



HAL
open science

Extremal systolic geometry on surfaces

Zeina Yassine

► **To cite this version:**

Zeina Yassine. Extremal systolic geometry on surfaces. General Mathematics [math.GM]. Université Paris-Est, 2016. English. NNT : 2016PESC1074 . tel-01552976

HAL Id: tel-01552976

<https://theses.hal.science/tel-01552976v1>

Submitted on 3 Jul 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

École Doctorale MSTIC

Laboratoire d'Analyse et de Mathématiques Appliquées

Thèse

Présentée pour l'obtention du grade de DOCTEUR

DE L'UNIVERSITÉ PARIS-EST

par

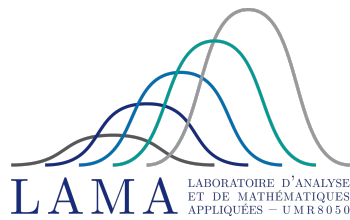
Zeina YASSINE

Géométrie systolique extrémale sur les surfaces

Spécialité : Mathématiques

Soutenue le 16 juin 2016 devant un jury composé de :

Rapporteur	M. BALACHEFF Florent	(Université Lille 1)
Rapporteur	M. BAVARD Christophe	(Université de Bordeaux)
Président	M. EL SOUFI Ahmad	(Université de Tours)
Examineur	M. KLOECKNER Benoît	(Université Paris-Est Créteil)
Directeur de thèse	M. SABOURAU Stéphane	(Université Paris-Est Créteil)
Examineur	M. VERNICOS Constantin	(Université de Montpellier)



Thèse effectuée au sein du **Laboratoire LAMA d'Analyse et de Mathématiques Appliquées**

de l'Université Paris-Est

Batiment P3

61 avenue du général de Gaulle

94010 Créteil cedex

France

Résumé

En 1949, C. Loewner a démontré dans un travail non publié que si \mathbb{T}^2 est un tore Riemannien, alors

$$\text{aire}(\mathbb{T}^2) \geq \frac{\sqrt{3}}{2} \text{sys}(\mathbb{T}^2)^2$$

où $\text{sys}(\mathbb{T}^2)$ est la systole du tore \mathbb{T}^2 , i.e. la longueur du plus court lacet non contractile de \mathbb{T}^2 . De plus, l'égalité est atteinte si et seulement si le tore est plat hexagonal. Ce résultat a donné naissance à la géométrie systolique. Dans cette thèse, nous étudions des inégalités de type systolique portant sur les longueurs minimales de différentes courbes et pas seulement la systole.

Dans un premier temps, nous démontrons trois inégalités géométriques optimales conformes sur la bouteille de Klein reliant l'aire au produit des longueurs des plus courts lacets noncontractiles dans des classes d'homotopie libres différentes. Pour chaque classe conforme, nous décrivons la métrique extrémale réalisant le cas d'égalité.

Nous établissons ensuite des inégalités géométriques optimales sur le ruban de Möbius muni d'une métrique de Finsler. Ces inégalités géométriques relient la systole et la hauteur du ruban de Möbius à son volume de Holmes-Thompson. Nous en déduisons une inégalité systolique optimale sur la bouteille de Klein munie d'une métrique de Finsler avec des symétries. Nous décrivons également une famille de métriques extrémales dans les deux cas.

Dans le troisième travail, nous démontrons une inégalité systolique critique sur la surface de genre deux. Plus précisément, il est connu que la surface de genre deux admet une métrique Riemannienne plate à singularités coniques qui est extrémale parmi les métriques à courbure nonpositive pour l'inégalité systolique. Nous montrons que cette métrique est en fait critique pour des variations lentes de métriques, cette fois-ci sans hypothèse de courbure, pour un autre problème systolique portant sur les longueurs des plus courts lacets non contractiles dans certaines classes d'homotopie libres données. Ces classes d'homotopie correspondent aux lacets systoliques et deux-systoliques de la surface extrémale.

Mots-clé:

Systole, aire, métrique extrémale, métrique de Finsler, bouteille de Klein, ruban de Möbius, surface de genre deux.

Extremal systolic geometry on surfaces

Abstract

In 1949, C. Loewner proved in an unpublished work that if \mathbb{T}^2 is a Riemannian two-torus, then it satisfies

$$\text{area}(\mathbb{T}^2) \geq \frac{\sqrt{3}}{2} \text{sys}(\mathbb{T}^2)^2$$

where $\text{sys}(\mathbb{T}^2)$ denotes the systole of the torus, i.e. the smallest length of a noncontractible loop in \mathbb{T}^2 . Furthermore, the equality is attained if and only if the torus is flat hexagonal. This result led to what was called later systolic geometry. In this thesis, we study several systolic-like inequalities. These inequalities involve the minimal length of various curves and not merely the systole.

First we obtain three optimal conformal geometric inequalities on Riemannian Klein bottles relating the area to the product of the lengths of the shortest noncontractible loops in different free homotopy classes. We describe the extremal metrics in each conformal class.

Then we prove optimal systolic inequalities on Finsler Möbius bands relating the systole and the height of the Möbius band to its Holmes-Thompson volume. We also establish an optimal systolic inequality for Finsler Klein bottles with symmetries. We describe extremal metric families in both cases.

Finally, we prove a critical systolic inequality on genus two surface. More precisely, it is known that the genus two surface admits a piecewise flat metric with conical singularities which is extremal for the systolic inequality among all nonpositively curved Riemannian metrics. We show that this piecewise flat metric is also critical for slow metric variations, this time without curvature restrictions, for another type of systolic inequality involving the lengths of the shortest noncontractible loops in different free homotopy classes. The free homotopy classes considered correspond to those of the systolic loops and the second-systolic loops of the extremal surface.

Keywords:

Systole, area, extremal metric, Finsler metric, Klein bottle, Möbius band, genus two surface.

Remerciements

D'abord, c'est avec une profonde gratitude que je remercie mon directeur de thèse Stéphane Sabourau. Son immense culture mathématique, sa disponibilité, sa patience et ses conseils ont été essentiels pour l'accomplissement de cette thèse. J'apprécie surtout sa bienveillance et sa gentillesse tout au long de ce travail.

Je remercie chaleureusement Christophe Bavard et Florent Balacheff pour avoir accepté de lire et de rapporter ma thèse et pour le temps qu'ils ont consacré. Je remercie également Ahmad El Soufi, Benoît Kloeckner et Constantin Vernicos de m'avoir fait l'honneur de participer au jury.

Cette thèse doit aussi à Chady ElMir. Sans lui je n'aurais pas pu venir en France. Il a su guider mes tous premiers pas en recherche en dirigeant mon mémoire de master. Sa confiance qu'il m'accordait toujours et son encouragement m'ont apporté beaucoup. C'est un plaisir de lui adresser ici mes remerciements les plus sincères.

Durant ce travail, j'ai bénéficié plusieurs fois du projet ANR Finsler. Les écoles d'été, les master class et les conférences dont j'ai participé étaient enrichissantes. La diversité des sujets m'a enrichi la connaissance et m'a ajouté des idées de recherche pour l'avenir. Je tiens ici à remercier les responsables du projet Athanase Papadopoulos et Ivan Babenko.

Tout au long de ce travail, j'ai bénéficié de conditions de travail très avantageuses au laboratoire LAMA, Créteil. J'apprécie les efforts de l'ancien directeur Raphael Danchin et de l'actuel Stéphane Sabourau. Je remercie également la secrétaire Anais Delgado.

La vie à l'étranger aurait été difficile sans l'entourage amical de mes collègues les doctorants et ATER dont je liste: Alaa, Cosmin, David, Hien, Houssam, Jean-Maxime, Jérémy, Juan, Khaled, Laurent, Marwa, Peng, Rana, Remy, Salwa, Victor, Xin, Xiaochuan... Tout particulièrement, je n'oublierai jamais les discussions culturelles avec Victor et Juan, la gentillesse de ma collègue de bureau Hien et l'aide amicale de Khaled et Marwa et les autres!

Je termine par remercier des gens qui m'ont soutenue à leur propre manière. Ce sont les personnes les plus proches de mon cœur, ... ma famille.

Contents

Introduction	1
1 Conformal geometric inequalities on the Klein bottle	11
1.1 Introduction, preliminaries and results	13
1.2 The main tools for the proofs	19
1.3 Proof of the geometric inequality of type $\ell_\sigma \ell_v$	25
1.4 Proof of the geometric inequality of type $L_\sigma \ell_v$	27
1.5 Proof of the geometric Inequality of type $\ell_\sigma \ell_v \ell_h$	29
1.6 Some remarks on the extremal metrics	31
2 Optimal systolic inequalities on Finsler Mobius bands	33
2.1 Introduction	35
2.2 Preliminaries	38
2.3 A systolic inequality on Finsler two-tori	40
2.4 Natural candidates for extremal metrics	42
2.5 Systolic inequalities on wide Finsler Mobius bands	47
2.6 Systolic inequalities on narrow Finsler Mobius bands	50
2.7 Systolic inequality on Finsler Klein bottles	54
2.8 A non-optimal systolic inequality on Finsler Klein bottles	57
3 A systolically critical genus two surface	59
3.1 Introduction	61
3.2 An extremal piecewise flat metric in genus two	64
3.3 Busemann functions and calibrating forms	66
3.4 Proof of Theorem 3.1.1	73

Bibliography

80

Introduction

We are interested in studying optimal geometric inequalities relating the area of a closed Riemannian surface Σ to the lengths of the shortest closed geodesics in certain homotopy classes. A typical example is given by the systolic inequalities, which relate the area of the surface to the systole, that is, the smallest length of a noncontractible loop in Σ . The first known systolic inequality is due to C. Loewner in 1949. He proved that every Riemannian two-torus \mathbb{T}^2 satisfies

$$\text{area}(\mathbb{T}^2) \geq \frac{\sqrt{3}}{2} \text{sys}(\mathbb{T}^2)^2,$$

where $\text{sys}(\mathbb{T}^2)$ denotes the systole of \mathbb{T}^2 . Furthermore, the equality holds if and only if the torus is flat hexagonal. C. Loewner did not publish his result, however it was mentioned by his student P. Pu, *cf.* [Pu52], who established a sharp systolic inequality on the projective plane \mathbb{RP}^2 . In this case, the equality is attained precisely by the round metrics. More than thirty years later, C. Bavard, *cf.* [Ba86] (see also [Sak88]) proved a sharp systolic inequality on the Klein bottle. Here, the extremal metrics are not smooth. These are the only manifolds for which an optimal systolic inequality is known. Optimal systolic-like inequalities, *i.e.*, inequalities that relate the area to the product of the lengths of the shortest loops or arcs in different (relative) homotopy classes are known only for the two-torus \mathbb{T}^2 , *cf.* [Ke67], the Klein bottle \mathbb{K} , *cf.* [Ba06, EY15], and the Mobius band \mathbb{M} , *cf.* [Pu52, Bl61, Ba06]. With the exception of some recent proofs of systolic inequalities, *cf.* [Iv02, Sa10], and systolic-like inequalities, *cf.* [SY16], on the projective plane \mathbb{RP}^2 , the two-torus \mathbb{T}^2 and the Mobius band \mathbb{M} , all the other known proofs of the aforementioned optimal geometric inequalities require the uniformization theorem as a main tool.

This thesis is divided into three chapters. In these chapters, we study various optimal systolic-like inequalities for different surfaces: the Klein bottle, the Mobius band, the torus and the genus two surface. Each chapter relies on specific techniques: extremal length methods in the first chapter, Finsler geometry in the second chapter and infinitesimal calibrating arguments in infinite-dimensional spaces in the third chapter.

In the first chapter, we present a joint work with Chady ElMir published in the Journal of Conformal Geometry and Dynamics, *cf.* [EY15].

In 1986, C. Bavard, *cf.* [Ba86], proved that every Riemannian Klein bottle \mathbb{K} satisfies

$$\text{area}(\mathbb{K}) \geq \frac{2\sqrt{2}}{\pi} \text{sys}(\mathbb{K})^2.$$

The extremal Klein bottle is the double of the $\frac{\pi}{4}$ -tubular neighborhood of the equator of the Euclidean sphere S^2 , with the identification described in Figure 1. In his proof, C. Bavard relies on P. Pu's inequality on the Mobius band \mathbb{M} , *cf.* [Pu52]. Later in 1988, T. Sakai [Sak88] proved the same result using C. Blatter's inequalities on \mathbb{M} , *cf.* [Bl61]. We refer to Section 2.4 for precise

definitions and details about P. Pu's and C. Blatter's inequalities on the Mobius band. A self-contained proof of the systolic volume of the Riemannian Klein bottle that do not require the use of inequalities on the Mobius band was given by C. Bavard in [Ba88]. The main tool in his proof is the method of extremal lengths, which is based also on the uniformization theorem. We refer to Section 1.2 for further details about it. This method is the main tool that we use in Chapter 1 to derive new optimal geometric inequalities on the Klein bottle relating the area to the length of loops in different free homotopy classes.

In order to describe these inequalities we introduce some notations. Recall that a flat Klein bottle \mathbb{K} is the quotient of \mathbb{C} by the group generated by a glide reflection σ and a translation t . The fundamental group $\pi_1(\mathbb{K})$ can be identified with the deck group $\langle \sigma, t \rangle$ generated by σ and t with their relation $\sigma t \sigma^{-1} t = e$. For every $a \in \pi_1(\mathbb{K})$, denote by γ_a a loop representing the homotopy class a . We distinguish three families of free noncontractible loops on the Klein bottle \mathbb{K} as follows.

1. $\mathcal{F}_\sigma = \{\gamma \mid \gamma \text{ is a loop freely homotopic to } \gamma_\sigma \text{ or } \gamma_{\sigma t}\},$
2. $\mathcal{F}_v = \{\gamma \mid \gamma \text{ is a loop freely homotopic to } \gamma_t\},$
3. $\mathcal{F}_h = \{\gamma \mid \gamma \text{ is a loop freely homotopic to } \gamma_{\sigma^2}\}.$

Furthermore, we define l_σ (resp. l_v , resp. l_h , resp. L_σ) to be the smallest length of a noncontractible free loop in \mathcal{F}_σ (resp. \mathcal{F}_v , resp. \mathcal{F}_h , resp. $\mathcal{F}_\sigma \cup \mathcal{F}_h$). Observe that the extremal metric g_e for the systole, cf. Figure 1, satisfies

$$l_\sigma(g_e) = l_h(g_e) = l_v(g_e) = L_\sigma(g_e).$$

We summarize the main results of the first chapter in the following theorem.

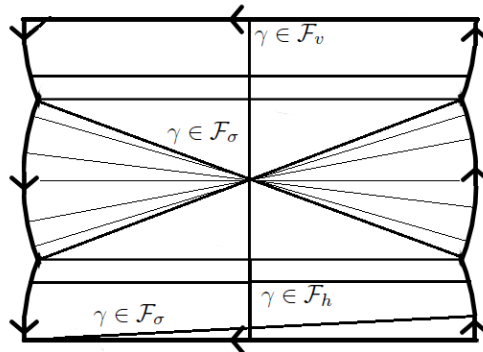


Figure 1: The double of the $\frac{\pi}{4}$ -tubular neighborhood of the equator of the Euclidean sphere S^2 is an extremal Klein bottle.

Theorem 0.0.1. *Let $\beta > 0$. For every Riemannian metric g on the Klein bottle \mathbb{K} of conformal type β , we have*

1. $\text{area}(\mathbb{K}, g) \geq C_\beta^1 \ell_\sigma(g) \ell_v(g)$,
2. $\text{area}(\mathbb{K}, g) \geq C_\beta^2 L_\sigma(g) \ell_v(g)$,
3. $\text{area}^{\frac{3}{2}}(\mathbb{K}, g) \geq C_\beta^3 \ell_\sigma(g) \ell_v(g) \ell_h(g)$,

where C_β^i , $i \in \{1, 2, 3\}$, are constants depending only on the conformal type β . Moreover, these inequalities are optimal.

In fact, the inequality 1. was proved by C. Bavard, cf. [Ba06], for the values of $\beta \in (0, 2 \ln(\frac{\pi}{4} + \frac{a_0}{2})]$, where $a_0 \in (0, \frac{\pi}{2})$ is the unique solution of the equation $\tan(a_0) = 2a_0$. We complete his work to cover all the remaining conformal classes. Observe also that the inequality 3. involves three different lengths. To our knowledge, there are no other known systolic-like inequalities, even non-optimal ones, for which the number of lengths is greater than the dimension of the manifold. We refer to Section 1.1.2 and Table 1.1 of Chapter 1 for details about the corresponding extremal metrics in each conformal class.

As a consequence of Theorem 0.0.1, we conclude that

Corollary 0.0.2. *There does not exist any positive universal constant c such that the inequality*

$$\text{area}(\mathbb{K}, g) \geq c \ell_\sigma(g) \ell_v(g),$$

holds for every Riemannian metric g on the Klein bottle. Similarly, there does not exist any positive universal constant c such that the inequality

$$\text{area}^{\frac{3}{2}}(\mathbb{K}, g) \geq c \ell_\sigma(g) \ell_v(g) \ell_h(g),$$

holds for every Riemannian metric g on the Klein bottle.

On the other hand, we have

Corollary 0.0.3. *For every Riemannian metric g on the Klein bottle \mathbb{K} ,*

$$\text{area}(\mathbb{K}, g) > \frac{1}{2} L_\sigma(g) \ell_v(g).$$

Moreover, the inequality is optimal.

In the second chapter, we present a joint work with Stéphane Sabourau, published in the Journal of Topology and Analysis, cf. [SY16].

This chapter is dedicated to the study of systolic inequalities for Finsler Mobius bands. Loosely speaking, a Finsler metric F is defined as a Riemannian metric except that its restriction to a tangent plane is no longer a Euclidean norm but a Minkowski norm. From a dynamical point of view, the function F^2 can be considered as a Lagrangian which induces a Lagrangian flow on the tangent bundle TM of M . Thus, Finsler manifolds can be considered as degree 2 homogeneous Lagrangian systems. The trajectories of the Lagrangian correspond to the geodesics of the Finsler metric. There exist several definitions of volume for Finsler manifolds which coincide in the Riemannian case. In this chapter, we consider the Holmes-Thompson volume vol_{HT} . Recall that the Holmes-Thompson volume of an n -dimensional Finsler manifold is equal to the symplectic volume of its unit co-disc bundle divided by the volume of the Euclidean unit ball of dimension n . We refer to Section 2.2 for further details about Finsler metrics.

Since there is no Finsler analog to the uniformization theorem, studying the systolic volume of Finsler surfaces is quite delicate. The first optimal Finsler systolic inequality has been obtained by S. Ivanov [Iv02]. He proved that the optimal systolic inequality on the projective plane for Riemannian metrics also holds for Finsler metrics. Namely, if \mathbb{RP}^2 is a Finsler projective plane, then

$$\text{area}(\mathbb{RP}^2) \geq \frac{2}{\pi} \text{sys}(\mathbb{RP}^2)^2.$$

The extremal Finsler metrics are those all of whose geodesics are closed and of the same length. Contrary to the Riemannian case, the Finsler systolic constant of the two-torus is the same as that of the projective plane. More precisely, S. Sabourau [Sa10] proved that if \mathbb{T}^2 is a Finsler two-torus, then

$$\text{area}(\mathbb{T}^2) \geq \frac{2}{\pi} \text{sys}(\mathbb{T}^2)^2.$$

Finsler two-tori that are homothetic to the quotient of \mathbb{R}^2 , endowed with a parallelogram norm $\|\cdot\|$, by a lattice whose unit disk of $\|\cdot\|$ is a fundamental domain are extremal. However, we do not know if those are the only extremal metrics. A sharp systolic inequality for asymmetric Finsler two-tori can be found in [ABT].

The main result of the second chapter is the following.

Theorem 0.0.4. *Let \mathbb{M} be a Finsler Mobius band. Let $\lambda := \frac{h(\mathbb{M})}{\text{sys}(\mathbb{M})}$. Then*

$$\frac{\text{area}(\mathbb{M})}{\text{sys}(\mathbb{M}) h(\mathbb{M})} \geq \begin{cases} \frac{2}{\pi} & \text{if } \lambda \in (0, 1] \\ \frac{1}{\pi} \frac{\lambda+1}{\lambda} & \text{otherwise.} \end{cases}$$

Moreover, the above inequalities are optimal for every value of $\lambda \in (0, +\infty)$.

Here, $h(\mathbb{M})$ represents the minimal length of arcs with endpoints on the boundary $\partial\mathbb{M}$ which are not homotopically trivial relative to $\partial\mathbb{M}$. We refer to Examples 2.5.4 and 2.6.2 for a description of an extremal metric family. As in the case of Finsler two-tori, we do not know if these are the only extremal metrics.

The proof of Theorem 0.0.4 is divided into two cases. The first case is when

$$h(\mathbb{M}) \geq \text{sys}(\mathbb{M}).$$

Its proof proceeds as follows. We first derive a Finsler analog of L. Keen's inequality for two-tori, cf. [Ke67]. More precisely,

Proposition 0.0.5. *Let \mathbb{T}^2 be a Finsler two-torus. There exist two closed curves of lengths a and b generating the first integral homology group of \mathbb{T}^2 such that*

$$\text{area}(\mathbb{T}^2) \geq \frac{2}{\pi}ab.$$

Equality holds if \mathbb{T}^2 is homothetic to the quotient of \mathbb{R}^2 , endowed with a parallelogram norm $\|\cdot\|$, by a lattice generated by two vectors of lengths a and b , parallel to the sides of the unit ball of $\|\cdot\|$.

The proof of Proposition 0.0.5 can be used as well in the Riemannian case and yields L. Keen's result without making use of the uniformization theorem as in the original proof. With the help of Proposition 0.0.5, we derive a similar result for Finsler cylinders C . More precisely, we obtain

$$\text{area}(C) \geq \frac{2}{\pi}\text{sys}(C)h(C)$$

where h denotes the height of the cylinder, that is, the distance between its two boundary components. Using this result and the Finsler systolic inequality on the projective plane, we obtain optimal systolic inequalities on wide Finsler Mobius bands, cf. Section 2.5.

Furthermore, in order to prove the systolic inequality for Finsler Mobius bands \mathbb{M} that satisfy

$$h(\mathbb{M}) < \text{sys}(\mathbb{M}),$$

we derive a systolic-height inequality which holds true whenever $\frac{h(\mathbb{M})}{\text{sys}(\mathbb{M})}$ is a dyadic rational in the interval $(0, 1)$ and deduce the result by continuity of the volume, the height and the systole over Finsler metrics. Hence the systolic inequality holds for narrow Mobius bands too, cf. Section 2.6.

The optimal systolic inequality for Finsler Klein bottles is still unknown. However, using Theorem 0.0.4, we deduce that Finsler Klein bottles with some natural symmetries satisfy the same systolic volume as Finsler two-tori \mathbb{T}^2 and Finsler projective planes $\mathbb{R}\mathbb{P}^2$. In particular, we prove

Theorem 0.0.6. *Let \mathbb{K} be a Finsler Klein bottle with a soul, soul-switching or rotational symmetry. Then*

$$\text{area}(\mathbb{K}) \geq \frac{2}{\pi}\text{sys}(\mathbb{K})^2. \tag{0.0.1}$$

Moreover, the inequality is optimal.

Recall that in the Riemannian case, Theorem 0.0.6 follows directly from the uniformization theorem, an average argument over the isometry group, and the Cauchy-Schwartz inequality. In

our proof, we do not use any of these arguments. Instead, we rely on the sharp systolic-like inequality we established for Finsler Mobius bands.

We end this chapter by proposing the following conjecture

Conjecture. *Let \mathbb{K} be a Finsler Klein bottle. Then*

$$\text{area}(\mathbb{K}) \geq \frac{2}{\pi} \text{sys}(\mathbb{K})^2. \quad (0.0.2)$$

A Finsler Klein bottle with systolic volume equal to $\frac{2}{\pi}$ is described in Example 2.7.7.

The third chapter deals with critical isosystolic genus two surfaces.

Recently, M. Katz and S. Sabourau, *cf.* [KS06], proved that there exists a piecewise flat metric g_0 on the genus two surface Σ_2 which is extremal among all nonpositively curved Riemannian metrics. This metric is composed of six regular flat octagons. It admits regions where only one systolic loop, *i.e.*, a noncontractible loop of length the systole, passes through every point. Hence this metric cannot be extremal for the general systolic inequality, *i.e.*, without restriction on the curvature. This follows from a result of E. Calabi, *cf.* [Ca96], which says that at least two systolic loops pass through every point of a systolically extremal surface. In fact, the non-extremality of the metric g_0 for the systolic inequality can also be deduced from a result of S. Sabourau, *cf.* [Sa04], which says that no flat metric with conical singularities is extremal for the systolic inequality in genus two.

Denote by $\ell_1(g)$ the length of the shortest noncontractible loop on (Σ_2, g) , that is,

$$\ell_1(g) = \text{sys}(g).$$

Recall that g_0 designates the extremal nonpositively curved genus two surface defined in [KS06]. Let Λ_1 be the subset of the free homotopy classes generated by the systolic loops of (Σ_2, g_0) and their multiples:

$$\Lambda_1 = \{\langle \gamma^k \rangle \mid \langle \gamma \rangle \neq 0, \ell_{g_0}(\gamma) = \ell_1(g_0) \text{ and } k \in \mathbb{Z}^*\}. \quad (0.0.3)$$

Similarly, let $\ell_2(g)$ be the length of the shortest noncontractible loop on (Σ_2, g) which is not homotopic to a systolic loop. That is,

$$\ell_2(g) = \inf_{\langle \gamma \rangle \notin \Lambda_1 \cup \{0\}} \ell_g(\gamma).$$

Denote also by

$$\Lambda_2 = \{\langle \gamma \rangle \mid \langle \gamma \rangle \notin \Lambda_1 \text{ and } \ell_{g_0}(\gamma) = \ell_2(g_0)\} \quad (0.0.4)$$

the subset of the free homotopy classes generated by the shortest noncontractible loops of (Σ_2, g_0) which are not freely homotopic to systolic loops.

We observe that the piecewise flat metric g_0 defined on the genus two surface Σ_2 may be a potential extremal metric for another systolic problem on Σ_2 . This observation follows from some geometric properties of the surface (Σ_2, g_0) . First, it is known that the systolic loops cover a systolically extremal surface. In our case, the ℓ_1 -loops and the ℓ_2 -loops cover the surface Σ_2 . Here, by an ℓ_i -loop we mean a loop of length $\ell_i(g_0)$ in (Σ_2, g_0) whose free homotopy class lies in Λ_i . Second, the unit tangent vectors of these ℓ_i -loops are well distributed on each tangent plane of the surface. More precisely, their convex hull forms a regular octagon on these tangent planes. In general, the convex hull of the unit tangent vectors of systolic loops of all the known systolically extremal surfaces is symmetric. Finally, an extremal surface of genus at least two tends to have flat regions, cf. [Ca96, Br96].

In [Ca96], E. Calabi described two piecewise flat genus three surfaces and conjectured that one of them is the global minimum for the systolic inequality. Later, S. Sabourau, cf. [Sa11], proved that these two genus three surfaces are critical for the systolic inequality with respect to slow metric variations. Of course, since the systolic volume functional is not necessarily differentiable, an adequate notion of criticality needs to be defined. The notion of criticality used in [Sa11] is the same as the one introduced in [Na96, EI00] to study the Riemannian surfaces that maximize the product of the area with the first nonzero eigenvalue of the Laplacian. Other notions of systolically critical metrics were used in [Bal06, Bal10, Sa10, AB14].

In this chapter we show that the metric g_0 on the genus two surface Σ_2 is critical in the sense of [Na96, EI00, Sa11] for slow metric variations. More precisely, we prove

Theorem 0.0.7. *Let g_t be a slow metric variation of g_0 defined on the genus two surface Σ_2 . If $\ell_1(g_t) \geq \ell_1(g_0)$ and $\ell_2(g_t) \geq \ell_2(g_0)$, then*

$$\text{area}(g_t) \geq \text{area}(g_0) + o(t).$$

The slow metric variations involved in Theorem 3.1.1 are analogous to those defined by S. Sabourau in [Sa11]. We refer to Definition 3.4.1 for a precise definition of these metric variations and to the last section of [Sa11] for examples. Observe for instance that deforming the regular octagons composing the extremal surface (Σ_2, g_0) into non-regular octagons gives rise to a slow metric variation.

To prove our result, we rely on recent calibrating methods, cf. [Iv02, Iv11, Sa11], that do not require the uniformization theorem classically used to establish sharp systolic inequalities on surfaces. Briefly, we embed the universal cover $\tilde{\Sigma}_2$ of the genus two surface Σ_2 in an infinite-dimensional space \mathbb{R}^∞ as follows. Given that $\ell_i(g) \geq \ell_i(g_0)$, we construct a 1-Lipschitz equivariant map

$$\Psi^g : \tilde{\Sigma}_2 \rightarrow \mathbb{R}^\infty$$

using the Busemann functions induced by the lifts of the ℓ_i -loops of Σ_2 , cf. Section 3.3. Then, we

introduce an appropriate infinitesimally calibrating two-form ω on \mathbb{R}^∞ whose pull-back passes to the quotient on the surface Σ_2 . This allows us to show that

$$\int_{\Sigma_2} (\Psi^g)^* \omega \leq c_0 \text{area}(\Sigma_2, g)$$

for every Riemannian metric g on Σ_2 , where c_0 is a sharp positive constant. Moreover, the equality holds if $g = g_0$. Finally, we prove that if $(g_t)_{t \geq 0}$ is a slow metric variation then

$$\int_{\Sigma_2} (\Psi^{g_t})^* \omega - \int_{\Sigma_2} (\Psi^{g_0})^* \omega = o(t).$$

This completes the proof, *cf.* Section 3.4.

Chapter 1

Conformal geometric inequalities on the Klein bottle

Nous démontrons trois inégalités géométriques optimales conformes sur la bouteille de Klein reliant l'aire au produit des longueurs des plus courts lacets noncontractiles dans des classes d'homotopie libres différentes. Pour chaque classe conforme, nous décrivons la métrique extrémale réalisant le cas d'égalité.

Abstract. We obtain three optimal conformal geometric inequalities on Riemannian Klein bottles relating the area to the product of the lengths of the shortest noncontractible loops in different free homotopy classes. We describe the extremal metrics in each conformal class.

1.1 Introduction, preliminaries and results

Among all Riemannian metrics on a given compact differentiable manifold, the most interesting ones are those that extremize some Riemannian invariant. An interesting problem, for example, is to study metrics which maximize the ratio $\frac{\text{sys}(M,g)^n}{\text{vol}(M,g)}$ over the set of all Riemannian metrics g on a given n -dimensional differentiable manifold M , where $\text{sys}(M, g)$ denotes the systole of (M, g) , i.e., the least length of a non-contractible closed curve. Concerning this problem, called isosystolic problem, it is known that the extremal metric for the 2-dimensional torus is the flat hexagonal metric (unique up to a homothety). It is due to C. Loewner in 1949 (unpublished). His student, P.M. Pu, showed in 1952 that the extremal metric on the projective plane is the spherical metric, cf. [Pu52]). Nevertheless, as in many works related to the isosystolic problem, various constraints can be put on the set of Riemannian metrics under consideration. For example, we may restrict ourselves to the set of nonpositive curvature metrics cf. [KS06], or to the set of metrics in a fixed conformal class cf. [Bl61, Ba88, Ba92, Ba06].

In 1961, C. Blatter proved, cf. [Bl61] optimal conformal lower bounds on the area of the Mobius band with boundary \mathbb{M} in terms of the product of the least lengths of two classes of curves. The first class consists of the family \mathcal{F} of arcs joining two points on the boundary with non-trivial intersection with the soul of the Mobius band. The second class consists of the family \mathcal{G} of loops in the homotopy class of a generator of $\pi_1(\mathbb{M})$. Let ℓ_v be the least length of an arc in \mathcal{F} and ℓ_σ be the least length of a loop in \mathcal{G} . Then, Blatter (in his notations, $\ell_v = \ell^*$ and $\ell_\sigma = \ell_1$) obtained the following optimal *conformal* lower bound on the area of \mathbb{M}

$$\ell_\sigma(g)\ell_v(g) \leq C_\beta \text{area}(g) \quad (1.1.1)$$

where C_β is a positive constant that depends only on the conformal type β of g . Note that the optimal constant C_β is not bounded from above over the set of conformal types β . Also, in the same article, Blatter proved the following conformal lower bound on the area of \mathbb{M}

$$\text{sys}(g)\ell_v(g) \leq C'_\beta \text{area}(g) \quad (1.1.2)$$

where C'_β is a positive constant that depends only on the conformal type β of g . Contrarily to the previous case, the optimal constant C'_β is bounded from above over the set of conformal types β . More precisely, $\sup_{\beta} C'_\beta = 2$. Hence, the optimal inequality

$$\text{sys}(g)\ell_v(g) \leq 2 \text{area}(g)$$

holds for every Riemannian metric g on \mathbb{M} .

Variants of C. Blatter's problem were studied by L. Keen in [Ke67] and J. Hebda in [He91] on the 2-dimensional torus. Keen obtained a lower bound on the area in terms of the product of the least lengths of two loops generating the fundamental group. On the other hand, Hebda obtained a lower bound on the square of the area in terms of the three primitive length spectrum. The equality in both cases is attained by the flat hexagonal metric.

In this chapter, we prove three types of inequalities on the Klein bottle \mathbb{K} in the same spirit as Blatter's inequalities. The classes of curves we consider are (free) homotopy classes of loops that are candidates to realize the systole. In particular, the role of the family \mathcal{F} will be taken by a free homotopy class of loops representing the vertical translation in $\pi_1(\mathbb{K})$. The main tool in our proof is the method of extremal length which was used by C. Bavard in 1988 to prove conformal *isosystolic* inequalities on the Klein bottle cf. [Ba88]. It can be applied to the Mobius band with boundary and provides an alternative proof of Blatter's inequalities (1.1.1) and (1.1.2).

1.1.1 The Klein bottle

A flat Klein bottle is the quotient of \mathbb{C} by the group generated by the maps $\sigma : z \mapsto \bar{z} + \pi$ and $t_v : z \mapsto z + 2i\beta$. The induced flat metric on the Klein bottle will be denoted by g_β . By the uniformization theorem, every Riemannian metric g on the Klein bottle is conformally equivalent to a unique flat metric g_β for some $\beta \in (0, +\infty)$. The parameter β represents the conformal type of the metric g . We call $t_h := \sigma^2$ a horizontal translation and identify the fundamental group $\pi_1(\mathbb{K})$ with the deck group $\langle \sigma, t_v \rangle$. Note that any Klein bottle \mathbb{K} is obtained by gluing two Mobius bands \mathbb{M}_1 and \mathbb{M}_2 along their boundaries. With the previous notations, fundamental domains of \mathbb{K} , \mathbb{M}_1 and \mathbb{M}_2 , are $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\beta, \beta]$, $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [\frac{-\beta}{2}, \frac{\beta}{2}]$ and $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\beta, \frac{-\beta}{2}] \cup [\frac{\beta}{2}, \beta]$ respectively.

By C. Bavard's theorem, cf. [Ba86], every Riemannian Klein bottle \mathbb{K} satisfies the isosystolic inequality

$$\text{sys}(\mathbb{K})^2 \leq \frac{\pi}{2\sqrt{2}} \text{area}(\mathbb{K})$$

where the equality is attained by a spherical metric outside a singular line (see [Ba86], [Sak88] and [EL08] p. 100 for a detailed description of the extremal Klein bottle). For details and (many) open problems in *systolic geometry* see the book of Katz [Ka07] and the paper of M. Gromov [Gr83].

In this chapter, we consider the following distinct families of homotopy classes of closed curves on the Klein bottle \mathbb{K} as follows.

Definition 1.1.1. For every $a \in \pi_1(\mathbb{K})$, denote by γ_a a loop representing the homotopy class a . Then we define

1. $\mathcal{F}_\sigma = \{\gamma \mid \gamma \text{ is a loop freely homotopic to } \gamma_\sigma \text{ or } \gamma_{\sigma t}\},$

2. $\mathcal{F}_v = \{\gamma \mid \gamma \text{ is a loop freely homotopic to } \gamma_{t_v}\},$

3. $\mathcal{F}_h = \{\gamma \mid \gamma \text{ is a loop freely homotopic to } \gamma_{\sigma^2}\}.$

Given a Riemannian metric on \mathbb{K} , the least length of a closed curve in the first (resp. second, resp. third) family will be denoted by ℓ_σ (resp. ℓ_v , resp. ℓ_h). Also, denote by L_σ the least length of a loop in $\mathcal{F}_\sigma \cup \mathcal{F}_h$.

Remark 1.1.2. An extremal metric g_{ext} for the isosystolic inequality on the Klein bottle \mathbb{K} satisfies

$$\ell_\sigma(g_{ext}) = L_\sigma(g_{ext}) = \ell_v(g_{ext}).$$

Its conformal type β is equal to $2 \ln(\tan(\frac{3\pi}{8}))$, cf. [Ba86].

Finally, denote by a_0 the unique real $x \in (0, \frac{\pi}{2})$ such that $\tan(x) = 2x$ ($a_0 \approx 1.1655$).

1.1.2 Two families of Riemannian metrics on the Klein bottle

We denote by $C^{0,\infty}(\mathbb{R})$ the set of continuous and piecewise smooth functions on \mathbb{R} . Let h^β be a Riemannian metric in $\mathbb{C} \simeq \mathbb{R}^2$, periodic with respect to the fundamental domain $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\beta, \beta]$ and satisfying

$$(h^\beta)_{(x,y)} = \varphi(y)(dx^2 + dy^2), \quad (1.1.3)$$

where $\varphi \in C^{0,\infty}(\mathbb{R})$ is a positive, even and 2β -periodic function. Then there exists a diffeomorphism $G : \mathbb{C} \rightarrow \mathbb{C}$ for which the Riemannian metric $h_b = (G^{-1})^* h^\beta$ satisfies

$$(h_b)_{(u,v)} = f^2(v)du^2 + dv^2, \quad (1.1.4)$$

where $f \in C^{0,\infty}(\mathbb{R})$ is positive, even and $2b$ -periodic function. To see this, let $\phi(y) = \int_0^y \sqrt{\varphi(t)} dt$ and $f = (\varphi \circ \phi^{-1})^{\frac{1}{2}}$ and define the map G by

$$G(x, y) = (x, \phi(y)). \quad (1.1.5)$$

The Riemannian metric h_b is then periodic with respect to the fundamental domain $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-2b, 2b]$. The quotient of (\mathbb{C}, h_b) by the subgroup of isometries $\langle \sigma, t_{v'} \rangle$, where $\sigma : z \mapsto \bar{z} + \pi$ and $t_{v'} : z \mapsto z + 4ib$, is a Riemannian Klein bottle. Its conformal type is

$$\beta = \int_0^{2b} \frac{dt}{f(t)}.$$

Now, we will introduce two families of metrics on \mathbb{C} . They induce families of metrics on the Klein bottle that will be used later in this chapter as conformally extremal metrics for three types of inequalities on the Klein bottle. For every $b \in (0, +\infty)$ and $\omega \in (0, b]$, they are as follows.

1. the metric S_b on \mathbb{C} periodic with respect to the fundamental domain $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-2b, 2b]$ and defined by

$$(S_b)_{(u,v)} = f_b^2(v)du^2 + dv^2$$

where f_b is the unique one-variable function invariant by the translation $(u, v) \mapsto (u, v + 2b)$ which agrees with *cosine* on $[-b, b]$. It is spherical outside the singular lines $v = nb$, where $n \in \mathbb{Z}$.

2. the metric $SF_{b,\omega}$ on \mathbb{C} periodic with respect to the fundamental domain $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-2b, 2b]$ and defined by

$$(SF_{b,\omega})_{(u,v)} = f_{b,\omega}^2(v)du^2 + dv^2$$

where $f_{b,\omega}$ is the unique one-variable function invariant by the translation $(u, v) \mapsto (u, v + 2b)$ which agrees with *cosine* on $[-\omega, \omega]$ and equal to the constant $\cos(\omega)$ on $[\omega, 2b - \omega]$, cf. [Ba88, Fig. 1]. It is spherical on the band $\mathbb{R} \times [-\omega, \omega]$ and its images by the translations $(u, v) \mapsto (u, v + 2nb)$, where $n \in \mathbb{Z}$, and flat elsewhere.

We will denote by (\mathbb{K}, S_b) the Riemannian Klein bottle obtained by taking the quotient of (\mathbb{C}, S_b) by the subgroup $\langle \sigma, t_{v'} \rangle$. Its conformal type is

$$\beta = 2 \ln(\tan(\frac{\pi}{4} + \frac{b}{2})).$$

Similarly, $(\mathbb{K}, SF_{b,\omega})$ will denote the Riemannian Klein bottle obtained by taking the quotient of $(\mathbb{C}, SF_{b,\omega})$ by the subgroup $\langle \sigma, t_{v'} \rangle$. Its conformal type is

$$\beta = 2 \ln(\tan(\frac{\pi}{4} + \frac{\omega}{2})) + \frac{1}{\cos(\omega)}(2b - 2\omega).$$

1.1.3 Geometric inequality of type $\ell_\sigma \ell_v$

Our first result studies optimal conformal inequalities of the form

$$\ell_\sigma \ell_v \leq C_\beta \text{ area}$$

where C_β is a constant depending only on the conformal type β . We call such a relation a “*geometric inequality of type $\ell_\sigma \ell_v$* ”. Actually, it extends the following result of C. Bavard to all the conformal classes of the Klein bottle.

Theorem 1.1.3 C. Bavard [Ba06]. *Let $a_0 \in (0, \frac{\pi}{2})$ be such that $\tan(a_0) = 2a_0$ and $0 < \beta \leq 2 \ln(\tan(\frac{\pi}{4} + \frac{a_0}{2}))$. Then, for every Riemannian metric g on the Klein bottle \mathbb{K} of conformal type β , we have the following optimal inequality*

$$\ell_\sigma(g) \ell_v(g) \leq \frac{\arcsin\left(\frac{e^\beta - 1}{e^\beta + 1}\right)}{\frac{e^\beta - 1}{e^\beta + 1}} \text{ area}(g).$$

The equality is attained if and only if g is proportional to the spherical metric S_b for b satisfying $\beta = 2 \ln(\tan(\frac{\pi}{4} + \frac{b}{2}))$.

We complete C. Bavard's study by providing an optimal inequality for the remaining conformal classes.

Theorem 1.1.4. *Let $a_0 \in (0, \frac{\pi}{2})$ be such that $\tan(a_0) = 2a_0$ and $\beta > 2 \ln(\tan(\frac{\pi}{4} + \frac{a_0}{2}))$. Let $\omega_1 \in [a_0, \frac{\pi}{2})$ be defined by the equation*

$$2 \sin(\omega_1) = \left(\beta - 2 \ln \left(\tan \left(\frac{\pi}{4} + \frac{\omega_1}{2} \right) \right) \right) \cos^2(\omega_1) + 4\omega_1 \cos(\omega_1).$$

Then, for every Riemannian metric g on the Klein bottle K of conformal type β , we have the following optimal inequality

$$\ell_\sigma(g) \ell_v(g) \leq \frac{1}{2 \cos(\omega_1)} \text{area}(g).$$

Moreover, the equality is attained if and only if g is proportional to the spherical-flat metric SF_{b, ω_1} for $b = \tan(\omega_1) - \omega_1$.

Remark 1.1.5. The bound $\beta \leq 2 \ln(\tan(\frac{\pi}{4} + \frac{a_0}{2}))$ found by C. Bavard in Theorem 1.1.3 is actually the critical value of the conformal type for the transition in the shape of the extremal metrics from *spherical* to *spherical-flat*.

Corollary 1.1.6. *There does not exist any (finite) positive universal constant c such that the inequality*

$$\ell_\sigma(g) \ell_v(g) \leq c \text{area}(g)$$

holds for every Riemannian metric g on the Klein bottle.

1.1.4 Geometric inequality of type $L_\sigma \ell_v$

The second part of this chapter is devoted to establishing optimal conformal inequalities of the form

$$L_\sigma \ell_v \leq C_\beta \text{area}.$$

Note that in the case of the Mobius band \mathbb{M} , L_σ is just the systole of \mathbb{M} . We call such a relation a “geometric inequality of type $L_\sigma \ell_v$ ”.

Theorem 1.1.7. *Let $\beta > 0$. For every Riemannian metric g on the Klein bottle \mathbb{K} of conformal type β , we have*

$$L_\sigma(g) \ell_v(g) \leq C_\beta \text{area}(g)$$

where

$$C_\beta = \begin{cases} \frac{e^\beta + 1}{e^\beta - 1} \arcsin \left(\frac{e^{\beta-1}}{e^\beta + 1} \right) & \text{if } 0 < \beta \leq 2 \ln(2 + \sqrt{3}) \\ \frac{2}{3} \cdot \frac{3\beta + 4\pi - 6 \ln(2 + \sqrt{3})}{4\sqrt{3} + \beta - 2 \ln(2 + \sqrt{3})} & \text{if } \beta > 2 \ln(2 + \sqrt{3}). \end{cases}$$

Moreover, the equality is attained if and only if g is proportional to the spherical metric S_b , for b satisfying $\beta = 2 \ln(\tan(\frac{\pi}{4} + \frac{b}{2}))$, in the first case and to the spherical-flat metric $SF_{b, \frac{\pi}{3}}$, for b satisfying $\beta = 2 \ln(2 + \sqrt{3}) + 4(b - \frac{\pi}{3})$, in the second case.

Since the supremum of the conformal constant C_β over β is equal to 2, we have

Corollary 1.1.8. *For every Riemannian metric g on the Klein bottle \mathbb{K} , we have*

$$L_\sigma(g)\ell_v(g) < 2 \text{ area}(g).$$

The inequality is optimal.

1.1.5 Geometric inequality of type $\ell_\sigma \ell_v \ell_h$

In the third part, we establish optimal conformal inequalities of the form

$$\ell_\sigma \ell_v \ell_h \leq C_\beta \text{ area}^{\frac{3}{2}}$$

which we call a “geometric inequality of type $\ell_\sigma \ell_v \ell_h$ ”. To our knowledge, this kind of inequality involving the product of three lengths has never been considered before. It is distinguished by the fact that the area to the power $\frac{3}{2}$ is bounded by the product of all the natural candidates for the systole

Theorem 1.1.9. *Let $\beta > 0$ and let $\omega_2 \in (0, \frac{\pi}{2})$ be defined by the equation*

$$\beta = 2 \ln\left(\tan\left(\frac{\pi}{4} + \frac{\omega_2}{2}\right)\right) + \frac{2}{\cos(\omega_2)} \left(\tan(\omega_2) - \omega_2 + \sqrt{\tan^2(\omega_2) - \omega_2 \tan(\omega_2) + \omega_2^2} \right).$$

Then, for every Riemannian metric g on the Klein bottle \mathbb{K} of conformal type β , we have the following optimal inequality

$$\ell_\sigma(g)\ell_v(g)\ell_h(g) \leq C_\beta \text{ area}(g)^{\frac{3}{2}}$$

where

$$C_\beta = \frac{\sqrt{\pi}}{3\sqrt{3}} \cdot \frac{(b^4 - 4b\omega_2 + \omega_2^2 + \omega_2^4 - 2b^2(-2 + \omega_2^2))^{\frac{1}{4}} (2b - \omega_2)}{(b - \omega_2)\sqrt{(b - \omega_2)b}}$$

with $b = \tan(\omega_2) + \sqrt{\tan^2(\omega_2) - \omega_2 \tan(\omega_2) + \omega_2^2}$. Moreover, the equality is attained if and only if g is proportional to the spherical-flat metric SF_{b, ω_2} .

Now, if we replace b by $\tan(\omega_2) + \sqrt{\tan^2(\omega_2) - \omega_2 \tan(\omega_2) + \omega_2^2}$ in

$$C(\omega_2) = \frac{(b^4 - 4b\omega_2 + \omega_2^2 + \omega_2^4 - 2b^2(-2 + \omega_2^2))^{\frac{1}{4}} (2b - \omega_2)}{(b - \omega_2)\sqrt{(b - \omega_2)b}}$$

we obtain a continuous increasing function $C :]\frac{2}{\pi}, +\infty[\rightarrow]\frac{2\sqrt{\pi}}{3\sqrt{3}}, +\infty[$ which tends to infinity as $\omega_2 \rightarrow 0$ (i.e. when $b \rightarrow 0$). Therefore, we derive

Corollary 1.1.10. *There does not exist any (finite) universal constant c such that the inequality*

$$\ell_\sigma(g)\ell_v(g)\ell_h(g) \leq c \operatorname{area}(g)^{\frac{3}{2}}$$

holds for every Riemannian metric g on the Klein bottle \mathbb{K} .

Remark 1.1.11. Unlike Corollaries 1.1.6 and 1.1.10, Corollary 1.1.8 provides a uniform upper bound on the Riemannian ratio $\frac{L_\sigma(g)\ell_v(g)}{\operatorname{area}(g)}$.

1.2 The main tools for the proofs

The key tool in our proofs is the method of extremal length initiated by B. Fuglede in [Fu57], J. A. Jenkins in [Je65] and M. Gromov in [Gr83]. It was used later by C. Bavard in the setting of isosystolic geometry, cf. [Ba88, Ba92]. This method characterizes a conformally extremal Riemannian manifold by means of its closed geodesics. See e.g. L. V. Ahlfors' book [Al73] and Rodin's paper [Ro68] and the references therein for more details and further applications of the method of extremal length.

1.2.1 A maximality criterion

Let (M, g) be a closed Riemannian manifold and Γ be a family of rectifiable curves on M . For every Radon measure μ on Γ , we associate a measure $^*\mu$ on M by setting, for $\varphi \in C^0(M, \mathbb{R})$,

$$\langle ^*\mu, \varphi \rangle = \langle \mu, \bar{\varphi} \rangle$$

where $\bar{\varphi}(\gamma) = \int \varphi \circ \gamma(s) ds$ and ds is the arc-length of γ with respect to g .

Theorem 1.2.1. ([Ba88],[Ba92] and [Je65]) *Let M be a closed manifold. Let \mathcal{S}_i , where $i \in \{1, \dots, p\}$, be some families of rectifiable curves on M such that $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$ for all $i \neq j$. Denote by $\ell_i(g)$ the least length of a curve in \mathcal{S}_i with respect to a Riemannian metric g on M . Suppose that g_e is a Riemannian metric on M for which there exists a positive measure μ on $\Gamma = \mathcal{S}_1 \cup \dots \cup \mathcal{S}_p$ satisfying the following three conditions:*

1. *for each $i \in \{1, \dots, p\}$, all the curves in \mathcal{S}_i have the same length with respect to g_e*
2. $m_1 \ell_1(g_e) = m_2 \ell_2(g_e) = \dots = m_p \ell_p(g_e)$
3. $^*\mu = dg_e$

where m_i is the mass of the measure μ on \mathcal{S}_i and dg_e is the volume measure of (M, g_e) . Then, for every Riemannian metric g on M conformal to g_e , we have

$$\frac{\ell_1(g) \cdots \ell_p(g)}{\text{vol}^{\frac{p}{2}}(g)} \leq \frac{\ell_1(g_e) \cdots \ell_p(g_e)}{\text{vol}^{\frac{p}{2}}(g_e)}. \quad (1.2.1)$$

Furthermore, the equality holds if and only if g is homothetic to g_e .

Since our version of Theorem 1.2.1 is slightly more general than in the aforementioned references (as it holds for an arbitrary number of curve families \mathcal{S}_i), we present a proof of it. It is straightforward by the following lemma.

Lemma 1.2.2. *Let \mathcal{S}_i , where $i \in \{1, \dots, p\}$, be families of rectifiable curves on a given closed manifold M such that $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$ for all $i \neq j$. Denote by $\ell_i(g)$ the least length of a curve in \mathcal{S}_i with respect to a Riemannian metric g on M . Let g_e be a Riemannian metric on M such that there exists a positive measure μ on $\Gamma = \mathcal{S}_1 \cup \dots \cup \mathcal{S}_p$ satisfying the three conditions of Theorem 1.2.1. Then,*

$$\ell_1(g) \cdots \ell_p(g) \leq \frac{\text{vol}^{\frac{p}{2}}(g_e)}{p^p m_1 \cdots m_p} \text{vol}^{\frac{p}{2}}(g) \quad (1.2.2)$$

where m_i is the mass of the measure μ on \mathcal{S}_i . Furthermore, the equality holds if and only if g is homothetic to g_e .

Proof. Let g be a Riemannian metric conformal to g_e , that is, $g = \phi^2 g_e$. We have

$$\begin{aligned} m_1 \ell_1(g) + \cdots + m_p \ell_p(g) &\leq \int_{\mathcal{S}_1} \bar{\phi}(\gamma) d\mu(\gamma) + \cdots + \int_{\mathcal{S}_p} \bar{\phi}(\gamma) d\mu(\gamma) & (1.2.3) \\ &= \int_{\Gamma} \bar{\phi}(\gamma) d\mu(\gamma) \\ &= \int_M \phi(x) d(*\mu)(x) \\ &= \int_M \phi(x) dg_e \\ &\leq \left(\int_M \phi^2(x) dg_e \right)^{\frac{1}{2}} \left(\int_M dg_e \right)^{\frac{1}{2}} & (1.2.4) \\ &= \sqrt{\text{vol}(g) \text{vol}(g_e)}. \end{aligned}$$

When g is equal to g_e , equality in (1.2.3) is attained because g_e satisfies condition (1). Moreover, it is straightforward that equality in (1.2.4) is attained if and only if g is equal to g_e by Cauchy-Schwartz. Using the arithmetic and geometric means inequality, we derive

$$p \cdot (m_1 \cdots m_p \ell_1(g) \cdots \ell_p(g))^{\frac{1}{p}} \leq m_1 \ell_1(g) + \cdots + m_p \ell_p(g)$$

with equality if and only if

$$m_1 \ell_1(g) = m_2 \ell_2(g) = \cdots = m_p \ell_p(g).$$

Finally, combining the two inequalities, we obtain that, under the required conditions, the following inequality holds

$$p^p \cdot (m_1 \cdots m_p) \cdot \ell_1(g) \cdots \ell_p(g) \leq (\text{vol}(g) \text{vol}(g_b))^{\frac{p}{2}},$$

with equality if and only if g is equal to g_e . □

Thus, proving the conformal extremality of a metric requires to find disjoint families of rectifiable curves $\mathcal{S}_1, \dots, \mathcal{S}_n$ on M and define a measure on them that satisfies the three conditions of the previous theorem.

1.2.2 Setting the curve families

To prove our results, we will make use of three families of curves in $(\mathbb{K}, SF_{b,\omega})$.

Definition 1.2.3. Let $(\mathbb{K}, SF_{b,\omega})$ be the spherical-flat Klein bottle defined in Section 1.1.2. Define three families of loops as follows.

1. For each $\theta \in \mathbb{R}/\pi\mathbb{Z}$ and each $a \in [-\omega, \omega] \cup [2b - \omega, 2b] \cup [-2b, -2b + \omega]$, the loop α_θ^a is the geodesic (image in $(\mathbb{K}, SF_{b,\omega})$ of a great circle) going through the points $(\theta - \pi/2, 0)$, (θ, a) . Let

$$\mathcal{S}_1(\omega) = \{\alpha_\theta^a \mid |a| \leq \omega \text{ or } 2b - \omega \leq |a| \leq 2b, \theta \in \mathbb{R}/\pi\mathbb{Z}\}.$$

2. For each $u \in \mathbb{R}/\pi\mathbb{Z}$, γ_u is the vertical loop defined by $\gamma_u(t) = (u, t)$ with $|t| \leq 2b$. Let

$$\mathcal{S}_2 = \{\gamma_u \mid u \in \mathbb{R}/\pi\mathbb{Z}\}.$$

3. For each $v \in \mathbb{R}$ satisfying $\omega \leq |v| \leq 2b - \omega$, δ_v is the horizontal loop defined by $\delta_v(t) = (t, |v|)$ with $|t| \leq \frac{\pi}{2}$. Let

$$\mathcal{S}_3(\omega) = \{\delta_v \mid \omega \leq |v| \leq 2b - \omega\}.$$

Remark 1.2.4. For each $i \in \{1, 2, 3\}$, the curves in \mathcal{S}_i have the same length with respect to the metric $SF_{b,\omega}$. Also note that a curve in \mathcal{S}_1 , resp. \mathcal{S}_2 , resp. \mathcal{S}_3 , belongs to the family \mathcal{F}_σ , resp. \mathcal{F}_v , resp. \mathcal{F}_h .

We move now to the next step, i.e., verifying the three conditions of Theorem 1.2.1. Conditions (1) and (2) are easy to verify while Condition (3) requires technical (but not simple) methods of calculation.

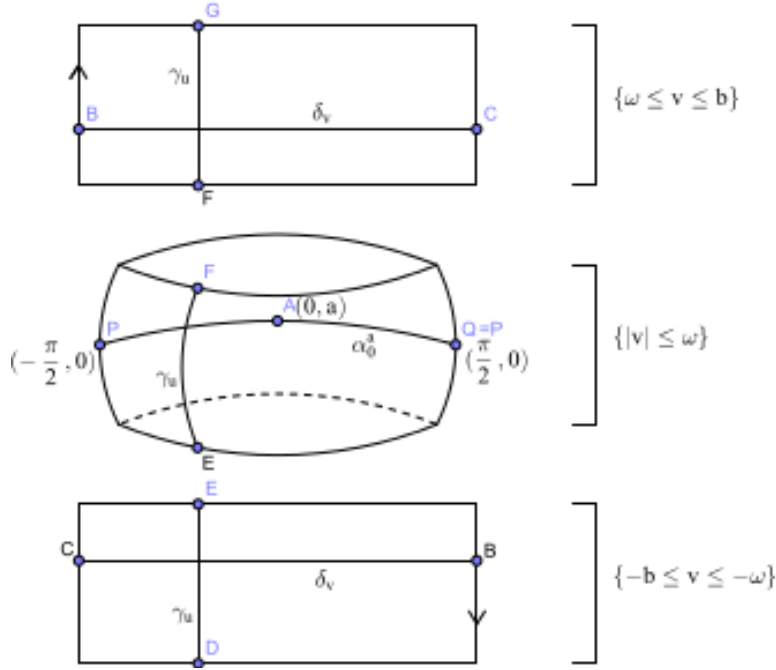


Figure 1.1: The curves α_θ^a , γ_u and δ_v in the Mobius band $\{(u, v) \mid |v| \leq b\}$ of $(\mathbb{K}, SF_{b,\omega})$.

In the following we will consider a subset of the family of curves $\mathcal{S}_1(\omega)$ in $(\mathbb{K}, SF_{b,\omega})$ and equip it with a measure μ depending on a function h of one parameter (since $SF_{b,\omega}$ has an isometry group of dimension 1). Then, we will calculate the measure ${}^*\mu$ in terms of the volume measure of $SF_{b,\omega}$.

1.2.3 The calculation of ${}^*\mu$ on a spherical region of \mathbb{K}

Let $b \in (0, +\infty)$ and $\omega \in (0, b]$. We consider on $(\mathbb{K}, SF_{b,\omega})$ the family of curves $\mathcal{S}'_1(\omega) \subset \mathcal{S}_1(\omega)$ defined by

$$\mathcal{S}'_1(\omega) = \{\alpha_\theta^a \subset (\mathbb{K}, SF_{b,\omega}) \mid -\omega \leq a \leq \omega, \theta \in \mathbb{R}/\pi\mathbb{Z}\}$$

where α_θ^a is the great circle introduced in Definition 1.2.3.

Lemma 1.2.5. *Let μ be a measure on $\mathcal{S}'_1(\omega)$ defined by*

$$d\mu(\alpha_\theta^a) = \begin{cases} h(a) da d\theta & \text{if } a \geq 0 \\ h(-a) da d\theta & \text{if } a < 0 \end{cases}$$

where $h : [0, \omega] \rightarrow \mathbb{R}$ is a continuous function. Then, we have

$${}^*\mu = 2\chi_{\{|v| \leq \omega\}} \left(\int_{|v|}^{\omega} (\cos^2(v) - \cos^2(a))^{-\frac{1}{2}} h(a) da \right) d(SF_{b,\omega})$$

where $d(SF_{b,\omega})$ is the volume measure of $SF_{b,\omega}$.

Proof. First, we compute the equation of α_θ^a . Suppose $\alpha_\theta^a(t) = (u(t), v(t))$. Then, from the classical geodesic equation $\frac{d^2 x^i}{dt^2} + \sum_{j,k} \Gamma_{j,k}^i \frac{dx^k}{dt} \frac{dx^j}{dt} = 0$ (see for example [GHL04] p. 81), where $\Gamma_{j,k}^i$ are the Christoffel symbols, $x^1 = u$ and $x^2 = v$, we derive

$$u'' - 2 \tan(v) u' v' = 0 \quad \text{and} \quad v'' + \sin(v) \cos(v) u'^2 = 0.$$

This shows that

$$\frac{du}{dv} = \frac{c}{\cos(v) \sqrt{\cos^2(v) - c^2}} \quad (1.2.5)$$

where c is a constant. The solution of the differential equation (1.2.5) is $\sin(u - d) = c \tan(v)$, where d is a constant. Using the fact that α_θ^a goes through $(\theta - \pi/2, 0)$ and (θ, a) , we obtain

$$\sin(u - \theta + \frac{\pi}{2}) = \frac{1}{\tan(a)} \tan(v). \quad (1.2.6)$$

Equation (1.2.6) shows that we can write $\alpha_\theta^a(u) = (u + \theta, v(u, a))$, with v verifying

$$v(u, -a) = -v(u, a).$$

Now, let $\phi \in C^0(\mathbb{K}, \mathbb{R})$. Then, by the definition of ${}^* \mu$, we have

$$\int_{\mathbb{K}} \phi(u, v) d({}^* \mu)(u, v) = \int_{S_1^*(\omega)} \bar{\phi}(\alpha_\theta^a) d\mu(\alpha_\theta^a)$$

where

$$\begin{aligned} \bar{\phi}(\alpha_\theta^a) &= \int_{\alpha_\theta^a} \phi(\alpha_\theta^a(s)) ds \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi(u + \theta, v(u, a)) \sqrt{\cos^2(v(u, a)) + \left(\frac{\partial v}{\partial u}(u, a)\right)^2} du. \end{aligned}$$

Then,

$$\begin{aligned} \int_{\mathbb{K}} \phi(u, v) d({}^* \mu)(u, v) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\omega}^0 \bar{\phi}(\alpha_\theta^a) h(-a) da d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\omega \bar{\phi}(\alpha_\theta^a) h(a) da d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\omega \int_{-\frac{\pi}{2}}^0 \phi(u + \theta, -v(u, a)) \sqrt{\cos^2(v(u, a)) + \left(\frac{\partial v}{\partial u}(u, a)\right)^2} h(a) du da d\theta \\ &\quad + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\omega \int_0^{\frac{\pi}{2}} \phi(u + \theta, -v(u, a)) \sqrt{\cos^2(v(u, a)) + \left(\frac{\partial v}{\partial u}(u, a)\right)^2} h(a) du da d\theta \\ &\quad + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\omega \int_{-\frac{\pi}{2}}^0 \phi(u + \theta, v(u, a)) \sqrt{\cos^2(v(u, a)) + \left(\frac{\partial v}{\partial u}(u, a)\right)^2} h(a) du da d\theta \end{aligned}$$

$$+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\omega \int_0^{\frac{\pi}{2}} \phi(u + \theta, v(u, a)) \sqrt{\cos^2(v(u, a)) + \left(\frac{\partial v}{\partial u}(u, a)\right)^2} h(a) du da d\theta$$

Next, we apply for the first two integrals in the previous expression the change of variables $y \mapsto -v(u, a)$, whose Jacobian is equal to $-\frac{\partial v}{\partial u}(u, a)$, and we write $k_1(y, a) = u$. For the last two integrals, we make use of the change of variables $y \mapsto v(u, a)$, whose Jacobian is equal to $\frac{\partial v}{\partial u}(u, a)$, and we write $k_2(y, a) = u$. In either case, let $z(y, a) = \frac{\partial v}{\partial u}(u, a)$. We derive

$$\begin{aligned} \int_{\mathbb{K}} \phi(u, v) d(*\mu)(u, v) &= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\omega \int_{-a}^0 \phi(k_1(y, a) + \theta, y) \sqrt{\cos^2(y) + z^2(y, a)} \frac{h(a)}{|z(y, a)|} dy da d\theta \\ &\quad + 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\omega \int_0^a \phi(k_2(y, a) + \theta, y) \sqrt{\cos^2(y) + z^2(y, a)} \frac{h(a)}{|z(y, a)|} dy da d\theta. \end{aligned}$$

Next, in the previous expression, we make the change of variables $x \mapsto k_1(y, a) + \theta$ for the first integral and $x \mapsto k_2(y, a) + \theta$ for the second integral. The Jacobian for both changes of variables is equal to 1. We obtain

$$\begin{aligned} \int_{\mathbb{K}} \phi(u, v) d(*\mu)(u, v) &= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\omega \int_{-a}^0 \phi(x, y) \sqrt{\cos^2(y) + z^2(y, a)} \frac{h(a)}{|z(y, a)|} dy da dx \\ &\quad + 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\omega \int_0^a \phi(x, y) \sqrt{\cos^2(y) + z^2(y, a)} \frac{h(a)}{|z(y, a)|} dy da dx \\ &= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\omega}^0 \int_{-y}^\omega \phi(x, y) \sqrt{\cos^2(y) + z^2(y, a)} \frac{h(a)}{|z(y, a)|} da dy dx \\ &\quad + 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\omega \int_y^\omega \phi(x, y) \sqrt{\cos^2(y) + z^2(y, a)} \frac{h(a)}{|z(y, a)|} da dy dx \\ &= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\omega}^\omega \phi(x, y) \int_{|y|}^\omega \sqrt{\left(\left(\frac{\cos(y)}{z(y, a)}\right)^2 + 1\right)} h(a) da dy dx \quad . \end{aligned}$$

Hence, we derive

$$*\mu = 2 \frac{\chi_{\{|v| \leq \omega\}}}{\cos(v)} \int_{|v|}^\omega \sqrt{\left(\left(\frac{\cos(v)}{z(v, a)}\right)^2 + 1\right)} h(a) da d(SF_{b, \omega})$$

where $d(SF_{b, \omega})$ is the volume element of $(\mathbb{K}, SF_{b, \omega})$. From equation (1.2.5), we obtain

$$z^2(v, a) = \frac{\cos^2(v)}{\cos^2(a)} (\cos^2(v) - \cos^2(a)).$$

This shows that

$$*\mu = 2 \chi_{\{|v| \leq \omega\}} \left(\int_{|v|}^\omega (\cos^2(v) - \cos^2(a))^{-\frac{1}{2}} h(a) da \right) d(SF_{b, \omega}).$$

□

1.3 Proof of the geometric inequality of type $\ell_\sigma \ell_v$

In the following, we prove the geometric inequality of type $\ell_\sigma \ell_v$ for $\beta > 2 \ln \left(\tan \left(\frac{\pi}{4} + \frac{a_0}{2} \right) \right)$. As in the case $\beta \leq 2 \ln \left(\tan \left(\frac{\pi}{4} + \frac{a_0}{2} \right) \right)$, we will consider both curve families $\mathcal{S}_1(\omega_1)$ and \mathcal{S}_2 , cf. [Ba06, Corollary 2].

Proof of Theorem 1.1.4. Let $\beta > 2 \ln \left(\tan \left(\frac{\pi}{4} + \frac{a_0}{2} \right) \right)$. There exists a unique $\omega_1 \in [a_0, \frac{\pi}{2})$ such that

$$2 \sin(\omega_1) = \left(\beta - 2 \ln \left(\tan \left(\frac{\pi}{4} + \frac{\omega_1}{2} \right) \right) \right) \cos^2(\omega_1) + 4\omega_1 \cos(\omega_1). \quad (1.3.1)$$

We endow \mathbb{K} with the metric SF_{b,ω_1} defined in Section 1.1.2, where

$$b = \tan(\omega_1) - \omega_1. \quad (1.3.2)$$

Consider the two families of curves $\mathcal{S}_1(\omega_1)$ and \mathcal{S}_2 , cf. Section 1.2.2. Each curve in $\mathcal{S}_1(\omega_1)$ (resp. \mathcal{S}_2) has length equal to π (resp. $4b$) with respect to SF_{b,ω_1} . Hence, condition (1) of Theorem 1.2.1 is satisfied. Now, let

$$\begin{aligned} h : [-\omega_1, \omega_1] &\longrightarrow \mathbb{R} \\ a &\longmapsto \frac{\sin(|a|)}{\pi \cos(a)} \sqrt{\cos^2(a) - \cos^2(\omega_1)} \end{aligned}$$

We define on the family $\mathcal{S}_1(\omega_1)$ the measure

$$\mu_1 = \tilde{h}(a) da \otimes d\theta$$

where

$$\tilde{h}(a) = \begin{cases} h(a) & \text{if } a \in [-\omega_1, \omega_1] \\ h(a - 2b) & \text{if } a \in [2b - \omega_1, 2b] \\ h(a + 2b) & \text{if } a \in [-2b, -2b + \omega_1] \end{cases}$$

On the family \mathcal{S}_2 , we define the measure

$$\mu_2 = \cos(\omega_1) du.$$

The mass m_1 of the measure μ_1 on $\mathcal{S}_1(\omega_1)$ is equal to $4 \sin(\omega_1) - 4\omega_1 \cos(\omega_1)$ and the mass m_2 of the measure μ_2 on \mathcal{S}_2 is equal to $\pi \cos(\omega_1)$. Now, since $\ell_\sigma(SF_{b,\omega_1}) = \pi$ and $\ell_v(SF_{b,\omega_1}) = 4b$, condition (2) of Theorem 1.2.1 is satisfied. Next, it can be easily verified that

$$*\mu_2 = \frac{\cos(\omega_1)}{f_{b,\omega_1}(v)} d(SF_{b,\omega_1})$$

where f_{b,ω_1} is the unique one-variable function invariant by the translation $(u, v) \mapsto (u, v + 2b)$ which agrees with *cosine* on $[-\omega_1, \omega_1]$ and equal to the constant $\cos(\omega_1)$ on $[\omega_1, 2b - \omega_1]$ (see Section 1.1.2). Moreover, by Lemma 1.2.5, we have

$$\chi_{\{|v| \leq \omega_1\}} \ast \mu_1 = 2\chi_{\{|v| \leq \omega_1\}} \left(\int_{|v|}^{\omega_1} (\cos^2(v) - \cos^2(a))^{-\frac{1}{2}} \tilde{h}(a) da \right) d(SF_{b,\omega_1})$$

Then, we derive

$$\chi_{\{|v| \leq \omega_1\}} \ast \mu_1 = \chi_{\{|v| \leq \omega_1\}} \left(1 - \frac{\cos(\omega_1)}{f_{b,\omega_1}(v)} \right) d(SF_{b,\omega_1}).$$

Finally, since f_{b,ω_1} and \tilde{h} are invariant by the translation $(u, v) \mapsto (u, v + 2b)$, we have

$$\ast \mu_1 = \left(1 - \frac{\cos(\omega_1)}{f_{b,\omega_1}(v)} \right) d(SF_{b,\omega_1}).$$

Hence, we get $\ast \mu_1 + \ast \mu_2 = d(SF_{b,\omega_1})$. This is condition (3) of Theorem 1.2.1. Finally, from Theorem 1.2.1 and inequality (1.2.2), we derive

$$\ell_\sigma(g)\ell_v(g) \leq \frac{1}{2 \cos(\omega_1)} \text{area}(g)$$

which holds for every Riemannian metric g conformal to SF_{b,ω_1} . The equality is attained if and only if g is homothetic to SF_{b,ω_1} , and the result follows. \square

Recall that \mathbb{M} denotes the Mobius band with boundary obtained by taking the quotient of $\mathbb{R} \times [-\beta, \beta]$ by the group generated by the map $\sigma : z \mapsto \bar{z} + \pi$. We denote by g_β the flat metric induced by such quotient. The parameter β represents the conformal type of any metric conformal to g_β .

We recover Blatter's result for the Mobius Band, cf. [Bl61, Theorem 2], by a simpler method. Indeed, consider in Bavard's proof of Theorem 1.1.3 and in our previous proof the restriction of S_b and SF_{b,ω_1} on the set $E_b = \{(u, v) \in \mathbb{C} \mid |v| \leq b\}$.

Corollary 1.3.1. [Bl61, Satz 2] Let $\beta > 0$ and $a_0 \in (0, \frac{\pi}{2})$ such that $\tan(a_0) = 2a_0$. Let $\omega_1 \in [a_0, \frac{\pi}{2})$ be defined by the equation

$$\sin(\omega_1) = \left(\beta - \ln \left(\tan \left(\frac{\pi}{4} + \frac{\omega_1}{2} \right) \right) \right) \cos^2(\omega_1) + 2\omega_1 \cos(\omega_1).$$

Then, for every Riemannian metric g on the Mobius band \mathbb{M} of conformal type β , we have

$$\ell_\sigma(g)\ell_v(g) \leq C_\beta \text{area}(g)$$

where

$$C_\beta = \begin{cases} \frac{e^{2\beta} + 1}{e^{2\beta} - 1} \arcsin \left(\frac{e^{2\beta} - 1}{e^{2\beta} + 1} \right) & \text{if } 0 < \beta \leq \ln \left(\frac{\pi}{4} + \frac{a_0}{2} \right) \\ \frac{1}{2 \cos(\omega_1)} & \text{if } \beta > \ln \left(\frac{\pi}{4} + \frac{a_0}{2} \right). \end{cases}$$

Moreover, the equality is attained if and only if g is proportional to the spherical metric S_b restricted to $E_b = \{(u, v) \in \mathbb{C} \mid |v| \leq b\}$, for b satisfying $\beta = 2 \ln(\tan(\frac{\pi}{4} + \frac{b}{2}))$, in the first case, and to the spherical-flat metric SF_{b, ω_1} restricted to $E_b = \{(u, v) \in \mathbb{C} \mid |v| \leq b\}$, for $b = \tan(\omega_1) - \omega_1$, in the second case.

1.4 Proof of the geometric inequality of type $L_\sigma \ell_v$

In the following, we prove the geometric inequality of type $L_\sigma \ell_v$ on the Klein bottle. The curve families we consider this time are $\mathcal{S}_1(\frac{\pi}{3})$, \mathcal{S}_2 and $\mathcal{S}_3(\frac{\pi}{3})$.

Proof of Theorem 1.1.7. The inequality in the first case can be deduced from Theorem 1.1.3 since when $0 < \beta \leq 2 \ln(2 + \sqrt{3})$, i.e. when $b \leq \frac{\pi}{3}$, we have $L_\sigma(S_b) = \ell_\sigma(S_b)$. Now, since the conformal type of (\mathbb{K}, S_b) is

$$\beta = 2 \ln(\tan(\frac{\pi}{4} + \frac{b}{2})),$$

we deduce that for $0 < \beta \leq 2 \ln(2 + \sqrt{3})$,

$$L_\sigma(g) \ell_v(g) \leq \frac{\arcsin\left(\frac{e^\beta - 1}{e^\beta + 1}\right)}{\frac{e^\beta - 1}{e^\beta + 1}} \text{area}(g).$$

Note that when b becomes greater than $\frac{\pi}{3}$, the horizontal geodesic loops closed by the horizontal translation t_h (corresponding to the singularity line of S_b) become shorter than the curves α_θ^a and therefore L_σ is attained by such lines.

To prove the inequality in the second case, we fix $\beta > 2 \ln(2 + \sqrt{3})$ and let b such that $\beta = 2 \ln(2 + \sqrt{3}) + 4(b - \frac{\pi}{3})$ (we have $b \geq \frac{\pi}{3}$). Then we endow \mathbb{K} with the metric $SF_{b, \frac{\pi}{3}}$. We consider the two families of curves $\mathcal{S} = \mathcal{S}_1(\frac{\pi}{3}) \cup \mathcal{S}_3(\frac{\pi}{3})$ and \mathcal{S}_2 , cf. Section 1.2.2. Each curve in \mathcal{S} , (resp. \mathcal{S}_2) has length equal to π , (resp. $4b$) with respect to the metric $SF_{b, \frac{\pi}{3}}$. Hence, condition (1) of Theorem 1.2.1 is satisfied. Now, let

$$\begin{aligned} k : \quad & [-\frac{\pi}{3}, \frac{\pi}{3}] \longrightarrow \mathbb{R} \\ & a \longmapsto \frac{\tan(|a|)}{24\pi b} \cdot \frac{24b \cos^2(a) - 3\sqrt{3} - 3b + \pi}{\sqrt{\cos^2(a) - \frac{1}{4}}} \end{aligned}$$

We define on the family \mathcal{S} the measure

$$\mu_1 = \tilde{k}(a) da \otimes d\theta + \frac{-3\sqrt{3} + 3b + \pi}{6b} dv.$$

where

$$\tilde{k}(a) = \begin{cases} k(a) & \text{if } a \in [-\frac{\pi}{3}, \frac{\pi}{3}] \\ k(a - 2b) & \text{if } a \in [2b - \frac{\pi}{3}, 2b] \\ k(a + 2b) & \text{if } a \in [-2b, -2b + \frac{\pi}{3}] \end{cases}$$

On the family \mathcal{S}_2 , we define the measure

$$\mu_2 = \frac{3\sqrt{3} + 3b - \pi}{12b} du.$$

Note that these measures are positive since $b \geq \frac{\pi}{3}$. The mass m_1 of the measure μ_1 on \mathcal{S} is equal to $\sqrt{3} + 3b - \frac{\pi}{3}$, and the mass m_2 of the measure μ_2 on \mathcal{S}_2 is equal to $\frac{3\sqrt{3} + 3b\pi - \pi^2}{12b}$. Now, since $L_\sigma(SF_{b, \frac{\pi}{3}}) = \pi$ and $\ell_v(SF_{b, \frac{\pi}{3}}) = 4b$, condition (2) of Theorem 1.2.1 is satisfied. Next, it can be easily verified that

$$*\mu_2 = \frac{3\sqrt{3} + 3b - \pi}{12bf_{b, \frac{\pi}{3}}(v)} d(SF_{b, \frac{\pi}{3}})$$

where $f_{b, \frac{\pi}{3}}$ is the unique one-variable function invariant by the translation $(u, v) \mapsto (u, v + 2b)$ which agrees with *cosine* on $[-\frac{\pi}{3}, \frac{\pi}{3}]$ and equal to the constant $\frac{1}{2}$ on $[\frac{\pi}{3}, 2b - \frac{\pi}{3}]$. Moreover, by Lemma 1.2.5 we obtain

$$\chi_{\{|v| \leq \frac{\pi}{3}\}} * \mu_1 = 2\chi_{\{|v| \leq \frac{\pi}{3}\}} \left(\int_{|v|}^{\frac{\pi}{3}} (\cos^2(v) - \cos^2(a))^{-\frac{1}{2}} \tilde{k}(a) da \right) d(SF_{b, \frac{\pi}{3}})$$

Then, we derive

$$\chi_{\{|v| \leq \frac{\pi}{3}\}} * \mu_1 = \chi_{\{|v| \leq \frac{\pi}{3}\}} \left(1 - \frac{3\sqrt{3} + 3b - \pi}{12bf_{b, \frac{\pi}{3}}(v)} \right) d(SF_{b, \frac{\pi}{3}}).$$

Finally, since $f_{b, \frac{\pi}{3}}$ and \tilde{k} are invariant by the translation $(u, v) \mapsto (u, v + 2b)$, we have

$$*\mu_1 = \begin{cases} \left(1 - \frac{3\sqrt{3} + 3b - \pi}{12bf_{b, \frac{\pi}{3}}(v)} \right) d(SF_{b, \frac{\pi}{3}}) & \text{if } |v| \leq \frac{\pi}{3} \text{ or } 2b - \frac{\pi}{3} \leq |v| \leq 2b \\ \frac{-3\sqrt{3} + 3b + \pi}{6b} d(SF_{b, \frac{\pi}{3}}) & \text{if } \frac{\pi}{3} \leq |v| \leq 2b - \frac{\pi}{3} \end{cases}$$

Hence, we get $*\mu_1 + *\mu_2 = d(SF_{b, \frac{\pi}{3}})$. Therefore, condition (3) of Theorem 1.2.1 is satisfied. It follows from Theorem 1.2.1 and inequality (1.2.2) that for every Riemannian metric g conformal to $SF_{b, \frac{\pi}{3}}$, we have

$$L_\sigma(g)\ell_v(g) \leq \frac{2b}{\sqrt{3} + b - \frac{\pi}{3}} \text{area}(g)$$

with equality if and only if g is homothetic to $SF_{b, \frac{\pi}{3}}$. □

We can also derive C. Blatter's similar result on the Mobius band \mathbb{M} (Theorem 3 in [Bl61]) using a minor adaptation of the previous proof.

Corollary 1.4.1. [Bl61, Satz 3] *Let $\beta > 0$. Then, for every Riemannian metric g on the Mobius band \mathbb{M} of conformal type β , we have*

$$\text{sys}(g)\ell_v(g) \leq C_\beta \text{area}(g)$$

where

$$C_\beta = \begin{cases} \frac{e^{2\beta}+1}{e^{2\beta}-1} \arcsin\left(\frac{e^{2\beta}-1}{e^{2\beta}+1}\right) & \text{if } 0 < \beta \leq \ln(2 + \sqrt{3}) \\ \frac{2}{3} \cdot \frac{3\beta + 2\pi - 3\ln(2 + \sqrt{3})}{2\sqrt{3} + \beta - \ln(2 + \sqrt{3})} & \text{if } \beta > \ln(2 + \sqrt{3}). \end{cases}$$

Moreover, the equality is attained if and only if g is proportional to the spherical metric S_b restricted to $E_b = \{(u, v) \in \mathbb{C} \mid |v| \leq b\}$, for b satisfying $\beta = 2 \ln(\tan(\frac{\pi}{4} + \frac{b}{2}))$, in the first case, and to the spherical-flat metric $SF_{b, \frac{\pi}{3}}$ restricted to $E_b = \{(u, v) \in \mathbb{C} \mid |v| \leq b\}$, for $b = \frac{1}{4}\beta + \frac{\pi}{3} - \frac{1}{2} \ln(2 + \sqrt{3})$, in the second case.

1.5 Proof of the geometric Inequality of type $\ell_\sigma \ell_V \ell_H$

In the following, we prove the geometric inequality of type $\ell_\sigma \ell_V \ell_H$ on the Klein bottle. We will consider the curve families $\mathcal{S}_1(\omega_2)$, \mathcal{S}_2 and $\mathcal{S}_3(\omega_2)$.

Proof of Theorem 1.1.9. Let $\beta > 0$. There exists a unique $\omega_2 \in (0, \frac{\pi}{2})$ such that

$$\beta = 2 \ln\left(\tan\left(\frac{\pi}{4} + \frac{\omega_2}{2}\right)\right) + \frac{2}{\cos(\omega_2)} \left(\tan(\omega_2) - \omega_2 + \sqrt{\tan^2(\omega_2) - \omega_2 \tan(\omega_2) + \omega_2^2} \right). \quad (1.5.1)$$

We equip \mathbb{K} with the metric SF_{b, ω_2} , where

$$b = \tan(\omega_2) + \sqrt{\tan^2(\omega_2) - \omega_2 \tan(\omega_2) + \omega_2^2}. \quad (1.5.2)$$

Then, we consider the families of curves $\mathcal{S}_1(\omega_2)$, \mathcal{S}_2 and $\mathcal{S}_3(\omega_2)$, cf. Section 1.2.2. Each curve in $\mathcal{S}_1(\omega_2)$, (resp. \mathcal{S}_2 , resp. $\mathcal{S}_3(\omega_2)$) has length equal to π , (resp. $4b$, resp. $2\pi \cos(\omega_2)$) with respect to the metric SF_{b, ω_2} . Hence, condition (1) of Theorem 1.2.1 is satisfied. Let

$$\begin{aligned} l : [-\omega_2, \omega_2] &\longrightarrow \mathbb{R} \\ a &\longmapsto \frac{\tan(|a|)}{\pi} \cdot \frac{\cos^2(a) - \cos^2(\omega_2)^{\frac{b-\omega_2}{2b-\omega_2}}}{(\cos^2(a) - \cos^2(\omega_2))^{\frac{1}{2}}} \end{aligned}$$

We define on the family $\mathcal{S}_1(\omega_2)$ the measure

$$\mu_1 = \tilde{l}(a) da \otimes d\theta$$

where

$$\tilde{l}(a) = \begin{cases} l(a) & \text{if } a \in [-\omega_2, \omega_2] \\ l(a - 2b) & \text{if } a \in [2b - \omega_2, 2b] \\ l(a + 2b) & \text{if } a \in [-2b, -2b + \omega_2] \end{cases}$$

On the family \mathcal{S}_2 , we define the measure

$$\mu_2 = \cos(\omega_2) \cdot \frac{b - \omega_2}{2b - \omega_2} du.$$

Finally, on $\mathcal{S}_3(\omega_2)$, we define the measure

$$\mu_3 = \frac{b}{2b - \omega_2} dv.$$

The masses m_i of the measures μ_i are $m_1 = 4 \sin(\omega_2) - \frac{4\omega_2(b-\omega_2)\cos(\omega_2)}{2b-\omega_2}$, $m_2 = \frac{\pi \cos(\omega_2)(b-\omega_2)}{2b-\omega_2}$, and $m_3 = \frac{2b(b-\omega_2)}{2b-\omega_2}$. Now, since $\ell_\sigma(SF_{b,\omega_2}) = \pi$, $\ell_v(SF_{b,\omega_2}) = 4b$, and $\ell_h(SF_{b,\omega_2}) = 2\pi \cos(\omega_2)$, condition (2) of Theorem 1.2.1 is satisfied. Next, it can be easily verified that

$${}^*\mu_2 = \frac{\cos(\omega_2)}{f_{b,\omega_2}(v)} \cdot \frac{b - \omega_2}{2b - \omega_2} d(SF_{b,\omega_2})$$

where f_{b,ω_2} is the unique one-variable function invariant by the translation $(u, v) \mapsto (u, v + 2b)$ which agrees with *cosine* on $[-\omega_2, \omega_2]$ and equal to the constant $\cos(\omega_2)$ on $[\omega_2, 2b - \omega_2]$. Moreover,

$${}^*\mu_3 = \begin{cases} 0 & \text{if } |v| \leq \omega_2 \text{ or } 2b - \omega_2 \leq |v| \leq 2b \\ \frac{b}{2b - \omega_2} \cdot d(SF_{b,\omega_2}) & \text{if } \omega_2 \leq |v| \leq 2b - \omega_2 \end{cases}$$

Next, by Lemma 1.2.5, we have

$$\chi_{\{|v| \leq \omega_2\}} {}^*\mu_1 = 2\chi_{\{|v| \leq \omega_2\}} \left(\int_{|v|}^{\omega_2} (\cos^2(v) - \cos^2(a))^{-\frac{1}{2}} \tilde{l}(a) da \right) d(SF_{b,\omega_2})$$

Then, we derive

$$\chi_{\{|v| \leq \omega_2\}} {}^*\mu_1 = \chi_{\{|v| \leq \omega_2\}} \left(1 - \frac{\cos(\omega_2)(b - \omega_2)}{f_{b,\omega_2}(v)(2b - \omega_2)} \right) d(SF_{b,\omega_2}).$$

Finally, since f_{b,ω_2} and \tilde{l} are invariant by the translation $(u, v) \mapsto (u, v + 2b)$, we have

$${}^*\mu_1 = \begin{cases} \left(1 - \frac{\cos(\omega_2)(b - \omega_2)}{f_{b,\omega_2}(v)(2b - \omega_2)} \right) d(SF_{b,\omega_2}) & \text{if } |v| \leq \omega_2 \text{ or } 2b - \omega_2 \leq |v| \leq 2b \\ 0 & \text{if } \omega_2 \leq |v| \leq 2b - \omega_2 \end{cases}$$

Thus, we obtain ${}^*\mu_1 + {}^*\mu_2 + {}^*\mu_3 = dSF_{b,\omega_2}$. Hence, condition (3) of Theorem 1.2.1 is satisfied. Hence,

$$\text{area}(g)^{\frac{3}{2}} \geq \frac{3\sqrt{3}}{\sqrt{\pi}} \cdot \frac{(b - \omega_2)\sqrt{(b - \omega_2)b}}{(b^4 - 4b\omega_2 + \omega_2^2 + \omega_2^4 - 2b^2(-2 + \omega_2^2))^{\frac{1}{4}} (2b - \omega_2)} \ell_\sigma(g)\ell_v(g)\ell_h(g)$$

with equality if and only if g is homothetic to SF_{b,ω_2} . \square

1.6 Some remarks on the extremal metrics

In this chapter, we present four families of conformally extremal metrics for different types of inequalities on the Klein bottle. They are as follows.

1. The spherical metric S_b , where $\beta = 2 \ln(\tan(\frac{\pi}{4} + \frac{b}{2}))$, which is conformally extremal for the inequality of type $\ell_\sigma \ell_v$ for $0 < \beta \leq 2 \ln(\tan(\frac{\pi}{4} + \frac{a_0}{2}))$, and for the inequality of type $L_\sigma \ell_v$ for $0 < \beta \leq 2 \ln(2 + \sqrt{3})$.
2. The spherical-flat metric SF_{b, ω_1} , where ω_1 and b verify the equations (1.3.1) and (1.3.2), which is conformally extremal for the inequality of type $\ell_\sigma \ell_v$ for $2 \ln(\tan(\frac{\pi}{4} + \frac{a_0}{2})) < \beta < \infty$.
3. The spherical-flat metric $SF_{b, \frac{\pi}{3}}$, where $\beta = 2 \ln(2 + \sqrt{3}) + 4(b - \frac{\pi}{3})$, which is conformally extremal for the inequality of type $L_\sigma \ell_v$ for $2 \ln(2 + \sqrt{3}) < \beta < \infty$.
4. The spherical-flat metric SF_{b, ω_2} , where ω_2 and b verify the equations (1.5.1) and (1.5.2), which is conformally extremal for the inequality of type $\ell_\sigma \ell_v \ell_h$ for any β .

Remark 1.6.1. We emphasize the fact that the metrics SF_{b, ω_1} and SF_{b, ω_2} do not have the same conformal type since ω_1 verifies the equation (1.3.1) whereas ω_2 verifies the equation (1.5.1). Note that the equality $\omega_1 = \omega_2$ leads to

$$\sqrt{\tan^2(\omega_1) - \omega_1 \tan(\omega_1) + \omega_1^2} + \omega_1 = 0$$

which is not possible.

Moreover, the β -extremal metric (extremal metric in the conformal class of β) for the inequality of type $\ell_\sigma \ell_v \ell_h$ agrees with the β -extremal metric for the inequality of type $L_\sigma \ell_v$ if and only if

$$\beta = \beta_0 = 2 \ln(2 + \sqrt{3}) + 4(\sqrt{3} + (3 - \frac{\pi}{\sqrt{3}} + \frac{\pi^2}{9})^{\frac{1}{2}} - \frac{\pi}{3})$$

(i.e., $b = \sqrt{3} + (3 - \frac{\pi}{\sqrt{3}} + \frac{\pi^2}{9})^{\frac{1}{2}}$ and $\omega_2 = \frac{\pi}{3}$). On the other hand, the β -extremal metric for the inequality of type $\ell_\sigma \ell_v$ agrees with the β -extremal metric for the inequality of type $L_\sigma \ell_v$ if and only if $\beta \leq 2 \ln(2 + \sqrt{3})$. Finally, the β -extremal metric for the inequality of $\ell_\sigma \ell_v$ and the β -extremal metric for the inequality of type $\ell_\sigma \ell_v \ell_h$ never agree.

Table 1.1 summarizes our study of the extremal metrics for the various inequality types relative to the conformal type β .

Table 1.1

β	$(0, 2\ln(2 + \sqrt{3})]$	$(2\ln(2 + \sqrt{3}), 2\ln(\tan(\frac{\pi}{4} + \frac{a_0}{2}))]$	$(2\ln(\tan(\frac{\pi}{4} + \frac{a_0}{2})), +\infty)$
$l_\sigma l_v \leq C_\beta \text{area}$	S_b	S_b	SF_{b,ω_1}
$L_\sigma l_v \leq C_\beta \text{area}$	S_b	$SF_{b,\frac{\pi}{3}}$	$SF_{b,\frac{\pi}{3}}$
$l_\sigma l_v l_h \leq C_\beta \text{area}^{\frac{3}{2}}$	SF_{b,ω_2}	SF_{b,ω_2}	SF_{b,ω_2}

Chapter 2

Optimal systolic inequalities on Finsler Mobius bands

Nous établissons des inégalités géométriques optimales sur le ruban de Möbius muni d'une métrique de Finsler. Ces inégalités géométriques relient la systole et la hauteur du ruban de Möbius à son volume de Holmes-Thompson. Nous en déduisons une inégalité systolique optimale sur la bouteille de Klein munie d'une métrique de Finsler avec des symétries. Nous décrivons également une famille de métriques extrémales dans les deux cas.

Abstract. We prove optimal systolic inequalities on Finsler Mobius bands relating the systole and the height of the Mobius band to its Holmes-Thompson volume. We also establish an optimal systolic inequality for Finsler Klein bottles with symmetries, which we conjecture to hold true for arbitrary Finsler metrics. We describe extremal metric families both in both cases.

2.1 Introduction

Optimal systolic inequalities were studied since the mid-twentieth century after C. Loewner proved in an unpublished work the following result, cf. [Ka07]. Every Riemannian two-torus \mathbb{T}^2 satisfies

$$\text{area}(\mathbb{T}^2) \geq \frac{\sqrt{3}}{2} \text{sys}^2(\mathbb{T}^2) \quad (2.1.1)$$

with equality if and only if \mathbb{T}^2 is a flat hexagonal torus. Recall that the *systole* of a nonsimply connected Riemannian surface M , denoted by $\text{sys}(M)$, represents the length of the shortest noncontractible loop of M . This inequality leads us to introduce the *systolic area* of M defined as

$$\sigma_R(M) := \inf_g \frac{\text{area}(M, g)}{\text{sys}^2(M, g)}. \quad (2.1.2)$$

where g runs over all the Riemannian metrics on M (hence the subscript R for Riemannian). Thus, $\sigma_R(\mathbb{T}^2) = \frac{\sqrt{3}}{2}$. Following this direction, P. Pu [Pu52] showed that $\sigma_R(\mathbb{RP}^2) = \frac{2}{\pi}$, where the infimum is attained exactly by the Riemannian metrics with constant (positive) curvature on the projective plane \mathbb{RP}^2 . In the eighties, C. Bavard [Ba86] proved that $\sigma_R(\mathbb{K}^2) = \frac{2\sqrt{2}}{\pi}$, where the infimum on the Klein bottle \mathbb{K}^2 is not attained by a smooth Riemannian metric. See also [Sak88], [Ba88] and [Ba06] for other proofs and variations on this inequality. These are the only nonsimply connected closed surfaces with a known systolic area. The existence of extremal metrics in higher dimension is wide open.

The original proofs of the optimal Riemannian systolic inequalities on \mathbb{T}^2 , \mathbb{RP}^2 and \mathbb{K}^2 rely on the conformal representation theorem (a consequence of the uniformization theorem on Riemann surfaces) and proceed as follows. By the uniformization theorem, every Riemannian metric g on a closed surface is conformally equivalent to a Riemannian metric g_0 of constant curvature. Taking the average of g over the isometry group of g_0 gives rise to a new metric \bar{g} with the same area as g . By the Cauchy-Schwarz inequality, the systole of \bar{g} is at most the systole of g . Thus, the new metric \bar{g} has a lower ratio area/sys^2 than the original metric g . Now, if the isometry group of g_0 is transitive, which is the case for \mathbb{T}^2 and \mathbb{RP}^2 , the metric \bar{g} has constant curvature. Hence the result for the projective plane. Then, it is not difficult to find the extremal metric among flat torus. The case of the Klein bottle requires an extra argument since the isometry group of g_0 is not transitive, cf. Section 2.7.

In this chapter, we consider Finsler systolic inequalities. Loosely speaking, a Finsler metric F

is defined as a Riemannian metric except that its restriction to a tangent plane is no longer a Euclidean norm but a Minkowski norm, *cf.* Section 2.2. From a dynamical point of view, the function F^2 can be considered as a Lagrangian which induces a Lagrangian flow on the tangent bundle TM of M . Thus, Finsler manifolds can be considered as degree 2 homogeneous Lagrangian systems. The trajectories of the Lagrangian correspond to the geodesics of the Finsler metric.

There exist several definitions of volume for Finsler manifolds which coincide in the Riemannian case. We will consider the Holmes-Thompson volume vol_{HT} , *cf.* Section 2.2. As previously, we can define the systolic area σ_F , with the subscript F for Finsler, by taking the infimum in (2.1.2) over all Finsler metrics on M .

Contrary to the Riemannian case, there is no uniformization theorem for Finsler surfaces. As a result, the classical Riemannian tools to prove optimal systolic inequalities on surfaces, which are based on the conformal length method described above, do not carry over to the Finsler case. New methods are thus required to deal with Finsler metrics.

The first optimal Finsler systolic inequality has been obtained by S. Ivanov [Iv02, Iv11] who extended Pu's systolic inequality to Finsler projective planes.

Theorem 2.1.1 [Iv02, Iv11]. *Let \mathbb{RP}^2 be a Finsler projective plane. Then*

$$\frac{\text{vol}_{HT}(\mathbb{RP}^2)}{\text{sys}^2(\mathbb{RP}^2)} \geq \frac{2}{\pi}.$$

Furthermore, equality holds if all the geodesics are closed of the same length.

In particular, the systolic area of the projective plane is the same in the Riemannian and Finsler settings, that is,

$$\sigma_R(\mathbb{RP}^2) = \sigma_F(\mathbb{RP}^2) = \frac{2}{\pi}.$$

Note that Theorem 2.1.1 provides an alternate proof of Pu's inequality in the Riemannian case which does not rely on the uniformization theorem.

Using a different method based on [Gr99] and [BI02], a Finsler version of Loewner's inequality (2.1.1) has been obtained by the first author [Sa10].

Theorem 2.1.2 [Sa10]. *Let \mathbb{T}^2 be a Finsler two-torus. Then*

$$\frac{\text{vol}_{HT}(\mathbb{T}^2)}{\text{sys}^2(\mathbb{T}^2)} \geq \frac{2}{\pi}.$$

Equality holds if \mathbb{T}^2 is homothetic to the quotient of \mathbb{R}^2 , endowed with a parallelogram norm $\|\cdot\|$, by a lattice whose unit disk of $\|\cdot\|$ is a fundamental domain.

Observe that $\sigma_F(\mathbb{T}^2) = \sigma_F(\mathbb{RP}^2)$ contrary to the Riemannian case. An optimal Finsler systolic inequality holds for non-reversible Finsler metrics on \mathbb{T}^2 , cf. [ABT]. Note also that there is no systolic inequality for non-reversible Finsler two-tori if one considers the Busemann volume instead of the Holmes-Thompson volume, cf. [AB].

No systolic inequality holds for manifolds with boundary either. However, P. Pu [Pu52] and C. Blatter [Bl61] obtained optimal Riemannian systolic inequalities in each conformal class of the Mobius band and described the extremal metrics, cf. Section 2.4. Later, these inequalities were used by C. Bavard [Ba86] and T. Sakai [Sak88] in their proofs of the systolic inequality on the Klein bottle in the Riemannian case. The proof of the optimal conformal Riemannian systolic inequalities on the Mobius band relies on the uniformization theorem and the conformal length method (as in the original proofs of the Riemannian systolic inequalities on \mathbb{T}^2 , \mathbb{RP}^2 and \mathbb{K}^2).

In this chapter, we first prove a Finsler generalization of the optimal systolic inequality on \mathbb{T}^2 extending Loewner's inequality, cf. [Ke67], and derive further optimal geometric inequalities on Finsler cylinders, cf. Section 2.3. These results allow us to establish an optimal inequality on every Finsler Mobius band \mathbb{M} relating its systole $\text{sys}(\mathbb{M})$, its height $h(\mathbb{M})$ and its (Holmes-Thompson) volume $\text{vol}_{HT}(\mathbb{M})$ at least when \mathbb{M} is wide enough, cf. Section 2.5. Here, the height $h(\mathbb{M})$ represents the minimal length of arcs with endpoints on the boundary $\partial\mathbb{M}$, which are not homotopic to an arc in $\partial\mathbb{M}$, cf. Definition 2.3.4. More precisely, we prove the following.

Theorem 2.1.3. *Let \mathbb{M} be a Finsler Mobius band. Let $\lambda := \frac{h(\mathbb{M})}{\text{sys}(\mathbb{M})}$. Then*

$$\frac{\text{vol}_{HT}(\mathbb{M})}{\text{sys}(\mathbb{M}) h(\mathbb{M})} \geq \begin{cases} \frac{2}{\pi} & \text{if } \lambda \in (0, 1] \\ \frac{1}{\pi} \frac{\lambda+1}{\lambda} & \text{otherwise.} \end{cases} \quad (2.1.3)$$

Moreover, the above inequalities are optimal for every value of $\lambda \in (0, +\infty)$.

We describe extremal and almost extremal metric families in details in Section 2.4, Example 2.5.4 and Example 2.6.2.

The optimal Finsler systolic inequality on the Klein bottle is still unknown. However, based on the inequality (2.1.3) on Finsler Mobius bands, we obtain a partial result for Finsler Klein bottles with nontrivial symmetries. We refer to Definition 2.7.1 for a description of the symmetries considered in the statement of the following theorem.

Theorem 2.1.4. *Let \mathbb{K} be a Finsler Klein bottle with a soul, soul-switching or rotational symmetry. Then*

$$\frac{\text{vol}_{HT}(\mathbb{K})}{\text{sys}^2(\mathbb{K})} \geq \frac{2}{\pi}. \quad (2.1.4)$$

Moreover, the inequality is optimal.

We also present some extremal metric family in Example 2.7.7.

Finally, we present as a conjecture that the inequality (2.1.4) should hold for every Finsler Klein bottle with or without symmetries. That is, $\sigma_F(\mathbb{K})$ should be equal to $\frac{2}{\pi}$ (as $\sigma_F(\mathbb{T}^2)$ and $\sigma_F(\mathbb{RP}^2)$). Note, however, that the Finsler systolic area σ_F is not the same for all the surfaces: it goes to infinity with the genus of the surface, cf. [Gr83].

2.2 Preliminaries

In this section, we introduce general definitions regarding Finsler manifolds.

A (reversible) *Finsler metric* $F : TM \rightarrow [0, +\infty)$ on the tangent bundle TM of a smooth n -dimensional manifold M is a continuous function satisfying the following conditions (for simplicity, let $F_x := F|_{T_xM}$):

1. Smoothness: F is smooth outside the zero section;
2. Homogeneity: $F_x(tv) = |t|F_x(v)$ for every $v \in T_xM$ and $t \in \mathbb{R}$;
3. Quadratic convexity: for every $x \in M$, the function F_x^2 has positive definite second derivatives on $T_xM \setminus 0$, i.e., if $p, u, v \in T_xM$, the symmetric bilinear form

$$g_p(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} (F_x^2(p + tu + sv)) \Big|_{t=s=0}$$

is an inner product.

The pair (M, F) is called a *Finsler manifold*. If F is only positive homogeneous instead of homogeneous, that is, (2) only holds for $t \geq 0$, we say that the Finsler metric is non-reversible. For simplicity, we will only consider reversible Finsler metrics.

Conditions (1), (2) and (3) imply that F is strictly positive outside the zero section and that for every $x \in M$ and $u, v \in T_xM$, we have

$$F_x(u + v) \leq F_x(u) + F_x(v),$$

with equality if and only if $u = \lambda v$ or $v = \lambda u$ for some $\lambda \geq 0$, cf. [BCS00, Theorem 1.2.2]. Hence, F induces a strictly convex norm F_x on each tangent space T_xM with $x \in M$. More specifically, it gives rise to a Minkowski norm F_x on each tangent symmetric space T_xM . Working with quadratically convex norms and not merely (strictly) convex norms provides nice dynamical features such as a geodesic flow and a Legendre transform, cf. [Be78, §1].

As in the Riemannian case, notions of length, distance, and geodesics extend to Finsler geometry.

Let $\gamma : [a, b] \rightarrow M$ be a piecewise smooth curve. The length of γ is defined as

$$\ell(\gamma) := \int_a^b F(\dot{\gamma}(t)) dt.$$

By condition (2), $\ell(\gamma)$ does not depend on the parametrization of γ . Moreover, the functional ℓ gives rise to a distance function $d_F : M \times M \rightarrow [0, \infty)$ defined as $d_F(x, y) = \inf_{\gamma} \ell(\gamma)$, where the infimum is taken over all piecewise smooth curves γ joining x to y . A geodesic is a curve which locally minimizes length. It is a critical point of the energy functional $\gamma \mapsto \int F^2(\dot{\gamma}(t)) dt$ (here the quadratic convexity condition (3) is necessary).

For $x \in M$, we denote by B_x the unit ball of the Minkowski norm F_x . Furthermore, the norm F_x is Euclidean if and only if B_x is an ellipsoid. The dual of B_x is defined as $B_x^* = \{f \in T_x^*M \mid F_x^*(f) \leq 1\}$ where F_x^* is the dual norm of F_x . Note that B_x^* identifies with the polar body $B_x^\circ = \{u \in T_xM \mid \langle u, v \rangle \leq 1 \text{ for every } v \in B_x\}$ of B_x , where $\langle \cdot, \cdot \rangle$ is a given scalar product on T_xM .

In the Riemannian case, there exists a unique notion of volume, up to normalization, which agrees both with the n -dimensional Hausdorff measure determined by the Riemannian metric and with the projection of the Liouville measure from the unit tangent bundle, cf. [BBI01, §5.5]. However, in the Finsler case, there is no notion of volume that satisfies both properties, cf. [BI12]. This leads to two distinct notions of Finsler volume presented below.

Denote by ε_n the Euclidean volume of the Euclidean unit ball in \mathbb{R}^n . Let dx represent a given volume form on M and m be the restriction of this volume form to each tangent space T_xM . Similarly, let m^* be the restriction of the volume form dual to dx to each cotangent space T_x^*M . The *Busemann volume*, cf. [Bu47], is defined as

$$\text{vol}_B(M) := \int_M \frac{\varepsilon_n}{m(B_x)} dx. \quad (2.2.1)$$

The Busemann volume is sometimes called the Busemann-Hausdorff volume as it agrees with the n -dimensional Hausdorff measure of M (at least when the Finsler metric F is reversible). Another volume frequently used in Finsler geometry is the *Holmes-Thompson volume*, cf. [HT79], defined as

$$\text{vol}_{HT}(M) := \int_M \frac{m^*(B_x^*)}{\varepsilon_n} dx. \quad (2.2.2)$$

It is equal to the Liouville volume of its unit co-disc bundle divided by ε_n , cf. [AT04]. Note that the integrals in (2.2.1) and (2.2.2) do not depend on the chosen volume form. Since the volume is a local notion, it is possible to extend this definition even when M is nonorientable, that is, when volume forms do not exist on M .

In [Du98], C. Durán proved the following volume comparison inequality for Finsler manifolds:

$$\text{vol}_{HT}(M) \leq \text{vol}_B(M) \tag{2.2.3}$$

with equality if and only if M is Riemannian. Hence, every systolic inequality for the Holmes-Thompson volume remains true for the Busemann volume. However, the inequality (2.2.3) may fail for non-reversible Finsler metrics.

2.3 A systolic inequality on Finsler two-tori

In this section we establish a Finsler version of the Minkowski second theorem for the two-torus. More precisely, L. Keen proved the following.

Proposition 2.3.1 [Ke67], [Ka07] §6.2. *Let \mathbb{T}^2 be a Riemannian two-torus. There exist two closed curves of lengths a and b generating the first integral homology group of \mathbb{T}^2 such that*

$$\text{vol}_{HT}(\mathbb{T}^2) \geq \frac{\sqrt{3}}{2}ab.$$

Equality holds if and only if \mathbb{T}^2 is homothetic to the flat torus obtained as the quotient of \mathbb{R}^2 by a hexagonal lattice.

The proof of Proposition 2.3.1 relies on the uniformization theorem and the Cauchy-Schwarz inequality.

A Finsler version of Proposition 2.3.1 is given by the following result.

Proposition 2.3.2. *Let \mathbb{T}^2 be a Finsler two-torus. There exist two closed curves of lengths a and b generating the first integral homology group of \mathbb{T}^2 such that*

$$\text{vol}_{HT}(\mathbb{T}^2) \geq \frac{2}{\pi}ab.$$

Equality holds if \mathbb{T}^2 is homothetic to the quotient of \mathbb{R}^2 , endowed with a parallelogram norm $\|\cdot\|$, by a lattice generated by two vectors of lengths a and b , parallel to the sides of the unit ball of $\|\cdot\|$.

Since there is no uniformization theorem for Finsler metrics, the proof of this proposition differs from the proof of Proposition 2.3.1.

Proof. Let α be a systolic loop of \mathbb{T}^2 and β be the shortest closed curve of \mathbb{T}^2 homologically independent with α . Denote by a and b the lengths of α and β . The loops α and β are simple and intersect each other at a single point. Cutting \mathbb{T}^2 open along α and β gives rise to a surface Δ isometric to a fundamental domain of \mathbb{T}^2 . Let L be a positive number greater than $\max\{a, b\}$. Denote by p and q the smallest integers such that $pa \geq L$ and $qb \geq L$. Then, glue pq copies of Δ

in such a way that the resulting shape is isometric to the fundamental domain of a Finsler torus of volume pq times the volume of \mathbb{T}^2 and of systole equal to $\min\{pa, qb\}$. By construction, this new Finsler torus is a degree pq cover of \mathbb{T}^2 . Then, by Theorem 2.1.2, we have

$$pq \operatorname{vol}_{HT}(\mathbb{T}^2) \geq \frac{2}{\pi} (\min\{pa, qb\})^2.$$

Hence,

$$\operatorname{vol}_{HT}(\mathbb{T}^2) \geq \frac{2}{\pi} \frac{L}{p} \frac{L}{q} \geq \frac{2}{\pi} \frac{p-1}{p} \frac{q-1}{q} ab.$$

By choosing L large enough, the integers p and q can be made arbitrarily large, which leads to the desired inequality.

Now, if \mathbb{T}^2 is the quotient of \mathbb{R}^2 , endowed with a parallelogram norm, by a lattice generated by two vectors of lengths a and b which are parallel to the sides of the unit ball of the parallelogram norm, then $\operatorname{vol}_{HT}(\mathbb{T}^2) = \frac{2}{\pi} ab$. \square

Remark 2.3.3. Briefly speaking, the idea of the proof of Proposition 2.3.2 is to use finite covers to get a quasi-isosystolic two-torus (*i.e.*, whose first homology group is generated by two loops of lengths nearly the systole) and to apply the systolic inequality of Theorem 2.1.2 to this two-torus. This argument also applies in the Riemannian case and gives an alternative proof of Proposition 2.3.1 without the use of the uniformization theorem.

We can apply Proposition 2.3.2 to prove a systolic inequality on Finsler cylinders. First, we give the following definition

Definition 2.3.4. Let M be a compact Finsler surface with boundary. The *height* $h(M)$ of M is the minimal length of arcs with endpoints on the boundary ∂M , which are not homotopic to an arc in ∂M . More formally,

$$h(M) := \inf\{\ell(\gamma) \mid \gamma : [0, 1] \rightarrow M \text{ with } \gamma(0), \gamma(1) \in \partial M \text{ and } [\gamma] \neq 0 \in \pi_1(M, \partial M)\}.$$

A *height arc* of M is a length-minimizing arc γ of M with endpoints in ∂M inducing a non-trivial class in $\pi_1(M, \partial M)$. By definition, the length of a height arc of M is equal to $h(M)$.

Proposition 2.3.5. *Let C be a Finsler cylinder. Then,*

$$\operatorname{vol}(C) \geq \frac{2}{\pi} \operatorname{sys}(C) h(C).$$

Proof. Let k be a positive even integer. We glue k copies of C by identifying the identical boundary components pairwise. The resulting space is a torus \mathbb{T}^2 . Every loop of \mathbb{T}^2 non freely homotopic to a multiple of a boundary component of C is of length at least $k h(C)$. Therefore, for symmetry reasons, if k satisfies $k h(C) \geq \operatorname{sys}(C)$, the systole of the torus \mathbb{T}^2 is equal to the

systole of the cylinder C . Applying Proposition 2.3.2 to this torus, we derive

$$\text{vol}(\mathbb{T}^2) = k \text{vol}(C) \geq \frac{2}{\pi} k \text{sys}(C)h(C).$$

Hence the result. □

We will make use of Proposition 2.3.5 in the proof of Theorem 2.1.3 for wide Finsler Mobius bands, cf. Section 2.5.

2.4 Natural candidates for extremal metrics

In this section, we first review the extremal Riemannian metrics for systolic inequalities on the Mobius band and the Klein bottle presented in [Pu52, Bl61, Ba86, Ba88, Sak88]. By analogy with the Riemannian metrics, we construct Finsler metrics which are natural to consider when studying optimal Finsler systolic inequalities.

Consider the standard sphere S^2 . Denote by u and v the longitude and the latitude on S^2 . Let $a \in (0, \frac{\pi}{2})$. The a -tubular neighborhood of the equator $\{v = 0\}$ is a spherical band S_a which can be represented as

$$S_a := \{(u, v) \mid -\pi \leq u \leq \pi, -a \leq v \leq a\}.$$

The quotient of S_a by the antipodal map is a Riemannian Mobius band with curvature 1 denoted by \mathbb{M}_a . The conformal modulus space of the Mobius band is parametrized by \mathbb{M}_a with $a \in (0, \frac{\pi}{2})$. More precisely, every conformal class on the Mobius band agrees with the conformal structure induced by some \mathbb{M}_a with $a \in (0, \frac{\pi}{2})$. Furthermore, the conformal classes of the \mathbb{M}_a 's are pairwise distinct. Indeed, the (conformal) stereographic projection sends S_a to a planar annulus whose conformal modulus varies from 0 to ∞ as a runs over $(0, \frac{\pi}{2})$, cf. [Bl61].

The spherical Mobius bands \mathbb{M}_a are involved in several extremal conformal systolic inequalities for Riemannian metrics. More precisely, we define the *orientable systole* of a Riemannian Mobius band \mathbb{M} as the shortest length of a noncontractible orientable loop in \mathbb{M} . It will be denoted by $\text{sys}_+(\mathbb{M})$. Similarly, we define the *nonorientable systole* of \mathbb{M} and denote it by $\text{sys}_-(\mathbb{M})$. Observe that $\text{sys}(\mathbb{M}) = \min \{\text{sys}_+(\mathbb{M}), \text{sys}_-(\mathbb{M})\}$. Moreover, we define $\ell_v(\mathbb{M})$ as the minimal length of the arcs joining $(u, -a)$ to (u, a) in the (u, v) -coordinates of \mathbb{M}_a , which are homotopic with fixed endpoints to the projections of the meridians in S_a . For instance, $\text{sys}_+(\mathbb{M}_a) = 2\pi \cos(a)$, $\text{sys}_-(\mathbb{M}_a) = \pi$ and $\ell_v(\mathbb{M}_a) = 2a$. Note that the definition of ℓ_v relies on conformal data, namely the longitude-latitude coordinates to define the endpoints of the arcs involved in the length minimization.

In [Bl61], C. Blatter obtained optimal lower bounds for the functionals

$$\sigma_1 := \frac{\text{vol}}{\text{sys}_-^2}, \quad \sigma_2 := \frac{\text{vol}}{\text{sys}_- \times \ell_v} \quad \text{and} \quad \sigma_3 := \frac{\text{vol}}{\text{sys} \times \ell_v}$$

in each conformal class of the Mobius band. More precisely, for every Riemannian metric conformally equivalent to \mathbb{M}_a , we have the sharp lower bound

$$\sigma_1(\mathbb{M}) \geq \sigma_1(\mathbb{M}_a). \quad (2.4.1)$$

We also have the sharp inequality

$$\sigma_2(\mathbb{M}) \geq \begin{cases} \sigma_2(\mathbb{M}_a) & \text{if } a \in (0, b] \\ \sigma_2(\mathbb{M}_{\alpha(a)} \cup C_{a,\alpha(a)}) & \text{if } a \in [b, \frac{\pi}{2}] \end{cases} \quad (2.4.2)$$

where b is the unique solution in $(0, \frac{\pi}{2})$ of the equation $\tan(x) = 2x$ and $\mathbb{M}_{\alpha(a)} \cup C_{a,\alpha(a)}$ is the Mobius band obtained by attaching a flat cylinder $C_{a,\alpha(a)}$ to the spherical Mobius band $\mathbb{M}_{a,\alpha(a)}$ along their boundary. Here, the angle $\alpha(a) \in [b, a]$ is implicitly given by a nonlinear equation depending on the conformal type a and the flat cylinder $C_{a,\alpha(a)}$ is defined as the product $\partial_+ S_\alpha(a) \times [0, \sin a - \sin \alpha(a)]$, where $\partial_+ S_\alpha(a)$ is a boundary component of $S_\alpha(a)$. Alternately, $C_{a,\alpha(a)}$ is the Mercator projection of a connected component of $S_a \setminus S_\alpha(a)$ to the vertical cylinder generated by $\partial S_\alpha(a)$.

Finally, we have the third sharp inequality

$$\sigma_3(\mathbb{M}) \geq \begin{cases} \sigma_3(\mathbb{M}_a) & \text{if } a \in (0, \frac{\pi}{3}] \\ \sigma_3(\mathbb{M}_{\frac{\pi}{3}} \cup C_{a,\frac{\pi}{3}}) & \text{if } a \in [\frac{\pi}{3}, \frac{\pi}{2}] \end{cases} \quad (2.4.3)$$

With the help of (2.4.1), C. Bavard [Ba86] established the optimal isosystolic inequality on the Klein bottle. Later, T. Sakai [Sak88] used the inequalities (2.4.2) and (2.4.3) to give an alternative proof of Bavard's isosystolic inequality for the Klein bottle. The extremal Riemannian metric on the Klein bottle is obtained by gluing two copies of the spherical Mobius band $\mathbb{M}_{\frac{\pi}{4}}$ along their boundary.

The closed geodesics in S_a project down to systolic loops in \mathbb{M}_a which differ only by rotations and can be described as follows. Let γ_0^0 be the equator $\{v = 0\}$ of S^2 parametrized by arclength. Every great circle γ in S_a different from γ_0^0 intersects γ_0^0 at a unique point $\gamma_0^0(s)$ with $s \in [0, \pi)$. The great circles γ_0^0 and γ form at $\gamma_0^0(s)$ an oriented angle $\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\vartheta \neq 0$. Such great circle γ is denoted by γ_ϑ^s . From the sinus formula in spherical trigonometry, the great circle γ_ϑ^s exactly lies between the circles of latitude $\pm\vartheta$. Thus, $\vartheta \in [-a, a]$.

For $\vartheta > 0$, let $v \in (0, \vartheta]$. By Clairaut's relation, cf. [dC76, §4.4], the positive angle θ_ϑ between the circle of latitude v and the great circle γ_ϑ^s satisfies $\theta_\vartheta(v) = \arccos\left(\frac{\cos(\vartheta)}{\cos(v)}\right)$. In particular,

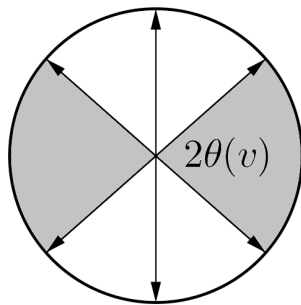


Figure 2.1: Systolic (in gray) and meridian directions in the unit tangent plane at a point of latitude v in \mathbb{M}_a .

the unit tangent vectors to the systolic loops of \mathbb{M}_a at a point of latitude v generate a symmetric cone of half angle

$$\theta(v) = \arccos\left(\frac{\cos(a)}{\cos(v)}\right)$$

in the tangent plane, *cf.* Figure 2.1. These unit tangent vectors of \mathbb{M}_a are referred to as *systolic directions*. The unit tangent vectors to the meridians are called *meridian directions*. Despite the risk of confusion, we will also call great circles of \mathbb{M}_a the projections of the great circles of S_a to \mathbb{M}_a .

The spherical Mobius band \mathbb{M}_a , which is extremal for some conformal systolic inequalities among Riemannian metrics, *cf.* (2.4.1), (2.4.2), and (2.4.3), is not extremal among Finsler metrics. Indeed, by slightly perturbing the quadratically convex norm in each tangent plane away from the systolic and meridian directions of the spherical metric, *cf.* Figure 2.1, we can decrease the area of the Mobius band without changing the systole and the height. This shows that any unit tangent vector to an extremal (quadratically convex) Finsler Mobius band is tangent either to a systolic loop or a height arc, *cf.* Definition 2.3.4. In other words, the unit tangent vectors induced by the systolic loops and the height arcs of an extremal (quadratically convex) Finsler Mobius band fill in its unit tangent bundle.

With this observation in mind, it is natural to consider the following (non-quadratically convex) Finsler metrics as potential extremal metrics. The idea is to adjust the shapes of the unit balls in the tangent bundle of the Mobius band so that the systolic and meridian directions fill in the unit tangent bundle. More precisely, define a Finsler metric F_a on S_a whose restriction to each tangent plane $T_x S_a$ is a norm $F_a|_x$ of unit ball B_x given by the convex hull of the systolic directions of \mathbb{M}_a , *cf.* Figure 2.2. In longitude and latitude coordinates, the ball B_x at $x = (u, v)$ can be represented as

$$B_x := \{(\xi_u, \xi_v) \in T_x S_a \mid \xi_u^2 + \xi_v^2 \leq 1, |\xi_v| \leq \sin \theta(v)\}.$$

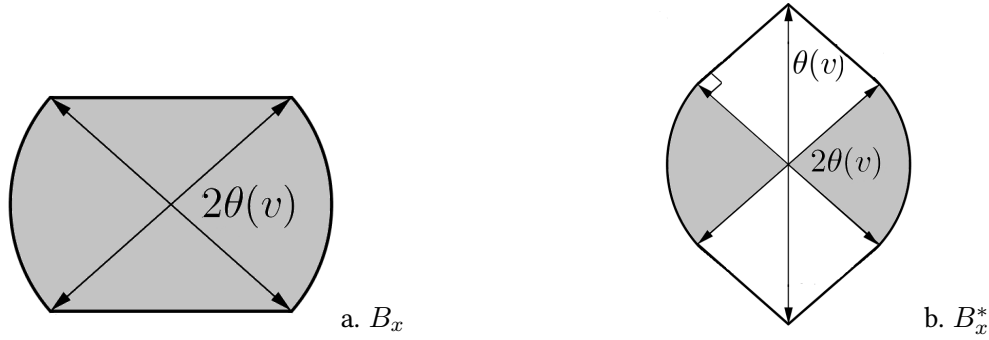


Figure 2.2: Unit ball B_x of F_a and its dual B_x^* at a point x of latitude v in S_a .

Hence, the Finsler metric F_a can be represented in local coordinates as

$$F_a = \begin{cases} \frac{1}{\sin(\theta(v))} |dv| & \text{if } \arctan\left(\frac{dv}{du}\right) \in [\theta(v), \frac{\pi}{2}] \\ \sqrt{du^2 + dv^2} & \text{if } \arctan\left(\frac{dv}{du}\right) \in [0, \theta(v)] \end{cases}$$

This metric passes to the quotient by the antipodal map to a Finsler metric still denoted by F_a . Denote by \mathbb{M}_{F_a} the Finsler Mobius band so obtained.

Instead of considering ℓ_v , whose definition relies on conformal data, we will work with the height $h(\mathbb{M})$, cf. Definition 2.3.4, in the Finsler case.

Some geometric features of the Finsler Mobius bands \mathbb{M}_{F_a} are summarized in the following two propositions.

Proposition 2.4.1. *Let $a \in (0, \frac{\pi}{2})$. Then, $\text{sys}_-(\mathbb{M}_{F_a}) = \pi$, $\text{sys}_+(\mathbb{M}_{F_a}) = 2\pi \cos(a)$ and $h(\mathbb{M}_{F_a}) = \pi$.*

In particular, if $a = \frac{\pi}{3}$ then $\text{sys}(\mathbb{M}_{F_a}) = \text{sys}_+(\mathbb{M}_{F_a}) = \text{sys}_-(\mathbb{M}_{F_a}) = h(\mathbb{M}_{F_a}) = \pi$.

Proof. Let us start with a useful observation. Denote by \mathcal{S} the interior of the domain of $U\mathbb{M}_{F_a}$ formed by the unit tangent vectors of the great circles of \mathbb{M}_a . The Finsler metric F_a coincides with the round Riemannian metric of \mathbb{M}_a on \mathcal{S} . Therefore, the subset \mathcal{S} is stable under the geodesic flow of F_a (which is well-defined on \mathcal{S}). Furthermore, the length of a great circle with respect to F_a is equal to π .

Let us show that $h(\mathbb{M}_{F_a}) = \pi$. Consider a height arc γ of \mathbb{M}_{F_a} . The arc γ can be parametrized with respect to the latitude. Otherwise, we could remove a subarc of γ joining two points at the same latitude and still make up an arc in the same relative homotopy class as γ with the remaining pieces using the rotational symmetry of \mathbb{M}_{F_a} . This would contradict the length-minimizing

property of γ . Hence,

$$\begin{aligned}
 h(\mathbb{M}_{F_a}) &= \ell(\gamma) \\
 &= \int_{-a}^a \frac{1}{\sin \theta(v)} dv \\
 &= \int_{-a}^a \frac{\cos(v)}{\sqrt{\cos^2(v) - \cos^2(a)}} dv \\
 &= 2 \arctan \left(\frac{\sqrt{2} \sin(v)}{\sqrt{\cos(2v) - \cos(2a)}} \right) \Big|_0^a \\
 &= \pi.
 \end{aligned}$$

Now, let us show that the systolic curves of \mathbb{M}_{F_a} agree with the great circles of \mathbb{M}_a in the nonorientable case and with the boundary of \mathbb{M}_a in the orientable case. Consider an orientable or nonorientable noncontractible loop γ of minimal length in \mathbb{M}_{F_a} .

If γ lies in the boundary of \mathbb{M}_{F_a} then the loop γ is orientable of length $2\pi \cos(a)$. Thus, we can assume that γ passes through an interior point p of \mathbb{M}_{F_a} .

If a tangent vector of γ lies in \mathcal{S} then the geodesic arc γ coincides with a great circle of \mathbb{M}_a in the nonorientable case and with a great circle run twice in the orientable case. (Recall that \mathcal{S} is stable by the geodesic flow of F_a .) In the former case, the curve γ is of length π , while in the latter, it is of length 2π . Thus, we can assume that the tangent vectors of γ do not lie in \mathcal{S} .

Consider the closed lift $\bar{\gamma}$ of γ in S_a . Let c_{\pm} be the two extreme great circles of S_a passing through the lifts of p and tangent to the boundary of S_a . That is, c_{\pm} are the great circles of S_a making an angle of $\pm\theta(v)$ with the curves of constant latitude $\pm v$ in S_a passing through the lifts of p . Since the tangent vectors of γ do not lie in \mathcal{S} , the curve $\bar{\gamma}$ does not intersect c_{\pm} in the interior of S_a , except at the lifts of p . Therefore, there exists a subarc of $\bar{\gamma}$ (actually two subarcs of $\bar{\gamma}$) joining the two boundary components of S_a in the region delimited by the great circles c_{\pm} and the boundary of S_a , see the gray region of Figure 2.3. Thus, $\ell(\gamma) \geq h(\mathbb{M}_{F_a}) = \pi$ with equality if γ agrees with c_{\pm} .

We conclude that $\text{sys}_-(\mathbb{M}_{F_a}) = \pi$ and $\text{sys}_+(\mathbb{M}_{F_a}) = \ell(\partial\mathbb{M}_{F_a}) = 2\pi \cos(a)$. Hence,

$$\text{sys}(\mathbb{M}_{F_a}) = \begin{cases} \text{sys}_-(\mathbb{M}_{F_a}) & \text{if } a \in (0, \frac{\pi}{3}] \\ \text{sys}_+(\mathbb{M}_{F_a}) & \text{if } a \in [\frac{\pi}{3}, \frac{\pi}{2}) \end{cases}$$

□

Remark 2.4.2. The Finsler Mobius bands \mathbb{M}_{F_a} are not pairwise isometric since they have distinct orientable systoles.

Proposition 2.4.3. *Let $a \in (0, \frac{\pi}{2})$. Then, $\text{vol}_{HT}(\mathbb{M}_{F_a}) = 2\pi$.*

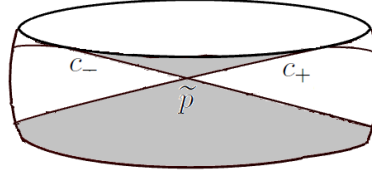


Figure 2.3: The extreme great circles c_{\pm} passing through a lift \tilde{p} of p in S_a .

Proof. The unit ball B_x^* coincides with the polar body of B_x described in Figure ?? . The area of B_x^* is equal to $2\theta(v) + \frac{2}{\tan\theta(v)}$. By definition of the Holmes-Thompson volume, cf. (2.2.2), we have

$$\begin{aligned} \text{vol}_{HT}(\mathbb{M}_{F_a}) &= \frac{1}{2\pi} \int_{S_a} m(B_x^*) dm(x) \\ &= \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-a}^a \left(\theta(v) + \frac{1}{\tan\theta(v)} \right) \cos(v) du dv \\ &= \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-a}^a \left(\arccos\left(\frac{\cos(a)}{\cos(v)}\right) + \frac{\cos(a)}{\sqrt{\cos^2(v) - \cos^2(a)}} \right) \cos(v) du dv \\ &= 2\pi. \end{aligned}$$

□

Remark 2.4.4. As a consequence of Propositions 2.4.1 and 2.4.3, we observe the following couple of points.

1. The orientable and nonorientable systoles of F_a agree with those of its dual F_a^* . Hence, $\text{sys}(\mathbb{M}_{F_a}) = \text{sys}(\mathbb{M}_{F_a^*})$. Moreover, computations similar to those in Propositions 2.4.1 and 2.4.3 show that $h(\mathbb{M}_{F_a^*}) = \pi(1 - \cos(a))$ and $\text{vol}_{HT}(\mathbb{M}_{F_a^*}) = 2\pi \sin^2(a)$. This means that for both F_a and its dual F_a^* , we have $\frac{\text{vol}_{HT}(\mathbb{M})}{\text{sys}_-(\mathbb{M})h(\mathbb{M})} \rightarrow \frac{2}{\pi}$, when $a \rightarrow \frac{\pi}{2}$.
2. The Finsler Mobius bands \mathbb{M}_{F_a} with $a \in (0, \frac{\pi}{3}]$ attain the equality case in (2.1.3) when $\text{sys}(\mathbb{M}) = h(\mathbb{M})$.

2.5 Systolic inequalities on wide Finsler Mobius bands

In this section, we give a proof of Theorem 2.1.3 for wide Finsler Mobius bands, that is, when $\lambda \geq 1$. More precisely, we prove the following result.

Proposition 2.5.1. *Let \mathbb{M} be a Finsler Mobius band with $h(\mathbb{M}) \geq \text{sys}(\mathbb{M})$. Then*

$$\text{vol}_{HT}(\mathbb{M}) \geq \frac{1}{\pi} \text{sys}(\mathbb{M}) (\text{sys}(\mathbb{M}) + h(\mathbb{M})). \quad (2.5.1)$$

We present examples showing this result is optimal at the end of this section, cf. Example 2.5.4.

Proof. Consider

$$\mathbb{U} := \{x \in \mathbb{M} \mid d(x, \partial\mathbb{M}) \leq \frac{\lambda - 1}{2} \text{sys}(\mathbb{M})\}.$$

Slightly perturbing the distance function $d(\cdot, \partial\mathbb{M})$ if necessary, we can assume that this distance function is a Morse function on \mathbb{M} for which $\frac{\lambda-1}{2} \text{sys}(\mathbb{M})$ is a regular value. In this case, \mathbb{U} is a surface with boundary. If \mathbb{M} has some “big bumps”, the surface \mathbb{U} may possibly have some holes. More precisely, the surface \mathbb{U} may not be a topological cylinder as some of its boundary components may bound topological disks in \mathbb{M} .

Let $\widehat{\mathbb{U}}$ be the union of \mathbb{U} with the topological disks of \mathbb{M} bounded by the boundary components of \mathbb{U} . Under this construction, $\widehat{\mathbb{U}}$ is a cylinder one of whose boundary components agrees with $\partial\mathbb{M}$. Clearly, the height of $\widehat{\mathbb{U}}$ is equal to $\frac{\lambda-1}{2} \text{sys}(\mathbb{M})$. Furthermore, since the inclusion $\widehat{\mathbb{U}} \subset \mathbb{M}$ induces a π_1 -isomorphism, we have $\text{sys}(\widehat{\mathbb{U}}) \geq \text{sys}(\mathbb{M})$. Applying Proposition 2.3.5 to the cylinder $\widehat{\mathbb{U}}$ yields

$$\text{vol}_{HT}(\widehat{\mathbb{U}}) \geq \frac{\lambda - 1}{\pi} \text{sys}(\mathbb{M})^2. \quad (2.5.2)$$

Now, consider the Finsler Mobius band $\mathbb{M}_- := \mathbb{M} \setminus \widehat{\mathbb{U}}$.

Lemma 2.5.2. *The height and systole of \mathbb{M}_- satisfy*

$$h(\mathbb{M}_-) = \text{sys}(\mathbb{M}) \quad \text{and} \quad \text{sys}(\mathbb{M}_-) \geq \text{sys}(\mathbb{M}).$$

Proof. Let γ_- be a height arc of \mathbb{M}_- , cf. Definition 2.3.4. By construction, $\partial\widehat{\mathbb{U}} = \partial\mathbb{M}_- \cup \partial\mathbb{M}$ and the points of $\partial\mathbb{M}_-$ are at distance $\frac{\lambda-1}{2} \text{sys}(\mathbb{M})$ from $\partial\mathbb{M}$. Therefore, the two endpoints of γ_- can be connected to $\partial\mathbb{M}$ by two arcs γ_1 and γ_2 of $\widehat{\mathbb{U}}$, each of length $\frac{\lambda-1}{2} \text{sys}(\mathbb{M})$. Moreover, the arc $\gamma := \gamma_- \cup \gamma_1 \cup \gamma_2$ with endpoints in $\partial\mathbb{M}$ induces a nontrivial class in $\pi_1(\mathbb{M}, \partial\mathbb{M})$. Therefore, since $h(\mathbb{M}) = \lambda \text{sys}(\mathbb{M})$, we obtain

$$\begin{aligned} h(\mathbb{M}_-) &= \ell(\gamma) - (\lambda - 1) \text{sys}(\mathbb{M}) \\ &\geq h(\mathbb{M}) - (\lambda - 1) \text{sys}(\mathbb{M}) \\ &\geq \text{sys}(\mathbb{M}). \end{aligned}$$

Now, let γ be a height arc of \mathbb{M} . By definition, we have $\ell(\gamma) = h(\mathbb{M}) = \lambda \text{sys}(\mathbb{M})$. The part $\gamma \cap \widehat{\mathbb{U}}$ of γ in $\widehat{\mathbb{U}}$ is made of two arcs, each of length at least $\frac{\lambda-1}{2} \text{sys}(\mathbb{M})$. Moreover, the arc $\gamma \cap \mathbb{M}_-$ with endpoints in $\partial\mathbb{M}_-$ induces a nontrivial class in $\pi_1(\mathbb{M}_-, \partial\mathbb{M}_-)$. Hence,

$$\begin{aligned} h(\mathbb{M}_-) &\leq \ell(\gamma) - \ell(\gamma \cap \widehat{\mathbb{U}}) \\ &\leq h(\mathbb{M}) - (\lambda - 1) \text{sys}(\mathbb{M}) \\ &\leq \text{sys}(\mathbb{M}). \end{aligned}$$

Since the inclusion $\mathbb{M}_- \subset \mathbb{M}$ induces a π_1 -isomorphism, we obtain

$$\text{sys}(\mathbb{M}_-) \geq \text{sys}(\mathbb{M}).$$

□

Consider the projective plane \mathbb{RP}^2 defined as the quotient $\mathbb{M}_-/\partial\mathbb{M}_-$, where the boundary $\partial\mathbb{M}_-$ is collapsed to a point. Strictly speaking, the Finsler metric on \mathbb{RP}^2 has a singularity at the point to which $\partial\mathbb{M}_-$ collapses, but we can smooth it out.

The following result allows us to derive the systole of \mathbb{RP}^2 .

Lemma 2.5.3. *Let \mathbb{RP}^2 be the projective plane defined as the quotient $\mathbb{M}/\partial\mathbb{M}$ of a Finsler Mobius band \mathbb{M} . Then,*

$$\text{sys}(\mathbb{RP}^2) = \min \{h(\mathbb{M}), \text{sys}(\mathbb{M})\}$$

where \mathbb{RP}^2 is endowed with the quotient metric.

Proof. Let γ be a noncontractible loop in \mathbb{RP}^2 . The curve γ lifts either to a noncontractible loop in \mathbb{M} or to a noncontractible arc in \mathbb{M} joining two points of the boundary $\partial\mathbb{M}$. In the former case, the length of γ is at least $\text{sys}(\mathbb{M})$, while in the latter, it is at least $h(\mathbb{M})$. On the other hand, we can easily construct noncontractible loops in \mathbb{RP}^2 of length $\text{sys}(\mathbb{M})$ or $h(\mathbb{M})$. □

From Lemma 2.5.3 and Lemma 2.5.2, the systole of \mathbb{RP}^2 is equal to $\text{sys}(\mathbb{M})$. Applying Theorem 2.1.1 to \mathbb{RP}^2 , we obtain

$$\text{vol}_{HT}(\mathbb{M}_-) = \text{vol}_{HT}(\mathbb{RP}^2) \geq \frac{2}{\pi} \text{sys}^2(\mathbb{M}).$$

This inequality combined with (2.5.2) yields

$$\begin{aligned} \text{vol}_{HT}(\mathbb{M}) &= \text{vol}_{HT}(\widehat{\mathbb{U}}) + \text{vol}_{HT}(\mathbb{M}_-) \\ &\geq \frac{1+\lambda}{\pi} \text{sys}(\mathbb{M})^2. \end{aligned}$$

Hence the result. □

We conclude this section by describing extremal and almost extremal Finsler metrics when $\lambda \geq 1$.

Example 2.5.4. Let $\lambda \in [1, +\infty)$.

(E.1) The horizontal translation τ of vector πe_x is an isometry of the plane \mathbb{R}^2 endowed with the sup-norm. The quotient of the strip $\mathbb{R} \times [0, (\lambda - 1)\frac{\pi}{2}]$ by the isometry group $\langle \tau \rangle$ generated by τ is a cylinder C . The Finsler mobius band \mathbb{M} obtained by gluing a boundary component of C to \mathbb{M}_{F_a} along $\partial\mathbb{M}_{F_a}$ with $a = \frac{\pi}{3}$, cf. Section 2.4, satisfies $\text{vol}_{HT}(\mathbb{M}) = (1 + \lambda)\pi$, $\text{sys}(\mathbb{M}) = \pi$ and $h(\mathbb{M}) = \lambda\pi$. See Figure 2.4.(A).

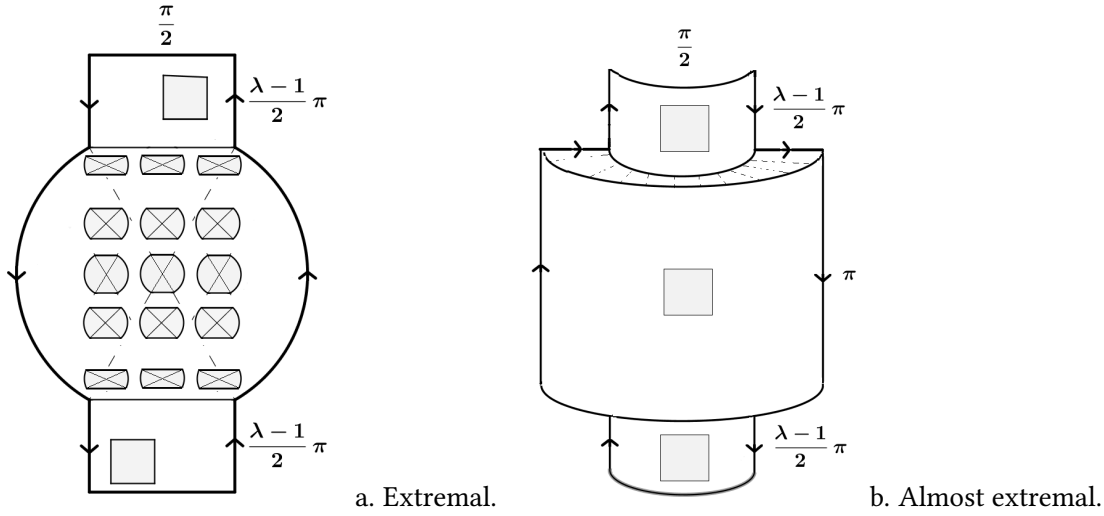


Figure 2.4: Almost extremal Finsler Mobius bands when $h(\mathbb{M}) \geq \text{sys}(\mathbb{M})$.

(E.2) Endow the plane \mathbb{R}^2 with the sup-norm. The quotient of the strip $\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ by the group generated by the map $(x, y) \mapsto (x + \pi, -y)$ is a Finsler Mobius band \mathbb{M}_π with $\text{vol}_{HT}(\mathbb{M}_\pi) = 2\pi$, $\text{sys}(\mathbb{M}_\pi) = \pi$ and $h(\mathbb{M}_\pi) = \pi$. Let C be the Finsler cylinder defined in (E.1). Attach C to \mathbb{M}_π via a cylindrical part of arbitrarily small area, cf. Figure 2.4.(B), so that the resulting space is a Finsler Mobius band \mathbb{M} with $\text{sys}(\mathbb{M}) = \pi$, $h(\mathbb{M}) = \nu_1 \lambda \pi$ and $\text{vol}_{HT}(\mathbb{M}) = \nu_2(\lambda + 1)\pi$, where $\nu_1, \nu_2 > 1$ are arbitrarily close to 1. This Finsler Mobius band is almost extremal for the inequality (2.5.1) when $\lambda \geq 1$.

2.6 Systolic inequalities on narrow Finsler Mobius bands

In this section, we give a proof of Theorem 2.1.3 for narrow Finsler Mobius bands, that is, when $\lambda < 1$. More precisely, we prove the following result.

Proposition 2.6.1. *Let \mathbb{M} be a Finsler Mobius band with $h(\mathbb{M}) < \text{sys}(\mathbb{M})$. Then*

$$\text{vol}_{HT}(\mathbb{M}) \geq \frac{2}{\pi} \text{sys}(\mathbb{M}) h(\mathbb{M}).$$

This inequality is optimal. Extremal Finsler metrics can be constructed as follows.

Example 2.6.2. Let $\lambda < 1$. Endow the plane \mathbb{R}^2 with the sup-norm. The quotient of the strip $\mathbb{R} \times [-\frac{\pi}{2}\lambda, \frac{\pi}{2}\lambda]$ by the group generated by the map $(x, y) \mapsto (x + \pi, -y)$ is a Finsler Mobius band \mathbb{M} with $\text{vol}_{HT}(\mathbb{M}) = 2\lambda\pi$, $\text{sys}(\mathbb{M}) = \pi$ and $h(\mathbb{M}) = \lambda\pi$.

Before proceeding to the proof of this proposition, we need to introduce a few definitions and notions.

Definition 2.6.3. The orientable double cover of a Finsler Mobius band \mathbb{M} is a cylinder denoted by C . The points of C which are at the same distance from each boundary component of C form a simple closed curve c invariant under deck transformations. The *soul* of \mathbb{M} , denoted by σ , is defined as the projection of c to \mathbb{M} . Note that σ is a nonorientable simple closed curve of \mathbb{M} whose homotopy class generates $\pi_1(\mathbb{M})$.

Let \mathbb{K} be the Finsler Klein bottle obtained by attaching to \mathbb{M} another copy \mathbb{M}' of \mathbb{M} along their boundary. (The Finsler metric on \mathbb{K} may have a singularity line along $\partial\mathbb{M}$.) The isometry of \mathbb{K} switching the souls of \mathbb{M} and \mathbb{M}' , and leaving $\partial\mathbb{M}$ pointwise fixed is called the *soul-switching symmetry* of \mathbb{K} .

Let $s, h \in \mathbb{R}_+$. The *systole-height inequality* on the Mobius band is said to be satisfied for (s, h) if for every Finsler Mobius band \mathbb{M} with $\text{sys}(\mathbb{M}) \geq s$ and $h(\mathbb{M}) \geq h$, we have

$$\text{vol}_{HT}(\mathbb{M}) \geq \frac{2}{\pi} sh.$$

By scale invariance, if the systole-height inequality is satisfied for (s, h) , then it is also satisfied for (s', h') with $\frac{h'}{s'} = \frac{h}{s}$.

We first prove the following preliminary result.

Lemma 2.6.4. *Let $\lambda \in (0, 1]$. Suppose that the systole-height inequality on the Mobius band is satisfied for (s, h) with $\frac{h}{s} = \lambda$. Then, it is also satisfied for (s, h) with $\frac{h}{s} = \frac{\lambda}{2}$.*

Proof. Let \mathbb{M} be a Finsler Mobius band with $\text{sys}(\mathbb{M}) \geq s$ and $h(\mathbb{M}) \geq h$, where $\frac{s}{h} = \frac{\lambda}{2}$. Consider the Klein bottle \mathbb{K} made of two copies of \mathbb{M} defined in Definition 2.6.3 and cut it open along the soul σ' of \mathbb{M}' . The resulting surface is a Finsler Mobius band denoted by $2\mathbb{M}$ whose boundary component double covers σ' in \mathbb{K} .

Let α be a noncontractible loop of $2\mathbb{M}$. Decompose α into two parts $a = \alpha \cap \mathbb{M}$ and $a' = \alpha \cap \mathbb{M}'$ with $\alpha = a \cup a'$. The parts a and a' form two collections of arcs with endpoints lying in $\partial\mathbb{M} = \partial\mathbb{M}'$. By construction, the image a'' of a' by the soul-switching symmetry lies in \mathbb{M} . Furthermore, the union $\bar{\alpha} = a \cup a''$ forms a closed curve lying in \mathbb{M} and homotopic to α in $2\mathbb{M}$ (and so noncontractible in \mathbb{M}). Since $\bar{\alpha}$ has the same length as α , we conclude that $\text{sys}(2\mathbb{M}) \geq \text{sys}(\mathbb{M}) \geq s$.

Actually, since the inclusion $\mathbb{M} \subset 2\mathbb{M}$ is a strong deformation retract, we derive the relation $\text{sys}(2\mathbb{M}) = \text{sys}(\mathbb{M})$. But we will not make use of this equality in the sequel.

By construction, the distance between the soul σ and $\partial\mathbb{M}$ (and between σ' and $\partial\mathbb{M}'$) is at least $\frac{1}{2}h(\mathbb{M})$. This implies that $h(2\mathbb{M}) \geq 2h(\mathbb{M}) \geq 2h$.

Actually, we can show that $h(2\mathbb{M}) = 2h(\mathbb{M})$ (but we will not make use of this relation afterwards). Indeed, let α be a height arc of \mathbb{M} . By definition, $\ell(\alpha) = h(\mathbb{M})$. Denote by α' its image

in \mathbb{M}' by the soul-switching symmetry of \mathbb{K} . The trace of the union $\alpha \cup \alpha'$ to $2\mathbb{M}$ defines an arc with endpoints in $\partial(2\mathbb{M})$ inducing a nontrivial class in $\pi_1(2\mathbb{M}, \partial(2\mathbb{M}))$. The length of this arc is twice the length of α . Therefore, $h(2\mathbb{M}) \leq 2h(\mathbb{M})$. Hence, the equality $h(2\mathbb{M}) = 2h(\mathbb{M})$.

In conclusion, the Mobius band $2\mathbb{M}$ satisfies $\text{sys}(2\mathbb{M}) \geq s$ and $h(2\mathbb{M}) \geq 2h$. Since $\frac{2h}{s} = \lambda$, the systole-height inequality is satisfied for $(s, 2h)$ by the lemma assumption. Therefore,

$$2 \text{vol}_{HT}(\mathbb{M}) = \text{vol}_{HT}(2\mathbb{M}) \geq \frac{4}{\pi} sh$$

and the result follows. \square

We establish a second preliminary result.

Lemma 2.6.5. *Let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $0 < \lambda_1 < \lambda_2 \leq 1$. Suppose that the systole-height inequality on the Mobius band is satisfied for (s, h) with $\frac{h}{s} = \lambda_1$ or λ_2 . Then, it is also satisfied for (s, h) with $\frac{h}{s} = \frac{\lambda_1 + \lambda_2}{2}$.*

Proof. Let \mathbb{M} be a Finsler Mobius band with $\text{sys}(\mathbb{M}) \geq s$ and $h(\mathbb{M}) \geq h$, where $\frac{h}{s} = \frac{\lambda_1 + \lambda_2}{2}$. Consider the Klein bottle \mathbb{K} made of two isometric Mobius bands \mathbb{M} and \mathbb{M}' with souls σ and σ' defined in Definition 2.6.3. Consider also

$$\mathbb{M}_1 = \{x \in \mathbb{K} \mid \lambda_2 d(x, \sigma) \leq \lambda_1 d(x, \sigma')\}$$

and

$$\mathbb{M}_2 = \{x \in \mathbb{K} \mid \lambda_2 d(x, \sigma) \geq \lambda_1 d(x, \sigma')\}$$

Note that if we drop the multiplicative constants λ_1 and λ_2 in the definitions of \mathbb{M}_1 and \mathbb{M}_2 , we obtain \mathbb{M} and \mathbb{M}' .

The subset \mathbb{M}_1 is a Finsler Mobius band contained in \mathbb{M} . Similarly, the subset \mathbb{M}_2 is a Finsler Mobius band containing \mathbb{M}' . Observe also that the Mobius bands \mathbb{M}_1 and \mathbb{M}_2 cover \mathbb{K} and that their interiors are disjoint.

Every point $z \in \partial\mathbb{M}_i$ satisfies the equality

$$\lambda_2 d(z, \sigma_1) = \lambda_1 d(z, \sigma_2). \tag{2.6.1}$$

By symmetry, the distance between $\sigma_1 = \sigma$ and $\sigma_2 = \sigma'$ is equal to $h(\mathbb{M})$. It follows by the triangle inequality that

$$d(z, \sigma_1) + d(z, \sigma_2) \geq h(\mathbb{M}). \tag{2.6.2}$$

As a result of the relations (2.6.1) and (2.6.2), we obtain

$$d(z, \sigma_i) \geq \frac{\lambda_i}{\lambda_1 + \lambda_2} h(\mathbb{M}). \tag{2.6.3}$$

Now, let α be an arc of \mathbb{M}_i with endpoints $x, y \in \partial\mathbb{M}_i$ inducing a nontrivial class in $\pi_1(\mathbb{M}_i, \partial\mathbb{M}_i)$. As α intersects σ_i , we deduce from (2.6.3) that

$$\ell(\alpha) \geq d(x, \sigma_i) + d(y, \sigma_i) \geq \frac{2\lambda_i}{\lambda_1 + \lambda_2} h(\mathbb{M}).$$

Therefore,

$$h(\mathbb{M}_i) \geq \frac{2\lambda_i}{\lambda_1 + \lambda_2} h(\mathbb{M}) \geq \frac{2\lambda_i}{\lambda_1 + \lambda_2} h. \quad (2.6.4)$$

In another direction, we can also bound from below the systole of \mathbb{M}_1 and \mathbb{M}_2 as follows.

For the systole of \mathbb{M}_1 , since the inclusion $\mathbb{M}_1 \subset \mathbb{M}$ induces a π_1 -isomorphism, we derive

$$\text{sys}(\mathbb{M}_1) \geq \text{sys}(\mathbb{M}) \geq s. \quad (2.6.5)$$

Note that the first inequality may be strict as the inclusion $\mathbb{M}_1 \subset \mathbb{M}$ is strict.

For the systole of \mathbb{M}_2 , we argue as in Lemma 2.6.4. Let α be a noncontractible loop of \mathbb{M}_2 . Decompose α into two parts $a = \alpha \cap \mathbb{M}$ and $a' = \alpha \cap \mathbb{M}'$ with $\alpha = a \cup a'$. The union $\bar{\alpha} = a^* \cup a'$, where a^* is the image of a by the soul-switching symmetry of \mathbb{K} , forms a closed curve of length $\ell(\alpha)$ lying in \mathbb{M}' and homotopic to α in \mathbb{M}_2 . Hence,

$$\text{sys}(\mathbb{M}_2) \geq \text{sys}(\mathbb{M}) \geq s. \quad (2.6.6)$$

The systole-height inequality on the Mobius band is satisfied for $(s, \frac{2\lambda_i}{\lambda_1 + \lambda_2} h)$ from the lemma assumption since $\frac{2\lambda_i}{\lambda_1 + \lambda_2} \frac{h}{s} = \lambda_i$. From the bounds (2.6.4), (2.6.5) and (2.6.6), this inequality applies to \mathbb{M}_i and yields

$$\text{vol}_{HT}(\mathbb{M}_i) \geq \frac{4}{\pi} \frac{\lambda_i}{\lambda_1 + \lambda_2} sh. \quad (2.6.7)$$

Finally, recall that the Mobius bands \mathbb{M}_1 and \mathbb{M}_2 cover \mathbb{K} and that their interiors are disjoint. By adding up (2.6.7) for $i = 1, 2$, we conclude that

$$\begin{aligned} 2 \text{vol}_{HT}(\mathbb{M}) &= \text{vol}_{HT}(\mathbb{K}) \\ &= \text{vol}_{HT}(\mathbb{M}_1) + \text{vol}_{HT}(\mathbb{M}_2) \geq \frac{4}{\pi} sh \end{aligned}$$

Hence the result. □

Remark 2.6.6. At first glance, it seems more natural to assume that $\text{sys}(\mathbb{M}) = s$ and $h(\mathbb{M}) = h$ in the definition of the systole-height inequality, cf. Definition 2.6.3. Observe that the proof of Lemma 2.6.4 carries over with this alternative notion. However, we have not been able to directly prove a result similar to Lemma 2.6.5 with this more restrictive notion. The reason is that the inequality (2.6.5), namely $\text{sys}(\mathbb{M}_1) \geq \text{sys}(\mathbb{M})$, may be strict as the inclusion $\mathbb{M}_1 \subset \mathbb{M}$ is strict. To get around this subtle difficulty, we relaxed the original definition and formulated the

systole-height inequality in terms of lower bounds for the systole and the height of the Mobius band.

We can now proceed to the proof of Proposition 2.6.1.

Proof of Proposition 2.6.1. By Lemma 2.6.4, for every nonnegative integer k , the systole-height inequality on the Mobius band is satisfied for (s, h) with $\frac{h}{s} = \frac{1}{2^k}$. Combined with Lemma 2.6.5, this implies that the systole-height inequality is satisfied for every (s, h) where $\frac{h}{s}$ is a dyadic rational of $(0, 1)$. Since the height, the systole and the volume are continuous over Finsler metrics, the result follows from the density of the dyadic rationals in $[0, 1]$. \square

2.7 Systolic inequality on Finsler Klein bottles

In this section, we show that the systolic area of Finsler Klein bottles with soul, soul-switching or rotational symmetries is equal to $\frac{2}{\pi}$.

Definition 2.7.1. Recall that every Riemannian Klein bottle is conformally equivalent to the quotient of \mathbb{R}^2 by the isometry group G generated by the glide reflection $(x, y) \mapsto (x + \pi, -y)$ and the vertical translation $(x, y) \mapsto (x, y + 2b)$ with $b > 0$.

The flat Klein bottle $\mathbb{K} = \mathbb{R}^2/G$ decomposes into two Mobius bands whose souls correspond to the projections of the lines $\{y = 0\}$ and $\{y = b\}$. The boundary of the two Mobius bands agrees with the projection of the line $\{y = \frac{b}{2}\}$ (or $\{y = -\frac{b}{2}\}$). A Finsler metric on \mathbb{K} has a *soul symmetry* if its lift to \mathbb{R}^2 is invariant by the map $(x, y) \mapsto (x, -y)$. Similarly, a Finsler metric on \mathbb{K} has a *soul-switching symmetry* if its lift to \mathbb{R}^2 is invariant by the map $(x, y) \mapsto (x, b - y)$. Finally, a Finsler metric on \mathbb{K} has a *rotational symmetry* if its lift to \mathbb{R}^2 is invariant by the map $(x, y) \mapsto (x + \theta, y)$ for every $\theta \in [0, 2\pi]$.

These definitions are consistent with the notions introduced in 2.6.

In 1986, C. Bavard established an optimal isosystolic inequality for Riemannian Klein bottles, cf. [Ba86]. Alternative proofs can be found in [Sak88, Ba88, Ba06]. All the proofs are based on the uniformization theorem. In fact, as mentioned in the introduction, the problem boils down to consider Riemannian Klein bottles invariant under soul (and rotational) symmetry. These Klein bottles are made of two isometric copies of Riemannian Mobius bands. Thus, in the end, the systolic inequality on Riemannian Klein bottles follows from optimal systolic inequalities on Riemannian Mobius bands, cf. [Pu52, Bl61].

There is no known optimal isosystolic inequality on the Klein bottle for Finsler metrics. However, we obtain the following partial result similar to the Riemannian case. Note that in the Riemannian case, the hypothesis is automatically satisfied by an average argument.

Theorem 2.7.2. *Let \mathbb{K} be a Finsler Klein bottle with a soul, soul-switching or rotational symmetry. Then*

$$\frac{\text{vol}_{HT}(\mathbb{K})}{\text{sys}^2(\mathbb{K})} \geq \frac{2}{\pi}. \quad (2.7.1)$$

Moreover, the inequality is optimal.

Let $\mathbb{K} = \mathbb{R}^2/G$ be a Finsler Klein bottle (with or without symmetry). Denote by \mathbb{M} the Finsler Moebius band obtained by cutting \mathbb{K} open along the soul given by the projection of the line $\{y = b\}$ to \mathbb{K} . The proof of the inequality (2.7.1) in Theorem 2.7.2 follows by combining the next three lemmas.

Lemma 2.7.3. *If $h(\mathbb{M}) \geq \text{sys}(\mathbb{K})$ then the inequality (2.7.1) holds true.*

Proof. The inclusion $\mathbb{M} \subset \mathbb{K}$ induces a π_1 -isomorphism. Hence, $\text{sys}(\mathbb{K}) \leq \text{sys}(\mathbb{M})$. Now, we only have two cases to consider.

First, if $h(\mathbb{M}) \leq \text{sys}(\mathbb{M})$, then we deduce from Theorem 2.1.3, first case, that

$$\begin{aligned} \frac{\text{vol}_{HT}(\mathbb{K})}{\text{sys}^2(\mathbb{K})} &\geq \frac{\text{vol}_{HT}(\mathbb{M})}{\text{sys}(\mathbb{M}) h(\mathbb{M})} \\ &\geq \frac{2}{\pi}. \end{aligned}$$

Second, if $h(\mathbb{M}) \geq \text{sys}(\mathbb{M})$, then we deduce from Theorem 2.1.3, second case, that

$$\begin{aligned} \frac{\text{vol}_{HT}(\mathbb{K})}{\text{sys}^2(\mathbb{K})} &\geq \frac{1}{\pi} \left(1 + \frac{\text{sys}(\mathbb{M})}{h(\mathbb{M})} \right) \frac{\text{sys}(\mathbb{M}) h(\mathbb{M})}{\text{sys}^2(\mathbb{K})} \\ &\geq \frac{1}{\pi} \left(\frac{h(\mathbb{M})}{\text{sys}(\mathbb{K})} + \frac{\text{sys}(\mathbb{M})}{\text{sys}(\mathbb{K})} \right) \frac{\text{sys}(\mathbb{M})}{\text{sys}(\mathbb{K})} \\ &\geq \frac{2}{\pi}, \end{aligned}$$

since both $\frac{h(\mathbb{M})}{\text{sys}(\mathbb{K})}$ and $\frac{\text{sys}(\mathbb{M})}{\text{sys}(\mathbb{K})}$ are greater or equal to 1. □

The next three lemmas show that the assumption of Lemma 2.7.3 is satisfied when the Finsler metric on \mathbb{K} has soul, soul-switching or rotational symmetries.

Lemma 2.7.4. *If \mathbb{K} is a Finsler Klein bottle with a soul symmetry then $h(\mathbb{M}) \geq \text{sys}(\mathbb{K})$.*

Proof. Observe that the soul symmetry of \mathbb{K} leaves both \mathbb{M} and $\partial\mathbb{M}$ invariant. Given an arc α of \mathbb{M} , we denote by α^* the arc of \mathbb{M} symmetric to α by the soul symmetry. Let γ be a height arc of \mathbb{M} , cf. Definition 2.3.4. This arc decomposes into two subarcs α and β connecting $\partial\mathbb{M}$ to the soul of \mathbb{M} with $\ell(\alpha) \leq \ell(\beta)$. The arc $\alpha \cup \alpha^*$ with endpoints in $\partial\mathbb{M}$ induces a nontrivial class in $\pi_1(\mathbb{M}, \partial\mathbb{M})$ of length at most the length of $\gamma = \alpha \cup \beta$. By definition of $h(\mathbb{M})$, we conclude that $\alpha \cup \alpha^*$ is as long as γ and so is length-minimizing in its relative homotopy class. In

particular, it is geodesic. Thus, the arc γ , which has the subarc α in common with $\alpha \cup \alpha^*$, agrees with $\alpha \cup \alpha^*$. In particular, it is invariant by the soul symmetry. As a result, the arc γ induces a noncontractible loop on \mathbb{K} after identification of the points of $\partial\mathbb{M}$ under the soul symmetry. Hence, $\ell(\alpha) \geq \text{sys}(\mathbb{K})$. \square

Lemma 2.7.5. *If \mathbb{K} is a Finsler Klein bottle with a soul-switching symmetry then $h(\mathbb{M}) \geq \text{sys}(\mathbb{K})$.*

Proof. Since the Finsler Klein bottle \mathbb{K} has a soul-switching symmetry, we can assume that it is composed of two isometric Finsler Mobius bands \mathbb{M}_1 and \mathbb{M}_2 . By symmetry, we have $h(\mathbb{M}) = 2h(\mathbb{M}_1)$. Given an arc α of \mathbb{M}_1 , we denote by α^* the arc of \mathbb{M}_2 symmetric to α by the soul-switching symmetry. Let γ be a height arc of \mathbb{M}_1 . This arc decomposes into two subarcs α_1 and β_1 connecting $\partial\mathbb{M}_1$ to the soul of \mathbb{M}_1 . The arc $\eta =: \alpha_1 \cup \alpha_1^* \cup \beta_1 \cup \beta_1^*$, i.e., the union of the arc γ and its symmetric image γ^* , has endpoints in $\partial\mathbb{M}$ and induces a nontrivial class in $\pi_1(\mathbb{M}, \partial\mathbb{M})$ of length equal to $2h(\mathbb{M}_1)$. Moreover, this arc η induces a noncontractible loop on \mathbb{K} after identification of the points of $\partial\mathbb{M}$ under the soul-switching symmetry. We conclude that $h(\mathbb{M}) \geq \text{sys}(\mathbb{K})$. \square

Lemma 2.7.6. *If \mathbb{K} is a Finsler Klein bottle with rotational symmetry then $h(\mathbb{M}) \geq \text{sys}(\mathbb{K})$.*

Proof. Observe that the rotational symmetries of \mathbb{K} leave both \mathbb{M} and $\partial\mathbb{M}$ invariant. Given an arc α of \mathbb{M} , we denote by α^θ the arc of \mathbb{M} symmetric to α by the rotational symmetry of angle θ . Let γ be a length-minimizing arc of \mathbb{M} parametrized (proportionally to its length) by $[0, 1]$ with endpoints in $\partial\mathbb{M}$ inducing a nontrivial class in $\pi_1(\mathbb{M}, \partial\mathbb{M})$. Note that γ is a geodesic arc of length $h(\mathbb{M})$. By the first variation formula for Finsler metrics, cf. [Sh01], the geodesic arc γ is perpendicular to $\partial\mathbb{M}$. It follows that the endpoints $\gamma(0)$ and $\gamma(1)$ of γ in $\partial\mathbb{M}$ are distinct. Since the Finsler metric is invariant under rotational symmetry, there exists $\theta \in (0, 2\pi)$ such that $\gamma^\theta(0) = \gamma(1)$. Both symmetric arcs γ and γ^θ are perpendicular to $\partial\mathbb{M}$. In particular, their tangent vectors at $\gamma^\theta(0) = \gamma(1)$ coincide up to sign. Therefore, the geodesic arcs γ and γ^θ agree up to reparametrization. More precisely, $\gamma^\theta(s) = \gamma(1 - s)$ for every $s \in [0, 1]$. Thus, $\gamma^{2\theta} = (\gamma^\theta)^\theta = \gamma$. Hence, $2\theta = 2\pi$, that is, $\theta = \pi$. Therefore, the arc γ projects to a closed curve in \mathbb{K} . It follows that the length of γ is at least $\text{sys}(\mathbb{K})$. \square

The following example shows that the inequality (2.7.1) is optimal.

Example 2.7.7. The quotient of \mathbb{R}^2 , endowed with the sup-norm, by the isometry group G generated by the glide reflection with parameter $b = \frac{\pi}{2}$, cf. Definition 2.7.1, is a Finsler Klein bottle with soul, soul-switching and rotational symmetries, of area 2π and systole π .

We believe that Theorem 2.7.2 holds true for every Finsler Klein bottle (not necessarily invariant under soul, soul-switching or rotational symmetries). More precisely, we state the following conjecture.

Conjecture. *Let \mathbb{K} be a Finsler Klein bottle. Then*

$$\frac{\text{vol}_{HT}(\mathbb{K})}{\text{sys}^2(\mathbb{K})} \geq \frac{2}{\pi}.$$

Moreover, the inequality is optimal.

Remark 2.7.8. If the conjecture is true, the Finsler systolic areas of \mathbb{RP}^2 , \mathbb{T}^2 and \mathbb{K} would be the same.

2.8 A non-optimal systolic inequality on Finsler Klein bottles

In this section, we present a non-optimal systolic inequality on Finsler Klein bottles.

Proposition 2.8.1. *Let \mathbb{K} be a Finsler Klein bottle. Then*

$$\frac{\text{vol}_{HT}(\mathbb{K})}{\text{sys}^2(\mathbb{K})} \geq \frac{\sqrt{2}}{\pi}.$$

Proof. Every symmetric convex body $C \subset \mathbb{R}^n$ admits a unique ellipsoid $E(C)$ of maximal volume among the ellipsoids contained in C . This ellipsoid, called John's ellipsoid, continuously varies with C for the Hausdorff topology. Furthermore, it satisfies the double inclusion, cf. [Gru07, Corollary 11.2] for instance,

$$E(C) \subset C \subset \sqrt{n} E(C). \quad (2.8.1)$$

Given a Klein bottle \mathbb{K} with a Finsler metric F , we define a continuous Riemannian metric g on \mathbb{K} by replacing the Minkowski norm F_x on each tangent space $T_x\mathbb{K}$ by the inner product induced by the John ellipsoid $E(B_x)$, where B_x is the unit ball of F_x . The double inclusion (2.8.1) satisfied by $E(B_x)$ implies that $\frac{1}{\sqrt{2}} \sqrt{g} \leq F \leq \sqrt{g}$. Hence,

$$\text{sys}(F) \leq \text{sys}(g) \quad \text{and} \quad \frac{1}{2} \text{vol}(g) \leq \text{vol}_{HT}(F) \quad (2.8.2)$$

From the optimal Riemannian systolic inequality on the Klein bottle [Ba86], we obtain

$$\text{vol}_{HT}(F) \geq \frac{1}{2} \text{vol}(g) \geq \frac{1}{2} \frac{2\sqrt{2}}{\pi} \text{sys}^2(g) \geq \frac{\sqrt{2}}{\pi} \text{sys}^2(F).$$

□

Remark 2.8.2. The naive volume bound in (2.8.2) can be improved into $\text{vol}(g) \leq \frac{\pi}{2} \text{vol}_{HT}(F)$, see the proof of [ABT, Theorem 4.11]. This leads to the better lower bound $\frac{4\sqrt{2}}{\pi^2}$ in Proposition 2.8.1.

Chapter 3

A systolically critical genus two surface

Nous démontrons une inégalité systolique critique sur la surface de genre deux. Plus précisément, il est connu que la surface de genre deux admet une métrique Riemannienne plate à singularités coniques qui est extrémale parmi les métriques à courbure nonpositive pour l'inégalité systolique. Nous montrons que cette métrique est en fait critique pour des variations lentes de métriques, cette fois-ci sans hypothèse de courbure, pour un autre problème systolique portant sur les longueurs des plus courts lacets non contractiles dans certaines classes d'homotopie libres données. Ces classes d'homotopie correspondent aux lacets systoliques et deux-systoliques de la surface extrémale.

Abstract. It is known that the genus two surface admits a piecewise flat metric with conical singularities which is extremal for the systolic inequality among all nonpositively curved metrics. We show that this piecewise flat metric is also critical for slow metric variations, without curvature restrictions, for another type of systolic inequality involving the lengths of the shortest noncontractible loops in different free homotopy classes. The free homotopy classes considered correspond to those of the systolic loops and the second-systolic loops of the extremal surface.

3.1 Introduction

We are interested in optimal geometric inequalities relating the area of a closed Riemannian surface Σ to the lengths of the shortest loops in certain homotopy classes. A typical example is given by the systolic inequalities, which relate the area of the surface to the systole, that is, the length of the shortest noncontractible loop in Σ . The first known systolic inequality is due to C. Loewner in 1949, who proved that every Riemannian two-torus (\mathbb{T}^2, g) satisfies

$$\text{area}(g) \geq \frac{\sqrt{3}}{2} \text{sys}^2(g),$$

where $\text{sys}(g)$ denotes the systole of the torus. Furthermore, the equality holds if and only if the torus is endowed with a flat hexagonal metric. C. Loewner did not publish his result, however it was mentioned by his student P. Pu, *cf.* [Pu52], who established a sharp systolic inequality on the projective plane \mathbb{RP}^2 . In this case, the equality is attained precisely by the round metrics. More than thirty years later, C. Bavard, *cf.* [Ba86] (see also [Sak88]) proved a sharp systolic inequality on the Klein bottle \mathbb{K} . Here, the extremal metrics are not smooth. These are the only manifolds for which an optimal systolic inequality is known. All these systolic inequalities can be written as follows. If

$$\text{sys}(g) \geq \text{sys}(g_0)$$

then

$$\text{area}(g) \geq \text{area}(g_0) \tag{3.1.1}$$

where g_0 is the extremal metric in each of these cases.

Optimal systolic-like inequalities, *i.e.*, inequalities that relate the area to the product of the lengths of the shortest loops or arcs in different relative homotopy classes are known only for the two-torus \mathbb{T}^2 , *cf.* [Ke67], the Klein bottle \mathbb{K} , *cf.* [Ba06, EY15], and the Mobius band \mathbb{M} , *cf.* [Pu52, Bl61, Ba06, EY15].

We can reformulate a systolic-like inequality as in (3.1.1). For instance, let $\ell_1(g)$ represent the length of the shortest noncontractible loop on (Σ, g) , that is,

$$\ell_1(g) = \text{sys}(g).$$

Denote by Λ_1 the subset of the free homotopy classes generated by the systolic loops of (Σ, g) and their multiples:

$$\Lambda_1 = \{\langle \gamma^k \rangle \mid \langle \gamma \rangle \neq 0, \ell_{g_0}(\gamma) = \ell_1(g_0) \text{ and } k \in \mathbb{Z}^*\}. \quad (3.1.2)$$

Similarly, let $\ell_2(g)$ be the length of the shortest noncontractible loop on (Σ, g) which is not homotopic to a systolic loop or its multiples. That is,

$$\ell_2(g) = \inf_{\langle \gamma \rangle \notin \Lambda_1 \cup \{0\}} \ell_g(\gamma).$$

Denote also by

$$\Lambda_2 = \{\langle \gamma \rangle \mid \langle \gamma \rangle \notin \Lambda_1 \text{ and } \ell_{g_0}(\gamma) = \ell_2(g_0)\} \quad (3.1.3)$$

the subset of the free homotopy classes generated by the shortest noncontractible loops of (Σ, g) which are not freely homotopic to systolic loops or their multiples. As a particular case of L. Keen's optimal systolic-like inequality for the two-torus in [Ke67], we have the following. If

$$\begin{cases} \ell_1(g) \geq \ell_1(g_0) \\ \ell_2(g) \geq \ell_2(g_0) \end{cases}$$

then

$$\text{area}(g) \geq \text{area}(g_0).$$

Recently, M. Katz and S. Sabourau, *cf.* [KS06], proved that there exists a piecewise flat metric g_0 on the genus two surface Σ_2 which is extremal among all nonpositively curved Riemannian metrics. This metric is composed of six regular octagons. It admits regions where only one systolic loop, *i.e.*, a noncontractible loop of length the systole, passes through every point. Hence this metric cannot be extremal for the general systolic inequality, *i.e.*, without restriction on the curvature. This follows from a result of E. Calabi, *cf.* [Ca96], which says that at least two systolic loops pass through every point of a systolically extremal surface. In fact, the non-extremality of the metric g_0 for the systolic inequality can also be deduced from a result of S. Sabourau, *cf.* [Sa04], which says that no flat metric with conical singularities is extremal for the systolic inequality in genus two.

We observe that the piecewise flat metric g_0 defined on the genus two surface Σ_2 may be a potential extremal metric for another systolic problem on Σ_2 . This observation follows from some geometric properties of the surface (Σ_2, g_0) . First, it is known that the systolic loops cover a systolically extremal surface. In our case, the ℓ_1 -loops and the ℓ_2 -loops cover the surface Σ_2 . Here, by an ℓ_i -loop we mean a loop of length $\ell_i(g_0)$ in (Σ_2, g_0) whose free homotopy class lies in Λ_i . Second, the unit tangent vectors of these ℓ_i -loops are well distributed on each tangent plane of the surface. More precisely, their convex hull forms a regular octagon on these tangent planes. In general, the convex hull of the unit tangent vectors of systolic loops on all the known

systolically extremal surfaces is symmetric. Finally, an extremal surface of genus at least two tends to have flat regions, *cf.* [Ca96, Br96].

In [Ca96], E. Calabi described two piecewise flat genus three surfaces and conjectured that one of them is the global minimum for the systolic inequality. Later, S. Sabourau, *cf.* [Sa11], proved that these two genus three surfaces are critical for the systolic inequality with respect to slow metric variations. Of course, the systolic volume functional is not necessarily differentiable and an adequate notion of criticality needs to be introduced. The notion of criticality used in [Sa11] was introduced in [Na96, EI00] to study the Riemannian surfaces that maximize the product of the area with the first nonzero eigenvalue of the Laplacian. Other notions of systolically critical metrics were used in [Bal06, Bal10, Sa10, AB14].

In this chapter we show that the metric g_0 on the genus two surface Σ_2 is critical in the sense of [Na96, EI00, Sa11] for slow metric variations. More precisely, we prove

Theorem 3.1.1. *Let g_t be a slow metric variation of g_0 defined on the genus two surface Σ_2 . If $\ell_1(g_t) \geq \ell_1(g_0)$ and $\ell_2(g_t) \geq \ell_2(g_0)$, then*

$$\text{area}(g_t) \geq \text{area}(g_0) + o(t).$$

The slow metric variations involved in Theorem 3.1.1 are analogous to those defined by S. Sabourau in [Sa11]. We refer to Definition 3.4.1 for a precise definition of these metric variations and to the last section of [Sa11] for examples. Observe for instance that deforming the regular octagons composing the extremal surface (Σ_2, g_0) into non-regular octagons gives rise to a slow metric variation.

With the exception of some recent proofs of systolic inequalities, *cf.* [Iv02, Sa10], and systolic-like inequalities, *cf.* [SY16], on the projective plane \mathbb{RP}^2 , the two-torus \mathbb{T}^2 and the Mobius band \mathbb{M} , all the other known proofs of the aforementioned optimal geometric inequalities require the uniformization theorem as a main tool. In our proof of Theorem 3.1.1, we do not make use of the uniformization theorem. Instead, we rely on recent calibrating methods, *cf.* [Iv02, Iv11, Sa11]. Briefly, we embed the universal cover $\tilde{\Sigma}_2$ of the genus two surface Σ_2 in an infinite-dimensional space \mathbb{R}^∞ as follows. Given that $\ell_i(g) \geq \ell_i(g_0)$, we construct a 1-Lipschitz equivariant map

$$\Psi^g : \tilde{\Sigma}_2 \rightarrow \mathbb{R}^\infty$$

using the Busemann functions induced by the lifts of the ℓ_i -loops of Σ_2 , *cf.* Section 3.3. Then, we introduce an appropriate infinitesimally calibrating two-form ω on \mathbb{R}^∞ whose pull-back passes to the quotient on the surface Σ_2 . This allows us to show that

$$\int_{\Sigma_2} (\Psi^g)^* \omega \leq c_0 \text{area}(\Sigma_2, g)$$

for every Riemannian metric g on Σ_2 , where c_0 is a sharp positive constant. Moreover, the equality holds if $g = g_0$. Finally, we prove that if $(g_t)_{t \geq 0}$ is a slow metric variation then

$$\int_{\Sigma_2} (\Psi^{g_t})^* \omega - \int_{\Sigma_2} (\Psi^{g_0})^* \omega = o(t).$$

This completes the proof, cf. Section 3.4.

3.2 An extremal piecewise flat metric in genus two

In this section we provide a description for the critical piecewise flat genus two surface and we introduce some notations and definitions.

The Bolza surface \mathfrak{B} is a genus two Riemann surface. It is the smooth completion of the smooth affine algebraic curve

$$y^2 = x^5 - x. \tag{3.2.1}$$

The set $\{-1, 0, 1, -i, i, \infty\}$ of roots of the polynomial (3.2.1) (including the point at infinity) corresponds to the set of vertices of the regular octahedral triangulation of the Riemann sphere S^2 under the conformal stereographic projection. Hence, these six points can be identified with the ramification points of the ramified conformal double cover $Q : \mathfrak{B} \rightarrow S^2$, or equivalently, with the Weierstrass points of \mathfrak{B} .

The conformal class of the Bolza surface \mathfrak{B} admits a piecewise flat nonpositively curved Riemannian metric g_0 with 16 conical singularities whose orientation-preserving isometry group is isomorphic to the automorphism group of \mathfrak{B} . This metric g_0 has been introduced by M. Katz and S. Sabourau in [KS06], where they prove that it is extremal for the systolic inequality among all nonpositively curved metrics on the genus two surface Σ_2 . The metric g_0 , defined up to homothety, on Σ_2 is isometric to the piecewise flat genus two surface composed of six identical flat regular octagons Ω_k with the identifications given in Figure 3.1. The Weierstrass points of \mathfrak{B} correspond to the centers of the octagons and the 16 conical singularities are located at their vertices.

Denote by \hbar the length of an edge of a regular octagon Ω_k of g_0 . We have

$$\text{area}(g_0) = 12(1 + \sqrt{2})\hbar^2.$$

The systole $\ell_1(g_0)$ of the surface (Σ_2, g_0) is computed in [KS06, Lemma 3.2]. Namely,

$$\ell_1(g_0) = 2(1 + \sqrt{2})\hbar.$$

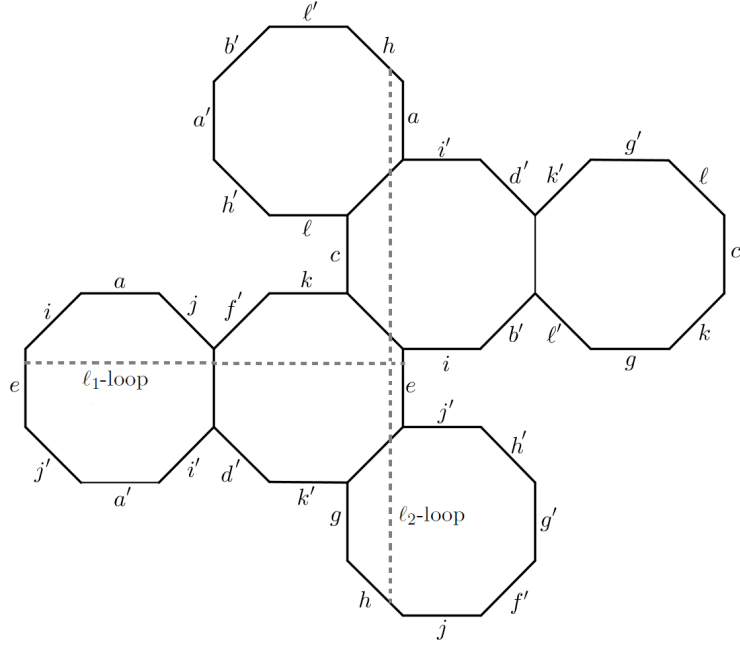


Figure 3.1: The critical surface (Σ_2, g_0) is isometric to the to the piecewise flat genus two surface composed of six identical regular octagons with the identifications given in the figure.

By using similar arguments, we calculate $\ell_2(g_0)$. In particular,

$$\ell_2(g_0) = 2(2 + \sqrt{2})h.$$

Recall that an ℓ_i -loop is a loop in (Σ_2, g_0) of length $\ell_i(g_0)$ and whose free homotopy class lies in Λ_i , cf. (3.1.2) and (3.1.3). We define an ℓ_i -band as follows.

Definition 3.2.1. Let $\langle \alpha \rangle$ be a free homotopy class in Λ_i . The ℓ_i -loops in $\langle \alpha \rangle$ are parallel to each other and form a flat cylinder $\mathcal{B}_{\langle \alpha \rangle}$ of height h if $i = 1$ and $\frac{\sqrt{2}}{2}h$ if $i = 2$. Such a cylinder $\mathcal{B}_{\langle \alpha \rangle}$ will be called an ℓ_i -band of (Σ_2, g_0) . The *soul* of an ℓ_i -band $\mathcal{B}_{\langle \alpha \rangle}$ is the ℓ_i -loop of $\mathcal{B}_{\langle \alpha \rangle}$ equidistant from the boundary components of the ℓ_i -band.

The intersections of the ℓ_i -bands decompose the surface Σ_2 into 150 polygons, cf. Figure 3.2. These polygons define four regions $\mathcal{R}_1, \dots, \mathcal{R}_4$ depending on the number of ℓ_i -loops that pass through every point in their interior. More precisely, such a polygon Δ_k lies in a region \mathcal{R}_k if through every point in its interior pass exactly $2k$ oriented ℓ_1 -loops and $8 - 2k$ oriented ℓ_2 -loops, cf. Figure 3.2. Hence, every region \mathcal{R}_k with $k \neq 4$ is composed of exactly 48 identical polygons Δ_k (with 8 polygons in each octagon), while the region \mathcal{R}_4 is composed of only 6 identical polygons Δ_4 . Observe that the polygons $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 have the shape of a right isosceles triangle, a kite, a right isosceles triangle and a (small) octagon. Furthermore, the right isosceles triangles Δ_1 are the only polygons which have an edge in common with the six regular flat octagons forming the critical surface (Σ_2, g_0) , namely their hypotenuses coincide with the

edges of the regular octagons Ω_k .

Fix an orientation on Σ_2 . Let Ω be one of the six regular octagons composing the surface (Σ_2, g_0) . Denote by $(\tilde{\Sigma}_2, \tilde{g}_0)$ the universal cover of (Σ_2, g_0) . Let $\tilde{\Omega}$ be a lift of Ω in the universal cover $\tilde{\Sigma}_2$ and let \mathcal{E} be an (oriented) edge of the regular octagon $\tilde{\Omega}$. Moreover, let $\tilde{\Delta}_k$ be a lift of the polygon Δ_k in $\tilde{\Sigma}_2$. The souls of the ℓ_i -bands lift to $\tilde{\Sigma}_2$ as follows.

Definition 3.2.2. An ℓ_1 -direction of $(\tilde{\Sigma}_2, \tilde{g}_0)$ is a \tilde{g}_0 -unit vector $\zeta = \zeta_{\tilde{\Omega}}^{\mathcal{E}}$ based at the center of $\tilde{\Omega}$ and pointing in the same direction as \mathcal{E} , cf. Figure 3.2. The unit vector ζ induces a geodesic line \tilde{c}_ζ on $(\tilde{\Sigma}_2, \tilde{g}_0)$ such that

$$\tilde{c}'_\zeta(0) = \zeta.$$

The projection c_ζ of \tilde{c}_ζ to (Σ_2, g_0) is a closed geodesic curve which coincides with the soul of the ℓ_1 -band $\mathcal{B}_{(c_\zeta)}$.

Now, denote by $x_{\mathcal{E}}$ the point of $\tilde{\Omega}$ lying in the perpendicular bisector of \mathcal{E} and at distance $\frac{\sqrt{2}}{4}h$ from \mathcal{E} . (Recall that h is the side length of $\tilde{\Omega}$.) Note that the point $x_{\mathcal{E}}$ lies in the right isosceles triangle $\tilde{\Delta}_1$ of $\tilde{\Omega}$ with hypotenuse \mathcal{E} .

Definition 3.2.3. An ℓ_2 -direction of $(\tilde{\Sigma}_2, \tilde{g}_0)$ is a \tilde{g}_0 -unit vector $\xi = \xi_{\tilde{\Omega}}^{\mathcal{E}}$ based at $x_{\mathcal{E}}$ and pointing in the same direction as the edge \mathcal{E} , cf. Figure 3.2. This unit vector ξ induces a geodesic line \tilde{c}_ξ on $(\tilde{\Sigma}_2, \tilde{g}_0)$ such that

$$\tilde{c}'_\xi(0) = \xi.$$

The projection c_ξ of \tilde{c}_ξ to (Σ_2, g_0) is a closed geodesic curve which coincides with the soul of the ℓ_2 -band $\mathcal{B}_{(c_\xi)}$.

Remark 3.2.4. In what follows, we will replace the symbols ζ and ξ by ν when there is no need to distinguish between an ℓ_1 -direction and an ℓ_2 -direction. Moreover, the indices $\tilde{\Omega}$ and \mathcal{E} in $\nu_{\tilde{\Omega}}^{\mathcal{E}}$ will be omitted.

3.3 Busemann functions and calibrating forms

In this section, we introduce some notations and preliminary results that will be useful in our proof of Theorem 3.1.1.

Definition 3.3.1. Let ν be an ℓ_i -direction. Extend ν to a map, still denoted by

$$\nu : \mathbb{R} \rightarrow U_{\tilde{g}_0} \tilde{\Sigma}_2$$

defined as $\nu(s) = \tilde{c}'_\nu(s)$. In particular, $\nu(0) = \nu$.

Along with the metric g_0 , consider another Riemannian metric g defined on Σ_2 and denote by \tilde{g} its lift to $\tilde{\Sigma}_2$.

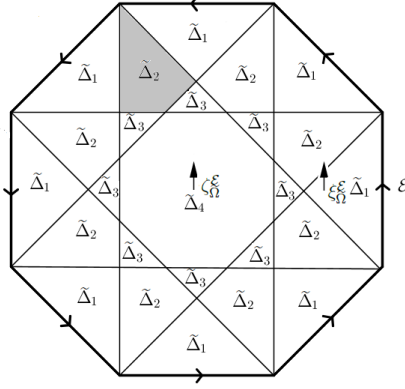


Figure 3.2: A flat regular octagon $\tilde{\Omega}$ of the metric \tilde{g}_0 .

Definition 3.3.2. Let ν be a \tilde{g}_0 -unit vector tangent to a geodesic line induced by an ℓ_i -direction. Define the *Busemann function* $b_\nu^g : \tilde{\Sigma}_2 \rightarrow \mathbb{R}$ as

$$b_\nu^g(x) = \limsup_{t \rightarrow \infty} d_{\tilde{g}}(x, \tilde{c}_\nu(t))\ell_i(g_0) - t \ell_g(\langle c_\nu \rangle)$$

where \tilde{c}_ν is the \tilde{g}_0 -geodesic line induced by ν and c_ν is its projection to Σ_2 .

Here, $\ell_g(\langle c_\nu \rangle)$ denotes the smallest length of a noncontractible loop in the free homotopy class $\langle c_\nu \rangle$ of c_ν with respect to the Riemannian metric g .

Remark 3.3.3. Contrary to the original definition of a Busemann function, cf. [BH99], the line \tilde{c}_ν is geodesic for \tilde{g}_0 but not for \tilde{g} . Moreover, the function

$$\begin{aligned} h : \mathbb{R} &\rightarrow \mathbb{R} \\ t &\mapsto d_{\tilde{g}}(x, \tilde{c}_\nu(t))\ell_i(g_0) - t \ell_g(\langle c_\nu \rangle) \end{aligned}$$

which appears in Definition 3.3.2, is not necessarily non-increasing.

Proposition 3.3.4. *The real function h is bounded.*

Proof. By the triangle inequality, we obtain

$$h(t) \geq d_{\tilde{g}}(\tilde{c}_\nu(0), \tilde{c}_\nu(t))\ell_i(g_0) - t \ell_g(\langle c_\nu \rangle) - d_{\tilde{g}}(x, \tilde{c}_\nu(0))\ell_i(g_0). \quad (3.3.1)$$

Let β be a length-minimizing loop of (Σ_2, g) in the homotopy class of c_ν , that is, $\ell_g(\beta) = \ell_g(\langle c_\nu \rangle)$. Let $[t]_i$ be the integer part of $\frac{t}{\ell_i(g_0)}$, that is,

$$[t]_i = \max\{k \in \mathbb{N} \mid t \geq k \ell_i(g_0)\}.$$

We have

$$[t]_i \ell_i(g_0) - t \geq -\ell_i(g_0). \quad (3.3.2)$$

Since the genus two surface is orientable, the $[t]_i$ -iterate of β is length-minimizing in its homotopy class with respect to g , cf. [Gr99]. Let $\tilde{\gamma}$ in $\tilde{\Sigma}_2$ be a length-minimizing arc connecting $\tilde{c}_\nu(0)$ to $\tilde{c}_\nu([t]_i)$. Its projection γ to Σ_2 is homotopic to $c_\nu^{[t]_i}$ and its length satisfies the two relations

$$\ell_g(\gamma) = d_{\tilde{g}}(\tilde{c}_\nu(0), \tilde{c}_\nu([t]_i))$$

and

$$\ell_g(\gamma) \geq [t]_i \ell_g(\langle c_\nu \rangle). \quad (3.3.3)$$

Combining (3.3.1), (3.3.2) and (3.3.3), we deduce that the map h is bounded from below.

Now, we show that the map h is bounded from above. Indeed, by the triangle inequality, we derive that

$$h(t) \leq \{d_{\tilde{g}}(x, \tilde{c}_\nu(0)) + d_{\tilde{g}}(\tilde{c}_\nu(0), \tilde{c}_\nu([t]_i)) + d_{\tilde{g}}(\tilde{c}_\nu([t]_i), \tilde{c}_\nu(t)) - [t]_i \ell_g(\langle c_\nu \rangle)\} \ell_i(g_0).$$

Moreover,

$$d_{\tilde{g}}(\tilde{c}_\nu(0), \tilde{c}_\nu([t]_i)) - [t]_i \ell_g(\langle c_\nu \rangle) \leq 2 \operatorname{diam}(\Sigma_2, g). \quad (3.3.4)$$

Indeed, let γ be a g -length minimizing loop in the free homotopy class $\langle c_\nu \rangle$, that is, $\ell_g(\gamma) = \ell_g(\langle c_\nu \rangle)$. Let α be a length-minimizing arc joining $c_\nu(0)$ to γ . We deduce that

$$d_{\tilde{g}}(\tilde{c}_\nu(0), \tilde{c}_\nu([t]_i)) \leq \ell_g(\alpha \cup \gamma^{[t]_i} \cup \alpha^{-1}) \leq \ell_g(\gamma^{[t]_i}) + \operatorname{diam}(\Sigma_2, g).$$

Then, the inequality (3.3.4) is satisfied. Furthermore,

$$d_{\tilde{g}}(\tilde{c}_\nu([t]_i), \tilde{c}_\nu(t)) \leq \operatorname{diam}(\Sigma_2, g).$$

This shows that h is bounded from above. \square

Definition 3.3.5. The map $\Psi_\nu^g : \tilde{\Sigma}_2 \rightarrow \mathbb{R}$ induced by an ℓ_i -direction ν , cf. Definitions 3.2.2 and 3.2.3, is defined as

$$\Psi_\nu^g(x) = \frac{1}{\ell_i(g_0) \ell_g(\langle c_\nu \rangle)} \int_{\mathbb{R}/\ell_i(g_0)\mathbb{Z}} b_{\nu(s)}^g(x) - b_{-\nu(s)}^g(x) ds,$$

where $\nu(s) = \tilde{c}_\nu(s)$, cf. Definition 3.3.1.

Example 3.3.6. If $g = g_0$, the g_0 -gradient $\nabla_x \Psi_\nu^{g_0}$ is a g_0 -unit tangent vector parallel to and pointing in the same direction as the ℓ_i -direction ν .

Proposition 3.3.7. Let ν be an ℓ_i -direction. The map Ψ_ν^g satisfies the following properties:

1. It is differentiable almost everywhere.
2. $\Psi_{-\nu}^g = -\Psi_\nu^g$.
3. The differential of the map Ψ_ν^g depends only on the oriented ℓ_i -band $\tilde{\mathcal{B}}$ generated by the vector ν .

Proof. For (1), the map Ψ_ν^g is 1-Lipschitz for the supremum norm as the Busemann function b_ν^g is $\ell_i(g_0)$ -Lipschitz and $\ell_i(g) \geq \ell_i(g_0)$. Hence, it is differentiable almost everywhere by the Rademacher theorem. Part (2) follows directly from Definition 3.3.5. For (3), let ν and η be two ℓ_i -directions generating the same oriented ℓ_i -band. That is, their basepoints lie in the same ℓ_i -band and the vectors point in the same direction. Recall that $\nu(s) = \tilde{c}'_\nu(s)$, cf. Definition 3.3.1, and the same with η . The definition of ℓ_i -directions and the assumption on the vectors ν and η imply that

$$\eta(s) = \nu\left(s + \frac{k}{n}\ell_i(g_0)\right)$$

for some integer k with $n = 2$ if $i = 1$ and $n = 4$ if $i = 2$. We derive easily from Definition 3.3.5 that

$$d\Psi_\eta^g = d\Psi_\nu^g.$$

This completes the proof. □

Let Γ be the deck transformation group of $\tilde{\Sigma}_2$. The \tilde{g}_0 -geodesic line \tilde{c}_ν induces a unique element γ_ν in Γ that leaves \tilde{c}_ν globally invariant and such that

$$\gamma_\nu.\tilde{c}_\nu(s) = \tilde{c}_\nu(s + \ell_i(g_0)).$$

Moreover, we define an action \star of Γ on the set of unit tangent vectors to the geodesic line \tilde{c}_ν as

$$\gamma \star \nu(s) = (\gamma.\tilde{c}_\nu)'(s)$$

where $\gamma \in \Gamma$.

We state now the following properties of the map Ψ_ν^g .

Proposition 3.3.8. *Let ν be an ℓ_i -direction. The map $\Psi_\nu^g : \tilde{\Sigma}_2 \rightarrow \mathbb{R}$ satisfies the following properties:*

1. It is equivariant, that is, for every γ in Γ ,

$$\Psi_{\gamma \star \nu}^g(\gamma.x) = \Psi_\nu^g(x).$$

2. It passes to the quotient by the cyclic subgroup $\langle \gamma_\nu \rangle$ and induces a map

$$\bar{\Psi}_\nu^g : \tilde{\Sigma}_2 / \langle \gamma_\nu \rangle \rightarrow \mathbb{R} / \ell_i(g_0)\mathbb{Z}$$

defined as

$$\bar{\Psi}_\nu^g(\gamma_\nu \cdot x) \equiv \Psi_\nu^g(x) \pmod{\ell_i(g_0)}.$$

Proof. Let $x \in \tilde{\Sigma}_2$ and $\gamma \in \Gamma$. Since γ is an isometry and $\gamma \cdot \tilde{c}_\nu = \tilde{c}_{\gamma \cdot \nu}$, we derive

$$\ell_g(\langle c_{\gamma \star \nu} \rangle) = \ell_g(\langle c_\nu \rangle).$$

Moreover, since the distance $d_{\tilde{g}}$ induced by the Riemannian metric \tilde{g} is Γ -invariant, we derive that $b_{\gamma \star \nu}^g(\gamma \cdot x) = b_\nu^g(x)$. Hence, the desired result

$$\Psi_{\gamma \star \nu}^g(\gamma \cdot x) = \Psi_\nu^g(x).$$

Now, since $\gamma_\nu^{-1} \cdot \tilde{c}_\nu(t) = \tilde{c}_\nu(t - \ell_i(g_0))$, we derive from the Γ -invariance of $d_{\tilde{g}}$ that

$$\begin{aligned} d_{\tilde{g}}(\gamma_\nu \cdot x, \tilde{c}_\nu(t)) &= d_{\tilde{g}}(x, \gamma_\nu^{-1} \cdot \tilde{c}_\nu(t)) \\ &= d_{\tilde{g}}(x, \tilde{c}_\nu(t - \ell_i(g_0))). \end{aligned}$$

Hence,

$$\begin{aligned} b_\nu^g(\gamma_\nu \cdot x) &= \limsup_{t \rightarrow \infty} \ell_i(g_0) d_{\tilde{g}}(x, \tilde{c}_\nu(t - \ell_i(g_0))) - t \ell_g(\langle c_\nu \rangle) \\ &= b_\nu^g(x) - \ell_i(g_0) \ell_g(\langle c_\nu \rangle). \end{aligned}$$

We deduce that

$$\Psi_\nu^g(\gamma_\nu \cdot x) = \Psi_\nu^g(x) - \ell_i(g_0). \quad (3.3.5)$$

□

Remark 3.3.9. As a consequence of Proposition 3.3.8 and the relation (3.3.5), we deduce the following couple of points:

1. $\gamma^*(d\Psi_{\gamma \star \nu}^g) = d\Psi_\nu^g$ for every $\gamma \in \Gamma$,
2. The map $\bar{\Psi}_\nu^g - \Psi_\nu^g$ passes to the quotient by the cyclic subgroup $\langle \gamma_\nu \rangle$ and induces a map

$$\bar{\Psi}_\nu^g - \bar{\Psi}_\nu^{g_0} : \tilde{\Sigma}_2 / \langle \gamma_\nu \rangle \rightarrow \mathbb{R}$$

such that $\bar{\Psi}_\nu^g(\gamma_\nu \cdot x) - \bar{\Psi}_\nu^{g_0}(\gamma_\nu \cdot x) = \Psi_\nu^g(x) - \Psi_\nu^{g_0}(x)$.

Actually, these two properties hold if we replace Ψ_ν^g with the Busemann function b_ν^g .

Let $\tilde{\Omega}$ be a regular octagon of the universal cover $(\tilde{\Sigma}_2, \tilde{g}_0)$ of the surface (Σ_2, g_0) . We order the (oriented) edges $(\mathcal{E}^j)_{1 \leq j \leq 8}$ of $\tilde{\Omega}$ with respect to the cyclic order induced by the orientation of $\tilde{\Omega}$

by fixing an initial edge \mathcal{E}^1 . Denote

$$\nu^j = \nu_{\tilde{\Omega}}^{\mathcal{E}^j}.$$

We state the following definition

Definition 3.3.10. Let $\mathbb{R}^\infty = \prod_{\tilde{\Omega}} \mathbb{R}^8$ be the infinite product of \mathbb{R}^8 where $\tilde{\Omega}$ runs over all the regular octagons of $(\tilde{\Sigma}_2, \tilde{g}_0)$. Let

$$\Psi^g : \tilde{\Sigma}_2 \rightarrow \mathbb{R}^\infty$$

be the map defined as

$$\begin{aligned} \pi_{\tilde{\Omega}} \circ \Psi^g|_{\tilde{\Omega}} : \tilde{\Omega} &\rightarrow \mathbb{R}^8 \\ x &\mapsto (\Psi_{\nu^j}^g(x))_{j=1}^8 \end{aligned}$$

where

$$\pi_{\tilde{\Omega}} : \mathbb{R}^\infty \rightarrow \mathbb{R}^8$$

is the canonical projection to the factor \mathbb{R}^8 of \mathbb{R}^∞ corresponding to $\tilde{\Omega}$. The vector ν^j is the ℓ_i -direction that is parallel and point in the same direction as the \tilde{g}_0 -unit vector based at x and tangent to the lift of an oriented ℓ_i -loop.

Example 3.3.11. Consider the regular octagon $\tilde{\Omega}$ in Figure 3.2. Suppose that the numbering of the edges of $\partial\tilde{\Omega}$ follows a cyclic order starting at \mathcal{E} , then we have

$$\Psi^g|_{\tilde{\Delta}_2} = (\Psi_{\zeta^1}^g, \Psi_{\zeta^2}^g, \Psi_{\zeta^3}^g, \Psi_{\zeta^4}^g, \Psi_{\zeta^5}^g, \Psi_{\zeta^6}^g, \Psi_{\zeta^7}^g, \Psi_{\zeta^8}^g)$$

where $\tilde{\Delta}_2$ is the gray polygon in Figure 3.2.

Remark 3.3.12. Note that the map Ψ^g is not well defined in the lift of the boundaries of the polygons Δ since there is some ambiguity in the choice of ν^j .

Definition 3.3.13. Consider the two-form

$$\omega_0 = \sum_{j=1}^8 dx_j \wedge dx_{j+2}$$

in \mathbb{R}^8 . We define a two-form $\tilde{\omega}$ on $\tilde{\Sigma}_2$ as the pull-back of ω_0 by the map

$$\pi_{\tilde{\Omega}} \circ \Psi^g|_{\tilde{\Omega}} : \tilde{\Omega} \rightarrow \mathbb{R}^8$$

on every regular octagon $\tilde{\Omega}$. It can be expressed on $\tilde{\Omega}$ as

$$\tilde{\omega} = \sum_{j=1}^8 d\Psi_{\nu^j}^g \wedge d\Psi_{\nu^{j+2}}^g$$

where $\nu^j = \nu_{\tilde{\Omega}}^j$ and all indices j are taken modulo 8. This defines a two-form on $\tilde{\Sigma}_2$ that we still denote by $\tilde{\omega}$.

Note that the two-form $\tilde{\omega}$ in Definition 3.3.13 is defined almost everywhere since the maps Ψ_{ν}^g are almost everywhere differentiable, cf. Proposition 3.3.7 (1).

Proposition 3.3.14. *The two-form $\tilde{\omega}$ passes to the quotient on Σ_2 .*

Proof. Let $\gamma \in \Gamma$. Observe that γ preserves the octagonal decomposition of the universal cover $(\tilde{\Sigma}_2, \tilde{g}_0)$ and its orientation. Therefore, for every regular octagon $\tilde{\Omega}$, there exists a permutation σ of $\{1, \dots, 8\}$ such that

$$\gamma^{-1}(\mathcal{E}_{\tilde{\Omega}}^j) = \mathcal{E}_{\gamma^{-1}(\tilde{\Omega})}^{\sigma(j)}.$$

That is,

$$\gamma^{-1}(\nu_{\tilde{\Omega}}^j) = \nu_{\gamma^{-1}(\tilde{\Omega})}^{\sigma(j)}.$$

Hence,

$$d\Psi_{\gamma^{-1}\star\nu^j}^g = d\Psi_{\nu^{\sigma(j)}}^g.$$

By the equivariance property of the map Ψ_{ν^j} , cf. Remark 3.3.9, we derive

$$\begin{aligned} \gamma^*\tilde{\omega} &= \sum_{j=1}^8 d\Psi_{\gamma^{-1}\star\nu^j}^g \wedge d\Psi_{\gamma^{-1}\star\nu^{j+2}}^g \\ &= \sum_{j=1}^8 d\Psi_{\nu^{\sigma(j)}}^g \wedge d\Psi_{\nu^{\sigma(j+2)}}^g. \end{aligned}$$

Finally, since the permutation σ arises from an orientation-preserving isometry γ , the angle between $\nabla_x \Psi_{\nu^{\sigma(j)}}^g$ and $\nabla_x \Psi_{\nu^{\sigma(j+2)}}^g$ is equal to the angle between $\nabla_{\gamma(x)} \Psi_{\nu^j}^g$ and $\nabla_{\gamma(x)} \Psi_{\nu^{j+2}}^g$ for every $x \in \tilde{\Sigma}_2$ at which $\tilde{\omega}$ is defined. Therefore,

$$\sigma(j+2) = \sigma(j) + 2.$$

Reindexing the sum in the expression of the two-forms ω_0 and $\tilde{\omega}$, we derive that

$$\gamma^*\tilde{\omega} = \tilde{\omega}.$$

□

Remark 3.3.15. The induced two-form on Σ_2 will be denoted by ω .

3.4 Proof of Theorem 3.1.1

We briefly restate the statement of Theorem 3.1.1. Let g_t be a slow metric variation of g_0 defined on Σ_2 . If $\ell_1(g_t) \geq \ell_1(g_0)$ and $\ell_2(g_t) \geq \ell_2(g_0)$, then

$$\text{area}(g_t) \geq \text{area}(g_0) + o(t).$$

Before proceeding to the proof, let us first define what we mean by a slow metric variation.

Definition 3.4.1. A *slow metric variation* is a family of Riemannian metrics g_t with $t \geq 0$ such that for every ℓ_i -direction ν , the following condition is satisfied

$$\|d(b_\nu^{g_t} - b_{-\nu}^{g_t}) - d(b_\nu^{g_0} - b_{-\nu}^{g_0})\|_{L^2(B_\nu)} = o(\sqrt{t}), \quad (3.4.1)$$

where B_ν is a fundamental domain in $(\tilde{\Sigma}_2, \tilde{g}_0)$ of the band $\mathcal{B}_\nu = \mathcal{B}_{\langle c_\nu \rangle}$.

Note that the map $b_\nu^{g_t} - b_\nu^{g_0}$ passes to the quotient by the cyclic subgroup $\langle c_\nu \rangle$, cf. Remark 3.3.9 (2).

Remark 3.4.2. A metric variation g_t should have conical singularities as the Riemannian metric g_0 has conical singularities, cf. Section 3.2.

Definition 3.4.3. The \mathcal{F} -functional of the map

$$\Psi^g : \tilde{\Sigma}_2 \rightarrow \mathbb{R}^\infty$$

introduced in Definition 3.3.10 is defined as

$$\mathcal{F}(\Psi^g) = \int_{\Sigma_2} \omega,$$

where ω is the quotient two-form on Σ_2 introduced in Section 3.3.

The proof of Theorem 3.1.1 follows directly from the next two lemmas.

Lemma 3.4.4. *The two-form ω is a calibrating form for the map Ψ^g . That is,*

$$\mathcal{F}(\Psi^g) \leq 8 \text{ area}(g)$$

with equality for $g = g_0$.

Proof. Let $x \in \tilde{\Delta}$ be a point at which the maps $\Psi_{\nu^j}^g$ are differentiable. Let v be a \tilde{g} -unit vector in the tangent plane $T_x \tilde{\Sigma}_2$. Since the map $\Psi_{\nu^j}^g$ is 1-Lipschitz, we derive that

$$|d_x \Psi_{\nu^j}^g(v)| \leq 1.$$

Hence

$$\|\tilde{\omega}\|_{\tilde{g}} = \left\| \sum_{j=1}^8 d\Psi_{\nu^j}^g \wedge d\Psi_{\nu^{j+2}}^g \right\|_{\tilde{g}} \leq 8.$$

The desired inequality directly follows.

Let \mathcal{D} denote a fundamental domain of the surface (Σ_2, g_0) in its universal cover made of regular octagons $\tilde{\Omega}$. By Definition 3.4.3, the \mathcal{F} -functional of the almost everywhere differentiable map Ψ^g is defined as

$$\begin{aligned} \mathcal{F}(\Psi^g) &= \int_{\Sigma_2} \omega \\ &= \sum_{\tilde{\Omega} \in \mathcal{D}} \int_{\tilde{\Omega}} \sum_{j=1}^8 d\Psi_{\nu^j}^g \wedge d\Psi_{\nu^{j+2}}^g. \end{aligned}$$

When $g = g_0$, the vectors ν^j and ν^{j+2} form an oriented orthonormal basis. The same holds for the covectors $d_x \Psi_{\nu^j}^{g_0}$ and $d_x \Psi_{\nu^{j+2}}^{g_0}$ from Example 3.3.6. Hence,

$$d_x \Psi_{\nu^j}^{g_0} \wedge d_x \Psi_{\nu^{j+2}}^{g_0} = d\tilde{g}_0$$

where $d\tilde{g}_0$ represents the area form of the Riemannian metric \tilde{g}_0 . In particular, we derive

$$\mathcal{F}(\Psi^{g_0}) = 8 \text{ area}(g_0)$$

after passing to the quotient. □

Remark 3.4.5. An analogous two-form ω along systolic directions could be defined on the genus three surface to recover the result of [Sa11] without using the Euclidean metrics.

Lemma 3.4.6. *Let g_t be a slow metric variation of g_0 . Then*

$$\mathcal{F}(\Psi^{g_t}) - \mathcal{F}(\Psi^{g_0}) = o(t).$$

Proof. Let $\nu = \nu_{\tilde{\Omega}}^{\xi}$, where $\tilde{\Omega}$ is a regular octagon of $(\tilde{\Sigma}_2, \tilde{g}_0)$ of side length \hbar . By definition, an ℓ_i -band $\mathcal{B}_\nu = \mathcal{B}_{\langle c_\nu \rangle}$ is the quotient of

$$\tilde{\mathcal{B}}_\nu = \{x \in \tilde{\Sigma}_2 \mid d_{\tilde{g}_0}(x, \tilde{c}_\nu) \leq \lambda\}$$

by the cyclic subgroup $\langle c_\nu \rangle$, where λ is equal to $\frac{1}{2}\hbar$ if $i = 1$ and equal to $\frac{\sqrt{2}}{4}\hbar$ if $i = 2$. We still denote by ν the vector given by the systolic direction after passing to the quotient.

Recall that the differential $d\Psi_\nu^g$ of the map Ψ_ν^g depends only on the oriented ℓ_i -band $\tilde{\mathcal{B}}$ generated by the vector ν (or rather the oriented geodesic line \tilde{c}_ν), but not on ν , cf. Proposition 3.3.7 (3). Moreover, by passing to the quotient, we can rearrange the sum in Definition 3.4.3 to be taken

over all the *oriented* ℓ_i -bands \mathcal{B} of (Σ_2, g_0) as follows

$$\mathcal{F}(\Psi^g) = \sum_{\mathcal{B}} \int_{\mathcal{B}} d\bar{\Psi}_{\nu_{\mathcal{B}}}^g \wedge d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^g$$

where $\nu_{\mathcal{B}}$ is an ℓ_i -direction that generates the ℓ_i -band \mathcal{B} and the vector $(\nu_{\mathcal{B}})^\perp$ is an ℓ_1 -direction or an ℓ_2 -direction orthogonal to the boundary $\partial\mathcal{B}$ and whose basepoint lies in the same regular octagon Ω as the vector $\nu_{\mathcal{B}}$. More precisely, if $\nu_{\mathcal{B}}$ points in the direction of an edge \mathcal{E}^j of Ω , then $(\nu_{\mathcal{B}})^\perp$ points in the direction of the edge \mathcal{E}^{j+2} , where the edges follow the cyclic order on $\partial\Omega$.

In particular, we have

$$\mathcal{F}(\Psi^{g_t}) - \mathcal{F}(\Psi^{g_0}) = I_1 + I_2,$$

where

$$I_1 = \sum_{\mathcal{B}} \int_{\mathcal{B}} d(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0}) \wedge d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0} - d(\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_t} - \bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0}) \wedge d\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0} \quad (3.4.2)$$

and

$$I_2 = \sum_{\mathcal{B}} \int_{\mathcal{B}} d(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0}) \wedge d(\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_t} - \bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0}).$$

First, we show that $I_1 = 0$. Recall that from Proposition 3.3.7(2), we have

$$-\Psi_{(\nu_{\mathcal{B}})^\perp}^g = \Psi_{-(\nu_{\mathcal{B}})^\perp}^g$$

and the same holds for $\bar{\Psi}$. Moreover, since the oriented angle between the vectors $\nu_{\mathcal{B}}$ and $(\nu_{\mathcal{B}})^\perp$ is the same as the one between the vectors $-(\nu_{\mathcal{B}})^\perp$ and $\nu_{\mathcal{B}}$, we can write

$$I_1 = 2 \sum_{\mathcal{B}} \int_{\mathcal{B}} d(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0}) \wedge d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0}$$

after arranging the oriented bands \mathcal{B} in the sum (3.4.2). Moreover, the one-forms $d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0}$ are constant over the bands \mathcal{B} . Therefore, we derive

$$\sum_{\mathcal{B}} \int_{\mathcal{B}} d(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0}) \wedge d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0} = \sum_{\mathcal{B}} \int_{\mathcal{B}} d\left((\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0}) d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0}\right).$$

Now, let v be a tangent vector to the boundary $\partial\mathcal{B}$. The vector v is parallel to the ℓ_i -direction $\nu_{\mathcal{B}}$. Moreover, by definition, the unit vectors $\nu_{\mathcal{B}}$ and $(\nu_{\mathcal{B}})^\perp$ form an orthonormal basis. Since the g_0 -gradient of $\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0}$ satisfies

$$\nabla \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0} = \nu_{\mathcal{B}}$$

for every ℓ_i -direction $\nu_{\mathcal{B}}$, cf. Example 3.3.6, and the same holds with $(\nu_{\mathcal{B}})^\perp$, we have

$$d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0}(v) = g_0((\nu_{\mathcal{B}})^\perp, v) = 0.$$

Then by Stokes' formula, we derive that

$$\begin{aligned} \sum_{\mathcal{B}} \int_{\mathcal{B}} d \left((\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0}) d\bar{\Psi}_{(\nu_{\mathcal{B}})^{\perp}}^{g_0} \right) &= \sum_{\mathcal{B}} \int_{\partial\mathcal{B}} (\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0}) d\bar{\Psi}_{(\nu_{\mathcal{B}})^{\perp}}^{g_0} \\ &= 0. \end{aligned}$$

Note that $\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0}$ is well defined as a real function, *cf.* Remark 3.3.9 (2). Hence, the integral I_1 vanishes.

Finally, we show that $I_2 = 0$. This follows directly by definition of a slow metric variation, *cf.* Definition 3.4.1, and the Cauchy-Schwartz inequality. \square

Bibliography

- [AR00] Adams, C.; Reid, A.: Systoles of hyperbolic 3-manifolds. *Math. Proc. Cambridge Philos. Soc.* 128 (2000), no. 1, 103–110.
- [Al73] Ahlfors, L.V., Conformal invariants: Topics in geometric function theory, MacGraw-Hill, New York, 1973.
- [AB] Álvarez Paiva, J.-C.; Balacheff, F.: Infinitesimal systolic rigidity of metrics all of whose geodesics are closed and of the same length. See arXiv:0912.3413.
- [AB14] Álvarez Paiva, J.-C.; Balacheff, F.: Contact geometry and isosystolic inequalities. *Geom. Funct. Anal.* 24 (2014), 648–669.
- [ABT] Álvarez Paiva, J.-C.; Balacheff, F.; Tzanev, K.: Isosystolic inequalities for optical hypersurfaces. See arXiv:1308.552.
- [AT04] Álvarez-Paiva J.-C.; Thompson A.-C.: Volumes on normed and Finsler spaces, A Sampler of Riemann-Finsler Geometry (D. Bao, R. Bryant, S.-S. Chern, and Z. Shen, eds.), Math. Sci. Res. Inst. Publ., no. 50, Cambridge Univ. Press, Cambridge, (2004), 1–48.
- [Bal06] Balacheff, F.: Sur la systole de la sphère au voisinage de la métrique standard, *Geom. Dedicata.* 121 (2006) 61–71.
- [Bal10] Balacheff, F.: A local optimal diastolic inequality on the two-sphere, *J. Topol. Anal.* 2 (2010), no. 1, 109–121.
- [BCS00] Bao, D.; Chern, S. S.; Shen, Z.: An Introduction to Riemann-Finsler Geometry, Graduate Texts in Mathematics 200, Springer-Verlag, New York, 2000.
- [Ba86] Bavard, C.: Inégalité isosystolique pour la bouteille de Klein, *Math. Ann.* 274 (1986) 439–441.
- [Ba88] Bavard, C.: Inégalités isosystoliques conformes pour la bouteille de Klein, *Geom. Dedicata.* 27 (1988) 349–355.
- [Ba92] Bavard, C.: Inégalités isosystoliques conformes, *Comment. Math. Helv.* 67 (1992) 146–166.

- [Ba06] Bavard, C.: Une remarque sur la géométrie systolique de la bouteille de Klein, *Arch. Math. (Basel)* 87 (2006) 72–74.
- [Be78] Besse, A.: *Manifolds all of whose geodesics are closed*, Springer-Verlag, Berlin-New York, 1978.
- [Bl61] Blatter, C.: Zur Riemannschen Geometrie im Grossen auf dem Möbiusband, *Compositio Math.* 15 (1961) 88–107.
- [BH99] Bridson, MR., Häfliger, A.: *Metric Spaces of Non-Positive Curvature*, Grundlehren der Mathematischen Wissenschaften, Springer, Berlin, Germany 1999.
- [Br96] Bryant, R.: On extremals with prescribed lagrangian densities, *Manifolds and geometry. de Bartolomeis (ed.) et al., Cambridge Univ. Press., Symp. Math.* 36 (1996) 86–111.
- [BBI01] Burago, D.; Burago, Y.; Ivanov, S.: *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001.
- [BI02] Burago, D.; Ivanov, S.: On asymptotic volume of Finsler tori, minimal surfaces in normed spaces, and symplectic filling volume, *Ann. of Math. (2)* 156 (2002), no. 3, 891–914.
- [BI12] Burago, D.; Ivanov, S.: Minimality of planes in normed spaces, *Geom. Funct. Anal.* 22 (2012), no. 3, 627–638.
- [Bu47] Busemann, H.: Intrinsic area, *Ann. of Math. (2)* 48 (1947) 234–267.
- [Ca96] Calabi, E.: Extremal isosystolic metrics for compact surfaces, *Actes de la Table Ronde de Géométrie Différentielle, Sémin. Congr.* 1 (1996), Soc. Math. France, 146–166.
- [dC76] M. do Carmo: *Differential geometry of curves and surfaces*. Prentice-Hall, Englewood Cliffs, NJ, 1976.
- [Du98] Durán, C. E.: A volume comparison theorem for Finsler manifolds, *Proc. Amer. Math. Soc.* 126 (1998) 3079–3082.
- [EL08] El Mir, C.; Lafontaine, J.: Sur la géométrie systolique des variétés de Bieberbach, *Geom. Dedicata.* 136, (2008) 95–110.
- [EY15] El Mir, C.; Yassine, Z.: Conformal geometric inequalities on the Klein bottle, *Conf. Geom. Dyn.* 19 (2015), 240–257.
- [EI00] El Soufi, A.; Ilias, S.: Riemannian manifolds admitting isometric immersions by their first eigenfunctions, *Pacific J. Math.* 195 (2000), no. 1, 91–99.
- [Fu57] Fuglede, B.: Extremal length and functional completion, *Acta Math.* 98 issue 1–4, (1957), 171–219.

- [GHL04] Gallot, S., Hulin, D., Lafontaine, J.: Riemannian Geometry, 3rd edition, Springer, Berlin Heidelberg 2004.
- [Gr83] Gromov, M.: Filling Riemannian manifolds, *J. Diff. Geom.* 18 (1983), 1–147.
- [Gr99] Gromov, M.: Metric structures for Riemannian and non-Riemannian spaces. Progr. in Mathematics 152, Birkhäuser, Boston, 1999.
- [Gru07] Gruber, P.: Convex and discrete geometry. Grundlehren der Mathematischen Wissenschaften 336, Springer, Berlin, 2007.
- [He82] Hebda, J.: Some lower bounds for the area of surfaces, *Invent. Math* 65 issue 3, (1982), 485-490
- [He91] Hebda, J.: Two Geometric Inequalities for the Torus, *Geom. Dedicata* 38, (1991), 101–106.
- [HT79] Holmes R. D.; Thompson A. C.: N-dimensional area and content in Minkowski spaces, *Pacific J. Math.* 85 (1979) 77–110.
- [Iv02] Ivanov, S.: On two-dimensional minimal fillings, *St. Petersburg Math. J.* 13 (2002), no. 1, 17–25.
- [Iv11] Ivanov, S.: Filling minimality of Finslerian 2-discs, *Proc. Steklov Inst. Math.* 273 (2011) 176–190.
- [Je65] Jenkins, J. A.: Univalent functions and conformal mapping. Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Heft 18. Reihe: Moderne Funktionentheorie. Berlin (1965).
- [Ka07] Katz, M.: Systolic geometry and topology, Math. Surveys and Monographs 137, Amer. Math. Soc., Providence, R.I. 2007.
- [KS06] Katz, M., Sabourau, S.: An optimal systolic inequality for CAT(0) metrics in genus two, *Pacific J. Math.*, 227 (2004) no. 1, 95–107.
- [Ke67] Keen, L.: An extremal length on a torus, *J. Analyse Math.* 19 (1967), no. 1, 203-206.
- [Na96] Nadirashvili, N.: Berger’s isoperimetric problem and minimal immersions of surfaces, *Geom. Funct. Anal.* 6 (1996), no. 5, 877–897.
- [Pu52] Pu, P.: Some inequalities in certain non-orientable Riemannian manifolds, *Pacific J. Math.* 2 (1952) 55–71.
- [Ro68] Rodin, B.: Extremal length and geometric inequalities, *Proc. Sympos. Pure Math.*, vol. 11, Amer. Math. Soc. Providence, R.I., 370-376 (1968).

-
- [Sa04] Sabourau, S.: Systoles des surfaces plates singulières de genre deux, *Math. Zeitschrift.* 247 (2004), no. 4, 693–709.
- [Sa10] Sabourau, S.: Local extremality of the Calabi-Croke sphere for the length of the shortest closed geodesic, *J. London Math. Soc.* 82 (2010), no. 3, 549–562.
- [Sa11] Sabourau, S.: Isosystolic genus three surfaces critical for slow metric variations, *Geom. Topol.* 15 (2011), 1477–1508.
- [SY16] Sabourau, S., Yassine, Z.: Optimal systolic inequalities on Finsler Mobius bands, *J. Topol. Anal.*, 8 (2016), no. 2, 349–372.
- [Sak88] Sakai, T.: A proof of the isosystolic inequality for the Klein bottle, *Proc. Amer. Math. Soc.* 104 (1988) 589–590.
- [Sc93] Schmutz, P.: Riemann surfaces with shortest geodesic of maximal length. *Geom. Funct. Anal.* 3 (1993), no. 6, 564–631.
- [Sh01] Shen, Z.: Lectures on Finsler geometry. World Scientific Publishing Co., Singapore, 2001.