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Giulia Cesari

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# THÈSE DE DOCTORAT

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Préparée dans le cadre d'une cotutelle entre  
l'Université Paris-Dauphine  
et le Politecnico di Milano

## GAME THEORETIC MODELS OF NETWORK FORMATION

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**Soutenue le 13-12-2016  
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December 13th 2016,  
Giulia CESARI



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## Preface

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This thesis has been carried out during my three years as a PhD student at Politecnico di Milano and Université Paris Dauphine.

Most of the results included in this thesis have been taken from articles that are currently submitted for publication; one of them has just been accepted by the International Journal of Game Theory, and another one has been recently published.

Chapter 3 and 4 contain part of the results included in “Cesari G., Lucchetti R., Moretti S. (2016) *Generalized Additive Games*, International Journal of Game Theory, doi:10.1007/s00182-016-0561-7”.

Chapter 5 presents the results included in “Cesari G., Ferrari M.M., (2016) *On the Position Value for Special Classes of Networks*. In Recent Advances in Game Theory and Applications (pp. 29-47). Springer International Publishing”.

Chapter 6 contains part of the analysis presented in “Cesari G., Fossati F., Moretti S., *A conflict index for arguments in an argumentation graph*, Submitted to the 2<sup>nd</sup> European Conference on Argumentation (ECA2017)”.

Lastly, the work in Chapter 7 has been presented as invited talk at the 28<sup>th</sup> European Conference on Operational Research, in the stream “Game Theory, Solutions and Structures” with the title “A relevance index for genes in a biological network” and contains the results included in “Cesari G., Algaba E., Moretti S., Nepomuceno J.A., *A game theoretic neighbourhood-based relevance index to evaluate nodes in gene co-expression networks*, Submitted to: Computer Methods and Programs in Biomedicine”.

A brief summary of the contents can be found at the end of this thesis, both in French and in Italian.

November 2016,  
Giulia CESARI



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## Abstract

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**T**HIS thesis deals with the theoretical analysis and the application of a new family of cooperative games, where the worth of each coalition can be computed from the contributions of single players via an additive operator describing how the individual abilities interact within groups. Specifically, we introduce a large class of games, namely the Generalized Additive Games, which encompasses several classes of cooperative games from the literature, and in particular of *graph games*, where a network describes the restriction of the interaction possibilities among players. Some properties and solutions of such class of games are studied, with the objective of providing useful tools for the analysis of known classes of games, as well as for the construction of new classes of games with interesting properties from a theoretic point of view. Moreover, we introduce a class of solution concepts for *communication situations*, where the formation of a network is described by means of an additive pattern, and in the last part of the thesis we present two approaches using our model to real-world problems described by graph games, to the fields of Argumentation Theory and Biomedicine.



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# CHAPTER 1

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## Introduction

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A Transferable Utility (TU) game with  $n$  players specifies a vector of  $2^n - 1$  real numbers, i.e., a number for each non-empty subset of players, which is difficult to handle for large  $n$ . Since the number of coalitions grows exponentially with respect to the number of players, it is computationally very interesting to single out classes of games that can be described in a concise way. Therefore, several models from the literature on cooperative games focus on interaction situations which are characterized by a compact representation of a TU-game, and such that the worth of each coalition can be easily computed. A compact representation not only allows to reduce the complexity of describing the game and computing solutions but also enables to collect a variety of real problems under a unified formalism.

Several classes of games describing in a compact way the synergism among players are found in the literature: among them, profit sharing and cost allocation games, market games, optimization games (spanning tree games, flow games and linear programming games) and voting games (see [17] and [54] for a survey on coalitional games and operation research games).

In particular, there exist several approaches for defining classes of games whose concise representation is derived by an additive pattern among coalitions. In some contexts, due to an underlying structure among the players, such as a network, an order, or a permission structure, the value of a coalition  $S \subseteq N$ , where  $N$  is a finite set of players, can be derived additively from a collection of subcoalitions  $\{T_1, \dots, T_k\}$ ,  $T_i \subseteq S \forall i \in \{1, \dots, k\}$ . Such situations are modelled, for example, by the graph-restricted games, introduced by Myerson in [75] and further studied by Owen in [78]; the component additive games [30], and the restricted component additive games [29].

Sometimes, the worth of each coalition is computed from the values that single players can guarantee themselves by means of a mechanism describing the interactions

of individuals within groups of players. In the simplest case we can consider that, when a coalition of players forms, each player brings his own value and the worth of the coalition is computed as the sum of the single contributions of players that form it. As an example, consider a cost game where  $n$  players want to buy online  $n$  different objects and the value of a single player in the game is defined as the price of the object he buys. Then if a group  $S$  of players agree to make the purchase together, the cost of the operation will simply be the sum of the  $s = |S|$  prices of the objects bought by players in  $S$ , i.e. the sum of the costs that the single players in  $S$  should bear if they bought the objects separately.

This situation can be described by means of an *additive game*, where the value of a coalition is computed as the sum of the disjoint coalitions that form it. An additive game is indeed determined by the vector of the  $n$  values of the singleton players and therefore provides a compact way to represent interaction situations among players.

However, such a model may fail to reflect the importance of a subset of players in contributing to the value of the coalition it belongs to. In the previous example, it is often the case that, by making a collective purchase, when a certain threshold price is reached, some of the objects will be sold for free and therefore the price that a coalition  $S$  should pay will depend only on the price of a subset of purchased objects.

In fact, in several cases the procedure used to assess the worth of a coalition  $S \subseteq N$  is strongly related to the sum of the individual values over another subset  $S' \subseteq N$ , not necessarily included in  $S$ .

Many examples from the literature fall into this category. As a simple example, consider the well-known *glove game*: the set of players  $N$  is divided into two categories, the players in  $L$  that own a left-hand glove, and those in  $R$  with a right-hand glove. The worth of a coalition of players  $S \subseteq N$  is defined as the number of pairs of gloves owned by the coalition  $S$ . In this context, the valuable players in a coalition are those whose class is represented by a minority of the players, since the value of  $S$  is given by the minimum between the number of players in  $S \cap L$  and in  $S \cap R$ . Therefore, we can represent this game by assigning value 1 to each player and by describing the worth of each coalition  $S$  as the sum of single players' values over the smaller subset among  $S \cap L$  and  $S \cap R$ . A similar approach can be used to describe several other classes of games from the literature and in particular some classes of *graph games*, among them the *airport games* [63, 64], the *connectivity game* and its extensions [2, 62], the *argumentation games* [16] and classes of operation research games, such as the *peer games* [19] and the *mountain situations* [73].

A coalitional game describes a situation in which all players can freely interact with each other, i.e. every coalition of players is able to form and cooperate. However, this is not the case in many real world scenarios and in many cases it is necessary to drop the assumption that all coalitions are feasible. A typical way to model the restriction of the interaction possibilities between players is through a network structure.

In *graph games*, a graph (or network) describes the interaction possibilities between players: the nodes of the network are the players of the game and there exists a link between two nodes if the corresponding players are able to interact directly. As an example, in argumentation games, an underlying directed graph describes the attack relations among the arguments in an opinion: there exists an edge from one argument to the other if the former attacks the latter; in peer games a rooted directed network describes

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the hierarchical structure among agents: there exists a direct link from one node to the other if the former is a superior of the latter in the hierarchy that has the leader of the organization as the root node; in mountain situations a rooted directed graph represents the connection possibilities among houses in a village and a source (e.g. a water purifier): there is a link between a house and a lower one if it is possible to connect them in order to create a channel that allows the water to reach the source.

The network structure models the restrictions of the interaction possibilities among players, thus determining how the individual abilities interact within groups of players: if we define the value of a coalition of arguments as its self-consistency, i.e. the number of arguments that are not attacked by another argument in the coalition, then a player in an argumentation game would contribute to a coalition it belongs to only if none of his attackers belongs to the coalition; an agent in a peer game would contribute with his individual value to the maintenance of the hierarchical organization if all players at an upper level in the hierarchy cooperate with him, in other words he would contribute only to those coalitions that contain all his superiors; a house in a mountain situation would contribute to the division of the cost of connection to the source only if it lies on the minimum cost tree connecting the players to the source.

In other words, in many cases, the network structure prescribes which players shall contribute to the value (or cost) of a given coalition, by bringing together their individual values.

In all the aforementioned models, the value of a coalition  $S$  of players is calculated as the sum of the single values of players in a subset of  $S$ . On the other hand, in some cases the worth of a coalition might be affected by external influences and players outside the coalition might contribute, either in a positive or negative way, to the worth of the coalition itself. This is the case, for example, of the bankruptcy games [5] and the maintenance problems [58].

The first part of this thesis is devoted to the introduction of a game-theoretical model that encompasses all the aforementioned classes of coalitional games. In Chapter 3, we introduce the class of *Generalized Additive Games (GAGs)*, where the worth of a coalition  $S \subseteq N$  is evaluated by means of an interaction filter, that is a map  $\mathcal{M}$  which returns the valuable players involved in the cooperation among players in  $S$ .

The objective of this model is to provide a general framework for describing several classes of games studied in the literature on coalitional games, and particularly on graph games, and to give a kind of taxonomy of coalitional games that are ascribable to this notion of additivity over individual values.

The general definition of the map  $\mathcal{M}$  allows various and wide classes of games to be embraced, as for example the simple games. Moreover, by making further hypothesis on  $\mathcal{M}$ , our approach enables to classify existing games based on the properties of  $\mathcal{M}$ .

In particular, we introduce the class of *basic GAGs*, which is characterized by the fact that the valuable players in a coalition  $S$  are selected on the basis of the presence, among the players in  $S$ , of their *friends* and *enemies*, that is, a player contributes to the value of  $S$  if and only if  $S$  contains at least one of his friends and none of his enemies is present.

Several of the aforementioned classes of games can be described as basic GAGs, as well as games deriving from real-world situations. As an example, this model turns out to be suitable for representing an online social network, where friends and enemies of

the web users are determined by their social profiles. Moreover, Chapter 7 presents an application of basic GAGs to the field of Biomedicine.

The interest of this classification is not only taxonomical, since it also allows to study the properties of solutions for classes of games known from the literature and provides some potentially useful tools for computing solutions of new classes of games that may fall into this classification.

In Chapter 4, we indeed provide results on classical solution concepts for basic GAGs, and we address the problem of how to guarantee that a basic GAG has a non-empty core. In particular, we give concise formulas for the semivalues for some families of basic GAGs and we provide sufficient conditions for the core of a basic GAG to be non empty.

In a GAG, the worth of each coalition is computed as the sum of the individual values of a subset of players. On the other hand, in many cases, when an underlying network describes the interaction among the players involved, it is possible to derive the worth of each coalition of players as the sum of the contributions that their pairwise interactions generate, that is as the sum of the individual values assigned to the edges in the underlying network. As an example, in *maintenance cost games* [58], a tree describes a maintenance system such as a computer network, with a service provider as root. The cost of connection of a set of computers to the provider is described by a coalitional game and computed as the sum of the costs of maintenance of all the connections among the computers lying on the corresponding minimum cost spanning tree, that is as the sum of the costs associated to the edges in the induced tree.

As for the maintenance cost game, in several other graph games the worth of a coalition can be additively computed starting from the values assigned to the edges in the underlying graph. Moreover, as in graph-restricted game the value of a coalition can be derived additively from a collection of subcoalitions of players, for the class of *link games*, introduced by Meessen [70] and further studied by Borm et al. [18], the value of a coalition of links can be derived additively from a collection of subcoalitions of links. Indeed, several approaches to coalitional games on networks rely on additive patterns among links, not only for what concerns the definition of a game, but also for the analysis of the relative solutions.

The second part of this thesis is indeed devoted to the introduction of a class of solution concepts for *communication situations* [75], where the payoff to each player is additively computed starting from the values generated by pairwise relations among players. More precisely, in Chapter 5, we consider a communication situation in which a network is produced by subsequent formation of links among players and at each step of the formation process, the surplus generated by a link is shared between the players involved, according to some rule. As a consequence, we obtain a family of solution concepts that we investigate on particular network structures. In particular, it turns out that the *position value*, introduced by Borm et al. [18] as a solution for communication situations, is obtained when a specific symmetric rule is considered. Moreover, in the same Chapter, we investigate the problem of computing this particular solution on special classes of communication situations.

The third and last part of this thesis is devoted to two applications of the game-theoretical models described so far. A first application, presented in Chapter 6, is to the field of argumentation theory [37]. Argumentation theory aims at formalizing decision

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systems and associated decision making processes. One of its objectives is the search for sets of accepted conclusions in an argumentation framework, which is modelled as a directed graph where nodes represent arguments, i.e. statements or series of statements, and direct edges represent attack relations, which express conflict between pairs of arguments.

In the literature, several *extension* semantics or labellings have been associated to the abstract argumentation framework with the objective to specify which arguments are accepted or not, and which are undecided [21, 37]. Different from extension semantics, the aim of *gradual* semantics is to assign a degree of acceptability to each argument [4, 13, 24, 49, 97].

Game theory has also been used to define intermediate level of acceptability of arguments. Specifically, in [67] a degree of acceptability is computed taking into account the minimax value of a zero-sum game between a ‘proponent’ and an ‘opponent’ and where the strategies and the payoffs of the players depend on the structure of an argumentation graph. More recently, coalitional games have been applied in [16] to measure the relative importance of arguments taking into account both preferences of an agent over the arguments and the information provided by the attack relations. In the aforementioned approaches, the weight attributed to each argument represents the strength of an argument to force its acceptability. On the other hand, acceptability is not the only arguments’ attribute that has been studied in literature from a gradual perspective. In [98] an index has been introduced to represent the *controversiality* of single arguments, where the most controversial arguments are those for which taking a decision on whether they are acceptable or not is difficult. In a similar direction, the problem of measuring the *disagreement* within an argumentation framework has been studied in [3], where the authors provide an axiomatic analysis of different disagreement measures for argumentation graphs.

In Chapter 6, we firstly show that the properties introduced in [3] for argumentation graphs can be reformulated for single arguments, and may drive the definition of a *conflict-based* ranking, that can be seen as an alternative ranking for measuring the controversiality of arguments. Secondly, we show that the conflict-based ranking we propose may be re-interpreted in terms of a classical solution for coalitional games, that is as the average marginal contribution of each argument to the disagreement induced by all possible coalitions of arguments in an argumentation graph. We do so by defining a cooperative game, where the players are the arguments in an argumentation graph and every coalition of arguments is assigned a value, which expresses the total disagreement within the coalition. In particular, every node and every link inside a coalition of arguments contributes to the value of the coalition with its individual share of the disagreement, as measured by the attack relations it brings to the coalition. The so-defined game is indeed representable in terms of basic GAGs, as a combination of the original model which will be introduced in Chapter 3 and its variant defined on links. We propose the Shapley value of such a game as a *conflict index* that measures the controversiality of arguments, since it measures the power of each argument in bringing conflict to the argumentation framework. Considering persuasion scenarios, we argue that our conflict-based ranking may drive agents to select those arguments that should be further developed in order to strengthen certain position in a debate, hence, responding to the question raised in [98] about the definition of a ranking representing

the *potential for development* of arguments.

While Chapter 6 describes a game-theoretic approach to the field of argumentation theory, Chapter 7 presents a real-world application of the model of GAGs to the field of Biomedicine, and in particular to the problem of assessing the relevance of genes in a biological network. Among biological networks, *gene regulatory networks* (or *pathways*) are of great interest in the field of molecular biology and epidemiology to better understand the interaction mechanisms between genes, proteins and other molecules within a cell and under certain biological condition of interest [20, 23, 31, 93]. A crucial point in the analysis of genes' interaction is the formulation of appropriate measures of the role played by each gene to influence the very complex system of genes' relationships in a network.

Centrality analysis represents an important tool for the interpretation of the interaction of genes in a gene regulatory network [12, 22, 44, 52, 53]. Classical centrality measures [41, 57] are used in network analysis to identify the relevant elements in a network, based on their position within the network structure. However, they are appropriate under the assumption that nodes behave independently and the system is sensible to the behaviour of each single node. On the contrary, in biological complex networks, assuming that the genes may express independently is not realistic and the consequences on the system can be appreciated only if many genes change their expression. Therefore, in a complex scenario, such as the pathogenesis of a genetic disease, we deal with the problem of quantifying the relative relevance of genes, taking into account not only the behaviour of single genes but most of all the level of their interaction.

Cooperative game theory has been proposed as a theoretical framework to face such limitations. Recently, several centrality measures based on coalitional game theory have been successfully applied to different kinds of biological networks, such as *brain networks* [55, 56, 59], *gene networks* [72], and *metabolic networks* [85].

We propose an approach, using coalitional games, and in particular basic GAGs to the problem of identifying relevant genes in a gene network. The problem has been firstly addressed by means of a game-theoretical model in [72], where the Shapley value for coalitional games is used to express the power of each gene in interaction with the others and to stress the centrality of certain hub genes in the regulation of biological pathways of interest. Our model represents a refinement of this approach, which generalizes the notion of degree centrality [77, 89], whose correlation with the relevance of genes for different biological functions is supported by several practical evidences in the literature (see [12], [22], [52], [53], [103]). We define a basic GAG with a biological interpretation on gene networks and propose the Shapley value of such a game as a new relevance index for genes, that evaluates the potential of genes in acting as intermediaries between hub nodes and leaf nodes and preserving the overall regulatory activity within gene networks. This approach is supported by an axiomatic characterization, where the set of properties satisfied by our index have a biological interpretation. Moreover, a formula for the computation of the new relevance index is provided, which can be directly derived from the theoretical results presented in Chapter 4. An experimental study is conducted on a gene expression dataset from microarrays, related to a lung cancer disease, as well as a comparison with classical centrality indices.

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The outline of the thesis is the following. Chapter 2 provides some preliminaries on coalitional games and networks. Chapter 3 introduces the main model of Generalized Additive Game (GAGs) and some possible extensions. Chapter 4 investigates the problem of computing the core and the semivalues for basic GAGs. In Chapter 5 a new family of solution concepts for communication situations is introduced and investigated on particular classes of networks. Chapter 6 presents a game-theoretical approach to measure conflict in argumentation graphs and, lastly, Chapter 7 describes an application of basic GAGs to the problem of assessing the relevance of genes in a biological network.



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# CHAPTER 2

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## Preliminaries

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This chapter is devoted to illustrate some preliminaries on coalitional games and networks. We refer to Maschler, Solan and Zamir [66] for a more accurate illustration of the following concepts.

### 2.1 Coalitional Games

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A *cooperative game with transferable utility (TU-game)*, also referred to as *coalitional game*, consists of a pair  $(N, v)$ , where:

- $N$  denotes the set of players;
- $v : 2^N \rightarrow \mathbb{R}$  is the *characteristic function*, a real-valued function on the family of subsets of  $N$ .

A group of players  $S \subseteq N$  is called *coalition* and the characteristic function associates to each coalition  $S$  a real number  $v(S)$ , which is called the *value* or *worth* of the coalition, representing the total payoff to the coalition of players when they cooperate, whatever the remaining players do. The value of a coalition may represent a gain, or a cost, depending on the situation modelled by the cooperative game. By convention, we assume  $v(\emptyset) = 0$ .

If the set  $N$  of players is fixed, we identify a coalitional game  $(N, v)$  with its characteristic function  $v$ . We shall assume throughout this monograph that  $N = \{1, \dots, n\}$  and denote by  $s$  the cardinality  $|S|$  of coalition  $S$ .

We shall denote by  $\mathcal{G}$  the class of all coalitional games and by  $\mathcal{G}^N$  the class of all coalitional games with players set  $N$ . Clearly,  $\mathcal{G}^N$  is a vector space of dimension  $2^n - 1$ . The *canonical basis* for this vector space is given by the family of canonical games  $\{e_S, S \subseteq N\}$ , where  $e_S$  is defined as:

$$e_S(T) = \begin{cases} 1 & \text{if } S = T \\ 0 & \text{otherwise} \end{cases} \quad \forall S \subseteq N, S \neq \emptyset.$$

Another basis for  $\mathcal{G}^N$  is the family of the *unanimity games*  $\{u_S, S \subseteq N\}$ , where  $u_S$  is defined as:

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise} \end{cases} \quad \forall S \subseteq N, S \neq \emptyset.$$

Every coalitional game  $v$  can be written as a linear combination of unanimity games as follows:

$$v = \sum_{S \subseteq N, S \neq \emptyset} c_S(v) u_S, \quad (2.1)$$

where the constants  $c_S(v)$ , referred to as *unanimity coefficients* of  $v$ , can be inductively defined in the following way: let  $c_{\{i\}}(v) = v(\{i\})$  and, for  $S \subseteq N$  of cardinality  $s \geq 2$ ,

$$c_S(v) = v(S) - \sum_{T \subsetneq S, T \neq \emptyset} c_T(v). \quad (2.2)$$

An equivalent formula for the unanimity coefficients is the following:

$$c_S(v) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T). \quad (2.3)$$

Let  $\mathcal{C} \subseteq \mathcal{G}$  be a subclass of coalitional games. Given a set of players  $N$ , we denote by  $\mathcal{C}^N \subseteq \mathcal{C}$  the class of coalitional games in  $\mathcal{C}$  with  $N$  as set of players. A particular class of games is that of *simple games*, where the characteristic function  $v$  can only assume values in  $\{0, 1\}$ .

A game  $(N, v)$  is said to be *monotonic* if it holds that  $v(S) \leq v(T)$  for all  $S, T \subseteq N$  such that  $S \subseteq T$  and it is said to be *superadditive* if it holds that

$$v(S \cup T) \geq v(S) + v(T)$$

for all  $S, T \subseteq N$  such that  $S \cap T = \emptyset$ .

Moreover, a game  $(N, v)$  is said to be *convex* or *supermodular* if it holds that

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$$

for all  $S, T \subseteq N$ .

Given a game  $(N, v)$ , where  $|N| = n$ , we call *allocation* an  $n$ -dimensional vector  $(x_1, \dots, x_n) \in \mathbb{R}^n$  assigning to player  $i \in N$  the amount  $x_i$ . A solution for a coalitional game prescribes how to convert the information on the worth of every coalition of players into a single attribution to each of the players. The solutions for coalitional games are classified into two groups: the ones which provide a (possibly empty) set of allocations (e.g. the core), called *set-valued* solutions, and the ones which provide only one allocation (e.g. the nucleolus and the power indices), called *one-point* solutions.

A subset of the set of allocations is that of imputations. An *imputation* is a vector  $x \in \mathbb{R}^n$  such that  $\sum_{i \in N} x_i = v(N)$  and  $x_i \geq v(\{i\})$  for all  $i \in N$ , that is the set of

allocations that satisfy *efficiency* and *individual rationality*. An important subset of the set of the imputations is the *core*, which represents a classical solution concept for TU-games. The *core* of  $v$  is defined as  $C(v) = \{x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \ \forall S \subset N\}$ , that is the set of imputations that also satisfy *coalitional rationality*. Note that all the definitions in this Section hold for TU-games where  $v$  represents a gain, while the inequalities should be replaced with  $\leq$  when  $v$  is a cost function.

A one-point solution (or simply solution) for a class  $\mathcal{C}^N$  of coalitional games is a function  $\psi : \mathcal{C}^N \rightarrow \mathbb{R}^n$  that assigns a payoff vector  $\psi(v) \in \mathbb{R}^n$  to every coalitional game in the class. A well-known solution is the Shapley value, that has been introduced in 1953 by an axiomatic approach, which consists of providing a set of properties that a solution for TU-games should satisfy [88].

Let  $\Sigma_N$  be the set of linear orders on the set  $N$ , that is the set of all bijections  $\sigma : N \rightarrow N$ , where  $\sigma(i) = j$  means that with respect to  $\sigma$ , player  $i$  is in the  $j$ -th position. For  $\sigma \in \Sigma_N$ , the *marginal vector*  $m^\sigma(v) \in \mathbb{R}^n$  is defined by  $m_i^\sigma(v) = v(\{j \in N : \sigma(j) \leq \sigma(i)\}) - v(\{j \in N : \sigma(j) < \sigma(i)\})$  for each  $i \in N$ , where  $m_i^\sigma(v)$  is the *marginal contribution* of player  $i$  to the coalition of players with lower positions in  $\sigma$ . The Shapley value  $\phi(v)$  of a game  $(N, v)$  is defined as the average of marginal vectors over all  $n!$  possible orders in  $\Sigma_N$ . In formula

$$\phi_i(v) = \sum_{\sigma \in \Sigma_N} \frac{m_i^\sigma(v)}{n!} \text{ for all } i \in N. \quad (2.4)$$

An alternative representation of the Shapley value for each  $i \in N$  is by the formula

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{1}{n \binom{n-1}{s}} (v(S \cup i) - v(S)). \quad (2.5)$$

It assigns to every player the averaged marginal contribution to all the coalitions he belongs to, with respect to a probability distribution that assigns equal probability to all the different sizes that a coalition of players can have.

The formula, despite the meaningfulness of its interpretation, has a great disadvantage: its computational complexity is exponential in the number of players, making it hard to be computed for a high number of players. For this reason it is very interesting to single out classes of games that lead to a concise formula, for which the Shapley value is easy to compute. Indeed, much effort in the literature has been driven in this direction and to the design of algorithms for computing the Shapley value efficiently.

We now introduce the classical characterization of the Shapley value. Indeed, the Shapley value of a coalitional game  $(N, v)$  is the only solution that satisfies the following four properties on the class of  $\mathcal{G}^N$  [1]:

- *efficiency* (EFF), i.e.  $\sum_{i \in N} \phi_i(v) = v(N)$ ;
- *symmetry* (SYM), i.e. if  $i, j \in N$  are such that  $v(S \cup \{i\}) = v(S \cup \{j\})$  for all  $S \subseteq N \setminus \{i, j\}$ , then  $\phi_i(v) = \phi_j(v)$ ;
- *dummy player property* (DPP), i.e. if  $i \in N$  is such that  $v(S \cup \{i\}) - v(S) = v(\{i\})$  for all  $S \subseteq N$ , then  $\phi_i(v) = v(\{i\})$ ;

1. Although this is a classical characterization for the Shapley value, note that it is not the one introduced by Shapley, which was on the class of superadditive games. We refer to [88] for further details.

— *additivity* (ADD), i.e.  $\phi(v) + \phi(w) = \phi(v + w)$  for each  $v, w \in \mathcal{C}^N$ , where the game  $v + w$  is such that  $v + w(S) = v(S) + w(S) \forall S \subseteq N$ .

This is not the only axiomatic characterization of the Shapley value. Among the others, we cite the one by Young [102], Hart and Mas-Colell [47] and the one on the class of simple games by Dubey [35]. We refer to [66] for further details.

Since its introduction, the Shapley value has been widely investigated and applied to a range of diverse fields and disciplines, including social sciences, medicine and biology, among the others [74]. Its success and application to a wide range of fields is due, among other factors, to the axiomatic approach that has characterized its introduction. The support for its employment in such a variety of fields of study often comes from an axiomatic characterization, where the properties it satisfies have a specific interpretation for the application under analysis. This will also be the approach carried out along this thesis, where the application of the Shapley value to the very different fields of Argumentation Theory and Biomedicine is supported by an axiomatic approach.

The Shapley value belongs to the broader class of *semivalues* [36]:

$$\Psi_i^{\mathbf{P}}(v) = \sum_{S \subseteq N \setminus \{i\}} p_s^n (v(S \cup \{i\}) - v(S)) \quad \forall i \in N, \quad (2.6)$$

where  $p_s^n$  is such that  $p_s^n \geq 0 \forall s = 0, 1, \dots, n-1$ ,  $\sum_{s=0}^{n-1} \binom{n-1}{s} p_s^n = 1$  and represents the probability that a coalition of size  $s+1$  forms. If  $p_s^n > 0$  for all  $s$ , then the semivalue is called *regular semivalue*. We shall write  $p_s$  instead of  $p_s^n$  when there is no ambiguity about the players set. In particular, the Shapley value is a regular semivalue with

$$p_s = \frac{1}{n \binom{n-1}{s}}. \quad (2.7)$$

and is the only semivalue that satisfies EFF, while all the semivalues satisfy the other three axioms: SYM, DPP, ADD.

Another well-known regular semivalue is the *Banzhaf index* [6], which is defined by (2.6) with  $p_s = \frac{1}{2^{n-1}}$ , that is the probability distribution that assigns the same probability to every coalition of players.

## 2.2 Networks

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In our context, a network describes the restriction of interactions among players in a cooperative game, and can be formally represented by a graph. In this thesis we will mainly deal with undirected graphs, that describe situations in which player  $i$  is able to interact directly with player  $j$  only if the converse is also true, i.e. the edges in the graph are not oriented. On the other hand, an argumentation framework as described in Chapter 6 is mathematically modelled by a directed graph, where the edges are oriented: it is possible that argument  $i$  attacks argument  $j$  but argument  $j$  does not attack argument  $i$ .

In this section we provide some basic notations and definitions for undirected graphs, that we shall simply call graphs or networks. We refer to Chapter 6 for a description of directed argumentation graphs.

A *graph* or *network*  $\Gamma$  is a pair  $\langle V, E \rangle$ , where  $V$  is a finite set of *vertices* or *nodes*

and  $E \subseteq \{\{i, j\} : i, j \in V, i \neq j\}$  is the set of *edges* or *links* between pairs of nodes. We define the set of *neighbours* of a node  $i$  in graph  $\langle V, E \rangle$  as the set  $N_i(E) = \{j \in N : \{i, j\} \in E\}$ , and the *degree* of  $i$  as the number  $d_i(E) = |N_i(E)|$  of neighbours of  $i$  in graph  $\langle V, E \rangle$ , i.e. the number of links incident to  $i$  in  $\Gamma$ . With a slight abuse of notation, we denote by  $N_S(E) = \{j \in N : \exists i \in S \text{ s.t. } j \in N_i(E)\}$  the set of neighbours of nodes in  $S \in 2^N, S \neq \emptyset$ , in the graph  $\langle V, E \rangle$ .

Given a subset  $S \subseteq V$  of nodes, we define the induced subgraph  $\Gamma_S = (S, E_S)$ , where  $E_S$  is the set of links  $\{i, j\} \in E$  such that  $i, j \in S$ . Similarly, we denote by  $\Gamma_A$  the graph  $(V_A, A)$  induced by a subset  $A \subseteq E$  of links, where  $V_A$  is the set of nodes incident to at least one link of  $A$ .

A *path* between nodes  $i$  and  $j$  in a graph  $\Gamma = \langle V, E \rangle$  is a finite sequence of distinct nodes  $(i_0, i_1, \dots, i_k)$  such that  $i_0 = i, i_k = j$  and  $\{i_s, i_{s+1}\} \in E$  for each  $s \in \{0, \dots, k-1\}$ . Two nodes  $i$  and  $j$  are said to be *connected* in  $\Gamma$  if  $i = j$  or if there exists a path between them in  $\Gamma$ . The *length* of a path between  $i$  and  $j$  is the number of edges in the path and a *shortest path* between  $i$  and  $j$  is a path between  $i$  and  $j$  with minimum length. We call *chain* the set of nodes on a path with different endpoints and we denote by *s-chain* a chain with  $s$  nodes.

A *connected component* in  $\Gamma$  is a maximal subset of  $V$  with the property that any two nodes of  $V$  are connected in  $\Gamma$ . We denote by  $C_\Gamma$  the set of connected components in  $\Gamma$ . A graph  $\Gamma$  is said to be *connected* if there exists a path between every two elements of  $V$ . A subset of nodes  $S \subseteq V$  (respectively a set of links  $A \in E$ ) is *connected* if the induced graph  $\Gamma_S$  (respectively  $\Gamma_A$ ) is connected.

A *cycle* in  $\Gamma$  is a path  $(i_0, i_1, \dots, i_k)$  such that  $i_0 = i_k$ . A *forest* is a graph without cycles. A *tree* is a forest with only one connected component.

A graph  $\langle V, E_S^i \rangle$ , where the set of edges is  $E_S^i = \{\{i, j\} : j \in S\}$  is said a *star on  $S$  with center in  $i$* . Notice that the set of neighbours of nodes in  $\langle V, E_S^i \rangle$  are such that  $N_i(E_S^i) = S, N_j(E_S^i) = \{i\}$ , for each  $j \in S$ , and  $N_j(E_S^i) = \emptyset$ , for each  $j \in V \setminus (S \cup \{i\})$ .



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# CHAPTER 3

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## Generalized Additive Games

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This Chapter is devoted to the introduction of the main model of *Generalized Additive Games (GAGs)*, where the worth of a coalition  $S \subseteq N$  is evaluated by means of an interaction filter, that is a map  $\mathcal{M}$  which returns the valuable players involved in the cooperation among players in  $S$ . In particular, we investigate the subclass of *basic GAGs*, where the filter  $\mathcal{M}$  selects, for each coalition  $S$ , those players that have *friends* but not *enemies* in  $S$ . We show that well-known classes of TU-games can be represented in terms of such basic GAGs, and we present some possible extensions of the model.

### 3.1 Introduction

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A Transferable Utility (TU) game with  $n$  players specifies a vector of  $2^n - 1$  real numbers, i.e. a number for each non-empty coalition, and this can be difficult to handle for large  $n$ . Therefore, several models from the literature focus on interaction situations which are characterized by a compact representation of a TU-game, and such that the worth of each coalition can be easily computed. In particular, there exist several approaches for defining classes of games whose concise representation is derived by an additive pattern among coalitions. In some contexts, due to an underlying structure among the players, such as a network, an order, or a permission structure, the value of a coalition  $S \subseteq N$  can be derived additively from a collection of subcoalitions  $\{T_1, \dots, T_k\}$ ,  $T_i \subseteq S \forall i \in \{1, \dots, k\}$ . Such situations are modelled, for example, by the graph-restricted games, introduced by Myerson in [75] and further studied by Owen in [78]; the component additive games [29, 30] and the non-negative additive games with an acyclic permission structure [100].

Sometimes, the worth of each coalition is computed from the values of single players by means of a mechanism describing how the individual abilities interact within

groups of players. In fact, in several cases the procedure used to assess the worth of a coalition  $S \subseteq N$  is strongly related to the sum of the individual values over another subset  $T \subseteq N$ , not necessarily included in  $S$ .

Many examples from the literature fall into this category, especially some classes of *graph games*, among them the *airport games* [63,64], the *connectivity game* and its extensions [2,62], the *argumentation games* [16] and some classes of operation research games, such as the *peer games* [19] and the *mountain situations* [73]. In all the aforementioned models, the value of a coalition  $S$  of players is calculated as the sum of the single values of players in a subset of  $S$ . On the other hand, in some cases the worth of a coalition might be affected by external influences and players outside the coalition might contribute, either in a positive or negative way, to the worth of the coalition itself. This is the case, for example, of the *bankruptcy games* [5] and the *maintenance cost games* [17,58]. In this chapter we introduce the class of *Generalized Additive Games* (GAGs), where the worth of a coalition  $S \subseteq N$  is evaluated by means of an interaction filter, that is a map  $\mathcal{M}$  which returns the valuable players involved in the cooperation among players in  $S$ . Our objective is to provide a general framework for describing several classes of games studied in the literature on coalitional games and to give a kind of taxonomy of coalitional games that are ascribable to this notion of additivity over individual values.

The general definition of the map  $\mathcal{M}$  allows various and wide classes of games to be embraced. Moreover, by making further hypothesis on  $\mathcal{M}$ , our approach enables to classify existing games based on the properties of  $\mathcal{M}$ . In particular, we investigate the subclass of *basic GAGs*, where the filter  $\mathcal{M}$  selects, for each coalition  $S$ , those players that have *friends* but not *enemies* in  $S$ . We also show that well-known classes of TU-games can be represented in terms of such basic GAGs, as well as games deriving from real-world situations. As an example, this model turns out to be suitable for representing an online social network, where friends and enemies of the web users are determined by their social profiles, as we shall see in Chapter 4. Furthermore, Chapter 7 is devoted to an application of basic GAGs to the field of Biomedicine.

The chapter is structured as follows. In Section 3.2, we introduce the class of Generalized Additive Games (GAGs) and provide examples of games falling into this category. In Section 3.3 we introduce some hypothesis on the map  $\mathcal{M}$  and describe the resulting subclass of basic GAGs, providing further examples from the literature. Moreover, in Section 3.4 we provide a characterization of basic GAGs. Some possible extensions of the model are finally presented in Section 3.5.

### 3.2 Generalized Additive Games (GAGs)

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In this section we define the class of games that is the object of the Chapter, and we provide some examples and basic properties.

The basic ingredients of our definition are the set  $N = \{1, \dots, n\}$ , representing the set of players, a map  $v : N \rightarrow \mathbb{R}$  that specifies the individual values of the players and a map  $\mathcal{M} : 2^N \rightarrow 2^N$ , called the *coalitional map*, which assigns a coalition  $\mathcal{M}(S)$  to each coalition  $S \subseteq N$  of players.

**Definition 1.** We shall call *Generalized Additive Situation* (GAS) any triple  $\langle N, v, \mathcal{M} \rangle$ ,

### 3.2. Generalized Additive Games (GAGs)

where  $N$  is the set of the players,  $v : N \rightarrow \mathbb{R}$  is a map that assigns to each player a real value and  $\mathcal{M} : 2^N \rightarrow 2^N$  is a coalitional map, which assigns a (possibly empty) coalition  $\mathcal{M}(S)$  to each coalition  $S \subseteq N$  of players and such that  $\mathcal{M}(\emptyset) = \emptyset$ .

**Definition 2.** Given the GAS  $\langle N, v, \mathcal{M} \rangle$ , the associated Generalized Additive Game (GAG) is defined as the TU-game  $(N, v^{\mathcal{M}})$  assigning to each coalition the value

$$v^{\mathcal{M}}(S) = \begin{cases} \sum_{i \in \mathcal{M}(S)} v(i) & \text{if } \mathcal{M}(S) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \quad (3.1)$$

**Example 1.** (simple games) Let  $w$  be a simple game. Then  $w$  can be described by the GAG associated to  $\langle N, v, \mathcal{M} \rangle$  with  $v(i) = 1$  for all  $i$  and

$$\mathcal{M}(S) = \begin{cases} \{i\} \subseteq S & \text{if } S \in \mathcal{W} \\ \emptyset & \text{otherwise} \end{cases}$$

where  $\mathcal{W}$  is the set of the winning coalitions in  $w$ .

In case there is a veto player, i.e. a player  $i$  such that  $S \in \mathcal{W}$  only if  $i \in S$ , then the game can also be described by  $v(i) = 1$ ,  $v(j) = 0 \forall j \neq i$  and

$$\mathcal{M}(S) = \begin{cases} T & \text{if } S \in \mathcal{W} \\ R & \text{otherwise} \end{cases}$$

with  $T, R \subseteq N$  such that  $i \in T$  and  $i \notin R$ .

From Example 1 it is clear that the description of a game as GAG need not be unique.

**Example 2.** (glove game) Let  $w$  be the glove game defined in the following way. A partition  $\{L, R\}$  of  $N$  is assigned. Define  $w(S) = \min\{|S \cap L|, |S \cap R|\}$ . Then  $w$  can be described as the GAG associated to  $\langle N, v, \mathcal{M} \rangle$  with  $v(i) = 1$  for all  $i$  and

$$\mathcal{M}(S) = \begin{cases} S \cap L & \text{if } |S \cap L| \leq |S \cap R| \\ S \cap R & \text{otherwise.} \end{cases}$$

**Example 3.** (connectivity games) [2, 62] Let  $\Gamma = (N, E)$  be a graph, where  $N$  is a finite set of vertices and  $E$  is a set of non-ordered pairs of vertices, i.e. the edges of the graph. Consider the (extended) connectivity game  $(N, v_{\Gamma})$ , where each node  $i$  of the underlying graph is assigned a weight  $w_i$ . The weighted connectivity game is defined as the game  $(N, w)$ , where

$$w(S) = \begin{cases} \sum_{i \in S} w_i & \text{if } S \subseteq N \text{ is connected in } \Gamma \text{ and } |S| > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then  $w$  can be described as the GAG associated to  $\langle N, v, \mathcal{M} \rangle$  with  $v(i) = w_i$  for all  $i$  and

$$\mathcal{M}(S) = \begin{cases} S & \text{if } S \subseteq N \text{ is connected in } \Gamma \\ \emptyset & \text{otherwise.} \end{cases}$$

**Example 4.** (*bankruptcy games*) Consider the bankruptcy game  $(N, w)$  introduced by Aumann and Maschler [5], where the value of a coalition  $S \subseteq N$  is given by

$$w(S) = \max\{E - \sum_{i \in N \setminus S} d_i, 0\}.$$

Here  $E \geq 0$  represents the estate to be divided and  $d \in \mathbb{R}_+^N$  is a vector of claims satisfying the condition  $\sum_{i \in N} d_i > E$ . It is easy to show that a bankruptcy game is the difference  $w = v_1^M - v_2^M$  of two GAGs  $v_1^M, v_2^M$  arising, respectively, from  $\langle N, v^1, \mathcal{M}^1 \rangle$  and  $\langle N, v^2, \mathcal{M}^2 \rangle$  with  $v^1(i) = E$  and  $v^2(i) = d_i$  for all  $i$ ,

$$\mathcal{M}^1(S) = \begin{cases} \{i\} \subseteq S & \text{if } S \in B \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$\mathcal{M}^2(S) = \begin{cases} N \setminus S & \text{if } S \in B \\ \emptyset & \text{otherwise} \end{cases}$$

for each  $S \in 2^N \setminus \{\emptyset\}$ , and where  $B = \{S \subseteq N : \sum_{i \in N \setminus S} d_i \leq E\}$ .

An extension of the bankruptcy games has been introduced in [80] and can be described as a sum of three GAGs. : an extended bankruptcy game  $(N, w)$  is defined as

$$w(S) = \max\{E - \sum_{i \in N \setminus S} d_i, \sum_{i \in S} r_i\}$$

where  $E \geq 0$  represents the estate to be divided,  $d \in \mathbb{R}_+^N$  is a vector of claims satisfying the condition  $\sum_{i \in N} d_i \geq E$  and  $r \in \mathbb{R}_+^N$  is a vector of objective entitlements satisfying the conditions  $0 \leq r_i \leq d_i$  for all  $i \in N$ . The extended bankruptcy game can be represented as the linear combination  $w = v_1^M - v_2^M + v_3^M$  of three GAGs  $v_1^M, v_2^M, v_3^M$  arising, respectively, from  $\langle N, v^1, \mathcal{M}^1 \rangle$ ,  $\langle N, v^2, \mathcal{M}^2 \rangle$  and  $\langle N, v^3, \mathcal{M}^3 \rangle$  with  $v^1(i) = E$ ,  $v^2(i) = d_i$  and  $v^3(i) = r_i$  for all  $i$  and

$$\mathcal{M}^1(S) = \begin{cases} \{i\} \subseteq S & \text{if } S \in R \\ \emptyset & \text{otherwise} \end{cases}$$

$$\mathcal{M}^2(S) = \begin{cases} N \setminus S & \text{if } S \in R \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$\mathcal{M}^3(S) = \begin{cases} \emptyset & \text{if } S \in R \\ S & \text{otherwise} \end{cases}$$

for each  $S \in 2^N \setminus \{\emptyset\}$ , where  $R = \{S \subseteq N : \sum_{i \in N \setminus S} d_i + \sum_{i \in S} r_i \leq E\}$ .

Observe that, clearly, despite the generality of the map  $\mathcal{M}$ , not every game can be described as a GAG. Obvious counterexamples can be provided for all  $n$ , in particular for  $n = 2$ . Indeed, given a game  $(N, v)$ , where  $N = \{1, 2\}$ , and given the values of the singletons  $v(\{1\})$  and  $v(\{2\})$ , in order for  $v$  to be described as a GAG  $v^M$ , the value of the grand coalition  $N$  can only assume values in the set  $\{0, v(\{1\}), v(\{2\}), v(\{1\}) + v(\{2\})\}$ , which of course is not the case for a generic game.

However, it is easy to show that every TU-game  $(N, v)$  can be described as a sum of  $k$  GAGs. For example, every game with 3 players can be represented as the sum of at most 3 GAGs. To see this, it suffices to consider 3 GAGs with maps, respectively,  $\mathcal{M}^1$ ,  $\mathcal{M}^2$  and  $\mathcal{M}^3$  such that  $\mathcal{M}^1(i) = \{i\}$  for all  $i$ ,  $\mathcal{M}^1(\{i, j\}) = \emptyset$  for all  $i, j$ ,  $\mathcal{M}^1(\{1, 2, 3\}) = \emptyset$ ,  $\mathcal{M}^2(i) = \emptyset$  for all  $i$ ,  $\mathcal{M}^2(\{1, 2\}) = \{1\}$ ,  $\mathcal{M}^2(\{1, 3\}) = \{3\}$ ,  $\mathcal{M}^2(\{2, 3\}) = \{2\}$ ,  $\mathcal{M}^2(\{1, 2, 3\}) = \emptyset$ ,  $\mathcal{M}^3(i) = \emptyset$  for all  $i$ ,  $\mathcal{M}^3(\{i, j\}) = \emptyset$  for all  $i, j$ ,  $\mathcal{M}^3(\{1, 2, 3\}) = \{1\}$  and  $v^1, v^2, v^3$  such that  $v^1(i) = v(\{i\})$  for all  $i$ ,  $v^2(1) = v(\{1, 2\})$ ,  $v^2(2) = v(\{2, 3\})$ ,  $v^2(3) = v(\{1, 3\})$ ,  $v^3(1) = v(\{1, 2, 3\})$  and  $v^3(2) = v^3(3) = 0$ . In general, at least  $\frac{2^n - 1}{n}$  GAGs are needed and such a description is therefore complicated, but less GAGs might be sufficient if there are some "additive" coalitions, i.e. coalitions such that their value can be derived as the sum of values of other coalitions, in which case the complexity of such a description would be reduced.

Some natural properties of the map  $\mathcal{M}$  can be translated into classical properties for the associated GAG.

**Definition 3.** *The map  $\mathcal{M}$  is said to be proper if  $\mathcal{M}(S) \subseteq S$  for each  $S \subseteq N$ ; it is said to be monotonic if  $\mathcal{M}(S) \subseteq \mathcal{M}(T)$  for each  $S, T$  such that  $S \subseteq T \subseteq N$ .*

Note that a map  $\mathcal{M}$  can be monotonic but not proper, or proper but not monotonic. An example of map  $\mathcal{M}$  which is not monotonic is the one relative to the glove game. Maps that are not proper will be seen later.

The following results are straightforward.

**Proposition 1.** *Let  $\langle N, v, \mathcal{M} \rangle$  be a GAS with  $v \in \mathbb{R}_+^N$  and  $\mathcal{M}$  monotonic. Then the associated GAG  $(N, v^{\mathcal{M}})$  is monotonic.*

**Proposition 2.** *Let  $\langle N, v, \mathcal{M} \rangle$  be a GAS with  $v \in \mathbb{R}_+^N$  and  $\mathcal{M}$  proper and monotonic. Then the associated GAG  $(N, v^{\mathcal{M}})$  is superadditive.*

*Proof.* Let  $S$  and  $T$  be two coalitions such that  $S \cap T = \emptyset$ . By properness it is  $\mathcal{M}(S) \cap \mathcal{M}(T) = \emptyset$ . By monotonicity it is

$$\mathcal{M}(S) \cup \mathcal{M}(T) \subseteq \mathcal{M}(S \cup T).$$

Thus, since  $v \in \mathbb{R}_+^N$ ,

$$v^{\mathcal{M}}(S \cup T) = \sum_{i \in \mathcal{M}(S \cup T)} v(i) \geq \sum_{i \in \mathcal{M}(S) \cup \mathcal{M}(T)} v(i) = v^{\mathcal{M}}(S) + v^{\mathcal{M}}(T).$$

□

Observe that Propositions 1 and 2 provide only sufficient conditions, for instance the glove game is monotonic and superadditive but the associated map  $\mathcal{M}$  is not monotonic.

The following Example shows that, if the map  $\mathcal{M}$  is proper and monotonic, the corresponding GAG needs not be convex.

**Example 5.** *Let  $N = \{1, 2, 3, 4\}$ ,  $v(i) > 0 \forall i \in N$  and let  $\mathcal{M}$  be such that  $\mathcal{M}(\{2\}) = \emptyset$ ,  $\mathcal{M}(\{2, 3\}) = \{3\}$  and  $\mathcal{M}(S) = S$  for all  $S \not\subseteq \{\{2\}, \{2, 3\}\}$ . Then  $\mathcal{M}$  is proper and monotonic but the corresponding GAG is not convex, since it holds that  $v^{\mathcal{M}}(S \cup T) + v^{\mathcal{M}}(S \cap T) < v^{\mathcal{M}}(S) + v^{\mathcal{M}}(T)$  for  $S = \{1, 2, 3\}$ ,  $T = \{2, 3, 4\}$ .*

The next Proposition shows that it is possible to provide sufficient conditions for a monotonic map  $\mathcal{M}$  to generate a convex GAG.

**Proposition 3.** *Let  $\langle N, v, \mathcal{M} \rangle$  be a GAS with  $v \in \mathbb{R}_+^N$  and  $\mathcal{M}$  such that*

$$\mathcal{M}(S) \cap \mathcal{M}(T) = \mathcal{M}(S \cap T), \quad (3.2)$$

for each  $S, T \in 2^N$ . Then the associated GAG  $(N, v^{\mathcal{M}})$  is convex.

*Proof.* The convexity condition for the GAG  $(N, v^{\mathcal{M}})$  can be written as follows:

$$\begin{aligned} \sum_{i \in \mathcal{M}(S \cup T)} v(i) + \sum_{i \in \mathcal{M}(S \cap T)} v(i) &\geq \sum_{i \in \mathcal{M}(S)} v(i) + \sum_{i \in \mathcal{M}(T)} v(i) \\ &= \sum_{i \in \mathcal{M}(S) \cup \mathcal{M}(T)} v(i) + \sum_{i \in \mathcal{M}(S) \cap \mathcal{M}(T)} v(i), \end{aligned} \quad (3.3)$$

for each  $S, T \in 2^N$ . It is easy to show that condition (3.2) implies monotonicity of  $\mathcal{M}$ . Then, we have that  $\mathcal{M}(S) \cup \mathcal{M}(T) \subseteq \mathcal{M}(S \cup T)$  and  $\mathcal{M}(S \cap T) \subseteq \mathcal{M}(S) \cap \mathcal{M}(T)$  for each  $S, T \in 2^N$ . Thus, if  $\mathcal{M}(S \cap T) = \mathcal{M}(S) \cap \mathcal{M}(T)$  for each  $S, T \in 2^N$ , then relation (3.3) holds. □

The condition provided by relation (3.2) can be useful to construct a monotonic map  $\mathcal{M}$  such that the corresponding GAG is convex when  $v \in \mathbb{R}_+^N$ . The most trivial example is the identity map  $\mathcal{M}(S) = S$  for each  $S \in 2^N$ . Another example is a map  $\mathcal{M}$  of a GAS  $\langle N, v, \mathcal{M} \rangle$  with  $N = \{1, 2, 3\}$  and  $v \in \mathbb{R}_+^N$  such that  $\mathcal{M}(\{1, 2, 3\}) = \{1, 2, 3\}$ ,  $\mathcal{M}(\{1, 2\}) = \{1, 2\}$ ,  $\mathcal{M}(\{2, 3\}) = \{2\}$ ,  $\mathcal{M}(\{2\}) = \{2\}$  and  $\mathcal{M}(\{1\}) = \mathcal{M}(\{3\}) = \mathcal{M}(\{1, 3\}) = \emptyset$ .

We conclude this Section with an example showing that relation (3.2) is not a necessary condition to have a convex GAG.

**Example 6.** *Let  $N = \{1, 2, 3\}$ ,  $v = (v(1), v(2), \alpha v(2))$  with  $v(1), v(2) \geq 0$  and  $\mathcal{M}$  be such that  $\mathcal{M}(\{1, 2, 3\}) = \{1, 2, 3\}$ ,  $\mathcal{M}(\{1, 2\}) = \{1, 2\}$ ,  $\mathcal{M}(\{2, 3\}) = \{2\}$  and  $\mathcal{M}(\{1\}) = \mathcal{M}(\{2\}) = \mathcal{M}(\{3\}) = \mathcal{M}(\{1, 3\}) = \emptyset$ . The map  $\mathcal{M}$  is monotonic, but relation (3.2) does not hold (to see this, just take  $S = \{1, 2\}$  and  $T = \{2, 3\}$ ). On the other hand, one can check that relation (3.3) holds for each  $S, T \in 2^N$  if and only if  $\alpha \geq 1$  (in particular note that for  $S = \{1, 2\}$  and  $T = \{2, 3\}$  relation (3.3) gives  $v(1) + v(2) + \alpha v(2) \geq v(1) + 2v(2)$ ).*

### 3.3 Basic GAGs

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We now define an interesting subclass of GASs. Consider a collection  $\mathcal{C} = \{\mathcal{C}_i\}_{i \in N}$ , where  $\mathcal{C}_i = \{F_i^1, \dots, F_i^{m_i}, E_i\}$  is a collection of subsets of  $N$  such that  $F_i^j \cap E_i = \emptyset$  for all  $i \in N$  and for all  $j = 1, \dots, m_i$ .

**Definition 4.** *We denote by  $\langle N, v, \mathcal{C} \rangle$  the basic GAS associated with the coalitional map  $\mathcal{M}$  defined, for all  $S \subseteq N$ , as:*

$$\mathcal{M}(S) = \{i \in N : S \cap F_i^1 \neq \emptyset, \dots, S \cap F_i^{m_i} \neq \emptyset, S \cap E_i = \emptyset\} \quad (3.4)$$

and by  $\langle N, v^{\mathcal{C}} \rangle$  the associated GAG, that we shall call basic GAG.

For simplicity of exposition, we assume w.l.o.g. that  $m_1 = m_2 = \dots = m_n := m$ . We shall call each  $F_i^k$ , for all  $i \in N$  and all  $k = 1, \dots, m$ , the  $k$ -th set of *friends* of  $i$ , while  $E_i$  is the set of *enemies* of  $i$ .

The basic GAG  $v^C$  associated with a basic GAS can be decomposed in the following sense: define the collection of  $n$  games  $v^{C_i}$ ,  $i = 1, \dots, n$ , as

$$v^{C_i}(S) = \begin{cases} v(i) & \text{if } S \cap E_i = \emptyset, S \cap F_i^k \neq \emptyset, k = 1, \dots, m \\ 0 & \text{otherwise.} \end{cases} \quad (3.5)$$

**Proposition 4.** *The basic GAG  $v^C$  associated with the map defined in (4) verifies:*

$$v^C = \sum_{i=1}^n v^{C_i}. \quad (3.6)$$

A particularly simple case is when every player has a unique set of friends, that we shall denote by  $F_i$ .

**Example 7.** (*airport games*) [63, 64] *Let  $N$  be the set of players. We partition  $N$  into groups  $N_1, N_2, \dots, N_k$  such that to each  $N_j$ ,  $j = 1, \dots, k$ , is associated a positive real number  $c_j$  with  $c_1 \leq c_2 \leq \dots \leq c_k$  (representing costs). Consider the game  $w(S) = \max\{c_i : N_i \cap S \neq \emptyset\}$ . This type of game (and variants) can be described by a basic GAS  $\langle N, (C_i = \{F_i, E_i\})_{i \in N}, v \rangle$  by setting for each  $i \in N_j$  and each  $j = 1, \dots, k$ :*

- the value  $v(i) = \frac{c_j}{|N_j|}$ ,
- the set of friends  $F_i = N_j$ ,

*and the set of enemies  $E_i = N_{j+1} \cup \dots \cup N_k$  for each  $i \in N_j$  and each  $j = 1, \dots, k-1$  and  $E_l = \emptyset$  for each  $l \in N_k$ .*

**Example 8.** (*top- $k$  nodes problem*) [1, 94] *Let  $(N, E)$  be a co-authorship network, where nodes represent researchers and there exists an edge between two nodes if the corresponding researchers have co-authored in a paper. Given a value  $k$ , the top- $k$  nodes problem consists in the search for a set of  $k$  researchers who have co-authored with the maximum number of other researchers. The problem, introduced in [94], is formalized as follows. For any  $S \subseteq N$ , we define the function  $g(S)$  as the number of nodes that are adjacent to nodes in the set  $S$ . Given a value  $k$ , the problem of finding a set  $S$  of cardinality  $k$  such that  $g(S)$  attains maximum value is NP-hard [94]. Therefore, in [94] and later in [1] a slightly different problem is studied through a game-theoretical approach, by using the Shapley value of a properly defined cooperative game as a measure of the importance of nodes in the network. In the corresponding game  $(N, w)$ , the worth of a coalition  $S$ , for each  $S \subseteq N$ ,  $S \neq \emptyset$ , is equal to the number of nodes that are connected to nodes in  $S$  via, at most, one edge. Formally,  $w(S) = |S \cup \bigcup_{i \in S} N_i(E)|$ , where  $N_i(E)$  is the set of neighbours of  $i \in N$  in the network  $(N, E)$ . It is easy to check that game  $w$  can be described as a basic GAS  $\langle N, v, (C_i = \{F_i, E_i\})_{i \in N} \rangle$ , where  $v(i) = 1$ ,  $F_i = \{i\} \cup N_i(E)$  and  $E_i = \emptyset \forall i \in N$ .*

**Example 9.** (*argumentation games*) [16]: *Consider a directed graph  $\langle N, \mathcal{R} \rangle$ , where the set of nodes  $N$  is a finite set of arguments and the set of arcs  $\mathcal{R} \subseteq N \times N$  is a binary defeat (or attack) relation (see Dung 1995). For each argument  $i$  we define the set of attackers of  $i$  in  $\langle N, \mathcal{R} \rangle$  as the set  $P(i) = \{j \in N : (j, i) \in \mathcal{R}\}$ . The meaning is*

the following:  $N$  is a set of arguments, if  $j \in P(i)$  this means that argument  $j$  attacks argument  $i$ . The value of a coalition  $S$  is the number of arguments in the opinion  $S$  which are not attacked by another argument of  $S$ . This type of game (and variants) can be described as a basic GAS  $\langle N, v, \{F_i, E_i\} \rangle$  by setting  $v(i) = 1$ , the set of friends  $F_i = \{i\}$  and the set of enemies  $E_i = P(i)$ . This example still falls in the setting of basic GAGs where each player has only one set of friends. However, there are also different, and natural as well, types of characteristic functions that can be considered. For instance, it is interesting to consider the game  $(N, v^{\mathcal{M}})$  such that for each  $S \subseteq N$ ,  $v^{\mathcal{M}}(S)$  is the sum of  $v(i)$  over the elements of the set  $D(S) = \{i \in N : P(i) \cap S = \emptyset \text{ and } \forall j \in P(i), P(j) \cap S \neq \emptyset\}$  of arguments that are not internally attacked by  $S$  and at the same time are defended by  $S$  from external attacks:

$$v^{\mathcal{M}}(S) = \sum_{i \in D(S)} v(i). \quad (3.7)$$

It is clear that such a situation cannot be described by a basic GAG where each player has a unique set of friends. The game in (3.7) can however be described as a basic GAG  $\langle N, v^{\mathcal{C}} \rangle$ , where, given a bijection  $k : P(i) \rightarrow \{1, \dots, |P(i)|\}$ ,  $\mathcal{C}_i = \{F_i^1, \dots, F_i^{|P(i)|}, E_i\}$  is such that  $F_i^{k(j)} = P(j) \setminus P(i)$  for all  $j \in P(i)$ , and  $E_i = P(i)$  for all  $i \in N$ .

Moreover, the following Examples show that the unanimity and canonical games can be represented as basic GAGs with more than one set of friends.

**Example 10.** (unanimity games) Let  $S = \{s_1, \dots, s_s\} \subseteq N$ . Consider the unanimity game  $(N, u_S)$  defined as

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise.} \end{cases}$$

This game can be described by a basic GAS  $\langle N, v, \{F_i^1, \dots, F_i^s, E_i\} \rangle$  by setting  $v(i) = 1$  for some  $i \in S$ ,  $v(j) = 0 \forall j \neq i$ ,  $F_i^k = \{s_k\} \forall k = 1, \dots, s$  and  $\forall i \in N$ , and  $E_i = \emptyset \forall i \in N$ .

**Example 11.** (canonical games) Let  $S \subseteq N$ . Consider the canonical game  $(N, e_S)$  defined as

$$e_S(T) = \begin{cases} 1 & \text{if } S = T \\ 0 & \text{otherwise.} \end{cases}$$

This game can be described as a basic GAS  $\langle N, v, \{F_i^1, \dots, F_i^s, E_i\} \rangle$  by setting  $v(i) = 1$  for some  $i \in S$ ,  $v(j) = 0 \forall j \neq i$ ,  $F_i^k = \{s_k\} \forall k = 1, \dots, s$  and  $\forall i \in N$ , and  $E_i = N \setminus S$ .

We now provide further examples of classes of coalitional games from the literature which can be represented as basic GAGs, where in general each player can have several sets of friends.

### 3.3.1 Peer games

In *peer games* [19] over a player set  $N$ , the economic relationships among players are represented by a hierarchy described by a directed rooted tree  $T$  with  $N$  as the set of nodes and with 1 as the root (representing the leader of the entire group). The individual features are agents' potential economic possibilities, described by a vector  $a \in \mathbb{R}^N$ , where  $a_i$  is the gain that player  $i$  can generate if all players at an upper level in the hierarchy cooperate with him. In other words, player  $i$  becomes effective and may produce a gain  $a_i$  only if his superiors cooperate with him.

For every  $i \in N$ , we denote by  $S(i)$  the set of all agents in the unique directed path connecting 1 to  $i$ , i.e. the set of superiors of  $i$ . Given a *peer group situation*  $(N, T, a)$  as described above, a *peer game* is defined as the game  $(N, v^P)$  such that for each non-empty coalition  $S \subseteq N$

$$v^P(S) = \sum_{i \in N: S(i) \subseteq S} a_i.$$

A peer game  $(N, v^P)$  can be represented as the GAG associated to the basic GAS on  $N$  where  $v(i) = a_i$  and where  $\mathcal{M}$  is defined by relation (3.4) with collections  $\mathcal{C}_i = \{F_i^1, \dots, F_i^n, E_i\}$  such that:

$$F_i^j = \begin{cases} \{j\} & \text{if } j \in S(i) \\ \{i\} & \text{otherwise} \end{cases}$$

and  $E_i = \emptyset$  for all  $i \in N$ .

An interesting example of peer games [19] (and indeed of GAGs) are coalitional games arising from sealed bid second price auctions, where there is a seller who wish to sell an object at price not smaller than a given  $r > 0$  (reservation price). Each player  $i \in N$  has his own evaluation  $w_i$  of the object and can submit a bid  $b_i$  in an envelope (not necessarily equal to  $w_i$ ). The mechanism of the auction is such that after the opening of the envelopes, the object is given to the player with the highest bid at the second highest price<sup>1</sup>. Suppose that  $w_1 > w_2 > \dots > w_n \geq r$ . It is easy to check that in such a situation, a dominant strategy for each player  $i$  who acts alone (i.e., without colluding with the other players) is to bid his own value  $b_i = w_i$ . This leads to a situation where player 1 obtains the object at the price  $w_2$ , so the players' payoffs are  $v(1) = w_1 - w_2$  and  $v(i) = 0$  if  $i \neq 1$ .

Now, consider the possibility of collusion among the players, which means that players may form coalitions and agree on the bid each player should put in the respective envelope. For a coalition  $S$ , the dominant strategy is that the player  $i(S) \in N$  with the highest evaluation in  $S$  bids  $w_{i(S)}$ , and the other players in  $S$  bid  $r$ , the reservation price. In this way, if all players collude, the worth of coalition  $N$  is  $v(N) = w_1 - r$ . In general, for every coalition  $S \subseteq N$ , we have that  $v(S) = 0$  if  $1 \notin S$  (it is still dominant for players in  $N \setminus S$  to play their true evaluation, and then  $1 \in N \setminus S$  gets the object), and  $v(S) = w_1 - w_{i(N \setminus S)}$  if  $1 \in S$ , where  $i(N \setminus S)$  is the player with highest evaluation in  $N \setminus S$ . Such a game  $(N, v)$  can be seen as the GAG associated to the basic GAS with collections  $\mathcal{C}_i = (F_i^1, \dots, F_i^i, E_i)$  where  $F_i^j = \{j\}$  for every  $j$  in  $\{1, \dots, i\}$ ,  $E_i = \emptyset$  and  $v(i) = w_i - w_{i+1}$  for every  $i \in N$ .

1. We do not consider here the case where players may submit equal bids.

### 3.3.2 Mountain situations

Consider a special version of a directed minimum cost spanning tree situation, introduced in [73], characterized by a group  $N$  of persons whose houses lie on mountains and are not yet connected to a water purifier downhill. It is possible but not necessary for every person (house) to be connected directly with the water purifier; being connected via others is sufficient. Only connections from houses to strictly lower ones are allowed. Such a situation can be represented by a rooted directed graph  $\langle N \cup \{0\}, A \rangle$  with  $N \cup \{0\}$  as set of vertices,  $A \subset N \times (N \cup \{0\})$  as set of edges and 0 as the root, and a weight function  $w : A \rightarrow \mathbb{R}^+$ , representing the cost associated to each edge. We assume that for each  $k \in N$ ,  $(k, 0) \in A$  (i.e., every node has the possibility to be directly connected with the source) and in order to impose the fact that only connection to lower houses are possible, no cycles are allowed.

Given such a mountain situation, the corresponding cooperative cost game  $(N, c)$  is given by  $c(\emptyset) = 0$  and the cost  $c(T) = \sum_{a \in \Gamma(T)} w(a)$  of a non-empty coalition  $T$  is the cost of an optimal 0-connecting tree  $\Gamma(T) \subseteq A(T)$  in the mountain problem on the directed graph  $\langle T \cup \{0\}, A(T) \rangle$ , i.e.  $\Gamma(T)$  is a tree of minimum cost connecting all players in  $T$  to the source 0.

It can be checked that for each optimal 0-connecting tree  $\Gamma(T) \subseteq A(T)$ , each node  $i \in T$  is directly connected with his best connection in  $T \cup \{0\}$ , that is a node  $b_T(i) \in \arg \min_{l \in T \cup \{0\}: (i,l) \in A} w(i, l)$  (see [71] for further details).

Now, assume that for each  $a \in A$ ,  $w(a)$  can assume only two values, let's say  $m$  or 0, with  $m > 0$ , and let  $B_i = \{j \in N : w(i, j) = 0\}$  the set of best connections for  $i \in N$  (actually, every mountain situation can be decomposed as a sum of simple dichotomous mountain situations like that). We can represent the cost game  $(N, c)$  as the GAG associated to the basic GAS on  $N$  where  $v(i) = 0$  if  $w(i, 0) = 0$ , and  $v(i) = m$  otherwise, and where  $\mathcal{M}$  is defined by relation (3.4) with collections  $\mathcal{C}_i = \{F_i, E_i\}$  such that  $F_i = \{i\}$  and  $E_i = B_i$  for every  $i \in N$ .

### 3.3.3 Maintenance cost games

A *maintenance problem* [17,58] arises when a group of players  $N$  is connected by a tree  $T$  (e.g., a computer network) to a root 0 (e.g., a service provider) and each edge of the tree has an associated maintenance cost; the problem is how to share in a fair way the cost of the entire network  $T$  among the players in  $N$ . More formally, a couple  $(T, t)$  is given, where  $T = (N \cup \{0\}, E)$  is a tree.  $N \cup \{0\}$  represents the set of vertices (or nodes) and  $E$  is the set of edges, i.e. the pairs  $\{i, j\}$  such that  $i, j \in N$ . 0 is the root of the tree having only one adjacent edge, and  $t : E \rightarrow \mathbb{R}^+$  is a non-negative cost function on the edges of the tree. Note that each vertex  $i \in N$  is connected to the root 0 by a unique path  $P_i$ ; we shall denote by  $e_i$  the edge in  $P_i$  that is incident to  $i$ . A *precedence relation*  $\preceq$  is defined by:  $j \preceq i$  if and only if  $j$  is on the path  $P_i$ . A *trunk*  $R \subseteq N \cup \{0\}$  is a set of vertices which is closed under the relation  $\preceq$ , i.e. if  $i \in R$  and  $j \preceq i$ , then  $j \in R$ . The set of *followers* of player  $i \in N$  is given by  $F(i) = \{j \in N | i \preceq j\}$  (note that  $i \in F(i)$  for each  $i \in N$ ). The cost of a trunk  $R$  is then defined as

$$C(R) = \sum_{i \in R \setminus \{0\}} t(e_i),$$

and the associated *maintenance cost game*  $(N, c)$  is defined by

$$c(S) = \min\{C(R) : S \subseteq R \text{ and } R \text{ is a trunk}\}. \quad (3.8)$$

Note that edge  $e_i$  is present in the cheapest trunk containing all members of  $S$  whenever a member of  $S$  is a follower of player  $i$ , i.e.  $S \cap F(i) \neq \emptyset$ . Therefore, we can represent the cost game  $(N, c)$  as the GAG associated to the basic GAS on  $N$  where  $v(i) = t(e_i)$  and where  $\mathcal{M}$  is defined by relation (3.4) with collections  $\mathcal{C}_i = \{F_i, E_i\}$  such that  $F_i = F(i)$  and  $E_i = \emptyset$  for every  $i \in N$ .

In a basic GAG, the worth of each coalition is computed as the sum of the individual values of a subset of players. On the other hand, in many of the aforementioned classes of games, when an underlying network describes the interaction among the players involved, it is possible to derive the worth of each coalition of players as the sum of the contributions that their pairwise interactions generate, that is as the sum of the individual values assigned to the edges in the underlying network. In the previous example, a *maintenance cost game* derives from a situation where a tree describes a maintenance system such as a computer network, with a service provider as root. The cost of connection of a set of computers to the provider is described by the coalitional game defined in (3.8) and computed as the sum of the costs of maintenance of all the connections among the computers lying on the corresponding minimum cost spanning tree, that is as the sum of the costs associated to the edges in the induced tree.

Formally, this idea may be described by defining a *link-based GAG*, corresponding to a *link-based GAS*  $\langle N, w, \mathcal{L} \rangle$ , where  $N = \{1, \dots, n\}$ , represent the set of nodes in a network  $\langle N, E \rangle$ ,  $w : E \rightarrow \mathbb{R}$  assigns a value to each edge in the graph and  $\mathcal{L} : 2^E \rightarrow 2^E$  assigns to each subset  $A \subseteq E$  of edges another subset  $\mathcal{L}(A) \subseteq E$ . Then the corresponding *link-based GAG*  $(N, v^{\mathcal{L}})$  would be defined by the following expression, for every  $S \subseteq N$ :

$$v^{\mathcal{L}}(S) = \sum_{e \in \mathcal{L}(E_S)} w(e),$$

where  $E_S$  are the edges in the graph induced by  $S$ .

Similarly, within this formalism, a *link-based basic GAS* would be defined as the triple  $\langle N, w, \mathcal{L} \rangle$  associated to the map

$$\mathcal{L}(A) = \{e \in E : A \cap F_e^1 \neq \emptyset, \dots, A \cap F_e^{m_e} \neq \emptyset, A \cap E_e = \emptyset\}$$

for every  $A \subseteq E$ , where  $\mathcal{C}_e = \{F_e^1, \dots, F_e^{m_e}, E_e\}$  is a collection of subsets of  $E$  such that  $F_e^j \cap E_e = \emptyset$  for all  $e \in N$  and for all  $j = 1, \dots, m_e$ .

Then a *link-based basic GAG*  $\langle N, w^{\mathcal{L}} \rangle$  would be defined as the associated GAG.

In the aforementioned example, a maintenance cost game  $(N, c)$  can be defined as the link-based basic GAG on  $N$ , where  $w(e) = t(e)$  is the cost associated to each edge  $e \in E$  in the tree, and the map  $\mathcal{L}$  is defined as above, with  $E_e = \emptyset$  and  $F_e = F(e)$  for every edge  $e \in E$ , where  $F(e) \subseteq E$  is the set of followers of edge  $e$ , which definition is the equivalent to the one given in the previous example for nodes.

The same formalism can be employed to describe the other classes of graph games presented in this Section. However, in what follows, and in Chapter 4, where we provide some theoretical results concerning solution concepts for the class of GAGs, we

shall always refer to the original model of basic GAGs, since it seems easier to deal with.

### 3.4 A characterization of basic GASs

As it has been shown in the previous sections, a variety of classes of games that have been widely investigated in the literature can be described using the formalism provided by basic GASs. Moreover, as we shall see in the next sections, it is possible to produce, for basic GAGs, results concerning important solution concepts, like the core and the semivalues. It is therefore interesting to study under which conditions a GAS can be described as a basic one. To this purpose, the following theorem provides a necessary and sufficient condition when the set of enemies of each player is empty.

**Theorem 1.** *Let  $\langle N, v, \mathcal{M} \rangle$  be a GAS. The map  $\mathcal{M}$  can be obtained by relation (3.4) via collections  $\mathcal{C}_i = \{F_i^1, \dots, F_i^{m_i}, E_i = \emptyset\}$ , for each  $i \in N$ , if and only if  $\mathcal{M}$  is monotonic.*

*Proof.* It is obvious that every map  $\mathcal{M}$  obtained by relation (3.4) over a collections  $\mathcal{C}_i = \{F_i^1, \dots, F_i^{m_i}, E_i = \emptyset\}$ , for each  $i \in N$ , is monotonic. Now, consider a monotonic map  $\mathcal{M}$  and, for each  $i \in N$ , define the set  $\mathcal{M}_i^{-1} = \{S \subseteq N : i \in \mathcal{M}(S)\}$  of all coalitions whose image in  $\mathcal{M}$  contains  $i$ . Let  $\mathcal{S}^{\mathcal{M},i}$  be the collection of minimal (with respect to set inclusion) coalitions in  $\mathcal{M}_i^{-1}$ , formally:

$$\mathcal{S}^{\mathcal{M},i} = \{S \in \mathcal{M}_i^{-1} : \text{it does not exist } T \in \mathcal{M}_i^{-1} \text{ with } T \subset S\}.$$

For each  $i \in N$ , consider the collection  $\mathcal{C}_i = \{F_i^1, \dots, F_i^{m_i}, E_i = \emptyset\}$  such that

$$\{F_i^1, \dots, F_i^{m_i}\} = \{T \subseteq N : |T \cap S| = 1 \forall S \in \mathcal{S}^{\mathcal{M},i} \text{ and } |T| \leq |\mathcal{S}^{\mathcal{M},i}|\}, \quad (3.9)$$

where each set of friends  $F_i^k$ ,  $k \in \{1, \dots, m_i\}$ , contains precisely one element in common with each coalition in  $S \in \mathcal{S}^{\mathcal{M},i}$  and no more than  $|\mathcal{S}^{\mathcal{M},i}|$  elements. Denote by  $\mathcal{M}^*$  the map obtained by relation (3.4) over such collections  $\mathcal{C}_i$ ,  $i \in N$ . We need to prove that  $\mathcal{M}(S) = \mathcal{M}^*(S)$  for each  $S \in 2^N$ ,  $S \neq \emptyset$ . First note that for each  $i \in N$  and for every coalition  $S \in \mathcal{M}_i^{-1}$ , we have  $i \in \mathcal{M}^*(S)$ . Let us prove now that  $i \notin \mathcal{M}^*(S)$  for each  $S \notin \mathcal{M}_i^{-1}$ . Suppose, by contradiction, that there exists  $T \subseteq N$  with  $T \notin \mathcal{M}_i^{-1}$  such that  $F_i^k \cap T \neq \emptyset$ , for each  $k \in \{1, \dots, m_i\}$ . Consequently, by the definition of  $\mathcal{S}^{\mathcal{M},i}$ , we have that for every  $S \in \mathcal{S}^{\mathcal{M},i}$ ,  $S \setminus T \neq \emptyset$ . Define a coalition  $U \subseteq N$  of no more than  $|\mathcal{S}^{\mathcal{M},i}|$  elements and such that  $U$  contains precisely one element of  $S \setminus T$  for each  $S \in \mathcal{S}^{\mathcal{M},i}$ , i.e.  $|U \cap (S \setminus T)| = 1$  for each  $S \in \mathcal{S}^{\mathcal{M},i}$  and  $|U| \leq |\mathcal{S}^{\mathcal{M},i}|$ . By relation (3.9),  $U$  must be a set of friends in the collection  $\{F_i^1, \dots, F_i^{m_i}\}$ , which yields a contradiction with the fact that  $U \cap T = \emptyset$ . It follows that for each  $i \in N$ ,  $i \in \mathcal{M}^*(S)$  if and only if  $i \in \mathcal{M}(S)$  for each  $S \subseteq N$ , which concludes the proof.  $\square$

Based on the arguments provided in the proof of Theorem 1, the following Example shows a procedure to represent a GAS with a monotonic map  $\mathcal{M}$  as a basic GAS.

**Example 12.** *Consider a GAS  $\langle N, v, \mathcal{M} \rangle$  with  $N = \{1, 2, 3, 4\}$  and  $\mathcal{M}$  such that  $\mathcal{M}(\{1, 2, 3\}) = \{3\}$ ,  $\mathcal{M}(\{3, 4\}) = \{2, 3\}$ ,  $\mathcal{M}(\{2, 3, 4\}) = \{2, 3\}$ ,  $\mathcal{M}(\{1, 3, 4\}) =$*

$\{2, 3, 4\}$ ,  $\mathcal{M}(N) = \{2, 3, 4\}$ , and  $\mathcal{M}(S) = \emptyset$  for all other coalitions. The sets of minimal coalitions are  $\mathcal{S}^{\mathcal{M},1} = \emptyset$ ,  $\mathcal{S}^{\mathcal{M},2} = \{\{3, 4\}\}$ ,  $\mathcal{S}^{\mathcal{M},3} = \{\{1, 2, 3\}, \{3, 4\}\}$ ,  $\mathcal{S}^{\mathcal{M},4} = \{\{1, 3, 4\}\}$ . Such a map can be represented via relation (3.4), where  $F_1^1 = \emptyset$ ,  $\{F_2^1, F_2^2\} = \{\{3\}, \{4\}\}$ ,  $\{F_3^1, \dots, F_3^3\} = \{\{1, 4\}, \{2, 4\}, \{3\}\}$ ,  $\{F_4^1, \dots, F_4^3\} = \{\{1\}, \{3\}, \{4\}\}$  and such collections of friends are obtained via relation (3.9).

The following proposition characterizes monotonic basic GAGs.

**Proposition 5.** *Let  $\langle N, v, \mathcal{C} \rangle$  be a basic GAS with  $v \in \mathbb{R}_+^N$  and  $\mathcal{C} = \{\mathcal{C}_i\}_{i \in N}$ . Then the associated GAG  $(N, v^{\mathcal{C}})$  is monotonic if and only if  $E_i = \emptyset \forall i \in N$ .*

*Proof.* The sufficient condition is obvious. Moreover, suppose  $E_i \neq \emptyset$  for some  $i$  and let  $j \in E_i$ . Consider  $S = F_i^1 \cup \dots \cup F_i^m$ . Then  $i \in \mathcal{M}(S)$ , while  $i \notin \mathcal{M}(S \cup j)$ .  $\square$

By Proposition 1, Theorem 1 and Proposition 5 we have the following corollary.

**Corollary 1.** *Let  $\langle N, v, \mathcal{M} \rangle$  be a basic GAS with  $v \in \mathbb{R}_+^N$ . Then the associated basic GAG  $(N, v^{\mathcal{C}})$  is monotonic if and only if  $\mathcal{M}$  is monotonic.*

### 3.5 Possible extensions

In this Section, we show that further extensions can be introduced, by generalizing the idea of coalitional map, in order to embrace a wider range of games that can be represented in a compact way into our framework.

The definition of GAG is based on the coalitional map  $\mathcal{M}$ , which is a multimap that assigns a coalition  $\mathcal{M}(S) \subseteq N$  to each coalition  $S \subseteq N$  of players. A further generalization is possible: think of the *graph-restricted game* introduced by Myerson [75], where the worth of a coalition is evaluated on the connected components induced by an underlying graph. This class of games can be represented by considering a multimap  $\mathcal{M} : 2^N \rightarrow 2^{2^N}$ , which assigns to each coalition  $S \subseteq N$ , a subset  $\mathcal{M}(S) \subseteq 2^N$ .

Analogously, the definition of a basic GAG is based on a collection of sets  $\mathcal{C} = \{\mathcal{C}_i\}_{i \in N}$ , one for each player, where  $\mathcal{C}_i = \{F_i^1, \dots, F_i^m, E_i\}$  is a collection of subsets of  $N$  that satisfy some particular properties.

If we provide each player  $i$  with multiple collections  $\{\mathcal{C}_i^1, \dots, \mathcal{C}_i^k\}$ , with  $\mathcal{C}_i^k = \{F_i^{k1}, \dots, F_i^{km}, E_i^k\}$ , we are then able to represent those games that are associated to marginal contribution nets (MC-nets), introduced in [50] (see also [25]).

The basic idea behind marginal contribution nets is to represent in a compact way the characteristic function of a game, as a set of rules of the form: *pattern*  $\rightarrow$  *value*, where a *pattern* is a Boolean formula over a set of  $n$  variables (one for each player) and a *value* is a real number. Here we restrict our attention to Boolean formulas that are conjunctions of literals, i.e. variables or their negations, and we shall call the corresponding rule a *basic rule*. A basic rule is said to *apply* to a coalition  $S$  if  $S$  contains all players whose variables appear unnegated in the pattern (represented by the literal  $x_i$ ), and does not contain any of the players that appear negated (represented by the literal  $\neg x_i$ ). For example, a rule with pattern  $x_1 \wedge x_2 \wedge \neg x_3$  applies to the coalition  $\{1, 2\}$  and  $\{1, 2, 3, 4\}$  but not to the coalitions  $\{1\}$  or  $\{1, 2, 3\}$ .

More formally, consider a collection of rules  $R = \{r^1, \dots, r^m\}$ , where  $r^k = \psi^k \rightarrow x^k$  for each  $k = 1, \dots, m$ , with  $\psi^k$  is being a basic rule over  $\{x_1, \dots, x_n\}$  and  $x^k \in \mathbb{R}$

for all  $k$ . A coalition  $S$  is said to *satisfy*  $\psi$  (and we write  $S \models \psi$ ) if and only if  $r : \psi \longrightarrow x$  applies to  $S$ . The set  $R$  defines a coalitional game  $(N, v^R)$ , introduced by Jeong and Shoham (2005), where  $N = \{1, \dots, n\}$  and the value of a coalition is computed by summing the values of all the rules that apply to it, i.e.  $v^R$  is given by

$$v^R(S) = \sum_{r^i \in R: S \models \psi^i} x^i.$$

As an example, consider the following MC-net:

$$\begin{aligned} r^1 &: x_1 \wedge x_2 \wedge \neg x_3 \longrightarrow 3 \\ r^2 &: x_2 \longrightarrow 4 \end{aligned}$$

The corresponding game is  $(N, v)$ , with  $N = \{1, 2, 3\}$  and  $v = 3e_{\{1,2\}} + 4u_{\{2\}}$ , where  $e_S$  and  $u_S$  are, respectively, the canonical and the unanimity game on  $S$ .

Indeed, every coalitional game can be represented through MC-nets by defining one rule for each coalition  $S \subseteq N$ , where the pattern contains all the variables corresponding to the players in  $S$  and the negation of all the other variables, and the corresponding value is equal to the value of  $S$  in the game.

We show here how a game deriving from a MC-nets representation can be described as a generalization of a basic GAG, where each player has multiple collections of set of friends and enemies, one for each rule, and is assigned a vector of values. For each rule  $r^k = \psi^k \longrightarrow x^k$  such that the variable  $x_i$  appears unnegated in  $\psi^k$ , we provide player  $i$  with a collection of sets of friends and enemies  $\mathcal{C}_i^k = \{F_i^{k1}, \dots, F_i^{km_k}, E_i^k\}$ , defined as follows:  $F_i^{kj} = \{j\}$  if  $x_j$  appears unnegated in  $\psi^k$  and  $E_i^k = \{j \in N \text{ such that } x_j \text{ appears negated in } \psi^k\}$ , where  $m_k$  is the number of variables that appear unnegated in the pattern  $\psi^k$ . Moreover, for each player  $i$  we define a vector of values  $v(i) = \{v_1(i), \dots, v_m(i)\}$ , where  $v_k(i) = 0$  if  $x_i$  appears negated or does not appear in  $\psi^k$  and, for each rule  $r^k$  where  $x_i$  appears unnegated,  $v_k(i)$  is given by:

$$v_k(i) = \frac{x^k}{c_k},$$

where  $c_k$  is the number of players whose corresponding variables appear unnegated in rule  $r^k$ . With the aforementioned definitions, a MC-nets game  $v^R$  can be described as

$$v^R = \sum_{i \in N} v^{C_i^R},$$

where  $v^{C_i^R}$  is defined as follows for every  $i \in N$ :

$$v^{C_i^R}(S) = \begin{cases} 0 & \text{if } S \cap F_i^{kj} = \emptyset \forall j, k \text{ or } S \cap E_i^k \neq \emptyset \forall k \\ \sum_{k=1}^m v_k(i) & \text{otherwise.} \end{cases}$$

For the previous example, the collections of friends and enemies would be defined as:

$$\begin{aligned} \mathcal{C}_1^1 &= \{F_1^{11} = \{1\}, F_1^{12} = \{2\}, E_1^1 = \{3\}\} \\ \mathcal{C}_2^1 &= \{F_1^{21} = \{1\}, F_1^{22} = \{2\}, E_1^2 = \{3\}\} \\ \mathcal{C}_2^2 &= \{F_2^{21} = \{2\}, E_2^2 = \emptyset\} \end{aligned}$$

Moreover,  $v(1) = (\frac{3}{2}, 0)$ ,  $v(2) = (\frac{3}{2}, 4)$  and  $v(3) = (0, 0)$ .

In this way, we are indeed able to describe every TU-game, since the representation of MC-nets is complete. The computational complexity of such representation is in general high. However, when a game can be described by a small collection of rules, and therefore the associated extended GAS is described in a relatively compact way, the complexity of its representation and of the computation of solutions is consequently reduced.



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## Solutions for Generalized Additive Games

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In Chapter 3, the class of *Generalized Additive Games* (GAGs) is introduced, where the worth of a coalition  $S \subseteq N$  is evaluated by means of an interaction filter, that is a map  $\mathcal{M}$  which returns the valuable players involved in the cooperation among players in  $S$ . Well-known classes of coalitional games are embraced by this model and several of them can be described in terms of *basic* GAGs, where the filter  $\mathcal{M}$  selects, for each coalition  $S$ , those players that have *friends* and not *enemies* in  $S$ . This model turns out to be suitable for representing an online social network, where friends and enemies of the web users are determined by their social profiles. The objective of this Chapter is to investigate the problem of computing solution concepts for this subclass of coalitional games. In particular, we address the problem of how to guarantee that a basic GAG has a non-empty core and we provide formulas for the semivalues for some families of basic GAGs.

### 4.1 Introduction

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It is well known that the problem of identifying influential users on a social networking web site plays a key role to find strategies aimed to increase the site's overall view [99]. The main issue is to target advertisement to the site members of the online social network whose activities' levels have a significant impact on the activity of the other site members. The overall influence of a user can be seen as the combination of two important ingredients: 1) the individual ability to get the attention of other site members, and 2) the personal characteristic of the social profile, that can be represented in terms of groups or communities to which users belong.

In Chapter 3, a new class of games, called *Generalized Additive Games*, has been introduced. Given a finite set of players  $N$  and a map  $v$  assigning to each element of  $N$  an individual value, a GAG on  $N$  assigns to each coalition  $S \subseteq N$  a worth that

is computed according to an interaction filter  $\mathcal{M}$ . More precisely, the interaction filter  $\mathcal{M}(S)$  specifies the valuable players involved in the cooperation among players in  $S$ ; in other words,  $\mathcal{M}(S)$  specifies which players of  $N$  are somehow “obliged” to contribute to the worth of  $S$ , and the worth of a coalition  $S$ , for every  $S \subseteq N$ , is computed as the sum of the individual values of players in  $\mathcal{M}(S)$ .

In this Chapter, we further investigate a particular class of GAGs, namely the *basic* GAGs, that seem to well represent an online social network as described at the beginning of this Section. According to this model, the players (e.g., members of a social network) are provided with a utility value that may represent their individual activity in a social networking web site (for instance, measured in terms of the productive time spent in uploading content files), and the participation of each player to the global activity of the social network is based on a coalitional structure of friends and enemies that is determined by their social profiles.

In the framework of coalition formation using hedonic games, in [34] a model where each player divides other players into friends and enemies is studied. In [34], the preferences over coalitions are based on the appreciation of friends and the aversion to enemies. Despite the analogy of using coalitions whose formation is based on the presence of friends and enemies of each player, in our model we deal with the problem of measuring the power of players in particular coalitional situations, and how to share a joint utility value when the grand coalition forms, and not specifically with the coalition formation problem.

The main objective of this Chapter is to analyse classical solutions from coalitional game theory, like the well known Shapley value [88], the Banzhaf value [6] and other semivalues [36], aimed at measuring the power (or influence) of players in basic GAGs. We also address the problem of how to guarantee that a basic GAG has a non-empty core.

With the aim of providing a tool for the analysis of a wide range of coalitional games, in Section 4.2 we analyse classical solutions from coalitional game theory, like the well known Shapley value [88], the Banzhaf Value [6] and other semivalues [36], while in Section 4.3 we address the problem of how to guarantee that a basic GAG has a non-empty core. Section 4.4 concludes.

## 4.2 Semivalues and GAGs

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In this section we focus on some formulas for the semivalues, in the context of basic GAGs. Since we are interested in evaluating additive power indices for the players in basic GAGs, it becomes interesting to evaluate the indices on the games  $v^{\mathcal{C}_i}$  defined via relation (3.5).

Indeed, we recall here that the basic GAG  $v^{\mathcal{C}}$  associated with the basic GAS  $\langle N, v, \mathcal{C} \rangle$ , where  $\mathcal{C} = \{\mathcal{C}_i\}_{i \in N}$ ,  $\mathcal{C}_i = \{F_i^1, \dots, F_i^m, E_i\} \forall i \in N$ , can be decomposed as the sum of  $n$  games:

$$v^{\mathcal{C}} = \sum_{i=1}^n v^{\mathcal{C}_i}, \quad (4.1)$$

where  $v^{\mathcal{C}_i}$  is defined for  $i = 1, \dots, n$  as

$$v^{\mathcal{C}_i}(S) = \begin{cases} v(i) & \text{if } S \cap E_i = \emptyset, S \cap F_i^k \neq \emptyset, k = 1, \dots, m \\ 0 & \text{otherwise.} \end{cases}$$

First of all, we present some results concerning the Shapley and Banzhaf values on interesting subclasses of basic GAGs, where each player has at most two sets of friends. Furthermore, we extend our analysis to a generic basic GAG, with multiple sets of friends.

Let us consider the basic GAS  $\langle N, v, \{\mathcal{C}_i = \{F_i^1 = \{i\}, F_i^2, E_i\}\}_{i \in N} \rangle$ . We recall that considering the set of friends  $F_i^1 = \{i\}$  for all  $i \in N$  is equivalent to imposing the properness property of the associated map  $\mathcal{M}$ . Therefore, letting  $F_i := F_i^2 \forall i \in N$ , the resulting basic GAG is defined as:

$$v^{\mathcal{C}}(S) = \sum_{i \in S, S \cap F_i \neq \emptyset, S \cap E_i = \emptyset} v(i), \quad (4.2)$$

that can be decomposed according to (4.1), where:

$$v^{\mathcal{C}_i}(S) = \begin{cases} v(i) & \text{if } i \in S, S \cap F_i \neq \emptyset, S \cap E_i = \emptyset \\ 0 & \text{otherwise.} \end{cases} \quad (4.3)$$

In what follows, in order to simplify the notations, we fix  $i \in N$  and denote by  $f$  the cardinality of  $F_i$  and by  $e$  the cardinality of  $E_i$  (in order to simplify the notations, if  $E_i = \emptyset$  we assume by convention that  $e = 0$  and  $\frac{1}{e} = 0$ ).

When  $i \notin F_i$ , the following proposition holds.

**Proposition 6.** *Let  $v^{\mathcal{C}_i}$  be as in (4.3), with  $i \notin F_i$ . Then the Shapley value is:*

$$\sigma_j(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if } j \in N \setminus (F_i \cup E_i \cup \{i\}) \\ v(i) \left( \frac{1}{e+1} - \frac{1}{f+e+1} \right) & \text{if } j = i \\ \frac{v(i)}{(f+e)(f+e+1)} & \text{if } j \in F_i \\ -v(i) \left( \frac{1}{e(e+1)} - \frac{1}{(f+e)(f+e+1)} \right) & \text{if } j \in E_i \end{cases}$$

On the other hand, the Banzhaf value is:

$$\beta_j(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if } j \in N \setminus (F_i \cup E_i \cup \{i\}) \\ v(i) \left( \frac{2^f - 1}{2^{n-1}} \right) & \text{if } j = i \\ \frac{v(i)}{2^{f+e-1}} & \text{if } j \in F_i \\ -v(i) \left( \frac{2^{n-e} - 1}{2^{n-1}} \right) & \text{if } j \in E_i \end{cases}$$

*Proof.* Clearly, players not in  $F_i \cup E_i \cup \{i\}$  are null players. We call a player *decisive* if he makes the worth of the coalition  $S$  changing, by joining it. This can happen only if the value changes from 0 to  $v(i)$  and conversely. Let us now consider  $j \in F_i$ . He is

decisive (with worth  $v(i)$ ) if and only if  $i$  is in the coalition (preceding him according to the characterization of Shapley index) and no other player  $k$  in  $F_i$  is in the coalition. This provides the formula for  $\beta_j$ , while for  $\sigma$  we need to consider all orderings of the form  $i, j, \dots$  and in this case the result follows. Let us now consider  $i$ . He becomes decisive (with worth  $v(i)$ ) if and only if *at least* one player  $j \in F_i$  is present, and *no* player in  $E_i$  is present. Counting all coalitions of this type provides the Banzhaf index. Moreover

$$\sigma_i(v^{\mathcal{C}_i}) = \frac{v(i)}{(f+e+1)!} \sum_{k=1}^f \binom{f}{k} k!(f+e-k)!$$

Finally, a player  $j$  in  $E_i$  is decisive (with *negative* worth  $v(i)$ ) if and only if  $i$ , a player  $k$  in  $F_i$  and no other player of  $E_i$  are present. Thus

$$\sigma_j(v^{\mathcal{C}_i}) = -\frac{v(i)}{(f+e+1)!} \sum_{k=1}^f \binom{f}{k} (k+1)!(f+e-k-1)!$$

The formulas for the Banzhaf index are proved, while the result for Shapley follows from Lemma 1 in Appendix A. □

Another interesting case is that of a basic GAG with a *single* set of friends for each player  $i$ <sup>1</sup>. In this case the game  $v^{\mathcal{C}_i}$  reduces to:

$$v^{\mathcal{C}_i}(S) = \begin{cases} v(i) & \text{if } S \cap F_i \neq \emptyset, S \cap E_i = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and the following proposition holds.

**Proposition 7.** *Let us consider a basic GAS on  $\langle N, v, \{\mathcal{C}_i = \{F_i, E_i\}\}_{i \in N} \rangle$ . Then the Shapley and Banzhaf values for the game  $v^{\mathcal{C}_i}$  are given, respectively, by:*

$$\sigma_j(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if } j \in N \setminus (F_i \cup E_i) \\ \frac{v(i)}{f+e} & \text{if } j \in F_i \\ -v(i) \frac{f}{e(f+e)} & \text{if } j \in E_i \end{cases}$$

and

$$\beta_j(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if } j \in N \setminus (F_i \cup E_i) \\ \frac{v(i)}{2^{f+e-1}} & \text{if } j \in F_i \\ -v(i) \frac{2^f-1}{2^{f+e-1}} & \text{if } j \in E_i. \end{cases}$$

*Proof.* Clearly, players not in  $F_i \cup E_i$  are null players. We call a player *decisive* if he makes the worth of the coalition  $S$  changing, by joining it. Let  $j \in F_i$ . He is decisive (with worth  $v(i)$ ) if and only if no player  $i$  in  $F_i$  and no player  $k$  in  $E_i$  is in the coalition (preceding him according to the characterization of Shapley index).

Let  $j \in E_i$ . He is decisive (with worth  $-v(i)$ ) if and only if at least one player  $i$  in  $F_i$  is

1. Note that this case does not coincide with the previous one, where we restricted our attention to basic GAGs associated with a proper map  $\mathcal{M}$ .

in the coalition (preceding him according to the characterization of Shapley index) and no player  $k$  in  $E_i$  is in the coalition.

This provides directly the formulas for  $\beta$ , and  $\sigma$  is derived by considering all the possible orderings of the players. □

Next, we introduce an example of computation of the Shapley value according to Proposition 7, together with some considerations about an associated strategic problem.

**Example 13.** *As a toy example, consider an online social network with four users  $N = \{1, 2, 3, 4\}$  where each user spends the same amount of time  $T$  (e.g., on a per month basis) in uploading new content files and, according to her/his social profile, each user  $i \in N$  belongs to a single community  $F_i \subseteq N$  (e.g., the set of users with whom  $i$  intends to share her/his content files) which is in conflict with the complementary one  $E_i = N \setminus F_i$  (here, enemies in  $E_i$  are interpreted as those members that have no permission to access the content files of player  $i$ ). Suppose, for instance, that  $F_1 = \{1, 2, 3\}$ ,  $F_2 = \{2, 3\}$ ,  $F_3 = \{3\}$  and  $F_4 = \{1, 2, 3, 4\}$ . Following the discussion about social networking web sites introduced in Section 4.1, we can represent such a situation as a basic GAS  $\langle N, v, \{\mathcal{C}_i = \{F_i, E_i = N \setminus F_i\}\}_{i \in N} \rangle$ . How to identify the most influential users? According to Proposition 7 the influence vector provided by the Shapley value is:  $\sigma(v^{\mathcal{C}}) = (\frac{T}{6}, \frac{4T}{6}, T, \frac{-5T}{6})$ . So, user 3 results the most influential one, followed by 2, then 1 and finally 4, who is the only user to get a negative index.*

*Suppose now that user 2 wants to improve her/his influence as measured by the Shapley value. It is worth noting that if user 2 eliminates 3 from her/his set of friends (and all the other sets of friends and enemies remain the same), then player 2 gets exactly the same Shapley value of user 3 (independently on whether 3 is in the set of enemies of 2 or not). Precisely, if now  $F_2 = \{2\}$  and  $E_2 = \{1, 3, 4\}$ , then  $\sigma_2(v^{\mathcal{C}}) = \sigma_3(v^{\mathcal{C}}) = \frac{2T}{3}$ , whereas if  $F_2 = \{2\}$  and  $E_2 = \{1, 4\}$ , then  $\sigma_2(v^{\mathcal{C}}) = \sigma_3(v^{\mathcal{C}}) = \frac{3T}{4}$ .*

Note that, when  $E_i = \emptyset$ , the formulas in Proposition 7 are further simplified and the following corollary holds.

**Corollary 2.** *Consider a basic GAS on  $\langle N, v, \{\mathcal{C}_i = \{F_i, E_i = \emptyset\}\}_{i \in N} \rangle$ . Then the Shapley value  $\sigma$  and the Banzhaf value  $\beta$  for the game  $v^{\mathcal{C}_i}$  are given, respectively, by:*

$$\sigma_j(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if } j \in N \setminus F_i \\ \frac{v(i)}{f} & \text{if } j \in F_i \end{cases}$$

and

$$\beta_j(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if } j \in N \setminus F_i \\ \frac{v(i)}{2^{f-1}} & \text{if } j \in F_i. \end{cases}$$

Proposition 7 and Corollary 2 represent a useful tool for computing the Shapley and Banzhaf values of a subclass of basic GAGs. Their advantage relies on the fact that, once a game is described in terms of basic GAGs, the formulas can be derived in a straightforward way from the individual values of the players and the cardinalities of the sets of friends and enemies. As an example, consider the game introduced in Example 8. The Shapley value of such a game has been proposed as a measure of centrality in networks in [94], where an approximate algorithm for its computation is

provided. Moreover, in [1], an exact formula for the Shapley value of such game is provided, but its proof relies on elaborate combinatorial and probabilistic arguments. On the other hand, the description of that game as a basic GAG, which can be easily derived from the definition of the game itself, leads to the same formula in a direct and intuitive way, since the Shapley value (and the Banzhaf value) can be directly derived from Corollary 2 and relation (3.6).

When generalizing the previous results to the case of a basic GAG with multiple sets of friends, it is natural to extend the analysis to other solutions, beyond the Shapley and Banzhaf values. In what follows, we focus on the class of semivalues.

Let  $F = \bigcup_{k=1}^m F_i^k$  and let  $\Gamma_i = \{1, \dots, f_i^1\} \times \dots \times \{1, \dots, f_i^m\}$  (if clear from the context, in the following we shall omit the lower index  $i$ ), where  $f$  and  $f_i^k$  are the cardinalities of  $F$  and  $F_i^k$ , for each  $k = 1, \dots, m$ .

The following Theorem generalizes the results in Proposition 6 and 7.

**Theorem 2.** Consider a GAS situation  $\langle N, v, \{F_i^1, \dots, F_i^m, E_i\}_{i \in N} \rangle$  with  $F_i^j \cap F_i^k = \emptyset$  for all  $i \in N$  and  $j, k = 1, \dots, m, j \neq k$ .

For all  $j \in N \setminus (F \cup E_i)$ , we have that  $\pi_j^P(v^{C_i}) = 0$ .

Take  $j \in F_i^b$ , with  $b \in \{1, \dots, m\}$ . Then  $\pi_j^P(v^{C_i})$  is equal to the following expression:

$$v(i) \sum_{(k_i^1, \dots, k_i^{b-1}, 0, k_i^{b+1}, \dots, k_i^m) \in \Gamma} \sum_{l=0}^{n-e-f} \binom{f_i^1}{k_i^1} \times \dots \times \binom{f_i^m}{k_i^m} \times \binom{n-e-f}{l} p_{h+l} \quad (4.4)$$

where  $e = |E_i|$  and  $h = \sum_{j=1}^m k_j$ .

Now, take  $j \in E_i$ . Then  $\pi_j^P(v^{C_i})$  is given by:

$$\pi_j^P(v^{C_i}) = -v(i) \sum_{(k_i^1, \dots, k_i^m) \in \Gamma} \sum_{l=0}^{n-t-f} \binom{f_i^1}{k_i^1} \times \dots \times \binom{f_i^m}{k_i^m} \times \binom{n-e-f}{l} p_{h+l}. \quad (4.5)$$

*Proof.* Players in  $N \setminus (F \cup E_i)$  are dummy players, so they receive nothing. Now, consider the case  $j \in F_i^b$  and take a coalition  $S \subseteq N \setminus \{j\}$  that does not contain  $j$ . The marginal contribution of  $j$  to coalition  $S$  is  $v^{C_i}(S \cup \{j\}) - v^{C_i}(S) = v(i)$  if  $S$  contains at least one friend from each set of friends  $F_i^t$  with  $t \neq b$  (i.e.,  $S \cap F_i^t \neq \emptyset$  for  $t \neq b$ ), and  $S$  does not contain neither any element of  $F_i^b$  nor any element of  $E_i$ ; otherwise,  $v^{C_i}(S \cup \{j\}) - v^{C_i}(S) = 0$ .

Given a vector  $(k_i^1, \dots, k_i^{b-1}, 0, k_i^{b+1}, \dots, k_i^m) \in \Gamma$  (i.e.,  $k_b = 0$ ) and  $l \in \{0, \dots, n-e-f\}$ , the product  $\binom{f_i^1}{k_i^1} \times \dots \times \binom{f_i^m}{k_i^m} \times \binom{n-e-f}{l}$  represents the number of sets  $S$  containing  $k_i^t$  elements of  $F_i^t$ , for each  $t \in \{1, \dots, m_i\}$  with  $t \neq b$ ,  $l$  elements of  $N \setminus (F \cup E_i)$  and such that  $v^{C_i}(S \cup \{j\}) - v^{C_i}(S) = v(i)$ . Of course, the probability of such a set  $S$  to form is  $p_{h+l}$ , and relation (4.4) follows. Now, consider the case  $j \in T$  and take a coalition  $S \subseteq N \setminus \{j\}$  that does not contain  $j$ . The marginal contribution of  $j$  to coalition  $S$  is  $v^{C_i}(S \cup \{j\}) - v^{C_i}(S) = -v(i)$  if  $S$  contains at least one friend from each set of friends  $F_i^t$  for each  $t$  (i.e.,  $S \cap F_i^t \neq \emptyset$  for each  $t = 1, \dots, m_i$ ), and  $S$  does not contain any element of  $E_i$ ; otherwise,  $v^{C_i}(S \cup \{j\}) - v^{C_i}(S) = 0$ . Given a vector  $(k_i^1, \dots, k_i^m) \in \Gamma$  and  $l \in \{0, \dots, n-e-f\}$ , the product  $\binom{f_i^1}{k_i^1} \times \dots \times \binom{f_i^m}{k_i^m} \times \binom{n-e-f}{l}$  represents the number of sets  $S$  containing  $k_i^t$  elements of  $F_i^t$ , for each  $t = 1, \dots, m_i$ ,  $l$  elements of  $N \setminus (F \cup E_i)$  and such that  $v^{C_i}(S \cup \{j\}) - v^{C_i}(S) = -v(i)$ . Of course, the

probability of such a set  $S$  to form is  $p_{h+l}$ , and relation (4.5) follows.  $\square$

Formulas for the semivalues on  $v^{C_i}$  when each  $i \in N$  has only one set of friends can be derived directly from the previous theorem. Indeed, the following corollaries hold.

**Corollary 3.** *Consider a basic GAS on  $\langle N, v, \{C_i = \{F_i, E_i\}\}_{i \in N}$ . Then a semivalue  $\pi^{\mathbf{p}}$  for the game  $v^{C_i}$  is given by:*

$$\pi_j^{\mathbf{p}}(v^{C_i}) = \begin{cases} 0 & \text{if } j \in N \setminus (F_i \cup E_i) \\ v(i) \sum_{k=0}^{n-f-e} \binom{n-f-e}{k} p_k & \text{if } j \in F_i \\ -v(i) \sum_{k=1}^f \binom{f}{k} \sum_{h=0}^{n-f-e} \binom{n-f-e}{h} p_{k+h} & \text{if } j \in E_i. \end{cases}$$

**Corollary 4.** *Consider a basic GAS on  $\langle N, v, \{C_i = \{F_i, E_i = \emptyset\}\}_{i \in N}$ . A semivalue  $\pi^{\mathbf{p}}$  for the game  $v^{C_i}$  is given by:*

$$\pi_j^{\mathbf{p}}(v^{C_i}) = \begin{cases} 0 & \text{if } j \in N \setminus F_i \\ v(i) \sum_{k=0}^{n-f} \binom{n-f}{k} p_k & \text{if } j \in F_i. \end{cases}$$

Note that, in the basic GASs  $\langle N, v, \{C_i = \{F_i, E_i\}\}_{i \in N}$  considered in Corollary 3, a semivalue  $\pi^{\mathbf{p}}$  for the game  $v^{C_i}$  assigns to a player  $j \in F_i$  a positive share of  $v_i$ , proportionally to the probability (according to  $\mathbf{p}$ ) that  $j$  enters in a coalition not containing any player of the set  $F_i \cup E_i$ ; on the contrary, each player  $l \in E_i$  receives a negative share of  $v_i$ , proportionally to the probability that  $l$  enters in a coalition containing at least one player of  $F_i$ . In particular, the unique semivalue such that  $\sum_{i \in F_i} \pi_i^{\mathbf{p}}(v^{C_i}) = \sum_{i \in E_i} \pi_i^{\mathbf{p}}(v^{C_i})$ , for every  $F_i, E_i \subseteq N$ ,  $F_i \cap E_i = \emptyset$ ,  $F_i \neq \emptyset$  and  $E_i \neq \emptyset$ , is the Shapley value.

Observe that from Corollary 3 we can derive a formula for the Shapley and Banzhaf value of a basic GAG when each player has a unique set of friends:

$$\sigma_j(v^{C_i}) = \begin{cases} v(i) \sum_{k=0}^{n-f-e} \binom{n-f-e}{k} \frac{k!(n-k-1)!}{n!} & \text{if } j \in F_i \\ -v(i) \sum_{k=1}^f \binom{f}{k} \sum_{h=0}^{n-f-e} \binom{n-f-e}{h} \frac{(k+h)!(n-k-h-1)!}{n!} & \text{if } j \in E_i \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta_j(v^{C_i}) = \begin{cases} v(i) \sum_{k=0}^{n-f-e} \binom{n-f-e}{k} \frac{1}{2^{n-1}} & \text{if } j \in F_i \\ -v(i) \sum_{k=1}^f \binom{f}{k} \sum_{h=0}^{n-f-e} \binom{n-f-e}{h} \frac{1}{2^{n-1}} & \text{if } j \in E_i \\ 0 & \text{otherwise.} \end{cases}$$

The equivalence with the formulas of Proposition 7 can be verified through Lemma 1 for the Shapley value (see Appendix 8).

Another interesting case, which is not covered by Theorem 2 since the set of friends are not disjoint, relates to the basic GAS  $\langle N, v, \mathcal{C} \rangle$ , where  $C_i = \{F_i^1 = \{i\}, F_i, E_i\}_{i \in N}$  and the set  $F_i$  does contain  $i$  for all  $i \in N$ .

In this case the following result holds.

**Proposition 8.** Let  $v^{C_i}$  be as in (4.3), with  $i \in F_i$ . Then the Shapley and Banzhaf value of each  $j \in N$  are given, respectively, by:

$$\sigma_j(v^{C_i}) = \begin{cases} 0 & \text{if } j \in N \setminus (F_i \cup E_i) \\ \frac{v(i)}{e+1} & \text{if } j = i \\ 0 & \text{if } j \in F_i \setminus \{i\} \\ -\frac{v(i)}{e(e+1)} & \text{if } j \in E_i \end{cases}$$

and

$$\beta_j(v^{C_i}) = \begin{cases} 0 & \text{if } j \in N \setminus (F_i \cup E_i) \\ \frac{v(i)}{2^e} & \text{if } j = i \\ 0 & \text{if } j \in F_i \setminus \{i\} \\ -\frac{v(i)}{2^e} & \text{if } j \in E_i \end{cases}$$

*Proof.* Observe that players different from player  $i$ , and not belonging to  $E_i$ , are dummy players. We call a player *decisive* if he make the worth of the coalition  $S$  changing, by joining it. This can happen only if the value changes form 0 to  $v(i)$  and conversely. Consider player  $i$ , and players in  $E_i$ . Then  $i$  is decisive if and only if he precedes all players in  $E_i$ . since the ordering of the players in  $E_i$  are irrelevant, the result follows. To conclude, use efficiency of the Shapley value and symmetry of the players in  $E_i$ .  $\square$

### 4.3 The Core of GAGs

In this section we consider GAGs where  $v(i) > 0$  represents a revenue for each  $i \in N$  and we present some results concerning the core of the GAGs. The first result we present in this section is quite simple, and relates to the core of general GAGs.

**Proposition 9.** Let  $\langle N, v, \mathcal{M} \rangle$  be a GAS with  $v \in \mathbb{R}_+^N$  where  $\mathcal{M}$  is proper and such that  $\mathcal{M}(N) = N$ . Then, the core of the associated (reward) GAG  $(N, v^{\mathcal{M}})$  is non-empty.

*Proof.* Let  $x \in \mathbb{R}^N$  be the allocation with  $x_i = v(i)$  for each  $i \in N$ . Consider the game  $w$  defined as:  $w(S) = \sum_{i \in S} v(i)$ . Notice that  $x \in C(w)$ . Moreover, it holds that  $v^{\mathcal{M}}(S) \leq w(S) \forall S \subseteq N$ , by properness of  $\mathcal{M}$ , and  $v^{\mathcal{M}}(N) = w(N)$ . Thus  $x \in C(v^{\mathcal{M}})$ .  $\square$

We now turn our attention to basic GAGs. An obvious way to find conditions under which the core of  $v^C$  is non empty is to look for conditions under which  $C(v^{C_i}) \neq \emptyset$  for every  $i \in N$  since, if  $x_i \in C(v^{C_i})$ , then  $x = \sum_{i \in N} x_i \in C(v^C)$ . As we shall see later, the condition is only sufficient. Condition  $\mathcal{M}(N) = N$  for a map  $\mathcal{M}$  defined via relation (3.4) is equivalent to impose  $E_i = \emptyset$  for each  $i \in N$ . We then focus on games associated to a GAG on  $N$  where  $\mathcal{M}$  is defined by relation (3.4) with collections  $\mathcal{C}_i = \{F_i^1, \dots, F_i^m, E_i = \emptyset\}$ . In this case, the game  $v^{C_i}$  defined in (3.5) becomes:

$$v^{C_i}(S) = \begin{cases} v(i) & \text{if } S \cap F_i^k \neq \emptyset, \forall k = 1, \dots, m \\ 0 & \text{otherwise.} \end{cases} \quad (4.6)$$

Denote by  $I_i = \{j \in N : \exists F_i^k \in \mathcal{C}_i \text{ s.t. } F_i^k = \{j\}\}$  the set of players that appear in collection  $\mathcal{C}_i$  as singletons. Note that  $I_i$  may be empty. Otherwise, players in  $I_i$  are veto players in the associated game  $v^{C_i}$ . From the above considerations the following Proposition holds, which characterizes the core of the game  $v^{C_i}$ .

**Proposition 10.** *Consider the game  $v^{C_i}$ , where  $\mathcal{M}$  is defined by relation (3.4) with collections  $\mathcal{C}_i = \{F_i^1, \dots, F_i^m, E_i = \emptyset\}$ . Then  $C(v^{C_i}) \neq \emptyset$  if and only if  $I_i \neq \emptyset$ . Moreover, if  $I_i \neq \emptyset$ , then it holds that:*

$$C(v^{C_i}) = \{x \in \mathbb{R}_+^N : \sum_{j \in I_i} x_j = v(i)\} \quad (4.7)$$

*Proof.* Note that  $I_i$  is the set of veto players in  $v^{C_i}$ . Therefore,  $C(v^{C_i}) \neq \emptyset$  if and only if  $I_i \neq \emptyset$ . Moreover, if  $I_i \neq \emptyset$ , relation (4.7) simply follows from the fact that  $v^{C_i}(N) = v(i)$ . □

It follows that if  $I_i \neq \emptyset \forall i \in N$ , then  $C(v^C) \neq \emptyset$ . However, when games  $v^{C_i}$  such that  $I_i = \emptyset$  are combined with games  $v^{C_j}$  such that  $I_j \neq \emptyset$ , the resulting GAG can have a nonempty core, as shown in the following example.

**Example 14.** *Consider a two-person basic GAS  $\langle N, v, \mathcal{C} \rangle$  with  $N = \{1, 2\}$  with no enemies such that  $v(1) = \alpha$ ,  $v(2) = 2$  and  $\mathcal{C}_1 = \{\{1\}, \{2\}, E_1 = \emptyset\}$ ,  $\mathcal{C}_2 = \{\{1, 2\}, E_2 = \emptyset\}$ . By Proposition 10 we have that  $C(v^{C_1}) \neq \emptyset$ , and  $C(v^{C_2}) = \emptyset$ . The core of the resulting GAG  $v^C = v^{C_1} + v^{C_2}$  is non-empty if and only if  $\alpha \geq 2$ .*

The situation described in the previous example can be generalized as follows. We first define the set  $\mathcal{I} = \{i \in N : I_i \neq \emptyset\}$ . The following proposition provides a necessary and sufficient condition for the non-emptiness of the core of a special class of basic GAGs of the type introduced in Example 14.

**Proposition 11.** *Let  $v^C$  be the GAG corresponding to a basic GAS  $\langle N, v, \mathcal{C} \rangle$  with  $v(i) \geq 0$  and  $\mathcal{C}_i = \{F_i^1, \dots, F_i^m, E_i = \emptyset\}$  for each  $i \in N$ . Suppose there exists a coalition  $S \subseteq N$ ,  $S \neq \emptyset$ , satisfying the following two conditions:*

- (i)  $S \subseteq I_i$  for each  $i \in \mathcal{I}$ ;
- (ii) for each  $i \in N \setminus \mathcal{I}$ , there exists  $k \in \{1, \dots, m\}$  such that  $F_i^k = S$ .

*Define the equal split allocation among players in  $S$  as the vector  $y$  such that*

$$y = e^S \frac{v^C(N)}{s},$$

*where  $s$  is the cardinality of  $S$  and where  $e^S \in \{0, 1\}^N$  is such that  $e_k^S = 1$ , if  $k \in S$  and  $e_k^S = 0$ , otherwise. The allocation  $y$  is in the core of the game  $v^C$  iff*

$$v^C(N) \geq s \sum_{i \in N \setminus \mathcal{I}} v(i). \quad (4.8)$$

*Proof.* First, we prove that condition (4.8) is necessary. Suppose that (4.8) does not hold, i.e.  $v^C(N) < s \sum_{i \in N \setminus \mathcal{I}} v(i)$ . Consider a coalition  $T \subseteq N$ , such that  $|T| \geq 2$ ,  $T \cap S = \{t\}$  and  $T \cap F_i^k \neq \emptyset \forall i \in N, \forall k = 1, \dots, m$ . It holds that:

$$\sum_{j \in T} y_j = y_t = \frac{v^C(N)}{s} < \sum_{i \in N \setminus \mathcal{I}} v(i) = v^C(T)$$

and therefore  $y \notin C(v^c)$ .

Moreover, we prove that condition (4.8) is also sufficient. Clearly,

$$\sum_{i \in N} y_i = \sum_{i \in S} y_i = v^c(N) = \sum_{i \in N} v(i). \quad (4.9)$$

In order to prove that  $y \in C(v^c)$  we have to show that if relation (4.8) holds then  $\sum_{i \in N} y_i \geq v^c(T)$  for each coalition  $T \subseteq N$ .

First consider a coalition  $T \subseteq N$  such that  $S \subseteq T$ . Note that  $\sum_{i \in N} v(i) \geq v^c(T)$ , and then by relation (4.9),  $\sum_{i \in T} y_i \geq \sum_{i \in S} y_i = \sum_{i \in N} v(i) \geq v^c(T)$ .

Now consider a coalition  $T \subseteq N$  such that  $S \cap T = \emptyset$ . By condition (i) and (ii), we have that  $v^c(T) = 0$ , which is not greater than  $\sum_{i \in T} y_i$  since  $y_i \geq 0$  for each  $i \in N$ .

Finally, consider a coalition  $T \subseteq N$  such that  $S \cap T \neq \emptyset$  and  $S \not\subseteq T$ . Since  $S \subseteq I_i$  for each  $i \in \mathcal{I}$ , then no term  $v(i)$  with  $i \in \mathcal{I}$  contributes to the worth of  $T$ . This means that

$$v^c(T) = \sum_{i \in \mathcal{M}(T)} v(i), \quad (4.10)$$

with  $\mathcal{M}(T) \subseteq N \setminus \mathcal{I}$  and where  $\mathcal{M}$  is defined by relation (3.4). Now consider a player  $i \in S \cap T$ . If condition (4.8) holds, then we have that

$$\sum_{j \in T} y_j \geq y_i \geq \sum_{i \in N \setminus \mathcal{I}} v(i) \geq v^c(T),$$

where the last inequality follows by relation (4.10), which concludes the proof.  $\square$

Observe that, even if the equal split allocation does not belong to the core, the core might be non empty, as the following example shows.

**Example 15.** Consider a GAS  $G = \langle N, v, M \rangle$  with  $N = \{1, 2, 3\}$  with no enemies such that  $\mathcal{C}_1 = \{\{1\}, \{2\}, \{3\}, E_1 = \emptyset\}$ ,  $\mathcal{C}_2 = \{\{2, 3\}, E_2 = \emptyset\}$  and  $\mathcal{C}_3 = \{\{2, 3\}, E_3 = \emptyset\}$ . The coalition  $\{2, 3\}$  satisfies the hypothesis in Proposition 11 but relation (4.8) is not satisfied. However, the allocation  $(v(1), v(2), v(3))$  belongs to the core of  $v^c$ .

The previous Propositions provide sufficient conditions for the non-emptiness of the core for basic GAGs. If a game can be represented in terms of a basic GAG, these conditions can be directly verified by considering only the collections of friends and enemies of each player, without having to check any further property of the characteristic function (for instance, the balancedness property [15, 86]), that in general involves much more complex procedures. Consequently, the results provided in this section can be used to construct non-trivial classes of games with a non-empty core, or to easily derive core allocations. For instance, even if a game is not itself representable in terms of a basic GAG, Proposition 10 may be applied to generate allocations in the core of games that can be described as the sum of proper basic GAGs where no player has enemies, as next example shows.

**Example 16.** Consider the game introduced in [33], in which the players are nodes of a graph with weights on the edges, and the value of a coalition is determined by the total weight of the edges contained in it. Formally, an undirected graph  $G = (N, E)$

is given, with weight  $w_{i,j}$  on the edge  $\{i, j\}$ <sup>2</sup> and the game  $v$  is defined, for every  $S \subseteq N$ , as  $v(S) = \sum_{i,j \in S} w_{i,j}$ . If all weights are non-negative, the game is convex and therefore the core is non empty. However, finding allocations in the core is not straightforward and in [33] necessary and sufficient conditions for the Shapley value to belong to it are provided.

Indeed, game  $v$  can be described as the sum of  $n$  basic GAGs, one for each player  $i \in N$ , where each other player  $j \neq i$  contributes to the worth of a coalition  $S \subseteq N$  with half of the weight  $w_{i,j}$  if and only if  $i$  and  $j$  belong to  $S$ , while  $i$  contributes to any coalition it belongs to with the weight  $w_{i,i}$ . Formally,  $v = \sum_{i \in N} v^{C_i}$ , where  $v^{C_i}$  is a proper basic GAG associated to collections  $\mathcal{C}_j^i = \{F_j^1 = \{i\}, F_j^2 = \{j\}, E_j = \emptyset\}$  and  $v(j) = \frac{w_{i,j}}{2}$ , for every  $j \in N$ ,  $j \neq i$ , while  $\mathcal{C}_i^i = \{F_j^1 = \{i\}, E_j = \emptyset\}$  and  $v(i) = w_{i,i}$ . As a sum of  $n$  proper GAGs such that  $\mathcal{M}(N) = N$ , Proposition 9 immediately implies the non-emptiness of the core of game  $v$ . Moreover, notice that according to Proposition 4 each basic GAG  $v^{C_i}$  can be decomposed as the sum  $v^{C_i} = \sum_{j \in N} v^{C_j^i}$ , for each  $i, j \in N$ , and then the repeated application of Proposition 10 on each  $v^{C_j^i}$  can be used to efficiently derive allocations in the core of the sum game  $v$ .

Following similar intuitions, we argue that the simple structure of basic GAGs could be useful to generalize some of the complexity results about the problem of finding core allocations provided in [33], for instance, considering classes of more sophisticated games that can be generated as a positive linear combination of basic GAGs.

## 4.4 Concluding remarks

In Chapters 3 and 4 we introduced and studied a class of coalitional games, namely the class of basic GAGs, where the worth of each coalition is calculated additively over the individual contributions and keeping into account social relationships among groups of players, that is by means of a map  $\mathcal{M}$  that selects the valuable players in the coalition.

Several examples from the literature of classical coalitional games that can be described within our approach have been presented. Our approach enables to classify existing games based on the properties of  $\mathcal{M}$ .

The interest of the classification is not only taxonomical, since it also allows to study the properties of solutions for classes of games known from the literature, which are studied in connection with the properties of the filtering map  $\mathcal{M}$  introduced and discussed in Chapter 3.

Indeed, we showed that, in many cases, basic GAGs allow for an easy computation of several classical solutions from cooperative game theory and, at the same time, provide quite simple representations of practical situations (for instance, arising from online social networks).

One of the goal of our future research is to apply these models on real social network data. As shown by Example 13, the information required to compute classical power indices on basic GAGs representing online social networks (like the users' activity time or the users' social profiles and social affinities) is not very demanding and can be obtained by available records and models from the literature [82]. Moreover, as it

2. Since the graph is undirected, we assume by convention that, if an edge between  $i$  and  $j$  is present,  $w_{i,j} \neq 0$  and  $w_{j,i} = 0$  for  $i < j$ .

has been stressed in the same example, it would be interesting to explore the strategic issues related to the attempt of players to increase their influence (as measured by the Shapley value or by other power indices) on a social network.

An interesting direction for future research is indeed that of coalition formation, since for generic basic GAGs associated to GASs with non-negative  $v$ , where the sets of enemies are not empty, the grand coalition is not likely to form. As an example of coalition formation problem that can be well represented in these terms, consider the following “triangle” situation on three researchers, namely, Alice, Bob and Carol. Due to their characters’ affinities, Alice and Bob love to do research together, but they do not like at all to be involved in research projects with Carol. Instead, Carol loves to do research with Bob, but not with Alice. On the other hand, in order to make a successful research, they need to perform a certain number of expensive experiments. Because of the bad financial status of their respective departments, Alice and Bob’s personal research funds are very limited, whereas Carol can rely on a conspicuous international funding. Such a situation can be represented by as basic GAS on the three researchers  $N = \{\text{Alice}, \text{Bob}, \text{Carol}\}$  where the set of friends of Alice contains only Bob, the set of friends of Bob and Carol is the same and coincides with the singleton Alice, the set of enemies of Alice and Bob is the singleton Carol and, finally, the set of enemies of Carol is the singleton Alice. In addition, the function  $v = (v(\text{Alice}), v(\text{Bob}), v(\text{Carol}))$  of their individual contribution is given by their respective research funds. The corresponding basic GAG is  $v^c(\text{Alice}, \text{Bob}) = v(\text{Alice}) + v(\text{Bob})$ ,  $v^c(\text{Bob}, \text{Carol}) = v(\text{Carol})$  and  $v^c(S) = 0$  for all the other coalitions  $S \subseteq N$ . It is quite natural to expect that if  $v(\text{Carol})$  is quite larger than  $v(\text{Alice}) + v(\text{Bob})$ , then the coalition  $\{\text{Bob}, \text{Carol}\}$  will form, despite the reciprocal friendship between Alice and Bob.

In general, we believe that the issue about which coalitions are more likely to form in a basic GAG is not trivial and deserves to be further explored.

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# CHAPTER 5

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## On the Position Value for Special Classes of Networks

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In this Chapter we deal with a particular class of TU-games, those whose cooperation is restricted by a network structure. We focus on the so-called *cooperative games with restricted communication*, where a cooperative game and a network that describes the restriction on the feasible coalitions lead to the definition of a *communication situation* and to the search for solution concepts that take into account the constraints imposed by the underlying network structure. In particular, we consider a communication situation in which a network is produced by subsequent formation of links among players and at each step of the network formation process, the surplus generated by a link is shared between the players involved, according to some rule. As a consequence, we obtain a family of solution concepts for communication situations that we investigate on particular network structures. This approach provides a different interpretation of the position value since it turns out that a specific symmetric rule leads to this solution concept.

### 5.1 Introduction

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In Chapter [3](#), we introduced the class of GAGs, where the worth of each coalition is computed as the sum of the individual values of a subset of players. We showed that several coalitional games on networks can be described within this formalism. On the other hand, in many cases, when an underlying network describes the interaction among the players involved, it is possible to derive the worth of each coalition of players as the sum of the contributions that their pairwise interactions generate, that is as the sum of the individual values assigned to the edges in the underlying network, as we have shown, as an example, for the class of maintenance cost games described in Section

**3.3.3** As it has been shown in Chapter **3**, this idea may be described by defining a *link-based GAG*, corresponding to a *link-based GAS*  $\langle N, w, \mathcal{L} \rangle$ , where  $N = \{1, \dots, n\}$ , represent the set of nodes in a network  $\langle N, E \rangle$ ,  $w : E \rightarrow \mathbb{R}$  assigns a value to each edge in the graph and  $\mathcal{L} : 2^E \rightarrow 2^E$  assigns to each subset  $A \subseteq E$  of edges another subset  $\mathcal{L}(A) \subseteq E$ . Then the corresponding *link-based GAG*  $(N, v^\mathcal{L})$  would be defined by the following expression, for every  $S \subseteq N$ :

$$v^\mathcal{L}(S) = \sum_{e \in \mathcal{L}(E_S)} w(e),$$

where  $E_S$  are the edges in the graph induced by  $S$ .

As for the maintenance cost games, in several other graph games the worth of a coalition can be additively computed starting from the values assigned to the edges in the underlying graph. Moreover, as in graph-restricted game the value of a coalition can be derived additively from a collection of subcoalitions of players, for the class of *link games*, introduced by Meessen **[70]** and further studied by Borm et al. **[18]**, the value of a coalition of links can be derived additively from a collection of subcoalitions of links. Indeed, several approaches to coalitional games on networks rely on additive patterns among links, not only for what concerns the definition of a game, but also for the analysis of the relative solutions.

This Chapter is indeed devoted to the introduction of a class of solution concepts for *communication situations* **[75]**, where the payoff to each player is additively computed starting from the values generated by pairwise relations among players. More precisely, we consider a communication situation in which a network is produced by subsequent formation of links among players and at each step of the formation process, the surplus generated by a link is shared between the players involved, according to some rule. As a consequence, we obtain a family of solution concepts that we investigate on particular network structures. In particular, it turns out that the *position value*, introduced by Borm et al. **[18]** as a solution for communication situations, is obtained when a specific symmetric rule is considered. Moreover, we investigate the problem of computing this particular solution on special classes of communication situations.

The Chapter is organized as follows. We first discuss some related literature in Section **5.2**. Then in Section **5.3** we describe the concept of communication situation and introduce the position value. In Section **5.4** we introduce the notion of allocation protocol and the class of solution concepts that derives. Furthermore, we focus on the computation of the position value. Section **5.5** presents some preliminary results and in Section **5.6** we give formulas for the position value on specific communication situations. Section **5.7** concludes.

## 5.2 Related Work

A TU-game describes a situation in which all players can freely interact with each other, i.e. every coalition of players is able to form and cooperate. However, this is not the case in many real world scenarios. A typical situation is when there exists a restriction on the communication possibilities among players, as in the context of social interactions between groups of people, political alliances within parties, economic exchange among firms, research collaborations and so on. In order to represent and study such situations it is necessary to drop the assumption that all coalitions are feasible. Then a natural question arises: *how can we model restrictions of the interaction possibilities between players?* Different approaches to model the restrictions on the interaction possibilities among players exist in literature. A typical way to do so is that of considering a *network structure* [1] that describes the interaction possibilities between the players: the nodes of the network are the players of the game and there exists a link between two nodes if the corresponding players are able to interact directly. In this context it is usual to refer to such networks as *communication networks*, since a typical situation they model is a restriction of communication possibilities between players. This approach leads to the definition of a so-called *communication situation* [2] [75] and to the search for solution concepts that take into account the constraints imposed by the underlying network structure. This corresponds to a flourishing stream in the literature on Game Theory, whose crucial point is to study how the communication constraints influence the allocation rules. There are at least two ways to measure this impact, that correspond to two different main streams in the recent literature.

In a first approach, the communication constraints determine how a coalition is evaluated. There is no actual restriction constraint on the set of feasible coalitions, but if a coalition is not connected through the communication graph, its worth is evaluated on the connected components in the induced graph. This approach is investigated in the seminal paper by Myerson [75], who introduces the Myerson value in order to generalize the Shapley value from TU-games to graph games. Jackson and Wolinsky [51] extend Myerson's model by considering a function assigning values to networks as a basic ingredient. Borm et al. [18] introduce the position value for communication situations. Like the Myerson value, the position value is based on the Shapley value, but it stresses the role of the pairwise connections in generating utility, rather than the role of the players. The value of a pairwise connection is derived as the Shapley value of a game on the set of links of the network and the position value equally divides the value of each link among the pair of players who form it. The position value has been extended in [90] to the setting of network situations introduced in Jackson and Wolinsky [51] and an axiomatic characterization in this context is given in [101].

In a second approach, the communication constraints determine which coalitions can actually form. The definition of the Shapley value relies on the idea of a one-by-one formation of the grand coalition: its interpretation assumes that the players gather one by one in a room; each player entering the room gets his marginal contribution to the coalition that was already there and all the different orders in which the players en-

1. Other models introduced in the literature are discussed in [92], including extensions of the interaction channels to hypergraphs and probabilistic networks, among others.

2. Note that the term *graph game* is sometimes used in the literature as a synonym for communication situation, while in this monograph we shall keep the two terms separate, to distinguish two different ways in which coalitional games and networks interact.

ter are equiprobable. To take into account the communication constraints, the orderings of the players that induce disconnected coalitions are ruled out: the formation of the grand coalition requires a communication at any stage. In order to satisfy the communication constraints Demange [32] proposes to model the sequential formation of the grand coalition by a rooted spanning tree of the communication graph.

Each rooted spanning tree represents a partial order on the players set such that the arrival of a new player forms a connected coalition. Demange [32] introduces the hierarchical outcome in order to extend the concept of marginal contribution from orderings of the players to rooted spanning trees. This second approach is also studied by Herings et al. [48] who introduce the average tree solution for graph games in which the communication graph is a forest (cycle-free graph). This allocation is the average of the hierarchical outcomes associated with all rooted spanning trees of the forest. Herings et al. [48] and Baron et al. [7] show how an extension of the average tree solution to arbitrary graph games can be seen as another generalization of the Shapley value. In Béal et al. [10], the principle of compensation formulated by Eisenman [39] is generalized from orderings of the players to rooted spanning trees and the compensation solution for graph games is introduced.

Based on the idea that the formation of the grand coalition requires a communication at any stage, our approach is different in spirit with respect to the aforementioned models. We assume indeed a different mechanism of coalition formation which results from subsequent connection of links among players, where the payoff to each player is additively computed starting from the values generated by the links that subsequently form. This idea naturally leads to consider a communication situation where a network between the players is produced by a permutation of links and we suppose that, at each step of the network formation process, the surplus generated by a link is shared between the players involved according to a certain protocol. Taking into account this mechanism, we propose a class of solution concepts where each solution corresponds to a different allocation protocol. In particular, at a certain step when a link between two players forms, it is reasonable to equally share the surplus between the players that are responsible for this connection, i.e. the two nodes incident to the link that is formed. It turns out that the solution obtained by this particular allocation protocol is indeed the position value introduced in [18]. Our model thus provides a different interpretation for this well-known solution concept and proposes a family of solution that embraces the basic principles of both the approaches described above, providing a bridge between two different ways of modelling the restriction of communication possibilities between players in a coalitional game.

### 5.3 Cooperative Games with Restricted Communication: the Position Value

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A coalitional game describes a situation in which every coalition of players is able to form and cooperate. If there exists a restriction on the interaction possibilities among players, not all coalitions are feasible. We can represent this situation by introducing a network structure that models the interactions between players. This leads to the definition of a communication situation.

Given a graph  $\Gamma$  and a coalitional game  $(N, v)$  we recall that a *communication sit-*

### 5.3. Cooperative Games with Restricted Communication: the Position Value

uation [75] is defined as the triple  $(N, v, \Gamma)$ , where  $N$  is the set of players,  $(N, v)$  is a coalitional game and  $\Gamma$  is an undirected graph with  $N$  as set of vertices. The graph  $\Gamma = (N, E)$  describes the communication possibilities between players: an indirect communication between  $i$  and  $j$  is possible if there is a path that connects them; if  $\{i, j\} \in E$ , then  $i$  and  $j$  can communicate directly.

In order to model the restriction of interactions among players described by a communication situation, Myerson introduced in [75] the so-called *graph-restricted game*  $v^\Gamma$ , defined by:

$$v^\Gamma(S) = \sum_{T \in C_{\Gamma_S}} v(T), \quad (5.1)$$

where  $C_{\Gamma_S}$  is the set of connected components in  $\Gamma_S$ . Note that, if  $S$  is connected in  $\Gamma$ , then  $v^\Gamma(S) = v(S)$ . Given a communication situation  $(N, v, \Gamma)$ , the *Myerson value*  $\mu(N, v, \Gamma)$  is defined as the Shapley value of the graph-restricted game, that is:

$$\mu(N, v, \Gamma) = \Phi(v^\Gamma).$$

Borm et al. introduced in [18] a solution concept for a communication situation based on the approach of Meessen [70]: given  $\Gamma = (N, E)$  and  $A \subseteq E$ , the *link game*  $v^L$  is defined by:

$$v^L(A) = \sum_{T \in C_{\Gamma_A}} v(T), \quad (5.2)$$

where  $C_{\Gamma_A}$  is the set of connected components in  $\Gamma_A$ . We denote by  $\mathcal{G}_L$  the vector space of all link games on  $\Gamma = (N, E)$ ,  $E \subseteq \{\{i, j\} : i, j \in V, i \neq j\}$ , where  $N$  is a fixed set of players. Note that the dimension of  $\mathcal{G}_L$  is equal to the number of connected subsets of  $E$ ; i.e. the cardinality of  $\{A \subseteq E : \Gamma_A \text{ is connected}\}$ .

Every link game  $v^L$  can be written as a linear combination of unanimity link games as follows:

$$v^L = \sum_{A \subseteq E} c_A(v^L) u_A, \quad (5.3)$$

where  $c_A$  are the unanimity coefficients of  $v^L$ :

$$c_A(v^L) = \sum_{B \subseteq A} (-1)^{|A|-|B|} v^L(B); \quad (5.4)$$

or equivalently  $c_{\{l\}}(v^L) = v^L(\{l\})$  and for  $A \subseteq E$ ,  $|A| \geq 2$ :

$$c_A(v^L) = v(A) - \sum_{B \subsetneq A, B \neq \emptyset} c_B(v^L). \quad (5.5)$$

Given a communication situation  $(N, v, \Gamma)$ , the *position value*  $\pi(N, v, \Gamma)$  is defined as:

$$\pi_i(N, v, \Gamma) = \frac{1}{2} \sum_{a \in A_i} \Phi_a(v^L) \quad \forall i \in N, \quad (5.6)$$

where  $A_i = \{\{i, j\} \in E, j \in N\}$  is the set of all links for which player  $i$  is an endpoint. Note that, since the players in  $v^L$  are the elements of  $E$ , i.e. the links of  $\Gamma$ , in formula

(5.6) we compute the Shapley value of a link. We shall write  $\pi(v)$  when there is no ambiguity about the underlying network.

We point out here a particular property satisfied by the position value that will be useful for our purpose, namely the *superfluous arc property* [101]. Given a communication situation  $(N, v, \Gamma)$ , with  $\Gamma = (N, E)$ , we call *superfluous* a link  $a$  such that  $v^L(A \cup \{a\}) = v^L(A) \forall A \subseteq E$ . The superfluous arc property states that if  $a$  is a superfluous arc, then  $\pi(N, v, \Gamma) = \pi(N, v, \Gamma')$ , where  $\Gamma' = (N, E \setminus \{a\})$ . The property follows directly by formula (5.6): the links (or arcs) that provide a marginal contribution equal to zero to every coalition of links (not containing the link itself) do not give contribution to the sum in (5.6), thus the position value does not change if they are removed from the network.

Note that, like the Shapley value, every semivalue  $\Psi^P$  induces a solution concept  $\psi^P$  for communication situations given by:

$$\psi_i^P(N, v, \Gamma) = \frac{1}{2} \sum_{a \in A_i} \Psi_a^P(v^L). \quad (5.7)$$

We write  $\psi(v)$  when there is no ambiguity about the underlying network. Note that, by definition of semivalue, the superfluous arc property still holds for every solution  $\psi$  corresponding to a given semivalue.

See [90], [91] and [101] for an axiomatic characterization of the position value for network situations, which generalize the context of communication situations.

## 5.4 Coalition formation and allocation protocols

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In Chapter 2 we introduced the Shapley value and gave a formula to compute it. Formula (2.5) has the following interpretation: suppose that the players gather one by one in a room to create the grand coalition. Each player entering the room gets his marginal contribution to the coalition that was already in the room. Assuming that all the different orders in which they enter are equiprobable, one gets the formula, where  $n!$  is the number of permutations on a set of  $n$  elements.

Let us consider a different mechanism of coalition formation: let us assume that a coalition forms by subsequent formation of links among players. This naturally leads to consider a communication situation, where a network between the players is produced by a permutation of links and all the different orders in which the links form are considered to be equiprobable. In this scenario, we can imagine that, when a link between two players forms, the players that are connected to each other receive a certain value according to some rule. Let us suppose that, at each step of the network formation process, when a link between two players  $i$  and  $j$  forms, the value of the coalition  $S$ , where  $S$  is the connected component containing  $i$  and  $j$ , reduced by the values of the connected components formed by the players of  $S$  at the previous step, is shared between the players involved according to a certain protocol. Then a natural question rises: *How to share this value?*

Given a communication situation  $(N, v, \Gamma)$ , let us consider a possible permutation  $\sigma$  of links. At each step  $k$  of the network formation process, when the  $k$ -th link  $a = \{i, j\}$

in the sequence determined by  $\sigma$  forms, let us consider the *surplus* produced by  $a$ :

$$S_k^\sigma = v(S) - v(C_{k-1,\sigma}^i) - v(C_{k-1,\sigma}^j) \quad (5.8)$$

where  $S$  is the connected component in  $\Gamma$  containing  $i$  and  $j$  at the step  $k$ , and  $C_{k-1,\sigma}^i$  and  $C_{k-1,\sigma}^j$  are the connected components in  $\Gamma$  at the step  $k - 1$ , containing  $i$  and  $j$  respectively.

An *allocation protocol* is a rule that specifies how to divide  $S_k^\sigma$  between the players in  $S$ . Given an allocation protocol  $r$  and a communication situation  $(N, v, \Gamma)$ , a solution of  $v$ , that we shall denote by  $\phi^r(v)$ , is given by:

$$\phi_i^r(v) = \frac{1}{|E|!} \sum_{\sigma \in \Sigma_E} \sum_{k=0}^{|E|} f_i^r(S_k^\sigma), \quad \forall i \in N, \quad (5.9)$$

where  $\Sigma_E$  is the set of possible orders on the set of links  $E$  in  $\Gamma$  and  $f_i^r$  is a function that assigns to each player  $i \in N$  a fixed amount of the surplus  $S_k^\sigma$ , depending on the allocation protocol  $r$ . In other words, the solution  $\phi^r(v)$  is computed by considering all possible permutations of links, and summing up, for each player  $i$ , all the contributions he gets with the allocation procedure  $r$ , averaged by the number of permutations over the set of links among the players, with the interpretation discussed at the beginning of this section.

This idea leads to the introduction of a class of solution concepts: different choices of the allocation protocol define different solutions for a communication situation. At a certain step, when a link  $a = \{i, j\}$  forms, it is possible to consider the allocation protocol that equally divides the surplus between players  $i$  and  $j$  only. The solution obtained by this particular allocation protocol is indeed the position value  $\pi$  defined in (5.6).

Note that other solution concepts can be achieved by sharing the surplus among the players involved in a different way.

The rest of the Chapter is devoted to the problem of computing such a solution on particular classes of communication situations.

## 5.5 Preliminary Results

In this Section we present some preliminary results that will be useful in the next Sections.

**Proposition 12.** *Let  $(N, v, \Gamma)$  be a communication situation and  $v^L$  the corresponding link game. Then  $c_A(v^L) = 0$  for any coalition  $A \subseteq E$  which is not connected in  $\Gamma$ , where  $c_A(v^L)$  are the unanimity coefficients of  $v^L$ .*

*Proof.* We prove the result by induction on  $a = |A|$ .

Suppose  $a = 2$ , i.e.  $A = \{l_1, l_2\}$ ,  $l_1, l_2 \in E$ , where  $l_1$  and  $l_2$  belong to two different connected components. From this hypothesis and from (5.2) and (5.5) we get:

$$\begin{aligned} v^L(A) &= v(\{l_1\}) + v(\{l_2\}) \\ &= c_{\{l_1\}}(v^L) + c_{\{l_2\}}(v^L) \end{aligned} \quad (5.10)$$

and

$$\begin{aligned} v^L(A) &= \sum_{B \subseteq A} c_B(v^L) \\ &= c_{\{l_1\}}(v^L) + c_{\{l_2\}}(v^L) + c_A(v^L) \end{aligned} \quad (5.11)$$

By comparing (5.10) and (5.11), we get that  $c_A(v^L) = 0$ .

Let us now consider  $k \geq 2$  and suppose by inductive hypothesis that  $c_B(v^L) = 0, \forall B$  such that  $|B| \leq k$  and  $B$  is not connected in  $\Gamma$ . We shall prove that  $c_A(v^L) = 0 \forall A$  such that  $A$  is not connected and  $|A| = k + 1$ . Let  $B_1 \subset A$  be a connected component in  $\Gamma$ , i.e.  $B_1 \in C_{\Gamma_A}$ . Then by hypothesis  $A \setminus B_1 \neq \emptyset$ . It follows that:

$$v^L(A) = v^L(B_1) + v^L(A \setminus B_1). \quad (5.12)$$

Moreover it holds:

$$\begin{aligned} v^L(A) &= \sum_{B \subseteq A} c_B(v^L) \\ &= \sum_{B \subseteq B_1} c_B(v^L) + \sum_{B \subseteq A \setminus B_1} c_B(v^L) + \sum_{B \subseteq A: A \cap B_1 \neq \emptyset \wedge B \cap (A \setminus B_1) \neq \emptyset} c_B(v^L) \\ &= v^L(B_1) + v^L(A \setminus B_1) + \sum_{B \subsetneq A: B \cap B_1 \neq \emptyset \wedge B \cap (A \setminus B_1) \neq \emptyset} c_B(v^L) + c_A(v^L) \\ &= v^L(B_1) + v^L(A \setminus B_1) + c_A(v^L), \end{aligned} \quad (5.13)$$

where (5.13) follows from the inductive hypothesis.

Then, by comparing (5.12) and (5.13) we get:  $c_A(v^L) = 0$ , which ends the proof.  $\square$

Note that an equivalent result has been proved by Van den Nouweland et al. [101] for a value function (i.e. a characteristic function over subsets of links).

**Corollary 5.** *The family of unanimity games  $\{u_A, A \subseteq E$ , where  $A$  is connected in  $\Gamma\}$  is a basis for  $\mathcal{G}_L$*

*Proof.* From Proposition 12, we get that  $\{u_A, A \subseteq E$ , where  $A$  is connected in  $\Gamma\}$  is a spanning set for the vector space  $\mathcal{G}_L$ . Moreover the cardinality of this set is equal to the dimension of  $\mathcal{G}_L$ .  $\square$

Equivalent results hold in the context of graph-restricted games and the proofs can be found, for example, in [46]. Moreover, the previous results hold for a generic value function  $v$  that satisfies the *component additivity property*, i.e. such that  $v(A) = \sum_{T \in C_{\Gamma_A}} v(T)$  for every network  $\Gamma$  over the set of nodes  $N$ .

## 5.6 The Position Value on Particular Classes of Communication Situations

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In general, given a communication situation  $(N, v, \Gamma)$  it is not easy to compute the position value. However, it is so for particular classes of games and graphs. In this

Section we give formulas to compute the position values on particular classes of communication situations, where the underlying network is described by a tree or a cycle. We assume w.l.o.g throughout the Chapter that  $v(\{i\}) = 0 \forall i \in N$ .

### 5.6.1 The Position Value on Trees

Let  $(N, v, \Gamma)$  be a communication situation, where  $\Gamma = (N, E)$  is a tree and  $|N| = n$ . Given a node  $i \in N$  and a coalition  $S \subseteq N$ , we define  $fringe(S) = \{j \in N \setminus S \text{ such that } \{i, j\} \in E \text{ for some } i \in S\}$ . Let  $f(S) := |fringe(S)|$ ,  $deg_S(i)$  the degree of  $i$  in  $S$ , i.e. the number of nodes in  $S$  that are directly connected to  $i$  in  $\Gamma$  and  $deg_{fringe(S)}(i)$  the number of nodes in  $fringe(S)$  that are directly connected to  $i$  in  $\Gamma$ .

We provide a formula for the position value on  $e_S$ , with  $S \subseteq N$  connected in  $\Gamma$  such that  $|S| \geq 2$ . If  $S$  is not connected, it doesn't make sense to consider the position value of  $e_S$ , since the associated link game  $e_S^L$  is the null game.

**Proposition 13.** *Let  $S \subseteq N$  connected in  $\Gamma$ , where  $\Gamma$  is a tree and  $|S| \geq 2$ . Then the position value on the canonical game  $e_S$  is given by:*

$$\pi_i(e_S) = \begin{cases} \frac{1}{2} \frac{(s-2)!(f(S)-1)!}{(m-1)!} \delta_i(s) & \text{if } i \in S \\ -\frac{1}{2} \frac{(s-1)!(m-s-1)!}{(m-1)!} & \text{if } i \in fringe(S) \\ 0 & \text{otherwise} \end{cases} \quad (5.14)$$

where  $m = s + f(S)$  and  $\delta_i(s) = f(S)deg_S(i) - (s-1)deg_{fringe(S)}(i)$ .

*Proof.* We observe that every link not in  $E_{S \cup fringe(S)}$  is superfluous. Therefore  $\pi_i(e_S) = 0$  for every  $i \notin S \cup fringe(S)$  and we can reduce the network to  $(S \cup fringe(S), E_{S \cup fringe(S)})$ .

Consider  $i \in S$ . Node  $i$  gets a positive contribution (equal to  $1/2$ ) every time a link incident to it is the last one to form inside  $S$  and no link outside  $S$  already formed. This happens  $deg_S(i)$  times. Moreover it gets a negative contribution (equal to  $-1/2$ ) when all the links in  $S$  already formed and a link incident to  $i$  is the first one to form outside  $S$ . This happens  $deg_{fringe(S)}(i)$  times. Therefore we get

$$\pi_i(e_S) = \frac{1}{2} \left[ \frac{(s-2)!(m-s)!}{(s-1)!} deg_S(i) - \frac{(s-1)!(m-s-1)!}{(m-1)!} deg_{fringe(S)}(i) \right],$$

where  $m = s + f(S)$  and the expression (5.14) follows directly.

Consider  $i \in fringe(S)$ . Node  $i$  gets a negative contribution when all the links in  $S$  already formed and the only link that connects  $i$  to  $S$  is the first one to form outside  $S$ . Therefore we get  $\pi_i(e_S) = -\frac{1}{2} \frac{(s-1)!(m-s-1)!}{(m-1)!}$ . □

Note that the formula holds also for  $S = N$ , with  $fringe(N) = \emptyset$ , which implies that  $f(N) = 0$ . On the other hand, when  $S = \{i\}$ , the associated link game  $e_S^L$  is the null game, as for  $S$  not connected.

Let us consider the unanimity games  $\{u_S, S \subseteq N\}$ . We also provide a formula for the position value on  $u_S$ , with  $S \subseteq N$  connected in  $\Gamma$  such that  $|S| \geq 2$ .

**Proposition 14.** Let  $S \subseteq N$  connected in  $\Gamma$ , where  $\Gamma$  is a tree and  $|S| \geq 2$ . Then the position value on the unanimity game  $u_S$  is given by:

$$\pi_i(u_S) = \begin{cases} \frac{1}{2} \deg_s(i) \frac{1}{s-1} & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases} \quad (5.15)$$

*Proof.* We observe that every link not in  $E_S$  is superfluous. Therefore  $\pi_i(u_S) = 0$  for every  $i \notin S$  and we can reduce the network to  $(S, E_S)$ .

Consider  $i \in S$ . Node  $i$  gets a positive contribution (equal to  $1/2$ ) every time a link incident to it is the last one to form inside  $S$ . This happens  $\deg_S(i)$  times. Therefore we get

$$\pi_i(e_S) = \frac{1}{2} \frac{(s-2)!(m-s)!}{(m-1)!} \deg_S(i),$$

where  $m = s$  and the result follows directly. □

Moreover, if  $S = \{j\}$ , easy calculations show that:

$$\pi_i(u_S) = \begin{cases} \frac{1}{2} & \text{if } i = j \\ \frac{1}{2f(S)} & \text{if } i \in \text{fringe}(\{j\}) \\ 0 & \text{otherwise.} \end{cases}$$

### 5.6.2 The Position Value on Cycles

Let  $(N, v, \Gamma)$  be a communication situation, where  $\Gamma = (N, E)$  is a cycle and  $|N| = n$ . We provide a formula for the position value on  $e_S$ , where  $S \subseteq N$  is a  $s$ -chain (i.e.  $S$  is connected in  $\Gamma$ ) with  $2 \leq s \leq n-2$ . If  $S$  is not connected, or  $S = \{i\}$ , it does not make sense to consider the position value of  $e_S$ , since the associated link game  $e_S^L$  is the null game.

**Proposition 15.** Let  $S \subseteq N$  be a  $s$ -chain in  $\Gamma$ , where  $\Gamma$  is a cycle and  $2 \leq s \leq n-2$ . Then the position value on the canonical game  $e_S$  is given by:

$$\pi_i(e_S) = \begin{cases} \frac{1}{2} \frac{(s-2)!(m-s-1)!}{(m-1)!} (m-2s+1) & \text{if } i \in S_e \\ \frac{(s-2)!(m-s)!}{(m-1)!} & \text{if } i \in S_i \\ -\frac{1}{2} \frac{(s-1)!(m-s-1)!}{(m-1)!} & \text{if } i \in \text{fringe}(S) \\ 0 & \text{otherwise} \end{cases} \quad (5.16)$$

where  $m = s + f(S)$ ,  $S_e$  is the set of the extremal nodes and  $S_i = S \setminus S_e$  is the set of the internal nodes.

*Proof.* We observe that every link not in  $E_{S \cup \text{fringe}(S)}$  is superfluous. Therefore  $\pi_i(e_S) = 0$  for every  $i \notin S \cup \text{fringe}(S)$  and we can reduce the network to  $(S \cup \text{fringe}(S), E_{S \cup \text{fringe}(S)})$ .

## 5.6. The Position Value on Particular Classes of Communication Situations

Consider  $i \in S$ . We shall distinguish between the internal and extremal nodes of the chain  $S$ . Let  $i \in S_e$  the set of endpoints in  $\Gamma_S$ . Node  $i$  gets a positive contribution when the link incident to it in the chain is the last one to form inside  $S$  and no link outside  $S$  already formed. Moreover it gets a negative contribution when all the links in  $S$  already formed and the link incident to  $i$  in  $fringe(S)$  is the first one to form outside  $S$ . Therefore we get

$$\pi_i(e_S) = \frac{1}{2} \left[ \frac{(s-2)!(m-s)!}{(m-1)!} - \frac{(s-1)!(m-s-1)!}{(m-1)!} \right],$$

where  $m = s + f(S)$  and the expression (5.16) follows directly.

Let  $i \in S_i = S \setminus S_e$ . Node  $i$  gets a positive contribution when one of the two links incident to it in  $S$  is the last one to form inside  $S$ . Therefore we get

$$\pi_i(e_S) = 2 \left[ \frac{1}{2} \frac{(s-2)!(m-s)!}{(m-1)!} \right].$$

Consider  $i \in fringe(S)$ . Node  $i$  gets a negative contribution when all the links in  $S$  already formed and the only link that connects  $i$  to  $S$  is the first one to form outside  $S$ . Therefore we get

$$\pi_i(e_S) = -\frac{1}{2} \frac{(s-1)!(m-s-1)!}{(m-1)!}.$$

□

On the other hand, if  $S = N \setminus \{j\}$ , the following proposition holds.

**Proposition 16.** *Let  $S \subseteq N$  be a  $s$ -chain in  $\Gamma$ , where  $\Gamma$  is a cycle and  $s = n - 1$ . Then the position value on the canonical game  $e_S$  is given by:*

$$\pi_i(e_S) = \begin{cases} \frac{4-n}{2n(n-1)(n-2)} & \text{if } i \in S_e \\ \frac{2}{n(n-1)(n-2)} & \text{if } i \in S_i \\ -\frac{1}{n(n-1)} & \text{if } i \in fringe(S) \end{cases} \quad (5.17)$$

where  $S_e$  is the set of the extremal nodes and  $S_i = S \setminus S_e$  is the set of the internal nodes.

*Proof.* Using the same argument of the previous proof, formulas for  $i \in S$  are derived by noting that there is no superfluous link and  $m = n$ . Moreover, the only node  $i \in fringe(S)$  gets twice the contribution he gets in the previous case since it is directly connected to  $S$  by its incident links.

□

Note that if  $S = N$ , there is no superfluous link and by symmetry  $\pi_i(e_S) = \frac{1}{n}$ , for all  $i \in N$ .

We provide a formula for the position value on  $u_S$ , with  $S \subseteq N$  a  $s$ -chain. If  $2 \leq |S| \leq n - 1$ , the following proposition holds.

**Proposition 17.** Let  $S \subseteq N$  be a  $s$ -chain in  $\Gamma$ , where  $\Gamma$  is a cycle and  $2 \leq s \leq n - 1$ . Then the position value on the unanimity game  $u_S$  is given by:

$$\pi_i(u_S) = \begin{cases} \frac{1}{2} \left[ \frac{(n-s+1)}{n(s-1)} + (2s-3) \frac{1}{n(n-1)} \right] & \text{if } i \in S_e \\ \frac{1}{2} \left[ 2 \frac{(n-s+1)}{n(s-1)} + 2(s-2) \frac{1}{n(n-1)} \right] & \text{if } i \in S_i \\ (s-1) \frac{1}{n(n-1)} & \text{if } i \notin S \end{cases}$$

where  $S_e$  is the set of the extremal nodes, i.e. the endpoints in  $\Gamma_S$ , and  $S_i = S \setminus S_e$  is the set of the internal nodes.

*Proof.* We observe that there is no superfluous link. Consider  $i \in S_e$ . Node  $i$  gets a positive contribution (equal to  $1/2$ ) every time the link incident to it in the chain is the last one to form inside  $S$  (no matter which links already formed outside  $S$ ).

Moreover it gets a positive contribution when the link incident to it outside the chain is the last one to form in  $E \setminus \{a\}$ , where  $a$  is the link incident to  $i$  in the chain  $S$  and every time one of the two links incident to  $i$  is the last one to form in  $E \setminus \{b\}$ , where  $b$  is one of the links in the chain  $S$  not incident to  $i$ . Note that the first case happens  $\sum_{k=0}^{n-s} \binom{n-s+1}{k}$  times; the second one only occurs once and the last case happens  $2(s-2)$  times. This yields the following formula for  $i \in S_e$ :  $\pi_i(u_S) = \frac{1}{2} \left[ \sum_{k=0}^{n-s} \binom{n-s+1}{k} \frac{(s-2+k)!(n-s-k+1)!}{n!} + (2s-3) \frac{1}{n(n-1)} \right]$ . Consider  $i \in S_i$ . Node  $i$  gets a positive contribution (equal to  $1/2$ ) every time one of the two links incident to it in the chain is the last one to form inside  $S$  (no matter which links already formed outside  $S$ ). Moreover it gets a positive contribution whenever one of the two links incident to it is the last one to form in  $E \setminus \{a\}$ , where  $a$  is other link incident to  $i$  in the chain  $S$  and every time one of the two links incident to  $i$  is the last one to form in  $E \setminus \{b\}$ , where  $b$  is one of the links in the chain  $S$  not incident to  $i$ . Note that the first case happens  $2 \sum_{k=0}^{n-s} \binom{n-s+1}{k}$  times; the second one only occurs twice and the last case happens  $2(s-3)$  times. This yields the following formula for  $i \in S_i$ :  $\pi_i(u_S) = \frac{1}{2} \left[ 2 \sum_{k=0}^{n-s} \binom{n-s+1}{k} \frac{(s-2+k)!(n-s-k+1)!}{n!} + 2(s-2) \frac{1}{n(n-1)} \right]$ . Consider  $i \notin S$ . Node  $i$  gets a positive contribution (equal to  $1/2$ ) every time one of the two links incident to it is the last one to form in  $E \setminus \{a\}$ , where  $a$  is one of the links in the chain  $S$ . This happens  $2(s-1)$  times. It follows that, for  $i \notin S$ ,  $\pi_i(u_S) = (s-1) \frac{1}{n(n-1)}$ .

This formula can be simplified by using Lemma [2](#) in Appendix B:

$$\begin{aligned} & \sum_{k=0}^{n-s} \binom{n-s+1}{k} \frac{(s-2+k)!(n-s-k+1)!}{n!} \\ &= \frac{1}{n} \sum_{k=0}^{n-s} \binom{n-s}{k} \frac{n-s+1}{n-s-k+1} \frac{(s-2+k)!(n-s-k+1)!}{(n-1)!} \\ &= \frac{n-s+1}{n} \sum_{k=0}^{n-s} \binom{n-s}{k} \frac{(s-2+k)!(n-s-k)!}{(n-1)!} \\ &= \frac{(n-s+1)}{n(s-1)}, \end{aligned} \tag{5.18}$$

where (5.18) follows from identity (8.3). This ends the proof.  $\square$

Note that if  $S = N$ , all players are symmetric and  $\pi_i(u_S) = \frac{1}{n}$ . On the other hand if  $S = \{j\}$ , the position value is very easy to compute. In fact, the links  $a = \{i, j\}$  and  $b = \{j, k\}$  are symmetric players in the link game, while all the remaining links are superfluous. This implies that  $\Phi_a = \Phi_b = \frac{1}{2}$  and  $\Phi_c = 0 \forall c \in E \setminus \{a, b\}$ .

$$\pi_i(u_S) = \begin{cases} 1/2 & \text{if } i = j \\ 1/4 & \text{if } i \neq j, \{i, j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

### 5.6.3 The Position Value For a Generic Coalitional Game

In the last two Sections we provided formulas for the position value on particular classes of games. We shall use those formulas and the results of Section 5.5 to derive an expression that shows the relation between the position value for a generic game and the position value of unanimity games.

**Proposition 18.** *Let  $(N, v, \Gamma)$  be a communication situation. Then the position value for  $i \in N$  is given by*

$$\pi_i(v) = \sum_{A \subseteq E \text{ connected}} c_A(v^L) \pi_i(w), \quad (5.19)$$

where  $w$  is such that  $w^L = u_A$ .

*Proof.* By definition of position value and by Corollary 5 we get that:

$$\begin{aligned} \pi_i(v) &= \frac{1}{2} \sum_{a \in A_i} \Phi_a(v^L) = \frac{1}{2} \sum_{a \in A_i} \sum_{A \subseteq E \text{ connected}} c_A(v^L) \Phi_a(u_A) \\ &= \sum_{A \subseteq E \text{ connected}} c_A(v^L) \pi_i(w) \end{aligned}$$

where  $w$  is such that  $w^L = u_A$ .  $\square$

This result implies that, in order to compute the position value of a generic game, we have to consider the position value on those games whose corresponding link game is a unanimity game on a connected subset of links.

However, when  $\Gamma$  is a tree, the formula (5.19) can be simplified and a direct relation between the position value for a generic game and the position value of unanimity games can be obtained.

**Corollary 6.** *Let  $(N, v, \Gamma)$  be a communication situation, where  $\Gamma$  is a tree. Then the position value for  $i \in N$  is given by*

$$\pi_i(v) = \sum_{S \subseteq N \text{ connected}} c_{E_S}(v^L) \pi_i(u_S). \quad (5.20)$$

*Proof.* Consider  $A \subseteq E$  connected in  $\Gamma$ . Let  $S$  be the set of nodes in  $\Gamma_A$ . This definition of  $S$  induces a bijection between the set  $\{w : w^L = u_A, A \text{ connected in } \Gamma\}$  and the set  $\{u_S : S \subseteq N, S \text{ connected in } \Gamma\}$ . Therefore the result follows directly.  $\square$

However, the computation of the position value for a generic game remains difficult even if the underlying graph is a tree. Indeed, deriving the position value using formula (5.20) requires listing all subtrees of a tree (the problem has been extensively addressed in the literature, see, for example, [81], [95] and [40]) and computing the corresponding unanimity coefficients.

### 5.7 Concluding remarks

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In this Chapter we proposed a family of solution concepts for communication situations that embraces the principles of the two main approaches existing in the related literature. We also provided a different interpretation of the position value, as the solution concept arising from a particular symmetric allocation protocol, which prescribes how to share the surplus generated by a link among the players involved in the network formation process. Moreover, we provide an expression for the position value of a game when the underlying network is a tree, which relates its computation to the one for unanimity games.

The computation of the position value and its complexity remains an open problem, which has not been studied in the literature and deserves, in our opinion, further investigation. Another interesting direction for future research is to provide a characterization of the family of solution concepts we introduced, based on some reasonable properties that an allocation protocol should satisfy. Moreover, it would be interesting to investigate the relationship between different allocation protocols and known solution concepts, besides the position value.

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## GAGs and Argumentation Theory

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This Chapter is devoted to the introduction, by means of a property-driven approach, of a *conflict-based* ranking of arguments in an argumentation graph. This approach is based on the observation that an argumentation graph, which consist of arguments and attack relations between pairs of arguments, is conflictual. We introduce a *conflict index* that measures the contribution of each argument to the total disagreement of the graph, therefore yielding a conflict-based ranking of arguments in an argumentation framework, and we provide a game-theoretical interpretation of such index in terms of coalitional games.

Indeed, we show that the conflict-based ranking we propose may be re-interpreted in terms of a classical solution for coalitional games, that is as the average marginal contribution of each argument to the disagreement induced by all possible coalitions of arguments in an argumentation graph. We do so by defining a cooperative game, where the players are the arguments in an argumentation graph and every coalition of arguments is assigned a value, which expresses the total disagreement within the coalition. In particular, every node and every link inside a coalition of arguments contributes to the value of the coalition with its individual share of the disagreement, as measured by the attack relations it brings to the coalition. The so-defined game is indeed representable in terms of basic GAGs, as a combination of the original model introduced in Chapter 3 and its variant defined on links, briefly described in Section 3.3. We propose the Shapley value of such a game as a *conflict index* that measures the controversiality of arguments, since it measures the power of each argument in bringing conflict to the argumentation framework.

## 6.1 Introduction

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Abstract argumentation [38] deals with the construction and the analysis of non-monotonic reasoning systems based on the complex interplay among distinct arguments. Basically, in this framework, arguments are represented as atomic entities (without any regard to their internal structure) whose interaction is modelled via a binary attack relation expressing a (possible) *disagreement* between pairs of arguments. In the literature, several *extension* semantics or labellings have been associated to the abstract argumentation framework with the objective to specify which arguments are accepted or not, and which are undecided [21,37].

Different from extension semantics, the aim of *gradual* semantics is to assign a degree of acceptability to each argument. An example of gradual semantic is the h-Categorizer introduced in [13], which is intended to quantify the relative strength of an argument taking into account how much such an argument is challenged by other arguments, and by recursion, how much it challenges its counter-arguments. Another gradual interaction-based evaluation reflecting the way in which arguments weaken each others has been introduced in [24] for a bipolar argumentation framework (i.e., supporting both attack and support relations between arguments). Other examples are the ranking-based semantics introduced in [4], where a procedure to transform an argumentation graph (i.e., a digraph where the nodes are the arguments and the arrows represent the attack relation) is introduced following a property-driven approach. Still different examples are the probabilistic approaches studied in [49,97], which interpret the probability of an argument as the degree to which the argument is believed to hold.

Game theory has also been used to define intermediate level of acceptability of arguments. Specifically, in [67] a degree of acceptability is computed taking into account the minimax value of a zero-sum game between a ‘proponent’ and an ‘opponent’ and where the strategies and the payoffs of the players depend on the structure of an argumentation graph. More recently, coalitional games have been applied in [16] to measure the relative importance of arguments taking into account both preferences of an agent over the arguments and the information provided by the attack relations.

In the aforementioned approaches, the weight attributed to each argument represents the strength of an argument to “force” its acceptability. On the other hand, acceptability is not the only arguments’ attribute that has been studied in literature from a “gradual” perspective. In [98] an index has been introduced to represent the *controversiality* of single arguments, where the most controversial arguments are those for which taking a decision on whether they are acceptable or not is difficult. In a similar direction, the problem of measuring the *disagreement* within an argumentation framework has been studied in [3], where the authors provided an axiomatic analysis of different disagreement measures for argumentation graphs. Both definitions of controversial-based ranking and disagreement measure are strictly related to the notion of *enforcement* introduced in [8], and aimed at identifying the minimal changes needed to enforce the acceptability of a set of arguments (see [98] for a discussion on the relation between controversiality and enforcement).

The objective of this Chapter is twofold. First, we want to show that the properties introduced in [3] for argumentation graphs can be reformulated for single arguments, and may drive the definition of a *conflict-based* ranking, that can be seen as an alter-

native ranking for measuring the controversiality of arguments. Our second goal, is to merge the abstract argumentation framework into a game theoretical coalitional framework similar to the one already proposed in [16], and re-interpret our conflict-based ranking in terms of a classical solution for coalitional games, that is as the average marginal contribution of each argument to the disagreement induced by all possible coalitions of arguments in an argumentation graph. Considering persuasion scenarios, we argue that the conflict-based ranking introduced in this Chapter may drive agents to select those arguments that should be further developed in order to strengthen certain positions in a debate, hence, responding to the question raised in [98] about the definition of a ranking representing the *potential for development* of arguments.

The Chapter is structured as follows. In Section 6.2 we introduce the concept of argumentation framework and the disagreement measure proposed in [3]. In Section 6.3 we focus on the properties for a conflict index for arguments. Section 6.4 is devoted to the property-driven analysis of conflict indices. Section 6.5 deals with the analysis of an associated coalitional framework and the reformulation of the conflict index introduced in Section 6.4 as a solution for these games.

## 6.2 Argumentation Graphs and Disagreement Measures

In this Section we introduce some preliminary notations and definitions on argumentation graphs and we introduce the disagreement measure proposed in [3].

An argumentation framework is a directed graph in which nodes represent arguments and direct edges represent attack relations [38]. Formally, an *argumentation framework* (or *argumentation graph*)  $\mathbf{A}$  is a pair  $\langle A, R \rangle$  where  $A = \{1, \dots, n\}$  is a non empty finite set of arguments and  $R \subseteq A \times A$  is an attack relation. Given two arguments  $a, b \in A$ ,  $(a, b) \in R$ , or equivalently  $aRb$ , if  $a$  attacks  $b$ . We denote by  $\mathcal{A}$  the set of all argumentation graphs and by  $\mathcal{A}^A$  the set of all argumentation graphs with  $A$  as the set of arguments. The number of arguments in the graph, that is  $n = |A|$ , is called the *size* of the graph. A *path* in  $\mathbf{A}$  is a sequence of arguments  $(a_1, \dots, a_k)$ , where  $a_i \in A$  for all  $i = 1, \dots, k$  such that  $a_i R a_{i+1}$  for all  $1 \leq i < k$  and  $a_i \neq a_j$  for all  $i \neq j$ . An *elementary cycle* is a path  $(a_1, \dots, a_k)$  such that  $a_k R a_1$ . A cycle represents a set of arguments which contradicts itself. Let  $i, j \in A$ ,  $i \neq j$ . The *distance* between  $i$  and  $j$   $d_{i,j}$  is defined as the length of the shortest path from  $i$  to  $j$  if such path exists, otherwise  $d_{i,j} = |A| + 1$ . If  $i = j$  then  $d_{i,i}$  is the length of the shortest elementary cycle in which  $i$  is involved, otherwise  $d_{i,i} = |A| + 1$ .

Let  $\mathbf{A} = \langle A, R \rangle$  and  $\mathbf{A}' = \langle A', R' \rangle$  be two argumentation graphs. An *isomorphism* from  $\mathbf{A}$  to  $\mathbf{A}'$  is a bijective function  $f : A \rightarrow A'$  such that for all  $a, b \in A$ ,  $aRb$  iff  $f(a)R'f(b)$ .

An argumentation graph is by definition conflictual, since it describes the attack relations among arguments in an argumentation system. In order to quantify this inner conflict, it is possible to associate to each argumentation framework a disagreement measure.

A *disagreement measure* [3] is a function  $\mathcal{K} : \mathcal{A} \rightarrow [0,1]$  with the interpretation that, for every  $\mathbf{A}, \mathbf{A}' \in \mathcal{A}$ ,  $\mathbf{A}$  is more conflicting than  $\mathbf{A}'$  if  $\mathcal{K}(\mathbf{A}) > \mathcal{K}(\mathbf{A}')$ . Note that  $\mathcal{K} = 0$  corresponds to the absence of disagreement in a graph, while the maximum disagreement is set to  $\mathcal{K} = 1$ .

Different measures can be proposed in order to quantify the disagreement in a graph. A property-driven approach has been proposed in [3], which leads to the introduction of a measure that is based on the concept of *global distance* among arguments in an argumentation framework.

Let  $\mathbf{A} = \langle A, R \rangle$  be an argumentation framework. The *global distance*  $D(\mathbf{A})$  is defined as

$$D(\mathbf{A}) = \sum_{i \in A} \sum_{j \in A} d_{i,j}.$$

To compute  $D(\mathbf{A})$  it is possible to build a matrix of distances  $\mathcal{D}$ :

$$\mathcal{D} = \begin{bmatrix} d_{1,1} & d_{1,2} & \dots \\ d_{2,1} & d_{2,2} & \dots \\ \dots & \dots & \dots \end{bmatrix},$$

in which the element  $d_{i,j}$  is the distance between argument  $i$  and argument  $j$ , and to sum all its elements.

The maximum value of  $D(\mathbf{A})$  corresponds to the case where  $R = \emptyset$  and it equals  $n^2(n + 1)$ . On the other hand, when  $R = A \times A$ ,  $D(\mathbf{A})$  takes minimum value, that is

### 6.3. A Property-driven Approach to Measure Conflict of Arguments

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$n^2$ . The maximum value of  $D(\mathbf{A})$  corresponds to the case in which the disagreement is minimal and the minimal value to the case in which the disagreement is maximal. For this reason it makes sense to define a disagreement measure that depends on the opposite value of  $D(\mathbf{A})$ .

**Definition 5** (Distance-based measure [3]). *Let  $\mathbf{A} = \langle A, R \rangle$  be an argumentation graph. The distance-based measure  $\mathcal{K}^D(\mathbf{A})$  is defined as:*

$$\mathcal{K}^D(\mathbf{A}) = \frac{\max - D(\mathbf{A})}{\max - \min}, \quad (6.1)$$

where  $\max = n^2(n + 1)$  and  $\min = n^2$ .

The distance-based measure is normalized by the value  $\max - \min$  in order to obtain a value between 0 and 1 that makes possible to easily compare different argumentation graphs. In [3] it is shown that this measure satisfies the following set of axioms: it depends only on the structure of the graph and not on the label of the nodes with the consequence that it assigns the same value to isomorphic graphs (*abstraction*); it assigns 0 to graphs without attack relations and 1 to complete graphs (*coherence* and *maximality*); the disagreement value of a graph does not increase if an isolated argument is added, i.e. if an argument is added without modifying the set  $R$ , and it does not decrease if an attack between two arguments is added (*free independence* and *monotonicity*); it detects cycles by assigning them a higher disagreement with respect to acyclic graphs of the same size (*cycle sensitivity*) and, when comparing two cycles, it assigns a higher value to the cycle with less arguments, following the idea that cycles are seen as dilemmas or paradoxes and the less arguments are needed to produce a cycle, the stronger is the disagreement (*size sensitivity*).

### 6.3 A Property-driven Approach to Measure Conflict of Arguments

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Given a disagreement measure, it is interesting to assess which are the arguments that contribute the most to the total disagreement in an argumentation graph. To this purpose, we introduce, by using an axiomatic approach, a conflict index that evaluates the contribution of each argument to the total disagreement.

A *conflict index*  $K : \mathcal{A}^A \rightarrow \mathbb{R}^n$  is a function that assigns to every argumentation graph with  $n = |A|$  nodes (arguments) a vector in  $\mathbb{R}^n$ , representing the contributions of each argument to the conflict in the graph. The higher the value that such index assigns to an argument, the larger is the disagreement brought by that argument to the graph. Let  $\mathbf{A} = \langle A, R \rangle$  be an argumentation framework. For every argument  $i \in A$ ,  $K_i(\mathbf{A})$  measures the contribution of argument  $i$  to the total disagreement in the graph: an argument  $i$  brings more disagreement than  $j$  if  $K_i(\mathbf{A}) > K_j(\mathbf{A})$ . In terms of controversiality, an argument with a large conflict index is highly controversial and is pointed as deserving further development in order to decide on its acceptability.

Following the approach proposed in [3], we introduce eight axioms that a conflict index should satisfy in order to describe the behaviour of arguments in a graph.

The first axiom states that the contribution to the disagreement of each argument depends only on the structure of the graph. This means that the label of an argument does

not add information about the contribution to the disagreement. Therefore if two argumentation graphs are isomorphic, the index assigns the same value to corresponding nodes.

**Axiom 1. (Abstraction)** *Let  $\mathbf{A} = \langle A, R \rangle$  and  $\mathbf{A}' = \langle A', R' \rangle$  be two argumentation graphs. If  $\mathbf{A}$  and  $\mathbf{A}'$  are isomorphic, then for all  $i \in A$ ,  $K_i(\mathbf{A}) = K_{f(i)}(\mathbf{A}')$ , where  $f(\cdot)$  is an isomorphism between  $\mathbf{A}$  and  $\mathbf{A}'$ .*

The second axiom set to zero the conflict index of arguments in a graph without attack relations. This is a natural request that follows from the absence of disagreement in the graph.

**Axiom 2. (Coherence)** *Let  $\mathbf{A} = \langle A, R \rangle$  be an argumentation graph. If  $R = \emptyset$ , then  $K_i(\mathbf{A}) = 0$  for all arguments  $i \in A$ .*

In order to state the third axiom, the concept of *star* must be introduced. An argumentation graph is a star if an argument  $i$  exists that attacks all the other arguments and receives attacks from all the arguments, included itself, and there is not other attack relation between arguments.  $i$  is called centre of a star.

The third axiom states that the argument that brings the maximum of disagreement in a graph is the centre of the star.

**Axiom 3. (Maximality)** *Let  $\mathbf{A} = \langle A, R \rangle$  be a star and let  $i$  be the center of the star. Then  $K_i(\mathbf{A}) > K_j(\mathbf{A}) \forall j \neq i \in A$  and for all the argumentation graphs  $\mathbf{A}'$  of the same size,  $\forall h \in A'$   
 $K_i(\mathbf{A}) \geq K_h(\mathbf{A}')$ .*

The fourth axiom states that adding isolated arguments to an argumentation graph which contains attacks does not increase the contribution of each argument to the total disagreement.

**Axiom 4. (Free independence)** *Let  $\text{Args}$  be the universe of arguments. Let  $\mathbf{A} = \langle A, R \rangle$  be an argumentation graph with  $R \neq \emptyset$  and let  $\mathbf{A}' = \langle A \cup a, R \rangle$  an argumentation graph with  $a \in \text{Args} \setminus A$ . Then  $K_i(\mathbf{A}) \geq K_i(\mathbf{A}') \forall i \in A$ .*

Next axiom states that if a new attack is added to an argumentation graph, no argument will decrease its contribution to the total disagreement.

**Axiom 5. (Monotony)** *Let  $\mathbf{A} = \langle A, R \rangle$  be an argumentation graph and let  $\mathbf{A}' = \langle A, R \cup R' \rangle$  be an argumentation graph with  $R' \subseteq (A \times A) \setminus R$ . Then, for all  $i \in A$ ,  $K_i(\mathbf{A}) \leq K_i(\mathbf{A}')$ .*

Next results for the centre of a star follows from the previous axioms.

**Proposition 19.** *If a conflict index satisfies Axioms 3 and 5, then adding attacks does not change the conflict index of the centre of the star.*

*Proof.* Let  $\mathbf{A} = \langle A, R \rangle$  be a star of size  $n$  and centre  $i$  and let  $\mathbf{A}'$  be a graph in which one attack is added to a star of size  $n$ . Thanks to axiom 3,  $K_i(\mathbf{A}) \geq K_h(\mathbf{A}') \forall h \in A'$ , but axiom 5 states that  $\forall j \in A$   $K_j(\mathbf{A}) \leq K_j(\mathbf{A}')$ , so the centre of the star does not change its value. □

Axiom 6 states that, among argumentation graphs of the same size, arguments that belong to an elementary cycle have larger contribution to the disagreement than arguments that belong to an acyclic graph.

**Axiom 6. (Cycle sensitivity)** *Let  $\mathbf{A} = \langle A, R \rangle$  be an acyclic argumentation graph and  $\mathbf{A}' = \langle A', R' \rangle$  an elementary cycle. If  $|A| = |A'|$  then, for all  $i \in A$  and for all  $j \in A'$ ,  $K_i(\mathbf{A}) < K_j(\mathbf{A}')$ .*

Axiom 7 states that the larger is the size of an elementary cycle, the smaller is the contribution of each argument to the total disagreement.

**Axiom 7. (Size sensitivity)** *Let  $\mathbf{A} = \langle A, R \rangle$  and  $\mathbf{A}' = \langle A', R' \rangle$  be two elementary cycles with  $|A| < |A'|$ . Then, for all  $i \in A$  and for all  $j \in A'$ ,  $K_i(\mathbf{A}) > K_j(\mathbf{A}')$ .*

Last axiom prescribes that the sum of the contributions of each node to the disagreement in a graph should be equal to the total disagreement in the graph, as measured by the disagreement measure in (6.1).

**Axiom 8. (Efficiency)** *Let  $\mathbf{A} = \langle A, R \rangle$  be an argumentation graph. and let  $\mathcal{K}^D$  be the distance-based disagreement measure defined by (6.1). Then  $\sum_{i \in N} K_i(\mathbf{A}) = \mathcal{K}^D(\mathbf{A})$ .*

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## 6.4 A Distance-based Conflict Index for Arguments

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In this Section we introduce a distance-based conflict index, which is linked to the disagreement measure introduced in [3], and show that it satisfies all the axioms stated in the previous Section.

**Definition 6** (Distance-based conflict index). *Let  $\mathbf{A} = \langle A, R \rangle$  be an argumentation graph of size  $n$ . We define the distance-based conflict index  $K_i^D$  as the conflict index that assigns to every  $i \in A$  the following value:*

$$K_i^D(\mathbf{A}) = \frac{1}{\Delta} \left( \frac{\max}{n} - \varphi_i \right), \quad (6.2)$$

where  $\max = n^2(n+1)$ ,  $\Delta = \max - \min = n^2(n+1) - n^2 = n^3$  and  $\varphi_i = \frac{1}{2} \sum_{j \in A \setminus i} d_{i,j} + \frac{1}{2} \sum_{j \in A \setminus i} d_{j,i} + d_{i,i}$ .

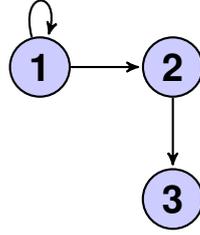
The value  $K_i^D(\mathbf{A})$  depends on  $\varphi_i$ , which takes into account all the distances from  $i$  to the other arguments and vice versa.

**Example 17.** *Let  $\mathbf{A} = \langle A, R \rangle$  be the argumentation graph where  $A = \{1, 2, 3\}$  and  $R = \{(1, 1), (1, 2), (2, 3)\}$ , as depicted in Figure 6.1.*

*The corresponding matrix of distances  $\mathcal{D}$  is*

$$\mathcal{D} = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 4 & 1 \\ 4 & 4 & 4 \end{bmatrix}$$

*and the value of  $\max$  and  $\min$  are 36 and 9 respectively. Thus  $\varphi_1 = \frac{13}{2}$ ,  $\varphi_2 = 9$ ,  $\varphi_3 = \frac{19}{2}$  and the conflict index are  $K_1 = 0.20$ ,  $K_2 = 0.11$ ,  $K_3 = 0.09$ . The most conflictual argument according to our measure is argument 1, followed respectively by arguments 2 and 3.*



**Figure 6.1:** An argumentation graph with three arguments.

**Theorem 3.** The distance-based conflict index defined by (6.2) satisfies all the eight axioms introduced in Section 6.3

*Proof.* Let  $\mathbf{A} = \langle A, R \rangle$  be an argumentation graph. **1.** If  $\mathbf{A}$  and  $\mathbf{A}'$  are isomorphic, then  $\varphi_i = \varphi_{f(i)}$  that implies  $K_i(\mathbf{A}) = K_{f(i)}(\mathbf{A}')$ . Abstraction is therefore satisfied. **2.** If  $R = \emptyset$  then  $\varphi_i = n(n+1)$ . It follows that

$$K_i(\mathbf{A}) = \frac{1}{\Delta} \left( \frac{\max}{n} - \varphi_i \right) = \frac{1}{n^3} \left( \frac{n^2(n+1)}{n} - n(n+1) \right) = 0.$$

Coherence is therefore satisfied. **3.** Let  $i \in A$  be an argument such that  $\forall j \in A \ jRi$  and  $iRj$  and such that there are not other attack relations between arguments. Then  $\varphi_i = n$  and  $\varphi_j = 2n - 1$  that implies

$$K_i(\mathbf{A}) = \frac{1}{n}$$

and

$$K_j(\mathbf{A}) = \frac{n^2 - n + 1}{n^3}.$$

It follows that  $K_i(\mathbf{A}) > K_j(\mathbf{A}) \ \forall j \in A$ . Furthermore  $n$  is the minimum possible value of  $\varphi$  for a node. This means that for all the argumentation graphs  $\mathbf{A}'$  of the same size,  $\forall h \in A' \ K_i(\mathbf{A}) \geq K_h(\mathbf{A}')$ . Maximality is therefore satisfied. **4.** Let  $\mathbf{A}' = \langle A \cup a, R \rangle$  be an argumentation graph with  $a \in \text{Args} \setminus A$ . For all the arguments  $i \in A \ \varphi_i(\mathbf{A}') \geq \varphi_i(\mathbf{A})$ . This implies  $K_i(\mathbf{A}) \geq K_i(\mathbf{A}') \ \forall i \in A$ . Free independence is therefore satisfied. **5.** Let  $\mathbf{A}' = \langle A, R' \rangle$  be an argumentation graph with  $R' \subseteq (A \times A) \setminus R$ . Then, for all the arguments  $i \in A, \varphi_i(\mathbf{A}) \geq \varphi_i(\mathbf{A}')$  and it follows that  $K_i(\mathbf{A}') \geq K_i(\mathbf{A}) \ \forall i \in A$ . Monotony is therefore satisfied. **6.** Let  $\mathbf{A} = \langle A, R \rangle$  be an acyclic argumentation graph and  $\mathbf{A}' = \langle A', R' \rangle$  an elementary cycle and let  $|A| = |A'|$ . In order to prove this axiom, first the acyclic configuration in which  $i$  gives the maximum possible disagreement is found and then it is shown that all the arguments in a cycle has a greater conflict index than  $i$ . The acyclic graph in which  $i$  gives the most possible disagreement is the one in which there is only one attack between  $i$  and all the other arguments  $h \in A \setminus i \ (\forall h \in A \ iRh \vee hRi)$ . In this case  $\varphi_i$  has the minimum value possible for a node in a acyclic graph because  $\forall j \neq i \in A \ (d_{i,j} = 1 \text{ and } d_{j,i} = n+1) \vee (d_{i,j} = n+1 \text{ and } d_{j,i} = 1)$ . Therefore

$$\varphi_i(\mathbf{A}) = \frac{1}{2}(n+1)(n-1) + \frac{1}{2}(n-1) + (n+1) = \frac{1}{2}n^2 + \frac{3}{2}n$$

It follows that

$$K_i(\mathbf{A}) = \frac{n-1}{2n^2}.$$

For an elementary cycle, for all the arguments  $j \in A'$  it holds that

$$\varphi_j(\mathbf{A}') = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

and

$$K_j(\mathbf{A}') = \frac{n+1}{2n^2}.$$

So for all  $i \in A$  and for all  $j \in A'$ ,  $K_i(\mathbf{A}) < K_j(\mathbf{A}')$ . Cycle sensitivity is therefore satisfied. **7.** Let  $\mathbf{A}$  be an elementary cycle. As it is proved in the cycle sensitive property,  $\forall i \in A$

$$\varphi_i(\mathbf{A}) = \frac{n(n+1)}{2},$$

so it follows:

$$K_i(\mathbf{A}) = \frac{n+1}{2n^2}.$$

Let  $\mathbf{A}' = \langle A', R' \rangle$  be another elementary cycle with  $|A| < |A'| = m$ , then  $\forall j \in A'$

$$K_j(\mathbf{A}') = \frac{m+1}{2m^2}.$$

It's easy to check that if  $n \geq 1$  and  $n < m$ ,  $K_i(\mathbf{A}) > K_j(\mathbf{A}')$  for all  $i \in A$  and for all  $j \in A'$ . Size sensitivity is therefore satisfied.

**8.** Let  $\mathcal{K}^D(\mathbf{A}) = \frac{max-D(A)}{max-min} = 1 + \frac{1}{n} - \frac{D(A)}{n^3}$  be the disagreement measure of the graph  $\mathbf{A}$ . Then

$$\sum_{i \in N} \frac{max}{n} - \varphi_i = max - D(\mathbf{A})$$

implies

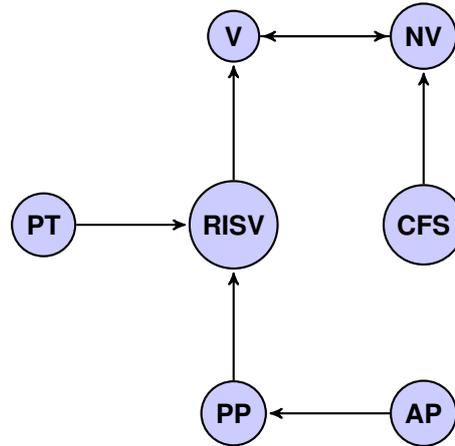
$$\sum_{i \in N} K_i^D(\mathbf{A}) = \frac{max - D(\mathbf{A})}{max - min}.$$

Efficiency is therefore satisfied and this concludes the proof. □

The distance-based conflict index associates to each argument a value that depends on the distance from an argument to the others and vice versa. Every isolated argument has measure equal to zero which means that it does not bring any disagreement. Furthermore, this index detects arguments that belong to cycles giving them a high value.

By assigning to each argument a real value, the index enables the comparison among arguments in contributing to the conflict in argumentation frameworks, which naturally leads to consider the conflict index as a ranking-based semantic, that is a function that transforms every argumentation graph into a ranking on the set of arguments.

A *ranking* on a set  $A$  is a binary relation  $\preceq$  on  $A$  that is  $\preceq$  is total, i.e.  $\forall a, b \in A, a \preceq b$  or  $b \preceq a$  and transitive, i.e.  $\forall a, b, c \in A, \text{if } a \preceq b \text{ and } b \preceq c \text{ then } a \preceq c$ .



**Figure 6.2:** An argumentation framework representing a debate between two transplant coordinators.

A ranking-based semantic [4] is a function  $S$  that transforms any argumentation framework  $\mathbf{A} = \langle A, R \rangle$  into a ranking on  $A$ .

The distance-based conflict index generates a ranking between arguments since it assigns a real number to each argument and  $\mathbb{R}$  is totally ordered. In particular, the higher is the conflict index of an argument, the least will be its position in the ranking, with the interpretation that an argument controversiality is directly proportional to the amount of disagreement it produces.

**Example 18.** Let  $A$  the argumentation graph in Figure 6.2. The graph summarizes a debate between two transplant coordinators that must take a decision about the viability of an organ. One coordinator is against the viability while the other is in favour of the viability of the organ. The first argues that the organ is not viable, since the donor had endocarditis due to streptococcus viridans and the recipient could then be infected by the same microorganism. In contrast, the other argues that the organ is viable, because the organ presents a correct function and a correct structure and the infection could be prevented with post-treatment with penicillin, even if the recipient is allergic to penicillin, since there is the option of post-treatment with teicoplanin. The following seven arguments constitutes the debate:

- $NV$ : organ is non viable;
- $V$ : organ is viable;
- $CFS$ : organ has correct function and correct structure;
- $RISV$ : recipient could be infected with streptococcus viridans;
- $PP$ : post-treatment with administer penicillin;
- $PT$ : post-treatment with administer teicoplanin;
- $AP$ : recipient is allergic to penicillin.

The distance-based conflict index defined in (6.2) provides the following ranking:

## 6.5. A Game-Theoretical Interpretation of the Conflict Index

<i>Ranking position</i>	<i>Argument</i>	$K_i^D(\mathbf{A})$
1	V	0.0816
2	NV	0.0772
3	RISV	0.0481
4	PP	0.0364
5	AP	0.0321
6	PT	0.0262
7	CFS	0.0190

Based on this ranking, we can conclude that V (the organ is viable) and NV (the organ is not viable) are the two arguments that bring more disagreement in the argumentation framework. In particular, V is the more controverse one. If a mediator is asked to evaluate the arguments in order to take a decision about the transplant, we argue that he should suggest, according to this ranking, to further develop these two arguments, by investigating the likelihood of a risk of infection on one hand, and the correct function and structure of the organ, on the other hand.

### 6.5 A Game-Theoretical Interpretation of the Conflict Index

In the previous Section, an index of conflict for arguments in an argumentation graph is proposed. The introduction of such index has been justified by means of an axiomatic approach: we proved that the index satisfies a number of interesting properties. Moreover, in this Section, we show that it coincides with the Shapley value of a properly defined game.

Let  $\mathbf{A} = \langle A, R \rangle$  be an argumentation graph, where  $A$  has cardinality  $n$ . We introduce a cooperative game  $(A, v)$ , where the set of players coincides with the set of arguments  $A$  in the argumentation graph and the characteristic function is defined as follows for every  $S \subseteq A$ :

$$v(S) = \frac{\max - D(S)}{\max - \min}, \quad (6.3)$$

where  $D(S) = \sum_{i,j \in S} d_{i,j}$ ,  $\max = n^2(n+1)$  is the maximal value that  $D$  can attain in an argumentation graph with  $n$  arguments and  $\min = n^2$  is the minimal one.  $D(S)$  measures the global distance among arguments in a coalition  $S$ , taken into account the attack relations that exist among them on the whole graph: it is defined as the sum of distances among the nodes of the coalition  $S$ , where the distance between two nodes is computed on the entire graph. The smaller is the global distance in a coalition, the higher is the value of that coalition in the game  $v$ , reflecting the fact that the overall conflict in a coalition of arguments inversely depends on the distance among arguments.

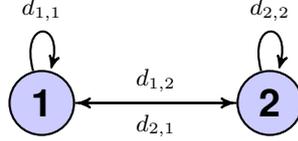
We observe that such a game can be expressed in terms of basic GAGs, as a combination of the original model introduced in Chapter 3 and its variant defined on links, briefly described in Section 3.3.

Indeed, it is possible to write  $v$  as follows

$$v(S) = \frac{\max}{\Delta} - \frac{D(S)}{\Delta},$$

where  $\Delta = \max - \min = n^3$ . Therefore,  $v$  can be written as the linear combination of two games:  $v = \frac{1}{\Delta}(v^{\max} + v^D)$ , where  $v^{\max}$  is a constant game that assumes value  $\max$

for every coalition and  $v^D(S) = D(S)$ . Moreover, we notice that  $v^D$  can be expressed in terms of unanimity games, by associating to the game a complete weighted graph, with  $|A| = n$  nodes, where the weight on the edge  $\{i, j\}$  is precisely the distance  $d_{i,j}$ , as depicted in the following figure for a graph with two arguments.



Given this graph, it is easy to observe that  $D$  can be reformulated in terms of unanimity games as follows:

$$v^D = \sum_{i,j \in N} \alpha_{i,j} u_{\{i,j\}} + \sum_{i \in N} \alpha_i u_{\{i\}},$$

where:

- $\alpha_{i,j} = d_{i,j} + d_{j,i}$ ;
- $\alpha_i = d_{i,i}$ ;
- $u_{\{i,j\}}$  and  $u_{\{i\}}$  are the unanimity games on  $\{i,j\}$  and  $\{i\}$ .

It follows that, for every  $S \subseteq A$ :

$$\begin{aligned} v^D(S) &= \sum_{e=\{i,j\} \in R_S} \alpha_{i,j} + \sum_{i \in S} \alpha_i \\ &:= v^1(S) + v^2(S), \end{aligned}$$

where  $R_S$  is the set of edges in the argumentation graph induced by coalition  $S$ .  $v^D$  is therefore the sum of two games,  $v^1$  and  $v^2$ , where  $v^2$  is a basic GAG associated to the GAS  $\langle A, v^2, \mathcal{M}^2 \rangle$ , where each player  $i \in A$  is assigned the value  $v^2(i) = \alpha_i$  and  $\mathcal{M}$  is associated to the collection of sets of friends and enemies defined as follows:  $F_i = \{i\}$  and  $E_i = \emptyset$  for all  $i \in A$ . On the other hand,  $v^1$  can be represented in terms of a link basic GAG  $\langle A, w, \mathcal{L} \rangle$ , as described in Section 3.3, where  $w(e) = \alpha_{i,j}$  for every  $e = \{i, j\} \in R$  and  $\mathcal{L}$  is associated to the collection of sets of friends and enemies defined as follows:  $F_e = \{e\}$  and  $E_e = \emptyset$  for all  $e \in R$ .

We show that the conflict index introduced in the previous section coincides with the Shapley value of the game defined in (8). Indeed, the following proposition holds.

**Proposition 20.** *The distance-based conflict index defined by (6.2) coincides with the Shapley value of the game  $(A, v)$  defined in (8).*

*Proof.* As we observed above,  $v$  can be written as the linear combination of two games:  $v = \frac{1}{\Delta}(v^{max} + v^D)$ , where  $v^{max}$  is a constant game that assumes value  $max$  for every coalition and  $v^D(S) = D(S)$ . Therefore, the Shapley value of  $v$ , for every  $i \in A$ , is given by

$$\sigma_i(v) = \frac{1}{\Delta}(\sigma_i(v^{max}) + \sigma_i(v^D)).$$

## 6.5. A Game-Theoretical Interpretation of the Conflict Index

The value  $\sigma_i(v^{max})$  is easy to calculate because  $v^{max}$  is a constant game and thus  $\sigma_i(v^{max}) = \frac{max}{n}$ . Moreover, the game  $v^D$  can be written in terms of unanimity games as follows:

$$v^D = \sum_{i,j \in N} \alpha_{i,j} u_{\{i,j\}} + \sum_{i \in N} \alpha_i u_{\{i\}},$$

where  $\alpha_{i,j} = d_{i,j} + d_{j,i}$  and  $\alpha_i = d_{i,i}$ . It follows that the Shapley value of  $v^D$  for every  $i \in A$  has the following expression:

$$\sigma_i(v^D) = \frac{1}{2} \sum_{j \in A \setminus \{i\}} d_{i,j} + \frac{1}{2} \sum_{j \in A \setminus \{i\}} d_{j,i} + d_{i,i}.$$

Therefore, the Shapley value of  $v$ , for every  $i \in A$ , is given by

$$\sigma_i(v) = \frac{1}{\Delta} \left( \frac{max}{n} \right) - \frac{\varphi_i}{\Delta},$$

which coincides with the expression in (6.2). □

**Example 19.** Let  $A$  the argumentation graph of Example 17. The matrix of distances  $\mathcal{D}$  in the entire graph is

$$\mathcal{D} = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 4 & 1 \\ 4 & 4 & 4 \end{bmatrix}$$

and  $D(S)$  is the sum of the elements of the submatrix whose rows and columns correspond to the arguments in  $S$ .

As an example, to compute the value of coalition  $\{1, 2\}$ , we shall sum up all the terms in the following submatrix:

$$\begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix},$$

which results in  $v(\{1, 2\}) = \frac{max-10}{max-min}$ . Thus the game associated to this argumentation graph is such that  $v(\emptyset) = 0$ ,  $v(\{1\}) = \frac{35}{27}$ ,  $v(\{2\}) = \frac{32}{27}$ ,  $v(\{3\}) = \frac{32}{27}$ ,  $v(\{1, 2\}) = \frac{26}{27}$ ,  $v(\{1, 3\}) = \frac{25}{27}$ ,  $v(\{2, 3\}) = \frac{23}{27}$  and  $v(A) = \frac{11}{27}$ . The Shapley value of the game coincides with the disagreement measure found in Example 17. According to the interpretation provided by the Shapley value, node 1 is the one which brings the most conflict to the graph, followed by node 2 and node 3.

Indeed, the distance-based conflict index we have introduced can be interpreted as an index of the contribution of each argument to the total disagreement in the graph, since it takes into account the marginal contribution, in terms of conflict, that each argument provides to any other coalition of arguments, weighting it according to a probabilistic coefficient that depends on the size of the coalition.

Note that it is possible to define other games on an argumentation graph, where the value of a coalition represents in some way the conflict among arguments in the coalition. The game we have considered takes into account the distance among arguments in

the original graph. However, one can imagine to restrict the attention to the only arguments in the coalition, and therefore compute the distances among them in the induced graph. This idea leads to the definition of the following game, for every  $S \subseteq A$ ,

$$w(S) = \frac{\max_{|S} - D_{|S}(S)}{\max_{|S} - \min_{|S}}, \quad (6.4)$$

where  $D_{|S}(S) = \sum_{i \in S} \sum_{j \in S} d_{|S}(i,j)$  is the global distance in the induced graph,  $\max_{|S} = |S|^2(|S| + 1)$  is the maximum value of the global distance in a graph with  $|S|$  nodes and  $\min_{|S} = |S|^2$  is the minimum one.

**Example 20.** Let  $\mathbf{A}$  the argumentation graph of Example 17. For each  $S \subseteq A$ , in order to compute  $D_{|S}(S)$ , the matrix of distances in the graph induced by coalition  $S$  has to be considered. As an example, to compute the value of coalition  $\{1, 2\}$ , we shall sum up all the terms in the following matrix:

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix},$$

which results in  $w(\{1, 2\}) = \frac{\max_{|S}-8}{\max_{|S}-\min_{|S}}$ .

Thus the game associated to this argumentation graph is such that  $w(\emptyset) = 0$ ,  $w(\{1\}) = 1$ ,  $w(\{2\}) = 0$ ,  $w(\{3\}) = 0$ ,  $w(\{1, 2\}) = \frac{1}{2}$ ,  $w(\{1, 3\}) = \frac{1}{4}$ ,  $w(\{2, 3\}) = \frac{1}{4}$  and  $w(A) = \frac{11}{27}$ . The Shapley value of the game is  $\sigma(w) = (\frac{331}{648}, \frac{7}{648}, -\frac{37}{324})$ , which differs from the Shapley value of  $v$  found in the previous example. However the ranking it induces among arguments is the same of the one induced by the Shapley value of  $v$ , the interpretation is different: according to the latter measure, node 3 brings a negative contribution to the conflict in the graph, indicating in a sense that it contrasts the inner disagreement among arguments.

We observe, however, that the Shapley value of game  $w$  as defined in (6.4) does not satisfy all the axiom stated in the previous Section. Indeed, the following proposition holds.

**Proposition 21.** The Shapley value of game  $w$  as defined in (6.4) does not satisfy Axioms 4 and 5.

*Proof.* First we show that Axiom 4 is not satisfied. Let  $\mathbf{A} = \langle A, R \rangle$  the argumentation graph with  $A = \{1, 2\}$ ,  $R = \{(1, 1), (1, 2)\}$  and let  $\mathbf{A}' = \langle A', R' \rangle$ , with  $A' = \{1, 2, 3\}$  and  $R' = R$ . On the graph  $\mathbf{A}$   $w$  assumes the following values:  $w(\{1\}) = 1$ ,  $w(\{2\}) = 0$  and  $w(\{1, 2\}) = \frac{1}{2}$ . The Shapley value is  $\sigma(w) = (\frac{3}{4}, -\frac{1}{4})$ . On the other hand, the game  $w$  on graph  $\mathbf{A}'$  assumes the following values:  $w(\{1\}) = 1$ ,  $w(\{2\}) = 0$ ,  $w(\{3\}) = 0$ ,  $w(\{1, 2\}) = \frac{1}{2}$ ,  $w(\{1, 3\}) = \frac{1}{4}$ ,  $w(\{2, 3\}) = 0$  and  $w(A) = \frac{2}{9}$  and the Shapley value of argument 2 is  $\sigma_2(w) = -\frac{10}{108} \simeq -0.09$ . Since  $\sigma_2(w)$  increases its value, free independence is not satisfied.

We now prove that Axiom 5 is not satisfied. Let  $\mathbf{A} = \langle A, R \rangle$  be such that  $A = \{1, 2, 3\}$ ,  $R = \{(1, 2)\}$  and  $\mathbf{A}' = \langle A', R' \rangle$  such that  $A' = A$ ,  $R' = R \cup \{(2, 3)\}$ . On the graph  $\mathbf{A}$   $w$  assumes the following values:  $w(\{1\}) = 1$ ,  $w(\{2\}) = 0$ ,  $w(\{3\}) = 0$ ,  $w(\{1, 2\}) = \frac{1}{4}$ ,  $w(\{1, 3\}) = w(\{2, 3\}) = 0$  and  $w(A) = \frac{1}{9}$ . The Shapley value of

node 1 is  $\sigma_1(w) = \frac{17}{216}$ . On the other hand, the game  $w$  on graph  $\mathbf{A}'$  assumes the following values:  $w(\{1\}) = 1$ ,  $w(\{2\}) = 0$ ,  $w(\{3\}) = 0$ ,  $w(\{1, 2\}) = w(\{2, 3\}) = \frac{1}{4}$ ,  $w(\{1, 3\}) = 0$ ,  $w(A) = \frac{2}{9}$  and the Shapley value of argument 1 is  $\sigma_1(w) = -\frac{7}{216}$ . Since  $\sigma_1(w)$  decreases its value, monotonicity is not satisfied. □

## 6.6 Conclusions

In this Chapter, we proposed a conflict index that quantifies the contribution of each argument to the total disagreement in an argumentation framework, as evaluated by the the disagreement measure introduced in [3]. Our index is introduced through a property-driven approach, where the properties introduced in [3] are translated in terms of properties for arguments, and results in a ranking on the set of arguments, from the most controversial to the less controversial, therefore leading to the definition of a ranking-based semantic. Moreover, we show that the index coincides with the Shapley value of a suitable game built on argumentation graphs and can be therefore interpreted as the average marginal contribution of each argument to the disagreement induced by all possible coalitions of arguments.

The study of ranking-based semantics is quite recent, and many contributions can be further provided in the future. In [3] and in this Chapter reasonable axioms have been identified in order to define a disagreement measure and consequently a conflict index for arguments. Using a similar approach, other axioms could be stated and new measures could be proposed in order to define new ranking-based semantics, whose comparison with the one proposed here may help in providing better support to the analysis of complex decision processes.



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# CHAPTER 7

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## GAGs and Biomedicine

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This Chapter is devoted to a real-world application of the model of GAGs to the field of Biomedicine. Game Theory has been employed ever since its introduction to solve many real-world problems. An important stream in the literature on coalitional games of networks is that of *graph games*, where a cooperative game is defined by assigning to each coalition of nodes a value that depends on the underlying network, in order to formally describe the interactions among nodes of a network, and game-theoretical tools and solutions are used with the goal of extracting some information from the network itself, for example, how to share costs or benefits, which are the influential nodes, the detection of communities and so on. Many graph games from the literature have been described in Chapter 3. On the other hand, an approach using graph games to the field of Biomedicine is presented in this Chapter. In particular, we propose an approach, using basic GAGs defined on graphs, to the problem of assessing the relevance of genes in a biological network such as gene co-expression networks. The problem has been firstly addressed by means of a game-theoretical model in [72]. We introduce a new relevance index, which is characterized by a set of axioms defined on gene networks and provide formula for its computation, which can be directly derived from the results in Chapter 4. Furthermore, an application to the analysis of gene expression data from microarrays is presented, as well as a comparison with classical centrality indices.

### 7.1 Introduction

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*Gene regulatory networks* and *co-expression networks* are of great interest in the field of molecular biology and epidemiology to better understand the interaction mechanisms between genes, proteins and other molecules within a cell and under certain

biological condition of interest [20,23,31,93]. A crucial point in the analysis of genes' interaction is the formulation of appropriate measures of the role played by each gene to influence the very complex system of genes' relationships in a network.

In our work, we will focus on a particular kind of networks, that are gene co-expression networks, but our approach may be used to analyse other kinds of networks, such as protein-protein interaction networks or cell-cell interaction networks. Co-expression networks [104] may be built from gene expression data collected by means of microarray technology and other high-throughput experimental techniques [79], which allows the simultaneous quantification of the expression of thousands of genes. The nodes in the co-expression network represent genes (or proteins) and their connection is determined by the co-expression of the genes in the data samples, often measured by the Pearson correlation coefficient between gene expression profiles. Co-expression hints at co-regulation [65]. This assumption is called the guilt-by-association heuristic: if two genes show similar expression profiles, they are supposed to follow the same regulatory regime. Since the coordinated co-expression of genes encode interacting proteins, studying co-expression patterns can provide insight into the underlying cellular processes and enable the reconstruction of gene regulatory networks.

Centrality analysis represents an important tool for the interpretation of the interaction of genes in a co-expression network [12,22,44,52,53]. The relationship between centrality of genes or proteins in a co-expression network and their relevance (measured by biological features such as *lethality* or *essentiality*) has been stressed in several works in the literature. Most central elements of protein networks have been found to be essential to predict lethal mutations [52]. Highly connected hub genes, largely responsible for maintaining network connectivity, have been discovered to be likely essential for yeast survival [22]. In [44] it has been shown how betweenness centrality is generally a positive marker for essential genes in *A. thaliana*. Similarly, the relationship between the degree centrality and the essentiality of genes in transcript co-expression networks has been highlighted in [12]. Moreover, other centrality measures have been investigated in this sense in the recent literature [53,103].

However, classical centrality measures [41,57] are appropriate under the assumption that nodes behave independently and the system is sensible to the behaviour of each single node. On the contrary, in biological complex networks, assuming that the genes may express independently is not realistic since the coordinated co-expression of genes is responsible for the regulatory mechanisms within cells and the consequences on the system can be appreciated only if many genes change their expression. Therefore, in a complex scenario, such as the pathogenesis of a genetic disease, we deal with the problem of quantifying the relative relevance of genes, taking into account not only the behaviour of single genes but most of all the level of their interaction.

Cooperative game theory has been proposed as a theoretical framework to face such limitations. Recently, several centrality measures based on coalitional games have been successfully applied to different kinds of biological networks, such as *brain networks* [55,56,59], *gene networks* [72], and *metabolic networks* [85].

We propose an approach, using basic GAGs, to the problem of identifying relevant genes in a gene network. The problem has been firstly addressed by means of a game-theoretical model in Moretti et al. [72], where the Shapley value for coalitional games is used to express the power of each gene in interaction with the others and to stress the

centrality of certain hub genes in the regulation of biological pathways of interest.

Our model represents a refinement of this approach, which generalizes the notion of degree centrality, whose correlation with the relevance of genes for different biological functions is supported by several practical evidences in the literature [12,22,52,53,103]. We define a coalitional game, where the value of a coalition of genes depends on the structure of the gene network as well as on a parameter that specifies the *a priori* importance (or weight) of each gene. The strength of a set of genes is measured by means of the weights of all the genes that directly interact with them in the network. We therefore propose the Shapley value [88] of such a coalitional game as a new relevance index that quantifies the potential of a gene in preserving the regulatory activity in a gene network. This approach is supported by a property-driven approach, where the properties satisfied by our index have a biological interpretation. Moreover, an experimental study is conducted on a gene expression dataset from microarrays, related to a lung cancer disease. Three relevance analyses are performed, for different choices of the genes' weights: firstly, no *a priori* knowledge is assumed, i.e. all genes are assigned the same weight; secondly, a list of known oncogenes is taken into consideration by dividing the set of genes in key-genes and non-key-genes and lastly, the game-theoretical approach is combined with clustering analysis in order to assess the relevance of genes in the network. A comparison among the three analysis, as well as a comparison of our index with classical centrality indices is presented and the results are investigated from a biological point of view.

The Chapter is organized as follows. Section 7.2 introduces some related work in the literature, by describing classical centrality measures and the game-theoretical centrality measure presented in [72]. Section 7.3 presents a motivating example, in order to clarify the significance and scope of our index and the difference with respect to classical centrality measures, as well as the target genes whose relevance we want to assess. In Section 7.4 we introduce our model and an axiomatic characterization of the game-theoretical relevance index in terms of biological properties. An application to gene expression data from microarray technology is presented in Section 7.5 and Section 7.6 concludes.

## 7.2 Related Work

The concept of *centrality* [41,57] plays an important role in many real-world applications and has been widely investigated in the field of *network analysis*. It is often natural to ask which are the more relevant nodes in a network; for example which are the most influential persons in a social network, the main roads in an infrastructure network or the most relevant genes in a gene regulatory network.

The idea of centrality was introduced by Bavelas in 1948 [9], and applied to human communication. The first studies on centrality were aimed at assessing the relationship between structural centrality and influence in group processes. His research highlighted that centrality was strongly linked to efficiency in problem-solving within groups of individuals. Applications of the concept of centrality, however, have not been confined to experimental studies of group problem-solving. Several studies were carried out in the following decades, in a variety of applications, and several centrality measures were introduced. In what follows we describe four of the main classical centrality measures.

### 7.2.1 Classical Centrality Measures

Centrality measures assign to each node in a network a value that corresponds to some extent to the relevance of that node within the network structure. The four classical centrality measures considered in this Chapter are the following:

- (1) *Degree centrality* [77, 89]: the degree centrality of  $i \in N$  is defined as  $|N_i(E)|$ , i.e. the number of neighbours of  $i$  in graph  $\langle N, E \rangle$ . It is an index of the potential communication activity of a node.
- (2) *Closeness centrality* [11, 83]: the closeness centrality of node  $i$  is defined as  $\frac{|N|-1}{\sum_{j \in N} d(i,j)}$ , where  $d(i, j)$  is the distance between  $i$  and  $j$ , i.e. the length of the shortest path between  $i$  and  $j$ . It measures to what extent a node can avoid the control potential of the others nodes.
- (3) *Betweenness centrality* [9, 42]: the betweenness centrality of a node  $k$  is defined as  $\sum_{i,j \in N} b_{ij}(k)$ , where  $b_{ij}(k) = \frac{g_{ij}(k)}{g_{ij}}$  and  $g_{ij}$  is the number of shortest paths between nodes  $i$  and  $j$ , while  $g_{ij}(k)$  is the number of shortest paths between nodes  $i$  and  $j$  that contain  $k$ . It is an index of the potential of a node for control of communication.
- (4) *Eigenvector centrality* [14]: the eigenvector centrality of a node  $i$  is defined as the  $i$ -th element of the principal eigenvector of the adjacency matrix  $A = (a_{ij})$  corresponding to  $\langle N, E \rangle$ , where  $a_{ij} = 1$  if  $\{i, j\} \in E$  and  $a_{ij} = 0$  otherwise. It assigns high centrality to nodes that are highly connected to nodes with high degree.

### 7.2.2 A Game-Theoretical Centrality Measure

An approach using coalitional games has been introduced in [72] to evaluate the centrality of genes in a biological network keeping into account genes' interactions. The Shapley value for coalitional games is used to express the power of each gene in interaction with the others and to stress the centrality of certain hub genes in the regulation of biological pathways of interest. We briefly describe here the game-theoretical model and refer to [72] for further details.

In [72], a set  $K$  of key genes is considered and these genes are assumed to be equally important for the regulation of a certain biological process. Let  $N$  be the set of genes who are studied together with genes in  $K$  on a sequence of (microarray) experiments under a condition of interest for instance, a genetic disorder. Let  $I \subseteq N \times K$  be the set of *interactions* between genes in  $N$  and key genes in  $K$ , that is, a gene  $i \in N$  and a key gene  $k \in K$  interact if and only if  $\{i, k\} \in I$ . The triple  $(N, K, I)$  is said a *gene-k-gene* (gkg) situation.

Given a gene-k-gene situation, a coalitional game  $(N, v)$  is introduced, in order to measure the strength of association of pathways of genes in  $N$ : for each group  $S \subseteq N$ ,  $v(S)$  is computed as the number of key genes interacting only with genes in  $S$ , based on the idea that the higher is the number of key genes which interact with genes in  $S$ , the higher is the likelihood that genes in  $S$  are also involved in the regulation of the biological process of interest.

Moreover, an *interaction network*  $\langle N, \Gamma \rangle$  is considered, which describes the interactions between genes in  $N$ . Given a gkg situation  $(N, K, I)$  with the corresponding

association game  $(N, v)$  and an interaction network  $\langle N, \Gamma \rangle$ , following the approach by Myerson [75], a new game  $(N, v^\Gamma)$  is considered, where  $v^\Gamma$  is the graph-restricted game introduced in Section 5.3.

Starting from the basic paper by Shapley and Shubik [87] (Shapley and Shubik, 1954), the Shapley value of a game has been considered as a player's power in several different applications [74]. Here, players are genes and the Shapley value is considered as a gene's power. The intuition behind the meaning of gene's power attributed to the Shapley value follows from this consideration. An order  $\sigma$  on a set of genes  $N$  may be interpreted as a sequence of activations of study genes and the corresponding marginal vector may be seen as a measure of the power of study genes to establish relevant interactions with key genes according to  $\sigma$ . However, in absence of information about which sequences of activations are more likely, it is reasonable to average the marginal vectors over all possible orders as an indication of the expected power of genes.

The difference between the power of a gene in the graph-restricted game and its power in the association one is proposed as a centrality measure for co-expression networks, following an approach introduced in [45] in the context of social networks.

### 7.3 A Motivating Example

Consider the network in Figure 7.1. All classical centrality measures assign the highest relevance to the hub of the graph, i.e. node 1. Such a node has maximum degree, is the closest node to all other nodes in the graph, lies on the highest number of shortest paths connecting the other nodes and is directly connected with the most nodes of high degree. These features correspond to four of the most known classical centrality measures: *degree centrality* [77, 89], *closeness centrality* [11, 83], *betweenness centrality* ([9, 42]) and *eigenvector centrality* [14], which give highest centrality to node 1.

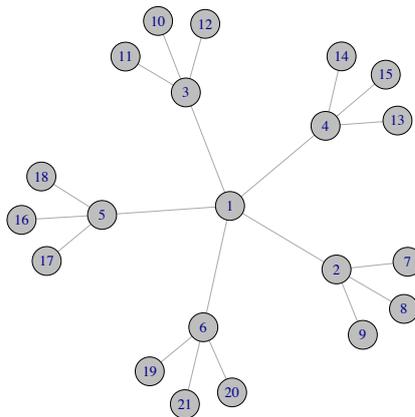


Figure 7.1: A network with 21 nodes.

However, in some cases, genes that lie in the periphery of a network might have an important role in the biological condition it represents. As an example, in [43] it has been shown that differentially expressed genes in major depression (i.e. those genes that present a statistically different behaviour in depressed patients compared to healthy patients) reside in the periphery of resilient gene co-expression networks, thus suggesting that the hub genes are not always the most relevant in the regulatory processes within gene networks.

In particular, the set of nodes  $\{2, \dots, 6\}$  has two characteristics that make them relevant genes when the network depicted in Figure 7.1 represents a gene regulatory network:

- (1) through their connections, the nodes in the set are able to influence the expression of all other genes in the network, i.e. they interact directly with all the other genes within the network;
- (2) its removal (or inhibition) breaks down the regulatory activity of the network, by leaving all the leaf nodes isolated and therefore not able to maintain their regulatory activity.

With these two features in mind, we introduce an index that aims at measuring the potential of a gene in preserving the regulatory activity within a gene network, by stressing the ability of a gene in influencing the overall expression of genes in the network and to absorb the effects of the inhibition of a correlated gene, or in another words its resilience to the removal of a connected node. In this sense, node 2 (as well as nodes 3,4, 5 and 6) is more relevant than node 1: when node 1 is removed, the network is divided into five components, whose overall regulation is maintained thanks to the presence of nodes 2,3,4,5 and 6 respectively. On the other hand, when one of these last nodes is removed, the network is split in four component, three of whom are no longer able (as being isolated nodes) to maintain their regulatory activity.

The index we propose aims at highlighting the role of genes in the overall "connectivity" of the network, by taking into account the effects that their inhibition have over the induced subnetworks.

A relevant set of genes to this extent would be able to interact directly with the maximum number of other nodes in the network and its removal would split the network in a maximum number of connected components with few genes, or eventually constituted by isolated genes.

To this purpose, we introduce in the next Section a cooperative game, specifically a basic GAG, where the value of a coalition of genes depends on the cardinality of the coalition itself and of its neighbourhood. The more the genes that are directly interacting in the network with genes in the coalition, and therefore the ability of the coalition to keep the network connected, the higher the strength of the coalition.

We propose the Shapley value of such a game as a relevance index for genes, which takes into account the marginal contributions of genes to the connectivity of all the coalitions of genes in the network, therefore assigning maximum relevance to nodes 2,3,4,5 and 6 in the example in Figure 7.1.

We introduce the index by an axiomatic characterization on gene networks and we provide a formula for its computation, which has a straightforward interpretation. Moreover, we use our index to assess the relevance of genes in a real dataset related to lung cancer, by means of three different analysis. On such a network, when no *a priori*

knowledge is assumed about the genes under analysis (first analysis), the index is able to highlight the role of genes in the overall connectivity of the network, by assigning the highest relevance to those genes that share the two aforementioned characteristics. Furthermore, these ideas are combined with other techniques to assess the relevance of genes: in the second and third analysis, further assumptions about the *a priori* importance of genes are taken into account, thus enabling to combine the considerations made in this Section with other techniques from network analysis.

## 7.4 A Relevance Index for Genes

Let  $\langle N, E \rangle$  be a *gene network*, that is a network where the set of nodes  $N$  represents a set of genes and the set of edges  $E$  describes the interaction among genes, i.e. there exists an edge between two genes if they are directly interacting in the biological condition under analysis. Moreover, let  $k \in \mathbb{R}^N$  be a parameter vector that specifies the *a priori* importance of each gene.

We define a coalitional game  $(N, v_E^k)$ , where  $N$  is the set of genes under study and the characteristic function  $v_E^k$  assigns a worth to each coalition of genes  $S \subseteq N$  representing the overall magnitude of the interaction between the genes in  $S$ , which takes into account the weight (*a priori* importance) of each gene directly connected to  $S$  in the biological network.

More precisely, the map  $v_E^k : 2^N \rightarrow \mathbb{N}$  assigns to each coalition  $S \in 2^N \setminus \{\emptyset\}$  the value

$$v_E^k(S) = \sum_{j \in S \cup N_S(E)} k_j \quad (7.1)$$

that is the sum of the weights associated to the genes in  $S$  and to the ones that are directly connected in  $\langle N, E \rangle$  to some genes in  $S$  (by convention,  $v_E^k(\emptyset) = 0$ ). The class of games  $(N, v)$  defined according to relation (7.1), on some gene network  $G \equiv \langle V, E \rangle$  and with parameter  $k \in \mathbb{R}^N$ , is denoted by  $\mathcal{EK}^N$ .

We observe that the so-defined game is clearly described as the basic GAG associated to the GAS  $\langle N, v, \mathcal{M} \rangle$ , where  $v(i) = k_i$  for every  $i \in N$  and  $\mathcal{M}$  is the map associated to the collections  $\mathcal{C}_i = \{F_i = N_i(E), E_i = \emptyset\} \forall i \in N$ . A gene  $i$  contributes to the worth of a coalition with its individual value, the weight  $k_i$ , if and only if it belongs to the coalition or if at least one of the genes which interacts directly with it are present.

Another way to keep into account the *a priori* importance of genes has been proposed in [72] by means of the so-called *association game*, where a set of key-genes  $K \subset N$  (e.g. a set of genes known *a priori* to be involved in biological pathways related to chromosome damage) is considered and the value assigned to a coalition  $S$  is the number of key-genes interacting only with  $S$ , as described in Section 7.2.

However, the definition proposed in relation (7.1) seems more flexible to explore all possibilities of reciprocal influence among genes. It generalises the game introduced in [94] for determining the "top- $k$  nodes" in a co-authorship network, by the introduction of a parameter that specifies the *a priori* importance of each node. The parameter vector  $k$  allows for an *a priori* ranking of the genes according to their importance, while in the previous model introduced in [72] only a two-level distinction was made between key-genes and non key-genes. Moreover, by measuring to what extent a coalition of genes

is connected to the rest of the network, relation (7.1) generalizes the notion of degree centrality for groups of genes, which is justified by some practical evidences showing a strong correlation between the degree centrality and genes that are essential for different biological functions (see, for instance, [12, 22, 52, 53, 103]). In fact, if only the weight of genes inside a coalition was to be considered (and not the one of the neighbours, as in our definition), the centrality measure obtained through the following approach would coincide with the weighted degree centrality.

We now introduce some properties for a *relevance index* for genes, that is a map  $\rho : \mathcal{EK}^N \rightarrow \mathbb{R}^N$ . We start with a reinterpretation of the classical properties of SYM, DPP and EFF on the class  $\mathcal{EK}^N$  (see Section 2.1 for a formal definition on the class of all TU-games).

Consider a gene network  $\langle N, E \rangle$  and a vector of weights  $k \in \mathbb{R}^N$ . The property of SYM implies that if two genes  $i, j \in N$  have the same weight ( $k_i = k_j$ ) and in addition, they are connected to the same set of neighbours ( $N_i(E) = N_j(E)$ ), then they should have the same relevance. For instance, nodes 2, 3, 4 and 5 in the star depicted in Figure 7.2 are symmetric.

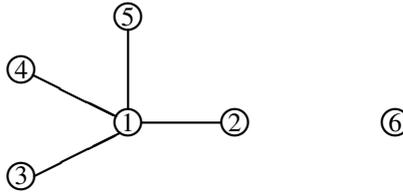


Figure 7.2: A star  $\langle \{1, 2, 3, 4, 5, 6\}, E_{\{2,3,4,5\}}^1 \rangle$ .

The property DPP also has an intuitive interpretation on the graph: every disconnected node  $i \in N$  (like node 6 in Figure 7.2) should have relevance  $k_i$ . Finally, the EFF property implies that the sum of the relevance of all genes should be equal to  $\sum_{i \in N} k_i$ , the total sum of weights.

We introduce now a new axiom, saying that the transformation of a node  $i$  with zero weight to a node with weight  $k_i$  should affect only the genes directly connected to  $i$ , and its impact on the relevance of its neighbours should be equal to the one had in an equivalent star of centre  $i$ .

**Axiom 9 (Star Additivity, SADD).** *Let  $\langle N, E \rangle$  be a gene network with parameter vector  $k_{-i} \in \mathbb{R}^N$  such that gene  $i$  has weight 0 and let  $v_E^{k_{-i}}$  be the corresponding game defined according to relation (7.1). Then consider the game  $v_E^k$  defined according to relation (7.1) on  $\langle N, E \rangle$  and with parameter vector  $k$  that assigns a positive weight  $k_i$  to gene  $i$  and the same weight as  $k_{-i}$  to all the other genes. An index  $\rho : \mathcal{EK}^N \rightarrow \mathbb{R}^N$  satisfies the SADD property iff*

$$\rho(v_E^k) = \rho(v_E^{k_{-i}}) + \rho(v_{E_{N_i(E)}}^{s^i}),$$

where  $v_{E_{N_i(E)}}^{s^i}$  is the game defined according to relation (7.1) on the star  $\langle N, E_{N_i(E)}^i \rangle$  on  $N_i(E)$  with centre  $i$  and  $s^i$  is the parameter vector that assigns  $k_i$  to  $i$  and 0 to  $j \neq i$ .

For instance, consider again the network of Figure 7.2, and suppose that  $\rho' \in \mathbb{R}^6$  is the relevance index corresponding to a parameter vector  $k_{-1}$ . Moreover let  $\rho'' \in \mathbb{R}^6$  be the relevance index on the same network with parameter  $s^1$  such that only node 1 has a positive weight  $k_1$ . Then, the SADD property says that in the situation where the parameter vector is given by  $k = k_{-1} + s^1$  and  $\rho''' \in \mathbb{R}^6$  the corresponding relevance index, then it must hold  $\rho''' = \rho' + \rho''$ .

Roughly speaking, Axiom SADD states that increasing the weight of a node  $i$  from 0 to a positive value should only affect the total relevance of gene  $i$  and its neighbours at the same extent for whatever graph. As a consequence, a positive change in the weight of a gene produces the same effect on its relevance and on the one of their neighbours independently from the topology of the network, and the effect of the changes is comparable along different networks.

**Proposition 22.** *The Shapley value is the unique relevance index  $\rho$  that satisfies SYM, DPP, EFF and SADD on the class  $\mathcal{EK}^N$ . Moreover, for each gene network  $\langle N, E \rangle$  with  $k \in \mathbb{R}^N$  as a vector of weights, it can be computed according to the following formula:*

$$\rho_i(v_E^k) = \sum_{j \in (N_i(E) \cup \{i\})} \frac{k_j}{d_j(E) + 1}, \quad (7.2)$$

for each  $i \in N$ .

*Proof.* Let  $\langle N, E_{N_i(E)}^i \rangle$  be a star on  $N_i(E)$  with centre in  $i$ , and such that only  $i$  has a positive weight equal to  $k_i$  and let  $v_{E_{N_i(E)}^i}^{s^i}$  be the corresponding game defined according to relation (7.1). It is easy to check that the unique index that satisfies the properties of SYM, DPP and EFF is the one such that

$$\rho_j(v_{E_{N_i(E)}^i}^{s^i}) = \phi_j(v_{E_{N_i(E)}^i}^{s^i}) = \begin{cases} \frac{k_i}{d_i(E)+1} & \text{if } j \in N_i(E) \cup \{i\}, \\ 0 & \text{otherwise.} \end{cases} \quad (7.3)$$

By the repeated application of axiom SADD, and since  $\sum_{i \in N} v_{E_{N_i(E)}^i}^{s^i} = v_E^k$ , we have that

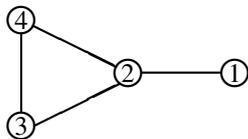
$$\rho(v_E^k) = \sum_{i \in N} \rho(v_{E_{N_i(E)}^i}^{s^i}). \quad (7.4)$$

Then, the proof follows by relation (7.3) and the additivity of the Shapley value.  $\square$

Moreover, we observe that, equivalently, the formula in (7.2) can be directly derived from Corollary 2 in Section 4.2.

The interpretation of the formula in (7.2) is straightforward: a gene is assigned a high relevance if it is connected to many genes which are in turn connected with few other genes, that is the more neighbours with low degree, the highest the relevance.

**Example 21.** *Consider the gene network  $\langle N, E \rangle$  depicted in the Figure 7.3 and with  $k = (1, 0, 3, 0)$ . The game  $v_E^k$  defined according to relation (7.1) is such that  $v_E^k(1) = v_E^k(\{1, 4\}) = 1$ ,  $v_E^k(\{1, 3\}) = v_E^k(\{1, 3, 4\}) = 4$ ,  $v_E^k(S) = 4$  if  $2 \in S$  and  $v_E^k(3) = v_E^k(4) = v_E^k(\{3, 4\}) = 3$ . By Proposition 22, we have that  $\rho(v_E^k) = (\frac{1}{2}, \frac{3}{2}, 1, 1)$ .*



**Figure 7.3:** A gene network with four nodes.

**Example 22.** Consider the gene network in Figure 7.1. Suppose all the genes have the same a priori importance, and let for simplicity  $k_i = 1 \forall i \in N$ . Then, by Proposition 22  $\rho(v_E^k) = (\frac{35}{30}, \frac{56}{30}, \frac{56}{30}, \frac{56}{30}, \frac{56}{30}, \frac{56}{30}, \frac{21}{30}, \dots, \frac{21}{30})$ . Therefore, our index gives the highest relevance to nodes 2, 3, 4, 5 and 6, followed by node 1 and the least relevance to the leaf nodes  $\{7, \dots, 21\}$ . On the other hand, all the other classical centrality measures defined in Section 7.2 provide the following ranking: node 1 has the maximum centrality, followed by nodes  $\{2, 3, 4, 5, 6\}$  and finally the leaf nodes.

## 7.5 An application to real data

The purpose of this Section is the validation of our model by analyzing a gene expression dataset and comparing the results with the related literature.

We consider a co-expression network, where the co-expression of two genes is measured by the correlation between their expression profiles (by means of the Pearson's correlation coefficient). In constructing such a network from a gene expression dataset, a cut-off has to be specified, that establishes which pairs of genes interact in the gene network: an edge between two genes is created if their correlation is above a certain threshold, while no edge is established otherwise [20, 23, 31, 104]. The choice of the threshold is critical to the analysis and has to rely on biological considerations and on the evaluation of different network parameters.

### 7.5.1 Robustness Evaluation

As a first result, the robustness of our model with respect to different choices of such threshold is shown. The model has been tested on a randomly generated symmetric matrix of size 1000 with entries in the range  $[0, 1]$ , that is a matrix where the element in row  $i$  and column  $j$  represents the correlation between gene  $i$  and gene  $j$  in a fictitious, randomly drawn dataset of 1000 genes. For the sake of this analysis the parameter vector  $k$  has been fixed in such a way that  $k_i = 1$  for every  $i$ . The matrix has been transformed in a boolean adjacency matrix (where 1 represents a connection in the network and 0 no connection) according to three different thresholds, 0.7, 0.8 and 0.9 respectively. A network has been generated in the three cases according to the aforementioned interpretation and the relevance index for each gene, i.e. the Shapley value of the game defined in (7.1), has been computed. A comparison between the results for the different threshold has been conducted. In particular, the list of the 5% of genes with the highest Shapley value has been selected for the three thresholds and

the comparison yields the following results: 18 genes are commonly selected by our relevance index for cut-off 0.7 and 0.8; 15 genes are commonly selected for cut-off 0.8 and 0.9 and 5 genes are commonly selected for cut-off 0.7 and 0.9, thus showing a discrete degree of robustness with respect to the choice of the cut-off. The previous results are summarized in Table 7.1.

	0.7	0.8	0.9
0.7	50	18	5
0.8	18	50	15
0.9	5	15	50

**Table 7.1:** Number of common genes by using a cutoff of 0.7, 0.8 and 0.9, respectively.

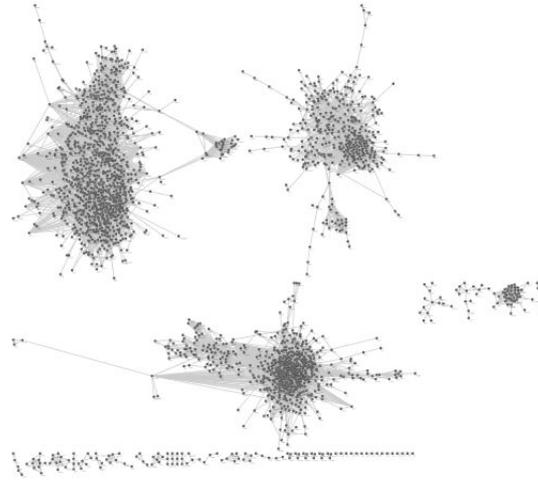
### 7.5.2 Relevance Analysis

We present here the application of our model to a lung cancer dataset. A description of the dataset and three different analysis are described in what follows: firstly, the dataset is analysed by assuming no a priori knowledge of the importance of the genes in the network; secondly, the knowledge about some key oncogenes is included in the analysis and lastly a method from clustering analysis is used to assess their a priori importance of genes.

#### Description of the Dataset

A gene expression dataset related with a very common kind of lung cancer called adenocarcinoma has been studied. Adenocarcinoma cancers are usually found in lung outer areas as the lining of the airways. This dataset with accession number GDS3257 is available in NCBI repository. These data were generated in a study where 107 samples of several tumour stages in a population of smoker and not smoker people were analysed [60]. These raw data has been preprocessed with Babelomics tool [69] using several standard filtering steps. Those gene profiles with a standard deviation under 0.5 were removed in order to only consider genes differentially expressed. The resulting gene expression matrix is composed by 2517 gene expression profiles (rows) and 107 samples (columns).

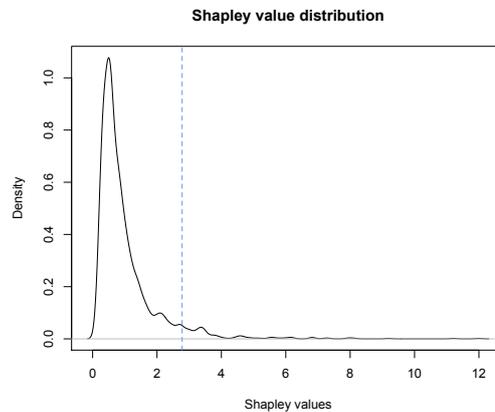
A gene co-expression network has been generated, by establishing a link between two genes if and only if the Pearson's correlation between their gene expression profiles is a higher than a fixed threshold. The choice of the threshold is based on the following considerations: a suitable network should consist in connected components with the highest possible cardinality and should also be as sparse as possible in order to better reveal the relationships between the nodes (genes). Therefore, the network must be experimentally built according to an equilibrium between connectivity and sparsification [27]. The BioLayout tool [96] has been used to conduct an experimental study, which has led to the choice of 0.8 as the value for the correlation threshold. The network so obtained is composed by 2154 nodes (genes) and 24821 edges. Figure 7.4 shows a picture of the network.



**Figure 7.4:** Gene co-expression network generated with lung cancer dataset.

**First Analysis**

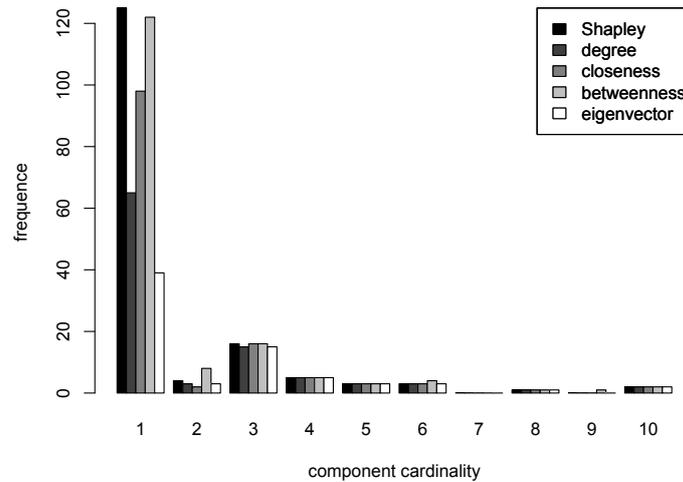
A first analysis has been carried out on the aforementioned network, with no a priori knowledge of the importance of the different genes, thus considering each gene equally important, i.e. setting  $k_i = 1$  for each gene  $i \in N$ . Following this approach, the relevance index  $\rho$  is computed. The density distribution of  $\rho$  is shown in Figure 7.5.



**Figure 7.5:** First analysis: The density distribution of the index  $\rho$  is shown, for  $k_i = 1$  for every gene  $i$ . The dotted vertical line represents the cut-off: the 5% of genes with highest index is selected.

In particular, we select the 5% of genes with highest relevance for further analysis. This list of genes is investigated with respect to the features described in the motivating example. It turns out that, in the comparison with the classical centrality measures, our index is able to highlight these characteristics. In particular, the lists of the 5% of genes with highest value according to the different centrality measures are compared with the following results:

- (i) the 108 genes selected by our index are directly interacting in the network with 1412 genes, comparably with the ones selected by the betweenness centrality, whose neighbourhood consists in 1423 genes. The other measures are much less effective in this sense: the genes selected by the degree centrality interact with 1062 genes, the ones by closeness centrality with 668 and the ones by eigenvector centrality with 383 genes.
- (ii) when the 108 genes selected by  $\rho$  are removed, the network is split in 165 connected components, 125 of which are isolated nodes. Three of them contain a high number of genes (550, 826 and 338), one of them 42 nodes, and the rest very few nodes (2 to 10 nodes each). A similar behaviour is observed after the removal of the 108 nodes selected by the betweenness centrality: the network is split in 170 components, 122 of which are isolated nodes. On the other hand, the effects of the removal of the genes selected by the other measures are definitively less severe. See Figure 7.6 for a comparison with the different measures. Note that the histogram has been constructed by considering only the components with less than ten nodes, since the bigger components have very similar frequencies for all measures.

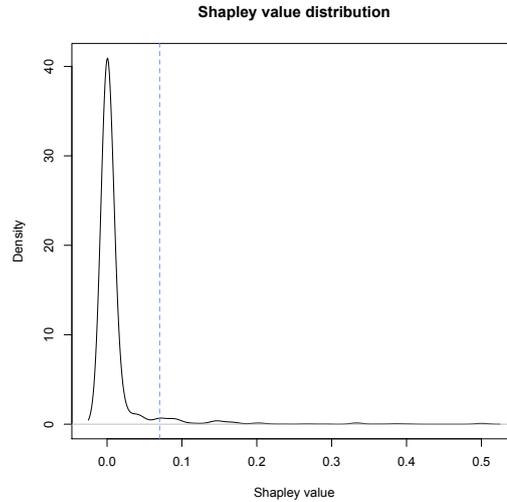


**Figure 7.6:** The histogram represents the frequency of components cardinality after the removal of the genes selected by the different centrality measures.

### Second Analysis

A second analysis has been conducted by taking into account the presence in the network of some known lung cancer key-genes, i.e. setting  $k_i = 1$  for each key-gene  $i \in N$  and  $k_i = 0$  otherwise. In particular, we consider a set of 23 known lung oncogenes found through the *Network Cancer of Genes* tool (NCG5.0).

The relevance index  $\rho$  is computed according to this selection and the 5% of genes with highest relevance is selected for further analysis. The density distribution of  $\rho$  is shown in Figure 7.7.



**Figure 7.7:** Second analysis: The density distribution of the index  $\rho$  is shown, for  $k_i = 1$  for every key-gene  $i$  and  $k_i = 0$  otherwise. The dotted vertical line represents the cut-off: the 5% of genes with highest index is selected.

### Third Analysis

A strength of our model relies on the possibility to integrate different tools from network analysis to assess the relevance of genes in a network. Indeed, even if the *a priori* weight of genes is not known, the freedom in the choice of parameter vector  $k$  allows for a variety of approaches. In particular, we use some techniques from cluster analysis to define the *a priori* importance of genes.

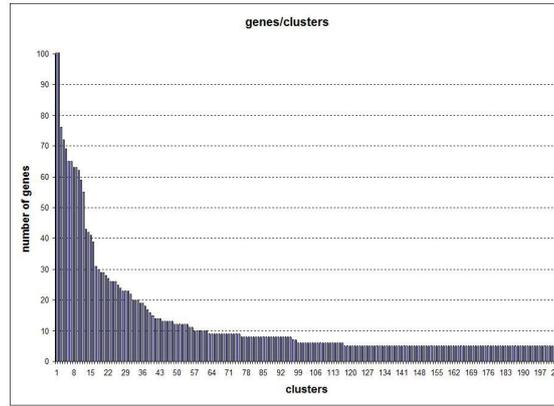
A third analysis has been conducted by measuring the *a priori* importance of genes by a parameter vector that depends on the clusters structure of the network.

The underlying idea is the following: the relevance of a gene is assessed by dividing the network in clusters, through the algorithm ClusterONE [76], and counting how many clusters a gene belongs to, i.e.  $k_i$  for every gene  $i \in N$  is defined as the number of cluster in the network it belongs to. This approach follows the idea [61] that overlapping genes among clusters are to some extent important in the network, and we use the aforementioned parameter vector to quantify their importance.

Traditional clustering algorithms report a partition of data such that all clusters are disjoint. However, the overlapping among clusters is interesting in the context of gene interaction networks, since genes are usually involved in several processes and might, as a consequence, belong to different clusters [61]. ClusterONE is a clustering algorithm that captures overlapping clusters of genes in a network. This algorithm is a greedy search process that finds groups of genes with a high cohesiveness among them [76].

The algorithm has been run using the default values for the advance parameters [76]. Basic parameters has been chosen as 5 and 0.5 for minimum cluster size and minimum cluster density respectively. The algorithm has finally reported 204 clusters. Figure 7.8 shows the number of genes per cluster. Note that the first and the second clusters have 306 and 107 genes respectively. The maximum value in the  $y$  axis has been chosen equal to 100 in order to improve the figure visualization.

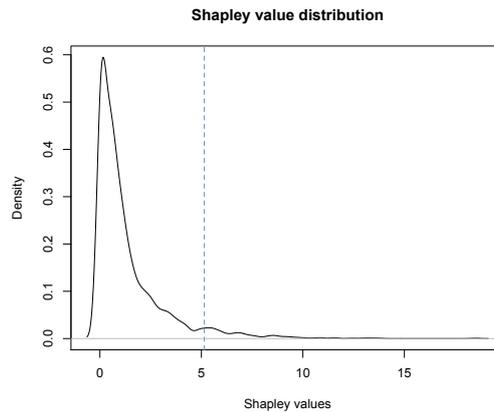
All the clusters generated through the aforementioned procedure are considered and



**Figure 7.8:** Number of genes per cluster.

each gene  $i$  belonging to these clusters (1444 out of 2154 in the whole dataset) is assigned a weight  $k_i$  equal to the number of clusters it belongs to.

The relevance index  $\rho$  is then computed and its density distribution is shown in Figure 7.9. Moreover, the 5% of genes with highest relevance is selected for further analysis.



**Figure 7.9:** Third analysis: The density distribution of the index  $\rho$  is shown, for  $k_i$  defined as the number of clusters gene  $i$  belongs to. The dotted vertical line represents the cutoff: the 5% of genes with highest index is selected.

### 7.5.3 Results Comparison

The results from the three different analyses have been compared with the results from classical centrality measures. Table 7.2 shows the number of common genes among the different lists and the correlation among the different measures is shown in Table 7.3.

Among the classical centrality measures, the relevance index computed according to the first analysis shows a maximum overlap with betweenness centrality, with 66 genes in common (out of the 108 selected with highest value) and a high positive correlation on the whole list of genes. In the second analysis, on the other hand, most of the genes selected by our index are not selected by other measures, with a maximum overlap

	$\rho(1)$	$\rho(2)$	$\rho(3)$	degree	closeness	betweenness	eigenvector
$\rho(1)$	108	27	61	49	40	66	28
$\rho(2)$	27	108	33	22	20	19	12
$\rho(3)$	61	33	108	31	49	55	10
degree	49	22	31	108	19	28	86
closeness	40	20	49	19	108	48	0
betweenness	66	19	55	28	48	108	7
eigenvector	28	12	10	86	0	7	108

**Table 7.2:** Number of common genes among the relevance vectors of 108 genes provided by the different relevance measures.

	$\rho(1)$	$\rho(2)$	$\rho(3)$	degree	closeness	betweenness	eigenvector
$\rho(1)$	1	0.265	0.808	0.694	0	0.804	0.269
$\rho(2)$	0.265	1	0.305	0.220	-0.016	0.178	0.073
$\rho(3)$	0.808	0.305	1	0.665	0.250	0.660	0.211
degree	0.694	0.220	0.665	1	-0.005	0.456	0.790
closeness	0	-0.016	0.250	-0.005	1	0.148	-0.145
betweenness	0.804	0.178	0.660	0.456	0.148	1	0.067
eigenvector	0.269	0.073	0.211	0.790	-0.145	0.067	1

**Table 7.3:** Correlation among the lists obtained by different centrality measures. Note that the correlation coefficients are computed on the entire lists of 2154 genes.

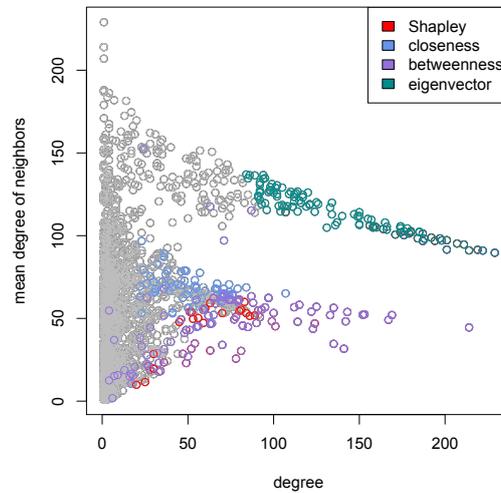
of 22 genes with degree centrality, suggesting that the introduction of a priori known key-genes strongly influences the analysis towards the selection of genes that interact with this particular set of genes. On the other hand, the third analysis seems to produce results that are more similar to the ones of the first analysis, with a maximum overlap with the list selected by betweenness centrality, followed by closeness and degree centrality.

Moreover, all the three analysis select very few genes in common with the eigenvector centrality, which is not surprising since our relevance index selects those genes that are co-expressed with many genes that have low degree, while on the contrary eigenvector centrality selects genes that are highly connected to genes with high degree.

The relationship among the degree of a node and the degree of its neighbours is highlighted in Figure 7.10. The coloured points represent the genes selected by the different measures. In particular, the red points are the ones selected only by our index, while the degree centrality selects all the nodes with degree higher than 100.

### 7.5.4 Biological Interpretation of the Results

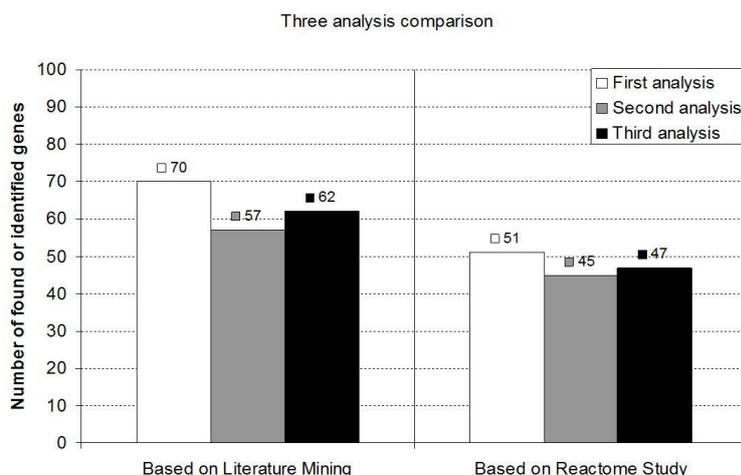
The results of the three analysis have also been compared from a biological point of view. The number of relevant genes stored in biological repositories has been considered as a quantitative criterion to compare them. Firstly, a Literature Mining approach has been used with a *Cytoscape* plugging called *Agilent Literature Search* [84]. Secondly, a Reactome study has also been performed with the same goal. It is important to note that the first 100 genes for each analysis have only been studied due to these tools limitations.



**Figure 7.10:** The points represent genes and their coordinates are given, respectively, by the degree of a gene (on the  $x$ -axis) and the mean degree of its neighbours (on the  $y$ -axis).

The Cytoscape plugging searches a set of genes in published papers available in public repositories such as *PubMed*. The search has been performed by taking as input the list of genes selected by our relevance index and a set of key-words, namely “Homo sapiens” and “Adenocarcinoma”. The tool provides as a result the subset of genes that are cited in the related literature. Note that this information is approximated, in the sense that the genes are cited in the literature but it is not known to what extent; false positives genes could be reported by this tool. Figure 7.11 shows on the left-hand side the results of the Literature Mining-based comparison. The first, second and third analysis report 70, 57 and 62 genes that are cited in the literature, respectively. The first analysis seems to report more known genes but the three analysis obtain comparable results, by finding in the literature more than a half of the genes selected by our relevance index.

Moreover, a study based on *Reactome* [28] has been performed in order to compare the three analysis. Reactome is a repository of biological pathways, namely groups of reactions among nucleic acids, proteins and another kind of molecules that interact as part of biological processes as for example the regulation of gene expression, metabolism, etc. The three lists of 100 genes have been analysed, yielding the following results: the first analysis identifies 51 genes, the second 45 and the third 47 (see the right-hand side of Figure 7.11). These results are coherent with the Literature Mining-based results. However, the Reactome study mainly shows the pathways where the input genes are identified. Although these pathways have not been carefully studied, it is interesting to note that the first analysis reports 329 pathways, the second 379 and the third 219. This information could indicate that the quality of the genes found by the second analysis is higher than the other two analysis.



**Figure 7.11:** Comparison of three analysis based on Literature Mining and Reactome study.

### Biological qualitative analysis

The *Network Cancer of Genes* tool (NCG5.0) has been used to further investigate the results of the analysis from a biological point of view. This tool only provide information about known cancer genes and it is therefore too restrictive to be used in a quantitative comparison as in Section 7.5.4. For example, a gene could be relevant as acting as a "switch" of a known oncogene (cancer gene) or co-regulate an important related process but it would not be reported by NGC, unless it is itself an oncogene. However, this tool provides some useful information from a qualitative point of view, allowing us to evaluate the results of our analysis and to compare them on the basis of the information it provides.

The first 100 genes for each analysis have been studied with NCG, with the objective of understanding their biological relevance from a qualitative perspective. The first analysis finds 5 oncogenes, the second 20 and the third 11. These results support the idea that the second analysis reports genes with a higher quality. However, it is important to emphasize that the second analysis uses a priori information, by considering as input 23 well-known lung cancer genes, precisely obtained using NCG. It must be noted that 15 genes out of this 20 were used as input key-genes. Therefore, we could state that each analysis identifies, respectively, 5,  $20 - 15 = 5$  and 11 not previously known oncogenes. The first and the third analysis do not use any a priori knowledge. Nevertheless, 4 out of 5 genes obtained by the first analysis are in the well-known set of 23 lung cancer genes, as well as 2 out of 11 genes obtained by the third analysis.

Moreover, it is interesting to further investigate those genes that are reported only by the proposed relevance index but not by the other (classical) centrality measures. With respect to this, the first analysis presents 19 genes, the second 71 and the third 33 that are selected only by our index. These sets of genes have also been analysed using the NCG tool. The first analysis does not show any cancer gene according to the information supported by the tool. However, the second analysis reports 18 of 71 genes as cancer genes and the third analysis 3 of 33. It could be inferred that the second analysis presents the best results in this sense, but it must be noted that 17

of the 18 genes are precisely part of the 23 lung cancer genes used as a priori information in the second analysis. Therefore, it only reports 1 cancer gene which is not previously known and used as input. The cancer genes reported by the second analysis that belongs to the set of key genes used as input are NRAS, PDIA4, DACH1, RUNX1T1, CDH104, HLA-A4, GRM84, ZMYND10, ATXN3L, DNAH3, PTPRD, PAK3, COL11A1, COL1A1, PPP1R3A, CTNNA3 and CCKBR. The gene G6PC is also reported by the second analysis but it is not included in the input set of key genes. This gene is a liver cancer gene with a functionality related with the regulation of intracellular processes and metabolism. Furthermore, the cancer genes reported by the third analysis are GNATI, CD1B and GML, which are respectively leukemia, lung and glioblastoma cancer genes. The gene CD1B is a lung cancer that belongs to the set of key genes used in the second analysis. It is important to note that this a priori information is not used in the third analysis. The gene G6PC reported by the second analysis and the aforementioned three genes are examples of genes that can be identified by using the proposed relevance index but not the other centrality measures.

## **7.6 Concluding remarks**

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In this Chapter, we proposed a relevance index for nodes in gene co-expression networks, with the objective of measuring the potential of genes in acting as intermediaries between hub nodes and leaf nodes and preserving the regulatory activity within gene networks. For this purpose, we used a game-theoretic approach, by defining a basic GAG where the strength of a coalition of genes depends on the *a priori* importance of the genes in its neighbourhood. The Shapley value of such a game is proposed as a new relevance index for genes. Our approach is supported by an axiomatic characterization, where the set of properties satisfied by our index have a biological interpretation. Moreover, an experimental study is conducted on a gene expression dataset from microarray technology, related to a lung cancer disease and the results of our index are compared with classical centrality measures.

The versatility of our model allows the combination of a game-theoretical approach with other techniques from network analysis. Indeed, we used an algorithm from cluster analysis that identifies overlapping clusters of genes, in order to assess the *a priori* importance of genes in the network under analysis. An interesting direction for future research is the further study of these techniques, in order to refine the relevance analysis, and the application of our model to other gene networks in order to provide new biological knowledge.



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## CHAPTER 8

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### General conclusions

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In recent times, *network analysis* has become a flourishing field of study, integrating knowledge from diverse disciplines, among them Social Sciences, Statistics, Game Theory and Computer Science. From a practical perspective, the amount of data collected from modern technology that is available to researchers gives potential new insights and opens new questions about the behaviour of interacting entities, such as users of a social network, or genes in a biological network. The interactions among entities within a network can be identified and analyzed using game theory models, with the aim of accounting for the properties of existing networks and predicting the emergence of new networks via the analysis of observed data.

Game Theory has been employed ever since its introduction to solve many real-world problems. A graph may be used, as an example, to formally describe an entity or object under analysis, like a power grid or a telecommunication network. In this context, a cooperative game may be defined by assigning to each coalition of nodes a value that depends on the underlying network, in order to formally describe the interactions among nodes of a network, and game-theoretical tools and solutions may be used with the goal of extracting some information from the network itself, for example, how to share costs or benefits, which are the influential nodes, the detection of communities and so on. The related literature is quite vast and only a portion of it is listed in the bibliography of this thesis. However, we noticed that many coalitional games on networks that have been studied in the literature share a common structure, that prescribes how the players involved contribute to the formation and maintenance of a network by bringing together their individual abilities.

Following this observation, we introduced in this thesis the model of Generalized Additive Games (GAGs), where the worth of a coalition is evaluated by means of an interaction filter that selects the valuable players involved in the cooperation. This gen-

eral framework is able to reveal the common structure of several types of known games, and has a twofold objective: on one hand, that of making their analysis more transparent and general, and on the other hand that of providing new tools for constructing classes of games whose properties and solutions can be analysed within this theoretical framework of reference. In particular, many classes of coalitional games on networks that has been introduced and studied in the literature of cooperative games can be described in terms of GAGs. In such games, the underlying network structure which describes the interactions among players "forces" the players to contribute to the worth of coalitions in a way that reflects the constraints imposed by the pairwise relations between players.

Moreover, our model highlights that several games introduced in the literature, for which it is possible to easily derive some solutions (for example, the Shapley value has a concise formula, or the non-emptiness of the core can be easily verified) share a common additive structure, that allows for a compact representation of the interactions among players and for a decrease in the complexity of computation of the relative solutions.

Besides the possible directions for future research highlighted in each chapter, we argue that our model represents a useful tool for the analysis of a variety of situations and fields, as suggested by the wide range of games that it encompasses, also thanks to the fact that specific subclasses of our model enable a simple and efficient analysis of the relative solutions.

An application of our model to two very different fields of research has been presented in Chapter 6 and 7, suggesting that it may be used as a flexible and simple tool for the investigation of a variety of problems within the framework of network analysis, such as the analysis of terrorist networks [62] or the detection of communities in a social network [26, 68]. Indeed, as suggested by Example 13, our model may be suitable for the analysis of other real data such as online social networks, whose collection however requires an effort which is beyond the scope of this thesis but represents an interesting direction for future research. Moreover, to the sake of the analysis of a wide range of real data, another direction for future work is the creation of software tools and libraries for implementing our model in its general version, in order to allow its customisation and employment on big datasets different from the ones already analysed in this thesis.

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## Appendix

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### Appendix A

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**Lemma 1.** For all  $n, t \in \mathbb{N}$  it holds:

$$\sum_{k=0}^n \binom{n}{k} (k)!(n+t-k)! = \frac{(n+t+1)!}{t+1}$$

$$\sum_{k=0}^n \binom{n}{k} (k+1)!(n+t-k-1)! = \frac{(n+t+1)!}{t(t+1)}.$$

*Proof.* We prove only the first, an analogous argument proves the second one. By using the product of formal series

$$\left( \sum_{n \geq 0} a_n \frac{x^n}{n!} \right) \left( \sum_{n \geq 0} b_n \frac{x^n}{n!} \right) = \sum_{n \geq 0} \left( \sum_{k=0}^n a_k b_{n-k} \right) \frac{x^n}{n!}$$

and by taking into account that

$$\sum_{n \geq 0} \binom{n+r}{r} x^n = \frac{1}{(1-x)^{r+1}},$$

we can compute:

$$\begin{aligned} A(x) &= \sum_{n \geq 0} \left[ \sum_{k=0}^n \binom{n}{k} (k)!(n+t-k)! \right] \frac{x^n}{n!} = \left( \sum_{n \geq 0} n! \frac{x^n}{n!} \right) \left( (n+t)! \frac{x^n}{n!} \right) \quad (8.1) \\ &= \left( \sum_{n \geq 0} x^n \right) \left( t! \sum_{n \geq 0} \frac{(n+t)!}{n!t!} x^n \right) = \\ &= \frac{1}{1-x} \left( t! \sum_{n \geq 0} \binom{n+t}{t} x^n \right) = \frac{1}{1-x} \frac{t!}{(1-x)^{t+1}} = \frac{t!}{(1-x)^{t+2}}. \end{aligned}$$

On the other hand

$$A(x) = \frac{t!}{(1-x)^{t+2}} = t! \sum_{n \geq 0} \binom{n+t+1}{t+1} x^n = \sum_{n \geq 0} \frac{(n+t+1)!}{t+1} \frac{x^n}{n!}. \quad (8.2)$$

By comparing (8.1) and (8.2) we get the result.  $\square$

## Appendix B

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**Lemma 2.** *Given  $n, s \in \mathbb{N}$  and  $2 \leq s \leq n$ , the following combinatorial identity holds:*

$$\sum_{k=0}^{n-s} \binom{n-s}{k} \frac{(s-2+k)!(n-s-k)!}{(n-1)!} = \frac{1}{s-1}. \quad (8.3)$$

*Proof.* To derive the position value on trees for the unanimity games  $u_S$ , we made use of the superfluous arc property. An equivalent formula can be obtained directly by considering all the possible coalitions to whom a given link provides a positive marginal contribution.

Each  $i \notin S$  gets a null contribution from every incident link, because of the superfluous arc property. Consider  $i \in S$ . Node  $i$  gets a positive contribution (equal to  $1/2$ ) every time a link incident to it is the last one to form inside  $S$  (no matter which links already formed outside  $S$ ). This happens  $\deg_S(i)$  times. It follows that:

$$\pi_i(u_S) = \begin{cases} \frac{1}{2} \deg_S(i) \sum_{k=0}^{n-s} \binom{n-s}{k} \frac{(s-2+k)!(n-s-k)!}{(n-1)!} & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases} \quad (8.4)$$

From the equivalence of formulas (5.15) and (8.4), the result follows directly.  $\square$

We point out that other combinatorial identities arise by considering a generic regular semivalue  $\Psi$  and computing the corresponding  $\psi(N, u_S, \Gamma)$  as in (5.7), when  $\Gamma$  is a tree.

As for the position value, the solution  $\psi(u_S)$  can be obtained directly or by using the superfluous arc property. From the equivalence of the corresponding formulas, it follows that:

$$\sum_{k=0}^{n-s} \binom{n-s}{k} p_{s+k-2}^{n-1} = p_{s-2}^{s-1} \quad (8.5)$$

where  $\{p_j^m\}_j$  is a probability distribution over the subsets of links in a network with  $m$  links. Precisely,  $p_j^m$  represents the probability for a link to join a coalition of cardinality  $j$ , with  $0 \leq j \leq m-1$ .

For example, by considering the Banzhaf index, we get the trivial identity

$$\sum_{k=0}^{n-s} \binom{n-s}{k} \frac{1}{2^{n-2}} = \frac{1}{2^{s-2}}. \quad (8.6)$$

Non-trivial identities can be derived by considering other regular semivalues, such as the p-binomial semivalues:

$$\sum_{k=0}^{n-s} \binom{n-s}{k} q^{s+k-2} (1-q)^{n-s-k} = q^{s-2}, \quad (8.7)$$

where  $q \in (0, 1)$ .

Note that the combinatorial identities that we derived can be easily obtained through classical game-theoretical tools by computing the corresponding power indices on the unanimity games.



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## Résumé

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Les jeux coalitionnels décrivent des situations dans lesquelles tous les joueurs sont libres d'interagir entre eux, c'est-à-dire que n'importe quelle coalition de joueurs peut se former et coopérer. Quand la restriction de possibilités d'interaction entre joueurs est décrite par une structure de réseau, on parle de *coalitional games on networks*, qui seront au centre de cette thèse. Un jeu coalitionnel, traditionnellement appelé jeu coopératif avec utilité transférable, consiste en une paire  $(N, v)$ , où  $N$  dénote un ensemble fini des joueurs et  $v : 2^N \rightarrow \mathbb{R}$  est la fonction caractéristique, une fonction réelle sur la famille de sous-ensembles de  $N$ . Un groupe de joueurs  $S \subseteq N$  est appelé *coalition* et la fonction caractéristique associe à chaque coalition  $S$  une valeur réelle  $v(S)$ , qui représente le total des profits de la coalition des joueurs quand ils coopèrent, quoi que fassent les joueurs restants. La valeur d'une coalition peut représenter un gain ou un coût, selon la situation modélisée par le jeu coopératif. Par convention, on pose  $v(\emptyset) = 0$ .

Un jeu coalitionnel avec  $n$  joueurs est ainsi caractérisé par un vecteur de  $2^n - 1$  réels, i.e. une valeur pour chaque sous-ensemble non vide des joueurs, ce qui devient difficile à traiter quand  $n$  est grand. Puisque le nombre de coalitions croît exponentiellement avec le nombre de joueurs, il est très intéressant, pour des raisons de calcul, de sélectionner des classes de jeux qui peuvent être décrites d'une façon concise. Par conséquent, de nombreux modèles dans la littérature sur les jeux coopératifs se concentrent sur des situations d'interaction caractérisées par une représentation compacte d'un jeu coalitionnel, de manière que la valeur de chaque coalition puisse être facilement calculée. Une représentation compacte permet non seulement de réduire la complexité de description d'un jeu et le calcul des solutions mais aussi de réunir de nombreux problèmes réels sous un formalisme unifié.

De nombreuses classes de jeux qui décrivent d'une façon compacte le synergisme entre joueurs se trouvent dans la littérature, entre autres : *profit sharing games*, *cost allocation games*, *market games*, *optimization games* (*spanning tree games*, *flow games* and *linear programming games*) et *voting games* (voir [17] et [54] pour une étude sur les jeux coalitionnels et les *operational research games*).

En particulier, il existe de nombreuses approches pour la définition des classes de jeux dont la représentation concise est dérivée d'un système additif entre coalitions. Dans certains contextes, en raison d'une structure sous-jacente entre les joueurs, telle qu'un

réseau, un ordre, ou une structure de permission, la valeur d'une coalition  $S \subseteq N$  peut être dérivée additivement à partir d'une collection des sous-coalitions  $\{T_1, \dots, T_k\}$ ,  $T_i \subseteq S \forall i \in \{1, \dots, k\}$ . De telles situations sont modélisées, par exemple, par les *graph-restricted games*, introduits par Myerson dans [75] et étudiés davantage par Owen dans [78]; les *component additive games* [30], et les *restricted component additive games* [29].

Parfois, la valeur de chaque coalition est calculée à partir des valeurs que les joueurs peuvent s'assurer par un mécanisme qui décrit les interactions entre les individus au sein d'un groupe de joueurs. Dans le cas le plus simple on peut considérer que, quand une coalition de joueurs se forme, chaque joueur apporte sa propre valeur et la valeur de la coalition est calculée comme la somme des contributions individuelles des joueurs qui la constituent. Par exemple, on peut considérer un jeu des coûts où  $n$  joueurs veulent acheter en ligne  $n$  objets différents et la valeur d'un joueur dans le jeu est définie comme le prix de l'objet qu'il achète. Ainsi, si un groupe de joueurs s'accorde pour faire un achat ensemble, le coût de l'opération sera simplement la somme du  $s = |S|$  prix des objets que les joueurs dans  $S$  achètent, c'est-à-dire la somme des coûts que les joueurs dans  $S$  devraient supporter individuellement s'ils achetaient les objets séparément.

Cette situation peut être décrite par un *jeu additif*, où la valeur d'une coalition est calculée comme la somme des coalitions disjointes qui la constituent. Un jeu additif est en effet déterminé par le vecteur des  $n$  valeurs individuelles des joueurs et constitue donc une façon compacte pour représenter les situations d'interactions entre joueurs.

Cependant, un tel modèle peut échouer à souligner l'importance d'un sous-ensemble des joueurs à contribuer à la valeur d'une coalition dont il fait partie. Dans l'exemple précédent, il arrive souvent qu'en faisant un achat collectif, quand un certain prix limite est atteint, des objets soient offerts et donc le prix qu'une coalition  $S$  doit payer dépendra seulement du prix d'un sous-ensemble des objets achetés.

En effet, dans certains cas la procédure utilisée pour estimer la valeur d'une coalition  $S \subseteq N$  est fortement liée à la somme des valeurs individuelles dans un autre sous-ensemble  $S' \subseteq N$ , non nécessairement inclus dans  $S$ .

De nombreux exemples dans la littérature rentrent dans cette catégorie. Par exemple, le célèbre *glove game* : l'ensemble des joueurs  $N$  est réparti en deux catégories, les joueurs dans  $L$  qui possèdent un gant gauche et ceux dans  $R$  qui possèdent un gant droit. La valeur d'une coalition des joueurs  $S \subseteq N$  est définie comme le nombre de paires de gants possédés par la coalition  $S$ . Dans ce contexte, les joueurs qui apportent une valeur à une coalition sont ceux dont la classe représente la minorité des joueurs, car la valeur de  $S$  est définie par le minimum entre le nombre des joueurs dans  $S \cap L$  et dans  $S \cap R$ . On peut donc représenter ce jeu en attribuant une valeur de 1 à chaque joueur et en calculant la valeur de chaque coalition  $S$  comme la somme des valeurs individuelles des joueurs dans le sous-ensemble le plus petit entre  $S \cap L$  et  $S \cap R$ . Une approche similaire peut être utilisée pour décrire de nombreuses classes de jeux dans la littérature et en particulier certaines classes de *graph games*, entre eux les *airport games* [63], [64], le *connectivity game* et ses extensions ([2], [62]), les *argumentation games* [16] et certaines classes de *operation research games*, comme les *peer games* [19] ou les *mountain situations* [73].

Dans les *graph games*, un graphe (ou réseau)  $(N, E)$  décrit les interactions entre joueurs : les nœuds (ou sommets) du réseau sont les joueurs dans  $N$  et il existe une arête (ou lien)  $e = \{i, j\} \in E$  entre deux nœuds  $i$  et  $j$  si les joueurs correspondants

sont capables d'interagir directement. Par exemple, dans les *argumentation games*, un graphe orienté décrit les relations d'attaque entre les arguments qui font partie d'une opinion : il existe une arête entre un argument et l'autre si l'un attaque l'autre ; dans les *peer games* un réseau orienté décrit la structure hiérarchique entre des agents : il existe un lien orienté entre un nœud et l'autre si l'un est supérieur à l'autre dans la hiérarchie et le chef de l'organisation est la source du graphe ; dans les *mountain situations* un graphe orienté représente les possibilités de connexion entre les maisons dans un village de montagne et une source d'eau : il existe un lien entre une maison et une autre plus en bas s'il est possible de les relier afin de créer une voie qui permet à l'eau d'arriver depuis la source.

Le réseau modélise les restrictions de possibilité d'interaction entre joueurs et établit la façon dont les compétences individuelles interagissent dans des groupes de joueurs : si on définit la valeur d'une coalition d'arguments comme le nombre d'arguments qui ne sont pas attaqués par un autre argument dans la coalition, alors un joueur dans un jeu d'argumentation contribue à la valeur d'une coalition dont il fait partie seulement si aucun de ses attaquants ne fait partie de la coalition ; un agent dans un jeu de pairs contribue avec sa propre compétence au fonctionnement de l'organisation si tous ses supérieurs dans la hiérarchie coopèrent avec lui, en d'autres termes il contribue seulement aux coalitions qui contiennent tous ses supérieurs ; une maison dans une mountain situation contribue à la division du coût de connexion à la source seulement si elle se trouve dans l'arbre de coût minimal qui relie les maisons à la source.

En d'autres termes, dans plusieurs cas, la structure de réseau détermine quels joueurs doivent contribuer à la valeur (ou coût) des coalitions, en assemblant leurs valeurs individuelles.

Dans tous les modèles mentionnés ci-dessus, la valeur d'une coalition  $S$  de joueurs est calculée comme la somme des valeurs individuelles des joueurs dans un sous-ensemble de  $S$ . D'un autre côté, dans certains cas la valeur d'une coalition peut être affectée par des influences extérieures et des joueurs en dehors de la coalition peuvent contribuer, soit d'une façon positive soit négative, à la valeur de la coalition même. C'est le cas, par exemple, des *bankruptcy games* [5] et des *maintenance problems* [58].

La première partie de cette thèse est dédiée à l'introduction d'un modèle de théorie des jeux qui embrasse toutes les classes de jeux coalitionnels mentionnées ci-dessus. Dans le Chapitre 3, on introduit la classe de *Generalized Additive Games (GAGs)*, où la valeur d'une coalition  $S \subseteq N$  est évaluée par un filtre d'interaction, c'est-à-dire une fonction  $\mathcal{M}$  qui élit les joueurs qui contribuent à la valeur de la coalition  $S$ .

L'objectif de ce modèle est de fournir un cadre général pour décrire de nombreuses classes des jeux étudiées dans la littérature sur les jeux coalitionnels, et en particulier sur les graph games, et de fournir une sorte de taxonomie des jeux coalitionnels qui sont attribuables à cette notion d'additivité sur les valeurs individuelles.

On appelle *Generalized Additive Situation (GAS)* un triplet  $\langle N, v, \mathcal{M} \rangle$ , où  $N$  est l'ensemble des joueurs,  $v : N \rightarrow \mathbb{R}$  est une fonction qui associe à chaque joueur une valeur réelle et  $\mathcal{M} : 2^N \rightarrow 2^N$  est une *fonction de coalition*, qui associe une coalition (qui peut être vide)  $\mathcal{M}(S)$  à chaque coalition  $S \subseteq N$  des joueurs et telle que  $\mathcal{M}(\emptyset) = \emptyset$ .

Étant donné le GAS  $\langle N, v, \mathcal{M} \rangle$ , le *Generalized Additive Game (GAG)* associé est défini comme le jeu coalitionnel  $(N, v^{\mathcal{M}})$  qui associe à chaque coalition la valeur

$$v^{\mathcal{M}}(S) = \begin{cases} \sum_{i \in \mathcal{M}(S)} v(i) & \text{si } \mathcal{M}(S) \neq \emptyset \\ 0 & \text{sinon.} \end{cases} \quad (1)$$

La définition générale de la fonction  $\mathcal{M}$  permet d'embrasser diverses classes de jeux, comme par exemple les *simple games*. Soit  $w$  un jeu simple.  $w$  peut être décrit par le GAG associé à  $\langle N, v, \mathcal{M} \rangle$  où  $v(i) = 1$  pour tous  $i$  et

$$\mathcal{M}(S) = \begin{cases} \{i\} \subseteq S & \text{si } S \in \mathcal{W} \\ \emptyset & \text{sinon} \end{cases}$$

où  $\mathcal{W}$  est l'ensemble des coalitions gagnantes dans  $w$ . Dans le cas où il y a un *veto player*, c'est-à-dire un joueur  $i$  tel que  $S \in \mathcal{W}$  seulement si  $i \in S$ , alors le jeu peut aussi être décrit par  $v(i) = 1, v(j) = 0 \forall j \neq i$  et

$$\mathcal{M}(S) = \begin{cases} T & \text{si } S \in \mathcal{W} \\ R & \end{cases}$$

avec  $T, R \subseteq N$  tels que  $i \in T$  et  $i \notin R$ . Ceci montre en particulier que la description d'un jeu comme GAG n'est pas nécessairement unique.

De plus, en faisant plus d'hypothèse sur  $\mathcal{M}$ , notre approche permet de classifier des jeux existants sur la base des propriétés de  $\mathcal{M}$ .

En particulier, on définit la classe de *basic GAGs*, qui est caractérisée par le fait que les joueurs qui contribuent à une coalition  $S$  sont sélectionnés sur la base de la présence, parmi les joueurs dans  $S$ , de leurs *amis* et *ennemis*, c'est-à-dire que un joueur contribue à la valeur de  $S$  si et seulement si  $S$  contient au moins un des ses amis et aucun des ses ennemis.

Soit  $\mathcal{C} = \{\mathcal{C}_i\}_{i \in N}$  une collection, dont  $\mathcal{C}_i = \{F_i^1, \dots, F_i^{m_i}, E_i\}$  est une collection de sous-ensembles de  $N$  telle que  $F_i^j \cap E_i = \emptyset$  pour tous  $i \in N$  et pour tous  $j = 1, \dots, m_i$ . On note  $\langle N, v, \mathcal{C} \rangle$  le *basic GAS* associé à la fonction de coalition  $\mathcal{M}$  définie par :

$$\mathcal{M}(S) = \{i \in N : S \cap F_i^1 \neq \emptyset, \dots, S \cap F_i^{m_i} \neq \emptyset, S \cap E_i = \emptyset\} \quad (2)$$

et par  $\langle N, v^{\mathcal{C}} \rangle$  le GAG associé, qu'on appelle *basic GAG*.

Pour simplifier, on peut supposer sans perte de généralité que  $m_1 = m_2 = \dots = m_n := m$ . On appelle chaque  $F_i^k$ , pour tous  $i \in N$  et tous  $k = 1, \dots, m$ , le  $k$ -ème ( $k$ -th) ensemble des *amis* de  $i$ , et  $E_i$  l'ensemble des *ennemis* de  $i$ .

Plusieurs classes de jeux susmentionnées peuvent être décrites comme *basic GAGs*, ainsi que des jeux qui dérivent de situations réelles. Par exemple, ce modèle, se révèle approprié pour représenter un réseau social en ligne, dont les amis et ennemis des utilisateurs du web sont déterminés par leurs profils sociaux. De plus, le Chapitre 7 présente une application des *basic GAGs* au domaine de la biomédecine.

Un cas particulièrement simple est celui où chaque joueur a un seul ensemble d'amis, qu'on note par  $F_i$ .

**Exemple 1.** (*airport games*) [63][64] : Soit  $N$  l'ensemble des joueurs. On divise  $N$  en groupes  $N_1, N_2, \dots, N_k$  tels que à chaque  $N_j, j = 1, \dots, k$ , est associé un nombre réel positif  $c_j$  avec  $c_1 \leq c_2 \leq \dots \leq c_k$ , qui représentent des coûts. Soit  $w$  le jeu dont la valeur d'une coalition  $S$  est définie par  $w(S) = \max\{c_i : i \in S\}$ . Tel jeu (et ses variantes) peut être décrit par un *basic GAS*  $\langle N, (\mathcal{C}_i = \{F_i, E_i\})_{i \in N}, v \rangle$  dont pour chaque  $i \in N_j$  et chaque  $j = 1, \dots, k$  :

- $v(i) = \frac{c_j}{|N_j|}$ ,
- $F_i = N_j$ ,

et  $E_i = N_{j+1} \cup \dots \cup N_k$  pour chaque  $i \in N_j$  et chaque  $j = 1, \dots, k-1$  et  $E_l = \emptyset$  pour chaque  $l \in N_k$ .

Par des arguments similaires, il est possible de montrer que les *maintenance games* [17, 58], qui généralisent les *airport games*, peuvent aussi être représentés comme basic GAGs. De plus, on fournit d'autres exemples de classes de jeux coalitionnels qui peuvent être décrits comme basic GAGs, où en générale chaque joueur a plusieurs ensembles d'amis.

**Exemple 2.** (*argumentation games*) [16] Soit  $\langle N, \mathcal{R} \rangle$  un graphe orienté, où l'ensemble des nœuds  $N$  est un ensemble fini d'arguments et l'ensemble des arêtes  $\mathcal{R} \subseteq N \times N$  est une relation binaire d'attaque [38]. Pour chaque argument  $i$ , on définit l'ensemble des attaquants de  $i$  dans  $\langle N, \mathcal{R} \rangle$  comme l'ensemble  $P(i) = \{j \in N : (j, i) \in \mathcal{R}\}$ . L'interprétation est la suivante : si  $j \in P(i)$  cela signifie que l'argument  $j$  attaque l'argument  $i$ . La valeur d'une coalition  $S$  est le nombre d'arguments dans l'opinion  $S$  qui ne sont pas attaqués par un autre argument dans  $S$ . Ce jeu (et ses variantes) peut être décrit comme un basic GAS  $\langle N, v, \{F_i, E_i\} \rangle$  en fixant  $v(i) = 1$ , l'ensemble des amis  $F_i = \{i\}$  et l'ensemble des ennemis  $E_i = P(i)$ . Cet exemple peut être décrit comme un basic GAG où chaque joueur a un seul ensemble d'amis. Pourtant, il existe aussi d'autres genres de fonctions caractéristiques qui peuvent être naturellement considérées. Par exemple, il est intéressant de considérer le jeu  $(N, v^M)$  tel que pour chaque  $S \subseteq N$ ,  $v^M(S)$  est la somme des  $v(i)$  sur les éléments de l'ensemble  $D(S) = \{i \in N : P(i) \cap S = \emptyset \text{ et } \forall j \in P(i), P(j) \cap S \neq \emptyset\}$  des arguments qui ne sont pas intérieurement attaqués par  $S$  et en même temps sont défendus par  $S$  des attaques extérieurs :

$$v^M(S) = \sum_{i \in D(S)} v(i). \quad (3)$$

Il est clair que cette situation ne peut pas être décrite par un basic GAG où chaque joueur a un seul ensemble d'amis. Le jeu dans [3] peut, cependant, être décrit comme un basic GAG  $\langle N, v^C \rangle$ , où, étant donnée une bijection  $k : P(i) \rightarrow \{1, \dots, |P(i)|\}$ ,  $\mathcal{C}_i = \{F_i^1, \dots, F_i^{|P(i)|}, E_i\}$  est tel que  $F_i^{k(j)} = P(j) \setminus P(i)$  pour tous  $j \in P(i)$ , et  $E_i = P(i)$  pour tous  $i \in N$ .

**Exemple 3.** (*peer games*) [19] Soit  $N = \{1, \dots, n\}$  l'ensemble des joueurs et  $T = (N, A)$  un arbre orienté avec une source, décrivant la hiérarchie parmi les joueurs, avec  $N$  comme ensemble des sommets, 1 comme source (représentant le chef du groupe) et  $A \subset N \times N$  comme ensemble des liens. Chaque agent  $i$  a un potentiel individuel  $a_i$  qui représente le gain que le joueur  $i$  peut produire si tous les joueurs aux niveaux supérieurs dans la hiérarchie coopèrent avec lui. Pour chaque  $i \in N$ , on note  $S(i)$  l'ensemble des tous les agents dans l'unique chemin orienté qui relie 1 à  $i$ , c'est-à-dire l'ensemble des supérieurs de  $i$ . Étant donné une peer group situation  $(N, T, a)$  décrite comme ci-dessus, un peer game est défini comme le jeu  $(N, v^P)$  tel que, pour chaque coalition non vide  $S \subseteq N$

$$v^P(S) = \sum_{i \in N: S(i) \subseteq S} a_i.$$

Un peer game  $(N, v^P)$  peut être décrit comme le GAG associé au basic GAS sur  $N$  où  $v(i) = a_i$  et où  $\mathcal{M}$  est définie par la relation (3.4) avec les collections  $\mathcal{C}_i = \{F_i^1, \dots, F_i^n, E_i\}$  telles que :

$$F_i^j = \begin{cases} \{j\} & \text{si } j \in S(i) \\ \{i\} & \text{sinon} \end{cases}$$

et  $E_i = \emptyset$  pour tous  $i \in N$ .

**Exemple 4.** (maintenance cost games) [17, 58] Soit  $N$  un groupe de joueurs reliés par un arbre  $T$  (par exemple, un réseau d'ordinateurs) à une source  $0$  (par exemple, un fournisseur de services) et supposons que chaque arête de l'arbre ait un coût de maintenance ; le problème qui se pose est comment diviser d'une façon convenable le coût du réseau entier parmi les joueurs dans  $N$ . Plus formellement, soit  $(T, t)$  une paire, où  $T = (N \cup \{0\}, E)$  est un arbre.  $N \cup \{0\}$  représente un ensemble de sommets et  $E$  l'ensemble de arêtes, les paires  $\{i, j\}$  telles que  $i, j \in N$ .  $0$  est la source de l'arbre, qui a une seule arête adjacente, et  $t : E \rightarrow \mathbb{R}^+$  est une fonction de coût non négative sur les arêtes de l'arbre. Notons que chaque sommet  $i \in N$  est relié à la source  $0$  par un unique chemin  $P_i$  ; on note  $e_i$  l'arête dans  $P_i$  qui est incidente à  $i$ . Une relation de précédence  $\preceq$  est définie par :  $j \preceq i$  si et seulement si  $j$  se trouve sur le chemin  $P_i$ . Un tronc  $R \subseteq N \cup \{0\}$  est un ensemble de sommets qui est fermé pour la relation  $\preceq$ , c'est-à-dire si  $i \in R$  et  $j \preceq i$ , alors  $j \in R$ . L'ensemble de followers du joueur  $i \in N$  est défini comme  $F(i) = \{j \in N \mid i \preceq j\}$  (notons que  $i \in F(i)$  pour tous  $i \in N$ ). Le coût d'un tronc  $R$  est défini comme

$$C(R) = \sum_{i \in R \setminus \{0\}} t(e_i),$$

et le maintenance cost game  $(N, c)$  associé est défini par

$$c(S) = \min\{C(R) : S \subseteq R \text{ and } R \text{ is a trunk}\}.$$

Notons que l'arête  $e_i$  est présente dans le tronc de coût minimal qui contient tous les membres de  $S$  chaque fois qu'un membre de  $S$  est un follower du joueur  $i$ , c'est-à-dire  $S \cap F(i) \neq \emptyset$ . On peut donc représenter le jeu de coût (cost game)  $(N, c)$  comme le GAG associé au basic GAS sur  $N$  où  $v(i) = t(e_i)$  et où  $\mathcal{M}$  est définie par la relation (3.4) avec des collections  $\mathcal{C}_i = \{F_i, E_i\}$  telles que  $F_i = F(i)$  et  $E_i = \emptyset$  pour tous  $i \in N$ .

L'intérêt de cette classification n'est pas seulement taxonomique, puisqu'elle permet aussi d'étudier les propriétés des solutions des classes de jeux connus dans la littérature et elle fournit des outils potentiellement efficaces pour calculer des solutions de nouvelles classes de jeux qui peuvent être décrites dans ce cadre formel. En particulier, il est possible de déduire, pour la classe de basic GAGs, des résultats concernant d'importants concepts de solutions, comme le *core* et les *semivalues*.

Il est donc intéressant d'étudier sous quelles conditions une GAS peut être décrite comme un basic GAS. Dans cette optique, on démontre le théorème suivant, qui décrit une condition nécessaire et suffisante quand l'ensemble des ennemis de chaque joueur est vide.

**Théorème 1.** Soit  $\langle N, v, \mathcal{M} \rangle$  une GAS. La fonction  $\mathcal{M}$  peut être obtenue par la relation (3.4) avec des collections  $\mathcal{C}_i = \{F_i^1, \dots, F_i^{m_i}, E_i = \emptyset\}$ , pour tous  $i \in N$ , si et seulement si  $\mathcal{M}$  est monotone.

Dans le Chapitre 4, on fournit quelques résultats sur des concepts des solutions classiques pour les basic GAGs, et on s'intéresse au problème de garantir le fait que le cœur d'un basic GAG soit non vide. En particulier, on fournit des conditions suffisantes pour que le cœur soit non vide et des formules concises pour les semivalues pour basic GAGs où chaque joueur a un seul ensemble d'amis, ou deux ensembles, dont un est le singleton  $F_i^k = \{i\}$ .

Dans un GAG, la valeur de chaque coalition est calculée comme la somme des valeurs individuelles d'un sous-ensemble de joueurs. D'autre part, dans plusieurs cas, quand une structure de réseau décrit les interactions parmi les joueurs concernés, il est possible de dériver la valeur de chaque coalition comme la somme des contributions qui sont produites par les interactions par paires, c'est-à-dire comme la somme des valeurs individuelles associées aux arêtes du réseau. A titre d'exemple, dans les *maintenance cost games* [58], un arbre décrit un système de maintenance, avec un fournisseur de services comme source. Le coût de connexion d'un ensemble d'ordinateurs au fournisseur est décrit par un jeu coalitionnel et calculé comme la somme de coûts de maintenance de toutes les connexions parmi les ordinateurs qui se trouvent dans l'arbre de coût minimal associé, c'est-à-dire comme la somme des coûts associés aux arêtes dans le graphe induit.

Comme pour les maintenance cost game, dans plusieurs autres graph games la valeur des coalitions peut être calculée additivement à partir des valeurs associées aux arêtes dans le graphe sous-jacent. De plus, comme dans les graph-restricted games la valeur d'une coalition peut être dérivée additivement d'une collection de sous-coalitions de joueurs, aussi pour la classe de *link games*, introduite par Meessen [70] et davantage étudié par Borm et al. [18], la valeur d'une coalition des liens peut être dérivée additivement à partir d'une collection des sous-coalitions des arêtes. En effet, plusieurs approches avec les *coalitional games on networks* s'appuient sur des structures additives parmi les arêtes, non seulement pour ce qui concerne la définition d'un jeu, mais aussi pour l'analyse des solutions respectives.

La deuxième partie de cette thèse est dédiée à l'introduction d'une classe de concepts des solutions pour les *communication situations* [75], où le paiement de chaque joueur est calculé additivement à partir des valeurs produites par les relations binaires parmi les joueurs. Dans le Chapitre 5, on considère une situation de communication  $(N, v, \Gamma)$  dans laquelle un réseau  $\Gamma$  est produit par la formation des liens parmi les joueurs et, à chaque étape du processus de formation, le surplus créé par un lien est partagé parmi les joueurs impliqués, selon une certaine règle. Plus précisément, on suppose qu'un réseau  $\Gamma$  parmi les joueurs dans un jeu coalitionnel  $(N, v)$  soit produit par une permutation des arêtes et que tous les différents ordres dans lesquels les liens se forment soient équiprobables. De plus, on suppose que à chaque étape du processus de formation du réseau, quand un lien entre deux joueurs  $i$  et  $j$  se crée, la valeur de la coalition  $S$ , où  $S$  est la composante connexe qui contient  $i$  et  $j$ , réduite par les valeurs des composants connexes formées par les joueurs dans  $S$  à l'étape précédent, soit partagée parmi les

joueurs impliqués selon un certain protocole. Plus formellement, étant donnée une situation de communication  $(N, v, \Gamma)$ , soit  $\sigma$  une permutation des liens. À chaque étape  $k$  du processus de formation du réseau, quand le  $k$ -ème lien  $a = \{i, j\}$  dans la séquence déterminée par  $\sigma$  se forme, soit  $S_k^\sigma$  le *surplus* produit par  $a$  :

$$S_k^\sigma = v(S) - v(C_{k-1,\sigma}^i) - v(C_{k-1,\sigma}^j) \quad (4)$$

où  $S$  est la composante connexe dans  $\Gamma$  qui contient  $i$  et  $j$  à l'étape  $k$ , et  $C_{k-1,\sigma}^i$  et  $C_{k-1,\sigma}^j$  sont les composantes connexes dans  $\Gamma$  à l'étape  $k-1$ , qui contient  $i$  et  $j$  respectivement. On appelle *protocole d'allocation* une règle qui spécifie comment partager  $S_k^\sigma$  parmi les joueurs dans  $S$ . Étant donnée un protocole d'allocation  $r$  et une situation de communication  $(N, v, \Gamma)$ , une solution pour  $v$ , que l'on note  $\phi^r(v)$ , est définie par :

$$\phi_i^r(v) = \frac{1}{|E|!} \sum_{\sigma \in \Sigma_E} \sum_{k=0}^{|E|} f_i^r(S_k^\sigma), \quad \forall i \in N, \quad (5)$$

où  $\Sigma_E$  est l'ensemble des toutes les permutations sur l'ensemble des liens  $E$  dans  $\Gamma$  et  $f_i^r$  est une fonction qui associe à chaque joueur  $i \in N$  une quantité fixée du surplus  $S_k^\sigma$ , qui dépend du protocole d'allocation  $r$ .

En d'autres termes, la solution  $\phi^r(v)$  est définie en considérant toutes les permutations possibles de liens, et en sommant, pour chaque joueur  $i$ , toutes les contributions qu'il reçoit avec la procédure d'allocation  $r$ , en faisant la moyenne pour toutes les permutations sur l'ensemble des liens parmi les joueurs. Cette idée amène à l'introduction d'une nouvelle classe de concepts de solutions : différents choix du protocole d'allocation définissent différentes solutions pour une situation de communication.

En particulier, lors d'une étape, quand le lien  $a = \{i, j\}$  se forme, il est possible de considérer le protocole d'allocation qui partage également le surplus entre les seuls joueurs  $i$  et  $j$ . La solution obtenue par ce protocole d'allocation coïncide avec le *position value*, introduit par Borm et al. [18], fournissant donc une interprétation différente de ce concept de solution.

De plus, dans le Chapitre 5, on enquête sur le problème de calculer cette solution pour des classes particulières de situations de communication. En particulier, on fournit une expression pour le position value d'un jeu quand le réseau sous-jacent est un arbre, qui relie son calcul à celui des jeux d'unanimité.

La troisième partie de cette thèse est dédiée à deux applications des modèles de théorie des jeux décrits jusqu'ici. Une première application, présentée dans le Chapitre 6, concerne le domaine de la théorie de l'argumentation. La théorie de l'argumentation a pour objectif de formaliser les systèmes de décision et les processus de décision associés. Un de ses objectifs est la recherche d'ensembles de conclusions acceptées dans un cadre d'argumentation, qui est modélisé par un graphe orienté où les nœuds représentent des arguments, c'est-à-dire des affirmations ou séries des affirmations, et les arêtes représentent des relations d'attaque, qui expriment le conflit parmi paires des arguments.

Dans la littérature, plusieurs *sémantiques d'extension* (*extension semantics*) ont été associées au cadre de l'argumentation abstraite avec l'objectif de spécifier quels argu-

ments sont acceptés, lesquels ne le sont pas et lesquels sont incertains [21, 38]. Différentes des sémantiques d’extensions, le but des *sémantiques graduelles* (*gradual semantics*) est d’associer à chaque argument un degré d’acceptabilité [4, 13, 24, 49, 97].

La théorie des jeux a aussi été employée pour la définition des niveaux intermédiaires d’acceptabilité des arguments. Précisément, dans [67] un degré d’acceptabilité est défini en prenant en compte la valeur de minmax d’un jeu à somme nulle entre un ‘partisan’ et un ‘opposant’ et où les stratégies et les paiements des joueurs dépendent de la structure du graphe d’argumentation. Plus récemment, les jeux coalitionnels ont été utilisés dans [16] pour mesurer l’importance relative des arguments en prenant en compte les préférences d’un agent sur les arguments et l’information fournie par les relations d’attaque. Dans les approches susmentionnées, le poids attribué à chaque argument représente la force d’un argument pour imposer son acceptabilité. D’autre part, l’acceptabilité n’est pas le seul attribut des arguments qui a été étudié dans la littérature dans une perspective graduelle. Dans [98] un index a été introduit pour représenter la *controversialité* individuelle des arguments, où les arguments les plus controversés sont ceux pour lesquels prendre une décision sur leur acceptabilité est difficile. Dans la même direction, le problème de mesurer le *désaccord* (*disagreement*) dans un cadre d’argumentation a été étudié dans [3], où les auteurs fournissent une analyse axiomatique des différents mesures pour les graphes d’argumentation.

Dans le Chapitre 6, on montre que les propriétés introduites dans [3] pour les arguments individuels, peuvent être reformulées pour les graphes d’argumentation et peuvent amener à la définition d’un classement basé sur le conflit (*conflict-based ranking*), qui peut être vu comme un classement alternatif pour mesurer la *controversialité* des arguments.

Plus formellement, soit  $\mathbf{A} = \langle A, R \rangle$  un *cadre d’argumentation* (ou *graphe d’argumentation*), où  $A = \{1, \dots, n\}$  est un ensemble fini non vide des arguments et  $R \subseteq A \times A$  est une relation d’attaque. Étant donnés deux arguments  $a, b \in N$ , on écrit  $(a, b) \in R$  (où  $aRb$ ) si  $a$  attaque  $b$ .

On note  $\mathcal{A}$  l’ensemble des tous les graphes d’argumentation et  $\mathcal{A}^A$  l’ensemble de tous les graphes d’argumentation avec  $A$  comme ensemble d’arguments.

Une *mesure de désaccord* (*disagreement measure*) [3] est une fonction  $\mathcal{K} : \mathcal{A} \rightarrow [0, 1]$  avec l’interprétation que, pour chaque  $\mathbf{A}, \mathbf{A}' \in \mathcal{U}$ ,  $\mathbf{A}$  est plus conflictuel de  $\mathbf{A}'$  si  $\mathcal{K}(\mathbf{A}) > \mathcal{K}(\mathbf{A}')$ . Notons que  $\mathcal{K} = 0$  correspond à l’absence de désaccord dans un graphe, alors que le maximum du désaccord est fixé par  $\mathcal{K} = 1$ .

En particulier, on considère une mesure basée sur les distances (*distance-based measure*) introduite dans [3], qui est définie comme :

$$\mathcal{K}^{\mathcal{D}}(\mathbf{N}) = \frac{\max - D(\mathbf{A})}{\max - \min} \quad (6)$$

où  $|N| = n$ ,  $\max = n^2(n+1)$ ,  $\min = n^2$  et  $D(\mathbf{A}) = \sum_{i \in A} \sum_{j \in A} d_{i,j}$  est la *distance globale* parmi les arguments dans  $\mathbf{A}$ , où  $d_{i,j}$  est la distance entre l’argument  $i$  et l’argument  $j$ , qui est définie comme la longueur du chemin le plus court entre  $i$  et  $j$  si un tel chemin existe ou  $n + 1$  autrement.

Étant donnée une mesure de désaccord, il est intéressant d’établir quels arguments contribuent le plus au désaccord total dans un graphe d’argumentation. Dans ce but, on introduit, par une approche axiomatique, un index de conflit (*conflict index*) qui évalue la

contribution de chaque argument au désaccord total. Un *index de conflit*  $K : \mathcal{A}^A \rightarrow \mathbb{R}^n$  est une fonction qui associe à chaque graphe d'argumentation avec  $n = |A|$  nœuds (arguments) un vecteur dans  $\mathbb{R}^n$ , qui représentent les contributions de chaque argument au conflit dans le graphe. Plus la valeur que tel index associe à un argument est haute, plus grand est le désaccord apporté par cet argument au graphe et par conséquent sa *controversialité*.

On introduit un index de conflit basé sur les distances (*distance-based conflict index*) défini comme l'index de conflit qui associe à chaque  $i \in N$  la valeur suivante :

$$K_i^D(\mathbf{A}) = \frac{1}{\Delta} \left( \frac{max}{n} - \varphi_i \right), \quad (7)$$

où  $max = n^2(n+1)$ ,  $\Delta = max - min = n^2(n+1) - n^2 = n^3$  et  $\varphi_i = \frac{1}{2} \sum_{j \in A \setminus i} d_{i,j} + \frac{1}{2} \sum_{j \in A \setminus i} d_{j,i}$ . Cet index de conflit associe à chaque argument une valeur qui dépend

de la distance d'un argument de tous les autres et vice versa. On montre que cet index de conflit satisfait huit propriétés : *abstraction*, *coherence*, *maximality*, *free independence*, *monotonicity*, *cycle sensitivity*, *size sensitivity* et *efficiency*, qui reformulent pour les arguments les propriétés introduites dans [3] pour une mesure de désaccord sur les graphes d'argumentation, et relie notre index de conflit à la mesure de désaccord basée sur les distances définies dans [6].

De plus, on montre que l'index de conflit dans [7] peut être re-interprété en termes de solution classique pour les jeux coalitionnels, c'est-à-dire comme la contribution marginale moyenne de chaque argument au désaccord induit par toutes les coalitions possibles de joueurs. On arrive à ce résultat en définissant un jeu coopératif, où les joueurs sont les arguments dans un graphe d'argumentation et à chaque coalition des arguments est associée une valeur, qui exprime le conflit total dans la coalition. En particulier, chaque nœud et chaque lien dans la coalition des arguments contribue à la valeur de la coalition avec la part du désaccord qu'il apporte à la coalition même, mesuré par les relations d'attaque qu'il provoque dans le graphe. Le jeu ainsi défini peut en effet être représenté en termes de basic GAGs, comme une combinaison du modèle original introduit dans le Chapitre 3 et sa variante définie sur les arêtes.

Soit  $\mathbf{A} = \langle A, R \rangle$  un graphe d'argumentation, où  $A$  a pour cardinal  $n$ . On introduit un jeu coopératif  $(A, v)$ , où l'ensemble des joueurs est l'ensemble des arguments  $A$  dans le graphe d'argumentation et la fonction caractéristique est définie comme suit pour chaque  $S \subseteq A$  :

$$v(S) = \frac{max - D(S)}{max - min}, \quad (8)$$

où  $D(S) = \sum_{i,j \in S} d_{i,j}$ ,  $max = n^2(n+1)$  est la valeur maximale que  $D$  peut atteindre dans

un graphe d'argumentation avec  $n$  arguments et  $min = n^2$  celui minimal.  $D(S)$  mesure la distance globale entre les arguments dans la coalition  $S$ , en prenant en compte les relations d'attaque qui existent parmi eux dans le graphe entier : elle est définie comme la somme des distances entre les nœuds dans la coalition  $S$ , où le distance entre deux nœuds est calculée sur le graphe entier. Plus la distance globale dans une coalition est grande, plus haute est la valeur de la coalition dans le jeu  $v$ , ce qui reflète le fait que le conflit global dans une coalition des arguments dépend inversement de la distance entre arguments.

On propose la valeur de Shapley [88] de ce jeu comme un *index de conflict* qui mesure la *controversialité* des arguments, puisqu'il mesure le pouvoir de chaque argument d'apporter du conflit dans le cadre d'argumentation, et on montre que il coïncide avec l'index de conflit défini dans (7).

Bien que le Chapitre 6 décrive une approche utilisant la théorie de jeux appliquée au domaine de la théorie de l'argumentation, le Chapitre 7 présente une application réelle du modèle de GAGs au domaine de la biomedécine, et en particulier au problème d'évaluer l'importance des gènes dans un réseau biologique. Parmi les réseaux biologiques, les *réseaux de régulation de gènes* (*gene regulatory networks* ou *pathways*) suscitent un grand intérêt dans le domaine de la biologie moléculaire et dans l'épidémiologie où il s'agit de mieux comprendre les mécanismes d'interaction entre gènes, protéines et d'autres molécules dans une cellule sous certaines conditions biologiques d'intérêt [20, 23, 31, 93]. Un point crucial de l'analyse des interactions entre gènes est la formulation des mesures appropriées de l'influence de chaque gène dans le système complexe des interactions dans un réseau.

*Centrality analysis* représente un outil important pour l'interprétation de l'interaction entre gènes dans un réseau de régulation de gènes [12, 22, 44, 52, 53]. Les mesures de centrality classiques [41, 57] sont utilisées dans l'analyse des réseaux pour identifier les éléments importants dans un réseau, sur la base de leur position dans la structure donnée par le réseau. Cependant, elles sont appropriées sous la condition que les nœuds agissent de manière indépendante les uns des autres et que le système soit sensible au comportement individuel de chaque nœud. Au contraire, dans des réseaux biologiques complexes, il n'est pas réaliste de supposer que les gènes puissent s'exprimer indépendamment et les conséquences sur le système peuvent être appréciées seulement si plusieurs gènes changent leur expression. Ainsi, dans un scénario complexe comme la pathogenèse d'une maladie génétique, on fait face au problème de quantifier l'importance relative des gènes, en tenant compte non seulement de leur comportement individuel mais surtout de leur niveau d'interaction.

La théorie des jeux coopératifs a été proposée comme cadre théorique pour faire face à ces limitations. Récemment, de nombreuses mesures basées sur les jeux coalitionnels ont été appliquées avec succès aux différentes sortes de réseaux biologiques, comme les *réseaux cérébraux* (*brain networks*) [55, 56, 59], les *réseaux des gènes* (*gene networks*) [72], et les *réseaux métaboliques* (*metabolic networks*) [85].

Nous proposons une approche, avec les jeux coalitionnels, et en particulier avec les basic GAGs au problème de l'identification des gènes importants dans un réseau de gènes. Le problème a été examiné pour la première fois par un modèle de théorie des jeux dans [72], où la valeur de Shapley pour les jeux coalitionnels est utilisée pour exprimer le pouvoir de chaque gène dans son interaction avec les autres et pour souligner l'importance des certains *hub* gènes dans la régulation des *pathways* biologiques d'intérêt. Notre modèle représente un raffinement de cet approche, qui généralise la notion de *degree centrality* [77, 89], dont la corrélation avec l'importance des gènes pour différentes fonctions biologiques est supportée par plusieurs indices pratiques dans la littérature [12, 22, 52, 53, 103]. Nous définissons un basic GAG avec une interprétation biologique sur les réseaux des gènes et nous proposons la valeur de Shapley de ce jeu comme un nouveau index d'importance pour les gènes. Cet approche est supportée par

une caractérisation axiomatique, où les propriétés satisfaites par notre index ont une interprétation biologique. De plus, on fournit une formule pour calculer le nouvel index, qui peut être directement dérivé des résultats théoriques présentés dans le Chapitre 4.

Soit  $\langle N, E \rangle$  un réseau de gènes (*gene network*), c'est-à-dire un réseau où l'ensemble des nœuds  $N$  représente l'ensemble des gènes et l'ensemble des arêtes  $E$  décrit l'interaction entre les gènes : il existe une arête entre deux gènes s'ils interagissent directement dans la condition biologique analysée. De plus, soit  $k \in \mathbb{R}^n$  un vecteur de paramètres qui spécifie l'importance *a priori* de chaque gène. On définit l'ensemble des *voisins* (*neighbours*) d'un nœud  $i$  dans le graphe  $\langle N, E \rangle$  comme l'ensemble  $N_i(E) = \{j \in N : \{i, j\} \in E\}$ , et le *degré* (*degree*) de  $i$  comme le nombre  $d_i(E) = |N_i(E)|$  de voisins de  $i$  dans le graphe  $\langle N, E \rangle$ , c'est-à-dire le nombre des liens incidents à  $i$  dans  $\Gamma$ . On note  $N_S(E) = \{j \in N : \exists i \in S \text{ s.t. } j \in N_i(E)\}$  l'ensemble des voisins des nœuds dans  $S \in 2^N$ ,  $S \neq \emptyset$ , et dans le graphe  $\langle N, E \rangle$ . Étant donné un sous-ensemble  $S \subseteq V$  des nœuds, on définit le sous-graphe induit  $\Gamma_S = (S, E_S)$ , où  $E_S$  est l'ensemble des liens  $\{i, j\} \in E$  tel que  $i, j \in S$ . Un graphe  $\langle N, E_S^i \rangle$ , où l'ensemble des arêtes est  $E_S^i = \{\{i, j\} : j \in S\}$  est appelé une *étoile* (*star*) sur  $S$  avec centre dans  $i$ . Notons que les ensembles de voisins des nœuds dans  $\langle N, E_S^i \rangle$  sont tels que  $N_i(E_S^i) = S$ ,  $N_j(E_S^i) = \{i\}$ , pour chaque  $j \in S$ , et  $N_j(E_S^i) = \emptyset$ , pour tous  $j \in N \setminus (S \cup \{i\})$ .

On définit un jeu coalitionnel  $(N, v_E^k)$ , où  $N$  est l'ensemble des gènes à l'étude et la fonction caractéristique  $v_E^k$  associe une valeur à chaque coalition de gènes  $S \subseteq N$  qui représente la magnitude globale de l'interaction entre les gènes dans  $S$ , qui prend en compte le poids (importance *a priori*) de chaque gène directement lié à  $S$  dans le réseau biologique.

Plus précisément, la fonction  $v_E^k : 2^N \rightarrow \mathbb{R}$  associe à chaque coalition  $S \in 2^N \setminus \{\emptyset\}$  la valeur

$$v_E^k(S) = \sum_{j \in S \cup N_S(E)} k_j \quad (9)$$

qui est la somme des poids associés aux gènes dans  $S$  et à ceux qui sont directement liés dans  $\langle N, E \rangle$  à quelques gènes dans  $S$  (par convention,  $v_E^k(\emptyset) = 0$ ). La classe de jeux  $(N, v)$  définie par la relation (9), sur un réseau de gènes  $G \equiv \langle V, E \rangle$  et avec paramètre  $k \in \mathbb{R}^n$ , est dénotée par  $\mathcal{EK}^N$ .

On observe que le jeu ainsi défini est clairement décrit comme le basic GAG associé à la GAS  $\langle N, v, \mathcal{M} \rangle$ , où  $v(i) = k_i$  pour tous  $i \in N$  et  $\mathcal{M}$  est la fonction associée aux collections  $\mathcal{C}_i = \{F_i = N_i(E), E_i = \emptyset\} \forall i \in N$ . Un gène  $i$  contribue à la valeur d'une coalition avec sa valeur individuelle, le poids  $k_i$ , si et seulement si il appartient à la coalition où au moins un des gènes qui interagissent directement avec lui est présent.

Par rapport au modèle précédent dans la littérature [72], la définition proposée par la relation (9) semble explorer toutes les possibilités d'influence mutuelle entre les gènes plus flexiblement. Elle généralise le jeu introduit dans [94] pour déterminer les "top- $k$  nodes" dans un *co-atorship network*, par l'introduction d'un vecteur de paramètres qui spécifie l'importance *a priori* de chaque nœud. Le vecteur de paramètres  $k$  permet de classifier *a priori* les gènes selon leur importance, alors que dans le modèle précédent [72] seulement une distinction à deux niveaux est faite, entre 'key-genes' et non 'key-genes'. De plus, en évaluant dans quelle mesure une coalition est liée au reste du réseau, la relation (9) généralise la notion de degree centrality pour les groupes des gènes,

qui est justifiée par plusieurs indices pratiques qui démontrent une forte corrélation entre la degree centrality et les gènes qui sont essentiels pour différentes fonctions biologiques [12, 22, 52, 53, 103]. En effet, si seulement les poids des gènes dans une coalition était considéré (et non celui des voisins, comme dans notre définition), la mesure d'importance obtenue par l'approche suivante coïnciderait avec la *weighted degree centrality*.

Nous proposons la valeur de Shapley de ce jeu comme une mesure de l'importance des gènes dans le maintien de l'activité de régulation globale dans un réseau de gènes (voir la Section 7.3 pour un exemple qui motive cette approche, en clarifiant l'objectif de notre index et la différence par rapport aux mesures d'importance classiques), et nous supportons cette idée avec une caractérisation axiomatique de notre index, où les propriétés ont une interprétation biologique sur les réseaux des gènes. On décrit quatre propriétés pour un index d'importance (*relevance index*) pour gènes, qui est une fonction  $\rho : \mathcal{EK}^N \rightarrow \mathbb{R}^n$ . On commence par une réinterprétation des propriétés classiques de SYM (symmetry), DPP (dummy player property) et EFF (efficiency) sur la classe  $\mathcal{EK}^N$  (voir la Section 2.1 pour une définition formelle sur la classe de tous les jeux coalitionnels).

Soit  $\langle N, E \rangle$  un réseau de gènes et  $k \in \mathbb{R}^n$  un vecteur des poids. La propriété de SYM implique que si deux gènes  $i, j \in N$  ont le même poids ( $k_i = k_j$ ) et si ils sont liés au même ensemble de voisins ( $N_i(E) = N_j(E)$ ), alors ils doivent avoir la même importance. La propriété de DPP a aussi une interprétation intuitive sur le graphe : chaque nœud  $i \in N$  qui est sans voisins doit avoir une importance égale à  $k_i$ . Enfin, la propriété de EFF impose que la somme de l'importance de tous les gènes doit être égale à  $\sum_{i \in N} k_i$ , la somme totale des poids.

De plus, on introduit un nouvel axiome, qui impose que la transformation d'un nœud  $i$  avec un poids nul dans un nœud avec poids non nul  $k_i$  doit affecter seulement les gènes directement liés à  $i$ , et son impact sur l'importance des ses voisins doit être égal à celui dans une étoile équivalente avec centre dans  $i$ .

**Axiome 1** (Star Additivity, SADD). *Soit  $\langle N, E \rangle$  un réseau de gènes avec un vecteur de paramètres  $k_{-i} \in \mathbb{R}^n$  tel que le gène  $i$  a poids nul et  $v_E^{k_{-i}}$  le jeu correspondant défini par la relation (9). Soit  $v_E^k$  le jeu défini par la relation (9) sur  $\langle N, E \rangle$  et avec vecteur de paramètres  $k$  qui associe un poids non nul  $k_i$  au gène  $i$  et les mêmes poids que  $k_{-i}$  à tous les autres gènes. Un index  $\rho : \mathcal{EK}^N \rightarrow \mathbb{R}^n$  satisfait la propriété SADD si et seulement si*

$$\rho(v_E^k) = \rho(v_E^{k_{-i}}) + \rho(v_{E_{N_i(E)}}^{s^i}),$$

où  $v_{E_{N_i(E)}}^{s^i}$  est le jeu défini par la relation (9) sur l'étoile  $\langle N, E_{N_i(E)}^i \rangle$  sur  $N_i(E)$  avec centre  $i$  et  $s^i$  est le vecteur des paramètres qui associe un poids  $k_i$  à  $i$  et 0 à chaque  $j \neq i$ .

L'Axiome SADD impose que l'augmentation de poids d'un nœud  $i$  de 0 à une valeur positive doit affecter l'importance totale du gène  $i$  et des ses voisins dans la même mesure pour tous graphes. En conséquence, le changement de poids d'un gène produit le même effet sur sa propre importance et sur celle de ses voisins, indépendamment de la topologie du réseau, et l'effet du changement est comparable parmi différents réseaux.

La proposition suivante fournit une caractérisation de notre index d'importance, ainsi qu'une formule pour son calcul, qui peut être démontré à partir des axiomes ou directement des résultats du Chapitre 4.

**Proposition 1.** *La valeur de Shapley est l'unique index d'importance  $\rho$  qui satisfait les propriétés de SYM, DPP, EFF et SADD sur la classe  $\mathcal{EK}^N$ . De plus, pour chaque réseau de gènes  $\langle N, E \rangle$  avec un vecteur des poids  $k \in \mathbb{R}^n$ , il peut être calculé à partir de la formule suivante :*

$$\rho_i(v_E^k) = \sum_{j \in (N_i(E) \cup \{i\})} \frac{k_j}{d_j(E) + 1}, \quad (10)$$

pour tous  $i \in N$ .

L'interprétation de la formule dans (10) est directe : un gène reçoit une grande importance s'il est lié à de nombreux gènes qui sont à leur tour liés à peu d'autres gènes, c'est-à-dire plus le nombre des voisins avec un degré faible est important, plus l'indice est élevé.

Enfin, on présente dans la Section 7.5 une application à des données réelles. Une étude expérimentale a été conduite sur un réseau d'expression des gènes, en lien avec le cancer du poumon, et une comparaison avec des mesures d'importance classiques conclut l'analyse.

Les deux applications décrites ci-dessus, qui concernent deux domaines de recherche très différents, comme la théorie de l'argumentation et la biomédecine, indiquent que notre modèle représente un important et flexible outil pour l'analyse d'une variété de situations et problèmes réels, aussi grâce au fait que pour certaines sous-classes il est possible d'étudier d'une façon simple et efficace les solutions correspondantes.

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## RIASSUNTO

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UN gioco di coalizione descrive una situazione in cui tutti i giocatori sono liberi di interagire tra di loro, ovvero ogni coalizione di giocatori può formarsi e cooperare. Quando le restrizioni delle interazioni tra giocatori sono descritte da una struttura di network, abbiamo a che fare con i cosiddetti *giochi di coalizione su grafi*, che sono l'oggetto centrale di questa tesi. Un gioco di coalizione, tradizionalmente chiamato *gioco cooperativo ad utilità trasferibile* (TU-game), è definito da una coppia  $(N, v)$ , dove  $N$  denota un insieme finito di giocatori e  $v : 2^N \rightarrow \mathbb{R}$  è la *funzione caratteristica*, una funzione a valori reali sulla famiglia dei sottoinsiemi di  $N$ . Un gruppo di giocatori  $S \subseteq N$  è detto *coalizione* e la funzione caratteristica associa ad ogni coalizione  $S$  un numero reale  $v(S)$ , che è detto *valore* della coalizione e rappresenta l'utilità totale che i giocatori all'interno della coalizione possono ricavare dalla cooperazione, indipendentemente dagli altri giocatori. Il valore di una coalizione può rappresentare un guadagno o un costo, a seconda della situazione descritta dal gioco cooperativo. Per convenzione, si assume che  $v(\emptyset) = 0$ .

Un gioco di coalizione con  $n$  giocatori è descritto da un vettore di  $2^n - 1$  numeri reali, ovvero il numero di sottoinsiemi non vuoti dell'insieme dei giocatori. Poiché il numero di coalizioni cresce esponenzialmente con il numero di giocatori, è interessante, dal punto di vista computazionale, trovare classi di giochi che possono essere descritti in maniera concisa. Di conseguenza, diversi modelli introdotti nella letteratura sui giochi cooperativi si concentrano su situazioni caratterizzate da una rappresentazione compatta di un TU-game, e tali che il valore di ogni coalizione può essere facilmente calcolato. Una rappresentazione compatta non solo permette di ridurre la complessità di descrizione del gioco e del calcolo delle soluzioni, ma permette anche di descrivere una varietà di problemi reali all'interno di un formalismo unificato.

In letteratura vi sono diverse classi di giochi che descrivono in maniera compatta il sinergismo tra i giocatori: tra di esse, i *profit sharing games*, *cost allocation games*, *market games*, *optimization games* (*spanning tree games*, *flow games* e *linear programming games*) e i *voting games* (si rimanda a [17] e [54] per un survey sui giochi di coalizione e gli operation research games).

In particolare, esistono diversi approcci in cui la rappresentazione concisa di classi di

giochi deriva da una struttura additiva tra le coalizioni. In alcuni casi, per via di una struttura sottostante il gioco, come un network, un ordine o una *permission structure*, il valore di una coalizione  $S \subseteq N$  può essere ricavato in maniera additiva a partire da una collezione di sottocoalizioni  $\{T_1, \dots, T_k\}$ ,  $T_i \subseteq S \forall i \in \{1, \dots, k\}$ . Tali situazioni sono descritte, per esempio, dai *graph-restricted games*, introdotti da Myerson in [75] e studiati successivamente da Owen in [78]; dai *component additive games* [30], e dai *restricted component additive games* [29].

Talvolta, il valore di ogni coalizione è calcolato a partire dai valori che i singoli giocatori sono in grado di garantirsi, tramite un meccanismo che descrive le interazioni degli individui all'interno di un gruppo. Nel caso più semplice possiamo considerare che, quando una coalizione di giocatori si forma, ognuno apporti il proprio valore individuale e che il valore della coalizione sia calcolato come la somma dei singoli contributi dei giocatori che la formano. Ad esempio, consideriamo un gioco di costo in cui  $n$  giocatori vogliono acquistare online  $n$  oggetti differenti e il valore di ogni giocatore all'interno del gioco è definito dal prezzo dell'oggetto che acquista. Se un gruppo di giocatori  $S$  decide di fare l'acquisto assieme, il costo dell'operazione è semplicemente la somma degli  $s = |S|$  costi degli oggetti comprati dai giocatori in  $S$ , cioè la somma dei costi che i singoli giocatori dovrebbero sostenere se acquistassero gli oggetti separatamente.

Questa situazione è descritta da un *gioco additivo*, in cui il valore di una coalizione è la somma dei valori delle coalizioni disgiunte che la costituiscono. Il vettore degli  $n$  valori dei singoli giocatori dunque sufficiente per rappresentare il gioco e descrivere in modo compatto l'interazione tra i giocatori.

Tuttavia, tale modello può risultare inefficace nel riflettere l'importanza che un sottoinsieme di giocatori può avere nel contribuire al valore di una coalizione a cui appartiene. Nel precedente esempio, accade spesso che, nel fare un acquisto collettivo, quando una certa soglia di prezzo è raggiunta, alcuni degli oggetti vengano ceduti gratuitamente e di conseguenza il prezzo che una coalizione  $S$  deve pagare dipenda solo dal prezzo di un sottoinsieme degli oggetti acquistati.

Di fatto, in molti casi la procedura utilizzata per definire il valore di una coalizione  $S \subseteq N$  è fortemente legata alla somma dei valori individuali in un altro sottoinsieme  $S \subseteq N$ , non necessariamente incluso in  $S$ .

Diversi esempi in letteratura ricadono in questa categoria. Un semplice esempio è il noto *gioco dei guanti*: l'insieme dei giocatori  $N$  è diviso in due categorie, i giocatori in  $L$  che possiedono un guanto sinistro, e quelli in  $R$  con un guanto destro. Il valore di una coalizione di giocatori  $S \subseteq N$  è definito come il numero di coppie di guanti posseduti dalla coalizione  $S$ . In questo contesto, i giocatori che contribuiscono al valore della coalizione sono quelli la cui categoria è costituita dalla minoranza dei giocatori, poiché il valore di  $S$  è dato dalla cardinalità dell'insieme più piccolo tra  $S \cap L$  e  $S \cap R$ . Il valore di una coalizione si può quindi ottenere considerando che ogni giocatore abbia un valore individuale pari a 1 e sommando i contributi dei singoli giocatori che appartengono al sottoinsieme meno numeroso tra  $S \cap L$  e  $S \cap R$ . Con un approccio analogo è possibile descrivere numerose altre classi di giochi in letteratura e in particolare alcune classi di *graph games*, tra cui gli *airport games* [63], [64], i *connectivity games* e loro estensioni ([2], [62]), gli *argumentation games* [16] e alcune classi di *operation research games*,

quali i *peer games* [19] e le *mountain situations* [73].

Nei *graph games*, un grafo (o network)  $(N, E)$  descrive le possibilità di interazione tra i giocatori: i nodi del network sono i giocatori in  $N$  e esiste un link  $e = \{i, j\} \in E$  tra due nodi  $i$  e  $j$  se i giocatori corrispondenti sono in grado di interagire direttamente tra di loro. Ad esempio, negli *argumentation games*, un grafo diretto descrive le relazioni di attacco tra argomenti che formano un'opinione: esiste un link tra un argomento e l'altro se il primo attacca il secondo; nei *peer games* un network descrive la struttura gerarchica tra agenti: esiste un link diretto tra un nodo e l'altro se il primo è superiore al secondo nella gerarchia dominata dal capo dell'organizzazione; nelle *mountain situations* un grafo diretto rappresenta le connessioni tra le case in un villaggio ed una fonte d'acqua: esiste un link tra due case se è possibile collegarle per creare un canale che consente il passaggio d'acqua dalla fonte.

La struttura di network impone una restrizione sulle possibilità di interazione dei giocatori, determinando di conseguenza come le abilità individuali si compenetrano all'interno di un gruppo di giocatori: se definiamo il valore di una coalizione di argomenti come il suo livello di coerenza interna, in particolare come il numero di argomenti che non sono attaccati da un altro argomento della coalizione, allora un giocatore in un *argumentation game* contribuisce ad una coalizione a cui appartiene solo se nessuno dei suoi attaccanti ne fa parte; un agente in un *peer game* contribuisce con il proprio potenziale individuale al mantenimento dell'organizzazione gerarchica se tutti gli agenti ad un livello superiore cooperano con lui, in altri termini un agente contribuisce solo a quelle coalizioni che contengono tutti i suoi superiori; una casa in una *mountain situation* contribuisce alla divisione dei costi di connessione alla fonte se e solo se giace sul cammino di costo minimo che connette i giocatori alla fonte.

In altre parole, in molti casi, la struttura di network determina quali giocatori devono fornire il proprio contributo individuale ad una data coalizione.

In tutti i modelli sopracitati, il valore di una coalizione di giocatori è calcolato come la somma dei singoli valori dei giocatori che appartengono ad un suo sottoinsieme. D'altro canto, in alcuni casi il valore di una coalizione può essere influenzato dai contributi di giocatori esterni alla coalizione stessa, sia positivamente che negativamente. È questo il caso, ad esempio, dei *bankruptcy games* [5] e dei *maintenance cost games* [58].

La prima parte di questa tesi è dedicata all'introduzione di un modello di teoria dei giochi che abbraccia tutte le sopracitate classi di giochi di coalizione. In particolare, nel Capitolo 3, viene introdotta la classe dei *Generalized Additive Games (GAGs)*, in cui il valore di una coalizione  $S \subseteq N$  è misurato attraverso un "filtro d'interazione", ovvero una mappa  $\mathcal{M}$  che seleziona quali giocatori contribuiscono al valore di  $S$ . L'obiettivo di questo modello è quello di fornire un quadro generale per la descrizione di numerose classi di giochi studiati in letteratura, e in particolare nell'ambito dei *graph games*, e di fornire una tassonomia dei giochi di coalizione ascrivibili a questa nozione di additività sui valori individuali. Definiamo *Generalized Additive Situation (GAS)* una terna  $\langle N, v, \mathcal{M} \rangle$ , dove  $N$  è l'insieme dei giocatori,  $v : N \rightarrow \mathbb{R}$  è una funzione che assegna ad ogni giocatore un valore reale e  $\mathcal{M} : 2^N \rightarrow 2^N$  è una *mappa di coalizione*, che associa una coalizione  $\mathcal{M}(S)$  (eventualmente vuota) ad ogni coalizione di giocatori  $S \subseteq N$  e tale che  $\mathcal{M}(\emptyset) = \emptyset$ .

Dato il GAS  $\langle N, v, \mathcal{M} \rangle$ , il *Generalized Additive Game (GAG)* associato è definito

come il gioco  $(N, v^{\mathcal{M}})$  che associa ad ogni coalizione il valore

$$v^{\mathcal{M}}(S) = \begin{cases} \sum_{i \in \mathcal{M}(S)} v(i) & \text{se } \mathcal{M}(S) \neq \emptyset \\ 0 & \text{altrimenti.} \end{cases} \quad (1)$$

La definizione generale della mappa  $\mathcal{M}$  consente di abbracciare numerose e ampie classi di giochi, tra cui ad esempio i giochi semplici. Sia  $w$  un gioco semplice.  $w$  può essere descritto dal GAG associato a  $\langle N, v, \mathcal{M} \rangle$  con  $v(i) = 1$  per ogni  $i$  e

$$\mathcal{M}(S) = \begin{cases} \{i\} \subseteq S & \text{se } S \in \mathcal{W} \\ \emptyset & \text{altrimenti} \end{cases}$$

dove  $\mathcal{W}$  è l'insieme delle coalizioni vincenti in  $w$ . Se esiste un *giocatore di veto*, ossia un giocatore  $i$  tale che  $S \in \mathcal{W}$  solo se  $i \in S$ , allora il gioco  $w$  può essere descritto anche attraverso  $v(i) = 1, v(j) = 0 \forall j \neq i$  e

$$\mathcal{M}(S) = \begin{cases} T & \text{se } S \in \mathcal{W} \\ R & \text{altrimenti} \end{cases}$$

con  $T, R \subseteq N$  tali che  $i \in T$  e  $i \notin R$ . Questo esempio mostra, in particolare, che la descrizione di un gioco in termini di GAG non è necessariamente unica.

Inoltre, facendo ulteriori ipotesi su  $\mathcal{M}$ , il nostro approccio permette di classificare giochi noti in letteratura sulla base delle proprietà di  $\mathcal{M}$ .

In particolare, introduciamo la classe dei *basic GAGs*, che è caratterizzata dal fatto che i giocatori che contribuiscono ad una coalizione  $S$  sono selezionati sulla base della presenza, all'interno di  $S$ , dei loro *amici* e *nemici*, ovvero un giocatore contribuisce al valore di  $S$  se e solo se  $S$  contiene almeno uno dei suoi amici e nessun nemico.

Sia data la collezione  $\mathcal{C} = \{\mathcal{C}_i\}_{i \in N}$ , dove  $\mathcal{C}_i = \{F_i^1, \dots, F_i^{m_i}, E_i\}$  è una collezione di sottoinsiemi di  $N$  tale che  $F_i^j \cap E_i = \emptyset$  per ogni  $i \in N$  e per ogni  $j = 1, \dots, m_i$ . Denotiamo con  $\langle N, v, \mathcal{C} \rangle$  il *basic GAS* associato alla mappa di coalizione  $\mathcal{M}$  definita da:

$$\mathcal{M}(S) = \{i \in N : S \cap F_i^1 \neq \emptyset, \dots, S \cap F_i^{m_i} \neq \emptyset, S \cap E_i = \emptyset\} \quad (2)$$

e con  $\langle N, v^{\mathcal{C}} \rangle$  il GAG associato, che chiameremo *basic GAG*.

Per semplicità, assumiamo, senza perdita di generalità, che  $m_1 = m_2 = \dots = m_n := m$ . Chiameremo ogni  $F_i^k$ , per ogni  $i \in N$  e ogni  $k = 1, \dots, m$ , il  $k$ -esimo insieme di *amici* di  $i$ , e  $E_i$  l'insieme dei *nemici* di  $i$ .

Molte delle sopracitate classi di giochi possono essere descritte come basic GAGs, così come alcuni giochi derivanti da situazioni reali. Ad esempio, tale modello può essere impiegato per rappresentare un social network, dove gli amici e i nemici degli utenti del web sono determinati dai loro profili sociali. In aggiunta, il Capitolo 7 presenta Un'applicazione dei GAGs al campo della Biomedicina.

Il caso più semplice è quello in cui ogni giocatore ha un unico insieme di amici, che denoteremo con  $F_i$ .

**Esempio 1.** (*airport games*) [63, 64]: Supponiamo che l'insieme dei giocatori  $N$  sia partizionato nei gruppi  $N_1, N_2, \dots, N_k$  tali che ad ogni  $N_j, j = 1, \dots, k$ , è associato un costo positivo  $c_j$  con  $c_1 \leq c_2 \leq \dots \leq c_k$ . Consideriamo il gioco  $w(S) = \max\{c_i : i \in S\}$ . Tale gioco può essere descritto da un basic GAS  $\langle N, (\mathcal{C}_i = \{F_i, E_i\})_{i \in N}, v \rangle$  definendo per ogni  $i \in N_j$  e per ogni  $j = 1, \dots, k$ :

- il valore  $v(i) = \frac{c_j}{|N_j|}$ ,
- l'insieme degli amici  $F_i = N_j$ ,

e l'insieme dei nemici  $E_i = N_{j+1} \cup \dots \cup N_k$  per ogni  $i \in N_j$  e ogni  $j = 1, \dots, k-1$  e  $E_i = \emptyset$  per ogni  $i \in N_k$ .

In maniera simile, è possibile mostrare che anche i maintenance games [17, 58], che generalizzano gli airport games, possono essere descritti come basic GAGs.

Di seguito, mostriamo altri esempi di classi di giochi di coalizione che possono essere rappresentati come basic GAGs, dove in generale ogni giocatore può avere diversi insiemi di amici.

**Esempio 2.** (argumentation games) [16] Sia  $\langle N, \mathcal{R} \rangle$  un grafo diretto, dove  $N$  è un insieme finito di argomenti e l'insieme dei link  $\mathcal{R} \subseteq N \times N$  è una relazione binaria di attacco [37]. Per ogni argomento  $i$ ,  $P(i) = \{j \in N : (j, i) \in \mathcal{R}\}$  è l'insieme degli argomenti che lo attaccano. Il valore di una coalizione  $S$  è il numero di argomenti nell'opinione  $S$  che non sono attaccati da nessun altro argomento appartenente ad  $S$ . Tale gioco può essere descritto come un basic GAS  $\langle N, v, \{F_i, E_i\} \rangle$  imponendo  $v(i) = 1$ ,  $F_i = \{i\}$  e  $E_i = P(i)$ . Tale esempio ricade ancora nella categoria dei basic GAGs in cui ogni giocatore ha un solo insieme di amici. Tuttavia, in questo contesto, possiamo considerare altre naturali definizioni di funzione caratteristica. Ad esempio, è interessante considerare il gioco  $(N, v^M)$  tale che per ogni  $S \subseteq N$ ,  $v^M(S)$  è la somma dei  $v(i)$  sugli elementi dell'insieme  $D(S) = \{i \in N : P(i) \cap S = \emptyset \text{ and } \forall j \in P(i), P(j) \cap S \neq \emptyset\}$ , ossia l'insieme degli argomenti che non sono internamente attaccati da  $S$  e allo stesso tempo sono difesi da parte di  $S$  da attacchi esterni:

$$v^M(S) = \sum_{i \in D(S)} v(i). \quad (3)$$

È chiaro che tale situazione non può essere rappresentata da un basic GAG in cui ogni giocatore ha un unico insieme di amici. Il gioco in (3) può tuttavia essere descritto come il basic GAG  $\langle N, v^C \rangle$ , in cui, data una biiezione  $k : P(i) \rightarrow \{1, \dots, |P(i)|\}$ ,  $\mathcal{C}_i = \{F_i^1, \dots, F_i^{|P(i)|}, E_i\}$  è tale che  $F_i^{k(j)} = P(j) \setminus P(i)$  per ogni  $j \in P(i)$ , e  $E_i = P(i)$  per ogni  $i \in N$ .

**Esempio 3.** (peer games) [19] Sia  $N = \{1, \dots, n\}$  l'insieme dei giocatori e  $T = (N, A)$  un albero (ossia un grafo aciclico) diretto che descrive la gerarchia tra i giocatori, dove  $N$  è l'insieme dei nodi, 1 la radice (il leader del gruppo) e  $A \subset N \times N$  l'insieme dei link. Ogni agente  $i$  ha un potenziale individuale  $a_i$  che rappresenta il guadagno che il giocatore  $i$  può generare se tutti i suoi superiori cooperano con lui. Per ogni  $i \in N$ , denotiamo con  $S(i)$  l'insieme di tutti gli agenti nell'unico cammino diretto che connette 1 a  $i$ , ovvero i superiori di  $i$ . Data una peer group situation  $(N, T, a)$  come quella sopra descritta, un peer game è definito come il gioco  $(N, v^P)$  tale che per ogni coalizione non vuota  $S \subseteq N$

$$v^P(S) = \sum_{i \in N: S(i) \subseteq S} a_i.$$

Un peer game  $(N, v^P)$  può essere descritto come il GAG associato al basic GAS su  $N$  in cui  $v(i) = a_i$  e  $\mathcal{M}$  è definita dalla relazione (3.4) tramite le collezioni  $\mathcal{C}_i = \{F_i^1, \dots, F_i^n, E_i\}$  tali che:

$$F_i^j = \begin{cases} \{j\} & \text{if } j \in S(i) \\ \{i\} & \text{altrimenti} \end{cases}$$

e  $E_i = \emptyset$  per ogni  $i \in N$ .

**Esempio 4.** (maintenance cost games) [17][58] Sia  $N$  un gruppo di giocatori connessi da un albero  $T$  (e.g., un network di computer) ad una fonte  $0$  (e.g., un fornitore di servizi) e supponiamo ad ogni lato dell'albero sia associato un costo. Si consideri la coppia  $(T, t)$ , dove  $T = (N \cup \{0\}, E)$  è un albero, con insieme di nodi  $N \cup \{0\}$  insieme di lati  $E$ , e  $t : E \rightarrow \mathbb{R}^+$  è una funzione di costo positiva sull'insieme dei lati.  $0$  è la radice ed ha un solo lato incidente. Osserviamo che ogni nodo  $i \in N$  è connesso alla radice  $0$  da un unico cammino  $P_i$ ; denotiamo con  $e_i$  il lato in  $P_i$  incidente ad  $i$ . Una relazione di precedenza  $\preceq$  è definita da:  $j \preceq i$  se e solo se  $j$  giace sul cammino  $P_i$ . Un tronco  $R \subseteq N \cup \{0\}$  è un insieme di nodi chiuso rispetto alla relazione  $\preceq$ , ovvero se  $i \in R$  e  $j \preceq i$ , allora  $j \in R$ . Sia  $F(i) = \{j \in N \mid i \preceq j\}$  l'insieme dei nodi che seguono il nodo  $i \in N$  nel cammino che lo congiunge alla radice. Osserviamo che  $i \in F(i)$  per ogni  $i \in N$ . Il costo di un tronco  $R$  è definito da

$$C(R) = \sum_{i \in R \setminus \{0\}} t(e_i),$$

e il maintenance cost game associato  $(N, c)$  è definito da

$$c(S) = \min\{C(R) : S \subseteq R \text{ e } R \text{ è un tronco}\}.$$

Notiamo che  $e_i$  appartiene al tronco di costo minimo che contiene tutti i membri di  $S$  ogniqualvolta  $S \cap F(i) \neq \emptyset$ . Possiamo quindi rappresentare il gioco  $(N, c)$  come il GAG associato al basic GAS su  $N$  dove  $v(i) = t(e_i)$  e dove  $\mathcal{M}$  è definita dalla relazione (3.4) tramite le collezioni  $\mathcal{C}_i = \{F_i, E_i\}$  tali che  $F_i = F(i)$  e  $E_i = \emptyset$  per ogni  $i \in N$ .

L'interesse di questa classificazione non è soltanto tassonomico, in quanto consente l'analisi delle proprietà delle soluzioni di classi di giochi note in letteratura e fornisce strumenti potenzialmente utili per il calcolo delle soluzioni di nuove classi di giochi che possono essere descritte all'interno di questo formalismo. In particolare, è possibile ricavare alcuni risultati per i basic GAG riguardanti due noti concetti di soluzione, il nucleo e i semivalues.

Risulta quindi interessante studiare sotto quali condizioni un GAG possa essere descritto come un basic GAG. Con questo obiettivo, presentiamo il Teorema seguente, che ne descrive una condizione necessaria e sufficiente, nel caso in cui l'insieme dei nemici di ogni giocatore è vuoto.

**Teorema 1.** Sia  $\langle N, v, \mathcal{M} \rangle$  un GAS. La mappa  $\mathcal{M}$  può essere ottenuta dalla relazione (3.4) per mezzo delle collezioni  $\mathcal{C}_i = \{F_i^1, \dots, F_i^{m_i}, E_i = \emptyset\}$ , per ogni  $i \in N$ , se e solo se  $\mathcal{M}$  è monotona.

Nel Capitolo 4, forniamo alcuni risultati riguardo a classici concetti di soluzione per i giochi cooperativi sulla classe dei basic GAGs, e affrontiamo il problema di come

garantire che il nucleo di un basic GAG sia non vuoto. In particolare, forniamo alcune condizioni sufficienti affinché ciò non accada e ricaviamo formule concise per i semi-values di basic GAGs in cui ogni giocatore ha un solo insieme di amici o al più due insiemi di amici, di cui uno è l'insieme  $F_i^k = \{i\}$ .

In un GAG, il valore di ogni coalizione è calcolato a partire dai valori individuali di un sottoinsieme di giocatori. D'altro canto, in molti casi, quando un network descrive le interazioni tra i giocatori, è possibile ricavare il valore di una coalizione di giocatori a partire dai contributi generati dalle interazioni binarie tra i giocatori, ossia come somma dei singoli valori associati ai lati del sottostante grafo. Ad esempio, nei *maintenance cost games* [58], il costo della connessione di un insieme di computer  $S$  al gestore di rete è calcolato come la somma dei costi di mantenimento di tutte le connessioni tra i computer che giacciono sull'albero di minimo costo che connette i computer in  $S$  al gestore, ovvero come la somma dei costi associati ai link nel grafo indotto.

Così come per i maintenance cost game, in molti altri graph games il valore di una coalizione di nodi può essere ricavato in maniera additiva a partire dai valori associati ai link nel sottostante network. Inoltre, così come per i graph-restricted game il valore di una coalizione si ricava in maniera additiva a partire da una collezione di sottocoalizioni di nodi, per la classe dei *link games*, introdotta da Meessen [70] e successivamente studiata da Borm et al. [18], il valore di una coalizione di link può essere ricavato additivamente da una collezione di sottocoalizioni di link. In effetti, diversi approcci nella letteratura sui giochi di coalizione su grafi si basano su strutture additive tra link, non solo per quanto riguarda la definizione di una classe di giochi, ma anche per l'analisi delle relative soluzioni.

La seconda parte di questa tesi è dedicata all'introduzione di una classe di soluzioni per le *communication situations* [75], in cui i payoff di ogni giocatore sono calcolati additivamente a partire dai valori generati dalle interazioni a coppie tra i giocatori. Nel Capitolo 5, consideriamo una situazione di comunicazione  $(N, v, \Gamma)$  dove un network  $\Gamma$  è generato dalla formazione successiva di link tra i giocatori e ad ogni step del processo di formazione, il surplus generato dalla cooperazione tra i nodi che stabiliscono un link tra di loro è diviso tra i giocatori coinvolti secondo una qualche regola. Più precisamente, assumiamo che un network  $\Gamma$  tra i giocatori nel gioco  $(N, v)$  sia prodotto da una permutazione di link e che tutti i possibili ordini in cui i link si formano a creare il network siano equiprobabili. Inoltre, supponiamo che ad ogni step del processo di formazione del network, quando un link tra i giocatori  $i$  e  $j$  si crea, il valore della coalizione  $S$ , dove  $S$  è la componente connessa contenente  $i$  e  $j$ , meno il valore delle componenti connesse formate dai giocatori di  $S$  allo step precedente, sia suddiviso tra i giocatori coinvolti seguendo un certo protocollo. Formalmente, data una situazione di comunicazione  $(N, v, \Gamma)$ , consideriamo una possibile permutazione di lati  $\sigma$ . Ad ogni step  $k$  del processo di formazione del network, quando il  $k$ -esimo link  $a = \{i, j\}$  nella sequenza  $\sigma$  si forma, consideriamo il *surplus* prodotto da  $a$ :

$$S_k^\sigma = v(S) - v(C_{k-1,\sigma}^i) - v(C_{k-1,\sigma}^j) \quad (4)$$

dove  $S$  è la componente connessa in  $\Gamma$  contenente  $i$  e  $j$  allo step  $k$ , e  $C_{k-1,\sigma}^i$  e  $C_{k-1,\sigma}^j$  sono le componenti connesse in  $\Gamma$  allo step  $k - 1$ , contenenti  $i$  e  $j$  rispettivamente. Chiamiamo *protocollo di allocazione* una regola che specifica come dividere  $S_k^\sigma$  tra i

giocatori di  $S$ . Dato un protocollo di allocazione  $r$  e una situazione di comunicazione  $(N, v, \Gamma)$ , una soluzione di  $v$ , che denotiamo con  $\phi^r(v)$ , è data da:

$$\phi_i^r(v) = \frac{1}{|E|!} \sum_{\sigma \in \Sigma_E} \sum_{k=0}^{|E|} f_i^r(S_k^\sigma), \forall i \in N, \quad (5)$$

dove  $\Sigma_E$  è l'insieme dei possibili ordini sull'insieme  $E$  in  $\Gamma$  e  $f_i^r$  è una funzione che assegna ad ogni giocatore  $i \in N$  una quantità fissata del surplus  $S_k^\sigma$ , che dipende dal protocollo di allocazione  $r$ .

In altre parole, la soluzione  $\phi^r(v)$  è calcolata considerando tutte le possibili permutazioni di link e sommando, per ogni giocatore  $i$ , tutti i contributi che riceve tramite la procedura di allocazione  $r$ , facendo la media su tutte le permutazioni dell'insieme dei link. Tale idea porta all'introduzione di una classe di concetti di soluzione: differenti scelte del protocollo di allocazione portano alla definizione di differenti soluzioni per una situazione di comunicazione.

In particolare, quando un link  $a = \{i, j\}$  si forma, è naturale considerare il protocollo di allocazione che divide equamente il surplus soltanto tra i giocatori  $i$  e  $j$ . La soluzione ottenuta da questo particolare protocollo di allocazione coincide con il *position value*, introdotto da Borm et al. [18]. Sulla base di queste osservazioni, nel Capitolo 5 forniamo quindi una diversa interpretazione di questo noto concetto di soluzione. Inoltre, analizziamo il problema di calcolare tale soluzione su alcune particolari classi di communication situations. In particolare, forniamo un'espressione per il position value di un gioco quando il network sottostante è un albero, che mette in relazione il calcolo di tale soluzione su un gioco generico con quello sui giochi di unanimità.

La terza e ultima parte di questa tesi è dedicata a due applicazioni dei modelli teorici descritti fin qui. Una prima applicazione, presentata nel Capitolo 6, è al campo della teoria dell'argomentazione [37]. La teoria dell'argomentazione ha lo scopo di formalizzare i sistemi decisionali e i processi di decisione associati. Uno dei suoi obiettivi è la ricerca di insiemi di conclusioni accettate in un quadro argomentativo, che è descritto da un grafo diretto in cui i nodi rappresentano argomenti, ovvero affermazioni o serie di affermazioni, e i lati diretti rappresentano relazioni di attacco, che esprimono il conflitto tra copie di argomenti.

In letteratura, numerose *semantiche d'estensione*, anche dette *labellings* sono state definite nell'ambito dell'argomentazione astratta con l'obiettivo di descrivere quali argomenti sono accettati o meno, e su quali non è possibile prendere una decisione [21, 38]. Diverse dalle semantiche d'estensione, lo scopo delle *semantiche graduali* è quello di assegnare un grado di accettabilità ad ogni argomento [4, 13, 24, 49, 97].

Anche la teoria dei giochi è stata utilizzata per definire livelli intermedi di accettabilità di argomenti. In particolare, in [67] il grado di accettabilità di un argomento è definito prendendo in considerazione il valore di minmax di un gioco a somma zero tra un 'sostenitore' e un 'oppositore' in cui le strategie e i payoff dipendono dalla struttura del grafo d'argomentazione. Più recentemente, i giochi di coalizione sono stati impiegati in [16] per misurare l'importanza relativa degli argomenti sulla base sia delle preferenze di un agente sugli argomenti sia dell'informazione fornita dalle relazioni d'attacco. Negli approcci descritti sopra, il peso attribuito ad ogni argomento riflette il potere di un

argomento nel forzare la sua accettabilità. D'altro canto, l'accettabilità non è l'unico attributo studiato in letteratura da un punto di vista graduale. In [98] viene introdotto un indice per misurare la *controversialità* di un argomento, dove gli argomenti più controversi sono quelli per cui è difficile prendere una decisione riguardo alla loro accettabilità. In una direzione simile, il problema di misurare il *disaccordo* all'interno di un grafo di argomentazione è stato studiato in [3]. In tale lavoro, viene presentata un'analisi assiomatica di diverse misure di disaccordo per grafi di argomentazione.

Nel Capitolo 6, mostriamo in primo luogo che le proprietà introdotte in [3] per grafi d'argomentazione possono essere riformulate per i singoli argomenti e portare ad una classificazione degli argomenti sulla base della loro conflittualità, che può essere vista come una classificazione alternativa per misurarne la controversialità.

Formalmente, sia  $\mathbf{A} = \langle A, R \rangle$  un *quadro d'argomentazione* (o *grafo d'argomentazione*), dove  $A = \{1, \dots, n\}$  è un insieme finito e non vuoto di argomenti e  $R \subseteq A \times A$  una relazione d'attacco. Dati due argomenti  $a, b \in N$ , scriviamo  $(a, b) \in R$  (o  $aRb$ ) se  $a$  attacca  $b$ .

Denotiamo con  $\mathcal{A}$  l'insieme di tutti i grafi d'argomentazione e con  $\mathcal{A}^A$  l'insieme di tutti i grafi d'argomentazione con  $A$  come insieme di argomenti.

Una *misura di disaccordo* [3] è una funzione  $\mathcal{K} : \mathcal{A} \rightarrow [0, 1]$  con l'interpretazione che, per ogni  $\mathbf{A}, \mathbf{A}' \in \mathcal{U}$ ,  $\mathbf{A}$  è più conflittuale di  $\mathbf{A}'$  se  $\mathcal{K}(\mathbf{A}) > \mathcal{K}(\mathbf{A}')$ . Osserviamo che  $\mathcal{K} = 0$  corrisponde all'assenza di disaccordo in un grafo, mentre il massimo disaccordo è fissato a  $\mathcal{K} = 1$ .

In particolare, consideriamo la misura basata sulle distanze introdotta in [3], definita come:

$$\mathcal{K}^D(\mathbf{N}) = \frac{\max - D(\mathbf{A})}{\max - \min} \quad (6)$$

dove  $|N| = n$ ,  $\max = n^2(n + 1)$  e  $\min = n^2$  and  $D(\mathbf{A}) = \sum_{i \in A} \sum_{j \in A} d_{i,j}$  è la *distanza globale* tra gli argomenti in  $\mathbf{A}$ , dove  $d_{i,j}$  è la distanza tra gli argomenti  $i$  e  $j$ , definita come la lunghezza del cammino più corto tra  $i$  e  $j$  se tale cammino esiste o  $n + 1$  altrimenti.

Data una misura di disaccordo, è interessante stabilire quali sono gli argomenti che contribuiscono maggiormente al disaccordo totale in un grafo di argomentazione. A tale scopo, introduciamo, attraverso un approccio assiomatico, un *indice di conflitto* che misura il contributo di ogni argomento al disaccordo totale. Un *indice di conflitto*  $K : \mathcal{A}^A \rightarrow \mathbb{R}^n$  è una funzione che assegna ad ogni grafo di argomentazione con  $n = |A|$  nodi (argomenti) un vettore di  $\mathbb{R}^n$ , che rappresenta i contributi di ogni argomento al conflitto all'interno del grafo. Più alto il valore di tale indice, maggiore l'apporto di disaccordo di un argomento all'interno del grafo e di conseguenza la sua controversialità.

Nel Capitolo 6 introduciamo in particolare un indice di conflitto basato sulle distanze, definito come l'indice di conflitto che associa ad ogni  $i \in N$  il seguente valore:

$$K_i^D(\mathbf{A}) = \frac{1}{\Delta} \left( \frac{\max}{n} - \varphi_i \right), \quad (7)$$

dove  $\max = n^2(n + 1)$ ,  $\Delta = \max - \min = n^2(n + 1) - n^2 = n^3$  e  $\varphi_i = \frac{1}{2} \sum_{j \in A \setminus i} d_{i,j} + \frac{1}{2} \sum_{j \in A \setminus i} d_{j,i} + d_{i,i}$ . Tale indice di conflitto associa ad ogni argomento un va-

lore che dipende dalla sua distanza da tutti gli altri argomenti e viceversa dalla distanza di tutti gli altri argomenti dall'argomento stesso. Mostriamo inoltre che tale indice di conflitto soddisfa otto proprietà: *abstraction, coherence, maximality, free independence, monotonicity, cycle sensitivity, size sensitivity e efficiency*, che riformulano per i singoli argomenti le proprietà introdotte in [3] per una misura di disaccordo, e mettono in relazione il nostro indice di conflitto con la misura di disaccordo basata sulle distanze definita in [6].

In secondo luogo, nello stesso capitolo, mostriamo che l'indice di conflitto da noi introdotto, può essere interpretato in termini di una classica soluzione per i giochi di coalizione, ossia come il contributo marginale medio di ogni argomento al disaccordo interno a tutte le possibili coalizioni di argomenti. Lo facciamo definendo un gioco cooperativo, in cui i giocatori sono gli argomenti all'interno di un grafo di argomentazione e ad ogni coalizione di argomenti è associato un valore che esprime il disaccordo totale all'interno della coalizione. In particolare, ogni nodo e ogni lato all'interno di una coalizione contribuiscono al valore della coalizione stessa con la propria porzione di disaccordo, misurata attraverso le relazioni d'attacco che apportano all'interno della coalizione. Il gioco così definito può in effetti essere rappresentato mediante i basic GAGs, come combinazione del modello originale introdotto nel Capitolo 3 e della sua variante definita sui link.

Sia  $\mathbf{A} = \langle A, R \rangle$  un grafo d'argomentazione, dove  $A$  ha cardinalità  $n$ . Introduciamo un gioco cooperativo  $(A, v)$ , in cui l'insieme dei giocatori coincide con l'insieme degli argomenti  $A$  e la funzione caratteristica è definita come segue, per ogni  $S \subseteq A$ :

$$v(S) = \frac{\max - D(S)}{\max - \min}, \quad (8)$$

dove  $D(S) = \sum_{i,j \in S} d_{i,j}$ ,  $\max = n^2(n+1)$  è il massimo valore che  $D$  può assumere in un grafo d'argomentazione con  $n$  argomenti e  $\min = n^2$  è quello minimo.  $D(S)$  misura la distanza globale tra gli argomenti all'interno della coalizione  $S$ , prendendo in considerazione le relazioni d'attacco che esistono tra di essi nell'intero grafo: è definita come la somma delle distanze tra i nodi della coalizione  $S$ , dove la distanza tra due nodi è calcolata a partire dall'intero grafo. Minore è la distanza globale all'interno di una coalizione, maggiore il valore di tale coalizione nel gioco  $v$ , in quanto il conflitto totale in una coalizione di argomenti dipende inversamente dalla distanza tra gli argomenti.

Nel Capitolo 6, proponiamo il valore Shapley [88] di tale gioco come indice di conflitto che quantifica la controversialità degli argomenti, in quanto misura il potere di ogni argomento nell'apportare conflitto all'interno del grafo di argomentazione, e dimostriamo che esso coincide con l'indice di conflitto definito in [7].

Considerando scenari di persuasione, sosteniamo che la nostra classificazione degli argomenti in base al conflitto che apportano in un grafo di argomentazione può guidare gli agenti nella scelta di quegli argomenti che meritano un maggiore sviluppo al fine di rafforzare certe posizioni all'interno di un dibattito, rispondendo quindi alla questione sollevata in [98] sulla definizione di un indice del potenziale di sviluppo degli argomenti.

Mentre il Capitolo 6 descrive un approccio con la teoria dei giochi al campo della teoria dell'argomentazione, il Capitolo 7 presenta un'applicazione del modello dei GAGs al campo della Biomedicina, e in particolare al problema di valutare la rilevanza di geni all'interno di un network biologico. Tra i network biologici, i *network*

(o pathway) di regolazione genica sono oggetto di grande interesse nel campo della biologia molecolare e dell'epidemiologia allo scopo di comprendere meglio i meccanismi di interazione tra geni, proteine a altre molecole all'interno di una cellula, in una condizione biologica di interesse. [20, 23, 31, 93]. Un punto cruciale nell'analisi delle interazioni tra geni è la formulazione di misure appropriate del ruolo ricoperto da ogni gene nell'influenzare i sistemi estremamente complessi che descrivono le relazioni tra geni all'interno di un network.

L'analisi di centralità rappresenta uno strumento importante per l'interpretazione delle interazioni tra geni in un network di regolazione genica [12, 22, 44, 52, 53]. Le misure di centralità classiche [41, 57] sono utilizzate per identificare gli elementi rilevanti all'interno di un network, sulla base della loro posizione all'interno della struttura del network. Tuttavia, esse rappresentano uno strumento appropriato se assumiamo che i nodi agiscano in maniera indipendente gli uni dagli altri e che il sistema in analisi sia sensibile al comportamento di ogni singolo nodo. Al contrario, nei complessi network biologici, assumere che i geni si esprimano indipendentemente non è realistico e le conseguenze sul sistema possono essere apprezzate solo se molti geni cambiano il loro livello di espressione. Dunque, in uno scenario complesso quale la patogenesi di una malattia genetica, abbiamo a che fare con il problema di quantificare l'importanza relativa dei geni, tenendo conto non solo del loro comportamento individuale, ma soprattutto del livello della loro interazione.

La teoria dei giochi cooperativi è stata proposta come quadro teorico per affrontare tali limitazioni. Di recente, numerose misure di centralità basate sui giochi cooperativi sono state applicate con successo a svariati tipi di network biologici, quali le reti neurali [55, 56, 59], i network di geni [72], e le reti metaboliche [85].

Nel Capitolo 7, proponiamo un approccio con i giochi di coalizione, e in particolare con i GAGs, al problema di identificare i geni rilevanti all'interno di un network di geni. Tale problema è stato affrontato per la prima volta per mezzo della teoria dei giochi in [72], dove il valore Shapley di un gioco di coalizione è usato per esprimere il potere di ogni gene nell'interazione con gli altri e per sottolineare l'importanza di particolari geni nella regolazione di pathway biologiche di interesse. Il nostro modello rappresenta un raffinamento di tale approccio, che generalizza la nozione di *degree centrality* [77, 89], la cui correlazione con la rilevanza di geni per diverse funzioni biologiche è supportata da diverse evidenze pratiche in letteratura [12, 22, 52, 53, 103]. Definiamo un basic GAG con interpretazione biologica su network di geni e proponiamo il valore Shapley di tale gioco come nuovo indice di rilevanza di geni. Tale approccio è supportato da una caratterizzazione assiomatica, in cui le proprietà soddisfatte dal nostro indice hanno un'interpretazione biologica. Inoltre, forniamo una formula per il calcolo del nuovo indice di rilevanza, che può essere direttamente dedotta dai risultati teorici presentati nel Capitolo 4.

Sia  $\langle N, E \rangle$  un network di geni, ossia un grafo il cui insieme di nodi  $N$  rappresenta un insieme di geni e l'insieme dei link  $E$  descrive le interazioni tra geni, ovvero esiste un link tra due geni se essi interagiscono direttamente all'interno di una condizione biologica in esame. Inoltre, sia  $k \in \mathbb{R}^N$  un vettore di parametri che specifica l'importanza *a priori* di ogni gene. Definiamo l'insieme dei vicini di un nodo  $i$  nel grafo  $\langle N, E \rangle$  come l'insieme  $N_i(E) = \{j \in N : \{i, j\} \in E\}$ , e il grado di  $i$  come il numero  $d_i(E) = |N_i(E)|$  di vicini di  $i$  nel grafo  $\langle N, E \rangle$ , ovvero il numero di link incidenti a

$i$  in  $\Gamma$ . Denotiamo inoltre con  $N_S(E) = \{j \in N : \exists i \in S \text{ s.t. } j \in N_i(E)\}$  l'insieme dei vicini dei nodi in  $S \in 2^N$ ,  $S \neq \emptyset$  nel grafo  $\langle N, E \rangle$ . Dato un sottoinsieme  $S \subseteq V$  di nodi, definiamo il sottografo indotto  $\Gamma_S = (S, E_S)$ , dove  $E_S$  è l'insieme dei link  $\{i, j\} \in E$  tali che  $i, j \in S$ . Un grafo  $\langle N, E_S^i \rangle$  in cui l'insieme dei link è dato da  $E_S^i = \{\{i, j\} : j \in S\}$  è detto *stella su  $S$  di centro  $i$* . Osserviamo che l'insieme dei vicini di un nodo in  $\langle N, E_S^i \rangle$  è tale che  $N_i(E_S^i) = S$ ,  $N_j(E_S^i) = \{i\}$ , per ogni  $j \in S$ , e  $N_j(E_S^i) = \emptyset$ , per ogni  $j \in N \setminus (S \cup \{i\})$ .

Definiamo quindi un gioco cooperativo  $(N, v_E^k)$ , dove  $N$  è l'insieme di geni sotto analisi e la funzione caratteristica  $v_E^k$  associa un valore ad ogni coalizione di geni  $S \subseteq N$  che rappresenta il livello di interazione globale all'interno della coalizione  $S$ , tenendo conto del peso di ogni gene direttamente connesso ad  $S$  nel network biologico.

Più precisamente, la mappa  $v_E^k : 2^N \rightarrow \mathbb{N}$  associa ad ogni coalizione  $S \in 2^N \setminus \{\emptyset\}$  il valore

$$v_E^k(S) = \sum_{j \in S \cup N_S(E)} k_j \quad (9)$$

ovvero la somma dei pesi associati ai geni in  $S$  e ai loro vicini (per convenzione,  $v_E^k(\emptyset) = 0$ ). La classe dei giochi  $(N, v)$  definiti dall'equazione (9), su di un network di geni  $G \equiv \langle V, E \rangle$  e con vettore di parametri  $k \in \mathbb{R}^N$ , è denotato con  $\mathcal{EK}^N$ .

Osserviamo inoltre che il gioco così definito può essere facilmente descritto come il basic GAG associato al GAS  $\langle N, v, \mathcal{M} \rangle$ , dove  $v(i) = k_i$  per ogni  $i \in N$  e  $\mathcal{M}$  è la mappa associata alle collezioni  $\mathcal{C}_i = \{F_i = N_i(E), E_i = \emptyset\} \forall i \in N$ . Un gene  $i$  contribuisce al valore di una coalizione con il proprio peso  $k_i$ , se e solo se esso appartiene alla coalizione o se almeno uno dei geni con cui interagisce direttamente all'interno del network ne fa parte.

Rispetto al precedente modello in letteratura [72], la definizione proposta in (9) appare più flessibile per esplorare tutte le possibilità di influenza reciproca tra geni. Essa generalizza il gioco introdotto in [94] per l'individuazione dei "top- $k$  nodes" in un *co-authorship network*, introducendo un parametro che specifica l'importanza a priori di ciascun nodo. Il vettore di parametri  $k$  consente un ordinamento a priori dei geni sulla base delle informazioni disponibili riguardo la loro importanza, mentre nel precedente modello introdotto in [72] veniva fatta solamente una distinzione a due livelli tra geni chiave, noti per avere un ruolo nella condizione biologica in esame, e tutti gli altri geni. Inoltre, valutando in che misura una coalizione di geni è connessa al resto del network, la relazione (9) generalizza la nozione di degree centrality per gruppi di geni, che è nota per avere una forte correlazione con l'essenzialità dei geni in diverse funzioni biologiche [12, 22, 52, 53, 103]. In effetti, se nella definizione del valore di una coalizione di geni si considerasse solamente il peso dei geni al suo interno (e non quello dei loro vicini, come nella nostra definizione), la misura di centralità che ne deriva coinciderebbe con la degree centrality pesata.

Sulla base delle precedenti considerazioni, proponiamo dunque il valore Shapley di tale gioco come misura dell'importanza di geni nel preservare l'attività di regolazione globale all'interno di un network di geni (nella Sezione 7.3 si trova un esempio che motiva tale approccio, il quale chiarisce l'obiettivo del nostro indice e le sue peculiarità rispetto ad alcune misure di centralità classiche). Un supporto a tale idea è fornito da una caratterizzazione assiomatica del nostro indice, in cui le proprietà hanno un significato biologico nel contesto dei network di geni. Di seguito descriviamo quattro

proprietà per un *indice di rilevanza* di geni, ossia una mappa  $\rho : \mathcal{EK}^N \rightarrow \mathbb{R}^N$ . Per prima cosa, forniamo una reinterpretazione delle classiche proprietà di *symmetry* (SYM), *dummy-player property* DPP e *efficiency* (EFF) sulla classe  $\mathcal{EK}^N$  (si veda la Sezione 2.1 per una definizione formale sulla classe di tutti i TU-games).

Consideriamo un network di geni  $\langle N, E \rangle$  e un vettore di pesi  $k \in \mathbb{R}^N$ . La proprietà di SYM implica che due geni  $i, j \in N$  che hanno lo stesso peso ( $k_i = k_j$ ) e che in aggiunta sono connessi allo stesso insieme di vicini ( $N_i(E) = N_j(E)$ ), abbiano la stessa rilevanza, ossia  $\rho_i(v_E^k) = \rho_j(v_E^k)$ . La proprietà di DPP ha anch'essa un'interpretazione intuitiva per un network di geni: ogni nodo disconnesso  $i \in N$  deve avere rilevanza  $k_i$ . Infine, la proprietà di EFF implica che la somma degli indici di rilevanza di tutti i geni sia uguale alla somma totale dei pesi  $\sum_{i \in N} k_i$ .

Inoltre, introduciamo un nuovo assioma, che afferma che la trasformazione di un nodo  $i$  con peso pari a zero in un nodo con peso positivo  $k_i$  deve influenzare solo i geni direttamente connessi a  $i$ , e il suo impatto sulla rilevanza dei suoi vicini deve essere lo stesso che avrebbe in una stella equivalente di centro  $i$ .

**Assioma 1** (Star Additivity, SADD). *Sia  $\langle N, E \rangle$  un network di geni con vettore di parametri  $k_{-i} \in \mathbb{R}^N$  tale che il gene  $i$  ha peso nullo e sia  $v_E^{k_{-i}}$  il gioco corrispondente definito dalla relazione (9). Consideriamo inoltre il gioco  $v_E^k$  definito da (9) sul network  $\langle N, E \rangle$  e con vettore di parametri  $k$  che associa un peso positivo  $k_i$  al gene  $i$  e lo stesso peso del vettore  $k_{-i}$  a tutti gli altri geni. Un indice  $\rho : \mathcal{EK}^N \rightarrow \mathbb{R}^N$  soddisfa la proprietà di SADD se e solo se*

$$\rho(v_E^k) = \rho(v_E^{k_{-i}}) + \rho(v_{E_{N_i(E)}}^{s^i}),$$

dove  $v_{E_{N_i(E)}}^{s^i}$  è il gioco definito dalla relazione (9) sulla stella  $\langle N, E_{N_i(E)}^i \rangle$  su  $N_i(E)$  di centro  $i$  e  $s^i$  è il vettore di parametri che assegna peso  $k_i$  ad  $i$  e peso nullo ad ogni  $j \neq i$ .

In altre parole, l'Assioma SADD afferma che l'incremento di peso di un nodo  $i$  da nullo ad un valore positivo dovrebbe influenzare solo la rilevanza del gene stesso e dei suoi vicini, allo stesso modo per ogni grafo. Di conseguenza, un cambio positivo nell'importanza a priori di un gene produce lo stesso effetto sulla sua rilevanza e su quella dei suoi vicini, indipendentemente dalla topologia del network, e l'effetto del cambiamento è confrontabile tra network differenti.

La Proposizione seguente caratterizza il nostro indice di rilevanza in termini delle sopracitate proprietà, e fornisce una formula per il suo calcolo, che può essere facilmente dimostrata a partire dalle proprietà stesse o, equivalentemente, dai risultati descritti nel Capitolo 4.

**Proposizione 1.** *Il valore Shapley del gioco definito dalla relazione (9) è l'unico indice di rilevanza  $\rho$  che soddisfa le proprietà di SYM, DPP, EFF e SADD sulla classe  $\mathcal{EK}^N$ . Inoltre, per ogni network di geni  $\langle N, E \rangle$  con  $k \in \mathbb{R}^N$  come vettore di pesi, esso può essere calcolato secondo la seguente formula:*

$$\rho_i(v_E^k) = \sum_{j \in (N_i(E) \cup \{i\})} \frac{k_j}{d_j(E) + 1}, \quad (10)$$

per ogni  $i \in N$ .

L'interpretazione della formula in (10) è diretta: un gene risulta molto rilevante se è connesso a molti geni che solo viceversa connessi a pochi altri geni, ovvero maggiore è il numero di vicini con grado basso, maggiore è l'indice di rilevanza di un gene.

Nel Capitolo 7 è inoltre presentata un'applicazione del modello sopra descritto ad un dataset reale. Un'analisi viene condotta su un network di co-espressione genica derivante da un esperimento di microarray, relativo ad un campione di pazienti affetti da adenocarcinoma, e infine un confronto dei risultati ottenuti dal nostro indice con alcune classiche misure di centralità conclude l'analisi.

Le due applicazioni sopra descritte, relative a due ambiti di ricerca molto diversi tra di loro come la teoria dell'argomentazione e la biomedicina, mostrano come la flessibilità del nostro modello lo renda un utile strumento per l'analisi di una varietà di situazioni e problemi reali, anche grazie al fatto che per particolari sottoclassi è possibile una semplice ed efficiente analisi delle relative soluzioni.

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## Résumé

Cette thèse traite de l'analyse théorique et l'application d'une nouvelle famille de jeux coopératifs, où la valeur de chaque coalition peut être calculée à partir des contributions des joueurs par un opérateur additif qui décrit comme les capacités individuelles interagissent au sein de groupes. Précisément, on introduit une grande classe de jeux, les Generalized Additive Games, qui embrasse plusieurs classes de jeux coopératifs dans la littérature, et en particulier de graph games, où un réseau décrit les restrictions des possibilités d'interaction entre les joueurs. Des propriétés et solutions pour cette classe de jeux sont étudiées, avec l'objectif de fournir des outils pour l'analyse de classes de jeux connues, ainsi que pour la construction de nouvelles classes de jeux avec des propriétés intéressantes d'un point de vue théorique. De plus, on introduit une classe de solutions pour les communication situations, où la formation d'un réseau est décrite par un mécanisme additif, et dans la dernière partie de cette thèse on présente des approches avec notre modèle à des problèmes réels modélisés par des graph games, dans les domaines de la théorie de l'argumentation et de la biomédecine.

## Mots Clés

théorie de jeux coopératifs, graph games, solutions pour jeux coalitionnels, approche axiomatique, communication situations, théorie de l'argumentation, centrality in gene networks

## Abstract

This thesis deals with the theoretical analysis and the application of a new family of cooperative games, where the worth of each coalition can be computed from the contributions of single players via an additive operator describing how the individual abilities interact within groups. Specifically, we introduce a large class of games, namely the Generalized Additive Games, which encompasses several classes of cooperative games from the literature, and in particular of graph games, where a network describes the restriction of the interaction possibilities among players. Some properties and solutions of such class of games are studied, with the objective of providing useful tools for the analysis of known classes of games, as well as for the construction of new classes of games with interesting properties from a theoretic point of view. Moreover, we introduce a class of solution concepts for communication situations, where the formation of a network is described by means of an additive pattern, and in the last part of the thesis we present two approaches using our model to real-world problems described by graph games, in the fields of Argumentation Theory and Biomedicine.

## Keywords

cooperative game theory, graph games, solutions for TU-games, axiomatic approach, communication situations, argumentation theory, centrality in gene networks