{\mathcal{M}^{j,k}}(t) \leq \frac{1}{\delta_t^{j,k} \beta_{j,k}(t)}$$

Proof. 1. Given $\delta > 1$. For every $t \in A_\delta^{j,k} \setminus J$, $|T_n^{j,k} - t| \leq |R_n^{j,k}|^\delta$ for infinitely many n , for those n verifying this inequality, we have

$$\begin{aligned} & \frac{\log |G(\mathcal{M}_{T_n^{j,k}-}, \Theta_n^{j,k}, R_n^{j,k})|}{\log |T_n^{j,k} - t|} \\ & \leq \frac{\log |G(\mathcal{M}_{T_n^{j,k}-}, \Theta_n^{j,k}, R_n^{j,k})|}{\delta \log |R_n^{j,k}|} \\ & \leq \frac{\log |G(\mathcal{M}_t, \Theta_n^{j,k}, R_n^{j,k})|}{\delta \log |R_n^{j,k}|} + \left| \frac{\log |G(\mathcal{M}_{T_n^{j,k}-}, \Theta_n^{j,k}, R_n^{j,k})| - \log |G(\mathcal{M}_t, \Theta_n^{j,k}, R_n^{j,k})|}{\delta \log |R_n^{j,k}|} \right|. \end{aligned}$$

By second item of class \mathcal{G}^d , for all $\varepsilon > 0$, the first term is bounded above by

$$\frac{1}{\delta(\tilde{\beta}(\mathcal{M}_t, \Theta_n^{j,k}) + \varepsilon)}$$

for n large enough. The second term converges to 0 by Lipschitz assumption and the continuity on the time t . Therefore, applying Jaffard's jump lemma to $\mathcal{M}^{j,k}$, we have

$$\begin{aligned} H_{\mathcal{M}^{j,k}}(t) & \leq \liminf_{n \rightarrow +\infty} \frac{\log |G(\mathcal{M}_{T_n^{j,k}-}, \Theta_n^{j,k}, R_n^{j,k})|}{\log |T_n^{j,k} - t|} \leq \liminf_{n \rightarrow +\infty} \frac{\log |G(\mathcal{M}_t, \Theta_n^{j,k}, R_n^{j,k})|}{\delta \log |R_n^{j,k}|} \\ & \leq \liminf_{n \rightarrow +\infty} \frac{1}{\delta(\tilde{\beta}(\mathcal{M}_t, \Theta_n^{j,k}) + \varepsilon)} \leq \frac{1}{\delta(\beta_{j,k}(t) + \varepsilon)}. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, we have

$$H_{\mathcal{M}^{j,k}}(t) \leq \frac{1}{\delta \beta_{j,k}(t)}.$$

By definition of the approximating rate, for all $\varepsilon > 0$, $t \notin J$,

$$H_{\mathcal{M}^{j,k}}(t) \leq \frac{1}{(\delta_t^{j,k} - \varepsilon) \beta_{j,k}(t)},$$

letting $\varepsilon \rightarrow 0$ yields the desired upper bound.

2. To obtain the lower bound, we use the localization argument in [120]. We cut $\mathcal{M}^{j,k}$ into pieces according to its index process $t \mapsto \beta^{j,k}(t)$, where every piece has a "constant" index, approximately. Every cut process will be a martingale that behaves like a Lévy pro-

cess given a certain index parameter. This technique allows to obtain uniform increment estimates for $\mathcal{M}^{j,k}$, hence the lower bound for its regularity. □

Now the regularity of \mathcal{M} follows.

Proposition 2.2. *Given $G \in \mathcal{G}^d$. With probability 1, for every $t \in [0, 1] \setminus J$, for every $j \in \mathbb{N}$,*

$$\lim_{j \rightarrow +\infty} \min_{0 \leq k \leq 2^j - 1} \frac{1}{\delta_t^{j,k} \beta_{j,k}(t)} \leq H_{\mathcal{M}}(t) \leq \lim_{j \rightarrow +\infty} \min_{0 \leq k \leq 2^j - 1} \frac{1}{\delta_t^{j,k} \beta_{j,k}(t)}$$

Proof. Clearly, upper and lower bound coincide due to the continuity of $\theta \mapsto \tilde{\beta}(x, \theta)$. Recall the classical property of Hölder exponent,

$$H_{f+g}(t) \geq \max(H_f(t), H_g(t))$$

and the equality holds if $H_f(t) \neq H_g(t)$. This together with Lemma 2.1 yields that for all $j \in \mathbb{N}$,

$$H_{\mathcal{M}}(t) \geq \min_{0 \leq k \leq 2^j - 1} \frac{1}{\delta_t^{j,k} \beta_{j,k}(t)}.$$

The following hierarchical structure

$$\begin{aligned} \delta_t^{j,k} &= \sup\{\delta^{j',k'} : I_{j',k'} \subset I_{j,k}\} \\ \beta_{j,k}(t) &= \sup\{\beta^{j',k'}(t) : I_{j',k'} \subset I_{j,k}\} \end{aligned}$$

yields that the limit in the lower bound exists.

Now let j be fixed, by virtue of Jaffard's jump lemma, using the jumps of process $\mathcal{M}^{j,k}$ which are jumps of \mathcal{M} as well, we get the upper bound $\frac{1}{\delta_t^{j,k} \beta_{j,k}(t)}$. Same argument for every k and letting $j \rightarrow +\infty$ yields the desired upper bound. □

2.1.4 Barral-Seuret's Theorem revisited

Barral-Seuret's localized ubiquity theorem [11] plays a crucial role in multifractal analysis of symmetric jump diffusions. As is seen in Proposition 2.2, in the anisotropic setting, one has to study simultaneously infinite number of Poisson system. We investigate here what we can do with a generalized version of their theorem.

The key question related to our problem is to determine the Hausdorff dimension of the set

$$\begin{aligned} E_{f,I}^j &= \left\{ t \in I : \delta_t^1 = \dots = \delta_t^j = f(t) \right\} \\ &= \bigcap_{1 \leq i \leq j} \left\{ t \in I : \delta_t^i = f(t) \right\}. \end{aligned}$$

where $f : \mathbb{R}^+ \rightarrow (1, +\infty)$ is a càdlàg function, I is an open interval in \mathbb{R}^+ , for every $1 \leq i \leq j \leq +\infty$, δ_t^i is the approximating rate at a given time $t \in [0, 1]$ by a system of points $\mathcal{P}^i = (x_n^i, r_n^i)$, satisfying the covering property

$$\limsup_{n \rightarrow +\infty} B(x_n^i, r_n^i) \supset [0, 1]$$

the weak redundancy and fine non-overlapping property, for definition of the latter notions, see Section 2.2 below. These three properties of a system of point will be referred to as \mathfrak{C} .

When $j = 1$, Barral-Seuret's Theorem asserts that

$$\dim E_{f,I}^1 = \sup_{t \in I} \frac{1}{f(t)}$$

A minor change in the proof of their theorem yields the following stronger assertion.

Theorem 2.1. *Suppose that every system of points \mathcal{P}^i satisfies \mathfrak{C} . Then for every $I \subset \mathbb{R}^+$, one has*

$$\dim E_{f,I}^\infty = \sup_{t \in I} \frac{1}{f(t)}$$

Proof. Clearly,

$$\dim E_{f,I}^\infty \leq \dim E_{f,I}^1 = \sup_{t \in I} \frac{1}{f(t)}.$$

When $j = 1$, the strategy of proving the lower bound of the dimension formula is to construct simultaneously a family of generalized Cantor set $(K_\varepsilon)_{\varepsilon > 0}$ included in the set $E_{f,I}^1$ which has the good dimension estimate up to a small $\varepsilon > 0$ (achieved via the construction of a family of measure supported on the family of Cantor sets). The key is that every point in the j -th generation of the Cantor set has the approximating rate $f(t)$ up to a small constant ε_j , which tends to 0 when $j \rightarrow +\infty$.

Due to the fact that every system of points \mathcal{P}^i is a "good" ubiquitous system, in j -th generation of the Cantor set, we can select the open intervals which are simultaneously included in union of $B(x_n^i, r_n^i)$ for $1 \leq i \leq j$ to form the $(j+1)$ -th generation of the Cantor set. Similarly, we construct the related measures having the right scaling properties. When $j \rightarrow +\infty$, every point in the obtained Cantor set has the approximating rate $f(t)$ with respect to every system of points $(\mathcal{P}^i)_{1 \leq i \leq +\infty}$. We thus obtain the result. \square

2.1.5 Multifractal analysis

As an application of the aforementioned theorem, we derive multifractal spectra for jump diffusions with anisotropic coefficient. The systems in question are $\mathcal{P}^{j,k}$ for $0 \leq k \leq 2^j - 1$ and $j \in \mathbb{N}$. We admit for the moment that each system $\mathcal{P}^{j,k}$ satisfies \mathfrak{C} , which is checked in the next section, after recalling the definition of the relevant notions. The key is that the intensity measure on \mathbb{S}^1 is the uniform probability measure.

Multifractal analysis concerns the Hausdorff dimension of the iso-Hölder sets

$$E_h = \{t \in I : H_{\mathcal{M}}(t) = h\}.$$

By Proposition 2.2, if $\delta_t^{j,k} = \delta_t$ for every couple (j, k) , we have

$$\frac{1}{\delta_t \beta^{0,0}(t)} \leq H_{\mathcal{M}}(t) \leq \lim_{j \rightarrow +\infty} \frac{1}{\delta_t \max_k \beta_{j,k}(t)}$$

which yields by continuity of $\theta \mapsto \tilde{\beta}(x, \theta)$,

$$H_{\mathcal{M}}(t) = \frac{1}{\delta_t \beta^{0,0}(t)},$$

recalling that $\beta^{0,0}(t) = \sup_{\theta \in [0, 2\pi)} \tilde{\beta}(\mathcal{M}_t, \theta)$. Hence

$$\begin{aligned} E_h &\supset \left\{ t \in I : \delta_t^{j,k} = \delta_t, h = \frac{1}{\delta_t \beta^{0,0}(t)} \right\} \\ &= \bigcap_{j,k} \left\{ t \in I : \delta_t^{j,k} = \frac{1}{h \beta^{0,0}(t)} \right\}. \end{aligned}$$

Without loss of generality, we suppose that I and h are such that $t \mapsto \frac{1}{h \beta^{0,0}(t)}$ takes values in $(1, +\infty)$ for every $t \in I$, then we can apply Theorem 2.1 to find

$$\dim E_h \geq \dim \bigcap_{j,k} \left\{ t \in I : \delta_t^{j,k} = \frac{1}{h \beta^{0,0}(t)} \right\} = h \cdot \sup_{t \in I} \beta^{0,0}(t).$$

On the other hand, by Lemma 2.1,

$$\begin{aligned} E_h &\subset \left\{ t \in I : \frac{1}{\delta_t \beta^{0,0}(t)} \leq h \right\} \\ &= \left\{ t \in I : \delta_t \geq \frac{1}{h \beta^{0,0}(t)} \right\} \end{aligned}$$

which yields by localized ubiquity theorem

$$\dim E_h \leq \dim \left\{ t \in I : \delta_t \geq \frac{1}{h\beta^{0,0}(t)} \right\} = h \cdot \sup_{t \in I} \beta^{0,0}(t).$$

The extension to general situation is straightforward.

2.2 General intensity measure

Another way to add asymmetry to the model consist in modifying the intensity measure of the Poisson point process driving the SDE.

In this section, we replace the smooth and symmetric Lévy measure on \mathbb{R}^d

$$\alpha \otimes \pi_0 = \alpha(d\theta) \otimes dr/r^2 = dz/|z|^{d+1}$$

by $\Pi(dz)$ which is a Lévy measure on \mathbb{R}^d having nearly the same asymptotic behavior at 0. In particular, purely discontinuous measure and measures with gap in their support are included. Recall that $C(a, b)$ is the annulus in \mathbb{R}^d with radius $0 \leq a < b$. Let

$$\beta_j := \frac{\log_2 \Pi(C(2^{-j-1}, 2^{-j}))}{j} \text{ and } \beta := \limsup_{j \rightarrow +\infty} \beta_j.$$

Lemma 2.2. *β defined above relates to the Blumenthal-Gettoor index of the Lévy measure $\Pi(dz)$.*

Proof. Recall that Blumenthal-Gettoor's upper index is defined as

$$\hat{\beta} := \inf\{\gamma \geq 0 : \int_{C(0,1)} |z|^\gamma \Pi(dz) < +\infty\}$$

Let $\varepsilon > 0$, by definition of β , $\beta_j \leq \beta + \varepsilon/2$ for all large j , thus for r small enough

$$\int_{C(0,r)} |z|^{\beta+\varepsilon} \Pi(dz) \leq \sum_j 2^{-j(\beta+\varepsilon)} \cdot 2^{j\beta_j} \leq \sum_j 2^{-j(\beta+\varepsilon)} \cdot 2^{j(\beta+\varepsilon/2)} = \sum_j 2^{-j\varepsilon/2} < +\infty,$$

which shows $\beta \geq \hat{\beta}$. The other inequality holds trivially if $\beta = 0$, so we suppose $\beta > 0$ (remark that $\beta \geq 0$ if $\Pi(\mathbb{R}^d) = +\infty$). For every $\varepsilon > 0$, there exists (j_n) a subsequence of (j) such that $\beta_{j_n} \geq \beta - \varepsilon$ for all n ,

$$\int_{C(0,1)} |z|^{\beta-\varepsilon} \Pi(dz) \geq \sum_j 2^{(-j-1)(\beta-\varepsilon)} \cdot 2^{j\beta_j} \geq \sum_{j_n} 2^{(-j-1)(\beta-\varepsilon)} \cdot 2^{j(\beta-\varepsilon)} = \sum_{j_n} 1 = +\infty$$

which yields $\beta \leq \hat{\beta}$. □

Main result of this section is the following

Proposition 2.3. *If $\beta = 1$ and*

$$\limsup_{j \rightarrow +\infty} (\beta_j - 1)^+ j < +\infty \quad (2.2)$$

then the weak redundancy and the fine non-overlapping property in Barral-Seuret's Theorem are a.s. satisfied for the Poisson system \mathcal{P} with intensity measure Π .

We will see in the proof that any Poisson system whose Lévy measure has index $\beta = 1$ (in fact, it is true for every $\beta \leq 1$, but we lost covering property if $\beta < 1$) satisfies the weak redundancy condition. Under the mild assumption (2.2) on the measure, which restricts the rate of exceeding the normal jumps intensity behavior (when there are too many jumps), the non-overlapping condition is also verified.

Trivial example of Π is given by $dz/|z|^{d+1}$, where the related sequence $(\beta_j)_{j \geq 1}$ is $(1, 1, \dots)$. We give other examples. The first one justifies the condition imposed in Theorem 2.1.

Example 2.2. 1. *Measure whose support is restricted in some angles in \mathbb{R}^2*

$$\Pi_1 = \alpha_{j_0, k_0} \otimes \pi_0$$

where $\alpha_{j_0, k_0}(d\theta) = \mathbf{1}_{I_{j_0, k_0}}(\theta) \alpha(d\theta)$, see Section 2.1.2. Here $\beta_j = 1 - \frac{j_0}{j}$, for every j .

2. *Measure with infinite many gaps.*

$$\Pi_2(dz) = \sum_{n \in \mathbb{N}} \mathbf{1}_{C(2^{-2n-1}, 2^{-2n})}(z) |z|^{-d-1} dz$$

3. *Purely discontinuous measure.*

$$\Pi_3(dz) = \sum_{j \in \mathbb{N}} 2^{-j} \delta_{\frac{1}{2}(2^{-j-1} + 2^{-j})}(dz)$$

where δ_x is a Dirac mass on the point x . This should not be confused with the approximating rate.

Remark 2.1. *By the definition of β , there exists a sequence ε_j decreasing to 0 such that $\beta_j < 1 + \varepsilon_j$ for all j . Enlarging the sequence ε_j if necessary, we also have $\varepsilon_j > 1/j$ for all j .*

The pointwise regularity and multifractal spectra of the jump diffusion with symmetric coefficient remain valid if intensity measure of N satisfies the conditions in Proposition 2.3.

2.2.1 Weak redundancy

Let us recall the weak redundancy introduced in Barral-Seuret [11]. We consider a system of points $\mathcal{S} = (x_n, l_n)$ in $\mathbb{R}^+ \times \mathbb{R}^d$ satisfying the covering property $[0, 1] \subset \limsup_{n \rightarrow +\infty} B(x_n, |l_n|)$. We say that \mathcal{S} is weakly redundant if for every j , for every index set $\mathcal{T}_j = \{n : l_n \in C(2^{-j-1}, 2^{-j})\}$, we can decompose \mathcal{T}_j into disjoint index sets $(\mathcal{T}_{j,i})_{i=1, \dots, N_j}$, such that the balls $B(x_n, |l_n|)$ with $n \in \mathcal{T}_{j,i}$ are disjoint and such that $\lim_{n \rightarrow \infty} \frac{\log_2 N_j}{j} = 0$.

To prove that the Poisson system \mathcal{P} satisfies almost surely the weak redundancy property, only $\beta = 1$ is needed.

The choice of N_j involves the values of β_j , and will be clear along with the computation. Let us introduce the events

$$\begin{aligned} A_j &= \{\exists t \in [0, 1], \#\{n \in \mathcal{T}_j : t \in B(T_n, |Z_n|)\} > N_j\} \\ A_{j,k} &= \{\exists t \in I_{j,k}, \#\{n \in \mathcal{T}_j : t \in B(T_n, |Z_n|)\} > N_j\}. \end{aligned}$$

with $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$. Clearly, if weak redundancy is satisfied for a fixed j , then the complementary of A_j is realized. Proving $\sum_j \mathbb{P}(A_j) < +\infty$ together with Borel-Cantelli Lemma yields the weak redundancy property of \mathcal{P} . We have

$$\mathbb{P}(A_j) \leq \sum_{k=0}^{2^j-1} \mathbb{P}(A_{j,k}) \leq \sum_{k=0}^{2^j-1} \mathbb{P}\left(\#\{n \in \mathcal{T}_j : T_n \in \widehat{I}_{j,k}\} > N_j\right)$$

with $\widehat{I}_{j,k} = I_{j,k-1} \cup I_{j,k} \cup I_{j,k+1}$ and the convention $I_{j,-1} = I_{j,2^j} = \emptyset$. In other words,

$$\mathbb{P}(A_j) \leq \sum_{k=0}^{2^j-1} \mathbb{P}(N(\widehat{I}_{j,k} \times C(2^{-j-1}, 2^{-j})) > N_j)$$

where $N(\widehat{I}_{j,k} \times C(2^{-j-1}, 2^{-j}))$ is a Poisson random variable with parameter

$$\lambda \otimes \Pi(\widehat{I}_{j,k} \times C(2^{-j-1}, 2^{-j})) = 3 \cdot 2^{-j} \cdot 2^{j\beta_j},$$

note that the parameter is independent of k , hence

$$\mathbb{P}(A_j) \leq 2^j \sum_{i > N_j} e^{-3 \cdot 2^{-j+j\beta_j}} \frac{(3 \cdot 2^{-j+j\beta_j})^i}{i!} \leq 2^j \sum_{i > N_j} \frac{(3 \cdot 2^{j\epsilon_j})^i}{i!}$$

By Stirling's formula, for all j large enough

$$\mathbb{P}(A_j) \leq 2^j \sum_{i > N_j} \left(\frac{3e \cdot 2^{j\epsilon_j}}{i}\right)^i \leq 2^j \sum_{i > N_j} \left(\frac{3e \cdot 2^{j\epsilon_j}}{N_j}\right)^i$$

Choose $N_j = 2^{j\sqrt{\varepsilon_j}}$ which satisfies $\frac{\log_2 N_j}{j} \rightarrow 0$ and $2^{j\varepsilon_j} \ll N_j$ for j large. Then

$$\mathbb{P}(A_j) \leq c \cdot 2^j \left(\frac{3e \cdot 2^{j\varepsilon_j}}{N_j} \right)^{N_j} = c 2^j \left(3e \cdot 2^{j(\varepsilon_j - \sqrt{\varepsilon_j})} \right)^{2^{j\sqrt{\varepsilon_j}}} \leq c 2^j \cdot 2^{\frac{1}{2}j(\varepsilon_j - \sqrt{\varepsilon_j})} 2^{j\sqrt{\varepsilon_j}}$$

which implies for J large

$$\sum_{j \geq J} \mathbb{P}(A_j) \leq c \sum_{j \geq J} 2^j \cdot 2^{-\frac{1}{4}j\sqrt{\varepsilon_j}} 2^{j\sqrt{\varepsilon_j}} \leq c \sum_{j \geq J} 2^j \cdot 2^{-\frac{1}{4}\sqrt{j}2^{\sqrt{j}}} \leq c \sum_{j \geq J} 2^j 2^{-2j} < +\infty,$$

where we used $\varepsilon_j > 1/j$ for the second inequality. The proof is complete.

2.2.2 Fine non-overlapping properties

Let us first recall the non-overlapping property \mathcal{C} in [11]. We say that the Poisson system \mathcal{P} satisfies the fine non-overlapping condition \mathcal{C} if a.s. for every $\delta \in (1, +\infty) \cap \mathbb{Q}$, for every dyadic interval U , say, of generation $g(U)$, there exist infinite many j larger than $g(U)$, such that we have the following : the number of dyadic intervals of generation j in U with the property $\mathcal{P}(\delta)$ exceeds $\kappa(\delta)2^{j-g(U)}$, where $\kappa(\delta) > 0$ is independent of j .

Recall that a given dyadic interval V of generation j satisfies $\mathcal{P}(\delta)$ if there exists $n_0 \in \mathcal{T}_j$ with $T_{n_0} \in V$ such that every Poisson point T_n with $n \in \cup_{\psi(j) \leq k \leq j\delta} \mathcal{T}_k$ is not covered by the interval $B(T_{n_0}, |Z_{n_0}|^\delta)$. As the value of $\psi(j)$ is not involved in the computations below, we do not give its explicit form, all we need to know is that $\psi(j) \leq j$. Clearly, this property involves not only the generation j of the Poisson system.

The desired non-overlapping property follows easily by applying Borel-Cantelli Lemma for a sequence of independent events whose probability have universal lower bound away from 0. The independence can be achieved by choosing a sequence j_n not too close with each other.

The main task is thus to establish a lower bound of probability, uniformly in j . Following Barral-Fournier-Jaffard-Seuret, it suffices to prove the following lemma.

Lemma 2.3. *There exists $\kappa_1(\delta) > 0$ such that, for every j , the probability that a dyadic interval V of generation j satisfies the property $\mathcal{P}(\delta)$ is larger than $\kappa_1(\delta)$.*

Indeed, we consider a smaller event : the proportion of odd dyadic intervals of generation j in U with property $\mathcal{P}(\delta)$ exceeds $\kappa(\delta) := \kappa_1(\delta)/4 > 0$. By odd dyadic interval of generation j , we mean intervals like $[2k \cdot 2^{-j}, (2k+1)2^{-j})$ for $2k \in [0, 2^j - 1]$. By considering these intervals, we have a family of independent Bernoulli random variables ($\mathbf{1}_{\mathcal{P}(\delta)}$ holds for V) indexed by all V odd dyadic interval of generation j . The common Bernoulli parameter is $\mathbb{P}(\mathcal{P}(\delta)$ holds for V) for one fixed, hence for every odd dyadic interval V . The property of binomial random variable

together with Lemma 2.3 yields that

$$\mathbb{P}\left(\sum_V \mathbf{1}_{\mathcal{P}(\delta)} \text{ holds for } V \geq \frac{1}{4}\kappa_1(\delta)2^{j-g(U)}\right) \geq \mathbb{P}\left(\sum_i^{2^{j-g(U)-1}} X_i \geq \frac{1}{2}\kappa_1(\delta)2^{j-g(U)-1}\right)$$

where the sum is taken for all V odd dyadic intervals in U , and (X_i) are i.i.d. Bernoulli random variables of parameter $\kappa_1(\delta)$. The law of large numbers implies that the right-hand side term converges to 1, hence larger than $1/2$ for all j large enough.

Finally, we prove Lemma 2.3. For every V dyadic interval, say, of generation j , we introduce the sets

$$S_V = V \times C(2^{-j-1}, 2^{-j}) \text{ and } \widehat{S}_V = V \times C(2^{-j\delta}, 2^{-\psi(j)}) \setminus S_V.$$

Denote by $(t_n, z_n)_{1 \leq n \leq N(S_V \cup \widehat{S}_V)}$ the Poisson points in $S_V \cup \widehat{S}_V$ where the first $N(S_V)$ points correspond to the points in S_V . The property $\mathcal{P}(\delta)$ holds for V if the Poisson points are configured as follows.

$$\mathbb{P}(\mathcal{P}(\delta) \text{ holds for } V) \geq \mathbb{P}(N(S_V) = 1 \text{ and } B(t_1, z_1) \cap \{t_2, \dots, t_{1+N(\widehat{S}_V)}\} = \emptyset)$$

Recall that conditioning on the value of a Poisson process up to time t , the time of jumps are i.i.d. uniformly distributed in the interval $[0, t]$. This, together with the independence between $N(S_V)$ and $N(\widehat{S}_V)$, yields

$$\begin{aligned} & \mathbb{P}(\mathcal{P}(\delta) \text{ holds for } V) \\ & \geq \mathbb{E}\left[\mathbb{P}\left(N(S_V) = 1, B(t_1, z_1) \cap \{t_2, \dots, t_{1+N(\widehat{S}_V)}\} = \emptyset \mid \{t_1\}, N(\widehat{S}_V)\right)\right] \\ & \geq \mathbb{E}\left[\mathbb{P}\left(B(t_1, z_1) \cap \{t_2, \dots, t_{1+N(\widehat{S}_V)}\} = \emptyset \mid N(S_V) = 1, \{t_1\}, N(\widehat{S}_V)\right) \mathbb{P}(N(S_V) = 1)\right] \\ & \geq \mathbb{E}\left[\mathbb{P}\left(B(t_1, 2^{-j}) \cap \{t_2, \dots, t_{1+N(\widehat{S}_V)}\} = \emptyset \mid N(S_V) = 1, \{t_1\}, N(\widehat{S}_V)\right) \mathbb{P}(N(S_V) = 1)\right] \\ & = e^{-1} \mathbb{E}\left[\left(1 - 2^{-j(\delta-1)}\right)^{N(\widehat{S}_V)}\right] \end{aligned}$$

which gives by using generating function of Poisson random variable with parameter $2^{-j} \cdot 2^{j\delta\beta_j\delta}$

$$\begin{aligned} \mathbb{P}(\mathcal{P}(\delta) \text{ holds for } V) & \geq e^{-1} e^{-2^{-j(\delta-1)} \cdot 2^{-j} \cdot 2^{j\delta\beta_j\delta}} = e^{-1} e^{-2^{j\delta(\beta_j\delta-1)}} \\ & \geq e^{-1} e^{-2^{j\delta(\beta_j\delta-1)^+}} \geq e^{-1} e^{-2^{\limsup_j j(\beta_j-1)^+}} := \kappa_1(\delta) > 0 \end{aligned}$$

by assumption. The fine non-overlapping property thus holds.

2.2.3 Regularity

Now we want to deduce the regularity results for SDE driven by N with general intensity measure. The condition imposed on G is given in the following definition.

Definition 2.2. *The set \mathcal{G}_{Π}^d is the set of those functions $G : \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, 1] \rightarrow C(0, 1)$ satisfying*

1. *Symmetry : For every $x \in \mathbb{R}^d, \theta \in \mathbb{S}^{d-1}, \theta_0 = (1, 0, \dots, 0)$,*

$$\langle G(x, \theta, r), \theta \rangle = |G(x, \theta, r)| \text{ and } G(x, \theta, r) = G(x, \theta_0, r)$$

2. *Asymptotically stable-like : for every $x \in \mathbb{R}^d, \theta \in \mathbb{S}^{d-1}$,*

$$\liminf_{r \rightarrow 0} \frac{\log |G(x, \theta, r)|}{\log r} \text{ exists and denoted by } \frac{1}{\widetilde{\beta}(x)}.$$

3. *Lipschitz condition : there exists $C > 0$ such that for all $(x, y) \in \mathbb{R}^{2d}$, for every $(\theta, r) \in \mathbb{S}^{d-1} \times (0, 1]$,*

$$\left| \frac{\log |G(y, \theta, r)| - \log |G(x, \theta, r)|}{\log r} \right| \leq C|x - y|.$$

4. *Boundedness : Range $\widetilde{\beta} \subset (0, 2)$ and for every $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for every $r \in (0, r_\varepsilon)$, every x and every θ ,*

$$|G(x, \theta, r)| \leq r^{\frac{1}{\widetilde{\beta}(x) + \varepsilon}}$$

\mathcal{G}_{Π}^d means that it is a condition on G in dimension d with respect to the general intensity measure Π .

Remark 2.2. *Due to the presence of anisotropy of the intensity measure, we do need symmetric coefficient G . Recall that the generalized Barral-Seuret Theorem in Section 2.1.4 holds if the sequence of Poisson system $(\mathcal{P}^i)_{0 \leq i \leq +\infty}$ have almost the same ubiquitous property for each i . Typically, when $\Pi = \alpha \otimes \pi_0$, by piecing out $(\Theta_n^{j,k})_{n \geq 1}$, the sequence $\{(T_n^{j,k}, R_n^{j,k})_{n \geq 1}\}_{0 \leq k \leq 2^j - 1}$ is i.i.d. random system of points. However, in the general intensity measure setting, one can privilege one angle without changing the ubiquitous nature of the system (the three properties are verified for the whole system). It can happen that even the basic covering condition is not satisfied in some angle.*

Now we can deduce the pointwise regularity of \mathcal{M} . Note $\beta(t) := \widetilde{\beta}(\mathcal{M}_t)$. Due to the symmetry of G , we do not need to decompose the Poisson system (T_n, Θ_n, R_n) . Introduce $A_\delta = \limsup_{n \rightarrow +\infty} B(T_n, (R_n)^\delta)$ and $\delta_t = \sup\{\delta \geq 1 : t \in A_\delta\}$.

Proposition 2.4. *For every $G \in \mathcal{G}_{\Pi}^d$, Π satisfying the condition of Proposition 2.3.*

1. *There exists a unique pathwise solution to (2.1).*

2. With probability 1, for all $t \in [0, 1] \setminus J$,

$$H_{\mathcal{M}}(t) = \frac{1}{\delta_t \beta(t)}$$

Proof. We only clarify several points. The condition (2.2) is neither used to prove the existence of the unique pathwise solution to (2.1), nor to deduce its regularity. Upper index of the intensity measure equals $\beta = 1$ is the key here. Note that the fourth item of \mathcal{G}_{Π}^d permits G to have some oscillation under $r^{\frac{1}{\beta(x)+\varepsilon}}$, changing ε to $\varepsilon/2$ if necessary, every step of the proof can be checked through. \square

Finally, for every $G \in \mathcal{G}_{\Pi}^d$, Π satisfying the condition of Proposition 2.3, the pointwise multifractal spectra can also be deduced following the methods in [120].

2.3 Tangent processes

In [120], we identify the tangent processes of \mathcal{M} with a specific choice for the coefficient

$$G_0(x, z) = \text{sign}(z)|z|^{\frac{1}{\tilde{\beta}(x)}}$$

where $\overline{\text{Range } \tilde{\beta}} \subset (0, 2)$ and $\tilde{\beta}$ is Lipschitz continuous. In this section, we extend it to a class of admissible functions given in the following definition.

Definition 2.3. The set \mathcal{G}_t^1 is the set of those functions $G : \mathbb{R} \times C(0, 1) \rightarrow [-1, 1]$ satisfying :

1. *Symmetry* : For every $x \in \mathbb{R}$, $z \in C(0, 1)$,

$$G(x, z) = \text{sign}(z)|G(x, |z|)|, \quad \text{sign } G(x, z) = \text{sign } z.$$

2. *Asymptotically stable-like* : $\forall x \in \mathbb{R}$,

$$\liminf_{z \rightarrow 0} \frac{\log |G(x, z)|}{\log |z|} \text{ exists and denoted by } \frac{1}{\tilde{\beta}(x)}.$$

3. *Lipschitz condition* : there exists $C > 0$ such that $\forall (x, y) \in \mathbb{R}^2, \forall z \in C(0, 1)$,

$$\left| \frac{\log |G(y, z)| - \log |G(x, z)|}{\log |z|} \right| \leq C|x - y|.$$

4. *Boundedness* : $\overline{\text{Range } \tilde{\beta}} \subset (0, 2)$ and for all $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for every $z \in C(0, r_\varepsilon)$ and every x ,

$$|G(x, z)| \leq |z|^{\frac{1}{\tilde{\beta}(x)+\varepsilon}}.$$

5. *Local oscillating restriction* : for all $x_0 \in \mathbb{R}$, there exists a positive function $\varepsilon_{x_0}(x)$ satisfying $\varepsilon_{x_0}(x) = O(|x - x_0|)$ such that $\forall x$ in a neighbourhood of $x_0 \forall z \in C(0, 1)$,

$$|z|^{\frac{1}{\beta(x_0) - \varepsilon_{x_0}(x)}} \leq |G(x, z)| \leq |z|^{\frac{1}{\beta(x_0) + \varepsilon_{x_0}(x)}}$$

Here \mathcal{G}_t^1 means that it is a condition on G in dimension one used for deriving tangent processes. Note that we add a strong condition (5). This is necessary due to the fact that tangent processes are scale invariant and have selfsimilarity (see Falconer). If the function G oscillates too much, we are not able to prove the existence of the tangent processes.

Main result of this section is the following. Recall that $\beta(t) := \tilde{\beta}(\mathcal{M}_t)$.

Proposition 2.5. *Given $G \in \mathcal{G}_t^1$. Let $t_0 \geq 0$, conditionally on \mathcal{F}_{t_0} , the family of processes $\left(\frac{\mathcal{M}_{t_0 + \alpha t} - \mathcal{M}_{t_0}}{\alpha^{1/\beta(t_0)}}\right)_{t \in [0, 1]}$ converges in law to a stable Lévy process with Lévy measure $\beta(t_0)u^{-1-\beta(t_0)} du$ when α tends to 0.*

To prove the limit theorem in this proposition, we should be able to compare the behavior of G with that of G_0 .

Lemma 2.4. *Given $G \in \mathcal{G}_t^1$ and $x_0 \in \mathbb{R}$. For all $\gamma > \tilde{\beta}(x_0)$, there exist a constant $0 < C_\gamma < +\infty$ such that*

$$\int_0^1 |G(x, z) - G_0(x_0, z)|^\gamma \pi_0(dz) \leq C_\gamma |x - x_0|^\gamma,$$

in a neighborhood of x_0 .

Proof. By local oscillating restriction,

$$\begin{aligned} & \int_0^1 |G(x, z) - G_0(x_0, z)|^\gamma \pi_0(dz) \\ & \leq \int_0^1 |z^{\frac{1}{\beta(x_0) + \varepsilon_{x_0}(x)}} - z^{\frac{1}{\beta(x_0)}}|^\gamma \pi_0(dz) + \int_0^1 |z^{\frac{1}{\beta(x_0) - \varepsilon_{x_0}(x)}} - z^{\frac{1}{\beta(x_0)}}|^\gamma \pi_0(dz), \end{aligned}$$

We only bound from above the first integral, the second can be dealt with similarly. Note that

$$\left| z^{\frac{1}{\beta(x_0) + \varepsilon_{x_0}(x)}} - z^{\frac{1}{\beta(x_0)}} \right| = z^{\frac{1}{\beta(x_0)}} \left(e^{\left| \frac{1}{\beta(x_0) + \varepsilon_{x_0}(x)} - \frac{1}{\beta(x_0)} \right| \log 1/z} - 1 \right)$$

and

$$\left| \frac{1}{\beta(x_0) + \varepsilon_{x_0}(x)} - \frac{1}{\beta(x_0)} \right| \leq c|x - x_0|$$

with some constant c , where we used the decreasing rate of the function ε_{x_0} . Splitting the integral

into two parts

$$I_1 := \int_0^{e^{-1/c'|x-x_0|}} \left| z^{\frac{1}{\beta(x_0)+\varepsilon_{x_0}(x)}} - z^{\frac{1}{\beta(x_0)}} \right|^\gamma \pi_0(dz),$$

$$I_2 := \int_{e^{-1/c'|x-x_0|}}^1 z^{\frac{\gamma}{\beta(x_0)}} \left(e^{\left| \frac{1}{\beta(x_0)+\varepsilon_{x_0}(x)} - \frac{1}{\beta(x_0)} \right| \log 1/z} - 1 \right)^\gamma \pi_0(dz),$$

we have for x sufficiently close to x_0 ,

$$\begin{aligned} I_1 &\leq \int_0^{e^{-1/c'|x-x_0|}} \left| z^{\frac{1}{\beta(x_0)+\varepsilon_{x_0}(x)}} \vee z^{\frac{1}{\beta(x_0)}} \right|^\gamma \pi_0(dz) \\ &\leq \int_0^{e^{-1/c'|x-x_0|}} z^{\frac{\gamma}{\beta(x_0)+\varepsilon}} \pi_0(dz) = C_\gamma \left(e^{-1/c'|x-x_0|} \right)^{\frac{\gamma}{\beta(x_0)+\varepsilon}-1} \\ &\leq C_\gamma e^{-c''|x-x_0|} \leq C_\gamma |x-x_0|^\gamma \end{aligned}$$

and

$$\begin{aligned} I_2 &\leq \int_{e^{-1/c'|x-x_0|}}^1 z^{\frac{\gamma}{\beta(x_0)}} (\log 1/z)^\gamma |x-x_0|^\gamma \pi(dz) \\ &\leq |x-x_0|^\gamma \int_0^1 z^{\frac{\gamma}{\beta(x_0)}+\varepsilon} \pi(dz) \leq C_\gamma |x-x_0|^\gamma, \end{aligned}$$

where we used $\gamma > \beta(x_0)$. The proof of the lemma is complete. \square

We need some moment estimate of \mathcal{M} near 0. Let us introduce the stopping times

$$\tau_\eta^1 := \inf\{t > 0 : \beta(t) > \beta(0) + \eta/3\}.$$

$$\tau_\eta^2 := \inf\{t > 0 : \varepsilon_0(\mathcal{M}_t) > \eta/3\}$$

$$\tau_\eta^3 := \inf\{t > 0 : |\mathcal{M}_t| > \eta/3\}$$

and denote $\tau_\eta = \tau_\eta^1 \wedge \tau_\eta^2 \wedge \tau_\eta^3$.

Lemma 2.5. *Given $G \in \mathcal{G}_t^1$.*

(i) *If $\beta(0) \geq 1$, for every $\gamma \in (\beta(0) + \eta, 2)$, there exists a constant c_γ such that $\forall \alpha > 0$,*

$$\mathbb{E}[|Z_{\alpha \wedge \tau_\eta}|^\gamma] \leq c_\gamma \alpha.$$

(ii) *If $\beta(0) < 1$, for every $\gamma \in (\beta(0) + \eta, 1 \wedge 2\beta(0))$, the same moment inequality holds.*

Here η is taken to be small enough such that the range of γ makes sense.

Proof. (i) \mathcal{M} being martingale, by Burkholder-Davis-Gundy inequality, symmetry and subaddi-

tivity, we have

$$\begin{aligned} \mathbb{E}[|\mathcal{M}_{\alpha \wedge \tau_\eta}|^\gamma] &\leq \mathbb{E}\left[\sup_{0 \leq t \leq \alpha \wedge \tau_\eta} |\mathcal{M}_t|^\gamma\right] \leq c_\gamma \mathbb{E}\left[\left|\int_0^{\alpha \wedge \tau_\eta} \int_{C(0,1)} |G(\mathcal{M}_{s-}, z)|^2 N(ds dz)\right|^{\gamma/2}\right] \\ &\leq c_\gamma \mathbb{E}\left[\int_0^{\alpha \wedge \tau_\eta} \int_0^1 |G(\mathcal{M}_{s-}, z)|^\gamma N(ds dz)\right] \\ &= c_\gamma \mathbb{E}\left[\int_0^{\alpha \wedge \tau_\eta} \int_0^1 |G(\mathcal{M}_{s-}, z)|^\gamma dz/z^2 ds\right] \end{aligned}$$

Since η can be taken arbitrarily small, by oscillating restriction, for every $s \in [0, \tau_\eta]$, one has

$$\int_0^1 |G(\mathcal{M}_{s-}, z)|^\gamma dz/z^2 \leq \int_0^1 |z|^{\frac{\gamma}{\beta(0)+\varepsilon_0(\mathcal{M}_{s-})}-2} dz \leq \int_0^1 |z|^{\frac{\gamma}{\beta(0)+\eta}-2} dz < +\infty$$

where we used $\gamma > \beta(0) + \eta$. Hence,

$$\mathbb{E}[|\mathcal{M}_{\alpha \wedge \tau_\eta}|^\gamma] \leq c_\gamma \mathbb{E}\left[\int_0^{\alpha \wedge \tau_\eta} ds\right] \leq c_\gamma \alpha.$$

(ii) For every $s \in [0, \tau_\eta]$ with η small enough, it makes sense to separate the Poisson measure and its compensator. By symmetry and subadditivity, we have

$$\begin{aligned} \mathbb{E}[|\mathcal{M}_{\alpha \wedge \tau_\eta}|^\gamma] &\leq c_\gamma \mathbb{E}\left[\left|\int_0^{\alpha \wedge \tau_\eta} \int_{C(0,1)} |G(\mathcal{M}_{s-}, z)| N(ds dz)\right|^\gamma\right] \\ &\quad + c_\gamma \mathbb{E}\left[\left|\int_0^{\alpha \wedge \tau_\eta} \int_{C(0,1)} |G(\mathcal{M}_{s-}, z)| dz/z^2 ds\right|^\gamma\right] \\ &\leq c_\gamma \mathbb{E}\left[\int_0^{\alpha \wedge \tau_\eta} \int_0^1 |G(\mathcal{M}_{s-}, z)|^\gamma dz/z^2 ds\right]. \end{aligned}$$

Repeating the lines at the end of the first point, one gets the result. \square

We only sketch the proof of Proposition 2.5, using the self-similarity of the limit process and last two lemmas.

Sketch of the proof: As in [120], we introduce

$$L_t = \int_0^t \int_{C(0,1)} G_0(0, z) \tilde{N}(ds dz), \quad S_t = \int_0^t \int_{C(0,+\infty)} G_0(0, z) \tilde{N}(ds dz).$$

which are Lévy processes whose Lévy measure are $\beta(0)|z|^{-\beta(0)-1} \mathbf{1}_{C(0,1)} dz$ and $\beta(0)|z|^{-\beta(0)-1} dz$, respectively. The self-similarity of S permits to compare re-scaled L with S as probability measure in Skorohod's space $D[0, 1]$. On the other hand, Lemma 2.4 permits us to compare re-scaled \mathcal{M} with re-scaled L . So every step of the proof can be checked easily via Lemma 2.4 and 2.5.

Chapitre 3

Dimension de Hausdorff de l'image et du graphe de diffusion à sauts de type stable

On détermine la dimension de Hausdorff de l'image d'une classe de processus de Markov à valeurs dans \mathbb{R}^d . Cette dimension s'avère être aléatoire et dépendre de l'intervalle du temps où on observe l'image. Les techniques développées permettent également de déduire la formule de dimension pour le graphe de ces processus.

Ce chapitre est basé sur l'article [\[121\]](#).

3.1 Introduction

Let $\mathcal{M} = \{\mathcal{M}_t, t \geq 0\}$ be a jump type Markov process taking value in \mathbb{R}^d , whose generator is

$$\mathcal{L}\varphi(x) = \int [\varphi(x+u) - \varphi(x) - \mathbf{1}_{|u| \leq 1} u \cdot \nabla \varphi(x)] n(x, du),$$

where $n(x, du)$ is the jump kernel, i.e. a family of Lévy measures on \mathbb{R}^d indexed by $x \in \mathbb{R}^d$ satisfying

$$\sup_{x \in \mathbb{R}^d} \int 1 \wedge |u|^2 n(x, du) < +\infty. \quad (3.1)$$

Here, $|\cdot|$ denotes the usual Euclidean metric in \mathbb{R}^d . The dependence on x of the jump kernel is an important feature because real life data (e.g. financial, geographical and meteorologic data) which have been modeled by Lévy processes often exhibit different characteristics along time. Hence modeling with this type of Markov processes can be relevant. When the jump kernel $n(x, du)$ does not depend on the value of x , \mathcal{M} is a Lévy process. Mandelbrot used Lévy model in [85], the so-called ‘‘Lévy flight’’, and proposed to find its fractal dimension. The question of finding the Hausdorff dimension of the range of Lévy processes was answered by Mckean [88], Blumenthal and Gettoor [27, 29] in the symmetric stable case, and Pruitt and Taylor [101], Pruitt [102] in the general case. We also mention the recent investigations by Khoshnevisan, Xiao and Zhong [70] and Khoshnevisan and Xiao [69] who proved the dimension formula in terms of characteristic exponent of a general Lévy process. The reference [117] is a systematic survey on this subject.

The question adressed in this article is the determination of the Hausdorff dimension of the range and the graph of sample paths of \mathcal{M} .

The sole condition (3.1) on $n(x, du)$ seems too general to get precise results. However, for a large class of Markov processes where the generator \mathcal{L} restricted on $C_c^\infty(\mathbb{R}^d)$ is a pseudo-differential operator, Schilling [105] (see also [31]) gave an upper bound for the dimension of the range of their sample paths. This upper bound appears as an index deduced from the symbol of the corresponding pseudo-differential operator. Recently, Knopova, Schilling and Wang [73] found a lower bound for the range of Markov processes whose transition density satisfies an upper bound in short time uniformly in x . Note that their upper and lower bounds do not coincide in general.

Our aim in this article is to give sharp dimension formula for the class of stable-like processes in the sense of Bass [15], whose jump kernel is given by

$$n(x, du) = \beta(x)|u|^{-d-\beta(x)} du \quad (3.2)$$

where $\beta : \mathbb{R}^d \mapsto (0, 2)$ is a Lipschitz continuous function whose range is included in a compact set in $(0, 2)$. Note $n^0(x, du) = C_{\beta(x)}|u|^{-d-\beta(x)} du$ and \mathcal{L}^0 the corresponding generator, where

$C_{\beta(x)}$ is the constant such that

$$\mathcal{L}^0(e^{i\xi \cdot x}) = |\xi|^{\beta(x)} e^{i\xi \cdot x}.$$

Immediately, the generator of our process is a variable order pseudo-differential operator with symbol $q(x, \xi) = \frac{\beta(x)}{C_{\beta(x)}} |\xi|^{\beta(x)}$. By the assumption on the range of β , one knows that $C_{\beta(\cdot)}$ is bounded uniformly away from 0 and $+\infty$. We recall the main features of such processes in Section 2. In this context, the stochastic process $t \mapsto \beta(\mathcal{M}_t)$ plays the role of a *local Blumenthal-Gettoor upper index*. Loosely speaking, a stable-like process behaves locally like a $\beta(\mathcal{M}_t)$ -stable process in a small neighbourhood of time t , see [120] for a precise interpretation in terms of tangent processes.

The main result of this article is the following.

Theorem 3.1. *Let \mathcal{M} be a stable-like process in \mathbb{R}^d , i.e. a jump-type Markov process whose jump kernel satisfies (3.2). Then a.s. for every open interval $I = (a, b) \subset \mathbb{R}^+$,*

$$\dim_{\mathcal{H}}(\mathcal{M}(I)) = d \wedge \sup_{s \in I} \beta(\mathcal{M}_s).$$

Here and after, $\dim_{\mathcal{H}} E$ denotes the Hausdorff dimension of the set E .

This result shows that the range of a variable order Markov process typically has a random Hausdorff dimension.

Let us comment on the method of proof. We combine the classical methods with the "slicing" technique introduced in [120] in order to compute the upper bound for the Hausdorff dimension of the range of stable-like processes. This technique allows to distinguish different local behavior of \mathcal{M} , see Section 3. On the other hand, a coupling argument is used to deduce the lower bound for the dimension of the range of \mathcal{M} .

As a by-product, we are also able to compute the dimension of the graph of stable-like processes.

Theorem 3.2. *Let \mathcal{M} satisfy the hypotheses in Theorem 3.1. Let $Gr_I(\mathcal{M}) = \{(t, \mathcal{M}_t) : t \in I\}$ be the graph of \mathcal{M} on the interval $I \subset \mathbb{R}^+$.*

1. *If $d \geq 2$, then a.s. for every open interval $I \subset \mathbb{R}^+$,*

$$\dim_{\mathcal{H}}(Gr_I(\mathcal{M})) = 1 \vee \sup_{t \in I} \beta(\mathcal{M}_t).$$

2. *If $d = 1$, then a.s. for every open interval $I \subset \mathbb{R}^+$,*

$$\dim_{\mathcal{H}}(Gr_I(\mathcal{M})) = 1 \vee \left(2 - \frac{1}{\sup_{t \in I} \beta(\mathcal{M}_t)}\right). \quad (3.3)$$

Jain-Pruitt [63] proved a Hausdorff measure result for the graph of a transient α -stable process, which implied that the Hausdorff dimension of its graph is $\alpha \vee 1$. Recall that a process X is said to be transient if $\lim_{t \rightarrow +\infty} |X_t| = +\infty$ and to be recurrent if $\liminf_{t \rightarrow +\infty} |X_t| = 0$. An α -stable process is transient if and only if $\alpha < d$, see for instance [104]. Their result covers all the α -stable processes in \mathbb{R}^d with $d \geq 2$ and all the one-dimensional α -stable processes with $0 < \alpha < 1$. Blumenthal-Gettoor [29] treated the recurrent case (necessarily $d = 1$ and $\alpha > 1$) and found that the dimension of the graph of a one-dimensional α -stable process is $2 - \frac{1}{\alpha}$. Their proof was performed by using the knowledge on the law of stable processes, thus seems hard to generalize. Later, Pruitt and Taylor [101] investigated, among others things, the asymptotic behavior of the sojourn time of a Lévy process with stable components and related the exact Hausdorff measure of the graph of such process to these results. We will follow and adapt, when necessary, the arguments in [101]. Theorem 3.2 generalizes several parts of [29, 63, 101] to the class of stable-like processes and we obtain the exact value for the dimension of their graph. Indeed, when the function $\beta(\cdot)$ is constant, we recover the classical results.

This paper is organized as follows. We first recall some basic properties of the stable-like processes in Section 2. We study the p -variation of \mathcal{M} in Section 3 to yield the upper bound for the dimension of the range of stable-like processes. The lower bound is proved in Section 4 using a coupling argument. Finally, we deal with the dimension of the graph of \mathcal{M} (Theorem 3.2) in Section 5.

In the whole paper, C is a positive finite constant independent of the problem, that may change from line to line.

From now on, we only consider the time interval $[0, 1]$, extension to any interval is straightforward.

3.2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let λ be the Lebesgue measure on \mathbb{R}^+ , H be the uniform probability measure on \mathbb{S}^{d-1} and $\pi(dr) = r^{-2}dr$ on \mathbb{R}^+ . Denote by N a Poisson random measure on the product space $\mathbb{R}^+ \times \mathbb{S}^{d-1} \times \mathbb{R}^+$ adapted to the filtration (\mathcal{F}_t) and with intensity $\lambda \otimes H \otimes \pi$. We denote by \tilde{N} the corresponding compensated Poisson measure. First we give the Poisson representation of stable-like processes which will be used in the sequel.

Proposition 3.1. *For every \mathcal{F}_0 -measurable random variable \mathcal{M}_0 , there exists a unique pathwise solution to the stochastic differential equation,*

$$\mathcal{M}_t = \mathcal{M}_0 + \int_0^t \int_{\mathbb{S}^{d-1}} \int_0^1 \theta r^{1/\beta(\mathcal{M}_{s-})} \tilde{N}(ds, d\theta, dr) + \int_0^t \int_{\mathbb{S}^d} \int_1^{+\infty} \theta r^{1/\beta(\mathcal{M}_{s-})} N(ds, d\theta, dr). \quad (3.4)$$

Furthermore, the solution to (3.4) is a càdlàg (\mathcal{F}_t) -adapted strong Markov process whose generator

is \mathcal{L} with jump kernel (3.2). We call the solution $(\mathcal{M}_t^x)_{t \geq 0}$ if $\mathcal{M}_0 = x$ a.s. for some $x \in \mathbb{R}^d$.

Proof. First we show the existence of a unique pathwise solution to

$$\mathcal{M}_t = \mathcal{M}_0 + \int_0^t \int_{\mathbb{S}^{d-1}} \int_0^1 \theta r^{1/\beta(\mathcal{M}_{s-})} \tilde{N}(ds, d\theta, dr). \quad (3.5)$$

Since the number of jumps larger than one for N is finite in any finite interval almost surely, the existence of a unique pathwise solution to (3.4) follows by an interlacing procedure as soon as it holds for (3.5), see e.g. [3, Chapter 6]. To this end, it suffices to check a growth and a Lipschitz condition, i.e. that there exists $C > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} \int_0^1 \left| \theta r^{\frac{1}{\beta(x)}} \right|^2 \frac{dr}{r^2} H(d\theta) &\leq C(1 + |x|^2) \\ \int_{\mathbb{S}^{d-1}} \int_0^1 \left| \theta r^{\frac{1}{\beta(x)}} - \theta r^{\frac{1}{\beta(y)}} \right|^2 \frac{dr}{r^2} H(d\theta) &\leq C|x - y|^2 \end{aligned}$$

see e.g. Chapter 6 of [3]. As the range of $\beta(\cdot)$ is included in a compact set of $(0, 2)$, there exists $\varepsilon > 0$ such that $x \mapsto \beta(x)$ is uniformly bounded from above by $2 - \varepsilon$, so that

$$\int_{\mathbb{S}^{d-1}} \int_0^1 \left| \theta r^{1/\beta(x)} \right|^2 \frac{dr}{r^2} H(d\theta) = \int_0^1 r^{\frac{2}{2-\varepsilon}} \frac{dr}{r^2} := C \leq C(1 + |x|^2);$$

and the growth condition is thus satisfied.

Let us now consider the Lipschitz condition. Let $x, y \in \mathbb{R}^d$. Without loss of generality, we assume $\beta(x) > \beta(y)$; then

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} \int_0^1 \left| \theta r^{1/\beta(x)} - \theta r^{1/\beta(y)} \right|^2 \frac{dr}{r^2} H(d\theta) &= \int_0^1 (r^{1/\beta(x)} - r^{1/\beta(y)})^2 \frac{dr}{r^2} \\ &= \int_0^1 r^{2/\beta(x)} \left(1 - e^{(\log \frac{1}{r}) \left(\frac{1}{\beta(x)} - \frac{1}{\beta(y)} \right)} \right)^2 \frac{dr}{r^2} \\ &\leq \int_0^1 r^{2/\beta(x)} \left(\log \frac{1}{r} \right)^2 \left(\frac{1}{\beta(x)} - \frac{1}{\beta(y)} \right)^2 \frac{dr}{r^2} \\ &\leq C|x - y|^2 \int_0^1 r^{2/\beta(x)} \left(\log \frac{1}{r} \right)^2 \frac{dr}{r^2}, \end{aligned}$$

where we used the Lipschitz continuity of the function β . Remark that $\log(1/r)^2 \leq Cr^{-\varepsilon_0}$ for every $r \in (0, 1)$ where $\varepsilon_0 = \frac{1}{2} \left(\frac{2}{\sup_{x \in \mathbb{R}^d} \beta(x)} - 1 \right)$. Hence the last integral is finite and independent of (x, y) . The Lipschitz condition follows.

Using (3.2) and a change of variable ($u = \theta r^{1/\beta(x)}$), the generator of stable-like processes can be written as

$$\mathcal{L}\varphi(x) = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^+} \left(\varphi(x + \theta r^{1/\beta(x)}) - \varphi(x) - \mathbf{1}_{0 < r < 1} r^{1/\beta(x)} \theta \cdot \nabla \varphi(x) \right) \frac{dr}{r^2} H(d\theta).$$

The SDE structure (3.4) together with Itô's formula for Poisson integral yields that, for every $\varphi \in C_c^2(\mathbb{R}^d)$,

$$\begin{aligned} \varphi(\mathcal{M}_t) = & \varphi(\mathcal{M}_0) + \int_0^t \int_{\mathbb{S}^{d-1}} \int_0^1 \left(\varphi(\mathcal{M}_{s-} + \theta r^{1/\beta(\mathcal{M}_{s-})}) - \varphi(\mathcal{M}_{s-}) \right) \tilde{N}(ds, d\theta, dr) \\ & + \int_0^t \int_{\mathbb{S}^{d-1}} \int_0^1 \left(\varphi(\mathcal{M}_{s-} + \theta r^{1/\beta(\mathcal{M}_{s-})}) - \varphi(\mathcal{M}_{s-}) \right. \\ & \quad \left. - \theta r^{1/\beta(\mathcal{M}_{s-})} \cdot \nabla \varphi(\mathcal{M}_{s-}) \right) \frac{dr}{r^2} H(d\theta) ds. \\ & + \int_0^t \int_{\mathbb{S}^{d-1}} \int_1^{+\infty} \left(\varphi(\mathcal{M}_{s-} + \theta r^{1/\beta(\mathcal{M}_{s-})}) - \varphi(\mathcal{M}_{s-}) \right) N(ds, d\theta, dr) \end{aligned}$$

Taking expectation, we get

$$\mathbb{E}[\varphi(\mathcal{M}_t)] = \varphi(\mathcal{M}_0) + \int_0^t \mathbb{E}[\mathcal{L}\varphi(\mathcal{M}_s)] ds$$

implying that \mathcal{L} is indeed the generator of \mathcal{M} , solution to (3.4). The strong Markov property follows from the pathwise uniqueness, see for instance [31]. □

We gather useful properties of stable-like processes in the following proposition. For more information about stable-like processes, we refer to a survey and a paper by Bass [27, 29].

Proposition 3.2. *Let \mathcal{M} be the solution to (3.4), then*

1. *the process*

$$t \mapsto \int_0^t \int_{\mathbb{S}^{d-1}} \int_0^1 \theta r^{1/\beta(\mathcal{M}_{s-})} \tilde{N}(ds, d\theta, dr)$$

is a (\mathcal{F}_t) -martingale in L^2 .

2. *\mathcal{M} is a Feller process whose generator is A with domain $\mathcal{D}(A) \supset C_c^\infty(\mathbb{R}^d)$, such that $A|_{C_c^\infty(\mathbb{R}^d)} = \mathcal{L}$ is a pseudo-differential operator with symbol $q(x, \xi) = (\beta(x)/C_{\beta(x)})|\xi|^{\beta(x)}$.*

Proof. To prove that \mathcal{M} is an L^2 -martingale, it suffices to check that

$$\forall t > 0, \quad I_t = \mathbb{E} \left[\int_0^t \int_{\mathbb{S}^{d-1}} \int_0^1 \left| \theta r^{1/\beta(\mathcal{M}_{s-})} \right|^2 \frac{dr}{r^2} H(d\theta) ds \right] < +\infty.$$

Since the range of β is included in a compact subset of $(0, 2)$, there exists $\varepsilon > 0$ such that $\beta(\cdot)$ is uniformly bounded above by $2 - \varepsilon$. So one has

$$I_t = \mathbb{E} \left[\int_0^t \int_0^1 r^{2/\beta(\mathcal{M}_{s-})} \frac{dr}{r^2} ds \right] \leq \int_0^t \int_0^1 r^{2/(2-\varepsilon)} \frac{dr}{r^2} ds < +\infty.$$

We refer to Chapter 4 of [31] or [17] for the computation of the symbol of the stable-like processes. \square

Now let us recall known results on the upper bound [31, 105] and the lower bound [73] for the dimension of the range of Feller processes.

Theorem 3.3 ([31, 73, 105]). *Let $(X_t)_{t \geq 0}$ be a Feller process with generator $(A, \mathcal{D}(A))$ such that $A|_{C_c^\infty(\mathbb{R}^d)}$ is a pseudo-differential operator with symbol $q(x, \xi)$ satisfying $|q(x, \xi)| \leq c(1 + |\xi|^2)$ for all x and $q(\cdot, 0) \equiv 0$. Then almost surely, $\dim_{\mathcal{H}}(X[0, 1]) \leq d \wedge \beta_\infty$ where*

$$\beta_\infty = \inf \left\{ \delta > 0 : \lim_{|\xi| \rightarrow \infty} \frac{\sup_{|\eta| \leq |\xi|} \sup_{x \in \mathbb{R}^d} |q(x, \eta)|}{|\xi|^\delta} = 0 \right\}. \quad (3.6)$$

If in addition the transition density $\mathfrak{p}(t, x, y)$ of the process X exists and satisfies for some constants c and $\alpha \in (0, 2)$,

$$\forall x, y \in \mathbb{R}^d, t \in [0, 1], \quad \mathfrak{p}(t, x, y) \leq ct^{-d/\alpha}, \quad (3.7)$$

then almost surely, $\dim_{\mathcal{H}}(X[0, 1]) \geq d \wedge \alpha$.

Remark 3.1. *The existence of the density for stable-like processes was proved by Negoro [93] under strong assumption on the regularity of β . Observe that the process \mathcal{M} satisfies the condition (3.7) with $\alpha = \inf_{x \in \mathbb{R}^d} \beta(x)$, see [74, Proposition 3.1 and Theorem 5.1] or [75, Chapter 7].*

A direct application of Theorem 3.3, Remark 3.1 and Proposition 3.2 to our process \mathcal{M} , yields a.s.

$$\dim_{\mathcal{H}}(\mathcal{M}([0, 1])) \in \left[d \wedge \inf_{x \in \mathbb{R}^d} \beta(x), d \wedge \sup_{x \in \mathbb{R}^d} \beta(x) \right].$$

3.3 Study of the p -variation of \mathcal{M} : upper bound of Theorem 3.1

The aim of this section is to prove that

$$\dim_{\mathcal{H}}(\mathcal{M}([0, 1])) \leq \beta^* \wedge d, \text{ where } \beta^* = \sup_{t \in [0, 1]} \beta(\mathcal{M}_t). \quad (3.8)$$

We use a slicing procedure for \mathcal{M} and the p -variation approach to tackle this problem. The use of p -variation in deducing an upper bound for the Hausdorff dimension of the range of sample paths goes back, at least, to McKean [88]. In this article we apply a Theorem by Lépingle [80] on the p -variation of semimartingales.

First let us introduce some notations for the p -variation of functions.

Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}^d$ be a càdlàg function and \mathcal{P} be a finite partition of the interval $[0, t]$ deduced naturally from a family of strictly ordered points $(0 = t_0 < \dots < t_n = t)$. Following the

notations in [80], for any $p \in (0, 2)$, let

$$V_p(f, \mathcal{P}) = \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|^p.$$

Then the (strong) p -variation of f in the interval $[0, t]$ is

$$W_p(f, [0, t]) = \sup \{V_p(f, \mathcal{P}) : \mathcal{P} \text{ finite partition of } [0, t]\}.$$

We also introduce the quantity corresponding to the jumps of f in the interval $[0, t]$,

$$S_p(f, [0, t]) = \sum_{0 < s \leq t} |\Delta f_s|^p,$$

where $\Delta f_s = f(s) - f(s-)$ and $f(s-) = \lim_{t \uparrow s} f(t)$.

Recall that a semimartingale is a process of the form $X_t = X_0 + M_t + A_t$, where X_0 is finite a.s. and is \mathcal{F}_0 measurable, M_t is a local martingale, and A_t is a process whose sample paths have bounded variation on $[0, t]$ for each t . Such a process can be written as $X_t = X_t^c + X_t^j$, the sum of a continuous part X^c and a pure jump part X^j . Let us state a part of Lépingle's result (see Theorem 1 of [80]) which is useful for our purpose.

Theorem 3.4 ([80]). *Let X be a semimartingale such that $\langle X^c \rangle \equiv 0$. Let $p > 0$. Then almost surely,*

$$S_p(X, [0, 1]) < +\infty \iff (\forall p' > p, W_{p'}(X, [0, 1]) < +\infty).$$

Following [120], we slice the process \mathcal{M} according to the different behavior of the local index process $t \mapsto \beta(\mathcal{M}_t)$. This induces a decomposition for the process \mathcal{M} . Precisely, for every $m \in \mathbb{N}_*$, we write $\mathcal{M} = \mathcal{M}_0 + \sum_{k=0}^{m-1} \mathcal{M}^{k,m} + \mathcal{M}^{\geq 1}$ where

$$\mathcal{M}_t^{k,m} = \int_0^t \int_{S^{d-1}} \int_0^1 \theta r^{1/\beta(\mathcal{M}_{s-})} \mathbf{1}_{\beta(\mathcal{M}_{s-}) \in [\frac{2k}{m}, \frac{2k+2}{m})} \tilde{N}(ds d\theta dr).$$

and

$$\mathcal{M}_t^{\geq 1} = \int_0^t \int_{\mathbb{S}^d} \int_1^{+\infty} \theta r^{1/\beta(\mathcal{M}_{s-})} N(ds, d\theta, dr).$$

From a trajectory point of view, each sliced process behaves exactly the same as \mathcal{M} when the index process takes value in the sliced interval, otherwise it is only a constant process. The process $\mathcal{M}^{\geq 1}$ is not relevant in the computation of p -variation for \mathcal{M} since it is piecewise constant with finite number of jumps in the unit interval.

Now we are ready to prove (3.8). For each (k, m) , we study the p -variation of $\mathcal{M}^{k,m}$, then deduce the finiteness of the p -variation of the whole process \mathcal{M} for any $p > \beta^*$. The desired inequality follows by a general argument by McKean [88] on the relation between Hausdorff dimension of the range of a function and its p -variation.

Lemma 3.1. *For every $m \in \mathbb{N}_*$ and every $k = 0, \dots, m$, almost surely,*

$$W_{\frac{2k+3}{m}}(\mathcal{M}^{k,m}, [0, 1]) < +\infty.$$

Proof. The method consists in applying Theorem 3.4 to each $\mathcal{M}^{k,m}$ since each of them is a semi-martingale satisfying $\langle (\mathcal{M}^{k,m})^c \rangle \equiv 0$. We start with the observation that

$$\begin{aligned} & S_{(2k+\frac{5}{2})/m}(\mathcal{M}^{k,m}, [0, 1]) \\ &= \int_0^1 \int_{\mathbb{S}^{d-1}} \int_0^1 r^{(2k+\frac{5}{2})/m} \beta(\mathcal{M}_{s-}) \mathbf{1}_{\beta(\mathcal{M}_{s-}) \in [2k/m, (2k+2)/m]} N(ds, d\theta, dr). \end{aligned}$$

Taking expectation, we see that

$$\begin{aligned} \mathbb{E}[S_{(2k+\frac{5}{2})/m}(\mathcal{M}^{k,m}, [0, 1])] &= \int_0^1 \int_0^1 \mathbb{E}[r^{(2k+\frac{5}{2})/m} \beta(\mathcal{M}_{s-}) \mathbf{1}_{\beta(\mathcal{M}_{s-}) \in [2k/m, (2k+2)/m]}] ds \frac{dr}{r^2} \\ &\leq \int_0^1 \int_0^1 r^{(2k+\frac{5}{2})/(2k+2)-2} dr ds < +\infty, \end{aligned}$$

which yields the result by Theorem 3.4. □

Now the finiteness of the p -variation of the whole process \mathcal{M} is proved as follows.

Lemma 3.2. *Almost surely, for any $p > \beta^*$,*

$$W_p(\mathcal{M}, [0, 1]) < +\infty.$$

Proof. Recall that β^* is defined in (3.8). Consider the events

$$A_{k,m} = \left\{ \beta^* + \frac{3}{m} \geq \frac{2k+3}{m} \right\}, \quad B_{k,m} = \left\{ W_{\beta^* + \frac{3}{m}}(\mathcal{M}^{k,m}, [0, 1]) < +\infty \right\}.$$

Since the mapping $p \mapsto \mathbf{1}_{W_p(f, [0,1]) < \infty}$ is non-decreasing, one has

$$\mathbb{P}(B_{k,m} \cap A_{k,m}) \geq \mathbb{P}\left(\left\{ W_{\frac{2k+3}{m}}(\mathcal{M}^{k,m}, [0, 1]) < +\infty \right\} \cap A_{k,m}\right).$$

Under $A_{k,m}^c$, i.e. the complementary of $A_{k,m}$, $\mathcal{M}^{k,m} \equiv 0$ by the definition of $\mathcal{M}^{k,m}$. Hence $B_{k,m}$ is also realized. This inclusion $A_{k,m}^c \subset B_{k,m}$ yields

$$\mathbb{P}(B_{k,m} \cap A_{k,m}^c) = \mathbb{P}(A_{k,m}^c).$$

Combining the previous two estimates, one obtains

$$\begin{aligned} \mathbb{P}(B_{k,m}) &= \mathbb{P}(B_{k,m} \cap A_{k,m}) + \mathbb{P}(B_{k,m} \cap A_{k,m}^c) \\ &\geq \mathbb{P}(\{W_{\frac{2k+3}{m}}(\mathcal{M}^{k,m}, [0, 1]) < +\infty\} \cap A_{k,m}) + \mathbb{P}(A_{k,m}^c) \\ &\geq \mathbb{P}(W_{\frac{2k+3}{m}}(\mathcal{M}^{k,m}, [0, 1]) < +\infty) = 1, \end{aligned}$$

where Lemma 3.1 has been used. By Jensen's inequality (when $p \geq 1$) or subadditivity (when $p < 1$), for any $p \in (0, 3)$, $n \in \mathbb{N}_*$ and $(a_1, \dots, a_n) \in \mathbb{R}^n$, one has $(\sum_{i=1}^n |a_i|)^p \leq (n^{p-1} \vee 1) \sum_{i=1}^n |a_i|^p$. This yields for any finite partition, every family of càdlàg functions $f_i : [0, 1] \rightarrow \mathbb{R}$ with $i = 1, \dots, n$, that

$$V_p\left(\sum_{i=1}^n f_i, \mathcal{P}\right) \leq C(n, p) \sum_{i=1}^n V_p(f_i, \mathcal{P}) \leq C(n, p) \sum_{i=1}^n W_p(f_i, [0, 1]),$$

where $C(n, p) = n^{p-1} \vee 1$. Therefore, since $\mathbb{P}(B_{k,m}) = 1$, one has a.s.

$$W_{\beta^* + \frac{3}{m}}(\mathcal{M}, [0, 1]) \leq C(m, \beta^* + \frac{3}{m}) \sum_{k=1}^m W_{\beta^* + \frac{3}{m}}(\mathcal{M}^{k,m}, [0, 1]) < +\infty$$

for every $m \in \mathbb{N}_*$, which yields the result. \square

Proof of (3.8) : Recall the following fact in [88] : if $f : [0, 1] \rightarrow \mathbb{R}^d$ is a càdlàg function with finite p -variation, then

$$\dim_{\mathcal{H}}(f[0, 1]) \leq p \wedge d. \quad (3.9)$$

Now (3.8) follows by combining the fact above and Lemma 3.2. \square

3.4 Lower bound of Theorem 3.1

To prove the lower bound, we introduce a suitable coupling of \mathcal{M} with a family of processes whose dimension of the range is known. This coupling is used in the proof of the following lemma, see (4.2) below.

Lemma 3.3. *Let $0 \leq t_0 < 1$. Almost surely,*

$$\dim_{\mathcal{H}}(\mathcal{M}[t_0, 1]) \geq \beta(\mathcal{M}_{t_0}) \wedge d.$$

Proof. For each $x \in \mathbb{R}^d$, denote by \mathbb{P}^x the law of \mathcal{M}^x . Consider the canonical space $(D, \mathcal{D}, (\mathcal{D}_t), X_t, \mathbb{P}^x)$ where D is the space of càdlàg functions, X is the coordinate process.

By the Markov property, one has

$$\mathbb{P}(\dim_{\mathcal{H}}(\mathcal{M}[t_0, 1]) \geq \beta(\mathcal{M}_{t_0}) | \mathcal{F}_{t_0}) = g(\mathcal{M}_{t_0})$$

where

$$g(x) = \mathbb{P}^x(\dim_{\mathcal{H}}(X[0, 1 - t_0]) \geq \beta(x)) = \mathbb{P}(\dim_{\mathcal{H}}(\mathcal{M}^x[0, 1 - t_0]) \geq \beta(x)).$$

Now one constructs a coupling with the process \mathcal{M}^x . Let $a \in (0, 2)$ and $\beta_a(\cdot) = \beta(\cdot) \vee a$. For each $\varepsilon > 0$ and any rational number $0 < a \leq \beta(x) - 2\varepsilon$, one introduces the process $\mathcal{M}^{x,a}$, solution to the SDE

$$\mathcal{M}_t^{x,a} = x + \int_0^t \int_{\mathbb{S}^{d-1}} \int_0^1 \theta r^{1/\beta_a(\mathcal{M}_{s-}^{x,a})} \tilde{N}(ds, d\theta, dr) + \int_0^t \int_{\mathbb{S}^d} \int_1^{+\infty} \theta r^{1/\beta_a(\mathcal{M}_{s-})} N(ds, d\theta, dr). \quad (3.10)$$

driven by the same Poisson random measure and endowed with the same initial condition, i.e. $\mathbb{P}(\mathcal{M}_0^{x,a} = \mathcal{M}_0^x = x) = 1$. Existence and pathwise uniqueness of these processes can be proved as in Proposition 3.1.

Define the stopping times

$$\tau^x = \inf\{t \geq 0 : \beta(\mathcal{M}_t^x) \leq \beta(x) - \varepsilon\} \quad \text{and} \quad \tau^{x,a} = \inf\{t \geq 0 : \beta(\mathcal{M}_t^{x,a}) \leq \beta(x) - \varepsilon\}.$$

Define also

$$\tau^{\geq 1} = \inf\{0 \leq t \leq 1 : N([0, t] \times [1, +\infty)) \geq 1\}$$

By the càdlàg property of the sample paths of $\mathcal{M}^{x,a}$ and \mathcal{M}^x , both stopping times are almost surely strictly positive. Note that $\tau^{\geq 1}$ is an exponential random variable with finite parameter, it is also strictly positive almost surely. Set $\tau = \min(\tau^x, \tau^{x,a}, \tau^{\geq 1})/2$. The following observation is fundamental :

$$\text{almost surely, } \forall t \geq 0, \mathcal{M}_{t \wedge \tau}^x = \mathcal{M}_{t \wedge \tau}^{x,a}. \quad (3.11)$$

Indeed, for every $t \geq 0$, using $\tau < \tau^{\geq 1}$, one remarks that the large jump term is identically zero

before time τ so that

$$\begin{aligned}
& \mathbb{E} \left[\left| \mathcal{M}_{t \wedge \tau}^{x,a} - \mathcal{M}_{t \wedge \tau}^x \right|^2 \right] \\
&= \mathbb{E} \left[\left| \int_0^{t \wedge \tau} \int_{S^{d-1}} \int_0^1 \theta \left(r^{1/\beta_a(\mathcal{M}_{s-}^{x,a})} - r^{1/\beta(\mathcal{M}_{s-}^x)} \right) \tilde{N}(dsd\theta dr) \right|^2 \right] \\
&\leq C \mathbb{E} \left[\int_0^{t \wedge \tau} \int_{S^{d-1}} \int_0^1 \left| r^{1/\beta_a(\mathcal{M}_{s-}^{x,a})} - r^{1/\beta(\mathcal{M}_{s-}^x)} \right|^2 N(ds d\theta dr) \right] \\
&= C \mathbb{E} \left[\int_0^{t \wedge \tau} \int_{S^{d-1}} \int_0^1 \left| r^{1/\beta_a(\mathcal{M}_{s-}^{x,a})} - r^{1/\beta(\mathcal{M}_{s-}^x)} \right|^2 \frac{dr}{r^2} H(d\theta) ds \right] \\
&= C \mathbb{E} \left[\int_0^{t \wedge \tau} \int_0^1 \left| r^{1/\beta_a(\mathcal{M}_{s-}^{x,a})} - r^{1/\beta(\mathcal{M}_{s-}^x)} \right|^2 \frac{dr}{r^2} ds \right] \\
&= C \mathbb{E} \left[\int_0^t \int_0^1 \left| r^{1/\beta(\mathcal{M}_{s-\wedge\tau}^{x,a})} - r^{1/\beta(\mathcal{M}_{s-\wedge\tau}^x)} \right|^2 \frac{dr}{r^2} ds \right] \\
&\leq C \mathbb{E} \left[\int_0^t |\mathcal{M}_{s \wedge \tau}^{x,a} - \mathcal{M}_{s \wedge \tau}^x|^2 ds \right] \\
&= C \int_0^t \mathbb{E}[|\mathcal{M}_{s \wedge \tau}^{x,a} - \mathcal{M}_{s \wedge \tau}^x|^2] ds,
\end{aligned}$$

where we used the Burkholder-Davis-Gundy inequality for the first inequality and the Lipschitz condition for the last inequality (see Proposition 3.1 where the existence of pathwise solution for SDE (3.4) was proved.) Hence, using Gronwall's Lemma, for every $t \geq 0$,

$$\mathbb{E} \left[|\mathcal{M}_{t \wedge \tau}^a - \mathcal{M}_{t \wedge \tau}|^2 \right] = 0.$$

This, along with the càdlàg property of the sample paths, yields (3.11).

To conclude, applying Theorem 3.3 to the Markov process $\mathcal{M}^{x,a}$, we obtain that for each $t \in (0, 1]$,

$$\text{almost surely, } \dim_{\mathcal{H}} \mathcal{M}^{x,a}([0, t]) \geq \inf_{x \in \mathbb{R}^d} \beta_a(x) \wedge d \geq a \wedge d.$$

This full probability set is indexed by t and is non-decreasing as t increases. Hence almost surely, for all $t \in (0, 1]$ and all rational $a \in (1, \beta(x) - 2\varepsilon)$, one has $\dim_{\mathcal{H}} \mathcal{M}^{x,a}([0, t]) \geq a \wedge d$. One deduces that a.s.

$$\dim_{\mathcal{H}} \mathcal{M}^x([0, 1 - t_0]) \geq \dim_{\mathcal{H}} \mathcal{M}^x([0, \tau \wedge (1 - t_0)]) = \dim_{\mathcal{H}} \mathcal{M}^{x,a}([0, \tau \wedge (1 - t_0)]) \geq a \wedge d,$$

where we used (3.11) for the equality and the fact that a.s. $\tau > 0$ for the last inequality. Letting $a \rightarrow \beta(x) - 2\varepsilon$ along a countable sequence, then letting $\varepsilon \rightarrow 0$, one obtains that

$$g(x) \equiv 1.$$

One concludes with $\mathbb{P}(\dim_{\mathcal{H}} \mathcal{M}([t_0, 1]) \geq \beta(\mathcal{M}_{t_0})) = \mathbb{E}[g(\mathcal{M}_{t_0})] = 1$. \square

Finally, we prove the lower bound in Theorem 3.1.

Proof. Using Lemma 3.3, we have for each $t_0 \in [0, 1)$ that a.s. $\dim_{\mathcal{H}}(\mathcal{M}([0, 1])) \geq \dim_{\mathcal{H}}(\mathcal{M}([t_0, 1])) \geq \beta(\mathcal{M}_{t_0}) \wedge d$, then a.s.

$$\dim_{\mathcal{H}}(\mathcal{M}([0, 1])) \geq \sup_{t_0 \in [0, 1) \cap \mathbb{Q}} \beta(\mathcal{M}_{t_0}) \wedge d = \sup_{t_0 \in [0, 1]} \beta(\mathcal{M}_{t_0}) \wedge d,$$

where we used the càdlàg property of the sample paths and the fact that $\Delta\mathcal{M}_1 = 0$ a.s. \square

3.5 Dimension of the graph of \mathcal{M} : proof of Theorem 3.2

3.5.1 Case $d \geq 2$

Since a projection never increases the dimension of a subset of \mathbb{R}^d , projecting the graph on the time axis then on the space axis yields the announced lower bound for dimension of the graph.

It remains us to prove the other inequality. Recall that $\beta^* = \sup_{t \in [0, 1]} \beta(\mathcal{M}_t)$. For every $p > \max(1, \beta^*) \geq \beta^* = \beta^* \wedge d$, consider the p -variation of the process $\mathcal{G}(t) = (\text{Id}(t), \mathcal{M}_t)$ in \mathbb{R}^{d+1} , where $\text{Id}(t) = t$. As $W_p(\text{Id}, [0, 1]) \leq 1$ for every $p > 1$, there exists a constant $C = C(d)$ such that

$$W_p(\mathcal{G}, [0, 1]) \leq C(1 + W_p(\mathcal{M}, [0, 1])) < +\infty.$$

by Lemma 3.2. Applying (3.9) yields the desired upper bound.

3.5.2 Case $d = 1$

The proof is split into several parts. The first one gives an upper bound for the upper box dimension of the graph of \mathcal{M} . For the definition of the upper box dimension, see Chapter 3 of [48]. The proof is quite standard, see for instance [94]. We prove it for completeness.

Proposition 3.3. *Almost surely,*

$$\overline{\dim}_B(\text{Gr}_{[0, 1]}(\mathcal{M})) \leq \max \left(1, 2 - \frac{1}{\sup_{t \in I} \beta(\mathcal{M}_t)} \right).$$

Proof. If the event $\{\sup_{t \in [0, 1]} \beta(\mathcal{M}_t) < 1\}$ is realized, Lemma 3.2 yields that the process \mathcal{M} has finite variation, a fortiori, the graph process \mathcal{G} has finite variation. Hence the dimension of the graph of \mathcal{M} (which is the range of \mathcal{G}) is 1 by the projection argument used in Section 3.5.1 and (3.9). The desired inequality is straightforward.

If $\{\sup_{t \in I} \beta(\mathcal{M}_t) \geq 1\}$ is realized, we consider $p > \sup_{t \in I} \beta(\mathcal{M}_t) \geq 1$ and relate the upper box dimension with the p -variation of the process. Denote the oscillation of the process \mathcal{M} in the

dyadic interval $[k2^{-j}, (k+1)2^{-j}]$ by

$$Osc(\mathcal{M}, I_{j,k}) := \sup\{|\mathcal{M}_s - \mathcal{M}_t| : s, t \in I_{j,k}\}.$$

For every k , $Gr_{I_{j,k}}(\mathcal{M})$ can be covered by at most $2^j Osc(\mathcal{M}, I_{j,k}) + 2$ squares of side length 2^{-j} . The number of squares of generation j required to cover the graph $Gr_{[0,1]}(\mathcal{M})$ satisfies

$$\begin{aligned} N_j &= \sum_{k=0}^{2^j-1} (2^j Osc(\mathcal{M}, I_{j,k}) + 2) \leq 2^j \sum_{k=0}^{2^j-1} W_p(\mathcal{M}, I_{j,k})^{\frac{1}{p}} + 2 \cdot 2^j \\ &\leq 2^j \left(\sum_{k=0}^{2^j-1} W_p(\mathcal{M}, I_{j,k}) \right)^{\frac{1}{p}} (2^j)^{1-\frac{1}{p}} + 2 \cdot 2^j \\ &\leq 2 \cdot 2^{j(2-\frac{1}{p})} W_p(\mathcal{M}, [0, 1])^{\frac{1}{p}} \end{aligned}$$

for all j large enough, where we used Hölder inequality for the second inequality. Therefore,

$$\overline{\dim}_B(Gr_{[0,1]}(\mathcal{M})) \leq \limsup_{j \rightarrow \infty} \frac{\log N_j}{\log 2^j} \leq 2 - \frac{1}{p},$$

where we used $W_p(\mathcal{M}, [0, 1]) < \infty$. Letting $p \rightarrow \sup_{t \in [0,1]} \beta(\mathcal{M}_t)$ yields the result. \square

The rest of this section is devoted to prove the lower bound in Theorem 3.2, i.e.

$$\dim_{\mathcal{H}}(Gr_{[0,1]}(\mathcal{M})) \geq 1 \vee \left(2 - \frac{1}{\sup_{t \in [0,1]} \beta(\mathcal{M}_t)} \right).$$

To prove it, we give a deterministic lower bound for the dimension of graph. This should be viewed as an analogue of Theorem 3.3 (the lower bound part) in the graph context.

Proposition 3.4. *Denote $\beta_* = \inf_{x \in \mathbb{R}^d} \beta(x)$. Almost surely,*

$$\dim_{\mathcal{H}}(Gr_{[0,1]}(\mathcal{M})) \geq 1 \vee \left(2 - \frac{1}{\beta_*} \right).$$

We prove this proposition in several steps. First we adapt the ideas in [101] to give tail estimates for the sojourn time of \mathcal{M} . This allows to understand the local behavior of the graph occupation measure. Then we use a density argument (also called mass distribution principle) to obtain the lower bound for the Hausdorff dimension of the graph of \mathcal{M} .

Following Pruitt-Taylor [101], we define the sojourn time of \mathcal{M} in the ball centered at \mathcal{M}_0 with radius a , up to time s as

$$T(a, s) = \int_0^s \mathbf{1}_{|\mathcal{M}_t - \mathcal{M}_0| \leq a} dt.$$

The main estimate is the following.

Lemma 3.4. *Assume that $\beta_* > 1$ and let $C = \frac{2}{1-1/\beta_*}$. For every $s \leq 1$, $\lambda > 0$, $a > 0$,*

$$\mathbb{P}(T(a, s) \geq \lambda a s^{1-\frac{1}{\beta_*}}) \leq e^{-\lambda/2C}$$

Proof. Recall the canonical space in the proof of Lemma 3.3. Define

$$T_X(a, s) = \int_0^s \mathbf{1}_{|X_t - X_0| \leq a} dt.$$

The Markov property entails that for any Borel set $A \subset \mathbb{R}$,

$$\mathbb{P}(T(a, s) \in A) = g_A(\mathcal{M}_0)$$

where $g_A(x) = \mathbb{P}^x(T_X(a, s) \in A)$.

We need to bound from above the exponential moment of the sojourn time. To this end, we study its n -th moment for all $n \geq 2$. For every $k \in \mathbb{N}$ and $s \in \mathbb{R}^+$, let

$$\Gamma_k = \Gamma_k(s) = \{(t_1, \dots, t_k) \in [0, s]^k : 0 \leq t_1 \leq \dots \leq t_k \leq s\}.$$

Using successively the Fubini argument, conditioning and the Markov property of \mathcal{M} ,

$$\begin{aligned} & \mathbb{E}[T(a, s)^n] \\ &= \int_0^s \dots \int_0^s \mathbb{P}\left(\bigcap_{i=1}^n \{|\mathcal{M}_{t_i}| \leq a\}\right) dt_1 \dots dt_n \\ &= n! \int_{\Gamma_n} \mathbb{P}\left(\bigcap_{i=1}^n \{|\mathcal{M}_{t_i}| \leq a\}\right) dt_1 \dots dt_n \\ &\leq n! \int_{\Gamma_n} \mathbb{P}\left(\bigcap_{i=1}^{n-1} \{|\mathcal{M}_{t_i}| \leq a\}, |\mathcal{M}_{t_n} - \mathcal{M}_{t_{n-1}}| \leq 2a\right) dt_1 \dots dt_n \\ &= n! \int_{\Gamma_n} \mathbb{E}\left[\mathbb{P}\left(\bigcap_{i=1}^{n-1} \{|\mathcal{M}_{t_i}| \leq a\} \mid \mathcal{F}_{t_{n-1}}\right) \mathbb{P}^{\mathcal{M}_{t_{n-1}}}\left(|X_{t_n} - X_{t_{n-1}}| \leq 2a\right)\right] dt_1 \dots dt_n \\ &= n! \int_{\Gamma_{n-1}} \mathbb{E}\left[\mathbb{P}\left(\bigcap_{i=1}^{n-1} \{|\mathcal{M}_{t_i}| \leq a\} \mid \mathcal{F}_{t_{n-1}}\right) \mathbb{E}^{\mathcal{M}_{t_{n-1}}}[T_X(2a, s - t_{n-1})]\right] dt_1 \dots dt_{n-1} \\ &\leq n \cdot \left(\sup_{x \in \mathbb{R}^d} \mathbb{E}^x[T_X(2a, s)]\right) \cdot (n-1)! \int_{\Gamma_{n-1}} \mathbb{P}\left(\bigcap_{i=1}^{n-1} \{|\mathcal{M}_{t_i}| \leq a\}\right) dt_1 \dots dt_{n-1} \\ &= n \cdot \left(\sup_{x \in \mathbb{R}^d} \mathbb{E}^x[T_X(2a, s)]\right) \cdot \mathbb{E}[T(a, s)^{n-1}]. \end{aligned}$$

We iterate this procedure to get

$$\mathbb{E}[T(a, s)^n] \leq n! \left(\sup_{x \in \mathbb{R}^d} \mathbb{E}^x [T_X(2a, s)] \right)^n.$$

Thus for all $u > 0$, the exponential moment of $\mathbb{E}[T(a, s)]$ is bounded from above by

$$\mathbb{E}[e^{uT(a, s)}] = \sum_{n=1}^{+\infty} \frac{u^n}{n!} \mathbb{E}[T(a, s)^n] \leq \sum_{n=1}^{+\infty} \left(u \sup_{x \in \mathbb{R}^d} \mathbb{E}^x [T_X(2a, s)] \right)^n.$$

Applying Remark 3.1 related to the density estimate (3.7) yields for all $x \in \mathbb{R}^d$

$$\mathbb{E}^x [T_X(2a, s)] = \int_0^s \mathbb{P}^x(|X_t - x| \leq 2a) dt \leq \int_0^s t^{-1/\beta_*} \cdot 2a dt \leq C \cdot as^{1-\frac{1}{\beta_*}}.$$

with $C = \frac{2}{1-1/\beta_*}$. Finally we choose

$$u = \frac{1}{2 \sup_{x \in \mathbb{R}^d} \mathbb{E}^x [T_X(2a, s)]}$$

to obtain

$$\mathbb{P}(T(a, s) \geq \lambda as^{1-\frac{1}{\beta_*}}) \leq e^{-u\lambda as^{1-\frac{1}{\beta_*}}} \mathbb{E}[e^{uT(a, s)}] \leq e^{-\frac{\lambda}{2C}}.$$

□

The following density lemma is useful for our purpose. We refer to Lemma 4 in [112] for a proof.

Lemma 3.5. *Suppose that ν is a probability measure supported on $E \subset \mathbb{R}^+ \times \mathbb{R}$ such that for ν -almost every (t, x) ,*

$$\limsup_{h \rightarrow 0} \frac{\nu([t, t+h] \times [x-h, x+h])}{\varphi(h)} \leq C < +\infty.$$

Then

$$\mathcal{H}^\varphi(E) \geq \frac{1}{C}$$

where \mathcal{H}^φ denotes the Hausdorff measure related to the gauge function φ , i.e. an increasing function from \mathbb{R}^+ to \mathbb{R}^+ with $\lim_{s \downarrow 0} \varphi(s) = 0$.

Proof of Proposition 3.4: By the projection argument used before, there is nothing to prove when $\beta_* \leq 1$. We thus assume that $\beta_* > 1$. For every $m \in \mathbb{N}_*$, using Lemma 3.4 with $a = s = 2^{-m}$ and $\lambda = m$ yields

$$\mathbb{P}\left(T(2^{-m}, 2^{-m}) \geq m2^{-m(2-1/\beta_*)}\right) \leq e^{-m/2C}.$$

We deduce using the Borel-Cantelli Lemma that a.s. for all m large enough,

$$T(2^{-m}, 2^{-m}) \leq m2^{-m(2-1/\beta_*)}.$$

For all a small enough, let m be the unique integer such that $2^{-m-1} \leq a < 2^{-m}$. Then

$$\frac{T(a, a)}{(\log |a|)a^{2-1/\beta_*}} \leq \frac{T(2^{-m}, 2^{-m})}{m2^{-m(2-1/\beta_*)}} \frac{m2^{-m(2-1/\beta_*)}}{(m+1)2^{-(m+1)(2-1/\beta_*)}} \leq C$$

where C is a positive finite constant independent of m . Thus a.s.

$$\limsup_{a \rightarrow 0} \frac{T(a, a)}{(\log |a|)a^{2-1/\beta_*}} \leq C. \quad (3.12)$$

Consider the probability measure μ , defined by $\mu(A) := \int_0^1 \mathbf{1}_A(t, \mathcal{M}_t) dt$ whose support is the graph $Gr_{[0,1]}(\mathcal{M})$. Applying the same arguments as for the estimate (3.12), we obtain that for all fixed $t_0 \in [0, 1)$, with the same constant as in (3.12),

$$\limsup_{a \rightarrow 0} \frac{\mu([t_0, t_0 + a] \times [\mathcal{M}_{t_0} - a, \mathcal{M}_{t_0} + a])}{\log(1/a)a^{2-1/\beta_*}} \leq C \quad \text{a.s.}$$

A Fubini argument yields that a.s.

$$\text{for } \mu - \text{almost every } t, \quad \limsup_{a \rightarrow 0} \frac{\mu([t, t + a] \times [\mathcal{M}_t - a, \mathcal{M}_t + a])}{\log(1/a)a^{2-1/\beta_*}} \leq C$$

This, together with Lemma 3.5 applied to μ , yields the desired lower bound for the dimension of $Gr_{[0,1]}(\mathcal{M})$. \square

Finally we prove Theorem 3.2 when $d = 1$.

Proof. The proof does not differ much from that of Theorem 3.1, thus we only sketch it. The upper bound is deduced by Proposition 3.3 since the Hausdorff dimension of a set is always smaller than its upper box dimension, see [48]. By Proposition 3.4 and the coupling argument used in Section 3.4, we have for every $t_0 \in [0, 1)$, almost surely,

$$\dim_{\mathcal{H}} Gr_{[t_0, 1]}(\mathcal{M}) \geq \max\left(1, 2 - \frac{1}{\beta(\mathcal{M}_{t_0})}\right).$$

So we have a.s.

$$\dim_{\mathcal{H}} (Gr_{[0,1]}(\mathcal{M})) \geq \sup_{t_0 \in [0, 1] \cap \mathbb{Q}} \max\left(1, 2 - \frac{1}{\beta(\mathcal{M}_{t_0})}\right) = \max\left(1, 2 - \frac{1}{\sup_{t \in [0, 1]} \beta(\mathcal{M}_t)}\right).$$

\square

3.6 Remarks and open questions

Recall that Blumenthal-Gettoor's approach [29] for determining the dimension of the graph of a recurrent stable process consists in finding the dimension of the level sets, then recovering the dimension of the graph by some geometric argument. It would be of great interest if their method could be generalized to the stable-like case.

One possible extension of this article is the study of $\dim_{\mathcal{H}} \mathcal{M}(E)$ with E being any Borel set in \mathbb{R}^+ . In [73], this question was considered and the authors obtained some bounds. The slicing and coupling argument of the present paper may certainly improve the bounds obtained in [73].

Another direction would be to obtain the Hausdorff measure result of stable-like processes. In Lemma 3.4 and 3.5, we already see some flavor of it. More precise estimates should be obtained in this direction.

Chapitre 4

Multifractalité de la mesure d'occupation de diffusion à sauts de type stable

Nous allons étudier le comportement local de la mesure d'occupation μ de diffusion à sauts de type stable \mathcal{M} . Cette mesure décrit le temps passé dans un ensemble quelconque $A \subset \mathbb{R}$ par ces processus. On calcule le spectre multifractal de μ , qui s'avère être aléatoire, dépendant des trajectoires. Cette propriété remarquable contraste fortement les résultats obtenus précédemment (comme par exemple des processus de Lévy), car le spectre multifractal est en général déterministe presque sûrement. De plus, l'allure de ce spectre est originale, qui reflète la richesse et la diversité de la régularité locale. La preuve est basée sur de nouvelles méthodes, qui procurent par exemple l'estimation fine sur la dimension de Hausdorff de certaines configurations de sauts dans le processus ponctuel de Poisson.

Ce chapitre est basé sur l'article [106] en collaboration avec Stéphane Seuret.

4.1 Introduction

The occupation measure of a \mathbb{R}^d -valued stochastic process $(X_t)_{t \geq 0}$ describes the time spent by X in any borelian set $A \subset \mathbb{R}^d$. It is the natural measure supported on the range of the process X , and plays an important role in describing the different fractal dimensions of the range of X . Local regularity results for the occupation measure and its density when it exists (often called local times if X is Markovian) yield considerable information about the path regularity of the process itself, see the survey article by Geman and Horowitz [52] on this subject.

We describe the local behaviors of this occupation measure via its multifractal analysis. Multifractal analysis is now identified as a fruitful approach to provide organized information on the fluctuation of the local regularity of functions and measures, see for instance [48, 62]. Its use in the study of pointwise regularity of stochastic processes and random measures has attracted much attention in recent years, e.g. (time changed) Lévy processes [6, 10, 43, 44, 61], stochastic differential equations with jumps [12, 119, 120], the branching measure on the boundary of a Galton-Watson tree [90, 91], local times of a continuous random tree [7, 20], SPDE [71, 92, 100], Brownian and stable occupation measure [36, 55, 56, 87, 108], amongst many other references.

In this article, we obtain the almost-sure multifractal spectrum of the occupation measure of stable-like jump diffusions, which turns out to be random, depending on the trajectory. This remarkable property is in sharp contrast with the results previously obtained on occupation measures of other processes (such as Lévy processes), since the multifractal spectrum is usually deterministic, almost surely. In addition, the shape of this multifractal spectrum is very original, reflecting the richness and variety of the local behaviors. The proof is based on new methods, which lead for instance to fine estimates on Hausdorff dimensions of certain jump configurations in Poisson point processes.

We first introduce the class of processes we focus on.

Definition 4.1. *Let $\varepsilon_0 > 0$, and $\beta : \mathbb{R} \rightarrow [\varepsilon_0, 1 - \varepsilon_0]$ be a nowhere constant non-decreasing, Lipschitz continuous map. The stable-like processes \mathcal{M} are pure jump Markov processes whose generator can be written as*

$$\mathfrak{L}f(x) = \int_0^1 (f(x+u) - f(x))\beta(x)u^{-1-\beta(x)}du. \quad (4.1)$$

Introduced by Bass [15] in the late 80's by solving a martingale problem, this class of processes has sample paths whose characteristics change as time passes, which is a relevant feature when modeling real data (e.g. financial, geographical data). Roughly speaking, the stable-like processes behave locally like a stable process, but the stability parameter evolves following the current position of the process, see [12] or [120] for an explanation from the tangent processes point of view.

Let $\mathcal{M} = \{\mathcal{M}_t, t \in [0, 1]\}$ be a stable-like process associated with a given function $x \mapsto \beta(x)$

as in Definition 4.1. Our purpose is to describe the local behaviors of the occupation measure of \mathcal{M} defined as

$$\mu(A) = \int_0^1 \mathbf{1}_A(\mathcal{M}_t) dt. \quad (4.2)$$

It depicts how long \mathcal{M} stays in any Borel set $A \subset \mathbb{R}$. We investigate the possible local dimensions for μ , as well as its multifractal spectrum. Let us recall these notions.

Definition 4.2. *Let ν be a positive measure on \mathbb{R} , whose support is $\text{Supp}(\nu) := \{x \in \mathbb{R} : \forall r > 0, \nu(B(x, r)) > 0\}$. The upper local dimension of ν at the point $x \in \text{Supp}(\nu)$ is defined by*

$$\overline{\dim}(\nu, x) = \limsup_{r \downarrow 0} \frac{\log \nu(B(x, r))}{\log r}.$$

Similarly, the lower local dimension of ν at x is

$$\underline{\dim}(\nu, x) = \liminf_{r \downarrow 0} \frac{\log \nu(B(x, r))}{\log r}.$$

When $\overline{\dim}(\nu, x)$ and $\underline{\dim}(\nu, x)$ coincide at x , the common value is denoted by $\dim(\nu, x)$, the local dimension of ν at x .

Our aim is to investigate two multifractal spectra of the occupation measure μ associated with stable-like processes, related to these local dimensions. Let $\dim_{\mathcal{H}}$ stand for the Hausdorff dimension in \mathbb{R} , with the convention that $\dim_{\mathcal{H}}(\emptyset) = -\infty$. The first multifractal spectrum (in *space*) is defined as follows.

Definition 4.3. *Let \mathcal{O} be an open set in \mathbb{R} , and ν a Borel measure on \mathbb{R} . Consider the level sets*

$$\begin{aligned} \overline{E}_{\nu}(\mathcal{O}, h) &= \{x \in \mathcal{O} \cap \text{Supp}(\nu) : \overline{\dim}(\nu, x) = h\} \\ \underline{E}_{\nu}(\mathcal{O}, h) &= \{x \in \mathcal{O} \cap \text{Supp}(\nu) : \underline{\dim}(\nu, x) = h\}. \end{aligned}$$

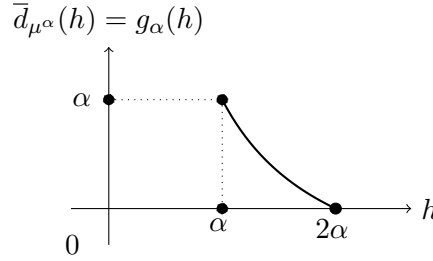
The upper and lower multifractal spectrum of ν are the mappings

$$\begin{aligned} \overline{d}_{\nu}(\mathcal{O}, \cdot) &: h \mapsto \dim_{\mathcal{H}} \overline{E}_{\nu}(\mathcal{O}, h), \\ \underline{d}_{\nu}(\mathcal{O}, \cdot) &: h \mapsto \dim_{\mathcal{H}} \underline{E}_{\nu}(\mathcal{O}, h). \end{aligned}$$

The famous paper by Hu and Taylor [55] states that for every α -stable subordinator \mathcal{L}^{α} whose occupation measure is denoted by μ^{α} , almost surely for all $x \in \text{Supp} \mu^{\alpha}$,

$$\underline{\dim}(\mu^{\alpha}, x) = \alpha \quad \text{and} \quad \overline{\dim}(\mu^{\alpha}, x) \in [\alpha, 2\alpha]. \quad (4.3)$$

It is a classical result [27] that when $\alpha \in (0, 1)$, the image of any interval I by \mathcal{L}^{α} has Hausdorff

Figure 4.1: Space upper multifractal spectrum of μ^α .

dimension α , almost surely. This implies that the support of μ^α has Hausdorff dimension α , almost surely.

With all this in mind, the lower spectrum of μ^α is trivial: for each open interval \mathcal{O} intersecting $\text{Supp}(\mu)$, one has $\underline{d}_{\mu^\alpha}(\mathcal{O}, h) = \begin{cases} \alpha & \text{when } h = \alpha, \\ -\infty & \text{when } h \neq \alpha. \end{cases}$

Hu and Taylor also prove that the upper spectrum is much more interesting (see Figure 4.1): Almost surely, for each open interval \mathcal{O} that intersects $\text{Supp}(\mu^\alpha)$,

$$\bar{d}_{\mu^\alpha}(\mathcal{O}, h) = g_\alpha(h) := \begin{cases} \alpha \left(\frac{2\alpha}{h} - 1 \right) & \text{when } h \in [\alpha, 2\alpha], \\ -\infty & \text{otherwise.} \end{cases} \quad (4.4)$$

Our first result gives the possible values for the local dimensions of the occupation measure μ associated with a stable-like process \mathcal{M} .

Theorem 4.1. *Consider a stable-like process \mathcal{M} associated to a non-decreasing mapping β , as in Definition 4.1, and the associated occupation measure μ . With probability 1, for every $x \in \text{Supp}(\mu)$,*

$$\underline{\dim}(\mu, x) = \beta(x) \text{ and } \overline{\dim}(\mu, x) \in [\beta(x), 2\beta(x)].$$

Hence, the support of the lower spectrum \underline{d}_μ is random, depending on the trajectory of \mathcal{M} .

The *space* lower multifractal spectrum is then quite easy to understand, since the level set $\underline{E}_\mu(\mathcal{O}, h)$ contains either one point or is empty, depending on whether h belongs to the closure of the range of the index process $\{\beta(\mathcal{M}_t) : \mathcal{M}_t \in \mathcal{O}\}$ or not. Theorem 4.1 indicates that the spectrum related to the upper local dimension $\overline{\dim}(\mu, \cdot)$ should be more interesting. This is indeed the case, as resumed in Theorem 4.2. Set

$$\hat{g}_\alpha(h) := \begin{cases} \alpha \left(\frac{2\alpha}{h} - 1 \right) & \text{when } h \in [\alpha, 2\alpha), \\ -\infty & \text{otherwise.} \end{cases}$$

Note that the only difference between g and \hat{g} is at the value $h = 2\alpha$.

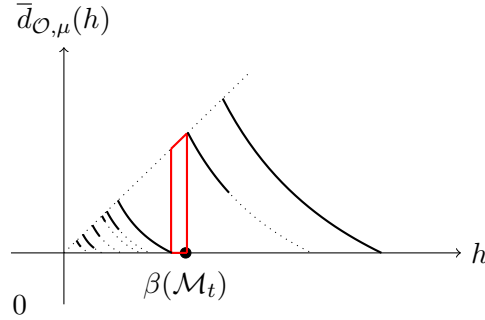


Figure 4.2: Space upper multifractal spectrum of μ on an open set \mathcal{O} . The spectrum (in thick) is obtained as the supremum of a random countable number of functions of the form $g_\alpha(h)$, for the values $\alpha \in \beta(\mathcal{M}_{t-})$, $\mathcal{M}_t \in \mathcal{O}$. It may happen that there is a hole in the support of $\bar{d}_{\mathcal{O},\mu}$ (in red in the figure). In this case, the value of $\bar{d}_{\mathcal{O},\mu}$ at $\beta(\mathcal{M}_t)$ is either 0 or $-\infty$.

Definition 4.4. For every monotone càdlàg function $\Upsilon : \mathbb{R}^+ \rightarrow \mathbb{R}$, we denote by $S(\Upsilon)$ the set of jumps of Υ .

Theorem 4.2. Set the (at most countable) sets of real numbers

$$\begin{aligned} \mathcal{E}_1 &= \{\beta(\mathcal{M}_t) : t \in S(\mathcal{M}) \text{ and } \beta(\mathcal{M}_t) \geq 2\beta(\mathcal{M}_{t-})\}, \\ \mathcal{E}_2 &= \{2\beta(\mathcal{M}_{t-}) : t \in S(\mathcal{M}) \text{ and } \beta(\mathcal{M}_t) \geq 2\beta(\mathcal{M}_{t-})\}, \\ \mathcal{E} &= \mathcal{E}_1 \cup \mathcal{E}_2. \end{aligned} \quad (4.5)$$

With probability 1, for every non-trivial open interval $\mathcal{O} \subset \mathbb{R}$, one has

$$\underline{d}_\mu(\mathcal{O}, h) = \begin{cases} 0 & \text{if } h \in \overline{\{\beta(\mathcal{M}_t) : \mathcal{M}_t \in \mathcal{O}\}}, \\ -\infty & \text{otherwise,} \end{cases} \quad (4.6)$$

and for every $h \in \mathbb{R}^+ \setminus \mathcal{E}$,

$$\bar{d}_\mu(\mathcal{O}, h) = \sup \{\hat{g}_\alpha(h) : \alpha \in \{\beta(\mathcal{M}_t) : \mathcal{M}_t \in \mathcal{O}\}\}. \quad (4.7)$$

Remark 4.1. If the range of $\beta(\cdot)$ is included in, for example, $[1/2, 9/10]$, then the set of exceptional values $\mathcal{E} = \emptyset$ a.s.

First, one shall notice that both spectra are random, depending on the trajectory *and* on the interval \mathcal{O} . In this sense, $\underline{d}_\mu(\mathcal{O}, \cdot)$ and $\bar{d}_\mu(\mathcal{O}, \cdot)$ are inhomogeneous, contrarily to what happens for the occupation measure μ^α of α -stable subordinators (the spectra do not depend on \mathcal{O}).

One shall interpret the space upper spectrum as the supremum of an infinite number of space multifractal spectra of "locally α -stable processes" for all values $\alpha \in \{\beta(\mathcal{M}_t) : \mathcal{M}_t \in \mathcal{O}\}$. This

formula finds its origin in the fact that locally, \mathcal{M} behaves around each continuous time t as an α -stable process with $\alpha = \beta(\mathcal{M}_t)$.

The allure of a typical space upper multifractal spectrum is depicted in Figure 4.2. This shape is very unusual in the literature.

First, observe that, since β and \mathcal{M} are increasing maps, when $t_0 \in S(\mathcal{M})$ is a jump time for \mathcal{M} , then the "local" index of \mathcal{M} jumps at t_0 from $\beta(\mathcal{M}_{t_0-})$ to $\beta(\mathcal{M}_{t_0})$, and for $t \geq t_0$, the only possibility to have $\bar{d}_\mu(\mathcal{O}, \mathcal{M}_t) = \beta(\mathcal{M}_{t_0})$ is when $t = t_0$. Similarly, when $t < t_0$, it is not possible to have $\bar{d}_\mu(\mathcal{O}, \mathcal{M}_t) = 2\beta(\mathcal{M}_{t_0-})$.

In particular there may be a "hole" in the support of $\bar{d}_\mu(\mathcal{O}, \cdot)$. Indeed, a quick analysis of the functions $g_\alpha(\cdot)$ shows that this happens when there is a time t_0 such that $\beta(\mathcal{M}_{t_0}) > 2\beta(\mathcal{M}_{t_0-})$, which occurs with positive probability for functions $\beta(\cdot)$ satisfying $2\varepsilon_0 < 1 - \varepsilon_0$.

All this explains the set of exceptional points \mathcal{E} in Theorem 4.2. We deal with these exceptional points in the following theorem, whose statement is rather long but whose proof follows directly from a careful analysis of the previous results.

Theorem 4.3. *With probability 1, for every non-trivial open interval $\mathcal{O} \subset \mathbb{R}$, when $h \in \mathcal{E}$, three cases may occur.*

1. $h = \beta(\mathcal{M}_t) > 2\beta(\mathcal{M}_{t-})$. If $\mathcal{M}_t \notin \mathcal{O}$, $\bar{d}_\mu(\mathcal{O}, h) = -\infty$, otherwise one has

$$\bar{d}_\mu(\mathcal{O}, h) = \begin{cases} 0 & \text{if } \bar{\dim}(\mu, \mathcal{M}_t) = h, \\ -\infty & \text{if } \bar{\dim}(\mu, \mathcal{M}_t) > h. \end{cases}$$

2. $h = 2\beta(\mathcal{M}_{t-}) < \beta(\mathcal{M}_t)$. If $\mathcal{M}_{t-} \notin \mathcal{O}$, $\bar{d}_\mu(\mathcal{O}, h) = -\infty$, otherwise one has

$$\bar{d}_\mu(\mathcal{O}, h) = \begin{cases} 0 & \text{if } \bar{\dim}(\mu, \mathcal{M}_{t-}) = h, \\ -\infty & \text{if } \bar{\dim}(\mu, \mathcal{M}_{t-}) < h. \end{cases}$$

3. $h = \beta(\mathcal{M}_t) = 2\beta(\mathcal{M}_{t-})$. If $\{\mathcal{M}_t, \mathcal{M}_{t-}\} \in \mathcal{O}$, one has

$$\bar{d}_\mu(\mathcal{O}, h) = \begin{cases} 0 & \text{if } \bar{\dim}(\mu, \mathcal{M}_{t-}) = h \text{ or } \bar{\dim}(\mu, \mathcal{M}_t) = h, \\ -\infty & \text{if } \bar{\dim}(\mu, \mathcal{M}_{t-}) < h \text{ and } \bar{\dim}(\mu, \mathcal{M}_t) > h. \end{cases}$$

If only one of \mathcal{M}_t and \mathcal{M}_{t-} belongs to \mathcal{O} (say, \mathcal{M}_{t-}), one has

$$\bar{d}_\mu(\mathcal{O}, h) = \begin{cases} 0 & \text{if } \bar{\dim}(\mu, \mathcal{M}_{t-}) = h, \\ -\infty & \text{if } \bar{\dim}(\mu, \mathcal{M}_{t-}) < h. \end{cases}$$

If neither \mathcal{M}_t nor \mathcal{M}_{t-} belongs to \mathcal{O} , one has $\bar{d}_\mu(\mathcal{O}, h) = -\infty$.

It is also interesting to consider the *time multifractal spectra* related to \mathcal{L}^α and \mathcal{M} , which describes the variation along time of the Hausdorff dimension of the set of times t such that μ^α (or μ) has a local dimension h at $x = \mathcal{L}_t^\alpha$ (or \mathcal{M}_t). By abuse of language, one says in this case that μ (or μ^α) has a dimension h at t . For this let us introduce other level sets.

Definition 4.5. For every open set $\mathcal{O} \subset [0, 1]$, set

$$\begin{aligned}\overline{E}_{\mu^\alpha}^t(\mathcal{O}, h) &= \{t \in \mathcal{O} : \overline{\dim}(\mu^\alpha, \mathcal{L}_t^\alpha) = h\}, \\ \overline{E}_\mu^t(\mathcal{O}, h) &= \{t \in \mathcal{O} : \overline{\dim}(\mu, \mathcal{M}_t) = h\},\end{aligned}$$

and the similar quantities for lower local dimensions $\underline{E}_{\mu^\alpha}^t(\mathcal{O}, h)$ and $\underline{E}_\mu^t(\mathcal{O}, h)$. The corresponding time multifractal spectra of μ^α and μ are

$$\begin{aligned}\overline{d}_{\mu^\alpha}^t(\mathcal{O}, \cdot) &: h \mapsto \dim_{\mathcal{H}} \overline{E}_{\mu^\alpha}^t(\mathcal{O}, h), \\ \overline{d}_\mu^t(\mathcal{O}, \cdot) &: h \mapsto \dim_{\mathcal{H}} \overline{E}_\mu^t(\mathcal{O}, h),\end{aligned}$$

and the similar quantities for lower local dimensions.

In the case of a stable subordinator, Hu and Taylor prove that a.s.,

$$\overline{d}_{\mu^\alpha}^t(\mathcal{O}, h) = \frac{g_\alpha(h)}{\alpha} = \begin{cases} 2\alpha/h - 1 & \text{when } h \in [\alpha, 2\alpha], \\ -\infty & \text{otherwise.} \end{cases} \quad (4.8)$$

This time upper multifractal spectrum is *homogeneous*, in the sense that *it does not depend on the choice of \mathcal{O}* . In this article, we also compute the time multifractal spectra of μ .

Theorem 4.4. Set the (at most countable) set of real numbers

$$\mathcal{E}' = \{\beta(\mathcal{M}_t) : t \in S(\mathcal{M}) \text{ and } \beta(\mathcal{M}_t) \geq 2\beta(\mathcal{M}_{t-})\}.$$

With probability 1, for every non-trivial open interval $\mathcal{O} \subset [0, 1]$,

$$\underline{d}_\mu^t(\mathcal{O}, h) = \begin{cases} 0 & \text{if } h \in \{\beta(\mathcal{M}_t) : t \in \mathcal{O}\}, \\ -\infty & \text{otherwise,} \end{cases} \quad (4.9)$$

and for every $h \in \mathcal{E}'$,

$$\overline{d}_\mu^t(\mathcal{O}, h) = \sup \left\{ \frac{\hat{g}_\alpha(h)}{\alpha} : \alpha \in \{\beta(\mathcal{M}_t) : t \in \mathcal{O}\} \right\}. \quad (4.10)$$

Remark 4.2. Note that $\mathcal{E}' \subset \mathcal{E}$. If, for example, the range of $\beta(\cdot)$ is included in $[1/2, 9/10]$, then almost surely $\mathcal{E}' = \emptyset$.

The first part is trivial. Observe that there is a subtle difference between (4.9) and (4.6), since at

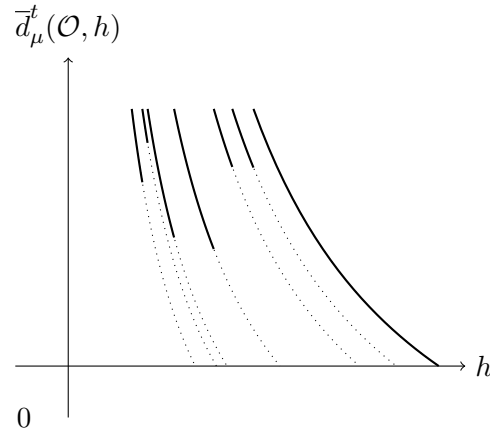


Figure 4.3: Time upper multifractal spectrum of μ on an open set \mathcal{O} . The spectrum (in thick) is obtained as the supremum of a random countable number of functions of the form $\frac{g_\alpha(h)}{\alpha}$ (drawn using dotted graphs), for the values $\alpha \in \beta(\mathcal{M}_t)$, $t \in \mathcal{O}$.

each jump time t for \mathcal{M} , there is no $s \in \mathbb{R}$ such that $\mathcal{M}_s = \mathcal{M}_{t-}$.

As for the space multifractal spectrum, there is some uncertainty about the value of $\bar{d}_\mu^t(\mathcal{O}, \mathcal{M}_t)$ at the jump times $t \in S(\mathcal{M})$, which is dealt with in the following theorem.

Theorem 4.5. *With probability 1, for every non-trivial open interval $\mathcal{O} \subset \mathbb{R}$, for every $h \in \{\beta(\mathcal{M}_t) : t \in S(\mathcal{M})\}$, writing $h = \beta(\mathcal{M}_t)$, one has*

$$\bar{d}_\mu^t(\mathcal{O}, h) = \frac{\bar{d}_\mu(\mathcal{O}, h)}{\beta(\mathcal{M}_{t-})}.$$

The correspondance between (4.4) and (4.8) follows from the fact that for every α -stable subordinator, almost surely for each measurable set $E \subset [0, 1]$,

$$\dim_{\mathcal{H}}(\mathcal{L}^\alpha(E)) = \alpha \cdot \dim_{\mathcal{H}}(E). \quad (4.11)$$

Up to a countable number of points, writing $\mathcal{O} = (g, d)$, one has the equality

$$\mathcal{L}^\alpha(\bar{E}_{\mu^\alpha}^t(\mathcal{O}, h)) = \bar{E}_{\mu^\alpha}((\mathcal{L}_g^\alpha, \mathcal{L}_d^\alpha), h).$$

The method developed by Hu and Taylor consists first in proving (4.8), and then in applying (4.11) to get (4.4).

Following these lines, we start by proving Theorem 4.4. The original methods by Hu and Taylor do not extend here, and an alternative way to compute the time multifractal spectrum of μ is needed. For this, some scenario leading to the fact that μ has exactly an upper local dimension equal to h at $x = \mathcal{M}_t$ is identified. More precisely, it will be proved that $d_\mu(\mathcal{O}, \mathcal{M}_t) = h$ when t is infinitely many times very closely surrounded by two "large" jump times for the Poisson

point process involved in the construction of \mathcal{M} . Using this property, we build in Section 4.6 a (random) Cantor set of such times t with the suitable Hausdorff dimension. The difficulty lies in the fact that the expected Hausdorff dimension is random and depends on the interval we are working on.

Further, it is natural to look for a uniform dimension formula such as (4.11) for the stable-like jump diffusion process \mathcal{M} . As is pointed out by Hu and Taylor [56] (see also page 94 of [24]), as long as the Laplace exponent of a general subordinator oscillates at infinity, i.e. it exhibits different power laws at infinity, one can never expect such an identity to hold. Nevertheless, using an argument based on coupling and time change, we find upper and lower dimension bounds for images of sets by stable-like processes.

Theorem 4.6. *Almost surely, for every measurable set $E \subset [0, 1]$, one has*

$$\dim_{\mathcal{H}} \mathcal{M}(E) \in \dim_{\mathcal{H}}(E) \cdot \left[\inf_{t \in E} \beta(\mathcal{M}(t)), \sup_{t \in E} \beta(\mathcal{M}(t-)) \right]. \quad (4.12)$$

Moreover, if the set E satisfies that for every non-trivial subinterval $\mathcal{O} \subset [0, 1]$, $\dim_{\mathcal{H}}(E) = \dim_{\mathcal{H}}(E \cap \mathcal{O})$, then

$$\dim_{\mathcal{H}} \mathcal{M}(E) = \dim_{\mathcal{H}}(E) \cdot \sup_{t \in E} \beta(\mathcal{M}(t-)).$$

These results are fine enough for us to deduce Theorem 4.2 from Theorem 4.4, even when the stability parameter of these processes varies as time passes. Also, this theorem solves partially a question left open in [121].

Let us end this introduction with a proposition describing the typical behavior of the occupation measure μ . We skip the proof since it can be deduced by adapting the arguments in the proof of our Theorem 4.7.

Proposition 4.1 (Typical Behavior). *With probability 1, one has:*

- for Lebesgue-almost every time $t \in [0, 1]$, $\dim(\mu, \mathcal{M}_t) = \beta(\mathcal{M}_t)$.
- for μ -almost every point $x \in \mathbb{R}$, $\dim(\mu, x) = \beta(x)$.

The rest of this paper is organized as follows. We start by recalling basic properties of the stable-like processes in Section 4.2. The local dimensions of the occupation measure (Theorem 4.1) are studied in Section 4.3. The time spectrum (Theorems 4.4 and 4.5) is obtained in Section 4.4 using a general result (Theorem 4.7), whose proof is given in Sections 4.5 and 4.6. Finally, the space spectrum (Theorem 4.2) is dealt with in Section 4.7, together with the dimension of images of arbitrary sets by stable-like processes (Theorem 4.6).

4.2 Preliminaries

First of all, stable-like processes admit a Poisson representation which was regularly used to study path properties of such processes, see for instance [12, 120, 121]. Let us recall this representation and a coupling associated with it which will be useful for our purposes.

Let $N(dt, dz)$ be a Poisson measure on $\mathbb{R}^+ \times \mathbb{R}$ with intensity $dt \otimes dz/z^2$. Such a measure can be constructed from a Poisson point process which is the set of jumps of a Lévy process with triplet $(0, 0, dz/z^2)$, see for instance Chapter 2 of [3]. We denote $\mathcal{F}_t = \sigma(\{N(A) : A \in \mathcal{B}([0, t] \times [0, +\infty))\})$.

Recall the definition of a stable-like process and formula (4.1). The existence and uniqueness of such jump diffusion processes is classical and recalled in the next proposition. Observe that by the substitution $u = z^{1/\beta(x)}$ (for each fixed x), the generator of a stable-like process is rewritten as

$$\mathfrak{L}f(x) = \int_0^1 \left(f\left(x + z^{\frac{1}{\beta(x)}}\right) - f(x) \right) \frac{dz}{z^2}. \quad (4.13)$$

Proposition 4.2. *Let N be as in the last paragraph.*

1. *There exists a unique càdlàg $(\mathcal{F}_t)_{t \in [0,1]}$ -adapted solution to*

$$\mathcal{M}_t = \int_0^t \int_0^1 z^{\frac{1}{\beta(\mathcal{M}_{u-})}} N(du, dz). \quad (4.14)$$

Furthermore, \mathcal{M} is an increasing strong Markov process with generator \mathfrak{L} given by (4.13).

2. *For every $\alpha \in (0, 1)$, we define*

$$\mathcal{L}_t^\alpha = \int_0^t \int_0^1 z^{\frac{1}{\alpha}} N(du, dz). \quad (4.15)$$

Then for all $\alpha \in (0, 1)$, $\{\mathcal{L}_t^\alpha, t \in [0, 1]\}$ is an α -stable subordinator whose jumps larger than 1 are truncated.

Classical arguments based on Gronwall inequality and Picard iteration yield the first item. For a proof, see Proposition 13 of [12] or Proposition 2.1-2.3 of [120] with some slight modifications. The second item is standard, see for instance Section 2.3 of [3].

Remark 4.3. *Recall that $S(\Upsilon)$ is the set of jumps of a monotone càdlàg function $\Upsilon : [0, 1] \rightarrow \mathbb{R}^+$.*

Observe that by construction, almost surely, the processes \mathcal{M} and the family of Lévy processes $(\mathcal{L}^\alpha)_{\alpha \in (0,1)}$ are purely discontinuous, increasing, with finite variation, and that they jump at the same times, i.e. $S(\mathcal{M}) = S(\mathcal{L}^\alpha)$.

Next observation is key for the study of the local dimensions of μ .

Proposition 4.3. *Consider the process \mathcal{M} and \mathcal{L}^α for all $\alpha \in (0, 1)$ introduced in Proposition 4.2. Almost surely, for all $0 \leq s \leq t \leq 1$,*

$$0 \leq \mathcal{L}_t^{\beta(\mathcal{M}_s)} - \mathcal{L}_s^{\beta(\mathcal{M}_s)} \leq \mathcal{M}_t - \mathcal{M}_s \leq \mathcal{L}_t^{\beta(\mathcal{M}_{t-})} - \mathcal{L}_s^{\beta(\mathcal{M}_{t-})}.$$

This is intuitively true because we construct simultaneously \mathcal{M} and \mathcal{L}^α such that they jump at the same times, and the jump size of \mathcal{L}^α is always larger than $\mathcal{L}^{\alpha'}$ whenever $\alpha > \alpha'$. See Proposition 14 of [12] for a proof.

4.3 Local dimensions of μ : Proof of Theorem 4.1

Observe that almost surely, for all $\alpha \in \mathbb{Q} \cap (0, 1)$, formula (4.3) is true. This, together with Proposition 4.3, leads to the local dimension of μ .

Proof of Theorem 4.1 : Three cases may occur.

1. $x = \mathcal{M}_t$ with t a continuous time of \mathcal{M} . Due to the coupling in their construction (Proposition 4.2), almost surely, for every α , the process \mathcal{L}^α is also continuous at t .

By continuity, for arbitrary rational numbers $\alpha, \alpha' \in (0, 1)$ satisfying $\alpha < \beta(\mathcal{M}_t) < \alpha'$, there exists a small $\delta > 0$ such that for all $s \in (t - \delta, t + \delta)$, $\alpha < \beta(\mathcal{M}_s) < \alpha'$. Using the occupation measure μ^α of the process \mathcal{L}^α , and applying Proposition 4.3 to \mathcal{L}^α and $\mathcal{L}^{\alpha'}$, one gets when r is small

$$\mu^{\alpha'}((\mathcal{L}_t^{\alpha'} - r, \mathcal{L}_t^{\alpha'} + r)) \leq \mu((\mathcal{M}_t - r, \mathcal{M}_t + r)) \leq \mu^\alpha((\mathcal{L}_t^\alpha - r, \mathcal{L}_t^\alpha + r)) \quad (4.16)$$

By formula (4.3) for the lower and upper local dimensions of μ^α , for all small $\varepsilon > 0$, almost surely, one has for r small enough that

$$\alpha - \varepsilon \leq \frac{\log \mu^\alpha((\mathcal{L}_t^\alpha - r, \mathcal{L}_t^\alpha + r))}{\log(r)} \leq 2\alpha + \varepsilon,$$

and the same for α' . Hence

$$\frac{\log \mu((\mathcal{M}_t - r, \mathcal{M}_t + r))}{\log(r)} \geq \frac{\log \mu^\alpha((\mathcal{L}_t^\alpha - r, \mathcal{L}_t^\alpha + r))}{\log(r)} \geq \alpha - \varepsilon.$$

and

$$\frac{\log \mu((\mathcal{M}_t - r, \mathcal{M}_t + r))}{\log(r)} \leq \frac{\log \mu^{\alpha'}((\mathcal{L}_t^{\alpha'} - r, \mathcal{L}_t^{\alpha'} + r))}{\log(r)} \leq 2\alpha' + \varepsilon.$$

Therefore, $\alpha - \varepsilon \leq \underline{\dim}(\mu, x) \leq \overline{\dim}(\mu, x) \leq 2\alpha' + \varepsilon$.

On the other hand, still by formula (4.3), $\underline{\dim}(\mu^{\alpha'}, x) = \alpha'$, so there exists a sequence (r_n) converging to 0 such that

$$\alpha' - \varepsilon \leq \frac{\log \mu^\alpha((\mathcal{L}_t^{\alpha'} - r_n, \mathcal{L}_t^{\alpha'} + r_n))}{\log(r_n)} \leq \alpha' + \varepsilon,$$

so $\alpha - \varepsilon \leq \underline{\dim}(\mu, x) \leq \alpha' + \varepsilon$.

Letting ε tend to zero and α, α' tend to $\beta(\mathcal{M}_t)$ with rational values yields $\beta(\mathcal{M}_t) = \underline{\dim}(\mu, x) \leq \overline{\dim}(\mu, x) \leq 2\beta(\mathcal{M}_t)$.

2. $x = \mathcal{M}_t$ with t a jump time for \mathcal{M} . Observe that in this case $\mu((x - r, x + r)) = \mu((\mathcal{M}_t, \mathcal{M}_t + r))$ for $r > 0$ small enough. For arbitrary rational numbers $\alpha < \beta(\mathcal{M}_t) < \alpha'$, the inequality (4.16) is straightforward using Proposition 4.3. We follow the same lines in the first case to obtain the desired result.
3. $x = \mathcal{M}_{t-}$ with t a jump time for \mathcal{M} . Now, $\mu((x - r, x + r)) = \mu((\mathcal{M}_{t-} - r, \mathcal{M}_{t-}))$ for $r > 0$ small. Then the proof goes like the previous items.

□

Let us end this Section with the proof of the easier part of Theorem 4.2 : space lower multifractal spectrum of μ .

Proof of Formula (4.6) of Theorem 4.2 : As noticed in Remark 4.3, $t \mapsto \beta(\mathcal{M}_t)$ is increasing due to the monotonicity of \mathcal{M} and β . Hence each level set of $t \mapsto \beta(\mathcal{M}_t)$ contains at most one point. This means that for each open interval \mathcal{O} that intersects $\text{Supp}(\mu)$,

$$\begin{aligned} \underline{E}_\mu(\mathcal{O}, h) &= \{x \in \text{Supp}(\mu) \cap \mathcal{O} : \beta(x) = h\} \\ &= \begin{cases} \{\beta(\mathcal{M}_t)\} & \text{if } h = \beta(\mathcal{M}_t) \text{ for some } t \text{ with } \mathcal{M}_t \in \mathcal{O}, \\ \{\beta(\mathcal{M}_{t-})\} & \text{if } h = \beta(\mathcal{M}_{t-}) \text{ for some } t \text{ with } \mathcal{M}_t \in \mathcal{O}, \\ \emptyset & \text{if } h \notin \overline{\{\beta(\mathcal{M}_t) : \mathcal{M}_t \in \mathcal{O}\}}, \end{cases} \end{aligned}$$

which completes the proof. □

4.4 A general result to get the time spectrum (Theorems 4.4 and 4.5)

Let us present a general result, proved in Sections 4.5 and 4.6. This theorem gives the dimension of the random set of times t where the local dimension mapping $s \mapsto \overline{\dim}(\mu, \mathcal{M}_s)$ coincides with a given function. The remarkable feature of this theorem is that it allows to determine these dimensions for all the monotone càdlàg function simultaneously, with probability one.

Theorem 4.7. *For every non-increasing càdlàg function $\Upsilon : [0, 1] \rightarrow [1, 2]$ and every open interval $\mathcal{O} \subset [0, 1]$, let us define $\Upsilon_{\min} = \inf_{u \in \mathcal{O}} \Upsilon(u)$ and the sets*

$$\begin{aligned}\overline{E}_\mu^t(\mathcal{O}, \Upsilon) &= \{t \in \mathcal{O} : \overline{\dim}(\mu, \mathcal{M}_t) = \Upsilon(t)\beta(\mathcal{M}_t)\}, \\ \overline{E}_\mu^{t, \geq}(\mathcal{O}, \Upsilon) &= \{t \in \mathcal{O} : \overline{\dim}(\mu, \mathcal{M}_t) \geq \Upsilon(t)\beta(\mathcal{M}_t)\}.\end{aligned}$$

With probability 1, for every non-increasing càdlàg function $\Upsilon : [0, 1] \rightarrow [1, 2]$ and every open interval $\mathcal{O} \subset [0, 1]$, we have

$$\dim_{\mathcal{H}} \overline{E}_\mu^t(\mathcal{O}, \Upsilon) = \dim_{\mathcal{H}} \overline{E}_\mu^{t, \geq}(\mathcal{O}, \Upsilon) = \dim_{\mathcal{H}} \overline{E}_\mu^{t, \geq}(\mathcal{O}, \Upsilon_{\min}) = \frac{2}{\Upsilon_{\min}} - 1. \quad (4.17)$$

The notation $\overline{E}_\mu^{t, \geq}(\mathcal{O}, \Upsilon_{\min})$ means that we consider the constant function $\Upsilon \equiv \Upsilon_{\min}$.

4.4.1 Proof for the time upper multifractal spectrum

Let us explain why Theorems 4.1 and 4.7 together imply Theorems 4.4 and 4.5. One wants to prove formula (4.10), which can be rewritten as

$$\overline{d}_\mu^t(\mathcal{O}, h) = \sup \left\{ \frac{2\alpha}{h} - 1 \in [0, 1] : \alpha \in \{\beta(\mathcal{M}_t) : t \in \mathcal{O}\} \right\}. \quad (4.18)$$

One combines Theorem 4.1 and Theorem 4.7 with the family of functions $\{\Upsilon^h(t) = h/\beta(\mathcal{M}_t) : h \geq 0\}$. With probability one, these functions are all càdlàg decreasing. Observe that for every open interval \mathcal{O} ,

$$\begin{aligned}\dim_{\mathcal{H}} \overline{E}_\mu^t(\mathcal{O}, \Upsilon^h) &= \dim_{\mathcal{H}} \{t \in \mathcal{O} : \overline{\dim}(\mu, \mathcal{M}_t) = \Upsilon^h(t)\beta(\mathcal{M}_t)\} \\ &= \dim_{\mathcal{H}} \{t \in \mathcal{O} : \overline{\dim}(\mu, \mathcal{M}_t) = h\} \\ &= \overline{d}_\mu^t(\mathcal{O}, h).\end{aligned}$$

We prove now that formula (4.17) applied to the family $\{\Upsilon^h : h \geq 0\}$ implies formula (4.18). Several cases may occur according to the value of h .

• If $\{1\} \not\subset \Upsilon^h(\mathcal{O})$:

1. **First case:** For all $t \in \mathcal{O}$, $\Upsilon^h(t) > 1$.

- If $\inf_{t \in \mathcal{O}} \Upsilon^h(t) \geq 2$, then for all $t \in \mathcal{O}$, $\Upsilon^h(t) > 2$. Theorem 4.1 entails $\overline{E}_\mu^t(\mathcal{O}, \Upsilon^h) = \emptyset$. So $\overline{d}_\mu^t(\mathcal{O}, h) = \dim_{\mathcal{H}} \overline{E}_\mu^t(\mathcal{O}, \Upsilon^h) = -\infty$, which coincides with (4.18).

- If $\inf_{t \in \mathcal{O}} \Upsilon^h(t) < 2$, consider the entrance time in $(1, 2)$ by Υ^h

$$\tau = \inf\{t \in \mathcal{O} : \Upsilon^h(t) < 2\}.$$

By construction, $\forall t \in (\tau, \infty) \cap \mathcal{O}$, $\Upsilon^h(t) \in (1, 2)$. By Theorems 4.1 and 4.7, one gets

$$\begin{aligned} \bar{d}_\mu^t(\mathcal{O}, h) &= \bar{d}_\mu^t((\tau, \infty) \cap \mathcal{O}, h) = \frac{2}{\inf_{t \in (\tau, \infty) \cap \mathcal{O}} \Upsilon^h(t)} - 1 \\ &= \frac{2 \sup_{t \in (\tau, \infty) \cap \mathcal{O}} \beta(\mathcal{M}_t)}{h} - 1, \end{aligned}$$

which coincides with (4.18).

2. **Second case:** There exists $t \in \mathcal{O}$, such that $\Upsilon^h(t) < 1$. Define the passage time of $(-\infty, 1)$ by Υ^h as

$$\sigma = \inf\{t \in \mathcal{O} : \Upsilon^h(t) < 1\}.$$

- If σ is the left endpoint of \mathcal{O} , then for all $t \in \mathcal{O}$, $\Upsilon^h(t) < 1$. Theorem 4.1 yields $\bar{E}_\mu^t(\mathcal{O}, \Upsilon^h) = \emptyset$. Again, this gives $\bar{d}_\mu^t(\mathcal{O}, h) = -\infty$, which coincides with (4.18).
- If σ belongs to the open interval \mathcal{O} , the proof goes along the same lines as in item 1. replacing \mathcal{O} by $(-\infty, \sigma) \cap \mathcal{O}$.

- **If $\{1\} \subset \Upsilon^h(\mathcal{O})$:** Let $t_0 \in \mathcal{O}$ be the unique time such that $\Upsilon^h(t_0) = 1$, i.e. $h = \beta(\mathcal{M}_{t_0})$. One distinguishes different cases according to the behavior of $t \mapsto \mathcal{M}_t$ at t_0 .

1. **If \mathcal{M} is continuous at t_0 :** $\beta(\mathcal{M}_\cdot)$ is also continuous at t_0 . By definition, the entrance time τ satisfies $\tau < t_0$. Theorem 4.7 entails

$$\bar{d}_\mu^t(\mathcal{O}, h) = \bar{d}_\mu^t((\tau, t_0) \cap \mathcal{O}, h) = \frac{2}{\inf_{t \in (\tau, t_0) \cap \mathcal{O}} \Upsilon^h(t)} - 1 = 1,$$

which coincides with (4.18).

2. **If t_0 is a jump time for \mathcal{M} and $\beta(\mathcal{M}_{t_0-}) < h = \beta(\mathcal{M}_{t_0}) < 2\beta(\mathcal{M}_{t_0-})$:** Then, using that $\Upsilon^h(t_0) = 1$, one deduces that $0 < \Upsilon^h(t_0-) - \Upsilon^h(t_0) < 1$, which implies $\inf_{t \in (\tau, t_0) \cap \mathcal{O}} \Upsilon^h(t) < 2$. The same computation as in item 1. with \mathcal{O} replaced by $(\tau, t_0) \cap \mathcal{O}$ yields formula (4.18).
3. **If t_0 is a jump time for \mathcal{M} and $h = \beta(\mathcal{M}_{t_0}) \geq 2\beta(\mathcal{M}_{t_0-})$:** Then $\Upsilon^h(t_0-) \geq 2$, thus $\tau = t_0$. One has

$$\bar{E}^t(\mathcal{O}, h) = \begin{cases} \{t_0\} & \text{if } \overline{\dim}(\mu, \mathcal{M}_{t_0}) = h, \\ \emptyset & \text{otherwise.} \end{cases}$$

This last formula coincides with the one claimed by Theorem 4.5.

4.4.2 Reduction of the problem

Observing that we have the obvious inclusion $\overline{E}_\mu^t(\mathcal{O}, \Upsilon) \subset \overline{E}_\mu^{t, \geq}(\mathcal{O}, \Upsilon) \subset \overline{E}_\mu^{t, \geq}(\mathcal{O}, \Upsilon_{\min})$, we proceed in two parts:

- first, in Section 4.5, we show that

$$\dim_{\mathcal{H}} \overline{E}_\mu^{t, \geq}(\mathcal{O}, \Upsilon_{\min}) \leq \frac{2}{\Upsilon_{\min}} - 1 \quad (4.19)$$

simultaneously for all Υ and \mathcal{O} .

In order to get (4.19), it is equivalent to show that, almost surely, for each $\gamma \in [1, 2]$ and open interval $(a, b) \subset [0, 1]$ with rational endpoints,

$$\dim_{\mathcal{H}} \overline{E}_\mu^{t, \geq}((a, b), \gamma) \leq \frac{2}{\gamma} - 1.$$

We will actually prove that for $\gamma \in (1, 2)$, almost surely,

$$\dim_{\mathcal{H}} \overline{E}_\mu^{t, \geq}((0, 1), \gamma) \leq \frac{2}{\gamma} - 1. \quad (4.20)$$

The extension to arbitrary $a, b \in [0, 1] \cap \mathbb{Q}$ and $\gamma \in \{1, 2\}$ is straightforward.

- second, in Section 4.6, we complete the result by proving that

$$\dim_{\mathcal{H}} \overline{E}_\mu^t(\mathcal{O}, \Upsilon) \geq \frac{2}{\Upsilon_{\min}} - 1 \quad (4.21)$$

also simultaneously for all Υ and \mathcal{O} , almost surely. It is also enough to get the result for $\mathcal{O} = (0, 1)$.

4.5 Proof of Theorem 4.7 : upper bound

Our aim is to prove (4.20). For notational simplicity, we write $\overline{E}_\mu^{t, \geq}((0, 1), \gamma) = E(\gamma)$. Let us first observe that the family of sets $\{E(\gamma), \gamma \in [1, 2]\}$ is non-increasing with respect to γ . Recall that $\gamma \in (1, 2)$ throughout this section.

The strategy is to find a natural limsup set which covers $E(\gamma)$.

For this, we start by pointing out a property satisfied by all points in $E(\gamma)$. Heuristically, it says that every $t \in E(\gamma)$ is infinitely many times surrounded very closely by two points which are large jumps of the Poisson point process generating N .

Proposition 4.4. *With probability 1, one has: for each $t \in E(\gamma)$ and $\varepsilon > 0$ small, there exists an infinite number of integers $n \geq 0$ such that $N((t - 2^{-n}, t] \times [2^{-n/(\gamma-\varepsilon)}, 1] \geq 1$ and $N((t, t + 2^{-n}] \times [2^{-n/(\gamma-\varepsilon)}, 1] \geq 1$.*

Proof Let $t \in E(\gamma)$. This implies that

$$\limsup_{r \rightarrow 0} \frac{\log \mu((\mathcal{M}_t - r, \mathcal{M}_t + r))}{\log r} \geq \gamma \cdot \beta(\mathcal{M}_t). \quad (4.22)$$

This equation is interpreted as the fact that the time spent by the process \mathcal{M} in the neighborhood of \mathcal{M}_t cannot be too large. The most likely way for μ to behave like this is that \mathcal{M} jumps into this small neighborhood of \mathcal{M}_t with a larger than normal jump, and quickly jumps out of that neighborhood with another big jump. This heuristic idea is made explicit by the following computations.

Lemma 4.1. *Let $0 < \varepsilon < \gamma - 1$. If $t \in E(\gamma)$, then there exist infinitely many integers n such that*

$$|\mathcal{M}_{t+2^{-n}} - \mathcal{M}_t| \wedge |\mathcal{M}_t - \mathcal{M}_{t-2^{-n}}| \geq 2^{-\frac{n}{(\gamma-\varepsilon/4)\beta(\mathcal{M}_t)}}. \quad (4.23)$$

Proof Let us prove that t satisfies

$$\limsup_{s \rightarrow 0^+} \frac{|\mathcal{M}_{t+s} - \mathcal{M}_t| \wedge |\mathcal{M}_t - \mathcal{M}_{t-s}|}{s^{1/((\gamma-\varepsilon/5)\beta(\mathcal{M}_t))}} \geq 1. \quad (4.24)$$

Assume first that \mathcal{M} is continuous at t . Assume toward contradiction that for all $s > 0$ sufficiently small, $|\mathcal{M}_{t+s} - \mathcal{M}_t| \leq s^{1/((\gamma-\varepsilon/5)\beta(\mathcal{M}_t))}$ or $|\mathcal{M}_t - \mathcal{M}_{t-s}| \leq s^{1/((\gamma-\varepsilon/5)\beta(\mathcal{M}_t))}$.

If $|\mathcal{M}_{t+s} - \mathcal{M}_t| \leq s^{1/((\gamma-\varepsilon/5)\beta(\mathcal{M}_t))}$, then setting $r = \mathcal{M}_{t+s} - \mathcal{M}_t$,

$$\mu((\mathcal{M}_t - r, \mathcal{M}_t + r)) \geq \mu((\mathcal{M}_t, \mathcal{M}_t + r)) = s \geq r^{(\gamma-\varepsilon/5)\beta(\mathcal{M}_t)}. \quad (4.25)$$

The same holds true when $|\mathcal{M}_t - \mathcal{M}_{t-s}| \leq s^{1/((\gamma-\varepsilon/5)\beta(\mathcal{M}_t))}$. We have thus proved that (4.25) holds for every small r by continuity of \mathcal{M} at t , this contradicts (4.22).

When t is a jump time for \mathcal{M} , the proof goes as above using the two obvious remarks : $\mu(\mathcal{M}_t - r, \mathcal{M}_t + r) = \mu(\mathcal{M}_t, \mathcal{M}_t + r)$ for all small $r > 0$, and $\mathcal{M}_t - \mathcal{M}_{t-s} > (\mathcal{M}_t - \mathcal{M}_{t-})/2$ which does not depend on s .

From (4.24) we deduce (4.23). □

Next technical lemma, proved in [120], shows that when (4.23) holds, there are necessarily at least two "large" jumps around (and very close to) t . Let us recall this lemma, adapted to our context.

Lemma 4.2 ([120]). *Let \tilde{N} stand for the compensated Poisson measure associated with the Poisson*

measure N . There exists a constant C such that for every $\delta > 1$, for all integers $n \geq 1$

$$\mathbb{P} \left(\sup_{\substack{0 \leq s < t \leq 1 \\ |s-t| \leq 2^{-n}}} 2^{\frac{n}{\delta(\beta(\mathcal{M}_{t+2^{-n}})+2/n)}} \left| \int_s^t \int_0^{2^{-\frac{n}{\delta}}} z^{\frac{1}{\beta(\mathcal{M}_{u-})}} \tilde{N}(du, dz) \right| \geq 6n^2 \right) \leq Ce^{-n}.$$

Remark 4.4. The formula looks easier than the one in [120] because in our context \mathcal{M} is increasing. When the function β is constant, the term $2/n$ in the previous inequality disappears, see [6].

Recall formula (4.14) of \mathcal{M} . Last Lemma allows us to control not exactly the increments of \mathcal{M} , but the increments of the "part of \mathcal{M} " constituted by the jumps of size less than $2^{-\frac{n}{\delta}}$. It essentially entails that these "restricted" increments over any interval of size less than 2^{-n} are uniformly controlled by $2^{-\frac{n}{\delta(\beta(\mathcal{M}_{t+2^{-n}})+2/n)}}$ with large probability.

More precisely, Borel-Cantelli Lemma applied to Lemma 4.2 with $\delta = \gamma - \varepsilon$ yields that for all integers n greater than some $n_{\gamma-\varepsilon}$,

$$\begin{aligned} \left| \int_t^{t+2^{-n}} \int_0^{2^{-\frac{n}{\gamma-\varepsilon}}} z^{\frac{1}{\beta(\mathcal{M}_{u-})}} \tilde{N}(du, dz) \right| &\leq 6n^2 \cdot 2^{-\frac{n}{(\gamma-\varepsilon)(\beta(\mathcal{M}_{t+2^{-n+1}})+2/n)}} \\ &\leq 2^{-\frac{n}{(\gamma-\varepsilon/2)\beta(\mathcal{M}_{t+2^{-n+1}})}}. \end{aligned}$$

On the other hand, for all integers n greater than some other $n'_{\gamma-\varepsilon}$, a direct computation gives

$$\begin{aligned} \int_t^{t+2^{-n}} \int_0^{2^{-\frac{n}{\gamma-\varepsilon}}} z^{\frac{1}{\beta(\mathcal{M}_{u-})}} du \frac{dz}{z^2} &\leq C 2^{-n} 2^{-\frac{n}{\gamma-\varepsilon}} \left(\frac{1}{\beta(\mathcal{M}_{t+2^{-n}})} - 1 \right) \\ &\leq 2^{-\frac{n}{(\gamma-\varepsilon)\beta(\mathcal{M}_{t+2^{-n}})}}. \end{aligned}$$

Therefore, for all large n ,

$$\begin{aligned} &\int_t^{t+2^{-n}} \int_0^{2^{-\frac{n}{\gamma-\varepsilon}}} z^{\frac{1}{\beta(\mathcal{M}_{u-})}} N(du, dz) \\ &\leq \left| \int_t^{t+2^{-n}} \int_0^{2^{-\frac{n}{\gamma-\varepsilon}}} z^{\frac{1}{\beta(\mathcal{M}_{u-})}} \tilde{N}(du, dz) \right| + \int_t^{t+2^{-n}} \int_0^{2^{-\frac{n}{\gamma-\varepsilon}}} z^{\frac{1}{\beta(\mathcal{M}_{u-})}} du \frac{dz}{z^2} \\ &\leq 2^{-\frac{n}{(\gamma-\varepsilon/3)\beta(\mathcal{M}_{t+2^{-n+1}})}}. \end{aligned} \tag{4.26}$$

Similarly, one establishes that

$$\int_{t-2^{-n}}^t \int_0^{2^{-\frac{n}{\gamma-4\varepsilon}}} z^{\frac{1}{\beta(\mathcal{M}_{u-})}} N(du, dz) \leq 2^{-\frac{n}{(\gamma-\varepsilon/3)\beta(\mathcal{M}_{t+2^{-n+1}})}}. \tag{4.27}$$

Let us introduce, for every integer $n \geq 1$, the process

$$\widetilde{\mathcal{M}}_t^n = \int_0^t \int_{2^{-\frac{n}{\gamma-\varepsilon}}}^1 z^{1/\beta(\mathcal{M}_{u-})} N(du, dz),$$

so that $\mathcal{M}_t = \widetilde{\mathcal{M}}_t^n + \int_0^t \int_0^{2^{-\frac{n}{\gamma-\varepsilon}}} z^{\frac{1}{\beta(\mathcal{M}_{u-})}} N(du, dz)$.

A direct estimate shows that by right-continuity of \mathcal{M} , when n becomes large, one has

$$3 \cdot 2^{-\frac{n}{(\gamma-\varepsilon/3)\beta(\mathcal{M}_{t+2^{-n}+1})}} < 2^{-\frac{n}{(\gamma-\varepsilon/4)\beta(\mathcal{M}_t)}}$$

Recalling formula (4.14), the three inequalities (4.23), (4.26) and (4.27) imply that for an infinite number of integers n

$$|\widetilde{\mathcal{M}}_{t+2^{-n}}^n - \widetilde{\mathcal{M}}_t^n| \wedge |\widetilde{\mathcal{M}}_t^n - \widetilde{\mathcal{M}}_{t-2^{-n}}^n| \geq 2^{-\frac{n}{(\gamma-\varepsilon/3)\beta(\mathcal{M}_{t+2^{-n}+1})}} \geq 2^{-\frac{n}{(\gamma-\varepsilon/2)\beta(\mathcal{M}_t)}}. \quad (4.28)$$

Since $\widetilde{\mathcal{M}}^n$ (and \mathcal{M}) are purely discontinuous and right continuous, this last inequality proves the existence of at least one time $t_1^n \in (t - 2^{-n}, t]$ and another time $t_2^n \in (t, t + 2^{-n}]$ such that $\widetilde{\mathcal{M}}^n$ (and \mathcal{M}) has a jump. The desired property on the Poisson measure N follows, and Proposition 4.4 is proved. \square

Further, in order to find an upper bound for the dimension of $E(\gamma)$, one constructs a suitable covering of it. For $n \in \mathbb{N}^*$ and $k = 0, \dots, 2^n - 1$, set

$$I_{n,k} = [k2^{-n}, (k+1)2^{-n}) \quad \text{and} \quad \widehat{I}_{n,k} = \bigcup_{\ell=k-1}^{k+1} I_{n,\ell}.$$

One introduces the collection of sets

$$E_n(\gamma, \varepsilon) = \left\{ \widehat{I}_{n,k} : N \left(\widehat{I}_{n,k} \times \left[2^{-\frac{n}{\gamma-\varepsilon}}, 1 \right] \right) \geq 2, \quad k = 0, \dots, 2^n - 1 \right\},$$

which is constituted by the intervals $\widehat{I}_{n,k}$ containing at least two jumps for N of size greater than $2^{-\frac{n}{\gamma-\varepsilon}}$. Finally, one considers the limsup set

$$E(\gamma, \varepsilon) = \limsup_{n \rightarrow +\infty} \bigcup_{\widehat{I} \in E_n(\gamma, \varepsilon)} \widehat{I}. \quad (4.29)$$

Proposition 4.4 states exactly that $E(\gamma) \subset E(\gamma, \varepsilon)$. So it is enough to find an upper bound for the Hausdorff dimension of $E(\gamma, \varepsilon)$. Next lemma estimates the number of intervals contained in $E_n(\gamma, \varepsilon)$.

Lemma 4.3. *With probability 1, there exists a constant C such that for all $n \geq 1$,*

$$\#E_n(\gamma, \varepsilon) \leq Cn^2 2^{n(\frac{2}{\gamma-\varepsilon}-1)}.$$

Proof For a fixed "enlarged" dyadic interval $\widehat{I}_{n,k}$, the inclusion $\widehat{I}_{n,k} \in E_n(\gamma, \varepsilon)$ corresponds to the event that a Poisson random variable with parameter $q_n = 3 \cdot 2^{-n} \cdot 2^{\frac{n}{\gamma-\varepsilon}}$ is larger than 2. Since $q_n \rightarrow 0$ exponentially fast, one has

$$p_n := \mathbb{P}(\widehat{I}_{n,k} \in E_n(\gamma, \varepsilon)) = C_n 2^{-n(2-\frac{2}{\gamma-\varepsilon})},$$

where C_n is a constant depending on n which stays bounded away from 0 and infinity.

The events $\{\widehat{I}_{n,3k} \in E_n(\gamma, \varepsilon)\}_{k \geq 0}$ being independent, the random variable $\#\{k \in \{1, \dots, \lfloor 2^n/3 \rfloor : \widehat{I}_{n,3k} \in E_n(\gamma, \varepsilon)\}$ is Binomial with parameters $(\lfloor 2^n/3 \rfloor, p_n)$. An application of Markov inequality yields

$$\mathbb{P}\left(\#\left\{k : \widehat{I}_{n,3k} \in E_n(\gamma, \varepsilon)\right\} \geq n^2 \lfloor 2^n/3 \rfloor p_n\right) \leq n^{-2}.$$

Further, Borel-Cantelli Lemma gives that almost surely, for n sufficiently large,

$$\#\left\{k : \widehat{I}_{n,3k} \in E_n(\gamma, \varepsilon)\right\} \leq n^2 \lfloor 2^n/3 \rfloor p_n.$$

The same holds for $\#\left\{k : \widehat{I}_{n,3k+1} \in E_n(\gamma, \varepsilon)\right\}$ and $\#\left\{k : \widehat{I}_{n,3k+2} \in E_n(\gamma, \varepsilon)\right\}$. One concludes that for all $n \in \mathbb{N}$ sufficiently large,

$$\#E_n(\gamma, \varepsilon) \leq 3n^2 \lfloor 2^n/3 \rfloor p_n,$$

which proves the claim. □

Now we are in position to prove the upper bound for the Hausdorff dimension of $E(\gamma)$.

Proof of (4.20): Let n_0 be so large that the previous inequalities hold true for all integers $n \geq n_0$. Recalling (4.29), one knows that for every $n_1 \geq n_0$, the union $\bigcup_{n \geq n_1} \bigcup_{\widehat{I} \in E_n(\gamma, \varepsilon)} \widehat{I}$ forms a covering of $E(\gamma, \varepsilon)$, thus a covering of $E(\gamma)$.

Let $s > \frac{2}{\gamma-\varepsilon} - 1$. Fix $\eta > 0$ and n_1 so large that all intervals $\widehat{I} \in E_{n_1}(\gamma, \varepsilon)$ have a diameter less than η . Using the covering just above, one sees that the s -Hausdorff measure of $E(\gamma)$ is bounded above by

$$\mathcal{H}_\eta^s(E(\gamma)) \leq \sum_{n \geq n_1} \sum_{\widehat{I} \in E_n(\gamma, \varepsilon)} |\widehat{I}|^s \leq \sum_{n \geq n_1} Cn^2 2^{n(\frac{2}{\gamma-\varepsilon}-1)} |3 \cdot 2^{-n}|^s$$

which is a convergent series. Therefore, $\mathcal{H}_\eta^s(E(\gamma)) = 0$ as n_1 can be chosen arbitrarily large.

This leads to $\mathcal{H}^s(E(\gamma)) = 0$ for every $s > \frac{2}{\gamma - \varepsilon} - 1$. We have thus proved almost surely,

$$\dim_{\mathcal{H}}(E(\gamma)) \leq \frac{2}{\gamma - \varepsilon} - 1.$$

Letting $\varepsilon \rightarrow 0$ yields the desired upper bound. \square

4.6 Proof of Theorem 4.7 : lower bound

The aim of this section is to get that with probability one, (4.21) holds with $\mathcal{O} = (0, 1)$ for all non-increasing càdlàg function $\Upsilon : [0, 1] \rightarrow [1, 2]$.

Recalling the notations in Theorem 4.7, for simplicity, we write

$$F(\Upsilon) = \overline{E}_{\mu}^t((0, 1), \Upsilon).$$

Let $\varepsilon > 0$ and $0 < b < \varepsilon$ be fixed until the end of Section 4.6.7. We construct simultaneously for all Υ with $1 + 2\varepsilon \leq \Upsilon_{\min} \leq 2 - 2\varepsilon$ and $\varepsilon' > 0$, a random Cantor set $C(\Upsilon, \varepsilon') \subset F(\Upsilon)$ with Hausdorff dimension larger than $2/(\Upsilon_{\min} + \varepsilon') - 1$. The lower bound for the Hausdorff dimension of $F(\Upsilon)$ follows.

We explain how to extend the proof to the functions Υ satisfying $\Upsilon_{\min} \in [1, 2] \setminus [1 + 2\varepsilon, 2 - 2\varepsilon]$ in subsection 4.6.8.

4.6.1 The time scales, and some notations

We aim at constructing Cantor sets inside $F(\Upsilon)$. Recalling Proposition 4.4, some configurations for the jump points are key in this problem. More precisely, one knows that every point in $F(\Upsilon)$ is infinitely often located in the middle of two large jumps which are really close to each other. So the Cantor set we are going to construct will focus on these behaviors.

Let us define a (deterministic) sequence of rapidly decreasing positive real numbers. First,

$$\begin{cases} \eta_{1,0} = 10^{-10}, \\ \eta_{1,\ell} = \eta_{1,\ell-1}^{1+\varepsilon} \text{ for } 1 \leq \ell \leq \ell_1 := \min\{\ell \geq 1 : \eta_{1,\ell} \leq e^{-\eta_{1,0}^{-1}}\}. \end{cases}$$

By induction one defines the sequence $\{\eta_{n,\ell} : n \in \mathbb{N}^*, 0 \leq \ell \leq \ell_n\}$ as

$$\begin{cases} \eta_{n,0} = \eta_{n-1,\ell_{n-1}}, \\ \eta_{n,\ell} = \eta_{n,\ell-1}^{1+\varepsilon} \text{ for } 1 \leq \ell \leq \ell_n := \min\{\ell \geq 1 : \eta_{n,\ell} \leq e^{-\eta_{n,0}^{-1}}\}, \end{cases}$$

which are our time scales. One also sets

$$\begin{aligned}\eta_{n,\ell_{n+1}} &= \eta_{n+1,1} \\ \eta_n &= \eta_{n,0}.\end{aligned}$$

The natural partition of $[0, 1]$ induced by this sequence is denoted by

$$\mathcal{J}_{n,\ell} = \left\{ J_{n,\ell,k} = [k\eta_{n,\ell}, (k+1)\eta_{n,\ell}) : k = 0, \dots, \left\lfloor \frac{1}{\eta_{n,\ell}} \right\rfloor \right\}.$$

By convention, $J_{n,\ell,-1} = J_{n,\ell,-2} = J_{n,\ell, \lfloor \frac{1}{\eta_n} \rfloor + 1} = J_{n,\ell, \lfloor \frac{1}{\eta_n} \rfloor + 2} = \emptyset$.

One needs the enlarged intervals

$$\widehat{J}_{n,\ell,k} = \bigcup_{i=k-1}^{k+1} J_{n,\ell,i}.$$

4.6.2 Zero jump and double jumps configuration

Two types of jump configuration along the scales are of particular interest, since they are the key properties used to build relevant Cantor sets. Recall that the Poisson random measure N has intensity $dt \otimes dz/z^2$.

Definition 4.6. For any $n \in \mathbb{N}^*$, $1 \leq \ell \leq \ell_n$ and $\gamma \in [1 + 2\varepsilon, 2 - 2\varepsilon]$, define

$$\mathcal{J}_{n,\ell}^z(\gamma) = \left\{ J_{n,\ell,k} \in \mathcal{J}_{n,\ell} : N\left(\widehat{J}_{n,\ell,k} \times [\eta_{n,\ell+1}^{1/\gamma}, \eta_{n,\ell}^{1/\gamma})\right) = 0 \right\} \quad (4.30)$$

$$\mathcal{J}_{n,\ell}^d(\gamma) = \left\{ J_{n,\ell,k} \in \mathcal{J}_{n,\ell} : \begin{cases} N\left(J_{n,\ell,k-2} \times [\eta_{n,\ell}^{1/\gamma}/2, \eta_{n,\ell}^{1/\gamma})\right) = 1 \\ N\left(J_{n,\ell,k+2} \times [\eta_{n,\ell}^{1/\gamma}/2, \eta_{n,\ell}^{1/\gamma})\right) = 1 \end{cases} \right\}. \quad (4.31)$$

Remark 4.5. The superscript “z” refers to “zero jump” while “d” refers to “double jump”.

Let us start with straightforward observations:

- for $(n, \ell) \neq (n', \ell')$, the composition (number and position of the intervals) of $\mathcal{J}_{n,\ell}^z(\gamma)$ and $\mathcal{J}_{n',\ell'}^z(\gamma)$ are independent thanks to the Poissonian nature of the measure N .
- The same holds true for the double jump configuration.
- Fixing (n, ℓ) , for $|k - k'| \geq 3$, the events $J_{n,\ell,k} \in \mathcal{J}_{n,\ell}^z(\gamma)$ and $J_{n,\ell,k'} \in \mathcal{J}_{n,\ell}^z(\gamma)$ are independent.
- The same holds for $\mathcal{J}_{n,\ell}^d(\gamma)$ if one assumes that $|k - k'| \geq 5$.
- For fixed (n, ℓ, k) , the events $J_{n,\ell,k} \in \mathcal{J}_{n,\ell}^z(\gamma)$ and $J_{n,\ell,k} \in \mathcal{J}_{n,\ell}^d(\gamma)$ are independent.

Next probability estimate is fundamental in the sequel.

Lemma 4.4. *For all $n \in \mathbb{N}^*$, $1 \leq \ell \leq \ell_n$, $\gamma \in [1 + 2\varepsilon, 2 - 2\varepsilon]$ and $J \in \mathcal{J}_{n,\ell}$,*

$$p_{n,\ell,\gamma} = \mathbb{P}(J \in \mathcal{J}_{n,\ell}^z(\gamma)) = \exp\left(-C_{n,\ell}\eta_{n,\ell}^{(1-\frac{1+\varepsilon}{\gamma})}\right) \quad (4.32)$$

$$q_{n,\ell,\gamma} = \mathbb{P}(J \in \mathcal{J}_{n,\ell}^d(\gamma)) = C'_{n,\ell}\eta_{n,\ell}^{2-\frac{2}{\gamma}} \quad (4.33)$$

where $C_{n,\ell}$, $C'_{n,\ell}$ are constants uniformly (with respect to n , ℓ and γ) bounded away from 0 and infinity.

Proof The value of $p_{n,\ell,\gamma}$ corresponds the probability that a Poisson random variable with parameter $\mathfrak{p} = 3\eta_{n,\ell} \left[\eta_{n,\ell+1}^{1/\gamma} - \eta_{n,\ell}^{1/\gamma} \right]$ equals zero, thus $p_{n,\ell,\gamma} = e^{-\mathfrak{p}}$. On the other hand, each condition in (4.31) relies on the probability that a Poisson variable with parameter $\mathfrak{q} = \eta_{n,\ell}^{1-1/\gamma}$ equals one. Hence, by independence, $q_{n,\ell,\gamma} = (e^{-\mathfrak{q}} \cdot \mathfrak{q})^2$. The result follows. \square

4.6.3 Random trees induced by the zero jump intervals and estimates of the number of their leaves

In this section, one constructs for a fixed integer $n \in \mathbb{N}^*$ a nested collection of intervals, indexed by $0 \leq \ell \leq \ell_n$. These intervals induce naturally a random tree with height $\ell_n + 1$.

One starts with any interval $J_n \in \mathcal{J}_n = \mathcal{J}_{n,0}$, which is the root of the tree, denoted by $\mathcal{T}_{n,0} = \{J_n\}$. Define by induction, for $1 \leq \ell \leq \ell_n$,

$$\mathcal{T}_{n,\ell} = \{J \in \mathcal{J}_{n,\ell} : J \in \mathcal{J}_{n,\ell}^z(\gamma) \text{ and } J \subset \tilde{J} \text{ for some } \tilde{J} \in \mathcal{T}_{n,\ell-1}\}.$$

One focuses on the J_n -rooted random tree $\mathbb{T}_{n,\gamma}(J_n) = (\mathcal{T}_{n,0}, \dots, \mathcal{T}_{n,\ell_n})$. The number of leaves of $\mathbb{T}_{n,\gamma}(J_n)$, denoted by $|\mathbb{T}_{n,\gamma}(J_n)|$, is the cardinality of \mathcal{T}_{n,ℓ_n} .

Fact: Every point belonging to the intervals indexed by the leaves of the tree have the remarkable property that "they do not see" large jump points between the scales η_n and η_{n+1} . This observation is made explicit in Lemma 4.9.

Remark 4.6. *Observe that we dropped the index γ in the definition of $\mathcal{T}_{n,\ell}$ to ease the notations, since these sets will not re-appear in the following sections.*

Our goal is to prove the following estimate on the number of leaves of $\mathbb{T}_{n,\gamma}(J_n)$.

Proposition 4.5. *With probability one, for every integer n large, for every $J_n \in \mathcal{J}_{n,0}$ and $\gamma \in [1 + 2\varepsilon, 2 - 2\varepsilon]$,*

$$|\mathbb{T}_{n,\gamma}(J_n)| \geq \left\lfloor \frac{\eta_n}{2\eta_{n+1}} \right\rfloor. \quad (4.34)$$

The estimate of $|\mathbb{T}_{n,\gamma}(J_n)|$ is divided into several short lemmas.

Lemma 4.5. *For all $n \in \mathbb{N}^*$, $J_n \in \mathcal{J}_{n,0}$ and $\gamma \in [1 + 2\varepsilon, 2 - 2\varepsilon]$, one has*

$$\begin{aligned} \mathbb{P} \left(\#\mathcal{T}_{n,1} \geq (1 - \log(1/\eta_{n,0})^{-2}) \left\lfloor \frac{\eta_{n,0}}{\eta_{n,1}} \right\rfloor p_{n,1,\gamma} \right) \\ \geq 1 - 3 \exp \left(-\log(1/\eta_{n,0})^{-4} \left\lfloor \frac{\eta_{n,0}}{3\eta_{n,1}} \right\rfloor p_{n,1,\gamma/2} \right) \end{aligned}$$

Proof For any (n, ℓ) and for $i \in \{0, 1, 2\}$, set

$$\mathcal{T}_{n,\ell}^i = \{J \in \mathcal{T}_{n,\ell} : J = J_{n,\ell,3k+i} \in \mathcal{J}_{n,\ell} : k \in \mathbb{N}\}.$$

By independence (see the observations before Lemma 4.4), for each $i \in \{0, 1, 2\}$, the number of vertices in $\mathcal{T}_{n,1}^i$ is binomial with parameter $(\lfloor \eta_{n,0}/(3\eta_{n,1}) \rfloor, p_{n,1,\gamma})$.

By Chernoff inequality, for every binomial random variable \mathfrak{X} with parameter (n, p) , for any $\delta \in (0, 1)$, one has

$$\mathbb{P}(\mathfrak{X} \leq (1 - \delta)np) \leq \exp(-\delta^2 np/2). \quad (4.35)$$

The result follows applying (4.35) with $\delta = \log(1/\eta_{n,0})^{-2}$ for every i . \square

Lemma 4.6. *Set $a(n, \ell) = (1 - \log(1/\eta_{n,\ell-1})^{-2}) \left\lfloor \frac{\eta_{n,\ell-1}}{\eta_{n,\ell}} \right\rfloor p_{n,\ell,\gamma}$.*

For all $n \in \mathbb{N}^$, $J_n \in \mathcal{J}_{n,0}$, $\gamma \in [1 - 2\varepsilon, 2 - 2\varepsilon]$ and $2 \leq \ell \leq \ell_n$, a.s.*

$$\begin{aligned} \mathbb{P}(\#\mathcal{T}_{n,\ell} \geq a(n, \ell)\#\mathcal{T}_{n,\ell-1}) \\ \geq 1 - 3 \exp \left(-\log(1/\eta_{n,\ell-1})^{-4} \left(\#\mathcal{T}_{n,\ell-1} \left\lfloor \frac{\eta_{n,\ell-1}}{3\eta_{n,\ell}} \right\rfloor \right) p_{n,\ell,\gamma/2} \right). \end{aligned}$$

Proof Using again the remarks before Lemma 4.4, for every $i \in \{0, 1, 2\}$, the law of the random variable $\#\mathcal{T}_{n,\ell}^i$ conditioning on $\#\mathcal{T}_{n,\ell-1}$ is binomial with parameter $(\#\mathcal{T}_{n,\ell-1} \lfloor \frac{\eta_{n,\ell-1}}{3\eta_{n,\ell}} \rfloor, p_{n,\ell,\gamma})$. Applying (4.35) gives the estimate. \square

Lemma 4.7. *Set*

$$b(n, \ell) = 1 - 3 \exp \left[-\frac{\log(1/\eta_{n,\ell-1})^{-4}}{6(1 - \log(1/\eta_{n,\ell-1})^{-2})} \prod_{k=1}^{\ell} a(n, k) \right].$$

For all $n \in \mathbb{N}^$, $J_n \in \mathcal{J}_{n,0}$ and $\gamma \in [1 - 2\varepsilon, 2 - 2\varepsilon]$, one has*

$$\mathbb{P} \left(|\mathbb{T}_{n,\gamma}(J_n)| \geq \prod_{\ell=1}^{\ell_n} a(n, \ell) \right) \geq \prod_{\ell=1}^{\ell_n} b(n, \ell).$$

Proof One has

$$\begin{aligned} & \mathbb{P} \left(|\mathbb{T}_{n,\gamma}(J_n)| \geq \prod_{\ell=1}^{\ell_n} a(n, \ell) \right) \\ & \geq \mathbb{P} \left(|\mathbb{T}_{n,\gamma}(J_n)| \geq \prod_{\ell=1}^{\ell_n} a(n, \ell), \# \mathcal{T}_{n,\ell_n-1} \geq \prod_{\ell=1}^{\ell_n-1} a(n, \ell) \right) \\ & \geq \mathbb{P} \left(|\mathbb{T}_{n,\gamma}(J_n)| \geq a(n, \ell_n) \# \mathcal{T}_{n,\ell_n-1}, \# \mathcal{T}_{n,\ell_n-1} \geq \prod_{\ell=1}^{\ell_n-1} a(n, \ell) \right). \end{aligned}$$

Conditioning on $\# \mathcal{T}_{n,\ell_n-1}$, and using Lemma 4.6 with $\ell = \ell_n$, this probability is greater than

$$\begin{aligned} & \mathbb{E} \left[\mathbb{E} \left[1 - 3 \exp \left(- \log(1/\eta_{n,\ell_n-1})^{-4} \# \mathcal{T}_{n,\ell_n-1} \left[\frac{\eta_{n,\ell_n-1}}{3\eta_{n,\ell_n}} \right] p_{n,\ell_n,\gamma}/2 \right) \middle| \# \mathcal{T}_{n,\ell_n-1} \right] \right. \\ & \quad \left. \times \mathbf{1}_{\# \mathcal{T}_{n,\ell_n-1} \geq \prod_{\ell=1}^{\ell_n-1} a(n,\ell)} \right] \\ & \geq \mathbb{E} \left[\mathbb{E} \left[1 - 3 \exp \left(- \frac{\log(1/\eta_{n,\ell_n-1})^{-4}}{6(1 - \log(1/\eta_{n,\ell_n-1})^{-2})} a(n, \ell_n) \# \mathcal{T}_{n,\ell_n-1} \right) \middle| \# \mathcal{T}_{n,\ell_n-1} \right] \right. \\ & \quad \left. \times \mathbf{1}_{\# \mathcal{T}_{n,\ell_n-1} \geq \prod_{\ell=1}^{\ell_n-1} a(n,\ell)} \right] \\ & \geq b(n, \ell_n) \mathbb{P} \left(\# \mathcal{T}_{n,\ell_n-1} \geq \prod_{\ell=1}^{\ell_n-1} a(n, \ell) \right) \end{aligned}$$

Iterating this computation yields the desired inequality. \square

We are now in position to prove Proposition 4.5.

Proof We are going to prove the following lemma:

Lemma 4.8. *For some constant c_1 , for all $n \in \mathbb{N}^*$ large enough, for every $J_n \in \mathcal{J}_{n,0}$ and $\gamma \in [1 + 2\varepsilon, 2 - 2\varepsilon]$, one has*

$$\mathbb{P} \left(|\mathbb{T}_{n,\gamma}(J_n)| \geq \left\lfloor \frac{\eta_n}{2\eta_{n+1}} \right\rfloor \right) \geq \exp \left(-c_1 \exp \left(-\eta_n^{-\varepsilon/2} \right) \right). \quad (4.36)$$

Proof One combines the estimates in Lemma 4.4 and Lemma 4.7. Let us first estimate $\prod_{\ell=1}^{\ell'} a(n, \ell)$ for $2 \leq \ell' \leq \ell_n$. Observe that for large n , one has

$$\begin{aligned} \ell_n &= \log \left(\frac{\log(1/\eta_{n,\ell_n})}{\log(1/\eta_{n,0})} \right) / \log(1 + \varepsilon) \leq \log \log(1/\eta_{n,\ell_n}) / \log(1 + \varepsilon) \\ &\leq \log \left(\frac{1 + \varepsilon}{\eta_{n,0}} \right) / \log(1 + \varepsilon) \leq 2 \log(1/\eta_{n,0}). \end{aligned}$$

Thus for n large enough,

$$\begin{aligned}
& \prod_{\ell=1}^{\ell'} (1 - \log(1/\eta_{n,\ell-1})^{-2}) \\
&= \exp\left(\sum_{\ell=1}^{\ell'} \log\left(1 - \log(1/\eta_{n,\ell-1})^{-2}\right)\right) \geq \exp\left(-2 \sum_{\ell=1}^{\ell'} \log(1/\eta_{n,\ell-1})^{-2}\right) \\
&\geq \exp(-2\ell' \log(1/\eta_{n,0})^{-2}) \geq \exp(-4 \log(1/\eta_{n,0})^{-1}) \geq 1/\sqrt{2}.
\end{aligned} \tag{4.37}$$

Using the rapid decay of $(\eta_{n,\ell})$ to zero and the uniform boundedness of $C_{n,\ell}$, one can find a constant $c_0 > 0$ such that for all n large enough,

$$\prod_{\ell=1}^{\ell'} p_{n,\ell,\gamma} = \exp\left(-\sum_{\ell=1}^{\ell'} C_{n,\ell} \eta_{n,\ell}^{1-\frac{1+\varepsilon}{\gamma}}\right) \geq \exp\left(-c_0 \eta_{n,1}^{1-\frac{1+\varepsilon}{\gamma}}\right) \geq 1/\sqrt{2}. \tag{4.38}$$

Combing (4.37) and (4.38), one concludes that for all large n

$$\prod_{\ell=1}^{\ell'} a(n, \ell) \geq \left\lfloor \frac{\eta_{n,0}}{2\eta_{n,\ell'}} \right\rfloor, \text{ and in particular, } \prod_{\ell=1}^{\ell_n} a(n, \ell) \geq \left\lfloor \frac{\eta_n}{2\eta_{n+1}} \right\rfloor. \tag{4.39}$$

Now we estimate the other product $\prod_{\ell=1}^{\ell_n} b(n, \ell)$. Using (4.37), (4.38) and (4.39), there exists a constant $c_1 > 0$ such that for every large n

$$\begin{aligned}
\prod_{\ell=1}^{\ell_n} b(n, \ell) &\geq \prod_{\ell=1}^{\ell_n} \left\{ 1 - 3 \exp\left(-\frac{\log(1/\eta_{n,\ell-1})^{-4}}{12} \left\lfloor \frac{\eta_{n,0}}{\eta_{n,\ell}} \right\rfloor\right) \right\} \\
&= \exp\left\{\sum_{\ell=1}^{\ell_n} \log\left(1 - 3 \exp\left(-\frac{\log(1/\eta_{n,\ell-1})^{-4}}{12} \left\lfloor \frac{\eta_{n,0}}{\eta_{n,\ell}} \right\rfloor\right)\right)\right\} \\
&\geq \exp\left\{-6 \sum_{\ell=1}^{\ell_n} \exp\left(-\frac{\log(1/\eta_{n,\ell-1})^{-4}}{12} \left\lfloor \frac{\eta_{n,0}}{\eta_{n,\ell}} \right\rfloor\right)\right\} \\
&\geq \exp\left\{-c_1 \exp\left(-\frac{\log(1/\eta_{n,0})^{-4}}{12} \left\lfloor \frac{\eta_{n,0}}{\eta_{n,1}} \right\rfloor\right)\right\} \\
&\geq \exp\left(-c_1 \exp\left(-\eta_{n,0}^{-\varepsilon/2}\right)\right) = \exp\left(-c_1 \exp\left(-\eta_n^{-\varepsilon/2}\right)\right).
\end{aligned}$$

where the fast decay rate of $(\eta_{n,\ell})$ to zero has been used for the third inequality.

These last equations prove exactly (4.36). \square

Finally, to prove Proposition 4.5, since the cardinality of \mathcal{J}_n is less than η_n^{-1} ,

$$\mathbb{P}\left(\exists J_n \in \mathcal{J}_n : |\mathbb{T}_{n,\gamma}(J_n)| < \left\lfloor \frac{\eta_n}{2\eta_{n+1}} \right\rfloor\right) \leq \eta_n^{-1} \left(1 - \exp\left(-c_1 \exp\left(-\eta_n^{-\varepsilon/2}\right)\right)\right).$$

Using the fast decay of η_n to zero, this is the general term of a convergent series, and the Borel-Cantelli lemma gives the result. \square

Remark 4.7. *Essentially, one needs to keep in mind that the number of leaves of the random tree $\mathbb{T}_{n,\gamma}(J_n)$ is the total number of intervals of \mathcal{J}_{n+1} inside J_n , up to a constant factor 1/2.*

One finishes this section by proving that every point belonging to a leaf of $\mathbb{T}_{n,\gamma}(J)$ "is not close" to large jumps.

Lemma 4.9. *Let $J \in \mathcal{J}_n$ and $r \in [\eta_{n+1}, \eta_n]$. Assume that $\mathbb{T}_{n,\gamma}(J)$ is not empty. Then for each $t \in \mathbb{T}_{n,\gamma}(J)$,*

$$N(B(t, r) \times [r^{1/\Upsilon_{J_n,0}^n(t)}, \eta_n^{1/\Upsilon_{J_n,0}^n(t)}]) = 0.$$

Proof For each $t \in \mathbb{T}_{n,\gamma}(J)$, denote by $J_{n,\ell}(t)$ the unique interval such that $t \in J_{n,\ell}(t)$ for all $0 \leq \ell \leq \ell_n$. Denote by ℓ_0 the unique integer such that $\eta_{n,\ell_0+1} \leq r < \eta_{n,\ell_0}$. By construction of the random tree $\mathbb{T}_{n,\gamma}(J)$, one has

$$N(B(t, r) \times [r^{1/\Upsilon_{J_n,0}^n(t)}, \eta_{n,\ell_0}^{1/\Upsilon_{J_n,0}^n(t)}]) \leq N(\widehat{J}_{n,\ell_0}(t) \times [\eta_{n,\ell_0+1}^{1/\Upsilon_{J_n,0}^n(t)}, \eta_{n,\ell_0}^{1/\Upsilon_{J_n,0}^n(t)}]) = 0.$$

Further, all ancestor interval of $J_{n,\ell_0}(t)$ has no large jumps around, in particular,

$$N(\widehat{J}_{n,\ell_0}(t) \times [\eta_{n,\ell_0}^{1/\Upsilon_{J_n,0}^n(t)}, \eta_n^{1/\Upsilon_{J_n,0}^n(t)}]) = 0$$

Combining these estimates yields the result. \square

4.6.4 Double jumps configuration around the leaves, and key lemma

In the previous section, we have seen that the "zero jump" configuration is quite frequent. The aim here is to estimate the number of intervals with "double jumps" amongst the leaves of the trees. To this end, we introduce further some notations. Set

$$M_n(\gamma) = \eta_{n+1}^{1-2/(\gamma+3 \cdot 2^{-n-1})} \eta_n^3.$$

Definition 4.7. *Let $J_0 \in \mathcal{J}_{n,0}$ and $\mathbb{T}_{n,\gamma}(J_0)$ be the random tree defined in last subsection. Consider its leaves that we denote $\{J'_i\}_{i=1, \dots, |\mathbb{T}_{n,\gamma}(J_0)|}$, which are intervals of length η_{n+1} .*

The families $\{\mathcal{F}(J_0, \gamma, m)\}_{m=1, \dots, \lfloor M_n(\gamma)/2 \rfloor}$ are defined as the following disjoint subfamilies of $\{J'_i\}_{i=1, \dots, |\mathbb{T}_{n,\gamma}(J_0)|}$:

$$\mathcal{F}(J_0, \gamma, m) = \left\{ J'_{5k+2m \lfloor \frac{|\mathbb{T}_{n,\gamma}(J_0)|}{M_n(\gamma)} \rfloor} : k \in \left\{ 0, \dots, \left\lfloor \frac{|\mathbb{T}_{n,\gamma}(J_0)|}{5M_n(\gamma)} \right\rfloor \right\} \right\}.$$

Hence, two families $\mathcal{F}(J_0, \gamma, m)$ and $\mathcal{F}(J_0, \gamma, m')$ are disjoint and separated by a distance equivalent to $\lfloor |\mathbb{T}_{n,\gamma}(J_0)|/M_n(\gamma) \rfloor \eta_{n+1}$, and the intervals belonging to the same $\mathcal{F}(J_0, \gamma, m)$ are separated by the distance at least $4\eta_{n+1}$.

Finally, denote by

$$\mathcal{D}_n = \{k2^{-n} : k \in \mathbb{Z}, n \in \mathbb{N}^*\}$$

the n -th generation dyadic numbers. One is ready to prove the key lemma.

Lemma 4.10. *The following holds with probability 1: there exists a (random) integer n_0 such that for all $n \geq n_0$, for every $J \in \mathcal{J}_n$, every $\gamma \in \mathcal{D}_n \cap [1+2\varepsilon, 2-2\varepsilon]$, every $\mathfrak{a} \in \{0, 1, 2, 3\}$, each family $\{\mathcal{F}(J, \gamma, m)\}_{m=1, \dots, \lfloor M_n(\gamma)/2 \rfloor}$ contains at least one interval belonging to $\mathcal{J}_{n+1}^d(\gamma + \mathfrak{a} \cdot 2^{-(n+1)})$.*

Remark that the intervals belonging to $\mathcal{F}(J, \gamma, m)$ come also from the tree $\mathbb{T}_{n,\gamma}(J)$ associated with J , so they also enjoy the "zero jump" property.

Proof Fix n a positive integer, $J \in \mathcal{J}_n$, $\gamma \in \mathcal{D}_n \cap [1+2\varepsilon, 2-2\varepsilon]$, $\mathfrak{a} \in \{0, 1, 2, 3\}$.

Recall that $\mathcal{J}_n = \mathcal{J}_{n,0}$ and $\mathcal{J}_{n+1} = \mathcal{J}_{n,\ell_n}$ with the notations of the previous sections.

By Lemma 4.4 and the observations made before this Lemma, there exists a positive finite constant c_2 such that for all n large

$$\begin{aligned} & \mathbb{P} \left(\exists m, \forall J' \in \mathcal{F}(J, \gamma + \mathfrak{a}2^{-n-1}, m), J' \notin \mathcal{J}_{n,\ell_n}^d(\gamma + \mathfrak{a}2^{-n-1}) \right. \\ & \qquad \qquad \qquad \left. \mid |\mathbb{T}_{n,\gamma+\mathfrak{a}2^{-n-1}}(J)| \geq \left\lfloor \frac{\eta_n}{2\eta_{n+1}} \right\rfloor \right) \\ & \leq \left\lfloor \frac{M_n(\gamma)}{2} \right\rfloor (1 - q_{n,\ell_n,\gamma+\mathfrak{a} \cdot 2^{-n-1}})^{\frac{1}{M_n(\gamma)} \cdot \lfloor \frac{\eta_n}{10\eta_{n+1}} \rfloor} \\ & \leq \eta_{n+1}^{1-\frac{2}{\gamma+3 \cdot 2^{-n-1}}} \eta_n^3 \left(1 - c_2 \eta_{n+1}^{\frac{2}{\gamma+\mathfrak{a}2^{-n-1}}} \right)^{\frac{\frac{\eta_n}{2}}{10\eta_{n+1}} \frac{2}{\gamma+3 \cdot 2^{-n-1}}} \eta_n^3 \\ & \leq \eta_{n+1}^{1-\frac{2}{\gamma+3 \cdot 2^{-n-1}}} \eta_n^3 \exp(-c_2 \eta_n^{-2}/10) \end{aligned}$$

Remark that $\eta_{n+1} \leq e^{-\eta_n^{-1}} \leq \eta_{n,\ell_n-1}$ implies $\log(1/\eta_{n+1}) \leq (1+\varepsilon)\eta_n^{-1}$. The above probability is thus bounded by above by

$$\eta_n^3 \exp \left(\left(\frac{2}{\gamma + 3 \cdot 2^{-n-1}} - 1 \right) (1 + \varepsilon) \eta_n^{-1} - \frac{c_2}{10} \eta_n^{-2} \right) \leq \eta_n^3.$$

On the other hand, by Lemma 4.8 one has

$$\begin{aligned} \mathbb{P} \left(\left| \mathbb{T}_{n, \gamma + \mathbf{a}2^{-n-1}}(J) \right| \leq \left\lfloor \frac{\eta_n}{2\eta_{n+1}} \right\rfloor \right) \\ \leq 1 - \exp \left(-c_1 \exp(-\eta_n^{-\varepsilon/2}) \right) \leq 2c_1 \exp(-\eta_n^{-\varepsilon/2}). \end{aligned}$$

Thus, $\mathbb{P} \left(\exists m, \forall J' \in \mathcal{F}(J, \gamma, m), J' \notin \mathcal{J}_{n+1}^d(\gamma + \mathbf{a}2^{-n-1}) \right) \leq 2\eta_n^3$. One deduces that

$$\begin{aligned} \mathbb{P}(\exists J \in \mathcal{J}_n, \exists m, \forall J' \in \mathcal{F}(J, \gamma, m), J' \notin \mathcal{J}_{n+1}^d(\gamma + \mathbf{a}2^{-n-1})) \\ \leq \eta_n^{-1} \cdot 2\eta_n^3 = 2\eta_n^2. \end{aligned}$$

There are less than 2^n possible choices for γ , and 4 choices for \mathbf{a} . Hence,

$$\begin{aligned} \mathbb{P}(\exists \gamma, \exists \mathbf{a}, \exists J \in \mathcal{J}_n, \exists m, \forall J' \in \mathcal{F}(J, \gamma, m), J' \notin \mathcal{J}_{n+1}^d(\gamma + \mathbf{a}2^{-n-1})) \\ \leq 2^{n+3} \eta_n^2, \end{aligned}$$

which is the general term of a convergent series. An application of Borel-Cantelli Lemma entails the result. \square

4.6.5 Construction of the Cantor sets

We are ready to construct the families of Cantor sets $\{C(\Upsilon, \varepsilon')\}$ associated with càdlàg non-increasing functions $\Upsilon : [0, 1] \rightarrow [1 + 2\varepsilon, 2 - 2\varepsilon]$, where ε' is any positive rational parameter. These sets are constituted by points which only see those double jump configurations studied before, and their Hausdorff dimension satisfies $\dim_{\mathcal{H}} C(\Upsilon, \varepsilon') \geq \frac{2}{\Upsilon_{\min} + 2\varepsilon'} - 1$.

Step 1 (localization). For each Υ as above and $\varepsilon' > 0$, there exist $t_{\varepsilon'} \in (0, 1)$, $r_{\varepsilon'} > 0$ such that $\forall t \in [t_{\varepsilon'} - r_{\varepsilon'}, t_{\varepsilon'} + r_{\varepsilon'}]$, we have $\Upsilon(t) < \Upsilon_{\min} + \varepsilon'$.

Let n_0 be the random integer obtained in Lemma 4.10. We assume that n_0 is so large that the conclusions (4.34) of Proposition 4.5 hold, and also that

$$2\varepsilon'/\eta_{n_0} > K_{\Upsilon} \cdot 2^{n_0}, \quad \text{where } K_{\Upsilon} = |\Upsilon(1-) - \Upsilon(0)| < +\infty.$$

For every interval J , let $\text{Osc}_{\Upsilon}(J) = \sup_{t \in J} \Upsilon(t) - \inf_{t \in J} \Upsilon(t)$ be the oscillation of Υ over J . By the monotonicity of Υ , for each $n \geq n_0$ one has

$$\#\{J \in \mathcal{J}_n : J \subset [t_{\varepsilon'} - r_{\varepsilon'}, t_{\varepsilon'} + r_{\varepsilon'}] \text{ and } \text{Osc}_{\Upsilon}(J) \geq 2^{-n}\} \leq K_{\Upsilon} \cdot 2^n \quad (4.40)$$

Step 2 (Initialization of the Cantor set). One chooses arbitrarily one interval $J^{n_0} \in \mathcal{J}_{n_0}$

contained in $[t_{\varepsilon'} - r_{\varepsilon'}, t_{\varepsilon'} + r_{\varepsilon'}]$ such that

$$\text{Osc}_{\Upsilon}(J^{n_0}) < 2^{-n_0}. \quad (4.41)$$

Set the generation "zero" of the Cantor set as $\mathcal{C}_{n_0}(\Upsilon, \varepsilon') = J^{n_0}$.

Simultaneously, we build a measure ν_{n_0} by setting $\nu_{n_0}(J^{n_0}) = 1$, and ν_{n_0} is uniformly distributed on J^{n_0} .

Step 3 (Next generation of the Cantor set).

Let us introduce the following notation: for each $n \in \mathbb{N}^*$ and $J \in \mathcal{J}_n$, set

$$\Upsilon_J^n = \max \left(\mathcal{D}_n \cap [1, \inf_{t \in J} \Upsilon(t)] \right).$$

We explain how to get the second generation of intervals $\mathcal{C}_{n_0+1}(\Upsilon, \varepsilon')$ of the Cantor set.

The oscillation restriction (4.41) for Υ on J^{n_0} implies that for every $J \in \mathcal{J}_{n_0+1}$ contained in J^{n_0} , the quantity $\Upsilon_J^{n_0+1}$ takes necessarily one of the four values $\{\Upsilon_{J^{n_0}}^{n_0} + \alpha 2^{-n_0-1} : \alpha = 0, 1, 2, 3\}$.

Moreover, applying Lemma 4.10 to J^{n_0} , one obtains that for each $\alpha \in \{0, 1, 2, 3\}$, each subfamily $\{\mathcal{F}(J^{n_0}, \Upsilon_{J^{n_0}}^{n_0}, m)\}_{m=1, \dots, \lfloor M_{n_0}(\Upsilon_{J^{n_0}}^{n_0})/2 \rfloor}$ contains at least one interval J belonging to $\mathcal{J}_{n_0+1}^d(\Upsilon_{J^{n_0}}^{n_0} + \alpha 2^{-n_0-1})$.

Recalling that Υ is non-increasing, the quantities $\Upsilon_J^{n_0+1}$ are also non-increasing when J ranges from left to right. Since there are $\lfloor M_{n_0}(\Upsilon_{J^{n_0}}^{n_0})/2 \rfloor$ disjoint families $\{\mathcal{F}(J_0, \Upsilon_{J_0}^{n_0}, m)\}$ which are organized in increasing order, one deduces that there is $\alpha \in \{0, 1, 2, 3\}$ such that there exist $\lfloor M_{n_0}(\Upsilon_{J^{n_0}}^{n_0})/8 \rfloor$ different integers $m \in \{1, \dots, \lfloor M_{n_0}(\Upsilon_{J^{n_0}}^{n_0})/2 \rfloor\}$ for which the family $\mathcal{F}(J_0, \Upsilon_{J_0}^{n_0}, m)$ contains (at least) one interval J satisfying $\Upsilon_J^{n_0+1} = \Upsilon_{J^{n_0}}^{n_0} + \alpha 2^{-n_0-1}$ and $J \in \mathcal{J}_{n_0+1}^d(\Upsilon_J^{n_0+1})$.

Remark that

$$2^{n_0+1} \ll \lfloor M_{n_0}(\Upsilon_{J^{n_0}}^{n_0})/8 \rfloor / 2$$

where we used that $\Upsilon_{\min} < 2 - 2\varepsilon$ and $J^{n_0} \subset [t_{\varepsilon'} - r_{\varepsilon'}, t_{\varepsilon'} + r_{\varepsilon'}]$. Then, applying (4.40) for $n = n_0 + 1$, one can choose the first $\lfloor M_{n_0}(\Upsilon_{J^{n_0}}^{n_0})/16 \rfloor$ intervals J which satisfy $\text{Osc}_{\Upsilon}(J) < 2^{-n_0-1}$ among those already selected in the last paragraph.

Finally, $\mathcal{C}_{n_0+1}(\Upsilon, \varepsilon')$ is the union of these intervals, which are called the basic intervals of generation n_0+1 . Observe that these intervals are separated by a distance larger than $\eta_{n_0}/(2M_{n_0}(\Upsilon_{J^{n_0}}^{n_0}))$ (thanks to Borel-Cantelli applied to Lemma 4.5), and they all have their length equal to η_{n_0+1} .

Simultaneously, one defines a refinement ν_{n_0+1} of the measure ν_0 by setting for every J^{n_0+1} basic interval of $\mathcal{C}_{n_0+1}(\Upsilon, \varepsilon')$

$$\nu_{n_0+1}(J^{n_0+1}) = \nu_{n_0}(J^{n_0}) \frac{1}{\lfloor M_{n_0}(\Upsilon_{J^{n_0}}^{n_0})/16 \rfloor},$$

and by saying that ν_{n_0+1} is uniformly distributed inside each J^{n_0+1} .

Step 4: Induction of the construction of the Cantor set:

Assume that for every $i = n_0, n_0 + 1, \dots, n_0 + n$, the generation $\mathcal{C}_i(\Upsilon, \varepsilon')$ has been constructed and satisfies the following:

1. $\mathcal{C}_i(\Upsilon, \varepsilon')$ is constituted by a finite number of basic disjoint intervals J^i belonging to \mathcal{J}_i ,
2. for every $i = n_0 + 1, \dots, n_0 + n$, each basic interval $J^i \in \mathcal{C}_i(\Upsilon, \varepsilon')$ is included in a unique basic interval $J^{i-1} \in \mathcal{C}_{i-1}(\Upsilon, \varepsilon')$.
3. for every $i = n_0, n_0 + 1, \dots, n_0 + n - 1$, each basic interval $J^i \in \mathcal{C}_i(\Upsilon, \varepsilon')$ contains $\lfloor M_i(\Upsilon_{J^i}^i)/16 \rfloor$ intervals $J^{i+1} \in \mathcal{C}_{i+1}(\Upsilon, \varepsilon')$. These intervals are separated by a distance at least equal to $\eta_i/(2M_i(\Upsilon_{J^i}^i))$. Moreover, each J^{i+1} belongs to $\mathcal{J}_{i+1}^d(\Upsilon_{J^{i+1}}^{i+1})$.
4. Each basic interval J^i of $\mathcal{C}_i(\Upsilon, \varepsilon')$ satisfies $\text{Osc}_\Upsilon(J^i) \leq 2^{-i}$.
5. for every $i = n_0 + 1, \dots, n_0 + n$, ν_i is a measure supported by the basic intervals J^i of $\mathcal{C}_i(\Upsilon, \varepsilon')$, and if J^{i-1} is the unique interval in $\mathcal{C}_{i-1}(\Upsilon, \varepsilon')$ such that $J^i \subset J^{i-1}$, then

$$\nu_i(J^i) = \nu_{i-1}(J^{i-1}) \frac{1}{\lfloor M_{i-1}(\Upsilon_{J^{i-1}}^{i-1})/16 \rfloor} \quad (4.42)$$

and ν_i is uniformly distributed inside each J^i .

We are now able to complete the induction.

For any basic interval $J^n \in \mathcal{C}_n(\Upsilon, \varepsilon')$, applying the same method as in step 3, one finds $\lfloor M_n(\Upsilon_{J^n}^n)/16 \rfloor$ intervals $J^{n+1} \in \mathcal{J}_{n+1}^d(\Upsilon_{J^{n+1}}^{n+1})$, also satisfying $\text{Osc}_\Upsilon(J^{n+1}) < 2^{-(n+1)}$.

Then, $\mathcal{C}_{n+1}(\Upsilon, \varepsilon')$ is the union of these intervals, which constitute the basic intervals of generation $n+1$. By construction, these basic intervals J^{n+1} are separated by at least $\eta_n/(2M_n(\Upsilon_{J^n}^n))$, (where J^n is the "parent" interval of J^{n+1} , i.e. the unique basic interval in $\mathcal{C}_n(\Upsilon, \varepsilon')$ such that $J^{n+1} \subset J^n$), and they all have their length equal to η_{n+1} .

Simultaneously, the refinement ν_{n+1} of the measure ν_n is defined by setting, for every J^n is the "parent" interval of J^{n+1} ,

$$\nu_{n+1}(J^{n+1}) = \nu_n(J^n) \frac{1}{\lfloor M_n(\Upsilon_{J^n}^n)/16 \rfloor},$$

and by saying that ν_{n+1} is uniformly distributed inside each J^{n+1} .

Proposition 4.6. *The Cantor set $\mathcal{C}(\Upsilon, \varepsilon')$ is defined as*

$$\mathcal{C}(\Upsilon, \varepsilon') = \bigcap_{n \geq n_0} \bigcup_{J \in \mathcal{C}_n(\Upsilon, \varepsilon')} J.$$

There exists a unique Borel probability measure $\nu_{\Upsilon, \varepsilon'}$ supported exactly by $\mathcal{C}(\Upsilon, \varepsilon')$ such that for all $n \geq n_0$, the measure $\nu_{\Upsilon, \varepsilon'}$ restricted to the σ -algebra generated by $\{\mathcal{J}_k : n_0 \leq k \leq n\}$ is ν_n .

The proof is immediate, since the step 4. of the construction ensures that the measure is a well-defined additive set function with total mass 1 on the algebra $\{\mathcal{J}_n : n \geq n_0\}$ which generates the Borel σ -algebra, thus extends to a unique probability measure on Borel sets.

Observe that the construction of the family of Cantor sets depends only on Lemma 4.10, which holds with probability one simultaneously for all functions Υ , as desired.

4.6.6 Properties of the Cantor sets

The following proposition is key, since it shows that our construction guarantees that we have built points in $F(\Upsilon)$.

Proposition 4.7. *Almost surely, for every non-increasing càdlàg function $\Upsilon : [0, 1] \rightarrow [1 + 2\varepsilon, 2 - 2\varepsilon]$ and for every small $\varepsilon' > 0$,*

$$\mathcal{C}(\Upsilon, \varepsilon') \setminus (S(\mathcal{M}) \cup S(\Upsilon)) \subset F(\Upsilon).$$

Proof Suppose that $t \in [0, 1] \cap \mathcal{C}(\Upsilon, \varepsilon')$ is a continuous time for \mathcal{M} and Υ . One wants to prove that $\overline{\dim}(\mu, \mathcal{M}_t) = \Upsilon(t)\beta(\mathcal{M}_t)$.

We start by bounding $\overline{\dim}(\mu, \mathcal{M}_t)$ from below.

By construction, for every $n \geq n_0$, t is covered by an interval $J^n \in \mathcal{J}_n^d(\Upsilon_{J^n}^n)$, a basic interval in $\mathcal{C}_n(\Upsilon, \varepsilon')$. Since $J^n \in \mathcal{J}_n^d(\Upsilon_{J^n}^n)$, property (4.31) entails that $t \in J^n$ is surrounded by two jumps of the Poisson point process located at t_n^1 and t_n^2 whose size belong to $\in [\eta_n^{1/\Upsilon_{J^n}^n}/2, \eta_n^{1/\Upsilon_{J^n}^n}]$, and whose mutual distance is at least $3\eta_n$, and at most $5\eta_n$.

The process \mathcal{M} jumps at t_n^1 and t_n^2 , with jump size $\eta_n^{1/(\Upsilon_{J^n}^n \beta(\mathcal{M}_{t_n^1-}))}$, and $\eta_n^{1/(\Upsilon_{J^n}^n \beta(\mathcal{M}_{t_n^2-}))}$. Since the process \mathcal{M} is increasing, if J^n is written $[k\eta_n, (k+1)\eta_n)$, both size of jumps at t_n^1 and t_n^2 are bounded by below by

$$r_n = \eta_n^{\frac{1}{\Upsilon_{J^n}^n \beta(\mathcal{M}_{(k-3)\eta_n})}}.$$

Hence, $\mu(B(\mathcal{M}_t, r_n)) \leq 5\eta_n$. Applying this when n becomes large, one gets

$$\overline{\dim}(\mu, \mathcal{M}_t) \geq \limsup_{n \rightarrow +\infty} \frac{\log \mu(B(\mathcal{M}_t, r_n))}{\log r_n} \geq \limsup_{n \rightarrow +\infty} \frac{\log 5\eta_n}{\log r_n} = \Upsilon(t)\beta(\mathcal{M}_t),$$

where we used the continuity of \mathcal{M} and Υ at t .

The rest of the proof is dedicated to prove the converse inequality, i.e. $\overline{\dim}(\mu, \mathcal{M}_t) \leq \Upsilon(t) \cdot \beta(\mathcal{M}_t)$, which is more delicate.

Let $\varepsilon_1 > 0$ be small. Thanks to the continuity of \mathcal{M} at t , there exists $r_0 > 0$ such that $r_0 \leq \eta_{n_0}$ and

$$N([t - r_0, t + r_0] \times [\eta_{n_0}^{1/\Upsilon_{J^{n_0}}}, 1]) = 0,$$

where J^{n_0} is the unique interval of $C_{n_0}(\Upsilon, \varepsilon')$ that contains t .

Now for any $0 < r < r_0/3$, there exists a unique integer $n \geq n_0$ such that $\eta_{n+1} \leq r < \eta_n$. Let us call $J_n(t)$ and $J_{n+1}(t)$ the unique intervals of \mathcal{J}_n and \mathcal{J}_{n+1} that contain t .

By construction of the random tree $\mathbb{T}_n(J_n(t))$, there is no large jump around t . More precisely, by Lemma 4.9,

$$N\left(B(t, r) \times [r^{1/\Upsilon_{J_n(t)}}, \eta_n^{1/\Upsilon_{J_n(t)}}]\right) = 0.$$

Applying same argument as in Lemma 4.9 to scales between n_0 and n , together with the fact that the sequence $n \mapsto \Upsilon_{J_{n+1}(t)}^{n+1}$ is increasing, yields that

$$N\left(B(t, r) \times [\eta_n^{1/\Upsilon_{J_n(t)}}, \eta_{n_0}^{1/\Upsilon_{J^{n_0}}}] \right) = 0.$$

One deduces that the increment of \mathcal{M} between $t - r$ and $t + r$ has the form

$$\mathcal{M}_{t+r} - \mathcal{M}_{t-r} = \int_{t-r}^{t+r} \int_0^r z^{1/\beta(\mathcal{M}_{s-})} N(ds, dz).$$

Denote by m the unique integer such that $2^{-m-1} \leq 2r < 2^{-m}$. One has

$$\begin{aligned} \mathcal{M}_{t+r} - \mathcal{M}_{t-r} &\leq \left| \int_{t-r}^{t+r} \int_0^{2^{-m/\Upsilon_{J_n(t)}}} z^{1/\beta(\mathcal{M}_{s-})} \tilde{N}(ds, dz) \right| \\ &\quad + \int_{t-r}^{t+r} \int_0^{2^{-m/\Upsilon_{J_n(t)}}} z^{1/\beta(\mathcal{M}_{s-})} \frac{dz}{z^2} ds := A_1(r) + A_2(r). \end{aligned}$$

Applying Lemma 4.2 entails

$$\begin{aligned} \mathbb{P} \left(\sup_{\substack{\gamma \in \mathcal{D}_m \\ \cap [1+2\varepsilon, 2-2\varepsilon]}} \sup_{\substack{0 \leq v < u \leq 1 \\ |v-u| \leq 2^{-m}}} 2^{\frac{m}{\gamma(\beta(\mathcal{M}_{u+2^{-m}}) + \frac{2}{m})}} \left| \int_v^u \int_0^{2^{-\frac{m}{\gamma}}} z^{1/\beta(\mathcal{M}_{s-})} \tilde{N}(ds, dz) \right| \geq 6m^2 \right) \\ \leq C \cdot 2^m \cdot e^{-m}. \end{aligned}$$

Borel-Cantelli Lemma yields that when m becomes larger than some m_0 , for every $\gamma \in \mathcal{D}_m \cap [1 + 2\varepsilon, 2 - 2\varepsilon]$ and $|v - u| \leq 2^{-m}$ (with $u > v$),

$$\left| \int_v^u \int_0^{2^{-\frac{m}{\gamma}}} z^{1/\beta(\mathcal{M}_{s-})} \tilde{N}(ds, dz) \right| \leq 6m^2 2^{\frac{-m}{\gamma(\beta(\mathcal{M}_{u+2^{-m}}) + \frac{2}{m})}}.$$

Assume that r_0 is so small that $2r_0 \leq 2^{-m_0}$. By our choices for n and m , one has $m > n$, so $\Upsilon_{J_{n+1}(t)}^{n+1} \in \mathcal{D}_m$. By choosing $\gamma = \Upsilon_{J_n(t)}^n$, $u = t + r$ and $v = t - r$, one gets

$$\begin{aligned} A_1(r) &\leq 6m^2 2^{\frac{\Upsilon_{J_n(t)}^n - m}{(\beta(\mathcal{M}_{t+r+2^{-m}}) + 2/m)}} \\ &\leq 12(\log_2(1/4r))^2 (2r)^{1/(\Upsilon_{J_n(t)}^n (\beta(\mathcal{M}_{t+r+2^{-m}}) + 2/m))}. \end{aligned}$$

In addition, by continuity of \mathcal{M} at t , when r_0 is small enough, one has

$$|\Upsilon_{J_n(t)}^n - \Upsilon(t)| < \varepsilon_1 \quad \text{and} \quad \beta(\mathcal{M}_{t+r+2^{-m}}) + 2/m \leq \beta(\mathcal{M}_t) + \varepsilon_1,$$

so finally

$$A_1(r) \leq r^{\frac{1}{(\Upsilon(t) + \varepsilon_1)(\beta(\mathcal{M}_t) + \varepsilon_1)}}.$$

On the other hand, recalling the constant $\varepsilon_0 > 0$ in Definition 4.1, an immediate computation shows that

$$\begin{aligned} A_2(r) &\leq \int_{t-r}^{t+r} \int_0^{2^{-m/(\Upsilon(t) + \varepsilon_1)}} z^{1/(\beta(\mathcal{M}_t) + \varepsilon_1)} \frac{dz}{z^2} ds \\ &\leq \frac{2r}{1/\varepsilon_0 - 1} 2^{\frac{-m}{(\Upsilon(t) + \varepsilon_1)} (\frac{1}{(\beta(\mathcal{M}_t) + \varepsilon_1)} - 1)} \\ &\leq \frac{1}{2/\varepsilon_0 - 2} (4r)^{\frac{1}{(\Upsilon(t) + \varepsilon_1)(\beta(\mathcal{M}_t) + \varepsilon_1)} - \frac{1}{\Upsilon(t) + \varepsilon_1} + 1} \\ &\leq r^{\frac{1}{(\Upsilon(t) + 2\varepsilon_1)(\beta(\mathcal{M}_t) + \varepsilon_1)}}, \end{aligned}$$

as soon as r_0 is small enough.

Combining these estimates, one obtains that for all $r \leq r_0$,

$$\mathcal{M}_{t+r} - \mathcal{M}_{t-r} \leq r^{\frac{1}{(\Upsilon(t) + 3\varepsilon_1)(\beta(\mathcal{M}_t) + \varepsilon_1)}},$$

which entails for all $0 < \mathfrak{r} < \mathcal{M}_{t+r_0} - \mathcal{M}_{t-r_0}$,

$$\mu(B(\mathcal{M}_t, \mathfrak{r})) \geq \mathfrak{r}^{(\Upsilon(t) + 3\varepsilon_1)(\beta(\mathcal{M}_t) + \varepsilon_1)}.$$

One concludes that

$$\overline{\dim}(\mu, \mathcal{M}_t) \leq (\Upsilon(t) + 3\varepsilon_1)(\beta(\mathcal{M}_t) + \varepsilon_1).$$

Letting $\varepsilon_1 \rightarrow 0$ yields the desired upper bound for $\overline{\dim}(\mu, \mathcal{M}_t)$. □

4.6.7 Dimension of $\mathcal{C}(\Upsilon, \varepsilon')$

Here we prove that $\dim_{\mathcal{H}} \mathcal{C}(\Upsilon, \varepsilon') \geq 2/(\Upsilon_{\min} + \varepsilon') - 1$.

Lemma 4.11. *With probability 1, for every Υ in Theorem 4.7 with $\Upsilon_{\min} \in [1 + 2\varepsilon, 2 - 2\varepsilon]$ and $\varepsilon' > 0$, there exists a finite positive constant $K_{\Upsilon, \varepsilon'}$ such that for all $B \in \mathcal{B}([0, 1])$,*

$$\nu_{\Upsilon, \varepsilon'}(B) \leq K_{\Upsilon, \varepsilon'} |B|^{\frac{2}{\Upsilon_{\min} + 2\varepsilon'} - 1} \quad (4.43)$$

Proof Let Υ and $\varepsilon' > 0$ be fixed.

Let B be an open interval in $[0, 1]$ such that $|B| \leq \eta_{n_0}$.

If $B \cap \mathcal{C}(\Upsilon, \varepsilon') = \emptyset$, (4.43) holds trivially.

If $B \cap \mathcal{C}(\Upsilon, \varepsilon') \neq \emptyset$, let n_1 be the largest integer such that B intersects $C_{n_1}(\Upsilon, \varepsilon')$ in exactly one basic interval, denoted by J^{n_1} .

Denote by $\delta_{n_1+1}(\Upsilon, \varepsilon', J^{n_1})$ the minimal distance between any two intervals of $\mathcal{C}_{n_1+1}(\Upsilon, \varepsilon')$ which are contained in J^{n_1} . Then $|B|$ contains at most

$$\min(M_{n_1}(\Upsilon_{J^{n_1}}), |B|/\delta_{n_1+1}(\Upsilon, \varepsilon', J^{n_1}))$$

intervals of generation $n_1 + 1$.

In addition, by construction, one has

$$\delta_{n_1+1}(\Upsilon, \varepsilon', J^{n_1}) \geq \frac{\eta_{n_1}}{2M_{n_1}(\Upsilon_{J^{n_1}})}. \quad (4.44)$$

Hence by (4.42), since all the intervals J^{n_1+1} of generation $n_1 + 1$ within J^{n_1} have the same ν -mass, one has (using (4.44))

$$\begin{aligned} & \nu_{\Upsilon, \varepsilon'}(B) \\ & \leq \min(M_{n_1}(\Upsilon_{J^{n_1}}), |B|/\delta_{n_1+1}(\Upsilon, \varepsilon', J^{n_1})) \cdot \nu_{n_1+1}(J^{n_1+1}) \\ & \leq \min\left(M_{n_1}(\Upsilon_{J^{n_1}}), |B| \frac{2M_{n_1}(\Upsilon_{J^{n_1}})}{\eta_{n_1,0}}\right) \cdot \left(\prod_{k=n_0}^{n_1-1} \frac{1}{M_k(\Upsilon_{\min} + \varepsilon')}\right) \cdot \frac{1}{M_{n_1}(\Upsilon_{J^{n_1}})} \\ & \leq 2 \left(\prod_{k=n_0}^{n_1-1} \frac{1}{M_k(\Upsilon_{\min} + \varepsilon')}\right) \cdot \eta_{n_1}^{-1} \cdot \min(\eta_{n_1}, |B|). \end{aligned}$$

Due to our choices for the sequence $(\eta_n)_{n \geq 1}$, when n_0 is large,

$$\prod_{k=n_0}^{n_1-1} \frac{1}{M_k(\Upsilon_{\min} + \varepsilon')} \leq (M_{n_1-1}(\Upsilon_{\min} + \varepsilon'))^{-1} \leq \eta_{n_1}^{\frac{2}{\Upsilon_{\min} + 2\varepsilon'} - 1},$$

so applying the inequality $x \wedge y \leq x^s y^{1-s}$ for $s \in (0, 1)$ and $0 < x, y < 1$ yields

$$\nu_{\Upsilon, \varepsilon'}(B) \leq 2\eta_{n_1}^{\frac{2}{\Upsilon_{\min} + 2\varepsilon'} - 1} \cdot \eta_{n_1}^{-1} \cdot \eta_{n_1}^{2 - \frac{2}{\Upsilon_{\min} + 2\varepsilon'}} \cdot |B|^{\frac{2}{\Upsilon_{\min} + 2\varepsilon'} - 1} = 2|B|^{\frac{2}{\Upsilon_{\min} + 2\varepsilon'} - 1}.$$

□

Finally the mass distribution principle applied to the measure $\nu_{\Upsilon, \varepsilon'}$, which is supported by the Cantor set $\mathcal{C}(\Upsilon, \varepsilon')$, allows one to conclude that

$$\dim_{\mathcal{H}} \mathcal{C}(\Upsilon, \varepsilon') \geq \frac{2}{\Upsilon_{\min} + 2\varepsilon'} - 1.$$

4.6.8 Extension to $\Upsilon_{\min} \in \{1, 2\}$

Letting $\varepsilon \rightarrow 0$ along a countable sequence yields that almost surely, for all Υ with $\Upsilon_{\min} \in (1, 2)$,

$$\dim_{\mathcal{H}} F(\Upsilon) \geq \frac{2}{\Upsilon_{\min}} - 1. \quad (4.45)$$

It remains to treat the extreme cases.

First case : $\Upsilon_{\min} = 1$. For each $\varepsilon_2 > 0$, there exists an open interval $\mathcal{O} \in (0, 1)$ such that every $t \in \mathcal{O}$ satisfies $\Upsilon(t) \geq 1 + \varepsilon_2 > 1$. Applying (4.45) yields that $\dim_{\mathcal{H}} F(\Upsilon) \geq \frac{2}{1 + \varepsilon_2} - 1$. Letting $\varepsilon_2 \rightarrow 0$ establishes that $\dim_{\mathcal{H}} F(\Upsilon) = 1$.

Second case : $\Upsilon_{\min} = 2$, i.e. $\Upsilon \equiv 2$. In order to prove $\dim_{\mathcal{H}} F(\Upsilon) \geq 0$, it suffices to show that there exists almost surely $t \in (0, 1)$ such that $\overline{\dim}(\mu, \mathcal{M}_t) = 2\beta(\mathcal{M}_t)$, i.e. $F(\Upsilon) \neq \emptyset$. To this end, some changes are needed for the construction of the Cantor set. We only sketch the proof since it is essentially the same as the one in the precedent sections (with simplification). Set

$$\begin{cases} \rho_0 = 1/2 \text{ and } \rho_n = \exp(-\rho_{n-1}^{-1}) \text{ for all } n \geq 1, \\ \eta_n = \rho_n \log(1/\rho_n)^{-1} \text{ for all } n \geq 0. \end{cases}$$

Let $\mathcal{J}_n(2)$ be the set composed of intervals $J_{n,k} = [k\eta_n, (k+1)\eta_n)$ that satisfy

$$\begin{cases} N \left([J_{n,k-2} \times \left[\rho_n^{1/2} \left(\log \frac{1}{\rho_n} \right)^{-3}, 1 \right] \right) = 1, \\ N \left([J_{n,k+2} \times \left[\rho_n^{1/2} \left(\log \frac{1}{\rho_n} \right)^{-3}, 1 \right] \right) = 1, \\ N \left([\hat{J}_{n,k} \times \left[\rho_n^{1/2} \left(\log \frac{1}{\rho_n} \right)^{-3}, 1 \right] \right) = 0, \end{cases} \quad (4.46)$$

It is easy to check that any point t covered by the collection $\mathcal{J}_n(2)$ infinitely often satisfy $\overline{\dim}(\mu, \mathcal{M}_t) \geq 2\beta(\mathcal{M}_t)$ (necessarily, one has equality thanks to Theorem 4.1). We construct as before, by induction, the collection $\mathcal{C}_n(\Upsilon \equiv 2)$ of basic intervals and the Cantor set $\mathcal{C}(\Upsilon \equiv 2) = \bigcap_n \mathcal{C}_n(\Upsilon \equiv 2)$

contained in $F(\Upsilon)$. The same arguments as in Lemma 4.4 give a constant C_n uniformly bounded below and above by 0 and $+\infty$ such that for any fixed $J_{n,k}$,

$$\mathbb{P}(J_{n,k} \in \mathcal{J}_n(2)) = C_n \cdot \rho_n \left(\log \frac{1}{\rho_n} \right)^4.$$

Thus one bounds from above the probability that there exists $J_{n,k}$ such that none of the intervals $J_{n+1,k'}$ contained in $J_{n,k}$ belongs to $\mathcal{J}_{n+1}(2)$ by

$$\frac{1}{\eta_n} \left(1 - C_{n+1} \cdot \rho_{n+1} \left(\log \frac{1}{\rho_{n+1}} \right)^4 \right)^{\frac{\eta_n}{\eta_{n+1}}}.$$

Observe that

$$\frac{\eta_n}{\eta_{n+1}} = C_{n+1} \rho_{n-1}^{-3} \cdot (C_{n+1})^{-1} \rho_{n+1}^{-1} \left(\log \frac{1}{\rho_{n+1}} \right)^{-3} \text{ with } C_{n+1} \rho_{n-1}^{-3} \gg 1.$$

So the probability in question is less than

$$\eta_n^{-1} e^{-C_{n+1} \rho_{n-1}^{-3}} \leq \eta_n^{-1} e^{-3\rho_{n-1}^{-1}} = \eta_n^{-1} \rho_n^3 \leq \rho_n.$$

Borel-Cantelli Lemma implies the existence of a sequence of embedded interval with length tending to 0 that satisfy (4.46). This justifies that $F(\Upsilon) \neq \emptyset$.

4.7 Space spectrum : proof of Theorem 4.2

4.7.1 A first theorem on dimensions, and the space spectrum

Throughout this section, we set $\varepsilon = \varepsilon_0$, which is defined in (4.1). We are going to prove the following theorem.

Theorem 4.8. *Let $\varepsilon > 0$. Denote by $\mathcal{P} = \{(T_n, Z_n)\}_{n \geq 1}$ a Poisson point process that generates the Poisson measure $N(dt, dz)$ with intensity $dt \otimes dz/z^2$. Consider the family (4.15) of stable processes $(\mathcal{L}^\alpha)_{\alpha \in (\varepsilon, 1-\varepsilon)}$. Also, for every non decreasing càdlàg function $f : [0, 1] \rightarrow [\varepsilon, 1 - \varepsilon]$, consider the process*

$$\mathcal{L}_t^f = \int_0^t \int_0^1 z^{\frac{1}{f(t-s)}} N(ds, dz). \quad (4.47)$$

Almost surely, for every set $E \subset [0, 1]$, for every function $f : [0, 1] \rightarrow [\varepsilon, 1 - \varepsilon]$, if $\alpha < \inf\{f(t) : t \in E\}$ and $\beta > \sup\{f(t) : t \in E\}$, then

$$\dim_{\mathcal{H}}(\mathcal{L}^\alpha(E)) \leq \dim_{\mathcal{H}} \mathcal{L}^f(E) \leq \dim_{\mathcal{H}}(\mathcal{L}^\beta(E)) = \frac{\beta}{\alpha} \dim_{\mathcal{H}}(\mathcal{L}^\alpha(E)).$$

Before proving Theorem 4.8 next subsection, let us explain how we deduce the space spectrum

of the occupation measure.

As mentioned in the introduction, almost surely, for every set $E \subset \mathbb{R}$, the image of E by an α -stable process \mathcal{L}^α has Hausdorff dimension $\alpha \dim_{\mathcal{H}}(E)$. Applying Theorem 4.8 to the function $f(t) = \beta(\mathcal{M}_t)$, which is almost surely càdlàg, one is now ready to prove Theorem 4.6.

Proof of Theorem 4.6 : The first part (the formula (4.12)) is immediate.

For the second part, let $E \subset [0, 1]$ be such that for every non-trivial subinterval $I \subset [0, 1]$, $\dim_{\mathcal{H}}(E) = \dim_{\mathcal{H}}(E \cap I)$.

For every $\eta > 0$, there exists an interval I of length less than η such that

$$\left[\inf_{t \in E \cap I} \beta(\mathcal{M}(t)), \sup_{t \in E \cap I} \beta(\mathcal{M}(t-)) \right] \subset \left[\sup_{t \in E} \beta(\mathcal{M}(t-)) - \eta, \sup_{t \in E} \beta(\mathcal{M}(t-)) \right].$$

This follows from the càdlàg regularity of $t \mapsto \beta(\mathcal{M}(t))$. Hence, applying (4.12) to $E \cap I$ gives

$$\dim_{\mathcal{H}} \mathcal{M}(E \cap I) \in \dim_{\mathcal{H}}(E \cap I) \cdot \left[\sup_{t \in E} \beta(\mathcal{M}(t-)) - \eta, \sup_{t \in E} \beta(\mathcal{M}(t-)) \right].$$

Since $\dim_{\mathcal{H}} \mathcal{M}(E \cap I) \leq \dim_{\mathcal{H}} \mathcal{M}(E)$ and $\dim_{\mathcal{H}}(E) = \dim_{\mathcal{H}}(E \cap I)$, the result follows by letting η tend to zero. \square

One deduces a corollary from Theorem 4.6, which will be used in the proof of Theorem 4.2.

Corollary 4.1. *For every open interval $I \subset [0, 1]$ and $h \geq 0$, consider the smallest interval $I_0 \subset I$ (it may be not open or reduced to a point) such that $\overline{E}_\mu^t(I, h) = \overline{E}_\mu^t(I_0, h)$. Denote by $d(I_0)$ the right endpoint of I_0 . Almost surely, for every open interval $I \subset [0, 1]$ and $h \geq 0$, one has*

$$\dim_{\mathcal{H}} \mathcal{M} \left(\overline{E}_\mu^t(I, h) \right) = \dim_{\mathcal{H}} \overline{E}_\mu^t(I_0, h) \cdot \sup_{t \in I_0 \setminus d(I_0)} \beta(\mathcal{M}_t). \quad (4.48)$$

with the convention that $0 \times (-\infty) = 0$ and $(-\infty) \times (-\infty) = -\infty$.

Proof If $\overline{E}_\mu^t(I, h)$ is empty or a singleton, there is nothing to prove. One thus assumes that $\overline{E}_\mu^t(I, h)$ is neither empty nor a singleton, so I_0 is a non-trivial interval. One could check the analysis in the proof of Theorem 4.4 for a construction of I_0 . Observe that the left-hand side of (4.48) is smaller than the right-hand side due to Theorem 4.6. The converse inequality follows by minimality of I_0 and a localization procedure as in the proof of Theorem 4.6. \square

Proof of Theorem 4.2 and 4.3 : To deduce the space spectrum, one needs some additional analysis other than the time spectrum. This is due to the following basic observation : for all $t \in S(\mathcal{M})$, \mathcal{M}_{t-} is not in the range of \mathcal{M} , but in the support of μ .

When \mathcal{O} does not intersect the range of \mathcal{M} , the level set $\overline{E}_\mu(\mathcal{O}, h) = \emptyset$, as is given in Theorem 4.2 and 4.3.

When \mathcal{O} intersects the range of \mathcal{M} , by the càdlàg property of \mathcal{M} , there is a non-trivial interval $\tilde{\mathcal{O}}$ such that

$$\mathcal{M}((0, 1)) \cap \mathcal{O} = \mathcal{M}(\tilde{\mathcal{O}}).$$

The set $\tilde{\mathcal{O}}$ is an open set (a, b) if \mathcal{M} enters \mathcal{O} continuously, or is a semi-open interval $[a, b)$ if \mathcal{M} enters \mathcal{O} with a jump. In any case, $\mathcal{M}_{a-} \notin \mathcal{O}$ and $\mathcal{M}_b \notin \mathcal{O}$ because \mathcal{O} is open. Observe that

$$\begin{aligned} \bar{E}_\mu(\mathcal{O}, h) &= \{x \in \text{Supp } \mu \cap \mathcal{O} : \bar{\dim}(\mu, x) = h\} \\ &= \{\mathcal{M}_t \in \mathcal{O} : \bar{\dim}(\mu, \mathcal{M}_t) = h\} \cup \{\mathcal{M}_{t-} \in \mathcal{O} : t \in S(\mathcal{M}) \text{ and } \bar{\dim}(\mu, \mathcal{M}_{t-}) = h\} \\ &= \mathcal{M}(\bar{E}_\mu^t(\tilde{\mathcal{O}}, h)) \cup \{\mathcal{M}_t : t \in S(\mathcal{M}) \cap \tilde{\mathcal{O}} \text{ and } \bar{\dim}(\mu, \mathcal{M}_{t-}) = h\} \end{aligned}$$

Since the Hausdorff dimension of the second set in the last union is at most 0, one distinguishes two types of situations according to the value of h .

• Type A. The time level set $\bar{E}_\mu^t(\tilde{\mathcal{O}}, h) \neq \emptyset$, so one ignores the second set in the last union when computing the Hausdorff dimension of $\bar{E}_\mu(\mathcal{O}, h)$. If $\#\bar{E}_\mu^t(\tilde{\mathcal{O}}, h) = 1$ (necessarily $h = \beta(\mathcal{M}_t)$ for some $t \in S(\mathcal{M})$ with $\beta(\mathcal{M}_t) \geq 2\beta(\mathcal{M}_{t-})$), one has $\bar{d}_\mu(\mathcal{O}, h) = 0$ which coincide with the formula in Theorem 4.3. Otherwise Corollary 4.1 applied to h and $\tilde{\mathcal{O}}$ entails the existence of a minimal $\tilde{\mathcal{O}}_0$ (that we can and will suppose open) such that

$$\begin{aligned} \bar{d}_\mu(\mathcal{O}, h) &= \bar{d}_\mu^t(\tilde{\mathcal{O}}_0, h) \cdot \sup_{t \in \tilde{\mathcal{O}}_0 \setminus d(\tilde{\mathcal{O}}_0)} \beta(\mathcal{M}_t) \\ &= \sup \left\{ \hat{g}_\alpha(h)/\alpha : \alpha \in \{\beta(\mathcal{M}_t) : t \in \tilde{\mathcal{O}}_0\} \right\} \cdot \sup_{t \in \tilde{\mathcal{O}}_0} \beta(\mathcal{M}_t) \\ &= \sup \left\{ \hat{g}_\alpha(h) : \alpha \in \{\beta(\mathcal{M}_t) : t \in \tilde{\mathcal{O}}_0\} \right\} \\ &= \sup \left\{ \hat{g}_\alpha(h) : \alpha \in \{\beta(\mathcal{M}_t) : t \in \tilde{\mathcal{O}}\} \right\} \\ &= \sup \left\{ \hat{g}_\alpha(h) : \alpha \in \{\beta(\mathcal{M}_t) : \mathcal{M}_t \in \mathcal{O}\} \right\}, \end{aligned}$$

as desired.

• Type B. The time level set

$$\bar{E}_\mu^t(\tilde{\mathcal{O}}, h) = \emptyset \tag{4.49}$$

so one has to consider the set $\mathcal{H} = \{\mathcal{M}_t : t \in S(\mathcal{M}) \cap \tilde{\mathcal{O}} \text{ and } \bar{\dim}(\mu, \mathcal{M}_{t-}) = h\}$ which is (at most) countable. Compared with the time spectrum, several cases may occur according to the value of h . Recall that $\tilde{\mathcal{O}} = [a, b)$ (when \mathcal{M} jumps into \mathcal{O}) or (a, b) (when \mathcal{M} enters \mathcal{O} continuously). In the following analysis, the case $\mathcal{M}_{t_0} \notin \mathcal{O}$ is trivial. We thus assume that every \mathcal{M}_{t_0} below belongs to \mathcal{O} .

1. (4.49) is due to $2\beta(\mathcal{M}_{t_0-}) < h < \beta(\mathcal{M}_{t_0})$ with $t_0 \in S(\mathcal{M})$. For all $t > t_0$, $\bar{\dim}(\mu, \mathcal{M}_{t-}) \geq \beta(\mathcal{M}_{t-}) > \beta(\mathcal{M}_{t_0}) > h$. For all $t \leq t_0$, $\bar{\dim}(\mu, \mathcal{M}_{t-}) \leq 2\beta(\mathcal{M}_{t-}) \leq 2\beta(\mathcal{M}_{t_0-}) < h$. So

$\mathcal{H} = \emptyset$ and $\bar{d}_\mu(\mathcal{O}, h) = -\infty$ as desired.

2. (4.49) is due to the fact that $2\beta(\mathcal{M}_{t_0-}) < \beta(\mathcal{M}_{t_0}) = h$ with $t_0 \in S(\mathcal{M})$ and $\bar{\dim}(\mu, \mathcal{M}_{t_0}) \neq \beta(\mathcal{M}_{t_0})$. As in the last item, $\mathcal{H} = \emptyset$ as desired.
3. (4.49) is due to $h = 2\beta(\mathcal{M}_{t_0-}) < \beta(\mathcal{M}_{t_0})$ or $h = 2\beta(\mathcal{M}_{t_0-}) = \beta(\mathcal{M}_{t_0})$. As before for all $t \neq t_0$, one has $\bar{\dim}(\mu, \mathcal{M}_{t-}) \neq h$. If $\bar{\dim}(\mu, \mathcal{M}_{t_0-}) = 2\beta(\mathcal{M}_{t_0-})$, $\mathcal{H} = \{t_0\}$, otherwise $\mathcal{H} = \emptyset$. This coincides with Theorem 4.3.
4. (4.49) is due to $h \geq 2\beta(\mathcal{M}_{b-})$. For all $t < b$, $\bar{\dim}(\mu, \mathcal{M}_{t-}) \leq 2\beta(\mathcal{M}_{t_0-}) < 2\beta(\mathcal{M}_{b-}) \leq h$, hence $\mathcal{H} = \emptyset$, as desired.
5. (4.49) is due to $h \leq \beta(\mathcal{M}_a)$. Recall first that $\beta(\mathcal{M}_a) \notin \mathcal{O}$. Further, for all $t > a$, $\bar{\dim}(\mu, \mathcal{M}_t) > h$. Hence, $\mathcal{H} = \emptyset$, as desired.

□

4.7.2 Proof of Theorem 4.8

We start with a Lemma describing the distribution properties of the Poisson point process $\mathcal{P} = \{(T_n, Z_n)\}_{n \geq 1}$.

Lemma 4.12. *For every $j \geq 1$, let $\mathcal{P}_j = \{n : Z_n \in [2^{-j-1}, 2^{-j}]\}$. Almost surely, there exist two positive decreasing sequences $(\varepsilon_j)_{j \geq 1}$ and $(\eta_j)_{j \geq 1}$ converging to zero such that for every integer J large enough, one has:*

1. $2^{J(1-\varepsilon_J)} \leq \#\mathcal{P}_J \leq 2^{J(1+\varepsilon_J)}$,
2. for every interval $I \subset [0, 1]$ with length 2^{-J} ,

$$1 \leq \# \bigcup_{j \leq J(1+\eta_j)} \{n \in \mathcal{P}_j : T_n \in I\} \leq 2^{J\varepsilon_J},$$

3. for every interval $I \subset [0, 1]$ with length 2^{-J} ,

$$0 \leq \# \bigcup_{j \leq J/3} \{n \in \mathcal{P}_j : T_n \in I\} \leq 1,$$

4. for every interval $I \subset [0, 1]$ with length 2^{-J} , for every $j \geq J(1 + \eta_j)$,

$$\#\{n \in \mathcal{P}_j : T_n \in I\} \leq 2^{j(1+\varepsilon_j)} 2^{-J}.$$

Routine computations as in Lemma 4.4 entail Lemma 4.12.

Let $E \subset [0, 1]$, $\alpha = \inf\{f(t) : t \in E\}$ and $\beta = \sup\{f(t) : t \in E\}$.

Call E^α (resp. E^f, E^β) the image of E by \mathcal{L}^α (resp. $\mathcal{L}^f, \mathcal{L}^\beta$).

Lemma 4.13. *Almost surely, the following holds. With each interval \tilde{B}^α such that $\tilde{B}^\alpha \cap E^\alpha \neq \emptyset$, one can associate an interval of the form $B^\alpha = \mathcal{L}^\alpha([T_m, T_n))$ such that $|B^\alpha| \leq 2|\tilde{B}^\alpha|$ and possibly a singleton of the form $\{\mathcal{L}^\alpha(T_n)\}$, such that*

$$E^\alpha \cap (B^\alpha \cup \{\mathcal{L}^\alpha(T_n)\}) = E^\alpha \cap \tilde{B}^\alpha.$$

The same holds true for every interval \tilde{B}^f such that $E^f \cap \tilde{B}^f \neq \emptyset$, which can be replaced by $B^f = \mathcal{L}^f([T_m, T_n))$ and possibly a singleton.

Proof

Almost surely all the processes $\mathcal{L}^\alpha, \mathcal{L}^\beta$ and \mathcal{L}^f are strictly increasing and càdlàg.

Let $\tilde{B}^\alpha = [x^\alpha, y^\alpha]$ be an interval satisfying $\tilde{B}^\alpha \cap E^\alpha \neq \emptyset$.

If x^α is not of the form $\mathcal{L}^\alpha(T_m)$, then two cases occur:

- when $x^\alpha \notin \mathcal{L}^\alpha(E)$: \tilde{B}^α can be replaced by $[x'^\alpha, y^\alpha]$, where $x'^\alpha = \inf(\tilde{B}^\alpha \cap E^\alpha)$, without altering the covering \mathcal{R}^α . Since \mathcal{L}^α is increasing and càdlàg, x'^α is necessarily the image of some jump point T_m by \mathcal{L}^α .
- when $x^\alpha \in \mathcal{L}^\alpha(E)$: x^α can be written as $\mathcal{L}^\alpha(t)$, for some t which is a continuous time for \mathcal{L}^α . Using the density of the jump points, there exists (T_m, Z_m) such that $T_m < t$ and $\mathcal{L}^\alpha(t) - \mathcal{L}^\alpha(T_m) < |\tilde{B}^\alpha|/2$. We then choose $x'^\alpha = \mathcal{L}^\alpha(T_m)$.

In all cases, \tilde{B}^α is replaced by $B'^\alpha = [x'^\alpha, y^\alpha]$, where $|B'^\alpha| \leq 3/2|B^\alpha|$.

Similarly, if y^α is not of the form $\mathcal{L}^\alpha(T_n-)$ (i.e. the left limit of \mathcal{L}^α at T_n for some jump point T_n), then :

- when $y^\alpha \notin \mathcal{L}^\alpha(E)$: B'^α can be replaced by $B^\alpha = [x'^\alpha, y'^\alpha]$, where $y'^\alpha = \sup(B'^\alpha \cap E^\alpha)$, without altering the covering \mathcal{R}^α . Since \mathcal{L}^α is increasing and càdlàg, y'^α is of the form $\mathcal{L}^\alpha(T_n-)$ for some jump point T_n .
- when $y^\alpha = \mathcal{L}^\alpha(T_n)$ for some jump time T_n : Then B'^α can be replaced by $\{\mathcal{L}^\alpha(T_n)\} \cup B^\alpha$, where $B^\alpha = [x'^\alpha, \mathcal{L}^\alpha(T_n-)]$. Indeed, there is no point of E^α between $\mathcal{L}^\alpha(T_n-)$ and $\mathcal{L}^\alpha(T_n)$.
- when $y = \mathcal{L}^\alpha(t)$ for some t which is a continuous time for \mathcal{L}^α . Using the same argument as above, there exists (T_n, Z_n) such that $T_n > t$ and $\mathcal{L}^\alpha(T_n) - \mathcal{L}^\alpha(t) < |B^\alpha|/2$. We then choose $y'^\alpha = \mathcal{L}^\alpha(T_n-)$.

This proves the claim. \square

Observe that the previous Lemma holds almost surely, for every interval B^α , for all α , since the randomness is only located in the distribution of the Point Poisson process and the strictly increasing and càdlàg properties of the processes, which hold simultaneously almost surely.

Next Lemma establishes that the increment of the process in an interval I is approximately the same order as the size of the largest jump in I , uniformly for all I and all the parameters.

Lemma 4.14. *With probability one, there exists a non-decreasing function $g : [0, 1] \rightarrow \mathbb{R}^+$ with $g(0) = 0$, continuous at 0, such that the following holds. Let (T_m, Z_m) and (T_n, Z_n) (with $T_m < T_n$) be two couples of the point Poisson process. Let $B^\alpha = [\mathcal{L}^\alpha(T_m), \mathcal{L}^\alpha(T_n-)]$, $B^\beta = [\mathcal{L}^\beta(T_m), \mathcal{L}^\beta(T_n-)]$ and $B^f = [\mathcal{L}^f(T_m), \mathcal{L}^f(T_n-)]$. Then when $B^\alpha = [\mathcal{L}^\alpha(T_m), \mathcal{L}^\alpha(T_n-)]$ is small enough, one has*

$$|B^\alpha|^{\alpha/\beta+g(|B^\alpha|)} \leq |B^\beta| \leq |B^\alpha|^{\alpha/\beta-g(|B^\alpha|)} \quad (4.50)$$

and

$$|B^\alpha| \leq |B^f| \leq |B^\beta|. \quad (4.51)$$

Proof

The three processes \mathcal{L}^α , \mathcal{L}^f and \mathcal{L}^β are almost surely pure jump processes with finite variations. One deduces that

$$|B^\alpha| = \sum_{p \in \mathbb{N}: T_p \in [T_m, T_n)} Z_p^{1/\alpha}, \quad (4.52)$$

and the same holds true for $|B^\beta|$ by replacing $1/\alpha$ by $1/\beta$. Similarly,

$$|B^f| = \sum_{p \in \mathbb{N}: T_p \in [T_m, T_n)} Z_p^{1/f(T_p-)}.$$

Then (4.51) follows immediately since f is monotone and $\alpha \leq f(t) \leq \beta$.

We write $B = [T_m, T_n)$, and consider J the unique integer such that $2^{-(J+1)} < |T_n - T_m| \leq 2^{-J}$. We assume that J is so large that $\varepsilon_J \leq (1/(1 - \varepsilon) - 1)/4 \leq (1/\alpha - 1)/4$.

We now make use of Lemma 4.12.

Let (T_N, Z_N) be the point Poisson process in the above sum (4.52) with largest jump Z_N . We write $Z_N = 2^{-J_N}$. Then one decomposes $|B^\alpha|$ into

$$|B^\alpha| = Z_N^{1/\alpha} + \sum_{j \leq J(1+\eta_J): T_p \in B \text{ and } p \in \mathcal{P}_j} Z_p^{1/\alpha} + \sum_{j > J(1+\eta_J): T_p \in B \text{ and } p \in \mathcal{P}_j} Z_p^{1/\alpha}. \quad (4.53)$$

Call S_1 and S_2 the two above sums.

Assume that $J_N < J/3$.

Observe that since B strictly contains an interval of length 2^{-J-1} , the left inequality part (2) of Lemma 4.12 yields that $J_N \leq (J+1)(1+\eta_{J+1})$.

Since B is contained in an interval of length 2^{-J} , one knows that all the jumps other than (T_N, Z_N) appearing in formula (4.53) are smaller than $2^{-J/(3\alpha)}$. Hence, the right inequality in part (2) of Lemma 4.12 yields

$$S_1 \leq 2^{J\varepsilon_J} 2^{-J/(3\alpha)}.$$

Similarly, applying part (4) of Lemma 4.12, S_2 is bounded by

$$\begin{aligned} S_2 &\leq \sum_{j \geq J(1+\eta_J)} 2^{j(1+\varepsilon_j)} 2^{-J} 2^{-j/\alpha} \leq \frac{2^{J(1+\eta_J)(1+\varepsilon_J-1/\alpha)-1}}{2^{3/4(1/\alpha-1)} - 1} \\ &\leq C_\varepsilon 2^{-J/\alpha} 2^{J(\varepsilon_J + \eta_J(1+\varepsilon_J-1/\alpha))}, \end{aligned} \quad (4.54)$$

where $C_\varepsilon := \frac{1}{2^{3/4(1/(1-\varepsilon))-1}-1}$. Recalling that $J_N \leq J/3$, one gets

$$\begin{aligned} |B^\alpha| &\leq (Z_N)^{1/\alpha} + 2^{-J/(3\alpha)-\varepsilon_J} + C_\varepsilon 2^{-J(1/\alpha-\varepsilon_J-\eta_J(1+\varepsilon_J-1/\alpha))} \\ &\leq (Z_N)^{1/\alpha} + (Z_N)^{1/(\alpha)-\varepsilon_{J_N}/3} + (Z_N)^{3(1/\alpha-\varepsilon_{J_N}-\eta_{J_N}(1+\varepsilon_{J_N}-1/(1-\varepsilon)))}. \end{aligned}$$

One concludes that

$$(Z_N)^{1/\alpha} \leq |B^\alpha| \leq (Z_N)^{1/\alpha-\tilde{\varepsilon}_{J_N}},$$

for some $\tilde{\varepsilon}_{J_N}$ which depends only on ε_{J_N} and η_{J_N} (not on α), is decreasing as a function of ε_{J_N} and η_{J_N} , and which tends to zero when J_N tends to infinity. In addition, the fact that $(Z_N)^{1/\alpha} \leq |B^\alpha|$ implies that $J_N \geq -\alpha \log_2 |B^\alpha| \geq \lfloor -\varepsilon \log_2 |B^\alpha| \rfloor$. Hence $\tilde{\varepsilon}_{J_N} \leq g_1(\lfloor -\varepsilon \log_2 |B^\alpha| \rfloor)$, where $g_1(r) = \tilde{\varepsilon}_{\lfloor -\varepsilon \log_2 r \rfloor}$. One can write finally

$$(Z_N)^{1/\alpha} \leq |B^\alpha| \leq (Z_N)^{1/\alpha-g_1(\lfloor |B^\alpha| \rfloor)}. \quad (4.55)$$

By construction, this mapping g_1 is non decreasing with r , and tends to 0 when r tends to 0.

Observe that since $\tilde{\varepsilon}_{J_N}$ is small (uniformly in α), one also has

$$(Z_N)^{1/\alpha} \leq |B^\alpha| \leq (Z_N)^{1/(2\alpha)}. \quad (4.56)$$

Assume now that $J_N \geq J/3$.

All the jumps other than (T_N, Z_N) involved in formula (4.53) are smaller than $2^{-J_N/\alpha}$. Hence, part (2) of Lemma 4.12 yields

$$S_1 \leq 2^{J\varepsilon_J} 2^{-J_N/\alpha} \leq 2^{-J_N(1/\alpha-3\varepsilon_{J_N})}.$$

The sum S_2 is still bounded by above by (4.54) with J replaced by $3J_N$. One deduces that

$$|B^\alpha| \leq (Z_N)^{1/\alpha} + (Z_N)^{1/(\alpha)-3\varepsilon J_N} + (Z_N)^{3(1/\alpha-\varepsilon J_N-\eta J_N(1+\varepsilon J_N^{-1/(1-\varepsilon)}))}.$$

One concludes that $|B^\alpha| \leq (Z_N)^{1/\alpha-\tilde{\varepsilon}_{J_N}}$ for some $\tilde{\varepsilon}_{J_N}$ which depends only on J_N (not on α), and which tends to zero when J_N tends to infinity. For the same reasons as above, equation (4.55) holds true.

Using that (4.55) holds true with β instead of α (but with the same mapping $\tilde{\varepsilon}$), one sees that

$$|B^\beta| \leq (Z_N)^{1/\beta-g_1(|B^\beta|)} \leq |B^\alpha|^{\alpha/\beta-\alpha g_1(|B^\beta|)}.$$

In addition, using (4.56) with β instead of α , one has $|B^\beta| \leq (Z_N)^{1/2\beta} \leq |B^\alpha|^{\alpha/(2\beta)}$. We deduce that $\alpha g_1(|B^\beta|) \leq (1-\varepsilon)g_1(|B^\alpha|^{\alpha/(2\beta)}) := g_2(|B^\alpha|)$, hence

$$|B^\beta| \leq |B^\alpha|^{\alpha/\beta-(1-\varepsilon)g_2(|B^\alpha|)}. \quad (4.57)$$

Similarly, recalling that $|B^\alpha|$ and $|B^\beta|$ are small quantities,

$$|B^\beta| \geq (Z_N)^{1/\beta} \geq |B^\alpha|^{1/(\beta(1/\alpha-g_1(|B^\alpha|)))} \geq |B^\alpha|^{\alpha/\beta+2\beta g_1(|B^\alpha|)} \geq |B^\alpha|^{\alpha/\beta+g_3(|B^\alpha|)}. \quad (4.58)$$

where $g_3(r) = 2(1-\varepsilon)\tilde{\varepsilon}(r)$. Finally, (4.57) and (4.58) gives the result, with $g(r) = \max(g_2(r), g_3(r))$.
□

Observe that one can also write

$$|B^\alpha|^{\alpha/\beta+\tilde{g}(|B^\beta|)} \leq |B^\beta| \leq |B^\alpha|^{\alpha/\beta-\tilde{g}(|B^\beta|)} \quad (4.59)$$

for some mapping \tilde{g} which enjoys the same properties as g .

One can now prove Theorem 4.8.

The following holds almost surely, since it depends only on Lemmas 4.13 and 4.14.

Let us denote by $d_\alpha = \dim_{\mathcal{H}} E^\alpha$, $d_\beta = \dim_{\mathcal{H}} E^\beta$, $d_f = \dim_{\mathcal{H}} E^f$.

Let $s > d_\alpha \beta/\alpha$, and let $\tilde{s} = s\alpha/\beta - (s\alpha/\beta - d_\alpha)/2$. One has $d_\alpha < \tilde{s} < s\alpha/\beta$.

By definition of d_α , there exists $\eta > 0$ such that $\mathcal{H}_{\eta/2}^{\tilde{s}}(E^\alpha) \leq 4^{-s}$. Hence, for some $\eta/2$ -covering \mathcal{R}^α of E^α , one has

$$\sum_{\tilde{B}^\alpha \in \mathcal{R}^\alpha} |\tilde{B}^\alpha|^{\tilde{s}} \leq 2^{-s}.$$

First, using 4.13, by slightly modifying the intervals $\tilde{B}^\alpha \in \tilde{\mathcal{R}}^\alpha$, one can replace these intervals with intervals of the form $B^\alpha = \mathcal{L}^\alpha([T_m, T_n))$ (plus at most a countable number of singletons), satisfying $|B^\alpha| \leq 2|\tilde{B}^\alpha|$, whose union is still covering E^α .

Hence, the initial $\eta/2$ -covering $\tilde{\mathcal{R}}^\alpha$ can be replaced by an η -covering \mathcal{R}^α , such that one has

$$\sum_{B^\alpha \in \mathcal{R}^\alpha} |B^\alpha|^{\tilde{s}} \leq 1.$$

Let us choose η so small that $g(\eta^{\alpha/\beta - g(\eta)}) < \frac{s\alpha/\beta - d_\alpha}{2s}$.

Each ball B^α is written $\mathcal{L}^\alpha(B)$, where $B = [T_m, T_n)$. As above, we write $B^\beta = \mathcal{L}^\beta(B)$ and $B^f = \mathcal{L}^f(B)$, and (4.50) and (4.51) hold true.

Since the balls (B^α) form an η -covering of E^α , the balls (B^β) form a $\tilde{\eta} := \eta^{\alpha/\beta - g(\eta)}$ -covering of E^β , and the balls (B^f) also form a $\tilde{\eta}$ -covering of E^f . We denote by \mathcal{R}^β and \mathcal{R}^f these two coverings. One has

$$\begin{aligned} \sum_{B^f \in \mathcal{R}^f} |B^f|^s &\leq \sum_{B^\beta \in \mathcal{R}^\beta} |B^\beta|^s \leq \sum_{B^\alpha \in \mathcal{R}^\alpha} |B^\alpha|^{s(\alpha/\beta - g(|B^\alpha|))} \\ &\leq \sum_{B^\alpha \in \mathcal{R}^\alpha} |B^\alpha|^{s\alpha/\beta - (s\alpha/\beta - d_\alpha)/2} = \sum_{B^\alpha \in \mathcal{R}^\alpha} |B^\alpha|^{\tilde{s}} \leq 1. \end{aligned}$$

Since \mathcal{R}^β is an $\tilde{\eta}$ -covering of E^β , the s -pre-Hausdorff measure of E^β , $\mathcal{H}_{\tilde{\eta}}^s(E^\beta)$ is less than 1. The same holds for $\mathcal{H}_{\tilde{\eta}}^s(E^f)$. This remains true for any sufficiently small $\tilde{\eta} > 0$, we conclude that both $\mathcal{H}^s(E^f)$ and $\mathcal{H}^s(E^\beta)$ are less than 1, hence d_f and d_β are smaller than s . Since this holds for any $s > d_\alpha \beta/\alpha$, one gets that $\max(d_f, d_\beta) \leq d_\alpha \beta/\alpha$.

Next, starting with a η -covering of E^f by balls B^f , one associates with every ball $B^f = \mathcal{L}^f([T_m, T_n))$ the ball $B^\beta = \mathcal{L}^\beta([T_m, T_n))$, the same lines of computation (simply using that $|B^f| \leq |B^\beta|$) yields that $d_f \leq d_\beta$.

The same argument shows that $d_\alpha \leq d_f$.

It remains us to prove the last inequality $d_\alpha \leq d_\beta \alpha/\beta$. The proof follows exactly the same lines, we write it without details.

Let $s > d_\beta \alpha/\beta$, and let $\tilde{s} = s\beta/\alpha - (s\beta/\alpha - d_\beta)/2$. One has $d_\beta < \tilde{s} < s\beta/\alpha$.

There exists an η -covering \mathcal{R}^β of E^β by intervals of the form $B^\beta = \mathcal{L}^\beta([T_m, T_n))$, such that

$$\sum_{B^\beta \in \mathcal{R}^\beta} |B^\beta|^{\tilde{s}} \leq 1.$$

One considers the associated intervals (B^α) and (B^f) , and the natural coverings \mathcal{R}^α and \mathcal{R}^f of E^α and E^f provided by these intervals.

Let η be so small that $\tilde{g}(|B^\beta|) \leq \tilde{g}(\eta) < \frac{s-d_\beta\alpha/\beta}{4s}$, where \tilde{g} is given by (4.59). One has

$$\begin{aligned} \sum_{B^\alpha \in \mathcal{R}^\alpha} |B^\alpha|^s &\leq \sum_{B^\beta \in \mathcal{R}^\beta} |B^\beta|^{s/(\alpha/\beta + \tilde{g}(|B^\beta|))} \leq \sum_{B^\beta \in \mathcal{R}^\beta} |B^\beta|^{s\beta/\alpha - 2s\tilde{g}(|B^\beta|)\beta/\alpha} \\ &\leq \sum_{B^\beta \in \mathcal{R}^\beta} |B^\beta|^{s\beta/\alpha - (s\beta/\alpha - d_\beta)/2} = \sum_{B^\beta \in \mathcal{R}^\beta} |B^\alpha|^{\tilde{s}} \leq 1. \end{aligned}$$

This holds true for any $\eta > 0$ small enough, so that $\mathcal{H}^s(E^\alpha) < +\infty$, hence $d_\alpha \leq s$. Since this holds true for any $s > d_\beta \alpha/\beta$, one gets that $d_\alpha \leq d_\beta \alpha/\beta$.

One concludes that $d_\alpha \leq d_f \leq d_\beta = d_\alpha \beta/\alpha$.

Remark 4.8. *It is certainly possible to short-cut the end of the proof, since one knows that α being fixed, almost surely, for every set $E \subset [0, 1]$, $\dim_{\mathcal{H}} \mathcal{L}^\alpha(E) = \alpha \dim_{\mathcal{H}}(E)$.*

But since we consider all α 's, it was easier to prove all inequalities at once.

Chapitre 5

Temps de séjour d'un mouvement brownien uni-dimensionnel sous des frontières glissantes

On étudie des propriétés de temps de séjour du mouvement brownien sous des frontières glissantes en utilisant la notion de dimension de grande échelle récemment introduite par Barlow et Taylor. Le résultat obtenu décrit la taille macroscopique de l'ensemble du temps où la croissance du mouvement brownien est beaucoup plus petite que normale et permet de quantifier la récurrence et la fluctuation du brownien à grande échelle.

5.1 Introduction

This chapter concerns the sojourn properties of the one-dimensional Brownian motion B with respect to some moving boundaries. More precisely, for an appropriate function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, we consider the sets

$$E^o(\varphi) = \{t \geq 0 : |B_t| \geq \varphi(t)\}, \quad (5.1)$$

$$E^i(\varphi) = \{t \geq 0 : |B_t| \leq \varphi(t)\}. \quad (5.2)$$

The first is called the set of Brownian sojourn outside the boundary $\varphi(\cdot)$ and the second inside the boundary.

Besides its obvious application in physics and finance, the understanding of these sets in different scales entails considerable informations on the path properties of Brownian motion. Two types of study are of particular interest.

- Geometric properties of $E^o(\varphi)$ and $E^i(\varphi)$ near $t_0 = 0$. The set $E^o(\varphi)$ is related to the irregularity of Brownian motion. Indeed, when 0 is an accumulation point of $E^o(\varphi)$, the local modulus of continuity at 0 can not be larger than φ . As a counterpart, the set $E^i(\varphi)$ corresponds to the regular behavior of Brownian path near zero. Concretely, local asymptotics of the Brownian motion such as Khintchine's law of iterated logarithm (LIL for short) might be described in terms of geometric properties of these sets around 0 with specific choices for φ . A natural question is under which condition on φ these sets admit an upper density with respect to Lebesgue measure (denoted by $|\cdot|$ throughout the chapter), i.e.

$$\limsup_{s \rightarrow 0} \frac{|E^{i/o}(\varphi) \cap [0, s]|}{s} > 0.$$

Uchiyama [113] treats the case $\varphi(t) = h(t)\sqrt{t}$ where h is taken from a whole class of correction functions (of logarithmic order), and he established density identities (for E^i) and bounds (for E^o) which are computed from the correction term.

- Geometric properties of these sets at infinity. This is related to the long time behaviors of the Brownian motion. As the Brownian motion scales like a square root function,

1. the set $E^o(\varphi)$, when φ grows faster than square root function, concerns the set of high peaks of Brownian motion. As is pointed out by Khoshnevisan, Kim and Xiao in [71], one can adapt Strassen's LIL [109] for random walks to Brownian motion in order to deduce upper density (at infinity) identities. More precisely, set $\varphi_\gamma(t) = \gamma\sqrt{2t \log \log(1/t)}$ with $\gamma > 0$, one has

$$\limsup_{s \rightarrow +\infty} \frac{|E^o(\varphi_\gamma) \cap [0, s]|}{s} = \max \left(1 - \exp \left\{ -4 \left(\frac{1}{\gamma^2} - 1 \right) \right\}, 0 \right).$$

Observe that $\gamma = 1$ is critical.

2. the set $E^i(\varphi)$, when φ grows slower than the square root function, concerns the lower than normal growth of Brownian motion. Uchiyama [113] established upper density bounds for $\varphi(t) = \sqrt{t}/h(t)$ with h belonging to a large class of logarithmic order correction functions.

Motivated by the study of Uchiyama [113] and Khoshnevisan, Kim and Xiao [71], we are interested in the asymptotics around infinity of the sojourn set of Brownian motion inside moving boundaries with much lower than normal growth. For this, we introduce the sets

$$\forall \gamma \in [0, 1/2], \quad E(\gamma) := E^i(\psi_\gamma) \text{ with } \psi_\gamma(t) = t^\gamma,$$

and our goal is to estimate the size of these sets for all values of γ . Our main tool for doing this is the theory of large scale dimensions developed by Barlow and Taylor [8, 9] in the late 80's, recently "refreshed" in the work of Xiao and Zheng [118] Georgiou *et al.* [53], Khoshnevisan *et al.* [71].

The initial motivation of Barlow and Taylor was to define a notion of fractals in discrete spaces such as \mathbb{Z}^d . This permits to describe the size properties of models in physical statistics, such as the infinite connected component of a percolation process, the range of a transient random walk for instance. To this end, they introduced and investigated several notions of dimension which describe different types of asymptotics of a set around infinity. Each dimension corresponds to an analogue in large scales of a classical fractal dimension. Among these dimensions, we are going to use the "macroscopic Hausdorff dimension" and the "mass dimensions", which are respectively the analogs of the classical Hausdorff dimension and Minkowski dimensions in large scales.

Let us state our main result. The macroscopic Hausdorff dimension of a set $E \subset \mathbb{R}^d$ is denoted by $\text{Dim}_H E$ and the upper mass dimension by $\text{Dim}_{UM} E$. Their definitions are recalled in Section 2.

Theorem 5.1. *Almost surely, for all $\gamma \in [0, 1/2]$,*

$$\text{Dim}_H E(\gamma) = \begin{cases} \frac{1}{2} & \text{if } \gamma \in [0, 1/2) \\ 1 & \text{if } \gamma = 1/2 \end{cases} \quad (5.3)$$

$$\text{Dim}_{UM} E(\gamma) = \frac{1}{2} + \gamma. \quad (5.4)$$

It is quite surprising that the macroscopic Hausdorff dimension of $E(\gamma)$ is constant for all $\gamma \in [0, 1/2)$. Further, one notices in the proof below that a.s. the Brownian zero set

$$\mathcal{Z} = \{t \geq 0 : B_t = 0\},$$

which is *a priori* thinner than $E(0)$ (hence all the $E(\gamma)$'s), also has macroscopic Hausdorff di-

mension $1/2$. The local structure of \mathcal{Z} is well understood since the works by Taylor and Wendel [112] and Perkins [97], who proved Hausdorff measure result for \mathcal{Z} using local times. In particular, the classical Hausdorff dimension of \mathcal{Z} is $1/2$. Our result gives the large scale structure of \mathcal{Z} and might be compared with an interesting result by Khoshnevisan [67] who states that zero set of a random walk ξ with finite variance in \mathbb{Z}^1

$$\{n \in \mathbb{N} : \xi_n = 0\}$$

has macroscopic Hausdorff dimension $1/2$. It is very interesting that the Brownian sojourn sets $E(\gamma)$, when they are measured by Dim_{UM} , describe somehow the large scale multifractal nature of Brownian motion. Our main theorem also gives a natural example of sets for which the macroscopic Hausdorff dimension and the upper mass dimension differ.

Our contribution to the formula (5.3) is for all $\gamma \in [0, 1/2)$. The case $\gamma = 1/2$ is deduced from Theorem 2 in [113] where Uchiyama obtained that a.s.

$$\limsup_{r \rightarrow \infty} \frac{|E(1/2) \cap [0, r]|}{r} > 0.$$

This inequality entails that a.s. $\text{Dim}_{\text{H}}E(1/2) = 1$ thanks to the following fact proved in [71] : for any $E \subset \mathbb{R}$

$$\left(\limsup_{r \rightarrow \infty} \frac{|E \cap [0, r]|}{r} > 0 \right) \Rightarrow \text{Dim}_{\text{H}}E = 1.$$

As Brownian motion fluctuates, it does return inside the boundary ψ_γ infinitely many times. In particular, the sets $E(\gamma)$ for $\gamma \in [0, 1/2]$ are unbounded. Our result allows to quantify the recurrence and fluctuation properties of Brownian motion at large scales.

This chapter is organized as follows. In Section 2, we recall the definition of large scale dimensions and establish the hitting probability estimate which is the probability that the Brownian motion hits on a moving boundary inside some arbitrary time interval. The first part of the main theorem (the macroscopic Hausdorff dimension) is proved in Section 3 and the second part (upper mass dimension) in Section 4.

Throughout this chapter, c, C are generic positive finite constants whose value may change from line to line. For two family of positive numbers $(a(x))$ and $(b(x))$, the equation $a(x) \asymp b(x)$ means that the ratio $a(x)/b(x)$ is uniformly bounded from below and above by some positive finite constant.

5.2 Preliminaries

5.2.1 Macroscopic dimensions

We adopt the notations in [71] that we recall now. Even if we work in one dimension, we still call the interval $Q(x, r) = [x, x + r)$ a cube with southwest corner x and side length r . For a given cube Q , its side length is denoted by $s(Q)$.

Define the annuli $\forall n \geq 1, \mathcal{S}_n = [2^{n-1}, 2^n]$ and $\mathcal{S}_0 = [0, 1)$. For any set $E \subset \mathbb{R}^+$, $n \in \mathbb{N}^*$, $\rho \geq 0$, introduce the quantity

$$\nu_\rho^n(E) = \inf \left\{ \sum_{i=1}^m \left(\frac{s(Q_i)}{2^n} \right)^\rho : E \cap \mathcal{S}_n \subset \bigcup_{i=1}^m Q_i \text{ with } s(Q_i) \geq 1 \text{ and } Q_i \subset \mathcal{S}_n \right\}.$$

Definition 5.1. Let $E \subset \mathbb{R}^+$. The macroscopic Hausdorff dimension of E is defined as

$$\text{Dim}_H E = \inf \left\{ \rho \geq 0 : \sum_{n \geq 0} \nu_\rho^n(E) < +\infty \right\}. \quad (5.5)$$

The upper and lower mass dimension of E are defined as

$$\begin{aligned} \text{Dim}_{UM} E &= \limsup_{n \rightarrow +\infty} \frac{\ln(|E \cap [0, n]|)}{\ln n}, \\ \text{Dim}_{LM} E &= \liminf_{n \rightarrow +\infty} \frac{\ln(|E \cap [0, n]|)}{\ln n}. \end{aligned}$$

The macroscopic Hausdorff dimension of a set does not depend on any of its bounded subsets, since the series in (5.5) converges if and only if its tail series converges. Further, the covering cubes are chosen to have side larger than 1, which explains why macroscopic Hausdorff dimensions does not rely on the local structure of the underlying set. The same remarks apply valid to mass dimensions.

It is known [8, 9] that for any set $E \subset \mathbb{R}$,

$$\text{Dim}_H E \leq \text{Dim}_{LM} E \leq \text{Dim}_{UM} E.$$

To bound $\text{Dim}_H E$ from above, one has to find an economic covering of E . To find the lower bound, there is an analog of the mass distribution principle, which is recalled in the following lemma.

Lemma 5.1. Let $E \subset \mathcal{S}_n$. Let μ be a finite Borel measure on \mathbb{R} with support included in E . Suppose that there exists finite positive constants C and ρ , such that for any cube $Q(x, r) \subset \mathcal{S}_n$ with $r \geq 1$, one has

$$\mu(Q(x, r)) \leq Cr^\rho.$$

Then

$$\nu_\rho^n(E) \geq C^{-1} 2^{-n\rho} \mu(\mathcal{S}_n).$$

5.2.2 Hitting probability estimates of Brownian motion to the moving boundaries

The following estimate is useful when one looks for an appropriate covering of $E(\gamma)$ with respect to different large scale dimensions.

Lemma 5.2. *Consider a cube $Q(a, r)$ inside \mathcal{S}_n , i.e. $a \in \mathcal{S}_n$ and $a + r \leq 2^n$. For each $0 \leq \gamma < 1/2$, define the event*

$$A(n, a, r, \gamma) = \{\exists t \in Q(a, r) : |B_t| \leq t^\gamma\}. \quad (5.6)$$

One has

$$\mathbb{P}(A(n, a, r, \gamma)) \leq \frac{2}{\sqrt{\pi}} 2^{n(\gamma-1/2)} + \frac{3}{\sqrt{2\pi}} \left(\frac{r}{a}\right)^{1/2}.$$

Let us summarize several basic properties of Brownian motion used in the proof of Lemma 5.2. For a proof, see for instance [103] or [89].

Lemma 5.3. *Let B be a standard Brownian motion.*

- *Symmetry* : $(B_t)_{t \geq 0}$ has the same law as $(-B_t)_{t \geq 0}$.
- *Self-similarity* : for any $\lambda > 0$, the law of $(\lambda^{-1/2} B_{\lambda t})_{t \geq 0}$ is identical to that of $(B_t)_{t \geq 0}$.
- *Markov property* : for each $t \geq 0$, the process $(B_{t+h} - B_t)_{h \geq 0}$ is independent of $\sigma\{B_s : 0 \leq s \leq t\}$, and has the same law as that of $(B_h)_{h \geq 0}$.
- *Reflexion principle* : for every $x \geq 0$, $\mathbb{P}(\sup_{0 \leq t \leq 1} B_t \geq x) = 2\mathbb{P}(B_1 \geq x)$.
- *Tail probability* : for all $x \geq 1$, one has

$$\mathbb{P}(B_1 \geq x) \leq \frac{1}{x\sqrt{2\pi}} \exp(-x^2/2).$$

Proof of Lemma 5.2 : One has

$$\begin{aligned} \mathbb{P}(A(n, a, r, \gamma)) &\leq \mathbb{P}\left(\inf_{t \in Q(a, r)} |B_t| \leq 2^{n\gamma}\right) \\ &= \mathbb{P}(|B_a| \leq 2^{n\gamma}) + \mathbb{P}\left(|B_a| > 2^{n\gamma}, \inf_{t \in Q(a, r)} |B_t| \leq 2^{n\gamma}\right) \\ &:= P_1 + P_2. \end{aligned}$$

By the self-similarity of B and recalling that $a \in \mathcal{S}_n$, one obtains

$$P_1 = \mathbb{P}(|B_1| \leq a^{-1/2}2^{n\gamma}) \leq \sqrt{\frac{2}{\pi}} a^{-1/2}2^{n\gamma} \leq \frac{2}{\sqrt{\pi}} 2^{n(\gamma-1/2)}.$$

Using the symmetry of B , one gets

$$\mathbb{P}\left(B_a > 2^{n\gamma}, \inf_{t \in Q(a,r)} B_t \leq 2^{n\gamma}\right) = \mathbb{P}\left(B_a < -2^{n\gamma}, \sup_{t \in Q(a,r)} B_t \geq -2^{n\gamma}\right).$$

Thus,

$$P_2 \leq 2\mathbb{P}\left(B_a > 2^{n\gamma}, \inf_{t \in Q(a,r)} B_t \leq 2^{n\gamma}\right) = 2\mathbb{P}\left(B_a > 2^{n\gamma}, \inf_{t \in Q(a,r)} (B_t - B_a) \leq 2^{2n\gamma} - B_a\right).$$

Set $X_h = B_{a+h} - B_a$ which is a Brownian motion independent of B_a . Using successively the self-similarity, the symmetry and the Markov property at a of B yields

$$\begin{aligned} P_2 &\leq 2\mathbb{P}\left(B_a > 2^{n\gamma}, \inf_{0 \leq h \leq r} X_h \leq 2^{n\gamma} - B_a\right) \\ &= 2\mathbb{P}\left(B_a > 2^{n\gamma}, \inf_{0 \leq h \leq 1} X_h \leq r^{-1/2}(2^{n\gamma} - B_a)\right) \\ &= 2\mathbb{P}\left(B_a > 2^{n\gamma}, \sup_{0 \leq h \leq 1} X_h \geq r^{-1/2}(B_a - 2^{n\gamma})\right) \\ &= 2 \int_{2^{n\gamma}}^{+\infty} \mathbb{P}\left(\sup_{0 \leq h \leq 1} X_h \geq r^{-1/2}(x - 2^{n\gamma})\right) e^{-\frac{x^2}{2a}} \frac{dx}{\sqrt{2\pi a}}. \end{aligned}$$

One splits the last integral into two parts. On one hand, bounding from above the probability inside the integral by 1, one obtains

$$\int_{2^{n\gamma}}^{2^{n\gamma}+r^{1/2}} e^{-\frac{x^2}{2a}} \frac{dx}{\sqrt{2\pi a}} = \mathbb{P}\left(X_1 \in \left[a^{-1/2}2^{n\gamma}, a^{-1/2}2^{n\gamma} + (r/a)^{1/2}\right]\right) \leq \frac{1}{\sqrt{2\pi}} \left(\frac{r}{a}\right)^{1/2}.$$

On the other hand, applying the reflexion principle to X and tail probability of standard Gaussian variable, one has

$$\begin{aligned} &\int_{(2^{n\gamma}+r^{1/2})}^{+\infty} \mathbb{P}\left(X_1 \geq r^{-1/2}(x - 2^{n\gamma})\right) e^{-\frac{x^2}{2a}} \frac{dx}{\sqrt{2\pi a}} \\ &\leq \int_{(2^{n\gamma}+r^{1/2})}^{+\infty} \frac{1}{r^{-1/2}(x - 2^{n\gamma})\sqrt{2\pi}} e^{-\frac{(x-2^{n\gamma})^2}{2r}} e^{-\frac{x^2}{2a}} \frac{dx}{\sqrt{2\pi a}} \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{2^{n\gamma}}^{+\infty} e^{-\frac{(x-2^{n\gamma})^2}{2r}} \frac{dx}{\sqrt{2\pi a}} = \frac{1}{\sqrt{2\pi}} \left(\frac{r}{a}\right)^{1/2} \int_{2^{n\gamma}}^{+\infty} e^{-\frac{(x-2^{n\gamma})^2}{2r}} \frac{dx}{\sqrt{2\pi r}} = \frac{1}{2\sqrt{2\pi}} \left(\frac{r}{a}\right)^{1/2}. \end{aligned}$$

Therefore, one has established that

$$T_2 \leq \frac{3}{\sqrt{2\pi}} \left(\frac{r}{a}\right)^{1/2}.$$

Combining the estimates ends the proof. \square

5.3 Proof of Theorem 5.1 : macroscopic Hausdorff dimension

In this section, we aim to prove the dimension formula (5.3). As is said in the introduction, by Uchiyama's upper density result [113], it is enough to compute $\text{Dim}_H E(\gamma)$ for all $\gamma \in [0, 1/2)$.

Let $0 \leq \gamma < 1/2$ be fixed throughout this section. Due to the monotonicity in γ of the sets $E(\gamma)$, and the fact that the zero set of Brownian motion

$$\mathcal{Z} = \{t \geq 0 : B_t = 0\} \subset E(0),$$

we divide the proof of (5.3) into two parts :

$$\text{Dim}_H E(\gamma) \leq \frac{1}{2} \quad \text{and} \quad \text{Dim}_H \mathcal{Z} \geq \frac{1}{2}.$$

Before proving these inequalities, one needs some background on \mathcal{Z} that we recall below. Our standard reference is [25], see also [24].

5.3.1 Facts on the Brownian zero set and 1/2-stable subordinator

It is well known that the Brownian zero set coincides with the range of a 1/2-stable subordinator via the Brownian local times at 0, see [24, 25]. A subordinator σ is a Lévy processes with increasing sample paths. It is said to be 1/2-stable if the Laplace transform of σ_1 is $e^{-\Phi(\lambda)}$ with

$$\Phi(\lambda) = \lambda^{1/2}, \quad \text{for all } \lambda > 0.$$

The renewal function $U(x)$ is the distribution function of the 0-potential measure of σ , i.e.

$$U(x) = \mathbb{E} \left[\int_0^{+\infty} \mathbf{1}_{\sigma_t \leq x} dt \right].$$

The 0-potential measure of σ , denoted by $U(dx)$, characterize the law of σ in the sense that its Laplace transform is $1/\Phi(\lambda)$ for all $\lambda > 0$. Tauberian theorems provide the relation between the Laplace exponent Φ of a subordinator and its renewal function. Precisely, one has

$$U(x) \asymp \frac{1}{\Phi(1/x)}, \quad (5.7)$$

see Section 1.3 in [25].

For a general subordinator σ , one has the following hitting probability estimate in terms of its renewal function. It gives bounds for the probability that the range of σ touches a deterministic set $A \subset \mathbb{R}$. Recall that $\mathcal{R} = \sigma([0, +\infty))$ is the range of σ .

Lemma 5.4 ([25], Lemma 5.5). *For every $0 < a < b$, one has*

$$\frac{U(b) - U(a)}{U(b-a)} \leq \mathbb{P}(\mathcal{R} \cap [a, b] \neq \emptyset) \leq \frac{U(2b-a) - U(a)}{U(b-a)}.$$

Next lemma gives a more precise hitting probability estimate for 1/2-stable subordinators.

Lemma 5.5. *Let σ be a 1/2-stable subordinator and $Q(a, r) = [a, a+r]$ be a cube in some annulus \mathcal{S}_n . One has*

$$c \left(\frac{r}{a}\right)^{1/2} \leq \mathbb{P}(\mathcal{R} \cap Q(a, r) \neq \emptyset) \leq C \left(\frac{r}{a}\right)^{1/2}.$$

where c, C is independent of a, r and n .

Proof. First, applying (5.7) yields

$$U((a+r) - a) = U(r) \geq Cr^{1/2}. \quad (5.8)$$

On the other hand, by the definition of $U(x)$, one obtains

$$U(2(a+r) - a) - U(a) = \int_0^{+\infty} \mathbb{P}(\sigma_t \in [a, a+2r]) dt = \int_0^{+\infty} \int_a^{a+2r} p_t(x) dx dt$$

where $p_t(x)$ is the density of σ_t , given by

$$p_t(x) = C \frac{t}{x^{3/2}} e^{-\frac{t^2}{2x}}.$$

As $Q(a, r) \subset \mathcal{S}_n$, one gets $r < a$ and deduces

$$\begin{aligned} U(a+2r) - U(a) &\leq C \int_0^{+\infty} \int_a^{a+2r} \frac{t}{a^{3/2}} e^{-\frac{t^2}{6a}} dx dt \\ &\leq C \frac{r}{a^{3/2}} \int_0^{+\infty} t e^{-t^2/(6a)} dt = C \frac{r}{a^{1/2}}. \end{aligned} \quad (5.9)$$

The second inequality follows by an application of Lemma 5.4 with $b = a+r$. Similarly, one gets by (5.7)

$$U(r) \leq Cr^{1/2},$$

and by $r < a$

$$\begin{aligned} U(a+r) - U(a) &= \int_0^{+\infty} \mathbb{P}(\sigma_t \in [a, a+r]) dt = \int_0^{+\infty} \int_a^{a+r} p_t(x) dx dt. \\ &\geq C \int_0^{+\infty} \int_a^{a+2r} \frac{t}{a^{3/2}} e^{-\frac{t^2}{2a}} dx dt = C \frac{r}{a^{1/2}}. \end{aligned} \quad (5.10)$$

By Lemma 5.4 with $b = a + r$, one gets the first inequality. \square

One also needs the following tail probability estimate of the occupation measure for subordinators, obtained by Pruitt and Taylor [101].

Lemma 5.6 ([101], Lemma 3.1). *Let σ be a $1/2$ -stable subordinator. Define the sojourn time of σ in the cube $Q(0, r)$,*

$$T(r) = \int_0^{+\infty} \mathbf{1}_{\sigma_t \leq r} dt.$$

One has for all $\lambda \geq 0$,

$$\mathbb{P}(T(r) \geq \lambda \mathbb{E}[T(2r)]) \leq e^{-\lambda/2}.$$

Remark 5.1. *Observe that by (5.7) $\mathbb{E}[T(2r)] = U(2r) \leq C_0 r^{1/2}$ for $0 < C_0 < +\infty$ independent of r .*

5.3.2 Proof of Theorem 5.1 : macroscopic Hausdorff dimension

Let $\gamma \in [0, 1/2)$ and $\rho > 1/2$ be fixed. First we show $\text{Dim}_{\mathbb{H}} E(\gamma) \leq 1/2$. Set

$$x_{n,i} = 2^{n-1} + i2^{2n\gamma} \text{ for } i \in \{0, \dots, \lfloor 2^{n-1}/2^{2n\gamma} \rfloor\}.$$

Consider the cubes $Q(x_{n,i}, 2^{2n\gamma})$ which form a partitions of S_n . Note that for each of these cubes, one has

$$E(\gamma) \cap Q(x_{n,i}, 2^{2n\gamma}) \subset Q(x_{n,i}, 2^{2n\gamma}) \mathbf{1}_{A(n, x_{n,i}, 2^{2n\gamma}, \gamma)}$$

by the definition of $A(n, x_{n,i}, 2^{2n\gamma}, \gamma)$ in Lemma 5.2. Thus,

$$\nu_{\rho}^n(E(\gamma)) \leq \sum_{i=1}^{\lfloor 2^{(1-2\gamma)n} \rfloor} \left(\frac{2^{2n\gamma}}{2^n} \right)^{\rho} \mathbf{1}_{A(n, x_{n,i}, 2^{2n\gamma}, \gamma)}.$$

By choosing the side length $2^{2n\gamma}$, one observes that the two terms in Lemma 5.2 are of the same order. Taking expectation in the above inequality, one obtains by Lemma 5.2 that there exists a

positive finite constant C such that for all $n \in \mathbb{N}^*$

$$\begin{aligned} \mathbb{E}[\nu_\rho^n(E(\gamma))] &\leq 2^{(1-2\gamma)n} \cdot 2^{(2\gamma-1)\rho n} \cdot \mathbb{P}(A(n, x_{n,i}, 2^{2n\gamma}, \gamma)) \\ &= 2^{(1-2\gamma)(1-\rho)n} \cdot C 2^{n(\gamma-1/2)} \\ &= C 2^{(1/2-\gamma)(1-2\rho)n}. \end{aligned}$$

Thus, Fubini Theorem entails $\mathbb{E}[\sum_{n \in \mathbb{N}^*} \nu_\rho^n(E(\gamma))] < +\infty$. This proves that $\text{Dim}_H E(\gamma) \leq \rho$ almost surely. Letting $\rho \rightarrow 1/2$ yields the upper bound.

Now one moves to the low bound for $\text{Dim}_H E(\gamma)$. We are going to prove that $\text{Dim}_H \mathcal{Z} \geq 1/2$. Since $\forall \gamma \geq 0$, $E(\gamma) \supset \mathcal{Z}$, we finally deduce that $\text{Dim}_H E(\gamma) = 1/2$. Recalling that \mathcal{R} is the range of the $1/2$ -stable subordinator σ , by Section 5.3.1, one knows that $\text{Dim}_H \mathcal{Z} = \text{Dim}_H \mathcal{R}$. So it is enough to prove that $\text{Dim}_H \mathcal{R} \geq 1/2$.

The natural measure supported by \mathcal{R} is the occupation measure μ of σ , defined for every Borel set E by

$$\mu(E) = \int_0^{+\infty} \mathbf{1}_E(\sigma_t) dt.$$

Let us prove a scaling property satisfied by μ , which enables us to invoke the mass distribution principle (Lemma 5.1).

Lemma 5.7. *Almost surely, there exists an integer n_0 , such that for all $n \geq n_0$, any cube $Q(a, r) \subset \mathcal{S}_n$ satisfies*

$$\mu(Q(a, r)) \leq 2C_0 n r^{1/2}. \quad (5.11)$$

where C_0 is the positive finite constant in Remark 5.1.

Proof. Let n be fixed. For any integer $0 \leq k \leq n-1$ and $0 \leq i \leq 2^{n-1-k} - 1$, denote by $Q_{n,k,i}$ the cube with south-west corner $x_{n,k,i} = 2^{n-1} + i2^k$ and side 2^k . For fixed (n, k) , one counts the number of cubes of type $Q_{n,k,i}$ which violate the scaling property (5.11) (up to a factor 2). Define the set of “bad” indices

$$I_{n,k} = \{i : \mu(Q_{n,k,i}) \geq C_0 n 2^{k/2}\}.$$

Note that for $i \in I_{n,k}$, the cube $Q_{n,k,i}$ should be hit by σ , and the sojourn of σ in that cube is longer than $C_0 n 2^{k/2}$. Applying the strong Markov property at the hitting time of the cube $Q_{n,k,i}$ by σ , along with the space homogeneity of Lévy processes, one obtains

$$\mathbb{P}(\mu(Q_{n,k,i}) \geq C_0 n 2^{k/2}) \leq \mathbb{P}(\mathcal{R} \cap Q_{n,k,i} \neq \emptyset) \mathbb{P}(T(2^k) \geq C_0 n 2^{k/2})$$

An application of Lemma 5.5, 5.6 and Remark 5.1 yields

$$\mathbb{P}(\mu(Q_{n,k,i}) \geq C_0 n 2^{k/2}) \leq C 2^{(k-n)/2} e^{-n/2}.$$

Therefore, one obtains

$$\begin{aligned} \sum_{k=0}^{n-1} \mathbb{E}[\#I_{n,k}] &= \sum_{k=0}^{n-1} \sum_{i=0}^{2^{n-1-k}-1} \mathbb{P}(\mu(Q_{n,k,i}) \geq C_0 n 2^{k/2}) \\ &\leq C \sum_{k=0}^{n-1} 2^{n-k} 2^{(k-n)/2} e^{-n/2} \\ &= C 2^{n/2} e^{-n/2} \end{aligned}$$

which is the general term of a convergent series. The Borel-Cantelli Lemma entails that almost surely, for all n large enough, for all (k, i) ,

$$\mu(Q_{n,k,i}) \leq C_0 n 2^{k/2}.$$

Finally, any $Q(a, r) \subset \mathcal{S}_n$ with $2^k \leq r < 2^{k+1}$ is covered by the union of two consecutive cubes $Q_{n,k,i}, Q_{n,k,i+1}$ of side 2^k , thus

$$\mu(Q(a, r)) \leq \mu(Q_{n,k,i}) + \mu(Q_{n,k,i+1}) \leq 2C_0 n 2^{k/2} \leq 2C_0 n r^{1/2}.$$

□

The last lemma estimates the moments for the occupation measure of σ in the annuli.

Lemma 5.8. *There exists finite positive constants C_1, C_1', C_2 such that for every $n \in \mathbb{N}^*$,*

$$\begin{aligned} C_1 2^{n/2} &\leq \mathbb{E}[\mu(\mathcal{S}_n)] \leq C_1' 2^{n/2} \\ \text{and } \mathbb{E}[\mu(\mathcal{S}_n)^2] &\leq C_2 2^n. \end{aligned}$$

Proof. Observe that $\mathbb{E}[\mu(\mathcal{S}_n)] = U(2^n) - U(2^{n-1})$ by the definition of the renewal function U . The first moment estimate follows from (5.9) and (5.13). We focus on the second moment

estimate. Using successively Fubini Theorem, symmetry and the Markov property of σ , one has

$$\begin{aligned}\mathbb{E}[\mu(\mathcal{S}_n)^2] &= \int_0^{+\infty} \int_0^\infty \mathbb{P}(\sigma_s \in \mathcal{S}_n, \sigma_t \in \mathcal{S}_n) dt ds \\ &= 2 \int_0^{+\infty} \int_s^{+\infty} \mathbb{P}(\sigma_s \in \mathcal{S}_n, \sigma_t \in \mathcal{S}_n) dt ds \\ &\leq 2 \int_0^{+\infty} \int_s^{+\infty} \mathbb{P}(\sigma_s \in \mathcal{S}_n) \mathbb{P}(\sigma_{t-s} \leq 2^{n-1}) dt ds \\ &= 2 \int_0^{+\infty} \mathbb{P}(\sigma_s \in \mathcal{S}_n) ds \int_0^{+\infty} \mathbb{P}(\sigma_u \leq 2^{n-1}) du.\end{aligned}$$

Applying (5.8) and (5.9) entails the result. \square

Let us finish the proof. By Paley-Zygmund inequality,

$$\mathbb{P}\left(\mu(\mathcal{S}_n) \geq \frac{1}{2} C_1' 2^{n/2}\right) \geq \mathbb{P}\left(\mu(\mathcal{S}_n) \geq \frac{1}{2} \mathbb{E}[\mu(\mathcal{S}_n)]\right) \geq \frac{\mathbb{E}[\mu(\mathcal{S}_n)]^2}{4\mathbb{E}[\mu(\mathcal{S}_n)^2]} \geq \frac{C_1^2}{4C_2} := p > 0.$$

Further, thanks to the strong Markov property at the hitting times of \mathcal{S}_n by σ and the monotonicity of σ , one knows that the sequence of random variables $(\mu(\mathcal{S}_n))_{n \geq 0}$ are independent. Thus, a variant of Borel-Cantelli Lemma (see Theorem 2.3.8 in [45]) yields that almost surely, for some sequence $\mathcal{N} = (n_k)_{k \geq 1}$ of integers with lower density at least p ,

$$\mu(\mathcal{S}_{n_k}) \geq \frac{1}{2} C_1' 2^{n_k/2}.$$

One concludes by Lemma 5.1 that almost surely,

$$\sum_{n \geq 0} \nu_{1/2}^n(\mathcal{R}) \geq C_0^{-1} \sum_{n \geq 0} n^{-1} 2^{-n/2} \mu(\mathcal{S}_n) \geq (2C_0)^{-1} C_1' \sum_{n \in \mathcal{N}} n^{-1} = +\infty.$$

Hence $\text{Dim}_{\text{H}} \mathcal{R} \geq 1/2$, almost surely.

5.4 Proof of Theorem 5.1 : upper mass dimension

Recall that we intend to prove that almost surely, for all $\gamma \in [0, 1/2]$,

$$\text{Dim}_{\text{UM}} E(\gamma) := \limsup_{n \rightarrow +\infty} \frac{\ln |E(\gamma) \cap [0, n]|}{\ln n} = \frac{1}{2} + \gamma.$$

We are going to show that the sojourn time $|E(\gamma) \cap \mathcal{S}_n|$ is larger than $2^{n(1/2+\gamma)}$ for infinite many n , but never exceeds $2^{n(1/2+\gamma+\varepsilon)}$ for all large n , where $\varepsilon > 0$ can be chosen arbitrarily small. As the underlying sequence of sojourn time are dependent random variables, one needs the following strong law of large numbers for dependent events, proved in Xiao and Zheng [118].

Lemma 5.9. [118, Lemma 2.9] Suppose that $\{A_k\}$ and $\{D_k\}$ are two sequences of events adapted to the same filtration $\{\mathcal{F}_k\}$ and are such that for some positive constants p , a and δ

$$\mathbb{P}(A_{k+1}|\mathcal{F}_k) \geq p \text{ on event } D_k, \text{ and } \mathbb{P}(D_k^c) \leq ae^{-\delta k}.$$

Then there exists $\varepsilon > 0$ such that almost surely,

$$\liminf_{n \rightarrow +\infty} \frac{\sum_{k=1}^n \mathbf{1}_{A_k}}{n} \geq \varepsilon.$$

Let us first prove the upper bound. For each annulus \mathcal{S}_n , consider the unit cubes $Q(k, 1)$ with integer south-west corner $k \in \mathcal{S}_n$. Using those $Q(k, 1)$ inside each \mathcal{S}_p ($0 \leq p \leq n$) satisfying the event in Lemma 5.2 as a covering of $E(\gamma) \cap [0, 2^n]$, one has

$$|E(\gamma) \cap [0, 2^n]| \leq \sum_{p=0}^n \sum_{k=2^{p-1}}^{2^p-1} \mathbf{1}_{A(p,k,1,\gamma)}.$$

Taking expectation, one obtains by Lemma 5.2 that for some finite positive constant C ,

$$\mathbb{E}[|E(\gamma) \cap [0, 2^n]|] \leq C \sum_{p=0}^n 2^p 2^{p(\gamma-1/2)} = C 2^{n(\gamma+1/2)}.$$

For $\rho > 1/2 + \gamma$, the first moment Markov inequality yields that for all $n \in \mathbb{N}^*$,

$$\mathbb{P}(|E(\gamma) \cap [0, 2^n]| \geq 2^{n\rho}) \leq C^{-1} 2^{-n(\rho-\gamma-1/2)}$$

which is the general term of a convergent series. Thus, an application of Borel-Cantelli Lemma gives almost surely, for all n large enough,

$$|E(\gamma) \cap [0, 2^n]| < 2^{n\rho}.$$

Hence, for all m large enough, denote by n the unique integer such that $m \in \mathcal{S}_n$, one gets

$$\frac{\ln |E(\gamma) \cap [0, m]|}{\ln m} \leq \frac{\ln |E(\gamma) \cap [0, 2^n]|}{\ln 2^{n-1}} \leq \frac{n}{n-1} \rho$$

Taking lim sup entails $\text{Dim}_{\text{UM}} E(\gamma) \leq \rho$, almost surely. The desired upper bound follows by letting ρ tend to $\gamma + 1/2$.

Now we prove the lower bound. Let us fix some notations before moving forward. Denote by $(C, \mathcal{C}, X, \mathbb{P}^x)$ the canonical space of Brownian motion, i.e. C is the space of continuous functions from \mathbb{R}^+ to \mathbb{R} , \mathcal{C} its Borel σ -algebra, X the coordinate process and \mathbb{P}^x is the law of Brownian

motion starting from $x \in \mathbb{R}$. Define the super-exponential increasing sequence of integers

$$\begin{cases} n_0 = 1, \\ n_{k+1} = 2^{n_k} \text{ for all } k \in \mathbb{N}^*. \end{cases}$$

Set

$$\widehat{\mathcal{S}}_n = \mathcal{S}_n \cup \mathcal{S}_{n+1} \cup \mathcal{S}_{n+2}.$$

Introduce two sequences of events

$$A_k = \left\{ |E(\gamma) \cap \widehat{\mathcal{S}}_{n_k}| \geq K 2^{n_k(1/2+\gamma)} \right\} \text{ and } D_k = \left\{ |B(2^{n_k+2})| \leq n_k \cdot 2^{(n_k+2)/2} \right\},$$

recalling that 2^{n_k+2} is the right endpoint of $\widehat{\mathcal{S}}_{n_k}$. Here K is a universal constant which will be chosen later.

One observes that A_k implies $|E(\gamma) \cap \mathcal{S}_n| \geq 2^{n(\gamma+1/2)}/3$ for some $n \in \{n_k, n_k + 1, n_k + 2\}$. As $n_{k+1} > n_k + 2$, one obtains the desired lower bound for $\text{Dim}_{\text{UM}} E(\gamma)$ as soon as A_k occurs for infinite many k , almost surely. To this end, one applies Lemma 5.9 to our $\{A_k\}$ and $\{D_k\}$, and verifies the conditions on these events.

The condition on $\{D_k\}$ is easy to check. One has by the self-similarity of Brownian motion, the tail probability of Gaussian variable, and the rapid increasing rate of $\{n_k\}$ that

$$\mathbb{P}(D_k^c) = \mathbb{P}(|B_1| > n_k) \leq \frac{2}{n_k \sqrt{2\pi}} e^{-n_k^2/2} \leq e^{-k}.$$

It remains to check the condition on $\{A_k\}$. By the simple Markov property of Brownian motion at time 2^{n_k+2} , one has

$$\mathbb{P}(A_{k+1} | \mathcal{F}_k) = g(B(2^{n_k+2}))$$

where

$$\begin{aligned} g(x) &= \mathbb{P}^x \left(\int_{2^{n_{k+1}-1} - 2^{n_k+2}}^{2^{n_{k+1}+2} - 2^{n_k+2}} \mathbf{1}_{|X_t| \leq (t+2^{n_k+2})^\gamma} dt \geq 2^{n_{k+1}(1/2+\gamma)} \right) \\ &\geq \mathbb{P}^x \left(\int_{2^{n_{k+1}-1}}^{2^{n_{k+1}+1}} \mathbf{1}_{|X_t| \leq t^\gamma} dt \geq K 2^{n_{k+1}(1/2+\gamma)} \right). \end{aligned}$$

Define the x -level set of X

$$\mathcal{Z}_x = \{t \geq 0 : X_t = x\}$$

and the stopping time

$$\tau_{k+1} = \inf\{t \in \mathcal{S}_{n_{k+1}} : X_t = x\}.$$

One has

$$g(x) \geq \mathbb{P}^x \left(\mathcal{Z}_x \cap (\mathcal{S}_{n_{k+1}} \cup \mathcal{S}_{n_{k+1}+1}) \neq \emptyset, \int_{\tau_{k+1}}^{\tau_{k+1}+2^{n_{k+1}}} \mathbf{1}_{|X_t| \leq t^\gamma} dt \geq K2^{n_{k+1}(1/2+\gamma)} \right)$$

By the strong Markov property of Brownian motion at τ_{k+1} ,

$$g(x) \geq \mathbb{P}^x \left(\mathcal{Z}_x \cap (\mathcal{S}_{n_{k+1}} \cup \mathcal{S}_{n_{k+1}+1}) \neq \emptyset \right) \mathbb{P}^x \left(\int_0^{2^{n_{k+1}}} \mathbf{1}_{|X_t| \leq 2^{(n_{k+1}-1)\gamma}} dt \geq K2^{n_{k+1}(1/2+\gamma)} \right) := P_k^1(x) P_k^2(x).$$

We estimate separately the two terms in the last product. Recall that \mathcal{Z}_x coincides with the range of a $1/2$ -stable subordinator, the hitting probability estimate Lemma 5.5 yields that

$$P_k^1(x) \geq C \tag{5.12}$$

uniformly for all k . Finally, one observes that uniformly for all $x \in [-n_k \cdot 2^{(n_k+2)/2}, n_k \cdot 2^{(n_k+2)/2}]$,

$$\begin{aligned} P_k^2(x) &\geq \mathbb{P}^0 \left(\int_0^{2^{n_{k+1}}} \mathbf{1}_{|X_t| \leq 2^{(n_{k+1}-1)\gamma-x}} dt \geq K2^{n_{k+1}(1/2+\gamma)} \right) \\ &\geq \mathbb{P}^0 \left(\int_0^{2^{n_{k+1}}} \mathbf{1}_{|X_t| \leq 2^{(n_{k+1}-2)\gamma}} dt \geq K2^{n_{k+1}(1/2+\gamma)} \right) := p_k \end{aligned} \tag{5.13}$$

The sequence $\{p_k\}$ will be uniformly controlled from below by the following lemma.

Lemma 5.10. *Let*

$$M_k = \int_{2^{n_{k+1}-1}}^{2^{n_{k+1}}} \mathbf{1}_{|X_t| \leq 2^{(n_{k+1}-2)\gamma}} dt.$$

There exist positive constants C_1, C'_1, C_2 such that

$$C'_1 2^{n_{k+1}(1/2+\gamma)} \geq \mathbb{E}[M_k] \geq C_1 2^{n_{k+1}(1/2+\gamma)}$$

and

$$\mathbb{E}[M_k^2] \leq C_2 2^{n_{k+1}(1+2\gamma)}$$

Proof. An application of Fubini theorem and the selfsimilarity of Brownian motion yields the

first moment estimate. For the second moment, one establishes as in Lemma 5.8 that

$$\begin{aligned}
\mathbb{E}^0[M_k^2] &= \int_{2^{n_{k+1}-1}}^{2^{n_{k+1}}} \int_{2^{n_{k+1}-1}}^{2^{n_{k+1}}} \mathbb{P}^0(|X_t| \leq 2^{(n_{k+1}-2)\gamma}, |X_s| \leq 2^{(n_{k+1}-2)\gamma}) ds dt \\
&= 2 \int_{2^{n_{k+1}-1}}^{2^{n_{k+1}}} \int_t^{2^{n_{k+1}}} \mathbb{P}^0(|X_t| \leq 2^{(n_{k+1}-2)\gamma}, |X_s| \leq 2^{(n_{k+1}-2)\gamma}) ds dt \\
&\leq 2 \int_{2^{n_{k+1}-1}}^{2^{n_{k+1}}} \int_0^{2^{n_{k+1}}} \mathbb{P}^0(|X_t| \leq 2^{(n_{k+1}-2)\gamma}) \mathbb{P}^0(|X_u| \leq 2 \cdot 2^{(n_{k+1}-2)\gamma}) du dt \\
&\leq 2 \int_{2^{n_{k+1}-1}}^{2^{n_{k+1}}} \mathbb{P}^0(|X_t| \leq 2^{(n_{k+1}-2)\gamma}) dt \sum_{i=1}^{n_{k+1}} \int_{2^{i-1}}^{2^i} \mathbb{P}^0(|X_u| \leq 2 \cdot 2^{(n_{k+1}-2)\gamma}) du.
\end{aligned}$$

Applying the first moment estimate yields the results. \square

Applying Paley-Zygmund inequality to p_k with $K = C_1/2$, one obtains that for all k ,

$$p_k \geq \frac{\mathbb{E}[M_k]^2}{4\mathbb{E}[M_k^2]} = \frac{C_1}{4C_2} := p_0 > 0.$$

Combining this, (5.12) and (5.13), one obtains that : there exists a positive constant p such that for all k ,

$$\mathbb{P}(A_{k+1} | \mathcal{F}_k) = g(B(2^{n_k+2})) \geq p$$

on the event D_k . This completes the proof for the lower bound.

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