Sur les représentations quantiques des groupes modulaires des surfaces

Julien Korinman

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Présentée par

Korinman Julien

Thèse dirigée par **Funar Louis**

préparée au sein **Institut Fourier**
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Mr. Christian Blanchet
Professeur, Université de Paris 7, Rapporteur
Mr. Francesco Costantino
Professeur, Université de Toulouse 3, Rapporteur
Mme. Christine Lescop
Directrice de recherches, Université de Grenoble 1, Examinateur
Mr. Julien Marché
Professeur, Université de Paris 6, Examinateur
Mr. Louis Funar
Directeur de recherches, Université de Grenoble 1, Directeur de thèse

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Introduction

L'objectif de cette thèse est l'étude de certaines familles de représentations projectives des groupes modulaires des surfaces.

Etant donnée une surface fermée orientée Σ, son groupe modulaire $\mathrm{Mod}(\Sigma)$ est défini comme le quotient du groupe des homéomorphismes de Σ préservant l'orientation et les points marqués, par le sous-groupe formé par la composante connexe contenant l'identité. En d'autres termes, il s'agit du groupe des classes d'isotopie des homéomorphismes de Σ :

$$\mathrm{Mod}(\Sigma) := \pi_0(\mathrm{Homeo}^+(\Sigma))$$

Nous considérerons deux familles de représentations projectives, indexées par un entier $p \geq 2$, définies sur des espaces de dimension finie :

$$\begin{aligned} \text{Les représentations de Weil :} \quad & \pi_p : \mathrm{Mod}(\Sigma) \to \mathrm{PGL}(U_p(\Sigma)) \\ \text{Les représentations de Reshetikhin-Turaev :} \quad & \rho_p : \mathrm{Mod}(\Sigma) \to \mathrm{PGL}(V_p(\Sigma)) \end{aligned}$$

Les représentations dites de *(Segal-Shale-)Weil* ou *représentations métaplectiques*, ne fournissent que peu d'information sur $\mathrm{Mod}(\Sigma)$.

Le premier groupe d'homologie entière $H_1(\Sigma_g, \mathbb{Z})$, muni de la forme d'intersection, est un espace symplectique. Les classes d'homéomorphismes de $\mathrm{Mod}(\Sigma_g)$ agissent dessus en préservant la forme d'intersection, ce qui fournit un morphisme surjectif de groupes :

$$f : \mathrm{Mod}(\Sigma_g) \to Sp_{2g}(\mathbb{Z})$$

Alors que le groupe symplectique $Sp_{2g}(\mathbb{Z})$ est bien connu, le noyau de f, appelé sous-groupe de Torelli , est encore mal compris. Les représentations de Weil agissent trivialement sur ce sous-groupe. Elles constituent cependant l'exemple le plus simple de représentations dites 'quantiques' et constituent un excellent modèle pour étudier les propriétés générales de ces représentations.

Les représentations de Weil sont apparues en 1946, dans les travaux de Kloosterman ([54]) où elles sont définies comme des transformations modulaires des fonctions theta de Jacobi. Leur étude a été reprise par les physiciens Segal et Shale en 1962 ([78, 80]) où elles proviennent de la quantification du tore symplectique, puis par Weil en 1964 ([88]).

La seconde famille de représentations que nous étudierons, les représentations dites de *(Witten-)Reshetikhin-Turaev* ou *représentations $SO(3)$ et $SU(2)$*, au contraire détectent le groupe de Torelli. Plus précisément, bien qu'aucune d'entre elles ne soient fidèles, la somme directe d'une infinité d'entre elles forme une représentation fidèle de $\mathrm{Mod}(\Sigma)$ modulo son

centre (voir [28, 3, 67]). L'image de ces représentations contient donc toute l'information de $\mathrm{Mod}(\Sigma)$.

Ces représentations tirent leur origine dans la découverte en 1985, par Jones, d'un invariant polynomial pour les noeuds et les entrelacs ([47]), et dans leur interprétation tridimensionelle donnée par Witten en 1989 dans [92]. Plus précisément, Witten définit la notion de théorie topologique quantique de champs (voir [6, 91]), ou TQFT , dans laquelle s'inscrivent le polynôme de Jones et les représentations de Reshetikhin-Turaev.

Une TQFT est un foncteur qui associe à toute surface Σ, un $\mathbb{C}$ espace vectoriel $V(\Sigma)$ (et $V(\emptyset) = \mathbb{C}$), et associe à tout cobordisme M entre Σ_1 et Σ_2, une application linéaire $V(M)$ entre $V(\Sigma_1)$ et $V(\Sigma_2)$. On autorise un tel cobordisme à contenir un entrelacs en rubans.

À une 3-variété fermée orientée, vue comme un cobordisme entre $\emptyset$ et $\emptyset$, un tel foncteur associe une application linéaire $\mathbb{C} \to \mathbb{C}$ qui peut être assimilée à un nombre complexe $\langle M \rangle \in \mathbb{C}$, appelé l'invariant quantique de M.

Dans [92], Witten annonce l'existence de TQFTs dont l'invariant quantique d'un entrelacs de S^3 correspond au polynôme de Jones évalué en une racine de l'unité. Un tel foncteur fournit une représentation de $\mathrm{Mod}(\Sigma)$ sur $V(\Sigma)$ dite représentation quantique. Bien que l'heuristique donnée par Witten pour l'existence d'une telle TQFT s'appuie sur la notion encore mal définie d'intégrales de chemins, l'auteur souligne que certaines propriétés de chirurgies et d'écheveaux permettent un calcul algébrique explicite de ces invariants et que les techniques de théories conformes de champs en dimension 2 permettent l'étude des espaces $V(\Sigma)$.

La première construction rigoureuse de ces TQFTs est faite par Reshetikhin et Turaev en 1991 dans [75]. Cette construction utilise des représentations de certains groupes quantiques définis par Drinfel'd et Jimbo.

Une construction plus combinatoire de ces invariants quantiques, s'appuyant sur les travaux de Kirby et Melvin ([51]), est proposée par Lickorish dans [63]. La connaissance de cet invariant permet de reconstruire toute la TQFT par une construction dite universelle définie par Blanchet, Habegger, Masbaum et Vogel dans [14]. Nous suivrons cette approche pour définir les TQFTs qui engendrent les représentations de Weil et de Reshetikhin-Turaev.

Les principaux résultats de cette thèse concernent la décomposition des représentations quantiques en représentations irréductibles. Décrivons brièvement le contenu de chaque chapitre.

Chapitre 1

Le premier chapitre sert d'introduction pour définir les notions et résultats classiques utilisés dans le reste de la thèse.

La première section sert à définir les TQFTs abéliennes, $SO(3)$ et $SU(2)$ en suivant [63, 14].

On définit d'abord la notion de TQFT sans anomalies. On montre comment on peut en extraire un invariant pour les 3 variétés fermés orientées munies d'un entrelacs et, pour chaque surface Σ, une représentation de $\mathrm{Mod}(\Sigma)$. On explique également la construction universelle de [14] qui permet de construir une TQFT depuis un invariant.

Dans un second temps, nous étudions la notion de module d'écheveaux d'une 3 variété ainsi que les algèbres de Temperley-Lieb et les polynômes de Jones colorés.

La troisième sous-section concerne le comportement des TQFTs vis-à-vis de la chirurgie. Nous énonçons les théorèmes de Lickorish-Wallace et de Kirby, qui stipulent que la définition d'un invariant pour les 3 variétés fermées orientées est équivalent à la donnée d'un invariant d'entrelacs en rubans stables sous certains mouvements locaux, dits de Kirby. Nous verrons que l'existence de TQFTs satisfaisant les propriétés d'écheveaux et de chirurgie, comme annoncée par Witten, implique que l'invariant quantique dépend de la signature de la 3-variété et nécessite d'étendre la catégorie de cobordismes.

Ceci nous amène, dans la quatrième sous-section, à définir une notion de TQFT avec anomalie, en munissant les surfaces de Lagrangiens dans leur premier groupe d'homologie rationelle, et les cobordismes de poids entiers. Avec cette définition, les propriétés d'écheveaux et de chirurgie définissent de manière unique les TQFTs abéliennes, $SO(3)$ et $SU(2)$. Les représentations quantiques ainsi obtenues sont alors projectives.

Enfin nous étudierons les espaces associés aux surfaces et donnerons des bases orthogonales pour certaines formes bilinéaires invariantes sous l'action de $\mathrm{Mod}(\Sigma)$.

La seconde section regroupe certains résultats classiques des représentations quantiques.

On démontre notamment une relation, dite d'Egorov, entre l'action de $\mathrm{Mod}(\Sigma)$ d'une part sur les espaces $V(\Sigma)$, d'autre part sur les algèbres $\mathrm{End}(V(\Sigma))$. On tire de cette relation certaines décompositions des représentations quantiques.

D'une part on obtient que les représentations de Weil se scindent en deux sous-représentations dites paires et impaires. On obtient également que les représentations de Weil aux niveaux $p = r^n$ sont des sous-représentations de celles aux niveaux $p = r^{n+2}$ lorsque r est un nombre premier. Enfin, en suivant [14], on démontre que si 4 divise p, alors les représentations de Reshetikhin-Turaev au niveau p se scindent en deux.

On démontre également, en suivant [27, 60], que lorsque $\Sigma_1 \cong S^1 \times S^1$, les représentations de Reshetikhin-Turaev de $\mathrm{Mod}(\Sigma_1) \cong SL_2(\mathbb{Z})$ sont des sous-représentations des représentations de Weil.

Nous verrons ensuite, en suivant [14], que, sous certaines conditions, lorsque l'invariant d'une TQFT est le produit des invariants de deux autres, la représentation quantique associée est produit tensoriel des deux autres. Nous dérivons ensuite de ce résultat deux décompositions tensorielles. D'une part si a et b sont premiers entre eux, la représentation de Weil au niveau ab est produit tensoriel des représentations aux niveaux a et b. D'autre part, lorsque r est impair, la représentations de Reshetikhin-Turaev au niveau $2r$ est produit tensoriel de celle au niveau r et de celle au niveau 6.

Enfin, en suivant Roberts ([76]), nous montrerons que les représentations de Reshetikhin-Turaev aux niveaux $p = r$ et $p = 2r$, avec r premier impair, sont irréductibles. Nous verrons également que la preuve fonctionne pour $r = 9$.

Chapitre 2

Ce chapitre contient la version non publiée de l'article [55] intitulé 'Decomposition of Weil representations into irreducible factors' dans lequel on donne la décomposition en

irréductibles des représentations de Weil.

Plus précisément, le module U_p de la représentation de Weil au niveau p se décompose ainsi :

Theorem (Théorème 2.1.1)**.**

1. *Si a et b sont premiers entre eux, alors $U_{ab} \cong U_a \otimes U_b$.*
2. *Si r est premier et $n \geq 1$, alors $U_{r^{n+2}} \cong U_{r^n} \oplus W_{r^{n+2}}$ où $W_{r^{n+2}}$ désigne un autre module.*
3. *Si r est premier impair, alors $U_{r^2} \cong \mathbb{1} \oplus W_{r^2}$ où $\mathbb{1}$ est la représentation triviale.*
4. *Les modules U_p et W_{r^n} avec $r > 2$ se scindent en deux sous-modules : $U_p \cong U_p^- \oplus U_p^+$, $W_{r^n} \cong W_{r^n}^+ \oplus W_{r^n}^-$.*
5. *Les modules $B_1 \otimes \ldots \otimes B_k$, où les B_i sont de la forme $U_r^+, U_r^-, U_2, U_4^+, U_4^-, W_{r^n}^+$ ou $W_{r^n}^-$ et sont associés à des entiers premiers entre eux, sont irréductibles.*

Cette décomposition était connue pour les représentations de Weil en genre 1 aux niveaux $p = r^n$, où r est un nombre premier impair (voir [54]). Notre étude généralise ces décompositions à tous les niveaux en genre arbitraire. On dérive de cette étude la décomposition des représentations de Reshetikhin-Turaev en genre 1 en irréductibles.

Bien que les quatre premiers points soient déjà démontrés au Chapitre 1, nous en donnons ici une preuve alternative plus directe en explicitant les isomorphismes utilisés. Le dernier point du théorème, l'irréductibilité des facteurs, est démontré de deux façons différentes.

Chapitre 3

Ce chapitre contient la version non publiée de l'article [56] intitulé 'Decomposition of some Reshetikhin-Turaev representations into irreducible factors', où l'on étudie la décomposition en irréductibles des représentations de Reshetikhin-Turaev lorsque le niveau est de la forme $p = 2r_1r_2$, $p = 2r^2$ et $p = 4r$, avec r, r_1, r_2 des nombres premiers impairs, modulo certaines conditions techniques. Plus précisément on démontre les deux théorèmes suivants :

Theorem (Théorème 3.1.1)**.**

1. *Si r est premier impair, alors $V_{4r,2}$ est somme de deux sous-représentations irréductibles.*
2. *Si r est premier impair, alors $V_{2r^2,2}$ est irréductible.*
3. *Si r_1, r_2 sont deux nombres premiers impairs distincts, alors $V_{2r_1r_2,2}$ est irréductible.*

A chaque niveau $p = 2r \geq 3$ est associé un ensemble de nombres complexes appelés $6j$-symboles. Un niveau $p = 2r \geq 3$, avec r impair, sera dit *générique* si les $6j$-symboles associés sont tous non nuls. Lorsque r est pair, la proposition 3.3.10 exhibe deux familles de $6j$-symboles nuls. Dans ce cas, p sera dit *générique* s'il n'en existe pas d'autres.

Theorem (Théorème 3.1.2)**.**

1. *Si 50 est générique , alors $V_{50,3}$ est irréductible.*

2. *Si $r \geq 7$ est premier impair, $p = 4r$ est générique et $g = 3$, alors $V_{4r,3}$ est somme de deux sous-représentations irréductibles.*
3. *Si r_1, r_2 sont deux nombres premiers impairs distincts, $p = 2r_1r_2$ est générique et $2g < min(r_1, r_2)$, alors $V_{2r_1r_2,g}$ est irréductible.*

La généricité d'un niveau assez petit donné peut se vérifier à l'aide d'un ordinateur.

Corollaire 0.0.1. *(preuve assistée par ordinateur)*
Le module $V_{50,3}$ est irreductible. Les modules $V_{28,3}$ et $V_{44,3}$ sont sommes de deux sous-modules irréductibles.

Chapitre 4

Ce chapitre regroupe pêle-mêle divers résultats sur les représentations quantiques.

Dans la première section, on adapte la preuve de [67] au cas abélien pour montrer que les normes au carré des représentations de Weil convergent, au sens de la topologie de Fell, vers la représentation homologique :

$$h : \mathrm{Mod}(\Sigma) \to \mathrm{GL}\left(\mathbb{C}[H_1(\Sigma, \mathbb{Z})]\right)$$

On démontre également qu'aux niveaux impairs, les représentations de Weil représentent fidèlement le groupe $Sp_{2g}(\mathbb{Z}/p\mathbb{Z})$.

Dans la seconde section, on étudie la finitude de l'image des représentations de Reshetikhin-Turaev lorsque le niveau est $p = r$ ou $p = 2r$ avec r premier impair. On retrouve notament les résultats de [29, 23], où il est prouvé que si $g \geq 2$, l'image est finie si et seulement si $r = 3$. On étudie également l'infinitude des images des représentations associées au tore percé, qui n'était connue qu'asymptotiquement dans [77].

Dans la troisième section, on exhibe des relevés linéaires explicites des représentations de Weil. Plus précisément, on relève les représentations de Weil en représentations linéaires de $Sp_{2g}(\mathbb{Z})$ lorsque $g = 1$ et lorsque $g \geq 2$ et que le niveau est impair. Lorsque $g \geq 2$ et que le niveau est pair, on relève les représentations de Weil en représentations d'une extension centrale non triviale de $Sp_{2g}(\mathbb{Z})$ par $\mathbb{Z}/2\mathbb{Z}$.

Chapitre 1

Topological Quantum Field Theories and quantum representations

Résumé

Ce chapitre sert à définir les objets, fixer les notations et énoncer les résultats classiques dont nous aurons besoin dans la thèse. Il ne contient aucun résultat original.

Dans la première section, nous introduisons la notion de TQFT. Il s'agit d'un foncteur entre une catégorie de cobordismes et une catégorie de modules, dont on peut extraire un invariant pour les 3-variétés fermées orientées munies d'entrelacs en rubans et une famille de représentations pour les groupes modulaires des surfaces fermées orientées munies de points marqués. En suivant [63, 14], nous définissons deux familles de TQFTs caractérisées d'une part par une propriété dite d'écheveaux pour les entrelacs en rubans et d'autre part par leur comportement vis-à-vis de la chirurgie.

La seconde section énonce des résultats classiques des représentations de Weil et Reshetikhin-Turaev issues de ces deux familles de TQFTs. Cela permettra de fixer les notations et préparer le terrain pour les résultats des prochains chapitres.

Abstract

In this chapter we define the objects, fix the notations and state the classical properties needed in the thesis. Nothing original is claimed here.

In the first section, we introduce the notion of TQFT. It is a functor between a cobordism category and a module category from which we can extract both an invariant for closed oriented 3-manifolds with an embedded link of ribbons, and a family of representations of the mapping class groups of closed oriented surfaces with marked points. Following [63, 14], we define two families of TQFTs caracterized one the one hand by a skein property for their link invariant and on the other hand by their behavior for surgery.

In the second section, we state some classical results concerning the Weil and Reshetikhin-Turaev representations arising from these families of TQFTs.

It permits to fix some notations and gives the background needed in the next chapters.

1.1 Construction of $U(1)$ and $SU(2)$ TQFTs

1.1.1 TQFT and universal construction

Definition of TQFTs without anomaly

A TQFT is a monoidal functor between a cobordism category and a category of k-modules.

We first define the *cobordism category* Cob_{2+1}.

The objects of Cob_{2+1} are *marked surfaces* $\mathbf{\Sigma} = (\Sigma, b)$, which are closed oriented surfaces Σ with an ordering on the set of its connected components and an ordered finite set $b = ((b_1, \epsilon_1), \ldots, (b_n, \epsilon_n))$ of co-oriented embedding b_i of $[0,1]$ in Σ called bands. We add the empty surface as an element.

A *banded cobordism* between two marked surfaces $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ is a $\mathbf{M} = (M, L, \phi_-, \phi_+)$ where M is a compact oriented 3-manifold with boundary $\partial M = \partial_- M \bigsqcup \partial_+ M$, L is a (possibly empty) framed tangle properly embedded in M and $\phi_- : -\Sigma \to \partial_- M$, $\phi_+ : \Sigma' \to \partial_+ M$ are orientation preserving homeomorphisms so that ϕ_+ (resp. ϕ_-), sends the bands of Σ' (resp. of $-\Sigma$) to $L \cap \partial_+ M$ (resp $L \cap \partial_- M$) by preserving the co-orientations. Two banded cobordisms are said equivalent if there exists a preserving orientation homeomorphism relatively to the boundary between the two underlying 3 manifolds which preserves the banded links.

The set of morphisms $\mathrm{Hom}(\mathbf{\Sigma}, \mathbf{\Sigma}')$ is the set of equivalence classes of banded cobordisms between $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$. The composition between $\mathbf{M_1} \in \mathrm{Hom}(\mathbf{\Sigma}, \mathbf{\Sigma}')$ and $\mathbf{M_2} \in \mathrm{Hom}(\mathbf{\Sigma}', \mathbf{\Sigma}'')$ is obtained by gluing the underlying cobordisms and links along $\phi_2^- \circ (\phi_1^+)^{-1} : \partial_+ M_1 \to \partial_- M_2$. Figure 1.1 illustrates this composition. The neutral element of $\mathrm{Hom}(\mathbf{\Sigma}, \mathbf{\Sigma})$ is the class of the cylinder $\Sigma \times [0,1]$ where we identify $-\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ with Σ using the identity map.

The category Cob_{2+1} has a *duality*, that is a functor $\mathrm{dual} : \mathrm{Cob}_{2+1} \to \mathrm{Cob}_{2+1}$ so that $\mathrm{dual} \circ \mathrm{dual} = \mathbb{1}$. It consists of changing the orientation of the surfaces and cobordisms and links.

Eventually the category Cob_{2+1} has a *tensor product* (we say it is a strict monoidal category), that is a functor $\bigsqcup : \mathrm{Cob}_{2+1} \times \mathrm{Cob}_{2+1} \to \mathrm{Cob}_{2+1}$ which takes the disjoint union of marked surfaces and cobordisms. It verifies the axioms of a tensor product, that is:

1. $\bigsqcup$ is associative for objects and morphisms.
2. There is a neutral object $\emptyset$ (the empty surface) so that:
$$\Sigma \bigsqcup \emptyset = \emptyset \bigsqcup \Sigma = \Sigma \text{ and } M \bigsqcup id_\emptyset = id_\emptyset \bigsqcup M = M$$
3. $id_\Sigma \bigsqcup id_{\Sigma'} = id_{\Sigma \bigsqcup \Sigma'}$
4. If $f \in \mathrm{Hom}(X, X')$, $f' \in \mathrm{Hom}(X', X'')$, $g \in \mathrm{Hom}(Y, Y')$, $g' \in \mathrm{Hom}(Y', Y'')$, then:
$$(f \circ f') \bigsqcup (g \circ g') = (f \bigsqcup g) \circ (f' \bigsqcup g')$$

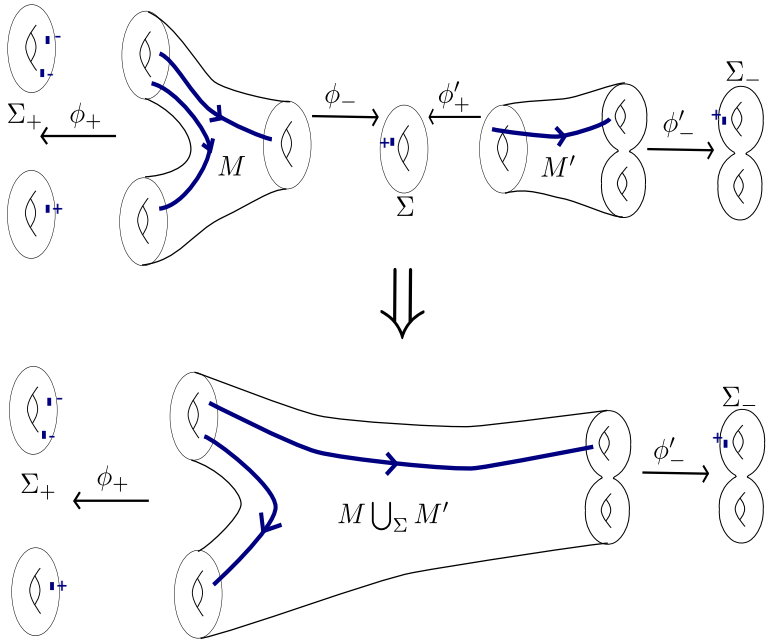


Figure 1.1: The composition of two morphisms in the cobordism category.

Let k be a commutative ring with a unit, an involution and a ring isomorphism $\langle\cdot\rangle : \mathrm{Hom}(k,k) \to k$. The main examples will be the field of complex numbers $\mathbb{C}$ with the complex conjugacy and the cyclotomic ring $\mathbb{Z}[A] \Big/ \phi_{2p}(A)$ with the involution sending A to A^{-1}.

Roughly speaking, the category of k-modules Mod_k is the category of free finite rank k-modules with the usual tensor product and duality. Unfortunately, this definition prevents the existence of functors $\mathrm{Cob}_{2+1} \to \mathrm{Mod}_k$ preserving tensor product. Indeed, the k-modules $(V_1 \otimes V_2) \otimes V_3$ and $V_1 \otimes (V_2 \otimes V_3)$ are isomorphic (with a unique isomorphism) but not equal, whereas the banded surfaces $(\Sigma_1 \bigsqcup \Sigma_2) \bigsqcup \Sigma_3$ and $\Sigma_1 \bigsqcup (\Sigma_2 \bigsqcup \Sigma_3)$ are equal. We solve the problem by defining the objects of Mod_k as finite (possibly empty) sequences $(V_1, \ldots, V_n)$ of free finite rank k-modules. The set $\mathrm{Hom}\left((V_1, \ldots, V_n), (W_1, \ldots, W_m)\right)$ is the set of linear map $V_1 \otimes \ldots \otimes V_n \to W_1 \otimes \ldots \otimes W_m$. The tensor product is the juxtaposition of sequences and duality given by $(V_1, \ldots, V_n)^* = (V_n^*, \ldots, V_1^*)$.

Note that a ring morphism $\nu : k_1 \to k_2$ that preserves involution, gives rise to a functor $\nu^* : \mathrm{Mod}_{k_1} \to \mathrm{Mod}_{k_2}$ that preserves tensor product and duality.

We finally state the:

Definition 1.1.1. A *Topological Quantum Field Theory* (or simply TQFT) is a functor $V : \mathrm{Cob}_{2+1} \to \mathrm{Mod}_k$ which preserves the duality and the tensor product (and the neutral object $V(\emptyset) = k$). A morphism between two TQFTs (V_1, k_1) and (V_2, k_2) is a ring morphism $\nu : k_1 \to k_2$ that preserves involution, together with a natural map between $\nu^* \circ V_1 : \mathrm{Cob}_{2+1} \to \mathrm{Mod}_{k_2}$ and V_2 preserving duality and tensor product.

We may also want the framed link inside cobordisms to be oriented. We define similarly a category Cob^o_{2+1} of cobordism where the objects are banded surfaces where each band b_i is equipped with a sign $\epsilon_i \in \{-1, +1\}$ and the morphisms are classes of banded cobordisms where the core of each framed link is oriented in such a way that is only allowed to enter

through a band with sign -1 and leave through a signed $+1$ band. A pair (U, k) with k a ring and U a functor $U : \mathrm{Cob}^{o}_{2+1} \to \mathrm{Mod}_k$ that preserves tensor product and duality will be called an *oriented TQFT* without anomaly. Figure 1.2 illustrates such a functor.

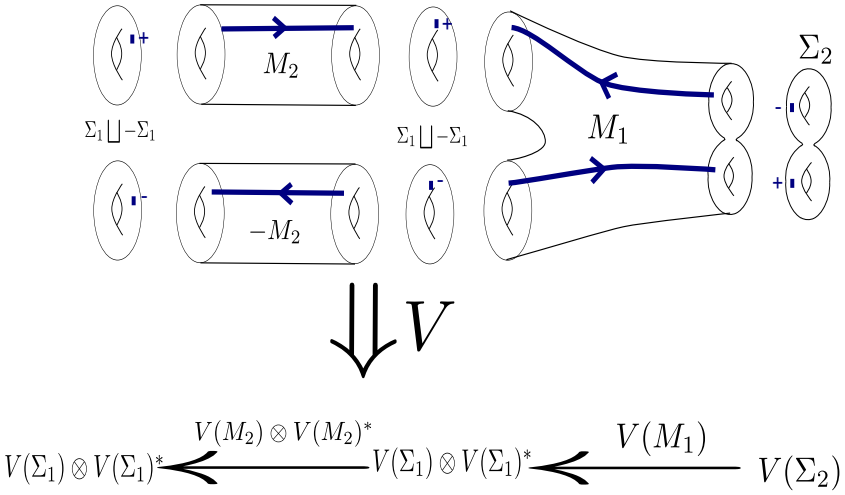


Figure 1.2: Illustration of a TQFT

Three-manifolds invariants

Let (M, L) be a closed oriented 3-manifold with an embedded link L. It defines an element of $\mathrm{Hom}(\emptyset, \emptyset)$. If (V, k) is a TQFT, then $V(M, L) \in \mathrm{Hom}(k, k) \cong k$.

Definition 1.1.2. If (V, k) is a TQFT without anomaly, the map

$$\langle \cdot \rangle_V : \{\text{closed oriented 3-manifolds with embedded link}\} /_{homeo^+} \to k$$

sending (M, L) to $\langle V(M, L) \rangle$, is the *quantum invariant* associated to (V, L).

This invariant satisfies the following properties:

1. $\langle \emptyset \rangle_V = 1_k$.
2. $\langle M_1 \bigsqcup M_2 \rangle_V = \langle M_1 \rangle_V \cdot \langle M_2 \rangle_V$.
3. $\langle -M \rangle_V = \langle \bar{M} \rangle_V$

An invariant that satisfies these three properties will be called a *good invariant*.

Mapping class group representations

Let $\mathbf{\Sigma}$ be a marked surface. We denote by $\mathrm{Mod}(\mathbf{\Sigma})$ the mapping class group of orientation-preserving homeomorphisms of Σ that preserve the bands together with their co-orientations. To $\phi \in \mathrm{Mod}(\mathbf{\Sigma})$ we associate the *cylinder* $C(\phi) \in \mathrm{Hom}((\Sigma), (\Sigma))$ which is the class of the cobordism $\Sigma \times [0, 1]$ with the link $\bigsqcup_i b_i \times [0, 1]$, where we identify $\Sigma \times \{0\}$ with $-\Sigma$ with the identity map and $\Sigma \times \{1\}$ with Σ with ϕ.

Definition 1.1.3. Let (V, k) be a TQFT without anomaly and $\mathbf{\Sigma}$ be a marked surface. The group morphism:

$$\rho_{V,\mathbf{\Sigma}} : \mathrm{Mod}(\mathbf{\Sigma}) \to \mathrm{GL}(V(\mathbf{\Sigma}))$$

defined by $\rho_{V,\mathbf{\Sigma}}(\phi) = V(C(\phi))$, is called the *quantum representation* of $\mathbf{\Sigma}$ associated to (V, k).

Non-degeneracy condition and the universal construction

Under certain non-degeneracy assumptions, a TQFT is completely determined by its quantum invariant. In [14], the authors gave a method to construct a TQFT from a good invariant.

Notation 1.1. When M is a compact oriented manifold with boundary $\partial M \cong \Sigma$, then M may be seen as a banded cobordism $\mathbf{M}$ between $\partial_- M = \emptyset$ and $\partial_+ M = \Sigma$. The morphism $V(\mathbf{M}) \in \mathrm{Hom}(k, V(\mathbf{\Sigma}))$ is completely determined by the image of the neutral element 1_k of k. We write:

$$Z(M) := V(\mathbf{M})(1_k) \in V(\mathbf{\Sigma})$$

the vector associated to M.

Definition 1.1.4. A TQFT without anomaly (V, k) is *non degenerate* if:

1. For every marked surface $\mathbf{\Sigma}$, the module $V(\mathbf{\Sigma})$ is spanned by the vectors $Z(M)$ with M a compact oriented manifold with boundary Σ.
2. The bilinear form $\langle \cdot, \cdot \rangle_{\mathbf{\Sigma},V} : V(\mathbf{\Sigma}) \times V(\mathbf{\Sigma}) \to k$ defined by:

$$\langle Z(M_1), Z(M_2) \rangle_{\mathbf{\Sigma},V} = \left\langle -M_1 \bigcup_{\Sigma} M_2 \right\rangle_V$$

   is non-degenerate, i.e. defines an isomorphism $V(\mathbf{\Sigma}) \cong (V(\mathbf{\Sigma}))^*$.

The *universal construction* is defined as follows. Let

$$I : \{\text{closed oriented 3-manifolds with embedded link}\} /_{homeo^+} \to k$$

be a good invariant. If $\mathbf{\Sigma}$ is a marked surface, we denote by $N_{\mathbf{\Sigma}}$ the free k-module generated by the classes of banded cobordisms $\mathbf{M}$ such that $\partial \mathbf{M} = \mathbf{\Sigma}$. We define the k-module $V_I(\mathbf{\Sigma})$ as the quotient of $N_{\mathbf{\Sigma}}$ by the kernel of the bilinear form:

$$\langle \cdot, \cdot \rangle_{\mathbf{\Sigma},I} : N_{\mathbf{\Sigma}} \times N_{\mathbf{\Sigma}} \to k$$

defined by $\langle M_1, M_2 \rangle_{\mathbf{\Sigma},I} = I(M_1 \bigcup_{\Sigma} -M_2)$.

A class of banded cobordism $\mathbf{M} \in \mathrm{Hom}(\mathbf{\Sigma}, \mathbf{\Sigma}')$ acts naturally on $V_I(\mathbf{\Sigma})$ by sending a class $[Z(X)] \in V_I(\mathbf{\Sigma})$ to the class $[Z(M \bigcup_{\Sigma} X)] \in V_I(\mathbf{\Sigma}')$. We get a functor $V_I : \mathrm{Cob}_{2+1} \to \mathrm{Mod}_k$ in this way.

It follows from definitions that V_I is a TQFT without anomaly if and only if:

1. The modules $V_I(\mathbf{\Sigma})$ have finite rank.
2. The bilinear form $\langle \cdot, \cdot \rangle_{\mathbf{\Sigma},V_I}$ induces an isomorphism $V(-\mathbf{\Sigma}) \cong (V(\mathbf{\Sigma}))^*$.
3. The map $V(\mathbf{\Sigma}) \otimes V(\mathbf{\Sigma}') \to V(\mathbf{\Sigma} \bigsqcup \mathbf{\Sigma}')$ is an isomorphism.

The second point follows from the first one if k is a field.

Remark. The definition of TQFT we gave suits well for the ones we want to study in this manuscript but is restrictive. Recent constructions involve also TQFTs with infinite dimensional vector spaces associated to surfaces. We might also simply ask the map $V(\mathbf{\Sigma}) \otimes V(\mathbf{\Sigma}') \to V(\mathbf{\Sigma} \bigsqcup \mathbf{\Sigma})$ to be injective and drop the non-degeneracy assumption as well.

1.1.2 Skein relations and colored Jones polynomials

In this section, k is a commutative ring with an involution so that there exists an isomorphism $\langle\cdot\rangle : \mathrm{Hom}(k,k) \to k$ and a ring morphism $\mu : \mathbb{Z}[A, A^{-1}] \to k$ with $\mu(\bar{A}) = A^{-1}$. When no confusion is possible, we will simply denote by A the element $\mu(A) \in k$.

Link invariants

Let $\Sigma_{0,4} \in \mathrm{obj}(\mathrm{Cob}_{2+1})$ and $\Sigma^0_{0,4} \in \mathrm{obj}(\mathrm{Cob}^o_{2+1})$ be the sphere S^2 with four bands where the signs of the bands of $\Sigma^o_{0,4}$ are $-1, -1, +1$ and $+1$. We consider the banded cobordisms  $\in \mathrm{Hom}(\emptyset, \Sigma_{0,4})$ and  $\in \mathrm{Hom}(\emptyset, S^3)$ and the cobordisms   $\in \mathrm{Hom}(\emptyset, \Sigma^o_{0,4})$ and  $\in \mathrm{Hom}^o(\emptyset, S^3)$.

Definition 1.1.5.

An oriented TQFT $U : \mathrm{Cob}^o_{2+1} \to \mathrm{Mod}_k$ *respects the abelian skein relations* if:

- $Z_U($  $) = AZ_U($ $)$,
- $Z_U($  $) = A^{-1}Z_U($ $)$,
- $Z_U($  $) = Z_U(\emptyset)$.

A non-oriented TQFT $V : \mathrm{Cob}_{2+1} \to \mathrm{Mod}_k$ *respects the Kauffman skein relations* if:

- $Z_V($  $) = A^{-1}Z_V($  $) + AZ_V($  $)$,
- $Z_V($  $) = -(A^2 + A^{-2})Z_V(\emptyset)$.

Recall that the linking number $lk(L)$ of a framed oriented link is defined as the number of positive crossing minus the number of negative ones of an arbitrary diagram representing the link. Note that the quantum invariant of a framed link $L \subset S^3$ for a TQFT which respects the abelian skein relations is equal to $A^{lk(L)}$, so is related to the linking number $lk(L)$ modulo p.

The quantum invariant of a framed link for a TQFT which respects the Kauffman skein relation is the Kauffman bracket as defined in [48] closely related to the Jones polynomial ([47]).

We will define two families $(U_p, k'_p)_{p\geq 2}$ and $(V_p, k_p)_{p\geq 3}$ of non degenerate TQFTs that respect the skein relations. They will be indexed by an integer p which specifies the order of $A \in k$. This is motivated by the following proposition, whose proof is postponed to the end of the subsection:

Proposition 1.1.6. *If k is the ring of a non-degenerate TQFT which respects the skein relations, then there exists an integer p, invertible in k, such that A is a primitive p-th root of unity if p is odd in the abelian case and such that A has order $2p$ in the other cases.*

The integer p of the previous proposition will be called the *level* of the TQFT.

Skein modules and skein algebras

The skein modules were introduced independently by V. Turaev ([84]) and J. Przytycki ([44]). They are central objects in the study of TQFTs which respect skein relations.

Definition 1.1.7.

Let M be an oriented compact 3-manifold, k a commutative ring with a morphism $\mu : \mathbb{Z}[A^{\pm}]$. The *skein modules* $\mathcal{K}_k(M)$ and $\mathcal{T}_k(M)$ are the quotient of the free k-module generated by isotopy classes of framed links in M, by the skein relations of Figure 1.3.

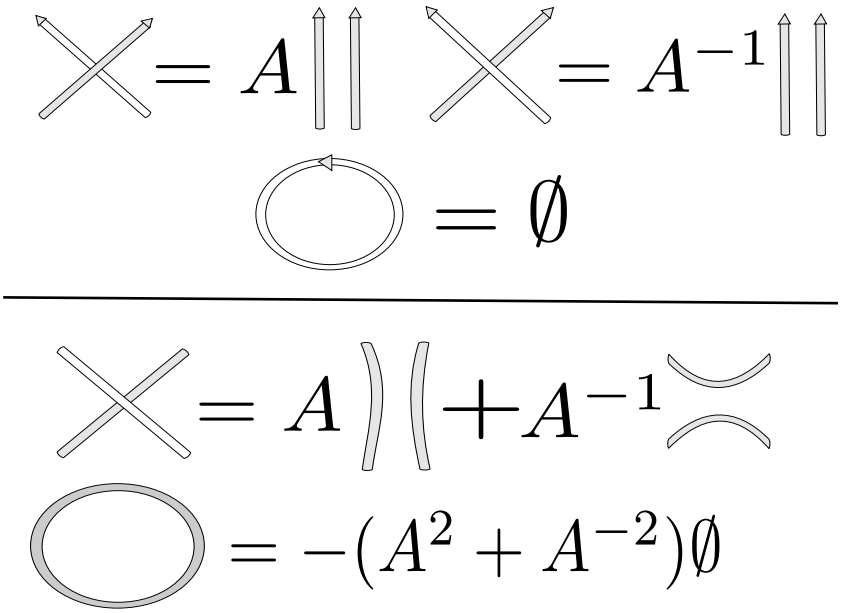


Figure 1.3: The skein relations defining $\mathcal{T}_k(M)$ on the top and $\mathcal{K}_k(M)$ on the bottom.

If Σ is an oriented compact surface, the *skein algebras* $\mathcal{T}_k(\Sigma)$ and $\mathcal{K}_k(\Sigma)$ are the k-modules $\mathcal{T}_k(\Sigma \times [0,1])$ and $\mathcal{K}_k(\Sigma \times [0,1])$. The product of the classes of two links L_1 and L_2 in $\Sigma \times [0,1]$ is obtained by isotoping L_1 in $\Sigma \times [0, \frac{1}{2})$ and L_2 in $\Sigma \times (\frac{1}{2}, 1]$ and then gluing the two parts in $\Sigma \times [0,1]$.

Remark.

1. The algebras $\mathcal{T}_k(\Sigma)$ are usually called *quantum tori*. Note that the class of a link embedded in $\Sigma \times \{\frac{1}{2}\}$ in an abelian skein algebra, only depends on the integer homology of the link. If $[L_1]$ and $[L_2]$ are the classes in $H_1(\Sigma, \mathbb{Z})$ of two links and $\omega([L_1], [L_2])$ represents the crossing number, the product in the abelian skein algebra is then given by:
$$[L_1] \cdot [L_2] = A^{\omega([L_1],[L_2])}[L_1 + L_2]$$
2. We introduce skein modules because if U and V are non-degenerate TQFTs that respect the abelian and Kauffman relations respectively, then $U(\Sigma)$ is a quotient of $\bigoplus_{\partial M = \Sigma} \mathcal{T}_k(M)$ and $V(\Sigma)$ a quotient of $\bigoplus_{\partial M = \Sigma} \mathcal{K}_k(M)$.

A *multicurve* in a surface Σ is a one-dimensional submanifold with no contractible components.

Proposition 1.1.8. *If Σ is an oriented compact surface, then:*

1. *The module $\mathcal{T}_k(\Sigma)$ is freely generated by the elements of $H_1(\Sigma, \mathbb{Z})$.*
2. *The module $\mathcal{K}_k(\Sigma)$ is freely generated by the multicurves of Σ.*

Proof. Any link in $\Sigma \times [0,1]$ can be isotoped and projected in $\Sigma \times \{\frac{1}{2}\}$ to give a regular diagram, that is an embedded graph (oriented in the abelian case) with vertices of degree four and with the choice for each vertex of a crossing  or  . Using the skein relations of Figure 1.3, we associate to the class of the link in the skein modules, a linear combination of regular diagram without crossing nor contractile components corresponding to multicurves. This shows that multicurves generate $\mathcal{K}_k(\Sigma)$ and that the elements of $H_1(\Sigma, \mathbb{Z})$ generate $\mathcal{T}_k(\Sigma)$.

The fact that the abelian skein relations preserve the class of a link in $H_1(\Sigma, \mathbb{Z})$, implies that the set of elements of $H_1(\Sigma, \mathbb{Z})$ in $\mathcal{T}_k(\Sigma)$ is free.

To show that the set of multicurves of Σ in $\mathcal{K}_k(\Sigma)$ is free, denote by K' the free k-module generated by isotopy classes of multicurves and by $\psi : K' \to \mathcal{K}_k(\Sigma)$ the map sending a multicurve to the skein class of the corresponding framed link in $\Sigma \times \{\frac{1}{2}\} \subset \Sigma \times [0,1]$.

To construct its reverse map $\phi : \mathcal{K}_k(\Sigma \times [0,1]) \to K'$, first note that $\mathcal{K}_k(\Sigma \times [0,1])$ is isomorphic to the free k-module generated by isotopy classes of regular diagrams, quotiented by the framed Reidemeister moves of Figure 1.4 and the Kauffman skein relations of Figure 1.3.

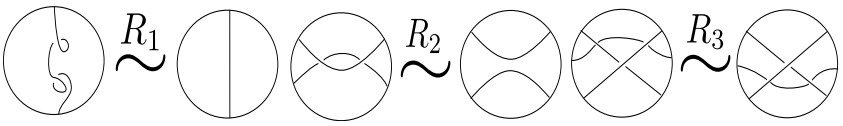


Figure 1.4: The three framed Reidemeister moves.

Let D be a regular diagram and $V(D)$ be the set of its vertices. To each map $c : V(D) \to \{-1, +1\}$, we associate a multicurve D_c obtained by replacing each vertex v with crossing  by the smoothing  if $c(v) = +1$ and by  if $c(v) = -1$, and then removing the contractible components. We also denote by $n(c)$ the number of contractible components removed and $m(c) := \sum_{v \in V(D)} c(v)$.

The map:

$$\phi(D) = \sum_c A^{m(c)} \left(-A^2 - A^{-2}\right)^{n(c)} D_c \in K'$$

is invariant under the framed Reidemeister moves and the Kauffman skein relations and gives the reverse map of ψ which is thus injective. This implies that the set of multicurves in $\mathcal{K}_k(\Sigma)$ is free. □

The Temperley-Lieb algebras and the Jones-Wenzl idempotents

Definition 1.1.9. Given $n \geq 2$, the *Temperley-Lieb algebra* TL_n is the quotient of the k-module generated by isotopy classes of (n, n)-tangles by the Kauffman skein relations (Figure 1.3). The composition is given by gluing two tangle vertically whereas the horizontal juxtaposition defines a morphism $TL_n \otimes TL_m \to TL_{n+m}$.

The algebra TL_n has a finite presentation with generators the neutral 1_n and the elements b_i, for $1 \leq i \leq n-1$, of Figure 1.5. The relations are given by the above relations and the fact that 1_n is the neutral:

1. $b_i^2 = -(A^2 + A^{-2})b_i$, for $1 \leq i \leq n-1$.
2. $b_i b_{i\pm1} b_i = b_i$, for $1 \leq i \leq n-1$.
3. $b_i b_j = b_j b_i$, when $|i-j| > 1$.

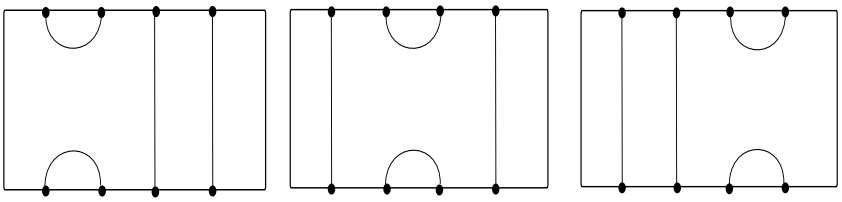


Figure 1.5: The elements b_1, b_2 and b_3 of TL_4 .

Definition 1.1.10.

– The *Chebychev polynomials* $T_n \in \mathbb{Z}[X]$ are defined by $T_1 = 1, T_2 = X$ and the recursive formula:

$$T_n(X) = XT_{n-1}(X) - T_{n-2}(X)$$

In particular $T_n(X + X^{-1}) = \frac{X^n - X^{-n}}{X - X^{-1}} = X^{n-1} + X^{n-3} + \ldots + X^{-n+1}$.

– The *quantum numbers* $[n] \in k$ are defined by $[2] := A^2 + A^{-2}$ and:

$$[n] := T_n([2]) = \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}$$

– The *Jones-Wenzl idempotents* $f_0, \ldots, f_n \in TL_n$ are defined by the following recursive formula, starting with $f_0 = 1$, and illustrated in Figure 1.6:

$$f_{n+1} = f_n \otimes 1_1 + \frac{[n]}{[n+1]} (f_n \otimes 1_1)(b_n)(f_n \otimes 1_1)$$

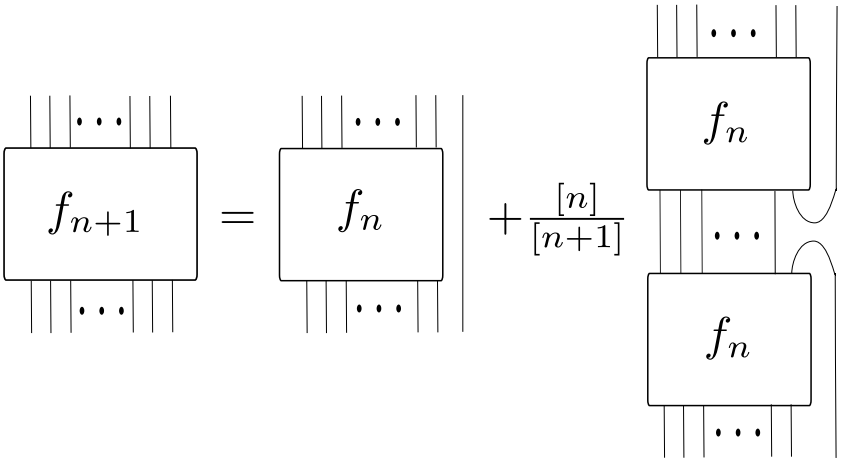


Figure 1.6: The recursive formula defining the Jones-Wenzl idempotents.

By abuse, we will identify an element $u \in TL_n$ with the element $u \otimes 1_m \in TL_{n+m}$ when no confusion is possible. We thus consider $f_0, \ldots, f_n$ as elements of TL_n.

Lemma 1.1.11. *The Jones-Wenzl idempotents $f_0, \ldots, f_n \in TL_n$ are characterized by the following properties:*

1. $f_n \neq 0$.
2. $f_n^2 = f_n$.
3. $b_i f_n = f_n b_i = 0$, *for all* $i \leq n-1$.

Proof. We show by induction the following properties which imply the ones of the lemma:

$$\begin{array}{ll} (1_n) & (f_n)^2 = f_n \\ (2_n) & b_i f_n = 0, \forall i \leq n-1 \\ (3_n) & (b_n f_n)^2 = -\frac{[n+1]}{[n]} b_n f_n \end{array}$$

We suppose these properties hold for some $n \geq 1$ and show (1_{n+1}) :

$$\begin{aligned} (f_{n+1})^2 &= (f_n + \frac{[n]}{[n+1]} f_n.b_n.f_n)^2 \\ &= f_n + 2\frac{[n]}{[n+1]} f_n.b_n.f_n + \left(\frac{[n]}{[n+1]}\right)^2 f_n b_n f_n b_n f_n \\ &= f_n + \frac{[n]}{[n+1]} f_n b_{n+1} f_n = f_{n+1} \end{aligned}$$

We show (2_{n+1})

$$\begin{aligned} b_i f_{n+1} &= b_i(f_n + \frac{[n]}{[n+1]} f_n b_{n+1} f_n) = 0 \\ b_n f_{n+1} &= b_n f_n + \frac{[n]}{[n+1]} (b_n f_n)^2 = 0 \end{aligned}$$

We show (3_{n+1})

$$\begin{aligned} (b_{n+1} f_{n+1})^2 &= b_{n+1}(f_n + \frac{[n]}{[n+1]} f_n b_n f_n) b_{n+1} f_{n+1} \\ &= -[2] b_{n+1} f_{n+1} + \frac{[n]}{[n+1]} b_{n+1} f_{n+1} \\ &= -\frac{[n+2]}{[n+1]} b_{n+1} f_{n+1} \end{aligned}$$

Denote by $\epsilon_0 : TL_n \to k$ the algebras morphism mapping 1_n to 1 and b_i to 0. It follows from the definition of the f_n that $\epsilon_0(f_n) = 1$. Now suppose $g_n \in TL_n$ satisfies the three properties of the lemma and write $x_n = g_n - \epsilon_0(g_n)1_n$. Since $g_n^2 = g_n$ and $g_n x_n = x_n g_n = 0$ we must have $\epsilon_0(g_n) = 1$ also. Now we have $f_n g_n = f_n(1_n + x_n) = f_n$ and similarly $g_n f_n = g_n$. Thus $f_n = g_n$ and the three properties characterize f_n. □

The previous lemma implies that:

$$f_n u = u f_n = \epsilon_n(u) f_n, \text{ for all } u \in TL_n$$

where $\epsilon_n : TL_n \to k$ is the algebras morphism defined by $\epsilon_n\left(\sum_k \alpha_k b_k\right) = \alpha_n$ and $\epsilon_n(1_n) = 0$.

Denote by T_n, C_n and Λ_n the elements of TL_n defined by Figure 1.7 and write $\mu_n := \epsilon_n(T_n)$ and $\lambda_n := \epsilon_n(C_n)$.

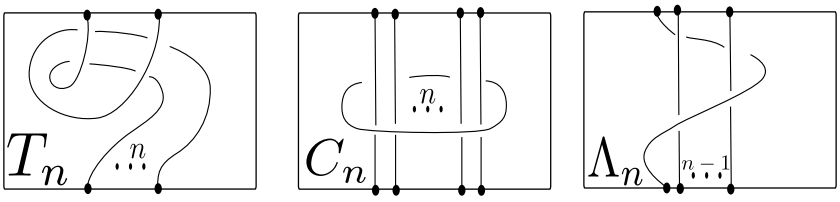


Figure 1.7: Three elements of TL_n.

Lemma 1.1.12. *We have* $\mu_n = (-1)^n A^{n(n+2)}$, $\lambda_n = -(A^{2(n+1)} + A^{-2(n+1)})$ *and* $\epsilon_n(\Lambda_n) = A^{2n-2}$.

Proof. Figure 1.8 shows the following three equalities:

1. $T_{n+1} \cdot (1_1 \otimes T_n) = -A^3 \Lambda_{n+1}$.
2. $C_{n+1} = A^{-2}(1_1 \otimes C_n) + (1 - A^4)\Lambda_{n+1}$.
3. $\Lambda_{n+1} = A^2(1_1 \otimes \Lambda_n) + h_n$ where $\epsilon_n(h_n) = 0$.

Taking the image by ϵ_n, we get the following system:

$$\begin{cases} \mu_{n+1} & = & -A^3 \epsilon_{n+1}(\Lambda_{n+1}) \cdot \mu_n \\ \lambda_{n+1} & = & A^{-2}\lambda_n + (1 - A^4)\epsilon_{n+1}(\Lambda_{n+1}) \\ \epsilon_{n+1}(\Lambda_{n+1}) & = & A^2 \epsilon_n(\Lambda_n) \end{cases}$$

We conclude by induction on n.

□

Colored Jones polynomials

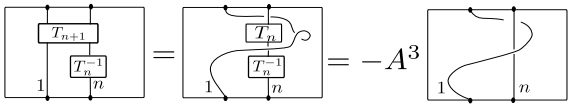
The tangles of TL_n can be seen as braids in $D^2 \times [0,1]$. When we identify $D^2 \times \{0\}$ with $D^2 \times \{1\}$, we obtain a framed link in $D^2 \times S^1$. We denote by $Tr_n : TL_n \to \mathcal{K}(D^2 \times S^1)$ the induced linear map and define the elements:

$$u_n := Tr_n(f_n) \in \mathcal{K}(D^2 \times S^1)$$

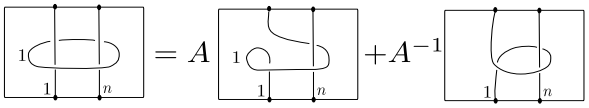
Let $K \subset S^3$ be a framed knot. We define a map

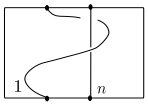
$$\phi_K : \mathcal{K}(D^2 \times S^1) \to k$$

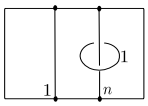
by embedding $D^2 \times S^1$ in S^3 into a tubular neighborhood of K using the framing of K to specify the gluing, and the isomorphism $\langle \cdot \rangle : \mathcal{K}(S^3) \to k$.

$$T_{n+1} \cdot (1 \otimes T_n^{-1}) =$$ 

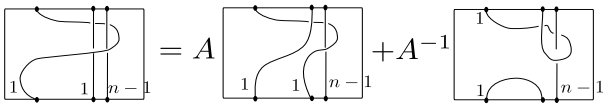
$$= -A^3\Lambda_{n+1}$$

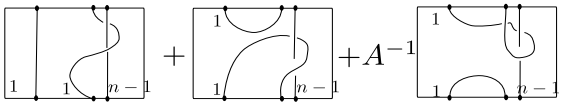
$$C_{n+1} =$$ 

$$= (1 - A^4)$$ 

$+A^{-2}$ 

$$= (1 - A^4)\Lambda_{n+1} + A^{-2} 1 \otimes C_n$$

$$\Lambda_{n+1} =$$ 

$$= A^2$$ 

$$= A^2\Lambda_n + h_n$$

Figure 1.8: A proof for three equalities in TL_n. The integers in front of the strands indicate the number of parallel copies we take.

Similarly if $L \subset S^3$ is a framed link with m components, we define a map:

$$\phi_L : \mathcal{K}(D^2 \times S^1)^{\otimes m} \to k$$

Definition 1.1.13. The elements $\phi_L(u_{i_1}, \ldots, u_{i_m}) \in k$, for $i_k \in \mathbb{N}$, are called the *colored Jones polynomials* of L.

Lemma 1.1.14. *The following three facts hold:*

1. *We have $u_i = T_i(u_1)$, for all $i \in \mathbb{N}$, where T_i is the i-th Chebychev polynomial defined in Definition 1.1.10.*
2. *If $\Omega_0 \subset S^3$ is the trivial framed knot, then:*

$$\phi_{\Omega_0}(u_i) = (-1)^i[i+1]$$

3. *If $H \subset S^3$ denotes the Hopf link, then:*

$$\phi_H(u_i, u_j) = (-1)^{i+j}[(i+1)(j+1)]$$

Proof. To prove the first point, we first derive from the recursive Definition 1.1.10 of the Jones-Wenzl idempotents the following equality:

$$\begin{aligned} e_i = Tr_n(f_i) &= Tr_n(f_{i-1} \otimes 1_1) + \frac{[i-1]}{[i]} Tr_n(b_i f_{i-1}) \\ &= u_1 Tr_n(f_{i-1}) + \frac{[i-1]}{[i]}(Tr_n(b_i f_{i-2}) + \frac{[i-2]}{[i-1]} Tr_n(b_i f_{i-2} b_{i-1} f_{i-2})) \end{aligned}$$

We then remark the following two facts which can be proved using simple drawings:

$$Tr_n(b_i f_{i-2}) = -(A^2 + A^{-2})Tr_n(f_{i-2})$$
$$Tr_n(b_i f_{i-2} b_{i-1} f_{i-2}) = Tr_n(b_i b_{i-1} f_{i-2}) = Tr_n(f_{i-2})$$

We deduce that:

$$Tr_n(f_i) = u_1 Tr_n(f_{i-1}) - \frac{[i-1]}{[i]}(A^2 + A^{-2} + \frac{[i-2]}{[i-1]})Tr_n(f_{i-2}) = u_1 Tr_n(f_{i-1}) - Tr_n(f_{i-2})$$

This is equivalent to $e_i = e_1 e_{i-1} - e_{i-2}$ and the result follows from the definition of the Tchebychev polynomials.

The second point is shown by induction. Suppose it holds for some $n - 1 \geq 1$, then:

$$\begin{aligned}
\phi_{\Omega_0}(u_n) &= \phi_{\Omega_0}(u_{n-1}) + \frac{[n-1]}{[n]}\phi_{\Omega_0}(Tr_n(f_{n-1} b_n f_{n-1})) \\
&= \frac{-[n][2] + [n-1]}{-[2][n]}\phi_{\Omega_0}(u_{n-1}) \\
&= -\frac{[n+1]}{[2][n]}\phi_{\Omega_0}(u_{n-1}) \\
&= (-1)^n[n+1]
\end{aligned}$$

To show the third point, write $T_j(X) = \sum_k s_{j,k} X^k$. We have:

$$\begin{aligned}
S_{i,j} := \phi_H(u_i, u_j) &= \sum_k s_{j,k} s_{j,k} \lambda_i^k \phi_{\Omega_0}(u_i) \\
&= T_j(-A^{2(i+1)} - A^{-2(i+1)})(-1)^i[i+1] \\
&= (-1)^{i+j}\frac{A^{2(i+1)(j+1)} - A^{-2(i+1)(j+1)}}{A^{2(i+1)} - A^{-2(i+1)}}\frac{A^{2(i+1)} - A^{-2(i+1)}}{A^2 - A^{-2}} \\
&= (-1)^{i+j}[(i+1)(j+1)]
\end{aligned}$$

□

Trivalent graphs, theta and tetrahedron coefficients

Definition 1.1.15.

A triple (i, j, k) of non negative integers is said admissible if:
– $i + j + k$ is even,
– $|i - j| \leq k \leq i + j$.

If Γ is a trivalent graph (the vertices have degree 3) with set of edges $E(\Gamma)$, an admissible coloring of Γ is a map $\sigma : E(\Gamma) \to \mathbb{N}$ so that if (e_1, e_2, e_3) are three edges adjacent to one vertex, then $(\sigma(e_1), \sigma(e_2), \sigma(e_3))$ is admissible.

If $\Gamma \subset M$ is a trivalent graph embedded in a compact 3-manifold M and σ a coloring, we associate an element of $\mathcal{K}_p(M)$, called the expansion of (Γ, σ), by replacing each edge $e \in E(\Gamma)$ by the idempotent $f_{\sigma(e)}$ and cabling three idempotents along a vertex using the planar tangle T_{e_1,e_2,e_3} of Figure 1.9.

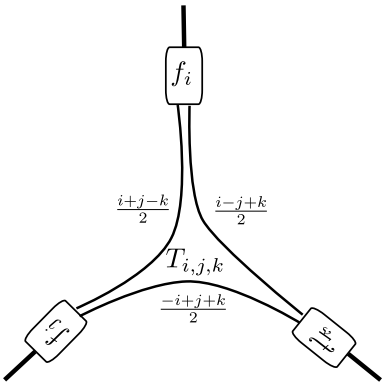
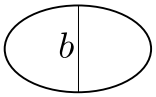


Figure 1.9: The link $T_{i,j,k}$ used to connect three idempotents f_i, f_j and f_k. The numbers above each three arcs denote the number of parallel copies of the arc used to define the link.

Consider the theta graph and the tetrahedron graph in S^3 of Figure 1.10, and denote by $\langle a, b, c\rangle$ and $\left\langle \begin{array}{ccc} A & B & E \\ C & D & F \end{array} \right\rangle$ the Kauffman bracket of their expansion using the colorings of Figure 1.10.

$\langle a, b, c\rangle =$



$\left\langle \begin{array}{ccc} A & B & E \\ C & D & F \end{array} \right\rangle =$

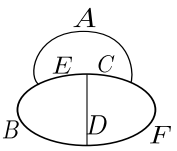


Figure 1.10: The colored tetha and tetrahedron graphs.

The computation of these theta and tetrahedron coefficients can be made using the recursive formula of the Jones-Wenzl idempotents (Definition 1.1.10). We refer to [70] for a proof.

Proposition 1.1.16. *Note $i = \frac{-a+b+c}{2}$, $j = \frac{a-b+c}{2}$, $k = \frac{a+b-c}{2}$, then we have:*

$$\langle a, b, c\rangle = (-1)^{i+j+k} \frac{[i+j+k+1]![i]![j]![k]!}{[i+j]![j+k]![i+k]!}$$

Proposition 1.1.17. *Note:*

$$\begin{array}{ll} a_1 = (A+B+E)/2 & b_1 = (B+C+E+F)/2 \\ a_2 = (B+D+F)/2 & b_2 = (A+B+C+D)/2 \\ a_3 = (C+D+E)/2 & b_3 = (A+D+E+F)/2 \\ a_4 = (A+C+F)/2 & \end{array}$$

Then we have:

$$\left\langle \begin{array}{ccc} A & B & E \\ D & C & F \end{array} \right\rangle := \frac{\prod_{i=1}^{3}\prod_{j=1}^{4}[b_i - a_j]!}{[A]![B]![C]![D]![E]![F]!} \sum_{max(a_j)\leq z\leq min(b_i)} \frac{(-1)^z[z+1]!}{\prod_{i=1}^{3}[b_i - z]!\prod_{j=1}^{4}[z-a_j]!}$$

Proof of Proposition 1.1.6

The Hopf link $H \subset S^3$ defines bilinear maps $\phi_H : \mathcal{K}_k(D^2 \times S^1) \times \mathcal{K}_k(D^2 \times S^1) \to k$ and $\tilde{\phi}_H : \mathcal{T}_k(D^2 \times S^1) \times \mathcal{T}_k(D^2 \times S^1) \to k$. If V and U are non degenerate TQFTs which respect the Kauffman skein relations and the abelian skein relations respectively, then $\mathcal{K}_k(D^2 \times S^1) \Big/ ker\phi_H$ is a quotient of a submodule of $V(\Sigma)$ and $\mathcal{T}_k(D^2 \times S^1) \Big/ ker\tilde{\phi}_H$ is a quotient of a submodule of $U(\Sigma)$. Denote by $e_i \in \mathcal{T}_k(D^2 \times S^1)$ the class of the link made of i parrallel copies of $\{0\} \times S^1$. Proposition 1.1.6 follows from the following:

Lemma 1.1.18. *The modules $\mathcal{K}_k(D^2 \times S^1) \Big/ ker\phi_H$ and $\mathcal{T}_k(D^2 \times S^1) \Big/ ker\tilde{\phi}_H$ have finite rank if and only if $A^2 \in k$ is a root of unity. If p denotes the order of A^2, then p divides the determinant of ϕ_H and $\tilde{\phi}_H$ in this quotient.*

If $p \in k$ is invertible, then:

1. *$\{e_0, e_1, \ldots, e_{p-1}\}$ is a basis of $\mathcal{T}_k(D^2 \times S^1) \Big/ ker\tilde{\phi}_H$,*
2. *$\{u_0, u_1, \ldots, u_{\frac{p-4}{2}}\}$ is a basis of $\mathcal{K}_k(D^2 \times S^1) \Big/ ker\phi_H$, when p is even,*
3. *$\{u_0, u_2, u_4, \ldots, u_{p-3}\}$ is a basis of $\mathcal{K}_k(D^2 \times S^1) \Big/ ker\phi_H$, when p is odd.*

Proof. The matrix $T_p := \left(\tilde{\phi}_H(e_i, e_j)\right)_{0 \leq i,j \leq p-1} = \left(A^{2ij}\right)_{0 \leq i,j \leq p-1}$ is a Vandermonde matrix of determinant $\det(T_p) = \prod_{i<j} (A^{2j} - A^{2i})$. If $\mathcal{T}_k(D^2 \times S^1) \Big/ ker\tilde{\phi}_H$ has finite rank, this determinant must vanish for some p which implies that A^2 is a root of unity. If p denotes its order, then T_p is invertible but none of the T_{p+i}, for $i > 1$, is invertible. This implies that $\{e_0, e_1, \ldots, e_{p-1}\}$ is a basis of $\mathcal{T}_k(D^2 \times S^1) \Big/ ker\tilde{\phi}_H$. Now consider the matrix $T'_p := \left(A^{-2ij}\right)_{0 \leq i,j \leq p-1}$. A simple computation shows that $T_p T'_p = p\mathbb{1}$, thus p divides the determinant of T_p.

In the non abelian case, consider the basis of $\mathcal{K}_k(D^2 \times S^1)$ given by the elements $Q_0 := 1$ and

$$Q_n := (u_1 - \lambda_0) \ldots (u_1 - \lambda_{n-1}) \in \mathcal{K}_k(D^2 \times S^1)$$

, where the $\lambda_i, i \geq 1$ are given by Lemma 1.1.12 and we put $\lambda_0 := -(A^2 + A^{-2})$.

A simple induction shows that:

$$\phi_H(Q_n, Q_m) = \begin{cases} 0 & \text{, if } n \neq m; \\ \prod_{1 \leq i \leq n} (\lambda_i - \lambda_{i-1})\sigma_{i-1} & \text{, if } n = m. \end{cases}$$

where $\sigma_{n-1} := \sum_{i \leq n} \lambda_i$ for $n \geq 1$.

Thus if $\mathcal{K}_k(D^2 \times S^1) \Big/ ker\phi_H$ has finite rank, then there exists $n \geq 1$ so that either $\lambda_n = 0$ or $\lambda_n = \lambda_{n-1}$. Both cases imply that A^2 is a root of unity. We denote by p its order.

Now recall that in Lemma 1.1.14, we have computed the colored Jones polynomials of the Hopf link and found that:

$$\phi_H(u_i, u_j) = (-1)^{i+j}[(i+1)(j+1)]$$

Simple computations on quantum numbers imply that in the quotient $\mathcal{K}_k(D^2 \times S^1) \Big/ ker\phi_H$, we have:

1. $u_{i+p} = (-1)^p u_p$.
2. $u_{n+i} = -u_{n-i}$, if $p = 2n$.
3. $u_{n+i} = u_{n-i-1}$, if $p = 2n+1$.

If $p = 2n$, it follows that $\{u_0, u_1, \ldots, u_{\frac{p-4}{2}}\}$ is a generating set of $\mathcal{K}_k(D^2 \times S^1) \Big/ ker\phi_H$. Denote by S_p the matrix of ϕ_H in this family. A simple computation shows that:

$$(A^2 - A^{-2})^2 (S_p)^2 = -p\mathbb{1}$$

thus $\{u_0, u_1, \ldots, u_{\frac{p-4}{2}}\}$ is a basis and p divides the determinant of S_p.

If $p = 2n+1$, we have that $\{u_0, u_2, u_4, \ldots, u_{p-3}\}$ is a generating set of $\mathcal{K}_k(D^2 \times S^1) \Big/ ker\phi_H$. Still denote by S_p the matrix of ϕ_H in this family and note $N := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. A direct computation shows that $S_{2p} = S_p \otimes N$. Using the result in the even case, we conclude that S_p is invertible and that p divides its determinant.

□

1.1.3 Surgery

Dehn surgery

Definition 1.1.19. Let M be a compact oriented 3-manifold and K a framed knot in M. Choose $N(K)$ a tubular neighborhood of K in M and a homeomorphism $\phi : \partial(D^2 \times S^1) \to \partial(M - N(K))$. The 3-manifold $(M - N(K)) \bigcup_\phi (D^2 \times S^1)$ is said obtained by *Dehn surgery* from M along K.

The obtained manifold only depends, modulo preserving orientation homeomorphism, of the isotopy class of ϕ in $SL_2(\mathbb{Z})$. Fix $([l], [m])$ a basis of $H_1(\partial(D^2 \times S^1), \mathbb{Z})$ so that $[m]$ is contractible in $D^2 \times S^1$. Fix also a basis $([L], [M])$ of $H_1(\partial(N(K)), \mathbb{Z})$ so that $[M] = [K]$ is obtained by isotoping K on the boundary of $N(K)$ by conserving its framing. There exists a unique class $[\phi] \in SL_2(\mathbb{Z})$ sending $[l]$ to $[M]$ and $[m]$ to $[L]$. We denote by $M(K)$ the 3-manifold obtained by Dehn surgery from M along K using $[\phi]$. This definition extends inductively to the framed links.

A similar definition can be given in the context of 4-manifolds.

Definition 1.1.20. Let W be a compact oriented 4-manifold. A 2-*handle* h is a copy of $D^2 \times D^2$ glued to W using an embedding $\psi : (\partial D^2) \times D^2 \to \partial W$. The 4-manifold $W \bigcup_\psi h$ is said to be obtained from W by adding a 2-handle.

Again $W \bigcup_\psi h$ only depends, modulo preserving orientation homeomorphism, on the isotopy class of ψ. The gluing data of 2-handle can be obtained from a framed knot in ∂W as follows. Consider $K \subset \partial W$ a framed knot and $N(K)$ a tubular neighborhood of K in ∂W. We define a homeomorphism $\phi_K : (\partial D^2) \times (\partial D^2) \to N(K)$ like previously, that is by choosing $[K]$ as a meridian of $\partial(N(K))$ and exchanging the longitudes and meridians. The homeomorphism ϕ_K extends uniquely to a homeomorphism $\psi_K : (\partial D^2) \times D^2 \to N(K)$. We denote by $W_K := W \bigcup_{\psi_K} h$ the 4-manifold obtained by adding a 2-handle to W using ψ_K and extend inductively this definition to framed links.

The 4-manifolds D^4_L, where $L \subset S^3$ is a framed link, are called *2-handlebodies*. The two constructions are related by the following:

Proposition 1.1.21. *Let L be a framed link in $M = \partial W$, then $M(K) \cong \partial W_K$.*

When $M = \partial W_L$, the homology and signature of M are related to the linking matrix of L through the following:

Proposition 1.1.22. *Let $W = D^4_L$ be a 2-handlebody, where $L = L_1 \bigsqcup \ldots \bigsqcup L_m$ and $M = \partial W$. Denote by B the intersection form on $H_2(W, \mathbb{Z})$. We have:*

1. *The module $H_2(W, \mathbb{Z})$ is free of rank m and has basis the set $\{[L_i]\}_{1 \leq i \leq m}$.*
2. *The matrix $([L_i] \bullet [L_j])_{i,j}$ of B in this basis is equal to the linking matrix $(lk(L_i, L_j))_{i,j}$ of L. In particular they have same signature.*
3. *We have an isomorphism $H_2(M, \mathbb{Z}) \cong ker(B)$.*
4. *We have an isomorphism $H_1(M, \mathbb{Z}) \cong coker(B)$.*

We refer to [40] for detailed proofs of the previous two propositions. The two last points of the last theorem follow from the exact sequence:

$$H_3(M, W) = 0 \to H_2(M) \to H_2(W) \to H_2(W, M) \to H_1(M) \to 0 = H_1(W)$$

We have a well defined map sending a framed link $L \subset S^3$ to an oriented closed 3-manifold $S^3(L)$. Lickorish and Wallace independently showed that this map is onto. Its kernel has been described by Kirby.

Theorem 1.1.23 (Lickorish [61], Wallace [87])**.** *Every oriented connected closed 3-manifold is obtained by Dehn surgery from S^3 along a framed link (so is the boundary of a 2-handlebody).*

The following so-called Kirby moves where introduced in [50]. We give a slightly modified definition following [25].

Definition 1.1.24. Two framed links L and L' in M are related by a sequence of *Kirby moves* if they differ locally by the transformations K^+ and K^- of Figure 1.11.

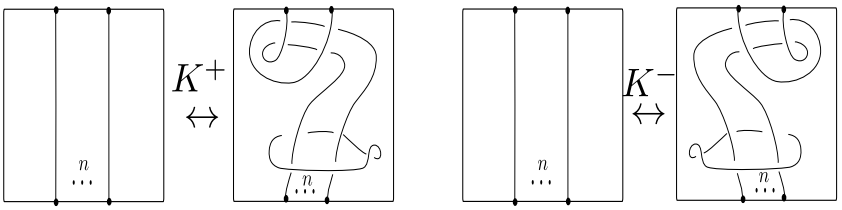


Figure 1.11: The two local transformations on links defining the Kirby moves.

We easily see that if L and L' in M are related by Kirby moves, then $M(L)$ is isomorphic to $M(L')$. We have better:

Theorem 1.1.25 (Kirby [50])**.** *Let M be a compact oriented 3-manifold and L, L' two framed links in M. The manifolds $M(L)$ and $M(L')$ are homeomorphic if and only if L and L' are related by a sequence of Kirby moves.*

As a consequence, we have a set bijection:

$$\left\{ \begin{array}{l} \text{framed links} \\ \text{in } S^3 \end{array} \right\} \Big/ \begin{array}{l} \text{isotopy} \\ \text{Kirby moves} \end{array} \leftrightarrow \left\{ \begin{array}{l} \text{oriented connected} \\ \text{closed 3-manifolds} \end{array} \right\} \Big/ \begin{array}{l} \text{preserving orientation} \\ \text{homeomorphisms} \end{array}$$

The surgery axioms for TQFTs

We will define two properties which, together with non degeneracy and the skein properties, completely characterize the TQFTs we will study in this manuscript.

We denote by $S^2 \times [0,1]$ and $-D^3 \bigsqcup D^3$ the cobordisms of $\mathrm{Hom}(S^2, S^2)$ with boundary identification by the identity map. We also note $D^2 \times S^1$ and $-S^1 \times D^2$ the cobordisms of $\mathrm{Hom}(\emptyset, S^1 \times S^1)$ where we identify the boundary with the identity map and the element $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$ respectively.

Definition 1.1.26. Let V be a TQFT with ring k. We say that V satisfies the surgery properties if:

1. There exists an invertible element $\eta \in k^\times$ so that:
$$(S1) \quad Z_V(S^2 \times [0,1]) = \eta^{-1} Z_V(-D^3 \bigsqcup D^3)$$
2. There exists $n \geq 0$, $(a_1, \ldots, a_n) \in k^n$ and $(L_1, \ldots, L_n)$ framed links in $D^2 \times S^1$, so that:
$$(S^2) \quad Z_V(-S^1 \times D^2) = \textstyle\sum_i a_i Z_V(D^2 \times S^1, L_i)$$

Remark. The surgery properties have immediate consequences. Suppose V in non degenerate and satisfies $(S1)$ and $(S2)$.

1. The property $(S1)$ implies that:
$$\langle M_1 \# M_2 \rangle_V = \eta^{-1} \langle M_1 \rangle_V \langle M_2 \rangle_V$$
where $\#$ denotes the connected sum.
Since $S^3 \# S^3 \cong S^3$, we must have:
$$\eta = \left\langle S^3 \right\rangle_V$$
Remark that $S^2 \times S^1$ is obtained from S^3 by 1-surgery, i.e. by removing two balls in S^3 and gluing $S^2 \times [0,1]$ on the boundary. This implies that:
$$\left\langle S^2 \times S^1 \right\rangle_V = 1$$
2. If V also respects one of the skein relations, denote by
$$\omega := \sum_i a_i \left(D^2 \times S^1, L_i \right)$$

the class in the skein module of $D^2 \times S^1$. If L is a framed link in S^3 and K a framed link in $S^3(L)$, then:

$$\left\langle S^3(L), K \right\rangle_V = \left\langle S^3, \phi_L(\omega, \dots, \omega) \cup K \right\rangle_V = \eta \left\langle \phi_L(\omega, \dots, \omega) \cup K \right\rangle$$

where $\langle \cdot \rangle$ denotes the skein bracket. Thus V is completely characterized by the choice of the ring k, the element ω in the skein module of $D^2 \times S^1$ and the element $\eta \in k^\times$.

3. Remark that performing Dehn surgery along the trivial framed link $\Omega_0 \subset S^3$ gives the manifold $S^2 \times S^1$. Since $\langle S^2 \times S^1 \rangle_V = 1$, we have:

$$\eta^{-1} = \langle \phi_{\Omega_0}(\omega) \rangle$$

and ω determines η.

4. Recall that if V is a non degenerate TQFT that respects the skein relations (say e.g. the Kauffman ones), then $V(\Sigma)$ is the quotient of $\bigoplus_{\partial M = \Sigma} \mathcal{K}_k(M)$ by the kernel of the gluing form as defined in the universal construction. This module is huge, thus the quotient is hard to study, but if V also satisfies the surgery axioms, then we can choose any cobordism $M \in \mathrm{Hom}(\emptyset, \Sigma)$, with arbitrary boundary identification, and $V(\Sigma)$ is the quotient of $\mathcal{K}_k(M)$ by the kernel of the corresponding gluing form. To study $V(\Sigma)$, we will choose for M a handlebody, whose skein module is described by Proposition 1.1.8.

Fix an integer $p \geq 2$ and, motivated by Proposition 1.1.6, consider the rings $k_p = \mathbb{Z}[A, \frac{1}{p}] \Big/ \phi_{2p(A)}$ and $k'_p = \mathbb{Z}[A, \frac{1}{p}] \Big/ \phi_{p(A)}$, where ϕ_p denotes the p-th cyclotomic polynomial. We will define our TQFTs on a ring in which k'_p can be injected when p is odd in the abelian case, and on which k_p is injected in the other cases.

Definition 1.1.27. Given $p \geq 2$ and $a, b \in \mathbb{Z}$, we define the Gauss sums:

1. $G(a, b, p) := \sum_{k \in \mathbb{Z}/p\mathbb{Z}} A^{ak^2+bk} \in k'_p$, when p is odd.
2. $G(a, b, 2p) := \sum_{k \in \mathbb{Z}/2p\mathbb{Z}} A^{ak^2+bk} \in k_p$.

The computation of the Gauss sums is detailed in [10]. For $x \in k_p$, denote by $\bar{x}$ the involution sending A to $A^{-1}$and write $|x|^2 := x\bar{x}$. We will use the facts that $|G(1, 0, p)|^2 = p$ when p is odd, $|G(1, 0, 2p)|^2 = 2p$ when p is even and $\overline{G(1, p, 2p)} = A^{\frac{p(p-1)}{2}} G(1, p, 2p)$. In particular, the fact that p is invertible in k_p and k'_p implies that the Gauss sums are invertible too.

Denote by t the endomorphism of the skein module of $D^2 \times S^1$ obtained by performing a positive Dehn twist along the meridian $(\partial D^2) \times \{0\}$. Lemma 1.1.12 implies that:

$$t(u_i) = \mu_i u_i$$

If b is an element of the skein module of $D^2 \times S^1$, we denote by $\langle u \rangle \in k_p$ the skein bracket of the element $\phi_{\Omega_0}(u)$. Lemma 1.1.14 implies that

$$\langle u_i \rangle = (-1)^i [i + 1]$$

Lemma 1.1.28.
Let

$$\Omega_p := \sum_{0\leq i\leq p-1} e_i \in \mathcal{T}_p(D^2\times S^1)$$

Then:

$$a := \langle t(\Omega_p)\rangle = \begin{cases} G(1,0,p) & \text{, if } p \text{ is odd.} \\ \frac{1}{2}G(1,0,2p) & \text{, if } p \text{ is even.} \end{cases}$$

Denote also

$$\Omega_p := \begin{cases} \sum_{0\leq i\leq \frac{p-4}{2}} \langle u_i\rangle\, u_i \in \mathcal{K}_p(D^2\times S^1) & \text{, when } p \text{ is even.} \\ \sum_{\substack{0\leq i\leq p-3 \\ i \text{ even}}} \langle u_i\rangle\, u_i \in \mathcal{K}_p(D^2\times S^1) & \text{, when } p \text{ is odd.} \end{cases}$$

Then:

$$a := \langle t(\Omega_p)\rangle = \frac{A^{-3}}{2(A^2-A^{-2})}G(1,p,2p)$$

Proof. In the abelian case, remark that $\langle e_i\rangle = 1$ and $t(e_i) = A^{i^2}e_i$, so:

$$\langle t(\Omega_p)\rangle = \sum_i A^{i^2} = \begin{cases} G(1,0,p) & \text{, if } p \text{ is odd.} \\ \frac{1}{2}G(1,0,2p) & \text{, if } p \text{ is even.} \end{cases}$$

In the non abelian case, remark that $\Omega_p = \frac{1}{4}\sum_{i=0}^{2p-1}\langle u_i\rangle\, u_i$ regardless of the parity of p. We compute:

$$\begin{aligned} < t(\Omega_p) > &= \frac{1}{4}\sum_{k=0}^{2p-1}\mu_k < e_k >^2 \\ &= \frac{1}{4}\frac{1}{(A^2-A^{-2})^2}\sum_{k=0}^{2p-1}(-1)^{k-1}A^{k^2-1}(A^{2k}-A^{-2k})^2 \\ &= \frac{A^{-3}}{2(A^2-A^{-2})}G(1,p,2p) \end{aligned}$$

□

Proposition 1.1.29. *If there exists a non degenerate TQFT V_p that preserves the skein relations at level p and satisfies the surgery properties, then the following equality holds in $V_p(\Sigma_1)$:*

$$\omega_p = a^{-1}\Omega_p$$

The above proposition and Remark 1.1.3 show that if there exists a non degenerate TQFT V_p on k_p that satisfies the skein and surgery properties, then it is unique. Moreover it is universal, in the sense that if V_p' is another TQFT that also satisfies the non degeneracy, skein and surgery properties at level p for some other ring k_p', then there exists a morphism between (V_p, k_p) and (V_p', k_p') as defined in Definition 1.1.1.

Proof. Remark that $U_p(S^1 \times S^1) \cong \mathcal{T}_{k_p}(D^2 \times S^1) \Big/ ker\tilde{\phi}_H$ in the abelian case and that $V_p(S^1 \times S^1) \cong \mathcal{K}_{k_p}(D^2 \times S^1) \Big/ ker\phi_H$ in the non abelian case. Thus Lemma 1.1.18 gives explicit basis for this modules. The fact that the skein bracket of a links colored by ω_p must be invariant under the Kirby moves of Definition 1.1.24, implies that:

$$\phi_H(t(\omega_p), t(b)) = \langle b \rangle \text{ , for all } b \in U_p(S^1 \times S^1) \text{ (resp. } b \in V_p(S^1 \times S^1) \tag{1.1}$$

Now remark that the non degeneracy property implies that the bilinear form $\phi_H(t(\cdot), t(\cdot))$ is non degenerate. So if ω_p does exist, then it is unique.

We now have to show that $a^{-1}\Omega_p$ is solution of equation (1.1). First compute:

$$\begin{aligned} \phi_H(t(a^{-1}\Omega_p), t(b)) &= a^{-1} \langle t(\Omega_p b) \rangle \\ &= \begin{cases} a^{-1} \sum_i \langle e_i \rangle \langle t(e_i b) \rangle & \text{, in the abelian case.} \\ a^{-1} \sum_i \langle u_i \rangle \langle t(u_i b) \rangle & \text{, in the non abelian case.} \end{cases} \end{aligned}$$

Denote by F the endomorphism of $U_p(S^1 \times S^1)$ (resp. $V_p(S^1 \times S^1)$) defined by $F(b) := \sum_i e_i t(e_i b)$ (resp $F(b) := \sum_i u_i t(u_i b)$). An induction on j shows that $F(e_j) = e_j F(1)$ (resp $F(u_j) = u_j F(1)$). Using Lemma 1.1.28, we have $\langle F(1) \rangle = a$, and:

$$\phi_H(t(a^{-1}\Omega_p), b) = a^{-1} \langle bF(1) \rangle = \langle b \rangle$$

Thus $\omega_p = a^{-1}\Omega_p$. □

We have shown that if a TQFT satisfying the non degeneracy, skein and surgery properties at level p, does exist, then it is unique. Conversely to show its existence, we first have to verify that the skein class of a link colored by ω_p is invariant under the Kirby moves.

Proposition 1.1.30. *Let $L = L_1 \cup \ldots \cup L_m$ be a framed link in S^3 and L^+, L^- be the links obtained from L by performing the Kirby moves K^+ and K^- of Figure 1.11. Then:*

$$\begin{aligned} \phi_{L^+}(\Omega, .., \Omega) &= a\phi_L(\Omega, .., \Omega) \\ \phi_{L_-}(\Omega, .., \Omega) &= \bar{a}\phi_L(\Omega, .., \Omega) \end{aligned}$$

Moreover, denote by $\sigma_+(L)$ and $\sigma_-(L)$ the number of positive and negative eigenvalues of the linking matrix of L. Then:

$$\begin{aligned} \sigma_+(L^+) = \sigma_+(L) + 1 \qquad \sigma_-(L^+) = \sigma_-(L) \\ \sigma_+(L^-) = \sigma_+(L) \quad \sigma_-(L^-) = \sigma_-(L) + 1 \end{aligned}$$

Proof. We prove the result for the Kirby move K^+ and leave the similar proof for K^- to the reader. Note $L = L_1 \cup \ldots \cup L_m$ and $L^+ = L_1^+ \cup \ldots \cup L_m^+ \cup T$, where T is the twisted knot added in the operation K^+. Choose a tubular neighborhood $N(T)$ of T. The class of $L_1^+(\omega) \cup \ldots \cup L_m^+(\omega)$ in $S^3 - N(K) \cong D^2 \times S^1$ gives an element $v \in V_p(S^1 \times S^1)$ such that $\phi_{L^+}(\Omega, \ldots, \Omega) = \phi_H(v, t(\Omega))$ and $\langle v \rangle = \langle \phi_L(t(\Omega), \ldots, t(\Omega)) \rangle$. So $\langle t^{-1}(v) \rangle = \langle \phi_L(\Omega, \ldots, \Omega) \rangle$ and:

$$\begin{aligned} \phi_{L^+}(\Omega, .., \Omega) &= < t(t^{-1}(v)), t(\Omega) >^H \\ &= < t^{-1}(v) >< t(\Omega) > \\ &= \phi_L(\Omega, .., \Omega) < t(\Omega) > \end{aligned}$$

Finally remark that the linking matrix of L^+ is bloc diagonal, with one block given by the linking matrix of L and a 1×1 bloc with value $+1$. This implies the property on the signature. □

The Proposition 1.1.30 implies that $\phi_L(\omega, \dots, \omega)$ is invariant under the Kirby moves if and only if $a = \bar{a} = 1$, which is never verified for any level. Thus:

There is no non degenerate TQFT which respects the skein relations and the surgery axioms...without anomaly

Proposition 1.1.30 still has a positive consequence: it shows that

$$\left\langle S^3(L), K \right\rangle := \frac{\langle \phi_L(\Omega, \dots, \Omega) \cup K \rangle}{a^{\sigma_+(L)} \bar{a}^{\sigma_-(L)}}$$

is invariant under the Kirby moves and gives a well defined good invariant for 3-manifolds. We can of course apply the universal construction ot this invariant but, as explained in Remark 1.1.3, the failure of the surgery axioms makes the study of the modules $V(\Sigma)$ almost impossible.

The trick is to extend the cobordism category so that cobordisms carry information that, once glued together, recover the signature of 3-manifolds. This justifies the introduction of anomaly in the next subsection.

1.1.4 TQFT with anomaly and definition of U_p, V_p

The dependance on the signature of the 3-manifold in the formulas defining the quantum invariants leads to the introduction of a so-called *anomaly*, that is we must change the cobordism category by adding some structure on the surfaces and cobordisms that permits to compute the signature by gluing such structure cobordisms. Different structures might be used such as Atiyah's 2^{nd} framing of 3 manifold ([7]), $p1$-structures ([14]), spin-structures ([15]). We will choose a structure defined by Walker ([86]), and further developed by Turaev ([83]) and studied in [38], in which surfaces are equipped with a Lagrangian in their first rational homology group, and cobordisms are equipped with an integer.

Maslov index

Let $\lambda_1, \lambda_2, \lambda_3$ be three Lagrangians of some symplectic space (H, ω). We define a symplectic bilinear form $\langle \cdot, \cdot \rangle$ on $(\lambda_1 + \lambda_2) \cap \lambda_3$ as follows. If $a = a_1 + a_2$ and $b = b_1 + b_2$ are in λ_3 with $a_1, b_1 \in \lambda_1$ and $a_2, b_2 \in \lambda_2$, then:

$$\langle a, b \rangle := \omega(a_2, b)$$

This form is well defined because the choice of a decomposition $a = a_1 + a_2$ may only differ by an element of $\lambda_1 \cap \lambda_2$ annihilated by $b \in \lambda_1 + \lambda_2 = (\lambda_1 \cap \lambda_2)^\perp$. This form is symmetric for:

$$< a, b > - < b, a >= \omega(a_2, b) - \omega(b_2, a) = \omega(a, b) - \omega(b, a) = 0$$

Definition 1.1.31. The signature of $\langle \cdot, \cdot \rangle$ (the number of positive eigenvalues minus the number of negative ones) is called the *Maslov index* $\mu(\lambda_1, \lambda_2, \lambda_3)$ of the three Lagrangians $\lambda_1, \lambda_2, \lambda_3$.

Proposition 1.1.32. *The Maslov index satisfies the following:*

1. *μ is anti-symmetric under permutation of the Lagrangians.*
2. *If H and H' are two symplectic spaces, $N : H \to H'$ is a relation and $\lambda_1, \lambda_2 \in \Lambda(H)$, $\lambda_1', \lambda_2' \in \Lambda(H')$ are Lagrangians, then:*

$$\mu(\lambda_1, \lambda_2, N^*(\lambda_1')) + \mu(N_*(\lambda_1), \lambda_1', \lambda_2') = \mu(\lambda_1, \lambda_2, N^*(\lambda_2')) + \mu(N_*(\lambda_2), \lambda_1', \lambda_2')$$

We refer to [85] for a proof.

TQFT with anomaly

We will consider the symplectic spaces $H_1(\Sigma, \mathbb{Q})$, where Σ is a closed oriented surface, together with the intersection form.

A compact oriented 3-manifold M gives rise to a Lagrangian λ_M of $H_1(\partial M, \mathbb{Q})$ defined as the kernel of the inclusion map $i_\Sigma : H_1(\partial M, \mathbb{Q}) \to H_1(M, \mathbb{Q})$.

The *structure cobordisms category* $\underline{\mathrm{Cob}}_{2+1}$ is defined as follows.

The objects are pairs $\underline{\Sigma} = (\mathbf{\Sigma}, \lambda)$ called *structure surfaces*, where $\mathbf{\Sigma}$ is a banded surface and λ a Lagrangian in $H_1(\Sigma, \mathbb{Q})$.

The morphisms of $\mathrm{Hom}(\underline{\Sigma}_1, \underline{\Sigma}_2)$ are pairs $\underline{M} = (M, w)$, called *structure cobordisms*, where M is a banded cobordism, and w an integer called *weight*. Such a structure cobordism defines a Lagrangian λ_M of $H_1(\Sigma_1, \mathbb{Q})$ by setting $\lambda_M := i_{\Sigma_1}^{-1}(i_{\Sigma_2}(\lambda_2))$.

If $(M_1, w_1) \in \mathrm{Hom}(\underline{\Sigma}_0, \underline{\Sigma}_1)$ and $(M_2, w_2) \in \mathrm{Hom}(\underline{\Sigma}_1, \underline{\Sigma}_2)$, the composition is defined by:

$$(M_1, w_1) \circ (M_2, w_2) = (M_1 \cup_\Sigma M_2, w_1 + w_2 - \mu(\lambda_{M_1}, \lambda, \lambda_{M_2}))$$

where $\underline{\Sigma}_1 = (\Sigma_1, \lambda)$. Proposition 1.1.32 ensures that this law is associative and commutative.

We endow $\underline{\mathrm{Cob}}_{2+1}$ with a tensor product and a duality by setting:

$$(\Sigma, \lambda)^* := (-\Sigma, -\lambda) \qquad (M, w)^* := (-M, w)$$
$$(\Sigma_1, \lambda_1) \bigsqcup (\Sigma_2, w_2) := (\Sigma_1 \bigsqcup \Sigma_2, \lambda_1 + \lambda_2) \quad (M_1, w_1) \bigsqcup (M_2, w_2) := (M_1 \bigsqcup M_2, w_1 + w_2)$$

We define similarly the oriented structure cobordisms category $\underline{\mathrm{Cob}}^{\mathrm{o}}_{2+1}$.

Definition 1.1.33. A TQFT with anomaly on a ring k is a functor $V : \underline{\mathrm{Cob}}_{2+1} \to \mathrm{Mod}_k$ (when non oriented) or $U : \underline{\mathrm{Cob}}^{\mathrm{o}}_{2+1} \to \mathrm{Mod}_k$ (when oriented), which respects tensor product and duality.

We define the morphisms between TQFTs with anomaly and the skein properties in the same manner than in the case without anomaly. Proposition 1.1.6 still holds so we can speak of the level of the TQFT.

A TQFT with anomaly defines a quantum invariant defined on the set of homeomorphisms classes of colored oriented 3-manifolds with a framed link, endowed with an integer.

If $\underline{\Sigma} = (\Sigma, w)$ is a structure surface, the group of structure cylinders $\underline{C}(\phi, w) := (C(\phi), w) \in \mathrm{Hom}(\underline{\Sigma}, \underline{\Sigma})$, where $\phi \in \mathrm{Mod}(\Sigma)$ and $w \in \mathbb{Z}$, will be denoted $\underline{\mathrm{Mod}}(\Sigma, \lambda)$. It has group law given by:

$$\underline{C}(\phi_1, w_1) \circ \underline{C}(\phi_2, w_2) = \underline{C}(\phi_1 \circ \phi_2, w_1 + w_2 - \mu(\phi_{1*}(\lambda), \lambda, \phi_{2*}^{-1}(\lambda)))$$

And we have a natural exact sequence:

$$0 \to \mathbb{Z} \to \underline{\mathrm{Mod}}(\Sigma, \lambda) \to \mathrm{Mod}(\Sigma) \to 1$$

where the second map is defined by $\underline{C}(\phi, w) \to C(\phi)$ whose kernel, homeomorphic to $\mathbb{Z}$, is generated by the central element $\underline{C}(\mathbb{1}, 1)$. We can show that $\underline{\mathrm{Mod}}(\Sigma, \lambda)$ does not depend, modulo isomorphism, on the choice of λ.

A TQFT V with anomaly gives quantum representations $\rho_V : \underline{\mathrm{Mod}}(\Sigma, \lambda) \to \mathrm{GL}(V(\underline{\Sigma}))$ defined by $\rho_V(\underline{C}(\phi, w)) := V(\underline{C}(\phi, w))$. They define **projective** representations of the mapping class group $\mathrm{Mod}(\Sigma)$.

Eventually we extend the notion of non degeneracy and the universal construction to TQFTs with anomaly.

Structure surgery

Consider the structure surface $\underline{S^1 \times S^1}$ with Lagrangian given by the meridian $\{0\} \times S^1$. Note $\underline{-S^1 \times D^2} \in \mathrm{Hom}(\emptyset, \underline{S^1 \times S^1})$ the structure cobordism with weight $w = 0$ and boundary identification with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$.

Definition 1.1.34. Let $\underline{M} = (\mathbf{M}, w)$ be a structure cobordism and $K \subset M$ a framed knot. Denote by $N(K)$ a tubular neighborhood of K. The structure cobordism

$$\underline{M}(K) := (M - N(K), w) \circ \underline{-S^1 \times D^2}$$

is said obtained from $\underline{M}$ by *structure surgery* along K.

By definition, $\underline{M}(K) = (M(K), w + w')$ where w' is a Maslov index on $H_1(S^1 \times S^1, \mathbb{Q})$. The space of Lagrangians on this space is one dimensional, thus w' is equal to $-1, 0$ or $+1$.

We extend inductively this definition to structure surgery on framed links.

The following theorem justifies the rather complicated and, until now, poorly motivated notions we introduced in this subsection.

Theorem 1.1.35 (Walker ([86])). *If L is a framed link in S^3 and $\underline{S^3} = (S^3, 0)$, then:*

$$\underline{S^3}(L) = (S^3(L), \sigma(L))$$

where $\sigma(L) = \sigma^+(L) - \sigma^-(L)$ is the signature of $S^3(L)$.

We can restate the surgery proposition:

Definition 1.1.36. A TQFT (V, k) with anomaly satisfies the surgery properties if:

1. There exists an invertible element $\eta \in k^\times$, so that:

$$(S_1): \quad V(S^2 \times [0,1], 0) = \eta^{-1} V(-D^3 \bigsqcup D^3, 0)$$

2. There exists $n \geq 0$, $(a_1, \ldots, a_n) \in k^n$ and $(L_1, \ldots, L_n)$ framed links in $D^2 \times S^1$, so that:

$$(S_2): \quad Z(-S^1 \times D^2, \oslash, 0) = \textstyle\sum_i a_i Z(D_2 \times S^1, L_i, 0)$$

Remark. Let (V, k) be a TQFT with anomaly satisfying the surgery properties.

1. The space $H_1(S^2 \bigsqcup S^2, \mathbb{Q})$ has null dimension, thus the Maslov index involved in doing 1-surgery is null. As a consequence, the quantum invariant satisfies:

$$\langle (M_1 \# M_2, 0) \rangle_V = \langle (M_1, 0) \rangle_V \langle (M_2, 0) \rangle_V$$

   So we recover the facts that $\eta = \langle (S^3, 0) \rangle_V$ and $1 = \langle (S^2 \times S^1, 0) \rangle_V$.

2. If $\partial M = \emptyset \bigsqcup \Sigma$, consider the composition along the empty surface: $(S^3, 1) \circ (M, w) = (M, w + 1)$, where the Maslov index is null. Setting:

$$\kappa := \left\langle (S^3, 1) \right\rangle_V \eta^{-1} \in k$$

   we have: $Z(M, w) = \kappa^w Z(M, 0)$.

3. If V satisfies one of the skein relations, denote by ω the class in the skein module of $\sum_i a_i L_i$. Theorem 1.1.35 implies that:

$$\left\langle (S^3, \phi_L(\omega), 0) \right\rangle_V = \kappa^{\sigma(L)} \left\langle S^3(L), \emptyset, 0 \right\rangle_V$$

   The term $\kappa^{\sigma(L)}$ is the missing part we needed to have an element invariant under Kirby moves.

Definition of the TQFTs U_p and V_p

Rephrasing the reasoning of the previous subsection, suppose that (V, k) is a non degenerate TQFT with anomaly that satisfies the skein and surgery properties.

The element ω is still completely determined by the equality:

$$\phi_H(t(\omega), t(b)) = \langle b \rangle, \text{ for all } b \in V(S^1 \times S^1)$$

This time, we must take into account the Maslov index appearing when we glue the two solid torus in $\phi_H(t(\cdot), t(\cdot))$ by identifying a twisted meridian with a twisted longitude. This index is equal to $+1$ and we find:

$$\omega = \kappa a^{-1}\Omega$$

We then use the fact that doing surgery in S^3 along the trivial knot twisted once negatively Ω_{-1}, gives again S^3. The Maslov index of the gluing is $+1$, so $\eta = \langle (S^3, 0) \rangle_V = \kappa^{-1} \langle (S^3, \Omega_{-1}(\omega), 0) \rangle_V = \eta\kappa\frac{\bar{a}}{a}$. Thus:

$$\kappa^2 = \frac{a}{\bar{a}} = \begin{cases} A^{-\frac{p(p-1)}{2}} & \text{, in the abelian case.} \\ A^{-6-\frac{p(p+1)}{2}} & \text{, in the non abelian case.} \end{cases}$$

Finally, doing surgery along the trivial framed knot $\Omega_0 \subset S^3$ gives $S^2 \times S^1$ and involves a null Maslov index. So $1 = \langle (S^2 \times S^1, \emptyset, 0) \rangle_V = \langle (S^3, \phi_{\Omega_0}(\omega), 0) \rangle_V = \eta(\kappa a^{-1})(a\bar{a}) = \eta a \kappa^{-1}$. Thus:

$$\eta = \kappa a^{-1}$$

Eventually, we proved that if a non degenerate TQFT with anomaly satifies the skein and surgery properties at some level p, then it is unique and its quantum invariant on connected manifolds is given by the formula:

$$\begin{aligned} \langle (M, K, w) \rangle_V &= \kappa^{w-\sigma(L)}\eta < \phi_L(\omega, .., \omega) \cup K > & (1.2) \\ &= (\kappa^w\eta)\eta^{\sigma^+ + \sigma^- + \sigma_0}\kappa^{\sigma^+ - \sigma^-} < \phi_L(\Omega, .., \Omega) \cup K > & (1.3) \\ &= (\kappa^w\eta)\eta^{b_1(M)} \frac{< \phi_L(\Omega, .., \Omega) \cup K >}{a^{\sigma^+}\bar{a}^{\sigma^-}} & (1.4) \end{aligned}$$

where we used that $\sigma_0(L) = b_1(M)$ is the first Betti number of M (see Proposition 1.1.22). Proposition 1.1.30 shows that this expression is invariant under Kirby moves on L, so the invariant is well defined. We extend it multiplicatively to non connected manifolds to get a good invariant.

Definition 1.1.37.

1. Let $p \geq 2$ and $k'_p := \begin{cases} \mathbb{Z}[A, \frac{1}{p}, \kappa] \Big/ \left(\phi_{2p}(A), \kappa^2 - A^{-\frac{p(p-1)}{2}} \right) & \text{, if } p \text{ is even.} \\ \mathbb{Z}[A, \frac{1}{p}, \kappa] \Big/ \left(\phi_p(A), \kappa^2 - A^{-\frac{p(p-1)}{2}} \right) & \text{, if } p \text{ is odd.} \end{cases}$
   We define the abelian TQFTs (U_p, k'_p) using the universal construction on the invariant given by the formula (1.4), where:
   – $\Omega = \sum_{0 \leq i \leq p-1} e_i$,
   – $a = \begin{cases} G(1, 0, p) & \text{, if } p \text{ is odd.} \\ \frac{1}{2}G(1, 0, 2p) & \text{, if } p \text{ is even.} \end{cases}$
   – $\eta = \kappa a^{-1}$.

2. Let $p \geq 3$ and $k_p := \mathbb{Z}[A, \frac{1}{p}, \kappa] \Big/ \left(\phi_{2p}(A), \kappa^2 - A^{-6-\frac{p(p+1)}{2}}\right)$.
   We define the TQFTs (V_p, k_p) using the universal construction on the invariant given by the formula (1.4), where:
   - $\Omega = \begin{cases} \sum_{0 \leq i \leq \frac{p-4}{2}} \langle u_i \rangle \, u_i \in \mathcal{K}_p(D^2 \times S^1) & \text{, when } p \text{ is even.} \\ \sum_{\substack{0 \leq i \leq p-3 \\ i \text{ even}}} \langle u_i \rangle \, u_i \in \mathcal{K}_p(D^2 \times S^1) & \text{, when } p \text{ is odd.} \end{cases}$
   - $a = \frac{A^{-3}}{2(A^2 - A^{-2})} G(1, p, 2p)$.
   - $\eta = \kappa a^{-1}$.

To show that the above functors are indeed TQFTs, we still have to show that:

1. The modules $V_p(\Sigma), U_p(\Sigma)$ have finite rank.
2. The gluings $\langle \cdot, \cdot \rangle_{\Sigma}$ are non degenerate.
3. The maps $V(\Sigma_1) \otimes V(\Sigma_2) \to V(\Sigma_1 \bigsqcup \Sigma_2)$ are isomorphisms.

It will be proved in the next subsection.

Remark that we proved that if $(\tilde{V}_p, \tilde{k}_p)$ is a non degenerate TQFT with anomaly that satisfies the Kauffman skein relations and surgery properties, then there exists an injective ring morphism $\nu : k_p \to \tilde{k}_p$ and a natural map from $\nu^* \circ V_p$ to $\tilde{V}_p$ that induces isomorphisms on each $\tilde{k}_p$-module $\tilde{V}_p(\Sigma)$.

In short, the TQFTs (V_p, k_p) and (U_p, k_p) are uniquely defined by the non degeneracy, skein, level and surgery condition, modulo a change of coefficient.

1.1.5 Study of the spaces $U_p(\Sigma), V_p(\Sigma)$

The abelian case

Denote by Σ_g a genus g closed oriented surface, by H_g a genus g oriented handlebody and by $U_{p,g}$ the module $U_p(\Sigma_g)$. Since H_g is isomorphic to the connected sum of g copies of $D^2 \times S^1$, the manifold $H_g \bigcup_{id} H_g$, glued using the identity map $id : \partial H_g \to \partial H_g$, is isomorphic to the connected sum of g copies of $S^2 \times S^1$. Since H_g is a thickened g-holed disc, Proposition 1.1.8 implies that its abelian skein module is $\mathcal{T}_p(H_g) \cong k_p\left[H_1(H_g, \mathbb{Z})\right] \cong k_p[\mathbb{Z}^g]$.

Denote by $\{e_{i_1} \otimes \ldots \otimes e_{i_g}, (i_1, \ldots, i_g) \in \mathbb{Z}^g\}$ its basis given by i_k parallel copies of the k-th oriented longitude as drawn in Figure 1.12. Note $\langle \cdot, \cdot \rangle_p^R$ the bilinear form on $\mathcal{T}_p(H_g)$ induced by the gluing $H_g \cup_{id} H_g$ using the identity map.

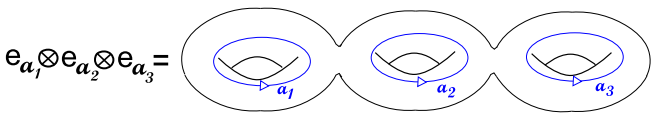


Figure 1.12: A basis for $\mathbb{C}\left[H_1(H_3, \mathbb{Z})\right]$.

By construction, $U_{p,g}$ is isomorphic to the quotient of $\mathcal{T}_p(H_g)$ by the kernel of $\langle \cdot, \cdot \rangle_p^R$.

Lemma 1.1.38. *Let $e'_i \in \mathcal{T}_p(S^2 \times S^1)$ be the skein class of i parallel copies of $\{0\} \times S^1$. Then:*

$$\left\langle (S^2 \times S^1, e'_i, 0) \right\rangle_{U_p} = \begin{cases} \eta^2 p & \text{, if } p \text{ divides } i; \\ 0 & \text{, else.} \end{cases}$$

Proof. Remark that $S^2 \times S^1$ is obtained by surgery from S^3 along the trivial framed knot and that the gluing $\langle \cdot, \cdot \rangle_p^R$ induces a null Maslov index. We thus have:

$$\begin{aligned} \left\langle (S^2 \times S^1, e'_i, 0) \right\rangle_{U_p} &= \eta \langle \phi_H(\omega, e_i) \rangle \\ &= \eta^2 \sum_{k=0}^{p-1} \langle \phi_H(e_k, e_i) \rangle \\ &= \eta^2 \sum_{k=0}^{p-1} A^{2ki} \end{aligned}$$

which permits to conclude. □

Still denote by $e_{i_1} \otimes \ldots \otimes e_{i_k}$ the class in $\mathcal{T}_p(H_g) \Big/ ker \langle \cdot, \cdot \rangle_p^R$ of the same element.

Theorem 1.1.39. *The vectors of $\{e_{i_1} \otimes \ldots \otimes e_{i_g}, (i_1, \ldots, i_g) \in (\mathbb{Z}/p\mathbb{Z})^g\}$ form a basis of $U_{p,g}$ orthogonal for $\langle \cdot, \cdot \rangle_p^R$. And:*

$$\langle e_{i_1} \otimes \ldots \otimes e_{i_g}, e_{i_1} \otimes \ldots \otimes e_{i_g} \rangle_p^R = (\eta p)^g$$

In particular, the $(U_p)_{p \geq 2}$ are TQFTs.

Proof. Using the 1-surgery property $(S1)$, we get:

$$\langle e_{i_1} \otimes \ldots \otimes e_{i_g}, e_{j_1} \otimes \ldots \otimes e_{j_g} \rangle_p^R = (\eta)^{-g} \prod_{1 \leq k \leq g} \left\langle (S^2 \times S^1, e'_{i_k - j_k}, 0) \right\rangle_{U_p}$$

We conclude using Lemma 1.1.38. □

Colored surfaces and the extended cobordisms categories

The isomorphism $U(\Sigma_1 \# \Sigma_2) \cong U(\Sigma_1) \otimes_k U(\Sigma_2)$ no longer holds in the non abelian case. It is replaced by a tensor formula given in Theorem 1.1.43 that permits to study any modules $V(\Sigma)$ by cutting Σ into elementary cobordisms. To state our formula, we need to add objects to the cobordisms category and extend the TQFT functors to this category. The objects added will be structure surfaces where the bands are colored by integers.

Definition 1.1.40. Let $p \geq 3$ and $I_p := \begin{cases} \{0, 1, 2, \ldots, r-2\} & \text{, if } p = 2r \text{ is even.} \\ \{0, 2, 4, \ldots, p-3\} & \text{, if } p \text{ is odd.} \end{cases}$

We define the *extended cobordism category* $\underline{\text{Cob}}_\text{p}$ at level p as follows.

The objects are couples $\underline{\Sigma}^c = (\underline{\Sigma}, c)$, called *colored surfaces*, where $\underline{\Sigma}$ is a structure surface with n bands and $c : \{1, \ldots, n\} \to I_p$ is a coloring of the bands by integers of I_p.

The *expansion* $e(\underline{\Sigma}^c)$ of a colored surface is the structure surface obtained from $\underline{\Sigma}$ by replacing each band b_i with color c_i, by a set of c_i bands.

If M bounds Σ, consider $\mathcal{K}_p(M)$ the quotient of the free k_p-module generated by properly embedded framed links in M intersecting ∂M along the bands of $e(\underline{\Sigma}^c)$ quotiented by the Kauffman relations. The algebra $TL_{c_1} \otimes \ldots \otimes TL_{c_n}$ naturally acts on $\mathcal{K}_p(M)$. Define the idempotents:

$$\epsilon_c := f_{c_1} \otimes \ldots \otimes f_{c_n}$$

We call skein module of M^c the image:

$$\mathcal{K}_p(M^c) := \epsilon_c \cdot \mathcal{K}_p(M)$$

The set of morphisms $\mathrm{Hom}(\underline{\Sigma}_1^{c_1}, \underline{\Sigma}_2^{c_2})$ is the k_p-module:

$$\mathrm{Hom}(\underline{\Sigma}_1^{c_1}, \underline{\Sigma}_2^{c_2}) := \oplus_{\underline{M} \in \mathrm{Hom}(e(\underline{\Sigma_1}^{c_1}), e(\underline{\Sigma_2}^{c_2}))} \epsilon_{c_1} \cdot \mathcal{K}_p(M) \cdot \epsilon_{c_2}$$

There is a monomorphism $I : \underline{\mathrm{Cob}}_{2+1} \to \underline{\mathrm{Cob}}_{\mathrm{p}}$ obtained by coloring the bands of structure surfaces with the color 1 is p is even and by $p-3$ if p is odd.

The universal construction applied to $\underline{\mathrm{Cob}}_{\mathrm{p}}$ gives a functor $V_p : \underline{\mathrm{Cob}}_{\mathrm{p}} \to \mathrm{Mod}_{k_p}$ such that $V_p \circ I$ is the TQFT defined in the previous subsection.

We can think of the skein module $\mathcal{K}_p(M^c)$ as the quotient of $\mathcal{K}_p(M)$ where we quotient by isotopy classes of links in M that connect two bands of the same set of c_i bands coming from the expansion of the same band colored by c_i.
The module $V_p(\underline{\Sigma}^c)$ is the quotient of $\mathcal{K}_p(M^c)$ by the kernel of the gluing form.

Definition 1.1.41. A triple $(i,j,k) \in (I_p)^3$ is said p-admissible if:
- $|i-j| \leq k \leq i+j$,
- $i+j+k$ is even and smaller than $p-2$ if p is even and $2p-2$ if p is odd.

Lemma 1.1.42.

1. *If $S^2(i,j)$ denotes the colored structured surface made of the sphere S^2 with two bands colored by i and j in I_p, then $V_p(S^2(i,j))$ has rank 1 if $i=j$ and is null if $i \neq j$.*
2. *If $S^2(i,j,k)$ denotes the sphere S^2 with three bands colored by i,j and k in I_p, then $V_p(S^2(i,j,k))$ has rank one if (i,j,k) is p-admissible and is null elsewhere.*

Proof. Fix a ball B^3 bounded by S^2.

If $i \neq j$ in $S^2(i,j)$ or if (i,j,k) is not admissible in $S^2(i,j,k)$, there is no link in the corresponding expansion which does not connect two bands coming from the expansion of the same colored band. Thus the corresponding skein modules are null.

If $i=j$ or if (i,j,k) is admissible, then there exists exactly one such link which is planar. The rank of the corresponding skein module is thus equal to one.

Consider $f_i \in V_p(S^2(i,i))$ the class of the i-th Jones-Wenzl idempotent inserted in B^3. Using Proposition 1.1.14, we get:

$$\langle f_i, f_i \rangle_p = \eta(-1)^i[i+1] \neq 0$$

thus $V_p(S^2(i,i))$ has rank one generated by f_i.

When (i,j,k) is admissible, consider $u_\sigma \in V_p(S^2(i,j,k))$ the class of the link $T_{i,j,k}$ of Figure 1.9 connecting three idempotents f_i, f_j and f_k in the sphere. Then:

$$\langle u_\sigma, u_\sigma \rangle_p = \eta \, \langle i,j,k \rangle$$

where $\langle i,j,k \rangle$ is the corresponding theta coefficient. It follows from Proposition 1.1.16 that this coefficient is non null if and only if (i,j,k) is p-admissible. This concludes the proof. □

Non abelian splitting theorem and study of $V_p(\Sigma)$

Let $\Gamma = \Gamma_1 \bigsqcup \ldots \bigsqcup \Gamma_m$ be a multicurve in Σ, so that $\Sigma - \Gamma$ has two connected components Σ_1 and Σ_2. If $i = (i_1, \ldots, i_m) \in (I_p)^m$, denote by Σ_1^i and Σ_2^i the colored surfaces obtained by gluing to Σ_1 and Σ_2 a disc with a band colored by i_k along each boundary curve Γ_k. The following theorem justifies the introduction of colored surfaces:

Theorem 1.1.43. *The gluing of colored cobordisms induces an isomorphism:*

$$V_p(\Sigma) \cong \oplus_{i \in (I_p)^m} V_p(\Sigma_1^i) \otimes_{k_p} V_p(\Sigma_2^i)$$

Before proving Theorem 1.1.43, we state its consequences. We fix a colored structure surface Σ with a handlebody H bounding Σ and a trivalent banded graph Γ properly embedded in H so that $\partial\Gamma \subset \Sigma$ consists of the colored bands of Σ. We denote by $col_p(\Gamma)$ the set of maps $\sigma : E(\Gamma) \to I_p$ so that:

– If (e_1, e_2, e_3) are three adjacent edges, then $(\sigma(e_1), \sigma(e_2), \sigma(e_3))$ is p-admissible.
– If $e \in E(\Gamma)$ crosses a band of Σ of color c_e, then $\sigma(e) = c_e$.

If Σ is a torus $S^1 \times S^1$ with no bands, we define Γ to be the graph with no vertices given by the meridian $\{0\} \times S^1 \in D^2 \times S^1$. If Σ is a sphere with two bands, Γ is the graph with one edge connecting the two bands in B^3.

As explained in subsection 1.1.2, to a graph Γ together with a coloring $\sigma \in col_p(\Gamma)$, we can associate a vector $u_\sigma \in V_p(\Sigma)$.

Theorem 1.1.44. *The set of elements $\{u_\sigma, \sigma \in col_p(\Gamma)\}$ forms a basis of $V_p(\Sigma)$, orthogonal for the form $\langle \cdot, \cdot \rangle_p^R$. Moreover we have:*

$$< u_\sigma, u_\sigma >_p = \eta^{\#v - \#e} \frac{\prod_v < \sigma(v) >}{\prod_e < \sigma(e) >}$$

where v runs through the vertices of Γ, e through its edges, $< \sigma(v) >$ is the corresponding theta coefficient and $< \sigma(e) > = (-1)^{\sigma(e)}[\sigma(e)+1]$.

This theorem implies that the functors V_p are TQFTs.

Proof. To each edge $e \in E(\Gamma)$, we properly embed a disc D_e in H intersecting Γ once transversely along e and we note $\gamma_e := \partial D_e \subset \Sigma$. Theorem 1.1.43 applied to Σ with the multicurve $\gamma = \bigsqcup_e \gamma_e$ together with Lemma 1.1.42, imply that $\{u_\sigma, \sigma \in col_p(\Gamma)\}$ forms an orthogonal basis. To show the formula for $\langle u_\sigma, u_\sigma \rangle_p^R$, we first remark, using Lemma 1.1.42, it is true for the elementary cobordisms given by a graph with shape Y for the three-banded spheres, a graph with shape I for the twice banded spheres and an O-shape graph for the torus $S^1 \times S^1$ with no bands. Then remark that the formula for $\langle u_\sigma, u_\sigma \rangle_p^R$ stated in the theorem is unchanged when we identify two vertices of degree 2 to have a edge. We conclude by induction on the number of elementary cobordisms in a given decomposition of Σ. □

Proof of Theorem 1.1.43

Fix M a colored cobordism bounding $\underline{\Sigma}^c$ in which the curves Γ_i bound discs $D = D_1 \bigsqcup \ldots \bigsqcup D_m$ so that, when cutting M along D, we get two cobordisms M_1 and M_2 with $M \cong M_1 \cup_D M_2$ and $\partial M_i = \Sigma_i$. As in the abelian case, we denote by $\langle \cdot, \cdot \rangle_p^R$ the gluing form obtained by gluing M with $-M$ along the identity map to get the double of M. We split the proof of Theorem 1.1.43 into two lemmas which easily imply the theorem.

Lemma 1.1.45. *The gluing of cobordisms induces an isomorphism of k_p-modules:*

$$F : \oplus_{i \in (I_p)^m} \mathcal{K}_p(M_1^i) \otimes_{k_p} \mathcal{K}_p(M_2^i) \to \mathcal{K}_p(M)$$

Proof. Note $\mathcal{A} := \mathcal{K}_p(M)$, $\mathcal{C} := \oplus_{i \in (I_p)^m} \mathcal{K}_p(M_1^i) \otimes_{k_p} \mathcal{K}_p(M_2^i)$ and introduce the module:

$$\mathcal{B} := \left(\oplus_i \mathcal{K}_p(M_1, i) \otimes \mathcal{K}_p(M_2, i)\right) /_\sim$$

where:

- $\mathcal{K}_p(M_a, i)$ denotes the skein module of M_a with i bands colored by 1.
  The modules $\mathcal{K}_p(M_1, i)$ and $\mathcal{K}_p(M_2, i)$ are equipped with a structure of right and left $TL_i(\Gamma)$ module respectively, with $TL_i(\Gamma) := TL_{i_1} \otimes \ldots \otimes TL_{i_m}$.
- The module $\mathcal{B}$ is defined as a quotient for the equivalence relation $ax \otimes b \sim a \otimes xb$ where $a \in \mathcal{K}_p(M_1, i)$, $b \in \mathcal{K}_p(M_2, i)$ and $x \in TL_i(\Gamma)$.

A morphism $f : \mathcal{B} \to \mathcal{A}$ is given by gluing cobordisms along D. To construct a reverse map $f^{-1} : \mathcal{A} \to \mathcal{B}$, choose a link $L \subset M$ isotoped to be transverse to D. It intersects each disc D_n a number i_n of times. Putting $i = (i_1, \ldots, i_m)$ and cutting M along D, we get an element of $\mathcal{K}_p(M_1, i) \otimes \mathcal{K}_p(M_2, i)$. The indetermination on the way we isotoped L in M before cutting disappears in the quotient by $\sim$.

The canonical embedding of $\mathcal{K}_p(M_a^i)$ in $\mathcal{K}_p(M_a, i)$ defines a morphism $g : \mathcal{C} \to \mathcal{B}$. We construct a reverse map as follows. Denote by ${}_aTL_b$ the quotient of the free k_p-module generated by isotopy classes of (a, b) tangles by the Kauffman skein relations (so ${}_nTL_n = TL_n$). Fix $n \geq 1$, there exists elements $a_i \in_n TL_i$ and $b_i \in_i TL_n$ so that:

$$1_n = \sum_i a_i f_i b_i \in TL_n$$

Similarly for $c = (c_1, \ldots, c_m)$ and $i \in (I_p)^m$, we have elements $a_i \in_c TL(Gamma)_i$ and $b_i \in_i TL(\Gamma)_c$ so that:

$$1_c = \sum_i a_i \epsilon_i b_i \in TL_c(\Gamma)$$

where $1_c = 1_{c_1} \otimes \ldots \otimes 1_{c_m}$. The reverse map $g^{-1} : \mathcal{B} \to \mathcal{C}$ is given by

$$g^{-1}(x \otimes y) := \sum_i x a_i \epsilon_i \otimes_{k_p} \epsilon_i b_i y, \text{ for } (x, y) \in \mathcal{K}_p(M_1^i) \otimes \mathcal{K}_p(M_2^i)$$

We thus have defined a reverse map $g^{-1} \circ f^{-1}$ for $F = f \circ g$.

□

Lemma 1.1.46. *Under the isomorphism F^{-1} of Lemma 1.1.45, the gluing form decomposes as :*

$$<,>_{\Sigma}^{R} \sim \oplus_{i \in (I_p)^m} <,>_{\Sigma_1^i}^{R} \otimes_k <,>_{\Sigma_2^i}^{R}$$

Proof. In the gluing $M \cup_\Sigma -M$, the discs $D^i \cup -D^i$ are glued along their boundaries to give spheres. Note $S := \bigsqcup_{i=1}^m D^i \cup -D^i$ the disjoint union of these spheres and $Y \subset M \cup_\Sigma -M$ a tubular neighborhood of S, homeomorphic to $S \times [0, 1]$. Let X denote the complementary of Y so that $M \cup_\Sigma -M \cong X \cup_{S \times \{0\} \bigsqcup S \times \{1\}} Y$. Apply Lemma 1.1.45 to this decomposition. We get:

$$\begin{aligned} \mathcal{K}_p(M \cup_\Sigma -M) &\cong \oplus_{i,j \in (I_p)^m} \mathcal{K}_p(X_{i,j} \otimes \mathcal{K}_p({}_{i,j}Y) \\ &\cong \oplus_{i \in (I_p)^m} \mathcal{K}_p(X_{i,i}) \otimes \mathcal{K}_p({}_{i,i}Y) \end{aligned}$$

where $X_{i,j}$ is the union of spheres $D_n \cup -D_n$ with a band colored by i_n in D_n and a band colored by j_n in $-D_n$. The second isomorphism states for $\mathcal{K}_p(X_{i,j})$ is null if $i \neq j$.

Denote by X_i the gluing of $X_{i,i}$ with ${}_{i,i}Y$ and by $G : \mathcal{K}_p(M \cup_\Sigma -M) \to \oplus_i X_i$ the resulting isomorphism. We conclude the proof by noticing that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{K}_p(M) \times \mathcal{K}_p(M) & \xrightarrow[\sim]{F \times F} & (\oplus_i \mathcal{K}_p(M_i^1) \otimes_k \mathcal{K}_p({}_iM^2))^{\times 2} \\ \downarrow R & & \downarrow \tilde{R} \\ \mathcal{K}_p(-M \cup_\Sigma M) & \xrightarrow[\sim]{G} & \oplus_i \mathcal{K}_p(X_{i,i}) \\ & \langle\cdot\rangle_p \searrow \quad \swarrow \langle\cdot\rangle_p & \\ & k_p & \end{array}$$

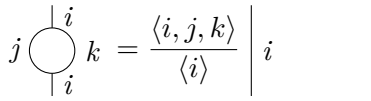
where $\tilde{R}$ is induced by the gluing $M \cong M_1 \cup M_2$. □

Fusion rules

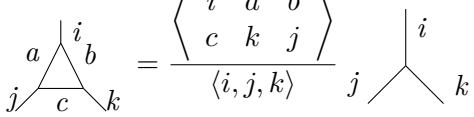
If $\Gamma \subset S^3$ is a planar trivalent graph and $\sigma \in col_p(\Gamma)$ is a p-admissible coloring, the couple (Γ, σ) defines an element in $\mathcal{K}_p(S^3) \cong k_p$ which can be computed using the following:

Lemma 1.1.47. *Let (i,j,k) be a p-admissible triple.*

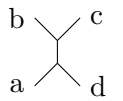
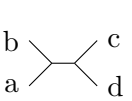
1. *In $V_p(S^2(i,i))$, we have:*



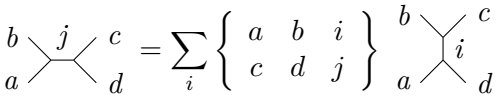
2. *In $V_p(S^2(i,j,k))$, we have:*



Proof. We proved in Lemma 1.1.42 that both modules $V_p(S^2(i,i))$ and $V_p(S^2(i,j,k))$ have rank one, so the vectors on the left and right hand side of both equalities differ by a scalar in k_p. We compute this scalar by taking the scalar product of both vectors with the vector of the right hand side. □

Consider the sphere S^2 with four bands colored by a, b, c and d and a ball B^3 bounded by S^2. Using the trivalent graphs  and  we get two basis of $V_p(S^2(a,b,c,d))$. They are related through the following:

Lemma 1.1.48. *If*

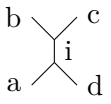


then:

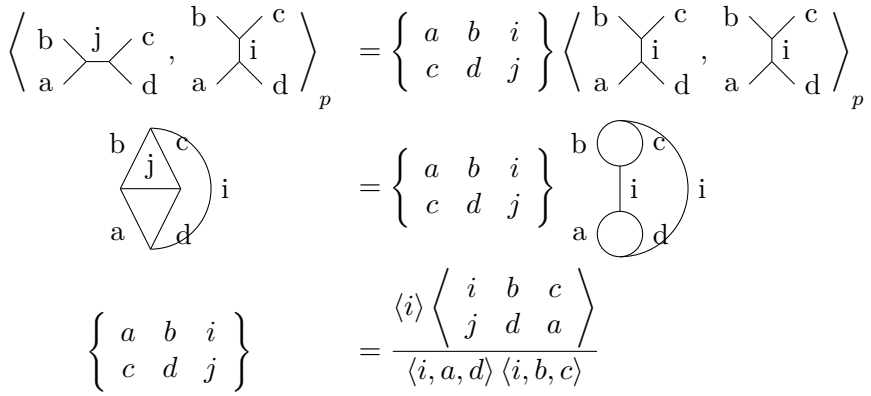
$$\left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\} = \frac{\langle i \rangle \left\langle \begin{array}{ccc} i & b & c \\ j & d & a \end{array} \right\rangle}{\langle i,a,d \rangle \langle i,b,c \rangle}$$

The coefficients $\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\}$ are called *recoupling coefficients* or *6j-symbols* . It follows from the above lemma that a $6j$-symbol is null if and only if the corresponding tetahedron coefficient is null.

If Σ represents an arbitrary colored surface bounding a handlebody H and Γ_1, Γ_2 represent two trivalent graphs embedded in H, we can decompose the basis of vectors $u_\sigma^{\Gamma_1}$ into the basis of vectors $u_\sigma^{\Gamma_2}$ using Lemma 1.1.48 for Γ_1 can be obtained from Γ_2 by a sequence of Whitehead moves.

Proof. We take the scalar product of both sides of the equality with the vector 

and get:



where we used Lemma 1.1.47 to get the last equality. □

1.2 First properties of the quantum representations

1.2.1 Generators and central extensions

Notations

Let Σ_g be an oriented closed genus g (structure) surface without bands and with an arbitrary Lagrangian. The TQFTs U_p and V_p define k'_p-modules $U_{p,g} := U_p(\Sigma_g)$ and k_p-modules $V_{p,g} := V_p(\Sigma_g)$ defined as the quotient of the free k_p-module of isotopy classes of framed links in a given handlebody H_g endowed with a preserving orientation homeomorphism $\alpha : \partial H_g \to \Sigma_g$, by the kernel of a certain bilinear form $\langle , \rangle_p^R$.

Denote by $\widetilde{\mathrm{Mod}(\Sigma_g)}$ the central extension of the mapping class group of Σ_g, defined by the structure cylinders. This group acts on $U_{p,g}$ and $V_{p,g}$ to give the so-called quantum representations. We note them:

$$\text{The Weil representations:} \quad \pi_{p,g} : \widetilde{\mathrm{Mod}(\Sigma_g)} \to \mathrm{GL}(U_{p,g})$$
$$\text{The Reshetikhin-Turaev representations:} \quad \rho_{p,g} : \widetilde{\mathrm{Mod}(\Sigma_g)} \to \mathrm{GL}(V_{p,g})$$

The modules $U_{p,g}$ and $V_{p,g}$ are endowed with two non degenerate bilinear forms. The *invariant form* $\langle \cdot, \cdot \rangle_p^R$ is induced by the gluing of H_g with $-H_g$ using the identity map on the boundary. This form is invariant under the action of $\widetilde{\mathrm{Mod}(\Sigma_g)}$. In particular, the image of the quantum representations are semi-simple: for any stable $\widetilde{\mathrm{Mod}(\Sigma_g)}$-submodule, its orthogonal for the invariant form is also stable. Moreover we know orthogonal basis for this form.

Another form called *Hopf pairing* and denoted $(\cdot, \cdot)_p^H$, is induced by an arbitrary Heegaard splitting of the sphere $H_g \cup_S -H_g \cong S^3$ for some $S \in \mathrm{Mod}(\Sigma_g)$. The matrix elements of the Hopf pairing in a trivalent graph basis can be computed using fusion rules.

Generators and computations

If $\phi \in \widetilde{\mathrm{Mod}(\Sigma_g)}$ is the lift of a homeomorphism of Σ_g that extends to H_g through α, the action of this extension on the framed links of H_g passes to the quotient to give the action of $\pi_{p,g}(\phi)$ on $U_{p,g}$ and of $\rho_{p,g}(\phi)$ on $V_{p,g}$.

If $\psi \in \widetilde{\mathrm{Mod}(\Sigma_g)}$ is the lift of a homeomorphism of Σ_g that extends to H_g through $S \circ \alpha$, the action of this extension on the framed links of H_g defines an operator. Its dual compare to the Hopf pairing gives the action of ψ by the quantum representations.

For the Weil representations, consider the Dehn twists $X_i, Z_{i,j}$ of Figure 1.13. They extend to H_g and their action on the basis $\{e_{a_1} \otimes \ldots \otimes e_{a_g}, a_i \in \mathbb{Z}/p\mathbb{Z}\}$ of $U_{p,g}$ is easily computed (see Figure 1.14. We find:

$$\begin{aligned} \pi_{p,1}(X) &= (A^{i^2}\delta_{i,j})_{i,j} \text{ and } \pi_{p,g}(X_i) = \mathbb{1}^{\otimes(i-1)} \otimes \pi_{p,1}(X) \otimes \mathbb{1}^{\otimes(g-i)} \\ \pi_{p,g}(Z_{i,j}) & \qquad (e_{a_1} \otimes \ldots \otimes e_{a_g}) = A^{(a_i - a_j)^2}(e_{a_1} \otimes \ldots \otimes e_{a_g}) \end{aligned}$$

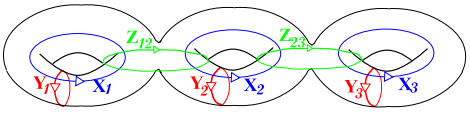


Figure 1.13: A set of Dehn twists generating the mapping class group and the symplectic group.

$$\pi_{p,1}(X) \cdot e_i = \quad = A^{i^2} e_i$$

$$\pi_{p,2}(Z_{1,2}) \cdot e_i \otimes e_j = \quad = A^{(i-j)^2} e_i \otimes e_j$$

Figure 1.14: The computation of the matrices associated to $\pi_{p,1}(X)$ and $\pi_{p,2}(Z_{ij})$.

The matrix elements of the Hopf pairing are easily computed:

$$\left(e_{a_1} \otimes \ldots \otimes e_{a_g}, e_{b_1} \otimes \ldots \otimes e_{b_g}\right)_p^H = (e_{a_1}, e_{b_1})_p^H \ldots \left(e_{a_g}, e_{b_g}\right)_p^H = \eta_p A^{-2\sum_i a_i b_i}$$

Since the Dehn twists Y_i are duals of the X_i for the Hopf pairing, their matrix elements can be computed. We find:

$$\begin{aligned} \pi_{p,1}(Y) &= \begin{cases} \frac{G(1,0,p)}{p}(A^{-(i-j)^2})_{i,j}, & \text{when } p \text{ is odd.} \\ \frac{G(1,0,2p)}{2p}(A^{-(i-j)^2})_{i,j}, & \text{when } p \text{ is even.} \end{cases} \\ \pi_{p,g}(Y_i) &= \mathbb{1}^{\otimes(i-1)} \otimes \pi_{p,1}(Y) \otimes \mathbb{1}^{\otimes(g-i)} \end{aligned}$$

For the Reshetikhin-Turaev representations, we choose a uni-trivalent banded graph $\Gamma \subset H_g$ on which H_g retracts by deformation and consider the associated basis $\{u_\sigma, \sigma \in col_p(\Gamma)\}$. A Dehn twist around an edge $e \in E(\Gamma)$ can be lifted in $\widetilde{\mathrm{Mod}(\Sigma_g)}$ to get an operator T_e so that:

$$\rho_{p,g}(T_e)u_\sigma = \mu_{\sigma(e)}u_\sigma$$

The matrix elements of the Hopf pairing can be computed using the fusion rules. Since S and the T_e generate $\widetilde{\mathrm{Mod}(\Sigma_g)}$, we can theoretically compute any matrices $\rho_{p,g}(\phi)$.

Linearisation of quantum representations

Since the class of a link in an abelian skein module depends only on its integer homology class, the Weil representations act trivially on the Torelli group and factorize as representations of a central extension of the symplectic group $\widetilde{Sp_{2g}(\mathbb{Z})}$. The above computations show that the image of any Dehn twist through $\pi_{p,g}$ has the same order than A, thus has order p if p is odd and $2p$ if p is even. So the Weil representations also act trivially on the congruence subgroups to give projective representations of $Sp_{2g}(\mathbb{Z}/p\mathbb{Z})$ if p is odd and of $Sp_{2g}(\mathbb{Z}/2p\mathbb{Z})$ if p is even.

In Chapter 4 we will show the following:

Proposition 1.2.1.

1. *The representations $\pi_{p,1}$ can be lifted to linear representations of $SL_2(\mathbb{Z}/p\mathbb{Z})$ if p is odd and of $SL_2(\mathbb{Z}/2p\mathbb{Z})$ if p is even.*
2. *If p is odd and $g \geq 2$, the Weil representation $\pi_{p,g}$ can be lifted to a linear representation of $Sp_{2g}(\mathbb{Z}/p\mathbb{Z})$.*
3. *If p is even and $g \geq 2$, the Weil representation $\pi_{p,g}$ cannot be lifted to a linear representation of $Sp_{2g}(\mathbb{Z}/2p\mathbb{Z})$ but represents a central extension by $\mathbb{Z}/2\mathbb{Z}$.*

Although this proposition is well known (see [1], [31]), we will give a direct proof by finding explicit lifts of the Weil representations. In addition we will show that these representations are faithfull when p is odd.

Concerning the Reshetikhin-Turaev representations, in genus one we will show that they can be lifted to linear representations of $SL_2(\mathbb{Z}/p\mathbb{Z})$ when p is odd and of $SL_2(\mathbb{Z}/2p\mathbb{Z})$ when p is even. More generally, in [38] the authors showed that the representations associated to a torus with one colored band can be lifted to linear representations of B_3. They also showed that in higher genus, the central extension $\widetilde{\mathrm{Mod}(\Sigma_g)}$ corresponds to a cohomology class which is a generator of $H^2(\mathrm{Mod}(\Sigma_g), \mathbb{Z})$.

As we saw previously, the image of any lift of a Dehn twist through $\rho_{p,g}$ has order $2p$. It is still an open problem to know if there exists more relations, that is if the representations

$$\rho_{p,g} : \widetilde{\mathrm{Mod}(\Sigma_g)} \Big/ (T_\gamma^{2p} - \mathbb{1}) \to \mathrm{GL}(V_{p,g})$$

are faithful or not when $g \geq 2$.

1.2.2 Egorov identity and first decompositions

Egorov identity

Let $L \subset \Sigma_g$ be a multicurve. Consider the cobordism $\Sigma_g \times [0,1]$ with L embedded in $\Sigma_g \times \{\frac{1}{2}\}$, with identification of the boundary using the identity map and null weight. The TQFTs associate to this cobordism some operators:

$$\mathrm{Add}_p^U(L) := Z_{U_p}(\Sigma \times [0,1], L, 0) \in \mathrm{End}(U_{p,g})$$
$$\mathrm{Add}_p^V(L) := Z_{V_p}(\Sigma \times [0,1], L, 0) \in \mathrm{End}(V_{p,g})$$

We thus define algebra representations $\mathrm{Add}_p^U : \mathcal{T}_p(\Sigma_g) \to \mathrm{End}(U_{p,g})$ and $\mathrm{Add}_p^V : \mathcal{K}_p(\Sigma_g) \to \mathrm{End}(V_{p,g})$. Usually the Add_p^U are called *Schrödinger representations* representations. It follows from the surgery properties of U_p and V_p that the morphisms Add_p are onto.

The following proposition gives an alternative definition of the quantum representations based on the skein representations Add_p.

Proposition 1.2.2. *For any $\phi \in \widetilde{\mathrm{Mod}(\Sigma_g)}$, the operators $\rho_{p,g}(\phi)$ and $\pi_{p,g}(\phi)$ are the unique solutions, up to multiplication by an invertible scalar, of the following identity:*

$$\pi_{p,g}(\phi)\,\mathrm{Add}_p^U(L)\pi_{p,g}(\phi)^{-1} = \mathrm{Add}_p^U(\phi(L)) \quad \text{, for all multicurve } L. \tag{1.5}$$
$$\rho_{p,g}(\phi)\,\mathrm{Add}_p^V(L)\rho_{p,g}(\phi)^{-1} = \mathrm{Add}_p^V(\phi(L)) \quad \text{, for all multicurve } L. \tag{1.6}$$

The equation 1.5 is called *Egorov identity*.

Proof. The proof is similar in the abelian and in the non abelian case, so we write it for the Reshetikhin-Turaev representations only.

The Egorov identity is clearly satisfied for a homeomorphism that extends to the handlebody H_g through α. By duality for the Hopf pairing, it is also true for homeomorphisms that extends to H_g through $S \circ \alpha$. These two families of homeomorphisms generate $\mathrm{Mod}(\Sigma_g)$. We now remark that if the Egorov identity 1.5 is satisfied for two elements ϕ_1 and ϕ_2, then it is also true for $\phi_1 \circ \phi_2$. This proves that all the operators $\rho_{p,g}(\phi)$ are solutions.

To prove the unicity, fix $\phi \in \mathrm{Mod}(\Sigma_g)$ and consider the two representations of the skein algebra $s_1, s_2 : \mathcal{K}_p(\Sigma_g) \to \mathrm{End}(V_{p,g})$ defined by $s_1(L) = \mathrm{Add}_p(L)$ and $s_2(L) = \mathrm{Add}_p(\phi(L))$.

The vector $v_0 \in V_{p,g}$ corresponding to the class of the empty link in H_g, is cyclic for both representations s_1 and s_2. So an intertwiner $\Theta \in \mathrm{GL}(V_{p,g})$ between s_1 and s_2 (i.e. a solution of 1.5) is completely determined by the image of v_0. Moreover for any $L \subset \Sigma_g$ which is contractible in H_g, then $\Theta \cdot v_0$ must be an eigenvector of $\mathrm{Add}_p(L)$ with eigenvalue $-(A^2 + A^{-2})$. Since $\lambda_i \neq \lambda_0$ when $i \neq 0$, the corresponding eigenspace is one dimensional, generated by $(\rho_{p,g}(\phi))^{-1}v_0$. This implies that $\Theta \cdot v_0$ is proportional to $(\rho_{p,g}(\phi))^{-1}v_0$, thus the space of intertwiners between s_1 and s_2 is one dimensional. This proves unicity. □

Decompositions of quantum representations

We deduce from the Egorov identity, a family of submodules of $U_{p,g}$ and $V_{p,g}$ stable under the action of $\widetilde{\mathrm{Mod}(\Sigma_g)}$.

Definition 1.2.3. Let b be an element of $U_{p,1}$ (resp of $V_{p,1}$).

We define the b-colored algebra $\mathcal{B}_p(b)$ as the subalgebra of $\mathrm{End}(U_{p,g})$ (resp of $\mathrm{End}(V_{p,g})$) generated by operators $\mathrm{Add}_p(L(b))$, where $L(b)$ is a multicurve of Σ_g whose components are colored by b.

We define the module $X_p(b)$ as the submodule of $U_{p,g}$ (resp $V_{p,g}$) of vectors stabilized by every elements of $\mathcal{B}_p(b)$.

Some decomposition of quantum representations can be deduced from the following:

Lemma 1.2.4. *For any b, the submodule $X_p(b)$ is stable for the action of $\widetilde{\mathrm{Mod}(\Sigma_g)}$.*

Proof. If $v \in X_p(b)$, $\phi \in \widetilde{\mathrm{Mod}(\Sigma_g)}$ and L is a multicurve in Σ_g, using Egorov identity we have:

$$\mathrm{Add}_p(L(b)) \cdot (\rho_p(\phi)v) = \rho_p(\phi)\,\mathrm{Add}_p(\phi^{-1}(L)(b)) \cdot v = \rho(\phi)v$$

thus $\rho_p(\phi)v \in X_p(b)$. □

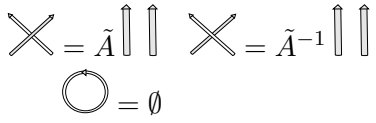
We derive two propositions from this lemma.

The following proposition was showed in [54] in the particular case where r is prime and $g = 1$.

Proposition 1.2.5. *If r is prime and $n \geq 1$, $U_{r^n,g}$ is a $\widetilde{\mathrm{Mod}(\Sigma_g)}$-submodule of $U_{r^{n+2},g}$.*

Proof. We show that the submodule $X(e_{r^{n+1}})$ is isomorphic to $U_{r^n,g}$ as a $\widetilde{\mathrm{Mod}(\Sigma_g)}$-module.

Consider the algebra $\mathcal{B}_{r^{n+2}}(e_r) \subset \mathrm{End}(U_{r^n,g})$ of links colored by e_r. Note $\tilde{A} := A^{(r^2)}$ the primitive r^n-th root of unity if r is odd and primitive 2^{n+1}-th root of unity if $r = 2$. The product in $\mathcal{B}_{r^{n+2}}(e_r)$ is described by the skein relations for links colored by e_r:



Thus we have a morphism of algebras $\Psi : \mathcal{T}_{r^n}(\Sigma_g) \to \mathcal{B}_{r^{n+2}}(e_r)$, where we turned $\mathcal{T}_{r^n}(\Sigma_g)$ into a $k'_{r^{n+2}}$ algebra with the ring morphism $\mu : k'_{r^n} \to k'_{r^{n+2}}$ defined by $\mu(A) = A^{(r^2)}$.

The submodule $X_{r^{n+2}}(e_{r^{n+1}})$ has basis $\{g_{a_1} \otimes \ldots \otimes g_{a_g}, a_i \in \mathbb{Z}/p\mathbb{Z}\}$ where $g_{a_i} := \sum_{k=0}^{r-1} e_{r(a_i+kr^n)}$.

Let $\Phi : X(e_{r^{n+1}}) \to V_{r^n,g}$ be the morphism defined by

$$\Phi(g_{a_1} \otimes \ldots \otimes g_{a_g}) := e_{a_1} \otimes \ldots \otimes e_{a_g}$$

A simple computation shows that Φ intertwines the two actions of $\mathcal{T}_{r^n}(\Sigma_g)$. The restriction to $X(e_{r^{n+1}})$ of $\pi_{r^{n+2},g}$ satisfies the Egorov identity at level r^n. It follows from the unicity of Proposition 1.2.2 that $X(e_{r^{n+1}})$ is isomorphic to $V_{r^n,g}$ as a projective $\widetilde{Sp(2g,\mathbb{Z})}$-module. □

We call *lollipop graph of genus g* a trivalent graph Γ made of g loops connected by a trivalent tree, called the *trunk* of Γ. We denote by $a_1, \dots, a_g$ the edges of Γ corresponding to the loops and by $b_1, \dots, b_g$ the edges of the trunk adjacent the a_i. See Figure 1.15.

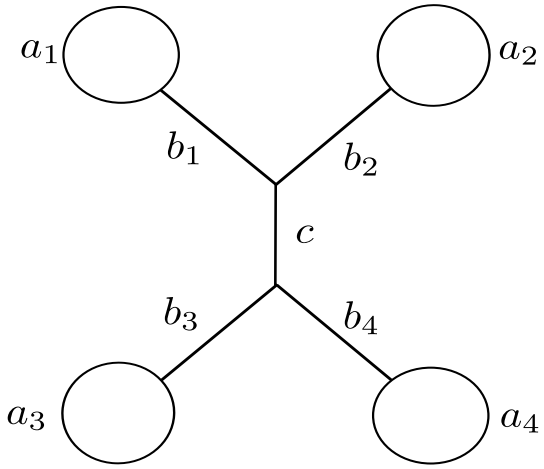


Figure 1.15: A genus 4 lollipop graph.

Note that if $\sigma \in col_p(\Gamma)$ then $\sigma(e)$ is even for any edge e of the trunk.

Suppose p is even and write $k := \frac{p-4}{2}$, so that $I_p = \{0, \dots, k\}$. Denote by $\epsilon(a,b) \in \{-1,+1\}$, the element:

$$\epsilon(a,b) := \frac{\left\langle \begin{array}{ccc} b & a & a \\ k & a-k & a-k \end{array} \right\rangle}{\langle b,a,a\rangle}$$

Note that $\epsilon(a,b) = \epsilon(k-a,b) = \pm 1$.

Let $\sigma \in col_p(\Gamma)$ and $i \in \{1, \dots, g\}$. The map $\{i\} \bullet \sigma : E(\Gamma) \to I_p$ defined by :

$$\{i\} \bullet \sigma(e) = \begin{cases} \sigma(e) & \text{, if } e \neq a_i, \\ k - \sigma(a_i) & \text{, if } e = a_i \end{cases}$$

is also a p-admissible coloring of Γ. This recursively defines an action of the subsets of $\{1, \dots, g\}$ on $col_p(\Gamma)$.

Definition 1.2.6. Let

$$v_\sigma := \sum_{X \subset \{1,\dots,g\}} \left(\prod_{i \in X} \epsilon(\sigma(a_i), \sigma(b_i)) \right) u_{X \bullet \sigma} \in V_{p,g}$$

We denote by $J_{p,g} \subset V_{p,g}$ the submodule spanned by the vectors v_σ for $\sigma \in col_p(\Gamma)$ a coloring so that $\sigma(a_i)$ is even for all $1 \leq i \leq g$.

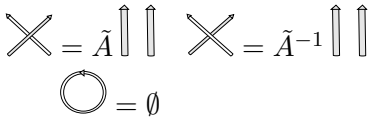
The following decomposition was found in [14].

Proposition 1.2.7. *If 4 divides p, then $J_{p,g}$ is a proper sub $\widetilde{\mathrm{Mod}(\Sigma_g)}$-module of $V_{p,g}$.*

Proof. We show that $J_{p,g} = X_p(u_k)$ to conclude.

Note $p = 2r$, $k = r - 2$ and $\tilde{A} := -A^{r^2} = \begin{cases} 1 & \text{, if } p \equiv 4 \pmod 8 \\ -1 & \text{, if 8 divides } p. \end{cases}$

The product in $\mathcal{B}(u_k)$ is given by the following skein relations satisfied by links colored by u_k:



Also two parallel framed knots colored by u_k can be removed. It results that $\mathcal{B}(u_k)$ is isomorphic to the algebra of the group $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ if $p \equiv 4 \pmod 8$ and of the group $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$ if 8 divides p, where the product is given by $[\gamma_1] \cdot [\gamma_2] = (-1)^{\omega(\gamma_1,\gamma_2)}[\gamma_1 + \gamma_2]$ with ω the mod 2 intersection form.

In particular a vector $v \in V_{p,g}$ belongs to $J_{p,g}$ if and only if $\mathrm{Add}(X_i(u_k)) \cdot v = v$ and $\mathrm{Add}(Y_i(u_k)) \cdot v = v$ for all $1 \leq i \leq g$. Here X_i, Y_i are the class in $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ of the Dehn twists of Figure 1.13.

Using Lemma 1.1.47 and 1.1.48, we get the two identities of Figure 1.16. They imply that:

$$\mathrm{Add}(X_i(u_k)) \cdot u_\sigma = (-1)^{\sigma(a_i)} u_\sigma$$

and:

$$\mathrm{Add}(Y_i(u_k) \cdot u_\sigma = \epsilon(\sigma(a_i), \sigma(b_i)) u_{\{i\} \bullet \sigma}$$

This proves that $J_{p,g} = X_p(u_k)$ and we conclude using Lemma 1.2.4.

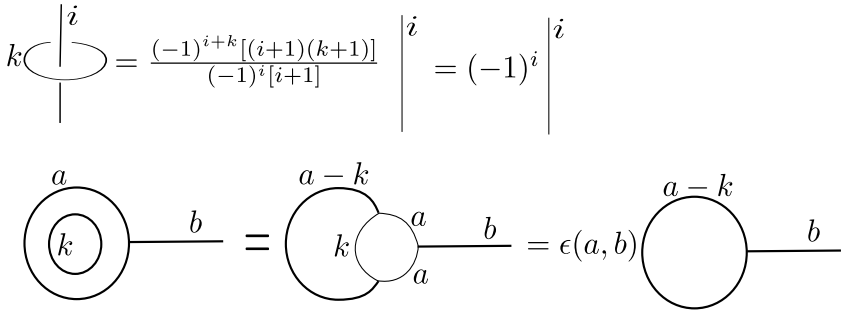


Figure 1.16: Computations derived from fusion rules to compute $X(u_k)$.

□

Note that the above proof also works for representations associated to surfaces with colored bands.

1.2.3 Relation between $U_{p,1}$ and $V_{p,1}$

Denote by $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ the two generators of $SL_2(\mathbb{Z})$.

Using the basis $\{e_i, i \in \mathbb{Z}/p\mathbb{Z}\}$ of $U_{p,1}$, the Weil representation in genus one is characterized by the projective classes of matrices:

$$\pi_p(T) = \left(A^{i^2}\delta_{i,j}\right)_{i,j} \quad \pi_p(S) = c_p\left(A^{-2ij}\right)_{i,j}$$

where $c_p := \frac{G(-1,0,2p)}{2p}$ when p is even and $c_p := \frac{G(-1,0,p)}{p}$ when p is odd.

We remark that the involution sending e_i to e_{-i} commutes with these two matrices. Thus the $SL_2(\mathbb{Z})$-projective module splits into two invariant submodules $U_{p,1} \cong U_{p,1}^+ \oplus U_{p,1}^-$ where $U_{p,1}^\pm := \text{Span}\{e_i \pm e_{-i}, i \in \mathbb{Z}/p\mathbb{Z}\}$.

Using the basis $\{u_i, i \in I_p\}$ of $V_{p,1}$, where $I_p := \begin{cases} \{0,1,2,\ldots,r-2\} & \text{, if } p = 2r \text{ is even.} \\ \{0,2,4,\ldots,p-3\} & \text{, if } p \text{ is odd.} \end{cases}$, the Reshetikhin-Turaev representations in genus one are characterized by the projective class of the matrices:

$$\rho_p(T) = \left(A^{i(i+2)}\delta_{i,j}\right)_{i,j} \quad \rho_p(S) = c_p\left((-1)^{i+j}[(i+1)(j+1)]\right)_{i,j}$$

where we used Lemmas 1.1.12 and 1.1.14.

Theorem 1.2.8. *For $p \geq 3$, the $SL_2(\mathbb{Z})$ projective modules $U_{p,1}^-$ and $V_{p,1}$ are projectively equivalent.*

When p is odd, the module $U_{p,1}$ is defined on the ring k'_p, where A is a primitive p-th root of unity, whereas $V_{p,1}$ is defined on k_p, where A is a primitive $2p$-th root of unity. In the preceding theorem, we turned $V_{p,1}^-$ into a k_p-module using the ring morphism $\mu : k_p \to k'_p$ defined by $\mu(A) = A^4$.

This theorem was shown in [27] when p is even and in [60] when $p \equiv 1 \pmod 4$. We will extend their proofs for arbitrary $p \geq 3$. Note that this equivalence was predicted in the geometric quantization setting in [8] (see also [66]).

Proof. When $p = 2r$ is even, we define an isomorphism of k_p-modules $\Psi : V_{p,1} \to U_{p,1}^-$ by $\Psi(u_i) = e_{r-i-1} - e_{r+i+1}$. We then compute the matrices of π_p^- in the basis $(\Psi(u_i), i = 0, 1, \ldots r-2)$:

$$\begin{aligned} \left\langle \Psi(u_j), \pi_p^-(T)\Psi(u_i) \right\rangle &= A^{(r-i-1)^2}\delta_{i,j} \\ &= A^{(r-1)^2} \cdot A^{-2ri} \cdot A^{i(i+2)} \\ &= A^{(r-1)^2}\rho_p(T)_{i,j} \end{aligned}$$

$$\begin{aligned} \left\langle \Psi(u_j), \pi_p^-(S)\Psi(u_i) \right\rangle &= c_p\left(A^{-2(r-i-1)(r-j-1)} - A^{2(r-i-1)(r-j-1)}\right) \\ &= c_p \cdot A^{2r(i+j)}\left(A^{-2(i+1)(j+1)} - A^{2(i+1)(j+1)}\right) \\ &= -\rho_p(S)_{i,j} \end{aligned}$$

So π_p^- and ρ_p are projectively equivalent when p is even.

Then when $p \geq 3$ is odd, we turn $U_{p,1}^-$ into a k_p-module via the ring morphism $\mu : k_p \to k'_p$ defined by $\mu(A) := A^4$. We define an isomorphism $\Psi : V_{p,1} \to U_{p,1}^-$ of k_p-modules via $\Psi(u_i) := e_{\frac{p-1-i}{2}} - e_{\frac{p+i+1}{2}}$. We then compute the matrices of π_p^- in the basis $(\Psi(u_i), i = 0, 2, 4, \dots p-3)$:

$$\begin{aligned}\left\langle \Psi(u_j), \pi_p^-(T)\Psi(u_i) \right\rangle &= \mu(A)^{\left(\frac{p-1-i}{2}\right)^2}\delta_{i,j} \\ &= A^{(p-i-1)^2}\delta_{i,j} \\ &= (-A)\cdot\rho_p(T)_{i,j}\end{aligned}$$

$$\begin{aligned}\left\langle \Psi(u_j), \pi_p^-(S)\Psi(u_i) \right\rangle &= c_p\left(\mu(A)^{-2(\frac{p-1-i}{2})(\frac{p-1-j}{2})} - \mu(A)^{2(\frac{p-1-i}{2})(\frac{p-1-j}{2})}\right) \\ &= c_p\left(A^{-2(p-i-1)(p-1-j)} - A^{2(p-i-1)(p-1-j)}\right) \\ &= -\rho_p(S)_{i,j}\end{aligned}$$

And the proof is completed. □

1.2.4 Tensor decompositions

Definition 1.2.9. Let V_1 and V_2 be two TQFTs with the same ring k. The tensor product $V_1 \otimes V_2$ is the TQFT defined by $V_1 \otimes V_2(\Sigma) := V_1(\Sigma) \otimes_k V_2(\Sigma)$ and $Z_{V_1\otimes V_2}(M) := Z_{V_1}(M) \otimes_k Z_{V_2}(M)$.

In particular the quantum invariant and quantum representations of a tensor product $V_1 \otimes V_2$ is the product and tensor product of the two quantum invariants and quantum representations of V_1 and V_2.

Lemma 1.2.10. *Let (V_1, k_1), (V_2, k_2) and (V_3, k_3) be three non degenerate TQFTs and $\mu : k_1 \to k_3$, $\nu : k_2 \to k_3$ be two ring morphisms preserving the involutions. Suppose that:*

1. $\mu \circ \langle\cdot\rangle_1 \times \nu \circ \langle\cdot\rangle_2 = \langle\cdot\rangle_3$
2. $\mathrm{rank}(V_1(\Sigma)) \cdot \mathrm{rank}(V_2(\Sigma)) = \mathrm{rank}(V_3(\Sigma))$ *for any structure surface Σ.*

Then we have $\mu_* V_1 \otimes \nu_* V_2 \cong V_3$.

Proof. Let Σ be a (non empty) structure surface and N_Σ denotes the set of classes of structure cobordisms bounding Σ. From the non degeneracy condition we have an isomorphism $V_3(\Sigma) \cong k_3[N_\Sigma] \Big/ ker\, \langle\cdot,\cdot\rangle_3$. Consider the linear map:

$$f : (k_3[N_\Sigma], \langle\cdot,\cdot\rangle_3) \to (\mu_* V_1(\Sigma) \otimes_{k_3} \nu_* V_2(\Sigma), \mu \circ \langle\cdot,\cdot\rangle_1 \otimes \nu \circ \langle\cdot,\cdot\rangle_2)$$

sending an element $M \in N_\Sigma$ to the corresponding class of $M \otimes M$ in $\mu_* V_1(\Sigma) \cong k_3[N_\Sigma] \Big/ ker\, (\mu \circ \langle\cdot,\cdot\rangle_1)$ and $\mu_* V_2(\Sigma) \cong k_3[N_\Sigma] \Big/ ker\, (\nu \circ \langle\cdot,\cdot\rangle_2)$.

The assumption on the quantum invariants implies that f preserves the bilinear forms, so sends the kernel of $\langle\cdot,\cdot\rangle_3$ inside the kernel of $\mu\circ\langle\cdot,\cdot\rangle_1\otimes\nu\circ\langle\cdot,\cdot\rangle_2$ which is trivial. Thus f induces an onto linear map:

$$\widetilde{f}: V_3(\Sigma)\to\mu_*V_1(\Sigma)\otimes_{k_3}\nu_*V_2(\Sigma)$$

The equality of the ranks of the modules implies that f is an isomorphism. By construction, this isomorphism induces an isomorphism between the TQFTs V_3 and $\mu_*V_1\otimes\nu_*V_2$. □

We give two applications of this lemma.

Let a and b be two coprime non negative integers and u,v be such that $au+bv=1$. We define morphisms $\mu: k_a\to k_{ab}$ and $\nu: k_b\to k_{ab}$ by setting $\mu(A)=A^{bv}$ and $\nu(A)=A^{au}$.

Proposition 1.2.11. *If a and b are coprime, then $\mu_*U_a\otimes\nu_*U_b\cong U_{ab}$.*

Proof. The abelian quantum invariants for a 3-manifold obtained by surgery along a framed link $L=L_1\cup\ldots\cup L_m\subset S^3$, has the following expression:

$$\left\langle S^3(L)\right\rangle_{U_p}=\kappa^{-\sigma(L)}\eta\sum_{i=(i_1,\ldots,i_m)\in(\mathbb{Z}/p\mathbb{Z})^m}A^{^tilk(L)i}$$

where $lk(L)$ is the linking matrix of L. Using this expression and the fact that $U_{p,g}$ has rank p^g, we easily see that the conditions of Lemma 1.2.10 are satisfied. □

Remark.

1. The fact that the abelian invariant at level ab is the product of the ones at levels a and b when a,b are coprime where noticed in [71].
2. This proposition implies that the Weil representations satisfy $\pi_{ab}\cong\pi_a\otimes\pi_b$ when a,b are coprime.

Proposition 1.2.12. *If r is odd, then $V_{2r}\cong V_6\otimes V_r$.*

The proof relies again on Lemma 1.2.10. We refer to [13] for a proof.

1.2.5 Roberts' proof for the prime cases and $p=18$

In this subsection, we briefly review the result of Roberts in [76] which shows that the Reshetikhin-Turaev representations at prime levels are irreducible. The proof also works for $p=18$.

We denote by $\mathcal{A}_{p,g}$ the subalgebra of $\mathrm{End}(V_{p,g})$ generated by the operators $\rho_{p,g}(\phi)$ for $\phi\in\widetilde{\mathrm{Mod}(\Sigma_g)}$.
Remark that if K is a framed knot in Σ_g and T_K the corresponding Dehn twist, the operator $\rho_{p,g}(T_K)$ is equal to the operator $\mathrm{Add}_p(K_{-1}(\omega))$ modulo an invertible scalar. Here we denote by K_{-1} the framed knot obtained from K by twisting once negatively.

The vector $v_0 \in V_{p,g}$, corresponding to the class of the empty link in the handlebody H_g, is said *cyclic* if $\mathcal{A}_{p,g} \cdot v_0 = V_{p,g}$.

Lemma 1.2.13. *Suppose $p \geq 3$ is so that if $i, j \in I_p$ then $i \neq j$ implies $\mu_i \neq \mu_j$. Then $v_0 \in V_{p,g}$ is cyclic for every $g \geq 1$.*

Proof. First when $g = 1$, write $d = rank(V_{p,1})$ and denote by L the longitude $\{0\} \times S^1$ in $D^2 \times S^1$. We define the vectors $t_i \in V_{p,1}$ by the formula $t_i := \mathrm{Add}(L(\omega))^i \cdot v_0 \in \mathcal{A}_{p,g} \cdot v_0$. We have:

$$(t_i, u_j)_p^H = \eta < j > \mu_j^i$$

Thus the matrix $\left((t_i, u_j)_p^H\right)_{0 \leq i \leq d-1, j \in I_p}$ is the product of a diagonal matrix $\left(\eta(-1)^j[j+1]\delta_{i,j}\right)_{i,j}$ with non null diagonal entries, with a Vandermonde matrix $(\mu_j^i)_{i,j}$ which is invertible for the μ_j are distincts.

Since the Hopf pairing is non degenerate, the set $\{t_0, t_1, \ldots, t_{d-1}\}$ is a basis of $V_{p,1}$ and v_0 is cyclic in genus one. Denote by $\alpha_0, \ldots, \alpha_{d-1}$ the scalars so that:

$$u_1 = \sum_i \alpha_i t_i$$

Let $g \geq 2$ and $L = L^1 \cup \ldots \cup L^m$ be a framed link in the g-holed disc D_g. Embedding L in $D_g \times \{\frac{1}{2}\} \subset D_g \times [0,1] \cong H_g$, we get a vector $Z_p(H_g, L) \in V_{p,g}$ equal to:

$$Z_p(H_g, L) = \sum_{i=(i_1,\ldots,i_m) \in \{0,\ldots,d-1\}^m} \alpha_{i_1} \ldots \alpha_{i_m} \mathrm{Add}(L^1_{-1}(\omega))^{i_1} \ldots \mathrm{Add}(L^m_{-1}(\omega))^{i_m} \cdot v_0 \in \mathcal{A}_{p,g} \cdot v_0$$

Lemma 1.1.8 implies that such vectors generate $V_{p,g}$, so v_0 is cyclic. □

Lemma 1.2.14. *Suppose $p \geq 3$ is so that if $i, j \in I_p$ then $i \neq j$ implies $\mu_i \neq \mu_j$. Then the representations $\rho_{p,g}$ are irreducible for every $g \geq 1$.*

Proof. Let $(\mathcal{A}_{p,g})'$ denote the commutant of $\mathcal{A}_{p,g}$ in $\mathrm{End}(V_{p,g})$, that is the subalgebra of operators that commute with every $\rho_{p,g}(\phi)$ for $\phi \in \widetilde{\mathrm{Mod}(\Sigma_g)}$.

Let $f : (\mathcal{A}_{p,g})' \to V_{p,g}$ be the linear map defined by $f(\Theta) = \Theta \cdot v_0$. The cyclicity of v_0 implies that f is injective.

Let $\Gamma \subset H_g$ be a trivalent graph and $\gamma = (\gamma_e)_{e \in E(\Gamma)}$ the corresponding pants decomposition. For each edge $e \in E(\Gamma)$, we choose a lift T_e of the Dehn twist around e. Since $\rho_{p,g}(T_e)v_0 = v_0$, then $\rho_{p,g}(T_e)\Theta \cdot v_0 = \Theta \cdot v_0$ for $\Theta \in (\mathcal{A}_{p,g})'$. This implies that $\Theta \cdot v_0$ belongs to the intersection of eigenspaces of the operators $\rho_{p,g}(T_e)$ with the eigenvalue one. Since $\mu_i \neq 1$ when $i \neq 0$, this intersection is one dimensional generated by v_0.

The injectivity of f implies that $(\mathcal{A}_{p,g})'$ is one dimensional and we conclude using the Schür lemma.

□

Proposition 1.2.15. *If $p = r$ or $p = 2r$ with r an odd prime or $r = 9$, then the representations $\rho_{p,g}$ are irreducible for every $g \geq 1$.*

Proof. We simply remark that under the hypothesis of the proposition, the μ_i are distinct and apply Lemma 1.2.14. □

Chapter 2

Decomposition of the Weil representations into irreducible factors

Résumé

Ce chapitre contient la version non publiée de l'article intitulé " Decomposition of the Weil representations into irreducible factors" ([55]). On y donne une décomposition explicite des représentations de Weil en représentations irréductibles. Certaines notations ont été changées pour coller avec le reste du manuscrit. Un résultat subsidiaire de fidélité est renvoyé à la première section du quatrième chapitre.

Abstract

This chapter contains the unpublished version of the article "Decomposition of the Weil representations into irreducible factors" ([55]). We give an explicit decomposition of the Weil representations into irreducible representations. Some notations have been changed to match with the rest of the manuscript. A subsidiary result concerning the faithfulness of Weil representations has been postponed to the first section of the fourth chapter.

2.1 Introduction and statements

2.1.1 A brief history

The Weil representations first appeared in the work of Kloosterman in 1946 (see [54]) where they arise as modular transformations of spaces of theta functions. They were rediscovered independently by Shale ([80]) following Segal ([78]) in 1962 when the authors studied the Weyl quantization of the symplectic torus. Their construction has been generalized to arbitrary locally compact abelian groups by Weil in 1964 (see [88]). The ones we consider in this paper are associated to $\mathbb{Z}/p\mathbb{Z}$. They also appeared independently in

the work of Igusa ([45]) and Shimura ([81]) on theta functions. See also [64] for another construction.

The generalization of the Weil representations for finite fields and $g = 1$ was studied in [59] for odd characteristic, and more recently in [41] for characteristic 2.

The mathematical physics community studied the semi-classical properties of the Weil representations associated to finite cyclic groups when the level p tends to infinity as a model for quantum chaotical behavior (see [11], [58], [24], [49]).

The relation between theta functions and the quantization of the torus was clarified through the geometrical quantization procedure of Souriau, Kostant and Kirillov (see [82, 57, 52]) where the Hilbert space associated to the torus is the space of holomorphic sections of a Hermitian line bundle over the torus endowed with a complex structure. This Hilbert space is naturally isomorphic to a space of theta functions (see [2, 89, 32]) and the symplectic group acts as monodromy operators on these sections providing a geometric interpretation for the Weil representations.

The Weil representations fit into the framework of Topological Quantum Field Theories, as defined by Atiyah and Witten in [6, 91]. Their definition for even levels and arbitrary genus first appeared in [30, 39] in relation with 3-manifold invariants which were studied in [71] and further explored in [21] in the more general context of abelian invariants.

The first rigorous construction of $SU(2)$-TQFTs was made by Reshetikhin and Turaev in [75] using a quotient of quantum groups $U_q\mathfrak{sl}_2$ (see also [85] for the general construction based on modular categories). A combinatorial construction of the $SU(2)$-TQFTs using skein theory was made in [14] using the construction in [63] of 3-manifold invariants. In this paper, we follow [32] where the authors adapted this point of view in order to construct the Weil representations at even levels.

2.1.2 Statements

Given two integers $p \geq 2$ and $g \geq 1$, the Weil representations are projective unitary representations of the symplectic group $Sp_{2g}(\mathbb{Z})$ into a free module of rank p^g over the ring:

$$k'_p := \begin{cases} \mathbb{Z}\left[A, \frac{1}{2p}\right]/(\phi_p(A)) \ , & \text{when } p \text{ is odd.} \\ \mathbb{Z}\left[A, \frac{1}{p}\right]/(\phi_{2p}(A)), & \text{when } p \text{ is even.} \end{cases}$$

where $\phi_p \in \mathbb{Z}[X]$ represents the cyclotomic polynomial of degree p.

The main result of this paper is the decomposition into irreducible factors of the Weil projective representations $\pi_{p,g} : Sp_{2g}(\mathbb{Z}) \to \mathrm{PGL}(U_p^{\otimes g})$ of the symplectic group. Here U_p is a free k'_p-module of rank p and we identified $U_{p,g}$ with $U_p^{\otimes g}$. The integer $p \geq 2$ will be called the level of the representation. The decomposition of the Weil representation for $g = 1$ and $p = r^n$ for r an odd prime into irreducible factors was given by Kloosterman in [54]. Our decompositions generalize his result to arbitrary genus and levels.

Let $a, b \geq 2$ be two coprime non negative integers with b odd, and let u and v be odd integers such that $au + bv = 1$ in the case where a is odd and such that $2au + bv = 1$ if a is even and b is odd. We define a ring isomorphism $\mu : k'_{ab} \to k'_a \otimes k'_b$ by $\mu(A) = (A^{vb}, A^{au})$ if a is odd and $\mu(A) = (A^{vb}, A^{2au})$ if a is even, which turns $U_a^{\otimes g} \otimes U_b^{\otimes g}$ into a k'_{ab}-module.

For r prime and $n \geq 1$, we define the ring homomorphism $\mu : k'_{r^{n+2}} \to k'_{r^n}$ by $\mu(A) = A^{r^2}$ which turns $U_{r^n}^{\otimes g}$ into a $k'_{r^{n+2}}$-module.

Set $\sigma(p)$ for the number of divisors of p including 1.

Main Theorem 2.1.1. *The level p Weil representation contains $\sigma(p)$ irreducible submodules, when p is odd and $\sigma(\frac{p}{2})$, when p is even. They decompose according to the following rules, where $\cong$ denotes an isomorphism of $Sp_{2g}(\mathbb{Z})$ projective modules:*

1. *If $a, b \geq 2$ are two coprime integers, then:*
$$U_a^{\otimes g} \otimes U_b^{\otimes g} \cong U_{ab}^{\otimes g}$$
2. *If r is prime and $n \geq 1$, then:*
$$U_{r^{n+2}}^{\otimes g} \cong U_{r^n}^{\otimes g} \oplus W_{r^{n+2}}^{\otimes g}$$
where $W_{r^{n+2}}$ is a free submodule of $U_{r^{n+2}}$.
3. *If r is an odd prime, then:*
$$U_{r^2}^{\otimes g} \cong \mathbb{1} \oplus W_{r^2}^{\otimes g}$$
where $\mathbb{1}$ denotes the trivial representation.
4. *Every factor $U_p^{\otimes g}, W_{r^n}^{\otimes g}$ for $p \geq 3$ decomposes into two invariant submodules,*
$$U_p^{\otimes g} \cong U_p^{g,+} \oplus U_p^{g,-}$$
$$W_{r^n}^{\otimes g} \cong W_{r^n}^{g,+} \oplus W_{r^n}^{g,-}$$
We call $U_p^{g,+}$ and $W_{r^n}^{g,+}$ the even modules and $U_p^{g,-}, W_{r^n}^{g,-}$ the odd modules.
5. *The application of the previous four rules decomposes any $U_p^{\otimes g}$ into a direct sum of modules of the form $B_{r_1} \otimes \ldots \otimes B_{r_k}$ with $r_1, \ldots, r_k$ distinct prime numbers and $B_{r_i} \in \{U_{r_i}^{g,\pm}, W_{r_i^n}^{g,\pm}\}$. These modules are all irreducible and pairwise distinct.*

When $g = 1$ the first point of this theorem was proved by Kurlberg and Rudnick in [58].

For instance, when r is an odd prime we have:

$$U_{r^n}^{\otimes g} \cong (W_{r^n}^{g,+} \oplus W_{r^n}^{g,-}) \oplus (W_{r^{n-2}}^{g,+} \oplus W_{r^{n-2}}^{g,-}) \oplus \ldots \oplus (W_{r^2}^{g,+} \oplus W_{r^2}^{g,-}) \oplus \mathbb{1}, \text{ if } n \text{ even.}$$
$$U_{r^n}^{\otimes g} \cong (W_{r^n}^{g,+} \oplus W_{r^n}^{g,-}) \oplus (W_{r^{n-2}}^{g,+} \oplus W_{r^{n-2}}^{g,-}) \oplus \ldots \oplus (W_{r^3}^{g,+} \oplus W_{r^3}^{g,-}) \oplus (U_r^{g,+} \oplus U_r^{g,-}), \text{ if } n \text{ odd.}$$

When $r = 2$ we obtain:

$$U_{2^n}^{\otimes g} \cong (W_{2^n}^{g,+} \oplus W_{2^n}^{g,-}) \oplus (W_{2^{n-2}}^{g,+} \oplus W_{2^{n-2}}^{g,-}) \oplus \ldots \oplus (W_4^{g,+} \oplus W_4^{g,-}), \text{ if } n \text{ is even.}$$
$$U_{2^n}^{\otimes g} \cong (W_{2^n}^{g,+} \oplus W_{2^n}^{g,-}) \oplus (W_{2^{n-2}}^{g,+} \oplus W_{2^{n-2}}^{g,-}) \oplus \ldots \oplus (W_8^{g,+} \oplus W_8^{g,-}) \oplus U_2^{\otimes g}, \text{ if } n \text{ is odd.}$$

We deduce from Theorem 3.3.1 the following:

Corollary 2.1.2. *We have the following decomposition into irreducible modules of the genus one $SO(3)$ and $SU(2)$ quantum representations at level p of $SL_2(\mathbb{Z})$:*

$$V_p \cong \bigoplus_{B \in E, B_1 \in E_1, \ldots, B_k \in E_k} B \otimes B_1 \otimes \ldots \otimes B_k, \text{ when p is even;}$$
$$V_p \cong \bigoplus_{B_1 \in E_1, \ldots, B_k \in E_k} B_1 \otimes \ldots \otimes B_k, \text{ when p is odd.}$$

where $p = 2^m r_1^{n_1} \ldots r_k^{n_k}$ is the factorization into primes and:

– *If j is such that n_j is odd,* $E_j = \left\{ W^+_{r_j^{n_j-2a_j}}, W^-_{r_j^{n_k-2a_j}}, U^+_{r_j}, U^-_{r_j} \mid 0 \leq a_j \leq \lceil \frac{n_k}{2} \rfloor - 1 \right\}$.

– *If j is such that n_j is even,* $E_j = \left\{ W^+_{r_j^{n_j-2a_j}}, W^-_{r_j^{n_k-2a_j}}, \mathbb{1} \mid 0 \leq a_j \leq \lceil \frac{n_k}{2} \rfloor - 1 \right\}$.

– *If m is odd,* $E = \left\{ W^+_{2^{m-2a}}, W^-_{2^{m-2a}}, U_2 \mid 0 \leq a \leq \lceil \frac{m}{2} \rfloor - 1 \right\}$.

– *If m is even,* $E = \left\{ W^+_{2^{m-2a}}, W^-_{2^{m-2a}}, U^+_4, U^-_4 \mid 0 \leq a \leq \lceil \frac{m}{2} \rfloor - 1 \right\}$.

with the condition that each summand $B \otimes B_1 \otimes \ldots \otimes B_k$ or $B_1 \otimes \ldots \otimes B_k$ contains an odd number of modules $U^-_p, W^-_{r^n}$.

Remark.
1. The irreducibility of the $SU(2)$ quantum representations in arbitrary genus at levels $2p$, for p an odd prime, was showed by Roberts in [76] and his proof easily extends to $SO(3)$ quantum representations at odd prime levels. This result was generalized to representations associated to surfaces with one marked point by Gilmer and Masbaum in [37].
2. The commutant of the $\mathbb{C}$-algebra generated by the image of the genus one Weil representations at even levels was studied by the physicists in [17] following the ideas of [33]. Our theorem recovers their result in a different manner and gives an explicit basis for the commutant formed by the projectors on each irreducible factors.

When p is odd, the projective Weil representation lift to linear representation of $Sp_{2g}(\mathbb{Z}/p\mathbb{Z})$ (see [1]). In [31], it is proved that when p is even and $g \geq 2$ the Weil representation does not lift to a linear representation of $Sp_{2g}(\mathbb{Z}/2p\mathbb{Z})$ but only of a non trivial central extension of $Sp_{2g}(\mathbb{Z}/2p\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$. For $g = 1$ however it is not difficult to show that the Weil representations lift to linear representation of $SL_2(\mathbb{Z})$. Explicit lifts are given in Chapter 4 section 3.

Finally, adapting the ideas of [67] to the abelian case, we will prove in Chapter 4 Theorem 4.1.1 the asymptotic faithfulness of the Weil representations already showed in [4] in a different way.

Theorem 2.1.3. *Let $(n_i)_i$ be an unbounded sequence of integers. Then:*

$$\bigcap_{n_i} Ker(\pi_{n_i,g}) = \begin{cases} \{\mathbb{1}, -\alpha.\mathbb{1}\}, & \text{when g} \geq 2 \text{ and all the n}_\text{i} \text{ are even;} \\ \{\mathbb{1}\}, & \text{if all n}_\text{i} \text{ are odd or g} = 1. \end{cases}$$

where α represents the central element added in the central extension $\widetilde{Sp_{2g}(\mathbb{Z})}$ defined in [31] and in Definition 4.3.3.

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2.2 Definition of the projective Weil representations

2.2.1 Abelian skein modules

The following section closely follows the definitions from [32]. To every compact oriented 3-manifold we associate a k'_p-module. The Weil representations arise by means of

the symplectic group action on the skein module associated to a genus g-holed handlebody.

Definition 2.2.1. Let $p \geq 2$ and M be a compact oriented 3-manifold possibly with boundary. The reduced abelian skein module $\widetilde{\mathcal{T}}_p(M)$ is the k'_p-module generated by the isotopy classes of oriented banded links of ribbons in M quotiented by the relations given by the abelian skein relations of Figure 2.1 and by the submodule generated by the links made of p parallel copies of the same ribbon.

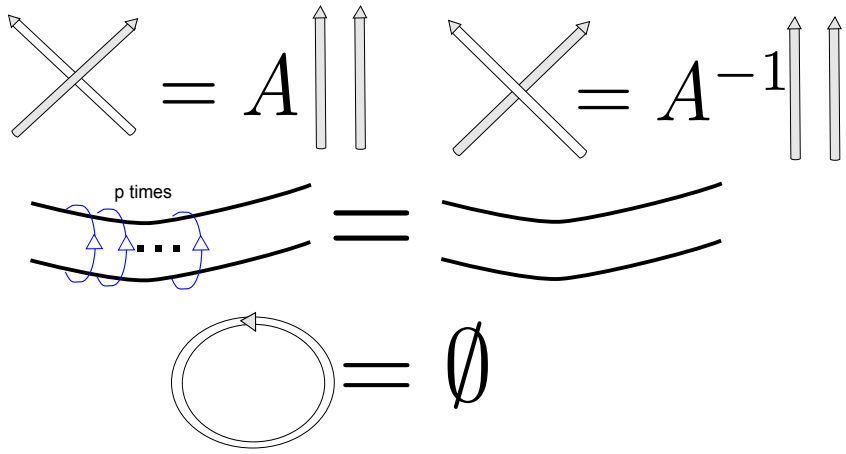


Figure 2.1: Skein relations defining the reduced abelian skein modules.

The reduced abelian skein module of the sphere S^3 has rank one. The class of a link $L \subset S^3$ in this module is equal to the class of the empty link multiplied by $A^{lk(L)}$ where $lk(L)$ represents the self-linking number of L. This gives a natural isomorphism $\widetilde{\mathcal{T}}_p(S^3) \cong k'_p$.

It is classic, see Proposition 1.1.8, that if $M \cong \Sigma \times [0,1]$ is a thickened surface, where Σ may have a boundary, then its skein module $\mathcal{T}_p(M)$ is isomorphic to $k'_p[H_1(\Sigma, \mathbb{Z}/p\mathbb{Z})]$.

2.2.2 Heisenberg groups and Schrödinger representations

Denote by H_g the genus g handlebody. Its abelian skein module is freely generated by the elements of $H_1(H_g, \mathbb{Z}/p\mathbb{Z})$. So, if we denote by U_p the module $\mathcal{T}_p(S^1 \times D^2)$, we have a natural k_p-isomorphism between $\widetilde{\mathcal{T}}_p(H_g)$ and $U_p^{\otimes g}$.

Definition 2.2.2. Let $g \geq 1$ and Σ_g be a closed oriented surface of genus g.

1. The module $\widetilde{\mathcal{T}}_p(\Sigma_g \times [0,1])$ has an algebra structure with product induced by superposition.
2. The *Heisenberg group* $\mathcal{H}_g$ is defined as the set $H_1(\Sigma_g, \mathbb{Z}) \times \mathbb{Z}$ with group law given by:
$$(X, z) \bullet (X', z') = (X + X', z + z' + \omega(X, X'))$$
where ω represents the intersection form.
A group morphism $\mathcal{H}_g \to \widetilde{\mathcal{T}}_p(\Sigma_g \times [0,1])$ is defined by sending a class $[\gamma] \in H_1(\Sigma_g, \mathbb{Z})$ to the skein class of the link γ inserted in $\Sigma_g \times \{\frac{1}{2}\}$ and by sending the central element

$(0,1) \in H_1(\Sigma_g, \mathbb{Z}) \times \mathbb{Z}$ to the element $c = A \cdot \mathbb{1} \in \widetilde{\mathcal{T}_p}(\Sigma_g \times [0,1])$ product of the class of the empty link by $A \in k_p$.

3. We choose a homeomorphism $\phi : \Sigma_g \to \Sigma_g$ so that $(\Sigma_g \times [0,1]) \bigcup_\phi H_g \cong H_g$. This gluing induces a linear action of the Heisenberg group on the skein module $\mathcal{T}_p(H_g) \cong U_p^{\otimes g}$. This representation is called the Schrödinger representation and will be denoted by $\mathrm{Add}_p : \mathcal{H}_g \to \mathrm{GL}(U_p^{\otimes g})$. Up to isomorphism, this representation does not depend on ϕ.

A key ingredient in the definition of the Weil projective representations is the Stone-Von Neumann theorem which states that the Schrödinger representation is the unique irreducible representation of the Heisenberg group sending c to the scalar operator $A \cdot \mathbb{1}$

2.2.3 The Weil representations

Every element of the mapping class group $\mathrm{Mod}(\Sigma_g)$ acts on $H_1(\Sigma_g, \mathbb{Z})$ by preserving the intersection form. Choosing a basis of $H_1(\Sigma_g, \mathbb{Z})$ we obtain a surjective morphism $f : \mathrm{Mod}(\Sigma_g) \to Sp_{2g}(\mathbb{Z})$.

Let $g \geq 1$, the mapping class group $\mathrm{Mod}(\Sigma_g)$ acts on the set of framed links of $\Sigma_g \times [0,1]$ and, by passing through the quotient by the reduced skein relations, acts on the Heisenberg group. We denote by $\bullet$ this action. Let $\phi \in \mathrm{Mod}(\Sigma_g)$ and consider the representation $s^\phi : \mathcal{H}_g \to \mathrm{GL}(U_p^{\otimes g})$ defined by $s^\phi(h) := \mathrm{Add}_p(\phi \bullet h)$ for all $h \in \mathcal{H}_g$. By the Stone-Von Neumann theorem, the representation s^ϕ is conjugate to the Schrödinger representation. Thus there exists $\pi_{p,g}(\phi) \in \mathrm{GL}(U_p^{\otimes g})$, uniquely determined up to multiplication by an invertible scalar, so that:

$$\pi_{p,g}(\phi)\, \mathrm{Add}_p(h) \pi_{p,g}(\phi)^{-1} = \mathrm{Add}_p(\phi \bullet h), \text{ for any } h \in \mathcal{H}_g \tag{2.1}$$

The equation (2.1) is also called the Egorov identity.

Definition 2.2.3. We call Weil representation at level p and genus g the projective representation $\underline{\pi}_{p,g} : \mathrm{Mod}(\Sigma_g) \to \mathrm{PGL}(U_p^{\otimes g})$.

It easily follows that the Weil representation factorizes through the Torelli group and through $Sp_{2g}(\mathbb{Z}/p\mathbb{Z})$ when p is odd and $Sp_{2g}(\mathbb{Z}/2p\mathbb{Z})$ when p is even.

The previous definition of the Weil representations as intertwining operators is not explicit. To manipulate it more easily, we choose the generators of $Sp_{2g}(\mathbb{Z})$ consisting of the image through f of the Dehn twists X_i, Y_i, Z_{ij} of Figure 3.6 (see [62] for a proof these Dehn twists generate the mapping class group). We define the basis $\{e_{a_1} \otimes \ldots \otimes e_{a_g} | a_1, \ldots, a_g \in \mathbb{Z}/p\mathbb{Z}\}$ of $U_p^{\otimes g}$ as in Figure 2.3, that means that $e_{a_1} \otimes \ldots \otimes e_{a_g}$ is the class of a link made of a_i parallel copies of an unframed ribbon encircling the i^{th} hole of H_g one time. To express the image of the generators in the basis, we will first need to define Gauss sums.

Definition 2.2.4. Let $p \geq 2$ and a, b be two integers. We define the Gauss sums by the formulas:

1. $G(a,b,p) := \sum_{k \in \mathbb{Z}/p\mathbb{Z}} A^{ak^2+bk} \in k'_p$ when p is odd.
2. $G(a,b,2p) := \sum_{k \in \mathbb{Z}/2p\mathbb{Z}} A^{ak^2+bk} = 2 \sum_{k \in \mathbb{Z}/p\mathbb{Z}} A^{ak^2+bk} \in k'_p$ when p is even.

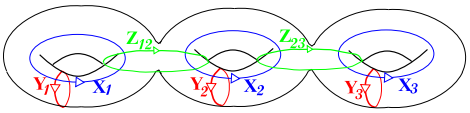


Figure 2.2: A set of Dehn twists generating the mapping class group and the symplectic group.

$$e_{a_1} \otimes e_{a_2} \otimes e_{a_3} =$$

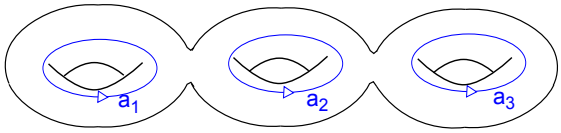


Figure 2.3: A basis for the abelian skein module of the genus g handlebody. Here an integer i in front of a ribbon means that we take i parallel copies of it.

The computation of the Gauss sums is detailed in [10].

Proposition 2.2.5. *The expression of the matrices of the Weil representation on the generators X_i, Y_i and $Z_{i,j}$ in the basis $\{e_{a_1} \otimes \ldots \otimes e_{a_g} | a_1, \ldots, a_g \in \mathbb{Z}/p\mathbb{Z}\}$ of $U_p^{\otimes g}$ is given by the projective class of the following matrices:*

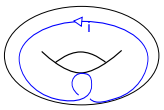
- $\pi_{p,1}(X) = (A^{2i^2}\delta_{i,j})_{i,j}$ *and* $\pi_{p,g}(X_i) = \mathbb{1}^{\otimes(i-1)} \otimes \pi_{p,1}(X) \otimes \mathbb{1}^{\otimes(g-i)}$.
- $\pi_{p,g}(Z_{i,j})(e_{a_1} \otimes \ldots \otimes e_{a_g}) = A^{(a_i - a_j)^2}(e_{a_1} \otimes \ldots \otimes e_{a_g})$.
- $\pi_{p,1}(Y) = \begin{cases} \frac{G(1,0,p)}{p}(A^{-(i-j)^2})_{i,j}, & \text{when } p \text{ is odd.} \\ \frac{G(1,0,2p)}{2p}(A^{-(i-j)^2})_{i,j}, & \text{when } p \text{ is even.} \end{cases}$

  $\pi_{p,g}(Y_i) = \mathbb{1}^{\otimes(i-1)} \otimes \pi_{p,1}(Y) \otimes \mathbb{1}^{\otimes(g-i)}$.

These generating matrices are unitary (they verify $\bar{U}^T U = \mathbf{1}$ where $\bar{U} = \left(\bar{U_{i,j}}\right)_{i,j}$ is defined by the involution of k_p' sending A to A^{-1}) so are the Weil representations.

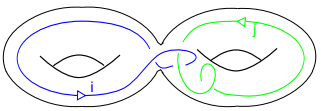
Proof. If $\phi \in \mathrm{Mod}(\Sigma_g)$ can be extended to a homeomorphism Φ of the handlebody H_g, the action of Φ on $\mathcal{T}_p(H_g) \cong U_p^{\otimes g}$ defines an operator which satisfies the Egorov identity (2.1) so is projectively equal to $\underline{\pi}_{p,g}(\phi)$. The generators X_i and $Z_{i,j}$ are such homeomorphisms and Figure 2.4 shows how we compute their action on the basis.

$$\pi_{p,1}(X) \cdot e_i =$$



$$= A^{i^2} e_i$$

$$\pi_{p,2}(Z_{1,2}) \cdot e_i \otimes e_j =$$



$$= A^{(i-j)^2} e_i \otimes e_j$$

Figure 2.4: The computation of the matrices associated to $\pi_{p,1}(X)$ and $\pi_{p,2}(Z_{1,2})$.

Then choose a Heegaard splitting of the sphere $H_g \bigcup_\phi H_g \cong S^3$ with $\phi \in \mathrm{Mod}(\Sigma_g)$. This splitting determines a pairing $\mathcal{T}_p(H_g) \times \mathcal{T}_p(H_g) \to \mathcal{T}_p(S^3)$. The associated bilinear pairing $(\cdot,\cdot)_p^H : U_p^{\otimes g} \otimes U_p^{\otimes g} \to k_p'$ is called the Hopf pairing. Figure 2.5 shows that:

$$\left(e_{a_1} \otimes \ldots \otimes e_{a_g}, e_{b_1} \otimes \ldots \otimes e_{b_g}\right)_p^H = A^{-2\sum_i a_i b_i}$$

Thus the Hopf pairing is non degenerate.

$$\left(e_a \otimes e_b, e_{a'} \otimes e_{b'}\right)_p^H = \quad = A^{-2ab-2a'b'}$$

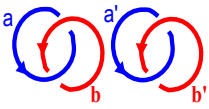


Figure 2.5: The computation of the matrix associated to the Hopf pairing when $g = 2$.

The dual of $\pi_{p,g}(X_i)$ for $\langle \cdot, \cdot \rangle^H$ satisfies the Egorov identity (2.1), so is projectively equal to $\underline{\pi}_{p,g}(Y_i)$. If $\pi p, 1(Y)$ is the dual of $\pi_{p,1}(X)$ for $(\cdot, \cdot)^H$, the previous expression of $\pi_{p,g}(X_i)$ implies that its dual for the Hopf pairing is $\mathbb{1}^{\otimes(i-1)} \otimes \pi_{p,1}(Y) \otimes \mathbb{1}^{\otimes(g-i)}$.

To compute the matrix of $\pi_{p,1}(Y)$, we remark that the matrix $S = \left(A^{-2ij}\right)_{i,j}$ of the Hopf pairing has inverse $S^{-1} = \frac{1}{p}\bar{S} = \frac{1}{p}(A^{2ij})_{i,j}$. A direct computation gives:

$$\pi_{p,1}(Y) = S\pi_{p,1}(X)S^{-1} = \begin{cases} \frac{G(1,2(j-i),p)}{p} = \frac{G(1,0,p)}{p}(A^{-(i-j)^2})_{i,j}, & \text{when p is odd;} \\ \frac{G(1,2(j-i),2p)}{2p} = \frac{G(1,0,2p)}{2p}(A^{-(i-j)^2})_{i,j}, & \text{when p is even.} \end{cases}$$

□

Remark. When p is even and $A = \exp\left(-\frac{i\pi}{p}\right)$, the projective representations we defined here coincide with the ones from [30] and [39] coming from theta functions.

2.2.4 Linearization of the Weil representations

When $g = 1$ and p is even, to have an explicit lift of the Weil representations as linear representation of $SL_2(\mathbb{Z}/p\mathbb{Z})$, we must add the ring k_p a primitive 24^{th} root of unity, so we change the ring to $k_p = \mathbb{Z}[A, \frac{1}{p}, \beta]/(\phi_{2p}(A), \phi_{24}(\beta))$. An explicit lift of the genus one Weil representations is given by the matrices:

$$\pi_{p,1}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) := \begin{cases} \frac{G(-1,0,p)}{p}(A^{-2ij})_{i,j}, & \text{if } p \text{ is odd.} \\ \beta^{-3}\frac{G(-1,0,2p)}{2p}(A^{-2ij})_{i,j}, & \text{if } p \text{ is even.} \end{cases} \tag{2.2}$$

$$\pi_{p,1}\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right) := \begin{cases} (A^{i^2}\delta_{i,j})_{i,j}, & \text{if } p \text{ is odd.} \\ \beta(A^{i^2}\delta_{i,j})_{i,j}, & \text{if } p \text{ is even.} \end{cases} \tag{2.3}$$

In Chapter 4 section 3, we prove that if $g \geq 2$ and p is even, the matrices $\pi_{p,g}(X_i), \pi_{p,g}(Y_i)$ and $\pi_{p,g}(Z_{i,j})$, defined in Proposition 2.2.5, generate a linear unitary representation of a non trivial extension of $Sp_{2g}(\mathbb{Z}/2p\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$.

When $g \geq 2$ and p is odd, it is known (see [1]) that the Weil representations lift to linear unitary representation of $Sp_{2g}(\mathbb{Z}/p\mathbb{Z})$. A direct computation similar to the one of [31] (see Theorem 4.3.4) shows that the matrices $\pi_{p,g}(X_i), \pi_{p,g}(Y_i)$ and $\pi_{p,g}(Z_{i,j})$ defined in Proposition 2.2.5 define an explicit lift.

We will denote by $\widetilde{Sp_{2g}(\mathbb{Z})}$ the central extension of $Sp_{2g}(\mathbb{Z}/2p\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$ defined in [31]. We have the following short exact sequence:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{i} \widetilde{Sp_{2g}(\mathbb{Z})} \xrightarrow{j} Sp_{2g}(\mathbb{Z}) \to 1$$

We will denote by α the image of -1 by i.

2.3 Decomposition of the Weil representations

In this section we prove the three first points of the Theorem 2.1.1. We first define:

$$U_p^{+,g} := \mathrm{Span}\{e_{a_1} \otimes \ldots \otimes e_{a_g} + e_{-a_1} \otimes \ldots \otimes e_{-a_g} | a_1, \ldots, a_g \in \mathbb{Z}/p\mathbb{Z}\}$$
$$U_p^{-,g} := \mathrm{Span}\{e_{a_1} \otimes \ldots \otimes e_{a_g} - e_{-a_1} \otimes \ldots \otimes e_{-a_g} | a_1, \ldots, a_g \in \mathbb{Z}/p\mathbb{Z}\}$$

where Span means that we consider the k'_p module generated by the elements in brackets.

Lemma 2.3.1. *The submodules $U_p^{+,g}$ and $U_p^{-,g}$ are $\pi_{p,g}$-stable.*

We will note $\pi_{p,g}^+$ and $\pi_{p,g}^-$ the corresponding restrictions of $\pi_{p,g}$ and call them the even and odd subrepresentations respectively.

Proof. A direct computation shows that the submodules $U_p^{g,+}$ and $U_p^{g,-}$ are stabilized by $\pi_{p,g}(X_i), \pi_{p,g}(Y_i)$ and $\pi_{p,g}(Z_{i,j})$.

□

Remark. To ensure that the injection $U_p^+ \oplus U_p^- \to U_p$ is an isomorphism of k'_p-modules, we need 2 to be invertible in k'_p.

Let $a, b \geq 2$ be two coprime non negative integers with b odd, and let u and v be odd integers such that $au + bv = 1$ in the case where a is odd and such that $2au + bv = 1$ if a is even and b is odd. We define a ring isomorphism $\mu : k'_{ab} \to k'_a \otimes k'_b$ by $\mu(A) = (A^{vb}, A^{au})$ if a is odd and $\mu(A) = (A^{vb}, A^{2au})$ if a is even, which turns $U_a^{\otimes g} \otimes U_b^{\otimes g}$ into a k'_{ab}-module. We also denote by $f : \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z} \to \mathbb{Z}/ab\mathbb{Z}$ the bijection sending (x, y) to $xv + yu$ when a is odd and to $xv + 2yu$ when a is even.

Lemma 2.3.2. *The isomorphism of k'_{ab}-modules $\psi : U_a^{\otimes g} \otimes U_b^{\otimes g} \to U_{ab}^{\otimes g}$ defined by*

$$\psi((e_{a_1} \otimes \ldots \otimes e_{a_g}) \otimes (e_{b_1} \otimes \ldots \otimes e_{b_g})) = e_{f(a_1,b_1)} \otimes \ldots \otimes e_{f(a_g,b_g)}$$

makes the following diagram commute for all $\phi \in Sp_{2g}(\mathbb{Z})$ (resp for all $\phi \in \widetilde{Sp_{2g}(\mathbb{Z})}$ when a is even):

$$\begin{array}{ccc} U_a^{\otimes g} \otimes U_b^{\otimes g} & \xrightarrow{\psi} & U_{ab}^{\otimes g} \\ {\scriptstyle \pi_{g,a}(\phi)\otimes\pi_{g,b}(\phi)}\uparrow & & \uparrow{\scriptstyle \pi_{g,ab}(\phi)} \\ U_a^{\otimes g} \otimes U_b^{\otimes g} & \xrightarrow{\psi} & U_{ab}^{\otimes g} \end{array}$$

Proof. We note $(A_1, A_2) := (A^{vb}, A^{au})$ when a and b are odd and $(A_1, A_2) = (A^{vb}, A^{2au})$ when a is even. It is enough to show the commutativity of the diagram for $\phi = X_i, Y_i$ and $Z_{i,j}$. For $\phi = X_i$, we compute:

$$\psi\left(\pi_{a,g}(X_i) \otimes \pi_{b,g}(X_i)((e_{a_1} \otimes \ldots \otimes e_{a_g}) \otimes (e_{b_1} \otimes \ldots \otimes e_{b_g}))\right) =$$
$$\psi\left(A_1^{a_i^2} A_2^{b_i^2}((e_{a_1} \otimes \ldots \otimes e_{a_g}) \otimes (e_{b_1} \otimes \ldots \otimes e_{b_g}))\right) =$$
$$A^{f(a_i,b_i)^2}(e_{f(a_1,b_1)} \otimes \ldots \otimes e_{f(a_g,b_g)})$$

Then for $\phi = Y_i$, we note $c_p = \frac{G(1,0,p)}{p}$ when p is odd and $c_p = \frac{G(1,0,2p)}{2p}$ when p is even:

$$\begin{aligned}
&\psi\left(\pi_{a,g}(Y_i)\otimes\pi_{b,g}(Y_i)((e_{a_1}\otimes\ldots\otimes e_{a_g})\otimes(e_{b_1}\otimes\ldots\otimes e_{b_g}))\right)\\
&=\psi\left(c_ac_b\sum_{\substack{k\in\mathbb{Z}/a\mathbb{Z}\\ l\in\mathbb{Z}/b\mathbb{Z}}}A_1^{-(a_i-k)^2}A_2^{-(b_i-l)^2}((e_{a_1}\otimes\ldots\otimes e_k\otimes\ldots\otimes e_{a_g})\otimes(e_{b_1}\otimes\ldots\otimes e_l\otimes\ldots\otimes e_{b_g}))\right)\\
&=\psi(c_ac_b)\sum_{m\in\mathbb{Z}/ab\mathbb{Z}}A^{-(f(a_i,b_i)-m)^2}(e_{f(a_1,b_1)}\otimes\ldots\otimes e_m\otimes\ldots\otimes e_{f(a_g,b_g)})
\end{aligned}$$

where we made the change of variable $m = f(k,l)$ to pass to the last line. It remains to show that $\psi(c_ac_b) = c_{ab}$ which is equivalent to show that $\psi(G(1,0,a)G(1,0,b)) = G(1,0,ab)$ when a is odd and $\psi(G(1,0,2a)G(1,0,b)) = G(1,0,2ab)$ when a is even. When a is even:

$$\begin{aligned}
\psi(G(1,0,2a)G(1,0,b))/2 &= \sum_{(x,y)\in\mathbb{Z}/a\mathbb{Z}\times\mathbb{Z}/b\mathbb{Z}}A_1^{x^2}A_2^{y^2}\\
&= \sum_{(x,y)}A^{vx^2+2uy^2} = \sum_{z\in\mathbb{Z}/ab\mathbb{Z}}A^{z^2} = G(1,0,2ab)/2
\end{aligned}$$

Finally for $\phi = Z_{i,j}$:

$$\begin{aligned}
&\psi\left(\pi_{a,g}(Z_{i,j})\otimes\pi_{b,g}(Z_{i,j})((e_{a_1}\otimes\ldots\otimes e_{a_g})\otimes(e_{b_1}\otimes\ldots\otimes e_{b_g}))\right)\\
&\quad=\psi\left(A_1^{(a_i-a_j)^2}A_2^{(b_i-b_j)^2}((e_{a_1}\otimes\ldots\otimes e_{a_g})\otimes(e_{b_1}\otimes\ldots\otimes e_{b_g}))\right)\\
&\quad=A^{f(a_i,b_i)^2}(e_{f(a_1,b_1)}\otimes\ldots\otimes e_{f(a_g,b_g)})
\end{aligned}$$

□

Remark. This lemma also follows from ([71], Proposition 2.3) where it is showed that the 3-manifold invariant coming from the abelian TQFT at level ab, with a coprime to b, is the product of the ones in level a and b.

Let r be a prime number and $n \geq 0$ if r is odd or $n \geq 1$ if $r = 2$. Let $\bar{U}_{r^n}^{\otimes g}$ be the submodule of $U_{r^{n+2}}^{\otimes g}$ spanned by the vectors $g_{a_1}\otimes\ldots\otimes g_{a_g}$ where $g_i := \sum_{0\leq k\leq r-1} e_{r(i+kr^n)}$.

Lemma 2.3.3. *The submodule $\bar{U}_{r^n}^{\otimes g}$ is stabilized by $\pi_{r^{n+2},g}$. Moreover the isomorphism of $k'_{r^{n+2}}$-modules $\psi : U_{r^n}^{\otimes g} \to \bar{U}_{r^n}^{\otimes g}$ sending $e_{a_1}\otimes\ldots\otimes e_{a_g}$ to $g_{a_1}\otimes\ldots\otimes g_{a_g}$ makes the following diagram commute for all $\phi \in Sp_{2g}(\mathbb{Z})$ (for all $\phi \in \widetilde{Sp_{2g}(\mathbb{Z})}$ when $r = 2$ respectively):*

$$\begin{array}{ccc}
\mathrm{GL}(U_{r^{n+2}}^{\otimes g})^{\bar{U}_{r^n}^{\otimes g}} & \xrightarrow{\pi_{r^{n+2},g}(\phi)} & \mathrm{GL}(U_{r^{n+2}}^{\otimes g})^{\bar{U}_{r^n}^{\otimes g}}\\
\uparrow & & \uparrow\\
\mathrm{GL}(U_{r^n}^{\otimes g}) & \xrightarrow{\pi_{r^n,g}(\phi)} & \mathrm{GL}(U_{r^n}^{\otimes g})
\end{array}$$

Proof. We generalize an argument of [19] to arbitrary genus to show that $\bar{U}_{r^n}^{\otimes g}$ is $\pi_{r^{n+2},g}$-stable. Denote by I the principal ideal $I := r^{n+1}H_1(\Sigma^g, \mathbb{Z}/r^{n+2}\mathbb{Z})$ of $H_1(\Sigma^g, \mathbb{Z}/r^{n+2}\mathbb{Z})$ and by D the subgroup $D := (I \times I, 0)$ of $\mathcal{H}_{r^{n+2},g}$. Since $I^2 = \{0\}$ and I is an ideal, D is a subgroup of $\mathcal{H}_{r^{n+2},g}$ stable under the action of $Sp_{2g}(\mathbb{Z})$. We deduce from the Egorov identity that the space $\{v \in U_{r^{n+2}}^{\otimes g} | \, \mathrm{Add}_p(\phi)v = v, \forall \phi \in D\}$ is preserved by $\pi_{r^{n+2},g}$. We now easily show that this space is $\bar{U}_{r^n}^{\otimes g}$.

We then verify the commutativity of the diagram for $\phi = X_i, Y_i$ and $Z_{i,j}$. When $\phi = X_i$ we have:

$$\pi_{r^{n+2},g}(X_i)(g_{a_1} \otimes \ldots \otimes g_{a_g}) = A^{(ri)^2}(g_{a_1} \otimes \ldots \otimes g_{a_g}) = \mu(A)^{i^2}(g_{a_1} \otimes \ldots \otimes g_{a_g})$$

When $\phi = Y_i$ we have:

$$\pi_{r^{n+2},g}(Y_i)(g_{a_1} \otimes \ldots \otimes g_{a_g}) = c_{r^{n+2}} \sum_{x \in \mathbb{Z}/r^{n+2}} \sum_{k \in \mathbb{Z}/r\mathbb{Z}} A^{-(r(a_i+kr^n)-x)^2} g_{a_1} \otimes \ldots \otimes e_x \otimes \ldots \otimes e_{a_g}$$

$$= c_{r^{n+2}} \sum_{x \in \mathbb{Z}/r^{n+2}} A^{-x^2-xra_i-r^2a_i^2} \left(\sum_{k \in \mathbb{Z}/r\mathbb{Z}} (A^{2r^{(n+1)}x})^k \right) g_{a_1} \otimes \ldots \otimes e_x \otimes \ldots \otimes g_{a_g}$$

$$= rc_{r^{n+2}} \sum_{y \in \mathbb{Z}/r^{n+1}\mathbb{Z}} (A^{r^2})^{-(y-a_i)^2} g_{a_1} \otimes \ldots \otimes e_{ry} \otimes \ldots \otimes g_{a_g}$$

$$= rc_{r^{n+2}}(\mu(A))^{-(z-a_i)^2} \sum_{z \in \mathbb{Z}/r^n\mathbb{Z}} g_{a_1} \otimes \ldots \otimes g_z \otimes \ldots \otimes g_{a_k}$$

We verify that $\mu(c_{r^n}) = rc_{r^{n+2}}$ to conclude in this case. Finally when $\phi = Z_{i,j}$:

$$\pi_{r^{n+2},g}(Z_{i,j})(g_{a_1} \otimes \ldots \otimes g_{a_g})$$

$$= \sum_{k,l \in \mathbb{Z}/p\mathbb{Z}} A^{(r(a_i+kr^n)-p(a_j+lr^n))^2}(g_{a_1} \otimes \ldots e_{r(a_i+kr^n)} \otimes \ldots \otimes e_{r(a_j+lr^n)} \otimes \ldots \otimes g_{a_g})$$

$$= \sum_{k,l \in \mathbb{Z}/p\mathbb{Z}} (A^{r^2})^{(a_i-a_j)^2}(g_{a_1} \otimes \ldots e_{r(a_i+kr^n)} \otimes \ldots \otimes e_{r(a_j+lr^n)} \otimes \ldots \otimes g_{a_g})$$

$$= (\mu(A)^{(a_i-a_j)^2}(g_{a_1} \otimes \ldots \otimes g_{a_g})$$

□

Let $W_{r^{n+2}}$ be the submodule of U_{r^n} orthogonal for the invariant form turning $\{e_0, \ldots, e_{r^{n+2}-1}\}$ into an orthogonal basis. It is freely generated by the vectors e_i when r does not divide i and by the vectors $e_{ri-r(i+k+r^n)}$ for $i \in \{0, \ldots, r^n - 1\}$ and $k \in \{1, \ldots, r-1\}$.

The orthogonal of $\bar{U}_{r^n}^{\otimes g}$ in $U_{r^{n+2}}^{\otimes g}$ is isomorphic to $W_{r^{n+2}}^{\otimes g}$ and is stabilized by $\pi_{r^{n+2},g}$. So are the two submodules $W_{r^{n+2}}^{g,\pm} := W_{r^{n+2}}^{\otimes g} \bigcap U_{r^{n+2}}^{g,\pm}$.

2.4 Irreducibility of the factors

2.4.1 The genus one cases

The irreducibility of the factors U_r^-, for r prime, has already been shown by Roberts in [76] by computing the commutator of the associated group algebra. The irreducibility of the factors $U_{r^n}^{\pm}$ and $W_{r^n}^{1,\pm}$, for r an odd prime, was proved by Kloosterman in [54].

When $g = 1$ the strategy for the proof lies on the computation of the sums:

$$\mathbf{S}_p := \frac{1}{|SL_2(\mathbb{Z}/p\mathbb{Z})|} \sum_{\phi \in SL_2(\mathbb{Z}/p\mathbb{Z})} |\operatorname{Tr}(\pi_p(\phi))|^2, \quad \text{when } p \text{ is odd.} \tag{2.4}$$

$$\mathbf{S}_{2p} := \frac{1}{|SL_2(\mathbb{Z}/2p\mathbb{Z})|} \sum_{\phi \in SL_2(\mathbb{Z}/2p\mathbb{Z})} |\operatorname{Tr}(\pi_p(\phi))|^2, \quad \text{when } p \text{ is even.} \tag{2.5}$$

It is a classical fact that if this sum is equal to the number of component in a decomposition of π_p then each factors appearing in this decomposition is irreducible and they are pairwise distinct (see [79], chapter 2).

Lemma 2.4.1. *If a is prime to b then $\boldsymbol{S}_{ab} = \boldsymbol{S}_a \times \boldsymbol{S}_b$ if they are both odd and $\boldsymbol{S}_{2ab} = \boldsymbol{S}_{2a} \times \boldsymbol{S}_b$ if a is even.*

Proof. This follows from the fact that we have a group isomorphism $SL_2(\mathbb{Z}/ab\mathbb{Z}) \cong SL_2(\mathbb{Z}/a\mathbb{Z}) \times SL_2(\mathbb{Z}/b\mathbb{Z})$ together with Proposition 2.3.2. □

In [54] Kloosterman showed that for an odd prime r and $n \geq 1$ then $\mathbf{S}_{r^n} = n + 1$. Thus, to complete the proof of Theorem 2.1.1 it remains to show the following:

Proposition 2.4.2. *For $n \geq 1$, we have:*

$$\boldsymbol{S}_{2^n} = n - 1$$

Since the summand $|\operatorname{Tr}(\pi_{2^n}(\phi))|^2$ only depends on the conjugacy class of ϕ we will first make a complete study of the conjugacy classes of $SL_2(\mathbb{Z}/2^n\mathbb{Z})$. Then we will compute the characters of the Weil representations on representatives of each conjugacy classes.

Conjugacy classes of $SL_2(\mathbb{Z}/2^n\mathbb{Z})$

We begin by defining three invariants of the conjugacy classes which almost classify the conjugacy classes:

Definition 2.4.3. For $A \in SL_2(\mathbb{Z}/2^n\mathbb{Z})$ there exists a unique integer $l \in \{0, \dots, n\}$ and $x \in \{0, \dots, 2^l - 1\}$ such that:

$$A \equiv x\mathbb{1} + 2^l U_1 \pmod{2^n}$$

for some matrix U_1 whose reduction modulo 2 is neither the identity, nor the null matrix. We define a third integer

$$\tau := \begin{cases} \operatorname{Tr}(A) \in \mathbb{Z}/2^n\mathbb{Z}, & \text{when } l = 0. \\ \det(U_1) \in \mathbb{Z}/2^{n-l}\mathbb{Z}, & \text{when } l \geq 1. \end{cases}$$

Note that $\det(U) = 1 \pmod{2^n}$ implies that $x^2 = 1 \pmod{2^l}$ hence if $l = 1$ then $x = 1$, when $l = 2$ then $x = 1$ or 3, when $l \geq 3$ we have four choices: $x = 1, 2^l - 1, 2^{l-1} + 1$ or $2^{l-1} - 1$.

Let us denote by $C(x, l, \tau)$ the set of matrices of $SL_2(\mathbb{Z}/2^n\mathbb{Z})$ having x, l and τ as invariants. Clearly $C(-1, l, \tau) = -C(1, l, \tau)$ and $C(2^{l-1} - 1, l, \tau) = -C(2^{l-1} + 1, l, \tau)$, thus we only need to study the conjugacy classes of $C(x, l, \tau)$ when $x = 1$ or $x = 2^{l-1} + 1$.

As example, the matrices with $l = 0$ are the matrices which are not equal to the identity matrix modulo 2 whereas those with $l = n$ are the four scalar matrices.

Definition 2.4.4. We define the following representatives of $C(x,l,\tau)$, where c_1 will denote an odd number:

– $l=0$, $A_0(\tau,c_1):=\begin{pmatrix}1 & c_1^{-1}(\tau-2)\\ c_1 & \tau-1\end{pmatrix}$.

– $l\geq 1$, $x=1$, $A_l(\tau,c_1):=\begin{pmatrix}1 & c_1^{-1}2^l\tau\\ c_1 2^l & 1+2^l\tau\end{pmatrix}$.

– $l\geq 3$, $x=1+2^{l-1}$, $B_l(\tau,c_1):=\begin{pmatrix}1+2^{l-1} & -c_1^{-1}2^l\tau\\ 2^l c_1 & 1+2^{l-1}-(1+2^{l-1})^{-1}(2^l+2^{2l-2}+2^{2l}\tau)\end{pmatrix}$.

Similar representative for $x=-1$ and $x=2^{l-1}-1$ are given by taking $-A_l$ and $-B_l$.

Proposition 2.4.5. *Each set $C(x,l,\tau)$ contains $1,2$ or 4 conjugacy classes each containing a matrix $\pm A_l(\tau,c_1)$ or $\pm B_l(\tau,c_1)$ for a suitable choice of c_1. The following table gives for every l,x,τ a set of $1,2$ or 4 representatives and the cardinal $m(A)$ of the corresponding conjugacy classes:*

l and x	*τ*	*Representatives of $C(x,l,\tau)$*	*$m(A)$*
$l=0$	$\mathrm{Tr}(U)=\tau$ *is odd*	$A_0(\tau,1)$	2^{2n-1}
	$\mathrm{Tr}(U)=\tau=2 \pmod 4$	$A_0(\tau,1),A_0(\tau,3),A_0(\tau,5),A_0(\tau,7)$	$3\cdot 2^{2n-4}$
	$\mathrm{Tr}(U)=\tau=0 \pmod 4$	$A_0(\tau,1),A_0(\tau,3)$	$3\cdot 2^{2n-3}$
$l=1$ *and* $x=1$	$\tau=1 \pmod 8$	$A_1(\tau,1),A_1(\tau,3),A_1(\tau,5),A_1(\tau,7)$	$3\cdot 2^{2n-6}$
	$\tau=3,5,7 \pmod 8$	$A_1(\tau,1),A_1(\tau,\tau)$	$3\cdot 2^{2n-5}$
	$\tau=2,4,6 \pmod 8$	$A_1(\tau,1),A_1(\tau,3)$	$3\cdot 2^{2n-5}$
	$\tau=0 \pmod 8$	$A_1(\tau,1),A_1(\tau,3),A_1(\tau,5),A_1(\tau,7)$	$3\cdot 2^{2n-6}$
$2\leq l\leq n-3$ *and* $x=1$	$\tau=1,4,5 \pmod 8$	$A_l(\tau,1),A_l(\tau,3)$	$3\cdot 2^{2n-2l-3}$
	$\tau=3,7 \pmod 8$	$A_l(\tau,1)$	$3\cdot 2^{2n-2l-2}$
	$\tau=2 \pmod 8$	$A_l(\tau,1),A_l(\tau,5)$	$3\cdot 2^{2n-2l-3}$
	$\tau=0 \pmod 8$	$A_l(\tau,1),A_l(\tau,3),A_l(\tau,5),A_l(\tau,7)$	$3\cdot 2^{2n-2l-4}$
$l=n-2$ *and* $x=1$	$\tau=0,1 \pmod 4$	$A_{n-2}(\tau,1),A_{n-2}(\tau,3)$	6
	$\tau=2,3 \pmod 4$	$A_{n-2}(\tau,1)$	12
$l=n-1$ *and* $x=1$	$\tau=0 \pmod 2$	$A_{n-1}(0,1)$	3
	$\tau=1 \pmod 2$	$A_{n-1}(1,1)$	3
$3\leq l\leq n-1$ *and* $x=1+2^{l-1}$	τ *odd*	$B_l(\tau,1)$	$2^{2n-2l-1}$
	τ *even*	$B_l(\tau,1)$	$3\cdot 2^{2n-2l-1}$
$l=n$		$\mathbb{1},-\mathbb{1},(2^{n-1}+1)\mathbb{1}$ *and* $(2^{n-1}-1)\mathbb{1}$	1

Proposition 2.4.5 gives the complete description of the conjugacy classes of $SL_2(\mathbb{Z}/2^n\mathbb{Z})$. The exact information needed for computing $\mathbf{S}_{2^n}$ is summarized in the following:

Corollary 2.4.6. *For $A\in SL_2(\mathbb{Z}/2^n\mathbb{Z})$ we define $s(A)\in\{2l,\dots,l+n\}$ to be the maximal s for which 2^{s-l} divides τ. Let $N(l,x)$, resp $N(l,x,s)$, be the number of matrices having l,x (resp s) as invariants. We deduce from Theorem 2.4.5 the following:*

1. $N(0,1,0) = 2^{3n-2}$.
2. *For* $1 \leq s \leq n-1$, $N(0,1,s) = 3 \cdot 2^{3n-s-3}$.
3. $N(0,1,n) = 3 \cdot 2^{2n-2}$.
4. *For* $l \geq 1$, $N(l,1,s) = 3.2^{3n-l-s-3}$ *if* $s \neq l+n$ *and* $N(l,1,n+l) = 3 \cdot 2^{2n-2l-2}$.
5. *For* $l \geq 2$, $N(l,-1) = 3 \cdot 2^{3n-3l-2}$.
6. *For* $l \geq 3$, $N(l,1+2^{l-1}) = N(l,2^{l-1}-1) = 2^{3n-3l}$.
7. $N(n,x) = 1$.

The proof of Proposition 2.4.5 will be deduced from the following:

Lemma 2.4.7. *Let* $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $U' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ *be two matrices of* $C(x,l,\tau)$. *If* $l = 0$, *we suppose that* c *and* c' *are odd. If* $l \geq 1$, *writing* $U = x\mathbb{1} + \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ *we suppose that* c_1 *and* c_1' *are odd. Note that each conjugacy class contains an element satisfying these conditions. We define* $E_{U,U'}$ *the following equation:*

$$\begin{aligned} c_1x^2 + (a_1 - d_1)xy - b_1y^2 &\equiv c_1' \pmod{2^{n-l}}, \quad \text{when } l \geq 1; \\ cx^2 + (a-d)xy - by^2 &\equiv c' \pmod{2^n}, \quad \text{when } l = 0. \end{aligned}$$

Then we have the two following properties:

1. *The matrix* U *is conjugate to* U' *if and only if* $E_{U,U'}$ *has solutions.*
2. *If* k *is the number of solutions of* $E_{U,U}$ *then the conjugacy class of* U *has* $m(U) = \frac{1}{k}3 \cdot 2^{3n-3l-2}$ *elements.*

Once this Lemma proved, the proof of Theorem 2.4.5 will follow from the study of the equations $E_{U,U'}$. We will need the Hensel's Lemma (see [18], section 3.2) which states that if $n \geq 1$, $x_0 \in \mathbb{Z}/2^n\mathbb{Z}$ and $P \in \mathbb{Z}[x]$ is a polynomial such that $P(x_0) \equiv 0 \pmod{2^n}$ and $P'(x_0)$ is odd, then there exists a unique element $\tilde{x_0} \in \mathbb{Z}/2^{n+1}\mathbb{Z}$ such that $\tilde{x_0} \equiv x_0 \pmod{2^n}$ and $P(\tilde{x_0}) \equiv 0 \pmod{2^{n+1}}$.

Lemma 2.4.8. *Let* $A \in SL_2(\mathbb{Z}/2^n\mathbb{Z})$, *then there exists exactly* 8 *matrices* $\tilde{A} \in SL_2(\mathbb{Z}/2^{n+1}\mathbb{Z})$ *such that* $\tilde{A} \equiv A \pmod{2^n}$.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then at least one entry of A must be odd. Suppose c is odd. There are exactly 8 ways to lift a, c and d into elements $\tilde{a}, \tilde{c}, \tilde{d}$ in $\mathbb{Z}/2^{n+1}\mathbb{Z}$. Using Hensel's Lemma to the polynomial $P(b) := -\tilde{c}b + \tilde{a}\tilde{d} - 1$ we show that for each of this 8 choices, there is exactly one way to lift b in $\mathbb{Z}/2^{n+1}\mathbb{Z}$ such that the corresponding matrix $\tilde{A}$ lies in $SL_2(\mathbb{Z}/2^{n+1}\mathbb{Z})$. □

Note that this lemma easily implies by induction that the cardinal of $SL_2(\mathbb{Z}/2^n\mathbb{Z})$ is $3 \cdot 2^{3n-2}$.

Proof of Lemma 2.4.7. Suppose that $X = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \in SL_2(\mathbb{Z}/2^n\mathbb{Z})$ is such that $XUX^{-1} = U'$. A simple computation shows that XUX^{-1} has the form $XUX^{-1} = \begin{pmatrix} * & * \\ cy_2^2 + (a-d)x_2y_2 - bx_2^2 & * \end{pmatrix}$. Thus (y_2, x_2) is solution of $E_{U,U'}$.

Conversely, let (y_2, x_2) be solution of $E_{U,U'}$. The equality $XU = U'X$ is equivalent to the following equations:

$$\begin{aligned} x_1a + cy_1 &= a'x_1 + b'x_2 && (2.6)\\ x_1b + y_1d &= a'y_1 + b'y_2 && (2.7)\\ x_2a + cy_2 &= c'x_1 + d'x_2 && (2.8)\\ x_2b + dy_2 &= c'y_1 + d'y_2 && (2.9) \end{aligned}$$

The equations (2.8) and (2.9) completely determine the values of x_1 and y_1, so of X, modulo 2^{n-l}. Direct computations show that this X is in $SL_2(\mathbb{Z}/2^{n-l}\mathbb{Z})$ and verifies (2.6) and (2.7).

Thus an element X in the stabilisator $\mathrm{Stab}(U)$ of U is completely determined modulo 2^{n-l} by a solution of $E_{U,U}$. Using Lemma 2.4.8, we see that there are exactly 2^{3l} ways to lift such a matrix in $SL_2(\mathbb{Z}/2^n\mathbb{Z})$. So, if k is the number of solutions of $E_{U,U}$ then $|\mathrm{Stab}(U)| = k2^{3l}$. The class formula concludes the proof.

□

It remains to compute the number of solutions of the equations $E_{U,U'}$.

Lemma 2.4.9. *Let $n \geq 1$ and A, B, C, D four integers so that ABD is odd. Let E_n be the following equation:*

$$Ax^2 + Bxy + Cy^2 \equiv D \pmod{2^n}$$

Then E_n has 2^{n-1} solutions if C is even and $3 \cdot 2^{n-1}$ solutions if C is odd.

Proof. We show the result by induction on n using Hensel's Lemma.

□

Lemma 2.4.10. *Let $n \geq 1$ and A, B, C, D be integers such that A and D are odd. Let (E) be the following equation with variables (x, y) both in $SL_2(\mathbb{Z}/p\mathbb{Z})$:*

$$Ax^2 + 2Bxy + Cy^2 \equiv D \pmod{2^n}$$

We note $\Delta := AC - B^2$. Then:

***(1)** If $n = 1$, (E) has 2 solutions.*

***(2)** If $n = 2$, when $\Delta \equiv 2, 3 \pmod 4$ then (E) has 4 solutions. When $\Delta \equiv 0, 1 \pmod 4$ then (E) has 8 solutions if $AD \equiv 1 \pmod 4$ and 0 otherwise.*

***(3)** If $n \geq 3$, we have the following cases:*

- ***(a)** If $\Delta \equiv 0 \pmod 8$ then (E) has 2^{n+2} solutions if $AD \equiv 1 \pmod 8$ and 0 otherwise.*

- ***(b)*** *If* $\Delta \equiv 2,4,6 \pmod 8$ *then* (E) *has* 2^{n+1} *solutions if* $AD \equiv 1 \pmod 8$ *or* $AD \equiv 1+\Delta \pmod 8$ *and* 0 *otherwise.*
- ***(c)*** *If* $\Delta \equiv 1,5 \pmod 8$ *then* (E) *has* 2^{n+1} *solutions if* $AD \equiv 1 \pmod 8$ *or* $AD \equiv 5 \pmod 8$ *and* 0 *otherwise.*
- ***(a)*** *If* $\Delta \equiv 3,7 \pmod 8$ *then* (E) *has* 2^n *solutions.*

Proof. First we put $z = Ax + By$. The map from $\mathbb{Z}/2^n\mathbb{Z} \times \mathbb{Z}/2^n\mathbb{Z}$ to itself sending (x,y) to (z,y) is bijective as A is odd and we remark that (x,y) is solution of (E) if and only if (z,y) is solution of the following equation, say (E'):

$$z^2 + \Delta y^2 \equiv AD \pmod{2^n}$$

Thus (E) and (E') have the same number of solutions. The number of solutions of (E') is easily computed using the fact (see [22], proposition 5.13) that if a is an odd number and $n \geq 3$, then the equation $x^2 \equiv a \pmod{2^n}$ has 4 solutions modulo 2^n if $a \equiv 1 \pmod 8$ and 0 otherwise.

□

End of the Proof of Proposition 2.4.5. We fix three invariants l, x and τ and study the conjugacy classes of $C(l,x,\tau)$. Let us take two matrices $U, U' \in C(l,x,\tau)$. We can always conjugate them so that they verify the hypothesis of Lemma 2.4.7. These two matrices are conjugate if and only if the set of solutions of $E_{U,U'}$ is not empty and the number of elements in the conjugacy class of U is computed by using Lemmas 2.4.7, 2.4.10 and 2.4.9.

□

Computation of the characters

Proposition 2.4.11. *Let* $A \in SL_2(\mathbb{Z}/2^n\mathbb{Z})$ *and* x, l, s *be its associated invariants. The definition of* s *has been given in Corollary* 2.4.6 *and will make sense now. The trace* $\mathrm{Tr}(\pi_{2^{n-1}}(A))$ *is given by:*

1. *If* $l = 0$, $|\mathrm{Tr}(\pi_{2^{n-1}}(A))|^2 = 2^s$ *if* $0 \leq s \leq n-2$, $\mathrm{Tr}(\pi_{2^{n-1}}(A)) = 0$ *if* $s = n-1$ *and* $|\mathrm{Tr}(\pi_{2^{n-1}}(A))|^2 = 2^{n-1}$ *if* $s = n$.
2. *If* $1 \leq l \leq n-2$ *and* $x = 1$ *then* $|\mathrm{Tr}(\pi_{2^{n-1}}(A))|^2 = 2^s$ *when* $2l \leq s \leq n+l-2$, $\mathrm{Tr}(\pi_{2^{n-1}}(A)) = 0$ *when* $s = n+l-1$ *and* $|\mathrm{Tr}(\pi_{2^{n-1}}(A))|^2 = 2^{n+l-1}$ *if* $s = n+l$.
3. *If* $l = n-1$ *and* $x = 1$ *then* $\mathrm{Tr}(\pi_{2^{n-1}}(A)) = 0$.
4. *If* $l = n$ *and* $x = 1$ *(*$A = I_2$*) then* $|\mathrm{Tr}(\pi_{2^{n-1}}(A))|^2 = 2^{2n-2}$.
5. *If* $2 \leq l \leq n$ *and* $x = -1$ *then* $|\mathrm{Tr}(\pi_{2^{n-1}}(A))|^2 = 4$.
6. *If* $3 \leq l \leq n$ *and* $x = 2^{l-1}+1$ *then* $|\mathrm{Tr}(\pi_{2^{n-1}}(A))|^2 = 2^{2l-2}$.
7. *If* $3 \leq l \leq n$ *and* $x = 2^{l-1}-1$ *then* $|\mathrm{Tr}(\pi_{2^{n-1}}(A))|^2 = 4$.

Lemma 2.4.12. *Let* a *be an odd integer and* $D_a := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SL_2(\mathbb{Z}/2^n\mathbb{Z})$. *Then we have* $\pi_{2^{n-1}}(D_a) = \epsilon(\delta_{ai,j})_{i,j}$ *where* ϵ *is a scalar such that* $|\epsilon|^2 = 1$.

Proof. It is proved by a direct computation using the fact that $D_a = T^{-a}ST^{-a^{-1}}ST^{-a}S$. □

Proof of Proposition 2.4.11. First when $l = 0$ or when $x = 1$, we can suppose that $A = \begin{pmatrix} 1 & b \\ c & 1+bc \end{pmatrix} = ST^cS^{-1}T^{-b}$ with $b = 2^{s-l}b_1$, $c = 2^l c_1$ where b_1 and c_1 are odd.

A simple computation gives:

$$\pi_{2^{n-1}}(A) = \beta^{\pm 3+x}\frac{G(-1,0,2^n)^2}{2^{2n}}\left(\sum_k A^{ck^2+2(j-i)k-bj^2}\right)_{i,j}$$

So:

$$|\operatorname{Tr}(\pi_{2^{n-1}}(A))| = \left|\left(\frac{G(-1,0,2^n)}{2^n}\right)^2\frac{G(c,0,2^n)}{2}\frac{G(-b,0,2^n)}{2}\right|$$

We conclude by using the fact that, if x is odd and $s \in \{0,\dots,n\}$ then (see [10]):

$$|G(x2^s,0,2^n)|^2 = \begin{cases} 2^{s+n}, & \text{when } s \leq n-2; \\ 0, & \text{when } s = n-1; \\ 2^n, & \text{when } s = n. \end{cases}$$

Then when $x = -1$ we can suppose $A = -\begin{pmatrix} 1 & b \\ c & 1+bc \end{pmatrix} = S^{-1}T^cS^{-1}T^{-b}$ with $b = 2^{s-l}b_1$, $c = 2^l c_1$ where b_1 and c_1 are odd. A similar computation gives:

$$\pi_{2^{n-1}}(A)_{i,i} = \epsilon\left(\frac{G(-1,0,2^n)}{2^n}\right)^2 A^{-bi^2}\frac{G(c,4i,2^n)}{2}$$

where $\epsilon = \beta^{c-b-6}$ is a norm one scalar. The Gauss sum $G(c,4i,2^n)$ is not null if and only if $i \in \{0, 2^{n-2}\}$ when $l = n$, 2^{n-3} divides i and 2^{n-2} does not when $l = n-1$ and 2^{l-1} divdes i when $2 \leq l \leq n-3$.

We conclude by summing $\pi_{2^{n-1}}(A)_{i,i}$ over these i.

Now to compute the traces when $x = 2^{l-1} \pm 1$, we write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a odd and $c = 2^l c_1$ with c_1 odd. We use the decomposition $A = ST^{ca^{-1}}SD_{-a}T^{-a^{-1}b}$ and Lemma 2.4.12 to find that:

$$(\pi_{2^{n-1}}(A))_{i,i} = \epsilon'\left(\frac{G(-1,0,2^n)}{2^n}\right)^2\frac{G(ca^{-1},2(a^{-1}-1)i,2^n)}{2}A^{a^{-1}bi^2}$$

where ϵ' is a norm one scalar. We conclude by summing $\pi_{2^{n-2}}(A)_{i,i}$ over every i and taking the norm. □

The computation of the sum $\mathbf{S}_{2^n}$

Proof of Proposition 2.4.2. Set $\mathbf{S}(x,l) := \sum_{A\in C(x,l)} |T(A)|^2$ and $\mathbf{S}(l) := \sum_{A\in C(l)} |T(A)|^2$. By using Propositions 2.4.6 and 2.4.11 together, we compute the following sums:

1. $\mathbf{S}(0) = 2^{3n-2} + 3\cdot 2^{3n-3}(n-1)$.
2. $\mathbf{S}(1,l) = 3\cdot 2^{3n-l-3}(n-l)$ if $1 \le l \le n-2$.
3. $\mathbf{S}(-1,l) = 3\cdot 2^{3n-3l}$ if $2 \le l \le n-1$.
4. $\mathbf{S}(1+2^{l-1},l) = 2^{3n-l-2}$ if $3 \le l \le n-1$.
5. $\mathbf{S}(-1+2^{l-1},l) = 2^{3n-3l+2}$ if $3 \le l \le n-1$.
6. $\mathbf{S}(1) = \mathbf{S}(1,1) = 3\cdot 2^{3n-4}(n-1)$.
7. $\mathbf{S}(2) = \mathbf{S}(1,2) + \mathbf{S}(-1,2) = 3\cdot 2^{3n-5}(n-2) + 3\cdot 2^{3n-6}$.
8. $\mathbf{S}(l) = 3\cdot 2^{3n-l-3}(n-l) + 3\cdot 2^{3n-3l} + 2^{3n-l-2} + 2^{3n-3l+2}$ if $3 \le l \le n-2$.
9. $\mathbf{S}(n-1) = 3\cdot 2^3 + 2^5 + 2^{2n-1}$.
10. $\mathbf{S}(n) = 2^3 + 2^{2n-1}$.

And we conclude by computing:

$$\begin{aligned} |SL_2(\mathbb{Z}/2^n\mathbb{Z})|\mathbf{S}_{2^n} &= \mathbf{S}(0) + \mathbf{S}(1) + \mathbf{S}(2) + \sum_{l=3}^{n-2} \mathbf{S}(l) + \mathbf{S}(n-1) + \mathbf{S}(n) \\ &= 3\cdot 2^{3n-2}(n-1) = |SL_2(\mathbb{Z}/2^n\mathbb{Z})| \times (n-1) \end{aligned}$$

□

2.4.2 Higher genus factors

Theorem 2.4.13. *If r is prime, the modules $U_r^{g,\pm}$ and $W_{r^n}^{g,\pm}$ are irreducible.*

Proof. First let us handle the $U_r^{g,\pm}$ modules, when r is prime. Denote by $\mathcal{A}$ the k_r'-subalgebra of $\mathrm{End}(U_r)$ generated by the operators $\pi_r(\phi)$ for $\phi \in SL_2(\mathbb{Z})$ and by $\mathcal{B}$ the k_r'-subalgebra of $\mathrm{End}(U_r^{\otimes g})$ generated by the operators $\pi_{r,g}(\phi)$ for $\phi \in Sp_{2g}(\mathbb{Z})$, when r is odd, and $\phi \in \widetilde{Sp_{2g}(\mathbb{Z})}$, when r is even.

We denote by $\mathcal{A}'$ and $\mathcal{B}'$ their commutant in $\mathrm{End}(U_r)$ and $\mathrm{End}(U_r^{\otimes g})$ respectively. We know from the genus one study that $\mathcal{A}'$ is generated by $\mathbb{1}$ and the symmetry $\theta \in \mathrm{GL}(U_r)$ sending e_i to e_{-i}. There is a natural injection $i : \mathcal{A}\otimes\ldots\otimes\mathcal{A} \hookrightarrow \mathcal{B}$. Now using the fact that the commutant of a tensor product is the tensor product of the commutant we get:

$$\mathcal{B}' \subset i((\mathcal{A}\otimes\ldots\otimes\mathcal{A}))' = i(\mathcal{A}'\otimes\ldots\otimes\mathcal{A}')$$

Note that when $r=2$ then $\theta = \mathbb{1}$ so $\mathcal{B}'$ consists of scalar elements and $\pi_{2,g} = \pi_{2;g}^+$ is irreducible. We can thus suppose that r is odd.

A generic element of $i(\mathcal{A}'\otimes\ldots\otimes\mathcal{A}')$ has the form:

$$C = \sum_{i\in I} \lambda_i a_{i_1}\otimes\ldots\otimes a_{i_g}, \text{ with } I \subset \{1,\ldots,p\}^g \text{ and } a_{i_k} = \mathbb{1} \text{ or } \theta$$

To conclude we must show that $\mathcal{B}'$ is generated by $\mathbb{1}\otimes\ldots\otimes\mathbb{1}$ and $\theta\otimes\ldots\otimes\theta$, that is show that if $C \in \mathcal{B}'$ then $a_{i_u} = a_{i_v}$ for all $i \in I$ and $u \neq v$.

Let us choose u, v and set $e := e_1 \otimes \ldots \otimes e_1$. We compute the commutator:

$$[C, \pi_{r,g}(Z_{u,v})](e) = \sum_{i \in I} \lambda_i (A^{4\epsilon_i} - 1)(a_{i_1} \otimes \ldots \otimes a_{i_g})(e)$$

where $\epsilon_i = 0$ if $a_{i_u} = a_{i_v}$ and $\epsilon_i = 1$ elsewhere. Since $A^4 \neq 1$ and the family $\{(a_{i_1} \otimes \ldots \otimes a_{i_g})(e), i \in I\}$ is free, the fact that C is in the commutant of $\mathcal{B}$ implies that $\epsilon_i = 0$ for all i so the two eigenspaces of $\theta \otimes \ldots \otimes \theta$ are irreducible.

Denote by $\mathcal{C}$ the k'_{r^n}-subalgebra of $\mathrm{End}(U_{r^n})$ generated by the operators $\pi_r(\phi)$ for $\phi \in SL_2(\mathbb{Z})$ and by the k'_{r^n}-subalgebra of $\mathrm{End}(U_{r^n}^{\otimes g})$ generated by the operators $\pi_{r,g}(\phi)$ for $\phi \in Sp_{2g}(\mathbb{Z})$, when r is odd, and $\phi \in \widetilde{Sp_{2g}(\mathbb{Z})}$, when r is even.

We denote by $\mathcal{C}'$ and $\mathcal{D}'$ their commutant in $\mathrm{End}(U_{r^n})$ and $\mathrm{End}(U_{r^n}^{\otimes g})$ respectively. We know from the genus one study that $\mathcal{A}'$ is generated by $\mathbb{1}$ and θ. The natural injection $i : \mathcal{C} \otimes \ldots \otimes \mathcal{C} \hookrightarrow \mathcal{D}$ implies that:

$$\mathcal{D}' \subset i((\mathcal{C} \otimes \ldots \otimes \mathcal{C})') = i(\mathcal{C}' \otimes \ldots \otimes \mathcal{C}')$$

Again we choose a generic element $C = \sum_{i \in I} \lambda_i a_{i_1} \otimes \ldots \otimes a_{i_g} \in i(\mathcal{C}' \otimes \ldots \otimes \mathcal{C}')$ with $I \subset \{1, \ldots, p^n\}^g$ and $a_{i_k} = \mathbb{1}$ or θ and suppose that $C \in B'$. Now remember that W_{r^n} is defined as the orthogonal of $\bar{U}_{r^{n-2}} = \mathrm{Span}(g_i)$ in U_{r^n} and since e_1 is orthogonal to all g_i we deduce that $e = e_1 \otimes \ldots \otimes e_1 \in W_{r^n}^{\otimes g}$. So the fact that the commutator $[C, \pi_{r^{n+2},g}(Z_{u,v})](e)$ is null if and only if C is a linear combination of $\mathbb{1} \otimes \ldots \otimes \mathbb{1}$ and $\theta \otimes \ldots \otimes \theta$ permits us to conclude. □

Finally the irreducibility of the factors coming from the decomposition at composite levels $p = r_1^{n_1} \ldots r_k^{n_k}$ follows, using the decomposition (1) from Theorem 2.1.1, exactly as in the genus one case:

Corollary 2.4.14. *All the modules of the form $B_{r_1} \otimes \ldots \otimes B_{r_k}$ with $r_1, \ldots, r_k$ distinct prime and $B_{r_i} = U_{r_i}^{g,\pm}$ or $W_{r_i^n}^{g,\pm}$, are irreducible and pairwise distinct.*

Proof. Let $p = 2^\alpha r_1^{n_1} \ldots r_k^{n_k}$ with r_i some distinct odd primes. There is a group isomorphism between $\widetilde{Sp_{2g}(\mathbb{Z}/2p\mathbb{Z})}$ and $\widetilde{Sp_{2g}(\mathbb{Z}/2^{\alpha+1}\mathbb{Z})} \times Sp_{2g}(\mathbb{Z}/r_1^{n_1}\mathbb{Z}) \times \ldots \times Sp_{2g}(\mathbb{Z}/r_k^{n_k}\mathbb{Z})$, if p is even, and between $Sp_{2g}(\mathbb{Z}/p\mathbb{Z})$ and $Sp_{2g}(\mathbb{Z}/r_1^{n_1}\mathbb{Z}) \times \ldots \times Sp_{2g}(\mathbb{Z}/r_k^{n_k}\mathbb{Z})$, if p is odd.

Denote by $\mathcal{A}_{p,g}$ the subalgebra of $\mathrm{End}(U_p^{\otimes g})$ generated by the operators $\pi_{p,g}(\phi)$. Using the first point of Theorem 2.1.1, we get an algebra isomorphism:

$$\mathcal{A}_{p,g} \cong \mathcal{A}_{2^\alpha,g} \otimes \mathcal{A}_{r_1^{n_1},g} \otimes \ldots \otimes \mathcal{A}_{r_k^{n_k},g}$$

We conclude using the fact that the commutant of a tensor product is the tensor product of the commutant and use Theorem 2.4.13.

□

2.4.3 An alternative proof

We give an alternative proof to show the irreducibility of the factors arising in the decomposition of the Weil representations. The proof is a generalization of the arguments of the physicists Capelli, Itzykson and Zuber in [16] inspired by the work of Qiu and Gepner in [33]. Denote by $(\mathcal{A}_{p,g})'$ the commutant in $\mathrm{End}(U_p^{\otimes g})$ of the k_p'-algebra generated by the operators $\pi_{p,g}(\phi)$.

Proposition 2.4.15. *The dimension of $(\mathcal{A}_{p,g})'$ is at most $\sigma(p)$ when p is odd and $\sigma(\frac{p}{2})$ when p is even.*

Since the dimension of $(\mathcal{A}_{p,g})'$ is equal to the number of irreducible subfactors of $U_p^{\otimes g}$, the previous proposition implies the irreducibility of the factors arising in the decomposition of Theorem 2.1.1.

Lemma 2.4.16. *The action $\bullet$ of $Sp_{2g}(\mathbb{Z})$ on $(\mathbb{Z}/p\mathbb{Z})^{2g}$ by matrix multiplication contains $\sigma(p)$ orbits. A set of representatives is given by the classes C_δ of the elements $(0,\delta,0,\delta,\ldots,0,\delta)$ where δ runs through the divisors of p.*

Proof. First remark that if δ_1 and δ_2 are two divisors of p such that there exists $\phi \in Sp_{2g}(\mathbb{Z})$ so that $\phi \bullet (0,\delta_1,\ldots,0,\delta_1) = (0,\delta_2,\ldots,0,\delta_2)$ then δ_1 divides δ_2. Since we can exchange δ_1 and δ_2 in the argument, then $\delta_1 = \delta_2$ and the classes C_δ are pairwise distinct.

Suppose $g = 1$ and let $a, b \geq 1$. Choose $u, v \in \mathbb{Z}$ so that $ua + vb = \mathrm{g.c.d(a,b)}$. Remark that $\begin{pmatrix} \frac{b}{\mathrm{g.c.d(a,b)}} & -\frac{a}{\mathrm{g.c.d(a,b)}} \\ u & v \end{pmatrix} \in SL_2(\mathbb{Z})$ and we have:

$$\begin{pmatrix} \frac{b}{\mathrm{g.c.d(a,b)}} & -\frac{a}{\mathrm{g.c.d(a,b)}} \\ u & v \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{g.c.d(a,b)} \end{pmatrix}$$

So (a,b) lies in the same orbit that $(0, \mathrm{g.c.d(a,b)})$.

Next choose d a divisor of p and x so that $\mathrm{g.c.d(x,p)} = 1$. Then $(0,d)$ and $(0,dx)$ lies in the same orbit for, if $xx' = 1 \pmod p$ then:

$$\begin{pmatrix} x' & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 0 \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ dx \end{pmatrix}$$

So any element (a,b) is in the orbit of $(0, \mathrm{g.c.d(a,b,p)})$ and we have proved the lemma when $g = 1$.

When $g \geq 2$, using the injection of $SL_2(\mathbb{Z})^g$ in $Sp_{2g}(\mathbb{Z})$ by diagonal blocks, we deduce from the previous study that any element of $(\mathbb{Z}/p\mathbb{Z})^{2g}$ lies in the orbit of an element $(0,\delta_1,\ldots,0,\delta_g)$, where the δ_k are divisors of p. We now show that when $g = 2$, the element $(0,\delta_1,0,\delta_2)$ is in the same orbit that $(0, \mathrm{g.c.d}(\delta_1,\delta_2), 0, \mathrm{g.c.d.}(\delta_1,\delta_2))$. The proof of the lemma will follow by induction on g using a suitable injection of $Sp_4(\mathbb{Z})$ in $Sp_{2g}(\mathbb{Z})$.

Choose four integers u, v, α, β such that:

$$u\delta_1 + v\delta_2 = \mathrm{g.c.d}(\delta_1,\delta_2)$$
$$\alpha u - \beta \frac{\delta_2}{\mathrm{g.c.d}(\delta_1,\delta_2)} = 1$$

Let $X := \begin{pmatrix} \alpha & \frac{\delta_2}{\text{g.c.d}(\delta_1,\delta_2)} \\ \beta & u \end{pmatrix}$ and $Y := \begin{pmatrix} 0 & -\frac{\delta_1}{\text{g.c.d}(\delta_1,\delta_2)} \\ -\frac{\delta_1}{\text{g.c.d}(\delta_1,\delta_2)} & v \end{pmatrix}$. The matrix X has determinant 1 and Y is symmetric so $\phi := \begin{pmatrix} X & Y \\ 0 & (X^{-1})^T \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$. A direct computation shows that $\phi \bullet (0, \delta_1, 0, \delta_2)$ lies in the same orbit that $(0, \text{g.c.d}(\delta_1, \delta_2), 0, \delta_2)$. Using the action of the matrix $\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$, we conclude. □

Proof of Proposition 2.4.15. Fix a basis $\{x_1, y_1, \ldots, x_g, y_g\}$ of $H_1(\Sigma_g, \mathbb{Z}/p\mathbb{Z})$ so that the intersection form verifies $\omega(x_i, y_j) = \delta_{i,j}$. This induces an isomorphism $f : H_1(\Sigma_g, \mathbb{Z}/p\mathbb{Z}) \to (\mathbb{Z}/p\mathbb{Z})^2$. The family $\{\text{Add}_p(\gamma, 0) | \gamma \in H_1(\Sigma_g, \mathbb{Z}/p\mathbb{Z})\}$, where Add_p denotes the Schrödinger representation, is a basis of $\text{End}(U_p^{\otimes g})$. Thus any element $\Theta \in \text{End}(U_p^{\otimes g})$ has a unique expression:

$$\Theta = \sum_{\gamma \in H_1(\Sigma_g, \mathbb{Z}/p\mathbb{Z})} \Theta_\gamma \, \text{Add}_p(\gamma, 0)$$

Now using the Egorov identity, we have:

$$\begin{aligned} \Theta \in (\mathcal{A}_{p,g})' &\Leftrightarrow \pi_{p,g}(\phi)^{-1} \Theta \pi_{p,g}(\phi) = \Theta, \text{ for all } \phi \in Sp_{2g}(\mathbb{Z}) \\ &\Leftrightarrow \sum_{\gamma \in H_1(\Sigma_g, \mathbb{Z}/p\mathbb{Z})} \Theta_\gamma \, \text{Add}_p(\gamma, 0) = \sum_{\gamma \in H_1(\Sigma_g, \mathbb{Z}/p\mathbb{Z})} \Theta_\gamma \, \text{Add}_p(\phi \bullet \gamma, 0), \\ &\quad \text{for all } \phi \in Sp_{2g}(\mathbb{Z}) \\ &\Leftrightarrow \Theta_{\phi \bullet \gamma} = \Theta_\gamma, \text{ for all } \phi \in Sp_{2g}(\mathbb{Z}) \text{ and } \gamma \in H_1(\Sigma_g, \mathbb{Z}/p\mathbb{Z}) \end{aligned}$$

Lemma 2.4.16 implies that $dim((\mathcal{A}_{p,g})') \leq \sigma(p)$ and the proof is completed when p is odd, for Θ is completely determined by the $\sigma(p)$ coefficients $\Theta_{f^{-1}(0,\delta,\ldots,0,\delta)}$, where δ/p.

When p is even, we will prove that $\Theta_{f^{-1}(0,\delta,\ldots,0,\delta)} = 0$ when δ is odd. This will imply that $\Theta \in (\mathcal{A}_{p,g})'$ is determined by the $\sigma(\frac{p}{2})$ coefficients $\Theta_{f^{-1}(0,\delta,\ldots,0,\delta)}$ where δ/p and δ is even.

We first suppose that $g = 1$. Let $\Delta_p := ker\,(SL_2(\mathbb{Z}/2p\mathbb{Z}) \to SL_2(\mathbb{Z}/p\mathbb{Z}))$. This subgroup contains the following eight elements:

$$\Delta_p = \left\{ \begin{pmatrix} 1+\alpha p & \beta p \\ \gamma p & 1+\alpha p \end{pmatrix} ; \alpha, \beta, \gamma \in \mathbb{Z}/2\mathbb{Z} \right\}$$

Now if $\Theta \in (\mathcal{A}_{p,1})'$ we have:

$$\begin{aligned} \Theta &= \frac{1}{|\Delta_p|} \sum_{\phi \in \Delta_p} \pi_p(\phi)^{-1} \Theta \pi_p(\phi) \\ &= \sum_{a,b} \Theta_{x^a y^b} \frac{1}{8} \sum_{\alpha,\beta,\gamma \in \mathbb{Z}/2\mathbb{Z}} \text{Add}_p(x^{(1+\alpha p)a + \beta p b} y^{\gamma p a + (1+\alpha p) b}) \\ &= \sum_{a,b} \Theta_{x^a y^b} \left(\frac{1}{8} \sum_{\alpha,\beta,\gamma \in \mathbb{Z}/2\mathbb{Z}} (-1)^{\gamma i + \beta j} \right) \text{Add}_p(x^a y^b) \end{aligned}$$

Where we used the fact that $\text{Add}_p(x)$ and $\text{Add}_p(y)$ have order p. Now we simply remark that:

$$\frac{1}{8} \sum_{\alpha,\beta,\gamma \in \mathbb{Z}/2\mathbb{Z}} (-1)^{\gamma i + \beta j} = \begin{cases} 1 & \text{, when } i, j \text{ are both even.} \\ 0 & \text{, elsewhere.} \end{cases}$$

And the proof is completed when $g = 1$. To handle the genus $g \geq 2$ cases, we choose an injection $i : SL_2(\mathbb{Z}) \hookrightarrow Sp_{2g}(\mathbb{Z})$ sending $SL_2(\mathbb{Z})$ in a diagonal block and completing by the identity matrix. A similar computation shows that $\Theta = \frac{1}{|i(\Delta_p)|} \sum_{\phi \in i(\Delta_p)} \pi_{p,g}(\phi)^{-1} \Theta \pi_{p,g}(\phi)$ implies that $\Theta_{f^{-1}(0,\delta,\ldots,0,\delta)} = 0$, when δ is odd. □

Remark. The proof of Proposition 2.4.15 shows that the operators:

$$\Omega_{\delta,p} := \sum_{\phi \in Sp_{2g}(\mathbb{Z})} \mathrm{Add}_p(\phi \bullet f^{-1}(0, \delta, \ldots, 0, \delta), 0)$$

form a generating set for $(\mathcal{A}_{p,g})'$. It is not difficult to deduce the decomposition of the Weil representations by showing that they form a free family and then study their eigenspaces. The relation between this basis of $(\mathcal{A}_{p,g})'$ and the decomposition of Theorem 2.1.1 is summarized in the following:

1. If a and b are coprime and δ_1/a, δ_2/b then $\Omega_{\delta_1\delta_1,ab} \cong \Omega_{\delta_1,a} \otimes \Omega_{\delta_2,b}$.
2. If r is prime, $n \geq 2$ and $k \leq \frac{n}{2}$ then Ω_{r^k,r^n} is the projector on the submodule $U_{r^{n-2k}}^{\otimes g} \subset U_{r^n}^{\otimes g}$. The operator Ω_{r^{n-k},r^n} is the composition of Ω_{r^k,r^n} with the symmetry sending $e_i \otimes \ldots \otimes e_i$ to $e_{-i} \otimes \ldots \otimes e_{-i}$.

Chapter 3

Decomposition of some Reshetikhin-Turaev representations into irreducible factors

Résumé

Ce chapitre contient la version non publiée de l'article intitulé " Decomposition of some Reshetikhin-Turaev representations into irreducible factors" ([56]). Le résultat principal est la décomposition en facteurs irréductibles des représentations de Reshetikhin-Turaev lorsque le niveau est de la forme $p = 4r$, $p = 2r^2$ ou $p = 2r_1r_2$ avec r, r_1, r_2 des nombres premiers impairs, modulo certaines hypothèses.

Abstract

This chapter contains the unpublished version of the article " Decomposition of some Reshetikhin-Turaev representations into irreducible factors" ([56]). The main result is the decomposition into irreducible factors of the Reshetikhin-Turaev representations when the level is $p = 4r$, $p = 2r^2$ or $p = 2r_1r_2$ with r, r_1, r_2 odd primes, under certain technical assumptions.

3.1 Introduction

Witten gave in [92] convincing arguments for the existence of Topological Field Theories, as defined in [6, 91], giving a three dimensional interpretation of the Jones polynomial when the gauge group is $SU(2)$. Each of these TQFTs gives a family of projective finite dimensional representations of the mapping class group $\mathrm{Mod}(\Sigma_g)$ of a genus g closed oriented surface Σ_g. Reshetikhin and Turaev gave a rigorous construction of these TQFTs [75] using representations of quantum groups. In this paper we will follow the skein theoretical construction of [63, 14] to define these representations.

We can lift these projective representations to linear representations of some central extension $\widetilde{\mathrm{Mod}(\Sigma_g)}$ of $\mathrm{Mod}(\Sigma_g)$ noted:

$$\rho_{p,g} : \widetilde{\mathrm{Mod}(\Sigma_g)} \to \mathrm{GL}(V_{p,g}).$$

Here $p = 2(k+2) \geq 3$ is an even integer indexing the representations and $V_{p,g}$ is a finite dimensional complex vector space. These representations are equipped with an invariant scalar product $\langle , \rangle_{p,g}$ with respect to which they are unitary.

The goal of this paper is to decompose some of these representations into irreducible factors. Only few results are known concerning their decomposition. In [14], an explicit proper submodule of $V_{p,g}$ is given whenever 4 divides p. In [76] it is shown that $V_{p,g}$ is irreducible when $\frac{p}{2}$ is an odd prime. Robert's proof extends word-by-word to show that the modules $V_{18,g}$ are also irreducible. In [5] the authors showed that for $p = 24, 36, 60$ then $V_{p,g}$ contains at least three invariant submodules. Finally we gave in [55] an explicit decomposition into irreducible factors of the modules $V_{p,1}$ for arbitrary level $p \geq 3$.

The main results of this paper are summarized in the two following theorems:

Theorem 3.1.1.

1. *If r is an odd prime, then $V_{4r,2}$ is the sum of two irreducible subrepresentations.*
2. *If r is an odd prime, then $V_{2r^2,2}$ is irreducible.*
3. *If r_1, r_2 are two distinct odd primes, then $V_{2r_1r_2,2}$ is irreducible.*

Given a level $p = 2r \geq 3$, there exists a set of complex numbers called $6j$-symbols at level p which will be defined in the next section. If r is odd, we call *generic* a level for which none of the level p $6j$-symbols is null. When r is even, we exhibit in Proposition 3.3.10 two families of vanishing $6j$-symbols at level p. We call such a p *generic* if no other $6j$-symbols vanish. Numerical computations suggest that every levels are generic.

Theorem 3.1.2.

1. *The modules $V_{18,g}$ are irreducible for arbitrary $g \geq 2$.*
2. *If 50 is generic then the module $V_{50,3}$ is irreducible.*
3. *If r is an odd prime, $p = 4r$ is generic and $g = 3$, then $V_{4r,3}$ is sum of two irreducible subrepresentations.*
4. *If r_1, r_2 are two distinct odd primes, $p = 2r_1r_2$ is generic and $2g < min(r_1, r_2)$, then $V_{2r_1r_2,g}$ is irreducible.*

The generecity of a given level p can be checked by numerical computations. This leads us to the following:

Corollary 3.1.3. *(Computer assisted proof)*
The module $V_{50,3}$ is irreducible. The modules $V_{28,3}, V_{44,3}$ are sum of two irreducible submodules.

Remark. In [14] some representations $\rho_{p,g}$ are also defined when p is odd. They verify $\rho_{2p,g} \cong \rho_{p,g} \otimes \rho_{6,g}$. In particular if an odd level r is such that $V_{2r,g}$ is irreducible, then so is $V_{r,g}$. This extends the two previous theorems to the $SO(3)$ cases as well.

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3.2 Skein construction of the Reshetikhin-Turaev representations

Following [14], we will briefly define the representations $\rho_{p,g}$ and fix some notations.

3.2.1 The spaces $V_{p,g}$

Given an even integer $p \geq 6$, we denote by $A \in \mathbb{C}$ an arbitrary primitive $2p - th$ root of unity. Using the Kauffman skein relation of Figure 3.1, we associate to any framed link $L \subset S^3$ an invariant $\langle L \rangle_p \in \mathbb{C}$.

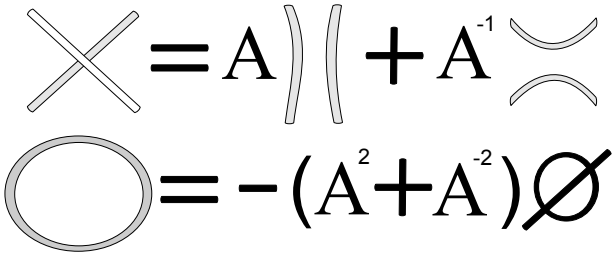


Figure 3.1: Skein relations defining the framed link invariants.

Choose $g \geq 1$ and denote by C_g the set of isotopy classes of framed links (including the empty link) in an oriented genus g handlebody H_g. We fix a genus g Heegaard splitting of the sphere, i.e. an element $S \in \mathrm{Mod}(\Sigma_g)$ and two handlebodies so that :

$$H_g^1 \bigcup_{S:\partial H_g^1 \to \partial H_g^2} H_g^2 \cong S^3$$

Take $L_1, L_2 \in C_g$ and embed L_1 in H_g^1 and L_2 in H_g^2 . The above gluing defines a link $L_1 \bigcup_S L_2 \subset S^3$. We call Hopf pairing the bilinear form:

$$(\cdot,\cdot)_{g,p}^H : \mathbb{C}[C_g] \times \mathbb{C}[C_g] \to \mathbb{C}$$

defined by

$$(L_1, L_2)_{p,g}^H := \left\langle L_1 \bigcup_S L_2 \right\rangle_p$$

Eventually we define the spaces $V_{p,g}$ as the quotients:

$$V_{p,g} := \mathbb{C}[C_g] \Big/ ker\left((\cdot,\cdot)_{g,p}^H\right)$$

The vector spaces $V_{p,g}$ are finite dimensional ([14]) and we can find explicit basis as follows. Let $g \geq 2$, choose a trivalent graph $\Gamma \subset H_g$ so that H_g retracts on Γ by deformation. If $g = 1$, Γ represents the circle $S^1 \times \{0\} \subset S^1 \times D^2 \cong H_1$. We denote by $E(\Gamma)$ the set of its edges.

A triple $(i, j, k) \in \left\{0, \ldots, \frac{p-4}{2}\right\}^3$ is said *p-admissible* if:

1. $|i - j| \leq k \leq i + j$,
2. $i + j + k$ is even and is smaller or equal to $p - 4$.

A map $\sigma : E(\Gamma) \to \{0, \dots, \frac{p-4}{2}\}$ is a *p-admissible coloring* of Γ if for every three edges $e_1, e_2, e_3 \in E(\Gamma)$ adjacent to a vertex, the triple $(\sigma(e_1), \sigma(e_2), \sigma(e_3))$ is p-admissible.

In [46, 90] the authors defined some idempotents $\{f_0, \dots, f_{\frac{p-4}{2}}\}$ of the Temperley-Lieb algebra with coefficient in $\mathbb{Q}(A)$ called Jones-Wenzl idempotents . To σ a p-admissible coloring of Γ we associate a vector $u_\sigma \in V_{p,g}$ as follows. We replace each edge $e \in E(\Gamma)$ by the Jones-Wenzl idempotent $f_{\sigma(e)}$. If (e_1, e_2, e_3) are three edges adjacent to a vertex of Γ, we connect the idempotents using the link $T_{\sigma(e_1),\sigma(e_2),\sigma(e_3)}$ defined in Figure 3.2.

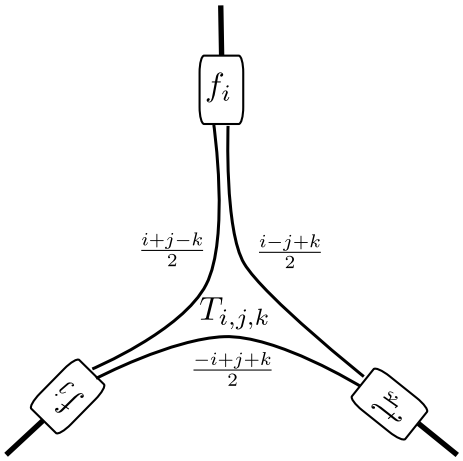


Figure 3.2: The link $T_{i,j,k}$ used to connect three idempotents f_i, f_j and f_k. The numbers above each three arcs denotes the number of parallel copies of the arc used to define the link.

Theorem 1.1.44 asserts that the elements u_σ, for σ a p-admissible coloring of Γ, form a basis of $V_{p,g}$. Moreover there exists a non-degenerate bilinear form $\langle , \rangle_{p,g}$ on $V_{p,g}$ invariant under the action of $\widetilde{\mathrm{Mod}(\Sigma_g)}$, for which the vectors u_σ are pairwise orthogonal.

The basis u_σ depends on the choice of the trivalent graph. We can transform a trivalent graph into one another by a sequence of Whitehead moves. Suppose that Γ_1 and Γ_2 are two trivalent graphs of genus $g \geq 2$, which only differ by a single Whitehead move, inside a ball B^3, as drawn in Figure 3.3.

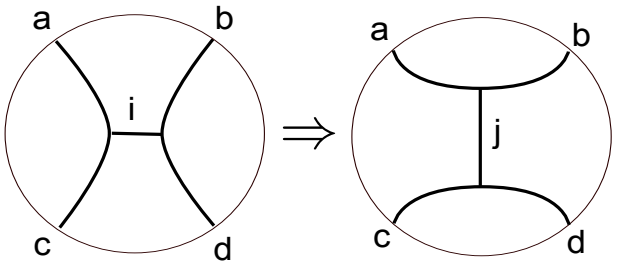


Figure 3.3: The two graphs Γ_1 on the left and Γ_2 on the right differ by a local Whitehead move.

Fix a p-admissible coloring of the graphs outside B^3 and denote by 

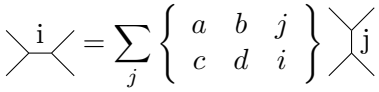
(resp. 

) the vector associated to the coloration of Γ_1 (resp of Γ_2) with the edge i colored by $\sigma(i)$ (resp with the edge j colored by $\sigma(j)$).

Then the vectors 

belong to the subspace spanned by the vectors 

and decompose using the so-called 'fusion rules' formula (Lemma 1.1.48) :



$$= \sum_j \left\{ \begin{array}{ccc} a & b & j \\ c & d & i \end{array} \right\}$$

(3.1)

where the sum runs through p-admissible colorings and the coefficient $\left\{ \begin{array}{ccc} a & b & j \\ c & d & i \end{array} \right\}$ only depends on the colors of the edges a, b, c, d, i and j and is called *recoupling coefficient* or $6j$*-symbol* in literature. The computation of $6j$-symbols follows from Lemma 1.1.48 and Proposition 1.1.17.

3.2.2 The Reshetikhin-Turaev representations

We fix an orientation preserving homeomorphism

$$\alpha : \Sigma_g \to \partial H_g$$

Choose a class $\phi \in \mathrm{Mod}(\Sigma_g)$ associated to a homeomorphism which extends to H_g through α. Then ϕ acts on C_g and preserves the kernel of the Hopf pairing so acts on $V_{p,g}$ by passing to the quotient. Denote by $\tilde{\rho}_{p,g}(\phi) \in \mathrm{GL}(V_{p,g})$ the resulting operator.

Now choose $\phi \in \mathrm{Mod}(\Sigma_g)$ so that the corresponding homeomorphisms extend to H_g through $\alpha \circ S$. This extension also defines, by quotient, an operator on $V_{p,g}$. We denote by $\tilde{\rho}_{p,g}(\phi)$ the dual of this operator for the Hopf pairing.

The elements of $\mathrm{Mod}(\Sigma_g)$ which extend to H_g either through α or through $\alpha \circ S$, generate the whole group $\mathrm{Mod}(\Sigma_g)$. It is a non trivial fact that the associated operators $\tilde{\rho}_{p,g}(\phi)$ generate a projective representation:

$$\tilde{\rho}_{p,g} : \mathrm{Mod}(\Sigma_g) \to \mathrm{PGL}(V_{p,g})$$

We consider a central extension $\widetilde{\mathrm{Mod}(\Sigma_g)}$ of $\mathrm{Mod}(\Sigma_g)$ that lifts the above projective representations to linear ones (see [69, 38]):

$$\rho_{p,g} : \widetilde{\mathrm{Mod}(\Sigma_g)} \to \mathrm{GL}(V_{p,g})$$

These are the so-called Reshetikhin-Turaev representations.

Now to each edge $e \in E(\Gamma)$, choose a disc D_e, properly embedded in H_g, that intersects Γ transversely once in e. Note that the set of boundary curves $\gamma_e := \partial D_e \subset \partial H_g \xrightarrow{\alpha} \Sigma_g$ forms a pants decomposition of Σ_g.

A classical property of the Jones-Wenzl idempotents (Lemma 1.1.12) asserts that, if $T_e \in \mathrm{Mod}(\Sigma_g)$ denotes the Dehn twist along γ_e, then:

$$\tilde{\rho}_{p,g}(T_e) \cdot u_\sigma = \mu_{\sigma(e)} u_\sigma$$

where $\mu_i := (-1)^i A^{i(i+2)}$.

We fix the lift of T_e in $\widetilde{\mathrm{Mod}(\Sigma_g)}$, still denoted T_e, so that $\rho_{p,g}(T_e) \cdot u_\sigma = \mu_{\sigma(e)} u_\sigma$.

We also fix the lift $S \in \widetilde{\mathrm{Mod}(\Sigma_g)}$ so that the matrix of $\rho_{p,g}(S)$ is the matrix of the Hopf pairing $(\cdot,\cdot)^H_{p,g}$ multiplied by an element $\eta \in \mathbb{C}$ which verifies $|\eta| = \frac{|A^2-A^{-2}|}{\sqrt{p}}$. We refer to [14], where η represents the quantum invariant of S^3, for a detailed discussion on η.

Since S and the $\{T_e\}_{e\in E(\Gamma)}$ generate $\widetilde{\mathrm{Mod}(\Sigma_g)}$ for some trivalent graphs, we have an explicit description of $\rho_{p,g}$.

3.3 Cyclicity of the vacuum vector

Denote by $\mathcal{A}_{p,g}$ the subalgebra of $\mathrm{End}(V_{p,g})$ generated by the operators $\rho_{p,g}(\phi)$ for $\phi \in \widetilde{\mathrm{Mod}(\Sigma_g)}$. The key ingredient to prove Theorem 3.1.1 is to show that the vacuum vector $v_0 \in V_{p,g}$, associated to the class of the empty link, is cyclic, i.e. that $\mathcal{A}_{p,g}\cdot v_0 = V_{p,g}$.

3.3.1 The genus one case

In [55] we gave an explicit decomposition of the Weil representations into irreducible factors (Theorem 2.1.1) . An easy generalization of the arguments of the proof of Lemma 3 of [27] leads to an explicit isomorphism of $SL_2(\mathbb{Z})$-modules between $V_{p,1}$ and the odd submodule of the Weil representation at level p (see Theorem 3.3.1) . Proving that $v_0 \in V_{p,1}$ is cyclic reduces to show that its projection on each irreducible submodule of $V_{p,1}$ is not null.

Denote by $\left\{u_0,\dots,u_{\frac{p-4}{2}}\right\}$ the basis of $V_{p,1}$ where u_i is the class of the closure of the $i-th$ Jones-Wenzl idempotent along a longitude in H_1. Also denote by $\{e_i, i \in \mathbb{Z}/p\mathbb{Z}\}$ the basis of the Weil $SL_2(\mathbb{Z})$-module U_p at level p as described in [55].

In this basis, the Weil projective representations in genus one are defined by the matrices:

$$\begin{aligned}\pi_p(S) &= \frac{1}{\sqrt{p}}\left(A^{-ij}\right)_{i,j\in\mathbb{Z}/p\mathbb{Z}}\\ \pi_p(T) &= \left(A^{i^2}\delta_{i,j}\right)_{i,j\in\mathbb{Z}/p\mathbb{Z}}\end{aligned}$$

Here the level is an integer $p \geq 2$ not necessary even. When p is even, we take A to be a primitive $2p-th$ root of unity. When p is odd, A is a primitive $p-th$ root of unity.

The vectors $\left\{e_i^- := e_i - e_{-i}, i \in \{1,\dots,\frac{p-2}{2}\}\right\}$ span a submodule $U_p^- \subset U_p$.

Lemma 3.3.1. *Let $p = 2r \geq 6$ be an even integer. Then the following map:*

$$\Psi : \left\{\begin{array}{ccc} U_p^- & \to & V_{p,1}\\ e_i^- & \longmapsto & u_{i+r-1}\end{array}\right.$$

is an isomorphism of $SL_2(\mathbb{Z})$-projective modules.

Proof. We compute the matrix elements:

$$\begin{aligned}\left\langle\psi(e_j^-),\rho_{p,1}(S)\psi(e_i^-)\right\rangle &= \frac{\eta\cdot(-1)^{i+j}}{A^2-A^{-2}}\left(A^{2(i+r)(j+r)}-A^{-2(i+r)(j+r)}\right)\\ &= \frac{\eta\cdot(-1)^r\sqrt{p}}{A^2-A^{-2}}\left\langle e_j^-,\pi_p(S)e_i^-\right\rangle\end{aligned}$$

where the scalar $\frac{\eta\cdot(-1)^r\sqrt{p}}{A^2-A^{-2}}$ has norm one.

$$\begin{aligned}\left\langle \psi(e_j^-), \rho_{p,1}(T)\psi(e_i^-)\right\rangle &= (-1)^{i+r-1}A^{(i+r-1)(i+r+1)}\delta_{i,j}\\ &= (-1)^{r-1}A^{-1}\left\langle e_j^-, \pi_p(T)e_i^-\right\rangle\end{aligned}$$

□

The decomposition into irreducible submodules of U_p is described by the following:

Proposition 3.3.2 ([55])**.** *We have the following decompositions where $\cong$ denotes an isomorphism of $SL_2(\mathbb{Z})$-modules:*

1. *If a and b are coprime, then $U_{ab} \cong U_a \otimes U_b$.*
2. *If r is prime and $n \geq 1$, then $U_{r^{n+2}} \cong U_{r^n} \oplus W_{r^{n+2}}$ where $W_{r^{n+2}}$ denotes another module.*
3. *If r is an odd prime, then $U_{r^2} \cong \mathbb{1} \oplus W_{r^2}$ where $\mathbb{1}$ is the trivial representation.*
4. *The modules U_p for $r > 2$ and W_{r^n} split into two submodules: $U_p \cong U_p^- \oplus U_p^+$, $W_{r^n} \cong W_{r^n}^+ \oplus W_{r^n}^-$.*
5. *The modules $B_1 \otimes \ldots \otimes B_k$, where the B_i have the form $U_r^+, U_r^-, U_2, U_4^+, U_4^-, W_{r^n}^+$ or $W_{r^n}^-$ and have pairwise coprime levels, are irreducible.*

We can now prove:

Proposition 3.3.3. *Let $p \geq 6$ be an even integer. Then the vacuum vector $v_0 \in V_{p,1}$ is cyclic if and only if one of the following three cases holds:*
- *$p = 2r_1 \ldots r_k$ with r_i distinct odd primes.*
- *$p = 2r^2$ with r prime.*
- *$p = 4r$ with r prime.*

Proof. We will use Proposition 3.3.2 and the explicit isomorphisms given in the main theorem of [55] to study whether the vector

$$v := \psi^{-1}(v_0) = e_{\frac{p-2}{2}} - e_{\frac{p+2}{2}} \in U_p^-$$

has non trivial projection on each submodule of U_p^- or not.

Given two integers x and n, we will denote by $[x]_n \in \mathbb{Z}/n\mathbb{Z}$ the class of x modulo n. We write

$$v = e_{[x]_p} - e_{[-x]_p} \in U_p^-$$

with $x = \frac{p-2}{2}$.

First, when $p = 2r^2$, with r prime, the module U_p^- is irreducible so the vector is cyclic.

When $p = 4r$, with r an odd prime, the module decomposes into two irreducible submodules:

$$U_{4r}^- \cong U_4^- \otimes U_r^+ \oplus U_4^+ \otimes U_r^-$$

The vector v decomposes as follows:

$$\begin{aligned} v &= e_{[x]_4} \otimes e_{[x]_r} + e_{[-x]_4} \otimes e_{[-x]_r} \\ &= \left(\frac{1}{2}(e_{[x]_4} - e_{[-x]_4}) \otimes (e_{[x]_r} + e_{[-x]_r})\right) \\ &\quad + \left(\frac{1}{2}(e_{[x]_4} + e_{[-x]_4}) \otimes (e_{[x]_r} - e_{[-x]_r})\right) \end{aligned}$$

Where the first term lies in $U_4^- \otimes U_r^+$ and the second in $U_4^+ \otimes U_r^-$.

Since $x = 2r - 1$, neither 4 nor r divide x, so $[x]_4 \neq [x]_4$ and $[x]_r \neq [x]_r$ and the two projections are not null.

When $p = 2r_1 \dots r_k$, with r_i distinct odd primes, we have the following decomposition:

$$U_p^- \cong \bigoplus_{\epsilon=(\epsilon_i)_i \in \{-1,+1\}^k} X_\epsilon$$

where

$$X_\epsilon := U_2 \otimes U_{r_1}^{\epsilon_1} \otimes \dots \otimes U_{r_k}^{\epsilon_k}$$

Let us fix ϵ and denote:

$$e_\epsilon := e_{[x]_2} \otimes e_{[x]_{r_1}}^{\epsilon_1} \otimes \dots \otimes e_{[x]_{r_k}}^{\epsilon_k} \in X_\epsilon$$

where we used the notation $e_i^\pm := e_i \pm e_{-i}$. By using the facts that $\langle e_i, e_i^\epsilon \rangle = 1$ and $\langle e_{-i}, e_i^\epsilon \rangle = (-1)^{\frac{1-\epsilon}{2}}$, we compute:

$$\begin{aligned} \langle v, e_\epsilon \rangle &= \left\langle e_{[x]_2} \otimes e_{[x]_{r_1}} \otimes \dots \otimes e_{[x]_{r_k}}, e_{[x]_{r_1}}^{\epsilon_1} \otimes \dots \otimes e_{[x]_{r_k}}^{\epsilon_k} \right\rangle \\ &\quad - \left\langle e_{[x]_2} \otimes e_{[-x]_{r_1}} \otimes \dots \otimes e_{[-x]_{r_k}}, e_{[x]_{r_1}}^{\epsilon_1} \otimes \dots \otimes e_{[x]_{r_k}}^{\epsilon_k} \right\rangle \\ &= 1 - (-1)^{\sum_i \frac{1-\epsilon_i}{2}} = 2 \neq 0 \end{aligned}$$

So the projection of v on each irreducible submodule X_ϵ is not null.

Now suppose that $p = 2r_1^{n_1} \dots r_k^{n_k}$ with $k \geq 2$, r_i distinct primes and $n_1 \geq 2$. Since r_1 does not divide x, the vector v has a null projection on the submodule:

$$\begin{cases} U^+_{r_1^{n_1-2}} \otimes U^-_{2r_2^{n_2}\dots r_k^{n_k}}, & \text{if } r_1 \neq 2, \\ U^+_{2^{n_1-1}} \otimes U^-_{r_2^{n_2}\dots r_k^{n_k}}, & \text{if } r_1 = 2 \end{cases}$$

Next if $p = 2r^n$, with r an odd prime and $n \geq 2$, the projection of v on $U_2 \otimes U^-_{r^{n-2}}$ is null.

Finally if $p = 2^n$, with $n \geq 3$, the projection of v on $U^-_{2^{n-2}}$ is null. □

3.3.2 Cyclicity in higher genus

We will denote by $Z_{p,g}$ the subspace of $V_{p,g}$ defined by:

$$Z_{p,g} := \text{Span}\{u_\sigma, \text{ so that } (u_\sigma, v_0)^H \neq 0\}$$

The goal of this subsection is to prove the following:

Proposition 3.3.4. *When $g \geq 2$, we have:*

1. *When $p = 4r$ with r an odd prime and if $g = 2$ or if p is generic and $g < r - 2$.*
2. *When $p = 2r^2$ with r an odd prime and $g = 2$ or if $p = 50$ is generic and $g = 3$, then the vacuum vector $v_0 \in V_{p,g}$ is cyclic.*
3. *When $p = 2r_1r_2$ with r_1, r_2 distinct odd primes and $g = 2$ or if p is generic and $2g < min(r_1, r_2)$, then the vacuum vector $v_0 \in V_{p,g}$ is cyclic.*
4. *When $p = 4r$ with r an odd prime and if $g = 2$ or if p is generic , and $g < r - 2$, then $Z_{p,g}$ is included in the cyclic subspace generated by v_0.*

Fix a trivalent graph $\Gamma \subset H_g$ as in section 2. Two p-admissible colorings σ_1, σ_2 of Γ will be said equivalent if:

$$(-1)^{\sigma_1(e)} A^{\sigma_1(e)(\sigma_1(e)+2)} = (-1)^{\sigma_2(e)} A^{\sigma_2(e)(\sigma_2(e)+2)}, \text{ for all } e \in E(\Gamma)$$

We denote by $\underline{col}_p(\Gamma)$ the set of equivalence classes of colorings for this relation. To $[\sigma] \in \underline{col}_p(\Gamma)$, we associate the subspace:

$$W_{[\sigma]} := \mathrm{Span}\{u_{\sigma'}, \sigma' \in [\sigma]\} \subset V_{p,g}$$

Lemma 3.3.5. *If $X \subset V_{p,g}$ is a $\widetilde{\mathrm{Mod}(\Sigma_g)}$-submodule, then:*

$$X = \bigoplus_{[\sigma] \in \underline{col}_p(\Gamma)} X \cap W_{[\sigma]}$$

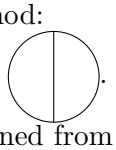
Proof. The matrices $\rho_{p,g}(T_e)$, for $e \in E(\Gamma)$, generate a commutative subalgebra of $\mathcal{A}_{p,g}$. The set $\underline{col}_p(\Gamma)$ indexes its characters and the spaces $W_{[\sigma]}$ are the associated common eigenspaces of the $\rho_{p,g}(T_e)$. The orthogonal projector on X must commute with the $\rho_{p,g}(T_e)$ and thus preserves the subspaces $W_{[\sigma]}$. □

The strategy to prove Proposition 3.3.4 is to apply Lemma 3.3.5 to

$$X := (\mathcal{A}_{p,g} \cdot v_0)^{\perp}$$

the orthogonal (for the invariant form) of the cyclic space generated by the vacuum vector.

Definition 3.3.6.

1. We call Γ_g a *fly eyes graph* of genus g if it is a trivalent graph obtained by the following inductive method:
   - Γ_2 is the Theta graph  .
   - A graph Γ_{g+1} is obtained from a Γ_g by choosing arbitrary a vertex and inserting a triangle as drawn on the left-hand side of Figure 3.4.

   The right-hand side gives an example of a genus 8 fly eyes graph.
2. The genus 3 fly eyes graph is unique and is called the tetrahedron graph. We say that a level of the form $p = 2r$, with r odd, is generic if for any coloring σ of Γ_3, we have:

$$(u_\sigma, v_0)^H_{p,3} \neq 0$$

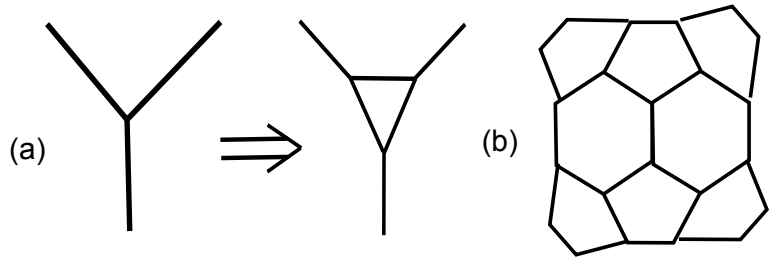


Figure 3.4: On the left: the operation transforming a fly eyes graph of genus g into a one of genus $g+1$. On the right: an example of genus 8 fly eye graph.

The complex numbers $(u_\sigma, v_0)^H_{p,3}$ are called tetrahedron coefficients in literature and are related to the $6j$-symbols defined in the previous section. In particular it is equivalent to say that the $6j$-symbols or the tetrahedron coefficients are not null for a level p. It follows from fusion-rules (equation (3.1)) that if p is generic , then for any $g \geq 3$, for any fly eyes graph Γ_g and for any p-admissible coloring σ of Γ_g, we have:

$$(u_\sigma, v_0)^H_{p,3} \neq 0$$

Fix $g \geq 2$ and embed a fly eyes graph Γ_g in S^3. Denote by H_g the embedded handlebody

$$H_g := S^3 \backslash V(\Gamma_g)$$

where $V(\Gamma_g)$ denotes a tubular neighborhood of Γ_g. For each edge $e \in E(\Gamma_g)$, fix a curve $\gamma_e \subset H_g$ which bounds a disc intersecting Γ_g only once along e.

We construct a map:

$$w : \mathbb{N}^{E(\Gamma_g)} \to V_{p,g}$$

as follows. To $f : E(\Gamma_g) \to \mathbb{N}$ we associate the class in $V_{p,g}$ of the link made of $f(e)$ parallel copies of γ_e for each edge $e \in E(\Gamma_g)$.

When $g = 2$, we will note $w_{a,b,c} \in V_{p,2}$ the class of the link made of a parallel copies of γ_1, b copies of γ_2 and c of γ_3.

Figure 3.5 shows the curves γ_e when $g = 2$ and $g = 3$.

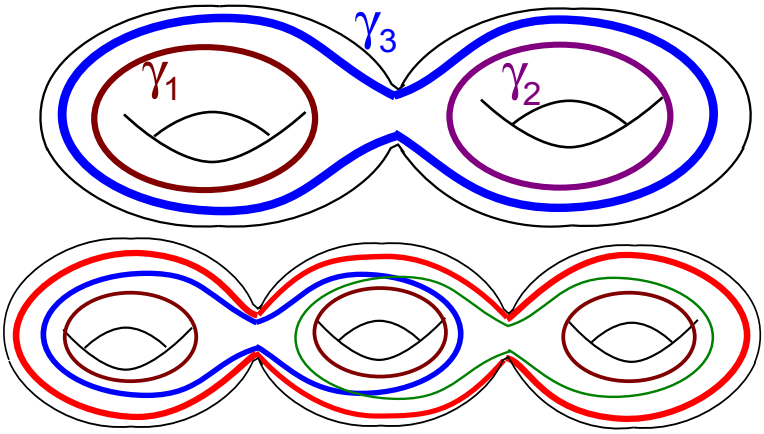
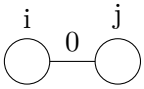
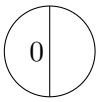


Figure 3.5: The curves γ_e defining the map w are drawn when $g = 2$ and $g = 3$.

Lemma 3.3.7. *If $p = 4r$, with r an odd prime, or if $p = 2r_1r_2$, with r_1, r_2 two distinct odd primes, then:*

$$w_{a,b,c} \in \mathcal{A}_{p,2} \cdot v_0, \text{ for any } a, b, c \in \{0, 1\}$$

Proof. The cyclicity of the vacuum vector $v_0 \in V_{p,1}$ in genus one, implies that the vectors 

and 

of $V_{p,2}$ belong to the cyclic space generated by $v_0 \in V_{p,2}$.

It remains to show that $w_{1,1,1} \in \mathcal{A}_{p,2} \cdot v_0$. It follows from the definition of Jones-Wenzl idempotents that:

$$w_{1,1,1} = 2 \text{ } \underset{2}{\bigcirc\!\!\!\!\bigcirc} \text{ } 2 + w_{2,0,0} + w_{0,2,0} + (A^2 + A^{-2})v_0$$

Thus we just have to show that $2 \text{ } \underset{2}{\bigcirc\!\!\!\!\bigcirc} \text{ } 2 \in \mathcal{A}_{p,2} \cdot v_0$.

Using Lemma 1.1.48, we have that:

$$2 \text{ } \underset{0}{\bigcirc\!\!\!\!\bigcirc} \text{ } 2 = \sum_{k=0,2,4} \begin{Bmatrix} 2 & 2 & k \\ 2 & 2 & 0 \end{Bmatrix} \overset{2}{\bigcirc} \overset{k}{-} \overset{2}{\bigcirc}$$

$$\rho_{p,2}(T_e) \cdot 2 \text{ } \underset{0}{\bigcirc\!\!\!\!\bigcirc} \text{ } 2 = \sum_{k=0,2,4} \begin{Bmatrix} 2 & 2 & k \\ 2 & 2 & 0 \end{Bmatrix} \mu_k \overset{2}{\bigcirc} \overset{k}{-} \overset{2}{\bigcirc}$$

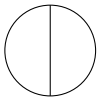
where T_e is (a lift of) the Dehn twist around the middle edge of the Theta graph (labeled 0).

The recoupling coefficients $\begin{Bmatrix} 2 & 2 & 2 \\ 2 & 2 & 0 \end{Bmatrix}$ and $\begin{Bmatrix} 2 & 2 & 4 \\ 2 & 2 & 0 \end{Bmatrix}$ are rational fraction in non null theta coefficients, thus are not null. Using the fact that $\mu_2 \neq 1$, we obtain that the family $\left\{ \overset{2}{\bigcirc} \overset{0}{-} \overset{2}{\bigcirc}, \; 2 \text{ } \underset{0}{\bigcirc\!\!\!\!\bigcirc} \text{ } 2 \; , \rho_{p,2}(T_e) \cdot 2 \text{ } \underset{0}{\bigcirc\!\!\!\!\bigcirc} \text{ } 2 \right\}$, made of vectors in $\mathcal{A}p, 2 \cdot v_0$, is free. Thus these three vectors generate the same subspace than the vectors $\overset{2}{\bigcirc} \overset{i}{-} \overset{2}{\bigcirc}$ for $i = 0, 2, 4$. Since $2 \text{ } \underset{2}{\bigcirc\!\!\!\!\bigcirc} \text{ } 2$ belongs to this space, it belongs to the cyclic space generated by the vacuum vector. So does $w_{1,1,1}$. □

Lemma 3.3.8. *When $p = 2r^2$, with r an odd prime, then*

$$w_{a,b,c} \in \mathcal{A}_{p,2} \cdot v_0, \text{ for all } 0 \leq a, b, c \leq \frac{r-3}{2}$$

.

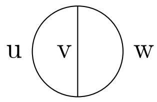
Moreover, if σ is a p-admissible coloring of $\Gamma =$  *such that:*

$$\sigma(e) \not\equiv -1 \pmod{r}, \text{ for all } e \in E(\Gamma)$$

then $u_\sigma \in \mathcal{A}_{p,2} \cdot v_0$.

Proof. Note first that $i, j \in \left\{0, \dots \frac{p-4}{2}\right\}$ are such that:
– $\mu_i = \mu_j$,
– $i \neq j$,
if and only if $i \equiv j \equiv -1 \pmod r$ and i and j have same parity (and are distinct). Thus when σ satisfies the condition of the lemma, the subspace $W_{[\sigma]}$ is one-dimensional. Lemma 3.3.5 implies that this subspace is either in $\mathcal{A}_{p,2} \cdot v_0$, or in its orthogonal. Now note that the Hopf pairing $(u_\sigma, v_0)^H_{p,2}$ is not zero for it is equal to a $3j$-symbol. This prove the second part of the lemma.

In particular, we just proved that:

Span (

$, 0 \leq u, v, w \leq r-2 \Big) \subset \mathcal{A}_{p,2} \cdot v_0$

We finish the proof by noticing that the vector $w_{a,b,c}$ belongs to this space whenever we have:

$$\begin{cases} a+c & \leq & r-2 \\ b+c & \leq & r-2 \\ a+b & \leq & r-2 \end{cases}$$

□

Lemma 3.3.9. *The vector* w_f *belongs to* $\mathcal{A}_{p,g} \cdot v_0$ *for* $f \in \{0,1\}^{E(\Gamma_g)}$ *when* p *is generic and:*
– $p = 4r$ *with* r *an odd prime such that* $g \leq r-2$.
– $p = 2r_1r_2$ *with* r_1, r_2 *distinct odd primes and* $2g \leq min(r_1, r_2)$.
– $p = 50$ *and* $g = 3$.

Proof. We proceed like in the proof of Lemma 3.3.8: first we note that if $f \in \{0,1\}^{E(\Gamma_g)}$ then:

$$w_f \in \mathrm{Span}\left(u_\sigma, 0 \leq \sigma(e) \leq g \text{ for all } e \in E(\Gamma_g)\right)$$

Then we note that if σ is such that $0 \leq \sigma(e) \leq g$ for all $e \in \Gamma_g$, then $W_{[\sigma]}$ is one-dimensional so is included in $\mathcal{A}_{p,g} \cdot v_0$ for $(u_\sigma, v_0)^H_{p,g} \neq 0$ by assumption. The fact that these $W_{[\sigma]}$ are one-dimensional is deduced from the following two facts:

1. When $p = 4r$, and $i, j \in \{0, \dots \frac{p-4}{2}\}$, then $\mu_i = \mu_j$ if and only if:
   – $i = j$,
   – or $i = \frac{p-4}{2} - j$ and i is even.
2. When $p = 2r_1r_2$, and $i, j \in \{0, \dots \frac{p-4}{2}\}$, then $\mu_i = \mu_j$ if and only if:
   – $i = j$,
   – or j is the only element satisfying
   $$\begin{cases} i \equiv j & \pmod{2r_1} \\ i \equiv -j-2 & \pmod{r_2} \end{cases} \text{ or } \begin{cases} i \equiv j & \pmod{2r_2} \\ i \equiv -j-2 & \pmod{r_1} \end{cases}$$

□

Proof of Proposition 3.3.4. Fix a fly eyes graph Γ, a class $[\sigma] \in \underline{col}_p(\Gamma)$, and choose a vector

$$v = \sum_{\sigma' \in [\sigma]} \alpha_{\sigma'} u_{\sigma'} \in W_{[\sigma]} \bigcap (A_{p,g} \cdot v_0)^{\perp}$$

By Lemma 3.3.5, we must show that $v = 0$ to conclude. We will find $\dim\left(W_{[\sigma]}\right)$ independent equations verified by the coefficients $\alpha_{\sigma'}$.

Note $F \subset \mathbb{N}^{E(\Gamma)}$ the set of functions f so that:
– $f(e) \in \{0, 1\}, \forall e \in E(\Gamma)$, if $p = 4r$ of $p = 2r_1 r_2$,
– $f(e) \in \left\{0, \dots, \frac{r-3}{2}\right\}, \forall e \in E(\Gamma)$, if $p = 2r^2$.

Using Lemmas 3.3.7, 3.3.8 and 3.3.9, we know that

$$w_f \in \mathcal{A}_{p,g} \cdot v_0, \text{ for all } f \in F$$

By definition of w_f, we have that:

$$(w_f, v_0)^H_{p,g} = 0, \qquad \text{for all } f \in F \tag{3.2}$$

$$\Leftrightarrow \quad \sum_{\sigma' \in [\sigma]} \left(\prod_{e \in E(\Gamma)} \lambda^{f(e)}_{\sigma'(e)}\right) \alpha_{\sigma'} \, (u_{\sigma'}, v_0)^H_{p,g} = 0, \quad \text{for all } f \in F \tag{3.3}$$

where $\lambda_i = -\left(A^{2(i+1)} + A^{-2(i+1)}\right)$.

Since the complex numbers $(u_{\sigma'}, v_0)^H_{p,g}$ are non null when $p = 2r^2$ or $p = 2r_1 r_2$ is generic or when $p = 4r$ and $u_\sigma \in Z_{p,g}$, it is enough to show that the matrix:

$$M := \left(\prod_{e \in E(\Gamma)} \lambda^{f(e)}_{\sigma'(e)} \right)_{\substack{\sigma' \in [\sigma] \\ f \in F}}$$

has independent lines to conclude the proof.

We now define an invertible square matrix $\tilde{M}$ such that M is obtained from $\tilde{M}$ by removing some lines.

When $i \in \left\{0, \dots, \frac{p-4}{2}\right\}$ we define the set:

$$\omega(i) := \left\{ j \in \left\{0, \dots, \frac{p-4}{2}\right\}, \text{ so that } \mu_i = \mu_j \right\}$$

And the Vandermonde matrix:

$$N[i] := \left(\lambda_j^n\right)_{\substack{j \in \omega(i) \\ 0 \leq n \leq \#\omega(i) - 1}}$$

Since $\lambda_a \neq \lambda_b$ when $a \neq b \in \omega(i)$, the matrix $N[i]$ is invertible.

Now note $E(\Gamma) = \{e_1, \dots, e_{3g-3}\}$ and choose $\sigma \in [\sigma]$ arbitrary. The matrix

$$\tilde{M} := N(e_1) \otimes \dots \otimes N(e_{3g-3})$$

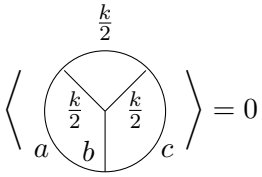
is clearly invertible and M is obtained from $\tilde{M}$ by removing the lines corresponding to non p-admissible colorings of Γ. □

3.3.3 Null $6j$-symbols when 4 divides p

When $p = 2r$ with r odd, numerical computations suggest that there is no null $6j$-symbols at level p. On the contrary, when 4 divides p, we have two families of $6j$-symbols that vanish at level p:

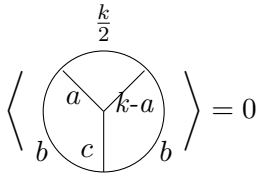
Proposition 3.3.10. *Suppose 4 divides $p \geq 8$ and write $p = 2(k+2)$, with k an even integer. Then the following tetrahedron coefficients vanish:*

1. *Type I:*



   when $a + b + c \equiv 2 \pmod 4$.

2. *Type II:*



   when $a + \frac{c+k}{2} \equiv 1 \pmod 2$.

Definition 3.3.11. If 4 divides $p \geq 8$, we say that p is generic if the only vanishing $6j$-symbols at level p are the ones given in Proposition 3.3.10

Numerical computations shows us that every level $p \leq 50$ is generic . We conjecture that every level is generic .

Lemma 3.3.12. *Let a, b be two integers such that $(a, k-a, b)$ is p-admissible. Set*

$$F(a,b) := \frac{\left\langle \begin{array}{c} a \\ k \quad k\text{-}a \\ k\text{-}a \quad b \quad a \end{array} \right\rangle < a >}{\left\langle \begin{array}{c} k\text{-}a \quad a \quad b \end{array} \right\rangle}$$

Then we have $F(a,b) = (-1)^{\frac{b+k}{2}+a}$.

Proof. A straightforward computation using the formulas of [70] gives:

$$F(a,b) = \frac{f(a)}{g(b)} \quad \text{, where} \quad f(a) = (-1)^a [a+1]![k-a]!$$

$$g(b) = (-1)^{\frac{k+b}{2}} \left[\frac{k-b}{2}\right]! \left[\frac{k+b}{2}+1\right]!$$

We now remark that

$$\frac{f(a+1)}{f(a)} = -\frac{[a+2]}{[k-a]} = -1$$

and

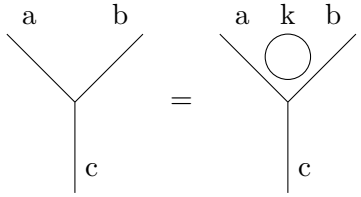
$$\frac{g(b+2)}{g(b)} = -\frac{\left[\frac{k+b}{2}+2\right]}{\left[\frac{k-b}{2}\right]} = -1$$

We conclude using the fact that $F(\frac{k}{2}, 2) = -1$. □

Lemma 3.3.13. *If (a, b, c) is a p-admissible triple, then we have:*

$$< k-a >< k-b > \frac{\left\langle \begin{array}{c} a \\ k \quad b \\ k\text{-}a \quad c \quad k\text{-}b \end{array} \right\rangle}{\left\langle \begin{array}{c} k\text{-}a \quad k\text{-}b \quad c \end{array} \right\rangle} \cdot \frac{\left\langle \begin{array}{c} k\text{-}a \\ k \quad k\text{-}b \\ a \quad c \quad b \end{array} \right\rangle}{\left\langle \begin{array}{c} a \quad k\text{-}b \quad c \end{array} \right\rangle} = 1$$

Proof. We use the fact that adding a trivial ribbon colored by k does not change the class of a vector. We work in the space associated to the sphere with three punctures colored by a, b and c:

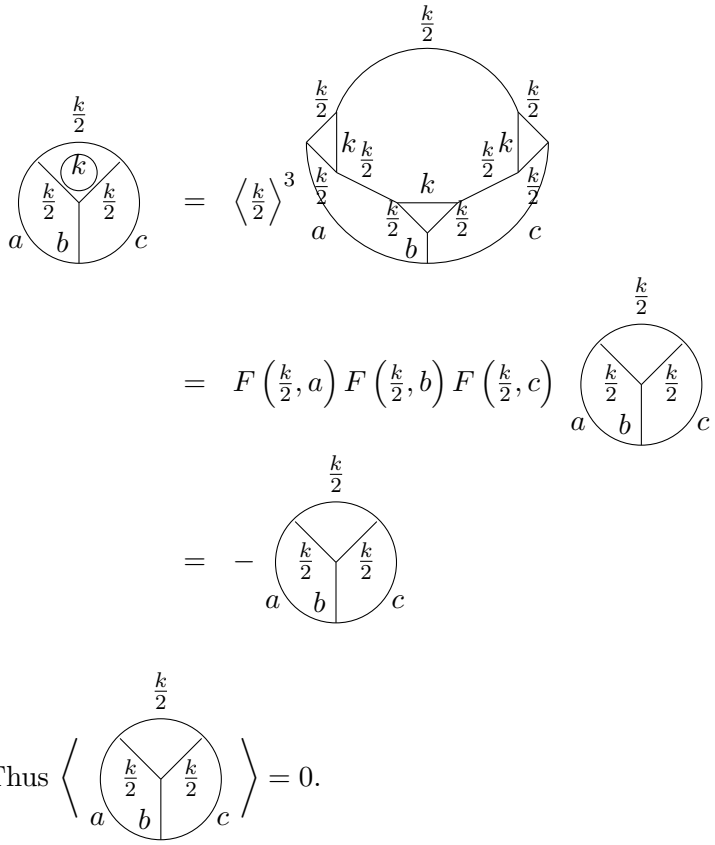


$$= < k-a >< k-b > \quad \begin{array}{ccccc} a & & & & b \\ & & k & & \\ & k-a & & k-b & \\ & & a \quad b & & \\ & & c & & \end{array}$$

$$= < k-a >< k-b > \frac{\left\langle \begin{array}{c} a \\ k \quad b \\ k\text{-a} \quad c \quad k\text{-b} \end{array} \right\rangle}{\left\langle \begin{array}{c} k\text{-a} \quad k\text{-b} \quad c \end{array} \right\rangle} \cdot \frac{\left\langle \begin{array}{c} k\text{-a} \\ k \quad k\text{-b} \\ a \quad c \quad b \end{array} \right\rangle}{\left\langle \begin{array}{c} a \quad k\text{-b} \quad c \end{array} \right\rangle} \quad \begin{array}{ccc} a & & b \\ & & \\ & c & \end{array}$$

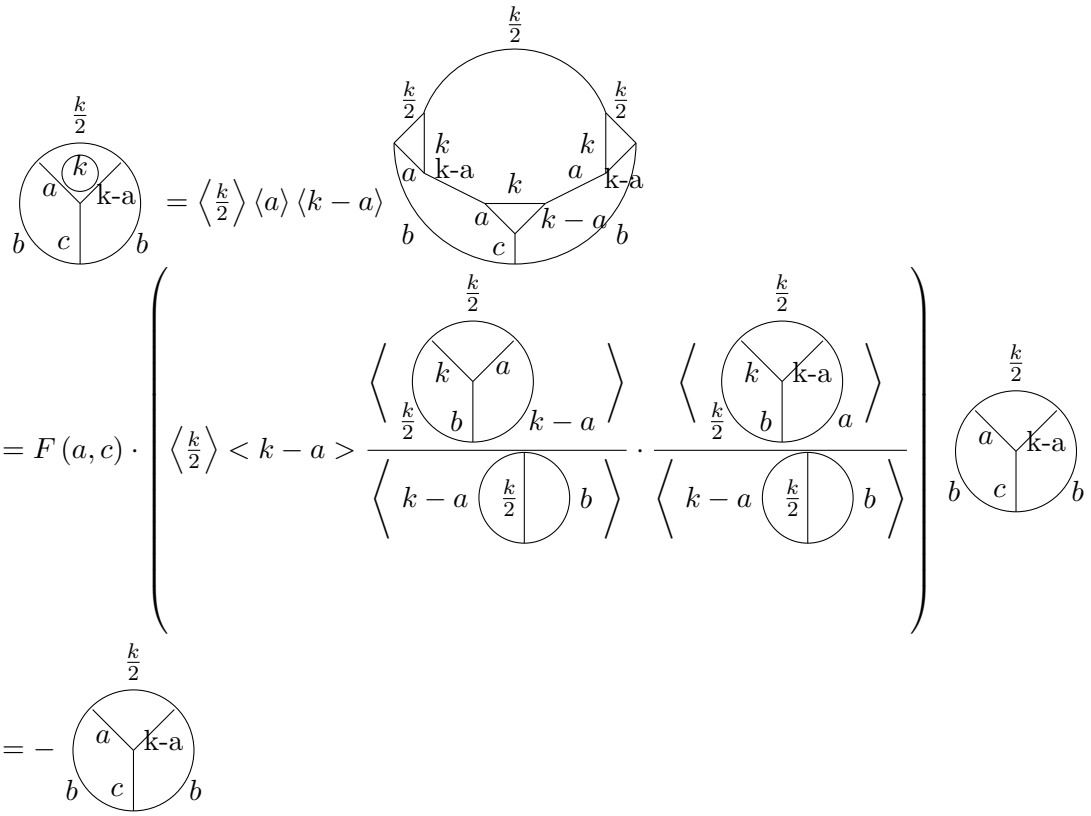
We conclude by identifying both vectors. □

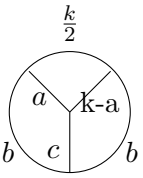
Proof of Proposition 3.3.10. We use the fact that the Kauffman bracket of a link in S^3 does not change if we add a trivial ribbon colored by k. First when σ is of type I, we use the fusion rules 1.1.48 and Lemma 3.3.12 to compute:



Thus $\left\langle \right\rangle = 0$.

When σ is of type II, a similar computation using Lemmas 3.3.12 and 3.3.13 gives:



Thus $\left\langle$ 

$\right\rangle = 0.$ □

The rest of this subsection is devoted to the proof of the following:

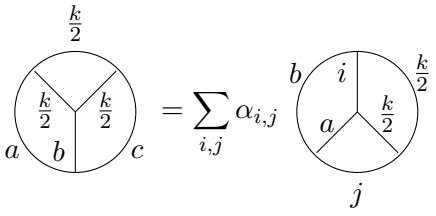
Proposition 3.3.14. *If $p = 4r$, with $r \leq 7$ an odd prime, is generic then the vacuum vector $v_0 \in V_{p,3}$ is cyclic in genus 3.*

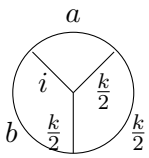
We already know from Proposition 3.3.4 that $Z_{p,3}$ is included in the cyclic subspace generated by the vacuum vector. When p is generic , its orthogonal is spanned by vectors u_σ with σ a coloration of the Tetahedron graph of type I or II given in Proposition 3.3.10. We must show that these vectors also belong to the cyclic space generated by v_0 to conclude.

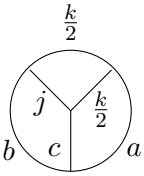
We split the proof into four lemmas which, together, imply Proposition 3.3.14.

Lemma 3.3.15. *Suppose $p = 4r$ with $r \geq 7$ an odd prime. Let σ be a coloration of the Tetrahedron graph of type I, as defined in Proposition 3.3.10, such that $a \neq \frac{k}{2}$ and $b \neq \frac{k}{2}$. Then u_σ belongs to the cyclic space generated by v_0.*

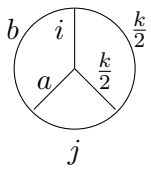
Proof. The proof relies on the following remark: embeded a colored Tetahedron graph in H_3, choose two opposite edges of the graph and perform two Whitehead moves on these edges as in the fusion rule 1.1.48. You get this way another embedding of the Tetahedron graph inside H_3. While choosing the edges colored by b and its opposite colored by $\frac{k}{2}$ in a type I coloration of the Tetahedron graph, we get:



where $\alpha_{i,j} =$ $\left\langle$ 

$\right\rangle \left\langle$ 

$\right\rangle c_{i,j}$ with $c_{i,j} \neq 0$.

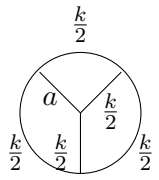
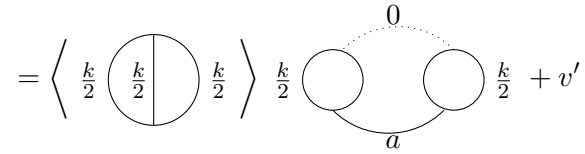
When $i = \frac{k}{2}$, we have $\alpha_{i,j} = 0$ by Proposition 3.3.10. When $i \neq \frac{k}{2}$, the vector 

belongs to $Z_{p,3}$ and thus to the cyclic space generated by v_0 by Proposition 3.3.4. This concludes the proof.

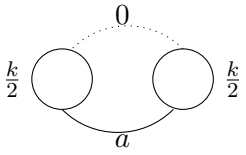
□

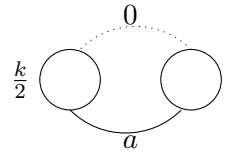
Lemma 3.3.16. *Suppose $p = 4r$ with $r \geq 7$ an odd prime. Let σ be a coloration of the Tetrahedron graph of type I, as defined in Proposition 3.3.10, such that $a = b = \frac{k}{2}$. Then u_σ belongs to the cyclic space generated by v_0.*

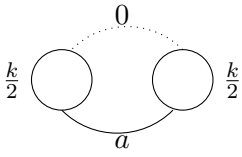
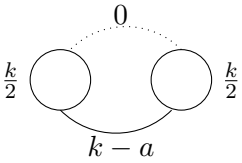
Proof. Using fusion rule 1.1.48, we get:

$v_a :=$  $=$  $+ v'$

, where v' is a vector orthogonal to the first one.

Now according to Proposition 3.3.4, we have that the vacuum vector is cyclic in genus 2. This implies that the vector  belongs to the cyclic space generated by the vacuum vector in genus 3.

When $a = \frac{k}{2}$, then $W_{[v_a]}$ is one dimensional, so according to Lemma 3.3.5, either $v_{\frac{k}{2}}$ belongs the cyclic space generated by v_0 or it belongs to its orthogonal. But its scalar product with the vector  is a non null $3j$-symbol. Thus $v_a \in \mathcal{A}_{p,3} \cdot v_0$.

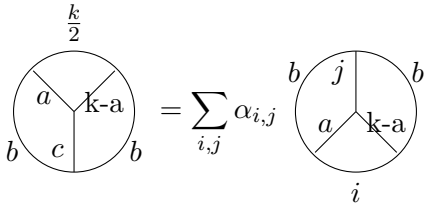
When $a \neq \frac{k}{2}$, then $W_{[v_a]}$ is two dimensional generated by v_a and v_{k-a}. If $v = \alpha_1 v_a + \alpha_2 v_{k-a}$ belongs to the orthogonal of the cyclic space generated by v_0, then v is orthogonal to both vectors  and  . This implies that $v = 0$ so $W_{[v_a]}$ is included in the cyclic space of the vacuum vector.

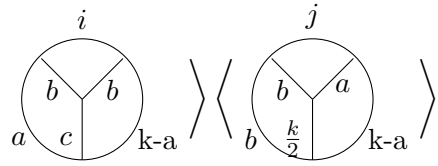
□

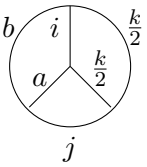
Lemma 3.3.17. *Suppose $p = 4r$ with $r \geq 7$ an odd prime. Let σ be a coloration of the Tetrahedron graph of type II, as defined in Proposition 3.3.10, such that either we have $a \neq b$ and $a \neq k - b$, or we have $c \equiv \frac{k}{2} \pmod 4$. Then u_σ belongs to the cyclic space generated by v_0.*

Proof. The proof is similar to the proof of Lemma 3.3.15. Using fusion rules 1.1.48 twice,

we get:



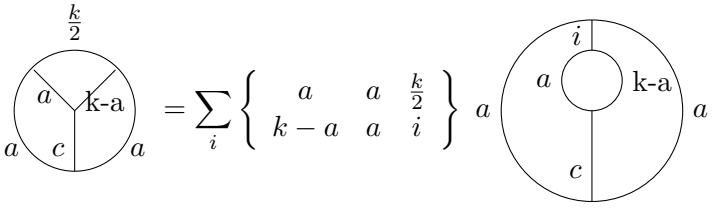
where $\alpha_{i,j} =$  $c_{i,j}$ with $c_{i,j} \neq 0$.

When $i = \frac{k}{2}$, we have $\alpha_{i,j} = 0$ by Proposition 3.3.10. When $i \neq \frac{k}{2}$, using the fact that either $a \neq b$ and $a \neq k - b$, or $c \equiv \frac{k}{2} \pmod 4$, we see that the vector  belongs

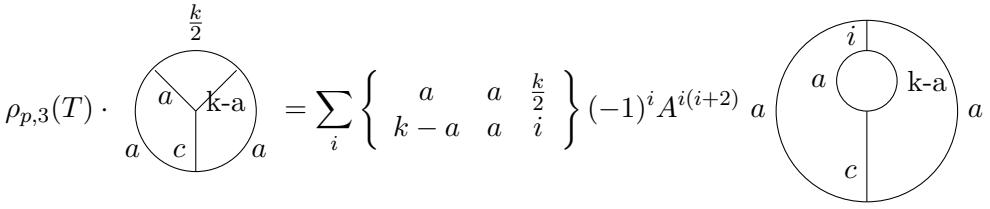
to $Z_{p,3}$ and thus to the cyclic space generated by v_0 by Proposition 3.3.4. This concludes the proof. □

Lemma 3.3.18. *Suppose $p = 4r$ with $r \geq 7$ an odd prime. Let σ be a coloration of the Tetrahedron graph of type II, as defined in Proposition 3.3.10, such that we have $a = b$ or $a = k - b$ and $c \equiv \frac{k}{2} + 2 \pmod 4$. Then u_σ belongs to the cyclic space generated by v_0.*

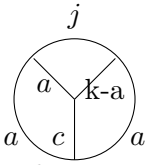
Proof. Using Lemma 1.1.48, we get:



Let $T \in \widetilde{\mathrm{Mod}(\Sigma_g)}$ represent a lift of the Dehn twist around the edge colored by i in the above graph. We have:



In particular, $\rho_{p,3}(T) \cdot u_\sigma$ belongs to the space generated by the vectors of the form



. Whenever $j \neq \frac{k}{2}$, these generating vectors belong to $Z_{p,3}$ and thus to the cyclic space generated by the vacuum vector. Denote by β the scalar product $\langle u_\sigma, \rho_{p,3}(T) u_\sigma \rangle$.

If $\beta = 0$, then $\rho_{p,3}(T) \cdot u_\sigma$ belongs to the cyclic space generated by v_0, so does u_σ for $\rho_{p,3}(T)$ is invertible.

If $\beta \neq 0$, then the operator $a := \beta \cdot \mathbb{1} + \left\{ \begin{array}{ccc} a & a & \frac{k}{2} \\ k-a & a & \frac{k}{2} \end{array} \right\} \rho_{p,3}(T) \in \mathcal{A}_{p,3}$ is invertible since $\rho_{p,3}(T)$ has finite order. Since $a \cdot u_\sigma$ belongs to the cyclic space generated by the vacuum vector, so does u_σ. □

3.4 Decomposition into irreducible factors

In this section, we will prove Theorems 3.1.1 and 3.1.2. Denote by $(\mathcal{A}_{p,g})'$ the commutant of the algebra $\mathcal{A}_{p,g}$, i.e. the subspace of $\mathrm{End}(V_{p,g})$ of operators commuting with all the $\rho_{p,g}(\phi)$ for $\phi \in \widetilde{\mathrm{Mod}(\Sigma_g)}$.

The dimension of $(\mathcal{A}_{p,g})'$ is equal to the number of irreducible submodules of $V_{p,g}$. We thus have to show that $\dim((\mathcal{A}_{p,g})')$ is one if $p = 2r^2$ and $p = 2r_1r_2$ and is two when $p = 4r$ with the additional assumptions of the two theorems.

Consider the following linear map:

$$f : \left\{ \begin{array}{ccc} (\mathcal{A}_{p,g})' & \hookrightarrow & V_{p,g} \\ \theta & \longmapsto & \theta \cdot v_0 \end{array} \right.$$

The cyclicity of v_0 (Proposition 3.3.4) implies that f is injective. Moreover if $\phi \in \widetilde{\mathrm{Mod}(\Sigma_g)}$ is the lift of a homeomorphism of Σ_g that extends to H_g through $\alpha : \Sigma_g \to \partial H_g$, then:

$$\rho_{p,g}(\phi) \cdot v_0 = v_0$$

Denote by $\widetilde{\mathrm{Mod}(H_g)} \subset \widetilde{\mathrm{Mod}(\Sigma_g)}$ the subgroup generated by these ϕ. By definition, we have:

$$\mathrm{Range}(f) \subset \left\{ v \in V_{p,g} \text{ so that } \rho_{p,g}(\phi) \cdot v = v, \text{ for all } \phi \in \widetilde{\mathrm{Mod}(H_g)} \right\}$$

In particular, for any trivalent graph Γ, we have $\mathrm{Range}(f) \subset W_{[0]}(\Gamma)$ where $[0]$ is the class of the coloring sending every edges of Γ to 0. As an immediate consequence, we get the:

Proof of Theorems 3.1.1 *and* 3.1.2 *when* $p = 2r^2$. When $p = 2r^2$, with r an odd prime, then $W_{[0]}$ is one-dimensional, generated by v_0 for $\mu_i \neq 1$ when $i \neq 0$. Thus $\mathrm{Range}(f) = \{v_0\}$ and $(\mathcal{A}_{2r^2,g})' = \{\mathbb{1}\}$. The Schur lemma implies that the module $V_{2r^2,g}$ is irreducible. □

Remark. When $p = 18$, we remark that the numbers $\mu_0, \mu_1, \ldots, \mu_{23}$ are pairwise distinct. The proof of Roberts [76] applies word-by-word in this case to show that $V_{18,g}$ is irreducible. Indeed the fact that the μ_i are distinct implies that the null vector $v_0 \in V_{18,1}$ is cyclic for the action of the group generated by the Dehn twist along the longitude of H_1. This easily implies that $v_0 \in V_{18,g}$ is cyclic for the action of $\widetilde{\mathrm{Mod}(\Sigma_g)}$ for arbitrary $g \geq 1$ and we conclude as above by noticing that $W_{[0]}$ is one dimensional generated by v_0.

3.4.1 The case where $p = 4r$

Let $p \geq 3$ be such that $p \equiv 4 \pmod 8$. Consider a link $L \subset \Sigma_g \times \{\frac{1}{2}\}$ inside the cylinder $\Sigma_g \times [0,1]$ and color L by p parallel copies of ω or, equivalently, by the $\left(\frac{p-4}{2}\right) - th$ Jones-Wenzl idempotent. The gluing of the above cobordism on H_g induces an operator acting on $V_{p,g}$. In [14] it is shown that this operator only depends the homology class of L in $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ and we get this way a morphism of algebras, whose injectivity is easily checked:

$$i : \mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})] \hookrightarrow \mathcal{A}_{p,g}$$

Its action on v_0 gives the space $W_{[0]} \cong \mathbb{C}[H_1(H_g, \mathbb{Z}/2\mathbb{Z})]$.

We denote by P the projector of $V_{p,g}$ on the subspace of vectors fixed by the operators of $i(\mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})])$. Clearly $P \in (\mathcal{A}_{p,g})'$.

Note $x_i, y_i \in H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ the meridian and longitude around the $i - th$ hole and note:

$$\Theta_i := \frac{1}{\sqrt{2}}(-1 + x_i + y_i + x_i y_i) \in \mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})]$$

The Θ_i's are symmetries which pairwise commute and

$$P = \frac{1}{2^g}\left(\sqrt{2}(\Theta_1 + \ldots + \Theta_g) + g + 1\right)$$

The symmetric group σ_g acts by permutation on the generators of $\mathbb{C}[\Theta_1, \ldots, \Theta_g]$. We note $\mathcal{W}_g \subset i(\mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})])$ the subalgebra of $\mathbb{C}[\Theta_1, \ldots, \Theta_g]$ of elements fixed by σ_g.

Finally we denote by $I \subset i(\mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})])$ the ideal generated by the elements $(x_i - 1)$ for $1 \leq i \leq g$. We have:

$$\mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})]\,/_I \cong \mathbb{C}[H_1(H_g, \mathbb{Z}/2\mathbb{Z})] \cong W_{[0]}$$

Lemma 3.4.1. *Consider the action of $Sp(2g, \mathbb{Z}/2\mathbb{Z})$ on $i(\mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})])$. Then:*

1. *The vectors fixed by this action are the ones of* $\mathrm{Span}(\mathbb{1}, P)$.
2. *For every $w \in \mathcal{W}_g$ and $\phi \in Sp(2g, \mathbb{Z}/2\mathbb{Z})$ we have:*

$$\phi \cdot w - w \in I$$

Proof. The first point follows from the fact that the action of $Sp(2g, \mathbb{Z}/2\mathbb{Z})$ on $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ has two orbits: the singleton containing the neutral element and the set containing the other elements. Indeed by taking an appropriate $\mathbb{Z}/2\mathbb{Z}$-basis of $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$, this action is described by the usual Birman generators of $Sp(2g, \mathbb{Z})$ ([12]) passed to the quotient in $Sp(2g, \mathbb{Z}/2\mathbb{Z})$, that is the $2g \times 2g$ matrices:

$$\begin{pmatrix} A & 0_g \\ 0_g & A^* \end{pmatrix}, \begin{pmatrix} 1_g & B \\ 0_g & 1_g \end{pmatrix} \text{ and } \begin{pmatrix} 0_g & 1_g \\ 1_g & 0_g \end{pmatrix}$$

where $A \in \mathrm{GL}(g, \mathbb{Z}/2\mathbb{Z})$ and B is symmetric. We just have to remark that the commutant of the algebra generated by these matrices consists of the scalar matrices to conclude.

To prove the second point, denote by $X_i, Y_i, Z_{i,j}$ for $1 \leq i, j \leq g$ the class in $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ of the Dehn twists of Figure 3.6 generating $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$. First note that the operators Θ_i are invariant under the action of the X_i and Y_i and that the element of the algebra $\mathcal{W}_g$

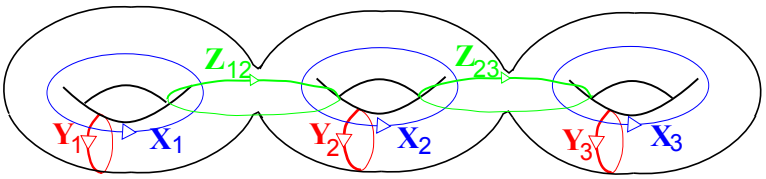


Figure 3.6: Some Dehn twists generating $Sp(2g, \mathbb{Z}/2\mathbb{Z})$ when $g = 3$ by passing to the quotient.

are invariant under permutation of the handles. We are reduced to show that for $w \in \mathcal{W}_g$, we have $Z_{1,2} \cdot w - w \in I$.

First note that $Z_{1,2} \cdot \Theta_i = \Theta_i$ when $i \notin \{1, 2\}$. Then we compute:

$$\begin{aligned}
Z_{1,2} \cdot \Theta_1 - \Theta_1 &= \frac{1}{\sqrt{2}}(y_1 + x_1 y_1)(x_2 - 1) \in I \\
Z_{1,2} \cdot \Theta_2 - \Theta_1 &= \frac{1}{\sqrt{2}}(y_2 + x_2 y_2)(x_1 - 1) \in I \\
Z_{1,2} \cdot (\Theta_1 \Theta_2) - (\Theta_1 \Theta_2) &= \frac{1}{2}((x_1 x_2 - 1)(x_1 y_1 + y_1)(x_2 y_2 + y_2) \\
&\quad + (x_2 - 1)(y_1 x_1 + y_1)(-1 + x_2) \\
&\quad + (x_1 - 1)(x_2 y_2 + y_2)(-1 + x_1)) \in I
\end{aligned}$$

□

The case $p = 4r$ of Theorems 3.1.1 and 3.1.2 are easily deduced from the:

Proposition 3.4.2. *If $p \equiv 4 \pmod 8$ and $v_0 \in V_{p,g}$ is cyclic, then*

$$f^{-1}\left(\mathbb{C}\left[H_1\left(H_g, \mathbb{Z}/2\mathbb{Z}\right)\right]\right) = \mathrm{Span}\left(\mathbb{1}, P\right)$$

Proof. Let $\Theta \in (\mathcal{A}_{p,g})'$. Since $\Theta \cdot v_0$ lies in $W_{[0]}$ and is invariant under permutation of the handles, there exists an element $w \in \mathcal{W}_g$ such that $w \cdot v_0 = \Theta \cdot v_0$. Now if $\phi \in \widetilde{\mathrm{Mod}(\Sigma_g)}$, then:

$$\Theta \circ \rho_{p,g}(\phi) \cdot v_0 = \rho_{p,g}(\phi) \circ \Theta \cdot v_0 = \rho_{p,g}(\phi) \circ w \cdot v_0 = w \circ \rho_{p,g}(\phi) \cdot v_0$$

where we used the second point of Lemma 3.4.1 in the last equality. Using the cyclicity of v_0 we get that $\Theta = w \in \mathcal{W}_g$. We conclude using the first point of Lemma 3.4.1. □

3.4.2 The case where $p = 2r_1r_2$

In this subsection, we suppose that $p = 2r_1r_2$ with r_1, r_2 distinct odd primes.
In this case, there exists an unique integer $x \in \{1, \ldots, r_1r_2 - 2\}$ such that $\mu_x = 1$. This integer is even and verifies either

$$\begin{cases} x \equiv -2 \pmod{r_1} \\ x \equiv 0 \pmod{r_2} \end{cases} \text{ or } \begin{cases} x \equiv 0 \pmod{r_1} \\ x \equiv -2 \pmod{r_2} \end{cases}.$$

We begin by stating a technical lemma whose proof will be the subject of the next subsection:

Lemma 3.4.3. *If (x, x, x) is p-admissible, then we have the following:*

$$\begin{Bmatrix} x & x & 2 \\ x & x & 0 \end{Bmatrix} \begin{Bmatrix} x & x & 4 \\ x & x & x \end{Bmatrix} \neq \begin{Bmatrix} x & x & 4 \\ x & x & 0 \end{Bmatrix} \begin{Bmatrix} x & x & 2 \\ x & x & x \end{Bmatrix}$$

Lemma 3.4.4. *Let $p \geq 3$ be such that (x, x, x) is p-admissible. Let Γ_1, Γ_2 be two trivalent graphs which only differ by a single Whitehead move inside a ball B^3 as drawn in Figure 3.3. Then:*

$$W_{[0]}(\Gamma_1) \cap W_{[0]}(\Gamma_2) \subset \text{Span}\left(u_\sigma^{\Gamma_1},\ \textit{such that } \sigma(a)\sigma(b)\sigma(c)\sigma(d) = 0\right)$$

Proof. Let σ_1, σ_2 be two p-admissible colorings of Γ_1, with colors 0 or x, such that:

$$\sigma_1(e) = \sigma_2(e), \forall e \in E(\Gamma_1) - \{i\}$$

and with $\sigma_i(a) = \sigma_i(b) = \sigma_i(c) = \sigma_i(d) = x$ and $\sigma_1(i) = 0$, $\sigma_2(i) = x$.

Suppose there exists $(\alpha, \beta) \in \mathbb{C}^2$ so that:

$$v := \alpha u_{\sigma_1} + \beta u_{\sigma_2} \in W_{[0]}(\Gamma_2)$$

We must show that $\alpha = \beta = 0$ to conclude. Using the fusion rule equation (3.1) of section 2.1, we get:

$$v = \left(\alpha \begin{Bmatrix} x & x & 2 \\ x & x & 0 \end{Bmatrix} + \beta \begin{Bmatrix} x & x & 2 \\ x & x & x \end{Bmatrix}\right) \; 2 \\ + \left(\alpha \begin{Bmatrix} x & x & 4 \\ x & x & 0 \end{Bmatrix} + \beta \begin{Bmatrix} x & x & 4 \\ x & x & x \end{Bmatrix}\right) \; 4 \; + v'$$

where 2 and 4 represent the vectors associated to colorations of Γ_2 by the same colors that σ_1, σ_2 outside the ball B^3 and with the edge j colored respectively by 2 and 4.

The vector v' is orthogonal to the two previous ones.

Now since $v \in W_{[0]}(\Gamma_2)$, we have the following system:

$$\begin{pmatrix} \begin{Bmatrix} x & x & 2 \\ x & x & 0 \end{Bmatrix} & \begin{Bmatrix} x & x & 2 \\ x & x & x \end{Bmatrix} \\ \begin{Bmatrix} x & x & 4 \\ x & x & 0 \end{Bmatrix} & \begin{Bmatrix} x & x & 4 \\ x & x & x \end{Bmatrix} \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We conclude using Lemma 3.4.3 □

If $i \in \{1, \ldots, g\}$, we note $b_i \in V_{p,g}$ the vector representing a single ribbon colored by x around the $i - th$ hole.

Lemma 3.4.5. *If $\mathcal{G}^g$ represents the set of all trivalent graph of genus g, then:*

$$\bigcap_{\Gamma \in \mathcal{G}^g} W_{[0]}(\Gamma) = \text{Span}\left(v_0, b_i, 1 \leq i \leq g\right)$$

Proof. Let σ be a coloring of the graph of Figure 3.7 such that:

1. $\sigma(e) \in \{0, x\}$, for all $e \in E(\Gamma)$,
2. There exists $i < j$ with $\sigma(a_i) = \sigma(b_i) = \sigma(a_j) = \sigma(b_j) = x$.

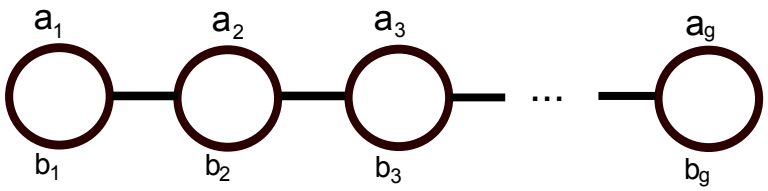


Figure 3.7: A trivalent graph of genus g.

We can suppose that for every $i < k < j$, then $\sigma(a_k)\sigma(b_k) = 0$.
Using Lemma 3.4.4 with $a = a_i$, $b = a_j$, $c = b_i$ and $d = b_j$, we have that the projection of u_σ on $\bigcap_{\Gamma\in\mathcal{G}^g} W_{[0]}(\Gamma)$ is null.

We conclude by noticing that if σ is a coloring of Γ, with colors in $\{0, x\}$, that does not satisfies 2, then $u_\sigma = b_i$ for some $i \in \{1, \dots, g\}$ or $u_\sigma = v_0$. □

Denote by $u_i \in V_{p,1}$ the class of the closure of the i^{th} Jones-Wenzl idempotent inside the longitude $\{0\} \times S^1$ of $D^2 \times S^1$.

Lemma 3.4.6. *There exists an element $a \in \mathcal{A}_{2r_1r_2,1}$ so that:*

$$\begin{cases} a \cdot u_0 = u_x \\ a \cdot u_x = u_0 \end{cases}$$

Proof. It is enough to show that there exists a symmetry $\psi \in (\mathcal{A}_{2r_1r_2,1})'$ so that:

$$\psi \cdot u_0 = u_x \text{ and } \psi \cdot u_x = u_0$$

Indeed, the cyclicity of u_0 (Proposition 3.3.3) implies the existence of $a \in \mathcal{A}_{2r_1r_2,1}$ so that

$$a \cdot u_0 = u_x$$

If such a ψ does exist, we then have:

$$a \cdot u_x = a \circ \psi \cdot u_0 = \psi \circ a \cdot u_0 = u_0$$

The symmetry ψ is defined as follows: choose $i \in \{0, \dots, r_1r_2 - 2\}$, then only one of the following two cases occurs:

– Either there exists $j \in \{0, \dots, r_1r_2 - 2\}$ so that

$$\begin{cases} j \equiv i & (\text{mod } 2r_1) \\ j \equiv -i - 2 & (\text{mod } r_2) \end{cases}$$

and we put $\psi(u_i) := +u_j$.

– Or there exists $j \in \{0, \dots, r_1r_2 - 2\}$ so that

$$\begin{cases} j \equiv i & (\text{mod } 2r_2) \\ j \equiv -i - 2 & (\text{mod } r_1) \end{cases}$$

and we put $\psi(u_i) := -u_j$.

A straightforward computation shows that ψ commutes with $\rho_{p,1}(T)$ and $\rho_{p,1}(S)$ and either ψ or $-\psi$ sends u_0 to u_x. □

The proof of Theorems 3.1.1 and 3.1.2 when $p = 2r_1r_2$ follows from the following:

Proposition 3.4.7. *Let r_1, r_2 be two distinct odd primes, $p = 2r_1r_2$ and $g \geq 2$ be such that $v_0 \in V_{p,g}$ is cyclic. Then $V_{p,g}$ is irreducible.*

Proof. Using Lemma 3.4.5 and the fact that the vectors of Range(f) must be invariant under permutation of the handles, we have that:

$$\text{Range}(f) \subset \text{Span}\,(v_0, b_1 + \ldots + b_g)$$

By contradiction, suppose there exists $\Theta \in (\mathcal{A}_{p,g})'$ so that:

$$\begin{aligned} \Theta \cdot v_0 &= b_1 + \ldots + b_g \\ &= (a \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} + \ldots + \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes a) \cdot v_0 \end{aligned}$$

where $a \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}$ denotes the embedding of the element $a \in \mathcal{A}_{p,1}$, seen as a linear combination of ω-colored link in $\Sigma_1 \times [0,1]$, in the first handle of $\Sigma_g \times [0,1]$. Note that this operator belongs to $\mathcal{A}_{p,g}$, so commutes with Θ.

Now we have:

$$\begin{aligned} \Theta^2 \cdot v_0 &= (a \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} + \ldots + \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes a)^2 \cdot v_0 \\ &= gv_0 + (a \otimes a \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}) \cdot v_0 + \ldots + (\mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes a \otimes a) \cdot v_0 \end{aligned}$$

We see that $\Theta^2 \cdot v_0$ does not belong to $\bigcap_{\Gamma \in \mathcal{G}^g} W_{[0]}(\Gamma)$ which contradicts the fact that $\Theta^2 \in (\mathcal{A}_{p,g})'$. □

3.4.3 Proof of Lemma 3.4.3

In this subsection we put $p = 2r_1r_2$, with r_1, r_2 two distinct odd primes. We suppose there exists $x \in \{1, \ldots, r_1r_2 - 2\}$ so that (x, x, x) is p-admissible and so that

$$\begin{cases} x \equiv 0 & (\text{mod } 2r_1) \\ x \equiv -2 & (\text{mod } r_2) \end{cases}$$

We also choose A_1 and A_2 some primitive $r_1 - th$ and $r_2 - th$ roots of unity, so that $A^2 = A_1A_2$. In particular we have $A^{2x} = A_2^{-2}$.

The goal of this subsection is to show that:

$$D := \left\{ \begin{array}{ccc} x & x & 2 \\ x & x & 0 \end{array} \right\} \left\{ \begin{array}{ccc} x & x & 4 \\ x & x & x \end{array} \right\} - \left\{ \begin{array}{ccc} x & x & 4 \\ x & x & 0 \end{array} \right\} \left\{ \begin{array}{ccc} x & x & 2 \\ x & x & x \end{array} \right\} \neq 0$$

A straightforward computation, using the formula of the recoupling coefficients ([70]), gives:

$$\begin{aligned} D &= (-1)^{\frac{x}{2}+1} \frac{[3][5]![x]\left[\frac{3}{2}x+1\right]!([\frac{x}{2}]!)^3}{[2][x+3]!([x+2]!)^2[x+3]\,[\frac{x}{2}+1]} \left(\left[\frac{x}{2}-1\right]^2 \left[\frac{x}{2}\right]^2 \left[\frac{x}{2}+1\right] \right. \\ &\quad -[2]^2 \left[\frac{x}{2}\right]^3 \left[\frac{3}{2}+2\right] \left[\frac{x}{2}+1\right] + \left[\frac{x}{2}-1\right] \left[\frac{x}{2}\right] \left[\frac{x}{2}+1\right] \left[\frac{x}{2}+2\right] \left[\frac{x}{2}+3\right] \\ &\quad \left. +[x-1][x+3] \left[\frac{x}{2}\right]^2 \left[\frac{x}{2}+1\right] - [x-1] \left[\frac{3}{2}x+2\right] [x+3] \right) \\ &= (-1)^{\frac{x}{2}+1} \frac{[3][5]![x]\left[\frac{3}{2}x+1\right]!([\frac{x}{2}]!)^3}{[2][x+3]!([x+2]!)^2[x+3]\,[\frac{x}{2}+1]\,(A^2-A^{-2})^7 A_1^{10}A_2^9} \cdot P(A_1, A_2) \end{aligned}$$

where we put:

$$\begin{aligned}P(x,y) := {} & x^{20}y^{16} - x^{17}y^{17} - x^{16}y^{18} + x^{19}y^{13} - 4x^{18}y^{14} + 3x^{17}y^{15} + 2x^{15}y^{17} - x^{19}y^{11} \\ & - 5x^{17}y^{13} - 4x^{15}y^{15} + 4x^{14}y^{16} - 2x^{13}y^{17} - x^{18}y^{10} + 2x^{17}y^{11} + 6x^{16}y^{12} + 2x^{15}y^{13} \\ & - x^{14}y^{14} + 2x^{13}y^{15} + x^{11}y^{17} + 2x^{15}y^{11} + x^{14}y^{12} + x^{13}y^{13} - 6x^{12}y^{14} + x^{10}y^{16} \\ & + x^{17}y^{7} + 4x^{16}y^{8} - x^{15}y^{9} - 4x^{14}y^{10} + 4x^{12}y^{12} + x^{15}y^{7} + x^{13}y^{9} - 4x^{12}y^{10} + x^{11}y^{11} \\ & + 4x^{10}y^{12} - 4x^{8}y^{14} - x^{7}y^{15} - 2x^{15}y^{5} - 6x^{14}y^{6} - 4x^{13}y^{7} + 2x^{12}y^{8} - 8x^{11}y^{9} - 6x^{10}y^{10} \\ & - 6x^{9}y^{11} - x^{7}y^{13} + x^{13}y^{5} + 6x^{11}y^{7} + 6x^{10}y^{8} + 8x^{9}y^{9} - 2x^{8}y^{10} + 4x^{7}y^{11} + 6x^{6}y^{12} + 2x^{5}y^{13} \\ & + x^{13}y^{3} + 4x^{12}y^{4} - 4x^{10}y^{6} - x^{9}y^{7} + 4x^{8}y^{8} - x^{7}y^{9} - x^{5}y^{11} - 4x^{8}y^{6} + 4x^{6}y^{8} + x^{5}y^{9} - 4x^{4}y^{10} - x^{3}y^{11} \\ & - x^{10}y^{2} + 6x^{8}y^{4} - x^{7}y^{5} - x^{6}y^{6} - 2x^{5}y^{7} - x^{9}y - 2x^{7}y^{3} + x^{6}y^{4} - 2x^{5}y^{5} - 6x^{4}y^{6} - 2x^{3}y^{7} + x^{2}y^{8} + 2x^{7}y \\ & - 4x^{6}y^{2} + 4x^{5}y^{3} + 5x^{3}y^{5} + xy^{7} - 2x^{5}y - 3x^{3}y^{3} + 4x^{2}y^{4} - xy^{5} + x^{4} + x^{3}y - y^{2}\end{aligned}$$

Note that $P(x,y)$ does not depend on r_1, r_2 or x. The proof reduces to show that $P(A_1, A_2) \neq 0$ for A_1, A_2 any primitive $r_1 - th$ and $r_2 - th$ roots of unity.

Consider the following algebraic curves in $\mathbb{C}^2$:

$$\begin{aligned}\mathcal{C} &:= \Big\{(z_1, z_2) \in \mathbb{C}^2 \text{ so that } P(z_1, z_2) = 0\Big\} \\ \mathcal{T} &:= \Big\{(z_1, z_2) \in \mathbb{C}^2 \text{ so that } |z_1|^2 = |z_2|^2 = 1\Big\}\end{aligned}$$

Note that $\mathcal{T}$ is a torus, has degree 3 and that these two curves share no irreducible components in common. The Bézout theorem (see [42] Chap I Corollary 7.8) implies that:

$$\#(\mathcal{C} \cap \mathcal{T}) \leq \deg(\mathcal{C}) \cdot \deg(\mathcal{T}) = 108$$

Now suppose there exist A_1 and A_2 some primitive r_1 and r_2 roots of unity so that $P(A_1, A_2) = 0$. Since $P(x,y) \in \mathbb{Z}[x,y]$, the equality $P(A_1, A_2) = 0$ must hold for every r_1 and r_2 roots of unity. Thus we have:

$$r_1 \cdot r_2 \leq \#(\mathcal{C} \cap \mathcal{T}) \leq \deg(\mathcal{C}) \cdot \deg(\mathcal{T}) = 108$$

So we just have the following possible cases:

$$\{r_1, r_2\} \in \{\{3,5\}, \{5,7\}, \{3,11\}, \{5,11\}, \{7,11\}, \{3,13\}, \{5,13\}, \{7,13\}\}$$

First if $\{r_1, r_2\} = \{3,5\}, \{5,7\}, \{3,11\}$ or $\{5,13\}$, then $x = 10, 28, 22$ and 50 respectively and we see that (x,x,x) is not $2r_1r_2$-admissible.

We handle the four remaining cases by checking that $P(e^{\frac{2i\pi}{5}}, e^{\frac{2i\pi}{11}}) \neq 0$, $P(e^{\frac{2i\pi}{7}}, e^{\frac{2i\pi}{11}}) \neq 0$, $P(e^{\frac{2i\pi}{3}}, e^{\frac{2i\pi}{13}}) \neq 0$ and $P(e^{\frac{2i\pi}{7}}, e^{\frac{2i\pi}{13}}) \neq 0$.

This concludes the proof of Lemma 3.4.3.

3.4.4 Maple code to test generecity

Here is the Maple code used to test genericity of small levels. Here the level is $p = 2r$. The fraction $q(n) = [n] \in \mathbb{Z}(x)$ represents the quantum number n seen as a rational fraction in $x = A^2$, the fraction $qfac(n) = [n]! \in \mathbb{Z}(x)$ represents the quantum factorial and $sixj(a,b,c,d,e,f) = \left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\} \in \mathbb{Z}(x)$ is (a renormalized version of) the $6j$ symbol. The loop tests for each 6-uplet of colors whether the associated coloration of the Tetrahedron graph is $2r$-admissible or not. If it is, it checks if the $2r$-th cyclotomic polynomial divides the corresponding $6j$-symbol. If so, the coloration is added to some list L. At the end of the loop, the list L contains every null $6j$-symbols at level $2r$ modulo symmetry of the Tetrahedron graph.

```
restart;
r := 77;
```

$$77 \tag{1}$$

```
q := n -> (x^n - x^(-n))/(x^1 - x^(-1));
```

$$n \to \frac{x^n - x^{-n}}{x - \frac{1}{x}} \tag{2}$$

```
qfac := n -> product(q(i), i=1..n); qfac(0) := 1;
```

$$n \to \prod_{i=1}^{n} q(i)$$
$$1 \tag{3}$$

```
Truc := (A1, A2, A3, A4, B1, B2, B3) -> sum((-1)^z*qfac(z+1)/(qfac(B1-z)*qfac(B2-z)*qfac(B3-z)*qfac(z-A1)*qfac(z-A2)*qfac(z-A3)*qfac(z-A4)), z = max(A1, A2, A3, A4)..min(B1, B2, B3));
```

$$(A1, A2, A3, A4, B1, B2, B3) \to \sum_{z=\max(A1,A2,A3,A4)}^{\min(B1,B2,B3)} \frac{(-1)^z\, qfac(z+1)}{qfac(B1-z)\, qfac(B2-z)\, qfac(B3-z)\, qfac(z-A1)\, qfac(z-A2)\, qfac(z-A3)\, qfac(z-A4)} \tag{4}$$

```
sixj := (A, B, C, D, E, F) -> Truc((A+B+E)/2, (B+D+F)/2, (C+D+E)/2, (A+C+F)/2, (B+C+E+F)/2, (A+B+C+D)/2, (A+D+E+F)/2);
```

$$(A, B, C, D, E, F) \to Truc\left(\frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}E, \frac{1}{2}B + \frac{1}{2}D + \frac{1}{2}F, \frac{1}{2}C + \frac{1}{2}D + \frac{1}{2}E, \frac{1}{2}A + \frac{1}{2}C + \frac{1}{2}F, \frac{1}{2}B + \frac{1}{2}C + \frac{1}{2}E + \frac{1}{2}F, \frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C + \frac{1}{2}D, \frac{1}{2}A + \frac{1}{2}D + \frac{1}{2}E + \frac{1}{2}F\right) \tag{5}$$

```
with(numtheory) :
cycl := cyclotomic(2*r, x);
```

$$1 + x - x^7 + x^{42} - x^{11} + x^{37} - x^{24} + x^{32} - x^{39} + x^{41} + x^{45} - x^{35} - x^{25} + x^{23} - x^{21} + x^{19} + x^{18} + x^{28} - x^{12} - x^8 - x^{26} - x^{52} - x^{49} - x^{48} - x^{30} + x^{60} + x^{46} + x^{59} - x^{53} + x^{15} + x^{14} \tag{6}$$

```
L := { } :
for a from 0 to r-2 do
for b from a to r-2 do
for c from 0 to r-2 do
for d from a to r-2 do
for e from b to r-2 do
for f from 0 to r-2 do

if |a-b|<= e <= a+b and a+b+e <= 2*r-4 and type(a+b+e, even)
and |a-c|<= f <= a+c and a+c+f <= 2*r-4 and type(a+c+f, even)
 and |b-d|<= f <= b+d and f+d+b <= 2*r-4 and type(f+d+b, even)
and |e-c|<= d <= e+c and e+c+d <= 2*r-4 and type(e+c+d, even)
then
if divide(numer(sixj(a, b, c, d, e, f)), cycl) then
L := L union {[a, b, e, d, c, f]};
 fi; fi;
od; od; od ;od; od ; od; L;
L;
```

Figure 3.8:

Chapter 4

Other results

Résumé

Ce chapitre contient des résultats subsidiaires concernant les représentations quantiques.

La première section contient le dernier chapitre de l'article "Decomposition of the Weil representations into irreducible factors", où l'on prouve d'une part que les representations de Weil aux niveaux impairs sont des représentations fidèles de $Sp_{2g}(\mathbb{Z}/p\mathbb{Z})$ et d'autre part que les représentations de Weil convergent vers la représentation homologique du groupe symplectique lorsque le niveau tend vers l'infini.

Dans la seconde section on donne un critère pour déterminer quelles représentations de Reshetikhin-Turaev sont d'image finie lorsque le niveau est de la forme $p = r$ ou $p = 2r$ avec r premier impair. La réponse à cette question pour les représentations associées aux surfaces fermées était donnée dans [29] lorsque $(p, g) \neq (20, 2)$ et [23] pour le cas $p = 20$ et $g = 2$. En plus de donner une démonstration alternative très simple de ces résultats, notre critère permet d'étudier la finitude des représentations associées au tore percé qui n'était connue que asymptotiquement dans [77].

Dans la troisième section, on exhibe des relevés explicites des représentations de Weil comme représentations de $Sp_{2g}(\mathbb{Z})$ lorsque $g = 1$ et lorsque $g \geq 2$ et que le niveau est impair. Lorsque $g \geq 2$ et p est pair, on relève les représentations de Weil en représentations linéaires d'une extension centrale de $Sp_{2g}(\mathbb{Z})$ par $\mathbb{Z}/2\mathbb{Z}$.

Abstract

This chapter contains some subsidiary results concerning quantum representations.

The first section contains the last chapter of the article "Decomposition of the Weil representations into irreducible factors", where we prove one the one hand that the Weil representations at odd levels faithfully represents $Sp_{2g}(\mathbb{Z}/p\mathbb{Z})$, and on the other that the Weil representations converge, in the Fell topology, towards the homological representation of the symplectic group when the level goes to infinity.

In the second section, we give a criterion to detect which Reshetikhin-Turaev representations have finite image, when the level has the form $p = r$ or $p = 2r$ with r an odd prime. The answer was known for the representations associated to a surface without marked points (see [29] when $(g, p) \neq (2, 10)$ and [23] when $(g, p) = (2, 10)$). Our criterion gives an alternative proof of these results and permit us to study the infiniteness of quantum representations associated to a one-holed torus, which was known only asymptotically ([77]).

In the third section, we give explicit lifts of the Weil representations as linear representations of $Sp_{2g}(\mathbb{Z})$ when $g = 1$ and when $g \geq 2$ and p is odd. When $g \geq 2$ and p is even, we lift the Weil representations to linear representations of a central extension of $Sp_{2g}(\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$.

4.1 Faithfulness and asymptotic of the Weil representations

The natural action of $Sp_{2g}(\mathbb{Z})$ on $H_1(\Sigma_g, \mathbb{Z})$ induces a faithful representation $\rho : Sp_{2g}(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[H_1(\Sigma_g, \mathbb{Z})])$. For each $p \geq 2$, we choose a particular primitive root of unity $A \in \mathbb{C}$ turning $\pi_{p,g}$ into a complex representation on $U_p^{\otimes g} \otimes_{k'_p} \mathbb{C}$.

In this section we show that the linear representations $\pi_{p,g} \otimes (\pi_{p,g})^* : Sp_{2g}(\mathbb{Z}) \to \mathrm{GL}(U_p^{\otimes g} \otimes (U_p^{\otimes g})^*)$ converge to ρ in the Fell topology, where $(\pi_{p,g})^*$ is the dual representation of $\pi_{p,g}$ for the Hopf pairing. We deduce the asymptotic faithfulness of the Weil representations already showed in [4] in a different way. The proof is an adaptation of the proof of [67] to the abelian case.

We independently show that the Weil representations of $Sp_{2g}(\mathbb{Z}/p\mathbb{Z})$ at odd level are faithful. The proof was suggested to us by L.Funar and is based on a classification made by Klingenberg ([53]) of the subgroups of $Sp_{2g}(\mathbb{Z}/r^n\mathbb{Z})$ for r an odd primes. It gives an alternative simpler proof for the asymptotic faithfulness in the odd case.

Theorem 4.1.1. *Let $(x_p)_p$ be a sequence of integers converging to $+\infty$ when $p \to \infty$. Then the representations $(\pi_{x_p,g} \otimes (\pi_{x_p,g})^*)_p$ converge to ρ in the Fell topology when p goes to $+\infty$. As a consequence:*

$$\bigcap_{p \geq 2} Ker(\pi_{x_p,g}) = \begin{cases} \{\mathbb{1}, -\alpha.\mathbb{1}\} & \text{, when } g \geq 2 \text{ and all the } x_p \text{ are even.} \\ \{\mathbb{1}\} & \text{, if all } x_p \text{are odd or if } g = 1. \end{cases}$$

where α represents the central element added in the central extension $\widetilde{Sp_{2g}(\mathbb{Z})}$.

Lemma 4.1.2. *There exist on $\mathbb{C}[H_1(\Sigma_g, \mathbb{Z})]$ and on $U_p^{\otimes g} \otimes (U_p^{\otimes g})^*$ some non degenerate hermitian forms, denoted $\langle , \rangle$ and $\langle , \rangle_p$ respectively, together with a $Sp_{2g}(\mathbb{Z})$-equivariant linear map $\phi_p : \mathbb{C}[H_1(\Sigma_g, \mathbb{Z})] \to U_p^{\otimes g} \otimes (U_p^{\otimes g})^*$ so that:*

$$\langle \phi_{x_p}(x), \phi_{x_p}(y) \rangle_{x_p} \xrightarrow[p \to +\infty]{} \langle x, y \rangle \text{ , for all } \mathrm{x, y} \in \mathbb{C}[\mathrm{H_1}(\Sigma_\mathrm{g}, \mathbb{Z})]$$

Proof of Theorem 4.1.1. It follows from the definition of the Fell topology (see [9], appendix C) that Lemma 4.1.2 implies the convergence of $\pi_{x_p} \otimes {\pi_{x_p}}^*$ towards ρ. To show the

asymptotic faithfulness, choose any two elements $\phi_1, \phi_2 \in Sp_{2g}(\mathbb{Z})$ with $\phi_1 \neq \phi_2$. There exists a couple $(x, y) \in \mathbb{C}[H_1(\Sigma_g, \mathbb{Z})]^2$ so that

$$\langle x, \rho(\phi_1 - \phi_2)y\rangle \neq 0$$

By Lemma 4.1.2 there exists a rank $p_0 \geq 0$ so that $p \geq p_0$ implies

$$\left\langle \phi_{x_p}(x), (\pi_{x_p} \otimes \pi^*_{x_p})(\phi_1 - \phi_2)\phi_{x_p}(y)\right\rangle_{x_p} \neq 0$$

so $\pi_{x_p}(\phi_1) \neq \pi_{x_p}(\phi_2)$ for $p \geq p_0$

□

Remark. Given a topological group G, the collection C of isomorphism classes of unitary representations of G on a Hilbert space, is not a set so we cannot define the Fell topology on C. It is customary to pass through this problem by defining the Fell topology on a suitable set in C such as the set of irreducible representations or the set of cyclic representations. Here, by Fell topology, we mean that we have chosen an arbitrary set in C containing ρ and the π_p. We refer the reader to the appendix F of [9] for a discussion on this subtlety.

We now define the hermitian forms $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_p$ and the morphism ϕ_p of Lemma 4.1.2:

The group law on $H_1(\Sigma_g, \mathbb{Z})$ induces a structure of commutative algebra on $\mathbb{C}[H_1(\Sigma_g, \mathbb{Z})] \cong \mathbb{C}[x_1^{\pm 1}, \ldots, x_g^{\pm 1}, y_1^{\pm 1}, \ldots, y_g^{\pm 1}]$ which is spanned by the meridians x_i and longitudes y_i, together with their inverse. An involutive anti-linear map $* : \mathbb{C}[H_1(\Sigma_g, \mathbb{Z})] \to \mathbb{C}[H_1(\Sigma_g, \mathbb{Z})]$ is defined by $1^* := 1$, $(x_i^n)^* := x_i^{-n}$ and $(y_i^n)^* := y_i^{-n}$. We define a morphism of algebras $\mathrm{Tr} : \mathbb{C}[H_1(\Sigma_g, \mathbb{Z})] \to \mathbb{C}$ with $\mathrm{Tr}(1) := 1$, $\mathrm{Tr}(x_i) = \mathrm{Tr}(y_i) = 0$. The non degenerate hermitian form $\langle\cdot,\cdot\rangle$ of Lemma 4.1.2 is defined by:

$$\langle x, y\rangle := \mathrm{Tr}(xy^*)\ ,\ \text{for all x, y} \in \mathbb{C}[\mathrm{H_1}(\Sigma_\mathrm{g}, \mathbb{Z})]$$

The vector space $U_p^{\otimes g} \otimes (U_p^{\otimes g})^*$ is naturally identified with $\mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/p\mathbb{Z})] \cong \mathcal{T}_p(\Sigma_g \times [0, 1])$ via the linear isomorphism $\Psi : U_p^{\otimes g} \otimes (U_p^{\otimes g})^* \to \mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/p\mathbb{Z})]$ sending $x_i \otimes 1$ to $[x_i]$ and $1 \otimes (x_i)^*$ to $[y_i]$. The Egorov identity implies that Ψ is $Sp_{2g}(\mathbb{Z})$-equivariant.

The composition $\mathbb{C}[H_1(\Sigma_g, \mathbb{Z})] \overset{\mathrm{pr}_p}{\to} \mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/p\mathbb{Z})] \overset{\Psi}{\to} U_p^{\otimes g} \otimes (U_p^{\otimes g})^*$ gives the $Sp_{2g}(\mathbb{Z})$-equivariant morphism ϕ_p of Lemma 4.1.2. We denoted by pr_p the quotient map.

Finally $\mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/p\mathbb{Z})]$ has an unique antilinear involution such that pr_p preserves the involutions. We define a trace $\mathrm{Tr}_p : \mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/p\mathbb{Z})] \to \mathbb{C}$ with:

$$\mathrm{Tr}_p(x) := \frac{1}{p^g}\,\mathrm{Tr}(\mathrm{Add}_p(x, 0)),\ \text{for all x} \in \mathbb{C}[\mathrm{H_1}(\Sigma_\mathrm{g}, \mathbb{Z}/\mathrm{p}\mathbb{Z})]$$

where Add_p is the Schrödinger representation and $(x, 0) \in \mathcal{H}_{p,g}$. The non degenerate hermitian form on $\mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/p\mathbb{Z})]$ is defined by:

$$\langle x, y\rangle_p := \mathrm{Tr}_p(xy^*)\ ,\ \text{for all x, y} \in \mathbb{C}[\mathrm{H_1}(\Sigma_\mathrm{g}, \mathbb{Z}/\mathrm{p}\mathbb{Z})]$$

gives, via Ψ, the hermitian form $\langle\cdot,\cdot\rangle_p$ on $U_p^{\otimes g} \otimes (U_p^{\otimes g})^*$ of Lemma 4.1.2.

Proof of Lemma 4.1.2. Since ϕ_p preserves the involution, we just have to show that:

$$\mathrm{Tr}_{x_p}(\phi_{x_p}(x)) \xrightarrow[p\to\infty]{} \mathrm{Tr}(x)\ ,\ \text{for all x} \in \mathbb{C}[\mathrm{H_1(\Sigma_g,\mathbb{Z})}]$$

First:

$$\mathrm{Tr}_{x_p}(\phi_{x_p}(1)) = \frac{1}{{x_p}^g}\,\mathrm{Tr}(\mathbb{1}_{{x_p}^g}) = 1 = \mathrm{Tr}(1)$$

Then choosing a couple $(n,m) \neq (0,0)$ and $i \in \{1,\dots,g\}$ we compute:

$$\mathrm{Add}_{x_p}(x_i^n y_i^m) = \mathbb{1}^{\otimes(i-1)} \otimes \left(A^{2nu^2}\delta_{u,v+m}\right)_{u,v} \otimes \mathbb{1}^{\otimes(g-i)}$$

So:

$$\mathrm{Tr}_{x_p}(\phi_{x_p}(x_i^n y_i^m)) = \begin{cases} \frac{G(2,0,x_p)}{x_p} & \text{, if } x_p \text{ is odd and divides } m; \\ \frac{G(1,0,x_p)}{x_p} & \text{, if } x_p \text{ is even and divides } m; \\ 0 & \text{, elsewhere.} \end{cases}$$

so $\lim\limits_{p\to\infty} \mathrm{Tr}_{x_p}(\phi_{x_p}(x_i^n y_i^m)) = 0 = \mathrm{Tr}(x_i^n y_i^m)$.

Now using the fact that $\mathrm{Tr}_{x_p}(x_1^{n_1}y_1^{m_1}\dots x_g^{n_g}y_g^{m_g}) = \mathrm{Tr}_{x_p}(x_1^{n_1}y_1^{m_1})\dots\mathrm{Tr}_{x_p}(x_g^{n_g}y_g^{m_g})$ we complete the proof. □

Remark. The involutive algebra $\mathbb{C}[H_1(\Sigma_g,\mathbb{Z})]$ together with the hermitian form $\langle\cdot,\cdot\rangle$ have a natural geometrical (or classical) interpretation. Let $\mathbf{M}(\Sigma_g) := H_1(\Sigma_g,\mathbb{R}) \Big/ H_1(\Sigma_g,\mathbb{Z}) \cong \mathbb{T}^{2g}$ be the $2g$-dimensional torus and $\mathbf{H}(\Sigma_g)$ be the algebra of regular functions on $\mathbf{M}(\Sigma_g)$. The action of the symplectic group on $\mathbf{M}(\Sigma_g)$ induces an action on $\mathbf{H}(\Sigma_g)$ via $\phi \bullet f(x) = f(\phi^{-1}\bullet x)$. We define a $Sp_{2g}(\mathbb{Z})$-equivariant isomorphism of involutive algebras

$$f : \mathbb{C}[H_1(\Sigma_g,\mathbb{Z})] \xrightarrow{\cong} \mathbf{H}(\Sigma_g)$$

by sending x_i to the projection on the i^{th} coordinate of $\mathbf{M}(\Sigma_g)$ and y_i to the projection on the $(i+g)^{th}$ one.

We verify that:

$$\mathrm{Tr}(x) = \int_{\mathbf{M}(\Sigma_g)} f_x dV$$

where $dV = dx^1 \wedge \dots \wedge dx^g \wedge dy^1 \wedge \dots \wedge dy^g$ is the natural volume form such that $\mathbf{M}(\Sigma_g)$ has volume 1.

As a consequence $\langle x,y\rangle = \int_{\mathbf{M}(\Sigma_g)} f_x \bar{f}_y dV$ has now a geometrical interpretation. The algebra $\mathbf{H}(\Sigma_g)$ is the algebra of classical observables on the symplectic torus and the morphism $\mathrm{Opp}_p := \phi_p \circ f^{-1} : \mathbf{H}(\Sigma_g) \to \mathrm{End}(U_p^{\otimes g})$ is a quantization operator. In this setting the asymptotic:

$$\Big\langle \mathrm{Opp}_p(f), \mathrm{Opp}_p(g) \Big\rangle_p \xrightarrow[p\to\infty]{} \int_{\mathbf{M}(\Sigma_g)} f\bar{g}dV$$

is classic in semi-classical analysis.

An alternative proof for the asymptotic faithfulness has been suggested to us by L. Funar; using a theorem of [53] we can show the stronger result:

Theorem 4.1.3. *The Weil representation $\pi_{p,g}$ at odd level $p \geq 3$, is a faithful representation of $Sp_{2g}(\mathbb{Z}/p\mathbb{Z})$.*

The asymptotic faithfulness for $(x_p)_p$ a sequence of odd integers converging to $+\infty$ follows for if $\phi \neq \mathbb{1}$ is an element in $Sp_{2g}(\mathbb{Z})$, there exists an $p \geq 1$ so that the class$[\phi] \neq [\mathbb{1}]$ in $Sp_{2g}(\mathbb{Z}/x_p\mathbb{Z})$ and Theorem 4.1.3 implies that $\phi \notin ker(\pi_{x_p,g})$. Thus we have that:

$$\bigcap_p Ker(\pi_{x_p,g}) = \{\mathbb{1}\}$$

The proof is based on the following fact proved (see [53]): Let r be an odd prime and $n, k \geq 2$ with $k < n$. Denote

$$\psi_k : Sp_{2g}(\mathbb{Z}/r^n\mathbb{Z}) \to Sp_{2g}(\mathbb{Z}/r^k\mathbb{Z})$$

the quotient map. We note $G_k := \psi_k^{-1}(\{-\mathbb{1}, +\mathbb{1}\})$ and $S_k := ker(\psi_k)$ the two proper normal subgroups of $Sp_{2g}(\mathbb{Z}/r^n\mathbb{Z})$. Then the only normal proper subgroups of $Sp_{2g}(\mathbb{Z}/r^n\mathbb{Z})$ are the G_k and S_k for $1 \leq k \leq n-1$.

Proof of Theorem 4.1.3. First using the first point of Theorem 2.1.1 and the group isomorphism between $Sp_{2g}(\mathbb{Z}/a\mathbb{Z}) \times Sp_{2g}(\mathbb{Z}/b\mathbb{Z})$ and $Sp_{2g}(\mathbb{Z}/ab\mathbb{Z})$ when a and b are coprime, we see that we only need to prove Theorem 4.1.3 for $p = r^n$ with p an odd prime.

For $1 \leq k \leq n-1$, the matrix $\pi_{r^n,g}(X_i)^{(r^{n-k})}$ is not the identity so the kernel of $\pi_{r^n,g}$ cannot be G_k or S_k. It results from Klingenberg's proposition, and the fact that $\pi_{r^n,g}$ is not trivial, that this kernel must be $\{\mathbb{1}\}$ when p is odd. □

4.2 Infiniteness of the image of Reshetikhin-Turaev representations

4.2.1 Finiteness of the image of Reshetikhin-Turaev representations

The Weil representations factor through congruence subgroups, thus their images are finite. Since in genus one the Reshetikhin-Turaev representations are subrepresentations of the Weil ones, they also have finite image. This fact was known in the conformal field theory community (see [20] and references herein) and has been proved independently by Gilmer in [35]. In higher genus, the Reshetikhin-Turaev representations have finite image at level 3 and 6 (see [93]) and 4 (it is the trivial representation). In any other cases, they have infinite image. It was proved by Funar in [29] when $(g, p) \neq (2, 10)$ (see also [68] for a proof that they contain an element of infinite order). The last remaining case $g = 2$, $p = 10$ was treated in [23]. Concerning the representation associated to a one holed torus, it was shown in [77] that they have infinite image for high enough level.

In this subsection, we develop a simple criterion to check whether a given Reshetikhin-Turaev representation has infinite image or not. Unfortunately, this criterion only works when $p = r$ or $p = 2r$ for r an odd prime. It permits us to recover the above results in these cases, and to study the finiteness of the representations associated to a one-holed torus, which could not be derived from previous papers.

4.2.2 Complete positivity

Let Σ be a colored surface and $p \geq 3$. The associated representations ρ_p act on a finite rank k_p-module $V_p(\Sigma)$ where $k_p = \mathbb{Z}[A, \frac{1}{p}, \kappa]$ quotiented by $\phi_{2p}(A) = 0$ and $\kappa^6 = A^{-6-\frac{p(p+1)}{2}}$. A choice of a $2p$ primitive root of unity A and a compatible complex κ gives a complex vector space $V_p^{A,\kappa}(\Sigma) = V_p(\Sigma) \otimes \mathbb{C}$ and complex representations $\rho_p^{A,\kappa}$ whose actions are preserved by non-degenerate hermitian forms $\langle \cdot, \cdot \rangle_p^{A,\kappa}$. Note that in the other chapters, we considered the invariant form as bilinear instead of hermitian.

Definition 4.2.1. We say that $V_p(\Sigma)$ is completely positive if for any choice of $A \in \mu(2p)$ and compatible $\kappa \in \mathbb{C}$, the hermitian form $(V_p^{A,\kappa}(\Sigma), \langle \cdot, \cdot \rangle_p^{A,\kappa})$ is positive definite or negative definite.

The only effect of a change of κ is eventually to change the eigenvalues of the hermitian form to its opposite so complete positivity only means that all eigenvalues have the same sign for a given κ and all A.

Proposition 4.2.2. *Let r be an odd prime number and $p = r$ or $p = 2r$ and let Σ be a colored surface. Then $(V_p(\Sigma), \langle \cdot, \cdot \rangle_p)$ is completely positive if and only if ρ_p has finite image.*

If $\rho_p(\widetilde{\mathrm{Mod}(\Sigma_g)})$ is finite, the following lemma, together with Roberts' results ([76], or Proposition 1.2.15), imply that $V_p(\Sigma)$ must be completely positive.

Lemma 4.2.3. *Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible group representation on a finite dimensional complex vector space V equipped with a non degenerate invariant hermitian form $\langle \cdot, \cdot \rangle_0$. If $\rho(G)$ is finite, then $\langle \cdot, \cdot \rangle_0$ is either definite positive or negative.*

Proof. If $\rho(G)$ is finite, the hermitian form on V defined by

$$\langle u, v \rangle_1 := \frac{1}{\#\rho(G)} \sum_{\rho(g) \in \rho(G)} \langle \rho(g)u, \rho(g)v \rangle$$

is definite positive and invariant under the action of G. Here $\langle , \rangle$ denotes an arbitrary scalar product on V.

If $t \in [0, 1]$, we define the invariant hermitian form

$$\langle u, v \rangle_t := t \langle u, v \rangle_0 + (1 - t) \langle u, v \rangle_1$$

Suppose there exists $v_0 \in V$ so that $\langle v_0, v_0 \rangle_0 < 0$. Since $\langle v_0, v_0 \rangle_1 > 0$, there exists $t_0 \in (0, 1)$ so that $\langle v_0, v_0 \rangle_{t_0} = 0$. The invariance of $\langle , \rangle_{t_0}$ under the action of G, implies that $\langle \rho(g)v_0, \rho(g)v_0 \rangle_{t_0} = 0$ for all $g \in G$. Since ρ is irreducible, the vector v_0 is cyclic so $\langle \cdot, \cdot \rangle_{t_0} = 0$ and $\langle \cdot, \cdot \rangle_0 = \frac{t_0 - 1}{t_0} \langle \cdot, \cdot \rangle_1$ is definite negative. □

Conversely, when $p = r$ or $p = 2r$ with r an odd prime, we note $\alpha_p := \begin{cases} p & , \text{if } p \equiv 3 \pmod 4 \\ 4r & , \text{if } p \equiv 1, 2 \pmod 4 \end{cases}$ and $\mathcal{O}_p := \mathbb{Z}[A] \Big/ \phi_{\alpha_p}(A)$.

It was showed in [36, 74] that $V_p(\Sigma)$ contains a free $\mathcal{O}_p$ lattice of maximal rank preserved by $\widetilde{\mathrm{Mod}(\Sigma_g)}$.

We denote by $\mu(\alpha_p) = \{q_1, \ldots, q_{\phi(\alpha_p)}\}$ the set of primitive α_p-th roots of unity and by $\phi(\alpha_p) = \#\mu(\alpha_p)$ the Euler characteristic.

Lemma 4.2.4. *Let $p = r$ or $p = 2r$ with r an odd prime. Then the injective linear map*

$$\Psi : \mathcal{O}_p \to \mathbb{C}^{\phi(\alpha_p)}$$

sending A^n to $(q_1^n, \ldots, q_{\alpha_p}^n)$, has a discrete image.

Proof. We endow $\mathbb{C}^{\phi(\alpha_p)}$ with the norm:

$$\left\|(z_1, \ldots, z_{\phi(\alpha_p)})\right\|^2 := \frac{1}{\phi(\alpha_p)} \sum_i |z_i|^2$$

where $|z|^2$ represents the Euclidian norm in $\mathbb{C}$.

If $P(X) = \sum_i n_i X^i \in \mathcal{O}_p$, then:

$$\begin{aligned} \|\Psi(P)\|^2 &= \frac{1}{\phi(\alpha_p)} \sum_{q \in \mu(\alpha_p)} \sum_{i,j} n_i n_j q^{i-j} \\ &= \sum_i n_i^2 + \sum_{i \neq j} n_i n_j \left(\frac{1}{\phi(\alpha_p)} \sum_{q \in \mu(\alpha_p)} q^{i-j} \right) \\ &= \begin{cases} \sum_i n_i^2 & \text{, if } p \text{ is prime.} \\ \sum_i n_i^2 - 2 \sum_{|i-j|=2r} n_i n_j & \text{, if } p = 2r. \end{cases} \end{aligned}$$

Thus we have $\|\Psi(\mathcal{O}_p)\|^2 \in \mathbb{N}$, so $\Psi(\mathcal{O}_p)$ is discrete in $\mathbb{C}^{\phi(\alpha_p)}$. □

Proof of Proposition 4.2.2. If $\rho_p(\widetilde{\mathrm{Mod}(\Sigma_g)})$ is finite, then the irreducibility of ρ_p (Proposition 1.2.15) together with Lemma 4.2.3, imply that every invariant forms $\langle \cdot, \cdot \rangle_p^{A,\kappa}$ must be either positive or negative, so $V_p(\Sigma)$ is completely positive.

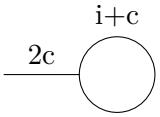
Conversely, if $V_p(\Sigma)$ is completely positive, using the $\mathcal{O}_p$ lattice of [36, 74], we have an injective group morphism from $\rho_p(G)$ to the group of matrices with coefficients in $\mathcal{O}_p$. Once composed with the map Ψ of Lemma 4.2.4, we get an injective group morphism $\widetilde{\Psi} : \rho(G) \to \mathrm{GL}_d(\mathbb{C}) \times \ldots \times \mathrm{GL}_d(\mathbb{C})$, where $d := dim(V_p(\Sigma))$. Lemma 4.2.4 implies that $\widetilde{\Psi}$ has a discrete image, and the complete positivity of $V_p(\Sigma)$ implies that the image lies in the compact product of $r-1$ unitary groups. This implies the finiteness of $\rho_p(\widetilde{\mathrm{Mod}(\Sigma_g)})$. □

4.2.3 (In)finiteness of the one holed torus representations

Theorem 4.2.5. *Let r be an odd prime and $p = r$ or $p = 2r$. Denote by ρ_p^c the Reshetikhin-Turaev representation associated to a torus $\mathcal{T}^c$ equipped with a band colored by $2c \in I_p$. We have:*

1. *If $2c = r - 3$, then ρ_{2r}^c has finite image.*
2. *If $c \equiv 1 \pmod 3$ and $r \neq 3, 5$, then ρ_{2r}^c has infinite image.*
3. *If $c = 1$ and $r \neq 3, 5$, then ρ_r^c has infinite image.*
4. *If ($c = 3[5]$ and $r = 2[5]$ or $r = 3[5]$) or ($c = 1[5]$ and $r = 3[5]$) or ($c = 2[5]$ and $r = 2[5]$), then ρ_{2r}^c has infinite image.*
5. *If $c \leq \frac{r-7}{3}$, then ρ_{2r}^c has infinite image.*

Proof. For $0 \leq i \leq r - 2 - 2c$, denote by $u_i^c \in V_p(\mathcal{T}^c)$ the vector associated to a lollipop graph whose stick is colored by $2c$ and loop colored by $i + c$, that is:

$u_i^c :=$ 

Using Theorem 1.1.44, we have:

$$\begin{aligned}
\frac{\langle u_{i+1}^c, u_{i+1}^c \rangle_p^R}{\langle u_i^c, u_i^c \rangle_p^R} &= \frac{[2c+i+2][i+1]}{[c+i+2][c+i+1]} \\
\frac{\langle u_{i+2}^c, u_{i+2}^c \rangle_p^R}{\langle u_i^c, u_i^c \rangle_p^R} &= \frac{[2c+i+3][2c+i+2][i+2][i+1]}{[c+i+1][c+i+3][c+i+2]^2}
\end{aligned}$$

Proposition 4.2.2 states that ρ_p^c has infinite image if and only if there exists $A \in \mu(2p)$ so that one of the above numbers is negative.

When $2c = r - 3$, choose $A = \exp\left(\frac{i\pi k}{r}\right)$ with $g.c.d.(k, 2r) = 1$. The vectors $\{u_0^c, u_1^c\}$ form a basis of $V_p(\mathcal{T}^c)$ and we have:

$$\frac{\langle u_1^c, u_1^c \rangle_p^R}{\langle u_0^c, u_0^c \rangle_p^R} = \frac{\sin(\frac{r-1}{r}\pi k)\sin(\frac{\pi k}{r})}{\sin(\frac{r+1}{2r}\pi k)\sin(\frac{r-1}{2r}\pi k)}$$

This product appears to be always positive for every k: indeed $\sin(\frac{r-1}{r}\pi k)$ and $\sin(\frac{\pi k}{r})$ are both positive if $k < r$ and both negative elsewhere whereas $\sin(\frac{r+1}{2r}\pi k)$ and $\sin(\frac{r-1}{2r}\pi k)$ are both positive if $\frac{k-1}{2}$ is even and both negative elsewhere. Thus ρ_p^c has finite image.

When $c \equiv 1 \pmod 3$, we take $k = \begin{cases} \frac{2r+1}{3} & \text{, if } r \equiv 1 \pmod 3 \\ \frac{2r-1}{3} & \text{, if } r \equiv 2 \pmod 3 \end{cases}$ and we have $\frac{\langle u_2^c, u_2^c \rangle_p^R}{\langle u_0^c, u_0^c \rangle_p^R} < 0$.

If $r \equiv 2 \pmod 5$, we put $k = \frac{2r+1}{5}$ and we have $\frac{\langle u_2^c, u_2^c \rangle_p^R}{\langle u_0^c, u_0^c \rangle_p^R} < 0$.

If $r \equiv 3 \pmod 5$, we put $k = \frac{2r-1}{5}$ and we have $\frac{\langle u_2^c, u_2^c\rangle_p^R}{\langle u_0^c, u_0^c\rangle_p^R} < 0$ when $c \equiv 1, 3 \pmod 5$.

When $c = 1$, then $\frac{\langle u_3^c, u_3^c\rangle_p^R}{\langle u_0^c, u_0^c\rangle_p^R} = \frac{[6]}{[2]} = \frac{\sin\left(\frac{2\pi k}{r}\right)}{\sin\left(\frac{4\pi k}{r}\right)}$ is negative if $\frac{r}{12} < k < \frac{r}{6}$. Such a k exists when $r \geq 12$.

For $r = 1$ or 7, we take $k = 1$ and for $r = 13, 17$, we take $k = 2$, and find again $\frac{\langle u_3^c, u_3^c\rangle_p^R}{\langle u_0^c, u_0^c\rangle_p^R} < 0$.

Finally, note that if there exist $0 \leq i \leq r-3$ and an odd $k < r$ so that $\frac{r}{2c+i+2} < k < \frac{r}{c+i+2}$ then $\frac{\langle u_{i+1}^c, u_{i+1}^c\rangle_p^R}{\langle u_i^c, u_i^c\rangle_p^R} < 0$ for $A = \exp\left(\frac{i\pi k}{r}\right)$. Then we can find such a i for $k = 3$ whenever $3 \in \bigcup_{i=0}^{r-3-2c} [\frac{r}{2c+i+2}, \frac{r}{c+i+2}] = [\frac{r}{r-1}, \frac{r}{c+2}]$, i.e. when $c \leq \frac{r-7}{3}$. □

Remark. We could have recovered the finiteness in this case by using a theorem of Formanek (see [26]) which states that the only 2-dimensional irreducible representations of B_3 are conjugate to one of the $\chi(y) \otimes \beta_3(q)$ where $\chi(y)$ is the character which sends both generators to $y \in \mathbb{C}^*$ and $\beta_3(q)$ is the reduced Burau representation in q. Since the eigenvalues of $\rho_p(t_1)$ are μ_c and μ_{c+1} we must have $y = -\mu_c$ and $q = A^r$ or $y = -\mu_{c+1}$ and $q = A^{-r}$ so q is a $4-th$ primitive root of unitity and $\beta_3(q)$ appears to have finite image (see [27] for a detailed discussion of finiteness of reduced Burau representations at roots of unity).

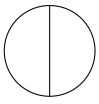
4.2.4 (In)finiteness of Reshetikhin-Turaev representations associated to closed surfaces

The following theorem results from [35, 29, 93, 23]. We derive another proof from Proposition 4.2.2.

Theorem 4.2.6. *Let $g \geq 1$ and $p = r$ or $p = 2r$ with r an odd prime. Then $\rho_{p,g}$ has finite image if and only if $g = 1$ or $r = 3$.*

Proof. If $g = 1$, then $\frac{\langle u_{i+1}, u_{i+1}\rangle_p^R}{\langle u_i, u_i\rangle_p^R} = 1$ for all i, so $\rho_{p,1}$ is completely positive.

If $p = 3$, since $I_3 = \{0\}$ then $V_3(\Sigma_g)$ is one dimensional so is completely positive. If $p = 6$, we have $I_6 = \{0, 1\}$ and $(1, 1, 1)$ is not 6-admissible so, if Γ is a trivalent graph and $\sigma \in col_p(\Gamma)$, then u_σ is a union of disjoint circles, say n ones, colored by 1. Thus $\langle u_\sigma, u_\sigma\rangle_6^R = 1$ and $V_6(\Sigma_g)$ is completely positive.

Now consider the case when $r = 5$ and $g = 2$. Take $\Gamma =$  and consider the associated vectors $u_{a,b,c}$ for (a, b, c) a p-admissible triple.

If $p = 5$, then $\frac{\langle u_{2,2,2}, u_{2,2,2}\rangle_5^R}{\langle u_{0,0,0}, u_{0,0,0}\rangle_5^R} = \frac{[4]}{[2]^2[3]^2}$ has the sign of $[4]$ which is negative for $A = \exp\left(\frac{3i\pi}{5}\right)$.

If $p = 10$, then $\frac{\langle u_{2,1,1}, u_{2,1,1}\rangle_{10}^R}{\langle u_{0,0,0}, u_{0,0,0}\rangle_{10}^R} = \frac{\langle 2,1,1\rangle}{\langle 1\rangle^2\langle 2\rangle}$ has the sign of $\langle 2\rangle = [3]$ which is negative for $A = \exp\left(\frac{3i\pi}{10}\right)$.

Now the fact that $V_p(\Sigma_2)$ is not completely positive for $r = 5$ implies that $V_p(\Sigma_g)$ is not completely positive when $g \geq 3$ as well, for we can embed the Theta graph into a genus $g \geq 3$ graph and define vectors $u_{a,b,c} \in V_p(\Sigma_g)$, with same norm as in genus 2, by coloring by 0 the complementary of the embedding.

Finally, if $r \neq 3, 5$ and $g \geq 2$, we isolate one handle of Σ_g and write $\Sigma_g \cong \Sigma_1 \bigcup_{S^1} \Sigma_{g-1}$. Then using Theorem 1.1.43, we have:

$$V_p(\Sigma_g) \cong \oplus_{c \in I_p} V_p(\mathcal{T}^c) \otimes V_p(\Sigma_{g-1}^c)$$

So the fact that $V_p(\mathcal{T}^1)$ is not completely positive (Theorem 4.2.5) implies that $V_p(\Sigma_g)$ is not as well for $g \geq 2$. We conclude using Proposition 4.2.2. □

4.3 Explicit lifts for the Weil representations

When $g = 1$, the Weil representations can be lifted to linear representations of $SL_2(\mathbb{Z})$. When $g \geq 2$ only the Weil representation at odd levels can be lifted to linear representations of $Sp_{2g}(\mathbb{Z})$. At even levels in genus $g \geq 2$, they can only be lifted to a representation of a central extension of $Sp_{2g}(\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$. This facts are well known and can be proved by elementary arguments. The goal of this subsection is to exhibit explicit lifts of the Weil representations. Note that when $g \geq 3$ and p is even, a different lift was given in [31] also. In [38] an explicit lift of the odd subrepresentations in genus one at even level of the Weil representations was also given.

4.3.1 The genus one case

Note $S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. It is well known that the group $SL_2(\mathbb{Z})$ is generated by S and T with relations $S^4 = I_2$ and $(ST)^3 = S^2$. In order to lift the Weil representations in genus one at even levels, we must add a primitive 24-th root of unity to the ring, thus we look at $k_p = \mathbb{Z}[A, \frac{1}{p}, \beta]/(\phi_{2p}(A), \phi_{24}(\beta))$.

Theorem 4.3.1. *An explicit lift of the genus one Weil representations is given by the matrices:*

$$\pi_p(S) := \begin{cases} \frac{G(-1,0,p)}{p}(A^{-2ij})_{i,j}, & \text{if } p \text{ is odd.} \\ \beta^{-3}\frac{G(-1,0,2p)}{2p}(A^{-2ij})_{i,j}, & \text{if } p \text{ is even.} \end{cases} \tag{4.1}$$

$$\pi_p(T) := \begin{cases} (A^{i^2}\delta_{i,j})_{i,j}, & \text{if } p \text{ is odd.} \\ \beta(A^{i^2}\delta_{i,j})_{i,j}, & \text{if } p \text{ is even.} \end{cases} \tag{4.2}$$

Proof. We check the two relations:

$$\pi_p(S)^2 = \begin{cases} \left(\frac{G(-1,0,p)}{p}\right)^2 p(\delta_{i+j,0})_{i,j} & \text{, when } p \text{ is odd.} \\ \alpha^{-6}\left(\frac{G(-1,0,2p)^2}{4p}\right)(\delta_{i+j,0})_{i,j} & \text{, when } p \text{ is even.} \end{cases}$$

$$\pi_p(S)^4 = \begin{cases} \left(\frac{G(-1,0,p)}{p}\right)^4 p^2.\mathbb{1} = \mathbb{1} & \text{, when } p \text{ is odd.} \\ \alpha^{-12}\left(\frac{G(-1,0,2p)}{2}\right)^4 \frac{1}{p^2}\mathbb{1} = \mathbb{1} & \text{, when } p \text{ is even.} \end{cases}$$

$$\pi_p(S)\pi_p(T) = \begin{cases} \frac{G(-1,0,p)}{p}(A^{j(j-2i)})_{i,j} & \text{, when } p \text{ is odd.} \\ \alpha^{-2}\left(\frac{G(-1,0,2p)}{2p}\right)(A^{-2ij+j^2})_{i,j} & \text{, when } p \text{ is even.} \end{cases}$$

$$\begin{aligned} (\pi_p(S)\pi_p(T))^2 &= \begin{cases} \left(\frac{G(-1,0,p)}{p}\right)^2 \left(\sum_{k\in\mathbb{Z}/p\mathbb{Z}} A^{k(k-2i)}A^{j(j-2k)}\right)_{i,j} & \text{, when } p \text{ is odd.} \\ \alpha^{-4}\left(\frac{G(-1,0,2p)}{2p}\right)^2 (A^{j^2}G(1,-2(i+j),2p))_{i,j} & \text{, when } p \text{ is even.} \end{cases} \\ &= \begin{cases} \left(\frac{G(-1,0,p)}{p}\right)(A^{-i(i+2j)})_{i,j} & \text{, when } p \text{ is odd.} \\ \alpha^{-4}\frac{G(-1,0,2p)}{2p}(A^{-2ij-i^2})_{i,j} & \text{, when } p \text{ is even.} \end{cases} \\ (\pi_p(S)\pi_p(T))^3 &= \begin{cases} \left(\frac{G(-1,0,p)^2}{p^2}\right)\left(\sum_k A^{-i(i+2k)}A^{j(j-2k)}\right)_{i,j} & \text{, when } p \text{ is odd.} \\ \alpha^{-6}\left(\frac{G(-1,0,2p)}{2p}\right)^2 (A^{j^2-i^2}p\delta_{i+j,0})_{i,j} & \text{, when } p \text{ is even.} \end{cases} \\ &= \begin{cases} \left(\frac{G(-1,0,p)^2}{p}\right)(\delta_{i+j,0})_{i,j} & \text{, when } p \text{ is odd.} \\ \alpha^{-6}\left(\frac{G(-1,0,2p)^2}{4p}\right)(\delta_{i+j,0})_{i,j} & \text{, when } p \text{ is even.} \end{cases} \\ &= \pi_p(S)^2 \end{aligned}$$

□

4.3.2 Presentation of the symplectic group and central extension

In order to give explicit lifts in higher genus, we define a finite presentation of $Sp_{2g}(\mathbb{Z})$ with generators the Dehn twists X_i, Y_i, Z_{ij} of Figure 3.6. This presentation will come from a presentation of the mapping class group of a closed oriented genus g surface, made by Gervais in [34] that we will pass to quotient through the Torelli group. Note that other presentations of $Sp_{2g}(\mathbb{Z})$ are known since the work of Birman [12] and Powell [72] (see also [65]).

Theorem 4.3.2. *The symplectic group $Sp_{2g}(\mathbb{Z})$ has presentation with generators $X_i, Y_j, Z_{i,j}$, $1 \leq i, j \leq g$ and relations:*

$$Z_{i,i} = 1 \tag{4.3}$$

Braids

$$[X_i, X_j] = [Y_i, Y_j] = [X_i, Y_j] = 0 \quad \textit{for } i \neq j. \tag{4.4}$$

$$X_iY_iX_i = Y_iX_iY_i. \tag{4.5}$$

$$[Z_{i,j}, X_k] = 0. \tag{4.6}$$

$$[Z_{i,j}, Y_k] = 0 \quad \textit{for } i \neq k, j \neq k. \tag{4.7}$$

$$Z_{i,j}Y_iZ_{i,j} = Y_iZ_{i,j}Y_i \quad \textit{for } i \neq j. \tag{4.8}$$

Stars

$$Z_{i,j}Z_{j,k}Z_{k,i} = (Z_{i,1}Z_{j,1}Z_{k,1}Y_1)^3 \quad \text{for } 2 \leq i,j,k \leq g. \tag{4.9}$$
$$Z_{i,j}X_iX_j = (Z_{i,1}Z_{j,1}X_1Y_1)^3 \quad \text{for } 2 \leq i \neq j \leq g. \tag{4.10}$$
$$X_iX_jZ_{i,j} = (Z_{i,1}X_1Z_{j,1}Y_1)^3 \quad \text{for } 2 \leq i \neq j \leq g. \tag{4.11}$$
$$X_iZ_{i,j}X_j = (X_1Z_{i,1}Z_{j,1}Y_1)^3 \quad \text{for } 2 \leq i \neq j \leq g. \tag{4.12}$$
$$X_i^2 = (Z_{i,1}X_1^2Y_1)^3 \quad \text{for } 2 \leq i \leq g. \tag{4.13}$$
$$1 = (X_1^3Y_1)^3. \tag{4.14}$$

Proof. In [34], Gervais made a presentation of the mapping class group with some generators α_i, β_i and $\gamma_{i,j}$ shown in Figure 4.1 and relations called 'Handles', 'Braids' and 'Stars'. To pass Gervais' presentation to the quotient through the Torelli group, we use the fact, proved by Birman [12] and Powell [72] (see also Putman [73] and Hatcher, Margalit [43] for more recent proofs) that the Torelli group is generated by the Dehn twists along separating curves and elements $T_{\gamma_1}T_{\gamma_2}^{-1}$ where γ_1, γ_2 are two homologous curves not isotopic.

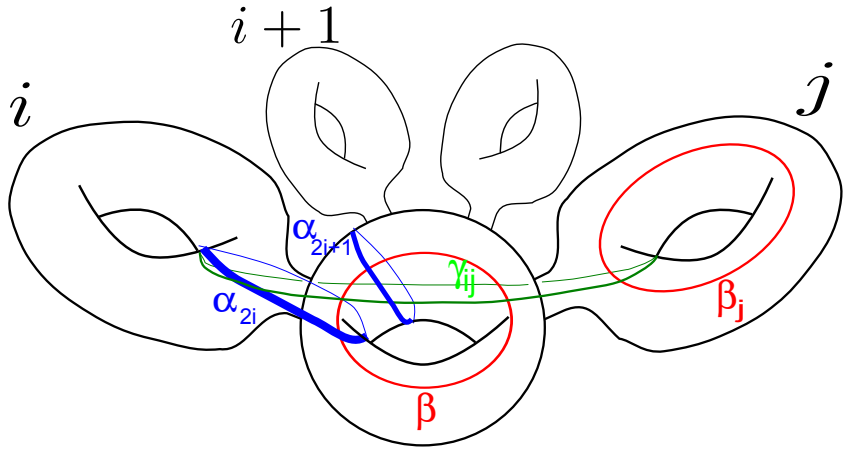


Figure 4.1: The curves defining the Dehn twists of Gervais' presentation of the mapping class group.

Let us now denote by $f : \mathrm{Mod}(\Sigma_g) \to Sp_{2g}(\mathbb{Z})$ the quotient map by the Torelli group. The image of Gervais' generators are:

- $f(\gamma_{2i+1,2j+1}) = 1$ for all i, j: these are the separating twists of the Torelli group.
- $f(\beta) = Y_1$, $f(\beta_i) = Y_{i+1}$, $2 \leq i \leq g$.
- $f(\gamma_{2i,2j}) = Z_{i,j}$, $2 \leq i, j \leq g$.
- $f(\alpha_{2i}) = Z_{1,i}$, $2 \leq i \leq g$.
- $f(\gamma_{2i,2j+1}) = X_i$, $2 \leq i \leq g$, $0 \leq j \leq g$ as the homology class of $\gamma_{2i,2j+1}$ doesn't depend on j.

and we just verify that passing Gervais' relations of type 'Handles', 'Braids' and 'Stars' through this quotient map gives the ones stated in the theorem. □

Using this presentation, we define a central extension of $Sp_{2g}(\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$ by modifying the stars relations $\gamma_{i,j}\gamma_{j,k}\gamma_{k,i} = (\alpha_i\alpha_j\alpha_k\beta)^3$ in Gervais' presentation by $\gamma_{i,j}\gamma_{j,k}\gamma_{k,i} = -(\alpha_i\alpha_j\alpha_k\beta)^3$:

Definition 4.3.3. Let $\widetilde{Sp_{2g}(\mathbb{Z})}$ be the group generated by the elements $X_i, Y_j, Z_{i,j}, 1 \leq i,j \leq g$ together with an element α and relations given by:

- The relations (4.3) and the Braids relations (4.4), (4.5), (4.6), (4.7), (4.8).
- α is central and $\alpha^2 = 1$.
- The modified stars relations:

$$\alpha Z_{i,j}Z_{j,k}Z_{k,i} = (Z_{i,1}Z_{j,1}Z_{k,1}Y_1)^3 \quad \text{for } 2 \leq i,j,k \leq g. \tag{4.15}$$
$$\alpha Z_{i,j}X_iX_j = (Z_{i,1}Z_{j,1}X_1Y_1)^3 \quad \text{for } 2 \leq i \neq j \leq g. \tag{4.16}$$
$$\alpha X_iX_jZ_{i,j} = (Z_{i,1}X_1Z_{j,1}Y_1)^3 \quad \text{for } 2 \leq i \neq j \leq g. \tag{4.17}$$
$$\alpha X_iZ_{i,j}X_j = (X_1Z_{i,1}Z_{j,1}Y_1)^3 \quad \text{for } 2 \leq i \neq j \leq g. \tag{4.18}$$
$$\alpha X_i^2 = (Z_{i,1}X_1^2Y_1)^3 \quad \text{for } 2 \leq i \leq g. \tag{4.19}$$
$$\alpha = (X_1^3Y_1)^3. \tag{4.20}$$

We have the following short exact sequence:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{i} \widetilde{Sp_{2g}(\mathbb{Z})} \xrightarrow{j} Sp_{2g}(\mathbb{Z}) \to 1$$

where i sends 1 to α and j sends $X_i, Y_j, Z_{i,j}$ to the generators with same name.

Note that in genus one the groups $Sp_2(\mathbb{Z})$ and $\widetilde{Sp_{2g}(\mathbb{Z})}$ are the same since there is no stars relation.

We will show that the Weil representations at even levels are linear representations of this central extension and that, when $g \geq 2$, they cannot be lifted as linear representations of $Sp_{2g}(\mathbb{Z})$. We will hence show that this extension is non trivial when $g \geq 2$.

4.3.3 Explicit lifts in higher genus

Theorem 4.3.4. *The following matrices define explicit lifts of the Weil representations in genus $g \geq 2$ to linear representation of $Sp_{2g}(\mathbb{Z})$ when p is odd, and of $\widetilde{Sp_{2g}(\mathbb{Z})}$ when p is even.*

- $\pi_{p,1}(X) = (A^{2i^2}\delta_{i,j})_{i,j}$ *and* $\pi_{p,g}(X_i) = \mathbb{1}^{\otimes(i-1)} \otimes \pi_{p,1}(X) \otimes \mathbb{1}^{\otimes(g-i)}$.
- $\pi_{p,g}(Z_{i,j})(e_{a_1} \otimes \ldots \otimes e_{a_g}) = A^{(a_i-a_j)^2}(e_{a_1} \otimes \ldots \otimes e_{a_g})$.
- $\pi_{p,1}(Y) = \begin{cases} \frac{G(1,0,p)}{p}(A^{-(i-j)^2})_{i,j}, & \textit{when } p \textit{ is odd.} \\ \frac{G(1,0,2p)}{2p}(A^{-(i-j)^2})_{i,j}, & \textit{when } p \textit{ is even.} \end{cases}$

  $\pi_{p,g}(Y_i) = \mathbb{1}^{\otimes(i-1)} \otimes \pi_p^1(Y) \otimes \mathbb{1}^{\otimes(g-i)}$.

Proof. We will show that the matrices $\pi_{p,g}(X_i), \pi_{p,g}(Y_i), \pi_{p,g}(Z_{i,j})$ satisfy the relations of the presentations of $Sp_{2g}(\mathbb{Z})$ of Theorem 4.3.2 in the odd case and of $\widetilde{Sp_{2g}(\mathbb{Z})}$ of Definition 4.3.3 in the even case. As the computations are quite similar, we will perform them in the even case only.

The relations (4.3), (4.4), (4.6) and (4.7) are clear from definition. For (4.5) we just have to prove that $\pi_p(X)\pi_p(Y)\pi_p(X) = \pi_p(Y)\pi_p(X)\pi_p(Y)$:

$$\pi_p(X)\pi_p(Y)\pi_p(X)e_i \quad = \quad \frac{G(1,0,2p)}{2p}\sum_k A^{k^2+i^2-(i-k)^2}e_k \quad = \quad \frac{G(1,0,2p)}{2p}\sum_k A^{2ik}e_k$$

$$\pi_p(Y)\pi_p(X)\pi_p(Y)e_i = \left(\frac{G(1,0,2p)}{2p}\right)^2 \sum_{k,l} A^{-k^2+2k(l+i)-i^2-l^2} e_l$$
$$= \left(\frac{G(1,0,2p)}{2p}\right)^2 \frac{G(-1,0,2p)}{2} \sum_l A^{2il} e_l = \pi_p(x)\pi_p(y)\pi_p(x)e_i$$

Now we must check (4.8) : $Z_{i,j}Y_iZ_{i,j} = Y_iZ_{i,j}Y_i$ for $i \neq j$.

$$\pi_{p,g}(Y_i)\pi_{p,g}(Z_{i,j})\pi_{p,g}(Y_i)(e_{a_1} \otimes \ldots \otimes e_{a_g})$$
$$= \left(\frac{G(1,0,2p)}{2p}\right)^2 \sum_{k,l} A^{-(a_i-k)^2+(k-a_j)^2-(k-l)^2} e_{a_1} \otimes \ldots \otimes e_l \otimes \ldots \otimes e_{a_g}$$
$$= \left(\frac{G(1,0,2p)}{2p}\right) A^{2a_j(a_j-a_i)} \sum_l A^{2(a_i-a_j)l} e_{a_1} \otimes \ldots \otimes e_l \otimes \ldots \otimes e_{a_g}$$

$$\pi_{p,g}(Z_{i,j})\pi_{p,g}(Y_i)\pi_{p,g}(Z_{i,j})(e_{a_1} \otimes \ldots \otimes e_{a_g})$$
$$= \left(\frac{G(1,0,2p)}{2p}\right) \sum_k A^{-(a_i-k)^2+(k-a_j)^2+(a_i-a_j)^2} e_{a_1} \otimes \ldots \otimes e_k \otimes \ldots \otimes e_{a_g}$$
$$= \left(\frac{G(1,0,2p)}{2p}\right) A^{2a_j^2-2a_ia_j} \sum_k A^{2(a_i-a_j)k} e_{a_1} \otimes \ldots \otimes e_k \otimes \ldots \otimes e_{a_g}$$
$$= \pi_{p,g}(Y_i)\pi_{p,g}(Z_{i,j})\pi_{p,g}(Y_i)(e_{a_1} \otimes \ldots \otimes e_{a_g})$$

Let us now check the (modified) stars relations. First (4.15) : $Z_{i,j}Z_{j,k}Z_{k,i} = -(Z_{i,1}Z_{j,1}Z_{k,1}Y_1)^3$ for $2 \leq i,j,k \leq g$:

$$\pi_{p,g}(Z_{i,j})\pi_{p,g}(Z_{j,k})\pi_{p,g}(Z_{k,i})(e_{a_1} \otimes \ldots \otimes e_{a_g})$$
$$= A^{(a_i-a_j)^2+(a_j-a_k)^2+(a_k-a_i)^2}(e_{a_1} \otimes \ldots \otimes e_{a_g})$$
$$= A^{2(a_i^2+a_j^2+a_k^2-a_ia_j-a_ja_k-a_ia_k)}(e_{a_1} \otimes \ldots \otimes e_{a_g})$$

$$(\pi_{p,g}(Z_{i,1})\pi_{p,g}(Z_{j,1})\pi_{p,g}(Z_{k,1})\pi_{p,g}(Y_1))^3(e_{a_1} \otimes \ldots \otimes e_{a_g}) = \left(\frac{G(1,0,2p)}{2p}\right)^3 \times$$
$$\sum_{u,v,w} A^{u^2+v^2+2w^2-a_1^2+3(i^2+j^2+k^2)+2(ua_1+uv+wv-u(i+j+k)-v(i+j+k)-w(i+j+k)} e_w \otimes e_{a_2} \otimes \ldots \otimes e_{a_g}$$
$$= \left(\frac{G(1,0,2p)}{2p}\right)^3 \frac{G(1,0,2p)}{2} A^{-a_1^2+3(i^2+j^2+k^2)} \times$$
$$\sum_w \left(\sum_v A^{2v(w-a_1)}\right) A^{-(a_1-i-j-k)^2-2w(i+j+k)+2w^2} e_w \otimes e_{a_2} \otimes \ldots \otimes e_{a_g}$$
$$= \left(\frac{G(1,0,2p)}{2}\right)^4 \frac{1}{p^2} A^{2(a_i^2+a_j^2+a_k^2-a_ia_j-a_ja_k-a_ia_k)}(e_{a_1} \otimes \ldots \otimes e_{a_g})$$
$$= -\pi_{p,g}(Z_{i,j})\pi_{p,g}(Z_{j,k})\pi_{p,g}(Z_{k,i})(e_{a_1} \otimes \ldots \otimes e_{a_g})$$

Now let us look at (4.16) : $Z_{i,j}X_iX_j = -(Z_{i,1}Z_{j,1}X_1Y_1)^3$ for $2 \leq i \neq j \leq g$, we leave the similar cases (4.17), (4.18) to the reader.

$$\pi_{p,g}(Z_{i,j})\pi_{p,g}(X_i)\pi_{p,g}(Y_j)(e_{a_1} \otimes \ldots \otimes e_{a_g}) = A^{2(a_i^2+a_j^2-a_ia_j}(e_{a_1} \otimes \ldots \otimes e_{a_g})$$

$$\pi_{p,g}(Z_{i,1})\pi_{p,g}(Z_{j,1})\pi_{p,g}(X_1)\pi_{p,g}(Y_1)(e_{a_1} \otimes \ldots \otimes e_{a_g})$$
$$= \left(\frac{G(1,0,2p)}{2p}\right) \sum_u A^{(a_i-u)^2+(a_j-u)^2+u^2-(u-a_1)^2} e_u \otimes e_{a_2} \otimes \ldots \otimes e_{a_g}$$

$$(\pi_{p,g}(Z_{i,1})\pi_{p,g}(Z_{j,1})\pi_{p,g}(X_1)\pi_{p,g}(Y_1))^2(e_{a_1} \otimes \ldots \otimes e_{a_g})$$
$$= \left(\frac{G(1,0,2p)}{2p}\right)^2 \sum_{u,v} A^{(a_i-u)^2+(a_j-u)^2+u^2-(u-a_1)^2-(u-v)^2+v^2+(a_j-v)^2+(a_i-v)^2} e_v \otimes e_{a_2} \otimes \ldots \otimes e_{a_g}$$
$$= \left(\frac{G(1,0,2p)}{2p}\right)^2 \frac{G(1,0,2p)}{2} \sum_v A^{v^2-2va_1+(a_i^2+a_j^2-2a_1^2+2a_1a_i+2a_1a_j-2a_ia_j} e_v \otimes e_{a_2} \otimes \ldots \otimes e_{a_g}$$

$$(\pi_{p,g}(Z_{i,1})\pi_{p,g}(Z_{j,1})\pi_{p,g}(X_1)\pi_{p,g}(Y_1))^3(e_{a_1} \otimes \ldots \otimes e_{a_g}) = \left(\frac{G(1,0,2p)}{2}\right)^4 \frac{1}{p^3} \times$$
$$\sum_w \left(\sum_v A^{2(w-a_1)v}\right) A^{(w-a_i)^2+(w-a_j)^2+(a_i^2+a_j^2-2a_1^2+2(a_1a_i+a_1a_j-a_ia_j))} e_w \otimes e_{a_2} \otimes \ldots \otimes e_{a_g}$$
$$= -A^{2(a_i^2+a_j^2-a_ia_j}(e_{a_1} \otimes \ldots \otimes e_{a_g})$$
$$= -\pi_{p,g}(Z_{i,j})\pi_{p,g}(X_i)\pi_{p,g}(Y_j)(e_{a_1} \otimes \ldots \otimes e_{a_g})$$

The relations (4.19) reads $X_i^2 = -(Z_{i,1}X_1^2Y_1)^3$ for $2 \leq i \leq g$ so we compute:

$$\pi_{p,g}(Z_{i,1})\pi_{p,g}(X_1)^2\pi_{p,g}(Y_1)(e_{a_1} \otimes \ldots \otimes e_{a_g}) = \left(\frac{G(1,0,2p)}{2p}\right) \sum_u A^{-(a_1-u)^2+2u^2+(a_i-u)^2} e_u \otimes \ldots \otimes e_{a_g}$$

$$(\pi_{p,g}(Z_{i,1})\pi_{p,g}(X_1)^2\pi_{p,g}(Y_1))^2(e_{a_1} \otimes \ldots \otimes e_{a_g})$$
$$= \left(\frac{G(1,0,2p)}{2p}\right)^2 \sum_v (\sum_u A^{u^2+2u(a_1-a_i+v)}) A^{2v^2-2va_i+2a_i^2-a_1^2} e_v \otimes \ldots \otimes e_{a_g}$$
$$= \left(\frac{G(1,0,2p)}{2p}\right)^2 \frac{G(1,0,2p)}{2} A^{a_i^2-2a_1^2+2a_1a_i} \sum_v A^{v^2-2va_1} e_v \otimes \ldots \otimes e_{a_g}$$

$$(\pi_{p,g}(Z_{i,1})\pi_{p,g}(X_1)^2\pi_{p,g}(Y_1))^3(e_{a_1} \otimes \ldots \otimes e_{a_g})$$
$$= \left(\frac{G(1,0,2p)}{2}\right)^4 \frac{1}{p^3} A^{a_i^2-2a_1^2+2a_1a_i} \sum_w (\sum_v A^{2v(w-a_1)}) A^{2w^2-2a_iw+a_i^2} e_w \otimes \ldots \otimes e_{a_g}$$
$$= -A^{2a_i^2} e_{a_1} \otimes \ldots \otimes e_{a_g} = -\pi_{p,g}(X_i)^2(e_{a_1} \otimes \ldots \otimes e_{a_g})$$

Finally we check the last relation (4.20) : $-1 = (X_1^3 Y_1)^3$.

$$
\begin{aligned}
&(\pi_{p,g}(X_1)^3 \pi_{p,g}(Y_1))^3 (e_{a_1} \otimes \ldots \otimes e_{a_g}) \\
&\quad = \left(\frac{G(1,0,2p)}{2p}\right)^3 \sum_{u,v,w} A^{-(u-a_1)^2+3u^2-(u-v)^2+3v^2-(v-w)^2+3w^2} e_w \otimes \ldots \otimes e_{a_g} \\
&\quad = \left(\frac{G(1,0,2p)}{2p}\right)^3 \frac{G(1,0,2p)}{2} \sum_{w,v} A^{-(a_1+v)^2-a_1^2+2v^2-(v-w)^2+3w^2} e_w \otimes \ldots \otimes e_{a_g} \\
&\quad = \left(\frac{G(1,0,2p)}{2}\right)^4 \frac{1}{p^3} \sum_{w} (\sum_{v} A^{2v(w-a_1)}) A^{-2a_1^2+2w^2} e_w \otimes \ldots \otimes e_{a_g} \\
&\quad = \left(\frac{G(1,0,2p)}{2}\right)^4 \frac{1}{p^2} (e_{a_1} \otimes \ldots \otimes e_{a_g}) = -(e_{a_1} \otimes \ldots \otimes e_{a_g})
\end{aligned}
$$

And that completes the proof. □

Corollary 4.3.5. *When $g \geq 2$ and p is even, the Weil representations cannot be lifted to linear representations of $Sp_{2g}(\mathbb{Z})$.*

Proof. Suppose that such a lift $\tilde{\pi}_{p,g}$ exists and is defined by $\tilde{\pi}_{p,g}(X_i) = \alpha_i \pi_{p,g}(X_i)$, $\tilde{\pi}_{p,g}(Y_i) = \beta_i \pi_{p,g}(Y_i)$ and $\tilde{\pi}_{p,g}(Z_{i,j}) = \gamma_{i,j} \pi_{p,g}(Z_{i,j})$ with α_i, β_i and $\gamma_{i,j}$ some invertible scalars of k_p. Then the braid relation (4.5) implies that $\alpha_i = \beta_i$ for all i. The braid relation (4.8) implies that $\gamma_{i,j} = \beta_i$ for all $i \neq j$. Hence the scalars $\alpha_i = \beta_i = \gamma_{i,j} = c$ are independent of i and j. Now the modified star relation (4.20) implies $c^{12} = -1$ whereas the modified star relation (4.19) gives $c^2 = -c^{12}$ and we have a contradiction. □

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Index

List of Symbols

Σ_g Closed oriented surface of genus g

H_g Closed oriented handlebody of genus g

$\mathrm{Mod}(\Sigma)$ Mapping class group of Σ

$\widetilde{\mathrm{Mod}(\Sigma_g)}$ Central extension of $\mathrm{Mod}(\Sigma_g)$

$Sp_{2g}(\mathbb{Z})$ Symplectic group

$\widetilde{Sp_{2g}(\mathbb{Z})}$ Central extension of $Sp_{2g}(\mathbb{Z})$

$\pi_{p,g}$ Weil representation at level p of a genus g closed surface

$\rho_{p,g}$ Reshetikhin-Turaev representation at level p of a genus g closed surface

Add (Schrödinger)-representation of the skein algebra induced by TQFT

$\mathcal{T}_k(M)$ Abelian skein module of M on the ring k

$\mathcal{K}_k(M)$ Kauffman skein module of M on the ring k

U_p Abelian TQFT at level p

V_p Reshetikhin-Turaev TQFT at level p

$V(\Sigma)$ Module associated to a closed oriented surface by the TQFT V

$Z_V(M)$ Vector associated to a closed oriented 3-manifold M by the TQFT V

$\langle M \rangle_V$ Quantum invariant of M associated to the TQFT V

$\langle \cdot, \cdot \rangle_V^R$ Invariant form associated to the TQFT V

$(\cdot, \cdot)_V^H$ Hopf pairing associated to the TQFT V

TL_n Temperley-Lieb algebra with n strands

f_n n-th Jones Wenzl idempotent

$[n]$ Quantum number $[n] = \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}$

$G(a, b, c)$ A Gauss sum

$\mathrm{Span}(X)$ Free module generated by the elements of X

$col_p(\Gamma)$ Set of p-admissible colorings of the graph Γ

$\underline{col}_p(\Gamma)$ Set of equivalence class of p-admissible colorings of the graph Γ

u_σ Vector associated to a coloring σ of a graph

$\sigma(L)$ Signature of a link L