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THÈSE

Pour obtenir le grade de

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Présentée par

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Thèse dirigée par **Greg McSHANE** et **Louis FUNAR**

préparée au sein de **l'Institut Fourier**
et de **l'Ecole Doctorale MSTII**

L'identité de Pleijel hyperbolique, la métrique de pression et l'extension centrale du groupe modulaire via quantification de Chekhov-Fock

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Introduction

Cette thèse consiste en trois parties que j'ai faites pendant ces trois ans.

La première partie (chapitre 2 et 3) va être constituée de l'étude de la distribution de la longueur de corde sur le plan hyperbolique. Elle est motivée par les travaux de Bridgeman et Dumas [18] et Bridgeman [17] où la distribution de la longueur de corde associée à une lamination sur une surface hyperbolique a été étudiée.

Soit D un domaine convexe compact dans le plan hyperbolique \mathbb{H} . On considère l'intersection entre D et une géodésique γ sur \mathbb{H} . On l'appelle la corde de γ par rapport à D . En considérant la longueur hyperbolique de corde, on définit une application ρ_D de $\mathcal{G}_{\mathbb{H}}$, l'ensemble des géodésiques de \mathbb{H} , dans \mathbb{R}^+ . Il y a une mesure μ sur $\mathcal{G}_{\mathbb{H}}$ unique à un facteur multiplicatif près qui est invariante par l'action du groupe d'isométrie de \mathbb{H} sur $\mathcal{G}_{\mathbb{H}}$. On l'appelle la mesure de Liouville. La mesure image de μ par ρ_D est une mesure sur \mathbb{R}^+ qui s'appelle la distribution de la longueur de corde.

L'outil principal est l'identité de Pleijel pour le plan hyperbolique. Sa version usuelle est pour le plan Euclidien. Dans [49], Pleijel a découvert une famille d'identités pour les domaines convexes planaires à bord C^1 . Elles sont associées aux inégalités isopérimétriques. En généralisant ces identités, Ambartzumian a donné l'identité de Pleijel dans [4]. Dans [5], il a donné une preuve combinatoire de cette identité. De plus, il a démontré une version généralisée de cette identité pour un polygone planaire convexe compact. On l'appelle l'identité d'Ambartzumian-Pleijel. Il a également souligné que cette identité peut servir à calculer la distribution de la longueur de corde en utilisant le δ -formalisme.

Soit \mathcal{G}_D l'ensemble de géodésiques passant par D . Avec ces notations, nous démontrons l'identité d'Ambartzumian-Pleijel pour \mathbb{H} énoncée dans le théorème suivant :

Théorème. *Soient D un polygone convexe compact sur \mathbb{H} et $f \in C^1(\mathbb{R}; \mathbb{R})$. Alors on a l'identité d'Ambartzumian-Pleijel suivante :*

$$\int_{\mathcal{G}_D} (f \circ \rho) d\mu = \int_{\mathcal{G}_D} (f' \circ \rho) \sinh \rho \cot \alpha_1 \cot \alpha_2 d\mu + \frac{1}{2} \sum_{i=1}^n \int_0^{|a_i|} f(x) dx,$$

où α_1 et α_2 , pour une géodésique dans \mathcal{G}_D , sont ses angles d'intersection avec le bord de D , dx désigne la mesure de Lebesgue sur \mathbb{R} , et $|a_i|$ est la longueur hyperbolique de la i -ième arête a_i du bord de D .

L'idée de la preuve vient de [19] où Cabo a utilisé la formule de Stokes pour montrer l'identité d'Ambartzumian-Pleijel pour le cas Euclidien.

Ensuite, considérons un domaine à bord C^1 . On peut utiliser les polygones inscrits dedans pour l'approcher. En appliquant le théorème précédent à ces polygones, nous démontrons l'identité de Pleijel pour \mathbb{H} comme suit :

Théorème. *Supposons que ∂D est C^1 . En utilisant les mêmes notations que dans le théorème précédent, alors on a l'identité suivante :*

$$\int_{\mathcal{G}_D} (f \circ \rho_D) d\mu = \int_{\mathcal{G}_D} (f' \circ \rho_D) \sinh \rho_D \cot \alpha_1 \cot \alpha_2 d\mu + \frac{1}{2} f(0) L(\partial D).$$

De plus, si $f(0) = 0$, alors on obtient l'identité de Pleijel pour \mathbb{H} .

En utilisant des fonctions f particulières, nous prouvons les deux corollaires suivants :

Corollaire. *Soit D un polygone convexe compact ou un domaine convexe compact à bord C^1 . La mesure de Liouville de \mathcal{G}_D est égale à la moitié de la longueur de ∂D .*

Corollaire. *Supposons que ∂D est C^1 . Alors on a l'inégalité isopérimétrique hyperbolique:*

$$L(\partial D)^2 \geq 4\pi A(D) + A(D)^2,$$

où l'égalité est réalisée si et seulement si D est un disque dans \mathbb{H} .

Bien que nous ne considérons que les domaines compacts dans les résultats ci-dessus, la stratégie peut s'étendre au cas non-compact. Grâce à cette observation, nous réussissons à calculer la distribution de la longueur de corde associée à un triangle idéal ou un quadrilatère idéal de \mathbb{H} . Les résultats sont énoncé comme suit :

Corollaire. *Soient T un triangle idéal de \mathbb{H} et μ la mesure de Liouville sur $\mathcal{G}_{\mathbb{H}}$. Alors, la distribution de la longueur de corde $dM_T = (\rho_T)_* d\mu$ est donnée par:*

$$dM_T = \frac{3\rho d\rho}{\sinh^2 \rho}.$$

Corollaire. *Soient Q un quadrilatère idéal de \mathbb{H} et μ la mesure de Liouville sur $\mathcal{G}_{\mathbb{H}}$. Soient $\gamma_1, \dots, \gamma_4$ les 4 arêtes de Q ordonnée dans le sens direct. Alors, la distribution de la longueur de corde $dM_Q = (\rho_Q)_* d\mu$ est donnée par :*

$$dM_Q = \frac{12\rho d\rho}{\sinh^2 \rho} + dM_{13} + dM_{24},$$

où dM_{13} est la distribution de la longueur de corde par rapport à γ_1 et γ_3 telle que :

$$\int_0^\rho dM_{13} = \frac{1}{2} \int_{[\eta]} \frac{\cot \alpha_1(\rho, \eta) \cot \alpha_3(\rho, \eta) \sinh \rho \cosh w(\rho, \eta)}{\sinh \rho_1(\rho, \eta) \cot \alpha_1(\rho, \eta) + \sinh \rho_3(\rho, \eta) \cot \alpha_3(\rho, \eta)} d\eta,$$

et dM_{24} est la distribution de la longueur de corde par rapport à γ_2 et γ_4 telle que :

$$\int_0^\rho dM_{24} = \frac{1}{2} \int_{[\eta]} \frac{\cot \alpha_2(\rho, \eta) \cot \alpha_4(\rho, \eta) \sinh \rho \cosh w(\rho, \eta)}{\sinh \rho_2(\rho, \eta) \cot \alpha_2(\rho, \eta) + \sinh \rho_4(\rho, \eta) \cot \alpha_4(\rho, \eta)} d\eta.$$

Remarque. Les notations $\rho_1, \rho_2, \rho_3, \rho_4$ et η dans le théorème ci-dessus sont expliquées dans le Chapitre 3.

D'autre part, nous avons une autre observation concernant les preuves des théorèmes ci-dessus. Les calculs que nous faisons sont basés sur les formules trigonométriques hyperboliques. En utilisant les formules trigonométriques générales, nous obtenons l'identité d'Ambartzumian-Pleijel générale pour une variété riemannienne \mathbb{X}_K en dimension 2 qui est simplement connexe et maximale symétrique et dont la courbure est constante et égale à K :

Théorème. Soit D un polygone convexe compact sur \mathbb{X}_K . Les notations f, α_1, α_2 et x sont les mêmes que dans les théorèmes ci-dessus. Alors on a l'identité d'Ambartzumian-Pleijel :

$$\int_{G_B^K} (f \circ \rho_K) d\mu_K = \int_{G_B^K} (f' \circ \rho_K) \sin_K(\rho_K) \cot \alpha_1 \cot \alpha_2 d\mu_K + \sum_{i=1}^n \int_0^{|\alpha_i|} f(x) dx,$$

où \sin_K est la fonction sinus générale pour \mathbb{X}_K .

Puisque la théorie de Teichmüller est un contexte commun aux deux dernières parties, le chapitre 4 va être consacré à donner une revue de cette théorie. Son point de départ a été le problème de module de Riemann. Soit Σ une surface topologique fermée orientée de genre g . D'après le théorème d'uniformisation de Riemann, il y a une unique structure complexe sur Σ dans chaque classe conforme. Riemann a demandé combien de structures complexes "différentes" existent sur Σ . Dans son article "Theorie der Abel'schen Functionen" (1857), Riemann a calculé le nombre de paramètres d'une classe d'isomorphisme des équations algébriques en deux variables. Chaque classe de ce type d'équations est équivalente à une classe de biholomorphisme de surface de Riemann compacte. L'espace qu'il considérait s'appelle maintenant l'espace de modules de Riemann. Plus tard, Teichmüller a considéré les applications quasiconformes et introduit l'espace de Teichmüller. C'est un espace de classes d'équivalences des structures complexes marquées. Deux structures complexes marquées sont dites équivalentes si et seulement s'il existe une fonction conforme entre eux qui est homotope à l'identité. En utilisant la dilatation d'applications quasiconformes entre les surfaces de Riemann, il a défini une métrique sur l'espace de Teichmüller qu'on appelle la métrique de Teichmüller. En gros, la théorie de Teichmüller est une théorie étudiant l'espace de Teichmüller et les sujets relatifs à cet espace. Les études de cette théorie profitent des idées fondamentales de domaines divers des mathématiques et de la physique en même temps, par exemple : la géométrie hyperbolique, l'analyse complexe, la géométrie algébrique, la théorie des représentations, le système dynamique, la théorie des cordes, etc.

La seconde partie (Chapitre 5) va contribuer aux études de la métrique de pression sur l'espace de Teichmüller $\mathcal{T}(\mathbb{T})$ de \mathbb{T} un tore privé d'un disque et la dégénération de \mathbb{T} quand la longueur du bord va à l'infini.

Soit Γ un groupe hyperbolique au sens de Gromov. La métrique de pression a été introduite dans [15] et définie sur l'espace $\mathcal{C}(\Gamma, m)$ des classes de conjugaison de représentation régulière irréductible convexe de Γ vers $SL(m, \mathbb{R})$. L'ingrédient principal est le formalisme thermodynamique pour un système dynamique symbolique associé à un système dynamique hyperbolique. Il a été développé par Bowen, Parry-Pollicott, Ruelle et d'autres. L'idée de la dynamique symbolique est de remplacer un système dynamique lisse (ou topologique) par un système discret semi-conjugué au dernier. En gros, on associe une partition de Markov au système dynamique lisse et on l'appelle l'ensemble des symboles. Alors une trajectoire dans le système lisse peut s'écrire comme une suite bi-infinie de symboles qui s'appelle le codage symbolique de cette trajectoire. L'espace de ces codages symboliques s'appelle l'espace de décalage. On peut lui associer une application de décalage qui simule la dynamique sur le système lisse. Par l'hyperbolicité d'un système dynamique, on entend que le flot associé à ce système dynamique se décompose en deux parties dont une partie est comprimée et l'autre partie est dilatée le long le flot. Dans ce cas, les informations d'une métrique sur ce système lisse sont traduites en fonctions de Hölder définies sur l'espace de décalage. Un concept important dans le formalisme thermodynamique est la pression d'une fonction de Hölder. Il a plusieurs définitions équivalentes. Une définition utilise l'opérateur de Ruelle associé à une fonction de Hölder qui est un opérateur linéaire borné défini sur l'espace des fonctions de Hölder. La pression est, par définition, le logarithme de la valeur propre la plus grande de l'opérateur de Ruelle. Par le théorème de perturbation, la pression est une fonction analytique. De plus, sa matrice hessienne, qu'on appelle la forme de pression, est semi-définie positive. Elle va devenir une métrique dans certains cas particuliers si l'on peut prouver en plus sa non-dégénération.

Notons par $\partial\Gamma$ le bord de Gromov de Γ et $U_0\Gamma$ le flot géodésique associé à Γ . Dans sa thèse, Sambarino a défini et puis étudié les représentations convexes de Γ dans $PSL(m, \mathbb{R})$. Ce type de représentation est une généralisation naturelle des représentations hyperconvexes étudiées par Labourie dans [42]. En combinant avec la propriété d'Anosov, on obtient l'objet principal dans [15], la représentation convexe Anosov de Γ dans $SL(m, \mathbb{R})$. Dans [15], les auteurs ont montré que à chaque telle représentation, on peut lui associer un flot transitif métrique Anosov $U_\rho\Gamma$ qui est une reparamétrisation Hölder de $U_0\Gamma$. D'après les travaux de Bowen dans [13, 14], le flot $U_\rho\Gamma$ admet un codage de Markov qui est plus fort qu'un codage symbolique.

Remark 0.0.1. *Le codage de Markov existe pour $U_\rho\Gamma$. Dans [22], Coornaert et Papadopoulos ont montré qu'il existe un codage symbolique pour $U_0\Gamma$. Mais ce codage n'est pas injectif dans un ensemble assez large pour qu'on puisse utiliser le formalisme thermodynamique.*

Ce codage induit un espace de décalage X tel que le flot $U_\rho\Gamma$ est identifié avec un flot suspension X_f défini par une fonction Hölder $f \in C^\alpha(X)$.

Notons h l'entropie topologique de $U_\rho(\Gamma)$. D'après le formalisme thermodynamique, la fonction $-hf$ a pression nulle. La correspondance entre ρ et $-hf$ induit l'application thermodynamique \mathcal{I} qui plonge l'espace des représentations $\mathcal{C}(\Gamma, m)$ dans l'espace des fonctions de Hölder avec pression nulle défini sur X où la forme de pression vit. En prouvant le fait que le tiré en arrière de la forme de pression par \mathcal{I} est non-dégénérée, on obtient une métrique sur la partie régulière de $\mathcal{C}(\Gamma, m)$. En particulier, on considère le cas où Γ est le groupe fondamental $\pi_1(\Sigma)$ d'une surface Σ fermée orientée de genre $g > 1$. En relevant $\mathrm{PSL}(m, \mathbb{R})$ à $\mathrm{SL}(m, \mathbb{R})$, la composante de Hitchin $H_m(\Sigma)$ peut se plonger dans $\mathcal{C}(\Gamma, m)$. Par conséquent, la restriction de la métrique de pression nous donne une métrique riemannienne sur $H_m(\Sigma)$ qui est invariante par l'action du groupe modulaire de Σ . De plus, la restriction de la métrique de pression sur le lieu fuchsien coïncide avec la métrique de Weil-Petersson. Le dernier fait est induit par les travaux de Bonahon [9] et Wolpert [61], d'après lesquels on peut prouver que la métrique riemannienne de Thurston définie sur l'espace de Teichmüller d'une surface fermée est équivalente à la métrique de Weil-Petersson et la métrique riemannienne de Thurston est exactement la métrique de pression dans le cas où $m = 2$.

La métrique de pression est aussi bien définie sur l'espace de Teichmüller d'une surface à bord. Mais dans ce cas, on ne sait pas encore si elle est équivalente à la métrique de Weil-Petersson. Dans le Chapitre 5, nous allons donner une approche pour répondre cette question. Bien que notre méthode ne donne pas de réponse définitive, nous obtenons quand même certaines informations intéressantes sur la métrique de pression et la fonction d'entropie. Dans ce chapitre, nous n'allons considérer que le cas du tore privé d'un disque \mathbb{T} , mais il est facile d'étendre les résultats aux cas où les surfaces à bord sont plus générales.

L'étape principale est de décrire la dégénération de \mathbb{T} à son graphe de ruban \mathbb{G} . Il y a déjà plusieurs façons de paramétriser cette dégénération. Mais pour voir plus clair dans le point de vue du système dynamique, nous allons introduire un "nouveau" système de coordonnées de l'espace de Teichmüller $\mathcal{T}(\mathbb{T})$ en utilisant les orthogodésiques. En fait, l'idée de cette construction a déjà existé dans [57]. Dans ces coordonnées, l'espace $\mathcal{T}(\mathbb{T})$ s'identifie avec $(\mathbb{R}^+)^3$. Un chemin de dégénération correspond à une demi-droite commençant d'origine de $(\mathbb{R}^+)^3$. En renormalisant les métriques associées aux points de cette demi-droite, nous montrons qu'une métrique sur \mathbb{G} peut se voir comme la limite projective ces métriques hyperboliques sur \mathbb{T} .

Ensuite, nous observons que le flot géodésique sur \mathbb{T} et celui sur \mathbb{G} ont le même codage de Markov. Par conséquent, les métriques sur \mathbb{T} et celles sur \mathbb{G} sont identifiées avec les fonctions de Hölder sur le même espace de décalage par une fonction thermodynamique \mathcal{I} . Soit $\mathcal{M}(\mathbb{G}, 1)$ l'espace des métriques sur \mathbb{G} avec la longueur totale égale à 1 et nous les appelons les métriques renormalisées. La dégénération au-dessus nous montre que $\mathcal{I}(\mathcal{M}(\mathbb{G}, 1))$ est une partie du bord de $\mathcal{I}(\mathcal{T}(\mathbb{T}))$. De l'autre côté, nous montrons que la construction de la forme de pression ne dépend pas de la renormalisation de la métrique. Ce fait implique que la forme de pression sur $\mathcal{M}(\mathbb{G}, 1)$ est la même que celle sur $\mathcal{M}_1(\mathbb{G})$

qui est étudiée par Sharp et Pollicott dans [51]. D'après leur travaux, nous montrons que la forme de pression sur $\mathcal{M}(\mathbb{G}, 1)$ est une métrique. En résumé, nous obtenons un espace connexe par arc $\mathcal{I}(\mathcal{T}(\mathbb{T}) \sqcup \mathcal{M}(\mathbb{G}, 1))$ où la forme de pression est bien définie et semi-définie positive et en plus ses restrictions sur $\mathcal{I}(\mathcal{T}(\mathbb{T}))$ et $\mathcal{I}(\mathcal{M}(\mathbb{G}, 1))$ sont non-dégénérées.

Une autre façon d'interpréter la métrique de pression dans [15] est d'utiliser la fonction d'intersection renormalisée $J : \mathcal{C}(\Gamma, m)^2 \rightarrow \mathbb{R}$. Cette fonction est une généralisation de la fonction d'intersection de Thurston définie pour l'espace de Teichmüller. Notons que la fonction d'entropie est utilisée dans la définition de J . Nous étudions aussi cette fonction et en particulier ses restrictions sur les feuilles symplectiques $\mathcal{T}(\Sigma_{g,r}, L_1, \dots, L_r)$ d'une surface à bord $\Sigma_{g,r}$. Nous prouvons le résultat suivant:

Proposition. *Soit $\Sigma = \Sigma_{g,r}$ une surface de genre $g > 0$ avec $r > 0$ composantes de bord. Si $g > 1$ ou $r > 1$, alors la fonction d'entropie n'est pas constante sur les feuilles symplectiques $\mathcal{T}(\Sigma, L_1, \dots, L_r)$; si $(g, r) = (1, 1)$, alors la fonction d'entropie n'est pas constante sur les feuilles symplectiques $\mathcal{T}(\Sigma, L_1)$ avec la longueur du bord L_1 assez grande.*

La preuve de cette proposition pour $\Sigma \neq \mathbb{T}$ vient d'un exemple construit par McMullen dans [44]. Cet exemple a été utilisé pour montrer qu'il existe une suite de groupes kleiniens qui converge géométriquement telle que la suite des dimensions de Hausdorff de leurs ensembles limites ne converge pas. Cet exemple implique qu'il existe une suite dans chaque feuille symplectique $\mathcal{T}(\Sigma, L_1, \dots, L_r)$ telle que les dimensions de Hausdorff des ensembles limites convergent vers 1. Par les travaux de Sullivan, la dimension de Hausdorff de l'ensemble limite d'un groupe fuchsien est égale à l'entropie topologique du flot géodésique associé à ce groupe. En même temps, l'entropie d'un groupe Fuchsien associé à une surface hyperbolique à bord est strictement plus petite que 1. Donc, la proposition pour $\Sigma \neq \mathbb{T}$ en découle. Pour le cas où $\Sigma = \mathbb{T}$, nous utilisons l'analyticité de la fonction de pression. Par le Théorème des Fonctions Implicites, la fonction d'entropie est aussi analytique. Par la dégénération décrite ci-dessus, les \mathcal{I} -images des feuilles symplectiques convergent vers la \mathcal{I} -image de $\mathcal{M}(\mathbb{G}, 1)$ quand la longueur du bord tend vers l'infini. La fonction d'entropie sur $\mathcal{M}(\mathbb{G}, 1)$ n'est pas constante par les calculs dans [51]. Ceci implique que la fonction d'entropie n'est pas constante sur les feuilles symplectiques telles que la longueur du bord associée est assez grande.

Finalement, la troisième partie (Chapitre 6) concerne le problème de quantification de l'espace de Teichmüller. La motivation initiale a été de comprendre la gravité quantique en dimension $2 + 1$. La théorie de Teichmüller quantique a été développée par Chekhov et Fock [20] et Kashaev [38] indépendamment et ensuite généralisée aux groupes de Lie de rang supérieur et aux algèbres amassées par Fock et Goncharov dans [25] et [26]. L'ingrédient principal est le dilogarithme quantique de Faddeev introduit par Faddeev dans [23]. Dans ce chapitre, nous nous intéressons à l'extension centrale du groupe modulaire via la quantification de Chekhov-Fock et notamment sa classe de cohomologie.

Cette partie est motivée par un résultat de Funar et Kashaev dans [27] où le même problème pour la quantification de Kashaev a été étudié.

Soient V un espace vectoriel et G un groupe. Une représentation projective de G sur V est un homomorphisme de G dans $\mathrm{PGL}(V)$. Il est connu que ce type d'homomorphisme est équivalent à une représentation sur V d'une extension centrale de G par \mathbb{C}^* . Plus précisément, soit h une représentation projective de G sur V . Soit \tilde{G} une extension centrale de G par un sous-groupe A de \mathbb{C}^* . Nous pouvons associer à \tilde{G} une représentation \tilde{h} de \tilde{G} sur V tel que le diagramme ci-dessous commute:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathrm{GL}(V) & \longrightarrow & \mathrm{PGL}(V) \longrightarrow 1 \\ & & \uparrow & & \uparrow \tilde{h} & & \uparrow h \\ 1 & \longrightarrow & A & \longrightarrow & \tilde{G} & \longrightarrow & G \longrightarrow 1 \end{array}$$

En particulier, soit \tilde{G}_0 une extension centrale de G par \mathbb{C}^* qui est le tiré en arrière de $\mathrm{GL}(V) \rightarrow \mathrm{PGL}(V)$ par h . Notons \tilde{h}_0 la représentation de \tilde{G}_0 sur V . Une réduction \tilde{G}_1 de \tilde{G}_0 est une extension centrale de G par un sous-groupe A_1 de \mathbb{C}^* telle que \tilde{G}_1 est un sous-groupe de \tilde{G}_0 et sa représentation associée \tilde{h}_1 est la restriction de \tilde{h}_0 . On dit qu'une réduction \tilde{G}_1 de \tilde{G}_0 est minimale si \tilde{G}_1 est minimale comme sous-groupe par rapport aux autres réductions à isomorphisme près.

Supposons que G est un quotient du groupe libre F de rang n par un sous groupe normal R engendré par un ensemble de relations. Soit \tilde{h} une représentation de F sur V telle que R est envoyé dans le centre de $\mathrm{GL}(V)$. Alors \tilde{h} induit une représentation projective de G sur V . L'homomorphisme \tilde{h} s'appelle une représentation presque linéaire de G sur V , afin de la distinguer d'une représentation projective. Dans cette partie, nous montrons que la quantification de Chekhov-Fock nous aide à construire une représentation presque linéaire du groupe modulaire qui induit une extension centrale de ce dernier.

Considérons la surface Σ_g^s de genre $g \leq 2$ avec $s > 0$ piqûres. Notons par $\mathbf{T}(\Sigma_g^s)$ l'ensemble des triangulations idéales étiquetées. Le groupe modulaire $\mathrm{Mod}(\Sigma_g^s)$ agit librement sur cet ensemble. On peut définir le groupoïde de Ptolemy comme une catégorie dont les objets sont les $\mathrm{Mod}(\Sigma_g^s)$ -orbites dans $\mathbf{T}(\Sigma_g^s)$ et les morphismes sont les orbites de l'action diagonale de $\mathrm{Mod}(\Sigma_g^s)$ sur $\mathbf{T}(\Sigma_g^s) \times \mathbf{T}(\Sigma_g^s)$. En particulier, les éléments de $\mathrm{Mod}(\Sigma_g^s)$ correspondent aux automorphismes d'un objet du groupoïde de Ptolemy.

D'après les travaux de Harer [35] et Penner [46, 47], le groupoïde de Ptolemy est aussi engendré par les actions des flips F et des permutations σ des étiquettes sur $\mathbf{T}(\Sigma_g^s)$. Ce résultat utilise des considérations sur les relations commutatives et les relations de pentagone entre les flips et les relations naturelles entre les deux types des actions. Les coordonnées de décalage associées à une triangulation idéale étiquetée induisent une $*$ -algèbre $A(T)$ munie d'un crochet de Poisson. Les flips et les permutations agissent comme les $*$ -isomorphismes entre les $A(T)$. La formule d'un $*$ -isomorphisme associé à un flip est donnée par la formule de changement des coordonnées de décalage associée au même flip. Ensuite, en déformant le crochet de Poisson, on obtient une famille d' $*$ -

algèbres $A^{\hbar}(T)$ qui dépend d'un paramètre \hbar positif réel pour chaque T . De plus, les $*$ -isomorphismes changent utilisant le dilogarithme de Faddeev. Par conséquent, on obtient une famille de foncteurs β^{\hbar} du groupoïde de Ptolemy dans la catégorie des $*$ -algèbres. Les foncteurs β^{\hbar} s'appellent la quantification de l'espace de Teichmüller. L'algèbre $A^{\hbar}(T)$ est isomorphe à une sous-algèbre de l'algèbre de Heisenberg H_n avec n égal au nombre des arcs dans T . La représentation intégrable irréductible ρ de H_n sur $\mathcal{H} = L^2(\mathbb{R}^{2n}, \mathbb{R})$ induit une représentation ρ de $A^{\hbar}(T)$ sur \mathcal{H} . D'après le théorème de Stone von Neumann, cette représentation est unique. Cette unicité induit l'existence d'un intertwiner entre deux représentations $\rho(A^{\hbar}(T))$ et $\rho^{\hbar}(T')$, ce qui nous donne une représentation presque linéaire du groupe modulaire. En particulier, les relations de pentagone deviennent les relations de pentagone quantique où un scalaire unitaire sort. En considérant les deux présentations du groupoïde de Ptolemy, l'action associée à un twist de Dehn peut s'écrire comme une composition des actions d'une suite de flips et de permutations. Relevant les flips et les permutations par le intertwiner, on obtient le relevé d'un twist de Dehn qui est un des générateurs du groupe modulaire. Le relevé d'un twist de Dehn obtenu de cette manière n'est pas unique. Nous montrons qu'en choisissant les relevés des twists de Dehn soigneusement, nous obtenons une extension centrale du groupe modulaire avec la présentation suivante :

Proposition. *En utilisant la quantification de Chekhov-Fock, nous obtenons une extension centrale du groupe modulaire Γ avec la présentation suivante:*

(1) *Générateurs:*

- (a) *Un élément central $w = z^{-12}$, où z est la constante qui vient de la quantification de Chekhov-Fock;*
- (b) *Un élément \tilde{D}_a associé à chaque twist de Dehn le long d'une courbe non-séparante.*

(2) *Relations:*

- (a) *La relation de tresse de type-0 : $\tilde{D}_a \tilde{D}_b = \tilde{D}_b \tilde{D}_a$;*
- (b) *La relation de tresse de type-1 : $\tilde{D}_a \tilde{D}_b \tilde{D}_a = \tilde{D}_b \tilde{D}_a \tilde{D}_b$;*
- (c) *La relation de Lantern : $\tilde{D}_{a_0} \tilde{D}_{a_1} \tilde{D}_{a_2} \tilde{D}_{a_3} = \tilde{D}_{a_{12}} \tilde{D}_{a_{23}} \tilde{D}_{a_{13}}$;*
- (d) *La relation de chaîne : $(\tilde{D}_a \tilde{D}_b \tilde{D}_c)^4 = w^{12} \tilde{D}_e \tilde{D}_f$;*
- (e) *La relation de piqûre : $\tilde{D}_{a_1} \tilde{D}_{a_2} \tilde{D}_{a_3} = w \tilde{D}_{a_{12}} \tilde{D}_{a_{23}} \tilde{D}_{a_{13}}$;*
- (f) *Si w est une racine unitaire d'ordre N , alors $w^N = 1$.*

La classe de cohomologie d'une extension centrale du groupe modulaire est caractérisé par un 2-cocycle. Le changement des relèvements des twists de Dehn ne change pas ce 2-cocycle, donc il ne change pas la classe de cohomologie. En utilisant l'extension centrale ci-dessus, nous montrons le théorème suivant :

Théorème. Soient $g \geq 2$ et $s \geq 4$. La réduction minimale de $\widetilde{\Gamma(\Sigma_g^s)}$ est obtenue par extension de $\Gamma(\Sigma_g^s)$ par A qui est un sous-groupe cyclique de \mathbb{C}^* engendré par z^{-12} . De plus, sa classe de cohomologie est

$$c_{\Gamma(\Sigma_g^s)} = 12\chi + \sum_{i=1}^s e_i \in H^2(\Gamma(\Sigma_g^s), A)$$

où χ est un quart de la classe de Meyer et e_i est la classe d'Euler associée à i -ième piqure.

Remarquons que cette extension centrale est dans la même classe de cohomologie que celle obtenue en utilisant la quantification de Kashaev dans [27] par Funar et Kashaev, bien qu'il n'existe pas de morphisme équivariant évident entre les deux quantifications.

Chapter 1

Introduction

This thesis consists of three parts corresponding to the three subjects that I have studied during the last three years.

The first part (Chapters 2 and 3) contains the study of the chord length distribution associated to a compact (or non-compact) domain in the hyperbolic plane. This is motivated by the work of Bridgeman and Dumas [18] and Bridgeman [17] where the chord length distribution associated to a lamination on a hyperbolic surface is studied.

More precisely, let D be a compact convex domain in the hyperbolic plane \mathbb{H} . We consider the intersection between D and a hyperbolic geodesic γ which we call the *chord* of γ with respect to D . Let $\mathcal{G}_{\mathbb{H}}$ denote the space of geodesics in \mathbb{H} . The hyperbolic length of the chord defines a function ρ_D from the set \mathcal{G}_D of geodesics in \mathbb{H} intersecting D to \mathbb{R}^+ . There is a unique measure up to scalar multiplication μ on $\mathcal{G}_{\mathbb{H}}$ which is invariant under the action of the isometry group of \mathbb{H} , called *Liouville measure*. We consider the restriction of μ to \mathcal{G}_D and push it forward by ρ_D . Then we obtain a distribution on \mathbb{R}^+ , called the *chord length distribution*.

The main tool in this part is the hyperbolic version of Pleijel's identity. The original Pleijel's identity is defined for the Euclidean plane. In [49], Pleijel discovered a family of identities associated with isoperimetric inequalities for planar convex domains with C^1 boundary. By generalizing these identities, Ambartzumian [4] gave the Pleijel's identity for the Euclidean plane. In [5], Ambartzumian gave a combinatorial proof of the Pleijel's identity. Moreover, he proved a general version of the Pleijel's identity for convex compact polygonal planar domains which we call the Ambartzumian-Pleijel identity. He also pointed out that Pleijel's identity can be used to find chord length distribution functions for convex domains by using the δ -formalism.

Back to the hyperbolic case, with the notation above, the Ambartzumian-Pleijel identity for the hyperbolic plane can be stated as follows:

Theorem. *Let D be a compact convex domain in \mathbb{H} whose boundary is a polygon and f be in $C^1(\mathbb{R}; \mathbb{R})$. Then we have the following hyperbolic version of the*

Ambartzumian-Pleijel identity:

$$\int_{\mathcal{G}_D} (f \circ \rho) d\mu = \int_{\mathcal{G}_D} (f' \circ \rho) \sinh \rho \cot \alpha_1 \cot \alpha_2 d\mu + \frac{1}{2} \sum_{i=1}^n \int_0^{|a_i|} f(x) dx,$$

where α_1 and α_2 are the intersection angles between the geodesic and the boundary of D , dx is the length element on \mathbb{R} , and $|a_i|$ is the hyperbolic length of the i -th boundary segment a_i of D .

The idea of the proof comes from [19] where Cabo used the Stokes' formula to prove the original Ambartzumian-Pleijel identity.

By using inscribed polygonal domains to approximate a convex compact domain with C^1 boundary, we prove the hyperbolic version of Pleijel's identity:

Theorem. *Suppose that ∂D is C^1 . With the same notation as above, we have the following identity:*

$$\int_{\mathcal{G}_D} (f \circ \rho_D) d\mu = \int_{\mathcal{G}_D} (f' \circ \rho_D) \sinh \rho_D \cot \alpha_1 \cot \alpha_2 d\mu + \frac{1}{2} f(0) L(\partial D).$$

In particular if $f(0) = 0$, then we have the hyperbolic version of the Pleijel's identity.

By choosing D and f carefully, we prove the two following corollaries:

Corollary. *The Liouville measure of \mathcal{G}_D is one half of the length of the boundary of D .*

Corollary. *Suppose that ∂D is C^1 . Then we have the hyperbolic isoperimetric inequality:*

$$L(\partial D)^2 \geq 4\pi A(D) + A(D)^2,$$

where the equality holds if and only if D is a disk in \mathbb{H} .

A priori, the Pleijel's identity and the Ambartzumian-Pleijel identity only hold for a compact domain D . But the strategy of the proof can be extended to the non compact case so that we are able to compute the chord length distributions for an ideal triangle and an ideal quadrilateral as follows:

Corollary. *Let T be an ideal triangle in \mathbb{H} and μ be the Liouville measure on $\mathcal{G}_{\mathbb{H}}$. The chord length distribution $dM_T = (\rho_T)_* d\mu$ is given by:*

$$dM_T = \frac{3\rho d\rho}{\sinh^2 \rho}.$$

Corollary. *Let Q be an ideal quadrilateral in \mathbb{H} and μ be the Liouville measure on $\mathcal{G}_{\mathbb{H}}$. Let $\gamma_1, \dots, \gamma_4$ be the 4 edges of Q ordered counter-clockwise. The chord length distribution $dM_Q = (\rho_Q)_* d\mu$ is given by:*

$$dM_Q = \frac{12\rho d\rho}{\sinh^2 \rho} + dM_{13} + dM_{24},$$

where dM_{13} is the chord length distribution with respect to γ_1 and γ_3 and satisfies:

$$\int_0^\rho dM_{13} = \frac{1}{2} \int_{[\eta]} \frac{\cot \alpha_1(\rho, \eta) \cot \alpha_3(\rho, \eta) \sinh \rho \cosh w(\rho, \eta)}{\sinh \rho_1(\rho, \eta) \cot \alpha_1(\rho, \eta) + \sinh \rho_3(\rho, \eta) \cot \alpha_3(\rho, \eta)} d\eta,$$

and dM_{24} is the chord length distribution with respect to γ_2 and γ_4 and satisfies:

$$\int_0^\rho dM_{24} = \frac{1}{2} \int_{[\eta]} \frac{\cot \alpha_2(\rho, \eta) \cot \alpha_4(\rho, \eta) \sinh \rho \cosh w(\rho, \eta)}{\sinh \rho_2(\rho, \eta) \cot \alpha_2(\rho, \eta) + \sinh \rho_4(\rho, \eta) \cot \alpha_4(\rho, \eta)} d\eta,$$

where η is the angle parameter in the polar parametrization of the set of geodesics in \mathbb{H} introduced later.

Another observation about the proof of the Ambartzumian-Pleijel identity is that as passing from the Euclidean case to the hyperbolic case, the only thing changed is the trigonometric functions. As a result of this observation, by using the general trigonometric functions, we obtain the general Ambartzumian-Pleijel identity for a maximally symmetric, simply connected, 2-dimensional Riemannian manifold with constant sectional curvature $K \in \mathbb{R}$, denoted by \mathbb{X}_K :

Theorem. *Let D be a convex compact domain in \mathbb{X}_K with geodesic polygon boundary and f be in $C^1(\mathbb{R}; \mathbb{R})$. Then we have the general Ambartzumian-Pleijel identity:*

$$\int_{\mathcal{G}_D^K} (f \circ \rho_K) d\mu_K = \int_{\mathcal{G}_D^K} (f' \circ \rho_K) \sin_K(\rho_K) \cot \alpha_1 \cot \alpha_2 d\mu_K + \sum_{i=1}^n \int_0^{|a_i|} f(x) dx,$$

where \sin_K is the general sinus function for \mathbb{X}_K , α_1 and α_2 are the intersection angles between the geodesic and the boundary of D , dx is the length element on \mathbb{R} , and $|a_i|$ is the length of the i -th boundary segment a_i of D .

Then in the same way as above, the Pleijel's identity for \mathbb{X}_K follows by which we prove the isoperimetric inequality for \mathbb{X}_K .

To appreciate the remaining two parts of the thesis, one must have a background of Teichmüller theory, thus we use Chapter 4 to give a review of this theory. The story of Teichmüller theory begins with Riemann's moduli problem. Let Σ be a closed oriented topological surface with genus $g > 1$. By the Uniformization Theorem, there is a unique complex structure on Σ in each conformal class. Riemann asked how many "different" complex structures that we can put on a fixed Σ . In his paper of "Theorie der Abel'schen Functionen" (1857), Riemann counted the number of parameters of isomorphism classes of algebraic equations in two variables which is equivalent to the biholomorphism classes of compact Riemann surfaces and found that the degree of freedom is $3g - 3$. The space that he considered is called *Riemann's moduli space*. Later Teichmüller

considered quasiconformal maps between Riemann surfaces and introduced *Teichmüller space* as the space of equivalent classes of complex structures up to homotopy class of conformal maps. He also defined a natural metric on this space for which the distance between two points is one half of the logarithm of the dilatation of the quasiconformal map associated to these two points. We call it the *Teichmüller metric*. The Teichmüller theory is a mathematics subject where Teichmüller space is studied. This wonderful subject brings together fundamentals ideas from different fields, both in mathematics and physics, such as hyperbolic geometry, complex analysis, algebraic geometry, representation theory, dynamical system, string theory, etc. Over the last half century, many beautiful results have been proved in these fields.

The second part of this thesis (Chapter 5) consists of the study of the pressure metric on the Teichmüller space $\mathcal{T}(\mathbb{T})$ of one-holed torus \mathbb{T} and the degeneration of \mathbb{T} as the boundary length goes to infinity.

The pressure metric is constructed in [15] for the space $\mathcal{C}(\Gamma, m)$ of conjugacy classes of regular irreducible convex representations of a word hyperbolic group Γ to $SL(m, \mathbb{R})$ using Thermodynamic formalism.

The Thermodynamic Formalism that we use was developed by Bowen, Parry-Pollicott, Ruelle and others for a symbolic dynamical system associated to a hyperbolic dynamical system. The idea of symbolic dynamics is to replace a smooth (or topological) dynamical system by a discrete system semi-conjugate to the former. Roughly speaking, one associates a finite Markov partition to a smooth dynamical system, called the set of symbols. Then the trajectory of the smooth dynamical system can be described as a bi-infinite sequence of symbols. The space of all such sequence is called the shift space with a shift map corresponding to the dynamics on trajectories. By hyperbolic, we mean that there exist the expanding and contracting directions along the flow associated to the smooth dynamical system. Under this setting, the metric information of the smooth dynamical system can be given by a Hölder function defined on the shift space and the flow associated to the smooth dynamical system is identified with a suspension flow of the shift space defined by this Hölder function. An important concept in this theory is the pressure of a Hölder function on a shift space. It has several equivalent definitions. One of them is given by the logarithm of the top eigenvalue of the Ruelle operator which is a bounded linear operator defined on the space of Hölder functions on the shift space. By sending each Hölder function to its pressure, we define the pressure function. The space of Hölder function on the shift space is a filtration of a sequence of Banach spaces. By using the perturbation theorem on each of these Banach space, one proves the analyticity of the pressure function on the Hölder function space. Moreover, restricting the pressure function to the pressure zero function space, one obtains that the Hessian of the pressure form is positive semi-definite. This is a good candidate for being a metric.

Back to our case, let $\partial\Gamma$ be the Gromov boundary and $U_0\Gamma$ be the Gromov geodesic flow associated to Γ . In his thesis, Sambarino defined and studied the convex representations of Γ into $PSL(m, \mathbb{R})$ which are a natural generalization of hyperconvex representations studied by Labourie [42]. In [42], Labourie

also studied the Anosov representation. Combining them together, one obtains the main object considered in [15], the convex Anosov representation of Γ in $\mathrm{SL}(m, \mathbb{R})$. In [15], the authors proved that to each such representation ρ , one can associate a transitive metric Anosov flow $U_\rho\Gamma$ which is a Hölder reparametrization of the Gromov geodesic flow $U_0\Gamma$. By a theorem of Bowen in [13, 14], one can find an Markov coding for $U_\rho\Gamma$.

Remark 1.0.2. *This coding is for $U_0\Gamma$ equipped with a convex Anosov representation. In [22], Coornaert and Papadopoulos showed that for a hyperbolic group, there exists a symbolic coding for its Gromov geodesic flow. But this coding is not one to one on a set large enough to apply the thermodynamic formalism.*

This coding induces a shift space X such that the flow on $U_\rho\Gamma$ is identified with a suspension flow X_f defined by some Hölder function $f \in C^\alpha(X)$. Denote by h the topological entropy of $U_\rho\Gamma$. By thermodynamic formalism, the Hölder function $-hf$ has pressure zero. The correspondence between ρ and $-hf$ induces the thermodynamic map \mathcal{I} which embeds the representation space $\mathcal{C}(\Gamma, m)$ to the space of pressure zero functions defined on X . This latter space comes equipped with a non-negative definite 2-form, called the *pressure form*. By pulling back the pressure form by \mathcal{I} and proving its non-degeneracy, one obtains a metric on the regular part of $\mathcal{C}(\Gamma, m)$. In particular, we take Γ to be the fundamental group $\pi_1(\Sigma)$ of a closed surface Σ . The Hitchin component $H_m(\Sigma)$ can be lifted to the regular part of $\mathcal{C}(\pi_1(\Sigma), m)$ by considering the lifts of $\mathrm{PSL}(m, \mathbb{R})$ in $\mathrm{SL}(m, \mathbb{R})$. Thus the restriction of the pressure metric on the lift induces a mapping class group invariant Riemannian metric on the Hitchin component $H_m(\Sigma)$. Moreover, when the pressure metric is restricted to the Fuchsian locus, it coincides with Thurston's Riemannian metric which is equivalent to the Weil-Petersson metric up to a constant factor by the work of Wolpert [61] and Bonahon [9]. In particular, when $m = 2$ the pressure metric on Teichmüller space is different from the Weil-Petersson metric by a constant factor.

It is true that their construction of pressure metric also works for the Teichmüller space of the bordered surface. However, in this setting we do not know whether it is still true that the pressure metric is equivalent to the Weil-Petersson metric. In Chapter 5, we give an approach to answer this question. Although our method does not give a definitive answer, we still obtain some interesting information of the pressure metric and the entropy function. We remark that we only consider the one-holed torus as an example and it is not hard to be extended to other bordered surfaces.

The main tool that we use here is the degeneration of \mathbb{T} to its fat graph \mathbb{G} . There are several obvious ways to parametrize this degeneration. But in order to describe the asymptotic of the geodesic flow during the degeneration in the point of view of symbolic dynamical system, we use a new way by introducing a "new" coordinate system of Teichmüller space $\mathcal{T}(\mathbb{T})$ using orthogeodesics. In fact, this idea of using orthogeodesics to define coordinates has already existed in [57]. Using this coordinate system, Teichmüller space $\mathcal{T}(\mathbb{T})$ can be identified with $(\mathbb{R}^+)^3$. Moreover, sequences of degenerations that we consider correspond

to rays starting from the origin in these coordinates. We renormalize a marked hyperbolic structure by rescaling it so that the associated length of boundary become 1. By renormalizing all points in $\mathcal{T}(\mathbb{T})$, we obtain the renormalized Teichmüller space. We show that the renormalization of the sequence of the degeneration converges to a metric on the associated fat graph.

Then we observe that the geodesic flow on \mathbb{T} and that on \mathbb{G} have the same Markov coding. This means that the metric on \mathbb{T} and that on \mathbb{G} can be seen as Hölder functions on the same shift space. Let $\mathcal{M}(\mathbb{G}, 1)$ denote the space of metrics on \mathbb{G} with total length 1. The above degeneration shows that $\mathcal{I}(\mathcal{M}(\mathbb{G}, 1))$ is a partial boundary of \mathcal{I} -image of the renormalized Teichmüller space. On the other hand, by a simple argument, we show that the construction of pressure metric does not depend on the renormalization of the metric. This implies that the renormalized Teichmüller space is isometric to $\mathcal{T}(\mathbb{T})$ with respect to the pressure metric and the moduli space $\mathcal{M}(\mathbb{G}, 1)$ is isometric to $\mathcal{M}_1(\mathbb{G})$ studied by Sharp and Pollicott in [51]. Thus we obtain a path connected space $\mathcal{I}(\mathcal{T}(\mathbb{T}) \sqcup \mathcal{M}_1(\mathbb{G}))$ where the pressure form is well defined and positive semi-definite and its restrictions to $\mathcal{I}(\mathcal{T}(\mathbb{T}))$ and to $\mathcal{I}(\mathcal{M}(\mathbb{G}, 1))$ are both positive definite.

Another result from [15] states that the pressure form can be interpreted as the Hessian of the renormalized intersection function $J : \mathcal{C}(\Gamma, m)^2 \rightarrow \mathbb{R}^+$. The function J is a generalization of Thurston's intersection function defined for Teichmüller space of a closed surface. In the definition of J , the entropy function is also involved. Thus, in this chapter, we also study the entropy function and in particular its property on each symplectic leaf $\mathcal{T}(\Sigma_{g,r}, L_1, \dots, L_r)$ of Teichmüller space $\mathcal{T}(\Sigma_{g,r})$ of a bordered surface Σ . We prove that:

Proposition. *Let $\Sigma = \Sigma_{g,r}$ be a bordered oriented surface of genus $g > 0$ with $r > 0$ boundary components and $\chi(\Sigma) < 0$. If $\Sigma \neq \mathbb{T}$, then the restrictions of the entropy function on symplectic leave $\mathcal{T}(\Sigma, L_1, \dots, L_r)$ are not constant; if $\Sigma = \mathbb{T}$, then the entropy function is not constant on symplectic leaves of $\mathcal{T}(\mathbb{T})$ whose associated boundary length is large enough.*

The proof of this proposition for $\Sigma \neq \mathbb{T}$ is given by repeating the construction of an example of McMullen in [44] where he used this example to prove that the geometrical convergence of a sequence of Kleinian groups does not guarantee the convergence of the Hausdorff dimensions of their limit sets. This example implies that there exists a sequence in each symplectic leaf $\mathcal{T}(\Sigma, L_1, \dots, L_r)$ such that the Hausdorff dimensions of their limit sets converges to 1. By the work of Sullivan, the Hausdorff dimension of the limit set of a Fuchsian group equals to the topological entropy of the associated geodesic flow. On the other hand, we know that the topological entropy of a geodesic flow associated to a bordered hyperbolic surface is strictly smaller than one. By combining all these results, we conclude our proposition for $\Sigma \neq \mathbb{T}$. For $\Sigma = \mathbb{T}$, we use the analyticity of the entropy function obtained from the analyticity of the pressure function by using implicit function theorem. Then by the degeneration described above, we see that the \mathcal{I} -image of the symplectic leaf converge to the \mathcal{I} -image of the moduli space of metric graph as the boundary length goes to infinity. The entropy

function on the latter space has an explicit formula by Sharp and Pollicott [51] by which we see that the entropy function is not constant on the moduli space of metric graph $\mathcal{M}(\mathbb{G}, 1)$. Thus the entropy function cannot be constant when the boundary length associated to the symplectic leaf is large enough.

Finally, the third part concerns another interesting problem related to the Teichmüller space, namely its quantization. It is motivated by understanding the quantum gravity in dimension 2+1. The quantum Teichmüller theory was developed by Chekhov and Fock [20] and Kashaev [38] independently, and then generalized to the higher rank Lie groups and cluster algebras by Fock and Goncharov in [25] and [26]. The main ingredient of both the constructions is Faddeev's quantum dilogarithm introduced by Faddeev in [23]. In this chapter, we are interested in the central extension of mapping class group coming from the Chekhov-Fock quantization and in particular we compute the cohomology class of central extensions of mapping class group coming from the Chekhov-Fock quantization. This work is motivated by the result in [27] where Funar and Kashaev studied the central extension of mapping class group coming from Kashaev's quantization. We remark that the Chekhov-Fock quantization, as well as the Kashaev quantization, is the infinite dimensional quantum Teichmüller theory. Meanwhile there is also the finite dimensional quantum Teichmüller theory. It has been developed by Bonahon and his collaborators (see [11] and [12]) where a problem analogue to that we consider in this part was studied.

More precisely, we consider the surface Σ_g^s of genus $g \geq 2$ with $s > 0$ punctures and $\chi(\Sigma_g^s) < 0$. Let $\mathbf{T}(\Sigma_g^s)$ be the set of all labeled ideal triangulation T . The mapping class group $\text{Mod}(\Sigma_g^s)$ acts on it freely. The Ptolemy groupoid associated to Σ_g^s is a category whose objects are the $\text{Mod}(\Sigma_g^s)$ -orbits in $\mathbf{T}(\Sigma_g^s)$ and morphisms are the orbits of diagonal $\text{Mod}(\Sigma_g^s)$ -action in $\mathbf{T}(\Sigma_g^s) \times \mathbf{T}(\Sigma_g^s)$. In particular, elements of $\text{Mod}(\Sigma_g^s)$ corresponds to automorphisms of an object in the Ptolemy groupoid.

By the work of Harer [35] and Penner [46, 47], the Ptolemy groupoid can also be generated by flips F and permutations σ of labels with commutative relations and pentagon relations between flips and natural relations between flips and permutations of labels. The shearing coordinates associated to one labeled ideal triangulation T is equipped with a Poisson bracket. It induces an associative algebra $A(T)$ which is a \mathbb{C} -vector space equipped with a multiplication rule coming from the Poisson bracket. A flip from T to T' acts as an isomorphism between $A(T)$ to $A(T')$. The formula of such isomorphism associated to a flip is given by the formula of change of shearing coordinates associated to this flip. Then by deforming the Poisson bracket, one obtains a family of algebra $A^{\hbar}(T)$ depending on a real positive parameter \hbar from $A(T)$. Moreover, the isomorphism associated to a flip also changes by using Faddeev's dilogarithm. Thus one obtains a family of functors \mathcal{Q}_{\hbar} from Ptolemy groupoid to the category of algebras. These functors are called the *quantization* of Teichmüller space. The algebra $A^{\hbar}(T)$ is isomorphic to a subalgebra of the Heisenberg algebra H_n with n equal to the number of edges in T . The irreducible integrable representation of H_n in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{2n}, \mathbb{R})$ induces a represen-

tation $\rho(A^{\hbar}(T))$ of $A^{\hbar}(T)$ in \mathcal{H} . By the Stone von Neumann Theorem, the isomorphism above between two algebras $A^{\hbar}(T)$ and $A^{\hbar}(T')$ acts as an intertwiner between $e^{i\rho(A^{\hbar}(T))}$ and $e^{i\rho(A^{\hbar}(T'))}$. The intertwiner functor induces an almost linear representation of Ptolemy groupoid. The intertwiners associated to the flips satisfying a pentagon relation satisfy the quantum pentagon relation where a unitary scalar appears. The action associated to each Dehn twist on an ideal triangulation can be rewritten as a composition of a sequence of actions corresponding to flips and permutations. By taking the composition of actions of the intertwiners associated to this sequence, we obtain the lift of each Dehn twist in the central extension of mapping class group. This lift obtained in this way for each Dehn twist is not unique. It depends on the sequence of flips and permutations that we use. Our first result is that by choosing carefully this sequence for each Dehn twist, one can construct the following central extension of mapping class group $\text{Mod}(\Sigma_g^s)$:

Proposition. *By using the Chekhov-Fock quantization, we obtain a central extension of Γ with the following presentation:*

(1) *Generators:*

- (a) *One central element: $w = z^{-12}$, where z is the constant coming from the Chekhov-Fock quantization;*
- (b) *One element \tilde{D}_a associated to each the Dehn twists D_a along all non separating simple closed geodesics a in S .*

(2) *Relations:*

- (a) *The type-0 braid relation: $\tilde{D}_a \tilde{D}_b = \tilde{D}_b \tilde{D}_a$;*
- (b) *The type-1 braid relation: $\tilde{D}_a \tilde{D}_b \tilde{D}_a = \tilde{D}_b \tilde{D}_a \tilde{D}_b$;*
- (c) *The Lantern relation: $\tilde{D}_{a_0} \tilde{D}_{a_1} \tilde{D}_{a_2} \tilde{D}_{a_3} = \tilde{D}_{a_{12}} \tilde{D}_{a_{23}} \tilde{D}_{a_{13}}$;*
- (d) *The chain relation: $(\tilde{D}_a \tilde{D}_b \tilde{D}_c)^4 = w^{12} \tilde{D}_e \tilde{D}_f$;*
- (e) *The puncture relation: $\tilde{D}_{a_1} \tilde{D}_{a_2} \tilde{D}_{a_3} = w \tilde{D}_{a_{12}} \tilde{D}_{a_{13}} \tilde{D}_{a_{23}}$;*
- (f) *If w is a root of unity with order N , then $w^N = 1$.*

A central extension of Mapping class group induces a 2-cocycle describing its cohomology class. Changing lifts of Dehn twists does not change this 2-cocycle. Thus by using the central extension that we obtain above, our main theorem can be stated as follows:

Theorem. *Let $g \geq 2$ and $s \geq 4$. The minimal reduction of $\widetilde{\Gamma(\Sigma_g^s)}$ can be obtained by centrally extending $\Gamma(\Sigma_g^s)$ by A which is a cyclic subgroup of \mathbb{C}^* generated by z^{-12} . Moreover, its cohomology class is*

$$c_{\Gamma(\Sigma_g^s)} = 12\chi + \sum_{i=1}^s e_i \in H^2(\Gamma(\Sigma_g^s), A)$$

where χ is one quarter of the Meyer signature class and e_i is the Euler class associated to the i -th puncture.

We remark that this central extension is in the same cohomology class as the central extension of mapping class group via Kashaev quantization proved by Funar and Kashaev in [27], although there is no obvious equivariant morphism between these two models.

Chapter 2

Hyperbolic geometry

Hyperbolic geometry is a non-Euclidean geometry where the Euclidean parallel postulate is replaced by "more than one parallel lines pass through a point outside a line". The hyperbolic plane is a two-dimensional simply connected Riemannian manifold with constant curvature -1 . There are several models of it. In this chapter, we will introduce three models of hyperbolic plane: the upper half plane model \mathbb{H} , the Poincaré disk model \mathbb{D} and the Minkowski model \mathbb{M} (or the hyperboloid model). The set $\mathcal{G}_{\mathbb{H}}$ of geodesics in the hyperbolic plane will be introduced and we will focus on the unique isometry invariant measure μ on $\mathcal{G}_{\mathbb{H}}$, namely Liouville measure and its different expressions under different parametrizations.

2.1 Upper half plane model

The **upper half plane model** is the set of points in the complex plane \mathbb{C} with strictly positive imaginary part:

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\},$$

equipped with the following metric:

$$ds = \frac{\sqrt{(dx)^2 + (dy)^2}}{y},$$

where the (x, y) are the Cartesian coordinates of \mathbb{C} .

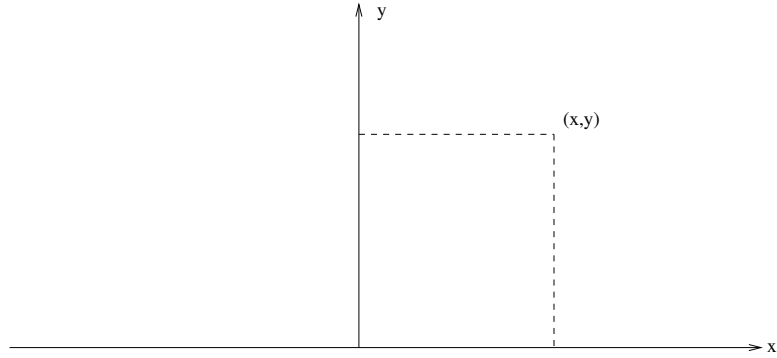


Figure 2.1: Cartesian coordinates

Given two points z and w , the distance $d_{\mathbb{H}}(z, w)$ between them is the following:

$$d_{\mathbb{H}}(z, w) = \log \frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|}. \quad (2.1)$$

The curvature of this metric is -1 . The orientation preserving isometry group of \mathbb{H} is isomorphic to the Lie group $\mathrm{PSL}(2, \mathbb{R})$ whose isometry action on \mathbb{H} is the Möbius transformation: let A be the following matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{R}),$$

the action of A on \mathbb{H} is given by:

$$\begin{aligned} A : \mathbb{H} &\rightarrow \mathbb{H}, \\ z &\mapsto \frac{az + b}{cz + d}. \end{aligned}$$

By considering the absolute value of the trace, the elements A in $\mathrm{PSL}(2, \mathbb{R})$ is classified into three types:

- If $|tr(A)| > 2$, the matrix A is called **hyperbolic**;
- If $|tr(A)| = 2$, the matrix A is called **parabolic**;
- If $|tr(A)| < 2$, the matrix A is called **elliptic**.

The **boundary at infinity** $\partial\mathbb{H}$ of \mathbb{H} can be identified with $\mathbb{R} \cup \infty$. The $\mathrm{PSL}(2, \mathbb{R})$ -action on \mathbb{H} is extended to $\partial\mathbb{H}$ in a natural way.

A geodesic in this model is either the intersection of \mathbb{H} with a circle perpendicular to the horizontal axis, or a vertical half-line. Let $\mathcal{G}_{\mathbb{H}}$ denote the set of non-oriented geodesics in \mathbb{H} . A geodesic $\gamma \in \mathcal{G}_{\mathbb{H}}$ is uniquely determined by its end points on $\partial\mathbb{H}$. This implies that the set $\mathcal{G}_{\mathbb{H}}$ is identified with $((\partial\mathbb{H} \times \partial\mathbb{H}) \setminus \Delta) / \mathbb{Z}_2$ where Δ is the diagonal of $\partial\mathbb{H} \times \partial\mathbb{H}$ and the \mathbb{Z}_2 -action is to exchange two end points of a geodesic. This identification induces a parametrization of $\mathcal{G}_{\mathbb{H}}$.

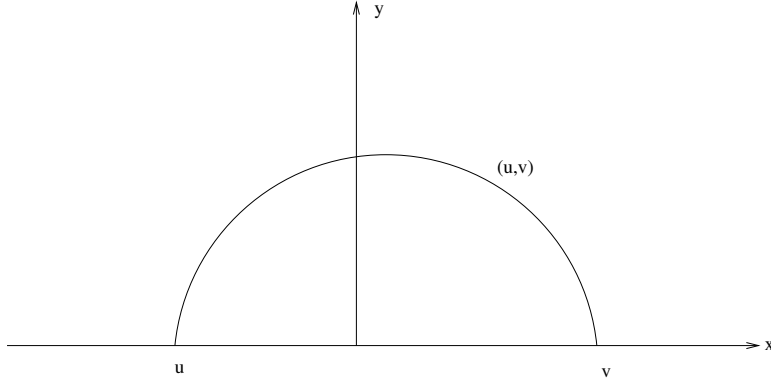


Figure 2.2: Parametrization of $\mathcal{G}_{\mathbb{H}}$ using boundary points

By considering the $\mathrm{PSL}(2, \mathbb{R})$ -action on the boundary, the group $\mathrm{PSL}(2, \mathbb{R})$ can also act on $\mathcal{G}_{\mathbb{H}}$. Moreover, there is a unique measure μ , up to scalar multiplication, on $\mathcal{G}_{\mathbb{H}}$ which is invariant under the $\mathrm{PSL}(2, \mathbb{R})$ -action, namely the **Liouville measure**. With the above parametrization of $\mathcal{G}_{\mathbb{H}}$, the Liouville measure μ is given by:

$$d\mu(u, v) = \frac{du dv}{|u - v|^2}.$$

Let $[a, b]$ and $[c, d]$ be two disjoint intervals in $\partial\mathbb{H}$. Then $[a, b] \times [c, d]$ is identified with a subset of $\mathcal{G}_{\mathbb{H}}$ consisting of geodesics having one end in $[a, b]$ and the other in $[c, d]$. Its Liouville measure is given in term of a cross-ratio:

$$\mu([a, b] \times [c, d]) = \left| \log \left| \frac{(a - c)(b - d)}{(a - d)(b - c)} \right| \right|.$$

Another important kind of curves in \mathbb{H} is the horocycle. To give its definition, we need to introduce the Busemann function:

Definition 2.1.1. *The **Busemann function** β is an application from $\mathbb{H} \times \mathbb{H} \times \partial\mathbb{H}$ to \mathbb{R} given by the following formula:*

$$\beta_u(z, w) = \lim_{z_n \rightarrow u} [d_{\mathbb{H}}(z, z_n) - d_{\mathbb{H}}(w, z_n)],$$

where z and w are two points in \mathbb{H} and u is a point in $\partial\mathbb{H}$.

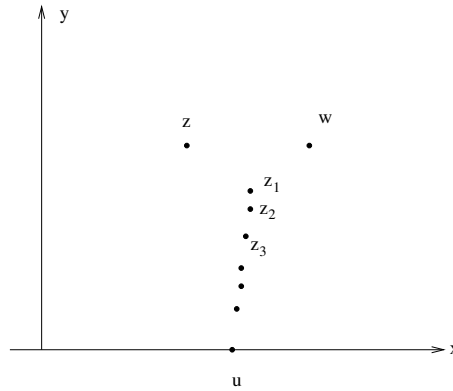


Figure 2.3: Busemann function

By considering the level set of β , the horocycle is defined as follows:

Definition 2.1.2. The *horocycle* $h_u(z)$ centered at u and passing through z is the following subset of \mathbb{H} :

$$h_u(z) = \{w \in \mathbb{H} : \beta_u(z, w) = 0\}.$$

A horocycle in \mathbb{H} is either an Euclidean circle tangent to the horizontal axis, or an Euclidean line parallel to the horizontal axis. The **base point** of horocycle is the tangent point in the first case and ∞ in the second case.

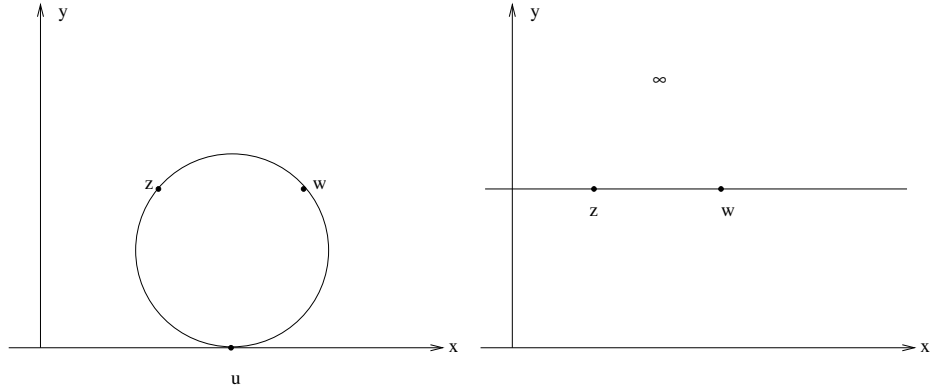


Figure 2.4: Horocycles

2.2 Poincaré disk

The **Poincaré disk** \mathbb{D} is the unit disk in \mathbb{C} equipped with the following metric:

$$ds = \frac{2\sqrt{(dR)^2 + (Rd\theta)^2}}{1 - R^2}, \quad (2.2)$$

where (R, θ) are the polar coordinates of \mathbb{C} with radius R and angle θ .

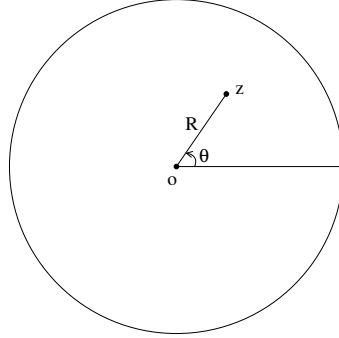


Figure 2.5: Polar coordinates of Poincaré disk

Let z and w be two points in \mathbb{D} . The hyperbolic distance $d_{\mathbb{D}}(z, w)$ between them is given by the following:

$$\cosh d_{\mathbb{D}}(z, w) = 1 + \frac{2|z - w|^2}{(1 - |z|^2)(1 - |w|^2)}.$$

where $|z|$ is the modulus of z as a point in \mathbb{C} .

The **boundary at infinity** $\partial\mathbb{D}$ of \mathbb{D} is the unit circle in \mathbb{C} . The geodesics in \mathbb{D} are intersections of \mathbb{D} with Euclidean circles perpendicular to $\partial\mathbb{D}$ and the diameters.

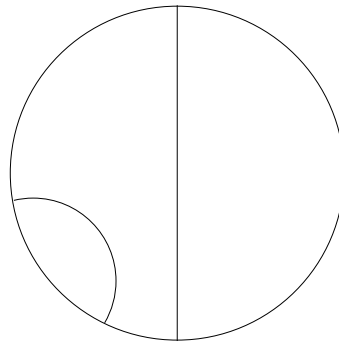
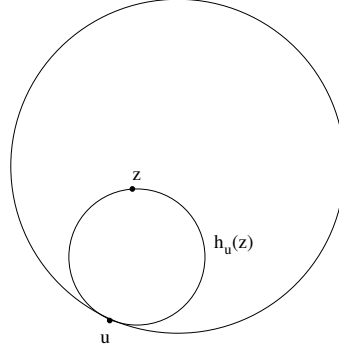


Figure 2.6: Geodesics in \mathbb{D}

The horocycles in \mathbb{D} are Euclidean circles tangent to $\partial\mathbb{D}$.

Figure 2.7: A horocycle in \mathbb{D}

The relation between \mathbb{D} and \mathbb{H} is given by the following Cayley transformation :

$$\begin{aligned} \omega : \mathbb{H} &\rightarrow \mathbb{D}, \\ z &\mapsto \frac{iz + 1}{z + i}. \end{aligned} \tag{2.3}$$

The map ω is conformal and also an isometry. All the formulas for \mathbb{H} are transformed to those for \mathbb{D} by using ω and vice-versa.

2.3 Minkowski model

By Klein, this model is also called the projective model of hyperbolic space. To give its definition, we need first to introduce the Minkowski space.

Definition 2.3.1. *The 3-dimensional Minkowski space V is a real vector space in dimension 3 equipped with a non-degenerated quadratic form \langle, \rangle with signature $(2, 1)$.*

A Minkowski space V has an orthonormal basis $\{e_0, e_1, e_2\}$ such that:

$$-\langle e_0, e_0 \rangle = \langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 1,$$

and

$$\langle e_i, e_j \rangle = 0,$$

if $i \neq j$.

Denoted by (p_0, p_1, p_2) the coordinates of a point $P \in V$ under this basis. Then the quadratic form is given by the following:

$$\langle P, P \rangle = -p_0^2 + p_1^2 + p_2^2.$$

By considering the signs of $\langle P, P \rangle$ and p_0 , we can separate V into 6 connected components:

- (1) the origin O ;
- (2) the space-like interval: $E = \{P \in \mathbb{V} : \langle P, P \rangle > 0\}$;
- (3) the time-like interval (future): $T^+ = \{P \in V : \langle P, P \rangle < 0 \text{ and } p_0 > 0\}$;
- (4) the time-like interval (past): $T^- = \{x \in V : \langle P, P \rangle < 0 \text{ and } p_0 < 0\}$;
- (5) the light cone (future) $L^+ = \{x \in V : \langle P, P \rangle = 0 \text{ and } p_0 > 0\}$;
- (6) the light cone (past) $L^- = \{x \in V : \langle P, P \rangle = 0 \text{ and } p_0 < 0\}$.

The **Minkowski model** \mathbb{M} is the following hyperboloid

$$\mathbb{M} = \{P \in L^+ : \langle P, P \rangle = -1\},$$

equipped with the metric induced by the restriction of quadratic form \langle, \rangle on \mathbb{M} . Let P and Q be two points in \mathbb{M} . The hyperbolic distance $d_{\mathbb{M}}(P, Q)$ between them is given by:

$$\cosh d_{\mathbb{M}}(P, Q) = -\langle P, Q \rangle.$$

The geodesics in \mathbb{M} are the intersections between \mathbb{M} and the planes passing through the origin. The horocycles in \mathbb{M} are the intersections between \mathbb{M} and the planes parallel to a ray in L^+ .

The relation between \mathbb{D} and \mathbb{M} is the stereographic projection from $(-1, 0, 0)$ to the unit disk in the plane containing e_1 and e_2 . The precise formula of this projection is the following:

$$\begin{aligned} \mathbb{M} &\rightarrow \mathbb{D}, \\ (p_0, p_1, p_2) &\mapsto \left(0, \frac{p_1}{p_0 + 1}, \frac{p_2}{p_0 + 1}\right). \end{aligned}$$

This map can be extended to L^+ so that each ray in L^+ is mapped to a point in $\partial\mathbb{D}$. There is a bijection between horocycles and the points on L^+ . Let u be a point in $\partial\mathbb{D}$. Denote by $\xi(u)$ the corresponding ray in L^+ . Then each point P_0 in $\xi(u)$ determines a unique horocycle h_{u, P_0} centered at u :

$$h_{u, P_0} = \{P \in \mathbb{M} : \langle P, P_0 \rangle = 1\}.$$

Remark 2.3.1. *This bijection plays an important role in the study of the lambda length in the decorated Teichmüller theory constructed by Penner [46, 48]. Roughly speaking, the lambda length is defined to be the exponential of one half of the signed hyperbolic distance between the intersection points of a geodesic γ with two horocycles h^+ and h^- based at γ^+ and γ^- respectively. Given such one geodesic and two horocycles, we can get two point on the light cone by the above bijection, denoted by P^+ and P^- . The lambda length given by such 3-uplet equals to $\sqrt{-\frac{1}{2}\langle P^+, P^- \rangle}$. This observation is an important tool to study the lambda length in the decorated Teichmüller theory, and in particular, one gets the Ptolemy relation between lambda lengths associated to an ideal quadrilateral.*

The group preserving the quadratic form \langle, \rangle is $O(1, 2)$. The identity component of $O(1, 2)$ is its subgroup $SO^+(1, 2)$ consisting all the orientation preserving isometry of \mathbb{M} . It is isomorphic to $PSL(2, \mathbb{R})$. The isomorphism can be given by describing the $PSL(2, \mathbb{R})$ -action on \mathbb{M} . To be more precise, we identify a point $P = (p_0, p_1, p_2) \in \mathbb{M}$ to a symmetric matrix B :

$$B = \begin{pmatrix} p_0 + p_1 & p_2 \\ p_2 & p_0 - p_1 \end{pmatrix}.$$

Let $A \in PSL(2, \mathbb{R})$. The A -action sends B to $A^t B A$. Then the image of A in $SO^+(1, 2)$ under the isomorphism is the element acting in the same way as A on \mathbb{M} .

Remark 2.3.2. *Another advantage of Minkowski model is that the hyperbolic structure on \mathbb{H} can be looked as a convex real projective structure on it. This induces a way to embed the Teichmüller space of a hyperbolic surface into the moduli space of the convex real projective structures equipped to the same surface which is identified with the rank 3 Hitchin component by the work of Goldman [31] and Choi and Goldman [21].*

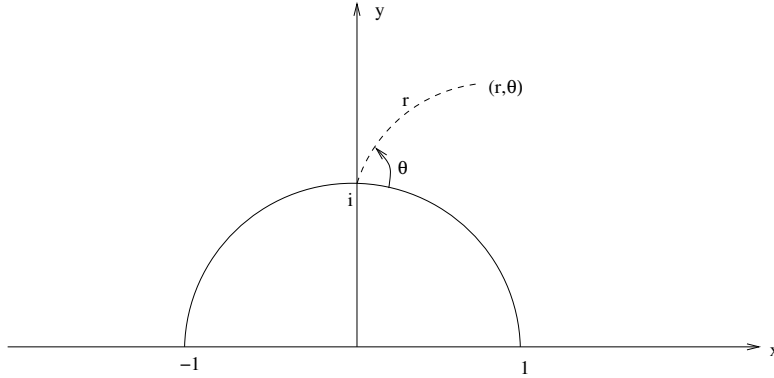
2.4 More formulas for hyperbolic metric and Liouville measure

We restrict ourselves to the upper half plane model \mathbb{H} in this section. The formulas appear in this section will be used in next chapter in the proof of our first result.

2.4.1 New coordinate systems for \mathbb{H}

Hyperbolic polar coordinates

Consider the polar coordinate system (R, θ) for \mathbb{D} introduced in the former section. Instead of the Euclidean radius R , we consider the hyperbolic radius r and obtain a new coordinate system (r, θ) for \mathbb{D} . The pullback of this coordinates of \mathbb{D} by the Cayley transformation (2.3) induce a coordinate system on \mathbb{H} which we call the hyperbolic polar coordinate system of \mathbb{H} .

Figure 2.8: Polar coordinates of \mathbb{H}

The metric and its volume form under this coordinate system of \mathbb{H} can be computed by using (2.2) for \mathbb{D} . The relation between r and R can be computed by integrating Formula (2.2) along the path where θ is fixed. Then we get:

$$r = \ln \frac{1+R}{1-R},$$

or equivalently:

$$\tanh \frac{r}{2} = R.$$

By changing variables, the hyperbolic metric under the polar coordinates is:

$$ds = \sqrt{\sinh^2 r d\theta + dr},$$

and its volume form is the following:

$$d\text{Vol} = \sinh r dr d\theta.$$

Remark 2.4.1. *By composing the Cayley transformation (2.3) with an element $A \in \text{PSL}(2, \mathbb{R})$, we can define such a coordinate system for each fixed pair (z_0, γ_0) where $z_0 = A(i)$ and γ_0 is the image of the half-geodesic $[i, 1[$ under A .*

Rectangular coordinate system

This coordinate system is similar to the Cartesian coordinate system for \mathbb{C} . Generally speaking we fix two geodesics as one horizontal axis and one vertical axis in \mathbb{H} which are orthogonal to each other. Each point in \mathbb{H} is described by a vertical coordinate and a horizontal coordinate.

More precisely, fix an oriented geodesic γ as vertical axis and one geodesic γ' of those orthogonal to γ as horizontal axis. Set the intersection point to be the origin. Then we have the parametrizations on both γ and γ' by using the directed hyperbolic distance to the origin. Denote by p the parameter on γ and by q the parameter on γ' . Then the points on γ have the coordinates $(p, 0)$ and

the points on γ' have the coordinates $(0, q)$. The horizontal line $p = p_0$ in this coordinate system is the geodesic orthogonal to γ at the point $(p_0, 0)$ and the vertical line $q = q_0$ consists of the points in \mathbb{H} have hyperbolic distance $|q_0|$ to γ and on the same side as $(0, q_0)$ with respect to γ .

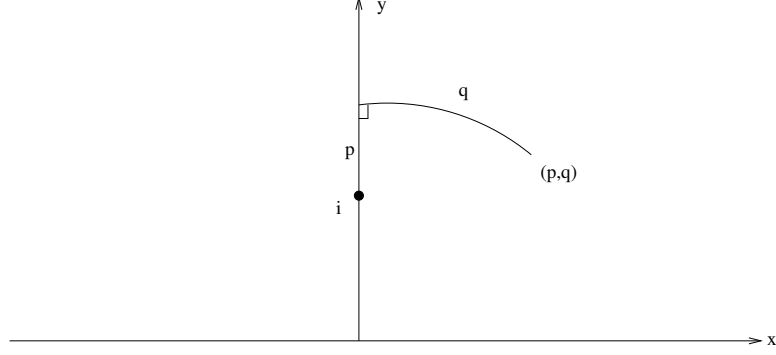


Figure 2.9: Rectangular coordinates of \mathbb{H}

We can see that this coordinate system depends on the pair of geodesic (γ, γ') that we chose. Denote by O the origin and let $z = (p, q) \in \mathbb{H}$. Let $A \in \text{PSL}(2, \mathbb{R})$. As A is the isometry of \mathbb{H} , the point $A(z)$ has the coordinates (p, q) in the rectangular coordinate system defined by using $(A(\gamma), A(\gamma'))$. In particular we can find a $A_0 \in \text{PSL}(2, \mathbb{R})$ such that $A_0(\gamma)$ is the geodesic ending at 1 and -1 with orientation from -1 to 1, and $A_0(\gamma')$ is the geodesic ending at 0 and ∞ with orientation from 0 to ∞ . We need only consider this case to find the formula of the hyperbolic metric at a point (p, q) in one rectangular coordinate system. Consider the polar coordinates (R, θ) of \mathbb{C} where R is the radius and θ is the angle. Then the point (x, y) in \mathbb{H} can be written as follows:

$$\begin{aligned} x &= R \cos \theta, \\ y &= R \sin \theta. \end{aligned}$$

The parameters (p, q) in rectangular coordinate system are described by the following formula:

$$\begin{aligned} p &= \ln R, \\ q &= \frac{1}{2} \ln \frac{1 + \cos \theta}{1 - \cos \theta}. \end{aligned}$$

Combining these four formulas, we get the following relation:

$$\begin{aligned} x &= e^p \tanh q, \\ y &= \frac{e^p}{\cosh q}. \end{aligned}$$

By changing variables, we get the expression of the hyperbolic metric and its volume form using the parameters p and q as follows:

$$ds = \sqrt{\cosh^2 q (dp)^2 + (dq)^2}, \quad (2.4)$$

$$d\text{Vol} = \cosh q dp dq. \quad (2.5)$$

Now consider a point $z \in \mathbb{H}$. Under two distinct rectangular coordinate system, it has two pairs of coordinates (p, q) and (p', q') . Then we have:

$$d\text{Vol}(z) = \cosh q' dp' dq' = \cosh q dp dq.$$

2.4.2 New parametrizations for $\mathcal{G}_{\mathbb{H}}$

Local parametrization by using oriented geodesics

Let γ_1 be an oriented geodesic in \mathbb{H} . Let \mathcal{G}_{γ_1} denote the set of geodesics intersecting γ_1 . We fix a point of γ_1 to be the origin and fix an orientation on γ_1 . We parametrize γ_1 using the directed hyperbolic distance from the origin to a point with respect to the chosen orientation. A geodesic $\gamma \in \mathcal{G}_{\gamma_1}$ is determined uniquely by the position of its intersection point l_1 and its intersection angle α_1 with γ_1 . As a convention, the intersection angle α_1 is measured from γ_1 to γ counter-clockwise.

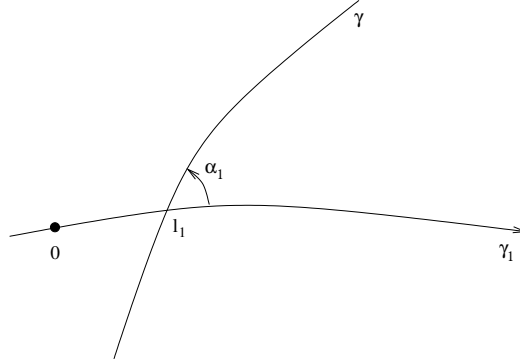


Figure 2.10: Local parametrization of \mathcal{G}_{γ_1}

Under this parametrization, the Liouville measure has the following local expression on the set \mathcal{G}_{γ_1} :

$$d\mu(l_1, \alpha_1) = F(l_1, \alpha_1) dl_1 d\alpha_1.$$

Proposition 2.4.1. *The density $F(l_1, \alpha_1)$ is independent of the parameter l_1 .*

Proof. Fixing an angle α_1 , we need to prove that for any two distinct real numbers l_1 and l'_1 , we have

$$F(l_1, \alpha_1) = F(l'_1, \alpha_1).$$

For any pair of distinct real numbers l_1 and l'_1 , there exists a hyperbolic element A of $PSL(2, \mathbb{R})$ fixing the end points of γ_1 such that A sends the geodesic $\gamma = (l_1, \alpha_1)$ to $\gamma' = (l'_1, \alpha_1)$. Let $L(A)$ be the directed translation length of A with respect to the orientation of γ_1 . Then we have the relation $l'_1 = l_1 + L(A)$. The invariance of Liouville measure implies the following equality:

$$F(l'_1, \alpha_1) dl'_1 d\alpha_1 = F(l_1, \alpha_1) dl_1 d\alpha_1.$$

By changing variables, we obtain:

$$F(l_1 + L(A), \alpha) dl_1 d\alpha_1 = F(l_1, \alpha_1) dl_1 d\alpha_1,$$

which implies that

$$F(l_1 + L(A), \alpha) = F(l_1, \alpha).$$

□

As a consequence of the above, we write $F(\alpha_1)$ short for $F(l_1, \alpha_1)$. To compute $F(\alpha_1)$, we choose another oriented geodesic γ_2 different from γ_1 . By an analogy construction, we can define the parameters (l_2, α_2) for the set \mathcal{G}_{g_2} of geodesics intersecting γ_2 .

Remark 2.4.2. *To simplify the computation, the definition of α_2 is slightly different from α_1 . It is the angle measured from γ_2 to h clockwise.*

Notice that to have both the parameters (l_1, α_1) and the parameters (l_2, α_2) , the geodesic γ need to intersect both γ_1 and γ_2 . The set of such geodesics is denoted by $\mathcal{G}_{\gamma_1 \cup \gamma_2}$.

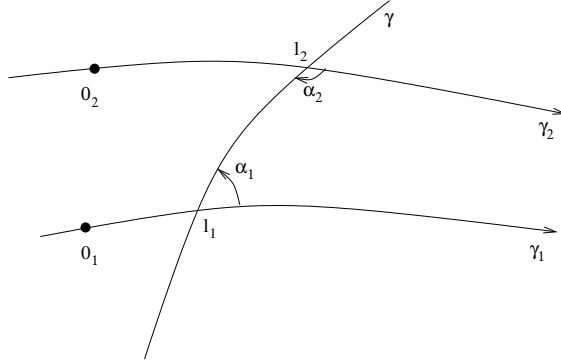


Figure 2.11: Local parametrization of $\mathcal{G}_{\gamma_1 \cup \gamma_2}$

By the same argument as above, we have:

$$d\mu(l_2, \alpha_2) = F(\pi - \alpha_2) dl_2 d\alpha_2.$$

The parameters (l_1, α_1) are evidently functions of (l_2, α_2) and vice-versa. By hyperbolic trigonometry, we have the following expressions for partial derivatives:

$$\frac{\partial \alpha_1}{\partial l_2} = \sigma_1(\gamma) \frac{\sin \alpha_2}{\sinh \rho(\gamma)},$$

and

$$\frac{\partial \alpha_2}{\partial l_1} = \sigma_2(\gamma) \frac{\sin \alpha_1}{\sinh \rho(\gamma)},$$

where $\rho(\gamma)$ is the chord length of γ with respect to γ_1 and γ_2 , and the values of $\sigma_1(\gamma)$ and $\sigma_2(\gamma)$ depend on the relative position of the chord of h with respect to γ_1 and γ_2 :

- (i) if the chord is on the left of γ_1 and left of γ_2 , then $\sigma_1(\gamma) = -1$ and $\sigma_2(\gamma) = 1$;
- (ii) if the chord is on the left of γ_1 and right of γ_2 , then $\sigma_1(\gamma) = 1$ and $\sigma_2(\gamma) = 1$;
- (iii) if the chord is on the right of γ_1 and left of γ_2 , then $\sigma_1(\gamma) = -1$ and $\sigma_2(\gamma) = -1$;
- (iv) if the chord is on the right of γ_1 and right of γ_2 , then $\sigma_1(\gamma) = 1$ and $\sigma_2(\gamma) = -1$.

Now changing variables yields:

$$d\mu = F(\alpha_1) dl_1 d\alpha_1 = \frac{F(\alpha_1) \sin \alpha_2}{\sinh \rho} dl_1 dl_2,$$

and

$$d\mu = F(\pi - \alpha_2) dl_2 d\alpha_2 = \frac{F(\pi - \alpha_2) \sin \alpha_1}{\sinh \rho} dl_1 dl_2.$$

By inspection, the function F has the form:

$$F(\alpha) = c \sin \alpha. \quad (2.6)$$

where c is a constant positive real number.

Taking $c = 1/2$ we obtain exactly the Liouville measure as before:

$$d\mu = \frac{1}{2} \sin \alpha dl d\alpha. \quad (2.7)$$

The above computation also yields another local expression for μ on $\mathcal{G}_{\gamma_1 \cup \gamma_2}$:

$$d\mu = \frac{\sin \alpha_1 \sin \alpha_2}{2 \sinh \rho} dl_1 dl_2. \quad (2.8)$$

Remark 2.4.3. *The Formula (2.7) was previously obtained by Bonahon in the appendix of [9] using the Poincaré disc model \mathbb{D} .*

Polar parametrization

The polar parameters for a geodesic $\gamma \in \mathcal{G}$ are a pair (w, η) . The parameter w is the hyperbolic distance from the point $i \in \mathbb{C}$ to γ . The parameter η is the angle between two geodesics: one is the geodesic passing i and orthogonal to γ ; the other one is the geodesic whose end points are 1 and -1 . The angle η is measured from the latter to the former counter-clockwise.

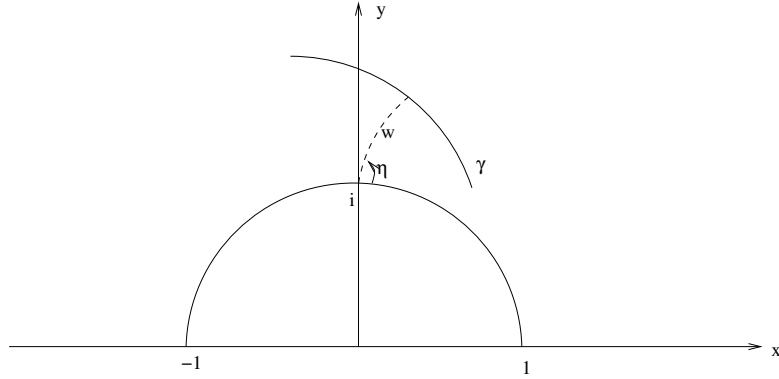


Figure 2.12: Polar parametrization of \mathcal{G}

The expression of μ in these parameters can be computed from the above local expression by using hyperbolic trigonometry.

Let (l, α) be the parameters with respect to the imaginary axis where the origin is the point i . The relation between (l, α) and (w, η) is given by the following formula:

$$\begin{aligned} \tanh l &= \frac{\tanh w}{\cos \eta}, \\ \cos \alpha &= \cosh w \sin \eta. \end{aligned}$$

Now by changing variables we obtain the formula of μ in terms of the parameters (u, η) :

$$d\mu = \frac{1}{2} \cosh w dw d\eta. \quad (2.9)$$

Chapter 3

Hyperbolic Pleijel's Identity and Its Applications

In this chapter, we first recall the original Pleijel's identity and its generalization in the Euclidean case. Then we describe the hyperbolic counterpart of them and give their proofs. We give four of its applications. At the end of this chapter, we consider a maximally symmetric, simply connected, 2-dimensional Riemannian manifold \mathbb{X}_K with constant sectional curvature K and give the analogous results for it.

3.1 Original Pleijel's identity and Ambartzumian-Pleijel identity

Let \mathbb{E} denote the Euclidean plane. Let $\mathcal{G}_{\mathbb{E}}$ denote the set of geodesics in \mathbb{E} and let $\mu_{\mathbb{E}}$ denote the measure on $\mathcal{G}_{\mathbb{E}}$ invariant under Euclidean motions. Let D be a compact convex domain in \mathbb{E} with C^1 boundary and consider the subset \mathcal{G}_D of $\mathcal{G}_{\mathbb{E}}$ consisting of all geodesics intersecting D . For each $\gamma \in \mathcal{G}_D$, the intersection $\gamma \cap D$ is called a chord of γ with respect to D . Denote the length of chord by $\rho_D(\gamma)$. Let $\alpha_1(\gamma)$ and $\alpha_2(\gamma)$ be the two angles between the boundary ∂D and the chord of γ lying on the same side of γ .

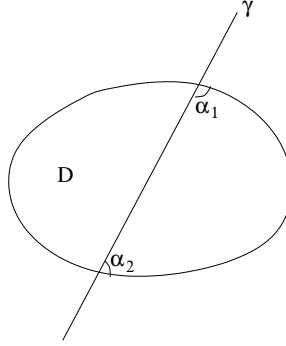


Figure 3.1: Euclidean Pleijel identity

With this notation, Pleijel's identity is the following:

$$\int_{\mathcal{G}_D} (f \circ \rho_D) d\mu_{\mathbb{E}} = \int_{\mathcal{G}_D} (f' \circ \rho_D) \rho_D \cot \alpha_1 \cot \alpha_2 d\mu_{\mathbb{E}},$$

where $f \in C^1(\mathbb{R}, \mathbb{R})$ with $f(0) = 0$.

In [5], Ambartzumian gave combinatorial proof of the Pleijel's identity. Moreover, he proved a general version of Pleijel's identity for convex compact polygonal planar domains and any C^1 -function f :

$$\int_{\mathcal{G}_D} (f \circ \rho_D) d\mu_{\mathbb{E}} = \int_{\mathcal{G}_D} (f' \circ \rho_D) \rho_D \cot \alpha_1 \cot \alpha_2 d\mu_{\mathbb{E}} + \sum_{i=1}^n \int_0^{|a_i|} f(x) dx, \quad (3.1)$$

where a_i is the i -th boundary segment of ∂D with its length denoted by $|a_i|$, and dx is the euclidean length element on \mathbb{R} . We call the equation (3.1) the Ambartzumian-Pleijel identity. In [19], Cabo gave another approach to this identity via Stokes' theorem which turns out to be the idea of our proof of the hyperbolic version of this identity.

3.2 The hyperbolic version identities

3.2.1 Hyperbolic Ambartzumian-Pleijel identity

The statement of the hyperbolic version of the Ambartzumian-Pleijel identity is the following:

Theorem 3.2.1. *Let f be in $C^1(\mathbb{R}; \mathbb{R})$. Then we have the following hyperbolic version of the Ambartzumian-Pleijel identity:*

$$\int_{\mathcal{G}_D} (f \circ \rho) d\mu = \int_{\mathcal{G}_D} (f' \circ \rho) \sinh \rho \cot \alpha_1 \cot \alpha_2 d\mu + \frac{1}{2} \sum_{i=1}^n \int_0^{|a_i|} f(x) dx,$$

where dx is the length element on \mathbb{R} , and $|a_i|$ is the hyperbolic length of the i -th boundary segment a_i of D .

Proof. Recall that D is a convex compact domain in \mathbb{H} whose boundary is a geodesic polygon and a_1, \dots, a_n are the edges of ∂D . Recall that \mathcal{G}_D is the subset of $\mathcal{G}_{\mathbb{H}}$ consisting of the geodesics intersecting D . We parametrize the geodesic in \mathcal{G}_D by an pair of distinct boundary points. Then there is a bijection between \mathcal{G}_D and $(\bigcup_{j>k} a_j \times a_k) \setminus Z$ where Z comes from the multiplicities of the diagonals of ∂D in $\bigcup_{j>k} a_j \times a_k$. As the Liouville measure has no atom, we have $\mu(Z) = 0$.

Let f be a C^1 function from \mathbb{R} to \mathbb{R} . We have the following equality:

$$\int_{\mathcal{G}_D} f(\rho) d\mu = \sum_{j>k} \left(\int_{a_j \times a_k} f(\rho) d\mu \right). \quad (3.2)$$

Consider the orientation on these geodesic segments such that D is on the left side of a_j for all j . Then the chord of $\gamma \in \mathcal{G}_D$ with respect to D is on the left side for each a_j . Consider a pair of edges (a_j, a_k) with $j > k$ and the local expression of μ with respect to them.

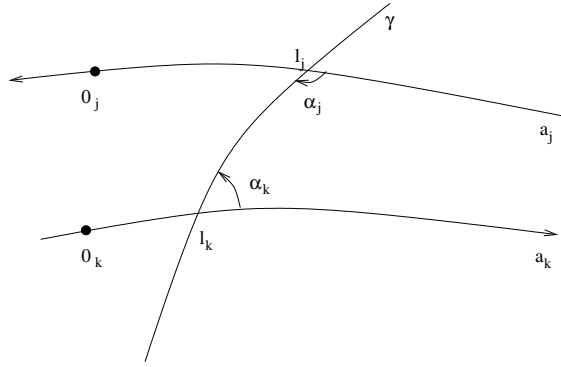


Figure 3.2: A pair of boundary segments of D

By hyperbolic trigonometry, we have:

$$\frac{\partial \rho}{\partial l_j} = \cos \alpha_j, \quad (3.3)$$

and

$$\frac{\partial \rho}{\partial l_k} = -\cos \alpha_k. \quad (3.4)$$

Now, consider the following 1-form on \mathbb{R}^2 :

$$\omega_{jk} = -\frac{\cos \alpha_j}{4} dl_j - \frac{\cos \alpha_k}{4} dl_k.$$

By changing variables, we have that:

$$d\omega_{jk} = \frac{\sin \alpha_j}{4} d\alpha_j \wedge dl_j + \frac{\sin \alpha_k}{4} d\alpha_k \wedge dl_k$$

$$\begin{aligned}
&= -\frac{\sin \alpha_j \sin \alpha_k}{4 \sinh \rho} dl_k \wedge dl_j + \frac{\sin \alpha_j \sin \alpha_k}{4 \sinh \rho} dl_j \wedge dl_k \\
&= \frac{\sin \alpha_j \sin \alpha_k}{2 \sinh \rho} dl_j \wedge dl_k \\
&= d\mu.
\end{aligned}$$

We will compute the right hand side of (3.2) term by term. For a pair of sides (a_j, a_k) with $j > k$, by Stokes' formula one has:

$$\int_{\partial(a_j \times a_k)} f(\rho) \omega_{jk} = \int_{a_j \times a_k} f'(\rho) d\rho \wedge \omega_{jk} + \int_{a_j \times a_k} f(\rho) d\omega_{jk}. \quad (3.5)$$

In the first term on the right hand side:

$$\begin{aligned}
d\rho \wedge \omega_{jk} &= (\cos \alpha_j dl_j - \cos \alpha_k dl_k) \wedge \left(-\frac{\cos \alpha_j}{4} dl_j - \frac{\cos \alpha_k}{4} dl_k\right) \\
&= -\frac{\cos \alpha_j \cos \alpha_k}{2} dl_j \wedge dl_k.
\end{aligned}$$

Comparing it with the formula for $d\mu$, we find the following relation:

$$\begin{aligned}
d\rho \wedge \omega_{jk} &= -\frac{\cos \alpha_j \cos \alpha_k}{2} dl_j \wedge dl_k \\
&= -\frac{\cos \alpha_j \cos \alpha_k}{2} \frac{2 \sinh \rho}{\sin \alpha_j \sin \alpha_k} d\mu \\
&= -\cot \alpha_j \cot \alpha_k \sinh \rho d\mu.
\end{aligned}$$

So this first term of the right hand side of (3.5) becomes:

$$\int_{a_j \times a_k} f'(\rho) d\rho \wedge \omega_{jk} = - \int_{a_j \times a_k} f'(\rho) \cot \alpha_j \cot \alpha_k \sinh \rho d\mu. \quad (3.6)$$

Now we turn to the left hand side of (3.5). We need to discuss two cases depending on the relative positions between a_j and a_k :

(I) The edges a_j and a_k are not adjacent (i.e $|j - k| \neq 1 \pmod n$).

We denote by A_j and B_j (resp. A_k and B_k) the starting and end points of a_j (resp. a_k). Then the left hand side of (3.5) can be computed as follows:

$$\begin{aligned}
\int_{\partial(a_j \times a_k)} f(\rho) \omega_{jk} &= - \int_{(A_j, A_k)}^{(A_j, B_k)} f(\rho) \omega_{jk} + \int_{(B_j, A_k)}^{(B_j, B_k)} f(\rho) \omega_{jk} \\
&\quad + \int_{(A_j, A_k)}^{(B_j, A_k)} f(\rho) \omega_{jk} - \int_{(A_j, B_k)}^{(B_j, B_k)} f(\rho) \omega_{jk} \\
&= \int_{(A_j, A_k)}^{(A_j, B_k)} f(\rho) \frac{\cos \alpha_k}{4} dl_k - \int_{(B_j, A_k)}^{(B_j, B_k)} f(\rho) \frac{\cos \alpha_k}{4} dl_k
\end{aligned}$$

$$\begin{aligned}
& - \int_{(A_j, A_k)}^{(B_j, A_k)} f(\rho) \frac{\cos \alpha_j}{4} dl_j + \int_{(A_j, A_k)}^{(A_j, B_k)} f(\rho) \frac{\cos \alpha_j}{4} dl_j \\
& = \frac{1}{4} \left(- \int_{\rho_3}^{\rho_1} + \int_{\rho_4}^{\rho_2} - \int_{\rho_3}^{\rho_4} + \int_{\rho_1}^{\rho_2} \right) f(\rho) d\rho. \tag{3.7}
\end{aligned}$$

where ρ_1 is the length of the diagonal (A_j, B_k) , ρ_2 is the length of the diagonal (B_j, B_k) , ρ_3 is the length of the diagonal (A_j, A_k) and ρ_4 is the length of the diagonal (B_j, A_k) . The last equality comes from the change of variable using (3.3) and (3.4).

(II) The edges a_j and a_k are adjacent (i.e. $|j - k| = 1 \pmod n$).

Without loss of generality, we can assume that $B_j = A_k$. In the same way as in (I), we get the following equality:

$$\int_{\partial(a_j \times a_k)} f(\rho) \omega_{jk} = \frac{1}{4} \left(- \int_{\rho_3}^{\rho_1} + \int_{\rho_4}^{\rho_2} - \int_{\rho_3}^{\rho_4} + \int_{\rho_1}^{\rho_2} \right) f(\rho) d\rho.$$

Moreover, we have the following relations:

$$\begin{aligned}
\rho_2 &= |a_k|, \\
\rho_3 &= |a_j|, \\
\rho_4 &= 0.
\end{aligned}$$

So in this case we obtain the following formula:

$$\int_{\partial(a_j \times a_k)} f(\rho) \omega_{jk} = \frac{1}{4} \left(- \int_{|a_j|}^{\rho_1} + \int_0^{|a_k|} + \int_0^{|a_j|} + \int_{\rho_1}^{|a_k|} \right) f(\rho) d\rho. \tag{3.8}$$

The last step is to sum up the formulas (3.5) for each (j, k) . Let us first compute the left hand side:

$$\sum_{j>k} \int_{\partial(a_j \times a_k)} f(\rho) \omega = \sum_{j>k} \frac{1}{4} \left(- \int_{\rho_3}^{\rho_1} + \int_{\rho_4}^{\rho_2} - \int_{\rho_3}^{\rho_4} + \int_{\rho_1}^{\rho_2} \right) f(\rho) d\rho$$

For any (j, k) in Case (I), the first term

$$- \int_{\rho_3}^{\rho_1} f(\rho) d\rho$$

on the right side of the associated Formula (3.7) will also appear on the right side of Formula (3.7) associated to $(j - 1, k)$, but with a different sign. The same happens for the other three terms for (j, k) . For any (j, k) in Case (II), this happens for the first and the last term in Formula (3.8).

After summing up Formula (3.7) and Formula (3.8) for all pairs (j, k) , the terms different by a sign will cancel each other and the rest is the following:

$$\sum_{j>k} \int_{\partial(a_j \times a_k)} f(\rho) \omega_{jk} = \frac{1}{2} \sum_{i=1}^n \int_0^{|a_i|} f(x) dx. \quad (3.9)$$

For the first term on the right hand side of Formula (3.5), by Formula (3.6), we have:

$$\begin{aligned} \sum_{j>k} \int_{a_j \times a_k} f'(\rho) d\rho \wedge \omega_{jk} &= \sum_{j>k} - \int_{a_j \times a_k} f'(\rho) \cot \alpha_j \cot \alpha_k \sinh \rho d\mu \\ &= - \int_{\mathcal{G}_D} f'(\rho) \cot \alpha_j \cot \alpha_k \sinh \rho d\mu. \end{aligned}$$

By moving it to the left hand side, we finally get the formula in Theorem 3.2.1. \square

3.2.2 Hyperbolic Pleijel's identity

Theorem 3.2.2. *Let f be in $C^1(\mathbb{R}; \mathbb{R})$. Suppose that ∂D is C^1 . With the same notation as in Theorem 3.2.1, we have the following identity:*

$$\int_{\mathcal{G}_D} (f \circ \rho) d\mu = \int_{\mathcal{G}_D} (f' \circ \rho) \sinh \rho \cot \alpha_1 \cot \alpha_2 d\mu + \frac{1}{2} f(0) L(\partial D).$$

In particular if $f(0) = 0$, then we have the hyperbolic version of the Pleijel's identity.

Proof. Given a convex compact domain D with C^1 boundary in \mathbb{H} , we can choose 3 points b_1, b_2 and b_3 in ∂D and get a triangle D_3 inscribed into D . Then by Theorem 3.2.1, we have the Ambartzumian-Pleijel identity for D_3 :

$$\int_{\mathcal{G}_{D_3}} (f \circ \rho_3) d\mu = \int_{\mathcal{G}_{D_3}} (f' \circ \rho_3) \sinh \rho_3 \cot \alpha_1 \cot \alpha_2 d\mu + \frac{1}{2} \sum_{i=1}^3 \int_0^{|a_i|} f(x) dx.$$

Each pair of the adjacent vertices (b_j, b_{j+1}) of D_3 separates ∂D into two parts and exactly one of which containing no vertices of D_3 . Denote this arc by γ_j . Then we consider a new set of points in ∂D consisting all b_j and the mid-point of γ_j for all j . The corresponding inscribed polygon denoted by D_6 will gives us another Ambartzumian-Pleijel identity:

$$\int_{\mathcal{G}_{D_6}} (f \circ \rho_6) d\mu = \int_{\mathcal{G}_{D_6}} (f' \circ \rho_6) \sinh \rho_6 \cot \alpha_1 \cot \alpha_2 d\mu + \frac{1}{2} \sum_{i=1}^6 \int_0^{|a_i|} f(x) dx.$$

Repeating the above construction for D_6 and so on, we can get a sequence of polygons D_{3n} . As D is compact convex with C^1 boundary and f is C^1 , the function $f \circ \rho_{3n}$ uniformly converges to $f \circ \rho$ where ρ is the chord length function for D . Also the maximum of lengths of boundary segments for each D_{3n} will go to 0 when n goes to ∞ . By passing to the limit, we obtain the required formula. \square

3.3 Applications

Corollary 3.3.1. *The Liouville measure of \mathcal{G}_D is one half of the length of the boundary of D .*

Proof. The proof of Corollary 3.3.1 is direct. We choose f to be the constant map: $f(x) = 1$. Since the derivative of f is identically zero, the first term of the right hand side is 0. The corollary follows. \square

Corollary 3.3.2. *We have the hyperbolic isoperimetric inequality:*

$$L(D)^2 \geq 4\pi A(D) + A(D)^2,$$

where the equality holds if and only if D is a disk in \mathbb{H} .

The idea of the proof comes from that for the Euclidean case using the Pleijel's identity given by Ambartzumian in [5].

Proof. In the proof of Theorem 3.2.1, we obtained the following two formulas for a polygon domain D :

$$\int_{\mathcal{G}_D} (f \circ \rho) d\mu = \frac{1}{2} \int_{\mathcal{G}_D} (f' \circ \rho) \cos \alpha_1 \cos \alpha_2 dl_1 dl_2 + \frac{1}{2} \sum_{i=1}^n \int_0^{|a_i|} f(x) dx,$$

$$\int_{\mathcal{G}_D} (f \circ \rho) d\mu = \frac{1}{2} \int_{\mathcal{G}_D} (f \circ \rho) \frac{\sin \alpha_1 \sin \alpha_2}{\sinh \rho} dl_1 dl_2.$$

By the same argument as in the proof of Theorem 3.2.2, we have the following two equalities for those D with C^1 boundary:

$$\int_{\mathcal{G}_D} (f \circ \rho) d\mu = \frac{1}{2} \int_{\mathcal{G}_D} (f' \circ \rho) \cos \alpha_1 \cos \alpha_2 dl_1 dl_2 + \frac{1}{2} f(0) L(\partial D), \quad (3.10)$$

$$\int_{\mathcal{G}_D} (f \circ \rho) d\mu = \frac{1}{2} \int_{\mathcal{G}_D} (f \circ \rho) \frac{\sin \alpha_1 \sin \alpha_2}{\sinh \rho} dl_1 dl_2. \quad (3.11)$$

In (3.10) we take f to be $f(x) = x$ and in (3.11) we take f to be $f(x) = \sinh x$, which yield the following two equalities:

$$\int_{\mathcal{G}_D} \rho d\mu = \frac{1}{2} \int_{\mathcal{G}_D} \cos \alpha_1 \cos \alpha_2 dl_1 dl_2, \quad (3.12)$$

$$\int_{\mathcal{G}_D} (\sinh \circ \rho) d\mu = \frac{1}{2} \int_{\mathcal{G}_D} \sin \alpha_1 \sin \alpha_2 dl_1 dl_2. \quad (3.13)$$

Adding these on the right hand side we get the following:

$$\begin{aligned} & \frac{1}{2} \int_{\mathcal{G}_D} (\cos \alpha_1 \cos \alpha_2 + \sin \alpha_1 \sin \alpha_2) dl_1 dl_2 \\ &= \frac{1}{2} \int_{\mathcal{G}_D} \cos(\alpha_1 - \alpha_2) dl_1 dl_2 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \int_{\partial D} \int_{\partial D} (1 - 2 \sin^2(\frac{\alpha_1 - \alpha_2}{2})) dl_1 dl_2 \\
&= \frac{1}{4} (L(\partial D))^2 - \frac{1}{2} \int_{\partial D} \int_{\partial D} \sin^2(\frac{\alpha_1 - \alpha_2}{2}) dl_1 dl_2,
\end{aligned}$$

whilst on the left hand side, we have the following:

$$\int_{\mathcal{G}_D} (\rho + \sinh(\rho)) d\mu = \int_{\mathcal{G}_D} 2\rho d\mu + \int_{\mathcal{G}_D} (\sinh(\rho) - \rho) d\mu. \quad (3.14)$$

The first term on the right side is the volume of the unit tangent bundle over D . So the integral equals $\pi A(D)$. To compute the second term on the right hand side of (3.14), we need to use the hyperbolic volume form that we found in the former sections: denote by P_1 and P_2 the points in D , then we have:

$$(A(D))^2 = \int_D \int_D d\text{Vol}(P_2) d\text{Vol}(P_1) \quad (3.15)$$

$$= \int_D \left(\int_D \sinh r_1(P_2) dr_1(P_2) d\theta_1(P_2) \right) d\text{Vol}(P_1), \quad (3.16)$$

where (r_1, θ_1) are the polar coordinates of P_2 with respect to P_1 .

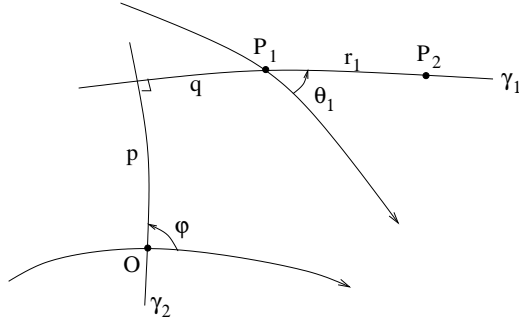


Figure 3.3: Different parameters of P_1 and P_2

Consider the geodesic γ_1 passing through the origin $O \in \mathbb{H}$ and orthogonal to the geodesic γ_2 passing through P_1 and P_2 . It has an angle ϕ to a fixed geodesic ray based on O . We can change the parameter from (r_1, θ_1) to (r_1, ϕ) and the resulting formula is:

$$(A(D))^2 = \int_D \int_D \sinh r_1 \frac{\cosh p}{\cosh q} dr_1 d\phi d\text{Vol}(P_1). \quad (3.17)$$

where p is the distance from the origin to γ_1 and q is the distance from the foot of orthogonal projection of the origin on γ_2 to P_1 . Consider the rectangular coordinates of P_1 with respect to γ_1 . Then we have the

$$\sinh r_1 \frac{\cosh p}{\cosh q} dr_1 d\phi d\text{Vol}(P_1) = \sinh r_1 \frac{\cosh p}{\cosh q} dr_1 d\phi \cosh q dp dq$$

$$\begin{aligned}
&= \sinh r_1 \cosh p dr_1 d\phi dp dq \\
&= 2 \sinh r_1 dr_1 dq d\mu.
\end{aligned}$$

where the last equality comes from the formula (2.9).

Consider a parametrization on geodesic γ_2 . Let p and p' denote the positions of P_1 and P_2 respectively. Then we have $r_1 = |p - p'|$. The above formula can be rewritten as follows:

$$(A(D))^2 = \int_{\mathcal{G}_D} \int_0^\rho \int_0^\rho 2 \sinh r_1 dq' dq d\mu \quad (3.18)$$

$$= \int_{\mathcal{G}_D} 4(\sinh \rho - \rho) d\mu. \quad (3.19)$$

By all the computations above, the sum of (3.12) and (3.13) can be rewritten as follows:

$$\frac{1}{4}(L(\partial D))^2 - \frac{1}{2} \int_{\partial D} \int_{\partial D} \sin^2\left(\frac{\alpha_1 - \alpha_2}{2}\right) dl_1 dl_2 = \pi A(D) + \frac{1}{4}(A(D))^2. \quad (3.20)$$

which implies the isoperimetric inequality. Also the formula (3.20) shows that the equality holds if and only if $\alpha_1 = \alpha_2$ for all $\gamma \in \mathcal{G}_D$ which implies ∂D is a circle. \square

Remark 3.3.1. *The computation for $(A(D))^2$ is due to Santalo in [55].*

Corollary 3.3.3. *Let T be an ideal triangle in \mathbb{H} and μ be the Liouville measure on $\mathcal{G}_{\mathbb{H}}$. The density of the measure $M_T = \rho_* \mu$ is given by:*

$$dM_T = \frac{6\rho d\rho}{\sinh^2 \rho}.$$

Proof. We first remark that this result has been previously obtained by Bridgeman and Dumas in [18]. Here we give a different approach.

It is well known that all ideal triangles in \mathbb{H}^2 are isometric. We can assume that the vertices of the ideal triangle are $0, 1$ and ∞ . These points separate the boundary of hyperbolic plane into three intervals: $I_1 =]\infty, 0[$, $I_2 =]0, 1[$ and $I_3 =]1, \infty[$. The set of the geodesics crossing the ideal triangle is $\bigcup_{j < k} I_j \times I_k$.

Because of the symmetry of the ideal triangle, we only need to consider $I_1 \times I_3$ and we denote it by \mathcal{G}_0 .

We parametrize $i\mathbb{R}$ such that the point z_1 with the coordinate l_1 is the point ie^{l_1} . In the same way, we parametrize $1 + i\mathbb{R}$ such that the point z_3 with the coordinate l_3 is $1 + ie^{l_3}$. Then a geodesic $\gamma \in \mathcal{G}_0$ can be parametrized by (l_1, l_3) . by the formula that we obtained in section 2, the Liouville measure can be expressed locally as follows:

$$d\mu = \frac{\sin \alpha_1 \sin \alpha_3}{2 \sinh \rho} dl_1 dl_3,$$

where α_1 (resp. α_2) is the angle between $\gamma \in \mathcal{G}_0$ and $i\mathbb{R}$ (reps. $1 + i\mathbb{R}$).

The chord length ρ of γ can be computed as follows:

$$\rho = \log \frac{|z_1 - \bar{z}_3| + |z_1 - z_3|}{|z_1 - \bar{z}_3| - |z_1 - z_3|}.$$

Equivalently we have:

$$\begin{aligned} \sinh \rho &= \frac{|z_1 - \bar{z}_3||z_1 - z_3|}{2e^{l_1}e^{l_3}}, \\ \cosh \rho &= \frac{1 + e^{2l_1} + e^{2l_3}}{2e^{l_1}e^{l_3}}. \end{aligned}$$

The center and the radius of γ are the following:

$$c = \frac{1 + e^{2l_3} - e^{2l_1}}{2},$$

$$r = \sinh \rho e^{l_1} e^{l_3}.$$

Then we can write the $\sin \alpha_1$ and $\sin \alpha_3$ as functions of ρ, l_1, l_3 :

$$\sin \alpha_1 = \frac{e^{l_1}}{r} = \frac{1}{\sinh \rho e^{l_3}},$$

$$\sin \alpha_3 = \frac{e^{l_3}}{r} = \frac{1}{\sinh \rho e^{l_1}}.$$

Then $d\mu$ is rewritten as follows:

$$d\mu = \frac{1}{\sinh^3 \rho e^{l_1} e^{l_3}} dl_1 dl_3.$$

By considering the equation:

$$e^{2l_3} - 2e^{l_1}e^{l_3} \cosh \rho + 1 + e^{2l_1} = 0,$$

the parameter k can be expressed by a function of ρ and l_1 :

$$e^{l_3} = e^{l_1} \cosh \rho \pm \sqrt{e^{2l_1} \sinh^2 \rho - 1}.$$

These two solutions correspond to two different cases. Fix the point z_1 and move the point z_3 from 1 to ∞ along the geodesic $1 + i\mathbb{R}$. The chord length ρ decreases from ∞ to a minimal value then increases back to ∞ where the minimal value is the distance between z_1 and the geodesic $1 + i\mathbb{R}$. We denote it by $d(z_1)$. This means that in generic case for a fixed z_1 and a fixed ρ , there are two points z_3 and z'_3 in $1 + i\mathbb{R}$ satisfying that their distances to z_1 are both ρ . These two points correspond to the two solutions above respectively.

For a fixed l_1 , to compute l_3 realizing $d(z_1)$, we consider the geodesic passing through z_1 and perpendicular to $1 + i\mathbb{R}$. The center of this geodesic is 1, and the

foot of the perpendicular geodesic is $1 + i\sqrt{1 + e^{2l_1}}$ which implies that $e^{2l_3} = 1 + e^{2l_1}$. Then we can separate \mathbb{R}^2 into two parts:

$$U_1 = \{(l_1, l_3) \in \mathbb{R}^2 \mid e^{2l_3} \geq 1 + e^{2l_1}\},$$

and

$$U_2 = \{(l_1, l_3) \in \mathbb{R}^2 \mid e^{2l_3} \leq 1 + e^{2l_1}\},$$

such that in each part, there is at most one l_3 for each pair (l_1, ρ) .

First we want to find which solution corresponds to which subset.

When $\rho = d(z_1)$, we have the equality $e^{l_3} = e^{l_1} \cosh d(z_1) = \sqrt{1 + e^{2l_1}}$. Then as l_3 increasing, we have the inequality:

$$e^{l_3} > e^{l_1} \cosh d(z_1),$$

and ρ increases at the same time. This tells us that the solution for the set U_1 is:

$$e^{l_3} = e^{l_1} \cosh \rho + \sqrt{e^{2l_1} \sinh^2 \rho - 1},$$

Then the solution for U_2 is:

$$e^{l_3} = e^{l_1} \cosh \rho - \sqrt{e^{2l_1} \sinh^2 \rho - 1}.$$

The integral splits into two parts:

$$\int_{\mathcal{G}_0} f(\rho) d\mu = \int_{-\infty}^{+\infty} \int_{\sqrt{1+e^{2l_1}}}^{+\infty} \frac{f(\rho)}{2e^{l_1} e^{l_3} \sinh^3 \rho} dl_3 dl_1 + \int_{-\infty}^{+\infty} \int_{-\infty}^{\sqrt{1+e^{2l_1}}} \frac{f(\rho)}{2e^{l_1} e^{l_3} \sinh^3 \rho} dl_3 dl_1$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is C^1 and with compact support. We use I and II to denote the first term and the second term of the right hand side.

Recall that in I we have

$$e^{l_3} = e^{l_1} \cosh \rho + \sqrt{e^{2l_1} \sinh^2 \rho - 1}.$$

from which we compute the determinant of Jacobi as follows:

$$\begin{aligned} \left| \frac{\partial l_3}{\partial \rho} \right| &= \frac{1}{e^{l_3}} \left[e^{l_1} \sinh \rho + \frac{e^{2l_1} \sinh \rho \cosh \rho}{\sqrt{e^{2l_1} \sinh^2 \rho - 1}} \right] \\ &= \frac{e^{l_1} \sinh \rho}{e^{l_3} \sqrt{e^{2l_1} \sinh^2 \rho - 1}} (\sqrt{e^{2l_1} \sinh^2 \rho - 1} + e^{l_1} \cosh \rho) \\ &= \frac{e^{l_1} \sinh \rho}{e^{l_3} \sqrt{e^{2l_1} \sinh^2 \rho - 1}} e^{l_3} \\ &= \frac{e^{l_1} \sinh \rho}{\sqrt{e^{2l_1} \sinh^2 \rho - 1}}. \end{aligned}$$

Then we have:

$$I = \int_{-\infty}^{+\infty} \int_{\sqrt{1+e^{2l_1}}}^{+\infty} \frac{f(\rho)}{2e^{l_1} e^{l_3} \sinh^3 \rho} dl_3 dl_1$$

$$\begin{aligned}
&= \int_{-\infty}^{+\infty} \int_{\sinh^{-1}(\frac{1}{e^{l_1}})}^{+\infty} \frac{f(\rho)}{2e^{l_1}e^{l_1^3} \sinh^3 \rho} \frac{e^{l_1} \sinh \rho}{\sqrt{e^{2l_1} \sinh^2 \rho - 1}} d\rho dl_1 \\
&= \int_{-\infty}^{+\infty} \int_{\sinh^{-1}(\frac{1}{e^{l_1}})}^{+\infty} \frac{f(\rho)}{2(\sqrt{e^{2l_1} \sinh^2 \rho - 1} + e^{l_1} \cosh \rho) \sinh^2 \rho \sqrt{e^{2l_1} \sinh^2 \rho - 1}} d\rho dl_1 \\
&= \int_0^{+\infty} \int_{-\ln \sinh \rho}^{+\infty} \frac{f(\rho)}{2(\sqrt{e^{2l_1} \sinh^2 \rho - 1} + e^{l_1} \cosh \rho) \sinh^2 \rho \sqrt{e^{2l_1} \sinh^2 \rho - 1}} dl_1 d\rho.
\end{aligned}$$

Let

$$v = (\sinh \rho + \cosh \rho)e^{l_1}(\sinh \rho e^{l_1} + \sqrt{\sinh^2 \rho e^{2l_1} - 1}).$$

We change the variables from (l_1, ρ) to (v, ρ) and the Jacobi is:

$$|J| = \left| \frac{\partial l_1}{\partial v} \right| = [(\sinh \rho + \cosh \rho)e^{2l_1}(\sinh \rho e^{l_1} + \sqrt{\sinh^2 \rho e^{2l_1} - 1})^2 \frac{\sinh \rho}{\sqrt{\sinh^2 \rho e^{2l_1} - 1}}]^{-1}.$$

So the integral I can be written as follows:

$$\begin{aligned}
I &= \int_0^{+\infty} \int_{\frac{\cosh \rho + \sinh \rho}{\sinh \rho}}^{+\infty} \frac{f(\rho)}{2(\sqrt{e^{2l_1} \sinh^2 \rho - 1} + e^{l_1} \cosh \rho) \sinh^2 \rho \sqrt{e^{2l_1} \sinh^2 \rho - 1}} \times \\
&\quad \times \frac{\sqrt{e^{2l_1} \sinh^2 \rho - 1}}{(\cosh \rho + \sinh \rho)e^{l_1}(\sinh \rho e^{l_1} + \sqrt{\sinh^2 \rho e^{2l_1} - 1})^2} dv d\rho \\
&= \int_0^{+\infty} \int_{\frac{\cosh \rho + \sinh \rho}{\sinh \rho}}^{+\infty} \frac{f(\rho)}{2 \sinh^2 \rho} \frac{1}{(\cosh \rho + \sinh \rho)e^{l_1}(\sinh \rho e^{l_1} + \sqrt{\sinh^2 \rho e^{2l_1} - 1})} \times \\
&\quad \times \frac{1}{(\sqrt{e^{2l_1} \sinh^2 \rho - 1} + e^{l_1} \cosh \rho)(\sinh \rho e^{l_1} + \sqrt{\sinh^2 \rho e^{2l_1} - 1})} dv d\rho \\
&= \int_0^{+\infty} \int_{\frac{\cosh \rho + \sinh \rho}{\sinh \rho}}^{+\infty} \frac{f(\rho)}{2 \sinh^2 \rho v(v-1)} dv d\rho.
\end{aligned}$$

We do the same for II . In this computation, we use:

$$v = (\sinh \rho + \cosh \rho)e^{l_1}(\sinh \rho e^{l_1} - \sqrt{\sinh^2 \rho e^{2l_1} - 1}),$$

to change variables.

Fix ρ and compute the partial differential of v with respect to l_1 , we may find that it is always negative. We compute the limit of v when l goes to infinity as follows:

$$\begin{aligned}
v &= (\sinh \rho + \cosh \rho)e^{l_1}(\sinh \rho e^{l_1} - \sqrt{\sinh^2 \rho e^{2l_1} - 1}) \\
&= (\sinh \rho + \cosh \rho)e^{l_1}(\sinh \rho e^{l_1} - \sinh \rho e^{l_1}(1 - \frac{1}{2 \sinh^2 \rho e^{2l_1}} + o(\frac{1}{\sinh^2 \rho e^{2l_1}}))), l_1 \rightarrow +\infty.
\end{aligned}$$

This implies that:

$$\lim_{l \rightarrow +\infty} v(\rho, l_1) = \frac{\sinh \rho + \cosh \rho}{2 \sinh \rho},$$

for all $\rho > 0$.

Then the second part of the integral is:

$$II = \int_0^{+\infty} \int_{\frac{\cosh \rho + \sinh \rho}{2 \sinh \rho}}^{\frac{\cosh \rho + \sinh \rho}{\sinh \rho}} \frac{f(\rho)}{2 \sinh^2 \rho v(v-1)} dv d\rho.$$

Putting I and II together:

$$\begin{aligned} \int_{\mathcal{G}_0} f(\rho) d\mu &= \int_0^{+\infty} \int_{\frac{\cosh \rho + \sinh \rho}{2 \sinh \rho}}^{\infty} \frac{f(\rho)}{2 \sinh^2 \rho v(v-1)} dv d\rho \\ &= \int_0^{+\infty} \frac{f(\rho)}{2 \sinh^2 \rho} \left(\int_{\frac{\cosh \rho + \sinh \rho}{2 \sinh \rho}}^{\infty} \frac{1}{v(v-1)} dv \right) d\rho \\ &= \int_0^{+\infty} \frac{f(\rho)}{2 \sinh^2 \rho} 2\rho d\rho \\ &= \int_0^{+\infty} \frac{f(\rho)\rho}{\sinh^2 \rho} d\rho. \end{aligned}$$

This implies that:

$$dM_T = \frac{3\rho d\rho}{\sinh^2 \rho}.$$

which is same as described in [18]. \square

Corollary 3.3.4. *Let Q be an ideal quadrilateral in \mathbb{H} and μ be the Liouville measure on $\mathcal{G}_{\mathbb{H}}$. Let $\gamma_1, \dots, \gamma_4$ be the 4 edges of Q ordered counter-clockwise. The chord length distribution $dM_Q = (\rho_Q)_* d\mu$ is given by:*

$$dM_Q = \frac{12\rho d\rho}{\sinh^2 \rho} + dM_{13} + dM_{24},$$

where dM_{13} is the chord length distribution with respect to γ_1 and γ_3 and satisfies:

$$\int_0^\rho dM_{13} = \frac{1}{2} \int_{[\eta]} \frac{\cot \alpha_1(\rho, \eta) \cot \alpha_3(\rho, \eta) \sinh \rho \cosh w(\rho, \eta)}{\sinh \rho_1(\rho, \eta) \cot \alpha_1(\rho, \eta) + \sinh \rho_3(\rho, \eta) \cot \alpha_3(\rho, \eta)} d\eta,$$

and dM_{24} is the chord length distribution with respect to γ_2 and γ_4 and satisfies:

$$\int_0^\rho dM_{24} = \frac{1}{2} \int_{[\eta]} \frac{\cot \alpha_2(\rho, \eta) \cot \alpha_4(\rho, \eta) \sinh \rho \cosh w(\rho, \eta)}{\sinh \rho_2(\rho, \eta) \cot \alpha_2(\rho, \eta) + \sinh \rho_4(\rho, \eta) \cot \alpha_4(\rho, \eta)} d\eta,$$

where η is the angle parameter in the polar parametrization of the set of geodesics in \mathbb{H} introduced later.

In this proof, the δ -formalism is the main tool and this idea comes from [28] for the Euclidean version of the Pleijel's identity.

Proof. Let Q be an ideal quadrilateral. Let $\gamma_1, \gamma_2, \gamma_3$ and γ_4 denote its 4 edges ordered counter-clockwise. The set of geodesics having at least one end at the vertices of Q has the Liouville measure 0. So we need only consider the following two types of geodesics intersecting Q : those intersecting two edges adjacent and those intersecting two opposite edges. The chord length distribution for the first type has been computed in the previous section and the main task in this section is to compute the chord length distribution for the second type.

Consider the geodesics γ_1 and γ_3 . Let \mathcal{G}_{13} denote the geodesic intersecting them. By using the δ -formalism to the Pleijel's identity, we can compute the Liouville measure of the following set:

$$\{\gamma \in \mathcal{G}_{13} : \rho(\gamma) \leq \rho_0\}.$$

By considering the distance to γ_3 , the geodesic γ_1 can be separated into three segments. The middle one consists of those points having distance to γ_3 smaller than ρ_0 . The other two consists of all points having the distance to γ_3 strictly bigger than ρ_0 . We choose one point in each of the latter two segments. Let A_1 and B_1 denote these two points then the geodesic segment $[A_1, B_1]$ contains the points in γ_1 having the distance to γ_3 smaller than ρ_0 and the distance of A_1 and B_1 to γ_3 are both bigger than ρ_0 . By considering the distance to γ_1 and the same argument as for γ_1 , we have two points A_3 and B_3 in γ_3 such that the geodesic segment $[A_3, B_3]$ contains the points in γ_3 having the distance to γ_1 smaller than ρ_0 and the distance of A_3 and B_3 to γ_1 are both bigger than ρ_0 .

We assume that the A_1, B_1, A_3 and B_3 are in the cyclic order in ∂Q . Let \mathcal{G}_0 be the set of geodesics intersecting γ_1 between A_1 and B_1 and intersecting γ_3 between A_3 and B_3 . Then we have that:

$$\mu(\{\gamma \in \mathcal{G}_{13} : \rho(\gamma) \leq \rho_0\}) = \mu(\{\gamma \in \mathcal{G}_0 : \rho(\gamma) \leq \rho_0\}).$$

Then by using the Stokes' theorem and repeating the proof of Theorem 3.2.1, we have:

$$\int_{\mathcal{G}_0} f(\rho) d\mu = \int_{\mathcal{G}_0} f'(\rho) \cot \alpha_1 \cot \alpha_3 \sinh \rho d\mu + \frac{1}{4} \left(- \int_{\rho_3}^{\rho_1} + \int_{\rho_4}^{\rho_2} - \int_{\rho_3}^{\rho_4} + \int_{\rho_1}^{\rho_2} \right) f(\rho) d\rho, \quad (3.21)$$

where ρ_1 is the length of the diagonal (A_3, B_1) , ρ_2 is the length of the diagonal (B_3, B_1) , ρ_3 is the length of the diagonal (A_3, A_1) and ρ_4 is the length of the diagonal (B_3, A_1) . By the assumptions for A_1, B_1, A_3 and B_3 , we have that ρ_1, \dots, ρ_4 all bigger than ρ_0 .

Instead of choosing f to be C^1 , we can formally choose f to be the step function:

$$f(x) = \begin{cases} 1 & \text{if } x \leq \rho_0 \\ 0 & \text{if } x > \rho_0 \end{cases}.$$

Then its derivative f' becomes a δ -function:

$$f'(x) = \delta(\rho_0 - x),$$

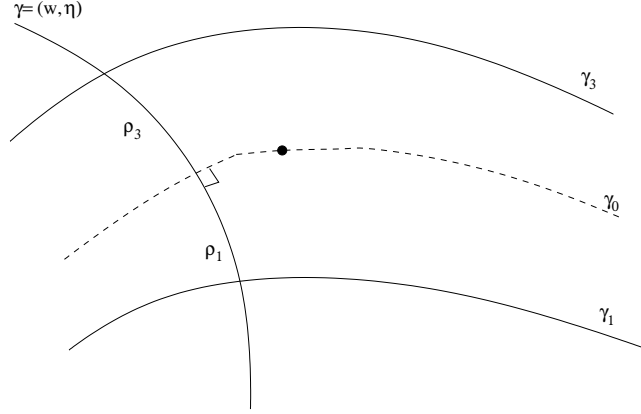
and the identity (3.21) becomes:

$$\mu(\{\gamma \in \mathcal{G}_{13} : \rho(\gamma) \leq \rho_0\}) = \int_{\mathcal{G}_D} \delta(\rho_0 - \rho) \cot \alpha_1 \cot \alpha_3 \sinh \rho \, d\mu. \quad (3.22)$$

As γ_1 and γ_3 are disjoint, there is a unique geodesic γ' orthogonal to both of them. By considering the mid point of the chord of γ' as the origin and choose one oriented geodesic passing it, we can define the polar coordinates (w, η) of $\mathcal{G}_{\mathbb{H}}$. Recall that under the polar coordinates, we have:

$$d\mu = \frac{\cosh w}{2} dw d\eta.$$

Let $\gamma = (w, \eta) \in \mathcal{G}_0$. Let γ_0 be the geodesic passing the origin orthogonal to γ at the point z_0 . Let z_1 be the intersection point of γ with γ_1 and z_3 be the intersection point of γ with γ_3 . Let ρ_1 denote the hyperbolic distance between z_0 and z_1 and ρ_3 denote the hyperbolic distance between z_0 and z_3 .



Fix η . Then by hyperbolic trigonometry, we have:

$$dw = -\frac{d\rho}{\sinh \rho_1 \cot \alpha_1 + \sinh \rho_3 \cot \alpha_3}.$$

By changing variables, we have:

$$\mu(\{\gamma \in \mathcal{G}_{13} : \rho(\gamma) \leq \rho_0\}) = -\frac{1}{2} \int_{\mathcal{G}_D} \delta(\rho_0 - \rho) \frac{\cot \alpha_1 \cot \alpha_3 \sinh \rho \cosh w}{\sinh \rho_1 \cot \alpha_1 + \sinh \rho_3 \cot \alpha_3} \, d\rho d\eta.$$

which implies:

$$\mu(\{\gamma \in \mathcal{G}_{13} : \rho(\gamma) \leq \rho_0\}) = \frac{1}{2} \int_{[\eta]} \frac{\cot \alpha_1(\rho_0, \eta) \cot \alpha_3(\rho_0, \eta) \sinh \rho_0 \cosh w(\rho_0, \eta)}{\sinh \rho_1(\rho_0, \eta) \cot \alpha_1(\rho_0, \eta) + \sinh \rho_3(\rho_0, \eta) \cot \alpha_3(\rho_0, \eta)} \, d\eta. \quad (3.23)$$

where $[\eta]$ is the set of η such that there exists $\gamma \in \mathcal{G}_0$ with angle parameter η and chord length ρ_0 .

Let M_{ij} be the chord length distribution with respect to γ_i and γ_j . The above formula yields that:

$$\int_0^{\rho_0} dM_{13} = \frac{1}{2} \int_{[\eta]} \frac{\cot \alpha_1(\rho_0, \eta) \cot \alpha_3(\rho_0, \eta) \sinh \rho_0 \cosh w(\rho_0, \eta)}{\sinh \rho_1(\rho_0, \eta) \cot \alpha_1(\rho_0, \eta) + \sinh \rho_3(\rho_0, \eta) \cot \alpha_3(\rho_0, \eta)} d\eta.$$

By the same argument, we have that:

$$\int_0^{\rho_0} dM_{24} = \frac{1}{2} \int_{[\eta]} \frac{\cot \alpha_2(\rho_0, \eta) \cot \alpha_4(\rho_0, \eta) \sinh \rho_0 \cosh w(\rho_0, \eta)}{\sinh \rho_2(\rho_0, \eta) \cot \alpha_2(\rho_0, \eta) + \sinh \rho_4(\rho_0, \eta) \cot \alpha_4(\rho_0, \eta)} d\eta,$$

The chord length distributions dM_{12} , dM_{23} , dM_{34} and dM_{14} have been computed in the former section. Then the chord length distribution $M_Q = (\rho_Q)_* \mu$ for Q is the following:

$$\begin{aligned} dM_Q(\rho_0) &= dM_{12}(\rho_0) + dM_{23}(\rho_0) + dM_{34}(\rho_0) + dM_{14}(\rho_0) + dM_{13}(\rho_0) + dM_{24}(\rho_0) \\ &= \frac{12\rho_0 d\rho_0}{\sinh^2 \rho_0} + dM_{13}(\rho_0) + dM_{24}(\rho_0). \end{aligned}$$

where dM_{13} and dM_{24} are described as above. \square

3.4 General case

From the proofs of the theorems and corollaries in this paper, we can see that the most important tool is the hyperbolic trigonometry. This inspires us that this method can be used to prove the similar results for \mathbb{X}_K .

We define the following function called general sine function for \mathbb{X}_K by the following series:

$$\sin_K(x) = x - \frac{Kx^3}{3!} + \frac{K^2x^5}{5!} - \dots, \quad (3.24)$$

or equivalently by the following formula:

$$\sin_K(x) = \begin{cases} \frac{1}{\sqrt{K}} \sin \sqrt{K}x & , \text{ if } K > 0, \\ x & , \text{ if } K = 0, \\ \frac{1}{\sqrt{-K}} \sinh \sqrt{-K}x & , \text{ if } K < 0. \end{cases}$$

We first give the general sines rules and the general cosine rules. Let T be a geodesic triangle in \mathbb{K} . Let β_1 , β_2 and β_3 be its three angles, and let γ_1 , γ_2 and γ_3 be the three edges opposite to β_1 , β_2 and β_3 respectively. Then we have:

$$\begin{aligned} \frac{\sin \beta_1}{\sin_K(\gamma_1)} &= \frac{\sin \beta_2}{\sin_K(\gamma_2)} = \frac{\sin \beta_3}{\sin_K(\gamma_3)}, \\ \cos \beta_1 &= -\cos \beta_2 \cos \beta_3 + \sin \beta_2 \sin \beta_3 (\sin_K)'(\gamma_1). \end{aligned}$$

Let μ_K to be the isometry invariant measure on the geodesics set \mathcal{G}^K of \mathbb{X}_K . We use 1 to normalize μ_K instead of $1/2$ for the Liouville measure in (2.6). Repeat the proof of Theorem 3.2.1 by replacing the hyperbolic sine rules by the general ones. Then we obtain the general Ambartzumian-Pleijel identity for \mathbb{X}_K :

Theorem 3.4.1. *Let D be a convex compact domain in \mathbb{X}_K with geodesic polygon boundary. Let f , α_1 , α_2 and x be the same as in Theorem 3.2.1. Then we have the general Ambartzumian-Pleijel identity:*

$$\int_{\mathcal{G}_D^K} (f \circ \rho_K) d\mu_K = \int_{\mathcal{G}_D^K} (f' \circ \rho_K) \sin_K(\rho_K) \cot \alpha_1 \cot \alpha_2 d\mu_K + \sum_{i=1}^n \int_0^{|\alpha_i|} f(x) dx. \quad (3.25)$$

By the same argument as in Theorem 3.2.2, we have the \mathbb{X}_K version of the Pleijel's identity:

Theorem 3.4.2. *Let D be a convex compact domain in \mathbb{X}_K with C^1 boundary. Let f , α_1 , α_2 and x be the same as above. Then we have the following identity:*

$$\int_{\mathcal{G}_D^K} (f \circ \rho_K) d\mu_K = \int_{\mathcal{G}_D^K} (f' \circ \rho_K) \sin_K(\rho_K) \cot \alpha_1 \cot \alpha_2 d\mu_K + f(0) L_K(\partial D). \quad (3.26)$$

Moreover, if $f(0) = 0$, we have the \mathbb{X}_K version of the Pleijel's identity.

By taking f to be the constant map, we can find that Corollary 3.3.1 still holds for \mathbb{X}_K (without the factor $1/2$). The \mathbb{X}_K version of Corollary 3.3.2 is slightly different from above. By replacing the function $\sinh x$ by $\sin_K(x)$ and $\cosh x$ by $(\sin_K)'(x)$ in the expressions of the volume form of the hyperbolic metric and the Liouville measure, we obtain for \mathbb{X}_K the general isoperimetric inequality:

$$L(D)^2 \geq 4\pi A(D) - KA(D)^2.$$

Chapter 4

Teichmüller theory

This chapter is a review of Teichmüller theory. We begin with the definitions of Fuchsian group, and Teichmüller space $\mathcal{T}(\Sigma)$ for a closed surface Σ . By considering the decomposition of Σ into pairs of pants, we introduce the Fenchel-Nielsen coordinates of $\mathcal{T}(\Sigma)$. Then we give the definition the mapping class group $\text{Mod}(\Sigma)$ and its presentation by using Dehn twists D_a associated to non-separating simple closed curves a on Σ . The Dehn-Nielsen-Baer Theorem and the Nielsen Realization Problem are also introduced. Later we briefly introduce the Weil-Petersson metric on the Teichmüller space and recall its properties. We also consider the Teichmüller theory for the case where $\Sigma_{g,r}^s$ has boundaries and punctures and we introduce the Poisson structure on the associated Teichmüller space $\mathcal{T}(\Sigma_{g,r}^s)$. Finally, we introduce the Hitchin component which is a generalization of Teichmüller space.

A good reference for Fuchsian groups and hyperbolic geometry is [7]; references for Teichmüller space are [1], [37] and [24]; a reference for mapping class group is [24].

4.1 Teichmüller space

4.1.1 Fuchsian group

Recall that the full orientation preserving isometry group of \mathbb{H} is isomorphic to $\text{PSL}(2, \mathbb{R})$. Throughout Γ will denote a non trivial subgroup of $\text{PSL}(2, \mathbb{R})$.

Definition 4.1.1. *The group Γ is said to be **discrete** if its relative topology with respect to $\text{PSL}(2, \mathbb{R})$ is discrete. Discrete subgroups of $\text{PSL}(2, \mathbb{R})$ are called **Fuchsian groups**.*

Definition 4.1.2. *The action of Γ on \mathbb{H} is said to be **properly discontinuous** if for any compact subset K of \mathbb{H} , we have:*

$$|\{A \in \Gamma : A(K) \cap K \neq \emptyset\}| < \infty.$$

The action of Γ on \mathbb{H} is said to be **free** if every non-identity element in Γ has no fixed point in \mathbb{H} .

The next proposition is well known:

Proposition 4.1.1. *The following are equivalent:*

- (1) *The group Γ acts properly discontinuously and freely on \mathbb{H} ;*
- (2) *The group Γ is a torsion free Fuchsian group;*
- (3) *The quotient space \mathbb{H}/Γ is Hausdorff and the quotient map is a covering map.*

Definition 4.1.3. *We say that a torsion free Fuchsian group Γ is **cocompact** if the quotient space \mathbb{H}/Γ is compact.*

Henceforth we assume that Γ acts properly discontinuously on \mathbb{H} .

Definition 4.1.4. *A **fundamental domain** D of Γ is an open set in \mathbb{H} satisfying:*

- (1) *If $A \neq id$, then $A(D) \cap D = \emptyset$;*
- (2) $\bigcup_{A \in \Gamma} A(\overline{D}) = \mathbb{H}$.

Example 4.1.1. *The first example of Fuchsian group is the subgroup $\langle B \rangle$ of $\text{PSL}(2, \mathbb{R})$ generated by a single hyperbolic element B . The action of B has two distinct fixed point on the boundary of \mathbb{H} . The geodesic γ ending at these two fixed points of B is fixed by B and B acts on γ by translation. Choose another geodesic γ' in \mathbb{H} intersecting γ . The domain in \mathbb{H} bounded by γ' and $B(\gamma')$ is a fundamental domain of B . The quotient of \mathbb{H} by $\langle B \rangle$ is an annulus with infinite area, thus the Fuchsian group $\langle B \rangle$ is not cocompact.*

There is a special kind of fundamental domain for Γ , called a Dirichlet domain defined as follows:

Definition 4.1.5. *Let z_0 be a point in \mathbb{H} . The **Dirichlet domain** of Γ centered at z_0 is:*

$$D_\Gamma(z_0) = \{z \in \mathbb{H} : d_{\mathbb{H}}(z_0, z) < d_{\mathbb{H}}(z_0, A(z)), \forall A \in \Gamma\}.$$

Moreover, if Γ acts freely and D is compact, then it is not hard to see that D is a finite sided compact polygonal domain in \mathbb{H} such that:

- (1) *The action of Γ on \mathbb{H} induces an identification in pairs among the sides of D ;*
- (2) *The identification among sides induces an identification among vertices. For each vertex, the sum of internal angles associated to those vertices identified with it is 2π .*

Reciprocally one has the celebrated Poincaré's theorem of fundamental polygon:

Theorem 4.1.1. *Let D be a compact polygon in \mathbb{H} . If the sides of D are identified in pairs by isometries of \mathbb{H} and the sum of internal angles associated to each cycle is 2π , then the subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ generated by the identification is a torsion free Fuchsian group and D is the fundamental domain of Γ .*

4.1.2 Hyperbolic structure on closed surfaces

Definition 4.1.6. *A **hyperbolic structure** on Σ is a maximal collection of charts $\{(U_j, \phi_j)\}_{j \in J}$ such that:*

- (1) *The collection of open sets $\{U_j\}_{j \in J}$ is an open cover of Σ ;*
- (2) *Each ϕ_j is continuous from U_j to \mathbb{H} such that ϕ_j is homeomorphism onto its image;*
- (3) *If $U_j \cap U_k$ is not empty, then the map:*

$$\phi_{jk} = \phi_j \circ \phi_k^{-1} : \phi_k(U_j \cap U_k) \rightarrow \phi_j(U_j \cap U_k),$$

is the restriction of an isometry of \mathbb{H} .

*The collection $\{(U_j, \phi_j)\}_{j \in J}$ is called an **atlas**.*

Definition 4.1.7. *The surface Σ equipped with a hyperbolic structure is called a **hyperbolic surface**, denoted by S .*

By Klein's Erlanger program, a geometry is a pair (G, X) where G is a Lie group and X is its symmetric space such that G acts on X by isometry with respect to a certain metric on X . From this point of view, a hyperbolic structure is a $(\mathrm{PSL}(2, \mathbb{R}), \mathbb{H})$ -structure.

Let $\tilde{\Sigma}$ denote the universal cover of Σ . Then there is a natural action of the fundamental group $\pi_1(\Sigma)$ of Σ on $\tilde{\Sigma}$. By the Cartan-Hadamard theorem, given a $(\mathrm{PSL}(2, \mathbb{R}), \mathbb{H})$ -structure we obtain a homeomorphism: $Dev : \tilde{\Sigma} \rightarrow \mathbb{H}$ and a homomorphism $hol : \pi_1(\Sigma) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ satisfying:

- (1) The associated hyperbolic surface S is isometric to $\mathbb{H}/hol(\pi_1(\Sigma))$.
- (2) For each $\gamma \in \pi_1(\Sigma)$, the following diagram commutes:

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{Dev} & \mathbb{H} \\ \gamma \downarrow & & \downarrow hol(\gamma) \\ \tilde{\Sigma} & \xrightarrow{Dev} & \mathbb{H} \end{array}$$

- (3) If (Dev', hol') is another pair arising from the same $(\mathrm{PSL}(2, \mathbb{R}), \mathbb{H})$ -structure on Σ , then there exists an element $A \in \mathrm{PSL}(2, \mathbb{R})$ such that $Dev' = A \circ Dev$ and $hol' = \iota_A \circ hol$ where ι_A is the inner automorphism of $\mathrm{PSL}(2, \mathbb{R})$ defined by A .

(4) The image of hol is a cocompact Fuchsian group.

The homeomorphism Dev is called the **developing map** and the homomorphism hol is called the **holonomy**.

From this point of view, equipping a hyperbolic structure to Σ is equivalent to giving a representation hol of the fundamental group $\pi_1(\Sigma)$ in $PSL(2, \mathbb{R})$ whose image is a cocompact Fuchsian group Γ .

The fundamental group $\pi_1(\Sigma)$ has the following presentation:

$$\pi_1(\Sigma) = \langle \gamma_1, \beta_1, \dots, \gamma_g, \beta_g \mid \prod_{j=1}^g [\gamma_j, \beta_j] = 1 \rangle, \quad (4.1)$$

where the generators γ_j and β_j are the distinct homotopic classes of simple closed curves in Σ such that they have represents satisfying: for each j the represents of γ_j and β_j have 1 intersection point and for $j \neq k$ the represents of γ_j and β_j are disjoint from those of γ_k and β_k . Then the holonomy hol map is determined by the images of these generators.

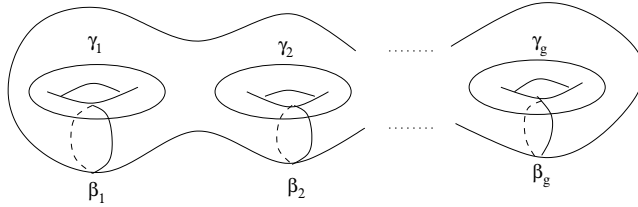


Figure 4.1: Generators of the fundamental group

4.1.3 Teichmüller space

The definition of the Teichmüller space $\mathcal{T}(\Sigma)$ from the point of view of hyperbolic geometry is the following:

Definition 4.1.8. *The Teichmüller space $\mathcal{T}(\Sigma)$ is defined by:*

$$\mathcal{T}(\Sigma) = \{(S, f)\} / \sim,$$

such that S is Σ equipped with a hyperbolic structure, the map $f : \Sigma \rightarrow S$ is a homeomorphism, called marking and two such pair (S_1, f_1) and (S_2, f_2) are said to be equivalent to each other if and only if there is an isometry $\iota : S_1 \rightarrow S_2$ such that $\iota \circ f_1$ is homotopic to f_2 .

Although the torus has no hyperbolic structure, it is useful to consider it as a first example in order to understand Teichmüller space.

Example 4.1.2. *Denote by S_1 the torus. Instead of hyperbolic structures, we consider the flat structures normalized so that the area is 1. The Teichmüller space $\mathcal{T}(S_1)$ of S_1 is the space of homotopy classes of marked normalized flat*

structures on S_1 where the marking is defined in the same way as above. Each flat structure on S_1 induces a lattice on \mathbb{R}^2 which is the universal cover of S_1 . By rotation, translation and rescaling, we can always assume that the origin of \mathbb{R}^2 is in the lattice and the point $(1,0)$ is a generator. Then the flat structure depends on where the second generator of the lattice is. Thus $\mathcal{T}(S_1)$ is identified with the upper half plane. Consider two lattices different from each other by an element in $\mathrm{PSL}(2, \mathbb{Z})$. Then such element induces an isometry between the two flat torus corresponding to the two lattices. This is an example where two toruses have isometric flat structures but different markings. Here is an example where two lattices different from each other by $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

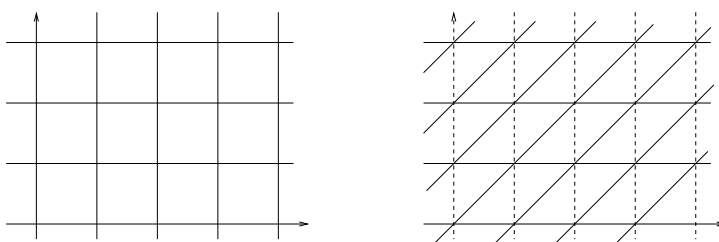


Figure 4.2: Action of A on the universal cover of a torus

By the discussion at the end of last section, another definition of $\mathcal{T}(\Sigma)$ is the following:

Definition 4.1.9. *The **Teichmüller space** $\mathcal{T}(\Sigma)$ is the space of conjugacy classes of discrete faithful representations of $\pi_1(\Sigma)$ in $\mathrm{PSL}(2, \mathbb{R})$, denoted by:*

$$\mathcal{T}(\Sigma) = \mathrm{Hom}_{d,f}(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbb{R})) / \mathrm{PSL}(2, \mathbb{R}).$$

We see from this definition that Teichmüller space $\mathcal{T}(\Sigma)$ is embedded into $\mathrm{PSL}(2, \mathbb{R})^{2g}$. This embedding induces a topology on $\mathcal{T}(\Sigma)$. Notice that $\mathrm{PSL}(2, \mathbb{R})^{2g}$ is a real algebraic variety while Teichmüller space is an analytic manifold.

Moreover, the dimension of $\mathcal{T}(\Sigma)$ can be counted by this definition. The dimension of $\mathrm{PSL}(2, \mathbb{R})$ is 3. As the number of generators is $2g$, the dimension of $\mathcal{T}(\Sigma)$ is at most $6g$. There is one relation in the definition of $\pi_1(\Sigma)$ which removes 3 dimension. The quotient by conjugacy removes another 3 dimension which yields that the dimension of $\mathcal{T}(\Sigma)$ is $6g - 6$.

4.1.4 Fenchel-Nielsen coordinates

Let γ be an essential simple closed curve on Σ , that is, one is not homotopic to a point on Σ . For each hyperbolic structure on Σ , there is a unique geodesic homotopic to γ . Thus we can define a function $l_\gamma : \mathcal{T}(\Sigma) \rightarrow \mathbb{R}^+$ by sending a marked hyperbolic surface to the length of associated geodesic homotopic to γ . We call l_γ the length function associated to γ .

Definition 4.1.10. Let γ_1 and γ_2 be two simple closed curves in Σ . We say that they are **compatible** if they have no intersection point.

Definition 4.1.11. A set \mathcal{P} of simple closed curves on Σ is called a **decomposition of pair of pants** if the curves in \mathcal{P} are pairwise compatible and \mathcal{P} is maximal.

The following picture is an example of one decomposition of pair of pants of Σ_2 :

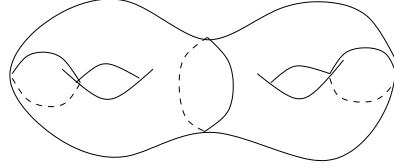


Figure 4.3: Decomposition of pair of pants

Remark 4.1.1. The number of curves in \mathcal{P} is topologically invariant. If the surface Σ has genus g , then $|\mathcal{P}|$ is $3g - 3$.

By cutting Σ along all curves in \mathcal{P} , one obtains $2g - 2$ connected components which are all homeomorphic to a three-holed sphere. Each such connected component is called a pair of pants, denoted by $S_{0,3}$. The marked hyperbolic structure on a pair of pants is given by a pair (P, f) where P is the pair of pants equipped with a hyperbolic structure such that the boundaries are all totally geodesic and $f : S_{0,3} \rightarrow P$ is a homeomorphism. Then the Teichmüller space $\mathcal{T}(S_{0,3})$ is defined to be the space of the homotopy classes of marked hyperbolic structures on $S_{0,3}$.

Lemma 4.1.1. Denote by γ_1, γ_2 and γ_3 the three boundary components of $S_{0,3}$. Then we have the following homeomorphism:

$$\begin{aligned} \mathcal{T}(S_{0,3}) &\rightarrow \mathbb{R}_+^3, \\ (P, f) &\mapsto (l_{\gamma_1}, l_{\gamma_2}, l_{\gamma_3}). \end{aligned}$$

Let $\mathcal{P} = \{\gamma_1, \dots, \gamma_{3g-3}\}$. We call (l_1, \dots, l_{3g-3}) the length parameters. The lengths (l_1, \dots, l_{3g-3}) determines the marked hyperbolic structures on the pairs of pants in $\Sigma \setminus \mathcal{P}$ which are possible to be glued back together.

In order to describe the gluing between pairs of pants, we introduce twist parameters. We first consider one pair of pants. Let P be a hyperbolic pair of pants with three geodesic boundary components γ_i, γ_j and γ_k . The orientation on P induces the orientations on three boundary components. Let α_{ij} be the unique simple geodesic orthogonal to γ_i and γ_j . Let α'_{ij} be an simple arc relating γ_i and γ_j .

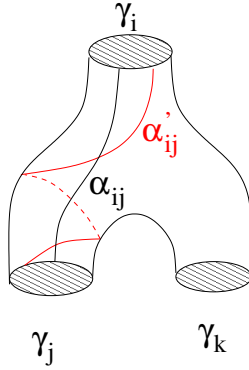


Figure 4.4: Twist parameter

Let δ be a positive number small enough such that γ_i and γ_j have disjoint cylinder neighborhoods N_i and N_j with height δ . We can modify α'_{ij} by isotopy relative to its end points such that between N_i and N_j , it goes along α_{ij} . By taking δ very small, the arc α'_{ij} is isotopic to an arc which first goes around γ_j , then along α_{ij} and finally around γ_j . The signed twist distance at each end with respect to the orientation chosen for each boundary yields the twist number $t_i(\alpha_{ij})$ and $t_j(\alpha_{ij})$ for α_{ij} . Now consider a set of simple geodesics $\{\beta_1, \dots, \beta_{3g-3}\}$ such that the intersections of $\{\beta_1, \dots, \beta_{3g-3}\}$ with each a pair of pants P are three arcs connecting each pair of boundary components of P . Assume β_j going through γ_k , then β_j will have two twist number on both sides of γ_k , denoted by $t_k^+(\beta_j)$ on the left side of γ_k and $t_k^-(\beta_j)$ on the right side. Then the twist parameter for γ_k is given by:

$$t_k = 2\pi \frac{t_k^+(\beta_j) - t_k^-(\beta_j)}{l_k}.$$

For fixed \mathcal{P} and $\{\beta_1, \dots, \beta_{3g-3}\}$, the Fenchel-Nielsen coordinates are given by the $(l_1, \dots, l_{3g-3}, t_1, \dots, t_{3g-3})$.

4.2 Mapping class group

Let $\text{Homeo}^+(\Sigma)$ denote the set of all homeomorphisms of Σ onto itself preserving the orientation. Let $\text{Homeo}_0^+(\Sigma)$ denote the connected component of $\text{Homeo}^+(\Sigma)$ containing the identity map.

Definition 4.2.1. *The **mapping class group** $\text{Mod}(\Sigma)$ of Σ is defined by:*

$$\text{Mod}(\Sigma) = \text{Homeo}^+(\Sigma) / \text{Homeo}_0^+(\Sigma).$$

One special kind of elements in $\text{Mod}(\Sigma)$ is the Dehn twist defined as follows:

Definition 4.2.2. *Let γ be a simple closed curve on Σ . The **Dehn twist** D_γ associated to γ is a homeomorphism induced by the following action on Σ :*

- (1) Cut Σ along γ which yields two boundary components;
- (2) Fix one boundary component and twist the other one by 2π , then glue the two boundaries back together.

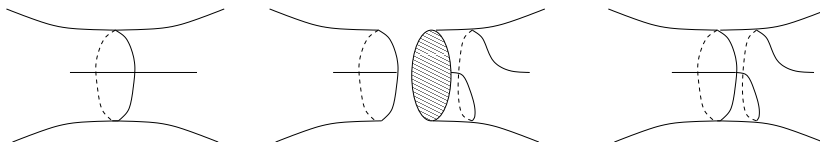


Figure 4.5: A left Dehn twist

Remark 4.2.1. *There are two choices of the 2π -twists. Choose and fix an orientations on $\Sigma \setminus \gamma$. This orientation induces orientations on the two boundary components. The twist along the positive direction is called the left Dehn twist and the twist along the negative direction is called the right Dehn twist. In this thesis, we only consider the right Dehn twist which is the inverse of the Dehn twist in the above picture.*

By the Dehn-Lickorish Theorem, the mapping class group $\text{Mod}(\Sigma)$ is generated by the Dehn twists associated to $3g + 1$ well chosen simple closed curves in Σ . This result has been improved by Humphries who stated that we only need the Dehn twists associated to $2g + 1$ well chosen simple closed curves to generate $\text{Mod}(\Sigma)$. Thus we have a "geometric" set of generators for $\text{Mod}(\Sigma)$, but it is easy to see that $\text{Mod}(\Sigma)$ is not free (having torsion elements), so it is natural to ask what the relations are.

Let a and b be two simple closed curves in Σ . If their geometric intersection number $i(a, b)$ is 0, then the associated Dehn twists commute:

$$D_a D_b = D_b D_a. \quad (4.2)$$

If $i(a, b)$ equals 1, then the following relation is satisfied:

$$D_a D_b D_a = D_b D_a D_b. \quad (4.3)$$

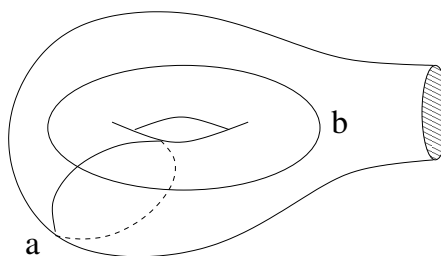


Figure 4.6: Type-1 braid relation

Relation (4.2) is called the **braid-0 relation** and Relation (4.3) is called the **braid-1 relation**.

Now consider a two holed torus embedded in Σ which is π_1 -injective. Let a, b, c, e and f be the five curves on it as follows:

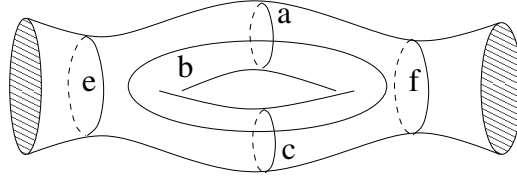


Figure 4.7: Chain relation

Then the associated Dehn twists D_a, D_b, D_c, D_e and D_f satisfy the following relation:

$$(D_a D_b D_c)^4 = D_e D_f,$$

and we call it a **chain relation**.

The last relation is called the **lantern relation**. It is described as follows: consider a four-holed sphere embedded in Σ which is π_1 -injective and denote by $a_0, a_1, a_2, a_3, a_{12}, a_{23}$ and a_{13} the following curves:

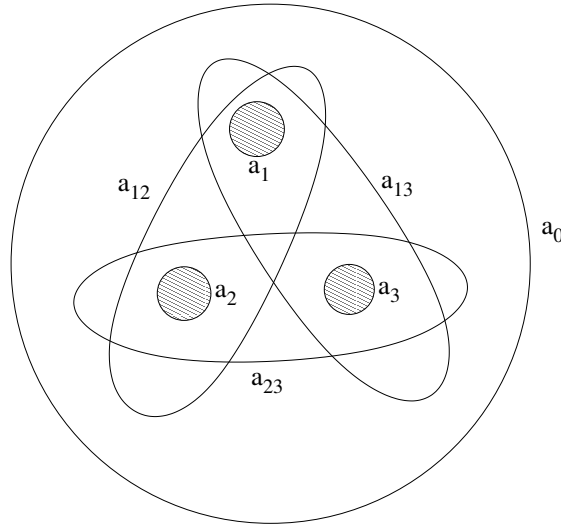


Figure 4.8: Lantern Relation

Then the associated Dehn twists $D_{a_0}, D_{a_1}, D_{a_2}, D_{a_3}, D_{a_{12}}, D_{a_{23}}$ and $D_{a_{13}}$ satisfy:

$$D_{a_0} D_{a_1} D_{a_2} D_{a_3} = D_{a_{13}} D_{a_{23}} D_{a_{12}}.$$

Gervais proved in [29] the following theorem:

Theorem 4.2.1. *For any closed oriented surface Σ of genus $g \geq 2$, the mapping class group has the following presentation:*

- (1) *The generators are the Dehn twists D_a along all non separating simple close geodesics a in S ;*
- (2) *The relation between them are the following:*
 - (a) *The type-0 braid relation;*
 - (b) *The type-1 braid relation;*
 - (c) *The lantern relation;*
 - (d) *The chain relation.*

Another result about mapping class group is the Dehn-Nielsen-Baer Theorem. The mapping class group $\text{Mod}(\Sigma)$ is an index-two subgroup of the extended mapping class group $\text{Mod}^\pm(\Sigma)$ where a homeomorphism may reverse the orientation of Σ . The Dehn-Nielsen-Baer Theorem relates $\text{Mod}^\pm(\Sigma)$ to a purely algebraic object. Let $\text{Aut}(\pi_1(\Sigma))$ be the automorphisms group of the fundamental group $\pi_1(\Sigma)$. Consider those automorphisms given by the conjugation and the normal subgroup of $\text{Aut}(\pi_1(\Sigma))$ generated by them is called the inner automorphism group of $\pi_1(\Sigma)$, denoted by $\text{Inn}(\pi_1(\Sigma))$. The outer-automorphisms group of $\pi_1(\Sigma)$ is then defined by the quotient:

$$\text{Out}(\pi_1(\Sigma)) = \text{Aut}(\pi_1(\Sigma))/\text{Inn}(\pi_1(\Sigma)).$$

The Dehn-Nielsen-Baer Theorem states that the extended mapping class group is isomorphic to the outer-automorphism group of the fundamental group:

$$\text{Mod}^\pm(\Sigma) = \text{Out}(\pi_1(\Sigma)).$$

The Nielsen Realization Problem is also an interesting problem about the mapping class group. It asks whether a finite subgroup of $\text{Mod}(\Sigma)$ can be realized as a group of isometries of a hyperbolic metric on Σ . This problem is answered by Kerckhoff positively in [40] using the convexity of the length function along the left earthquake defined by Thurston.

4.3 Weil-Petersson geometry on Teichmüller space

Let M_0 and M be two Riemann surfaces defined by equipping complex structures with Σ where M_0 is fixed . By a marking on M we mean a quasiconformal map $f : M_0 \rightarrow M$. Then Teichmüller space $\mathcal{T}(\Sigma)$ can also be defined as the space of equivalent classes of marked complex structures on Σ where two marked complex structure (M_1, f_1) and (M_2, f_2) are equivalent if there exists a biholomorphism $f : M_1 \rightarrow M_2$ such that $f \circ f_1$ is homotopic to f_2 . Then by the deformation theory introduced by Kodaira and Spencer, the tangent space of $\mathcal{T}(\Sigma)$ at a point (M, f) is the infinitesimal deformation of complex structure

on M defined as the first cohomology group of holomorphic vector field which turns out to be $H^1(M, K^{-1})$ where K is the canonical bundle over M . By Serre duality, its dual is $H^0(M, K^2)$ which is the space of holomorphic quadratic differential forms.

In [58], Weil introduced a cometric on $\mathcal{T}(\Sigma)$ using Petersson inner product on $H^0(M, K^2)$ which is identified with the cotangent space of $\mathcal{T}(\Sigma)$ by the discussion above. More precisely, let ϕ_1 and ϕ_2 be two holomorphic quadratic differential. Let $d \text{Vol}$ be the hyperbolic volume form of M . Then the Weil-Petersson cometric is given by:

$$\langle \phi_1 \phi_2 \rangle = \int_M \frac{\phi_1 \bar{\phi}_2}{d \text{Vol}}.$$

By duality, the Weil-Petersson metric is defined.

In [3], [2] and [8], the authors constructed the complex structure on the Teichmüller space and proved that the Weil-Petersson metric is Kähler and its Ricci curvature and holomorphic sectional curvature are both negative. In [60], Wolpert proved that the sectional curvature of Weil-Petersson metric is also negative. In [59], by estimating the Weil-Petersson length of the path associated to the pinching of a simple closed geodesic, Wolpert proved that the Weil-Petersson metric is not complete. By the work of Masur in [43], the completion of Weil-Petersson metric is the augmented Teichmüller space $\overline{\mathcal{T}(\Sigma)}$ constructed by adding the stratas to the non complete part of the boundary of Teichmüller space. The quotient of $\overline{\mathcal{T}(\Sigma)}$ by the mapping class group is the Deligne-Mumford compactification of the moduli space. In [62], Wolpert proved that the length function is strictly convex along a Weil-Petersson geodesic.

4.4 Surfaces with boundaries and punctures

4.4.1 Definitions of Teichmüller space and mapping class group

In this section, we consider the compact oriented surface with marked points and boundary components. We denote by $\Sigma_{g,r}^s$ the surface of genus g with s marked points and r boundary components.

Example 4.4.1. *The following picture represents the surface $\Sigma_{2,1}^3$:*

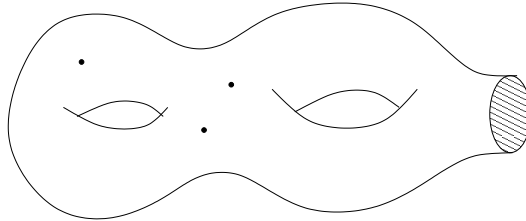


Figure 4.9: Surface $\Sigma_{2,1}^3$

The hyperbolic structure on $\Sigma_{g,r}^s$ that we consider is such that the boundary components are totally geodesic and the punctures are at infinite distance from any other points inside. The marking homeomorphism for $\Sigma_{g,r}^s$ is required to be identity on each boundary component and send punctures to punctures. Notice that punctures can be permuted by a such homeomorphism. With these conventions, the Teichmüller space $\mathcal{T}(\Sigma_{g,r}^s)$ and the mapping class group $\text{Mod}(\Sigma_{g,r}^s)$ are defined in the same way as for the closed surface case.

Remark 4.4.1. *When $s > 1$, the mapping class group $\text{Mod}(\Sigma_{g,r}^s)$ can no longer be generated only by Dehn twists, since no Dehn twist can permute punctures. We denote by $\text{PMod}(\Sigma_{g,r}^s)$ the subgroup of $\text{Mod}(\Sigma_{g,r}^s)$ generated by the Dehn twists on $\Sigma_{g,r}^s$ and call it the pure mapping class group.*

4.4.2 Shearing coordinates

Shearing coordinates were introduced by Thurston in [56]. They were systematically studied by Bonahon in [10] and developed for the decorated Teichmüller theory by Penner in [46] and [48], and to the higher Teichmüller theory by Fock-Goncharov in [25]. In this section, we recall the shearing coordinate system of the Teichmüller space $\mathcal{T}(\Sigma_{g,r}^s)$ of $\Sigma_{g,r}^s$ and the associated expression of the Poisson structure on $\mathcal{T}(\Sigma_{g,r}^s)$. To simplify the notation, we use Σ instead of $\Sigma_{g,r}^s$.

Definition 4.4.1. *An **arc** is an isotopy class of a simple curve on Σ ending either at a puncture or spiraling to a boundary component which is non homotopic to a point or a puncture of Σ .*

Definition 4.4.2. *An **ideal triangulation** of Σ is a maximal collection of distinct arcs which have pairwise disjoint represents.*

Let T be an ideal triangulation. Let α be an arc of T which is the common boundary of two distinct ideal triangles whose union is an embedded quadrilateral in Σ .

Definition 4.4.3. *A **flip** (or diagonal exchange) on α in T is to substitute α by the other diagonal α' of this ideal quadrilateral to get a new ideal triangulation T' of Σ .*

Given a hyperbolic structure on Σ , each arc has a unique geodesic representative. In the following, by ‘‘arc’’ we mean its geodesic representative. Fix an orientation on α . Let $\tilde{\alpha}$ be one of its lifts. Then $\tilde{\alpha}$ will be an oriented diagonal in an ideal quadrilateral Q with one triangle Δ_l on its left side and another triangle Δ_r on its right side. In each triangle, the vertex not lying on $\tilde{\alpha}$ can be orthogonally projected to $\tilde{\alpha}$. We denote by v_l and v_r respectively the images of the vertices of Δ_l and Δ_r .

Definition 4.4.4. *We call the directed hyperbolic distance from v_l to v_r the **shearing coordinate** on α .*

An explicit formula is given by using the cross-ratios of the vertices of the associated ideal quadrilateral. Denote by $\{p_1, p_2, p_3, p_4\}$ the four vertices of the associated ideal quadrilateral Q with a counter-clockwise order. Assuming that p_1 and p_3 are the vertices of $\tilde{\alpha}$ with the orientation such that p_2 is on the left and p_4 is on the right. Then the shearing coordinate on α is defined by:

$$t(\alpha) = \log -[p_1, p_3; p_2, p_4] = \log - \frac{(p_1 - p_2)(p_3 - p_4)}{(p_1 - p_4)(p_3 - p_2)}$$

The shearing coordinate system depends on the choice of the ideal triangulation. By doing a flip we get another shearing coordinate system. Let T' be the ideal triangulation coming from T by flipping α to α' . By comparing the formulas of cross-ratios before and after the flip, we obtain the following relation:

$$t'(\beta) = \begin{cases} -t(\alpha) & \text{if } \beta = \alpha' \\ t(\beta) + \epsilon_T(\alpha, \beta)\phi(\text{sign}(\epsilon_T(\alpha, \beta)))t(\alpha) & \text{if } \beta \text{ and } \alpha' \text{ are adjacent but } \beta \neq \alpha' \\ t(\beta) & \text{otherwise} \end{cases},$$

where $\phi(z) = \log(1 + \exp(z))$ and the function ϵ_T is defined in the following. Fix an orientation on Σ . Let Δ be an ideal triangle on Σ which is a connected component of $\Sigma \setminus T$. Let $E(\Delta)$ be the set of its edges. We can define an anti-symmetric map

$$\epsilon_{T,\Delta} : E(\Delta) \times E(\Delta) \rightarrow \{0, \pm 1\}$$

in the following way:

$$\epsilon_{T,\Delta}(\alpha, \beta) = \begin{cases} -1 & \text{if } \beta \text{ comes after } \alpha' \text{ counter-clockwise} \\ 1 & \text{if } \beta \text{ comes after } \alpha' \text{ clockwise} \\ 0 & \text{otherwise} \end{cases},$$

where (α, β) is in $E(\Delta) \times E(\Delta)$. By taking the sum of $\epsilon_{T,\Delta}$ over all ideal triangles Δ , we obtain the following anti-symmetric map ϵ_T :

$$\epsilon_T : E(T) \times E(T) \rightarrow \{0, \pm 1, \pm 2\},$$

where $E(T)$ is the set of arcs in T .

This anti-symmetric map ϵ_T also gives us the Poisson structure on the Teichmüller space by the following bi-vector field:

$$P(T) = \sum_{\alpha, \beta} \epsilon_T(\alpha, \beta) \frac{\partial}{\partial t(\alpha)} \wedge \frac{\partial}{\partial t(\beta)}.$$

4.4.3 Weil-Petersson geometry

The Weil-Petersson metric is also well-defined in this case. It has also the non-completeness and the negative curvature. But there exists a difference in the

Weil-Petersson geometry when $r > 0$. Instead of the whole Teichmüller space, the Kähler property of the Weil-Petersson metric only exists on the symplectic leaves in $\mathcal{T}(\Sigma_{g,r}^s)$ with respect to its Poisson structure. Assume that $r > 0$ and denote by $\gamma_1, \dots, \gamma_r$ the r boundary components. Let L_1, \dots, L_r be r positive real numbers. A symplectic leaf $\mathcal{T}(\Sigma_{g,r}^s, L_1, \dots, L_r)$ in $\mathcal{T}(\Sigma_{g,r}^s)$ consists of all marked hyperbolic metric on $\Sigma_{g,r}^s$ such that γ_i has the length L_i .

Property 4.4.1. *A symplectic leaf is not totally geodesic in $\mathcal{T}(\Sigma_{g,r}^s)$ with respect to the Weil-Petersson metric.*

The proof is a simple argument by using the strict convexity of the length function associated to a boundary component. Moreover, by applying this argument to the length function of any simple closed geodesic on the surface, we conclude that the length of any simple closed geodesic cannot be constant along a Weil-Petersson geodesic.

4.5 Hitchin component

Let Σ be an oriented closed surface. In [36], Hitchin studied the connected components of the representation space $\text{Hom}(\pi_1(\Sigma), \text{PSL}(n, \mathbb{R}))$ using Higgs bundle techniques. The representation space is a singular real algebraic variety and there is a natural action of the Lie group $\text{PSL}(n, \mathbb{R})$ by conjugation. This action is not free. Consider the quotient space $\text{Hom}(\pi_1(\Sigma), \text{PSL}(n, \mathbb{R}))/\text{PSL}(n, \mathbb{R})$. It is also a real singular algebraic variety, but its quotient topology is not Hausdorff. One may consider a representation whose images are contained in a unipotent subgroup of $\text{PSL}(n, \mathbb{R})$, then the identity representation is contained in all its neighborhoods. To adjust this problem, we consider the following identification of its points: two points in $\text{Hom}(\pi_1(\Sigma), \text{PSL}(n, \mathbb{R}))/\text{PSL}(n, \mathbb{R})$ are identified with each other if and only if one point is in every neighborhood of the other point. We denote by $\text{Hom}(\pi_1(\Sigma), \text{PSL}(n, \mathbb{R}))/\text{PSL}(n, \mathbb{R})$ the quotient space of $\text{Hom}(\pi_1(\Sigma), \text{PSL}(n, \mathbb{R}))$ by the conjugacy action and the identification. This space is proved to be identified with the space of conjugacy class of all semi-simple representations denoted by:

$$\text{Rep}_n(\Sigma) = \text{Hom}^{s.s.}(\pi_1(\Sigma), \text{PSL}(n, \mathbb{R}))/\text{PSL}(n, \mathbb{R}).$$

The two spaces $\text{Rep}_n(\Sigma)$ and $\text{Hom}(\pi_1(\Sigma), \text{PSL}(n, \mathbb{R}))$ have the same number of connected components. Hitchin investigated the connected components for $n \geq 2$ and, in particular showed:

Theorem 4.5.1. *If $n > 2$, the space $\text{Rep}_n(\Sigma)$ has 3 components if n is odd, and 6 components if n is even. Moreover, each of these components is homeomorphic to a ball of real dimension $(2g - 2)(n^2 - 1)$.*

The Lie group $\text{PSL}(2, \mathbb{R})$ has a unique irreducible representation ι in $\text{PSL}(n, \mathbb{R})$. Let ρ be in $\mathcal{T}(\Sigma)$. The composition $\iota \circ \rho$ gives a representation of $\pi_1(\Sigma)$ into $\text{PSL}(n, \mathbb{R})$. We call it a **n -Fuchsian representation**. Hitchin also studied the components containing n -Fuchsian representation.

Theorem 4.5.2. *There is a single component containing all n -Fuchsian representation in $\text{Rep}_n(\Sigma)$ if n is odd and exactly 2 such components if n is even.*

Definition 4.5.1. *The **Hitchin component** $H_n(\Sigma)$ is a connected component in $\text{Rep}_n(\Sigma)$ containing n -Fuchsian representation.*

Remark 4.5.1. *The definition of n -Fuchsian representation induces an embedding of $\mathcal{T}(\Sigma)$ in $H_n(\Sigma)$. The image of this embedding is called **Fuchsian locus**.*

The above results are the generalization of [30] where Goldman gave a full description of the connected components for the representation space $\text{Hom}(\pi_1(\Sigma), \text{PSL}(2, \mathbb{R}))$.

A point in Teichmüller space corresponds to a marked hyperbolic structure on Σ . There is an analogous result for $H_3(\Sigma)$ due to Choi and Goldman (see [31] and [21]):

Theorem 4.5.3. *For $n = 3$, the Hitchin component $H_3(\Sigma)$ consists of the holonomies of real convex projective structures on Σ .*

From this point of view, each point in $H_3(\Sigma)$ determines a strictly convex C^1 -curve in \mathbb{RP}^2 with an action of $\pi_1(\Sigma)$ on it given by the holonomy. In [42] Labourie gave a generalization of this correspondence for $H_n(\Sigma)$:

Theorem 4.5.4. *For each representation ρ in the Hitchin component $H_n(\Sigma)$, there exists a ρ -equivariant hyperconvex Frenet curve in \mathbb{RP}^{n-1} :*

$$\theta : \partial_\infty \pi_1(\Sigma) \rightarrow \mathbb{P}(\mathbb{R}^n).$$

The reciprocal of this theorem was proved by Guichard in [33].

Chapter 5

Pressure metric

In [15], the authors defined the pressure metric on the space $\mathcal{C}(\Gamma, m)$ of conjugacy classes of regular irreducible convex representations of a word hyperbolic group Γ in $\mathrm{SL}(m, \mathbb{R})$. The main ingredient is the thermodynamic formalism developed by Bowen, Parry-Pollicott, Ruelle and others. A representation in $\mathcal{C}(\Gamma, m)$ can be identified with a Hölder reparametrization of the Gromov geodesic flow $U_0\Gamma$ associated to Γ obtained by integrating a Hölder continuous positive function f defined on $U_0\Gamma$ along geodesics. Thus we can embed $\mathcal{C}(\Gamma, m)$ into the space $P_0(U_0\Gamma)$ of pressure zero Hölder functions on the shift space associated to $U_0\Gamma$. By the thermodynamic formalism, the pressure function is analytic and this in turn implies the analyticity of the entropy function. Moreover, the Hessian of pressure function on $P_0(U_0\Gamma)$ is positive semi-definite. By proving its non-degeneracy on the image of the embedding of $\mathcal{C}(\Gamma, m)$ and pulling back to $\mathcal{C}(\Gamma, m)$, we obtain the pressure metric on $\mathcal{C}(\Gamma, m)$. Since this construction only depend on the image of the representation, this metric is $\mathrm{Out}(\Gamma)$ -invariant. In particular, we obtain a mapping class group invariant Riemannian metric on Hitchin component. In the beginning of this chapter, we briefly recall the thermodynamic formalism and the construction of pressure metric in [15]. In the following part we consider a special case where Γ is the fundamental group of one holed torus and $m = 2$. Thus by the result in [15], we obtain the pressure metric on Teichmüller space $\mathcal{T}(\mathbb{T})$. We describe a degeneration of hyperbolic structures on \mathbb{T} to metrics on its fat graph \mathbb{G} which has two vertices connected by three edges. By this degeneration, we are able to glue the moduli space $\mathcal{M}(\mathbb{G}, 1)$ of metrics on \mathbb{G} to the boundary of $\mathcal{T}(\mathbb{T})$. By the work of Sharp and Pollicott in [51], the pressure metric is also well defined on $\mathcal{M}(\mathbb{G}, 1)$ which is called the Weil-Petersson type metric in [51]. The above gluing shows that the two pressure metrics on the $\mathcal{T}(\mathbb{T})$ and $\mathcal{M}(\mathbb{G}, 1)$ are both the pullback of the pressure form defined on $P_0(U_0\Gamma)$. The entropy function is involved in the other interpretation of the pressure metric by using the renormalized intersection function. In the end of this chapter, we prove that the entropy function is non-constant on each symplectic leaf $\mathcal{T}(\Sigma_{g,r}, L_1, \dots, L_r)$ for $\Sigma_{g,r}$ with $g > 1$ and $r > 1$ and non-constant on the symplectic leaf $\mathcal{T}(\mathbb{T}, L)$ with L large enough.

This work is motivated by the comparison between the Weil-Petersson metric and the pressure metric on $\mathcal{T}(\mathbb{T})$.

References for thermodynamic formalism on symbolic dynamical system are [45] and [53].

5.1 Thermodynamic formalism

Let $\{1, \dots, k\} \subset \mathbb{N}$ with $k \geq 2$. Let A be a $k \times k$ matrix with all entries taken values in $\{0, 1\}$. We define the **two-sided shift space** X_A as follows:

$$X_A = \{x = (x_n)_{n \in \mathbb{Z}} : \forall n \in \mathbb{Z}, x_n \in \{1, \dots, k\} \text{ and } A(x_n, x_{n+1}) = 1\},$$

equipped with a **shift map** $\sigma : X_A \rightarrow X_A$ such that if $\sigma(x) = y$, then $y_n = x_{n+1}$.

The matrix A is said to be **irreducible** if for each pair i and j in $\{1, \dots, k\}$, there exists a $n \in \mathbb{N}$ such that $A^n(i, j) > 0$. Its period d is the highest common factor of

$$\inf\{n : A^n(i, i) > 0\},$$

for all $i \in \{1, \dots, k\}$. The irreducible matrix A is said to be **aperiodic** if $d = 1$. We shall always assume the matrix A to be aperiodic.

To the shift space X_A , we can associate a **one-sided shift space** X_A^+ defined by,

$$X_A^+ = \{x = (x_n)_{n \in \mathbb{N}} : \forall n \in \mathbb{N}, x_n \in \{1, \dots, k\} \text{ and } A(x_n, x_{n+1}) = 1\},$$

equipped with a shift map $\sigma^+ : X_A^+ \rightarrow X_A^+$ defined in the same way as σ . We equip X_A^+ with the Tychonov topology which makes it compact. Let $0 < \xi < 1$. We define a distance function d_ξ by:

$$d_\xi(x, y) = \xi^{-m}, \tag{5.1}$$

where $m = \inf\{n : x_n \neq y_n\}$. The topology induced by d_ξ is equivalent to the Tychonov topology.

Let F_ξ^+ denote the space of complex valued Lipschitz continuous functions with respect to d_ξ on X_A^+ . For $f \in F_\xi^+$, we define the following quantity:

$$\text{var}_n(f) = \{|f(x) - f(y)| : x, y \in X_A^+, \forall i < n, x_i = y_i\}.$$

Definition 5.1.1. *The least Hölder constant $|f|_\xi$ of f is defined as follows:*

$$|f|_\xi = \sup\left\{\frac{\text{var}_n(f)}{\xi^n} : n \in \mathbb{N}\right\}.$$

It is not hard to see that $|f|_\xi < \infty$ if and only if there exists a positive constant C such that:

$$|f(x) - f(y)| \leq C d_\xi(x, y),$$

for all x and y in X_A^+ .

Notice that $|\cdot|_\xi$ is only a semi-norm, since $|f|_\xi = 0$ if f is a constant function on X_A^+ . By adding the uniform norm $|f|_\infty = \sup\{f(x) : x \in X_A^+\}$, one obtains a norm $\|\cdot\|_\xi$ on F_ξ^+ given by:

$$\|f\|_\xi = |f|_\infty + |f|_\xi.$$

An important result in [45] is the following proposition:

Proposition 5.1.1. *The space $(F_\xi^+, \|\cdot\|_\xi)$ is a Banach space.*

Remark 5.1.1. *In the whole theory, we are allowed to change the value of ξ in $]0, 1[$ to define different metrics on X_A^+ by using Formula (5.1). The above proposition is true for every $\xi \in]0, 1[$. Moreover, let $0 < \xi < \xi' < 1$, then we have $F_{\xi'}^+ \supset F_\xi^+$.*

Remark 5.1.2. *If a function f is a Hölder continuous of exponent α on X_A^+ with respect to d_ξ , then it is Lipschitz continuous on X_A^+ with respect to d_{ξ^α} . As we consider all Hölder functions for all Hölder exponents, there is no need to distinguish these two concept and we follow the reference [45] and use the Lipschitz continuity.*

For each $f \in F_\xi^+$ real valued, we define its **Ruelle operator** $L_f : F_\xi^+ \rightarrow F_\xi^+$ by the following formula:

$$L_f(g)(x) = \sum_{x=\sigma^+(y)} e^{f(y)} g(y).$$

Then we have the Ruelle-Perron-Frobenius theorem:

Theorem 5.1.1. *Let $f \in F_\xi^+$ be real valued. Then,*

- (1) *The Ruelle operator L_f has a simple maximal positive eigenvalue β with a corresponding strictly positive eigenfunction $h \in F_\xi^+$;*
- (2) *The spectrums of L_f other than β are contained in a disc with the radius strictly smaller than β ;*
- (3) *There is a unique probability measure μ such that $(L_f)^* \mu = \beta \mu$;*
- (4) *$\frac{1}{\beta^n} L_f^n(g) \rightarrow h \int_{X_A^+} g d\mu$ uniformly for all $g \in F_\xi^+$ where h is as above and $\int_{X_A^+} h d\mu = 1$.*

Let p be a σ^+ -invariant probability measure on X_A^+ . Let γ be a p -measurable finite partition of X_A^+ . Denote by

$$\bigvee_{i=0}^{n+1} (\sigma^+)^i \gamma$$

the sigma algebra generated by the collection of $(\sigma^+)^i \gamma$ for $0 \leq i \leq (n+1)$. We define $H(\sigma^+, p, \gamma)$ to be the following quantity:

$$- \lim_{n \rightarrow +\infty} \frac{1}{n} \left(\sum_{A \in \mathcal{V}_{i=0}^{n+1}(\sigma^+)^i \gamma} p(A) \log p(A) \right).$$

To verify that the limit exists, we consider the sequence:

$$H_n = - \sum_{A \in \mathcal{V}_{i=0}^{n+1}(\sigma^+)^i \gamma} p(A) \log p(A).$$

The strict concavity of the function $-x \log x$ implies the sub-additivity of H_n that is $H_{n+m} \leq H_n + H_m$. As H_n is a positive for all n , we conclude that the limit of H_n/n exists and is finite as n goes to infinity.

Then we define the entropy of σ^+ with respect to γ as follows:

$$h_p(\sigma^+) = \sup_{\gamma} H(\sigma^+, p, \gamma).$$

Let $M(\sigma^+)$ denote the set of σ^+ -invariant probability measures. By taking the supremum over $M(\sigma^+)$, we define the topological entropy of σ^+ as follows:

$$h(\sigma^+) = \sup\{h_p(\sigma^+) : p \in M(\sigma^+)\}.$$

We have the variational principle:

Proposition 5.1.2. *Let $f \in F_{\xi}^+$ real valued, then there exists a unique σ^+ -invariant probability measure μ such that for any σ^+ -invariant probability measure $p \in M(\sigma^+)$, we have the following inequality:*

$$h_p(\sigma^+) + \int_{X_A^+} f dp \leq h_{\mu}(\sigma^+) + \int_{X_A^+} f d\mu.$$

Moreover, the equality holds if and only if $p = \mu$.

The measure μ is called the **equilibrium state** of f .

We denote by $P(f)$ the quantity:

$$P(f) = \sup_{p \in M(\sigma^+)} \{h_p(\sigma^+) + \int_{X_A^+} f dp\} = h_{\mu}(\sigma^+) + \int_{X_A^+} f d\mu.$$

Then $P(f) = \log \beta(f)$ where $\beta(f)$ is the maximal eigenvalue of L_f . The quantity $P(f)$ is called the **pressure** of f . In particular, let $f = 0$, then we have that $P(0) = h(\sigma^+)$.

Definition 5.1.2. *Let $f, g \in F_{\xi}^+$ real valued. We say that f and g are Livšic cohomologous if there exists a function $h \in F_{\xi}^+$ such that $f = g + h \circ \sigma^+ - h$.*

Then we can verify that two cohomologous functions has the same pressure.

Remark 5.1.3. *This definition is also well-defined for complex valued functions in F_ξ^+ .*

Now we consider the Ruelle operator L_f associated to $f \in F_\xi^+$ which is no longer required to be real valued. The following result gives us the information about the eigenvalue with maximal modulus of L_f :

Theorem 5.1.2. *Let $f = u + iv \in F_\xi^+$. Then the spectral radius of L_f is less than or equal to the top eigenvalue $\exp(P(u))$ of u . If L_f has an eigenvalue of modulus $\exp(P(u))$, then it is simple and unique and the rest of the spectrum is contained in a disc of radius strictly smaller than $\exp(P(u))$. If L_f has no eigenvalue of modulus $\exp(P(u))$, then the spectral radius is strictly smaller than $\exp(P(u))$.*

If L_f has the unique eigenvalue of modulus $\exp(P(u))$, we call this eigenvalue the **maximal eigenvalue** of L_f . Recall that $(F_\xi^+, \|\cdot\|_\xi)$ is a Banach space and the Ruelle operator is a bounded linear operator. Denote by $B(F_\xi^+)$ the Banach space of bounded linear operators on F_ξ^+ . By perturbation theorem (see [39] for more details), the map sending a Ruelle operator to its maximal eigenvalue is analytic on its domain.

The definition of pressure can also be extended to $f \in F_\xi^+$ whose associated Ruelle operator has a maximal eigenvalue λ by defining $P(f) = \log \lambda$ by requiring that if f is cohomologous to $g + c + 2\pi Mi$ where c is a real number and M is a continuous function with integer value, then $P(f) = P(g) + c$. By using the analyticity for the Ruelle operator, we have the analyticity of the pressure function:

Theorem 5.1.3. *The pressure function is defined on an open subset of F_ξ^+ and it is analytic from its domain to \mathbb{C} .*

Remark 5.1.4. *By analytic, we mean the following. Let B be a complex Banach space. A map $f : \mathbb{C} \rightarrow B$ is **analytic**, if for any linear bounded functional $l : B \rightarrow \mathbb{C}$, the composition $l \circ f : \mathbb{C} \rightarrow \mathbb{C}$ is an analytic function in the usual sense. Let B_1 and B_2 be two Banach spaces. A map $g : B_1 \rightarrow B_2$ is **analytic** if the composition $g \circ f : \mathbb{C} \rightarrow B_2$ is analytic for any analytic map $f : \mathbb{C} \rightarrow B_1$. These notions can be defined similarly for real Banach spaces.*

The pressure function has the following properties:

Theorem 5.1.4. (1) *Let $f, g \in F_\xi^+$ are real valued. If $P(f) = 0$, then*

$$P'(0) = \left. \frac{dP(f + sg)}{ds} \right|_{s=0} = \int_{X_A^+} g dm,$$

where m is the equilibrium state of f .

(2) *Let $f, g \in F_\xi^+$ are real valued. If $P(f) = 0$ and $\int_{X_A^+} g dm = 0$ where m is the equilibrium state of f , then*

$$P''(0) = \left. \frac{d^2 P(f + sg)}{ds^2} \right|_{s=0} = \lim_{n \rightarrow +\infty} \int_{X_A^+} \left(\sum_{i=1}^{n-1} g \circ (\sigma^+)^i \right)^2 dm.$$

The Hessian of P on the pressure zero Hölder function space $P_0(X_A)$ is called the **pressure form**. By (2) in Theorem 5.1.4, we can deduce that the pressure form is positive semi-definite.

Given a strictly positive function $f \in F_A^+$, we define the suspension space $X_{A,f}^+$ from X_A^+ as follows:

$$X_{A,f}^+ = \{(x, s) \in X_A^+ \times \mathbb{R}^+ : 0 \leq s \leq f(x)\} / (\sigma(x)^+, 0) \sim (x, f(x)).$$

The suspension space $X_{A,f}^+$ admits a suspension flow ϕ_f^t defined by:

$$\phi_f^t(x, s) = \begin{cases} (x, s+t) & , \text{ if } s+t \leq f(x) \\ (\sigma^+(x), s+t-f(x)) & , \text{ if } s+t \geq f(x) \end{cases} .$$

A ϕ_f -invariant measure p_f is always equivalent to a product measure of σ^+ invariant measure p on X_A^+ and the Lebesgue measure on the "vertical line" for each $x \in X_A^+$. Consider the Hölder continuous functions F on $X_{A,f}^+$. We define the entropy $h_{p_f}(F)$, the topological entropy $h(\phi_f)$ of the suspension flow and the pressure of F in a similar way to the above.

Let $x \in X_A^+$ be in a periodic orbit for the σ^+ action with period n . Then $(x, 0) \in X_{A,f}^+$ is also in a periodic orbit for ϕ_f and the period is $f(x) + \dots + f((\sigma^+)^{n-1}x)$ denoted by $\lambda(x)$. By using Ruelle's zeta function, it has been proved (see Chapter 6 of [45]) that $P(-h(\phi_f)f) = 0$ and the entropy $h(\phi_f)$ has the following expression:

$$h(\phi_f) = \lim_{T \rightarrow +\infty} \frac{\log |R_T(\phi_f)|}{T},$$

where $R_T(\phi_f) = \{\text{Orb}(x, 0) : \lambda(x) < T\}$.

Remark 5.1.5. *The theory above is also well defined for a two-sided shift space in a natural way. (see Chapters 1, 2 and 3 of [45])*

5.2 Pressure metric

The pressure metric was studied in [15] by Bridgeman, Canary, Labourie and Sambarino. Given a word hyperbolic group Γ , they consider the space $\mathcal{C}(\Gamma, m)$ of regular irreducible convex representations of Γ in $\text{SL}(m, \mathbb{R})$. The pressure metric is an $\text{Out}(\Gamma)$ -invariant Riemannian metric on the smooth generic points of $\mathcal{C}(\Gamma, m)$. In particular, when Γ is the fundamental group, the space $\mathcal{C}(\Gamma, m)$ contains the Hitchin component $H_m(\Sigma)$ in its generic part, thus one obtains a mapping class group invariant Riemannian metric on $H_m(\Sigma)$. In fact, the idea of its construction is inspired by earlier work in Fuchsian case by McMullen in [44] and in quasi-Fuchsian case by Bridgeman in [16] where the thermodynamic formalism is also used.

More precisely, let Γ be the word hyperbolic group. Let $\partial_\infty \Gamma$ be its Gromov boundary. We have the following definition due to Sambarino in [54]:

Definition 5.2.1. A representation ρ of Γ in $\mathrm{SL}(m, \mathbb{R})$ is said to be **convex** if there exist two ρ -equivariant continuous maps:

$$\begin{aligned}\eta &: \partial_\infty \Gamma \rightarrow \mathbb{RP}(m), \\ \theta &: \partial_\infty \Gamma \rightarrow (\mathbb{RP}(m))^*,\end{aligned}$$

such that for distinct points x and y in $\partial_\infty \Gamma$, we have that

$$\eta(x) \oplus \theta(y) = \mathbb{R}^m.$$

In [32], Gromov associated to a hyperbolic group Γ a locally compact finite dimensional hyperbolic metric space $\widetilde{U}_0\Gamma$ satisfying:

(1) It is homeomorphic to $\partial_\infty \Gamma^{(2)} \times \mathbb{R}$, where

$$\partial_\infty \Gamma^{(2)} = \partial_\infty \Gamma \times \partial_\infty \Gamma \setminus \{(x, x) : x \in \partial_\infty \Gamma\};$$

(2) There is a proper cocompact action of Γ on it by a diagonal action on $\partial_\infty \Gamma^{(2)}$ which is an isometry with respect to the metric on $\widetilde{U}_0\Gamma$;

(3) Using the identification in (1), the \mathbb{R} action on $\widetilde{U}_0\Gamma$ is given by translation on the last factor and the orbit of this action induces a quasi-isometry embedding of \mathbb{R} ;

(4) The geodesic flow acts by Lipschitz homeomorphism.

Thus the flow on $\widetilde{U}_0\Gamma$ descends to a flow on the quotient:

$$U_0\Gamma = \widetilde{U}_0\Gamma / \Gamma.$$

Let ρ be a convex representation of Γ in $\mathrm{SL}(m, \mathbb{R})$. Let E_ρ be a flat bundle over $U_0\Gamma$ with fiber \mathbb{R}^m defined by:

$$E_\rho = \widetilde{U}_0\Gamma \times \mathbb{R}^m / \Gamma,$$

where Γ acts on \mathbb{R}^m by considering the action of its ρ -image. By defining the action to be trivial on each fiber, the \mathbb{R} -action on $\widetilde{U}_0\Gamma$ extends to an \mathbb{R} -action on $\widetilde{U}_0\Gamma \times \mathbb{R}^m$ and so we have a flow on the latter space. It is easy to see that this flow descends to a flow ϕ_ρ^t on E_ρ . The limit maps η and θ induce a splitting of E_ρ into the direct sum of Ξ and Θ . The flow ϕ_ρ^t respects this splitting.

Definition 5.2.2. A convex representation $\rho : \Gamma \rightarrow \mathrm{SL}(m, \mathbb{R})$ is **Anosov** if the flow ϕ_ρ^t on $\mathrm{Hom}(\Xi, \Theta)$ is contracted, i.e. given a metric $\|\cdot\|$ on $\mathrm{Hom}(\Xi, \Theta)$, there exists $t_0 > 0$ such that if $v \in \mathrm{Hom}(\Xi, \Theta)$, then we have:

$$\|\phi_\rho^{t_0}(v)\| \leq \frac{1}{2}\|v\|.$$

Remark 5.2.1. The number t_0 depends on the choice of the metric, but the Anosov property does not, because the base space $U_0\Gamma$ is compact and all metrics on it are equivalent.

We denote by $\mathcal{C}(\Gamma, m)$ the space of the conjugacy classes of convex Anosov representations of Γ in $\mathrm{SL}(m, \mathbb{R})$. The pressure metric is defined on the smooth points of this space.

Remark 5.2.2. *By the work of Labourie in [42], when $\Gamma = \pi_1(\Sigma)$, the Hitchin component $H_m(\Sigma)$ can be lifted to the smooth part of $\mathcal{C}(\pi_1(\Sigma), m)$. Thus the restriction of the pressure metric on $\mathcal{C}(\pi_1(\Sigma), m)$ to the lift of $H_m(\Sigma)$ induces a metric on $H_m(\Sigma)$.*

Let $\rho \in \mathcal{C}(\Gamma, m)$ and let η and θ be the corresponding limit maps. Let F be the total space of the \mathbb{R} -principle bundle over $\mathbb{RP}(m) \times (\mathbb{RP}(m))^*$ whose fiber is the set of metrics on the first factor. The \mathbb{R} -action on each fiber is given by sending the metric u to the metric $e^{-t}u$ at time t . Let F_ρ be the \mathbb{R} -principle bundle over $\partial_\infty \Gamma^{(2)}$ which is the pullback of F by (η, θ) . Then the \mathbb{R} -action on F gives rise a flow on F_ρ . In [15], the authors proved the following theorem:

Theorem 5.2.1. *The Γ -action on F_ρ is proper and cocompact. Moreover, the \mathbb{R} -action on $U_\rho F = F_\rho/\Gamma$ is topologically transitive metric Anosov flow which is Hölder orbit equivalent to $U_0\Gamma$.*

Here the "Hölder orbit equivalent" means that the flow on $U_\rho F$ is reparametrization of that on $U_0\Gamma$ by integrating a positive Hölder function defined on $U_0\Gamma$ along the orbit of the flow. In [15], they also proved the following rigidity result:

Theorem 5.2.2. *Let ρ_1 et ρ_2 be two irreducible convex Anosov representations of Γ in $\mathrm{SL}(m, \mathbb{R})$. If for all periodic orbits in $U_0\Gamma$, the reparametrized periods associated to $U_{\rho_1}F$ and $U_{\rho_2}F$ are the same, then the two representations are conjugate to each other by an element in $\mathrm{SL}(m, \mathbb{R})$.*

The results of Bowen [13, 14], Pollicott [50] and Ratner [52] tell us that a topologically transitive metric Anosov flow on a compact space has a Markov coding and this flow can be identified with a suspension flow associated to a Hölder function on the coding space which is a shift space. Thus there exists a shift space X for the flow on $U_\rho F$. The flow on $U_\rho F$ given by the \mathbb{R} -action is identified with a suspension flow associated to X by a Hölder function f_ρ on X . Then the function $-h_{f_\rho} f_\rho$ has pressure zero where h_{f_ρ} is the topological entropy of the suspension flow associated to f_ρ . By sending a convex Anosov representation ρ to the corresponding pressure zero function $-h_{f_\rho} f_\rho$, we define the thermodynamic map:

$$\mathcal{I} : \mathcal{C}(\Gamma, m) \rightarrow P_0(U_0\Gamma),$$

where $P_0(U_0\Gamma)$ denote the space of pressure zero Hölder functions on X . Then the pressure metric is the pullback of the pressure form on $P_0(U_0\Gamma)$ by \mathcal{I} .

Remark 5.2.3. *For more details about the non-degeneracy of the above pullback of the pressure form, see Section 10 of [15].*

For readers' convenience, we recall Remark 1.0.2:

Remark 1.0.1. *This coding is for $U_0\Gamma$ equipped with a convex Anosov representation. In [22], Coornaert and Papadopoulos showed that for a hyperbolic group, there exists a symbolic coding for its Gromov geodesic flow. But this coding is not one to one on a set large enough to apply thermodynamic formalism.*

A periodic orbit O corresponds to a pair of limit points $(x, y) \in \partial_\infty\Gamma^{(2)}$ fixed by some $\gamma \in \Gamma$. The period of O in flow F_ρ equal to the spectral radius of $\rho(\gamma)$ denoted by $\lambda_\gamma(\rho)$. Let T be a positive real number. We denote by $R_T(\rho)$ the set of $\gamma \in \Gamma$ such that $\log(\lambda_\gamma(\rho)) < T$. Then we have that:

$$h_{f_\rho} = \lim_{T \rightarrow +\infty} \frac{\log |R_T(\rho)|}{T}.$$

Let ρ_1 and ρ_2 be two convex Anosov representations of Γ in $\mathrm{SL}(m, \mathbb{R})$. We define their intersection to be:

$$I(\rho_1, \rho_2) = \lim_{T \rightarrow +\infty} \frac{1}{|R_T(\rho_1)|} \sum_{\gamma \in R_T(\rho_1)} \frac{\log(\lambda_\gamma(\rho_1))}{\log(\lambda_\gamma(\rho_2))}.$$

Further define the renormalized intersection function J by:

$$J(\rho_1, \rho_2) = \frac{h_{f_{\rho_2}}}{h_{f_{\rho_1}}} I(\rho_1, \rho_2).$$

Another result in [15] states that the pressure metric is given by the Hessian of J .

Remark 5.2.4. *A fixed flow may have several different Markov codings, but they induce the same pressure metric. This is also true for the flow on a graph in the next section.*

5.3 Moduli space of metric graph

Inspired by the work of McMullen in [44], in [51] Sharp and Pollicott studied the moduli space of metrics on a graph and defined a Weil-Petersson type metric on this moduli space. The main ingredient is also thermodynamic formalism described in the first section of this chapter.

The graph that we consider here is non-oriented and the valence of each vertex is at least 3. A metric on it is a function from the set of edges to \mathbb{R}^+ . Let $\mathbb{G} = \{\mathbb{V}, \mathbb{E}\}$ be a graph. We denote by $\mathcal{M}(\mathbb{G})$ the moduli space of all metrics on \mathbb{G} .

Denote by $\overline{\mathbb{G}} = \{\mathbb{V}, \overline{\mathbb{E}}\}$ the oriented graph with the same vertices as \mathbb{G} . The edges in $\overline{\mathbb{E}} = \{\overline{e}\}$ come from replacing each edge e in \mathbb{G} by its two oriented versions $+e$ and $-e$. An oriented geodesic on \mathbb{G} is a sequence of $(\overline{e}_n)_{n \in \mathbb{Z}}$ such that for each n , the edges \overline{e}_{n+1} follows \overline{e}_n and $\overline{e}_{n+1} \neq -\overline{e}_n$. Then the geodesic flow on \mathbb{G} is well defined. Let l be a metric on \mathbb{G} . As in the first section, we denote by $R_T(l)$ the set of periodic orbits of the geodesic flow with period less

than T . Then the topological entropy $h(l)$ of the associated geodesic flow is given by:

$$h(l) = \lim_{T \rightarrow +\infty} \frac{\log |R_T(l)|}{T}.$$

In [51], the moduli space of metrics on \mathbb{G} is proposed as an analogue of the Riemann's moduli space. As for the case of surfaces, to rescale a metric by a constant factor does not change much the dynamics associated to the geodesic flow. To avoid this, a renormalization is necessary. In the case of closed surfaces, the renormalization of the volume is the same as the renormalization of the topological entropy of the geodesic flow. But for graphs, this is no longer true. In [51], all metrics are normalized such that the entropy is 1. We denote by $\mathcal{M}_1(\mathbb{G})$ the space of all such renormalized metrics. Meanwhile we also consider the moduli space of renormalized metrics on \mathbb{G} with volume 1 and we denote it by $\mathcal{M}(\mathbb{G}, 1)$. To define the pressure metric on $\mathcal{M}_1(\mathbb{G})$, let A be a $\{0, 1\}$ -matrix in a size $|\bar{\mathbb{E}}|$ defined by:

$$A(\bar{e}, \bar{e}') = \begin{cases} 1 & \text{if } \bar{e}' \text{ follows } \bar{e} \text{ and } \bar{e}' \neq -\bar{e} \\ 0 & \text{otherwise.} \end{cases}$$

Then the following shift space gives the codings of all geodesics on \mathbb{G} :

$$\bar{E}_A = \{(\bar{e}_n)_{n \in \mathbb{Z}} : \forall n \in \mathbb{Z}, \bar{e}_n \in \bar{\mathbb{E}} \text{ and } A(\bar{e}_n, \bar{e}_{n+1}) = 1\}, \quad (5.2)$$

A metric l on \mathbb{G} will be looked as a locally constant function f_l on \bar{E}_A such that:

$$f_l((\bar{e}_n)_{n \in \mathbb{Z}}) = l(\bar{e}_0),$$

which is Hölder.

By [51], one can repeat all process in Section 1 to define a pressure form on $\mathcal{M}_1(\mathbb{G})$ which turns out to be a metric. The non-degeneracy comes from the definition of the tangent space of $\mathcal{M}_1(\mathbb{G})$.

We can repeat the above process and obtain a pressure form for $\mathcal{M}(\mathbb{G}, 1)$.

An interesting problem is to compare this metric with the Weil-Petersson metric on Riemann moduli space. We have seen that the latter is negatively curved, geodesically convex but non complete. Sharp and Pollicott proved that the metric can be non complete but can possibly have positive curvature at some point depending on the graph that we choose.

Before going on, we state one fact which is not discussed in [51], but will be useful later:

Fact 5.3.1. *The pressure metric does not depend on the way to renormalize, i.e. given two renormalizations, the pressure metrics on the two corresponding moduli space are isometric. In particular, the pressure form on $\mathcal{M}(\mathbb{G}, 1)$ is positive definite.*

Proof. The proof comes from the following two observation. The first observation is that there is an bijection between $\mathcal{M}(\mathbb{G}, 1)$ and $\mathcal{M}_1(\mathbb{G})$. Given a metric $l \in \mathcal{M}(\mathbb{G}, 1)$. If we rescale it by a factor $\lambda > 0$, the result metric λl will have the

associated topological entropy $h_{\lambda l} = \frac{1}{\lambda} h_l$. Then by choosing $\lambda = h_l$, we obtain a unique metric $l' = \lambda l$ in $\mathcal{M}_1(\mathbb{G})$. The reciprocal is similar.

The second observation is the following: the images of l and l' under the thermodynamic map are the same. By the definition of f_l , we obtain that $f_{\lambda l} = \lambda f_l$. On the other hand, we have $h_{\lambda l} = \frac{1}{\lambda} h_l$. Combining these two relations, we conclude that $-h_{\lambda l} f_{\lambda l} = -h_l f_l$. Thus the embeddings of $\mathcal{M}(\mathbb{G}, 1)$ and $\mathcal{M}_1(\mathbb{G})$ by the thermodynamic map have the same image.

By the work of Sharp and Pollicott in [51], the pullback of the pressure form gives a Riemannian metric on $\mathcal{M}_1(\mathbb{G})$. Thus by the second observation, the pullback of pressure form on $\mathcal{M}(\mathbb{G}, 1)$ also gives a metric. Moreover, the bijection in the first observation is an isometry with respect to the pressure metric. \square

5.4 Pressure metric and degeneration of \mathbb{T}

5.4.1 “New“ coordinate system on Teichmüller space

We should remark here that the idea of the construction of a coordinate system using orthogeodesics has already appeared in [57].

Henceforth \mathbb{T} will denote a hyperbolic one-holed torus whose boundary is a simple closed geodesic.

Definition 5.4.1. An *orthogeodesic* α on \mathbb{T} is a geodesic arc perpendicular to the boundary. Its length $l(\alpha)$ is an *element of orthospectrum* of \mathbb{T} and we refer to it as an *otholength*.

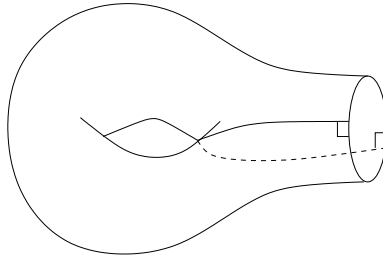


Figure 5.1: Orthogeodesics

Remark 5.4.1. These notions were introduced by Basmajian in [6] for a finite volume hyperbolic manifold with totally geodesic boundary. There he proved an identity relating the orthospectrum to the volume of the boundary.

As a convention, when the boundary length goes to zero, in the limit we obtain a once-punctured torus, and an orthogeodesic will become a bi-infinite geodesic connecting the puncture to itself.

Definition 5.4.2. An *ideal triangulation* T for \mathbb{T} is a maximal collection of simple pairwise disjoint orthogeodesic of \mathbb{T} .

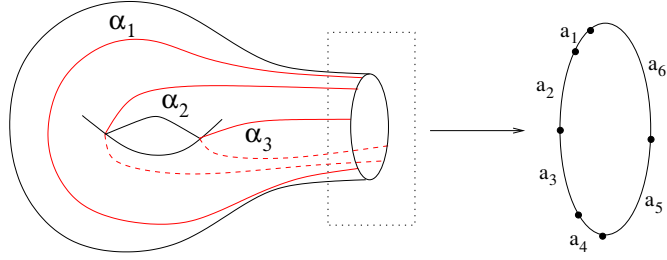


Figure 5.2: Ending points of an ideal triangulation

Remark 5.4.2. *This is a definition extended from the usual ideal triangulation. The complement of T in \mathbb{T} is a disjoint union of two right angled hexagons instead of ideal triangle. Notice that when the boundary length goes to 0, in the limit we obtain an ideal triangulation in the usual sens for the punctured torus whose sides are bi-infinite simple geodesics.*

Let T be an ideal triangulation of \mathbb{T} . Denote by α_1 , α_2 and α_3 the three orthogeodesics in T . Their end points separate the boundary into 6 segments, denoted by a_1, \dots, a_6 in a cyclic order. By the hyperelliptic involution of \mathbb{T} , we have that the length of a_i equals to that of a_{i+3} .

By hyperbolic geometry, the hyperbolic structure on a right angled hexagon is determined by the lengths of 3 of its edges. This implies the following two fact:

- (1) The hyperbolic structure on each hexagon in $\mathbb{T} \setminus T$ determined by the lengths of a_1, \dots, a_6 ;
- (2) The two hexagon in $\mathbb{T} \setminus T$ are isometric.

By gluing them together, we get a hyperbolic structure on \mathbb{T} . By the convention of the marking homeomorphism that we made in Chapter 4, we can see that the lengths of a_1 , a_2 and a_3 also determine the marking. Thus we obtain a parametrization of $\mathcal{T}(\mathbb{T})$ given by the following map:

$$\mathcal{O} : \mathcal{T}(\mathbb{T}) \rightarrow (\mathbb{R}^+)^3,$$

where the image of a marked hyperbolic structure is its lengths of a_1 , a_2 and a_3 . The map \mathcal{O} is the new coordinate system that we use to describe the degeneration.

Remark 5.4.3. *In the following, an ideal triangulation will be chosen one time for all and we will use the notation a_1 , a_2 and a_3 to denote the corresponding coordinates.*

5.4.2 Degenerations of bordered surfaces

We consider the coordinate system that we introduced in the preceding section. We will restrict ourselves in the following cone in $\mathcal{T}(\mathbb{T})$:

$$\mathcal{C} = \{g = (a_1, a_2, a_3) \in \mathcal{T}(\mathbb{T}) : \forall i, j, k \in \{1, 2, 3\}, a_i < a_j + a_k\}.$$

Let $g = (a_1, a_2, a_3)$ be a point in \mathcal{C} . We assume that $a_1 + a_2 + a_3 = 1$. We consider the sequence $g_n = (\lambda_n a_1, \lambda_n a_2, \lambda_n a_3)_{n \in \mathbb{N}}$ where $(\lambda_n)_{n \in \mathbb{N}}$ is a sequence of positive number and the limit of λ_n is infinite as n goes to infinity. We renormalize each marked hyperbolic structure $(\lambda_n a_1, \lambda_n a_2, \lambda_n a_3)$ by rescaling it by a factor λ_n^{-1} and we denote the renormalized metric by g'_n . Notice that in each hexagon, the renormalized lengths of a_1 , a_2 and a_3 are constant. By elementary hyperbolic geometry, the lengths of the three orthogeodesics go to 0 when n goes to infinity. As the curvature goes to $-\infty$ when n goes to ∞ , we can see that in the limit, each hexagon retracts to a graph with one vertex and three branches. Thus the universal cover $\overline{\mathbb{T}}$ of \mathbb{T} degenerates to a binary tree in the limit which is the dual graph of the ideal triangulation of $\overline{\mathbb{T}}$. By quotient the action of fundamental group, the surface \mathbb{T} retracts to \mathbb{G} . The following picture shows the degeneration of one fundamental domain of \mathbb{T} :

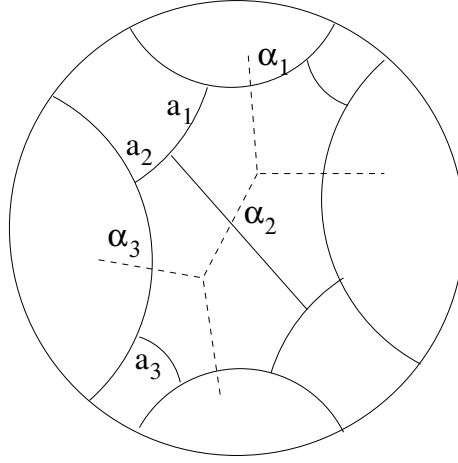


Figure 5.3: Degeneration of one fundamental domain of \mathbb{T}

To describe the topology, we use the idea of the construction of Thurston's compactification using measured laminations. Let C be the set of isotopy classes of simple closed curves in \mathbb{T} . Let \mathbb{R}_+^C be the space of functions from C to \mathbb{R}_+ . The renormalized metric g'_n on \mathbb{T} and the metric l on \mathbb{G} can be both embedded in \mathbb{R}_+^C where the \mathbb{G} is looked as the dual graph of the ideal triangulation that we choose to define our coordinates. Let l be a metric on \mathbb{G} satisfying the following relations:

$$\begin{aligned} a_1 &= \frac{l(e_1) + l(e_2)}{2}; \\ a_2 &= \frac{l(e_2) + l(e_3)}{2}; \\ a_3 &= \frac{l(e_1) + l(e_3)}{2}. \end{aligned}$$

Then we can verify that the limit of the images of g'_n in \mathbb{R}_+^C is the image of l

in \mathbb{R}_+^C . Let $\mathbb{P}(\mathbb{R}_+^C)$ be the projective space. We can embed both the cone \mathcal{C} and $\mathcal{M}(\mathbb{G}, 1)$ in it and the above argument shows that the image of later is contained in the set of accumulation points of the image of the former.

Let γ be an element in the fundamental group of \mathbb{G} which is the same as that of \mathbb{T} . The length of γ is the linear combination of the length of the generators appearing in the presentation of γ which is not the case for hyperbolic metric. This means that the image of \mathcal{C} and the image of $\mathcal{M}(\mathbb{G}, 1)$ in \mathbb{R}_+^C are disjoint.

This concludes that $\mathcal{M}(\mathbb{G}, 1)$ is a part of the boundary of \mathcal{C} under the topology induced by that on $\mathbb{P}(\mathbb{R}_+^C)$.

By considering the action of fundamental group on the fundamental domain, the geodesic flows on \mathbb{T} and \mathbb{G} have the same coding space X_A . This implies that by the thermodynamic map \mathcal{I} , the cone \mathcal{C} and $\mathcal{M}(\mathbb{G}, 1)$ can be embedded to the same pressure zero function space $P_0(X_A)$. Moreover, the pressure zero functions associated to the renormalized metrics g'_n converge to that associated to a metric l uniformly.

By using the same argument in the proof of Fact 5.3.1, we obtain that the \mathcal{I} -image of Teichmüller space $\mathcal{T}(\mathbb{T})$ and that of the renormalized Teichmüller space are the same and this yields an isometry between them with respect to the pressure metric. Moreover, the above argument for the renormalized Teichmüller space shows that $\mathcal{I}(\mathcal{M}(\mathbb{G}, 1))$ is a part of the boundary of $\mathcal{I}(\mathcal{C})$ in $P_0(X_A)$. The pressure form is well-defined on $\mathcal{I}(\mathcal{M}(\mathbb{G}, 1) \sqcup \mathcal{C})$, thus its pullback by \mathcal{I} well defined on $\mathcal{M}(\mathbb{G}, 1) \sqcup \mathcal{C}$ and it is positive definite on both the two moduli spaces by [15] and [51].

5.5 Asymptotic properties of entropy function

The main result in this section is the following:

Proposition 5.5.1. *Let $\Sigma = \Sigma_{g,r}$ be a bordered oriented surface of genus $g > 0$ with $\chi(\Sigma) < 0$ and $r > 0$. If $\chi(\Sigma) < -1$ meaning that it is different from one-holed torus, then the restriction of the entropy function on symplectic leave $\mathcal{T}(\Sigma, L_1, \dots, L_r)$ are not constant; if $\Sigma = \mathbb{T}$, then the entropy function is not constant on the symplectic leave $\mathcal{T}(\mathbb{T}, L)$ with L large enough.*

Proof. The proof for the one-holed torus case is based on the degeneration that we studied above. The entropy function h is defined on $\mathcal{T}(\mathbb{T}, L)$ and $\mathcal{M}(\mathbb{G}, 1)$ by sending a metric g on \mathbb{T} or \mathbb{G} to the topological entropy of the associated geodesic flow. It can be described by:

$$P(-h(\phi_f)f) = 0,$$

where f is the Hölder function on a shift space whose associated suspension flow is identified with the geodesic flow of g . By the analyticity of pressure and implicit function theorem, we conclude the analyticity of the entropy function. Let \mathcal{C}' be the renormalization of \mathcal{C} in the way that we described before. The degeneration of \mathbb{T} helps us to define h as a continuous function on $\mathcal{C}' \sqcup \mathcal{M}(\mathbb{G}, 1)$.

Assume that there exists a positive number L_0 such that h is constant on $\mathcal{C}(L)$ with $L > L_0$. The continuity of h implies that h is constant on $\mathcal{M}(\mathbb{G}, 1)$. By the computation in [51], this is not true which yields the contradiction.

The proof of Proposition 5.5.1 for the bordered surface S has genus bigger than 1 is quite different.

This proof repeats the construction of an example of McMullen in [44].

Consider the bordered surface S equipped with a hyperbolic metric. There is an infinite volume hyperbolic surface S' which is a quotient space of \mathbb{H} by the Fuchsian group Γ determined by the holonomy of S such that S is isometric to the convex core of S' . The bottom spectrum of Laplacian of S' is defined by:

$$\lambda_0(S') = \inf\left\{\frac{\int_{S'} |\nabla f|^2 dS'}{\int_{S'} |f|^2 dS'} : f \in C_0^\infty(S')\right\}, \quad (5.3)$$

where $C_0^\infty(S')$ is the space of smooth function on S' with compact support.

The result of McMullen shows that:

$$\lambda_0(S') = \begin{cases} \frac{1}{4} & \text{if } \delta(\Gamma) \leq \frac{1}{2} \\ \delta(\Gamma)(1 - \delta(\Gamma)) & \text{if } \delta(\Gamma) \geq \frac{1}{2} \end{cases} \quad (5.4)$$

where $\delta(\Gamma)$ is the critical exponent of Γ . By Sullivan's result, we know that $\delta(\Gamma)$ equals to the topological entropy of the no wandering part of the geodesic flow on S and the Hausdorff dimension of the limit set of Γ .

By assumption, there is a separating simple closed geodesic α on S' which is not a boundary component of the convex core of S' such that one connected component in its complement is a one-holed torus. The pinching of α produces a sequence S'_n in the symplectic leaf of $\mathcal{T}(S')$. For each hyperbolic surface S'_n , we construct a smooth function f_n with compact support on it in the following way: $f_n = 0$ on the infinite side of α and $f_n = 1$ on the one holed torus part except a cylinder neighborhood of α with height 1; $|\nabla f| = 1$ on the cylinder neighborhood. It is easy to see that the ratio $\int_{S'_n} |\nabla f|^2 dS'_n / \int_{S'_n} |f_n|^2 dS'_n$ goes to 0 as n goes to infinity. This implies that $\lambda_0(S'_n)$ goes to 0 as n goes to infinity. By the formula (5.4), we conclude that $\delta(\Gamma_n)$ goes to 1 when n goes to infinity, so is the entropy. We also know that for a bordered hyperbolic surface, the limit set is a Cantor set on S^1 whose Hausdorff dimension is strictly smaller than 1. Combining these two facts, we conclude that the entropy function is not constant on each symplectic leaf of $\mathcal{T}(S)$. \square

Remark 5.5.1. *This example of McMullen is used to prove that there is a sequence of Kleinian groups which converges geometrically such that the Hausdorff dimension does not converge.*

Chapter 6

Central extension of mapping class group via Chekhov-Fock quantization of Teichmüller space

In this chapter, we give the construction of the Chekhov-Fock quantization of the Teichmüller space. Then we present a construction of a central extension of mapping class group by using the Chekhov-Fock quantization, and show that the cohomology class of this central extension of mapping class groups of punctured surfaces of finite type is 12 times the Meyer class plus the Euler classes of the punctures which is same as for Kashaev quantization.

We remark that the Chekhov-Fock quantization, as well as the Kashaev quantization, is infinite dimensional quantum Teichmüller theory. Meanwhile there is also the finite dimensional quantum Teichmüller theory. It has been developed by Bonahon and his collaborators (see [11] and [12]) where a problem analogue to that we consider in this chapter was studied.

6.1 Projective representation and almost linear representation

Let V be a vector space and G be a group. A **projective representation** of G on V is a homomorphism from G to $\mathrm{PGL}(V)$. It is well-known that projective representations of one group are equivalent to representations of the central extensions of the same group by subgroups of \mathbb{C}^* . More precisely, let h be a projective representation of G on V . To a central extension \tilde{G} of G by a subgroup A of \mathbb{C}^* , one can associate a representation \tilde{h} of \tilde{G} on V such that the following commutative diagram holds :

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathrm{GL}(V) & \longrightarrow & \mathrm{PGL}(V) \longrightarrow 1 \\
& & \uparrow & & \uparrow \tilde{h} & & \uparrow h \\
1 & \longrightarrow & A & \longrightarrow & \tilde{G} & \longrightarrow & G \longrightarrow 1
\end{array}$$

In particular, let \tilde{G}_0 be a central extension of G by \mathbb{C}^* which is the pullback of $\mathrm{GL}(V) \rightarrow \mathrm{PGL}(V)$ by h . Denote by \tilde{h}_0 the associated representation of \tilde{G}_0 on V . A **reduction** of \tilde{G}_0 is a central extension \tilde{G}_1 of G by a subgroup A_1 of \mathbb{C}^* , such that \tilde{G}_1 is a subgroup of \tilde{G}_0 and the associated representation $h_1 : \tilde{G}_1 \rightarrow \mathrm{GL}(V)$ is the restriction of h_0 . We say that \tilde{G}_1 is the **minimal reduction** of \tilde{G}_0 if \tilde{G}_1 is minimal among all reductions of \tilde{G}_0 up to isomorphism.

Suppose that G is a quotient of a free group F by a set of relation R . A projective representation of G on V can be induced by a representation \bar{h} of F on V such that normal subgroup generated by the relation set R are mapped into the center of $\mathrm{GL}(V)$. The homomorphism \bar{h} will be called an **almost linear representation** of G on V , in order to distinguish the projective representation.

6.2 Ptolemy groupoid

A **groupoid** is a category such that all morphisms are invertible and for each pair of objects there exists at least one morphism between them. The automorphisms of an object form a group. Reciprocally, if a group G acts freely on a set X , we can define an associated groupoid for which the objects are the G -orbits in X and the morphisms are the orbits of the diagonal G -action on $X \times X$.

Let $\Sigma = \Sigma_g^s$ be the oriented surface of genus g with s punctures and $\Gamma = \Gamma(\Sigma_g^s)$ its mapping class group. Let $|\mathbf{T}(\Sigma)|$ be the set of ideal triangulations of Σ . By labeling the arcs of an ideal triangulation, one obtains a **labeled ideal triangulation**. Let $\mathbf{T}(\Sigma)$ be the set of labeled ideal triangulations on Σ . The action of Γ on $\mathbf{T}(\Sigma)$ is free.

Remark 6.2.1. *The Γ -action on $|\mathbf{T}(\Sigma)|$ is not free. In particular, if an ideal triangulation T has some symmetries, then the set of arcs in T will be fixed by an element of Γ that permutes these arcs (see [20]).*

Definition 6.2.1. *The **Ptolemy groupoid** is defined as follows:*

- (1) *the objects are the Γ -orbits in $\mathbf{T}(\Sigma)$;*
- (2) *the morphisms are the Γ -orbits in $\mathbf{T}(\Sigma) \times \mathbf{T}(\Sigma)$.*

Given a labeled ideal triangulation T of Σ , we will denote by $F_\alpha(T)$ the flip on the arc α of T . Also we denote by S_n , where $n = -3\chi(\Sigma)$, the permutation group on the set of labels of the arcs of T . Then the Ptolemy groupoid can also be defined by using flips and permutations of labels. Moreover, the Ptolemy groupoid has the following presentation due to Harer in [35] and Penner in [46] and [47]:

Theorem 6.2.1. *If Σ is different from the three-holed sphere or the one-holed torus, then any pair of labeled ideal triangulations can be connected by a chain of flips and permutations.*

The Ptolemy groupoid is generated by the action of flips and symmetry group. The relations between them are the following:

- (1) *For any arc α in T , we have that $F_\alpha^2 = 1$;*
- (2) *If α and β are two arcs in T having no common end point, we have that $F_\alpha F_\beta = F_\beta F_\alpha$;*
- (3) *For any two arcs α and β contained in an ideal pentagon as diagonals, the pentagon relation holds:*

$$F_\alpha F_\beta F_\alpha F_\beta F_\alpha = \sigma(\alpha, \beta),$$

where $\sigma \in S_n$ is the permutation of the labels of α and β ;

- (4) *Let $\sigma \in S_n$ and let α be a labeled arc, then we have that $F_\alpha \sigma = \sigma F_{\sigma(\alpha)}$.*

6.3 Quantum Teichmüller space

To introduce the Chekhov-Fock quantization, let us first recall the definition of a $*$ -algebra.

Definition 6.3.1. *A $*$ -ring R is a ring with a map $*$: $R \rightarrow R$ satisfying that for all x and y in R , we have:*

- (1) $(x + y)^* = x^* + y^*$;
- (2) $(xy)^* = y^* x^*$;
- (3) $1^* = 1$;
- (4) $(x^*)^* = x$.

The map $$ is called an **involution** of R .*

Definition 6.3.2. *A $*$ -algebra is a $*$ -ring with an involution $*$ that is an associative algebra over a commutative $*$ -ring R with involution $x \mapsto \bar{x}$ such that $(xa)^* = \bar{x} a^*$, for all $a \in A$ and $x \in R$.*

Remark 6.3.1. *In the rest of this chapter, the $*$ -algebras that we will consider are \mathbb{C} -vector spaces associated with a multiplication rule and a trivial involution $*$. Thus, as a convention, we will simply call them algebras omitting the notation $*$.*

The quantization of a Poisson manifold equivariant with respect to a discrete group G -action is a family of algebras A^\hbar depending smoothly on a positive real parameter \hbar satisfying the following properties:

- (1) All A^{\hbar} are isomorphic to each other as vector spaces ;
- (2) The group G acts as the outer automorphisms on each algebra ;
- (3) For $\hbar = 0$, the algebra A^0 is isomorphic as a G -module to the algebra of the complex-valued function on the Poisson manifold ;
- (4) The Poisson bracket $\{, \}$ on A^0 is the limit of $\{, \}_{\hbar}/(2\pi i\hbar)$ as \hbar is going to zero. It coincides with the one on the original Poisson manifold.

Remark 6.3.2. *By a family of algebras depending smoothly on the parameter \hbar , we mean that the multiplication rule varies smoothly with respect to \hbar .*

By the discussion in the Section 4.4.2, we have seen that the Teichmüller space of a punctured surface has a Poisson structure. Generally speaking, the Chekhov-Fock quantization is the quantization of the Teichmüller space with respect to the mapping class group action.

More precisely, we associate an algebra $A^{\hbar}(T)$ to each ideal triangulation T on Σ , generated by $\{Z_{\hbar}(\alpha) : \alpha \in T\}$. The Poisson bracket on $A^{\hbar}(T)$ is obtained by deforming the Poisson bracket for the shearing coordinates associated to T by the following formula:

$$\{Z_{\hbar}(\alpha), Z_{\hbar}(\beta)\}_{\hbar} = 2\pi i\hbar\{t(\alpha), t(\beta)\}.$$

The flip $F_{\alpha} : T \rightarrow T'$ acts on the algebras $A^{\hbar}(T)$ by the formula

$$Z'_{\hbar}(\beta) = \begin{cases} -Z_{\hbar}(\alpha) & \text{if } \beta = \alpha' \\ Z_{\hbar}(\beta) + \epsilon_T(\alpha, \beta)\phi^{\hbar}(\text{sign}(\epsilon_T(\alpha, \beta))Z_{\hbar}(\alpha)) & \text{if } \beta \text{ and } \alpha' \text{ are adjacent but } \beta \neq \alpha' \\ Z_{\hbar}(\beta) & \text{otherwise} \end{cases},$$

where the $Z'_{\hbar}(\beta)$'s with β in T' are the generators of $A^{\hbar}(T')$, $\epsilon_T(\alpha, \beta)$ is defined in the Section 4.4.2 and a real function,

$$\phi^{\hbar}(z) = -\frac{\pi\hbar}{2} \int_{\Omega} \frac{\exp(-iuz)}{\sinh(\pi u) \sinh(\pi\hbar u)} du, \quad (6.1)$$

where Ω is the path goes along the real axis from $-\infty$ to $+\infty$ and passing the origin from above. The symmetric group acts as the permutation of the labels.

Remark 6.3.3. *In the next section, we will describe a way to represent each generator $Z_{\hbar}(\alpha)$ as an operator acting on some Hilbert space. And we will use the holomorphic functional calculus to evaluate the function $\phi^{\hbar}(z)$ on such operator. Notice that the one parameter groups $\exp(iuZ_{\hbar}(\alpha))$ are unitary so that the integral makes sense by functional calculus.*

For each \hbar , the construction above gives us a functor \mathcal{Q}_{\hbar} from the Ptolemy groupoid to the category of algebra.

Definition 6.3.3. *The family of functors \mathcal{Q}_{\hbar} are called the **Chekhov-Fock quantization** of Teichmüller space.*

6.4 Almost linear representation of Ptolemy groupoid

A Heisenberg algebra H_n is generated by $2n+1$ generators $P_1, \dots, P_n, Q_1, \dots, Q_n$ and C satisfying the following relations:

- (1) The generator C is a central element ;
- (2) For any two index j and k , we have the relations:

$$\{P_j, P_k\} = \{Q_j, Q_k\} = 0;$$

- (3) For any two index j and k , we have the relations:

$$\{P_j, Q_k\} = C\delta_{jk}.$$

where the bracket is the standard commutator.

It has a irreducible integrable representation in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ described as follows: let x_1, \dots, x_n denote the real coordinates of \mathbb{R}^n , then we represent the generators by the following operators:

$$\begin{aligned} \rho(P_j)(f)(x_1, \dots, x_n) &= x_j f(x_1, \dots, x_n), \\ \rho(Q_j)(f)(x_1, \dots, x_n) &= -2\pi i \hbar \frac{\partial f}{\partial x_j}(x_1, \dots, x_n), \\ \rho(C)(f)(x_1, \dots, x_n) &= 2\pi i \hbar f(x_1, \dots, x_n). \end{aligned}$$

for all $f \in \mathcal{H}$. Let $n = -\chi(\Sigma)$ which is the number of arcs of an ideal triangulation of Σ . Then the representation of H_n induces a representation of $A^\hbar(T)$ in \mathcal{H} . More precisely, by the assumption of n , there is a bijection between the generators Q_j 's and the arcs in T . Let Q_α denote the generator associated to the arc α by the bijection. Then the representation ρ of $A^\hbar(T)$ is given by the following formulas:

$$\rho(Z_\hbar(\alpha)) = \frac{1}{2}\rho(Q_\alpha) + \sum_{\beta \in T} \epsilon_T(\alpha, \beta)\rho(P_\beta),$$

for all $\alpha \in T$.

The Stone von Neumann theorem holds true for $A^\hbar(T)$. In particular, consider the representations $\rho(A^\hbar(T))$ and $\rho(A^\hbar(T'))$ of $A^\hbar(T)$ and $A^\hbar(T')$ respectively. The uniqueness of representations yields the existence of an intertwiner K between the two representations. It acts in the following way:

$$e^{i\rho(A^\hbar(T'))} = K^{-1}(T, T')e^{i\rho(A^\hbar(T))}K(T, T'). \quad (6.2)$$

To give the explicit formula of K , consider two arcs α and β in an labeled ideal triangulation T having one common vertex. The quantization functor \mathcal{Q}_\hbar sends the associated shearing coordinates t_α and t_β to two elements $Z_\hbar(\alpha)$ and $Z_\hbar(\beta)$ in $A^\hbar(T)$ which generate a subalgebra isomorphic to H_1 the Heisenberg

algebra with 3 generators. Let P , Q and C be the three generators of H_1 satisfying: $\{P, Q\} = C$ and $\{P, C\} = \{Q, C\} = 0$ where the bracket is the standard commutator. Any unitary irreducible representation of H_1 is equivalent to its following representation ρ on $L^2(\mathbb{R})$:

$$\begin{aligned}\rho(P)(f)(x) &= 2\pi i\hbar \frac{\partial f(x)}{\partial x}; \\ \rho(Q)(f)(x) &= xf(x); \\ \rho(C)(f)(x) &= 2\pi i\hbar f(x).\end{aligned}$$

Let $U = \exp(i\rho(P))$ and $V = \exp(i\rho(Q))$. The flip F_α on α sends (P, Q) to $(Q', P') = (-P, Q + \phi^\hbar(P))$ by the quantization functor β^\hbar . Let $U' = \exp(i\rho(P'))$ and $V' = \exp(i\rho(Q'))$.

Consider following function:

$$\Phi^\hbar(z) = \exp\left(-\frac{1}{4} \int_\Omega \frac{\exp(-iuz)}{u \sinh(\pi u) \sinh(\pi\hbar u)} du\right),$$

where Ω is a path defined in the same way as in the function (6.1). Then the operator K is given by:

$$K(f)(x) = \int_{-\infty}^{+\infty} f(z) \Phi^\hbar(z) \exp\left(\frac{-xz}{2\pi i\hbar}\right) dz.$$

Moreover we have that $K^{-1}UK = U'$ and $K^{-1}VK = V'$. For the representation of $A^\hbar(T)$, we have an operator K for each variable of $f \in L^2(\mathbb{R}^n)$. They are independent of each other. We simply take their composition to get the operator K in Formula (6.2).

The following result is proved in [26]:

Proposition 6.4.1. *The intertwiner K has the following property:*

- (1) *For any disjoint arcs α and β in the labeled ideal triangulation T , the operators $K(F_\alpha)$ and $K(F_\beta)$ commute with each other ;*
- (2) $K(F_\alpha)^2 = 1$;
- (3) *The pentagon relation: for α and β which are two diagonals in a pentagon,*

$$K(F_\alpha)K(F_\beta)K(F_\alpha)K(F_\beta)K(F_\alpha) = e^{2\pi i\hbar} \sigma,$$

where σ is the permutation of the labels α and β .

By the intertwiner K , all morphisms in the Ptolemy groupoid are sent to $GL(\mathcal{H})$ and all relations are sent to the center of $GL(\mathcal{H})$. Thus we obtain an almost linear representation of the Ptolemy groupoid which induces almost linear representations of Γ .

6.5 Presentation of central extension via Chekhov-Fock quantization

In [27], the authors gave the following presentation of Γ which is a consequence of Gervais' result in [29].

Lemma 6.5.1. *For any oriented surface S of genus $g \geq 2$ and $s \geq 4$ punctures, the mapping class group has the following presentation:*

- (1) *The generators are the Dehn twists D_a along all non separating simple closed geodesics a in S ;*
- (2) *The relation between them are the following:*
 - (a) *The type-0 braid relation: for each pair of disjoint non-separating simple closed geodesics a and b , we have that $D_a D_b = D_b D_a$;*
 - (b) *The type-1 braid relation: for each pair of non-separating simple closed geodesics a and b with the geometric intersection number $i(a, b) = 1$ (see Figure 4.6), we have that:*

$$D_a D_b D_a = D_b D_a D_b;$$

- (c) *The chain relation: for each two-holed torus embedded in the surface (see Figure 4.7), we have that:*

$$(D_a D_b D_c)^4 = D_e D_f;$$

- (d) *The lantern relation: for each four-holes sphere embedded in the surface whose boundary a_0, a_1, a_2, a_3 are the non-separating simple closed geodesics (see Figure 4.8), we have that:*

$$D_{a_0} D_{a_1} D_{a_2} D_{a_3} = D_{a_{13}} D_{a_{23}} D_{a_{12}};$$

- (e) *The puncture relation: for each sphere with three holes and one puncture embedded in the surface, we have*

$$D_{a_1} D_{a_2} D_{a_3} = D_{a_{13}} D_{a_{23}} D_{a_{12}}.$$

By using this presentation, we are able to prove the following proposition which is the main step of the proof of the theorem:

Proposition 6.5.1. *By using the Chekhov-Fock quantization, we obtain a central extension of Γ with the following presentation:*

- (1) *Generators:*
 - (a) *One central element: $w = z^{-12}$, where $z = e^{2\pi i \hbar}$ is the constant coming from the pentagon relation in the Proposition 6.4.1;*

(b) One element \tilde{D}_a associated to each the Dehn twists D_a along all non separating simple closed geodesics a in S .

(2) Relations (same hypothesis on curves as in Lemme 6.5.1):

- (a) The type-0 braid relation: $\tilde{D}_a \tilde{D}_b = \tilde{D}_b \tilde{D}_a$;
- (b) The type-1 braid relation: $\tilde{D}_a \tilde{D}_b \tilde{D}_a = \tilde{D}_b \tilde{D}_a \tilde{D}_b$;
- (c) The Lantern relation: $\tilde{D}_{a_0} \tilde{D}_{a_1} \tilde{D}_{a_2} \tilde{D}_{a_3} = \tilde{D}_{a_{12}} \tilde{D}_{a_{23}} \tilde{D}_{a_{13}}$;
- (d) The chain relation: $(\tilde{D}_a \tilde{D}_b \tilde{D}_c)^4 = w^{12} \tilde{D}_e \tilde{D}_f$;
- (e) The puncture relation: $\tilde{D}_{a_{12}} \tilde{D}_{a_{13}} \tilde{D}_{a_{23}} = w \tilde{D}_{a_1} \tilde{D}_{a_2} \tilde{D}_{a_3}$;
- (f) If w is a root of unity with order N , then $w^N = 1$.

Remark 6.5.1. By using the Chekhov-Fock quantization, we can construct more than one central extension of Γ , but they are all in the same cohomology class. The one in the proposition above makes it easy to compute this cohomology class.

We prove Proposition 6.5.1 by proving a sequence of lemmas.

Lemma 6.5.2. For the type-0 braid relation, we have that $\tilde{D}_a \tilde{D}_b = \tilde{D}_b \tilde{D}_a$.

Proof. The lifts of two commutative elements are commutative in the central extension. \square

Remark 6.5.2. This lemme holds true for any lifts of any pairs of Dehn twists D_a and D_b satisfying the type-0 braid relation.

Lemma 6.5.3. For the type-1 braid relation, by choosing the lifts, we have that:

$$\tilde{D}_a \tilde{D}_b \tilde{D}_a = \tilde{D}_b \tilde{D}_a \tilde{D}_b.$$

Proof. Let a and b be two non-separating simple closed geodesics in Σ with $i(a, b) = 1$. Suppose that we have their lifts \tilde{D}_a and \tilde{D}_b such that:

$$\tilde{D}_b \tilde{D}_a \tilde{D}_b = z^k \tilde{D}_a \tilde{D}_b \tilde{D}_a.$$

Then by changing the lift \tilde{D}_b to $\tilde{D}'_b = z^k \tilde{D}_b$ we have that:

$$\tilde{D}_a \tilde{D}'_b \tilde{D}_a = \tilde{D}'_b \tilde{D}_a \tilde{D}'_b.$$

Let x and y be another pair of non-separating simple closed geodesics with $i(x, y) = 1$. Then there is a homeomorphism ϕ of Σ sending a and b to x and y respectively. Let $\tilde{\phi}$ be its arbitrary lift. Consider the lifts of D_x and D_y as follows:

$$\begin{aligned} \tilde{D}_x &= \tilde{\phi}^{-1} \tilde{D}_a \tilde{\phi}, \\ \tilde{D}_y &= \tilde{\phi}^{-1} \tilde{D}'_b \tilde{\phi}. \end{aligned}$$

Then the corresponding type-1 braid relation is

$$\tilde{D}_x \tilde{D}_y \tilde{D}_x = \tilde{D}_y \tilde{D}_x \tilde{D}_y.$$

We begin by choosing a lift for D_a , then there is a unique lift of D_b satisfying the trivial type-1 braid relation. By using the homeomorphisms of Σ , the lifts of the other D_x and D_y appearing in the type-1 braid relations can be fixed too. Then we have the trivial type-1 braid relation everywhere. \square

Consider the Dehn twists as the homomorphisms in Ptolemy groupoid. Then they can be expressed as the compositions of the flip actions and the permutation actions. By using the almost linear representation K , we obtain the lifts of Dehn twists. Notice that a Dehn twist may have several expressions different from each other by the relations in Proposition 6.2.1. Different expressions may induce different lifts. In the following part, we will prove that by well choosing the expression, the induced lifted Dehn twists satisfy the relations in Proposition 6.5.1. We will use \tilde{F}_α to denote $K(F_\alpha)$. Fixing an expression of a Dehn twist D_a , we will use \tilde{D}_a to denote the composition of the K -images of the flips and permutations in that expression.

Convention 6.5.1. *For our convenience, we make two conventions for the proof in the rest of this section:*

- (1) *For the underlined part, we use either the pentagon relation or the commutation relations;*
- (2) *In the proof, we compose the lifts of flips and permutations from the left to the right which is contrast to the usual way.*

Lemma 6.5.4. *By choosing the lift for each Dehn twist in the lantern relation, we have that:*

$$\tilde{D}_{a_0} \tilde{D}_{a_1} \tilde{D}_{a_2} \tilde{D}_{a_3} = z^{-12} \tilde{D}_{a_{13}} \tilde{D}_{a_{23}} \tilde{D}_{a_{12}}.$$

Proof. The proof is similar to the one in [27]. Consider the the four-holed sphere with one puncture on each boundary component. The ideal triangulation and the labels are given as follows:

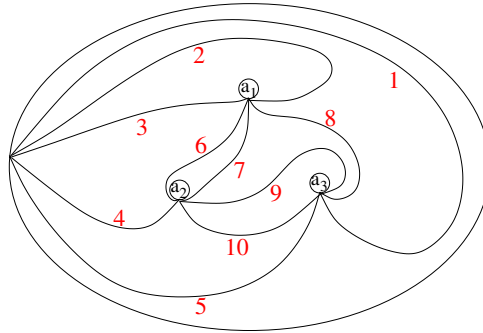


Figure 6.1: Lantern relation

Then the Dehn twist D_0, D_1, D_2 and D_3 have the following expression:

$$\begin{aligned}
 D_0 &= F_5 F_4 F_3 F_2 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}, \\
 D_1 &= F_3 F_6 F_7 F_8 \begin{pmatrix} 2 & 3 & 6 & 7 & 8 \\ 3 & 6 & 7 & 8 & 2 \end{pmatrix}, \\
 D_2 &= F_6 F_4 F_{10} F_9 \begin{pmatrix} 4 & 6 & 7 & 9 & 10 \\ 10 & 4 & 6 & 7 & 9 \end{pmatrix}, \\
 D_3 &= F_{10} F_5 F_1 F_8 \begin{pmatrix} 1 & 5 & 8 & 9 & 10 \\ 8 & 1 & 9 & 10 & 5 \end{pmatrix}.
 \end{aligned}$$

For D_{12}, D_{23} and D_{13} , we use the same strategy as in [27]. First transform the triangulation so that there are only two arcs intersecting the geodesic associated to the Dehn twist. Then do the Dehn twist and transform the triangulation back. Follow this idea we get the following formulas:

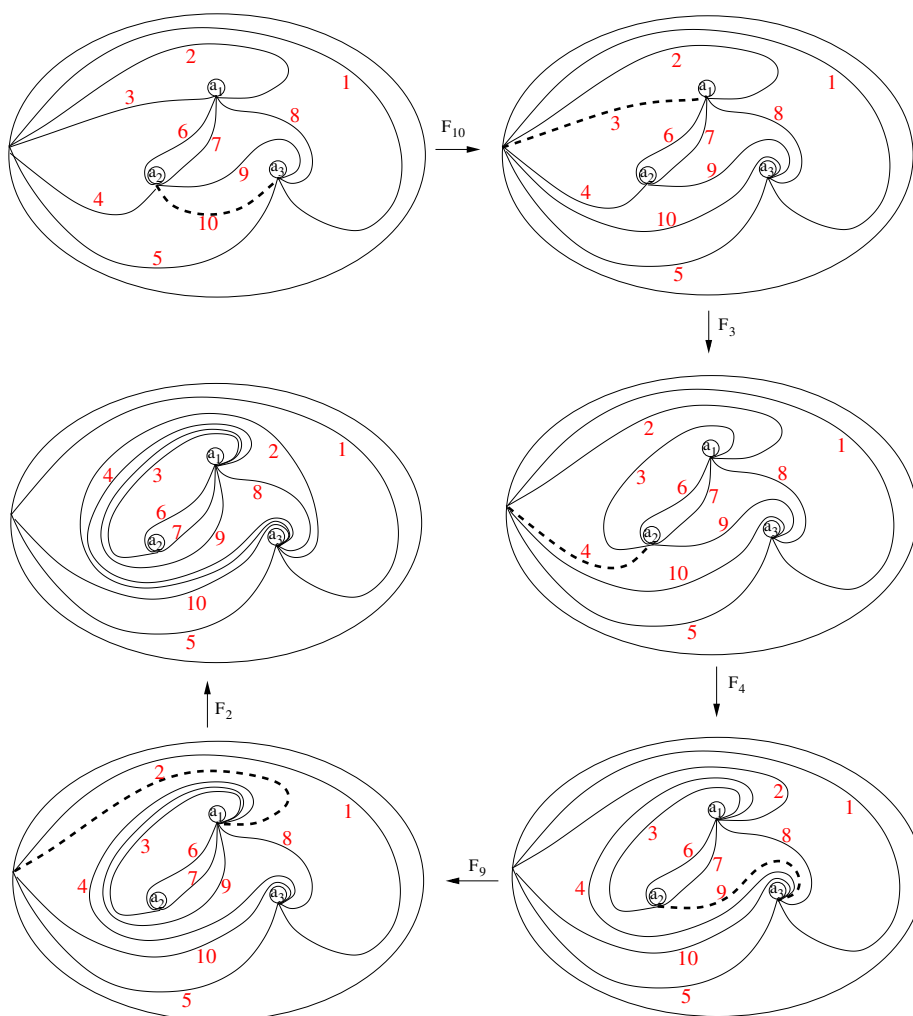


Figure 6.2: D_{12}

$$D_{12} = \text{Ad}(F_{10}F_3F_4F_9F_2)F_8 \begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix},$$

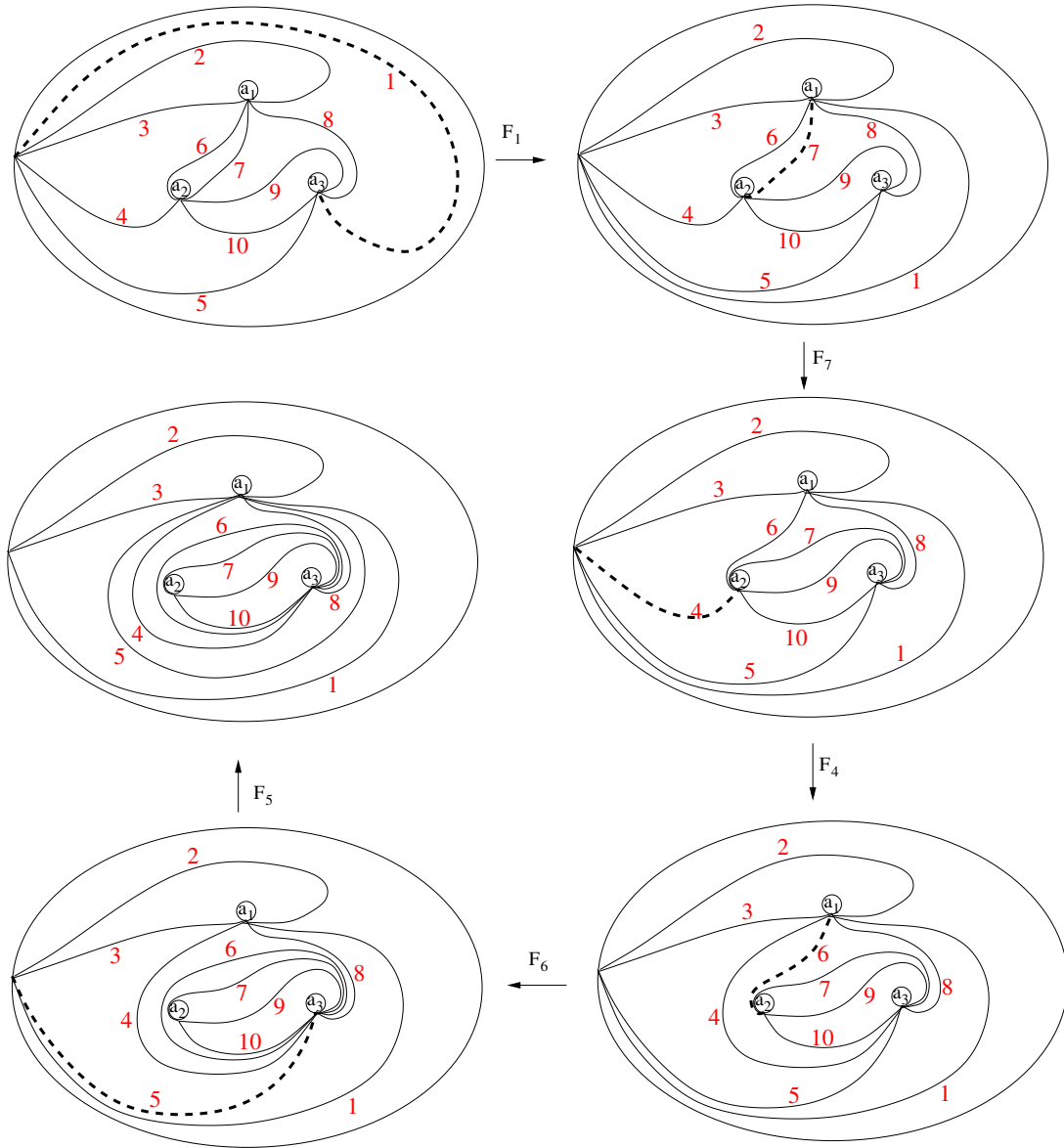


Figure 6.3: D_{13}

$$D_{13} = \text{Ad}(F_1F_7F_4F_6F_5)F_4 \begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix},$$

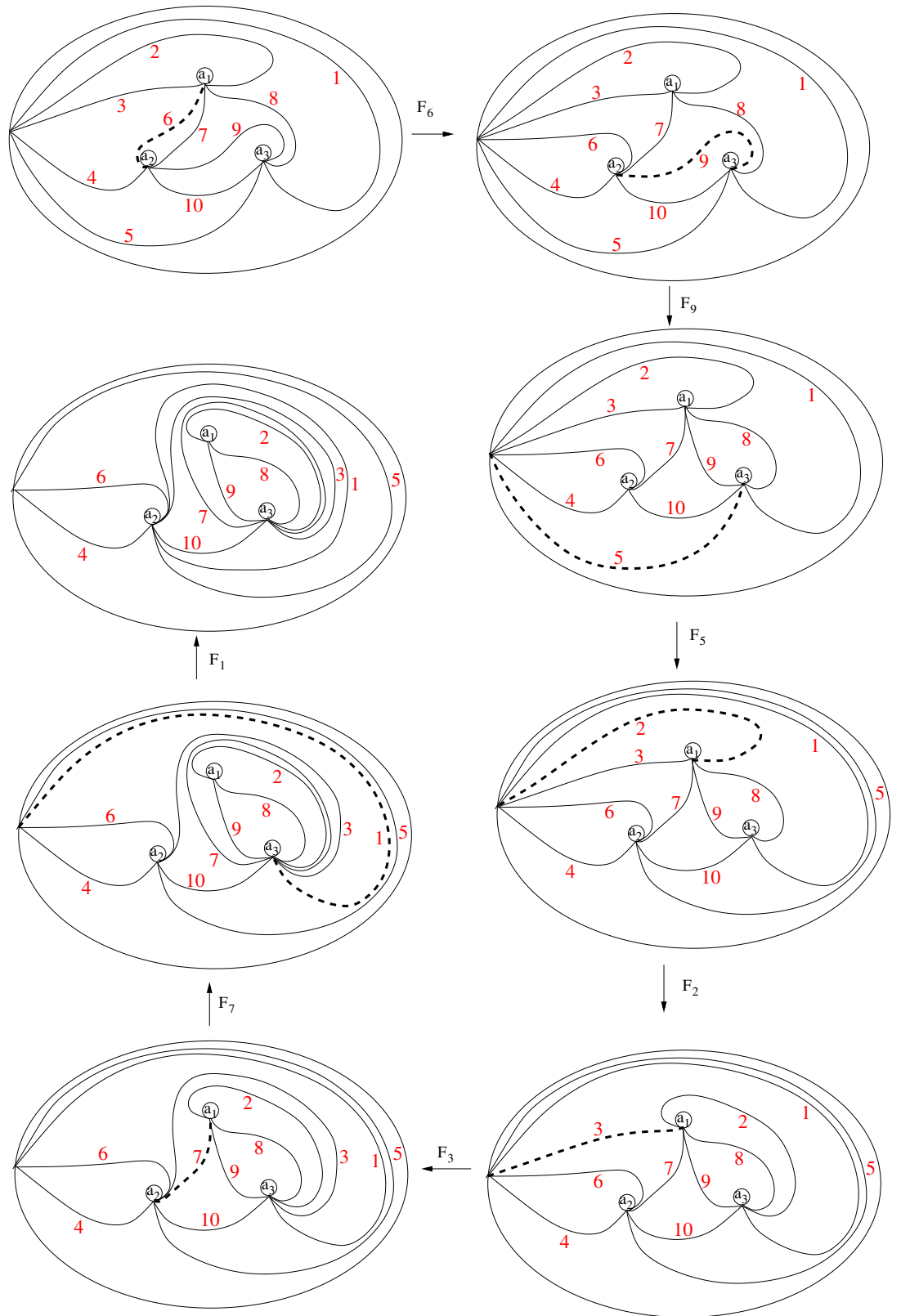


Figure 6.4: D_{23}

$$D_{23} = \text{Ad}(F_6 F_9 F_5 F_2 F_3 F_7 F_1) F_{10} \left(\begin{smallmatrix} 3 & 10 \\ 10 & 3 \end{smallmatrix} \right).$$

They can be simplified as follows:

$$\begin{aligned} D_{12} &= F_{10} F_3 F_4 F_9 F_2 F_8 \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) F_2 F_9 F_4 F_3 F_{10} = \\ &= F_{10} F_3 F_4 F_2 F_9 F_8 F_9 F_2 F_8 F_3 F_{10} \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = \\ &= F_{10} F_3 F_4 F_2 F_8 F_9 \left(\begin{smallmatrix} 8 & 9 \\ 9 & 8 \end{smallmatrix} \right) F_2 F_8 F_3 F_{10} \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = \\ &= F_{10} F_3 F_4 F_2 F_8 F_2 F_9 \left(\begin{smallmatrix} 2 & 9 \\ 9 & 2 \end{smallmatrix} \right) F_3 F_{10} \left(\begin{smallmatrix} 8 & 9 \\ 9 & 8 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = \\ &= F_{10} F_3 F_4 F_8 F_2 \left(\begin{smallmatrix} 2 & 8 \\ 8 & 2 \end{smallmatrix} \right) F_9 F_3 F_{10} \left(\begin{smallmatrix} 2 & 9 \\ 9 & 2 \end{smallmatrix} \right) \left(\begin{smallmatrix} 8 & 9 \\ 9 & 8 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = \\ &= \underline{F_{10} F_3 F_4 F_8 F_2 F_9 F_3 F_{10}} \left(\begin{smallmatrix} 2 & 8 \\ 8 & 2 \end{smallmatrix} \right) \left(\begin{smallmatrix} 2 & 9 \\ 9 & 2 \end{smallmatrix} \right) \left(\begin{smallmatrix} 8 & 9 \\ 9 & 8 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = \\ &= \underline{F_3 F_{10} F_8 F_4 F_2 F_{10} F_9 F_3} \left(\begin{smallmatrix} 2 & 8 \\ 8 & 2 \end{smallmatrix} \right) \left(\begin{smallmatrix} 2 & 9 \\ 9 & 2 \end{smallmatrix} \right) \left(\begin{smallmatrix} 8 & 9 \\ 9 & 8 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = \\ &= \underline{F_3 F_8 F_{10} F_4 F_{10} F_2 F_{10} F_9 F_3} \left(\begin{smallmatrix} 2 & 8 \\ 8 & 2 \end{smallmatrix} \right) \left(\begin{smallmatrix} 2 & 9 \\ 9 & 2 \end{smallmatrix} \right) \left(\begin{smallmatrix} 8 & 9 \\ 9 & 8 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = \\ &= \underline{F_3 F_8 F_4 F_{10}} \left(\begin{smallmatrix} 4 & 10 \\ 10 & 4 \end{smallmatrix} \right) \underline{F_2 F_{10}} \left(\begin{smallmatrix} 2 & 10 \\ 10 & 2 \end{smallmatrix} \right) \underline{F_9 F_3} \left(\begin{smallmatrix} 2 & 9 \\ 9 & 2 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = \\ &= F_3 F_8 F_4 F_{10} F_2 F_4 F_9 F_3 \left(\begin{smallmatrix} 2 & 4 & 8 & 9 \\ 10 & 9 & 4 & 2 & 8 \end{smallmatrix} \right), \end{aligned}$$

$$\begin{aligned} D_{13} &= F_1 F_7 F_4 F_6 F_5 F_4 \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) F_5 F_6 F_4 F_7 F_1 = \\ &= F_1 F_7 F_4 F_5 F_6 F_4 F_6 F_5 F_8 F_7 F_1 \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = \\ &= F_1 F_7 F_4 F_5 F_4 F_6 \left(\begin{smallmatrix} 4 & 6 \\ 6 & 4 \end{smallmatrix} \right) F_5 F_8 F_7 F_1 \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = \\ &= F_1 F_7 F_5 F_4 \left(\begin{smallmatrix} 4 & 5 \\ 5 & 4 \end{smallmatrix} \right) F_6 F_5 F_8 F_7 F_1 \left(\begin{smallmatrix} 4 & 6 \\ 6 & 4 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = \\ &= \underline{F_1 F_7 F_5 F_6 F_4} \left(\begin{smallmatrix} 4 & 6 \\ 6 & 4 \end{smallmatrix} \right) \underline{F_8 F_7 F_1} \left(\begin{smallmatrix} 4 & 5 \\ 5 & 4 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 6 \\ 6 & 4 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = \\ &= F_1 F_5 F_7 F_6 F_7 F_7 F_4 F_7 F_8 F_1 \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) \left(\begin{smallmatrix} 5 & 6 \\ 6 & 5 \end{smallmatrix} \right) = \\ &= F_1 F_5 F_6 F_7 F_4 F_6 F_8 F_1 \left(\begin{smallmatrix} 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 8 & 5 & 4 \end{smallmatrix} \right), \end{aligned}$$

$$\begin{aligned} D_{23} &= F_6 F_9 F_5 F_2 F_3 F_7 F_1 F_{10} \left(\begin{smallmatrix} 3 & 10 \\ 10 & 3 \end{smallmatrix} \right) F_1 F_7 F_3 F_2 F_5 F_9 F_6 = \\ &= F_6 F_9 F_5 F_2 F_3 F_1 F_7 F_{10} F_7 F_1 F_{10} F_2 F_5 F_9 F_6 \left(\begin{smallmatrix} 3 & 10 \\ 10 & 3 \end{smallmatrix} \right) = \\ &= F_6 F_9 F_5 F_2 F_3 F_1 F_{10} F_7 F_1 F_7 F_2 F_5 F_9 F_6 \left(\begin{smallmatrix} 7 & 10 \\ 10 & 7 \end{smallmatrix} \right) \left(\begin{smallmatrix} 3 & 10 \\ 10 & 3 \end{smallmatrix} \right) = \\ &= F_6 F_9 F_5 F_2 F_3 F_1 F_{10} F_1 F_7 F_2 F_5 F_9 F_6 \left(\begin{smallmatrix} 1 & 7 \\ 7 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 7 & 10 \\ 10 & 7 \end{smallmatrix} \right) \left(\begin{smallmatrix} 3 & 10 \\ 10 & 3 \end{smallmatrix} \right) = \\ &= F_6 F_9 F_5 F_2 F_3 F_{10} F_1 F_7 F_2 F_5 F_9 F_6 \left(\begin{smallmatrix} 1 & 10 \\ 10 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 7 \\ 7 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 7 & 10 \\ 10 & 7 \end{smallmatrix} \right) \left(\begin{smallmatrix} 3 & 10 \\ 10 & 3 \end{smallmatrix} \right) = \\ &= F_6 F_9 F_5 F_2 F_3 F_{10} F_1 F_7 F_2 F_5 F_9 F_6 \left(\begin{smallmatrix} 1 & 7 \\ 7 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 3 & 10 \\ 10 & 3 \end{smallmatrix} \right). \end{aligned}$$

The expressions which we use to lift Dehn twists are the following:

$$\begin{aligned} D_0 &= F_5 F_4 F_3 F_2 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{smallmatrix} \right), \\ D_1 &= F_3 F_6 F_7 F_8 \left(\begin{smallmatrix} 2 & 3 & 6 & 7 & 8 \\ 3 & 6 & 7 & 8 & 2 \end{smallmatrix} \right), \\ D_2 &= F_6 F_4 F_{10} F_9 \left(\begin{smallmatrix} 4 & 6 & 7 & 9 & 10 \\ 10 & 4 & 6 & 7 & 9 \end{smallmatrix} \right), \\ D_3 &= F_{10} F_5 F_1 F_8 \left(\begin{smallmatrix} 1 & 5 & 8 & 9 & 10 \\ 8 & 1 & 9 & 10 & 5 \end{smallmatrix} \right), \\ D_{12} &= F_3 F_8 F_4 F_{10} F_2 F_4 F_9 F_3 \left(\begin{smallmatrix} 2 & 4 & 8 & 9 & 10 \\ 10 & 9 & 4 & 2 & 8 \end{smallmatrix} \right), \\ D_{13} &= F_1 F_5 F_6 F_7 F_4 F_6 F_8 F_1 \left(\begin{smallmatrix} 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 8 & 5 & 4 \end{smallmatrix} \right), \end{aligned}$$

$$\begin{aligned}
& \times \tilde{F}_1 \tilde{F}_2 \tilde{F}_4 \tilde{F}_5 \tilde{F}_1 \tilde{F}_8 \tilde{F}_9 \tilde{F}_{10} \tilde{F}_5 \left(\begin{smallmatrix} 1 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 9 & 1 & 10 & 6 & 7 & 8 & 4 \end{smallmatrix} \right) = \\
& = z^{-8} \tilde{D}_1 \tilde{D}_2 \tilde{F}_7 \tilde{F}_8 \tilde{F}_2 \tilde{F}_4 \tilde{F}_{10} \tilde{F}_3 \tilde{F}_7 \tilde{F}_5 \tilde{F}_3 \tilde{F}_1 \tilde{F}_2 \tilde{F}_4 \tilde{F}_5 \tilde{F}_1 \tilde{F}_8 \tilde{F}_7 \tilde{F}_{10} \tilde{F}_5 \left(\begin{smallmatrix} 1 & 5 \\ 5 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 7 & 8 \\ 8 & 7 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 9 & 10 \\ 9 & 10 & 4 \end{smallmatrix} \right) = \\
& = z^{-7} \tilde{D}_1 \tilde{D}_2 \tilde{F}_8 \tilde{F}_7 \left(\begin{smallmatrix} 7 & 8 \\ 8 & 7 \end{smallmatrix} \right) \tilde{F}_2 \tilde{F}_4 \tilde{F}_{10} \tilde{F}_3 \tilde{F}_5 \tilde{F}_3 \tilde{F}_1 \tilde{F}_2 \tilde{F}_4 \tilde{F}_5 \tilde{F}_1 \tilde{F}_8 \tilde{F}_7 \tilde{F}_{10} \tilde{F}_5 \left(\begin{smallmatrix} 1 & 5 \\ 5 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 7 & 8 \\ 8 & 7 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 9 & 10 \\ 9 & 10 & 4 \end{smallmatrix} \right) = \\
& = z^{-8} \tilde{D}_1 \tilde{D}_2 \tilde{F}_8 \tilde{F}_7 \left(\begin{smallmatrix} 7 & 8 \\ 8 & 7 \end{smallmatrix} \right) \tilde{F}_2 \tilde{F}_4 \tilde{F}_{10} \tilde{F}_5 \tilde{F}_3 \left(\begin{smallmatrix} 3 & 5 \\ 5 & 3 \end{smallmatrix} \right) \tilde{F}_1 \tilde{F}_2 \tilde{F}_4 \tilde{F}_5 \tilde{F}_1 \tilde{F}_8 \tilde{F}_7 \tilde{F}_{10} \tilde{F}_5 \left(\begin{smallmatrix} 1 & 5 \\ 5 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 7 & 8 \\ 8 & 7 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 9 & 10 \\ 9 & 10 & 4 \end{smallmatrix} \right) = \\
& = z^{-9} \tilde{D}_1 \tilde{D}_2 \tilde{F}_8 \tilde{F}_7 \tilde{F}_2 \tilde{F}_4 \tilde{F}_{10} \tilde{F}_5 \tilde{F}_3 \tilde{F}_2 \tilde{F}_4 \tilde{F}_1 \left(\begin{smallmatrix} 1 & 4 \\ 4 & 1 \end{smallmatrix} \right) \tilde{F}_3 \tilde{F}_7 \tilde{F}_8 \tilde{F}_{10} \tilde{F}_3 \left(\begin{smallmatrix} 1 & 3 & 4 & 5 & 9 & 10 \\ 5 & 1 & 9 & 3 & 10 & 4 \end{smallmatrix} \right) = \\
& = z^{-10} \tilde{D}_1 \tilde{D}_2 \tilde{F}_8 \tilde{F}_7 \tilde{F}_4 \tilde{F}_{10} \tilde{F}_5 \tilde{F}_3 \tilde{F}_2 \left(\begin{smallmatrix} 2 & 3 \\ 3 & 2 \end{smallmatrix} \right) \tilde{F}_4 \tilde{F}_1 \tilde{F}_3 \tilde{F}_7 \tilde{F}_8 \tilde{F}_{10} \tilde{F}_3 \left(\begin{smallmatrix} 1 & 3 & 4 & 5 & 9 & 10 \\ 9 & 1 & 5 & 3 & 10 & 4 \end{smallmatrix} \right) = \\
& = z^{-10} \tilde{D}_1 \tilde{D}_2 \tilde{F}_8 \tilde{F}_7 \tilde{F}_4 \tilde{F}_{10} \tilde{F}_5 \tilde{F}_3 \tilde{F}_4 \tilde{F}_1 \tilde{F}_7 \tilde{F}_8 \tilde{F}_{10} \tilde{F}_2 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 9 & 10 \\ 9 & 1 & 2 & 5 & 3 & 10 & 4 \end{smallmatrix} \right) = \\
& = z^{-10} \tilde{D}_1 \tilde{D}_2 \tilde{F}_8 \tilde{F}_4 \tilde{F}_{10} \tilde{F}_5 \tilde{F}_3 \tilde{F}_4 \tilde{F}_1 \tilde{F}_8 \tilde{F}_{10} \tilde{F}_2 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 9 & 10 \\ 9 & 1 & 2 & 5 & 3 & 10 & 4 \end{smallmatrix} \right) = \\
& = z^{-11} \tilde{D}_1 \tilde{D}_2 \tilde{F}_4 \tilde{F}_{10} \tilde{F}_5 \tilde{F}_3 \tilde{F}_4 \tilde{F}_1 \tilde{F}_8 \left(\begin{smallmatrix} 1 & 8 \\ 8 & 1 \end{smallmatrix} \right) \tilde{F}_{10} \tilde{F}_2 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 9 & 10 \\ 9 & 1 & 2 & 5 & 3 & 10 & 4 \end{smallmatrix} \right) = \\
& = z^{-11} \tilde{D}_1 \tilde{D}_2 \tilde{D}_0 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{smallmatrix} \right) \tilde{F}_2 \tilde{F}_3 \tilde{F}_4 \tilde{F}_5 \tilde{F}_4 \tilde{F}_{10} \tilde{F}_5 \tilde{F}_3 \tilde{F}_4 \tilde{F}_1 \tilde{F}_8 \left(\begin{smallmatrix} 1 & 8 \\ 8 & 1 \end{smallmatrix} \right) \tilde{F}_{10} \tilde{F}_2 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 9 & 10 \\ 9 & 1 & 2 & 5 & 3 & 10 & 4 \end{smallmatrix} \right) = \\
& = z^{-12} \tilde{D}_1 \tilde{D}_2 \tilde{D}_0 \tilde{F}_1 \tilde{F}_2 \tilde{F}_4 \tilde{F}_3 \left(\begin{smallmatrix} 3 & 4 \\ 4 & 3 \end{smallmatrix} \right) \tilde{F}_{10} \tilde{F}_4 \tilde{F}_2 \tilde{F}_3 \tilde{F}_5 \tilde{F}_8 \tilde{F}_{10} \tilde{F}_1 \left(\begin{smallmatrix} 3 & 4 & 5 & 8 & 9 & 10 \\ 5 & 3 & 8 & 9 & 10 & 4 \end{smallmatrix} \right) = \\
& = z^{-12} \tilde{D}_1 \tilde{D}_2 \tilde{D}_0 \tilde{F}_1 \tilde{F}_2 \tilde{F}_4 \tilde{F}_{10} \tilde{F}_2 \tilde{F}_4 \tilde{F}_5 \tilde{F}_8 \tilde{F}_{10} \tilde{F}_1 \left(\begin{smallmatrix} 4 & 5 & 8 & 9 & 10 \\ 5 & 8 & 9 & 10 & 4 \end{smallmatrix} \right) = \\
& = z^{-12} \tilde{D}_1 \tilde{D}_2 \tilde{D}_0 \tilde{F}_1 \tilde{F}_4 \tilde{F}_{10} \tilde{F}_4 \tilde{F}_5 \tilde{F}_8 \tilde{F}_{10} \tilde{F}_1 \left(\begin{smallmatrix} 4 & 5 & 8 & 9 & 10 \\ 5 & 8 & 9 & 10 & 4 \end{smallmatrix} \right) = \\
& = z^{-11} \tilde{D}_1 \tilde{D}_2 \tilde{D}_0 \tilde{F}_1 \tilde{F}_{10} \tilde{F}_4 \left(\begin{smallmatrix} 4 & 10 \\ 10 & 4 \end{smallmatrix} \right) \tilde{F}_5 \tilde{F}_8 \tilde{F}_{10} \tilde{F}_1 \left(\begin{smallmatrix} 4 & 5 & 8 & 9 & 10 \\ 5 & 8 & 9 & 10 & 4 \end{smallmatrix} \right) = \\
& = z^{-11} \tilde{D}_1 \tilde{D}_2 \tilde{D}_0 \tilde{F}_1 \tilde{F}_{10} \tilde{F}_5 \tilde{F}_8 \tilde{F}_1 \left(\begin{smallmatrix} 5 & 8 & 9 & 10 \\ 8 & 9 & 10 & 5 \end{smallmatrix} \right) = \\
& = z^{-12} \tilde{D}_1 \tilde{D}_2 \tilde{D}_0 \tilde{F}_{10} \tilde{F}_5 \tilde{F}_1 \left(\begin{smallmatrix} 1 & 5 \\ 5 & 1 \end{smallmatrix} \right) \tilde{F}_8 \left(\begin{smallmatrix} 5 & 8 & 9 & 10 \\ 8 & 9 & 10 & 5 \end{smallmatrix} \right) = \\
& = z^{-12} \tilde{D}_1 \tilde{D}_2 \tilde{D}_0 \tilde{F}_{10} \tilde{F}_5 \tilde{F}_1 \tilde{F}_8 \left(\begin{smallmatrix} 1 & 5 & 8 & 9 & 10 \\ 8 & 1 & 9 & 10 & 5 \end{smallmatrix} \right) = \\
& = z^{-12} \tilde{D}_1 \tilde{D}_2 \tilde{D}_0 \tilde{D}_3.
\end{aligned}$$

□

Lemma 6.5.5. *There are the lifts of Dehn twists such that for the puncture relation we have that:*

$$\tilde{D}_{a_{12}} \tilde{D}_{a_{13}} \tilde{D}_{a_{23}} = z^{-12} \tilde{D}_{a_1} \tilde{D}_{a_2} \tilde{D}_{a_3}.$$

Proof. As this is a degenerated case of the lantern relation, the proof is the same as above. □

Lemma 6.5.6. *By choosing the lifts of the Dehn twists in the chain relation, we have that:*

$$(\tilde{D}_a \tilde{D}_b \tilde{D}_c)^4 = z^{-24} \tilde{D}_e \tilde{D}_f.$$

Proof. Consider the two-holed torus with one puncture on each boundary component. The simple closed geodesics a , b , c , e and f are as follows:

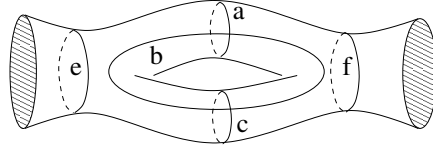


Figure 6.5: Chain relation

We consider the following ideal triangulation of the two-holed torus:

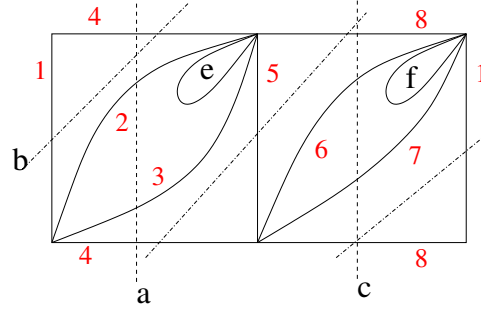


Figure 6.6: An ideal triangulation of a two-holed torus

The Dehn twists corresponding to a , b , c , e and f can be presented as follows:

$$\begin{aligned}
 D_a &= F_3 F_4 F_3 \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}, \\
 D_b &= \text{Ad}(F_8 F_1 F_5) F_8 \begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix}, \\
 D_c &= F_7 F_8 F_7 \begin{pmatrix} 6 & 8 \\ 8 & 6 \end{pmatrix}, \\
 D_e &= \text{Ad}(F_5 F_4 F_8 F_6 F_7) F_3 \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \\
 D_f &= \text{Ad}(F_1 F_8 F_4 F_2 F_3) F_7 \begin{pmatrix} 6 & 7 \\ 7 & 6 \end{pmatrix}.
 \end{aligned}$$

The Dehn twist D_e can be simplified as follows:

$$\begin{aligned}
 D_e &= F_5 F_4 F_8 F_6 F_7 F_3 \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} F_7 F_6 F_8 F_4 F_5 = \\
 &= F_5 F_4 F_8 F_6 F_7 F_3 F_7 F_6 F_8 F_4 F_5 \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} = \\
 &= F_5 F_4 F_8 F_6 F_3 F_7 \begin{pmatrix} 3 & 7 \\ 7 & 3 \end{pmatrix} F_6 F_8 F_4 F_5 \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} = \\
 &= F_5 F_4 F_8 F_3 F_7 F_6 F_8 F_4 F_5 \begin{pmatrix} 6 & 7 \\ 7 & 6 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} = \\
 &= F_5 F_8 F_3 F_7 F_6 F_4 F_8 F_5 \begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 6 & 7 \\ 7 & 6 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} = \\
 &= F_5 F_3 F_8 F_7 F_6 F_4 F_5 \begin{pmatrix} 3 & 8 \\ 8 & 3 \end{pmatrix} \begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 6 & 7 \\ 7 & 6 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} = \\
 &= F_3 F_5 F_3 F_8 F_7 F_6 F_4 F_3 \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 3 & 8 \\ 8 & 3 \end{pmatrix} \begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 6 & 7 \\ 7 & 6 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} = \\
 &= F_3 F_5 F_8 F_7 F_6 F_3 F_4 \begin{pmatrix} 3 & 6 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 3 & 8 \\ 8 & 3 \end{pmatrix} \begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 6 & 7 \\ 7 & 6 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} = \\
 &= F_3 F_5 F_8 F_7 F_6 F_3 F_4 \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 8 & 5 & 6 & 7 \end{pmatrix},
 \end{aligned}$$

and D_f can be simplified in a similar way as follows:

$$D_f = F_7 F_1 F_4 F_3 F_2 F_7 F_8 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 6 & 7 & 8 \\ 4 & 1 & 2 & 3 & 7 & 8 & 6 \end{smallmatrix} \right),$$

by sending $(2, 3, 4, 5, 6, 7, 8)$ to $(6, 7, 8, 1, 2, 3, 4)$.

We will use the following expression of these Dehn twists to get their lifts:

$$\begin{aligned} D_a &= F_3 F_4 F_3 \left(\begin{smallmatrix} 2 & 4 \\ 4 & 2 \end{smallmatrix} \right), \\ D_b &= \text{Ad}(F_8 F_1 F_5) F_8 \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right), \\ D_c &= F_7 F_8 F_7 \left(\begin{smallmatrix} 6 & 8 \\ 8 & 6 \end{smallmatrix} \right), \\ D_e &= F_5 F_3 F_8 F_7 F_6 F_4 F_5 \left(\begin{smallmatrix} 2 & 3 & 4 & 6 & 7 & 8 \\ 3 & 8 & 2 & 4 & 6 & 7 \end{smallmatrix} \right), \\ D_f &= F_1 F_7 F_4 F_3 F_2 F_8 F_1 \left(\begin{smallmatrix} 2 & 3 & 4 & 6 & 7 & 8 \\ 8 & 2 & 3 & 7 & 4 & 6 \end{smallmatrix} \right). \end{aligned}$$

We need to check if the induced lifts of the Dehn twists satisfy the trivial type-1 braid relation. As there is a symmetry between D_a and D_c with respect to D_b , we only need to verify the trivial type-1 braid relation for \tilde{D}_a and \tilde{D}_b .

We rewrite \tilde{D}_b as follows:

$$\begin{aligned} \tilde{D}_b &= \tilde{F}_8 \tilde{F}_1 \tilde{F}_5 \tilde{F}_8 \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) \tilde{F}_5 \tilde{F}_1 \tilde{F}_8 = z^{-1} \tilde{F}_8 \tilde{F}_1 \tilde{F}_5 \tilde{F}_8 \left(\begin{smallmatrix} 5 & 8 \\ 8 & 5 \end{smallmatrix} \right) \tilde{F}_1 \tilde{F}_4 \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = \\ &= z^{-2} \tilde{F}_1 \tilde{F}_8 \left(\begin{smallmatrix} 1 & 8 \\ 8 & 1 \end{smallmatrix} \right) \tilde{F}_5 \tilde{F}_1 \tilde{F}_4 \left(\begin{smallmatrix} 5 & 8 \\ 8 & 5 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = \\ &= z^{-3} \tilde{F}_1 \tilde{F}_5 \tilde{F}_8 \left(\begin{smallmatrix} 5 & 8 \\ 8 & 5 \end{smallmatrix} \right) \tilde{F}_4 \left(\begin{smallmatrix} 1 & 8 \\ 8 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 5 & 8 \\ 8 & 5 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = z^{-3} \tilde{F}_1 \tilde{F}_5 \tilde{F}_8 \tilde{F}_4 \left(\begin{smallmatrix} 1 & 5 \\ 5 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right). \end{aligned}$$

By using once pentagon relation, we have the following equality:

$$\tilde{D}_a = \tilde{F}_3 \tilde{F}_4 \tilde{F}_3 \left(\begin{smallmatrix} 2 & 4 \\ 4 & 2 \end{smallmatrix} \right) = z^{-1} \tilde{F}_4 \tilde{F}_3 \left(\begin{smallmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \end{smallmatrix} \right).$$

Then the type-1 braid relation can be verified as follows:

$$\begin{aligned} \tilde{D}_b \tilde{D}_a \tilde{D}_b &= z^{-7} \tilde{F}_1 \tilde{F}_5 \tilde{F}_8 \tilde{F}_4 \left(\begin{smallmatrix} 1 & 5 \\ 5 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) \tilde{F}_4 \tilde{F}_3 \left(\begin{smallmatrix} 2 & 3 & 4 \\ 3 & 4 & 2 \end{smallmatrix} \right) \tilde{F}_1 \tilde{F}_5 \tilde{F}_8 \tilde{F}_4 \left(\begin{smallmatrix} 1 & 5 \\ 5 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) = \\ &= z^{-7} \tilde{F}_1 \tilde{F}_5 \tilde{F}_8 \tilde{F}_4 \tilde{F}_8 \tilde{F}_3 \tilde{F}_5 \tilde{F}_1 \tilde{F}_4 \tilde{F}_2 \left(\begin{smallmatrix} 2 & 3 & 8 \\ 8 & 2 & 3 \end{smallmatrix} \right) = \\ &= z^{-7} \tilde{F}_1 \tilde{F}_5 \tilde{F}_4 \tilde{F}_3 \tilde{F}_5 \tilde{F}_1 \tilde{F}_4 \tilde{F}_2 \left(\begin{smallmatrix} 2 & 3 & 8 \\ 8 & 2 & 3 \end{smallmatrix} \right) = \\ &= z^{-6} \tilde{F}_5 \tilde{F}_4 \tilde{F}_1 \left(\begin{smallmatrix} 1 & 4 \\ 4 & 1 \end{smallmatrix} \right) \tilde{F}_3 \tilde{F}_5 \tilde{F}_4 \tilde{F}_2 \left(\begin{smallmatrix} 2 & 3 & 8 \\ 8 & 2 & 3 \end{smallmatrix} \right) = \\ &= z^{-5} \tilde{F}_5 \tilde{F}_4 \tilde{F}_3 \tilde{F}_5 \tilde{F}_1 \left(\begin{smallmatrix} 1 & 5 \\ 5 & 1 \end{smallmatrix} \right) \tilde{F}_2 \left(\begin{smallmatrix} 1 & 4 \\ 4 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 2 & 3 & 8 \\ 8 & 2 & 3 \end{smallmatrix} \right) = \\ &= z^{-5} \tilde{F}_5 \tilde{F}_4 \tilde{F}_5 \tilde{F}_5 \tilde{F}_3 \tilde{F}_5 \tilde{F}_1 \left(\begin{smallmatrix} 1 & 5 \\ 5 & 1 \end{smallmatrix} \right) \tilde{F}_2 \left(\begin{smallmatrix} 1 & 4 \\ 4 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 2 & 3 & 8 \\ 8 & 2 & 3 \end{smallmatrix} \right) = \\ &= z^{-5} \tilde{F}_4 \tilde{F}_5 \left(\begin{smallmatrix} 4 & 5 \\ 5 & 4 \end{smallmatrix} \right) \tilde{F}_3 \tilde{F}_5 \left(\begin{smallmatrix} 3 & 5 \\ 5 & 3 \end{smallmatrix} \right) \tilde{F}_1 \tilde{F}_2 \left(\begin{smallmatrix} 1 & 5 \\ 5 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 4 \\ 4 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 2 & 3 & 8 \\ 8 & 2 & 3 \end{smallmatrix} \right) = \\ &= z^{-5} \tilde{F}_4 \tilde{F}_3 \tilde{F}_5 \tilde{F}_4 \tilde{F}_1 \tilde{F}_2 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 8 \\ 5 & 8 & 4 & 2 & 1 & 3 \end{smallmatrix} \right) = \\ &= z^{-4} \tilde{D}_a \left(\begin{smallmatrix} 2 & 3 & 4 \\ 3 & 4 & 2 \end{smallmatrix} \right) \tilde{F}_5 \tilde{F}_4 \tilde{F}_1 \tilde{F}_2 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 8 \\ 5 & 8 & 4 & 2 & 1 & 3 \end{smallmatrix} \right) = \\ &= z^{-4} \tilde{D}_a \tilde{F}_5 \tilde{F}_3 \tilde{F}_1 \tilde{F}_4 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 8 \\ 5 & 4 & 2 & 8 & 1 & 3 \end{smallmatrix} \right) = \\ &= z^{-4} \tilde{D}_a \tilde{F}_1 \tilde{F}_5 \tilde{F}_3 \tilde{F}_4 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 8 \\ 5 & 4 & 2 & 8 & 1 & 3 \end{smallmatrix} \right) = \\ &= z^{-1} \tilde{D}_a \tilde{D}_b \left(\begin{smallmatrix} 1 & 5 \\ 5 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix} \right) \tilde{F}_4 \tilde{F}_8 \tilde{F}_3 \tilde{F}_4 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 8 \\ 5 & 4 & 2 & 8 & 1 & 3 \end{smallmatrix} \right) = \\ &= \tilde{D}_a \tilde{D}_b \tilde{D}_a. \end{aligned}$$

$$\begin{aligned}
& \times \begin{pmatrix} 7 & 8 \\ 8 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 5 & 7 & 4 & 2 & 1 & 3 \end{pmatrix} = \\
= & z^{-23} \widetilde{F}_7 \widetilde{F}_1 \widetilde{F}_3 \widetilde{F}_5 \widetilde{F}_8 \widetilde{F}_5 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 5 & 7 & 3 & 2 & 1 & 4 \end{pmatrix} \widetilde{F}_7 \widetilde{F}_8 \widetilde{F}_3 \widetilde{F}_2 \widetilde{F}_3 \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \widetilde{F}_1 \widetilde{F}_2 \widetilde{F}_5 \widetilde{F}_6 \times \\
& \times \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 5 & 7 & 4 & 2 & 3 & 1 \end{pmatrix} = \\
= & z^{-23} \widetilde{F}_7 \widetilde{F}_1 \widetilde{F}_3 \widetilde{F}_5 \widetilde{F}_8 \widetilde{F}_5 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 5 & 7 & 3 & 2 & 1 & 4 \end{pmatrix} \widetilde{F}_7 \widetilde{F}_8 \widetilde{F}_3 \widetilde{F}_2 \widetilde{F}_3 \widetilde{F}_1 \widetilde{F}_3 \widetilde{F}_5 \widetilde{F}_6 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 8 & 7 & 4 & 2 & 3 & 1 \end{pmatrix} = \\
= & z^{-24} \widetilde{F}_7 \widetilde{F}_1 \widetilde{F}_3 \widetilde{F}_5 \widetilde{F}_8 \widetilde{F}_5 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 5 & 7 & 3 & 2 & 1 & 4 \end{pmatrix} \widetilde{F}_7 \widetilde{F}_8 \widetilde{F}_3 \widetilde{F}_2 \widetilde{F}_1 \widetilde{F}_3 \widetilde{F}_5 \widetilde{F}_6 \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \times \\
& \times \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 8 & 7 & 4 & 2 & 3 & 1 \end{pmatrix} = \\
= & z^{-24} \widetilde{F}_7 \widetilde{F}_1 \widetilde{F}_3 \widetilde{F}_5 \widetilde{F}_8 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 5 & 7 & 3 & 2 & 1 & 4 \end{pmatrix} \widetilde{F}_7 \widetilde{F}_8 \widetilde{F}_2 \widetilde{F}_1 \widetilde{F}_3 \widetilde{F}_5 \widetilde{F}_6 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 6 & 7 & 4 & 2 & 3 & 1 \end{pmatrix} = \\
= & z^{-24} \widetilde{F}_7 \widetilde{F}_1 \widetilde{F}_3 \widetilde{F}_4 \widetilde{F}_5 \widetilde{F}_8 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 5 & 7 & 3 & 2 & 1 & 4 \end{pmatrix} \widetilde{F}_8 \widetilde{F}_2 \widetilde{F}_1 \widetilde{F}_3 \widetilde{F}_5 \widetilde{F}_6 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 6 & 7 & 4 & 2 & 3 & 1 \end{pmatrix} = \\
= & z^{-25} \widetilde{F}_7 \widetilde{F}_1 \widetilde{F}_4 \widetilde{F}_3 \widetilde{F}_4 \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \widetilde{F}_5 \widetilde{F}_8 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 5 & 7 & 3 & 2 & 1 & 4 \end{pmatrix} \widetilde{F}_8 \widetilde{F}_2 \widetilde{F}_1 \widetilde{F}_3 \widetilde{F}_5 \widetilde{F}_6 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 6 & 7 & 4 & 2 & 3 & 1 \end{pmatrix} = \\
= & z^{-25} \widetilde{F}_7 \widetilde{F}_1 \widetilde{F}_4 \widetilde{F}_3 \widetilde{F}_4 \widetilde{F}_5 \widetilde{F}_2 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 7 & 5 & 3 & 2 & 1 & 4 \end{pmatrix} \widetilde{F}_4 \widetilde{F}_1 \widetilde{F}_2 \widetilde{F}_3 \widetilde{F}_5 \widetilde{F}_6 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 6 & 7 & 4 & 2 & 3 & 1 \end{pmatrix} = \\
= & z^{-27} \widetilde{F}_7 \widetilde{F}_1 \widetilde{F}_4 \widetilde{F}_3 \widetilde{F}_4 \widetilde{F}_5 \widetilde{F}_2 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 7 & 5 & 3 & 2 & 1 & 4 \end{pmatrix} \widetilde{F}_1 \widetilde{F}_4 \widetilde{F}_1 \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \widetilde{F}_3 \widetilde{F}_2 \widetilde{F}_3 \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \widetilde{F}_5 \widetilde{F}_6 \times \\
& \times \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 6 & 7 & 4 & 2 & 3 & 1 \end{pmatrix} = \\
= & z^{-27} \widetilde{F}_7 \widetilde{F}_1 \widetilde{F}_4 \widetilde{F}_3 \widetilde{F}_4 \widetilde{F}_2 \widetilde{F}_7 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 7 & 5 & 3 & 2 & 1 & 4 \end{pmatrix} \widetilde{F}_3 \widetilde{F}_4 \widetilde{F}_3 \widetilde{F}_1 \widetilde{F}_2 \widetilde{F}_3 \widetilde{F}_5 \widetilde{F}_6 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 8 & 4 & 2 & 3 & 1 \end{pmatrix} = \\
= & z^{-26} \widetilde{F}_7 \widetilde{F}_1 \widetilde{F}_4 \widetilde{F}_3 \widetilde{F}_4 \widetilde{F}_2 \widetilde{F}_7 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 7 & 5 & 3 & 2 & 1 & 4 \end{pmatrix} \widetilde{F}_4 \widetilde{F}_3 \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \widetilde{F}_1 \widetilde{F}_2 \widetilde{F}_3 \widetilde{F}_5 \widetilde{F}_6 \times \\
& \times \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 8 & 4 & 2 & 3 & 1 \end{pmatrix} = \\
= & z^{-25} \widetilde{D}_f \begin{pmatrix} 6 & 7 & 8 \\ 8 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \widetilde{F}_4 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 7 & 5 & 3 & 2 & 1 & 4 \end{pmatrix} \widetilde{F}_3 \widetilde{F}_1 \widetilde{F}_2 \widetilde{F}_4 \widetilde{F}_5 \widetilde{F}_6 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 8 & 5 & 4 & 2 & 3 & 1 \end{pmatrix} = \\
= & z^{-25} \widetilde{D}_f \widetilde{F}_3 \widetilde{F}_5 \widetilde{F}_8 \widetilde{F}_7 \widetilde{F}_6 \widetilde{F}_3 \widetilde{F}_4 \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 8 & 5 & 6 & 7 \end{pmatrix} = \\
= & z^{-24} \widetilde{D}_f \widetilde{D}_e.
\end{aligned}$$

The chain relation becomes:

$$(\widetilde{D}_a \widetilde{D}_b \widetilde{D}_c)^4 = z^{-24} \widetilde{D}_e \widetilde{D}_f$$

□

We say that the lifts of Dehn twists are **normalized** if all braid type relations and all lantern relations are lifted in the trivial way.

By the lemmas above, we normalize the lift of each Dehn twist D from \widetilde{D} to $z^{-12} \widetilde{D}$. Then we get the presentation in Proposition (6.5.1).

6.6 Second cohomology group $H^2(\Gamma, \mathbb{Z})$

The second cohomology group of the mapping class group was first computed by Harer in [34] for $g \geq 5$ and further completed by Korkmaz and Stipsicz in [41] for $g \geq 4$. By these results, we have

$$H^2(\Gamma_g^s) = \mathbb{Z}^{s+1},$$

for $g \geq 4$. Moreover the generators are the one quarter of the Meyer class χ' and s Euler classes e_i associated to s punctures respectively. We recall their

definitions in this section. Notice that by the Universal Coefficient Theorem, we can define the Meyer class and the Euler class in $H^2(\Sigma_g^s, A)$ for any abelian group A .

6.6.1 Meyer class χ'

The Meyer class χ' is given by the Meyer signature cocycle $C_{\chi'}$ defined as follows. Let P denote a pair of pants. Let A , B and C denote the three elements of $\pi_1(P)$ associated to the three boundary components of P . Thus A , B and C generate $\pi_1(P)$:

$$\pi_1(P) = \langle A, B, C \mid ABC = Id \rangle.$$

Let ϕ and ψ be two homeomorphisms of Σ . We consider the surface bundle over P whose fiber is Σ and whose holonomy sends the A to ϕ , B to ψ and C to $\psi^{-1}\phi^{-1}$. We obtain a 4 dimensional manifold M . Then we consider the second cohomology group $H^2(M, \mathbb{Z})$. Notice that there is a cup product \smile defined as follows:

$$\begin{aligned} \smile: H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) &\rightarrow H^4(M, \mathbb{Z}), \\ (x, y) &\mapsto x \smile y. \end{aligned}$$

As M is compact by construction, the group $H^4(M, \mathbb{Z})$ identifies with $H_0(M, \mathbb{Z})$ by Poincaré duality. On the other hand $H_0(M, \mathbb{Z})$ is isomorphic to \mathbb{Z} . Thus we obtain a symmetric bilinear form Q on $H^2(M, \mathbb{Z})$. Moreover Q is non-degenerated. Let $Sign(Q)$ denote the signature of Q . Then the Meyer signature cocycle is defined by:

$$C_{\chi'}(\phi, \psi) = Sign(Q).$$

6.6.2 Euler class e^i

The second cohomology group $H^2(\Gamma_g^s, \mathbb{Z})$ describes the central extension of Γ_g^s by \mathbb{Z} up to isomorphism. We will introduce the Euler class by describing one central extension of Γ_g^s in this class.

We first consider the homeomorphism of S^1 preserving the orientation. Let $Hom^+(S_1)$ denote this group. It has a central extension by \mathbb{Z} given by lifting a homeomorphism of S^1 to a homeomorphism of \mathbb{R} which has period 1. We denote this central extension by $\widetilde{Hom}^+(S^1)$ which is a subgroup of $Hom^+(\mathbb{R})$.

Consider the i -th puncture of Σ_g^s . To define the Euler class e_i associated to this puncture, we consider the surface Σ_g^{s-1} given by adding a point p to the i -th puncture of Σ_g^s and we treat p as a marked point in Σ_g^{s-1} . There is a natural homeomorphism $f: \Sigma_g^s \rightarrow \Sigma_g^{s-1} \setminus \{p\}$ which implies an injective homomorphism from Γ_g^s to Γ_g^{s-1} by sending a homeomorphism ϕ of Σ_g^s to a homeomorphism ψ of Σ_g^{s-1} which fixes p and equals to $f \circ \phi \circ f^{-1}$ on $\Sigma_g^{s-1} \setminus \{p\}$.

Fix a hyperbolic structure on Σ_g^{s-1} . Its universal cover is isometric to \mathbb{H} . Let \tilde{p} be a lift of p . A homeomorphism ψ of Σ_g^{s-1} fixing p can be lifted to a homeomorphism of \mathbb{H} and we consider the one which fix \tilde{p} and denote it by

$\tilde{\psi}$. It has a natural extension to the boundary of \mathbb{H} which is identified with S^1 . By this correspondence, we obtain an injective homomorphism from Γ_g^s to $\widehat{Hom}^+(S^1)$. Using this homomorphism to pull back the central extension $\widehat{Hom}^+(S^1) \rightarrow Hom^+(S^1)$, we obtain a central extension of Γ_g^s giving an element e_i in $H^2(\Sigma_g^s, \mathbb{Z})$. This element e_i is called the Euler class associated to the i -th puncture.

6.7 Cohomology class of $\widehat{\Gamma}$

We are now ready to state our main theorem in this section as follows:

Theorem 6.7.1. *Let $g \geq 2$ and $s \geq 4$. The minimal reduction of $\widehat{\Gamma}(\Sigma_g^s)$ can be obtained by centrally extending $\Gamma(\Sigma_g^s)$ by A which is a cyclic subgroup of \mathbb{C}^* generated by z^{-12} . Moreover, its cohomology class is*

$$c_{\Gamma(\Sigma_g^s)} = 12\chi + \sum_{i=1}^s e_i \in H^2(\Gamma(\Sigma_g^s), A),$$

where χ is one quarter of the Meyer signature class χ' and e_i is the Euler class associated to the i -th puncture.

As we obtain a central extension of mapping class group isomorphic to that in [27], we can prove this theorem by the same argument. In the following, we assume that z is not a root of unity, so that A is isomorphic to \mathbb{Z} , and we first prove the theorem in this case.

We denote by $\widehat{\Gamma}_g^s$ the group defined by Proposition (6.5.1) for any s and g .

Lemma 6.7.1. *If $g \geq 2$ and $s = 0$, then we have $c_{\widehat{\Gamma}_g} = 12\chi \in H^2(\Gamma(\Sigma_g), \mathbb{Z})$.*

Proof. As $s = 0$, there is no more puncture relation in the presentation. Let $\Gamma_g(1)$ denote the subgroup of $\widehat{\Gamma}_g$ generated by the lifts \tilde{D}_a of Dehn twists and the central element $u = w^{12}$. Then $\Gamma_g(1)$ is the universal central extension considered by Harer in [34] and thus $c_{\Gamma_g(1)}$ is the generator χ of $H^2(\Gamma_g(1)) \cong \mathbb{Z}$ where χ is one quarter of the Meyer signature class.

Let $C_{\Gamma_g(1)} : \Gamma_g \times \Gamma_g \rightarrow \mathbb{Z}$ denote the 2-cocycle associated to $c_{\Gamma_g(1)}$. It arises as follows. Let $S : \Gamma_g \rightarrow \Gamma_g(1)$ be a set-wise section. Let $\pi : \Gamma_g(1) \rightarrow \Gamma_g$ be the projection. We define the isomorphism $i : \ker(\pi) \rightarrow \mathbb{Z}$ by setting $i(u) = 1$. Then the 2-cocycle $C_{\Gamma_g(1)}$ is given by:

$$C_{\Gamma_g(1)}(x, y) = i(S(xy)S(x^{-1})S(y^{-1})).$$

We repeat this for $\widehat{\Gamma}_g$ to find its associated 2-cocycle. Let $\iota : \Gamma_g(1) \rightarrow \widehat{\Gamma}_g$ be the inclusion. The above section S induces a section $S' = \iota \circ S : \Gamma_g \rightarrow \widehat{\Gamma}_g$. Let $\pi' : \widehat{\Gamma}_g \rightarrow \Gamma_g$ be the projection. We define the isomorphism $j : \ker(\pi') \rightarrow \mathbb{Z}$ by setting $j(w) = 1$. Then the 2-cocycle $C_{\widehat{\Gamma}_g}$ is given by:

$$C_{\widehat{\Gamma}_g}(x, y) = j(S'(xy)S'(x^{-1})S'(y^{-1}))$$

$$\begin{aligned}
&= j \circ \iota(S(xy)S(x^{-1})S(y^{-1})) \\
&= j \circ \iota \circ i^{-1}(C_{\Gamma_g(1)}(x, y)).
\end{aligned}$$

By definitions of j , ι and i , the application $j \circ \iota \circ i^{-1}$ is an injective homomorphism from \mathbb{Z} to \mathbb{Z} . Moreover, by considering the relation $u = w^{12}$, we have that $j \circ \iota \circ i^{-1}(1) = j \circ \iota(u) = j(w^{12}) = 12$, which implies that:

$$C_{\widehat{\Gamma}_g} = 12C_{\Gamma_g(1)}.$$

Thus $c_{\widehat{\Gamma}_g} = 12\chi$. □

By adding one point on each puncture, we obtain Σ_g from Σ_g^s . This induces an embedding of Σ_g^s in Σ_g , thus an surjective f from Γ_g^s to Γ_g . The pullback $f^*\widehat{\Gamma}_g$ by f is a central extension of Γ_g^s which is in the Meyer class of $H^2(\Gamma_g^s)$.

Assume now $s > 0$. The central extension of Γ_g^s in the Euler class $e_i \in H^2(\Gamma_g^s)$ associated to the i -th puncture can be identified with the mapping class group $\Gamma_{g,1}^{s-1}$ where the surface $\Sigma_{g,1}^{s-1}$ is obtained from Σ_g^s by replacing the i -th puncture by a boundary component.

Definition 6.7.1. *Let $s > 0$ and $(m_1, \dots, m_s) \in \mathbb{Z}$. We define the central extension $\Gamma_g^s(m_1, \dots, m_s)$ of Γ_g^s by A having the following presentation:*

(1) *The generators are:*

- (a) *The lifts \widetilde{D}_a of the Dehn twists D_a associated to non-separating curves a ;*
- (b) *The central element $w = z^{-12}$;*

(2) *The relation are:*

- (a) *Trivial braid relation, trivial chain relation and trivial lantern relation;*
- (b) *The puncture relation becomes:*

$$\widetilde{D}_{a_1}\widetilde{D}_{a_2}\widetilde{D}_{a_3} = w^{m_i}\widetilde{D}_{a_{12}}\widetilde{D}_{a_{13}}\widetilde{D}_{a_{23}},$$

for the i -th puncture.

Lemma 6.7.2. *The central extension $\Gamma_g^s(0, \dots, 1, \dots, 0)$ with 1 on the i -th position is isomorphic to $\Gamma_{g,1}^{s-1}$.*

Proof. By collapsing the boundary of $\Sigma_{g,1}^{s-1}$ to a puncture, we can define a homeomorphism $f : \Sigma_g^s \rightarrow \Sigma_{g,1}^{s-1} \setminus \partial\Sigma_{g,1}^{s-1}$. It induces a homomorphism $\widetilde{f} : \Gamma_g^s(0, \dots, 1, \dots, 0) \rightarrow \Gamma_{g,1}^{s-1}$ by sending a generator \widetilde{D}_a to $\widetilde{D}_{f(a)}$ and the generator w to \widetilde{D}_{b_i} where b_i is the boundary component. This homomorphism is well-defined and in fact is an isomorphism. We can verify this easily by checking the relations between the generators. □

Now we are ready to prove Theorem 6.7.1.

Proof of Theorem 6.7.1. Let \oplus denote the fibered product between central extensions of Γ_g^s . Consider the following central extension:

$$\overline{\Gamma}_g^s := f^* \widehat{\Gamma}_g \oplus \left(\bigoplus_{i=1}^s \Gamma_g^s(0, \dots, 1, \dots, 0) \right),$$

of Γ_g^s by \mathbb{Z}^{s+1} . As $H^2(\Sigma_g^s, \mathbb{Z}^{s+1}) = \bigoplus_{i=1}^{s+1} H^2(\Sigma_g^s, \mathbb{Z})$, by the above two lemma, we can conclude that the 2-cohomology class $c_{\overline{\Gamma}_g^s}$ is $(12\chi, e_1, \dots, e_s)$. Define $L : \mathbb{Z}^{s+1} \rightarrow \mathbb{Z}$ by $L(m_1, \dots, m_{s+1}) = m_1 + \dots + m_{s+1}$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}^{s+1} & \longrightarrow & \overline{\Gamma}_g^s & \longrightarrow & \Gamma_g^s \longrightarrow 1 \\ & & \downarrow L & & \downarrow \pi & & \downarrow id \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widehat{\Gamma}_g^s & \longrightarrow & \Gamma_g^s \longrightarrow 1 \end{array}$$

The group $\overline{\Gamma}_g^s$ is naturally isomorphic to $\widehat{\Gamma}_g^s$, while an isomorphism π is defined by the identification, for each curve, of the corresponding lifts of Dehn twists. The class $c_{\overline{\Gamma}_g^s}$ is sent to the class $c_{\widehat{\Gamma}_g^s}$ by L . As a result, we have $c_{\overline{\Gamma}_g^s} = c_{f^* \widehat{\Gamma}_g} + \sum_{i=1}^s c_{\Gamma_g^s(0, \dots, 1, \dots, 0)}$ which completes the proof. \square

When z is a root of unity, the group A is isomorphic to $\mathbb{Z}/N\mathbb{Z}$ where N is the order of z^{-12} . To prove the result in this case, it is sufficient to replace \mathbb{Z} by $\mathbb{Z}/N\mathbb{Z}$ everywhere. All arguments above go through without essential modifications.

Bibliography

- [1] *Travaux de Thurston sur les surfaces*, volume 66 of *Astérisque*. Société Mathématique de France, Paris, 1979. Séminaire Orsay, With an English summary.
- [2] Lars V. Ahlfors. The complex analytic structure of the space of closed Riemann surfaces. In *Analytic functions*, pages 45–66. Princeton Univ. Press, Princeton, N.J., 1960.
- [3] Lars V. Ahlfors. Some remarks on Teichmüller’s space of Riemann surfaces. *Ann. of Math. (2)*, 74:171–191, 1961.
- [4] R. V. Ambartzumian. *Combinatorial integral geometry*. Wiley Series in Probability and Mathematical Statistics: Tracts on Probability and Statistics. John Wiley & Sons, Inc., New York, 1982. With applications to mathematical stereology, Edited and with an appendix by Adrian Baddeley.
- [5] R. V. Ambartzumian. *Factorization calculus and geometric probability*, volume 33 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1990.
- [6] Ara Basmajian. The orthogonal spectrum of a hyperbolic manifold. *Amer. J. Math.*, 115(5):1139–1159, 1993.
- [7] Alan F. Beardon. *The geometry of discrete groups*, volume 91 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Corrected reprint of the 1983 original.
- [8] Lipman Bers. Spaces of Riemann surfaces. In *Proc. Internat. Congress Math. 1958*, pages 349–361. Cambridge Univ. Press, New York, 1960.
- [9] Francis Bonahon. The geometry of Teichmüller space via geodesic currents. *Invent. Math.*, 92(1):139–162, 1988.
- [10] Francis Bonahon. Shearing hyperbolic surfaces, bending pleated surfaces and Thurston’s symplectic form. *Ann. Fac. Sci. Toulouse Math. (6)*, 5(2):233–297, 1996.

- [11] Francis Bonahon and Xiaobo Liu. Representations of the quantum Teichmüller space and invariants of surface diffeomorphisms. *Geom. Topol.*, 11:889–937, 2007.
- [12] Francis Bonahon and Helen Wong. Quantum traces for representations of surface groups in $SL_2(\mathbb{C})$. *Geom. Topol.*, 15(3):1569–1615, 2011.
- [13] Rufus Bowen. Periodic orbits for hyperbolic flows. *Amer. J. Math.*, 94:1–30, 1972.
- [14] Rufus Bowen. Symbolic dynamics for hyperbolic flows. *Amer. J. Math.*, 95:429–460, 1973.
- [15] M. Bridgeman, R. Canary, F. Labourie, and A. Sambarino. The pressure metric for convex representations. *ArXiv e-prints*, January 2013.
- [16] Martin Bridgeman. Hausdorff dimension and the Weil-Petersson extension to quasifuchsian space. *Geom. Topol.*, 14(2):799–831, 2010.
- [17] Martin Bridgeman. Orthospectra of geodesic laminations and dilogarithm identities on moduli space. *Geom. Topol.*, 15(2):707–733, 2011.
- [18] Martin Bridgeman and David Dumas. Distribution of intersection lengths of a random geodesic with a geodesic lamination. *Ergodic Theory Dynam. Systems*, 27(4):1055–1072, 2007.
- [19] A. J. Cabo. An elementary proof of the Ambartzumian-Pleijel identity. *Math. Proc. Cambridge Philos. Soc.*, 112(3):535–538, 1992.
- [20] L. Chekhov and V. V. Fock. Quantum Teichmüller spaces. *Teoret. Mat. Fiz.*, 120(3):511–528, 1999.
- [21] Suhyoung Choi and William M. Goldman. Convex real projective structures on closed surfaces are closed. *Proc. Amer. Math. Soc.*, 118(2):657–661, 1993.
- [22] Michel Coornaert and Athanase Papadopoulos. Symbolic coding for the geodesic flow associated to a word hyperbolic group. *Manuscripta Math.*, 109(4):465–492, 2002.
- [23] L. D. Faddeev. Discrete Heisenberg-Weyl group and modular group. *Lett. Math. Phys.*, 34(3):249–254, 1995.
- [24] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.
- [25] V. V. Fock and A. B. Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.*, 103(1):1–211, 2006.

- [26] V. V. Fock and A. B. Goncharov. The quantum dilogarithm and representations of quantum cluster varieties. *Invent. Math.*, 175(2):223–286, 2009.
- [27] Louis Funar and Rinat M. Kashaev. Centrally extended mapping class groups from quantum Teichmüller theory. *Adv. Math.*, 252:260–291, 2014.
- [28] V. M. Gasparyan. Pleijel identity and distribution of chord length for planar convex domains. In *Stochastic geometry, geometric statistics, stereology (Oberwolfach, 1983)*, volume 65 of *Teubner-Texte Math.*, pages 91–94. Teubner, Leipzig, 1984.
- [29] Sylvain Gervais. Presentation and central extensions of mapping class groups. *Trans. Amer. Math. Soc.*, 348(8):3097–3132, 1996.
- [30] William M. Goldman. Topological components of spaces of representations. *Invent. Math.*, 93(3):557–607, 1988.
- [31] William M. Goldman. Convex real projective structures on compact surfaces. *J. Differential Geom.*, 31(3):791–845, 1990.
- [32] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.
- [33] Olivier Guichard. Composantes de Hitchin et représentations hyperconvexes de groupes de surface. *J. Differential Geom.*, 80(3):391–431, 2008.
- [34] John L. Harer. The second homology group of the mapping class group of an orientable surface. *Invent. Math.*, 72(2):221–239, 1983.
- [35] John L. Harer. The virtual cohomological dimension of the mapping class group of an orientable surface. *Invent. Math.*, 84(1):157–176, 1986.
- [36] N. J. Hitchin. Lie groups and Teichmüller space. *Topology*, 31(3):449–473, 1992.
- [37] Y. Imayoshi and M. Taniguchi. *An introduction to Teichmüller spaces*. Springer-Verlag, Tokyo, 1992. Translated and revised from the Japanese by the authors.
- [38] R. M. Kashaev. Quantization of Teichmüller spaces and the quantum dilogarithm. *Lett. Math. Phys.*, 43(2):105–115, 1998.
- [39] Tosio Kato. *Perturbation theory for linear operators*. Die Grundlehren der mathematischen Wissenschaften, Band 132. Springer-Verlag New York, Inc., New York, 1966.
- [40] Steven P. Kerckhoff. The Nielsen realization problem. *Bull. Amer. Math. Soc. (N.S.)*, 2(3):452–454, 1980.
- [41] Mustafa Korkmaz and András I. Stipsicz. The second homology groups of mapping class groups of oriented surfaces. *Math. Proc. Cambridge Philos. Soc.*, 134(3):479–489, 2003.

- [42] François Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.*, 165(1):51–114, 2006.
- [43] Howard Masur. Extension of the Weil-Petersson metric to the boundary of Teichmüller space. *Duke Math. J.*, 43(3):623–635, 1976.
- [44] Curtis T. McMullen. Thermodynamics, dimension and the Weil-Petersson metric. *Invent. Math.*, 173(2):365–425, 2008.
- [45] William Parry and Mark Pollicott. Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Astérisque*, (187-188):268, 1990.
- [46] R. C. Penner. The decorated Teichmüller space of punctured surfaces. *Comm. Math. Phys.*, 113(2):299–339, 1987.
- [47] R. C. Penner. Universal constructions in Teichmüller theory. *Adv. Math.*, 98(2):143–215, 1993.
- [48] R. C. Penner. Decorated Teichmüller theory of bordered surfaces. *Comm. Anal. Geom.*, 12(4):793–820, 2004.
- [49] Arne Pleijel. Zwei kurze Beweise der isoperimetrischen Ungleichung. *Arch. Math. (Basel)*, 7:317–319, 1956.
- [50] Mark Pollicott. Symbolic dynamics for Smale flows. *Amer. J. Math.*, 109(1):183–200, 1987.
- [51] Mark Pollicott and Richard Sharp. A weil-petersson type metric on spaces of metric graphs. *Geometriae Dedicata*, pages 1–16, 2013.
- [52] M. Ratner. Markov partitions for Anosov flows on n -dimensional manifolds. *Israel J. Math.*, 15:92–114, 1973.
- [53] David Ruelle. *Thermodynamic formalism*, volume 5 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1978. The mathematical structures of classical equilibrium statistical mechanics, With a foreword by Giovanni Gallavotti and Gian-Carlo Rota.
- [54] Andrés Sambarino. Quantitative properties of convex representations. *Comment. Math. Helv.*, 89(2):443–488, 2014.
- [55] L. A. Santaló. Integral geometry on surfaces of constant negative curvature. *Duke Math. J.*, 10:687–709, 1943.
- [56] W.P. Thurston. Minimal stretch maps between hyperbolic surfaces. *ArXiv Mathematics e-prints*, January 1998.
- [57] Akira Ushijima. A canonical cellular decomposition of the Teichmüller space of compact surfaces with boundary. *Comm. Math. Phys.*, 201(2):305–326, 1999.

- [58] André Weil. Modules des surfaces de Riemann. In *Séminaire Bourbaki; 10e année: 1957/1958. Textes des conférences; Exposés 152 à 168; 2e éd. corrigée, Exposé 168*, page 7. Secrétariat mathématique, Paris, 1958.
- [59] Scott Wolpert. Noncompleteness of the Weil-Petersson metric for Teichmüller space. *Pacific J. Math.*, 61(2):573–577, 1975.
- [60] Scott A. Wolpert. Chern forms and the Riemann tensor for the moduli space of curves. *Invent. Math.*, 85(1):119–145, 1986.
- [61] Scott A. Wolpert. Thurston’s Riemannian metric for Teichmüller space. *J. Differential Geom.*, 23(2):143–174, 1986.
- [62] Scott A. Wolpert. Geodesic length functions and the Nielsen problem. *J. Differential Geom.*, 25(2):275–296, 1987.