# Extremal systolic geometry on surfaces 

Zeina Yassine

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UNIVERSITÉ
— PARIS-EST

## École Doctorale MSTIC

Laboratoire d'Analyse et de Mathématiques Appliquées

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# Géométrie systolique extrémale sur les surfaces 

Spécialité : Mathématiques

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## Résumé

En 1949, C. Loewner a demontré dans un travail non publié que si $\mathbb{T}^{2}$ est un tore Riemannien, alors

$$
\operatorname{aire}\left(\mathbb{T}^{2}\right) \geq \frac{\sqrt{3}}{2} \operatorname{sys}\left(\mathbb{T}^{2}\right)^{2}
$$

où sys $\left(\mathbb{T}^{2}\right)$ est la systole du tore $\mathbb{T}^{2}$, i.e. la longeur du plus court lacet non contractile de $\mathbb{T}^{2}$. De plus, l'égalité est atteinte si et seulement si le tore est plat hexagonal. Ce résultat a donné naissance à la géométrie systolique. Dans cette thèse, nous étudions des inégalités de type systolique portant sur les longueurs minimales de différentes courbes et pas seulement la systole.

Dans un premier temps, nous démontrons trois inégalités géométriques optimales conformes sur la bouteille de Klein reliant l'aire au produit des longueurs des plus courts lacets noncontractiles dans des classes d'homotopie libres différentes. Pour chaque classe conforme, nous décrivons la métrique extrémale réalisant le cas d'égalité.

Nous établissons ensuite des inégalités géométriques optimales sur le ruban de Möbius muni d'une métrique de Finsler. Ces inégalités géométriques relient la systole et la hauteur du ruban de Möbius à son volume de Holmes-Thompson. Nous en déduisons une inégalité systolique optimale sur la bouteille de Klein munie d'une métrique de Finsler avec des symétries. Nous décrivons également une famille de métriques extrémales dans les deux cas.

Dans le troisième travail, nous démontrons une inégalité systolique critique sur la surface de genre deux. Plus précisément, il est connu que la surface de genre deux admet une métrique Riemannienne plate à singularités coniques qui est extrémale parmi les métriques à courbure nonpositive pour l'inégalité systolique. Nous montrons que cette métrique est en fait critique pour des variations lentes de métriques, cette fois-ci sans hypothèse de courbure, pour un autre problème systolique portant sur les longueurs des plus courts lacets non contractiles dans certaines classes d'homotopie libres données. Ces classes d'homotopie correspondent aux lacets systoliques et deux-systoliques de la surface extrémale.

## Mots-clé:

Systole, aire, métrique extrémale, métrique de Finsler, bouteille de Klein, ruban de Möbius, surface de genre deux.

## Extremal systolic geometry on surfaces

## Abstract

In 1949, C. Loewner proved in an unpublished work that if $\mathbb{T}^{2}$ is a Riemannian two-torus, then it satisfies

$$
\operatorname{area}\left(\mathbb{T}^{2}\right) \geq \frac{\sqrt{3}}{2} \operatorname{sys}\left(\mathbb{T}^{2}\right)^{2}
$$

where $\operatorname{sys}\left(\mathbb{T}^{2}\right)$ denotes the systole of the torus, i.e. the smallest length of a noncontractible loop in $\mathbb{T}^{2}$. Furthermore, the equality is attained if and only if the torus is flat hexagonal. This result led to what was called later systolic geometry. In this thesis, we study several systolic-like inequalities. These inequalities involve the minimal length of various curves and not merely the systole.

First we obtain three optimal conformal geometric inequalities on Riemannian Klein bottles relating the area to the product of the lengths of the shortest noncontractible loops in different free homotopy classes. We describe the extremal metrics in each conformal class.

Then we prove optimal systolic inequalities on Finsler Möbius bands relating the systole and the height of the Möbius band to its Holmes-Thompson volume. We also establish an optimal systolic inequality for Finsler Klein bottles with symmetries. We describe extremal metric families in both cases.

Finally, we prove a critical systolic inequality on genus two surface. More precisely, it is known that the genus two surface admits a piecewise flat metric with conical singularities which is extremal for the systolic inequality among all nonpositively curved Riemannian metrics. We show that this piecewise flat metric is also critical for slow metric variations, this time without curvature restrictions, for another type of systolic inequality involving the lengths of the shortest noncontractible loops in different free homotopy classes. The free homotopy classes considered correspond to those of the systolic loops and the second-systolic loops of the extremal surface.

## Keywords:

Systole, area, extremal metric, Finsler metric, Klein bottle, Möbius band, genus two surface.

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## Introduction

We are interested in studying optimal geometric inequalities relating the area of a closed Riemannian surface $\Sigma$ to the lengths of the shortest closed geodesics in certain homotopy classes. A typical example is given by the systolic inequalities, which relate the area of the surface to the systole, that is, the smallest length of a noncontractible loop in $\Sigma$. The first known systolic inequality is due to C. Loewner in 1949. He proved that every Riemannian two-torus $\mathbb{T}^{2}$ satisfies

$$
\operatorname{area}\left(\mathbb{T}^{2}\right) \geq \frac{\sqrt{3}}{2} \operatorname{sys}\left(\mathbb{T}^{2}\right)^{2}
$$

where $\operatorname{sys}\left(\mathbb{T}^{2}\right)$ denotes the systole of $\mathbb{T}^{2}$. Furthermore, the equality holds if and only if the torus is flat hexagonal. C. Loewner did not publish his result, however it was mentioned by his student P . $\mathrm{Pu}, c f$. [Pu52], who established a sharp systolic inequality on the projective plane $\mathbb{R}^{2}$. In this case, the equality is attained precisely by the round metrics. More than thirty years later, C. Bavard, cf. [Ba86] (see also [Sak88]) proved a sharp systolic inequality on the Klein bottle. Here, the extremal metrics are not smooth. These are the only manifolds for which an optimal systolic inequality is known. Optimal systolic-like inequalities, i.e., inequalities that relate the area to the product of the lengths of the shortest loops or arcs in different (relative) homotopy classes are known only for the two-torus $\mathbb{T}^{2}$, cf. [Ke67], the Klein bottle $\mathbb{K}$, cf. [Ba06, EY15], and the Mobius band $\mathbb{M}, c f$. [Pu52, Bl61, Ba06]. With the exception of some recent proofs of systolic inequalities, cf. [Iv02, Sa10], and systolic-like inequalities, cf. [SY16], on the projective plane $\mathbb{R}^{2}$, the two-torus $\mathbb{T}^{2}$ and the Mobius band $\mathbb{M}$, all the other known proofs of the aforementioned optimal geometric inequalities require the uniformization theorem as a main tool.

This thesis is divided into three chapters. In these chapters, we study various optimal systolic-like inequalities for different surfaces: the Klein bottle, the Mobius band, the torus and the genus two surface. Each chapter relies on specific techniques: extremal length methods in the first chapter, Finsler geometry in the second chapter and infinitesimal calibrating arguments in infinitedimensional spaces in the third chapter.

In the first chapter, we present a joint work with Chady ElMir published in the Journal of Conformal Geometry and Dynamics, cf. [EY15].

In 1986, C. Bavard, cf. [Ba86], proved that every Riemannian Klein bottle $\mathbb{K}$ satisfies

$$
\operatorname{area}(\mathbb{K}) \geq \frac{2 \sqrt{2}}{\pi} \operatorname{sys}(\mathbb{K})^{2}
$$

The extremal Klein bottle is the double of the $\frac{\pi}{4}$-tubular neighborhood of the equator of the Euclidean sphere $S^{2}$, with the identification described in Figure 1. In his proof, C. Bavard relies on P. Pu's inequality on the Mobius band $\mathbb{M}$, $c f$. [Pu52]. Later in 1988, T. Sakai [Sak88] proved the same result using C. Blatter's inequalities on $\mathbb{M}$, $c f$. [Bl61]. We refer to Section 2.4 for precise
definitions and details about P. Pu's and C. Blatter's inequalities on the Mobius band. A selfcontained proof of the systolic volume of the Riemannian Klein bottle that do not require the use of inequalities on the Mobius band was given by C. Bavard in [Ba88]. The main tool in his proof is the method of extremal lengths, which is based also on the uniformization theorem. We refer to Section 1.2 for further details about it. This method is the main tool that we use in Chapter 1 to derive new optimal geometric inequalities on the Klein bottle relating the area to the length of loops in different free homotopy classes.

In order to describe these inequalities we introduce some notations. Recall that a flat Klein bottle $\mathbb{K}$ is the quotient of $\mathbb{C}$ by the group generated by a glide reflection $\sigma$ and a translation $t$. The fundamental group $\pi_{1}(\mathbb{K})$ can be identified with the deck group $\langle\sigma, t\rangle$ generated by $\sigma$ and $t$ with their relation $\sigma t \sigma^{-1} t=e$. For every $a \in \pi_{1}(\mathbb{K})$, denote by $\gamma_{a}$ a loop representing the homotopy class $a$. We distinguish three families of free noncontractible loops on the Klein bottle $\mathbb{K}$ as follows.

1. $\mathcal{F}_{\sigma}=\left\{\gamma \mid \gamma\right.$ is a loop freely homotopic to $\gamma_{\sigma}$ or $\left.\gamma_{\sigma t}\right\}$,
2. $\mathcal{F}_{v}=\left\{\gamma \mid \gamma\right.$ is a loop freely homotopic to $\left.\gamma_{t}\right\}$,
3. $\mathcal{F}_{h}=\left\{\gamma \mid \gamma\right.$ is a loop freely homotopic to $\left.\gamma_{\sigma^{2}}\right\}$.

Furthermore, we define $\ell_{\sigma}$ (resp. $\ell_{v}$, resp. $\ell_{h}$, resp. $L_{\sigma}$ ) to be the smallest length of a noncontractible free loop in $\mathcal{F}_{\sigma}$ (resp. $\mathcal{F}_{v}$, resp. $\mathcal{F}_{h}$, resp. $\mathcal{F}_{\sigma} \cup \mathcal{F}_{h}$ ). Observe that the extremal metric $g_{e}$ for the systole, $c f$. Figure 1, satisfies

$$
\ell_{\sigma}\left(g_{e}\right)=\ell_{h}\left(g_{e}\right)=\ell_{v}\left(g_{e}\right)=L_{\sigma}\left(g_{e}\right)
$$

We summarize the main results of the first chapter in the following theorem.


Figure 1: The double of the $\frac{\pi}{4}$-tubular neighborhood of the equator of the Euclidean sphere $S^{2}$ is an extremal Klein bottle.

Theorem 0.0.1. Let $\beta>0$. For every Riemannian metric $g$ on the Klein bottle $\mathbb{K}$ of conformal type $\beta$, we have

1. $\operatorname{area}(\mathbb{K}, g) \geq C_{\beta}^{1} \ell_{\sigma}(g) \ell_{v}(g)$,
2. $\operatorname{area}(\mathbb{K}, g) \geq C_{\beta}^{2} L_{\sigma}(g) \ell_{v}(g)$,
3. $\operatorname{area}^{\frac{3}{2}}(\mathbb{K}, g) \geq C_{\beta}^{3} \ell_{\sigma}(g) \ell_{v}(g) \ell_{h}(g)$,
where $C_{\beta}^{i}, i \in\{1,2,3\}$, are constants depending only on the conformal type $\beta$. Moreover, these inequalities are optimal.

In fact, the inequality 1 . was proved by C. Bavard, $c f$. [Ba06], for the values of $\beta \in\left(0,2 \ln \left(\frac{\pi}{4}+\frac{a_{0}}{2}\right)\right]$, where $a_{0} \in\left(0, \frac{\pi}{2}\right)$ is the unique solution of the equation $\tan \left(a_{0}\right)=2 a_{0}$. We complete his work to cover all the remaining conformal classes. Observe also that the inequality 3. involves three different lengths. To our knowledge, there are no other known systolic-like inequalities, even non-optimal ones, for which the number of lengths is greater than the dimension of the manifold. We refer to Section 1.1.2 and Table 1.1 of Chapter 1 for details about the corresponding extremal metrics in each conformal class.

As a consequence of Theorem 0.0.1, we conclude that
Corollary 0.0.2. There does not exist any positive universal constant $c$ such that the inequality

$$
\operatorname{area}(\mathbb{K}, g) \geq c \ell_{\sigma}(g) \ell_{v}(g)
$$

holds for every Riemannian metric $g$ on the Klein bottle. Similarly, there does not exist any positive universal constant $c$ such that the inequality

$$
\operatorname{area}^{\frac{3}{2}}(\mathbb{K}, g) \geq c \ell_{\sigma}(g) \ell_{v}(g) \ell_{h}(g)
$$

holds for every Riemannian metric $g$ on the Klein bottle.

On the other hand, we have
Corollary 0.0.3. For every Riemannian metric $g$ on the Klein bottle $\mathbb{K}$,

$$
\operatorname{area}(\mathbb{K}, g)>\frac{1}{2} L_{\sigma}(g) \ell_{v}(g)
$$

Moreover, the inequality is optimal.

In the second chapter, we present a joint work with Stéphane Sabourau, published in the Journal of Topology and Analysis, cf. [SY16].

This chapter is dedicated to the study of systolic inequalities for Finsler Mobius bands. Loosely speaking, a Finsler metric $F$ is defined as a Riemannian metric except that its restriction to a tangent plane is no longer a Euclidean norm but a Minkowski norm. From a dynamical point of view, the function $F^{2}$ can be considered as a Lagrangian which induces a Lagrangian flow on the tangent bundle $T M$ of $M$. Thus, Finsler manifolds can be considered as degree 2 homogeneous Lagrangian systems. The trajectories of the Lagrangian correspond to the geodesics of the Finsler metric. There exist several definitions of volume for Finsler manifolds which coincide in the Riemannian case. In this chapter, we consider the Holmes-Thompson volume vol $H T$. Recall that the Holmes-Thompson volume of an $n$-dimensional Finsler manifold is equal to the symplectic volume of its unit co-disc bundle divided by the volume of the Euclidean unit ball of dimension $n$. We refer to Section 2.2 for further details about Finsler metrics.

Since there is no Finsler analog to the uniformization theorem, studying the systolic volume of Finsler surfaces is quite delicate. The first optimal Finsler systolic inequality has been obtained by S. Ivanov [Iv02]. He proved that the optimal systolic inequality on the projective plane for Riemannian metrics also holds for Finsler metrics. Namely, if $\mathbb{R P}^{2}$ is a Finsler projective plane, then

$$
\operatorname{area}\left(\mathbb{R P}^{2}\right) \geq \frac{2}{\pi} \operatorname{sys}\left(\mathbb{R P}^{2}\right)^{2}
$$

The extremal Finsler metrics are those all of whose geodesics are closed and of the same length. Contrary to the Riemannian case, the Finsler systolic constant of the two-torus is the same as that of the projective plane. More precisely, S. Sabourau [Sa10] proved that if $\mathbb{T}^{2}$ is a Finsler two-torus, then

$$
\operatorname{area}\left(\mathbb{T}^{2}\right) \geq \frac{2}{\pi} \operatorname{sys}\left(\mathbb{T}^{2}\right)^{2}
$$

Finsler two-tori that are homothetic to the quotient of $\mathbb{R}^{2}$, endowed with a parallelogram norm $\|$.$\| , by a lattice whose unit disk of \|$.$\| is a fundamental domain are extremal. However, we$ do not know if those are the only extremal metrics. A sharp systolic inequality for asymmetric Finsler two-tori can be found in [ABT].

The main result of the second chapter is the following.
Theorem 0.0.4. Let $\mathbb{M}$ be a Finsler Mobius band. Let $\lambda:=\frac{h(\mathbb{M})}{\operatorname{sys}(\mathbb{M})}$. Then

$$
\frac{\operatorname{area}(\mathbb{M})}{\operatorname{sys}(\mathbb{M}) h(\mathbb{M})} \geq \begin{cases}\frac{2}{\pi} & \text { if } \lambda \in(0,1] \\ \frac{1}{\pi} \frac{\lambda+1}{\lambda} & \text { otherwise }\end{cases}
$$

Moreover, the above inequalities are optimal for every value of $\lambda \in(0,+\infty)$.
Here, $h(\mathbb{M})$ represents the minimal length of arcs with endpoints on the boundary $\partial \mathbb{M}$ which are not homotopically trivial relative to $\partial \mathbb{M}$. We refer to Examples 2.5.4 and 2.6.2 for a description of an extremal metric family. As in the case of Finsler two-tori, we do not know if these are the only extremal metrics.

The proof of Theorem 0.0 .4 is divided into two cases. The first case is when

$$
h(\mathbb{M}) \geq \operatorname{sys}(\mathbb{M})
$$

Its proof proceeds as follows. We first derive a Finsler analog of L. Keen's inequality for two-tori, cf. [Ke67]. More precisely

Proposition 0.0.5. Let $\mathbb{T}^{2}$ be a Finsler two-torus. There exist two closed curves of lengths $a$ and $b$ generating the first integral homology group of $\mathbb{T}^{2}$ such that

$$
\operatorname{area}\left(\mathbb{T}^{2}\right) \geq \frac{2}{\pi} a b
$$

Equality holds if $\mathbb{T}^{2}$ is homothetic to the quotient of $\mathbb{R}^{2}$, endowed with a parallelogram norm $\|$.$\| ,$ by a lattice generated by two vectors of lengths $a$ and $b$, parallel to the sides of the unit ball of $\|$.$\| .$

The proof of Proposition 0.0 .5 can be used as well in the Riemannian case and yields L. Keen's result without making use of the uniformization theorem as in the original proof. With the help of Proposition 0.0 .5 , we derive a similar result for Finsler cylinders $C$. More precisely, we obtain

$$
\operatorname{area}(C) \geq \frac{2}{\pi} \operatorname{sys}(C) h(C)
$$

where $h$ denotes the height of the cylinder, that is, the distance between its two boundary components. Using this result and the Finsler systolic inequality on the projective plane, we obtain optimal systolic inequalities on wide Finsler Mobius bands, cf. Section 2.5.

Furthermore, in order to prove the systolic inequality for Finsler Mobius bands $\mathbb{M}$ that satisfy

$$
h(\mathbb{M})<\operatorname{sys}(\mathbb{M})
$$

we derive a systolic-height inequality which holds true whenever $\frac{h(\mathbb{M})}{\operatorname{sys}(\mathbb{M})}$ is a dyadic rational in the interval $(0,1)$ and deduce the result by continuity of the volume, the height and the systole over Finsler metrics. Hence the systolic inequality holds for narrow Mobius bands too, $c f$. Section 2.6.

The optimal systolic inequality for Finsler Klein bottles is still unknown. However, using Theorem 0.0.4, we deduce that Finsler Klein bottles with some natural symmetries satisfy the same systolic volume as Finsler two-tori $\mathbb{T}^{2}$ and Finsler projective planes $\mathbb{R P}^{2}$. In particular, we prove

Theorem 0.0.6. Let $\mathbb{K}$ be a Finsler Klein bottle with a soul, soul-switching or rotational symmetry. Then

$$
\begin{equation*}
\operatorname{area}(\mathbb{K}) \geq \frac{2}{\pi} \operatorname{sys}(\mathbb{K})^{2} \tag{0.0.1}
\end{equation*}
$$

Moreover, the inequality is optimal.
Recall that in the Riemannian case, Theorem 0.0 .6 follows directly from the uniformization theorem, an average argument over the isometry group, and the Cauchy-Schwartz inequality. In
our proof, we do not use any of these arguments. Instead, we rely on the sharp systolic-like inequality we established for Finsler Mobius bands.

We end this chapter by proposing the following conjecture
Conjecture. Let $\mathbb{K}$ be a Finsler Klein bottle. Then

$$
\begin{equation*}
\operatorname{area}(\mathbb{K}) \geq \frac{2}{\pi} \operatorname{sys}(\mathbb{K})^{2} \tag{0.0.2}
\end{equation*}
$$

A Finsler Klein bottle with systolic volume equal to $\frac{2}{\pi}$ is described in Example 2.7.7.

The third chapter deals with critical isosystolic genus two surfaces.
Recently, M. Katz and S. Sabourau, cf. [KS06], proved that there exists a piecewise flat metric $g_{0}$ on the genus two surface $\Sigma_{2}$ which is extremal among all nonpositively curved Riemannian metrics. This metric is composed of six regular flat octagons. It admits regions where only one systolic loop, i.e., a noncontractible loop of length the systole, passes through every point. Hence this metric cannot be extremal for the general systolic inequality, i.e., without restriction on the curvature. This follows from a result of E. Calabi, cf. [Ca96], which says that at least two systolic loops pass through every point of a systolically extremal surface. In fact, the non-extremality of the metric $g_{0}$ for the systolic inequality can also be deduced from a result of S. Sabourau, cf. [Sa04], which says that no flat metric with conical singularities is extremal for the systolic inequality in genus two.

Denote by $\ell_{1}(g)$ the length of the shortest noncontractible loop on $\left(\Sigma_{2}, g\right)$, that is,

$$
\ell_{1}(g)=\operatorname{sys}(g)
$$

Recall that $g_{0}$ designates the extremal nonpositively curved genus two surface defined in [KS06]. Let $\Lambda_{1}$ be the subset of the free homotopy classes generated by the systolic loops of $\left(\Sigma_{2}, g_{0}\right)$ and their multiples:

$$
\begin{equation*}
\Lambda_{1}=\left\{\left\langle\gamma^{k}\right\rangle \mid\langle\gamma\rangle \neq 0, \ell_{g_{0}}(\gamma)=\ell_{1}\left(g_{0}\right) \text { and } k \in \mathbb{Z}^{*}\right\} \tag{0.0.3}
\end{equation*}
$$

Similarly, let $\ell_{2}(g)$ be the length of the shortest noncontractible loop on $\left(\Sigma_{2}, g\right)$ which is not homotopic to a systolic loop. That is,

$$
\ell_{2}(g)=\inf _{\langle\gamma\rangle \notin \Lambda_{1} \cup\{0\}} \ell_{g}(\gamma)
$$

Denote also by

$$
\begin{equation*}
\Lambda_{2}=\left\{\langle\gamma\rangle \mid\langle\gamma\rangle \notin \Lambda_{1} \text { and } \ell_{g_{0}}(\gamma)=\ell_{2}\left(g_{0}\right)\right\} \tag{0.0.4}
\end{equation*}
$$

the subset of the free homotopy classes generated by the shortest noncontractible loops of $\left(\Sigma_{2}, g_{0}\right)$ which are not freely homotopic to systolic loops.

We observe that the piecewise flat metric $g_{0}$ defined on the genus two surface $\Sigma_{2}$ may be a potential extremal metric for another systolic problem on $\Sigma_{2}$. This observation follows from some geometric properties of the surface $\left(\Sigma_{2}, g_{0}\right)$. First, it is known that the systolic loops cover a systolically extremal surface. In our case, the $\ell_{1}$-loops and the $\ell_{2}$-loops cover the surface $\Sigma_{2}$. Here, by an $\ell_{i}$-loop we mean a loop of length $\ell_{i}\left(g_{0}\right)$ in $\left(\Sigma_{2}, g_{0}\right)$ whose free homotopy class lies in $\Lambda_{i}$. Second, the unit tangent vectors of these $\ell_{i}$-loops are well distributed on each tangent plane of the surface. More precisely, their convex hull forms a regular octagon on these tangent planes. In general, the convex hull of the unit tangent vectors of systolic loops of all the known systolically extremal surfaces is symmetric. Finally, an extremal surface of genus at least two tends to have flat regions, $c f$. [Ca96, Br96].

In [Ca96], E. Calabi described two piecewise flat genus three surfaces and conjectured that one of them is the global minimum for the systolic inequality. Later, S. Sabourau, cf. [Sa11], proved that these two genus three surfaces are critical for the systolic inequality with respect to slow metric variations. Of course, since the systolic volume functional is not necessarily differentiable, an adequate notion of criticality needs to be defined. The notion of criticality used in [Sa11] is the same as the one introduced in [Na96, EI00] to study the Riemannian surfaces that maximize the product of the area with the first nonzero eigenvalue of the Laplacian. Other notions of systolically critical metrics were used in [Bal06, Bal10, Sa10, AB14].

In this chapter we show that the metric $g_{0}$ on the genus two surface $\Sigma_{2}$ is critical in the sense of [Na96, EI00, Sa11] for slow metric variations. More precisely, we prove

Theorem 0.0.7. Let $g_{t}$ be a slow metric variation of $g_{0}$ defined on the genus two surface $\Sigma_{2}$. If $\ell_{1}\left(g_{t}\right) \geq \ell_{1}\left(g_{0}\right)$ and $\ell_{2}\left(g_{t}\right) \geq \ell_{2}\left(g_{0}\right)$, then

$$
\operatorname{area}\left(g_{t}\right) \geq \operatorname{area}\left(g_{0}\right)+o(t) .
$$

The slow metric variations involved in Theorem 3.1.1 are analogous to those defined by S. Sabourau in [Sa11]. We refer to Definition 3.4.1 for a precise definition of these metric variations and to the last section of [Sa11] for examples. Observe for instance that deforming the regular octagons composing the extremal surface $\left(\Sigma_{2}, g_{0}\right)$ into non-regular octagons gives rise to a slow metric variation.

To prove our result, we rely on recent calibrating methods, cf. [Iv02, Iv11, Sa11], that do not require the uniformization theorem classically used to establish sharp systolic inequalities on surfaces. Briefly, we embed the universal cover $\widetilde{\Sigma}_{2}$ of the genus two surface $\Sigma_{2}$ in an infinitedimensional space $\mathbb{R}^{\infty}$ as follows. Given that $\ell_{i}(g) \geq \ell_{i}\left(g_{0}\right)$, we construct a 1-Lipschitz equivariant map

$$
\Psi^{g}: \widetilde{\Sigma}_{2} \rightarrow \mathbb{R}^{\infty}
$$

using the Busemann functions induced by the lifts of the $\ell_{i}$-loops of $\Sigma_{2}$, cf. Section 3.3. Then, we
introduce an appropriate infinitesimally calibrating two-form $\omega$ on $\mathbb{R}^{\infty}$ whose pull-back passes to the quotient on the surface $\Sigma_{2}$. This allows us to show that

$$
\int_{\Sigma_{2}}\left(\Psi^{g}\right)^{*} \omega \leq c_{0} \operatorname{area}\left(\Sigma_{2}, g\right)
$$

for every Riemannian metric $g$ on $\Sigma_{2}$, where $c_{0}$ is a sharp positive constant. Moreover, the equality holds if $g=g_{0}$. Finally, we prove that if $\left(g_{t}\right)_{t \geq 0}$ is a slow metric variation then

$$
\int_{\Sigma_{2}}\left(\Psi^{g_{t}}\right)^{*} \omega-\int_{\Sigma_{2}}\left(\Psi^{g_{0}}\right)^{*} \omega=o(t)
$$

This completes the proof, $c f$. Section 3.4.

## Chapter 1

# Conformal geometric inequalities on the Klein bottle 

Nous démontrons trois inégalités géométriques optimales conformes sur la bouteille de Klein reliant l'aire au produit des longueurs des plus courts lacets noncontractiles dans des classes d'homotopie libres différentes. Pour chaque classe conforme, nous décrivons la métrique extrémale réalisant le cas d'égalité.


#### Abstract

We obtain three optimal conformal geometric inequalities on Riemannian Klein bottles relating the area to the product of the lengths of the shortest noncontractible loops in different free homotopy classes. We describe the extremal metrics in each conformal class.


### 1.1 Introduction, preliminaries and results

Among all Riemannian metrics on a given compact differentiable manifold, the most interesting ones are those that extremize some Riemannian invariant. An interesting problem, for example, is to study metrics which maximize the ratio $\frac{\operatorname{sys}(M, g)^{n}}{\operatorname{vol}(M, g)}$ over the set of all Riemannian metrics $g$ on a given $n$-dimensional differentiable manifold $M$, where $\operatorname{sys}(M, g)$ denotes the systole of $(M, g)$, i.e., the least length of a non-contractible closed curve. Concerning this problem, called isosystolic problem, it is known that the extremal metric for the 2-dimensional torus is the flat hexagonal metric (unique up to a homothety). It is due to C. Loewner in 1949 (unpublished). His student, P.M. Pu, showed in 1952 that the extremal metric on the projective plane is the spherical metric, cf. [Pu52]). Nevertheless, as in many works related to the isosystolic problem, various constraints can be put on the set of Riemannian metrics under consideration. For example, we may restrict ourselves to the set of nonpositive curvature metrics $c f$. [KS06], or to the set of metrics in a fixed conformal class $c f$. [B161, Ba88, Ba92, Ba06].

In 1961, C. Blatter proved, cf. [Bl61] optimal conformal lower bounds on the area of the Mobius band with boundary $\mathbb{M}$ in terms of the product of the least lengths of two classes of curves. The first class consists of the family $\mathcal{F}$ of arcs joining two points on the boundary with non-trivial intersection with the soul of the Mobius band. The second class consists of the family $\mathcal{G}$ of loops in the homotopy class of a generator of $\pi_{1}(\mathbb{M})$. Let $\ell_{v}$ be the least length of an arc in $\mathcal{F}$ and $\ell_{\sigma}$ be the least length of a loop in $\mathcal{G}$. Then, Blatter (in his notations, $\ell_{v}=\ell^{*}$ and $\ell_{\sigma}=\ell_{1}$ ) obtained the following optimal conformal lower bound on the area of $\mathbb{M}$

$$
\begin{equation*}
\ell_{\sigma}(g) \ell_{v}(g) \leq C_{\beta} \operatorname{area}(g) \tag{1.1.1}
\end{equation*}
$$

where $C_{\beta}$ is a positive constant that depends only on the conformal type $\beta$ of $g$. Note that the optimal constant $C_{\beta}$ is not bounded from above over the set of conformal types $\beta$. Also, in the same article, Blatter proved the following conformal lower bound on the area of $\mathbb{M}$

$$
\begin{equation*}
\operatorname{sys}(g) \ell_{v}(g) \leq C_{\beta}^{\prime} \operatorname{area}(g) \tag{1.1.2}
\end{equation*}
$$

where $C_{\beta}^{\prime}$ is a positive constant that depends only on the conformal type $\beta$ of $g$. Contrarily to the previous case, the optimal constant $C_{\beta}^{\prime}$ is bounded from above over the set of conformal types $\beta$. More precisely, $\sup _{\beta} C_{\beta}^{\prime}=2$. Hence, the optimal inequality

$$
\operatorname{sys}(g) \ell_{v}(g) \leq 2 \operatorname{area}(g)
$$

holds for every Riemannian metric $g$ on $\mathbb{M}$.
Variants of C. Blatter's problem were studied by L. Keen in [Ke67] and J. Hebda in [He91] on the 2-dimensional torus. Keen obtained a lower bound on the area in terms of the product of the least lengths of two loops generating the fundamental group. On the other hand, Hebda obtained a lower bound on the square of the area in terms of the three primitive length spectrum. The equality in both cases is attained by the flat hexagonal metric.

In this chapter, we prove three types of inequalities on the Klein bottle $\mathbb{K}$ in the same spirit as Blatter's inequalities. The classes of curves we consider are (free) homotopy classes of loops that are candidates to realize the systole. In particular, the role of the family $\mathcal{F}$ will be taken by a free homotopy class of loops representing the vertical translation in $\pi_{1}(\mathbb{K})$. The main tool in our proof is the method of extremal length which was used by C. Bavard in 1988 to prove conformal isosystolic inequalities on the Klein bottle $c f$. [Ba88]. It can be applied to the Mobius band with boundary and provides an alternative proof of Blatter's inequalities (1.1.1) and (1.1.2).

### 1.1.1 The Klein bottle

A flat Klein bottle is the quotient of $\mathbb{C}$ by the group generated by the maps $\sigma: z \mapsto \bar{z}+\pi$ and $t_{v}: z \mapsto z+2 i \beta$. The induced flat metric on the Klein bottle will be denoted by $g_{\beta}$. By the uniformization theorem, every Riemannian metric $g$ on the Klein bottle is conformally equivalent to a unique flat metric $g_{\beta}$ for some $\beta \in(0,+\infty)$. The parameter $\beta$ represents the conformal type of the metric $g$. We call $t_{h}:=\sigma^{2}$ a horizontal translation and identify the fundamental group $\pi_{1}(\mathbb{K})$ with the deck group $\left\langle\sigma, t_{v}\right\rangle$. Note that any Klein bottle $\mathbb{K}$ is obtained by gluing two Mobius bands $\mathbb{M}_{1}$ and $\mathbb{M}_{2}$ along their boundaries. With the previous notations, fundamental domains of $\mathbb{K}, \mathbb{M}_{1}$ and $\mathbb{M}_{2}$, are $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times[-\beta, \beta],\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times\left[\frac{-\beta}{2}, \frac{\beta}{2}\right]$ and $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times\left[-\beta, \frac{-\beta}{2}\right] \cup\left[\frac{\beta}{2}, \beta\right]$ respectively.

By C. Bavard's theorem, cf. [Ba86], every Riemannian Klein bottle $\mathbb{K}$ satisfies the isosystolic inequality

$$
\operatorname{sys}(\mathbb{K})^{2} \leq \frac{\pi}{2 \sqrt{2}} \operatorname{area}(\mathbb{K})
$$

where the equality is attained by a spherical metric outside a singular line (see [Ba86], [Sak88] and [EL08] p. 100 for a detailed description of the extremal Klein bottle). For details and (many) open problems in systolic geometry see the book of Katz [Ka07] and the paper of M. Gromov [Gr83].

In this chapter, we consider the following distinct families of homotopy classes of closed curves on the Klein bottle $\mathbb{K}$ as follows.

Definition 1.1.1. For every $a \in \pi_{1}(\mathbb{K})$, denote by $\gamma_{a}$ a loop representing the homotopy class $a$. Then we define

1. $\mathcal{F}_{\sigma}=\left\{\gamma \mid \gamma\right.$ is a loop freely homotopic to $\gamma_{\sigma}$ or $\left.\gamma_{\sigma t}\right\}$,
2. $\mathcal{F}_{v}=\left\{\gamma \mid \gamma\right.$ is a loop freely homotopic to $\left.\gamma_{t v}\right\}$,
3. $\mathcal{F}_{h}=\left\{\gamma \mid \gamma\right.$ is a loop freely homotopic to $\left.\gamma_{\sigma^{2}}\right\}$.

Given a Riemannian metric on $\mathbb{K}$, the least length of a closed curve in the first (resp. second, resp. third) family will be denoted by $\ell_{\sigma}$ (resp. $\ell_{v}$, resp. $\ell_{h}$ ). Also, denote by $L_{\sigma}$ the least length of a loop in $\mathcal{F}_{\sigma} \cup \mathcal{F}_{h}$.

Remark 1.1.2. An extremal metric $g_{e x t}$ for the isosystolic inequality on the Klein bottle $\mathbb{K}$ satisfies

$$
\ell_{\sigma}\left(g_{e x t}\right)=L_{\sigma}\left(g_{e x t}\right)=\ell_{v}\left(g_{e x t}\right)
$$

Its conformal type $\beta$ is equal to $2 \ln \left(\tan \left(\frac{3 \pi}{8}\right)\right)$, cf. [Ba86].

Finally, denote by $a_{0}$ the unique real $x \in\left(0, \frac{\pi}{2}\right)$ such that $\tan (x)=2 x\left(a_{0} \approx 1.1655\right)$.

### 1.1.2 Two families of Riemannian metrics on the Klein bottle

We denote by $C^{0, \infty}(\mathbb{R})$ the set of continuous and piecewise smooth functions on $\mathbb{R}$. Let $h^{\beta}$ be a Riemannian metric in $\mathbb{C} \simeq \mathbb{R}^{2}$, periodic with respect to the fundamental domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times$ $[-\beta, \beta]$ and satisfying

$$
\begin{equation*}
\left(h^{\beta}\right)_{(x, y)}=\varphi(y)\left(d x^{2}+d y^{2}\right) \tag{1.1.3}
\end{equation*}
$$

where $\varphi \in C^{0, \infty}(\mathbb{R})$ is a positive, even and $2 \beta$-periodic function. Then there exists a diffeomorphism $G: \mathbb{C} \rightarrow \mathbb{C}$ for which the Riemannian metric $h_{b}=\left(G^{-1}\right)^{*} h^{\beta}$ satisfies

$$
\begin{equation*}
\left(h_{b}\right)_{(u, v)}=f^{2}(v) d u^{2}+d v^{2} \tag{1.1.4}
\end{equation*}
$$

where $f \in C^{0, \infty}(\mathbb{R})$ is positive, even and $2 b$-periodic function. To see this, let $\phi(y)=\int_{0}^{y} \sqrt{\varphi(t)} d t$ and $f=\left(\varphi \circ \phi^{-1}\right)^{\frac{1}{2}}$ and define the map $G$ by

$$
\begin{equation*}
G(x, y)=(x, \phi(y)) \tag{1.1.5}
\end{equation*}
$$

The Riemannian metric $h_{b}$ is then periodic with respect to the fundamental domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times$ $[-2 b, 2 b]$. The quotient of $\left(\mathbb{C}, h_{b}\right)$ by the subgroup of isometries $<\sigma, t_{v^{\prime}}>$, where $\sigma: z \mapsto \bar{z}+\pi$ and $t_{v^{\prime}}: z \mapsto z+4 i b$, is a Riemannian Klein bottle. Its conformal type is

$$
\beta=\int_{0}^{2 b} \frac{d t}{f(t)}
$$

Now, we will introduce two families of metrics on $\mathbb{C}$. They induce families of metrics on the Klein bottle that will be used later in this chapter as conformally extremal metrics for three types of inequalities on the Klein bottle. For every $b \in(0,+\infty)$ and $\omega \in(0, b]$, they are as follows.

1. the metric $S_{b}$ on $\mathbb{C}$ periodic with respect to the fundamental domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times[-2 b, 2 b]$ and defined by

$$
\left(S_{b}\right)_{(u, v)}=f_{b}^{2}(v) d u^{2}+d v^{2}
$$

where $f_{b}$ is the unique one-variable function invariant by the translation $(u, v) \mapsto(u, v+$ $2 b$ ) which agrees with cosine on $[-b, b]$. It is spherical outside the singular lines $v=n b$, where $n \in \mathbb{Z}$.
2. the metric $S F_{b, \omega}$ on $\mathbb{C}$ periodic with respect to the fundamental domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times[-2 b, 2 b]$ and defined by

$$
\left(S F_{b, \omega}\right)_{(u, v)}=f_{b, \omega}^{2}(v) d u^{2}+d v^{2}
$$

where $f_{b, \omega}$ is the unique one-variable function invariant by the translation $(u, v) \mapsto(u, v+$ $2 b)$ which agrees with cosine on $[-\omega, \omega]$ and equal to the constant $\cos (\omega)$ on $[\omega, 2 b-\omega]$, $c f$. [Ba88, Fig. 1]. It is spherical on the band $\mathbb{R} \times[-\omega, \omega]$ and its images by the translations $(u, v) \mapsto(u, v+2 n b)$, where $n \in \mathbb{Z}$, and flat elsewhere.

We will denote by $\left(\mathbb{K}, S_{b}\right)$ the Riemannian Klein bottle obtained by taking the quotient of $\left(\mathbb{C}, S_{b}\right)$ by the subgroup $<\sigma, t_{v^{\prime}}>$. Its conformal type is

$$
\beta=2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{b}{2}\right)\right)
$$

Similarly, $\left(\mathbb{K}, S F_{b, \omega}\right)$ will denote the Riemannian Klein bottle obtained by taking the quotient of $\left(\mathbb{C}, S F_{b, \omega}\right)$ by the subgroup $<\sigma, t_{v^{\prime}}>$. Its conformal type is

$$
\beta=2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{\omega}{2}\right)\right)+\frac{1}{\cos (\omega)}(2 b-2 \omega)
$$

### 1.1.3 Geometric inequality of type $\ell_{\sigma} \ell_{v}$

Our first result studies optimal conformal inequalities of the form

$$
\ell_{\sigma} \ell_{v} \leq C_{\beta} \text { area }
$$

where $C_{\beta}$ is a constant depending only on the conformal type $\beta$. We call such a relation a " geometric inequality of type $\ell_{\sigma} \ell_{v}$ ". Actually, it extends the following result of C. Bavard to all the conformal classes of the Klein bottle.

Theorem 1.1.3 C. Bavard [Ba06]. Let $a_{0} \in\left(0, \frac{\pi}{2}\right)$ be such that $\tan \left(a_{0}\right)=2 a_{0}$ and $0<\beta \leq$ $2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{a_{0}}{2}\right)\right)$. Then, for every Riemannian metric $g$ on the Klein bottle $\mathbb{K}$ of conformal type $\beta$, we have the following optimal inequality

$$
\ell_{\sigma}(g) \ell_{v}(g) \leq \frac{\arcsin \left(\frac{e^{\beta}-1}{e^{\beta}+1}\right)}{\frac{e^{\beta}-1}{e^{\beta}+1}} \operatorname{area}(g)
$$

The equality is attained if and only if $g$ is proportional to the spherical metric $S_{b}$ for $b$ satisfying $\beta=2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{b}{2}\right)\right)$.

We complete C. Bavard's study by providing an optimal inequality for the remaining conformal classes.

Theorem 1.1.4. Let $a_{0} \in\left(0, \frac{\pi}{2}\right)$ be such that $\tan \left(a_{0}\right)=2 a_{0}$ and $\beta>2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{a_{0}}{2}\right)\right)$. Let $\omega_{1} \in\left[a_{0}, \frac{\pi}{2}\right)$ be defined by the equation

$$
2 \sin \left(\omega_{1}\right)=\left(\beta-2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{\omega_{1}}{2}\right)\right)\right) \cos ^{2}\left(\omega_{1}\right)+4 \omega_{1} \cos \left(\omega_{1}\right)
$$

Then, for every Riemannian metric $g$ on the Klein bottle $K$ of conformal type $\beta$, we have the following optimal inequality

$$
\ell_{\sigma}(g) \ell_{v}(g) \leq \frac{1}{2 \cos \left(\omega_{1}\right)} \operatorname{area}(g)
$$

Moreover, the equality is attained if and only if $g$ is proportional to the spherical-flat metric $S F_{b, \omega_{1}}$ for $b=\tan \left(\omega_{1}\right)-\omega_{1}$.

Remark 1.1.5. The bound $\beta \leq 2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{a_{0}}{2}\right)\right)$ found by C. Bavard in Theorem 1.1.3 is actually the critical value of the conformal type for the transition in the shape of the extremal metrics from spherical to spherical-flat.

Corollary 1.1.6. There does not exist any (finite) positive universal constant $c$ such that the inequality

$$
\ell_{\sigma}(g) \ell_{v}(g) \leq c \operatorname{area}(g)
$$

holds for every Riemannian metric $g$ on the Klein bottle.

### 1.1.4 Geometric inequality of type $L_{\sigma} \ell_{v}$

The second part of this chapter is devoted to establishing optimal conformal inequalities of the form

$$
L_{\sigma} \ell_{v} \leq C_{\beta} \text { area. }
$$

Note that in the case of the Mobius band $\mathbb{M}, L_{\sigma}$ is just the systole of $\mathbb{M}$. We call such a relation a "geometric inequality of type $L_{\sigma} \ell_{v}$ ".

Theorem 1.1.7. Let $\beta>0$. For every Riemannian metric $g$ on the Klein bottle $\mathbb{K}$ of conformal type $\beta$, we have

$$
L_{\sigma}(g) \ell_{v}(g) \leq C_{\beta} \operatorname{area}(g)
$$

where

$$
C_{\beta}= \begin{cases}\frac{e^{\beta}+1}{e^{\beta}-1} \arcsin \left(\frac{e^{\beta-1}}{e^{\beta}+1}\right) & \text { if } 0<\beta \leq 2 \ln (2+\sqrt{3}) \\ \frac{2}{3} \cdot \frac{3 \beta+4 \pi-6 \ln (2+\sqrt{3})}{4 \sqrt{3}+\beta-2 \ln (2+\sqrt{3})} & \text { if } \beta>2 \ln (2+\sqrt{3})\end{cases}
$$

Moreover, the equality is attained if and only if $g$ is proportional to the spherical metric $S_{b}$, for $b$ satisfying $\beta=2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{b}{2}\right)\right)$, in the first case and to the spherical-flat metric $S F_{b, \frac{\pi}{3}}$, for $b$ satisfying $\beta=2 \ln (2+\sqrt{3})+4\left(b-\frac{\pi}{3}\right)$, in the second case.

Since the supremum of the conformal constant $C_{\beta}$ over $\beta$ is equal to 2 , we have

Corollary 1.1.8. For every Riemannian metric $g$ on the Klein bottle $\mathbb{K}$, we have

$$
L_{\sigma}(g) \ell_{v}(g)<2 \operatorname{area}(g)
$$

The inequality is optimal.

### 1.1.5 Geometric inequality of type $\ell_{\sigma} \ell_{v} \ell_{h}$

In the third part, we establish optimal conformal inequalities of the form

$$
\ell_{\sigma} \ell_{v} \ell_{h} \leq C_{\beta} \text { area }^{\frac{3}{2}}
$$

which we call a "geometric inequality of type $\ell_{\sigma} \ell_{v} \ell_{h}$ ". To our knowledge, this kind of inequality involving the product of three lengths has never been considered before. It is distinguished by the fact that the area to the power $\frac{3}{2}$ is bounded by the product of all the natural candidates for the systole

Theorem 1.1.9. Let $\beta>0$ and let $\omega_{2} \in\left(0, \frac{\pi}{2}\right)$ be defined by the equation

$$
\beta=2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{\omega_{2}}{2}\right)\right)+\frac{2}{\cos \left(\omega_{2}\right)}\left(\tan \left(\omega_{2}\right)-\omega_{2}+\sqrt{\tan ^{2}\left(\omega_{2}\right)-\omega_{2} \tan \left(\omega_{2}\right)+\omega_{2}^{2}}\right)
$$

Then, for every Riemannian metric $g$ on the Klein bottle $\mathbb{K}$ of conformal type $\beta$, we have the following optimal inequality

$$
\ell_{\sigma}(g) \ell_{v}(g) \ell_{h}(g) \leq C_{\beta} \operatorname{area}(g)^{\frac{3}{2}}
$$

where

$$
C_{\beta}=\frac{\sqrt{\pi}}{3 \sqrt{3}} \cdot \frac{\left(b^{4}-4 b \omega_{2}+\omega_{2}^{2}+\omega_{2}^{4}-2 b^{2}\left(-2+\omega_{2}^{2}\right)\right)^{\frac{1}{4}}\left(2 b-\omega_{2}\right)}{\left(b-\omega_{2}\right) \sqrt{\left(b-\omega_{2}\right) b}}
$$

with $b=\tan \left(\omega_{2}\right)+\sqrt{\tan ^{2}\left(\omega_{2}\right)-\omega_{2} \tan \left(\omega_{2}\right)+\omega_{2}^{2}}$. Moreover, the equality is attained if and only if $g$ is proportional to the spherical-flat metric $S F_{b, \omega_{2}}$.

Now, if we replace $b$ by $\tan \left(\omega_{2}\right)+\sqrt{\tan ^{2}\left(\omega_{2}\right)-\omega_{2} \tan \left(\omega_{2}\right)+\omega_{2}^{2}}$ in

$$
C\left(\omega_{2}\right)=\frac{\left(b^{4}-4 b \omega_{2}+\omega_{2}^{2}+\omega_{2}^{4}-2 b^{2}\left(-2+\omega_{2}^{2}\right)\right)^{\frac{1}{4}}\left(2 b-\omega_{2}\right)}{\left(b-\omega_{2}\right) \sqrt{\left(b-\omega_{2}\right) b}}
$$

we obtain a continuous increasing function $C:] \frac{2}{\pi},+\infty[\rightarrow] \frac{2 \sqrt{\pi}}{3 \sqrt{3}},+\infty[$ which tends to infinity as $\omega_{2} \rightarrow 0$ (i.e. when $b \rightarrow 0$ ). Therefore, we derive

Corollary 1.1.10. There does not exist any (finite) universal constant $c$ such that the inequality

$$
\ell_{\sigma}(g) \ell_{v}(g) \ell_{h}(g) \leq c \operatorname{area}(g)^{\frac{3}{2}}
$$

holds for every Riemannian metric $g$ on the Klein bottle $\mathbb{K}$.
Remark 1.1.11. Unlike Corollaries 1.1.6 and 1.1.10, Corollary 1.1 .8 provides a uniform upper bound on the Riemannian ratio $\frac{L_{\sigma}(g) \ell_{v}(g)}{\text { area }(g)}$.

### 1.2 The main tools for the proofs

The key tool in our proofs is the method of extremal length initiated by B. Fuglede in [Fu57], J. A. Jenkins in [Je65] and M. Gromov in [Gr83]. It was used later by C. Bavard in the setting of isosystolic geometry, cf. [Ba88, Ba92]. This method charaterizes a conformally extremal Riemannian manifold by means of its closed geodesics. See e.g. L. V. Ahlfors' book [Al73] and Rodin's paper [Ro68] and the references therein for more details and further applications of the method of extremal length.

### 1.2.1 A maximality criterion

Let $(M, g)$ be a closed Riemannian manifold and $\Gamma$ be a family of rectifiable curves on $M$. For every Radon measure $\mu$ on $\Gamma$, we associate a measure ${ }^{*} \mu$ on $M$ by setting, for $\varphi \in C^{0}(M, \mathbb{R})$,

$$
\left\langle{ }^{*} \mu, \varphi\right\rangle=\langle\mu, \bar{\varphi}\rangle
$$

where $\bar{\varphi}(\gamma)=\int \varphi \circ \gamma(s) d s$ and $d s$ is the arc-length of $\gamma$ with respect to $g$.
Theorem 1.2.1. ([Ba88],[Ba92] and [fe65]) Let $M$ be a closed manifold. Let $\mathcal{S}_{i}$, where $i \in\{1, \cdots, p\}$, be some families of rectifiable curves on $M$ such that $\mathcal{S}_{i} \cap \mathcal{S}_{j}=\emptyset$ for all $i \neq j$. Denote by $\ell_{i}(g)$ the least length of a curve in $\mathcal{S}_{i}$ with respect to a Riemannian metric $g$ on $M$. Suppose that $g_{e}$ is a Riemannian metric on $M$ for which there exists a positive measure $\mu$ on $\Gamma=\mathcal{S}_{1} \cup \cdots \cup \mathcal{S}_{p}$ satisfying the following three conditions:

1. for each $i \in\{1, \cdots, p\}$, all the curves in $\mathcal{S}_{i}$ have the same length with respect to $g_{e}$
2. $m_{1} \ell_{1}\left(g_{e}\right)=m_{2} \ell_{2}\left(g_{e}\right)=\cdots=m_{p} \ell_{p}\left(g_{e}\right)$
3. ${ }^{*} \mu=d g_{e}$
where $m_{i}$ is the mass of the measure $\mu$ on $\mathcal{S}_{i}$ and $d g_{e}$ is the volume measure of $\left(M, g_{e}\right)$. Then, for every Riemannian metric $g$ on $M$ conformal to $g_{e}$, we have

$$
\begin{equation*}
\frac{\ell_{1}(g) \cdots \ell_{p}(g)}{\operatorname{vol}^{\frac{p}{2}}(g)} \leq \frac{\ell_{1}\left(g_{e}\right) \cdots \ell_{p}\left(g_{e}\right)}{\operatorname{vol}^{\frac{p}{2}}\left(g_{e}\right)} . \tag{1.2.1}
\end{equation*}
$$

Furthermore, the equality holds if and only ifg is homothetic to $g_{e}$.

Since our version of Theorem 1.2.1 is slightly more general than in the aforementioned references (as it holds for an arbitrary number of curve families $\mathcal{S}_{i}$ ), we present a proof of it. It is straightforward by the following lemma.

Lemma 1.2.2. Let $\mathcal{S}_{i}$, where $i \in\{1, \cdots, p\}$, be families of rectifiable curves on a given closed manifold $M$ such that $\mathcal{S}_{i} \cap \mathcal{S}_{j}=\emptyset$ for all $i \neq j$. Denote by $\ell_{i}(g)$ the least length of a curve in $\mathcal{S}_{i}$ with respect to a Riemannian metric $g$ on $M$. Let $g_{e}$ be a Riemannian metric on $M$ such that there exists a positive measure $\mu$ on $\Gamma=\mathcal{S}_{1} \cup \cdots \cup \mathcal{S}_{p}$ satisfying the three conditions of Theorem 1.2.1. Then,

$$
\begin{equation*}
\ell_{1}(g) \cdots \ell_{p}(g) \leq \frac{\operatorname{vol}^{\frac{p}{2}}\left(g_{e}\right)}{p^{p} m_{1} \cdots m_{p}} \operatorname{vol}^{\frac{p}{2}}(g) \tag{1.2.2}
\end{equation*}
$$

where $m_{i}$ is the mass of the measure $\mu$ on $\mathcal{S}_{i}$. Furthermore, the equality holds if and only if $g$ is homothetic to $g_{e}$.

Proof. Let $g$ be a Riemannian metric conformal to $g_{e}$, that is, $g=\phi^{2} g_{e}$. We have

$$
\begin{align*}
m_{1} \ell_{1}(g)+\cdots+m_{p} \ell_{p}(g) & \leq \int_{\mathcal{S}_{1}} \bar{\phi}(\gamma) d \mu(\gamma)+\cdots+\int_{\mathcal{S}_{p}} \bar{\phi}(\gamma) d \mu(\gamma)  \tag{1.2.3}\\
& =\int_{\Gamma} \bar{\phi}(\gamma) d \mu(\gamma) \\
& =\int_{M} \phi(x) d\left({ }^{*} \mu\right)(x) \\
& =\int_{M} \phi(x) d g_{e} \\
& \leq\left(\int_{M} \phi^{2}(x) d g_{e}\right)^{\frac{1}{2}}\left(\int_{M} d g_{e}\right)^{\frac{1}{2}}  \tag{1.2.4}\\
& =\sqrt{\operatorname{vol}(g) \operatorname{vol}\left(g_{e}\right)} .
\end{align*}
$$

When $g$ is equal to $g_{e}$, equality in (1.2.3) is attained because $g_{e}$ satisfies condition (1). Moreover, it is straightforward that equality in (1.2.4) is attained if and only if $g$ is equal to $g_{e}$ by CauchySchwartz. Using the arithmetic and geometric means inequality, we derive

$$
p .\left(m_{1} \cdots m_{p} \ell_{1}(g) \cdots \ell_{p}(g)\right)^{\frac{1}{p}} \leq m_{1} \ell_{1}(g)+\cdots+m_{p} \ell_{p}(g)
$$

with equality if and only if

$$
m_{1} \ell_{1}(g)=m_{2} \ell_{2}(g)=\cdots=m_{p} \ell_{p}(g)
$$

Finally, combining the two inequalities, we obtain that, under the required conditions, the following inequality holds

$$
p^{p} \cdot\left(m_{1} \cdots m_{p}\right) \cdot \ell_{1}(g) \cdots \ell_{p}(g) \leq\left(\operatorname{vol}(g) \operatorname{vol}\left(g_{b}\right)\right)^{\frac{p}{2}}
$$

with equality if and only if $g$ is equal to $g_{e}$.

Thus, proving the conformal extremality of a metric requires to find disjoint families of rectifiable curves $\mathcal{S}_{1}, \cdots, \mathcal{S}_{n}$ on $M$ and define a measure on them that satisfies the three conditions of the previous theorem.

### 1.2.2 Setting the curve families

To prove our results, we will make use of three families of curves in $\left(\mathbb{K}, S F_{b, \omega}\right)$.
Definition 1.2.3. Let $\left(\mathbb{K}, S F_{b, \omega}\right)$ be the spherical-flat Klein bottle defined in Section 1.1.2. Define three families of loops as follows.

1. For each $\theta \in \mathbb{R} / \pi \mathbb{Z}$ and each $a \in[-\omega, \omega] \cup[2 b-\omega, 2 b] \cup[-2 b,-2 b+\omega]$, the loop $\alpha_{\theta}^{a}$ is the geodesic (image in $\left(\mathbb{K}, S F_{b, \omega}\right)$ of a great circle) going through the points $(\theta-\pi / 2,0)$, $(\theta, a)$. Let

$$
\mathcal{S}_{1}(\omega)=\left\{\alpha_{\theta}^{a}| | a \mid \leq \omega \text { or } 2 b-\omega \leq|a| \leq 2 b, \theta \in \mathbb{R} / \pi \mathbb{Z}\right\}
$$

2. For each $u \in \mathbb{R} / \pi \mathbb{Z}, \gamma_{u}$ is the vertical loop defined by $\gamma_{u}(t)=(u, t)$ with $|t| \leq 2 b$. Let

$$
\mathcal{S}_{2}=\left\{\gamma_{u} \mid u \in \mathbb{R} / \pi \mathbb{Z}\right\}
$$

3. For each $v \in \mathbb{R}$ satisfying $\omega \leq|v| \leq 2 b-\omega, \delta_{v}$ is the horizontal loop defined by $\delta_{v}(t)=$ $(t,|v|)$ with $|t| \leq \frac{\pi}{2}$. Let

$$
\mathcal{S}_{3}(\omega)=\left\{\delta_{v}|\omega \leq|v| \leq 2 b-\omega\} .\right.
$$

Remark 1.2.4. For each $i \in\{1,2,3\}$, the curves in $\mathcal{S}_{i}$ have the same length with respect to the metric $S F_{b, \omega}$. Also note that a curve in $\mathcal{S}_{1}$, resp. $\mathcal{S}_{2}$, resp. $\mathcal{S}_{3}$, belongs to the family $\mathcal{F}_{\sigma}$, resp. $\mathcal{F}_{v}$, resp. $\mathcal{F}_{h}$.

We move now to the next step, i.e., verifying the three conditions of Theorem 1.2.1. Conditions (1) and (2) are easy to verify while Condition (3) requires technical (but not simple) methods of calculation.

## 22CHAPTER 1: CONFORMAL GEOMETRIC INEQUALITIES ON THE KLEIN BOTTLE



Figure 1.1: The curves $\alpha_{\theta}^{a}, \gamma_{u}$ and $\delta_{v}$ in the Mobius band $\left\{(u, v)||v| \leq b\}\right.$ of $\left(\mathbb{K}, S F_{b, \omega}\right)$.

In the following we will consider a subset of the family of curves $\mathcal{S}_{1}(\omega)$ in $\left(\mathbb{K}, S F_{b, \omega}\right)$ and equip it with a measure $\mu$ depending on a function $h$ of one parameter (since $S F_{b, \omega}$ has an isometry group of dimension 1). Then, we will calculate the measure ${ }^{*} \mu$ in terms of the volume measure of $S F_{b, \omega}$.

### 1.2.3 The calculation of * $\mu$ on a spherical region of $\mathbb{K}$

Let $b \in(0,+\infty)$ and $\omega \in(0, b]$. We consider on $\left(\mathbb{K}, S F_{b, \omega}\right)$ the family of curves $\mathcal{S}_{1}^{\prime}(\omega) \subset \mathcal{S}_{1}(\omega)$ defined by

$$
\mathcal{S}_{1}^{\prime}(\omega)=\left\{\alpha_{\theta}^{a} \subset\left(\mathbb{K}, S F_{b, \omega}\right) \mid-\omega \leq a \leq \omega, \theta \in \mathbb{R} / \pi \mathbb{Z}\right\}
$$

where $\alpha_{\theta}^{a}$ is the great circle introduced in Definition 1.2.3.
Lemma 1.2.5. Let $\mu$ be a measure on $\mathcal{S}_{1}^{\prime}(\omega)$ defined by

$$
d \mu\left(\alpha_{\theta}^{a}\right)= \begin{cases}h(a) d a d \theta & \text { if } \quad a \geq 0 \\ h(-a) d a d \theta & \text { if } \quad a<0\end{cases}
$$

where $h:[0, \omega] \rightarrow \mathbb{R}$ is a continuous function. Then, we have

$$
{ }^{*} \mu=2 \chi_{\{|v| \leq \omega\}}\left(\int_{|v|}^{\omega}\left(\cos ^{2}(v)-\cos ^{2}(a)\right)^{-\frac{1}{2}} h(a) d a\right) d\left(S F_{b, \omega}\right)
$$

where $d\left(S F_{b, \omega}\right)$ is the volume measure of $S F_{b, \omega}$.

Proof. First, we compute the equation of $\alpha_{\theta}^{a}$. Suppose $\alpha_{\theta}^{a}(t)=(u(t), v(t))$. Then, from the classical geodesic equation $\frac{d^{2} x^{i}}{d t^{2}}+\sum_{j, k} \Gamma_{j, k}^{i} \frac{d x^{k}}{d t} \frac{d x^{j}}{d t}=0$ (see for example [GHL04] p. 81), where $\Gamma_{j, k}^{i}$ are the Christoffel symbols, $x^{1}=u$ and $x^{2}=v$, we derive

$$
u^{\prime \prime}-2 \tan (v) u^{\prime} v^{\prime}=0 \quad \text { and } \quad v^{\prime \prime}+\sin (v) \cos (v) u^{\prime 2}=0
$$

This shows that

$$
\begin{equation*}
\frac{d u}{d v}=\frac{c}{\cos (v) \sqrt{\cos ^{2}(v)-c^{2}}} \tag{1.2.5}
\end{equation*}
$$

where $c$ is a constant. The solution of the differential equation (1.2.5) is $\sin (u-d)=c \tan (v)$, where $d$ is a constant. Using the fact that $\alpha_{\theta}^{a}$ goes through $(\theta-\pi / 2,0)$ and $(\theta, a)$, we obtain

$$
\begin{equation*}
\sin \left(u-\theta+\frac{\pi}{2}\right)=\frac{1}{\tan (a)} \tan (v) \tag{1.2.6}
\end{equation*}
$$

Equation (1.2.6) shows that we can write $\alpha_{\theta}^{a}(u)=(u+\theta, v(u, a))$, with $v$ verifying

$$
v(u,-a)=-v(u, a)
$$

Now, let $\phi \in C^{0}(\mathbb{K}, \mathbb{R})$. Then, by the definition of * $\mu$, we have

$$
\int_{\mathbb{K}} \phi(u, v) d\left(^{*} \mu\right)(u, v)=\int_{\mathcal{S}_{1}^{\prime}(\omega)} \bar{\phi}\left(\alpha_{\theta}^{a}\right) d \mu\left(\alpha_{\theta}^{a}\right)
$$

where

$$
\begin{aligned}
\bar{\phi}\left(\alpha_{\theta}^{a}\right) & =\int_{\alpha_{\theta}^{a}} \phi\left(\alpha_{\theta}^{a}(s)\right) d s \\
& =\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi(u+\theta, v(u, a)) \sqrt{\cos ^{2}(v(u, a))+\left(\frac{\partial v}{\partial u}(u, a)\right)^{2}} d u .
\end{aligned}
$$

Then,

$$
\begin{aligned}
& \int_{\mathbb{K}} \phi(u, v) d\left({ }^{*} \mu\right)(u, v)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\omega}^{0} \bar{\phi}\left(\alpha_{\theta}^{a}\right) h(-a) d a d \theta+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\omega} \bar{\phi}\left(\alpha_{\theta}^{a}\right) h(a) d a d \theta \\
& =\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\omega} \int_{-\frac{\pi}{2}}^{0} \phi(u+\theta,-v(u, a)) \sqrt{\cos ^{2}(v(u, a))+\left(\frac{\partial v}{\partial u}(u, a)\right)^{2}} h(a) d u d a d \theta \\
& +\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\omega} \int_{0}^{\frac{\pi}{2}} \phi(u+\theta,-v(u, a)) \sqrt{\cos ^{2}(v(u, a))+\left(\frac{\partial v}{\partial u}(u, a)\right)^{2}} h(a) d u d a d \theta \\
& +\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\omega} \int_{-\frac{\pi}{2}}^{0} \phi(u+\theta, v(u, a)) \sqrt{\cos ^{2}(v(u, a))+\left(\frac{\partial v}{\partial u}(u, a)\right)^{2}} h(a) d u d a d \theta
\end{aligned}
$$

$$
+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\omega} \int_{0}^{\frac{\pi}{2}} \phi(u+\theta, v(u, a)) \sqrt{\cos ^{2}(v(u, a))+\left(\frac{\partial v}{\partial u}(u, a)\right)^{2}} h(a) d u d a d \theta
$$

Next, we apply for the first two integrals in the previous expression the change of variables $y \mapsto-v(u, a)$, whose Jacobian is equal to $-\frac{\partial v}{\partial u}(u, a)$, and we write $k_{1}(y, a)=u$. For the last two integrals, we make use of the change of variables $y \mapsto v(u, a)$, whose Jacobian is equal to $\frac{\partial v}{\partial u}(u, a)$, and we write $k_{2}(y, a)=u$. In either case, let $z(y, a)=\frac{\partial v}{\partial u}(u, a)$. We derive

$$
\begin{aligned}
\int_{\mathbb{K}} \phi(u, v) d\left({ }^{*} \mu\right)(u, v) & =2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\omega} \int_{-a}^{0} \phi\left(k_{1}(y, a)+\theta, y\right) \sqrt{\cos ^{2}(y)+z^{2}(y, a)} \frac{h(a)}{|z(y, a)|} d y d a d \theta \\
& +2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\omega} \int_{0}^{a} \phi\left(k_{2}(y, a)+\theta, y\right) \sqrt{\cos ^{2}(y)+z^{2}(y, a)} \frac{h(a)}{|z(y, a)|} d y d a d \theta
\end{aligned}
$$

Next, in the previous expression, we make the change of variables $x \mapsto k_{1}(y, a)+\theta$ for the first integral and $x \mapsto k_{2}(y, a)+\theta$ for the second integral. The Jacobian for both changes of variables is equal to 1 . We obtain

$$
\begin{aligned}
\int_{\mathbb{K}} \phi(u, v) d\left({ }^{*} \mu\right)(u, v) & =2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\omega} \int_{-a}^{0} \phi(x, y) \sqrt{\cos ^{2}(y)+z^{2}(y, a)} \frac{h(a)}{|z(y, a)|} d y d a d x \\
& +2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\omega} \int_{0}^{a} \phi(x, y) \sqrt{\cos ^{2}(y)+z^{2}(y, a)} \frac{h(a)}{|z(y, a)|} d y d a d x \\
& =2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\omega}^{0} \int_{-y}^{\omega} \phi(x, y) \sqrt{\cos ^{2}(y)+z^{2}(y, a)} \frac{h(a)}{|z(y, a)|} d a d y d x \\
& +2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\omega} \int_{y}^{\omega} \phi(x, y) \sqrt{\cos ^{2}(y)+z^{2}(y, a)} \frac{h(a)}{|z(y, a)|} d a d y d x \\
& =2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\omega}^{\omega} \phi(x, y) \int_{|y|}^{\omega} \sqrt{\left(\left(\frac{\cos (y)}{z(y, a)}\right)^{2}+1\right)} h(a) d a d y d x
\end{aligned}
$$

Hence, we derive

$$
{ }^{*} \mu=2 \frac{\chi_{\{|v| \leq \omega\}}}{\cos (v)} \int_{|v|}^{\omega} \sqrt{\left(\left(\frac{\cos (v)}{z(v, a)}\right)^{2}+1\right)} h(a) d a d\left(S F_{b, \omega}\right)
$$

where $d\left(S F_{b, \omega}\right)$ is the volume element of $\left(\mathbb{K}, S F_{b, \omega}\right)$. From equation (1.2.5), we obtain

$$
z^{2}(v, a)=\frac{\cos ^{2}(v)}{\cos ^{2}(a)}\left(\cos ^{2}(v)-\cos ^{2}(a)\right)
$$

This shows that

$$
{ }^{*} \mu=2 \chi_{\{|v| \leq \omega\}}\left(\int_{|v|}^{\omega}\left(\cos ^{2}(v)-\cos ^{2}(a)\right)^{-\frac{1}{2}} h(a) d a\right) d\left(S F_{b, \omega}\right)
$$

### 1.3 Proof of the geometric inequality of type $\ell_{\sigma} \ell_{v}$

In the following, we prove the geometric inequality of type $\ell_{\sigma} \ell_{v}$ for $\beta>2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{a_{0}}{2}\right)\right)$. As in the case $\beta \leq 2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{a_{0}}{2}\right)\right)$, we will consider both curve families $\mathcal{S}_{1}\left(\omega_{1}\right)$ and $\mathcal{S}_{2}$, cf. [Ba06, Corollary 2].

Proof of Theorem 1.1.4. Let $\beta>2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{a_{0}}{2}\right)\right)$. There exists a unique $\omega_{1} \in\left[a_{0}, \frac{\pi}{2}\right)$ such that

$$
\begin{equation*}
2 \sin \left(\omega_{1}\right)=\left(\beta-2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{\omega_{1}}{2}\right)\right)\right) \cos ^{2}\left(\omega_{1}\right)+4 \omega_{1} \cos \left(\omega_{1}\right) \tag{1.3.1}
\end{equation*}
$$

We endow $\mathbb{K}$ with the metric $S F_{b, \omega_{1}}$ defined in Section 1.1.2, where

$$
\begin{equation*}
b=\tan \left(\omega_{1}\right)-\omega_{1} \tag{1.3.2}
\end{equation*}
$$

Consider the two families of curves $\mathcal{S}_{1}\left(\omega_{1}\right)$ and $\mathcal{S}_{2}$, cf. Section 1.2.2. Each curve in $\mathcal{S}_{1}\left(\omega_{1}\right)$ (resp. $\mathcal{S}_{2}$ ) has length equal to $\pi$ (resp. 4b) with respect to $S F_{b, \omega_{1}}$. Hence, condition (1) of Theorem 1.2.1 is satisfied. Now, let

$$
\begin{array}{ccc}
h:\left[-\omega_{1}, \omega_{1}\right] & \longrightarrow & \mathbb{R} \\
a & \mapsto & \frac{\sin (|a|)}{\pi \cos (a)} \sqrt{\cos ^{2}(a)-\cos ^{2}\left(\omega_{1}\right)}
\end{array}
$$

We define on the family $\mathcal{S}_{1}\left(\omega_{1}\right)$ the measure

$$
\mu_{1}=\widetilde{h}(a) d a \otimes d \theta
$$

where

$$
\widetilde{h}(a)=\left\{\begin{array}{cll}
h(a) & \text { if } & a \in\left[-\omega_{1}, \omega_{1}\right] \\
h(a-2 b) & \text { if } & a \in\left[2 b-\omega_{1}, 2 b\right] \\
h(a+2 b) & \text { if } & a \in\left[-2 b,-2 b+\omega_{1}\right]
\end{array}\right.
$$

On the family $\mathcal{S}_{2}$, we define the measure

$$
\mu_{2}=\cos \left(\omega_{1}\right) d u
$$

The mass $m_{1}$ of the measure $\mu_{1}$ on $\mathcal{S}_{1}\left(\omega_{1}\right)$ is equal to $4 \sin \left(\omega_{1}\right)-4 \omega_{1} \cos \left(\omega_{1}\right)$ and the mass $m_{2}$ of the measure $\mu_{2}$ on $\mathcal{S}_{2}$ is equal to $\pi \cos \left(\omega_{1}\right)$. Now, since $\ell_{\sigma}\left(S F_{b, \omega_{1}}\right)=\pi$ and $\ell_{v}\left(S F_{b, \omega_{1}}\right)=4 b$, condition (2) of Theorem 1.2.1 is satisfied. Next, it can be easily verified that

$$
{ }^{*} \mu_{2}=\frac{\cos \left(\omega_{1}\right)}{f_{b, \omega_{1}}(v)} d\left(S F_{b, \omega_{1}}\right)
$$

where $f_{b, \omega_{1}}$ is the unique one-variable function invariant by the translation $(u, v) \mapsto(u, v+2 b)$ which agrees with cosine on $\left[-\omega_{1}, \omega_{1}\right]$ and equal to the constant $\cos \left(\omega_{1}\right)$ on $\left[\omega_{1}, 2 b-\omega_{1}\right]$ (see Section 1.1.2). Moreover, by Lemma 1.2.5, we have

$$
\chi_{\left\{|v| \leq \omega_{1}\right\}}^{*} \mu_{1}=2 \chi_{\left\{|v| \leq \omega_{1}\right\}}\left(\int_{|v|}^{\omega_{1}}\left(\cos ^{2}(v)-\cos ^{2}(a)\right)^{-\frac{1}{2}} \widetilde{h}(a) d a\right) d\left(S F_{\left.b, \omega_{1}\right)}\right)
$$

Then, we derive

$$
\chi_{\left\{|v| \leq \omega_{1}\right\}}^{*} \mu_{1}=\chi_{\left\{|v| \leq \omega_{1}\right\}}\left(1-\frac{\cos \left(\omega_{1}\right)}{f_{b, \omega_{1}}(v)}\right) d\left(S F_{\left.b, \omega_{1}\right)}\right)
$$

Finally, since $f_{b, \omega_{1}}$ and $\widetilde{h}$ are invariant by the translation $(u, v) \mapsto(u, v+2 b)$, we have

$$
{ }^{*} \mu_{1}=\left(1-\frac{\cos \left(\omega_{1}\right)}{f_{b, \omega_{1}}(v)}\right) d\left(S F_{b, \omega_{1}}\right) .
$$

Hence, we get ${ }^{*} \mu_{1}+{ }^{*} \mu_{2}=d\left(S F_{b, \omega_{1}}\right)$. This is condition (3) of Theorem 1.2.1. Finally, from Theorem 1.2 .1 and inequality (1.2.2), we derive

$$
\ell_{\sigma}(g) \ell_{v}(g) \leq \frac{1}{2 \cos \left(\omega_{1}\right)} \operatorname{area}(g)
$$

which holds for every Riemannian metric $g$ conformal to $S F_{b, \omega_{1}}$. The equality is attained if and only if $g$ is homothetic to $S F_{b, \omega_{1}}$, and the result follows.

Recall that $\mathbb{M}$ denotes the Mobius band with boundary obtained by taking the quotient of $\mathbb{R} \times$ $[-\beta, \beta]$ by the group generated by the map $\sigma: z \mapsto \bar{z}+\pi$. We denote by $g_{\beta}$ the flat metric induced by such quotient. The parameter $\beta$ represents the conformal type of any metric conformal to $g_{\beta}$.

We recover Blatter's result for the Mobius Band, cf. [Bl61, Theorem 2], by a simpler method. Indeed, consider in Bavard's proof of Theorem 1.1.3 and in our previous proof the restriction of $S_{b}$ and $S F_{b, \omega_{1}}$ on the set $E_{b}=\{(u, v) \in \mathbb{C}| | v \mid \leq b\}$.
Corollary 1.3.1. [Bl61, Satz 2] Let $\beta>0$ and $a_{0} \in\left(0, \frac{\pi}{2}\right)$ such that $\tan \left(a_{0}\right)=2 a_{0}$. Let $\omega_{1} \in$ $\left[a_{0}, \frac{\pi}{2}\right)$ be defined by the equation

$$
\sin \left(\omega_{1}\right)=\left(\beta-\ln \left(\tan \left(\frac{\pi}{4}+\frac{\omega_{1}}{2}\right)\right)\right) \cos ^{2}\left(\omega_{1}\right)+2 \omega_{1} \cos \left(\omega_{1}\right) .
$$

Then, for every Riemannian metric $g$ on the Mobius band $\mathbb{M}$ of conformal type $\beta$, we have

$$
\ell_{\sigma}(g) \ell_{v}(g) \leq C_{\beta} \operatorname{area}(g)
$$

where

$$
C_{\beta}= \begin{cases}\frac{e^{2 \beta}+1}{e^{2 \beta}-1} \arcsin \left(\frac{e^{2 \beta}-1}{e^{2 \beta}+1}\right) & \text { if } 0<\beta \leq \ln \left(\frac{\pi}{4}+\frac{a_{0}}{2}\right) \\ \frac{1}{2 \cos \left(\omega_{1}\right)} & \text { if } \beta>\ln \left(\frac{\pi}{4}+\frac{a_{0}}{2}\right) .\end{cases}
$$

Moreover, the equality is attained if and only ifg is proportional to the spherical metric $S_{b}$ restricted to $E_{b}=\{(u, v) \in \mathbb{C}| | v \mid \leq b\}$, forb satisfying $\beta=2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{b}{2}\right)\right)$, in the first case, and to the spherical-flat metric $S F_{b, \omega_{1}}$ restricted to $E_{b}=\{(u, v) \in \mathbb{C}| | v \mid \leq b\}$, for $b=\tan \left(\omega_{1}\right)-\omega_{1}$, in the second case.

### 1.4 Proof of the geometric inequality of type $L_{\sigma} \ell_{v}$

In the following, we prove the geometric inequality of type $L_{\sigma} \ell_{v}$ on the Klein bottle. The curve families we consider this time are $\mathcal{S}_{1}\left(\frac{\pi}{3}\right), \mathcal{S}_{2}$ and $\mathcal{S}_{3}\left(\frac{\pi}{3}\right)$.

Proof of Theorem 1.1.7. The inequality in the first case can be deduced from Theorem 1.1.3 since when $0<\beta \leq 2 \ln (2+\sqrt{3})$, i.e. when $b \leq \frac{\pi}{3}$, we have $L_{\sigma}\left(S_{b}\right)=\ell_{\sigma}\left(S_{b}\right)$. Now, since the conformal type of $\left(\mathbb{K}, S_{b}\right)$ is

$$
\beta=2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{b}{2}\right)\right)
$$

we deduce that for $0<\beta \leq 2 \ln (2+\sqrt{3})$,

$$
L_{\sigma}(g) \ell_{v}(g) \leq \frac{\arcsin \left(\frac{e^{\beta}-1}{e^{\beta}+1}\right)}{\frac{e^{\beta}-1}{e^{\beta}+1}} \operatorname{area}(g)
$$

Note that when $b$ becomes greater than $\frac{\pi}{3}$, the horizontal geodesic loops closed by the horizontal translation $t_{h}$ (corresponding to the singularity line of $S_{b}$ ) become shorter than the curves $\alpha_{\theta}^{a}$ and therefore $L_{\sigma}$ is attained by such lines.

To prove the inequality in the second case, we fix $\beta>2 \ln (2+\sqrt{3})$ and let $b$ such that $\beta=$ $2 \ln (2+\sqrt{3})+4\left(b-\frac{\pi}{3}\right)$ (we have $b \geq \frac{\pi}{3}$ ). Then we endow $\mathbb{K}$ with the metric $S F_{b, \frac{\pi}{3}}$. We consider the two families of curves $\mathcal{S}=\mathcal{S}_{1}\left(\frac{\pi}{3}\right) \cup \mathcal{S}_{3}\left(\frac{\pi}{3}\right)$ and $\mathcal{S}_{2}$, cf. Section 1.2.2. Each curve in $\mathcal{S}$, (resp. $\mathcal{S}_{2}$ ) has length equal to $\pi$, (resp. 4b) with respect to the metric $S F_{b, \frac{\pi}{3}}$. Hence, condition (1) of Theorem 1.2.1 is satisfied. Now, let

$$
\begin{array}{rlc}
k:\left[-\frac{\pi}{3}, \frac{\pi}{3}\right] & \longrightarrow & \mathbb{R} \\
a & \mapsto & \frac{\tan (|a|)}{24 \pi b} \cdot \frac{24 b \cos ^{2}(a)-3 \sqrt{3}-3 b+\pi}{\sqrt{\cos ^{2}(a)-\frac{1}{4}}}
\end{array}
$$

We define on the family $\mathcal{S}$ the measure

$$
\mu_{1}=\widetilde{k}(a) d a \otimes d \theta+\frac{-3 \sqrt{3}+3 b+\pi}{6 b} d v
$$

where

$$
\widetilde{k}(a)=\left\{\begin{array}{cll}
k(a) & \text { if } & a \in\left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \\
k(a-2 b) & \text { if } & a \in\left[2 b-\frac{\pi}{3}, 2 b\right] \\
k(a+2 b) & \text { if } & a \in\left[-2 b,-2 b+\frac{\pi}{3}\right]
\end{array}\right.
$$

On the family $\mathcal{S}_{2}$, we define the measure

$$
\mu_{2}=\frac{3 \sqrt{3}+3 b-\pi}{12 b} d u .
$$

Note that these measures are positive since $b \geq \frac{\pi}{3}$. The mass $m_{1}$ of the measure $\mu_{1}$ on $\mathcal{S}$ is equal to $\sqrt{3}+3 b-\frac{\pi}{3}$, and the mass $m_{2}$ of the measure $\mu_{2}$ on $\mathcal{S}_{2}$ is equal to $\frac{3 \sqrt{3}+3 b \pi-\pi^{2}}{12 b}$. Now, since $L_{\sigma}\left(S F_{b, \frac{\pi}{3}}\right)=\pi$ and $\ell_{v}\left(S F_{b, \frac{\pi}{3}}\right)=4 b$, condition (2) of Theorem 1.2.1 is satisfied. Next, it can be easily verified that

$$
{ }^{*} \mu_{2}=\frac{3 \sqrt{3}+3 b-\pi}{12 b f_{b, \frac{\pi}{3}}(v)} d\left(S F_{b, \frac{\pi}{3}}\right)
$$

where $f_{b, \frac{\pi}{3}}$ is the unique one-variable function invariant by the translation $(u, v) \mapsto(u, v+2 b)$ which agrees with cosine on $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ and equal to the constant $\frac{1}{2}$ on $\left[\frac{\pi}{3}, 2 b-\frac{\pi}{3}\right]$. Moreover, by Lemma 1.2.5 we obtain

$$
\chi_{\left\{|v| \leq \frac{\pi}{3}\right\}}^{*} \mu_{1}=2 \chi_{\left\{|v| \leq \frac{\pi}{3}\right\}}\left(\int_{|v|}^{\frac{\pi}{3}}\left(\cos ^{2}(v)-\cos ^{2}(a)\right)^{-\frac{1}{2}} \widetilde{k}(a) d a\right) d\left(S F_{b, \frac{\pi}{3}}\right)
$$

Then, we derive

$$
\chi_{\left\{|v| \leq \frac{\pi}{3}\right\}}{ }^{*} \mu_{1}=\chi_{\left\{|v| \leq \frac{\pi}{3}\right\}}\left(1-\frac{3 \sqrt{3}+3 b-\pi}{12 b f_{b, \frac{\pi}{3}}(v)}\right) d\left(S F_{b, \frac{\pi}{3}}\right) .
$$

Finally, since $f_{b, \frac{\pi}{3}}$ and $\widetilde{k}$ are invariant by the translation $(u, v) \mapsto(u, v+2 b)$, we have

$$
{ }^{*} \mu_{1}=\left\{\begin{array}{cll}
\left(1-\frac{3 \sqrt{3}+3 b-\pi}{12 b f_{b} \frac{\pi}{3}(v)}\right) d\left(S F_{b, \frac{\pi}{3}}\right) & \text { if } & |v| \leq \frac{\pi}{3} \text { or } 2 b-\frac{\pi}{3} \leq|v| \leq 2 b \\
\frac{-3 \sqrt{3}+3 b+\pi}{6 b} d\left(S F_{b, \frac{\pi}{3}}\right) & \text { if } & \frac{\pi}{3} \leq|v| \leq 2 b-\frac{\pi}{3}
\end{array}\right.
$$

Hence, we get ${ }^{*} \mu_{1}+{ }^{*} \mu_{2}=d\left(S F_{b, \frac{\pi}{3}}\right)$. Therefore, condition (3) of Theorem 1.2.1 is satisfied. It follows from Theorem 1.2.1 and inequality (1.2.2) that for every Riemannian metric $g$ conformal to $S F_{b, \frac{\pi}{3}}$, we have

$$
L_{\sigma}(g) \ell_{v}(g) \leq \frac{2 b}{\sqrt{3}+b-\frac{\pi}{3}} \text { area }(g)
$$

with equality if and only if $g$ is homothetic to $S F_{b, \frac{\pi}{3}}$.

We can also derive C . Blatter's similar result on the Mobius band $\mathbb{M}$ (Theorem 3 in [Bl61]) using a minor adaptation of the previous proof.

Corollary 1.4.1. [Bl61, Satz 3] Let $\beta>0$. Then, for every Riemannian metric $g$ on the Mobius band $\mathbb{M}$ of conformal type $\beta$, we have

$$
\operatorname{sys}(g) \ell_{v}(g) \leq C_{\beta} \operatorname{area}(g)
$$

where

$$
C_{\beta}= \begin{cases}\frac{e^{2 \beta}+1}{e^{2 \beta}-1} \arcsin \left(\frac{e^{2 \beta}-1}{e^{2 \beta}+1}\right) & \text { if } 0<\beta \leq \ln (2+\sqrt{3}) \\ \frac{2}{3} \cdot \frac{3 \beta+2 \pi-3 \ln (2+\sqrt{3})}{2 \sqrt{3}+\beta-\ln (2+\sqrt{3})} & \text { if } \beta>\ln (2+\sqrt{3})\end{cases}
$$

Moreover, the equality is attained if and only ifg is proportional to the spherical metric $S_{b}$ restricted to $E_{b}=\{(u, v) \in \mathbb{C}| | v \mid \leq b\}$, for $b$ satisfying $\beta=2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{b}{2}\right)\right)$, in the first case, and to the spherical-flat metric $S F_{b, \frac{\pi}{3}}$ restricted to $E_{b}=\{(u, v) \in \mathbb{C}| | v \mid \leq b\}$, for $b=\frac{1}{4} \beta+\frac{\pi}{3}-\frac{1}{2} \ln (2+$ $\sqrt{3}$ ), in the second case.

### 1.5 Proof of the geometric Inequality of type $\ell_{\sigma} \ell_{v} \ell_{h}$

In the following, we prove the geometric inequality of type $\ell_{\sigma} \ell_{v} \ell_{h}$ on the Klein bottle. We will consider the curve families $\mathcal{S}_{1}\left(\omega_{2}\right), \mathcal{S}_{2}$ and $\mathcal{S}_{3}\left(\omega_{2}\right)$.

Proof of Theorem 1.1.9. Let $\beta>0$. There exists a unique $\omega_{2} \in\left(0, \frac{\pi}{2}\right)$ such that

$$
\begin{equation*}
\beta=2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{\omega_{2}}{2}\right)\right)+\frac{2}{\cos \left(\omega_{2}\right)}\left(\tan \left(\omega_{2}\right)-\omega_{2}+\sqrt{\tan ^{2}\left(\omega_{2}\right)-\omega_{2} \tan \left(\omega_{2}\right)+\omega_{2}^{2}}\right) \tag{1.5.1}
\end{equation*}
$$

We equip $\mathbb{K}$ with the metric $S F_{b, \omega_{2}}$, where

$$
\begin{equation*}
b=\tan \left(\omega_{2}\right)+\sqrt{\tan ^{2}\left(\omega_{2}\right)-\omega_{2} \tan \left(\omega_{2}\right)+\omega_{2}^{2}} \tag{1.5.2}
\end{equation*}
$$

Then, we consider the families of curves $\mathcal{S}_{1}\left(\omega_{2}\right), \mathcal{S}_{2}$ and $\mathcal{S}_{3}\left(\omega_{2}\right)$, cf. Section 1.2.2. Each curve in $\mathcal{S}_{1}\left(\omega_{2}\right)$, (resp. $\mathcal{S}_{2}$, resp. $\mathcal{S}_{3}\left(\omega_{2}\right)$ ) has length equal to $\pi$, (resp. $4 b$, resp. $2 \pi \cos \left(\omega_{2}\right)$ ) with respect to the metric $S F_{b, \omega_{2}}$. Hence, condition (1) of Theorem 1.2.1 is satisfied. Let

$$
\begin{array}{ccc}
l:\left[-\omega_{2}, \omega_{2}\right] & \longrightarrow & \mathbb{R} \\
a & \mapsto & \frac{\tan (|a|)}{\pi} \cdot \frac{\cos ^{2}(a)-\cos ^{2}\left(\omega_{2}\right) \frac{b-\omega_{2}}{2 b-\omega_{2}}}{\left(\cos ^{2}(a)-\cos ^{2}\left(\omega_{2}\right)\right)^{\frac{1}{2}}}
\end{array}
$$

We define on the family $\mathcal{S}_{1}\left(\omega_{2}\right)$ the measure

$$
\mu_{1}=\widetilde{l}(a) d a \otimes d \theta
$$

where

$$
\widetilde{l}(a)=\left\{\begin{array}{cll}
l(a) & \text { if } & a \in\left[-\omega_{2}, \omega_{2}\right] \\
l(a-2 b) & \text { if } & a \in\left[2 b-\omega_{2}, 2 b\right] \\
l(a+2 b) & \text { if } & a \in\left[-2 b,-2 b+\omega_{2}\right]
\end{array}\right.
$$

On the family $\mathcal{S}_{2}$, we define the measure

$$
\mu_{2}=\cos \left(\omega_{2}\right) \cdot \frac{b-\omega_{2}}{2 b-\omega_{2}} d u
$$

Finally, on $\mathcal{S}_{3}\left(\omega_{2}\right)$, we define the measure

$$
\mu_{3}=\frac{b}{2 b-\omega_{2}} d v .
$$

The masses $m_{i}$ of the measures $\mu_{i}$ are $m_{1}=4 \sin \left(\omega_{2}\right)-\frac{4 \omega_{2}\left(b-\omega_{2}\right) \cos \left(\omega_{2}\right)}{2 b-\omega_{2}}, m_{2}=\frac{\pi \cos \left(\omega_{2}\right)\left(b-\omega_{2}\right)}{2 b-\omega_{2}}$ and $m_{3}=\frac{2 b\left(b-\omega_{2}\right)}{2 b-\omega_{2}}$. Now, since $\ell_{\sigma}\left(S F_{b, \omega_{2}}\right)=\pi, \ell_{v}\left(S F_{b, \omega_{2}}\right)=4 b$, and $\ell_{h}\left(S F_{b, \omega_{2}}\right)=2 \pi \cos \left(\omega_{2}\right)$, condition (2) of Theorem 1.2.1 is satisfied. Next, it can be easily verified that

$$
{ }^{*} \mu_{2}=\frac{\cos \left(\omega_{2}\right)}{f_{b, \omega_{2}}(v)} \cdot \frac{b-\omega_{2}}{2 b-\omega_{2}} d\left(S F_{b, \omega_{2}}\right)
$$

where $f_{b, \omega_{2}}$ is the unique one-variable function invariant by the translation $(u, v) \mapsto(u, v+$ $2 b)$ which agrees with cosine on $\left[-\omega_{2}, \omega_{2}\right]$ and equal to the constant $\cos \left(\omega_{2}\right)$ on $\left[\omega_{2}, 2 b-\omega_{2}\right]$. Moreover,

$$
{ }^{*} \mu_{3}=\left\{\begin{array}{ccl}
0 & \text { if } & |v| \leq \omega_{2} \text { or } 2 b-\omega_{2} \leq|v| \leq 2 b \\
\frac{b}{2 b-\omega_{2}} \cdot d\left(S F_{b, \omega_{2}}\right) & \text { if } & \omega_{2} \leq|v| \leq 2 b-\omega_{2}
\end{array}\right.
$$

Next, by Lemma 1.2.5, we have

$$
\chi_{\left\{|v| \leq \omega_{2}\right\}}{ }^{*} \mu_{1}=2 \chi_{\left\{|v| \leq \omega_{2}\right\}}\left(\int_{|v|}^{\omega_{2}}\left(\cos ^{2}(v)-\cos ^{2}(a)\right)^{-\frac{1}{2}} \widetilde{l}(a) d a\right) d\left(S F_{b, \omega_{2}}\right)
$$

Then, we derive

$$
\chi_{\left\{|v| \leq \omega_{2}\right\}}{ }^{*} \mu_{1}=\chi_{\left\{|v| \leq \omega_{2}\right\}}\left(1-\frac{\cos \left(\omega_{2}\right)\left(b-\omega_{2}\right)}{f_{b, \omega_{2}}(v)\left(2 b-\omega_{2}\right)}\right) d\left(S F_{b, \omega_{2}}\right) .
$$

Finally, since $f_{b, \omega_{2}}$ and $\widetilde{l}$ are invariant by the translation $(u, v) \mapsto(u, v+2 b)$, we have

$$
{ }^{*} \mu_{1}=\left\{\begin{array}{cll}
\left(1-\frac{\cos \left(\omega_{2}\right)\left(b-\omega_{2}\right)}{f_{b, \omega_{2}}(v)\left(2 b-\omega_{2}\right)}\right) d\left(S F_{b, \omega_{2}}\right) & \text { if } & |v| \leq \omega_{2} \text { or } 2 b-\omega_{2} \leq|v| \leq 2 b \\
0 & \text { if } & \omega_{2} \leq|v| \leq 2 b-\omega_{2}
\end{array}\right.
$$

Thus, we obtain ${ }^{*} \mu_{1}+{ }^{*} \mu_{2}+{ }^{*} \mu_{3}=d S F_{b, \omega_{2}}$. Hence, condition (3) of Theorem 1.2.1 is satisfied. Hence,

$$
\operatorname{area}(g)^{\frac{3}{2}} \geq \frac{3 \sqrt{3}}{\sqrt{\pi}} \cdot \frac{\left(b-\omega_{2}\right) \sqrt{\left(b-\omega_{2}\right) b}}{\left(b^{4}-4 b \omega_{2}+\omega_{2}^{2}+\omega_{2}^{4}-2 b^{2}\left(-2+\omega_{2}^{2}\right)\right)^{\frac{1}{4}}\left(2 b-\omega_{2}\right)} \ell_{\sigma}(g) \ell_{v}(g) \ell_{h}(g)
$$

with equality if and only if $g$ is homothetic to $S F_{b, \omega_{2}}$.

### 1.6 Some remarks on the extremal metrics

In this chapter, we present four families of conformally extremal metrics for different types of inequalities on the Klein bottle. They are as follows.

1. The spherical metric $S_{b}$, where $\beta=2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{b}{2}\right)\right)$, which is conformally extremal for the inequality of type $\ell_{\sigma} \ell_{v}$ for $0<\beta \leq 2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{a_{0}}{2}\right)\right)$, and for the inequality of type $L_{\sigma} \ell_{v}$ for $0<\beta \leq 2 \ln (2+\sqrt{3})$.
2. The spherical-flat metric $S F_{b, \omega_{1}}$, where $\omega_{1}$ and $b$ verify the equations (1.3.1) and (1.3.2), which is conformally extremal for the inequality of type $\ell_{\sigma} \ell_{v}$ for $2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{a_{0}}{2}\right)\right)<$ $\beta<\infty$.
3. The spherical-flat metric $S F_{b, \frac{\pi}{3}}$, where $\beta=2 \ln (2+\sqrt{3})+4\left(b-\frac{\pi}{3}\right)$, which is conformally extremal for the inequality of type $L_{\sigma} \ell_{v}$ for $2 \ln (2+\sqrt{3})<\beta<\infty$.
4. The spherical-flat metric $S F_{b, \omega_{2}}$, where $\omega_{2}$ and $b$ verify the equations (1.5.1) and (1.5.2), which is conformally extremal for the inequality of type $\ell_{\sigma} \ell_{v} \ell_{h}$ for any $\beta$.

Remark 1.6.1. We emphasize the fact that the metrics $S F_{b, \omega_{1}}$ and $S F_{b, \omega_{2}}$ do not have the same conformal type since $\omega_{1}$ verifies the equation (1.3.1) whereas $\omega_{2}$ verifies the equation (1.5.1). Note that the equality $\omega_{1}=\omega_{2}$ leads to

$$
\sqrt{\tan ^{2}\left(\omega_{1}\right)-\omega_{1} \tan \left(\omega_{1}\right)+\omega_{1}^{2}}+\omega_{1}=0
$$

which is not possible.

Moreover, the $\beta$-extremal metric (extremal metric in the conformal class of $\beta$ ) for the inequality of type $\ell_{\sigma} \ell_{v} \ell_{h}$ agrees with the $\beta$-extremal metric for the inequality of type $L_{\sigma} \ell_{v}$ if and only if

$$
\beta=\beta_{0}=2 \ln (2+\sqrt{3})+4\left(\sqrt{3}+\left(3-\frac{\pi}{\sqrt{3}}+\frac{\pi^{2}}{9}\right)^{\frac{1}{2}}-\frac{\pi}{3}\right)
$$

(i.e., $b=\sqrt{3}+\left(3-\frac{\pi}{\sqrt{3}}+\frac{\pi^{2}}{9}\right)^{\frac{1}{2}}$ and $\omega_{2}=\frac{\pi}{3}$ ). On the other hand, the $\beta$-extremal metric for the inequality of type $\ell_{\sigma} \ell_{v}$ agrees with the $\beta$-extremal metric for the inequality of type $L_{\sigma} \ell_{v}$ if and only if $\beta \leq 2 \ln (2+\sqrt{3})$. Finally, the $\beta$-extremal metric for the inequality of $\ell_{\sigma} \ell_{v}$ and the $\beta$-extremal metric for the inequality of type $\ell_{\sigma} \ell_{v} \ell_{h}$ never agree.

Table 1.1 summarizes our study of the extremal metrics for the various inequality types relative to the conformal type $\beta$.

Table 1.1

| $\beta$ | $(0,2 \ln (2+\sqrt{3})]$ | $\left(2 \ln (2+\sqrt{3}), 2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{a_{0}}{2}\right)\right)\right]$ | $\left(2 \ln \left(\tan \left(\frac{\pi}{4}+\frac{a_{0}}{2}\right)\right),+\infty\right)$ |
| :--- | :--- | :--- | :--- |
| $\ell_{\sigma} \ell_{v} \leq C_{\beta}$ area | $S_{b}$ | $S_{b}$ | $S F_{b, \omega_{1}}$ |
| $L_{\sigma} \ell_{v} \leq C_{\beta}$ area | $S_{b}$ | $S F_{b, \frac{\pi}{3}}$ | $S F_{b, \frac{\pi}{3}}$ |
| $\ell_{\sigma} \ell_{v} \ell_{h} \leq C_{\beta}$ area $a^{\frac{3}{2}}$ | $S F_{b, \omega_{2}}$ | $S F_{b, \omega_{2}}$ | $S F_{b, \omega_{2}}$ |

## Chapter 2

# Optimal systolic inequalities on Finsler Mobius bands 

Nous établissons des inégalités géométriques optimales sur le ruban de Möbius muni d'une métrique de Finsler. Ces inégalités géométriques relient la systole et la hauteur du ruban de Möbius à son volume de Holmes-Thompson. Nous en déduisons une inégalité systolique optimale sur la bouteille de Klein munie d'une métrique de Finsler avec des symétries. Nous décrivons également une famille de métriques extremales dans les deux cas.


#### Abstract

We prove optimal systolic inequalities on Finsler Mobius bands relating the systole and the height of the Mobius band to its Holmes-Thompson volume. We also establish an optimal systolic inequality for Finsler Klein bottles with symmetries, which we conjecture to hold true for arbitrary Finsler metrics. We describe extremal metric families both in both cases.


### 2.1 Introduction

Optimal systolic inequalities were studied since the mid-twentieth century after C. Loewner proved in an unpublished work the following result, cf. [Ka07]. Every Riemannian two-torus $\mathbb{T}^{2}$ satisfies

$$
\begin{equation*}
\operatorname{area}\left(\mathbb{T}^{2}\right) \geq \frac{\sqrt{3}}{2} \operatorname{sys}^{2}\left(\mathbb{T}^{2}\right) \tag{2.1.1}
\end{equation*}
$$

with equality if and only if $\mathbb{T}^{2}$ is a flat hexagonal torus. Recall that the systole of a nonsimply connected Riemannian surface $M$, denoted by $\operatorname{sys}(M)$, represents the length of the shortest noncontractible loop of $M$. This inequality leads us to introduce the systolic area of $M$ defined as

$$
\begin{equation*}
\sigma_{R}(M):=\inf _{g} \frac{\operatorname{area}(M, g)}{\operatorname{sys}^{2}(M, g)} \tag{2.1.2}
\end{equation*}
$$

where $g$ runs over all the Riemannian metrics on $M$ (hence the subscript $R$ for Riemannian). Thus, $\sigma_{R}\left(\mathbb{T}^{2}\right)=\frac{\sqrt{3}}{2}$. Following this direction, P. Pu [Pu52] showed that $\sigma_{R}\left(\mathbb{R} \mathbb{P}^{2}\right)=\frac{2}{\pi}$, where the infimum is attained exactly by the Riemannian metrics with constant (positive) curvature on the projective plane $\mathbb{R}^{2} \mathbb{P}^{2}$. In the eighties, C. Bavard [Ba86] proved that $\sigma_{R}\left(\mathbb{K}^{2}\right)=\frac{2 \sqrt{2}}{\pi}$, where the infimum on the Klein bottle $\mathbb{K}^{2}$ is not attained by a smooth Riemannian metric. See also [Sak88], [Ba88] and [Ba06] for other proofs and variations on this inequality. These are the only nonsimply connected closed surfaces with a known systolic area. The existence of extremal metrics in higher dimension is wide open.

The original proofs of the optimal Riemannian systolic inequalities on $\mathbb{T}^{2}, \mathbb{R P}^{2}$ and $\mathbb{K}^{2}$ rely on the conformal representation theorem (a consequence of the uniformization theorem on Riemann surfaces) and proceed as follows. By the uniformization theorem, every Riemannian metric $g$ on a closed surface is conformally equivalent to a Riemannian metric $g_{0}$ of constant curvature. Taking the average of $g$ over the isometry group of $g_{0}$ gives rise to a new metric $\bar{g}$ with the same area as $g$. By the Cauchy-Schwarz inequality, the systole of $\bar{g}$ is at most the systole of $g$. Thus, the new metric $\bar{g}$ has a lower ratio area/sys ${ }^{2}$ than the original metric $g$. Now, if the isometry group of $g_{0}$ is transitive, which is the case for $\mathbb{T}^{2}$ and $\mathbb{R} \mathbb{P}^{2}$, the metric $\bar{g}$ has constant curvature. Hence the result for the projective plane. Then, it is not difficult to find the extremal metric among flat torus. The case of the Klein bottle requires an extra argument since the isometry group of $g_{0}$ is not transitive, $c f$. Section 2.7.

In this chapter, we consider Finsler systolic inequalities. Loosely speaking, a Finsler metric $F$
is defined as a Riemannian metric except that its restriction to a tangent plane is no longer a Euclidean norm but a Minkowski norm, cf. Section 2.2. From a dynamical point of view, the function $F^{2}$ can be considered as a Lagrangian which induces a Lagrangian flow on the tangent bundle $T M$ of $M$. Thus, Finsler manifolds can be considered as degree 2 homogeneous Lagrangian systems. The trajectories of the Lagrangian correspond to the geodesics of the Finsler metric.

There exist several definitions of volume for Finsler manifolds which coincide in the Riemannian case. We will consider the Holmes-Thompson volume vol $_{H T}$, cf. Section 2.2. As previously, we can define the systolic area $\sigma_{F}$, with the subscript $F$ for Finsler, by taking the infimum in (2.1.2) over all Finsler metrics on $M$.

Contrary to the Riemannian case, there is no uniformization theorem for Finsler surfaces. As a result, the classical Riemannian tools to prove optimal systolic inequalities on surfaces, which are based on the conformal length method described above, do not carry over to the Finsler case. New methods are thus required to deal with Finsler metrics.

The first optimal Finsler systolic inequality has been obtained by S. Ivanov [Iv02, Iv11] who extended Pu's systolic inequality to Finsler projective planes.

Theorem 2.1.1 [Iv02, Iv11]. Let $\mathbb{R P}^{2}$ be a Finsler projective plane. Then

$$
\frac{\operatorname{vol}_{H T}\left(\mathbb{R P}^{2}\right)}{\operatorname{sys}^{2}\left(\mathbb{R P}^{2}\right)} \geq \frac{2}{\pi}
$$

Furthermore, equality holds if all the geodesics are closed of the same length.

In particular, the systolic area of the projective plane is the same in the Riemannian and Finsler settings, that is,

$$
\sigma_{R}\left(\mathbb{R P}^{2}\right)=\sigma_{F}\left(\mathbb{R P}^{2}\right)=\frac{2}{\pi}
$$

Note that Theorem 2.1.1 provides an alternate proof of Pu's inequality in the Riemannian case which does not rely on the uniformization theorem.

Using a different method based on [Gr99] and [BI02], a Finsler version of Loewner's inequality (2.1.1) has been obtained by the first author [Sa10].

Theorem 2.1.2 [Sa10]. Let $\mathbb{T}^{2}$ be a Finsler two-torus. Then

$$
\frac{\operatorname{vol}_{H T}\left(\mathbb{T}^{2}\right)}{\operatorname{sys}^{2}\left(\mathbb{T}^{2}\right)} \geq \frac{2}{\pi}
$$

Equality holds if $\mathbb{T}^{2}$ is homothetic to the quotient of $\mathbb{R}^{2}$, endowed with a parallelogram norm $\|$.$\| ,$ by a lattice whose unit disk of $\|$.$\| is a fundamental domain.$

Observe that $\sigma_{F}\left(\mathbb{T}^{2}\right)=\sigma_{F}\left(\mathbb{R} \mathbb{P}^{2}\right)$ contrary to the Riemannian case. An optimal Finsler systolic inequality holds for non-reversible Finsler metrics on $\mathbb{T}^{2}, c f$. [ABT]. Note also that there is no systolic inequality for non-reversible Finsler two-tori if one considers the Busemann volume instead of the Holmes-Thompson volume, cf. [AB].

No systolic inequality holds for manifolds with boundary either. However, P. Pu [Pu52] and C. Blatter [Bl61] obtained optimal Riemannian systolic inequalities in each conformal class of the Mobius band and described the extremal metrics, cf. Section 2.4. Later, these inequalities were used by C. Bavard [Ba86] and T. Sakai [Sak88] in their proofs of the systolic inequality on the Klein bottle in the Riemannian case. The proof of the optimal conformal Riemannian systolic inequalities on the Mobius band relies on the uniformization theorem and the conformal length method (as in the original proofs of the Riemannian systolic inequalities on $\mathbb{T}^{2}, \mathbb{R P}^{2}$ and $\mathbb{K}^{2}$ ).

In this chapter, we first prove a Finsler generalization of the optimal systolic inequality on $\mathbb{T}^{2}$ extending Loewner's inequality, cf. [Ke67], and derive further optimal geometric inequalities on Finsler cylinders, $c f$. Section 2.3. These results allow us to establish an optimal inequality on every Finsler Mobius band $\mathbb{M}$ relating its systole sys $(\mathbb{M})$, its height $h(\mathbb{M})$ and its (HolmesThompson) volume $\operatorname{vol}_{H T}(\mathbb{M})$ at least when $\mathbb{M}$ is wide enough, $c f$. Section 2.5. Here, the height $h(\mathbb{M})$ represents the minimal length of arcs with endpoints on the boundary $\partial \mathbb{M}$, which are not homotopic to an arc in $\partial \mathbb{M}$, $c f$. Definition 2.3.4. More precisely, we prove the following.

Theorem 2.1.3. Let $\mathbb{M}$ be a Finsler Mobius band. Let $\lambda:=\frac{h(\mathbb{M})}{\operatorname{sys}(\mathbb{M})}$. Then

$$
\frac{\operatorname{vol}_{H T}(\mathbb{M})}{\operatorname{sys}(\mathbb{M}) h(\mathbb{M})} \geq \begin{cases}\frac{2}{\pi} & \text { if } \lambda \in(0,1]  \tag{2.1.3}\\ \frac{1}{\pi} \frac{\lambda+1}{\lambda} & \text { otherwise }\end{cases}
$$

Moreover, the above inequalities are optimal for every value of $\lambda \in(0,+\infty)$.

We describe extremal and almost extremal metric families in details in Section 2.4, Example 2.5.4 and Example 2.6.2.

The optimal Finsler systolic inequality on the Klein bottle is still unknown. However, based on the inequality (2.1.3) on Finsler Mobius bands, we obtain a partial result for Finsler Klein bottles with nontrivial symmetries. We refer to Definition 2.7.1 for a description of the symmetries considered in the statement of the following theorem.

Theorem 2.1.4. Let $\mathbb{K}$ be a Finsler Klein bottle with a soul, soul-switching or rotational symmetry. Then

$$
\begin{equation*}
\frac{\operatorname{vol}_{H T}(\mathbb{K})}{\operatorname{sys}^{2}(\mathbb{K})} \geq \frac{2}{\pi} \tag{2.1.4}
\end{equation*}
$$

Moreover, the inequality is optimal.

We also present some extremal metric family in Example 2.7.7.

Finally, we present as a conjecture that the inequality (2.1.4) should hold for every Finsler Klein bottle with or without symmetries. That is, $\sigma_{F}(\mathbb{K})$ should be equal to $\frac{2}{\pi}$ (as $\sigma_{F}\left(\mathbb{T}^{2}\right)$ and $\sigma_{F}\left(\mathbb{R} \mathbb{P}^{2}\right)$ ). Note, however, that the Finsler systolic area $\sigma_{F}$ is not the same for all the surfaces: it goes to infinity with the genus of the surface, $c f$. [Gr83].

### 2.2 Preliminaries

In this section, we introduce general definitions regarding Finsler manifolds.

A (reversible) Finsler metric $F: T M \rightarrow[0,+\infty)$ on the tangent bundle $T M$ of a smooth ndimensional manifold $M$ is a continuous function satisfying the following conditions (for simplicity, let $\left.F_{x}:=\left.F\right|_{T_{x} M}\right)$ :

1. Smoothness: $F$ is smooth outside the zero section;
2. Homogeneity: $F_{x}(t v)=|t| F_{x}(v)$ for every $v \in T_{x} M$ and $t \in \mathbb{R}$;
3. Quadratic convexity: for every $x \in M$, the function $F_{x}^{2}$ has positive definite second derivatives on $T_{x} M \backslash 0$, i.e., if $p, u, v \in T_{x} M$, the symmetric bilinear form

$$
g_{p}(u, v):=\left.\frac{1}{2} \frac{\partial^{2}}{\partial s \partial t}\left(F_{x}^{2}(p+t u+s v)\right)\right|_{t=s=0}
$$

is an inner product.

The pair $(M, F)$ is called a Finsler manifold. If $F$ is only positive homogeneous instead of homogeneous, that is, (2) only holds for $t \geq 0$, we say that the Finsler metric is non-reversible. For simplicity, we will only consider reversible Finsler metrics.

Conditions (1), (2) and (3) imply that $F$ is strictly positive outside the zero section and that for every $x \in M$ and $u, v \in T_{x} M$, we have

$$
F_{x}(u+v) \leq F_{x}(u)+F_{x}(v)
$$

with equality if and only if $u=\lambda v$ or $v=\lambda u$ for some $\lambda \geq 0, c f$. [BCS00, Theorem 1.2.2]. Hence, $F$ induces a strictly convex norm $F_{x}$ on each tangent space $T_{x} M$ with $x \in M$. More specifically, it gives rise to a Minkowski norm $F_{x}$ on each tangent symmetric space $T_{x} M$. Working with quadratically convex norms and not merely (strictly) convex norms provides nice dynamical features such as a geodesic flow and a Legendre transform, $c f$. [Be78, §1].

As in the Riemannian case, notions of length, distance, and geodesics extend to Finsler geometry.

Let $\gamma:[a, b] \rightarrow M$ be a piecewise smooth curve. The length of $\gamma$ is defined as

$$
\ell(\gamma):=\int_{a}^{b} F(\dot{\gamma}(t)) d t
$$

By condition (2), $\ell(\gamma)$ does not depend on the parametrization of $\gamma$. Moreover, the functional $\ell$ gives rise to a distance function $d_{F}: M \times M \rightarrow[0, \infty)$ defined as $d_{F}(x, y)=\inf _{\gamma} \ell(\gamma)$, where the infimum is taken over all piecewise smooth curves $\gamma$ joining $x$ to $y$. A geodesic is a curve which locally minimizes length. It is a critical point of the energy functional $\gamma \mapsto \int F^{2}(\gamma(t)) d t$ (here the quadratic convexity condition (3) is necessary).

For $x \in M$, we denote by $B_{x}$ the unit ball of the Minkowski norm $F_{x}$. Furthermore, the norm $F_{x}$ is Euclidean if and only if $B_{x}$ is an ellipsoid. The dual of $B_{x}$ is defined as $B_{x}^{*}=$ $\left\{f \in T_{x}^{*} M \mid F_{x}^{*}(f) \leq 1\right\}$ where $F_{x}^{*}$ is the dual norm of $F_{x}$. Note that $B_{x}^{*}$ identifies with the polar body $B_{x}^{\circ}=\left\{u \in T_{x} M \mid\langle u, v\rangle \leq 1\right.$ for every $\left.v \in B_{x}\right\}$ of $B_{x}$, where $\langle.,$.$\rangle is a given scalar$ product on $T_{x} M$.

In the Riemannian case, there exists a unique notion of volume, up to normalization, which agrees both with the n -dimensional Hausdorff measure determined by the Riemannian metric and with the projection of the Liouville measure from the unit tangent bundle, cf. [BBI01, §5.5]. However, in the Finsler case, there is no notion of volume that satisfies both properties, cf. [BI12]. This leads to two distinct notions of Finsler volume presented below.

Denote by $\varepsilon_{n}$ the Euclidean volume of the Euclidean unit ball in $\mathbb{R}^{n}$. Let $d x$ represent a given volume form on $M$ and $m$ be the restriction of this volume form to each tangent space $T_{x} M$. Similarly, let $m^{*}$ be the restriction of the volume form dual to $d x$ to each cotangent space $T_{x}^{*} M$. The Busemann volume, cf. [Bu47], is defined as

$$
\begin{equation*}
\operatorname{vol}_{B}(M):=\int_{M} \frac{\varepsilon_{n}}{m\left(B_{x}\right)} d x \tag{2.2.1}
\end{equation*}
$$

The Busemann volume is sometimes called the Busemann-Hausdorff volume as it agrees with the $n$-dimensional Hausdorff measure of $M$ (at least when the Finsler metric $F$ is reversible). Another volume frequently used in Finsler geometry is the Holmes-Thompson volume, cf. [HT79], defined as

$$
\begin{equation*}
\operatorname{vol}_{H T}(M):=\int_{M} \frac{m^{*}\left(B_{x}^{*}\right)}{\varepsilon_{n}} d x \tag{2.2.2}
\end{equation*}
$$

It is equal to the Liouville volume of its unit co-disc bundle divided by $\varepsilon_{n}, c f$. [AT04]. Note that the integrals in (2.2.1) and (2.2.2) do not depend on the chosen volume form. Sine the volume is a local notion, it is possible to extend this definition even when $M$ is nonorientable, that is, when volume forms do not exist on $M$.

In [Du98], C. Durán proved the following volume comparison inequality for Finsler manifolds:

$$
\begin{equation*}
\operatorname{vol}_{H T}(M) \leq \operatorname{vol}_{B}(M) \tag{2.2.3}
\end{equation*}
$$

with equality if and only if $M$ is Riemannian. Hence, every systolic inequality for the HolmesThompson volume remains true for the Busemann volume. However, the inequality (2.2.3) may fail for non-reversible Finsler metrics.

### 2.3 A systolic inequality on Finsler two-tori

In this section we establish a Finsler version of the Minkowski second theorem for the two-torus. More precisely, L. Keen proved the following.

Proposition 2.3.1 [Ke67], [Ka07] §6.2. Let $\mathbb{T}^{2}$ be a Riemannian two-torus. There exist two closed curves of lengths $a$ and $b$ generating the first integral homology group of $\mathbb{T}^{2}$ such that

$$
\operatorname{vol}_{H T}\left(\mathbb{T}^{2}\right) \geq \frac{\sqrt{3}}{2} a b
$$

Equality holds if and only if $\mathbb{T}^{2}$ is homothetic to the flat torus obtained as the quotient of $\mathbb{R}^{2}$ by a hexagonal lattice.

The proof of Proposition 2.3.1 relies on the uniformization theorem and the Cauchy-Schwarz inequality.

A Finsler version of Proposition 2.3.1 is given by the following result.
Proposition 2.3.2. Let $\mathbb{T}^{2}$ be a Finsler two-torus. There exist two closed curves of lengths $a$ and $b$ generating the first integral homology group of $\mathbb{T}^{2}$ such that

$$
\operatorname{vol}_{H T}\left(\mathbb{T}^{2}\right) \geq \frac{2}{\pi} a b
$$

Equality holds if $\mathbb{T}^{2}$ is homothetic to the quotient of $\mathbb{R}^{2}$, endowed with a parallelogram norm $\|$.$\| ,$ by a lattice generated by two vectors of lengths $a$ and $b$, parallel to the sides of the unit ball of $\|$.$\| .$

Since there is no uniformization theorem for Finsler metrics, the proof of this proposition differs from the proof of Proposition 2.3.1.

Proof. Let $\alpha$ be a systolic loop of $\mathbb{T}^{2}$ and $\beta$ be the shortest closed curve of $\mathbb{T}^{2}$ homologically independent with $\alpha$. Denote by $a$ and $b$ the lengths of $\alpha$ and $\beta$. The loops $\alpha$ and $\beta$ are simple and intersect each other at a single point. Cutting $\mathbb{T}^{2}$ open along $\alpha$ and $\beta$ gives rise to a surface $\Delta$ isometric to a fundamental domain of $\mathbb{T}^{2}$. Let $L$ be a positive number greater than $\max \{a, b\}$. Denote by $p$ and $q$ the smallest integers such that $p a \geq L$ and $q b \geq L$. Then, glue $p q$ copies of $\Delta$
in such a way that the resulting shape is isometric to the fundamental domain of a Finsler torus of volume $p q$ times the volume of $\mathbb{T}^{2}$ and of systole equal to $\min \{p a, q b\}$. By construction, this new Finsler torus is a degree $p q$ cover of $\mathbb{T}^{2}$. Then, by Theorem 2.1.2, we have

$$
p q \operatorname{vol}_{H T}\left(\mathbb{T}^{2}\right) \geq \frac{2}{\pi}(\min \{p a, q b\})^{2}
$$

Hence,

$$
\operatorname{vol}_{H T}\left(\mathbb{T}^{2}\right) \geq \frac{2}{\pi} \frac{L}{p} \frac{L}{q} \geq \frac{2}{\pi} \frac{p-1}{p} \frac{q-1}{q} a b
$$

By choosing $L$ large enough, the integers $p$ and $q$ can be made arbitrarily large, which leads to the desired inequality.

Now, if $\mathbb{T}^{2}$ is the quotient of $\mathbb{R}^{2}$, endowed with a parallelogram norm, by a lattice generated by two vectors of lengths $a$ and $b$ which are parallel to the sides of the unit ball of the parallelogram norm, then $\operatorname{vol}_{H T}\left(\mathbb{T}^{2}\right)=\frac{2}{\pi} a b$.

Remark 2.3.3. Briefly speaking, the idea of the proof of Proposition 2.3.2 is to use finite covers to get a quasi-isosystolic two-torus (i.e., whose first homology group is generated by two loops of lengths nearly the systole) and to apply the systolic inequality of Theorem 2.1.2 to this two-torus. This argument also applies in the Riemannian case and gives an alternative proof of Proposition 2.3.1 without the use of the uniformization theorem.

We can apply Proposition 2.3.2 to prove a systolic inequality on Finsler cylinders. First, we give the following definition

Definition 2.3.4. Let $M$ be a compact Finsler surface with boundary. The height $h(M)$ of $M$ is the minimal length of arcs with endpoints on the boundary $\partial M$, which are not homotopic to an arc in $\partial M$. More formally,

$$
h(M):=\inf \left\{\ell(\gamma) \mid \gamma:[0,1] \rightarrow M \text { with } \gamma(0), \gamma(1) \in \partial M \text { and }[\gamma] \neq 0 \in \pi_{1}(M, \partial M)\right\}
$$

A height arc of $M$ is a length-minimizing arc of $\gamma$ of $M$ with endpoints in $\partial M$ inducing a nontrivial class in $\pi_{1}(M, \partial M)$. By definition, the length of a height arc of $M$ is equal to $h(M)$.

Proposition 2.3.5. Let $C$ be a Finsler cylinder. Then,

$$
\operatorname{vol}(C) \geq \frac{2}{\pi} \operatorname{sys}(C) h(C)
$$

Proof. Let $k$ be a positive even integer. We glue $k$ copies of $C$ by identifiying the identical boundary components pairwise. The resulting space is a torus $\mathbb{T}^{2}$. Every loop of $\mathbb{T}^{2}$ non freely homotopic to a multiple of a boundary component of $C$ is of length at least $k h(C)$. Therefore, for symmetry reasons, if $k$ satisfies $k h(C) \geq \operatorname{sys}(C)$, the systole of the torus $\mathbb{T}^{2}$ is equal to the
systole of the cylinder $C$. Applying Proposition 2.3.2 to this torus, we derive

$$
\operatorname{vol}\left(\mathbb{T}^{2}\right)=k \operatorname{vol}(C) \geq \frac{2}{\pi} k \operatorname{sys}(C) h(C)
$$

Hence the result.

We will make use of Proposition 2.3.5 in the proof of Theorem 2.1.3 for wide Finsler Mobius bands, cf. Section 2.5.

### 2.4 Natural candidates for extremal metrics

In this section, we first review the extremal Riemannian metrics for systolic inequalities on the Mobius band and the Klein bottle presented in [Pu52, Bl61, Ba86, Ba88, Sak88]. By analogy with the Riemannian metrics, we construct Finsler metrics which are natural to consider when studying optimal Finsler systolic inequalities.

Consider the standard sphere $S^{2}$. Denote by $u$ and $v$ the longitude and the latitude on $S^{2}$. Let $a \in\left(0, \frac{\pi}{2}\right)$. The $a$-tubular neighborhood of the equator $\{v=0\}$ is a spherical band $S_{a}$ which can be represented as

$$
S_{a}:=\{(u, v) \mid-\pi \leq u \leq \pi,-a \leq v \leq a\}
$$

The quotient of $S_{a}$ by the antipodal map is a Riemannian Mobius band with curvature 1 denoted by $\mathbb{M}_{a}$. The conformal modulus space of the Mobius band is parametrized by $\mathbb{M}_{a}$ with $a \in\left(0, \frac{\pi}{2}\right)$. More precisely, every conformal class on the Mobius band agrees with the conformal structure induced by some $\mathbb{M}_{a}$ with $a \in\left(0, \frac{\pi}{2}\right)$. Furthermore, the conformal classes of the $\mathbb{M}_{a}$ 's are pairwise distinct. Indeed, the (conformal) stereographic projection sends $S_{a}$ to a planar annulus whose conformal modulus varies from 0 to $\infty$ as $a$ runs over ( $0, \frac{\pi}{2}$ ), cf. [Bl61].

The spherical Mobius bands $\mathbb{M}_{a}$ are involved in several extremal conformal systolic inequalities for Riemannian metrics. More precisely, we define the orientable systole of a Riemannian Mobius band $\mathbb{M}$ as the shortest length of a noncontractible orientable loop in $\mathbb{M}$. It will be denoted by $\operatorname{sys}_{+}(\mathbb{M})$. Similarly, we define the nonorientable systole of $\mathbb{M}$ and denote it by sys_( $\mathbb{M}$ ). Observe that $\operatorname{sys}(\mathbb{M})=\min \left\{\operatorname{sys}_{+}(\mathbb{M}), \operatorname{sys}_{-}(\mathbb{M})\right\}$. Moreover, we define $\ell_{v}(\mathbb{M})$ as the minimal length of the arcs joining $(u,-a)$ to $(u, a)$ in the $(u, v)$-coordinates of $\mathbb{M}_{a}$, which are homotopic with fixed endpoints to the projections of the meridians in $S_{a}$. For instance, $\operatorname{sys}_{+}\left(\mathbb{M}_{a}\right)=2 \pi \cos (a)$, $\operatorname{sys}_{-}\left(\mathbb{M}_{a}\right)=\pi$ and $\ell_{v}\left(\mathbb{M}_{a}\right)=2 a$. Note that the definition of $\ell_{v}$ relies on conformal data, namely the longitude-latitude coordinates to define the endpoints of the arcs involved in the length minimization.

In [Bl61], C. Blatter obtained optimal lower bounds for the functionals

$$
\sigma_{1}:=\frac{\mathrm{vol}}{\mathrm{sys}_{-}^{2}}, \quad \sigma_{2}:=\frac{\mathrm{vol}}{\mathrm{sys}_{-} \times \ell_{v}} \quad \text { and } \quad \sigma_{3}:=\frac{\mathrm{vol}}{\mathrm{sys} \times \ell_{v}}
$$

in each conformal class of the Mobius band. More precisely, for every Riemannian metric conformally equivalent to $\mathbb{M}_{a}$, we have the sharp lower bound

$$
\begin{equation*}
\sigma_{1}(\mathbb{M}) \geq \sigma_{1}\left(\mathbb{M}_{a}\right) \tag{2.4.1}
\end{equation*}
$$

We also have the sharp inequality

$$
\sigma_{2}(\mathbb{M}) \geq \begin{cases}\sigma_{2}\left(\mathbb{M}_{a}\right) & \text { if } a \in(0, b]  \tag{2.4.2}\\ \sigma_{2}\left(\mathbb{M}_{\alpha(a)} \cup C_{a, \alpha(a)}\right) & \text { if } a \in\left[b, \frac{\pi}{2}\right)\end{cases}
$$

where $b$ is the unique solution in $\left(0, \frac{\pi}{2}\right)$ of the equation $\tan (x)=2 x$ and $\mathbb{M}_{\alpha(a)} \cup C_{a, \alpha(a)}$ is the Mobius band obtained by attaching a flat cylinder $C_{a, \alpha(a)}$ to the spherical Mobius band $\mathbb{M}_{a, \alpha(a)}$ along their boundary. Here, the angle $\alpha(a) \in[b, a]$ is implicitly given by a nonlinear equation depending on the conformal type $a$ and the flat cylinder $C_{a, \alpha(a)}$ is defined as the product $\partial_{+} S_{\alpha}(a) \times[0, \sin a-\sin \alpha(a)]$, where $\partial_{+} S_{\alpha(a)}$ is a boundary component of $S_{\alpha}(a)$. Alternately, $C_{a, \alpha(a)}$ is the Mercator projection of a connected component of $S_{a} \backslash S_{\alpha(a)}$ to the vertical cylinder generated by $\partial S_{\alpha(a)}$.
Finally, we have the third sharp inequality

$$
\sigma_{3}(\mathbb{M}) \geq \begin{cases}\sigma_{3}\left(\mathbb{M}_{a}\right) & \text { if } a \in\left(0, \frac{\pi}{3}\right]  \tag{2.4.3}\\ \sigma_{3}\left(\mathbb{M}_{\frac{\pi}{3}} \cup C_{a, \frac{\pi}{3}}\right) & \text { if } a \in\left[\frac{\pi}{3}, \frac{\pi}{2}\right)\end{cases}
$$

With the help of (2.4.1), C. Bavard [Ba86] established the optimal isosystolic inequality on the Klein bottle. Later, T. Sakai [Sak88] used the inequalities (2.4.2) and (2.4.3) to give an alternative proof of Bavard's isosystolic inequality for the Klein bottle. The extremal Riemannian metric on the Klein bottle is obtained by gluing two copies of the spherical Mobius band $\mathbb{M}_{\frac{\pi}{4}}$ along their boundary.

The closed geodesics in $S_{a}$ project down to systolic loops in $\mathbb{M}_{a}$ which differ only by rotations and can be described as follows. Let $\gamma_{0}^{0}$ be the equator $\{v=0\}$ of $S^{2}$ parametrized by arclength. Every great circle $\gamma$ in $S_{a}$ different from $\gamma_{0}^{0}$ intersects $\gamma_{0}^{0}$ at a unique point $\gamma_{0}^{0}(s)$ with $s \in[0, \pi)$. The great circles $\gamma_{0}^{0}$ and $\gamma$ form at $\gamma_{0}^{0}(s)$ an oriented angle $\vartheta \in\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\vartheta \neq 0$. Such great circle $\gamma$ is denoted by $\gamma_{\vartheta}^{s}$. From the sinus formula in spherical trigonometry, the great circle $\gamma_{\vartheta}^{S}$ exactly lies between the circles of latitude $\pm \vartheta$. Thus, $\vartheta \in[-a, a]$.
For $\vartheta>0$, let $v \in(0, \vartheta]$. By Clairaut's relation, $c f$. [dC76, §4.4], the positive angle $\theta_{\vartheta}$ between the circle of latitude $v$ and the great circle $\gamma_{\vartheta}^{s}$ satisfies $\theta_{\vartheta}(v)=\arccos \left(\frac{\cos (\vartheta)}{\cos (v)}\right)$. In particular,


Figure 2.1: Systolic (in gray) and meridian directions in the unit tangent plane at a point of latitude $v$ in $\mathbb{M}_{a}$.
the unit tangent vectors to the systolic loops of $\mathbb{M}_{a}$ at a point of latitude $v$ generate a symmetric cone of half angle

$$
\theta(v)=\arccos \left(\frac{\cos (a)}{\cos (v)}\right)
$$

in the tangent plane, $c f$. Figure 2.1. These unit tangent vectors of $\mathbb{M}_{a}$ are referred to as systolic directions. The unit tangent vectors to the meridians are called meridian directions. Despite the risk of confusion, we will also call great circles of $\mathbb{M}_{a}$ the projections of the great circles of $S_{a}$ to $\mathbb{M}_{a}$.

The spherical Mobius band $\mathbb{M}_{a}$, which is extremal for some conformal systolic inequalities among Riemannian metrics, $c f$. (2.4.1), (2.4.2), and (2.4.3), is not extremal among Finsler metrics. Indeed, by slightly perturbing the quadratically convex norm in each tangent plane away from the systolic and meridian directions of the spherical metric, cf. Figure 2.1, we can decrease the area of the Mobius band without changing the systole and the height. This shows that any unit tangent vector to an extremal (quadratically convex) Finsler Mobius band is tangent either to a systolic loop or a height arc, $c f$. Definition 2.3.4. In other words, the unit tangent vectors induced by the systolic loops and the height arcs of an extremal (quadratically convex) Finsler Mobius band fill in its unit tangent bundle.

With this observation in mind, it is natural to consider the following (non-quadratically convex) Finsler metrics as potential extremal metrics. The idea is to adjust the shapes of the unit balls in the tangent bundle of the Mobius band so that the systolic and meridian directions fill in the unit tangent bundle. More precisely, define a Finsler metric $F_{a}$ on $S_{a}$ whose restriction to each tangent plane $T_{x} S_{a}$ is a norm $\left.F_{a}\right|_{x}$ of unit ball $B_{x}$ given by the convex hull of the systolic directions of $\mathbb{M}_{a}$, cf. Figure 2.2. In longitude and latitude coordinates, the ball $B_{x}$ at $x=(u, v)$ can be represented as

$$
B_{x}:=\left\{\left(\xi_{u}, \xi_{v}\right) \in T_{x} S_{a}\left|\xi_{u}^{2}+\xi_{v}^{2} \leq 1,\left|\xi_{v}\right| \leq \sin \theta(v)\right\}\right.
$$


b. $B_{x}^{*}$

Figure 2.2: Unit ball $B_{x}$ of $F_{a}$ and its dual $B_{x}^{*}$ at a point $x$ of latitude $v$ in $S_{a}$.

Hence, the Finsler metric $F_{a}$ can be represented in local coordinates as

$$
F_{a}= \begin{cases}\frac{1}{\sin (\theta(v))}|d v| & \text { if } \arctan \left(\frac{d v}{d u}\right) \in\left[\theta(v), \frac{\pi}{2}\right] \\ \sqrt{d u^{2}+d v^{2}} & \text { if } \arctan \left(\frac{d v}{d u}\right) \in[0, \theta(v)]\end{cases}
$$

This metric passes to the quotient by the antipodal map to a Finsler metric still denoted by $F_{a}$. Denote by $\mathbb{M}_{F_{a}}$ the Finsler Mobius band so obtained.

Instead of considering $\ell_{v}$, whose definition relies on conformal data, we will work with the height $h(\mathbb{M}), c f$. Definition 2.3.4, in the Finsler case.
Some geometric features of the Finsler Mobius bands $\mathbb{M}_{F_{a}}$ are summarized in the following two propositions.

Proposition 2.4.1. Let $a \in\left(0, \frac{\pi}{2}\right)$. Then, $\operatorname{sys}_{-}\left(\mathbb{M}_{F_{a}}\right)=\pi$, $\operatorname{sys}_{+}\left(\mathbb{M}_{F_{a}}\right)=2 \pi \cos (a)$ and $h\left(\mathbb{M}_{F_{a}}\right)=\pi$.
In particular, if $a=\frac{\pi}{3}$ then $\operatorname{sys}\left(\mathbb{M}_{F_{a}}\right)=\operatorname{sys}_{+}\left(\mathbb{M}_{F_{a}}\right)=\operatorname{sys}_{-}\left(\mathbb{M}_{F_{a}}\right)=h\left(\mathbb{M}_{F_{a}}\right)=\pi$.

Proof. Let us start with a useful observation. Denote by $\mathcal{S}$ the interior of the domain of $U \mathbb{M}_{F_{a}}$ formed by the unit tangent vectors of the great circles of $\mathbb{M}_{a}$. The Finsler metric $F_{a}$ coincides with the round Riemannian metric of $\mathbb{M}_{a}$ on $\mathcal{S}$. Therefore, the subset $\mathcal{S}$ is stable under the geodesic flow of $F_{a}$ (which is well-defined on $\mathcal{S}$ ). Furthermore, the length of a great circle with respect to $F_{a}$ is equal to $\pi$.

Let us show that $h\left(\mathbb{M}_{F_{a}}\right)=\pi$. Consider a height arc $\gamma$ of $\mathbb{M}_{F_{a}}$. The arc $\gamma$ can be parametrized with respect to the latitude. Otherwise, we could remove a subarc of $\gamma$ joining two points at the same latitude and still make up an arc in the same relative homotopy class as $\gamma$ with the remaining pieces using the rotational symmetry of $\mathbb{M}_{F_{a}}$. This would contradict the length-minimizing
property of $\gamma$. Hence,

$$
\begin{aligned}
h\left(\mathbb{M}_{F_{a}}\right) & =\ell(\gamma) \\
& =\int_{-a}^{a} \frac{1}{\sin \theta(v)} d v \\
& =\int_{-a}^{a} \frac{\cos (v)}{\sqrt{\cos ^{2}(v)-\cos ^{2}(a)}} d v \\
& =\left.2 \arctan \left(\frac{\sqrt{2} \sin (v)}{\sqrt{\cos (2 v)-\cos (2 a)}}\right)\right|_{0} ^{a} \\
& =\pi
\end{aligned}
$$

Now, let us show that the systolic curves of $\mathbb{M}_{F_{a}}$ agree with the great circles of $\mathbb{M}_{a}$ in the nonorientable case and with the boundary of $\mathbb{M}_{a}$ in the orientable case. Consider an orientable or nonorientable noncontractible loop $\gamma$ of minimal length in $\mathbb{M}_{F_{a}}$.

If $\gamma$ lies in the boundary of $\mathbb{M}_{F_{a}}$ then the loop $\gamma$ is orientable of length $2 \pi \cos (a)$. Thus, we can assume that $\gamma$ passes through an interior point $p$ of $\mathbb{M}_{F_{a}}$.

If a tangent vector of $\gamma$ lies in $\mathcal{S}$ then the geodesic arc $\gamma$ coincides with a great circle of $\mathbb{M}_{a}$ in the nonorientable case and with a great circle run twice in the orientable case. (Recall that $\mathcal{S}$ is stable by the geodesic flow of $F_{a}$.) In the former case, the curve $\gamma$ is of length $\pi$, while in the latter, it is of length $2 \pi$. Thus, we can assume that the tangent vectors of $\gamma$ do not lie in $\mathcal{S}$.

Consider the closed lift $\bar{\gamma}$ of $\gamma$ in $S_{a}$. Let $c_{ \pm}$be the two extreme great circles of $S_{a}$ passing through the lifts of $p$ and tangent to the boundary of $S_{a}$. That is, $c_{ \pm}$are the great circles of $S_{a}$ making an angle of $\pm \theta(v)$ with the curves of constant latitude $\pm v$ in $S_{a}$ passing through the lifts of $p$. Since the tangent vectors of $\gamma$ do not lie in $\mathcal{S}$, the curve $\bar{\gamma}$ does not intersect $c_{ \pm}$in the interior of $S_{a}$, except at the lifts of $p$. Therefore, there exists a subarc of $\bar{\gamma}$ (actually two subarcs of $\bar{\gamma}$ ) joining the two boundary components of $S_{a}$ in the region delimited by the great circles $c_{ \pm}$and the boundary of $S_{a}$, see the gray region of Figure 2.3. Thus, $\ell(\gamma) \geq h\left(\mathbb{M}_{F_{a}}\right)=\pi$ with equality if $\gamma$ agrees with $c_{ \pm}$.

We conclude that sys_ $\left(\mathbb{M}_{F_{a}}\right)=\pi$ and $\operatorname{sys}_{+}\left(\mathbb{M}_{F_{a}}\right)=\ell\left(\partial \mathbb{M}_{F_{a}}\right)=2 \pi \cos (a)$. Hence,

$$
\operatorname{sys}\left(\mathbb{M}_{F_{a}}\right)= \begin{cases}\operatorname{sys}_{-}\left(\mathbb{M}_{F_{a}}\right) & \text { if } a \in\left(0, \frac{\pi}{3}\right] \\ \operatorname{sys}_{+}\left(\mathbb{M}_{F_{a}}\right) & \text { if } a \in\left[\frac{\pi}{3}, \frac{\pi}{2}\right)\end{cases}
$$

Remark 2.4.2. The Finsler Mobius bands $\mathbb{M}_{F_{a}}$ are not pairwise isometric since they have distinct orientable systoles.

Proposition 2.4.3. Let $a \in\left(0, \frac{\pi}{2}\right)$. Then, $\operatorname{vol}_{H T}\left(\mathbb{M}_{F_{a}}\right)=2 \pi$.


Figure 2.3: The extreme great circles $c_{ \pm}$passing through a lift $\widetilde{p}$ of $p$ in $S_{a}$.

Proof. The unit ball $B_{x}^{*}$ coincides with the polar body of $B_{x}$ described in Figure ??. The area of $B_{x}^{*}$ is equal to $2 \theta(v)+\frac{2}{\tan \theta(v)}$. By definition of the Holmes-Thompson volume, cf. (2.2.2), we have

$$
\begin{aligned}
\operatorname{vol}_{H T}\left(\mathbb{M}_{F_{a}}\right) & =\frac{1}{2 \pi} \int_{S_{a}} m\left(B_{x}^{*}\right) d m(x) \\
& =\frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-a}^{a}\left(\theta(v)+\frac{1}{\tan \theta(v)}\right) \cos (v) d u d v \\
& =\frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-a}^{a}\left(\arccos \left(\frac{\cos (a)}{\cos (v)}\right)+\frac{\cos (a)}{\sqrt{\cos ^{2}(v)-\cos ^{2}(a)}}\right) \cos (v) d u d v \\
& =2 \pi
\end{aligned}
$$

Remark 2.4.4. As a consequence of Propositions 2.4 .1 and 2.4.3, we observe the following couple of points.

1. The orientable and nonorientable systoles of $F_{a}$ agree with those of its dual $F_{a}^{*}$. Hence, $\operatorname{sys}\left(\mathbb{M}_{F_{a}}\right)=\operatorname{sys}\left(\mathbb{M}_{F_{a}^{*}}\right)$. Moreover, computations similar to those in Propositions 2.4.1 and 2.4.3 show that $h\left(\mathbb{M}_{F_{a}^{*}}\right)=\pi(1-\cos (a))$ and $\operatorname{vol}_{H T}\left(\mathbb{M}_{F_{a}^{*}}\right)=2 \pi \sin ^{2}(a)$. This means that for both $F_{a}$ and its dual $F_{a}^{*}$, we have $\frac{\operatorname{vol}_{H T}(\mathbb{M})}{\operatorname{sys}_{-}(\mathbb{M}) h(\mathbb{M})} \rightarrow \frac{2}{\pi}$, when $a \rightarrow \frac{\pi}{2}$.
2. The Finsler Mobius bands $\mathbb{M}_{F_{a}}$ with $a \in\left(0, \frac{\pi}{3}\right]$ attain the equality case in (2.1.3) when $\operatorname{sys}(\mathbb{M})=h(\mathbb{M})$.

### 2.5 Systolic inequalities on wide Finsler Mobius bands

In this section, we give a proof of Theorem 2.1.3 for wide Finsler Mobius bands, that is, when $\lambda \geq 1$. More precisely, we prove the following result.

Proposition 2.5.1. Let $\mathbb{M}$ be a Finsler Mobius band with $h(\mathbb{M}) \geq \operatorname{sys}(\mathbb{M})$. Then

$$
\begin{equation*}
\operatorname{vol}_{H T}(\mathbb{M}) \geq \frac{1}{\pi} \operatorname{sys}(\mathbb{M})(\operatorname{sys}(\mathbb{M})+h(\mathbb{M})) \tag{2.5.1}
\end{equation*}
$$

We present examples showing this result is optimal at the end of this section, cf. Example 2.5.4.

Proof. Consider

$$
\mathbb{U}:=\left\{x \in \mathbb{M} \left\lvert\, d(x, \partial \mathbb{M}) \leq \frac{\lambda-1}{2} \operatorname{sys}(\mathbb{M})\right.\right\}
$$

Slightly perturbing the distance function $d(., \partial \mathbb{M})$ if necessary, we can assume that this distance function is a Morse function on $\mathbb{M}$ for which $\frac{\lambda-1}{2} \operatorname{sys}(\mathbb{M})$ is a regular value. In this case, $\mathbb{U}$ is a surface with boundary. If $\mathbb{M}$ has some "big bumps", the surface $\mathbb{U}$ may possibly have some holes. More precisely, the surface $\mathbb{U}$ may not be a topological cylinder as some of its boundary components may bound topological disks in $\mathbb{M}$.

Let $\widehat{\mathbb{U}}$ be the union of $\mathbb{U}$ with the topological disks of $\mathbb{M}$ bounded by the boundary components of $\mathbb{U}$. Under this construction, $\widehat{\mathbb{U}}$ is a cylinder one of whose boundary components agrees with $\partial \mathbb{M}$. Clearly, the height of $\widehat{\mathbb{U}}$ is equal to $\frac{\lambda-1}{2} \operatorname{sys}(\mathbb{M})$. Furthermore, since the inclusion $\widehat{\mathbb{U}} \subset \mathbb{M}$ induces a $\pi_{1}$-isomorphism, we have $\operatorname{sys}(\widehat{\mathbb{U}}) \geq \operatorname{sys}(\mathbb{M})$. Applying Proposition 2.3.5 to the cylinder $\widehat{\mathbb{U}}$ yields

$$
\begin{equation*}
\operatorname{vol}_{H T}(\widehat{\mathbb{U}}) \geq \frac{\lambda-1}{\pi} \operatorname{sys}(\mathbb{M})^{2} \tag{2.5.2}
\end{equation*}
$$

Now, consider the Finsler Mobius band $\mathbb{M}_{-}:=\mathbb{M} \backslash \widehat{\mathbb{U}}$.
Lemma 2.5.2. The height and systole of $\mathbb{M}_{-}$satisfy

$$
h\left(\mathbb{M}_{-}\right)=\operatorname{sys}(\mathbb{M}) \quad \text { and } \quad \operatorname{sys}\left(\mathbb{M}_{-}\right) \geq \operatorname{sys}(\mathbb{M})
$$

Proof. Let $\gamma_{-}$be a height arc of $\mathbb{M}_{-}$, cf. Definition 2.3.4. By construction, $\partial \widehat{\mathbb{U}}=\partial \mathbb{M}_{-} \cup \partial \mathbb{M}$ and the points of $\partial \mathbb{M}_{-}$are at distance $\frac{\lambda-1}{2} \operatorname{sys}(\mathbb{M})$ from $\partial \mathbb{M}$. Therefore, the two endpoints of $\gamma_{-}$ can be connected to $\partial \mathbb{M}$ by two arcs $\gamma_{1}$ and $\gamma_{2}$ of $\widehat{\mathbb{U}}$, each of length $\frac{\lambda-1}{2} \operatorname{sys}(\mathbb{M})$. Moreover, the $\operatorname{arc} \gamma:=\gamma_{-} \cup \gamma_{1} \cup \gamma_{2}$ with endpoints in $\partial \mathbb{M}$ induces a nontrivial class in $\pi_{1}(\mathbb{M}, \partial \mathbb{M})$. Therefore, since $h(\mathbb{M})=\lambda \operatorname{sys}(\mathbb{M})$, we obtain

$$
\begin{aligned}
h\left(\mathbb{M}_{-}\right) & =\ell(\gamma)-(\lambda-1) \operatorname{sys}(\mathbb{M}) \\
& \geq h(\mathbb{M})-(\lambda-1) \operatorname{sys}(\mathbb{M}) \\
& \geq \operatorname{sys}(\mathbb{M})
\end{aligned}
$$

Now, let $\gamma$ be a height arc of $\mathbb{M}$. By definition, we have $\ell(\gamma)=h(\mathbb{M})=\lambda \operatorname{sys}(\mathbb{M})$. The part $\gamma \cap \widehat{\mathbb{U}}$ of $\gamma$ in $\widehat{\mathbb{U}}$ is made of two arcs, each of length at least $\frac{\lambda-1}{2} \operatorname{sys}(\mathbb{M})$. Moreover, the arc $\gamma \cap \mathbb{M}_{-}$with endpoints in $\partial \mathbb{M}_{-}$induces a nontrivial class in $\pi_{1}\left(\mathbb{M}_{-}, \partial \mathbb{M}_{-}\right)$. Hence,

$$
\begin{aligned}
h\left(\mathbb{M}_{-}\right) & \leq \ell(\gamma)-\ell(\gamma \cap \widehat{\mathbb{U}}) \\
& \leq h(\mathbb{M})-(\lambda-1) \operatorname{sys}(\mathbb{M}) \\
& \leq \operatorname{sys}(\mathbb{M})
\end{aligned}
$$

Since the inclusion $\mathbb{M}_{-} \subset \mathbb{M}$ induces a $\pi_{1}$-isomorphism, we obtain

$$
\operatorname{sys}\left(\mathbb{M}_{-}\right) \geq \operatorname{sys}(\mathbb{M})
$$

Consider the projective plane $\mathbb{R} \mathbb{P}^{2}$ defined as the quotient $\mathbb{M}_{-} / \partial \mathbb{M}_{-}$, where the boundary $\partial \mathbb{M}_{-}$ is collapsed to a point. Strictly speaking, the Finsler metric on $\mathbb{R P}^{2}$ has a singularity at the point to which $\partial \mathbb{M}_{-}$collapses, but we can smooth it out.

The following result allows us to derive the systole of $\mathbb{R} \mathbb{P}^{2}$.
Lemma 2.5.3. Let $\mathbb{R}^{2}$ be the projective plane defined as the quotient $\mathbb{M} / \partial \mathbb{M}$ of a Finsler Mobius band $\mathbb{M}$. Then,

$$
\operatorname{sys}\left(\mathbb{R} \mathbb{P}^{2}\right)=\min \{h(\mathbb{M}), \operatorname{sys}(\mathbb{M})\}
$$

where $\mathbb{R P}^{2}$ is endowed with the quotient metric.
Proof. Let $\gamma$ be a noncontractible loop in $\mathbb{R P}^{2}$. The curve $\gamma$ lifts either to a noncontractible loop in $\mathbb{M}$ or to a noncontractible $\operatorname{arc}$ in $\mathbb{M}$ joining two points of the boundary $\partial \mathbb{M}$. In the former case, the length of $\gamma$ is at least $\operatorname{sys}(\mathbb{M})$, while in the latter, it is at least $h(\mathbb{M})$. On the other hand, we can easy construct noncontractible loops in $\mathbb{R P}^{2}$ of length sys $(\mathbb{M})$ or $h(\mathbb{M})$.

From Lemma 2.5.3 and Lemma 2.5.2, the systole of $\mathbb{R} \mathbb{P}^{2}$ is equal to sys $(\mathbb{M})$. Applying Theorem 2.1.1 to $\mathbb{R P}^{2}$, we obtain

$$
\operatorname{vol}_{H T}\left(\mathbb{M}_{-}\right)=\operatorname{vol}_{H T}\left(\mathbb{R P}^{2}\right) \geq \frac{2}{\pi} \operatorname{sys}^{2}(\mathbb{M})
$$

This inequality combined with (2.5.2) yields

$$
\begin{aligned}
\operatorname{vol}_{H T}(\mathbb{M}) & =\operatorname{vol}_{H T}(\widehat{\mathbb{U}})+\operatorname{vol}_{H T}\left(\mathbb{M}_{-}\right) \\
& \geq \frac{1+\lambda}{\pi} \operatorname{sys}(\mathbb{M})^{2}
\end{aligned}
$$

Hence the result.

We conclude this section by describing extremal and almost extremal Finsler metrics when $\lambda \geq 1$.
Example 2.5.4. Let $\lambda \in[1,+\infty)$.
(E.1) The horizontal translation $\tau$ of vector $\pi \boldsymbol{e}_{x}$ is an isometry of the plane $\mathbb{R}^{2}$ endowed with the sup-norm. The quotient of the strip $\mathbb{R} \times\left[0,(\lambda-1) \frac{\pi}{2}\right]$ by the isometry group $\langle\tau\rangle$ generated by $\tau$ is a cylinder $C$. The Finsler mobius band $\mathbb{M}$ obtained by gluing a boundary component of $C$ to $\mathbb{M}_{F_{a}}$ along $\partial \mathbb{M}_{F_{a}}$ with $a=\frac{\pi}{3}$, cf. Section 2.4, satisfies $\operatorname{vol}_{H T}(\mathbb{M})=(1+\lambda) \pi$, $\operatorname{sys}(\mathbb{M})=\pi$ and $h(\mathbb{M})=\lambda \pi$. See Figure 2.4.(A).


Figure 2.4: Almost extremal Finsler Mobius bands when $h(\mathbb{M}) \geq \operatorname{sys}(\mathbb{M})$.
(E.2) Endow the plane $\mathbb{R}^{2}$ with the sup-norm. The quotient of the strip $\mathbb{R} \times\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ by the group generated by the map $(x, y) \mapsto(x+\pi,-y)$ is a Finsler Mobius band $\mathbb{M}_{\pi}$ with $\operatorname{vol}_{H T}\left(\mathbb{M}_{\pi}\right)=2 \pi, \operatorname{sys}\left(\mathbb{M}_{\pi}\right)=\pi$ and $h\left(\mathbb{M}_{\pi}\right)=\pi$. Let $C$ be the Finsler cylinder defined in (E.1). Attach $C$ to $\mathbb{M}_{\pi}$ via a cylindrical part of arbitrarily small area, cf. Figure 2.4.(B), so that the resulting space is a Finsler Mobius band $\mathbb{M}$ with $\operatorname{sys}(\mathbb{M})=\pi, h(\mathbb{M})=\nu_{1} \lambda \pi$ and $\operatorname{vol}_{H T}(\mathbb{M})=\nu_{2}(\lambda+1) \pi$, where $\nu_{1}, \nu_{2}>1$ are arbitrarily close to 1 . This Finsler Mobius band is almost extremal for the inequality (2.5.1) when $\lambda \geq 1$.

### 2.6 Systolic inequalities on narrow Finsler Mobius bands

In this section, we give a proof of Theorem 2.1.3 for narrow Finsler Mobius bands, that is, when $\lambda<1$. More precisely, we prove the following result.

Proposition 2.6.1. Let $\mathbb{M}$ be a Finsler Mobius band with $h(\mathbb{M})<\operatorname{sys}(\mathbb{M})$. Then

$$
\operatorname{vol}_{H T}(\mathbb{M}) \geq \frac{2}{\pi} \operatorname{sys}(\mathbb{M}) h(\mathbb{M})
$$

This inequality is optimal. Extremal Finsler metrics can be constructed as follows.
Example 2.6.2. Let $\lambda<1$. Endow the plane $\mathbb{R}^{2}$ with the sup-norm. The quotient of the strip $\mathbb{R} \times\left[-\frac{\pi}{2} \lambda, \frac{\pi}{2} \lambda\right]$ by the group generated by the map $(x, y) \mapsto(x+\pi,-y)$ is a Finsler Mobius band $\mathbb{M}$ with $\operatorname{vol}_{H T}(\mathbb{M})=2 \lambda \pi, \operatorname{sys}(\mathbb{M})=\pi$ and $h(\mathbb{M})=\lambda \pi$.

Before proceeding to the proof of this proposition, we need to introduce a few definitions and notions.

Definition 2.6.3. The orientable double cover of a Finsler Mobius band $\mathbb{M}$ is a cylinder denoted by $C$. The points of $C$ which are at the same distance from each boundary component of $C$ form a simple closed curve $c$ invariant under deck transformations. The soul of $\mathbb{M}$, denoted by $\sigma$, is defined as the projection of $c$ to $\mathbb{M}$. Note that $\sigma$ is a nonorientable simple closed curve of $\mathbb{M}$ whose homotopy class generates $\pi_{1}(\mathbb{M})$.

Let $\mathbb{K}$ be the Finsler Klein bottle obtained by attaching to $\mathbb{M}$ another copy $\mathbb{M}^{\prime}$ of $\mathbb{M}$ along their boundary. (The Finsler metric on $\mathbb{K}$ may have a singularity line along $\partial \mathbb{M}$.) The isometry of $\mathbb{K}$ switching the souls of $\mathbb{M}$ and $\mathbb{M}^{\prime}$, and leaving $\partial \mathbb{M}$ pointwise fixed is called the soul-switching symmetry of $\mathbb{K}$.

Let $s, h \in \mathbb{R}_{+}$. The systole-height inequality on the Mobius band is said to be satisfied for $(s, h)$ if for every Finsler Mobius band $\mathbb{M}$ with $\operatorname{sys}(\mathbb{M}) \geq s$ and $h(\mathbb{M}) \geq h$, we have

$$
\operatorname{vol}_{H T}(\mathbb{M}) \geq \frac{2}{\pi} s h
$$

By scale invariance, if the systole-height inequality is satisfied for $(s, h)$, then it is also satisfied for $\left(s^{\prime}, h^{\prime}\right)$ with $\frac{h^{\prime}}{s^{\prime}}=\frac{h}{s}$.

We first prove the following preliminary result.
Lemma 2.6.4. Let $\lambda \in(0,1]$. Suppose that the systole-height inequality on the Mobius band is satisfied for $(s, h)$ with $\frac{h}{s}=\lambda$. Then, it is also satisfied for $(s, h)$ with $\frac{h}{s}=\frac{\lambda}{2}$.

Proof. Let $\mathbb{M}$ be a Finsler Mobius band with $\operatorname{sys}(\mathbb{M}) \geq s$ and $h(\mathbb{M}) \geq h$, where $\frac{s}{h}=\frac{\lambda}{2}$. Consider the Klein bottle $\mathbb{K}$ made of two copies of $\mathbb{M}$ defined in Definition 2.6.3 and cut it open along the soul $\sigma^{\prime}$ of $\mathbb{M}^{\prime}$. The resulting surface is a Finsler Mobius band denoted by $2 \mathbb{M}$ whose boundary component double covers $\sigma^{\prime}$ in $\mathbb{K}$.

Let $\alpha$ be a noncontractible loop of $2 \mathbb{M}$. Decompose $\alpha$ into two parts $a=\alpha \cap \mathbb{M}$ and $a^{\prime}=$ $\alpha \cap \mathbb{M}^{\prime}$ with $\alpha=a \cup a^{\prime}$. The parts $a$ and $a^{\prime}$ form two collections of arcs with endpoints lying in $\partial \mathbb{M}=\partial \mathbb{M}^{\prime}$. By construction, the image $a^{\prime \prime}$ of $a^{\prime}$ by the soul-switching symmetry lies in $\mathbb{M}$. Furthermore, the union $\bar{\alpha}=a \cup a^{\prime \prime}$ forms a closed curve lying in $\mathbb{M}$ and homotopic to $\alpha$ in $2 \mathbb{M}$ (and so noncontractible in $\mathbb{M}$ ). Since $\bar{\alpha}$ has the same length as $\alpha$, we conclude that $\operatorname{sys}(2 \mathbb{M}) \geq$ $\operatorname{sys}(\mathbb{M}) \geq s$.
Actually, since the inclusion $\mathbb{M} \subset 2 \mathbb{M}$ is a strong deformation retract, we derive the relation $\operatorname{sys}(2 \mathbb{M})=\operatorname{sys}(\mathbb{M})$. But we will not make use of this equality in the sequel.

By construction, the distance between the soul $\sigma$ and $\partial \mathbb{M}$ (and between $\sigma^{\prime}$ and $\partial \mathbb{M}^{\prime}$ ) is at least $\frac{1}{2} h(\mathbb{M})$. This implies that $h(2 \mathbb{M}) \geq 2 h(\mathbb{M}) \geq 2 h$.
Actually, we can show that $h(2 \mathbb{M})=2 h(\mathbb{M})$ (but we will not make use of this relation afterwards). Indeed, let $\alpha$ be a height arc of $\mathbb{M}$. By definition, $\ell(\alpha)=h(\mathbb{M})$. Denote by $\alpha^{\prime}$ its image
in $\mathbb{M}^{\prime}$ by the soul-switching symmetry of $\mathbb{K}$. The trace of the union $\alpha \cup \alpha^{\prime}$ to $2 \mathbb{M}$ defines an arc with endpoints in $\partial(2 \mathbb{M})$ inducing a nontrivial class in $\pi_{1}(2 \mathbb{M}, \partial(2 \mathbb{M}))$. The length of this arc is twice the length of $\alpha$. Therefore, $h(2 \mathbb{M}) \leq 2 h(\mathbb{M})$. Hence, the equality $h(2 \mathbb{M})=2 h(\mathbb{M})$.

In conclusion, the Mobius band $2 \mathbb{M}$ satisfies $\operatorname{sys}(2 \mathbb{M}) \geq s$ and $h(2 \mathbb{M}) \geq 2 h$. Since $\frac{2 h}{s}=\lambda$, the systole-height inequality is satisfied for $(s, 2 h)$ by the lemma assumption. Therefore,

$$
2 \operatorname{vol}_{H T}(\mathbb{M})=\operatorname{vol}_{H T}(2 \mathbb{M}) \geq \frac{4}{\pi} s h
$$

and the result follows.
We establish a second preliminary result.
Lemma 2.6.5. Let $\lambda_{1}, \lambda_{2} \in \mathbb{R}$ such that $0<\lambda_{1}<\lambda_{2} \leq 1$. Suppose that the systole-height inequality on the Mobius band is satisfied for $(s, h)$ with $\frac{h}{s}=\lambda_{1}$ or $\lambda_{2}$. Then, it is also satisfied for $(s, h)$ with $\frac{h}{s}=\frac{\lambda_{1}+\lambda_{2}}{2}$.

Proof. Let $\mathbb{M}$ be a Finsler Mobius band with $\operatorname{sys}(\mathbb{M}) \geq s$ and $h(\mathbb{M}) \geq h$, where $\frac{h}{s}=\frac{\lambda_{1}+\lambda_{2}}{2}$. Consider the Klein bottle $\mathbb{K}$ made of two isometric Mobius bands $\mathbb{M}$ and $\mathbb{M}^{\prime}$ with souls $\sigma$ and $\sigma^{\prime}$ defined in Definition 2.6.3. Consider also

$$
\mathbb{M}_{1}=\left\{x \in \mathbb{K} \mid \lambda_{2} d(x, \sigma) \leq \lambda_{1} d\left(x, \sigma^{\prime}\right)\right\}
$$

and

$$
\mathbb{M}_{2}=\left\{x \in \mathbb{K} \mid \lambda_{2} d(x, \sigma) \geq \lambda_{1} d\left(x, \sigma^{\prime}\right)\right\}
$$

Note that if we drop the multiplicative constants $\lambda_{1}$ and $\lambda_{2}$ in the definitions of $\mathbb{M}_{1}$ and $\mathbb{M}_{2}$, we obtain $\mathbb{M}$ and $\mathbb{M}^{\prime}$.

The subset $\mathbb{M}_{1}$ is a Finsler Mobius band contained in $\mathbb{M}$. Similarly, the subset $\mathbb{M}_{2}$ is a Finsler Mobius band containing $\mathbb{M}^{\prime}$. Observe also that the Mobius bands $\mathbb{M}_{1}$ and $\mathbb{M}_{2}$ cover $\mathbb{K}$ and that their interiors are disjoint.
Every point $z \in \partial \mathbb{M}_{i}$ satisfies the equality

$$
\begin{equation*}
\lambda_{2} d\left(z, \sigma_{1}\right)=\lambda_{1} d\left(z, \sigma_{2}\right) \tag{2.6.1}
\end{equation*}
$$

By symmetry, the distance between $\sigma_{1}=\sigma$ and $\sigma_{2}=\sigma^{\prime}$ is equal to $h(\mathbb{M})$. It follows by the triangle inequality that

$$
\begin{equation*}
d\left(z, \sigma_{1}\right)+d\left(z, \sigma_{2}\right) \geq h(\mathbb{M}) \tag{2.6.2}
\end{equation*}
$$

As a result of the relations (2.6.1) and (2.6.2), we obtain

$$
\begin{equation*}
d\left(z, \sigma_{i}\right) \geq \frac{\lambda_{i}}{\lambda_{1}+\lambda_{2}} h(\mathbb{M}) . \tag{2.6.3}
\end{equation*}
$$

Now, let $\alpha$ be an arc of $\mathbb{M}_{i}$ with endpoints $x, y \in \partial \mathbb{M}_{i}$ inducing a nontrivial class in $\pi_{1}\left(\mathbb{M}_{i}, \partial \mathbb{M}_{i}\right)$. As $\alpha$ intersects $\sigma_{i}$, we deduce from (2.6.3) that

$$
\ell(\alpha) \geq d\left(x, \sigma_{i}\right)+d\left(y, \sigma_{i}\right) \geq \frac{2 \lambda_{i}}{\lambda_{1}+\lambda_{2}} h(\mathbb{M})
$$

Therefore,

$$
\begin{equation*}
h\left(\mathbb{M}_{i}\right) \geq \frac{2 \lambda_{i}}{\lambda_{1}+\lambda_{2}} h(\mathbb{M}) \geq \frac{2 \lambda_{i}}{\lambda_{1}+\lambda_{2}} h \tag{2.6.4}
\end{equation*}
$$

In another direction, we can also bound from below the systole of $\mathbb{M}_{1}$ and $\mathbb{M}_{2}$ as follows.
For the systole of $\mathbb{M}_{1}$, since the inclusion $\mathbb{M}_{1} \subset \mathbb{M}$ induces a $\pi_{1}$-isomorphism, we derive

$$
\begin{equation*}
\operatorname{sys}\left(\mathbb{M}_{1}\right) \geq \operatorname{sys}(\mathbb{M}) \geq s \tag{2.6.5}
\end{equation*}
$$

Note that the first inequality may be strict as the inclusion $\mathbb{M}_{1} \subset \mathbb{M}$ is strict.
For the systole of $\mathbb{M}_{2}$, we argue as in Lemma 2.6.4. Let $\alpha$ be a noncontractible loop of $\mathbb{M}_{2}$. Decompose $\alpha$ into two parts $a=\alpha \cap \mathbb{M}$ and $a^{\prime}=\alpha \cap \mathbb{M}^{\prime}$ with $\alpha=a \cup a^{\prime}$. The union $\bar{\alpha}=$ $a^{*} \cup a^{\prime}$, where $a^{*}$ is the image of $a$ by the soul-switching symmetry of $\mathbb{K}$, forms a closed curve of length $\ell(\alpha)$ lying in $\mathbb{M}^{\prime}$ and homotopic to $\alpha$ in $\mathbb{M}_{2}$. Hence,

$$
\begin{equation*}
\operatorname{sys}\left(\mathbb{M}_{2}\right) \geq \operatorname{sys}(\mathbb{M}) \geq s \tag{2.6.6}
\end{equation*}
$$

The systole-height inequality on the Mobius band is satisfied for $\left(s, \frac{2 \lambda_{i}}{\lambda_{1}+\lambda_{2}} h\right)$ from the lemma assumption since $\frac{2 \lambda_{i}}{\lambda_{1}+\lambda_{2}} \frac{h}{s}=\lambda_{i}$. From the bounds (2.6.4), (2.6.5) and (2.6.6), this inequality applies to $\mathbb{M}_{i}$ and yields

$$
\begin{equation*}
\operatorname{vol}_{H T}\left(\mathbb{M}_{i}\right) \geq \frac{4}{\pi} \frac{\lambda_{i}}{\lambda_{1}+\lambda_{2}} s h \tag{2.6.7}
\end{equation*}
$$

Finally, recall that the Mobius bands $\mathbb{M}_{1}$ and $\mathbb{M}_{2}$ cover $\mathbb{K}$ and that their interiors are disjoint. By adding up (2.6.7) for $i=1,2$, we conclude that

$$
\begin{aligned}
2 \operatorname{vol}_{H T}(\mathbb{M}) & =\operatorname{vol}_{H T}(\mathbb{K}) \\
& =\operatorname{vol}_{H T}\left(\mathbb{M}_{1}\right)+\operatorname{vol}_{H T}\left(\mathbb{M}_{2}\right) \geq \frac{4}{\pi} s h
\end{aligned}
$$

Hence the result.
Remark 2.6.6. At first glance, it seems more natural to assume that $\operatorname{sys}(\mathbb{M})=s$ and $h(\mathbb{M})=h$ in the definition of the systole-height inequality, cf. Definition 2.6.3. Observe that the proof of Lemma 2.6.4 carries over with this alternative notion. However, we have not been able to directly prove a result similar to Lemma 2.6 .5 with this more restrictive notion. The reason is that the inequality (2.6.5), namely $\operatorname{sys}\left(\mathbb{M}_{1}\right) \geq \operatorname{sys}(\mathbb{M})$, may be strict as the inclusion $\mathbb{M}_{1} \subset \mathbb{M}$ is strict. To get around this subtle difficulty, we relaxed the original definition and formulated the
systole-height inequality in terms of lower bounds for the systole and the height of the Mobius band.

We can now proceed to the proof of Proposition 2.6.1.

Proof of Proposition 2.6.1. By Lemma 2.6.4, for every nonnegative integer $k$, the systole-height inequality on the Mobius band is satisfied for $(s, h)$ with $\frac{h}{s}=\frac{1}{2^{k}}$. Combined with Lemma 2.6.5, this implies that the systole-height inequality is satisfied for every $(s, h)$ where $\frac{h}{s}$ is a dyadic rational of $(0,1)$. Since the height, the systole and the volume are continuous over Finsler metrics, the result follows from the density of the dyadic rationals in $[0,1]$.

### 2.7 Systolic inequality on Finsler Klein bottles

In this section, we show that the systolic area of Finsler Klein bottles with soul, soul-switching or rotational symmetries is equal to $\frac{2}{\pi}$.

Definition 2.7.1. Recall that every Riemannian Klein bottle is conformally equivalent to the quotient of $\mathbb{R}^{2}$ by the isometry group $G$ generated by the glide reflection $(x, y) \mapsto(x+\pi,-y)$ and the vertical translation $(x, y) \mapsto(x, y+2 b)$ with $b>0$.

The flat Klein bottle $\mathbb{K}=\mathbb{R}^{2} / G$ decomposes into two Mobius bands whose souls correspond to the projections of the lines $\{y=0\}$ and $\{y=b\}$. The boundary of the two Mobius bands agrees with the projection of the line $\left\{y=\frac{b}{2}\right\}$ (or $\left\{y=-\frac{b}{2}\right\}$ ). A Finsler metric on $\mathbb{K}$ has a soul symmetry if its lift to $\mathbb{R}^{2}$ is invariant by the map $(x, y) \mapsto(x,-y)$. Similarly, a Finsler metric on $\mathbb{K}$ has a soul-switching symmetry if its lift to $\mathbb{R}^{2}$ is invariant by the map $(x, y) \mapsto(x, b-y)$. Finally, a Finsler metric on $\mathbb{K}$ has a rotational symmetry if its lift to $\mathbb{R}^{2}$ is invariant by the map $(x, y) \mapsto(x+\theta, y)$ for every $\theta \in[0,2 \pi]$.
These definitions are consistent with the notions introduced in 2.6.

In 1986, C. Bavard established an optimal isosystolic inequality for Riemannian Klein bottles, cf. [Ba86]. Alternative proofs can be found in [Sak88, Ba88, Ba06]. All the proofs are based on the uniformization theorem. In fact, as mentioned in the introduction, the problem boils down to consider Riemannian Klein bottles invariant under soul (and rotational) symmetry. These Klein bottles are made of two isometric copies of Riemannian Mobius bands. Thus, in the end, the systolic inequality on Riemannian Klein bottles follows from optimal systolic inequalities on Riemannian Mobius bands, cf. [Pu52, Bl61].

There is no known optimal isosystolic inequality on the Klein bottle for Finsler metrics. However, we obtain the following partial result similar to the Riemannian case. Note that in the Riemannian case, the hypothesis is automatically satisfied by an average argument.

Theorem 2.7.2. Let $\mathbb{K}$ be a Finsler Klein bottle with a soul, soul-switching or rotational symmetry. Then

$$
\begin{equation*}
\frac{\operatorname{vol}_{H T}(\mathbb{K})}{\operatorname{sys}^{2}(\mathbb{K})} \geq \frac{2}{\pi} \tag{2.7.1}
\end{equation*}
$$

Moreover, the inequality is optimal.

Let $\mathbb{K}=\mathbb{R}^{2} / G$ be a Finsler Klein bottle (with or without symmetry). Denote by $\mathbb{M}$ the Finsler Mobius band obtained by cutting $\mathbb{K}$ open along the soul given by the projection of the line $\{y=b\}$ to $\mathbb{K}$. The proof of the inequality (2.7.1) in Theorem 2.7.2 follows by combining the next three lemmas.

Lemma 2.7.3. If $h(\mathbb{M}) \geq \operatorname{sys}(\mathbb{K})$ then the inequality (2.7.1) holds true.

Proof. The inclusion $\mathbb{M} \subset \mathbb{K}$ induces a $\pi_{1}$-isomorphism. Hence, $\operatorname{sys}(\mathbb{K}) \leq \operatorname{sys}(\mathbb{M})$. Now, we only have two cases to consider.
First, if $h(\mathbb{M}) \leq \operatorname{sys}(\mathbb{M})$, then we deduce from Theorem 2.1.3, first case, that

$$
\begin{aligned}
\frac{\operatorname{vol}_{H T}(\mathbb{K})}{\operatorname{sys}^{2}(\mathbb{K})} & \geq \frac{\operatorname{vol}_{H T}(\mathbb{M})}{\operatorname{sys}(\mathbb{M}) h(\mathbb{M})} \\
& \geq \frac{2}{\pi}
\end{aligned}
$$

Second, if $h(\mathbb{M}) \geq \operatorname{sys}(\mathbb{M})$, then we deduce from Theorem 2.1.3, second case, that

$$
\begin{aligned}
\frac{\operatorname{vol}_{H T}(\mathbb{K})}{\operatorname{sys}^{2}(\mathbb{K})} & \geq \frac{1}{\pi}\left(1+\frac{\operatorname{sys}(\mathbb{M})}{h(\mathbb{M})}\right) \frac{\operatorname{sys}(\mathbb{M}) h(\mathbb{M})}{\operatorname{sys}^{2}(\mathbb{K})} \\
& \geq \frac{1}{\pi}\left(\frac{h(\mathbb{M})}{\operatorname{sys}(\mathbb{K})}+\frac{\operatorname{sys}(\mathbb{M})}{\operatorname{sys}(\mathbb{K})}\right) \frac{\operatorname{sys}(\mathbb{M})}{\operatorname{sys}(\mathbb{K})} \\
& \geq \frac{2}{\pi}
\end{aligned}
$$

since both $\frac{h(\mathbb{M})}{\operatorname{sys}(\mathbb{K})}$ and $\frac{\operatorname{sys}(\mathbb{M})}{\operatorname{sys}(\mathbb{K})}$ are greater or equal to 1 .
The next three lemmas show that the assumption of Lemma 2.7.3 is satisfied when the Finsler metric on $\mathbb{K}$ has soul, soul-switching or rotational symmetries.

Lemma 2.7.4. If $\mathbb{K}$ is a Finsler Klein bottle with a soul symmetry then $h(\mathbb{M}) \geq \operatorname{sys}(\mathbb{K})$.

Proof. Observe that the soul symmetry of $\mathbb{K}$ leaves both $\mathbb{M}$ and $\partial \mathbb{M}$ invariant. Given an arc $\alpha$ of $\mathbb{M}$, we denote by $\alpha^{*}$ the arc of $\mathbb{M}$ symmetric to $\alpha$ by the soul symmetry. Let $\gamma$ be a height arc of $\mathbb{M}$, cf. Definition 2.3.4. This arc decomposes into two subarcs $\alpha$ and $\beta$ connecting $\partial \mathbb{M}$ to the soul of $\mathbb{M}$ with $\ell(\alpha) \leq \ell(\beta)$. The arc $\alpha \cup \alpha^{*}$ with endpoints in $\partial \mathbb{M}$ induces a nontrivial class in $\pi_{1}(\mathbb{M}, \partial \mathbb{M})$ of length at most the length of $\gamma=\alpha \cup \beta$. By definition of $h(\mathbb{M})$, we conclude that $\alpha \cup \alpha^{*}$ is as long as $\gamma$ and so is length-minimizing in its relative homotopy class. In
particular, it is geodesic. Thus, the arc $\gamma$, which has the subarc $\alpha$ in common with $\alpha \cup \alpha^{*}$, agrees with $\alpha \cup \alpha^{*}$. In particular, it is invariant by the soul symmetry. As a result, the arc $\gamma$ induces a noncontractible loop on $\mathbb{K}$ after identification of the points of $\partial \mathbb{M}$ under the soul symmetry. Hence, $\ell(\alpha) \geq \operatorname{sys}(\mathbb{K})$.

Lemma 2.7.5. If $\mathbb{K}$ is a Finsler Klein bottle with a soul-switching symmetry then $h(\mathbb{M}) \geq \operatorname{sys}(\mathbb{K})$.

Proof. Since the Finsler Klein bottle $\mathbb{K}$ has a soul-switching symmetry, we can assume that it is composed of two isometric Finsler Mobius bands $\mathbb{M}_{1}$ and $\mathbb{M}_{2}$. By symmetry, we have $h(\mathbb{M})=$ $2 h\left(\mathbb{M}_{1}\right)$. Given an arc $\alpha$ of $\mathbb{M}_{1}$, we denote by $\alpha^{*}$ the arc of $\mathbb{M}_{2}$ symmetric to $\alpha$ by the soulswitching symmetry. Let $\gamma$ be a height arc of $\mathbb{M}_{1}$. This arc decomposes into two subarcs $\alpha_{1}$ and $\beta_{1}$ connecting $\partial \mathbb{M}_{1}$ to the soul of $\mathbb{M}_{1}$. The arc $\eta=: \alpha_{1} \cup \alpha_{1}^{*} \cup \beta_{1} \cup \beta_{1}^{*}$, i.e., the union of the arc $\gamma$ and its symmetric image $\gamma^{*}$, has endpoints in $\partial \mathbb{M}$ and induces a nontrivial class in $\pi_{1}(\mathbb{M}, \partial \mathbb{M})$ of length equal to $2 h\left(\mathbb{M}_{1}\right)$. Moreover, this arc $\eta$ induces a noncontractible loop on $\mathbb{K}$ after identification of the points of $\partial \mathbb{M}$ under the soul-switching symmetry. We conclude that $h(\mathbb{M}) \geq \operatorname{sys}(\mathbb{K})$.

Lemma 2.7.6. If $\mathbb{K}$ is a Finsler Klein bottle with rotational symmetry then $h(\mathbb{M}) \geq \operatorname{sys}(\mathbb{K})$.

Proof. Observe that the rotational symmetries of $\mathbb{K}$ leave both $\mathbb{M}$ and $\partial \mathbb{M}$ invariant. Given an $\operatorname{arc} \alpha$ of $\mathbb{M}$, we denote by $\alpha^{\theta}$ the $\operatorname{arc}$ of $\mathbb{M}$ symmetric to $\alpha$ by the rotational symmetry of angle $\theta$. Let $\gamma$ be a length-minimizing arc of $\mathbb{M}$ parametrized (proportionally to its length) by $[0,1]$ with endpoints in $\partial \mathbb{M}$ inducing a nontrivial class in $\pi_{1}(\mathbb{M}, \partial \mathbb{M})$. Note that $\gamma$ is a geodesic arc of length $h(\mathbb{M})$. By the first variation formula for Finsler metrics, $c f$. [Sh01], the geodesic arc $\gamma$ is perpendicular to $\partial \mathbb{M}$. It follows that the endpoints $\gamma(0)$ and $\gamma(1)$ of $\gamma$ in $\partial \mathbb{M}$ are distinct. Since the Finsler metric is invariant under rotational symmetry, there exists $\theta \in(0,2 \pi)$ such that $\gamma^{\theta}(0)=\gamma(1)$. Both symmetric arcs $\gamma$ and $\gamma^{\theta}$ are perpendicular to $\partial \mathbb{M}$. In particular, their tangent vectors at $\gamma^{\theta}(0)=\gamma(1)$ coincide up to sign. Therefore, the geodesic arcs $\gamma$ and $\gamma^{\theta}$ agree up to reparametrization. More precisely, $\gamma^{\theta}(s)=\gamma(1-s)$ for every $s \in[0,1]$. Thus, $\gamma^{2 \theta}=\left(\gamma^{\theta}\right)^{\theta}=\gamma$. Hence, $2 \theta=2 \pi$, that is, $\theta=\pi$. Therefore, the arc $\gamma$ projects to a closed curve in $\mathbb{K}$. It follows that the length of $\gamma$ is at least $\operatorname{sys}(\mathbb{K})$.

The following example shows that the inequality (2.7.1) is optimal.
Example 2.7.7. The quotient of $\mathbb{R}^{2}$, endowed with the sup-norm, by the isometry group $G$ generated by the glide reflection with parameter $b=\frac{\pi}{2}$, cf. Definition 2.7.1, is a Finsler Klein bottle with soul, soul-switching and rotational symmetries, of area $2 \pi$ and systole $\pi$.

We believe that Theorem 2.7.2 holds true for every Finsler Klein bottle (not necessarily invariant under soul, soul-switching or rotational symmetries). More precisely, we state the following conjecture.

Conjecture. Let $\mathbb{K}$ be a Finsler Klein bottle. Then

$$
\frac{\operatorname{vol}_{H T}(\mathbb{K})}{\operatorname{sys}^{2}(\mathbb{K})} \geq \frac{2}{\pi}
$$

Moreover, the inequality is optimal.
Remark 2.7.8. If the conjecture is true, the Finsler systolic areas of $\mathbb{R} \mathbb{P}^{2}, \mathbb{T}^{2}$ and $\mathbb{K}$ would be the same.

### 2.8 A non-optimal systolic inequality on Finsler Klein bottles

In this section, we present a non-optimal systolic inequality on Finsler Klein bottles.
Proposition 2.8.1. Let $\mathbb{K}$ be a Finsler Klein bottle. Then

$$
\frac{\operatorname{vol}_{H T}(\mathbb{K})}{\operatorname{sys}^{2}(\mathbb{K})} \geq \frac{\sqrt{2}}{\pi}
$$

Proof. Every symmetric convex body $C \subset \mathbb{R}^{n}$ admits a unique ellipsoid $E(C)$ of maximal volume among the ellipsoids contained in $C$. This ellipsoid, called John's ellipsoid, continuously varies with $C$ for the Hausdorff topology. Furthermore, it satisfies the double inclusion, $c f$. [Gru07, Corollary 11.2] for instance,

$$
\begin{equation*}
E(C) \subset C \subset \sqrt{n} E(C) \tag{2.8.1}
\end{equation*}
$$

Given a Klein bottle $\mathbb{K}$ with a Finsler metric $F$, we define a continuous Riemannian metric $g$ on $\mathbb{K}$ by replacing the Minkowski norm $F_{x}$ on each tangent space $T_{x} \mathbb{K}$ by the inner product induced by the John ellipsoid $E\left(B_{x}\right)$, where $B_{x}$ is the unit ball of $F_{x}$. The double inclusion (2.8.1) satisfied by $E\left(B_{x}\right)$ implies that $\frac{1}{\sqrt{2}} \sqrt{g} \leq F \leq \sqrt{g}$. Hence,

$$
\begin{equation*}
\operatorname{sys}(F) \leq \operatorname{sys}(g) \quad \text { and } \quad \frac{1}{2} \operatorname{vol}(g) \leq \operatorname{vol}_{H T}(F) \tag{2.8.2}
\end{equation*}
$$

From the optimal Riemannian systolic inequality on the Klein bottle [Ba86], we obtain

$$
\operatorname{vol}_{H T}(F) \geq \frac{1}{2} \operatorname{vol}(g) \geq \frac{1}{2} \frac{2 \sqrt{2}}{\pi} \operatorname{sys}^{2}(g) \geq \frac{\sqrt{2}}{\pi} \operatorname{sys}^{2}(F) .
$$

Remark 2.8.2. The naive volume bound in (2.8.2) can be improved into $\operatorname{vol}(g) \leq \frac{\pi}{2} \operatorname{vol}_{H T}(F)$, see the proof of [ABT, Theorem 4.11]. This leads to the better lower bound $\frac{4 \sqrt{2}}{\pi^{2}}$ in Proposition 2.8.1.

## Chapter 3

## A systolically critical genus two surface

Nous démontrons une inégalité systolique critique sur la surface de genre deux. Plus précisement, il est connu que la surface de genre deux admet une métrique Riemannienne plate à singularités coniques qui est extrémale parmi les métriques à courbure nonpositive pour l'inégalité systolique. Nous montrons que cette métrique est en fait critique pour des variations lentes de métriques, cette fois-ci sans hypothèse de courbure, pour un autre problème systolique portant sur les longueurs des plus courts lacets non contractiles dans certaines classes d'homotopie libres données. Ces classes d'homotopie correspondent aux lacets systoliques et deux-systoliques de la surface extrémale.


#### Abstract

It is known that the genus two surface admits a piecewise flat metric with conical singularities which is extremal for the systolic inequality among all nonpositively curved metrics. We show that this piecewise flat metric is also critical for slow metric variations, without curvature restrictions, for another type of systolic inequality involving the lengths of the shortest noncontractible loops in different free homotopy classes. The free homotopy classes considered correspond to those of the systolic loops and the second-systolic loops of the extremal surface.


### 3.1 Introduction

We are interested in optimal geometric inequalities relating the area of a closed Riemannian surface $\Sigma$ to the lengths of the shortest loops in certain homotopy classes. A typical example is given by the systolic inequalities, which relate the area of the surface to the systole, that is, the length of the shortest noncontractible loop in $\Sigma$. The first known systolic inequality is due to C. Loewner in 1949, who proved that every Riemannian two-torus $\left(\mathbb{T}^{2}, g\right)$ satisfies

$$
\operatorname{area}(g) \geq \frac{\sqrt{3}}{2} \operatorname{sys}^{2}(g)
$$

where $\operatorname{sys}(g)$ denotes the systole of the torus. Furthermore, the equality holds if and only if the torus is endowed with a flat hexagonal metric. C. Loewner did not publish his result, however it was mentioned by his student P. Pu, cf. [Pu52], who established a sharp systolic inequality on the projective plane $\mathbb{R P}^{2}$. In this case, the equality is attained precisely by the round metrics. More than thirty years later, C. Bavard, cf. [Ba86] (see also [Sak88]) proved a sharp systolic inequality on the Klein bottle $\mathbb{K}$. Here, the extremal metrics are not smooth. These are the only manifolds for which an optimal systolic inequality is known. All these systolic inequalities can be written as follows. If

$$
\operatorname{sys}(g) \geq \operatorname{sys}\left(g_{0}\right)
$$

then

$$
\begin{equation*}
\operatorname{area}(g) \geq \operatorname{area}\left(g_{0}\right) \tag{3.1.1}
\end{equation*}
$$

where $g_{0}$ is the extremal metric in each of these cases.

Optimal systolic-like inequalities, i.e., inequalities that relate the area to the product of the lengths of the shortest loops or arcs in different relative homotopy classes are known only for the twotorus $\mathbb{T}^{2}$, cf. [Ke67], the Klein bottle $\mathbb{K}$, $c f$. [Ba06, EY15], and the Mobius band $\mathbb{M}$, $c f$. [Pu52, Bl61, Ba06, EY15].

We can reformulate a systolic-like inequality as in (3.1.1). For instance, let $\ell_{1}(g)$ represent the length of the shortest noncontractible loop on $(\Sigma, g)$, that is,

$$
\ell_{1}(g)=\operatorname{sys}(g)
$$

Denote by $\Lambda_{1}$ the subset of the free homotopy classes generated by the systolic loops of $(\Sigma, g)$ and their multiples:

$$
\begin{equation*}
\Lambda_{1}=\left\{\left\langle\gamma^{k}\right\rangle \mid\langle\gamma\rangle \neq 0, \ell_{g_{0}}(\gamma)=\ell_{1}\left(g_{0}\right) \text { and } k \in \mathbb{Z}^{*}\right\} \tag{3.1.2}
\end{equation*}
$$

Similarly, let $\ell_{2}(g)$ be the length of the shortest noncontractible loop on $(\Sigma, g)$ which is not homotopic to a systolic loop or its multiples. That is,

$$
\ell_{2}(g)=\inf _{\langle\gamma\rangle \notin \Lambda_{1} \cup\{0\}} \ell_{g}(\gamma)
$$

Denote also by

$$
\begin{equation*}
\Lambda_{2}=\left\{\langle\gamma\rangle \mid\langle\gamma\rangle \notin \Lambda_{1} \text { and } \ell_{g_{0}}(\gamma)=\ell_{2}\left(g_{0}\right)\right\} \tag{3.1.3}
\end{equation*}
$$

the subset of the free homotopy classes generated by the shortest noncontractible loops of $(\Sigma, g)$ which are not freely homotopic to systolic loops or their multiples. As a particular case of L. Keen's optimal systolic-like inequality for the two-torus in [Ke67], we have the following. If

$$
\left\{\begin{array}{l}
\ell_{1}(g) \geq \ell_{1}\left(g_{0}\right) \\
\ell_{2}(g) \geq \ell_{2}\left(g_{0}\right)
\end{array}\right.
$$

then

$$
\operatorname{area}(g) \geq \operatorname{area}\left(g_{0}\right)
$$

Recently, M. Katz and S. Sabourau, cf. [KS06], proved that there exists a piecewise flat metric $g_{0}$ on the genus two surface $\Sigma_{2}$ which is extremal among all nonpositively curved Riemannian metrics. This metric is composed of six regular octagons. It admits regions where only one systolic loop, i.e., a noncontractible loop of length the systole, passes through every point. Hence this metric cannot be extremal for the general systolic inequality, i.e., without restriction on the curvature. This follows from a result of E. Calabi, cf. [Ca96], which says that at least two systolic loops pass through every point of a systolically extremal surface. In fact, the non-extremality of the metric $g_{0}$ for the systolic inequality can also be deduced from a result of S. Sabourau, cf. [Sa04], which says that no flat metric with conical singularities is extremal for the systolic inequality in genus two.

We observe that the piecewise flat metric $g_{0}$ defined on the genus two surface $\Sigma_{2}$ may be a potential extremal metric for another systolic problem on $\Sigma_{2}$. This observation follows from some geometric properties of the surface $\left(\Sigma_{2}, g_{0}\right)$. First, it is known that the systolic loops cover a systolically extremal surface. In our case, the $\ell_{1}$-loops and the $\ell_{2}$-loops cover the surface $\Sigma_{2}$. Here, by an $\ell_{i}$-loop we mean a loop of length $\ell_{i}\left(g_{0}\right)$ in $\left(\Sigma_{2}, g_{0}\right)$ whose free homotopy class lies in $\Lambda_{i}$. Second, the unit tangent vectors of these $\ell_{i}$-loops are well distributed on each tangent plane of the surface. More precisely, their convex hull forms a regular octagon on these tangent planes. In general, the convex hull of the unit tangent vectors of systolic loops on all the known
systolically extremal surfaces is symmetric. Finally, an extremal surface of genus at least two tends to have flat regions, cf. [Ca96, Br96].

In [Ca96], E. Calabi described two piecewise flat genus three surfaces and conjectured that one of them is the global minimum for the systolic inequality. Later, S. Sabourau, cf. [Sa11], proved that these two genus three surfaces are critical for the systolic inequality with respect to slow metric variations. Of course, the systolic volume functional is not necessarily differentiable and an adequate notion of criticality needs to be introduced. The notion of criticality used in [Sa11] was introduced in [Na96, EI00] to study the Riemannian surfaces that maximize the product of the area with the first nonzero eigenvalue of the Laplacian. Other notions of systolically critical metrics were used in [Bal06, Bal10, Sa10, AB14].

In this chapter we show that the metric $g_{0}$ on the genus two surface $\Sigma_{2}$ is critical in the sense of [Na96, EI00, Sa11] for slow metric variations. More precisely, we prove

Theorem 3.1.1. Let $g_{t}$ be a slow metric variation of $g_{0}$ defined on the genus two surface $\Sigma_{2}$. If $\ell_{1}\left(g_{t}\right) \geq \ell_{1}\left(g_{0}\right)$ and $\ell_{2}\left(g_{t}\right) \geq \ell_{2}\left(g_{0}\right)$, then

$$
\operatorname{area}\left(g_{t}\right) \geq \operatorname{area}\left(g_{0}\right)+o(t)
$$

The slow metric variations involved in Theorem 3.1.1 are analogous to those defined by S. Sabourau in [Sa11]. We refer to Definition 3.4.1 for a precise definition of these metric variations and to the last section of [Sa11] for examples. Observe for instance that deforming the regular octagons composing the extremal surface $\left(\Sigma_{2}, g_{0}\right)$ into non-regular octagons gives rise to a slow metric variation.

With the exception of some recent proofs of systolic inequalities, cf. [Iv02, Sa10], and systolic-like inequalities, $c f$. [SY16], on the projective plane $\mathbb{R P}^{2}$, the two-torus $\mathbb{T}^{2}$ and the Mobius band $\mathbb{M}$, all the other known proofs of the aforementioned optimal geometric inequalities require the uniformization theorem as a main tool. In our proof of Theorem 3.1.1, we do not make use of the uniformization theorem. Instead, we rely on recent calibrating methods, cf. [Iv02, Iv11, Sa11]. Briefly, we embed the universal cover $\widetilde{\Sigma}_{2}$ of the genus two surface $\Sigma_{2}$ in an infinite-dimensional space $\mathbb{R}^{\infty}$ as follows. Given that $\ell_{i}(g) \geq \ell_{i}\left(g_{0}\right)$, we construct a 1 -Lipschitz equivariant map

$$
\Psi^{g}: \widetilde{\Sigma}_{2} \rightarrow \mathbb{R}^{\infty}
$$

using the Busemann functions induced by the lifts of the $\ell_{i}$-loops of $\Sigma_{2}$, cf. Section 3.3. Then, we introduce an appropriate infinitesimally calibrating two-form $\omega$ on $\mathbb{R}^{\infty}$ whose pull-back passes to the quotient on the surface $\Sigma_{2}$. This allows us to show that

$$
\int_{\Sigma_{2}}\left(\Psi^{g}\right)^{*} \omega \leq c_{0} \operatorname{area}\left(\Sigma_{2}, g\right)
$$

for every Riemannian metric $g$ on $\Sigma_{2}$, where $c_{0}$ is a sharp positive constant. Moreover, the equality holds if $g=g_{0}$. Finally, we prove that if $\left(g_{t}\right)_{t \geq 0}$ is a slow metric variation then

$$
\int_{\Sigma_{2}}\left(\Psi^{g_{t}}\right)^{*} \omega-\int_{\Sigma_{2}}\left(\Psi^{g_{0}}\right)^{*} \omega=o(t) .
$$

This completes the proof, cf. Section 3.4.

### 3.2 An extremal piecewise flat metric in genus two

In this section we provide a description for the critical piecewise flat genus two surface and we introduce some notations and definitions.

The Bolza surface $\mathfrak{B}$ is a genus two Riemann surface. It is the smooth completion of the smooth affine algebraic curve

$$
\begin{equation*}
y^{2}=x^{5}-x . \tag{3.2.1}
\end{equation*}
$$

The set $\{-1,0,1,-i, i, \infty\}$ of roots of the polynomial (3.2.1) (including the point at infinity) corresponds to the set of vertices of the regular octahedral triangulation of the Riemann sphere $S^{2}$ under the conformal stereographic projection. Hence, these six points can be identified with the ramification points of the ramified conformal double cover $Q: \mathfrak{B} \rightarrow S^{2}$, or equivalently, with the Weierstrass points of $\mathfrak{B}$.

The conformal class of the Bolza surface $\mathfrak{B}$ admits a piecewise flat nonpositively curved Riemannian metric $g_{0}$ with 16 conical singularities whose orientation-preserving isometry group is isomorphic to the automorphism group of $\mathfrak{B}$. This metric $g_{0}$ has been introduced by M. Katz and S. Sabourau in [KS06], where they prove that it is extremal for the systolic inequality among all nonpositively curved metrics on the genus two surface $\Sigma_{2}$. The metric $g_{0}$, defined up to homothety, on $\Sigma_{2}$ is isometric to the piecewise flat genus two surface composed of six identical flat regular octagons $\Omega_{k}$ with the identifications given in Figure 3.1. The Weierstrass points of $\mathfrak{B}$ correspond to the centers of the octagons and the 16 conical singularities are located at their vertices.

Denote by $\kappa$ the length of an edge of a regular octagon $\Omega_{k}$ of $g_{0}$. We have

$$
\operatorname{area}\left(g_{0}\right)=12(1+\sqrt{2}) h^{2} .
$$

The systole $\ell_{1}\left(g_{0}\right)$ of the surface $\left(\Sigma_{2}, g_{0}\right)$ is computed in [KS06, Lemma 3.2]. Namely,

$$
\ell_{1}\left(g_{0}\right)=2(1+\sqrt{2}) \hbar .
$$



Figure 3.1: The critical surface $\left(\Sigma_{2}, g_{0}\right)$ is isometric to the to the piecewise flat genus two surface composed of six identical regular octagons with the identifications given in the figure.

By using similar arguments, we calculate $\ell_{2}\left(g_{0}\right)$. In particular,

$$
\ell_{2}\left(g_{0}\right)=2(2+\sqrt{2}) \hbar
$$

Recall that an $\ell_{i}$-loop is a loop in $\left(\Sigma_{2}, g_{0}\right)$ of length $\ell_{i}\left(g_{0}\right)$ and whose free homotopy class lies in $\Lambda_{i}$, cf. (3.1.2) and (3.1.3). We define an $\ell_{i}$-band as follows.

Definition 3.2.1. Let $\langle\alpha\rangle$ be a free homotopy class in $\Lambda_{i}$. The $\ell_{i}$-loops in $\langle\alpha\rangle$ are parallel to each other and form a flat cylinder $\mathcal{B}_{\langle\alpha\rangle}$ of height $\kappa$ if $i=1$ and $\frac{\sqrt{2}}{2} \hbar$ if $i=2$. Such a cylinder $\mathcal{B}_{\langle\alpha\rangle}$ will be called an $\ell_{i}$-band of $\left(\Sigma_{2}, g_{0}\right)$. The soul of an $\ell_{i}$-band $\mathcal{B}_{\langle\alpha\rangle}$ is the $\ell_{i}$-loop of $\mathcal{B}_{\langle\alpha\rangle}$ equidistant from the boundary components of the $\ell_{i}$-band.

The intersections of the $\ell_{i}$-bands decompose the surface $\Sigma_{2}$ into 150 polygons, cf. Figure 3.2. These polygons define four regions $\mathcal{R}_{1}, \ldots, \mathcal{R}_{4}$ depending on the number of $\ell_{i}$-loops that pass through every point in their interior. More precisely, such a polygon $\Delta_{k}$ lies in a region $\mathcal{R}_{k}$ if through every point in its interior pass exactly $2 k$ oriented $\ell_{1}$-loops and $8-2 k$ oriented $\ell_{2}$ loops, cf. Figure 3.2. Hence, every region $\mathcal{R}_{k}$ with $k \neq 4$ is composed of exactly 48 identical polygons $\Delta_{k}$ (with 8 polygons in each octagon), while the region $\mathcal{R}_{4}$ is composed of only 6 identical polygons $\Delta_{4}$. Observe that the polygons $\Delta_{1}, \Delta_{2}, \Delta_{3}$ and $\Delta_{4}$ have the shape of a right isosceles triangle, a kite, a right isosceles triangle and a (small) octagon. Furthermore, the right isosceles triangles $\Delta_{1}$ are the only polygons which have an edge in common with the six regular flat octagons forming the critical surface $\left(\Sigma_{2}, g_{0}\right)$, namely their hypothenuses coincide with the
edges of the regular octagons $\Omega_{k}$.
Fix an orientation on $\Sigma_{2}$. Let $\Omega$ be one of the six regular octagons composing the surface $\left(\Sigma_{2}, g_{0}\right)$. Denote by $\left(\widetilde{\Sigma}_{2}, \widetilde{g}_{0}\right)$ the universal cover of $\left(\Sigma_{2}, g_{0}\right)$. Let $\widetilde{\Omega}$ be a lift of $\Omega$ in the universal cover $\widetilde{\Sigma}_{2}$ and let $\mathcal{E}$ be an (oriented) edge of the regular octagon $\widetilde{\Omega}$. Moreover, let $\widetilde{\Delta}_{k}$ be a lift of the polygon $\Delta_{k}$ in $\widetilde{\Sigma}_{2}$. The souls of the $\ell_{i}$-bands lift to $\widetilde{\Sigma}_{2}$ as follows.

Definition 3.2.2. An $\ell_{1}$-direction of $\left(\widetilde{\Sigma}_{2}, \widetilde{g}_{0}\right)$ is a $\widetilde{g}_{0}$-unit vector $\zeta=\zeta_{\widetilde{\Omega}}^{\mathcal{E}}$ based at the center of $\widetilde{\Omega}$ and pointing in the same direction as $\mathcal{E}$, $c f$. Figure 3.2. The unit vector $\zeta$ induces a geodesic line $\widetilde{c}_{\zeta}$ on $\left(\widetilde{\Sigma}_{2}, \widetilde{g}_{0}\right)$ such that

$$
\widetilde{c}_{\zeta}(0)=\zeta
$$

The projection $c_{\zeta}$ of $\widetilde{c}_{\zeta}$ to $\left(\Sigma_{2}, g_{0}\right)$ is a closed geodesic curve which coincides with the soul of the $\ell_{1}$-band $\mathcal{B}_{\left\langle c_{\zeta}\right\rangle}$.

Now, denote by $x_{\mathcal{E}}$ the point of $\widetilde{\Omega}$ lying in the perpendicular bisector of $\mathcal{E}$ and at distance $\frac{\sqrt{2}}{4} \kappa$ from $\mathcal{E}$. (Recall that $\kappa$ is the side length of $\widetilde{\Omega}$.) Note that the point $x_{\mathcal{E}}$ lies in the right isosceles triangle $\widetilde{\Delta}_{1}$ of $\widetilde{\Omega}$ with hypotenuse $\mathcal{E}$.

Definition 3.2.3. An $\ell_{2}$-direction of $\left(\widetilde{\Sigma}_{2}, \widetilde{g}_{0}\right)$ is a $\widetilde{g}_{0}$-unit vector $\xi=\xi_{\widetilde{\Omega}}^{\mathcal{E}}$ based at $x_{\mathcal{E}}$ and pointing in the same direction as the edge $\mathcal{E}$, cf. Figure 3.2. This unit vector $\xi$ induces a geodesic line $\widetilde{c}_{\xi}$ on $\left(\widetilde{\Sigma}_{2}, \widetilde{g}_{0}\right)$ such that

$$
\widetilde{c}_{\xi}(0)=\xi
$$

The projection $c_{\xi}$ of $\widetilde{c}_{\xi}$ to $\left(\Sigma_{2}, g_{0}\right)$ is a closed geodesic curve which coincides with the soul of the $\ell_{2}$-band $\mathcal{B}_{\left\langle c_{\xi}\right\rangle}$.

Remark 3.2.4. In what follows, we will replace the symbols $\zeta$ and $\xi$ by $\nu$ when there is no need to distinguish between an $\ell_{1}$-direction and an $\ell_{2}$-direction. Moreover, the indices $\widetilde{\Omega}$ and $\mathcal{E}$ in $\nu_{\widetilde{\Omega}}^{\mathcal{E}}$ will be omitted.

### 3.3 Busemann functions and calibrating forms

In this section, we introduce some notations and preliminary results that will be useful in our proof of Theorem 3.1.1.

Definition 3.3.1. Let $\nu$ be an $\ell_{i}$-direction. Extend $\nu$ to a map, still denoted by

$$
\nu: \mathbb{R} \rightarrow U_{\widetilde{g}_{0}} \widetilde{\Sigma}_{2}
$$

defined as $\nu(s)=\widetilde{c}_{\nu}(s)$. In particular, $\nu(0)=\nu$.
Along with the metric $g_{0}$, consider another Riemannian metric $g$ defined on $\Sigma_{2}$ and denote by $\widetilde{g}$ its lift to $\widetilde{\Sigma}_{2}$.


Figure 3.2: A flat regular octagon $\widetilde{\Omega}$ of the metric $\widetilde{g}_{0}$.

Definition 3.3.2. Let $\nu$ be a $\widetilde{g}_{0}$-unit vector tangent to a geodesic line induced by an $\ell_{i}$-direction. Define the Busemann function $b_{\nu}^{g}: \widetilde{\Sigma}_{2} \rightarrow \mathbb{R}$ as

$$
b_{\nu}^{g}(x)=\limsup _{t \rightarrow \infty} d_{\tilde{g}}\left(x, \widetilde{c}_{\nu}(t)\right) \ell_{i}\left(g_{0}\right)-t \ell_{g}\left(\left\langle c_{\nu}\right\rangle\right)
$$

where $\widetilde{c}_{\nu}$ is the $\widetilde{g}_{0}$-geodesic line induced by $\nu$ and $c_{\nu}$ is its projection to $\Sigma_{2}$.

Here, $\ell_{g}\left(\left\langle c_{\nu}\right\rangle\right)$ denotes the smallest length of a noncontractible loop in the free homotopy class $\left\langle c_{\nu}\right\rangle$ of $c_{\nu}$ with respect to the Riemannian metric $g$.

Remark 3.3.3. Contrary to the original definition of a Busemann function, $c f$. [BH99], the line $\widetilde{c}_{\nu}$ is geodesic for $\widetilde{g}_{0}$ but not for $\widetilde{g}$. Moreover, the function

$$
\begin{aligned}
h: \mathbb{R} & \rightarrow \mathbb{R} \\
t & \mapsto d_{\tilde{g}}\left(x, \widetilde{c}_{\nu}(t)\right) \ell_{i}\left(g_{0}\right)-t \ell_{g}\left(\left\langle c_{\nu}\right\rangle\right)
\end{aligned}
$$

which appears in Definition 3.3.2, is not necessarily non-increasing.

Proposition 3.3.4. The real function $h$ is bounded.

Proof. By the triangle inequality, we obtain

$$
\begin{equation*}
h(t) \geq d_{\tilde{g}}\left(\widetilde{c}_{\nu}(0), \widetilde{c}_{\nu}(t)\right) \ell_{i}\left(g_{0}\right)-t \ell_{g}\left(\left\langle c_{\nu}\right\rangle\right)-d_{\tilde{g}}\left(x, \widetilde{c}_{\nu}(0)\right) \ell_{i}\left(g_{0}\right) \tag{3.3.1}
\end{equation*}
$$

Let $\beta$ be a length-minimizing loop of $\left(\Sigma_{2}, g\right)$ in the homotopy class of $c_{\nu}$, that is, $\ell_{g}(\beta)=\ell_{g}\left(\left\langle c_{\nu}\right\rangle\right)$. Let $[t]_{i}$ be the integer part of $\frac{t}{\ell_{i}\left(g_{0}\right)}$, that is,

$$
[t]_{i}=\max \left\{k \in \mathbb{N} \mid t \geq k \ell_{i}\left(g_{0}\right)\right\}
$$

We have

$$
\begin{equation*}
[t]_{i} \ell_{i}\left(g_{0}\right)-t \geq-\ell_{i}\left(g_{0}\right) . \tag{3.3.2}
\end{equation*}
$$

Since the genus two surface is orientable, the $[t]_{i}$-iterate of $\beta$ is length-minimizing in its homotopy class with respect to $g$, $c f$. [Gr99]. Let $\widetilde{\gamma}$ in $\widetilde{\Sigma}_{2}$ be a length-minimizing arc connecting $\widetilde{c}_{\nu}(0)$ to $\widetilde{c}_{\nu}\left([t]_{i}\right)$. Its projection $\gamma$ to $\Sigma_{2}$ is homotopic to $c_{\nu}^{[t]]}$ and its length satisfies the two relations

$$
\ell_{g}(\gamma)=d_{\tilde{g}}\left(\widetilde{c}_{\nu}(0), \widetilde{c}_{\nu}\left([t]_{i}\right)\right)
$$

and

$$
\begin{equation*}
\ell_{g}(\gamma) \geq[t]_{i} \ell_{g}\left(\left\langle c_{\nu}\right\rangle\right) . \tag{3.3.3}
\end{equation*}
$$

Combining (3.3.1), (3.3.2) and (3.3.3), we deduce that the map $h$ is bounded from below.
Now, we show that the map $h$ is bounded from above. Indeed, by the triangle inequality, we derive that

$$
h(t) \leq\left\{d_{\widetilde{g}}\left(x, \widetilde{c}_{\nu}(0)\right)+d_{\widetilde{g}}\left(\widetilde{c}_{\nu}(0), \widetilde{c}_{\nu}\left([t]_{i}\right)\right)+d_{\widetilde{g}}\left(\widetilde{c}_{\nu}\left([t]_{i}\right), \widetilde{c}_{\nu}(t)\right)-[t]_{i} \ell_{g}\left(\left\langle c_{\nu}\right\rangle\right)\right\} \ell_{i}\left(g_{0}\right) .
$$

Moreover,

$$
\begin{equation*}
d_{\widetilde{g}}\left(\widetilde{c}_{\nu}(0), \widetilde{c}_{\nu}\left([t]_{i}\right)\right)-[t]_{i} \ell_{g}\left(\left\langle c_{\nu}\right\rangle\right) \leq 2 \operatorname{diam}\left(\Sigma_{2}, g\right) . \tag{3.3.4}
\end{equation*}
$$

Indeed, let $\gamma$ be a $g$-length minimizing loop in the free homotopy class $\left\langle c_{\nu}\right\rangle$, that is, $\ell_{g}(\gamma)=$ $\ell_{g}\left(\left\langle c_{\nu}\right\rangle\right)$. Let $\alpha$ be a length-minimizing arc joining $c_{\nu}(0)$ to $\gamma$. We deduce that

$$
d_{\widetilde{g}}\left(\widetilde{c}_{\nu}(0), \widetilde{c}_{\nu}\left([t]_{i}\right)\right) \leq \ell_{g}\left(\alpha \cup \gamma^{[t]_{i}} \cup \alpha^{-1}\right) \leq \ell_{g}\left(\gamma^{[t]_{i}}\right)+\operatorname{diam}\left(\Sigma_{2}, g\right) .
$$

Then, the inequality (3.3.4) is satisfied. Furthermore,

$$
d_{\tilde{g}}\left(\widetilde{c}_{\nu}\left([t]_{i}\right), \widetilde{c}_{\nu}(t)\right) \leq \operatorname{diam}\left(\Sigma_{2}, g\right) .
$$

This shows that $h$ is bounded from above.
Definition 3.3.5. The map $\Psi_{\nu}^{g}: \widetilde{\Sigma}_{2} \rightarrow \mathbb{R}$ induced by an $\ell_{i}$-direction $\nu$, cf. Definitions 3.2.2 and 3.2.3, is defined as

$$
\Psi_{\nu}^{g}(x)=\frac{1}{\ell_{i}\left(g_{0}\right) \ell_{g}\left(\left\langle c_{\nu}\right\rangle\right)} \int_{\mathbb{R} / \ell_{i}\left(g_{0}\right) \mathbb{Z}} b_{\nu(s)}^{g}(x)-b_{-\nu(s)}^{g}(x) d s
$$

where $\nu(s)=\widetilde{c}_{\nu}(s), c f$. Definition 3.3.1.
Example 3.3.6. If $g=g_{0}$, the $g_{0}$-gradient $\nabla_{x} \Psi_{\nu}^{g_{0}}$ is a $g_{0}$-unit tangent vector parallel to and pointing in the same direction as the $\ell_{i}$-direction $\nu$.

Proposition 3.3.7. Let $\nu$ be an $\ell_{i}$-direction. The map $\Psi_{\nu}^{g}$ satisfies the following properties:

1. It is differentiable almost everywhere.
2. $\Psi_{-\nu}^{g}=-\Psi_{\nu}^{g}$.
3. The differential of the map $\Psi_{\nu}^{g}$ depends only on the oriented $\ell_{i}$-band $\widetilde{\mathcal{B}}$ generated by the vector $\nu$.

Proof. For (1), the map $\Psi_{\nu}^{g}$ is 1-Lipschitz for the supremum norm as the Busemann function $b_{\nu}^{g}$ is $\ell_{i}\left(g_{0}\right)$-Lipschitz and $\ell_{i}(g) \geq \ell_{i}\left(g_{0}\right)$. Hence, it is differentiable almost everywhere by the Rademacher theorem. Part (2) follows directly from Definition 3.3.5. For (3), let $\nu$ and $\eta$ be two $\ell_{i}$-directions generating the same oriented $\ell_{i}$-band. That is, their basepoints lie in the same $\ell_{i}$-band and the vectors point in the same direction. Recall that $\nu(s)=\widetilde{c}_{\nu}(s), c f$. Definition 3.3.1, and the same with $\eta$. The definition of $\ell_{i}$-directions and the assumption on the vectors $\nu$ and $\eta$ imply that

$$
\eta(s)=\nu\left(s+\frac{k}{n} \ell_{i}\left(g_{0}\right)\right)
$$

for some integer $k$ with $n=2$ if $i=1$ and $n=4$ if $i=2$. We derive easily from Definition 3.3.5 that

$$
d \Psi_{\eta}^{g}=d \Psi_{\nu}^{g}
$$

This completes the proof.
Let $\Gamma$ be the deck transformation group of $\widetilde{\Sigma}_{2}$. The $\widetilde{g}_{0}$-geodesic line $\widetilde{c}_{\nu}$ induces a unique element $\gamma_{\nu}$ in $\Gamma$ that leaves $\widetilde{c}_{\nu}$ globally invariant and such that

$$
\gamma_{\nu} \cdot \widetilde{c}_{\nu}(s)=\widetilde{c}_{\nu}\left(s+\ell_{i}\left(g_{0}\right)\right)
$$

Moreover, we define an action $\star$ of $\Gamma$ on the set of unit tangent vectors to the geodesic line $\widetilde{c}_{\nu}$ as

$$
\gamma \star \nu(s)=\left(\gamma \cdot \widetilde{c}_{\nu}\right)^{\prime}(s)
$$

where $\gamma \in \Gamma$.
We state now the following properties of the map $\Psi_{\nu}^{g}$.
Proposition 3.3.8. Let $\nu$ be an $\ell_{i}$-direction. The $\operatorname{map} \Psi_{\nu}^{g}: \widetilde{\Sigma_{2}} \rightarrow \mathbb{R}$ satisfies the following properties:

1. It is equivariant, that is, for every $\gamma$ in $\Gamma$,

$$
\Psi_{\gamma \star \nu}^{g}(\gamma \cdot x)=\Psi_{\nu}^{g}(x)
$$

2. It passes to the quotient by the cyclic subgroup $\left\langle\gamma_{\nu}\right\rangle$ and induces a map

$$
\bar{\Psi}_{\nu}^{g}: \widetilde{\Sigma}_{2} /\left\langle\gamma_{\nu}\right\rangle \rightarrow \mathbb{R} / \ell_{i}\left(g_{0}\right) \mathbb{Z}
$$

defined as

$$
\bar{\Psi}_{\nu}^{g}\left(\gamma_{\nu} \cdot x\right) \equiv \Psi_{\nu}^{g}(x) \bmod \ell_{i}\left(g_{0}\right)
$$

Proof. Let $x \in \widetilde{\Sigma}_{2}$ and $\gamma \in \Gamma$. Since $\gamma$ is an isometry and $\gamma \cdot \widetilde{c}_{\nu}=\widetilde{c}_{\gamma . \nu}$, we derive

$$
\ell_{g}\left(\left\langle c_{\gamma \star \nu}\right\rangle\right)=\ell_{g}\left(\left\langle c_{\nu}\right\rangle\right) .
$$

Moreover, since the distance $d_{\widetilde{g}}$ induced by the Riemannian metric $\widetilde{g}$ is $\Gamma$-invariant, we derive that $b_{\gamma \star \nu}^{g}(\gamma \cdot x)=b_{\nu}^{g}(x)$. Hence, the desired result

$$
\Psi_{\gamma * \nu}^{g}(\gamma \cdot x)=\Psi_{\nu}^{g}(x)
$$

Now, since $\gamma_{\nu}^{-1} . \widetilde{c}_{\nu}(t)=\widetilde{c}_{\nu}\left(t-\ell_{i}\left(g_{0}\right)\right)$, we derive from the $\Gamma$-invariance of $d_{\widetilde{g}}$ that

$$
\begin{aligned}
d_{\tilde{g}}\left(\gamma_{\nu} \cdot x, \widetilde{c}_{\nu}(t)\right) & =d_{\tilde{g}}\left(x, \gamma_{\nu}^{-1} \cdot \widetilde{c}_{\nu}(t)\right) \\
& =d_{\tilde{g}}\left(x, \widetilde{c}_{\nu}\left(t-\ell_{i}\left(g_{0}\right)\right)\right)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
b_{\nu}^{g}\left(\gamma_{\nu} \cdot x\right) & =\limsup _{t \rightarrow \infty} \ell_{i}\left(g_{0}\right) d_{\tilde{g}}\left(x, \widetilde{c}_{\nu}\left(t-\ell_{i}\left(g_{0}\right)\right)\right)-t \ell_{g}\left(\left\langle c_{\nu}\right\rangle\right) \\
& =b_{\nu}^{g}(x)-\ell_{i}\left(g_{0}\right) \ell_{g}\left(\left\langle c_{\nu}\right\rangle\right)
\end{aligned}
$$

We deduce that

$$
\begin{equation*}
\Psi_{\nu}^{g}\left(\gamma_{\nu} \cdot x\right)=\Psi_{\nu}^{g}(x)-\ell_{i}\left(g_{0}\right) \tag{3.3.5}
\end{equation*}
$$

Remark 3.3.9. As a consequence of Proposition 3.3 .8 and the relation (3.3.5), we deduce the following couple of points:

1. $\gamma^{*}\left(d \Psi_{\gamma \star \nu}^{g}\right)=d \Psi_{\nu}^{g}$ for every $\gamma \in \Gamma$,
2. The map $\Psi_{\nu}^{g}-\Psi_{\nu}^{g}$ passes to the quotient by the cyclic subgroup $\left\langle\gamma_{\nu}\right\rangle$ and induces a map

$$
\bar{\Psi}_{\nu}^{g}-\bar{\Psi}_{\nu}^{g_{0}}: \widetilde{\Sigma}_{2} /\left\langle\gamma_{\nu}\right\rangle \rightarrow \mathbb{R}
$$

such that $\bar{\Psi}_{\nu}^{g}\left(\gamma_{\nu} \cdot x\right)-\bar{\Psi}_{\nu}^{g_{0}}\left(\gamma_{\nu} \cdot x\right)=\Psi_{\nu}^{g}(x)-\Psi_{\nu}^{g_{0}}(x)$.

Actually, these two properties hold if we replace $\Psi_{\nu}^{g}$ with the Busemann function $b_{\nu}^{g}$.

Let $\widetilde{\Omega}$ be a regular octagon of the universal cover $\left(\widetilde{\Sigma}_{2}, \widetilde{g}_{0}\right)$ of the surface $\left(\Sigma_{2}, g_{0}\right)$. We order the (oriented) edges $\left(\mathcal{E}^{j}\right)_{1 \leq j \leq 8}$ of $\widetilde{\Omega}$ with respect to the cyclic order induced by the orientation of $\widetilde{\Omega}$
by fixing an initial edge $\mathcal{E}^{1}$. Denote

$$
\nu^{j}=\nu_{\widetilde{\Omega}}^{\mathcal{E}^{j}}
$$

We state the following definition
Definition 3.3.10. Let $\mathbb{R}^{\infty}=\Pi_{\widetilde{\Omega}} \mathbb{R}^{8}$ be the infinite product of $\mathbb{R}^{8}$ where $\widetilde{\Omega}$ runs over all the regular octagons of $\left(\widetilde{\Sigma}_{2}, \widetilde{g}_{0}\right)$. Let

$$
\Psi^{g}: \widetilde{\Sigma}_{2} \rightarrow \mathbb{R}^{\infty}
$$

be the map defined as

$$
\begin{aligned}
\left.\pi_{\widetilde{\Omega}} \circ \Psi^{g}\right|_{\tilde{\Omega}}: \widetilde{\Omega} & \rightarrow \mathbb{R}^{8} \\
x & \mapsto\left(\Psi_{\nu^{j}}^{g}(x)\right)_{j=1}^{8}
\end{aligned}
$$

where

$$
\pi_{\widetilde{\Omega}}: \mathbb{R}^{\infty} \rightarrow \mathbb{R}^{8}
$$

is the canonical projection to the factor $\mathbb{R}^{8}$ of $\mathbb{R}^{\infty}$ corresponding to $\widetilde{\Omega}$. The vector $\nu^{j}$ is the $\ell_{i}$ direction that is parallel and point in the same direction as the $\widetilde{g}_{0}$-unit vector based at $x$ and tangent to the lift of an oriented $\ell_{i}$-loop.

Example 3.3.11. Consider the regular octagon $\widetilde{\Omega}$ in Figure 3.2. Suppose that the numbering of the edges of $\partial \widetilde{\Omega}$ follows a cyclic order starting at $\mathcal{E}$, then we have

$$
\Psi_{\tilde{\Delta}_{2}}^{g}=\left(\Psi_{\zeta^{1}}^{g}, \Psi_{\zeta^{2}}^{g}, \Psi_{\xi^{3}}^{g}, \Psi_{\xi^{4}}^{g}, \Psi_{\zeta^{5}}^{g}, \Psi_{\zeta^{6}}^{g}, \Psi_{\xi^{7}}^{g}, \Psi_{\xi^{8}}^{g}\right)
$$

where $\widetilde{\Delta}_{2}$ is the gray polygon in Figure 3.2.
Remark 3.3.12. Note that the map $\Psi^{g}$ is not well defined in the lift of the boundaries of the polygons $\Delta$ since there is some ambiguity in the choice of $\nu^{j}$.

Definition 3.3.13. Consider the two-form

$$
\omega_{0}=\sum_{j=1}^{8} d x_{j} \wedge d x_{j+2}
$$

in $\mathbb{R}^{8}$. We define a two-form $\widetilde{\omega}$ on $\widetilde{\Sigma}_{2}$ as the pull-back of $\omega_{0}$ by the map

$$
\pi_{\widetilde{\Omega}} \circ \Psi_{\left.\right|_{\tilde{\Omega}}}^{g}: \widetilde{\Omega} \rightarrow \mathbb{R}^{8}
$$

on every regular octagon $\widetilde{\Omega}$. It can be expressed on $\widetilde{\Omega}$ as

$$
\widetilde{\omega}=\sum_{j=1}^{8} d \Psi_{\nu^{j}}^{g} \wedge d \Psi_{\nu^{j+2}}^{g}
$$

where $\nu^{j}=\nu_{\widetilde{\Omega}}^{j}$ and all indices $j$ are taken modulo 8 . This defines a two-form on $\widetilde{\Sigma}_{2}$ that we still denote by $\widetilde{\omega}$.

Note that the two-form $\widetilde{\omega}$ in Definition 3.3.13 is defined almost everywhere since the maps $\Psi_{\nu}^{g}$ are almost everywhere differentiable, cf. Proposition 3.3.7 (1).

Proposition 3.3.14. The two-form $\widetilde{\omega}$ passes to the quotient on $\Sigma_{2}$.

Proof. Let $\gamma \in \Gamma$. Observe that $\gamma$ preserves the octagonal decomposition of the universal cover $\left(\widetilde{\Sigma}_{2}, \widetilde{g}_{0}\right)$ and its orientation. Therefore, for every regular octagon $\widetilde{\Omega}$, there exists a permutation $\sigma$ of $\{1, \cdots, 8\}$ such that

$$
\gamma^{-1}\left(\mathcal{E}_{\widetilde{\Omega}}^{j}\right)=\mathcal{E}_{\gamma^{-1}(\widetilde{\Omega})}^{\sigma(j)}
$$

That is,

$$
\gamma^{-1}\left(\nu_{\widetilde{\Omega}}^{j}\right)=\nu_{\gamma^{-1}(\widetilde{\Omega})}^{\sigma(j)}
$$

Hence,

$$
d \Psi_{\gamma^{-1} \star \nu^{j}}^{g}=d \Psi_{\nu^{\sigma(j)}}^{g}
$$

By the equivariance property of the map $\Psi_{\nu^{j}}$, cf. Remark 3.3.9, we derive

$$
\begin{aligned}
\gamma^{*} \widetilde{\omega} & =\sum_{j=1}^{8} d \Psi_{\gamma^{-1} \star \nu^{j}}^{g} \wedge d \Psi_{\gamma^{-1} \star \nu^{j+2}}^{g} \\
& =\sum_{j=1}^{8} d \Psi_{\nu^{\sigma(j)}}^{g} \wedge d \Psi_{\nu^{\sigma(j+2)}}^{g}
\end{aligned}
$$

Finally, since the permutation $\sigma$ arises from an orientation-preserving isometry $\gamma$, the angle between $\nabla_{x} \Psi_{\nu^{\sigma(j)}}^{g}$ and $\nabla_{x} \Psi_{\nu^{\sigma(j+2)}}^{g}$ is equal to the angle between $\nabla_{\gamma(x)} \Psi_{\nu^{j}}^{g}$ and $\nabla_{\gamma(x)} \Psi_{\nu^{j+2}}^{g}$ for every $x \in \widetilde{\Sigma}_{2}$ at which $\widetilde{\omega}$ is defined. Therefore,

$$
\sigma(j+2)=\sigma(j)+2
$$

Reindexing the sum in the expression of the two-forms $\omega_{0}$ and $\widetilde{\omega}$, we derive that

$$
\gamma^{*} \widetilde{\omega}=\widetilde{\omega}
$$

Remark 3.3.15. The induced two-form on $\Sigma_{2}$ will be denoted by $\omega$.

### 3.4 Proof of Theorem 3.1.1

We briefly restate the statement of Theorem 3.1.1. Let $g_{t}$ be a slow metric variation of $g_{0}$ defined on $\Sigma_{2}$. If $\ell_{1}\left(g_{t}\right) \geq \ell_{1}\left(g_{0}\right)$ and $\ell_{2}\left(g_{t}\right) \geq \ell_{2}\left(g_{0}\right)$, then

$$
\operatorname{area}\left(g_{t}\right) \geq \operatorname{area}\left(g_{0}\right)+o(t)
$$

Before proceeding to the proof, let us first define what we mean by a slow metric variation.

Definition 3.4.1. A slow metric variation is a family of Riemannian metrics $g_{t}$ with $t \geq 0$ such that for every $\ell_{i}$-direction $\nu$, the following condition is satisfied

$$
\begin{equation*}
\left\|d\left(b_{\nu}^{g_{t}}-b_{-\nu}^{g_{t}}\right)-d\left(b_{\nu}^{g_{0}}-b_{-\nu}^{g_{0}}\right)\right\|_{L^{2}\left(B_{\nu}\right)}=o(\sqrt{t}) \tag{3.4.1}
\end{equation*}
$$

where $B_{\nu}$ is a fundamental domain in $\left(\widetilde{\Sigma}_{2}, \widetilde{g}_{0}\right)$ of the band $\mathcal{B}_{\nu}=\mathcal{B}_{\left\langle c_{\nu}\right\rangle}$.

Note that the map $b_{\nu}^{g_{t}}-b_{\nu}^{g_{0}}$ passes to the quotient by the cyclic subgroup $\left\langle c_{\nu}\right\rangle$, cf. Remark 3.3.9 (2).

Remark 3.4.2. A metric variation $g_{t}$ should have conical singularities as the Riemannian metric $g_{0}$ has conical singularities, cf. Section 3.2.

Definition 3.4.3. The $\mathcal{F}$-functional of the map

$$
\Psi^{g}: \widetilde{\Sigma}_{2} \rightarrow \mathbb{R}^{\infty}
$$

introduced in Definition 3.3.10 is defined as

$$
\mathcal{F}\left(\Psi^{g}\right)=\int_{\Sigma_{2}} \omega
$$

where $\omega$ is the quotient two-form on $\Sigma_{2}$ introduced in Section 3.3.

The proof of Theorem 3.1.1 follows directly from the next two lemmas.

Lemma 3.4.4. The two-form $\omega$ is a calibrating form for the map $\Psi^{g}$. That is,

$$
\mathcal{F}\left(\Psi^{g}\right) \leq 8 \operatorname{area}(g)
$$

with equality for $g=g_{0}$.

Proof. Let $x \in \widetilde{\Delta}$ be a point at which the maps $\Psi^{g}{ }_{\nu j}$ are differentiable. Let $v$ be a $\widetilde{g}$-unit vector in the tangent plane $T_{x} \widetilde{\Sigma}_{2}$. Since the map $\Psi^{g}{ }_{\nu}$ is 1-Lipschitz, we derive that

$$
\left|d_{x} \Psi_{\nu^{j}}^{g}(v)\right| \leq 1
$$

Hence

$$
\|\widetilde{\omega}\|_{\tilde{g}}=\left\|\sum_{j=1}^{8} d \Psi_{\nu^{j}}^{g} \wedge d \Psi_{\nu^{j+2}}^{g}\right\|_{\tilde{g}} \leq 8 .
$$

The desired inequality directly follows.
Let $\mathcal{D}$ denote a fundamental domain of the surface $\left(\Sigma_{2}, g_{0}\right)$ in its universal cover made of regular octagons $\widetilde{\Omega}$. By Definition 3.4.3, the $\mathcal{F}$-functional of the almost everywhere differentiable map $\Psi^{g}$ is defined as

$$
\begin{aligned}
\mathcal{F}\left(\Psi^{g}\right) & =\int_{\Sigma_{2}} \omega \\
& =\sum_{\tilde{\Omega} \in \mathcal{D}} \int_{\tilde{\Omega}} \sum_{j=1}^{8} d \Psi_{\nu^{j}}^{g} \wedge d \Psi_{\nu^{j+2}}^{g} .
\end{aligned}
$$

When $g=g_{0}$, the vectors $\nu^{j}$ and $\nu^{j+2}$ form an oriented orthonormal basis. The same holds for the covectors $d_{x} \Psi_{\nu j}^{g_{0}}$ and $d_{x} \Psi_{\nu j+2}^{g_{0}}$ from Example 3.3.6. Hence,

$$
d_{x} \Psi_{\nu^{j}}^{g_{0}} \wedge d_{x} \Psi_{\nu^{j+2}}^{g_{0}}=d \widetilde{g}_{0}
$$

where $d \widetilde{g}_{0}$ represents the area form of the Riemannian metric $\widetilde{g}_{0}$. In particular, we derive

$$
\mathcal{F}\left(\Psi^{g_{0}}\right)=8 \operatorname{area}\left(g_{0}\right)
$$

after passing to the quotient.
Remark 3.4.5. An analogous two-form $\omega$ along systolic directions could be defined on the genus three surface to recover the result of [Sa11] without using the Euclidean metrics.

Lemma 3.4.6. Let $g_{t}$ be a slow metric variation of $g_{0}$. Then

$$
\mathcal{F}\left(\Psi^{g_{t}}\right)-\mathcal{F}\left(\Psi^{g_{0}}\right)=o(t) .
$$

Proof. Let $\nu=\nu_{\widetilde{\Omega}}^{\mathcal{E}}$, where $\widetilde{\Omega}$ is a regular octagon of $\left(\widetilde{\Sigma}_{2}, \widetilde{g}_{0}\right)$ of side length $\kappa$. By definition, an $\ell_{i}$-band $\mathcal{B}_{\nu}=\mathcal{B}_{\left\langle c_{\nu}\right\rangle}$ is the quotient of

$$
\widetilde{\mathcal{B}}_{\nu}=\left\{x \in \widetilde{\Sigma}_{2} \mid d_{\widetilde{g}_{0}}\left(x, \widetilde{c}_{\nu}\right) \leq \lambda\right\}
$$

by the cyclic subgroup $\left\langle c_{\nu}\right\rangle$, where $\lambda$ is equal to $\frac{1}{2} \pi$ if $i=1$ and equal to $\frac{\sqrt{2}}{4} \hbar$ if $i=2$. We still denote by $\nu$ the vector given by the systolic direction after passing to the quotient.

Recall that the differential $d \Psi_{\nu}^{g}$ of the map $\Psi_{\nu}^{g}$ depends only on the oriented $\ell_{i}$-band $\widetilde{\mathcal{B}}$ generated by the vector $\nu$ (or rather the oriented geodesic line $\widetilde{c}_{\nu}$ ), but not on $\nu$, cf. Proposition 3.3.7 (3). Moreover, by passing to the quotient, we can rearrange the sum in Definition 3.4.3 to be taken
over all the oriented $\ell_{i}$-bands $\mathcal{B}$ of $\left(\Sigma_{2}, g_{0}\right)$ as follows

$$
\mathcal{F}\left(\Psi^{g}\right)=\sum_{\mathcal{B}} \int_{\mathcal{B}} d \bar{\Psi}_{\nu_{\mathcal{B}}}^{g} \wedge d \bar{\Psi}_{\left(\nu_{\mathcal{B}}\right)^{\perp}}^{g}
$$

where $\nu_{\mathcal{B}}$ is an $\ell_{i}$-direction that generates the $\ell_{i}$-band $\mathcal{B}$ and the vector $\left(\nu_{\mathcal{B}}\right)^{\perp}$ is an $\ell_{1}$-direction or an $\ell_{2}$-direction orthogonal to the boundary $\partial \mathcal{B}$ and whose basepoint lies in the same regular octagon $\Omega$ as the vector $\nu_{\mathcal{B}}$. More precisely, if $\nu_{\mathcal{B}}$ points in the direction of an edge $\mathcal{E}^{j}$ of $\Omega$, then $\left(\nu_{\mathcal{B}}\right)^{\perp}$ points in the direction of the edge $\mathcal{E}^{j+2}$, where the edges follow the cyclic order on $\partial \Omega$.

In particular, we have

$$
\mathcal{F}\left(\Psi^{g_{t}}\right)-\mathcal{F}\left(\Psi^{g_{0}}\right)=I_{1}+I_{2}
$$

where

$$
\begin{equation*}
I_{1}=\sum_{\mathcal{B}} \int_{\mathcal{B}} d\left(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{t}}-\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{0}}\right) \wedge d \bar{\Psi}_{\left(\nu_{\mathcal{B}}\right)^{\perp}}^{g_{0}}-d\left(\bar{\Psi}_{\left(\nu_{\mathcal{B}}\right)^{\perp}}^{g_{t}}-\bar{\Psi}_{\left(\nu_{\mathcal{B}}\right)^{\perp}}^{g_{0}}\right) \wedge d \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{0}} \tag{3.4.2}
\end{equation*}
$$

and

$$
I_{2}=\sum_{\mathcal{B}} \int_{\mathcal{B}} d\left(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{t}}-\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{0}}\right) \wedge d\left(\bar{\Psi}_{\left(\nu_{\mathcal{B}}\right)^{\perp}}^{g_{t}}-\bar{\Psi}_{\left(\nu_{\mathcal{B}}\right)^{\perp}}^{g_{0}}\right)
$$

First, we show that $I_{1}=0$. Recall that from Proposition 3.3.7(2), we have

$$
-\Psi_{\left(\nu_{\mathcal{B}}\right)^{\perp}}^{g}=\Psi_{-\left(\nu_{\mathcal{B}}\right)^{\perp}}^{g}
$$

and the same holds for $\bar{\Psi}$. Moreover, since the oriented angle between the vectors $\nu_{\mathcal{B}}$ and $\left(\nu_{\mathcal{B}}\right)^{\perp}$ is the same as the one between the vectors $-\left(\nu_{\mathcal{B}}\right)^{\perp}$ and $\nu_{\mathcal{B}}$, we can write

$$
I_{1}=2 \sum_{\mathcal{B}} \int_{\mathcal{B}} d\left(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{t}}-\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{0}}\right) \wedge d \bar{\Psi}_{\left(\nu_{\mathcal{B}}\right)^{\perp}}^{g_{0}}
$$

after arranging the oriented bands $\mathcal{B}$ in the sum (3.4.2). Moreover, the one-forms $d \bar{\Psi}_{(\nu \mathcal{B})^{\perp}}^{g_{0}}$ are constant over the bands $\mathcal{B}$. Therefore, we derive

$$
\sum_{\mathcal{B}} \int_{\mathcal{B}} d\left(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{t}}-\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{0}}\right) \wedge d \bar{\Psi}_{\left(\nu_{\mathcal{B}}\right)^{\perp}}^{g_{0}}=\sum_{\mathcal{B}} \int_{\mathcal{B}} d\left(\left(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{t}}-\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{0}}\right) d \bar{\Psi}_{\left(\nu_{\mathcal{B}}\right)^{\perp}}^{g_{0}}\right)
$$

Now, let $v$ be a tangent vector to the boundary $\partial \mathcal{B}$. The vector $v$ is parallel to the $\ell_{i}$-direction $\nu_{\mathcal{B}}$. Moreover, by definition, the unit vectors $\nu_{\mathcal{B}}$ and $\left(\nu_{\mathcal{B}}\right)^{\perp}$ form an orthonormal basis. Since the $g_{0}$-gradient of $\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{0}}$ satisfies

$$
\nabla \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{0}}=\nu_{\mathcal{B}}
$$

for every $\ell_{i}$-direction $\nu_{\mathcal{B}}$, cf. Example 3.3.6, and the same holds with $\left(\nu_{\mathcal{B}}\right)^{\perp}$, we have

$$
d \bar{\Psi}_{\left(\nu_{\mathcal{B}}\right)^{\perp}}^{g_{0}}(v)=g_{0}\left(\left(\nu_{\mathcal{B}}\right)^{\perp}, v\right)=0
$$

Then by Stokes' formula, we derive that

$$
\begin{aligned}
\sum_{\mathcal{B}} \int_{\mathcal{B}} d\left(\left(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{t}}-\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{0}}\right) d \bar{\Psi}_{\left(\nu_{\mathcal{B}}\right)^{\perp}}^{g_{0}}\right) & =\sum_{\mathcal{B}} \int_{\partial \mathcal{B}}\left(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{t}}-\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{0}}\right) d \bar{\Psi}_{\left(\nu_{\mathcal{B}}\right)^{\perp}}^{g_{0}} \\
& =0
\end{aligned}
$$

Note that $\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{t}}-\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_{0}}$ is well defined as a real function, $c f$. Remark 3.3.9 (2). Hence, the integral $I_{1}$ vanishes.

Finally, we show that $I_{2}=0$. This follows directly by definition of a slow metric variation, $c f$. Definition 3.4.1, and the Cauchy-Schwartz inequality.

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